# New Rayleigh Flexible Weibull Extension (RFWE) Distribution with Applications to Real and Simulated Data 

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#### Abstract

The Rayleigh flexible Weibull extension (RFWE) distribution, a new three-parameter model introduced in this paper, is a generalization of the flexible Weibull extension. This model produces best fit for failure time of electronic device obtained from power-linkage voltage spikes during electronic storms. We derive the statistical properties of the RFWE distribution. The parameters of this new distribution are estimated using the maximum likelihood method, which also yielded asymptotic confidence bounds. This model is examined using both real and simulated data. Under various priors, an additional Bayesian estimate is also carried out. The Bayes estimates and other posterior results are calculated using simulations.


## 1. Introduction

To model the lifetime components, Weibull distribution is very useful in fields like physics and engineering. Farooq et al. [1] focused on investigations to derive a new probability model for data sets with extreme values in engineering. Ijaz et al. [2] developed a new modification with three parameters of the Lomax distribution. Kumaraswamy Weibull distribution is studied by Corderio et al. [3]. Moreover, El-Morshedy et al. [4] proposed a three-parameter model by exponentiating the inverse flexible Weibull extension distribution. They called it exponentiated inverse flexible Weibull extension (EIFW) distribution. Manisha and Tiensuwan [5] introduced a beta transmuted Weibull distribution, which contains several distributions as special cases, and properties of the distribution are also discussed. Mustafa et al. [6] introduced a four-parameter model called the Weibull generalized flexible Weibull extension (WGFWE) distribution which exhibits a bathtub-shaped hazard rate. Bebbington et al. [7] discussed applications of the flexible Weibull distribution that includes life testing experiments and applied statistics. Nadarajah and Kotz [8] and Murthy et al. [9] discussed the extensions of the Weibull distribution.

In this article, a new generalization of the flexible Weibull extension (FWE) distribution called Rayleigh flexible Weibull extension (RFWE) distribution is constructed by using a method developed by Alzaatreh et al. [10] to generate a family of distributions. This class of distributions is defined as

$$
\begin{equation*}
G(z)=\int_{0}^{-\ln (1-F(z))} g(x) d x \tag{1}
\end{equation*}
$$

Alzaatreh et al. [10] derived the Weibull-Pareto distribution by taking $g(x)$ to be the probability density function (pdf) of the Weibull distribution and $F(\mathrm{z})$ to be the cumulative density function (cdf) of the Pareto distribution, and Alzaatreh et al. [11] derived the gamma-normal distribution by taking $g(x)$ to be the pdf of the gamma distribution and $F(\mathrm{z})$ to be the cdf of the normal distribution.

Now, we consider pdf $g(x)$ of Rayleigh distribution as a parent distribution given as

$$
\begin{equation*}
g(x)=2 \theta x \mathrm{e}^{-\theta x^{2}} \tag{2}
\end{equation*}
$$

Substituting Equation (2) in Equation (1), we get

$$
\begin{equation*}
G(z)=1-\exp \left[-\lambda[-\ln \{1-F(z)\}]^{2}\right] \tag{3}
\end{equation*}
$$

and pdf

$$
\begin{equation*}
g(z)=\frac{2 \lambda f(z)}{\{1-F(z)\}}[-\ln \{1-F(z)\}] \exp \left[-\lambda[-\ln \{1-F(z)\}]^{2}\right] \tag{4}
\end{equation*}
$$

In the above expression, the FWE distribution is used as $f(\mathrm{z})$ as follows:
$f(z)=\left(\beta+\frac{\gamma}{z^{2}}\right) \exp \left(\beta \mathrm{z}-\frac{\gamma}{z}\right) \exp \left\{-\exp \left(\beta \mathrm{z}-\frac{\gamma}{z}\right)\right\}, \quad z>0, \beta, \gamma>0$,
and cdf

$$
\begin{equation*}
F(z)=1-\exp \left\{-\exp \left(\beta z-\frac{\gamma}{z}\right)\right\}, \quad z>0, \beta, \gamma>0 \tag{6}
\end{equation*}
$$

## 2. Rayleigh Flexible Weibull Extension Distribution (RFWE)

If $Z$ follows the flexible Weibull extension distribution with the pdf given in Equation (5). Then, from Equation (4), the pdf of RFWE distribution is defined as

$$
\begin{align*}
g(z)= & 2 \lambda\left(\beta+\frac{\gamma}{z^{2}}\right) \exp \left\{2\left(\beta z-\frac{\gamma}{z}\right)\right\} \exp \\
& \cdot\left[-\lambda\left\{\exp \left(\beta z-\frac{\gamma}{z}\right)\right\}^{2}\right], \quad z>0, \beta, \gamma, \lambda>0 \tag{7}
\end{align*}
$$

From (3), we obtain the cdf of RFWE distribution as
$G(z)=1-\exp \left[-\lambda\left\{\exp \left(\beta z-\frac{\gamma}{z}\right)\right\}^{2}\right], \quad z>0, \beta, \gamma, \lambda>0$.
2.1. Survival and Hazard Functions. From Equation (8), we can define the survival function (sf) of RFWE distribution as follows:
$S(z)=1-G(z)=\exp \left[-\lambda\left\{\exp \left(\beta z-\frac{\gamma}{z}\right)\right\}^{2}\right], \quad z>0, \beta, \gamma, \lambda>0$.

From Equations (7)-(9), we can define the hazard function (hf) of RFWE distribution as follows:

$$
\begin{equation*}
h(z)=\frac{g(z)}{S(z)}=2 \lambda\left(\beta+\frac{\gamma}{z^{2}}\right) \exp \left\{2\left(\beta z-\frac{\gamma}{z}\right)\right\}, \quad z>0, \beta, \gamma, \lambda>0 . \tag{10}
\end{equation*}
$$

This hazard function (hf) is plotted along with the pdf (7), cdf (8), and survival function (9) in Figure 1 for different
values of the parameters to depict the nature of the proposed distribution.

## 3. Statistical Properties of RFWE Distribution

In this section, we study the quantile function, $r$ th moment, moment generating function, characteristic function, probability generating function, and factorial moment generating function.
3.1. Quantile Function. For a positive continuous random variable, Let $Z$ follows the RFWE distribution, then, the quantile function of $Z$ is derived as

$$
\begin{align*}
& P\left(Z \leq z_{q}\right)=q, \quad 0<q<1 \\
& 1-\exp \left[-\lambda\left\{\exp \left(\beta z_{q}-\frac{\gamma}{z_{q}}\right)\right\}^{2}\right]=q . \tag{11}
\end{align*}
$$

By solving the above equation, we get $z_{q}$ as follows:
$z_{q}=\frac{1}{2 \beta}\left[\ln \left\{\sqrt{\frac{\ln (1 / 1-q)}{\lambda}}\right\} \pm \sqrt{\left\{\ln \left(\sqrt{\frac{\ln (1 / 1-q)}{\lambda}}\right)\right\}^{2}+4 \beta \gamma}\right]$.

Since quantile $z_{q}$ is positive, then we get $z_{q}$ as follows:
$z_{q}=\frac{1}{2 \beta}\left[\ln \left\{\sqrt{\frac{\ln (1 / 1-q)}{\lambda}}\right\}+\sqrt{\left\{\ln \left(\sqrt{\frac{\ln (1 / 1-q)}{\lambda}}\right)\right\}^{2}+4 \beta \gamma}\right]$.

The median of RFWE can be obtained from Equation (13) by taking $q=1 / 2$. That is,

Median $=\frac{1}{2 \beta}\left[\ln \left\{\sqrt{\frac{\ln (1 / 1-(1 / 2))}{\lambda}}\right\}+\sqrt{\left\{\ln \left(\sqrt{\frac{\ln (1 / 1-(1 / 2))}{\lambda}}\right)\right\}^{2}+4 \beta \gamma}\right]$.

Now, to generate data from the cdf of RFWE, first generate the random numbers from standard uniform distribution then transform the random numbers into inverse cdf function. The inverse cdf function of RFWE can be written as

$$
\begin{equation*}
G_{\mathrm{RFWE}}(z ; \beta, \gamma, \lambda)=u \tag{15}
\end{equation*}
$$

where $U$ is a standard uniform random variable; from (8), we get

$$
\begin{equation*}
1-\exp \left[-\lambda\left\{\exp \left(\beta z-\frac{\gamma}{z}\right)\right\}^{2}\right]=u \tag{16}
\end{equation*}
$$



Figure 1: Visual plots for the pdf, cdf, sf, and hf of the proposed model.

In Equation (13), if we replace $q$ by $u$, we get the inverse function
$z_{u}=\frac{1}{2 \beta}\left[\ln \left\{\sqrt{\frac{\ln (1 / 1-u)}{\lambda}}\right\}+\sqrt{\left\{\ln \left(\sqrt{\frac{\ln (1 / 1-u)}{\lambda}}\right)\right\}^{2}+4 \beta \gamma}\right]$.
3.2. The Moments. Moments are very important for the statistical analysis to study the average, variation, skewness, and kurtosis.

The $r$ th moment of RFWE distribution is introduced by the following theorem.

Theorem 1. The $r^{\text {th }}$ moment of a random variable $Z \sim$ $\operatorname{RFWE}(\Theta)$, where $\Theta=(\beta, \gamma, \lambda)$ is given by

$$
\begin{align*}
\mu_{r}^{\prime}= & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j} \Gamma(r-j-1)}{i!j!(\beta)^{r-j} 2^{\gamma-2 j}(i+1)^{r-2 j+1}}  \tag{18}\\
& \cdot\left\{(r-j)(r-j-1)+4 \gamma \beta(i+1)^{2}\right\} .
\end{align*}
$$

Proof. The $r$ th moment of the random variable $Z$ with $g(z)$ is defined as

$$
\begin{equation*}
\mu_{r}^{\prime}=\int_{0}^{\infty} z^{r} g(z) d z \tag{19}
\end{equation*}
$$

By putting Equation (7) into Equation (19), we get

$$
\begin{align*}
& \mu_{r}^{\prime}= 2 \lambda \int_{0}^{\infty} z^{r}\left(\beta+\frac{\gamma}{z^{2}}\right) \exp \left\{2\left(\beta z-\frac{\gamma}{z}\right)\right\} \exp  \tag{20}\\
& \cdot\left[-\lambda\left\{\exp \left(\beta z-\frac{\gamma}{z}\right)\right\}^{2}\right] d z
\end{align*}
$$

using the Taylor series

$$
\begin{equation*}
\exp (-\lambda x)=\sum_{i=0}^{\infty}(-1)^{i} \frac{(x \lambda)^{i}}{i!} \tag{21}
\end{equation*}
$$

Take $x=\{\exp (\beta z-(\gamma / z))\}^{2}$ in the above equation:
$\exp \left[-\lambda\left\{\exp \left(\beta z-\frac{\gamma}{z}\right)\right\}^{2}\right]=\sum_{i=0}^{\infty}(-1)^{i} \frac{\lambda^{i}\left[\{\exp (\beta z-(\gamma / z))\}^{2}\right]^{i}}{i!}$,

Equation (20) becomes

$$
\begin{align*}
\mu_{r}^{\prime}= & 2 \sum_{i=0}^{\infty}(-1)^{i} \frac{\lambda^{i+1}}{i!} \int_{0}^{\infty} z^{r}\left(\beta+\frac{\gamma}{z^{2}}\right) \exp \\
& \cdot\left\{2\left(\beta z-\frac{\gamma}{z}\right)\right\}\left[\left\{\exp \left(\beta z-\frac{\gamma}{z}\right)\right\}^{2}\right]^{i} d z \tag{23}
\end{align*}
$$

By solving the above equation, we obtain

$$
\begin{align*}
\mu_{r}^{\prime}= & 2 \sum_{i=0}^{\infty}(-1)^{i} \frac{\lambda^{i+1}}{i!} \int_{0}^{\infty} z^{r}\left(\beta+\frac{\gamma}{z^{2}}\right) \exp  \tag{24}\\
& \cdot\{2(i+1)(\beta \mathrm{z})\} \exp \left\{-2(i+1)\left(\frac{\gamma}{z}\right)\right\} d z
\end{align*}
$$

Using the series expansion of $\exp \{-2(i+1)(\gamma / z)\}$, we obtain

$$
\begin{align*}
\mu_{r}^{\prime}= & 2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j} 2^{j}(i+1)^{j} \gamma^{j} \lambda^{i+1}}{i!j!}  \tag{25}\\
& \cdot \int_{0}^{\infty} z^{r-j}\left(\beta+\frac{\gamma}{z^{2}}\right) \exp \{2(i+1)(\beta z)\} d z
\end{align*}
$$

by using the gamma function in the form

$$
\begin{equation*}
\Gamma(\alpha)=\beta^{\alpha} \int_{0}^{\infty} \exp (t \beta) t^{\alpha-1} d t, \quad \alpha, \beta>0 \tag{26}
\end{equation*}
$$

Finally, we get the $r$ th moment of RFWE in the form

$$
\begin{align*}
\mu_{r}^{\prime}= & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j} \Gamma(r-j-1)}{i!j!(\beta)^{r-j} 2^{\gamma-2 j}(i+1)^{r-2 j+1}}  \tag{27}\\
& \cdot\left\{(r-j)(r-j-1)+4 \gamma \beta(i+1)^{2}\right\} .
\end{align*}
$$

3.3. Moment Generating Function. In this section, we derive the moment generating function of RFWE distribution.

Theorem 2. The moment generating function $M_{Z}(t)$ of a random variable $Z$, that is, $Z \sim \operatorname{RFWE}(\Theta)$, where $\Theta=(\beta, \gamma$, $\lambda$ ) is given by

$$
\begin{align*}
M_{z}(t)= & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j} t^{m} \Gamma(m-j-1)}{i!!m!(\beta)^{m-j}(i+1)^{m-2 j+1} 2^{m-2 j}} \\
& \cdot\{(m-j)(m-j-1)\} \\
& +\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j} t^{m} \Gamma(m-j-1)}{i!j!m!(\beta)^{m-j}(i+1)^{m-2 j+1} 2^{m-2 j}}\left\{4 \gamma \beta(i+1)^{2}\right\} . \tag{28}
\end{align*}
$$

Proof. The moment generating function $M_{Z}(t)$ is defined as

$$
\begin{equation*}
M_{Z}(t)=\int_{0}^{\infty} \exp (t z) g(z) d z \tag{29}
\end{equation*}
$$

Using series expansion of $e^{t z}$, we obtain

$$
\begin{equation*}
M_{Z}(t)=\sum_{m=0}^{\infty} \frac{t^{m}}{m!} \int_{0}^{\infty} z^{m} g(z) d z=\sum_{m=0}^{\infty} \frac{t^{m}}{m!} \mu_{m}^{\prime} \tag{30}
\end{equation*}
$$

Substituting from Equation (27) into Equation (30), we get

$$
\begin{align*}
M_{z}(t)= & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j} t^{m} \Gamma(m-j-1)}{i!!m!(\beta)^{m-j}(i+1)^{m-2 j+1} 2^{m-2 j}} \\
& \cdot\{(m-j)(m-j-1)\} \\
& +\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j} t^{m} \Gamma(m-j-1)}{i!j!m!(\beta)^{m-j}(i+1)^{m-2 j+1} 2^{m-2 j}}\left\{4 \gamma \beta(i+1)^{2}\right\} . \tag{31}
\end{align*}
$$

Hence, the proof.
3.4. Characteristic Function. The characteristic function of RFWE distribution can be given as follows:

$$
\begin{align*}
\phi_{z}(t)= & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j}(i t)^{m} \Gamma(m-j-1)}{i!j!m!(\beta)^{m-j}(i+1)^{m-2 j+1} 2^{m-2 j}} \\
& \cdot\{(m-j)(m-j-1)\} \\
& +\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j}(i t)^{m} \Gamma(m-j-1)}{i!j!m!(\beta)^{m-j}(i+1)^{m-2 j+1} 2^{m-2 j}}\left\{4 \gamma \beta(i+1)^{2}\right\} . \tag{32}
\end{align*}
$$

3.5. Probability Generating Function. The probability generating function of RFWE distribution can be given as follows:

$$
\begin{align*}
G(\alpha)= & E\left(\alpha^{z}\right) \\
G(\alpha)= & \int_{0}^{\infty} \exp \{z \ln (\alpha)\} g(z) d z \\
G(\alpha)= & 2 \lambda \int_{0}^{\infty} \exp \{z \ln (\alpha)\}\left(\beta+\frac{\gamma}{z^{2}}\right) \exp \left\{2\left(\beta \mathrm{z}-\frac{\gamma}{z}\right)\right\} \exp \\
& \cdot\left[-\lambda\left\{\exp \left(\beta \mathrm{z}-\frac{\gamma}{z}\right)\right\}^{2}\right] d z \\
G(\alpha)= & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j}\left(\ln ^{m}(\alpha)\right) \Gamma(m-j-1)}{i!j!m!(\beta)^{m-j}(i+1)^{m-2 j+1} 2^{m-2 j}} \\
& \cdot\{(m-j)(m-j-1)\} \\
& +\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j}\left(\ln ^{m}(\alpha)\right) \Gamma(m-j-1)}{i!j!m!(\beta)^{m-j}(i+1)^{m-2 j+1} 2^{m-2 j}} \\
& \cdot\left\{4 \gamma \beta(i+1)^{2}\right\} . \tag{33}
\end{align*}
$$

3.6. Factorial Moment Generating Function. The factorial moment generating function of RFWE distribution can be given as follows:

$$
\begin{align*}
H_{0}(\delta)= & E\left((1+\delta)^{z}\right), \\
H_{0}(\delta)= & \int_{0}^{\infty} \exp \{z \ln (1+\delta)\} g(z ; \beta, \gamma, \lambda) d z, \\
H_{0}(\delta)= & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j}\left(\ln ^{m}(1+\delta)\right) \Gamma(m-j-1)}{i!j!m!(\beta)^{m-j}(i+1)^{m-2 j+1} 2^{m-2 j}} \\
& \cdot\{(m-j)(m-j-1)\}+\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \\
& \times \frac{(-1)^{i+j}(\lambda)^{i+1}(\gamma)^{j}\left(\ln ^{m}(1+\delta)\right) \Gamma(m-j-1)}{i!j!m!(\beta)^{m-j}(i+1)^{m-2 j+1} 2^{m-2 j}} \\
& \cdot\left\{4 \gamma \beta(i+1)^{2}\right\} . \tag{34}
\end{align*}
$$

## 4. Renyi Entropy

Entropy is widely used in physics; Renyi entropy is one of the famous measures introduced by Renyi [12]. It is used to measure the unpredictability of a distribution, high entropy exhibits more uncertainty, and low entropy shows minimum uncertainty or another word more informative. Zero value of entropy indicates surety of completely certain information.

Let $Z \sim \operatorname{RFWE}(\beta, \gamma, \lambda)$; then, the corresponding Renyi entropy can be obtained as

$$
\begin{align*}
H(\rho)= & \frac{1}{1-\rho} \log \left[\int_{0}^{\infty}\{g(z ; \beta, \gamma, \lambda)\}^{\rho} d z\right] \\
H(\rho)= & \frac{1}{1-\rho} \log \sum_{i=0}^{\rho} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \\
& \times \frac{\binom{\rho}{i}(-1)^{k} \gamma^{i}(2 \lambda)^{\rho+k} \rho^{k} \beta^{\rho-i+j}(2 \rho+k)^{j}}{j!k!}  \tag{35}\\
& \cdot\left(\frac{\Gamma(i-j+1)}{\gamma(2 \rho+k)^{i-j+1}}\right)
\end{align*}
$$

## 5. Classical Estimation

In this section, we discussed the estimation of RFWE parameters by using the method of maximum likelihood and asymptotic confidence bounds.
5.1. Maximum Likelihood Estimators (MLEs). Let $Z_{1}, Z_{2}, \cdots$ , $Z_{k}$ be independent and identical random sample of size $k$ from RFWE $(\beta, \gamma, \lambda)$ with observed values $z_{1}, z_{2}, \cdots, z_{k}$; then the likelihood function can be written as

$$
\begin{equation*}
L=\prod_{i=1}^{k} g\left(z_{i} ; \beta, \gamma, \lambda\right) \tag{36}
\end{equation*}
$$

Substituting from Equation (7) into Equation (36), we get

$$
\begin{align*}
L= & \prod_{i=1}^{k}\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp \\
& \cdot\left[-\lambda\left\{\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\}^{2}\right] . \tag{37}
\end{align*}
$$

The log-likelihood function can be written as

$$
\begin{align*}
\ln L= & k \ln 2+k \ln \lambda+\sum_{i=1}^{k}\left\{\ln \left(\beta+\frac{\gamma}{z_{i}^{2}}\right)\right\}+2 \beta \sum_{i=1}^{k} z_{i} \\
& -2 \gamma \sum_{i=1}^{k}\left(\frac{1}{z_{i}}\right)-2 \lambda \sum_{i=1}^{k}\left\{\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} . \tag{38}
\end{align*}
$$

The MLEs of the parameters are obtained by differentiating the log-likelihood function with respect to the parameters $\beta, \gamma, \lambda$ and setting the result to zero.

$$
\begin{aligned}
& \frac{\partial \ln L}{\partial \beta}=\sum_{i=1}^{k}\left(\frac{z_{i}^{2}}{\beta z_{i}^{2}+\gamma}\right)+2 \sum_{i=1}^{k} z_{i}-2 \lambda \sum_{i=1}^{k}\left\{z_{i} \exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\}, \\
& \frac{\partial \ln L}{\partial \gamma}=\sum_{i=1}^{k}\left(\frac{1}{\beta z_{i}^{2}+\gamma}\right)-2 \sum_{i=1}^{k}\left(\frac{1}{z_{i}}\right)+2 \lambda \sum_{i=1}^{k}\left\{\frac{\exp \left(\beta z_{i}-\left(\gamma / z_{i}\right)\right)}{z_{i}}\right\},
\end{aligned}
$$

$$
\begin{equation*}
\frac{\partial \ln L}{\partial \lambda}=\frac{k}{\lambda}-2 \sum_{i=1}^{k}\left\{\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} . \tag{39}
\end{equation*}
$$

There is no closed-form solution to the above equations. Therefore, R software and Mathematica are used to get the numerical solutions.
5.2. Asymptotic Confidence Bounds. Now, we obtain the asymptotic confidence interval of the unknown parameters $\beta, \gamma, \lambda$. We assume that the $\operatorname{MLEs}(\beta, \gamma, \lambda)$ are approximately multivariate normal with mean $(\beta, \gamma, \lambda)$ and covariance matrix $I_{0}^{-1}$, where $I_{0}^{-1}$ is the inverse of the observed information matrix defined as

$$
\begin{align*}
& I_{0}^{-1}=-\left(\begin{array}{lll}
\frac{\partial^{2} \ln L}{\partial \beta^{2}} & \frac{\partial^{2} \ln L}{\partial \beta \partial \gamma} & \frac{\partial^{2} \ln L}{\partial \beta \partial \lambda} \\
\frac{\partial^{2} \ln L}{\partial \gamma \partial \beta} & \frac{\partial^{2} \ln L}{\partial \gamma^{2}} & \frac{\partial^{2} \ln L}{\partial \gamma \partial \lambda} \\
\frac{\partial^{2} \ln L}{\partial \lambda \partial \beta} & \frac{\partial^{2} \ln L}{\partial \lambda \partial \gamma} & \frac{\partial^{2} \ln L}{\partial \lambda^{2}}
\end{array}\right)^{-1}, \\
&\left(\begin{array}{lll}
\operatorname{var}(\widehat{\beta}) & \operatorname{cov}(\widehat{\beta}, \widehat{\gamma}) & \operatorname{cov}(\widehat{\beta}, \widehat{\lambda}) \\
\operatorname{cov}(\widehat{\gamma}, \widehat{\beta}) & \operatorname{var}(\widehat{\gamma}) & \operatorname{cov}(\widehat{\gamma}, \widehat{\lambda}) \\
\operatorname{cov}(\hat{\lambda}, \widehat{\beta}) & \operatorname{cov}(\hat{\lambda}, \widehat{\gamma}) & \operatorname{var}(\widehat{\lambda})
\end{array}\right) . \tag{40}
\end{align*}
$$

The second partial derivatives included in $I_{0}^{-1}$ are given as follows:

$$
\begin{align*}
& \frac{\partial^{2} \ln L}{\partial \lambda^{2}}=-\frac{k}{\lambda^{2}} \\
& \frac{\partial^{2} \ln L}{\partial \lambda \partial \gamma}=2 \sum_{i=1}^{k}\left\{\frac{\exp \left(\beta z_{i}-\left(\gamma / z_{i}\right)\right)}{\mathrm{z}_{i}}\right\}, \\
& \frac{\partial^{2} \ln L}{\partial \beta^{2}}=\sum_{i=1}^{k}\left\{\frac{z_{i}^{4}}{\left(\beta z_{i}^{2}+\gamma\right)^{2}}\right\}-2 \lambda \sum_{i=1}^{k} z_{i}^{2} \exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right), \\
& \frac{\partial^{2} \ln L}{\partial \gamma^{2}}=-\sum_{i=1}^{k}\left\{\frac{1}{\left(\beta z_{i}^{2}+\gamma\right)^{2}}\right\}-2 \lambda \sum_{i=1}^{k}\left\{\frac{\exp \left(\beta z_{i}-\left(\gamma / z_{i}\right)\right)}{z_{i}^{2}}\right\}, \\
& \frac{\partial^{2} \ln L}{\partial \beta \partial \gamma}=-\sum_{i=1}^{k}\left\{\frac{z_{i}^{2}}{\left(\beta z_{i}^{2}+\gamma\right)^{2}}\right\}+\lambda \sum_{i=1}^{k}\left\{\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\}, \\
& \frac{\partial^{2} \ln L}{\partial \lambda \partial \beta}=-2 \sum_{i=1}^{k}\left\{z_{i} \exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} . \tag{41}
\end{align*}
$$

The above expressions are used to derive the $(1-\delta)$ $100 \%$ confidence intervals for the parameters $\beta, \gamma$, and $\lambda$ as in the following forms:

$$
\begin{align*}
& \widehat{\beta} \pm z_{(\delta / 2)} \sqrt{\operatorname{var}(\hat{\beta})} \\
& \widehat{\gamma} \pm z_{\delta / 2} \sqrt{\operatorname{var}(\widehat{\gamma})}  \tag{42}\\
& \widehat{\lambda} \pm z_{\delta / 2} \sqrt{\operatorname{var}(\hat{\lambda})}
\end{align*}
$$

where $z_{\delta / 2}$ is the upper $(\delta / 2)$ th percentile of the standard normal distribution.

## 6. Order Statistics

Let $Z_{1: k}, Z_{2: k}, \cdots, Z_{k: k}$ denote the order statistics obtained from a random sample $Z_{1}, Z_{2}, \cdots, Z_{k}$ taken from a continuous population with $\operatorname{cdf} G(z, \Phi)$ and $\operatorname{pdf} g(z, \Phi)$ as follows:

$$
\begin{equation*}
g_{i: k}(z, \Phi)=\frac{1}{B(i, n-i+1)} g(z, \Phi)\{G(z, \Phi)\}^{i-1}\{1-G(z, \Phi)\}^{k-i} \tag{43}
\end{equation*}
$$

where $B(i, n-i+1)$ is the beta function.
6.1. PDF of Minimum, Median, and Maximum Order Statistics. In this subsection, we will consider the expression for the sample distribution of the minimum, median, and the maximum order statistics when a random sample of size $k$ is drawn from the $\operatorname{RFWE}(\beta, \gamma, \lambda)$ distribution. These pdfs can be obtained by solving Equation (43).

The pdf of the minimum order statistics is as follows:

$$
\begin{align*}
g_{1: k}(z)= & 2 \lambda k\left(\beta+\frac{\gamma}{z^{2}}\right) \exp \left\{2\left(\beta z-\frac{\gamma}{z}\right)\right\} \\
& \cdot\left[\exp \left\{-\lambda\left(\exp \left(\beta z-\frac{\gamma}{z}\right)\right)^{2}\right\}\right]^{k} \tag{44}
\end{align*}
$$

The pdf of the median order statistics is as follows:

$$
\begin{align*}
g_{m+1: k}(\tilde{z})= & \frac{(2 m+1)!}{m!m!} 2 \lambda\left(\beta+\frac{\gamma}{z^{2}}\right) \exp \left\{2\left(\beta z-\frac{\gamma}{z}\right)\right\} \\
& \cdot\left[\exp \left\{-\lambda\left(\exp \left(\beta z-\frac{\gamma}{z}\right)\right)^{2}\right\}\right]^{m+1} \\
& \times\left\{1-\exp \left\{-\lambda\left(\exp \left(\beta z-\frac{\gamma}{z}\right)\right)^{2}\right\}\right\}^{m} \tag{45}
\end{align*}
$$

Finally, the pdf of the maximum order statistics is as follows:

$$
\begin{align*}
g_{k: k}(z)= & 2 k \lambda\left(\beta+\frac{\gamma}{z^{2}}\right) \exp \left\{2\left(\beta z-\frac{\gamma}{z}\right)\right\} \exp \\
& \cdot\left\{-\lambda\left(\exp \left(\beta z-\frac{\gamma}{z}\right)\right)^{2}\right\}  \tag{46}\\
& \times\left\{1-\exp \left\{-\lambda\left(\exp \left(\beta z-\frac{\gamma}{z}\right)\right)^{2}\right\}\right\}^{k-1}
\end{align*}
$$

6.2. Joint PDF of the Minimum and Maximum Order Statistics. In this subsection, we will consider the expression for the joint pdf of the minimum and the maximum order statistics when a random sample of size $k$ is drawn from the RFWE $(\beta, \gamma, \lambda)$ distribution. These pdfs can be obtained by solving the following equation:

$$
\begin{align*}
g_{i: j: k}\left(z_{i}, z_{j}\right)= & C\left[G\left(z_{i}\right)\right]^{i-1}\left[G\left(z_{j}\right)-G\left(z_{i}\right)\right]^{j-i-1}  \tag{47}\\
& \cdot\left[1-G\left(z_{j}\right)\right]^{k-j} g\left(z_{i}\right) g\left(z_{j}\right),
\end{align*}
$$

where

$$
\begin{equation*}
C=\frac{k!}{(i-1)!(j-i-1)!(k-j)!} \tag{48}
\end{equation*}
$$

The joint pdf of the minimum and maximum, that is, $i^{\text {th }}$ and $j^{\text {th }}$-order statistics from RFWE distribution, is

$$
\begin{align*}
g_{i \cdot \mathrm{j}: k}\left(z_{i}, z_{j}\right)= & 4 C \lambda^{2}\left[1-\exp \left(-\lambda\left\{\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\}^{2}\right)\right]^{i-1} \\
& \times\left[\exp \left[-\lambda\left\{\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\}^{2}\right]\right. \\
& \left.-\exp \left\{-\lambda\left(\exp \left(\beta z_{j}-\frac{\gamma}{z_{j}}\right)\right)^{2}\right\}\right]^{j-i-1} \\
& \times\left[\exp \left\{-\lambda\left(\exp \left(\beta z_{j}-\frac{\gamma}{z_{j}}\right)\right)^{2}\right\}\right]^{k-j} \\
& \cdot\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \\
& \times \exp \left[-\lambda\left\{\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\}^{2}\right]\left(\beta+\frac{\gamma}{z_{j}^{2}}\right) \tag{49}
\end{align*}
$$

For special case, let us suppose $i=1$ and $j=k$; the joint pdf of minimum and maximum order statistics is as follows:

$$
\begin{align*}
g_{1: k k k}\left(z_{1}, z_{k}\right)= & k(k-1) 4 \lambda^{2}\left[\exp \left(-\lambda\left\{\exp \left(\beta z_{1}-\frac{\gamma}{z_{1}}\right)\right\}^{2}\right)\right. \\
& \left.-\exp \left(-\lambda\left\{\exp \left(\beta z_{k}-\frac{\gamma}{z_{k}}\right)\right\}^{2}\right)\right]^{k-2} \\
& \times\left(\beta+\frac{\gamma}{z_{1}^{2}}\right) \exp \left\{2\left(\beta z_{1}-\frac{\gamma}{z_{1}}\right)\right\} \exp \\
& \cdot\left(-\lambda\left\{\exp \left(\beta z_{1}-\frac{\gamma}{z_{1}}\right)\right\}^{2}\right) \times\left(\beta+\frac{\gamma}{z_{k}^{2}}\right) \exp \\
& \cdot\left\{2\left(\beta z_{k}-\frac{\gamma}{z_{k}}\right)\right\} \exp \left(-\lambda\left\{\exp \left(\beta z_{k}-\frac{\gamma}{z_{k}}\right)\right\}^{2}\right) \\
& \times \exp \left\{2\left(\beta z_{j}-\frac{\gamma}{z_{j}}\right)\right\} \exp \left\{-\lambda\left(\exp \left(\beta z_{j}-\frac{\gamma}{z_{j}}\right)\right)^{2}\right\} . \tag{50}
\end{align*}
$$

## 7. Data Analysis

In this section, we use a real data set to show that the RFWE distribution can be a better model, compared with many known distributions such as the FWE distribution, Weibull distribution, exponential distribution, and Rayleigh distribution. Consider that the data obtained by Khan and Jan [13] represents the failure time of electronic devices obtained from power-line voltage spikes during electronic storms. The times are $2.75,0.13,1.47,0.23,1.81,0.30,0.65,0.10$, $3.00,1.73,1.06,3.00,3.00,2.12,3.00,3.00,3.00,0.02,2.61$, $2.93,0.88,2.47,0.28,1.43,3.00,0.23,3.00,0.80,2.45$, and 2.66 .

The MLEs of the unknown parameters $\beta, \gamma, \lambda$ for the different distributions are given in Table 1. We considered Anderson-Darling (AD) test statistic, Cramer-von-Misses (CM) test statistic, and Kolmogorov-Smirnov (KS) test statistic. Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC), Akaike's Information

Table 1: Goodness of fit results using AD, CM, and KS.

| Distributions | MLE's | AD | CM | KS |
| :--- | :---: | :---: | :---: | :---: |
| RFWE | $\widehat{\beta}=0.45, \widehat{\gamma}=0.04, \widehat{\lambda}=0.147$ | 1.14 | 0.17 | 0.19 |
| FEW | $\widehat{\beta}=0.328, \widehat{\gamma}=0.158$ | 2.04 | 0.32 | 0.39 |
| Weibull | $\widehat{\alpha}=1.264, \widehat{\gamma}=1.820$ | 1.82 | 0.30 | 0.21 |
| Exponential | $\widehat{\lambda}=0.564$ | 1.90 | 0.32 | 0.21 |
| Rayleigh | $\widehat{\sigma}=1.485$ | 1.64 | 0.26 | 0.21 |

Table 2: Goodness of fit results using AIC, BIC, CAIC, and HQIC.

| Distribution | AIC | BIC | CAIC | HQIC |
| :--- | :---: | :---: | :---: | :---: |
| RFWE | 82.75 | 86.95 | 83.67 | 84.09 |
| FEW | 111.31 | 114.11 | 111.75 | 112.20 |
| Weibull | 96.317 | 99.119 | 96.761 | 97.213 |
| Exponential | 96.270 | 97.671 | 96.412 | 96.718 |
| Rayleigh | 103.76 | 105.16 | 103.90 | 104.21 |

Criterion (AIC), and Consistent Akaike's Information Criterion (CAIC) as investigative measures are given in Table 2.
7.1. Discussion. From the above tables, the test statistics above mentioned showed smaller values for the RFWE distribution as compared with the rest of the distributions that indicate better performance of the model.

## 8. Simulations

The accuracy and the performance in estimating the model parameters with different sample sizes are investigated by using Equation (17) to generate the data from the RFWE distribution. We take $n=20,60$, and 200, and the true values of the parameters are given in Tables 3 and 4; for each $n$, we simulate 100 random data sets from the RFWE distribution and for different values of the parameters, average bias (bias), mean square errors (MSE), estimated mean (Est.), and variance (Var) are calculated for different sample sizes.
8.1. Discussion. As we increase the sample size, it is clearly noticed from the above tables that the bias and MSE tend to decrease, Est. tends towards the true parameter values, and Var also tends to decrease.

## 9. Bayesian Analyses

In this section, we are going to estimate the parameters of the proposed distribution and that of exponential, Rayleigh, Weibull, and FWEx distributions under different prior distributions. The posterior variances are calculated, and the credible intervals are provided for each parameter. To compare our proposed distribution with the others considered, we calculate difference information criteria (DIC). Muhammad et al. [14] used the DIC, and their scheme is better with a minimum number of parameters.

Table 3: True value of parameters with MSE and bias of $\operatorname{RFWE}(\beta, \gamma, \lambda)$.

| $(\beta, \gamma, \lambda)$ | $n$ | $\operatorname{Bias}(\widehat{\beta})$ | $\operatorname{Bias}(\widehat{\gamma})$ | $\operatorname{Bias}(\widehat{\lambda})$ | $\operatorname{MSE}(\widehat{\beta})$ | $\operatorname{MSE}(\widehat{\gamma})$ | $\operatorname{MSE}(\widehat{\lambda})$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(0.008,0.009,0.3)$ | 20 | 0.010132 | 0.0059832 | 0.24598 | 0.0013112 | 0.00523868 | 0.125635 |
|  | 60 | 0.005630 | 0.0043496 | 0.237024 | 0.0007432 | 0.0055031 | 0.094754 |
|  | 200 | 0.000846 | -0.004023 | 0.205216 | 0.0000008 | 0.00001689 | 0.042515 |
| $(0.008,0.008,0.3)$ | 20 | 0.005679 | 0.0037737 | 0.225885 | 0.0006367 | 0.00303306 | 0.080190 |
|  | 60 | 0.005270 | -0.002586 | 0.215227 | 0.0005765 | 0.00015290 | 0.070337 |
|  | 200 | 0.002294 | -0.003548 | 0.216339 | 0.0002232 | 0.00001552 | 0.053182 |
|  | 20 | 0.051085 | 0.0658686 | 0.650023 | 0.0054512 | 0.0345581 | 0.6694 |
|  | $0.0008,0.0009,0.3)$ | 60 | 0.048204 | 0.0234407 | 0.449733 | 0.0044114 | 0.00126066 |
|  | 200 | 0.029875 | 0.0226881 | 0.319649 | 0.0018333 | 0.00119996 | 0.1859507 |
|  | 20 | 0.055087 | 0.0441444 | 0.481222 | 0.0056520 | 0.0150503 | 0.457649 |
| $(0.0008,0.0009,0.4)$ | 60 | 0.045361 | 0.0210023 | 0.259504 | 0.0036151 | 0.00104289 | 0.17942 |
|  | 200 | 0.035612 | 0.0123426 | 0.127486 | 0.0024561 | 0.00036699 | 0.015043 |

Table 4: True value of parameters with mean (Est.) and variance (Var).

| R-FWEx $(\beta, \gamma, \lambda)$ | $n$ | Est. | $\beta$ | Var | Est. | $\gamma$ | Var | Est. |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |$]$| Var |
| :--- |

The likelihood function of RFWE distribution is

$$
\begin{align*}
L= & \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp \\
& \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\} \tag{51}
\end{align*}
$$

First, we consider the uninformative uniform prior distribution for the parameters involved in Equation (51) as

$$
\begin{equation*}
p(\lambda, \beta, \gamma) \propto 1 \tag{52}
\end{equation*}
$$

The joint posterior distribution under uniform prior distribution given in Equation (52) is as follows:

$$
\begin{align*}
& p(\lambda, \beta, \gamma \mid z) \propto \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp \\
& \quad \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\} \tag{53}
\end{align*}
$$

The constant of proportionality, $C$, is obtained as

$$
\begin{align*}
C= & \int_{\gamma} \int_{\beta} \int_{\lambda} \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp \\
& \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\} d \lambda d \beta d \gamma . \tag{54}
\end{align*}
$$

The marginal posterior distribution of the parameter given data may be obtained as follows:

$$
\begin{align*}
p(\gamma \mid z)= & \frac{1}{C} \int_{\beta} \int_{\lambda} \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \\
& \cdot\left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp \\
& \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\} d \lambda d \beta, \quad 0 \leq \gamma \leq \infty,  \tag{55}\\
p(\lambda \mid z)= & \frac{1}{C} \int_{\beta} \int_{\gamma} \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \\
& \cdot\left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp \\
& \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\} d \gamma d \beta, \quad 0 \leq \lambda \leq \infty,  \tag{56}\\
p(\beta \mid z)= & \frac{1}{C} \\
& \cdot\left\{\int_{\lambda} \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \right. \\
& \left.\left\{\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp  \tag{57}\\
& \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\} d \gamma d \lambda, \quad 0 \leq \beta \leq \infty .
\end{align*}
$$

As the posterior distribution given in Equations (55)-(57) are not analytically tractable, so we use MCMC simulations for the calculation of the Bayes estimates (BE) and other posterior results.

Furthermore, we now consider an informative prior distribution, gamma distribution, for the parameters. The joint gamma prior is the simple product of the marginal gamma distribution and is given as

$$
\begin{align*}
p(\lambda, \beta, \gamma)= & \frac{b_{1}^{a_{1}}}{\Gamma\left(a_{1}\right)} \frac{b_{2}^{a_{2}}}{\Gamma\left(a_{2}\right)} \frac{b_{3}^{a_{3}}}{\Gamma\left(a_{3}\right)} \lambda^{a_{1}-1} \exp \\
& \cdot\left(-\lambda b_{1}\right) \beta^{a_{2}-1} \exp \left(-\beta b_{2}\right) \gamma^{a_{3}-1} \exp \left(-\gamma b_{3}\right) \tag{58}
\end{align*}
$$

where, in (58), $\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right)$, and $\left(a_{3}, b_{3}\right)$ are the hyperparameters of the prior distribution of $\lambda, \beta$, and $\gamma$, respectively. The joint posterior distribution under Equation (58) is as follows:

$$
\begin{align*}
p(\lambda, \beta, \gamma \mid z)= & \frac{1}{C_{1}} \frac{b_{1}^{a_{1}}}{\Gamma\left(a_{1}\right)} \frac{b_{2}^{a_{2}}}{\Gamma\left(a_{2}\right)} \frac{b_{3}^{a_{3}}}{\Gamma\left(a_{3}\right)} \lambda^{a_{1}-1} \exp \\
& \cdot\left(-\lambda b_{1}\right) \beta^{a_{2}-1} \exp \left(-\beta b_{2}\right) \gamma_{3}^{a_{3}-1} \exp \left(-\gamma b_{3}\right) \\
& \times \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp  \tag{59}\\
& \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\},
\end{align*}
$$

where $C_{1}$ is known as the constant of proportionality and obtained as follows:

$$
\begin{align*}
C_{1}= & \int_{\gamma} \int_{\beta} \int_{\lambda} \frac{b_{1}^{a_{1}}}{\Gamma\left(a_{1}\right)} \frac{b_{2}^{a_{2}}}{\Gamma\left(a_{2}\right)} \frac{b_{3}^{a_{3}}}{\Gamma\left(a_{3}\right)} \lambda^{a_{1}-1} \exp \\
& \cdot\left(-\lambda b_{1}\right) \beta^{a_{2}-1} \exp \left(-\beta b_{2}\right) \gamma^{a_{3}-1} \exp \left(-\gamma b_{3}\right) \\
& \times \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp  \tag{60}\\
& \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\} d \lambda \quad d \beta \quad d \gamma .
\end{align*}
$$

The marginal posterior distribution of the parameter given data, under gamma prior distribution, is obtained by integrating Equation (59) as follows:

$$
\begin{align*}
& p(\gamma \mid z)= \frac{1}{C_{1}} \int_{\beta} \int_{\lambda} \frac{b_{1}^{a_{1}}}{\Gamma\left(a_{1}\right)} \frac{b_{2}^{a_{2}}}{\Gamma\left(a_{2}\right)} \frac{b_{3}^{a_{3}}}{\Gamma\left(a_{3}\right)} \lambda^{a_{1}-1} \exp \\
& \cdot\left(-\lambda b_{1}\right) \beta^{a_{2}-1} \exp \left(-\beta b_{2}\right) \gamma^{a_{3}-1} \exp \left(-\gamma b_{3}\right) \\
& \times \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp \\
& \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\} d \lambda d \beta, 0 \leq \gamma \leq \infty, \\
& \times \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp \\
& \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\} d \beta d \gamma, 0 \leq \lambda \leq \infty, \\
& \cdot\left(-\lambda b_{1}\right) \beta^{a_{2}-1} \exp \left(-\beta b_{2}\right) \gamma^{a_{3}-1} \exp \left(-\gamma b_{3}\right) \\
& \Gamma\left(a_{1}\right) b_{2}^{a_{1}} \\
& \Gamma\left(a_{2}\right) b_{3}^{a_{3}} \\
& \Gamma\left(a_{3}\right) \\
& C_{1}-1 \exp  \tag{61}\\
& p(\beta \mid z)= \frac{1}{C_{1}} \int_{\lambda} \int_{\gamma} \frac{b_{1}^{a_{1}}}{\Gamma\left(a_{1}\right)} \frac{b_{2}^{a_{2}}}{\Gamma\left(a_{2}\right)} \frac{b_{3}^{a_{3}}}{\Gamma\left(a_{3}\right)} \lambda^{a_{1}-1} \exp \\
& \cdot\left(-\lambda b_{1}\right) \beta^{a_{2}-1} \exp \left(-\beta b_{2}\right) \gamma^{a_{3}-1} \exp \left(-\gamma b_{3}\right) \\
& \times \prod_{i=1}^{k} 2 \lambda\left(\beta+\frac{\gamma}{z_{i}^{2}}\right) \exp \left\{2\left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right\} \exp \\
& \cdot\left\{-\lambda\left(\exp \left(\beta z_{i}-\frac{\gamma}{z_{i}}\right)\right)^{2}\right\} d \gamma d \lambda .0 \leq \beta \leq \infty .
\end{align*}
$$

These posterior distributions are summarized in Tables 5 and 6. For numerical calculation, we set the hyperparameters as $a_{1}=2, b_{1}=5, a_{1}=3, b_{1}=6$, and $a_{1}=2, b_{1}=4$.
9.1. Discussion. We noticed a reduction in the posterior standard deviations (SDs) while changing the prior from noninformative to informative prior distribution in Equation (58), and consequently, the credible intervals, that is, lower credible interval (LCL) and upper credible interval (UCL), become narrower for the all parameters in all of the distributions. Further, the DIC is minimum in gamma prior.

Table 5: Posterior results under uniform prior distribution.

| Distribution | Parameter | BE | SD | [LCL, UCL] | DIC |
| :--- | :---: | :---: | :---: | :---: | :---: |
| RFWE | $\lambda$ | 0.1612 | 0.0733 | $[0.0565,0.3393]$ |  |
|  | $\beta$ | 0.4438 | 0.0791 | $[0.3038,0.6088]$ | 219.5698 |
|  | $\gamma$ | 0.0430 | 0.0201 | $[0.0123,0.0882]$ |  |
| Rayleigh | $\lambda$ | 0.5863 | 0.1030 | $[0.4053,0.8038]$ | 248.0342 |
| Weibull | $\theta$ | 0.2341 | 0.04074 | $[0.1594,0.3162]$ | 266.7312 |
|  | $\phi$ | 1.2655 | 0.2028 | $[0.8922,1.6879]$ | 256.7191 |
|  | $\alpha$ | 1.9662 | 0.2988 | $[1.4206,2.6207]$ | 293.6861 |

Table 6: Posterior results under gamma prior distribution.

| Distribution | Parameter | BE | S.D | [LCL, UCL] | DIC |
| :--- | :---: | :---: | :---: | :---: | :---: |
| RFWE | $\lambda$ | 0.0581 | 0.0715 | $[0.0771,0.3508]$ |  |
|  | $\beta$ | 0.4100 | 0.0685 | $[0.2871,0.5583]$ | 205.1394 |
|  | $\gamma$ | 0.0581 | 0.0213 | $[0.0239,0.1065]$ |  |
| Rayleigh | $\lambda$ | 0.5484 | 0.0966 | $[0.3734,0.7625]$ | 238.2889 |
| Weibull | $\theta$ | 0.2332 | 0.0409 | $[0.1576,0.3194]$ | 253.4404 |
|  | $\phi$ | 1.0480 | 0.1702 | $[0.7487,1.4153]$ | 269.9739 |
|  | $\alpha$ | 1.5462 | 0.2596 | $[1.0749,2.0872]$ | 266.0848 |

So we conclude that the gamma prior is more suitable than the uniform prior. While using the uniform prior distribution for our proposed model, RFWE produced a minimum DIC as compared to the others considered, namely, exponential, Weibull, Rayleigh, and FWE distributions.

In fact, this value of 219.5698 is better than the DICs produced by other models under uninformative prior distribution. A decrease in the DIC is observed when we considered gamma prior distribution; and this DIC, 205.1394, is the smallest among all.

## 10. Conclusions

We propose a new three-parameter distribution, based on the idea which is to add parameters to the flexible Weibull extension distribution; this new distribution is called the Rayleigh flexible Weibull extension (RFWE) distribution. Its statistical properties are studied. We use the maximum likelihood method for estimating the parameters of the distribution. Finally, the advantage of the RFWE distribution is concluded by an application using real data set and a simulation study is conducted. From the simulation study, as the sample size
increases, the bias, MSE, and variance tend to decrease, and the estimated means tend towards the true parameter value.

Moreover, it is shown that the RFWE distribution fits better than existing known distributions in the classical approach, while our proposed, RFWE, distribution has minimum DIC under uniform and gamma priors under the Bayesian paradigm. This showed that our proposed model is better than the other consideration. Furthermore, our model performs better when informative gamma prior distributions were considered, better in terms of narrower credible intervals and minimum posterior standard variations. In this study, we used a single real data set and two priors; in the future, one can consider the other uninformative prior, like Jeffrey's prior, and in informative priors like normal, lognormal can be used for comparison purposes.

## Data Availability

All the data are available and properly cited in the paper.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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