Supplementary Material for "*BQ-Bank*: A quantum software for finance and banking"

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I. OPTION PRICING & BLACK-SCHOLES MODEL

In the field of finance, option is a kind of financial derivatives that started in the European and American securities markets in the 18th century. Until the year of 1973, with the foundation of the Chicago Board Options Exchange (CBOE) and the standardization of options contract trading, options gradually became an important derivative in financial investment and asset allocation. In 2021, the total volume of global options traded reached a record of 33.31 billion contracts. Option, as the name implies, is a right of one party to an option contract (the owner) for buying or selling the underlying asset at a certain price before a specific time, which is given by the other contracting party. The price specified in the option contract is called the strike price, also known as exercise price. An option contract usually has a pre-determined valid period of time after which the option expires (called expiration date). According to the division of the underlying transaction in options, an option to buy the underlying asset at a certain price is called a call option, often simply labeled a "call"; an option to sell the underlying asset at a certain price is called a put option, often simply labeled a "put". According to the difference in the exercising stage of options, options can be divided into European-style option and American-style option. In other words, a European-style option can only be exercised on its expiration date, while the American-style option can be exercised at any time before or on the option expiration date. In addition, there are also other types of options, such as company employee options, which are the rights granted from a company to its employees to purchase company stock at a certain price in the future. It is not a circulating financial product but is used to motivate employees, which is more and more common in IT companies.

Generally, the Black-Scholes option pricing model requires the following assumptions:

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- The revenues of the underlying asset follows a normal distribution.
- The risk-free rate is constant and investors can borrow unlimitedly at this rate.
- The transaction cost in the market is 0.
- The underlying asset has no dividends and other income during the validity period of the option.
- The option exercising date is exactly the option expiration date.
- The underlying asset is divisible and freely tradable, and short selling is allowed.
- There are no risk-free arbitrage opportunities in the market.

II. QUANTUM AMPLITUDE ESTIMATION

Quantum Amplitude Estimation (QAE) is to estimate the amplitude of a specific component of a quantum state by designing quantum operations and measurements [1–4], the complexity of gate operations required by the QAE algorithm is the square root of which required by direct multiple projection measurements, with a quadratic speedup advantage. The QAE algorithm can be constructed in the next few steps.

1. A unitary operator $\mathcal{A}$ acting on a $(n+1)$-qubit quantum system with each qubit initialized at $|0\rangle$ is defined as

$$\mathcal{A}\,|0\rangle_{n+1} = \sqrt{1-a}\,|\psi_0\rangle_n\,|0\rangle + \sqrt{a}\,|\psi_1\rangle_n\,|1\rangle\,, \tag{1}$$

where $a \in [0,1]$ is an unknown parameter to be determined.

2. By defining two rotation operators $\mathcal{S}_0 = I - 2\,|0\rangle_{n+1}\,\langle 0|_{n+1}$ and $\mathcal{S}_{\psi_0} = I - 2\,|\psi_0\rangle_n\,\langle\psi_0|_n \otimes |0\rangle\,\langle 0|$, another unitary operator $\mathcal{Q}$ can thus be defined as

$$\mathcal{Q} = \mathcal{A}\mathcal{S}_0\mathcal{A}^\dagger\mathcal{S}_{\psi_0}. \tag{2}$$

3. Note $a = \sin^2(\theta_a)$, similar to the process of Grover's algorithm, after applying $\mathcal{A}$ and $\mathcal{Q}^k$ operators, the $(n+1)$-qubit quantum system evolutes from the initial state $|0\rangle_{n+1}$ to

$$\mathcal{Q}^k\mathcal{A}\,|0\rangle_{n+1} = \cos((2k+1)\theta_a)\,|\psi_0\rangle_n\,|0\rangle + \sin((2k+1)\theta_a)\,|\psi_1\rangle_n\,|1\rangle\,. \tag{3}$$

4. After simple calculation, we show that $\mathcal{Q}$ is a rotation operation in a two-dimensional space spanned by $\{|\psi_0\rangle_n\,|0\rangle\,, |\psi_1\rangle_n\,|1\rangle\}$ as same as $\mathcal{S}_{\psi_0}$, and the two eigenvalues and their corresponding eigenstates of $\mathcal{Q}$

expressed in this space can be obtained as,

$$\begin{aligned} e^{-i2\theta_a} &\rightarrow \left|\Psi_+\right\rangle = \tfrac{1}{\sqrt{2}}(\left|\psi_0\right\rangle_n \left|0\right\rangle + i \left|\psi_1\right\rangle_n \left|1\right\rangle), \\ e^{i2\theta_a} &\rightarrow \left|\Psi_-\right\rangle = \tfrac{1}{\sqrt{2}}(\left|\psi_0\right\rangle_n \left|0\right\rangle - i \left|\psi_1\right\rangle_n \left|1\right\rangle). \end{aligned} \tag{4}$$

In this sense, The initial state $\left|0\right\rangle_{n+1}$ after acting operator $\mathcal{A}$ can be expanded in the above eigenstates and written as

$$\mathcal{A}\left|0\right\rangle_{n+1} = \frac{1}{\sqrt{2}}(e^{-i\theta_a}\left|\Psi_+\right\rangle + e^{i\theta_a}\left|\Psi_-\right\rangle), \tag{5}$$

and then we reexpress Eq. (3) in eigenstates of $\mathcal{Q}$ as

$$\mathcal{Q}^k\mathcal{A}\left|0\right\rangle_{n+1} = \frac{1}{\sqrt{2}}(e^{-i(2k+1)\theta_a}\left|\Psi_+\right\rangle + e^{i(2k+1)\theta_a}\left|\Psi_-\right\rangle). \tag{6}$$

5. From the last step, we can see the amplitude a is encoded in the phases of two eigenstates of $\mathcal{Q}$, and the phases can be extracted via Quantum Phase Estimation [5] algorithm, which includes a few controlled gates and an inverse Quantum Fourier Transform. As shown in Figure 1.

Since the absolute phase values of the two eigenvalues in the phase estimation stage are the same, one of the approaches to distinguishing them is setting one of the qubits (simplly take the first one) as an indicator of sign. Then the estimation of the eigenvalue phase θ_a corresponding to the two eigenstates can be obtained through the quantum phase estimation algorithm, thereby obtaining the estimate of a with $\tilde{a}$. If we denote $M = 2^m$, where m is the number of auxiliary bits for phase estimation. Then in the algorithm when M is constant, the error between $\tilde{a}$ and a has at least $8/\pi^2$ probability less than

$$|a - \tilde{a}| \leq \frac{\pi}{M} + \frac{\pi^2}{M^2} = \mathcal{O}(M^{-1}). \tag{7}$$

This shows that the complexity of this quantum algorithm is in the order of $\mathcal{O}(1/\epsilon)$, compared with the $\mathcal{O}((1/\epsilon)^{1/2})$ convergence rate of the classical Monte Carlo algorithm, where ϵ is the accuracy of the solution.

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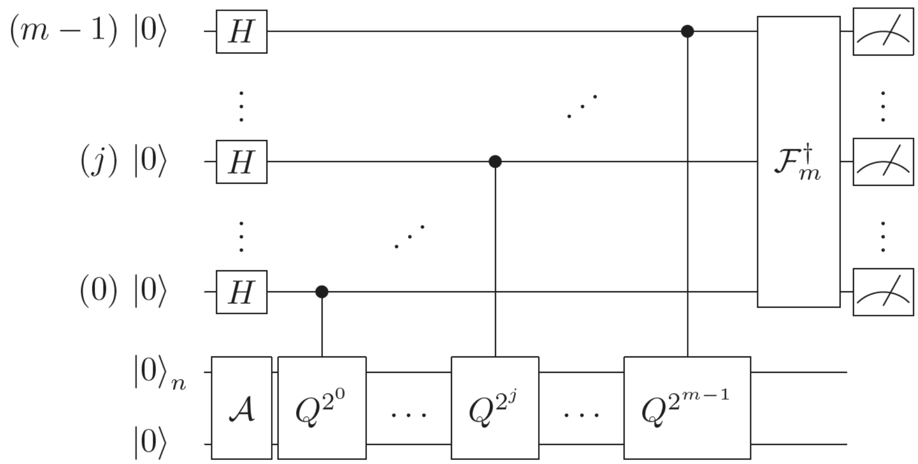

Figure 1. Quantum circuit for Quantum Amplitude Estimation algorithm, where the bottom $(n + 1)$ qubits act as the work system, and the top m qubits act as the phase estimation ancillary system [5].

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