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Research Article

Determination Rule for α, β Directions and φ in Teaching of Slip-Line Theory

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In the teaching of plastic mechanics and applications of slip-line theory using conventional methods, multivalued results are usually caused by the uncertain direction of the slip line and dip angles. Determination rules for the α and β directions and φ values are proposed to improve slip-line theory according to the particle flow law under the effect of principal stress, and slip lines and dip angles suitable for a typical stress boundary problem are described. The α and β slip lines should simultaneously point to or away from the intersection, and the synthetic direction of the slip lines should point to the first principal stress σ_1 or away from the direction of the third principal stress σ_3. When the Hencky stress equation of the α line is applied, two points on the α line should maintain the same direction, and the absolute value of the φ difference should be less than or equal to π. Moreover, the α line of two points should simultaneously point to the inner and outer normal direction of the β line when the Hencky stress equation of the β line is used. The average stress and critical load of plastic deformation in the plane lath V-notch tension are solved using slip-line theory. Both the calculated critical stress and the load maintain uniformity using different slip lines and dip angles, and the proposed determination rule reliably avoids multivalued solutions. This is important for students and researchers in correctly understanding and applying slip-line theory.

1. Introduction

The maximum shear stress of metal particles appears in orthogonal pairs in a plastic-deformation body. The trajectories of the maximum shear stress of each point are slip lines; the two groups of slip lines, α and β, form orthogonal networks known as slip-line fields. Slip-line theory is applicable mainly to plane strain problems of ideal rigid–plastic materials and is also used to solve problems concerning plane stress, axisymmetric, and hardened materials under certain conditions [1, 2].

As a main method of plastic mechanics, slip-line theory is widely used in analysis of mechanical and energy parameters of plastic deformation through determination of a typical stress boundary, α and β slip-line fields, dip angle φ, and application of the Hencky stress equations [3–5]. An analytical model is proposed for the thread-rolling process simplified as single-tooth rolling [6]. Modeling with the slip-line field theory is used to calculate the protrusion height of the workpiece; the force and torque of thread-rolling are analytically calculated using the classical slab method. The plastic deformation during the external spline-rolling process (ESRP) is concentrated in the superficial area of the workpiece [7]. The analytical models for ESRP were modeled using the slip-line field method with the Coulomb friction model, Tresca friction model, and Coulomb–Tresca friction model. Compared with the experimental and numerical results, the error of the slip-line field method (SLFM) results, including the undeformed zone of the workpiece, was less than 10% between the SLFM results and the experimental and finite-element method results.

In the past, many researchers usually used the conventional method to only determine the α, β slip lines and dip angles, and the direction of α, β slip lines and the range of φ value were not received significant attention and consideration in most textbooks and references [8–12]. However, the lack of rule description of α, β directions and range of φ value leads to the multivalued solutions easily, which makes the slip-line theory difficult for students to understand accurately in teaching and analyze correctly in engineering application. Thus, determination rules for slip-line direction and dip angle are proposed in this study to improve slip-line theory understanding and engineering application by students and researchers, and then the typical stress boundary conditions for slip lines are described in detail. Finally, the tension of a plane lath with a V-notch is solved, and a unique solution is obtained according to the determination rules. This study is significant supplement and improvement for teaching and application of the slip-line theory.

2. Conventional Questions and Establishment of Determination Rules

2.1. Hencky Stress Equations and the Multivalued Solution Questions. When the slip-line method is used to solve plastic engineering problems, the slip-line fields are drawn first, and the slip lines of α and β are determined. Subsequently, the dip angle φ and average stress σ_m of the two known points along the α and β lines are determined according to typical boundary conditions. The stress distribution law of the corresponding points is solved using the Hencky stress equations along the α or β of the slip lines.

The Hencky stress equations are described by the average stress σ_m, shear yield strength k, and dip angle φ as follows.

$$\left.\begin{aligned} \sigma_m - 2k\varphi &= C_1(\text{along lines } \alpha) \\ \sigma_m + 2k\varphi &= C_2(\text{along lines } \beta) \end{aligned}\right\}, \quad (1)$$

where C_1 and C_2 are constants along lines α and β, respectively.

When the average stress σ_m and dip angle φ of a certain point are determined using Equation (1), the stress state of the point can be determined using Equation (2).

$$\left.\begin{aligned} \sigma_x &= \sigma_{\mathrm{m}} - k\sin 2\varphi \\ \sigma_y &= \sigma_{\mathrm{m}} + k\sin 2\varphi \\ \tau_{xy} &= k\cos 2\varphi \\ \sigma_z &= \left(\sigma_x + \sigma_y\right)/2 \end{aligned}\right\}, \quad (2)$$

where σ_x is the x-direction component of the normal stress; σ_y is the y-direction component of the normal stress; σ_z is the z-direction component of the normal stress; and τ_{xy} is the shear stress in the x–y plane.

From Equations (1) and (2), the elastoplastic critical loads and plastic-deformation loads are obtained according to the stress state of the typical points. To avoid multivalued solutions, it is critical to investigate and clarify the slip-line direction and the range of dip angles. In most classical textbooks and references, there are two methods for determining the α and β of slip-line fields and one method for determining the dip angle φ. α or β lines in slip-line fields are usually determined using the following two methods [8–10].

(1) The vector of maximum shear stress on both sides of line α is clockwise, and the vector of maximum shear stress on both sides of line β is counterclockwise.

(2) The horizontal and vertical axes of a right-handed coordinate system are formed according to lines α and β, respectively. The action direction of the first principal stress σ_1 is located in the first and third quadrants, and the action direction of the third principal stress σ_3 is located in the second and fourth quadrants. The dip angle φ of each point on the slip line is determined mainly by the angle between the tangent of each point of line α and the positive direction of the x-axis. The positive direction of ox is used as the angle measurement starting line; the counterclockwise rotation angle φ is positive, and the clockwise rotation angle φ is negative.

It is easy to determine line α or β and the subsequent dip angle φ in the slip-line fields using these methods. However, a lack of clear rules for determining slip-line direction and the scope of the dip angle in these methods lead to multivalued solutions in slip-line theory, in teaching, and in actual engineering problems. As an example, Figure 1 shows a diagram of a slip-line field with known lines α and β, where the red and blue lines represent α and β, respectively. There is no clear direction for the slip lines, as shown in Figure 1(a). The dip angle of point M can be selected as φ, $\varphi - 2\pi$, $\varphi - \pi$, or $\pi + \varphi$ according to the methods for determining the dip angle. Even if the direction of the slip lines is clear, as in Figure 1(a), the dip angle values of point M can be selected as φ or $\varphi - \pi$. Multiple dip angles in a given condition lead to multivalued solutions for the stress state of point and deformed loads, leading to difficulty in correctly understanding and applying the theory.

2.2. Determination Rules. The hypothesis of element N shown in Figure 2 is only subjected to the first and third principal stresses σ_1 and σ_3; the deformation tendency and maximum shear stress k of the element body can be predicted, as shown in Figure 2(b). The slip lines are determined according to the clockwise direction formed by the maximum shear stress k on both sides of the slip line. The maximum shear stress on both sides of line α is clockwise but is counterclockwise for line β (Figure 2(c)). Thus, lines α and β at the intersection point of the boundary should point to or away from the intersection point according to the theorem that the shear stresses are equal and appear in pairs. The synthetic direction of lines α and β should be consistent with the direction of the first principal stress according to the flow tendency and stress condition of the deformed metal. In other words, lines α and β at the boundary simultaneously point in the direction of the first principal stress or away from the direction of the third principal stress, as shown in Figure 2(d).

Figure 3 shows the determination of inclination angle φ. The slip line is the track of the maximum shear stress; the rotation angle from point P_1 to point P_2 should be less than π for a continuous graded slip line without velocity singularities (Figure 3(a)). Thus, only a dip angle φ with an absolute value less than π can be used in the Hencky equations, although angles φ and $\varphi - 2\pi$ can be selected according to

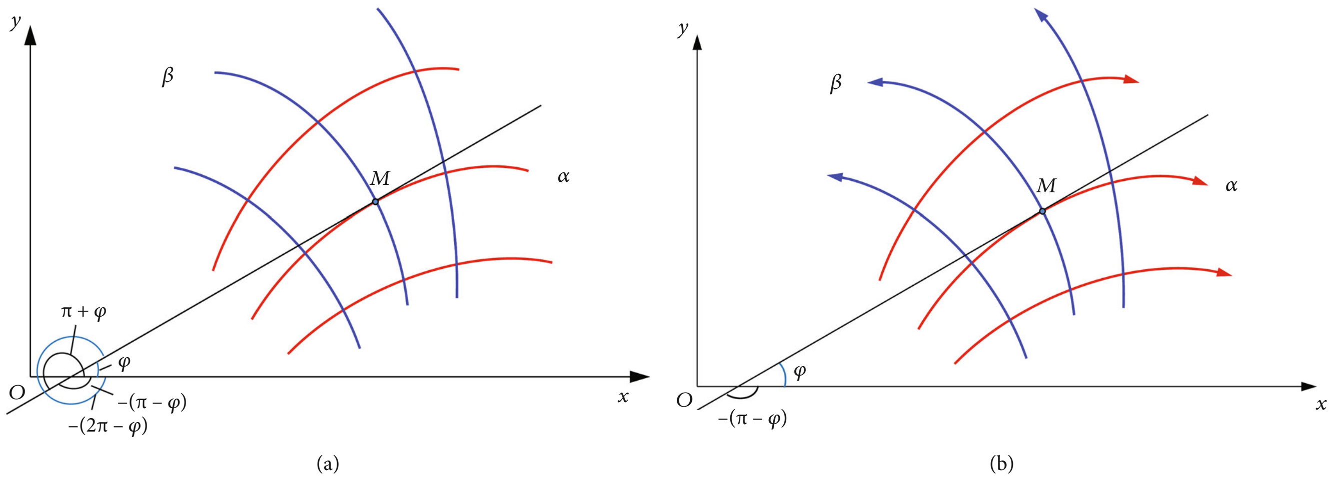

FIGURE 1: Diagram of slip-line field: (a) with no direction; (b) with clear direction.

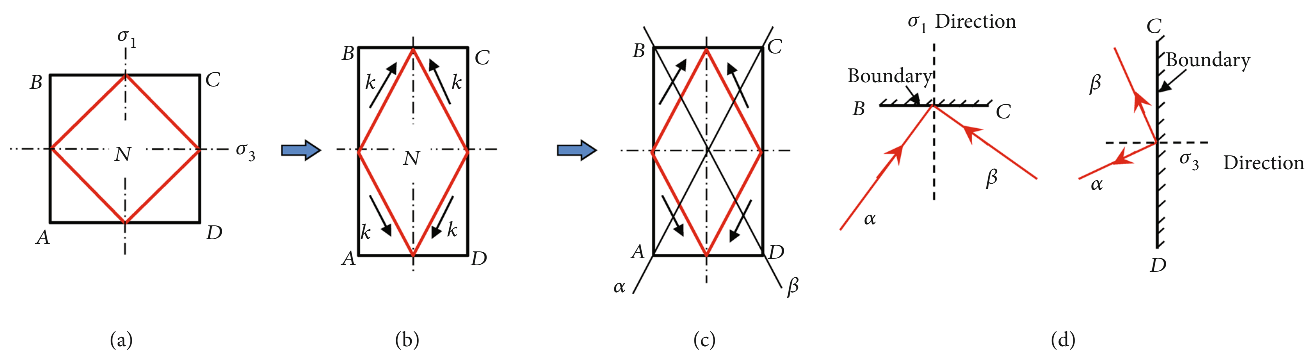

FIGURE 2: Determination of slip-line direction at stress boundary: (a) direction of principal stress; (b) predicted deformation tendency; (c) determined type of slip line; (d) direction of slip lines at boundary.

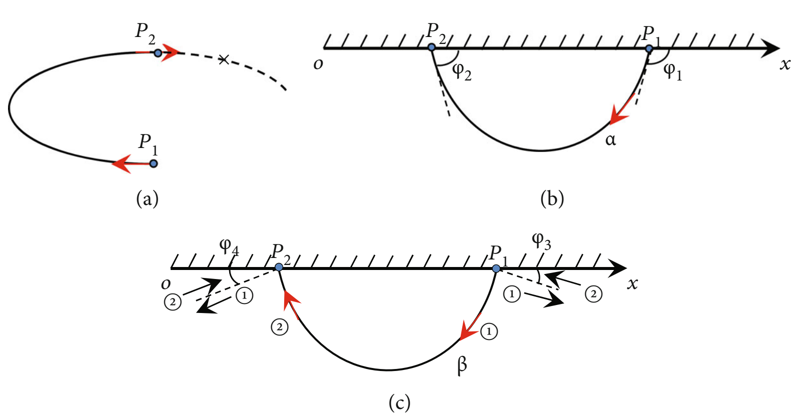

FIGURE 3: Determination of inclination angle φ: (a) continuous slip line; (b) along slip line α; (c) along slip line β.

the datum coordinate axis ox with the known direction of the slip line (Figure 1(b)). When the Hencky stress equation of slip line α with a known direction is used to solve an engineering problem, as shown in Figure 3(b), the dip angles at points P_1 and P_2 must be φ_1 and $\pi - \varphi_2$, respectively, according to the rule that the absolute value of the difference between the dip angles at the two points is less than or equal to π. However, when the Hencky stress equation of slip line β with known direction is used to solve an engineering problem, as shown in Figure 3(c), the slip line α at two points should simultaneously point in the direction of the inner or outer normal of line β at the intersection point, independent of the actual direction of α. As an example, the direction of slip line α at point P_1 (marked as 1, the outer normal direction of slip line β) is assumed to be the reference vector of the determined dip angles; the direction of slip line α at point P_2 should maintain the same orientation as that of point P_1. In this condition, the dip angles of points P_1 and P_2 are $-\varphi_3$ and $\pi - \varphi_4$, respectively. However, the direction of slip line α at point P_2 (marked as 2, the inner normal direction of slip line β) is assumed as the reference vector of the determined dip angles; the direction of slip line α at point P_1 should maintain the same orientation as that of point P_2. In this condition, the dip angles of points P_1 and P_2 are $\pi - \varphi_3$ and φ_4, respectively.

From the investigation, the rules for determining the direction of slip lines α and β and dip angle φ with stress boundary conditions are summarized as follows.

(1) Lines α and β at the intersection point of the boundary should simultaneously point to or away from the intersection point, and the synthetic direction of α and β at the boundary should maintain the direction of the first principal stress σ_1 or the opposite direction of the third principal stress σ_3.

(2) The absolute value of the dip angles should be less than π, based on the ox-axis.

(3) When points A and B on line α are substituted into the Hencky stress equations, the difference in dip angles between the two points must be less than π ($|\varphi_A - \varphi_B| = |\Delta \varphi_{AB}| \leq \pi$).

(4) When points A and B on line β are substituted into the Hencky stress equations, line α crossing the two points should simultaneously point in the direction of the outer normal or inner normal of line β. The actual direction of line α can be ignored.

2.3. Typical Stress Boundary Conditions. In solving engineering problems, three typical stress boundary conditions (free surface, contact surface without friction, and contact surface with adhesive friction) are usually used to determine the deformation loads and stress distribution. Determining the direction of slip lines and dip angles of typical points at the stress boundary is essential for correctly solving engineering problems based on slip-line theory. Thus, it is necessary to define and describe the characteristics in detail, including the direction of the slip lines and the dip angles at the boundary, according to the proposed rules.

2.3.1. Free Surface. It is possible for plastic deformation to extend near the free surface; the normal stress σ_n and shear stress τ_n on the free surface are zero. Thus, the free surface is the main plane, and the normal direction of the free surface is the main direction. According to the methods for determining the slip-line type and dip angle, both the slip lines α and β form angles equal to $\pi/4$ with the surface, as shown in Figure 4. Figure 4(a) shows that σ_y is the first principal stress σ_1 when the stress state is compressive. According to the proposed rules, the directions of slip lines α and β simultaneously point to the intersection, and the dip angle φ at the interface is $\pi/4$. Substituting the dip angle into Equation (2), the average pressure $\sigma_m = -k$ on the free boundary is obtained in the compressive stress state. Here, σ_y is the third principal stress σ_3 when the stress state is tensile stress. According to the proposed rules, the directions of slip lines α and β both depart from the intersection, and the dip angle φ at the interface is $-\pi/4$. Substituting the dip angle into Equation (2), the average pressure $\sigma_m = k$ on the free boundary is obtained in the compressive stress state.

2.3.2. Contact Surface without Friction. When the contact surface is smooth and lubricity is good, the shear stress τ_n resulting from the contact friction can be considered to be zero, and the contact surface and normal direction are the main plane and main direction, respectively. According to the methods for determining the slip-line type and dip angle, both the slip lines α and β form angles equal to $\pi/4$ with the surface, as shown in Figure 5. The normal stress σ_n on the surface is the first principal stress σ_1. According to the proposed rules, the directions of slip lines α and β simultaneously point to the intersection, and the dip angle φ at the interface is $\pi/4$, as shown in Figure 5(a). The normal stress σ_n on the surface is the first principal stress σ_3. According to the proposed rules, the directions of slip lines α and β simultaneously depart from the intersection, and the dip angle φ at the interface is $-\pi/4$, as shown in Figure 5(b).

2.3.3. Contact Surface with Adhesive Friction. Adhesion between the workpiece and tool is common in hot-rolling, extrusion, and forging processes without lubrication, leading to a maximum friction stress τ_n equal to k generated on the contact surface; the normal pressure of the surface is σ_n. According to the methods of determining the slip-line type and dip angle, both the slip lines α and β form angles equal to zero with the surface, as shown in Figure 6. The shear stress τ on the surface equals $-k$, and the direction of the slip lines is independent of the direction of shear stress. According to the proposed rules, the directions of slip lines α and β simultaneously depart from the intersection, and the dip angle φ at the interface is $-\pi/2$ or zero, as shown in Figures 6(a) and 6(b), respectively. Substituting the dip angle into Equation (2), the average pressure $\sigma_m = \sigma_n$ on the free boundary is obtained in the compressive stress state.

3. Solution of V-Notch Tensile Loads

Tensile experiments on a plane lath with a V-notch are often used to study the elastic–plastic and fracture properties of materials [11, 12]; engineering problems have been solved using slip-line theory [13]. However, a controversial conclusion has been presented indicating that the direction of slip line α can be selected with no effect on the calculation results. Thus, slip-line theory was used to solve a V-notch problem in this study to verify the reliability of the proposed rules and the inaccuracy of the conclusion presented in the literature. Furthermore, the influences of geometry and size on the critical load were investigated according to the force balance calculation results.

Figure 7 shows the external load p applied at both ends of the plane lath with a V-notch and slip-line fields. The plane lath has a symmetrical left–right and up–down structure; there is no shear stress on the upper–lower symmetrical

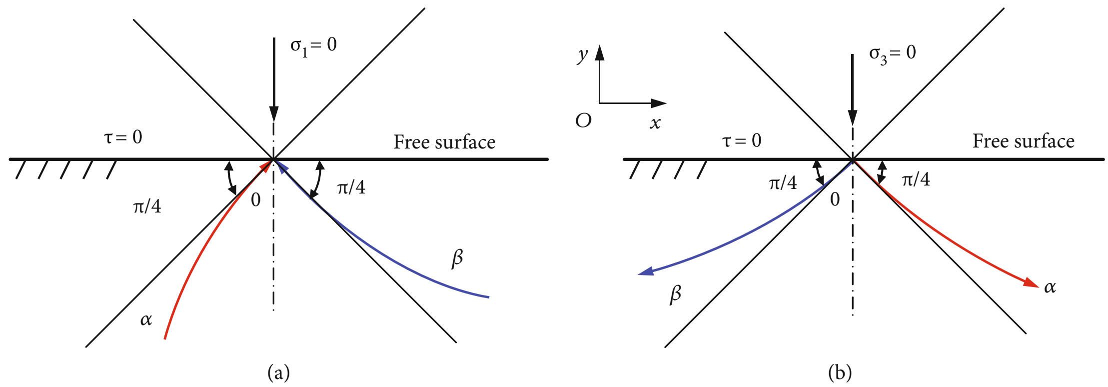

FIGURE 4: Slip-line characteristics on free surface: (a) compressive stress state; (b) tensile stress state.

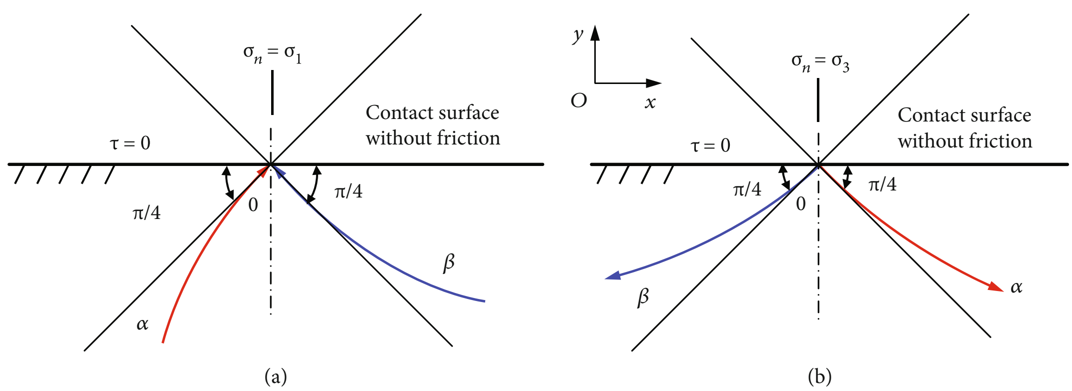

FIGURE 5: Slip-line characteristics on contact surface without friction: (a) normal stress is σ_1; (b) normal stress is σ_3.

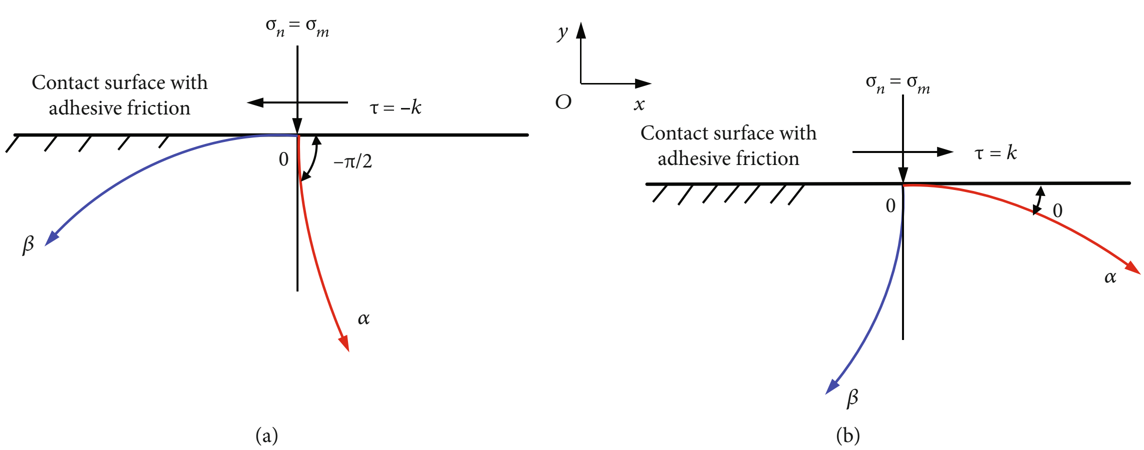

FIGURE 6: Slip-line characteristics on contact surface with adhesive friction: (a) $\tau_n = -k$; (b) $\tau_n = k$.

center, and the normal of AB is the main direction (Figure 7(a)). The slip-line fields are symmetric to the symmetry axis. The angles between the slip lines and symmetry axis are $\pi/4$ according to the relationship between the principal and shear stresses. The angles between the slip line and lines AD and BE on the free surface are also $\pi/4$, and points A and B are velocity singularities. Thus, the slip-line fields consist of two straight linear fields and a fan field, as shown in Figure 7(b).

The stress state in the deformation of the plane lath tension is tensile stress; the vector directions of the first and third principal stresses on the symmetrical line AB remain the same as for the y- and x-axes, respectively. The lines CGFD and CHIE are the α and β lines, respectively, according to the slip-line fields and the right-handed coordinate system. There are two points, D and E, on the free surface, and an average stress equal to k according to the boundary conditions.

The average stress of point D on the free surface is k; the stress of typical point D can be used to solve the average stress of typical point C based on the Hencky equation along

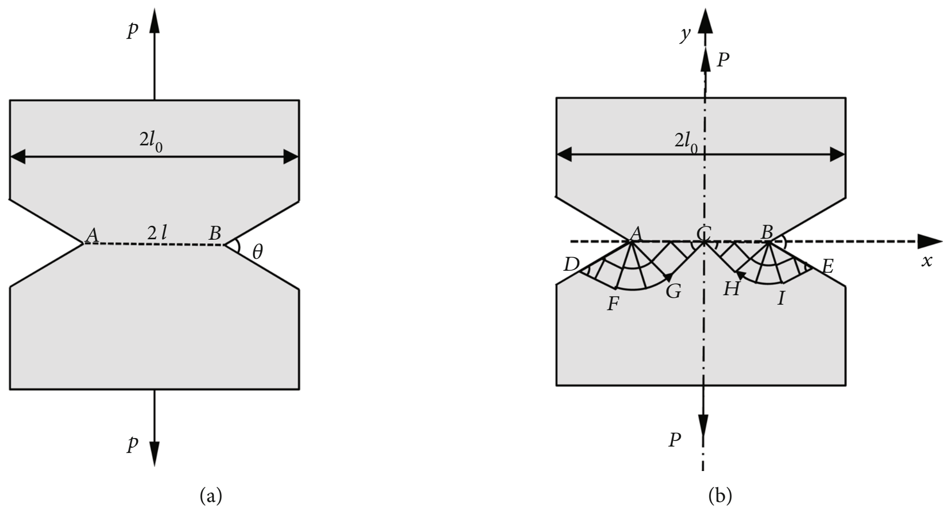

FIGURE 7: Diagram of (a) plane lath with V-notch tension; (b) slip-line fields.

the α line. When the Hencky stress equation is applied along the α line *CGFD,* as shown in Equation (1), according to the conventional method without considering the direction of the slip line, two groups of dip angles φ of typical points C and D can be determined.

$$\begin{cases} \varphi_C = -\dfrac{3\pi}{4} \\ \varphi_D = \pi - \left(\dfrac{\pi}{4} - \theta\right) = \dfrac{3\pi}{4} + \theta \end{cases}$$

$$\begin{cases} \varphi_C = \dfrac{\pi}{4} \\ \varphi_D = -\left(\dfrac{\pi}{4} - \theta\right) = \theta - \dfrac{\pi}{4} \end{cases} \quad (3)$$

Substituting the angles of points C and D, $-3\pi/4$ and $(3\pi/4 + \theta)$, into the Hencky stress Equation (1) along the α lines, the average stress can be obtained as

$$\sigma_{mC} - 2k\varphi_C = \sigma_{mD} - 2k\varphi_D \Rightarrow \sigma_{mC} = k \cdot (1 - 3\pi - 2\theta). \quad (4)$$

Substituting the average stress of point C into Equation (2), the y-direction stress can be obtained as

$$\sigma_{yC} = \sigma_{mC} + k \sin 2\varphi_C = k \cdot (2 - 3\pi - 2\theta). \quad (5)$$

If the angles of points C and D, $\pi/4$ and $(\theta\text{-}\pi/4)$, are substituted into the Hencky stress Equation (1) along the α lines, the average stress can be obtained as

$$\sigma_{mC} - 2k\varphi_C = \sigma_{mD} - 2k\varphi_D \Rightarrow \sigma_{mC} = k \cdot (1 + \pi - 2\theta). \quad (6)$$

Substituting the average stress of point C into Equation (2), the y-direction stress can be obtained as

$$\sigma_{yC} = \sigma_{mC} + k \sin 2\varphi_C = k \cdot (2 + \pi - 2\theta). \quad (7)$$

Equations (5) and (7) show that different dip angles result in different calculated results. The direction of slip line α has an important influence on the solution; the results are inconsistent with the conclusion presented in the literature [12].

The result in Equation (5) is incorrect because a difference in dip angles greater than π between points C and D does not meet the rules proposed in this study.

$$|\varphi_C - \varphi_D| = \frac{3\pi}{2} + \theta \geq \pi. \quad (8)$$

However, the differences in the dip angles of points C and D ($\pi/4$ and $(\theta - \pi/4)$) satisfy the rules; the calculated results using Equations (6) and (7) are consistent with the values reported in the previous study.

According to the rules proposed to determine the slip-line direction, the direction of the α line points from point D to point C, and the direction of the β line points from point E to point C. The dip angles $\pi/4$ and $(\theta - \pi/4)$ can be used to correctly solve the problem according to the rule that the absolute value of the dip angles should be less than π, based on the ox-axis. The direction of the slip lines has an important influence on the solution; the rules for determining the direction of the slip lines and dip angles are reliable in applying the Hencky equation along the α line.

To verify the reliability of the proposed rules, the same problem was solved using the Hencky stress equation along the β lines. When the Hencky stress equation is used along the β line CHIE, as shown in Equation (1), the lines α crossing points C and E should simultaneously point in the direction of the outer normal or inner normal of line β.

According to the determined rules, two groups of dip angles φ of typical points C and E can be determined as

$$\begin{cases} \varphi_C = \dfrac{\pi}{4} \\ \varphi_E = \left(\dfrac{3\pi}{4} - \theta\right) \end{cases} (1) \cdot \begin{cases} \varphi_C = -\dfrac{3\pi}{4} \\ \varphi_E = -\left(\dfrac{\pi}{4} + \theta\right) \end{cases} (2). \tag{9}$$

Both groups of dip angles ($\pi/4$ and $(3\pi/4 - \theta)$ and $-3\pi/4$ and $-(\pi/4 + \theta)$) of points C and E are substituted into Hencky stress Equation (1) along the β lines. The average stress of point C can always be obtained as

$$\sigma_{mC} = (\pi + 1 - 2\theta) \cdot k. \tag{10}$$

As the value of $\sin(\pi/2)$ is equal to $\sin(-3\pi/2)$, when Equation (10) is substituted into Equation (2), the y-direction stress of point C can be obtained as

$$\sigma_{yC} = \sigma_{mC} + k \sin 2\varphi_C = k \cdot (2 + \pi - 2\theta). \tag{11}$$

The same result in Equation (11) is obtained compared with that of the value described in reference [13]. However, the accurate results can always be obtained using the different equations and dip angles along the α or β lines, and thus one can reliably determine the direction of the slip line and the dip angles according to the rules. In addition, the boundary value characteristics of the slip-line theory are further clarified based on the description of the typical stress boundary conditions. The rules proposed to determine slip-line directions and dip angles further improve slip-line theory and have significance for students and researchers in correctly understanding and accurately applying slip-line theory in studying mechanics and in engineering solutions.

To further investigate the influence of the geometric size on the critical tensile load, the upper part is regarded as a whole body. The critical tensile load can be obtained using Equation (12), based on the balance of the y-direction force:

$$p = \sigma_{yC} \frac{l}{l_0} = \frac{k(2 + \pi - 2\theta)l}{l_0}. \tag{12}$$

Equation (12) shows that both the depth and the angle of the V-notch have an important influence on the critical tensile load. With an increase in notch depth, the critical loads of plastic deformation and material fracture decrease. Furthermore, the critical load increases with a decrease in the V-notch angle. Both the critical stress in Equation (11) and critical tensile load in Equation (12) equal to the $2k$ with the θ equals to $\pi/2$, which meets with the plastic yield condition of un-notched specimen in tension [13]. The unique solution of plastic mechanics obtained according to the slip-line fields theory, and the proposed determination rule satisfies all the conditions of the plastic plane strain problem, and it is a complete solution of plastic mechanics.

4. Conclusions

(1) The directions of slip lines α and β have an important influence in application of slip-line theory. Lines α and β at the intersection point of the boundary should simultaneously point to or away from the intersection point, and the synthetic direction of α and β at the boundary should maintain the direction of the first principal stress σ_1 or the opposite direction of the third principal stress σ_3. The absolute values of the dip angles should be less than π, based on the ox-axis.

(2) When points A and B on line α are substituted into the Hencky stress equations to solve an engineering problem, the difference in dip angles between the two points must be less than π. When points A and B on line β are substituted into the Hencky stress equations to solve an engineering problem, line α crossing the two points should simultaneously point in the direction of the outer and inner normals of line β.

(3) The critical tensile load in a plane lath with V-notch tension was successfully solved using the drawn slip-line fields and Hencky stress equations. The average stress value of the y-direction symmetric center can always be obtained accurately using the different equations along the α and β lines and groups of dip angles. The depth and angle of the V-notch have an important influence on the critical tensile load; the calculated results are consistent with plastic-deformation theory. The proposed rules are reliable for determining the direction of the slip line and the dip angles to avoid multivalued solutions. Slip-line theory is improved, which is important for students and researchers seeking to understand and accurately apply the method.

Data Availability

Data supporting this research article are available from the corresponding author or first author on reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] B. Y. Xu, "Mechanics method in plastic deformation processes," *Mechanics in Engineering*, vol. 2, pp. 1–8, 1993.

[2] R. B. Mei, L. Bao, and X. H. Liu, "Research on non-Prandtl solution of coarse plane punch pin pressed in the semi-unfinite body in slip line teaching," *Mechanics in Engineering*, vol. 44, no. 5, pp. 1202–1205, 2022.

[3] A. Uysal and I. S. Jawahir, "Analysis of slip-line model for serrated chip formation in orthogonal machining of AISI 304

stainless steel under various cooling/lubricating conditions," *Journal of Manufacturing Processes*, vol. 67, pp. 447–460, 2021.

[4] C. C. Li, Y. F. Guan, Z. Y. Cai et al., "Ultimate bearing capacity of strip footings placed near slopes determined by rigorous slip line field theory," *Chinese Journal of Geotechical Engineering*, vol. 43, no. 8, pp. 1408–1416, 2021.

[5] Y. F. Zhang, H. S. Di, X. Li et al., "Combined analysis of vertical rolling force with slab method and slip-field," *Journal of Plasticity Engineering*, vol. 27, no. 1, pp. 153–158, 2020.

[6] S. W. Zhang, S. D. Zhao, F. Jiang, Y. Ren, D. Zhang, and M. G. Lee, "Analysis of thread rolling process using an analytical single tooth rolling model based on the slip line field theory and slab method," *Journal of Manufacturing Processes*, vol. 76, pp. 675–686, 2022.

[7] D. W. Zhang, F. F. Xu, Z. C. Yu, K. Y. Lu, Z. B. Zheng, and S. D. Zhao, "Coulomb, Tresca and Coulomb-Tresca friction models used in analytical analysis for rolling process of external spline," *Journal of Materials Processing Technology*, vol. 292, no. 5, pp. 117059–117071, 2021.

[8] P. Wang, *Mechanics of Plastic Forming of Metals*, pp. 127–141, Metallurgical Industry Press, Beijing, 2013.

[9] Z. R. Wang, *Fundamentals of Plastic Processing Mechanics*, pp. 256–272, National Defense Industry Press, Beijing, 1989.

[10] R. Wang, *Fundamentals of Plastic Mechanics*, pp. 244–272, Science Press, Beijing, 1982.

[11] V. Ashish, B. Atul, and K. Y. Hari, "Stress rupture studies of V-notched grade 92 steel for high temperature applications," *Procedia Structural Integrity*, vol. 14, pp. 410–415, 2019.

[12] H. L. Zhang, M. Liu, and Y. H. Qi, "Stress concentration factor of specimen with V-notch under uniaxial tensile conditions," *China Measurement and Test*, vol. 43, no. 8, pp. 14–18, 2020.

[13] B. Y. Xu and X. S. Liu, "Theory and application of slip line," *Journal of Mechanical Strength*, vol. 4, pp. 78–88, 1983.