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Research Article

Transformation of Superposed Quantum States Using Measurement Operators

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Quantum computation based on a gate model is described. This model initially creates a superposition $|\psi_0\rangle$ consisting of $N = 2^n$ states, and these states are labeled by an n qubit index value j. Two working qubits $|0\rangle_{wk0}$ and $|0\rangle_{wk1}$ are added for a measurement. Moreover, one marking qubit $|0\rangle_{mk}$ is added to discriminate between states in a superposition. Thus, $|\psi_1\rangle = (1/\sqrt{N})(\sum_{j=0}^{N-1}|j\rangle)\otimes|0\rangle_{wk0}\otimes|0\rangle_{wk1}\otimes|0\rangle_{mk}$. The Hadamard transformation is applied to $|0\rangle_{wk0}$ and $|0\rangle_{wk1}$. $|\psi_2\rangle = (1/\sqrt{N})(1/\sqrt{2})^2(\sum_{j=0}^{N-1}|j\rangle)\otimes(|0\rangle_{wk0}+|1\rangle_{wk0})\otimes(|0\rangle_{wk1}+|1\rangle_{wk1})\otimes|0\rangle_{mk}$. After a computation, a set of states is divided into two subsets; one is a subset bad (B) and the other is a subset good (G). $|\psi_3\rangle = (1/\sqrt{N})(1/\sqrt{2})^2\{(\sum_{j\in B}|j\rangle)\otimes(|0\rangle_{wk0}+|1\rangle_{wk0})\otimes(|0\rangle_{wk1}+|1\rangle_{wk1})\otimes|0\rangle_{mk} + (\sum_{j\in G}|j\rangle)\otimes(|0\rangle_{wk0}+|1\rangle_{wk0})\otimes(|0\rangle_{wk1}+|1\rangle_{wk1})\otimes|1\rangle_{mk}\}$. After a marking, a superposition is measured by POVM. The measurement is described by a collection of four measurement operators. The measurement transforms $|\psi_3\rangle$ into $|\psi_4\rangle = (1/\sqrt{N})\{(\sum_{j\in B}|j\rangle)\otimes|0\rangle_{wk0}\otimes|0\rangle_{wk1}\otimes(p_0 \sin\theta)|0\rangle_{mk} + (\sum_{j\in G}|j\rangle)\otimes|0\rangle_{wk0}\otimes|0\rangle_{wk1}\otimes(p_1\cos\theta)|1\rangle_{mk}\}/D$; here, $D = \sqrt{\text{card}(B)p_0^* p_0 \sin^2\theta + \text{card}(G)p_1^* p_1 \cos^2\theta}$, and $p_0^* p_0 \sin^2\theta + p_1^* p_1 \cos^2\theta = 2$ which is derived from the completeness equation. The state $|0\rangle_{mk}$ and the state $|1\rangle_{mk}$ before the measurement are transformed into $p_0\sin\theta|0\rangle_{mk}$ and $p_1\cos\theta|1\rangle_{mk}$, respectively. This paper describes these measurement operators.

1. Introduction

Quantum computation based on a gate model initially creates a superposition $|\psi_0\rangle$ that comprises of $N = 2^n$ states from n qubits. These states are labeled by an n qubit index value j.

$$|\psi_0\rangle = \left(\frac{1}{\sqrt{N}}\right)\sum_{j=0}^{N-1}|j\rangle. \tag{1}$$

For the measurement, two working qubits $|0\rangle_{wk0}$ and $|0\rangle_{wk1}$ are added to $|\psi_0\rangle$. Moreover, to discriminate between states in a superposition, a marking qubit $|0\rangle_{mk}$ is added.

$$|\psi_1\rangle = \left(\frac{1}{\sqrt{N}}\right)\left(\sum_{j=0}^{N-1}|j\rangle\right)\otimes|0\rangle_{wk0}\otimes|0\rangle_{wk1}\otimes|0\rangle_{mk}. \tag{2}$$

The Hadamard transformation H,

$$H = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix}, \tag{3}$$

is applied to two working qubits. Then,

$$|\psi_2\rangle = \left(\frac{1}{\sqrt{N}}\right)\left(\frac{1}{\sqrt{2}}\right)^2 \left(\sum_{j=0}^{N-1} |j\rangle\right) \otimes \left(|0\rangle_{wk0} + |1\rangle_{wk0}\right) \otimes \left(|0\rangle_{wk1} + |1\rangle_{wk1}\right) \otimes |0\rangle_{mk}. \quad (4)$$

Starting with a state $|\psi_2\rangle$, a computation proceeds. After this computation, a superposition is divided into two subsets; one is a subset bad (B), and the other is a subset good (G) [1, 2].

To map bad or good states to a qubit $|0\rangle_{mk}$, a mapping function f is introduced as follows:

$$f(j) = |0\rangle_{mk}, \quad j \in B, \quad (5)$$

$$f(j) = |1\rangle_{mk}, \quad j \in G, \quad (6)$$

where $B \cap G = \varnothing$ and $\mathrm{card}(B) + \mathrm{card}(G) = N$.

Using a mapping function f, $|\psi_2\rangle$ is transformed into [3]

$$|\psi_3\rangle = \left(\frac{1}{\sqrt{N}}\right)\left(\frac{1}{\sqrt{2}}\right)^2 \left\{\left(\sum_{j\in B} |j\rangle\right) \otimes \left(|0\rangle_{wk0} + |1\rangle_{wk0}\right) \otimes \left(|0\rangle_{wk1} + |1\rangle_{wk1}\right) \otimes |0\rangle_{mk} + \left(\sum_{j\in G} |j\rangle\right) \otimes \left(|0\rangle_{wk0} + |1\rangle_{wk0}\right) \otimes \left(|0\rangle_{wk1} + |1\rangle_{wk1}\right) \otimes |1\rangle_{mk}\right\}. \quad (7)$$

Therefore, a discrimination between bad and good states is equivalent to an amplitude transformation of $|0\rangle_{mk}$ and $|1\rangle_{mk}$.

Next, a superposition $|\psi_3\rangle$ is measured by POVM. The measurement M is described in a collection of four measurement operators $M = \{P, Q, R, S\}$. In Section 2, P, Q, R, and S are defined.

After the measurement, a superposition $|\psi_3\rangle$ is transformed into

$$|\psi_4\rangle = \frac{(1/\sqrt{N})\left\{\left(\sum_{j\in B}|j\rangle\right) \otimes |0\rangle_{wk0} \otimes |0\rangle_{wk1} \otimes (p_0 \sin\theta)|0\rangle_{mk} + \left(\sum_{j\in G}|j\rangle\right) \otimes |0\rangle_{wk0} \otimes |1\rangle_{wk1} \otimes (p_1 \cos\theta)|1\rangle_{mk}\right\}}{D}, \quad (8)$$

$$D = \sqrt{\mathrm{card}(B) p_0^* p_0 \sin^2\theta + \mathrm{card}(G) p_1^* p_1 \cos^2\theta}. \quad (9)$$

The relation

$$p_0^* p_0 \sin^2\theta + p_1^* p_1 \cos^2\theta = 2, \quad (10)$$

is derived from the completeness equation [4] which is described in Section 3. The states $|0\rangle_{mk}$ and $|1\rangle_{mk}$ before the measurement are transformed into $p_0 \sin\theta |0\rangle_{mk}$ and $p_1 \cos\theta |1\rangle_{mk}$, respectively.

On the amplitude transformation of superposed quantum states, Grover's algorithm [5] is well known, which makes use of unitary transformations. However, quantum measurements play an important role in a one-way computation model [6] and a quantum teleportation model [7, 8]. These models have a wide applicability to solve NP-hard problems [9].

2. Measurement Operators

In this section, the measurement operators P, Q, R, and S are described. P, Q, R, and S are defined as follows:

$$P = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} p_0 \sin\theta & 0 \\ 0 & p_1 \cos\theta \end{vmatrix}, \quad (11)$$

$$Q = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} q_0 \cos\theta & 0 \\ 0 & q_1 \sin\theta \end{vmatrix}, \quad (12)$$

$$R = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} r_0\sin\phi & 0 \\ 0 & r_1\cos\phi \end{vmatrix}, \quad (13)$$

$$S = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} s_0\cos\phi & 0 \\ 0 & s_1\sin\phi \end{vmatrix}. \quad (14)$$

Each operator is a tensor product of three components. From the left, two components control an occurrence of an operator by acting on two qubits $wk0$ and $wk1$, and the remainder transforms an amplitude of a qubit mk. When the state $|\psi_3\rangle$ is given, the occurrence is limited to P.

3. POVM

Let the adjoint operators of P, Q, R, and S be equal to $P^\dagger, Q^\dagger, R^\dagger$, and $S^\dagger$, respectively. POVM requires the following two conditions [4]:

(1) $P^\dagger P$, $Q^\dagger Q$, $R^\dagger R$, and $S^\dagger S$ are positive operators

(2) $P^\dagger P + Q^\dagger Q + R^\dagger R + S^\dagger S = I$, where I is an identity operator

Condition 1 is easily proved. Condition 2 is the completeness equation and proved in the following. P, Q, R, and S belong to Kraus operators [10] because of Condition 2.

We search for the conditions where $C = P^\dagger P + Q^\dagger Q + R^\dagger R + S^\dagger S$ is equal to an identity operator. Let C be equal to

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} & c_{18} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & c_{27} & c_{28} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} & c_{37} & c_{38} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} & c_{47} & c_{48} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} & c_{57} & c_{58} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} & c_{67} & c_{68} \\ c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} & c_{77} & c_{78} \\ c_{81} & c_{82} & c_{83} & c_{84} & c_{85} & c_{86} & c_{87} & c_{88} \end{vmatrix}. \quad (15)$$

Then,

$$c_{11} = c_{33} = c_{55} = c_{77} = \frac{\left(p_0^* p_0 \sin^2\theta + q_0^* q_0 \cos^2\theta + r_0^* r_0 \sin^2\phi + s_0^* s_0 \cos^2\phi\right)}{4} = 1, \quad (16)$$

$$c_{13} = c_{31} = c_{57} = c_{75} = \frac{\left(p_0^* p_0 \sin^2\theta - q_0^* q_0 \cos^2\theta + r_0^* r_0 \sin^2\phi - s_0^* s_0 \cos^2\phi\right)}{4} = 0, \quad (17)$$

$$c_{15} = c_{37} = c_{51} = c_{73} = \frac{\left(p_0^* p_0 \sin^2\theta + q_0^* q_0 \cos^2\theta - r_0^* r_0 \sin^2\phi - s_0^* s_0 \cos^2\phi\right)}{4} = 0, \quad (18)$$

$$c_{17} = c_{35} = c_{53} = c_{71} = \frac{\left(p_0^* p_0 \sin^2\theta - q_0^* q_0 \cos^2\theta - r_0^* r_0 \sin^2\phi + s_0^* s_0 \cos^2\phi\right)}{4} = 0, \quad (19)$$

$$c_{22} = c_{44} = c_{66} = c_{88} = \frac{\left(p_1^* p_1 \cos^2\theta + q_1^* q_1 \sin^2\theta + r_1^* r_1 \cos^2\phi + s_1^* s_1 \sin^2\phi\right)}{4} = 1, \quad (20)$$

$$c_{24} = c_{42} = c_{68} = c_{86} = \frac{\left(p_1^* p_1 \cos^2\theta - q_1^* q_1 \sin^2\theta + r_1^* r_1 \cos^2\phi - s_1^* s_1 \sin^2\phi\right)}{4} = 0, \quad (21)$$

$$c_{26} = c_{48} = c_{62} = c_{84} = \frac{\left(p_1^* p_1 \cos^2\theta + q_1^* q_1 \sin^2\theta - r_1^* r_1 \cos^2\phi - s_1^* s_1 \sin^2\phi\right)}{4} = 0, \quad (22)$$

$$c_{28} = c_{46} = c_{64} = c_{82} = \frac{\left(p_1^* p_1 \cos^2\theta - q_1^* q_1 \sin^2\theta - r_1^* r_1 \cos^2\phi + s_1^* s_1 \sin^2\phi\right)}{4} = 0. \quad (23)$$

The value of the matrix element that does not appear in the above expressions is unconditionally equal to 0.

Expression (16) plus Expression (20) is equal to the following expression:

$$(p_0^* p_0 + q_1^* q_1)\sin^2\theta + (r_0^* r_0 + s_1^* s_1)\sin^2\phi + (p_1^* p_1 + q_0^* q_0)\cos^2\theta + (r_1^* r_1 + s_0^* s_0)\cos^2\phi = 8. \quad (24)$$

Expression (17) plus Expression (21) is equal to the following expression:

$$(p_0^* p_0 - q_1^* q_1)\sin^2\theta + (r_0^* r_0 - s_1^* s_1)\sin^2\ \phi + (p_1^* p_1 - q_0^* q_0)\cos^2\theta + (r_1^* r_1 - s_0^* s_0)\cos^2\phi = 0. \tag{25}$$

Expression (18) plus Expression (22) is equal to the following expression:

$$(p_0^* p_0 + q_1^* q_1)\sin^2\theta - (r_0^* r_0 + s_1^* s_0)\sin^2\phi + (p_1^* p_1 + q_0^* q_0)\cos^2\theta - (r_1^* r_1 + s_0^* s_0)\cos^2\phi = 0. \tag{26}$$

Expression (19) plus Expression (23) is equal to the following expression:

$$(p_0^* p_0 - q_1^* q_1)\sin^2\theta - (r_0^* r_0 - s_1^* s_1)\sin^2\phi + (p_1^* p_1 - q_0^* q_0)\cos^2\theta - (r_1^* r_1 - s_0^* s_0)\cos^2\phi = 0. \tag{27}$$

We assume that

$$p_0^* p_0 = q_1^* q_1, p_1^* p_1 = q_0^* q_0, r_0^* r_0 = s_1^* s_1 \text{ and } r_1^* r_1 = s_0^* s_0. \tag{28}$$

Then, Expressions (25) and (27) are always satisfied.

Expression (24) plus Expression (26) is equal to the following expression:

$$(p_0^* p_0 + q_1^* q_1)\sin^2\theta + (p_1^* p_1 + q_0^* q_0)\cos^2\theta = 4. \tag{29}$$

Expression (24) minus Expression (26) is equal to the following expression:

$$(r_0^* r_0 + s_1^* s_1)\sin^2\phi + (r_1^* r_1 + s_0^* s_0)\cos^2\phi = 4. \tag{30}$$

By using Expression (28), Expressions (29) and (30) change as follows:

$$p_0^* p_0 \sin^2\theta + p_1^* p_1 \cos^2\theta = 2, \tag{31}$$

$$r_0^* r_0 \sin^2\phi + r_1^* r_1 \cos^2\phi = 2. \tag{32}$$

Expressions (28), (31), and (32) are the conditions that the measurement operators P, Q, R, and S satisfy the completeness equation.

4. Conclusion

In quantum computation, computation begins with a superposition which consists of 2^n states. Usually, an equal amplitude is given between states. After computation, a set of states is divided into two subsets; one is a subset bad (B), and the other is a subset good (G). The discrimination between bad and good states is enhanced by the measurement which is POVM. Then, amplitudes representing bad and good states correspond to $p_0\sin\theta$ and $p_1\cos\theta$, respectively. The relation $p_0^* p_0 \sin^2\theta + p_1^* p_1 \cos^2\theta = 2$ is derived from the completeness equation. Thus, the measurement well discriminate between bad and good states.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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