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Research Article

Quantum Codes Obtained from Skew ϱ-λ-Constacyclic Codes over $\mathfrak{R}_k$

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Let $\mathfrak{R}_k = \mathbb{F}_q[u_1, u_2, \cdots, u_k]/ < u_i^2 = \alpha_i u_i, u_i u_j = u_j u_i = 0 >$, where $q = p^m$, p is an odd prime, α_i is a unit over $\mathbb{F}_q$, and $i, j = 1, 2, \cdots, k$. In this article, we define a Gray map from $\mathfrak{R}_k^n$ to $\mathbb{F}_q^{(k+1)n}$, we study the structure of skew ϱ-λ-constacyclic codes over $\mathfrak{R}_k$, and then we give the necessary and sufficient conditions for skew ϱ-λ-constacyclic codes over $\mathfrak{R}_k$ to satisfy dual containing. Further, we have obtained some new nonbinary quantum codes from skew ϱ-λ-constacyclic over $\mathfrak{R}_k$ by using the CSS construction.

1. Introduction

Since Calderbank and Shor [1] and Steane [2] introduced a simple construction of quantum error-correcting code in 1996, many quantum error-correcting codes have been obtained from classical error-correcting codes by using the CSS construction [3–6]. In recent years, many researchers constructed quantum codes from constacyclic codes over finite nonchain rings [7–14]. In [15, 16], Boucher et al. proposed skew cyclic codes as a new kind of generalized cyclic codes by applying skew polynomial rings. Siap et al. [17] studied the structure of skew cyclic codes for an arbitrary length over finite fields. In [18, 19], skew constacyclic codes were studied over finite fields and finite chain rings. Bag et al. constructed quantum codes from skew $(1 - 2u_1 - 2u_2 - \cdots - 2u_m)$-constacyclic codes over $\mathbb{F}_q + u_1\mathbb{F}_q + \cdots + u_{2m}\mathbb{F}_q$ and Θ-λ-skew constacyclic codes over $\mathbb{F}_q[u, v]/\langle u^2 - 1, v^2 - 1, uv = vu\rangle$ by applying the CSS construction [20, 21]. In [22, 23], some good quantum codes were obtained from linear skew constacyclic over $\mathbb{F}_{q^2}R$ and $\mathbb{F}_{q^2}[v_1, v_2, \ldots, v_\ell]/\langle v_i^2 - 1, v_i v_j - v_j v_i\rangle_{1\le i,j\le \ell}$ by using the Hermitian construction. In [24, 25], some new quantum codes were obtained from skew constacyclic codes over $R_{e,q}[u]/\langle u^e = 1\rangle$, and some MDS quantum codes were construed from skew cyclic codes over $\mathbb{F}_q[u]/\langle u^{k+1} - u\rangle$ by applying the CSS construction. Dinh et al. [26] obtained some optimal codes and near-optimal codes from skew θ-cyclic codes and discussed the advantages of quantum codes from skew θ-cyclic codes than from cyclic codes over $\mathbb{F}_q$. In this article, we study the algebraic structures of skew ϱ-λ-constacyclic codes over $\mathfrak{R}_k = \mathbb{F}_q[u_1, u_2, \cdots, u_k]/\langle u_i^2 = \alpha_i u_i, u_i u_j = u_j u_i = 0\rangle$; as an application, we give some new quantum codes from skew ϱ-λ-constacyclic codes over $\mathfrak{R}_k$ by using the CSS construction.

The rest of this article is arranged as follows: In Section 2, we define a new nonchain ring $\mathfrak{R}_k$ and a Gray map from $\mathfrak{R}_k^n$ to $\mathbb{F}_q^{(k+1)n}$ and introduce some basic knowledge of skew constacyclic code over $\mathfrak{R}_k$. In Section 3, we give the necessary and sufficient conditions for skew ϱ-λ-constacyclic codes over $\mathfrak{R}_k$ to satisfy dual containing. In Section 4, we give some examples and obtain some new quantum codes from skew ϱ-λ-constacyclic codes over $\mathfrak{R}_k$.

2. Preliminaries

Let $\mathfrak{R}_k = \mathbb{F}_q[u_1, u_2, \cdots, u_k]/\langle u_i^2 = \alpha_i u_i, u_i u_j = u_j u_i = 0\rangle$ be a nonchain ring, where $q = p^m$, p is an odd prime and α_i is a unit over $\mathbb{F}_q$, $i, j = 1, 2, \cdots, k$.

Clearly, $\mathfrak{R}_k$ is semilocal and has $q^{(k+1)}$ elements.

Let $\varsigma_1 = u_1/\alpha_1, \varsigma_2 = u_2/\alpha_2, \cdots, \varsigma_k = u_k/\alpha_k, \varsigma_{k+1} = 1 - u_1/\alpha_1 - u_2/\alpha_2 - \cdots - u_k/\alpha_k$. We can get that $\varsigma_i\varsigma_j = 0$, when

$i \neq j$, $\varsigma_i \varsigma_i = \varsigma_i$, when $i, j = 1, 2, \cdots, k+1$, and $1 = \varsigma_1 + \varsigma_2 + \cdots + \varsigma_{k+1}$. Thus, $\mathfrak{R}_k = \varsigma_1 \mathfrak{R}_k \oplus \varsigma_2 \mathfrak{R}_k \oplus \cdots \oplus \varsigma_{k+1} \mathfrak{R}_k$.

For any $r \in \mathfrak{R}_k$, r can only be said to $r = r_1 \varsigma_1 + r_2 \varsigma_2 + \cdots + r_{k+1} \varsigma_{k+1}$, where $r_j \in \mathbb{F}_q$ and $j = 1, 2, \cdots, k+1$.

Let θ_t be $\mathbb{F}_q$ automorphism, $\theta_t: \mathbb{F}_q \longrightarrow \mathbb{F}_q$ by $\theta_t(a) = a^{p^t}$. We define the automorphism of $\mathfrak{R}_k$ as follows:

$$\begin{aligned} &\varrho: \mathfrak{R}_k \rightarrow \mathfrak{R}_k, \\ &a_0 + a_1 u_1 + \cdots + a_k u_k \mapsto \theta_t(a_0) + \theta_t(a_1) u_1 + \cdots + \theta_t(a_k) u_k. \end{aligned} \tag{1}$$

By the above definition, the order of ϱ is m/t.

Let the set $\mathfrak{R}_k[x, \varrho] = \{a_0 + a_1 x + \cdots + a_n x^n, a_i \in \mathfrak{R}_k, i = 0, 1, 2, \cdots, n\}$, the addition on $\mathfrak{R}_k[x, \varrho]$ is defined as the general form of polynomials and the multiplication of polynomials is $(ax^i)(bx^j) = a\varrho^i(b) x^{i+j}$.

By the above definition, it is easy to know that the set $\mathfrak{R}_k[x, \varrho]$ is a noncommutative ring and a skew polynomial ring. $\forall f(x), g(x) \in \mathfrak{R}_k[x, \varrho]$, $g(x)$ is a right divisor of $f(x)$ if there exists $q(x) \in \mathfrak{R}_k[x, \varrho]$ subject to $f(x) = q(x) * g(x)$. Similarly, the left divisor can be given as above.

Let λ be a unit of $\mathfrak{R}_k$, the skew constacyclic shift $\sigma_{\varrho,\lambda}$ of $c = (c_0, c_1, \cdots, c_{n-1}) \in \mathfrak{R}_k^n$ is defined by $\sigma_{\varrho,\lambda}(c) = (\lambda \varrho(c_{n-1}), \varrho(c_0), \cdots, \varrho(c_{n-2}))$. Then, C is called a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$ if C is invariant under $\sigma_{\varrho,\lambda}: \mathfrak{R}_k^n \longrightarrow \mathfrak{R}_k^n$. In particular, C is called a skew ϱ-cyclic code and skew ϱ-negacyclic code of length n over $\mathfrak{R}_k$, when $\lambda = 1$ and $\lambda = -1$.

A map is defined as follows:

$$\begin{aligned} &\psi: \mathfrak{R}_k^n \rightarrow \frac{\mathfrak{R}_k[x, \varrho]}{\langle x^n - \lambda \rangle}, \\ &(a_0, a_1, \cdots, a_{n-1}) \mapsto a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}. \end{aligned} \tag{2}$$

Then, $\forall (a_0, a_1, \cdots, a_{n-1}) \in \mathfrak{R}_k^n$ is identified as a polynomial $a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ over $\mathfrak{R}_k[x, \varrho]/\langle x^n - \lambda \rangle$. Let the order of ϱ, $|\varrho| = l$; if $l|n$, we define a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$ as a left ideal of $\mathfrak{R}_k[x, \varrho]/\langle x^n - \lambda \rangle$.

By the same method of Lemma 7 in [17], we can have the following lemma.

Lemma 1. *If $f(x)g(x) \in Z(\mathfrak{R}_k[x, \varrho])$, the centre of $\mathfrak{R}_k[x, \varrho]$ is $Z(\mathfrak{R}_k[x, \varrho])$ and then $f(x)g(x) = g(x) f(x) \in \mathfrak{R}_k[x, \varrho]$.*

For any $a = a_1 \varsigma_1 + a_2 \varsigma_2 + \cdots + a_{k+1} \varsigma_{k+1} \in \mathfrak{R}_k$, the Gray map ϕ_k is defined as follows:

$$\begin{aligned} &\phi_k: \mathfrak{R}_k \rightarrow \mathbb{F}_q^{k+1}, \\ &a \mapsto (a_1, a_2, \cdots, a_{k+1}). \end{aligned} \tag{3}$$

We extend ϕ_k as follows:

$$\begin{aligned} &\phi_k: \mathfrak{R}_k^n \rightarrow \mathbb{F}_q^{(k+1)n}, \\ &(a_0, a_1, \cdots, a_{n-1}) \mapsto \big(a_{1,0}, \cdots, a_{1,n-1}, a_{2,0}, \cdots, a_{2,n-1}, \\ &\cdots, a_{k+1,0}, \cdots, a_{k+1,n-1}\big), \end{aligned} \tag{4}$$

where $a_i = a_{1,i} \varsigma_1 + a_{2,i} \varsigma_2 + \cdots + a_{k+1,i} \varsigma_{k+1} \in \mathfrak{R}_k, i = 0, 1, 2, \cdots, n-1$.

For any $c = (c_0, c_1, \cdots, c_{n-1}) \in C$, c can be said to be as follows:

$$c = (c_0, c_1, \cdots, c_{n-1}) \leftrightarrow c(x) = \sum_{i=0}^{n-1} c_i x^i \in \mathfrak{R}_k[x]. \tag{5}$$

Let C be a linear code of length n over $\mathfrak{R}_k$, and

$$C_j = \left\{ x_j \in \mathbb{F}_q^n \,\middle|\, \sum_{j=1}^{k+1} x_j \varsigma_j \in C, x_1, x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{k+1} \in \mathbb{F}_q^n \right\}, \tag{6}$$

for $j = 1, 2, \cdots, k+1$. One can quickly verify that C_j is a linear code of length n over $\mathbb{F}_q$ for $j = 1, 2, \cdots, k+1$, and $C = \oplus_{j=1}^{k+1} \varsigma_j C_j$, $|C| = \prod_{j=1}^{k+1} |C_j|$.

Lemma 2 (see [14]). *An element $\lambda = \lambda_1 \varsigma_1 + \lambda_2 \varsigma_2 + \cdots + \lambda_{k+1} \varsigma_{k+1}$ is a unit in $\mathfrak{R}_k$ if and only if λ_j is a unit in $\mathbb{F}_q$ for $j = 1, 2, \cdots, k+1$.*

3. Skew ϱ-λ-Constacyclic Codes over $\mathfrak{R}_k$

Lemma 3. *Let $C = \oplus_{j=1}^{k+1} \varsigma_j C_j$ be a linear code of length n over $\mathfrak{R}_k$, and $\lambda = \lambda_1 \varsigma_1 + \lambda_2 \varsigma_2 + \cdots + \lambda_{k+1} \varsigma_{k+1}$ is a unit in $\mathfrak{R}_k$, $\mathrm{ord}(\varrho) \mid n$. Then, $\varrho(\lambda) = \lambda$ if and only if $\theta_t(\lambda_j) = \lambda_j$ and $j = 1, 2, \cdots, k+1$, where $\theta_t(\alpha_i) = \alpha_i$ and $i = 1, 2, \cdots, k$.*

Proof. Suppose $\varrho(\lambda) = \lambda$, we have

$$\begin{aligned} \lambda &= \lambda_1 \varsigma_1 + \lambda_2 \varsigma_2 + \cdots + \lambda_{k+1} \varsigma_{k+1} \\ &= \lambda_{k+1} + u_1 \left(\frac{\lambda_1 - \lambda_{k+1}}{\alpha_1} \right) + \cdots + u_k \left(\frac{\lambda_k - \lambda_{k+1}}{\alpha_k} \right) \\ &= \varrho \left(\lambda_{k+1} + u_1 \left(\frac{\lambda_1 - \lambda_{k+1}}{\alpha_1} \right) + \cdots + u_k \left(\frac{\lambda_k - \lambda_{k+1}}{\alpha_k} \right) \right) \\ &= \theta_t(\lambda_{k+1}) + u_1 \theta_t \left(\frac{\lambda_1 - \lambda_{k+1}}{\alpha_1} \right) + \cdots + u_k \theta_t \left(\frac{\lambda_k - \lambda_{k+1}}{\alpha_k} \right) \\ &= \lambda_{k+1}^{p^t} + u_1 \left(\frac{\lambda_1 - \lambda_{k+1}}{\alpha_1} \right)^{p^t} + \cdots + u_k \left(\frac{\lambda_k - \lambda_{k+1}}{\alpha_k} \right)^{p^t} \\ &= \lambda_{k+1}^{p^t} + u_1 \left(\frac{\lambda_1^{p^t} - \lambda_{k+1}^{p^t}}{\alpha_1^{p^t}} \right) + \cdots + u_k \left(\frac{\lambda_k^{p^t} - \lambda_{k+1}^{p^t}}{\alpha_k^{p^t}} \right) \\ &= \theta_t(\lambda_{k+1}) + \left(\frac{\theta_t(\lambda_1) - \theta_t(\lambda_{k+1})}{\theta_t(\alpha_1)} \right) + \cdots + \left(\frac{\theta_t(\lambda_k) - \theta_t(\lambda_{k+1})}{\theta_t(\alpha_k)} \right). \end{aligned} \tag{7}$$

On comparing the coefficients, we have

$$\lambda_{k+1}^{p^t} = \lambda_{k+1}, \frac{\theta_t(\lambda_1) - \theta_t(\lambda_{k+1})}{\theta_t(\alpha_1)} = \frac{\lambda_1 - \lambda_{k+1}}{\alpha_1}, \cdots, \frac{\theta_t(\lambda_k) - \theta_t(\lambda_{k+1})}{\theta_t(\alpha_k)} = \frac{\lambda_k - \lambda_{k+1}}{\alpha_k}. \tag{8}$$

Note that $\theta_t(\alpha_i) = \alpha_i$ for $i = 1, 2, \cdots, k$, we can get that $\theta_t(\lambda_j) = \lambda_j$ for $j = 1, 2, \cdots, k+1$.

Conversely, if $\theta_t(\lambda_j) = \lambda_j$ for $j = 1, 2, \cdots, k+1$, note that $\theta_t(\alpha_i) = \alpha_i$ for $i = 1, 2, \cdots, k$, then we can have $\theta_t(\varsigma_i) = \theta_t(u_i/\alpha_i) = \varsigma_i$ and $\theta_t(\varsigma_{k+1}) = \theta_t(1 - u_1/\alpha_1 - \cdots - u_k/\alpha_k) = \varsigma_{k+1}$.

So, $\varrho(\lambda) = \varrho(\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1}) = \lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1} = \lambda$. □

Theorem 1. *Let $C = \oplus_{j=1}^{k+1}\varsigma_j C_j$ be a linear code of length n over $\mathfrak{R}_k$ and $\lambda = \lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1}$ is a unit in $\mathfrak{R}_k$, $\mathrm{ord}(\varrho)|n$, $\varrho(\lambda) = \lambda$. Then, C is a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$ if and only if C_j is a skew θ_t-λ_j-constacyclic code of length n over $\mathbb{F}_q$ for $j = 1, 2, \cdots, k+1$.*

Proof. For any $c_j = (c_{j,0}, c_{j,1}, \cdots, c_{j,n-1}) \in C_j$, $j = 1, 2, \cdots, k+1$. Then, $c = \varsigma_1 c_1 + \varsigma_2 c_2 + \cdots + \varsigma_{k+1}c_{k+1} = (\sum_{j=1}^{k+1}\varsigma_j c_{j,0}, \sum_{j=1}^{k+1}\varsigma_j c_{j,1}, \cdots, \sum_{j=1}^{k+1}\varsigma_j c_{j,n-1}) \in C$

If C_j is a θ_t-λ_j-constacyclic code of length n over $\mathbb{F}_q$, then

$$\begin{aligned}
\sigma_{\theta_t,\lambda_j}(c_j) &= \sigma_{\theta_t,\lambda_j}(c_{j,0}, c_{j,1}, \cdots, c_{j,n-1}) = \left(\lambda_j (c_{j,n-1})^{\theta_t}, (c_{j,0})^{\theta_t}, \cdots, (c_{j,n-2})^{\theta_t}\right) \in C_j,\\
\sigma_{\varrho,\lambda}(c) &= \left((\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1})\varrho\left(\sum_{i=1}^{k+1}\varsigma_i c_{i,n-1}\right), \varrho\left(\sum_{i=1}^{k+1}\varsigma_i c_{i,0}\right), \cdots, \varrho\left(\sum_{i=1}^{k+1}\varsigma_i c_{i,n-2}\right)\right)\\
&= \left((\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1})\theta_t\left(\sum_{i=1}^{k+1}\varsigma_i c_{i,n-1}\right), \theta_t\left(\sum_{i=1}^{k+1}\varsigma_i c_{i,0}\right), \cdots, \theta_t\left(\sum_{i=1}^{k+1}\varsigma_i c_{i,n-2}\right)\right)\\
&= \left((\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1})\sum_{i=1}^{k+1}\varsigma_i\theta_t(c_{i,n-1}), \sum_{i=1}^{k+1}\varsigma_i\theta_t(c_{i,0}), \cdots, \sum_{i=1}^{k+1}\varsigma_i\theta_t(c_{i,n-2})\right)\\
&= \varsigma_1\sigma_{\theta_t,\lambda_1}(c_1) + \varsigma_2\sigma_{\theta_t,\lambda_2}(c_2) + \cdots + \varsigma_{k+1}\sigma_{\theta_t,\lambda_{k+1}}(c_{k+1}) \in C.
\end{aligned} \tag{9}$$

So, C is a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$.

On the other hand, if C is a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$, we have

$$\sigma_{\varrho,\lambda}(c) = \varsigma_1\sigma_{\theta_t,\lambda_1}(c_1) + \varsigma_2\sigma_{\theta_t,\lambda_2}(c_2) + \cdots + \varsigma_{k+1}\sigma_{\theta_t,\lambda_{k+1}}(c_{k+1}) \in C = \oplus_{j=1}^{k+1}\varsigma_j C_j. \tag{10}$$

So, $\sigma_{\theta_t,\lambda_j}(c_j) \in C_j$, C_j is a skew θ_t-λ_j-constacyclic code of length n over $\mathbb{F}_q$ for $j = 1, 2, \cdots, k+1$. □

Theorem 2. *Let $C = \oplus_{j=1}^{k+1}\varsigma_j C_j$ be a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$, $\lambda = \lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1}$ is a unit in $\mathfrak{R}_k$, $\mathrm{ord}(\varrho)|n$, $\varrho(\lambda) = \lambda$. Then, $C^\perp = \sum_{j=1}^{k+1}\varsigma_j C_j^\perp$ is a skew θ_t-λ^{-1}-constacyclic code of length n over $\mathfrak{R}_k$, and $C_j^\perp$ is a skew θ_t-λ_j^{-1}-constacyclic code over $\mathbb{F}_q$ for $j = 1, 2, \cdots, k+1$, where $\lambda^{-1} = \lambda_1^{-1}\varsigma_1 + \lambda_2^{-1}\varsigma_2 + \cdots + \lambda_{k+1}^{-1}\varsigma_{k+1}$.*

Proof. Let $C = \oplus_{j=1}^{k+1}\varsigma_j C_j$ be a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$, where $\lambda = \lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1}$ is a unit in $\mathfrak{R}_k$. For any $x = (x_0, x_1, \cdots, x_{n-1}) \in C^\perp$, $y = (y_0, y_1, \cdots, y_{n-1}) \in C$, then

$$\sigma_{\varrho,\lambda}^{n-1}(y) = (\lambda\varrho(y_1)^{n-1}, \lambda\varrho(y_2)^{n-1}, \cdots, \lambda\varrho(y_{n-1})^{n-1}, \varrho(y_0{}^{n-1})) \in C. \tag{11}$$

We can get that

$$\begin{aligned}
0 &= x \cdot \sigma_{\varrho,\lambda}^{n-1}(y)\lambda x_0\varrho(y_1)^{n-1} + \lambda x_1\varrho(y_2)^{n-1} + \cdots + \lambda x_{n-2}\varrho(y_{n-1})^{n-1} + x_{n-1}\varrho(y_0)^{n-1}\\
&= \lambda\left(x_0\varrho(y_1)^{n-1} + x_1\varrho(y_2)^{n-1} + \cdots + x_{n-2}\varrho(y_{n-1})^{n-1} + \lambda^{-1}x_{n-1}\varrho(y_0{}^{n-1})\right),\\
0 &= \varrho(0)\\
&= \varrho\left(x_0\varrho(y_1)^{n-1} + x_1\varrho(y_2)^{n-1} + \cdots + x_{n-2}\varrho(y_{n-1}{}^{n-1} + \lambda^{-1}x_{n-1}\varrho(y_0{}^{n-1})\right)\\
&= \varrho(x_0)y_1 + \varrho(x_1)y_2 + \cdots + \varrho(x_{n-2})y_{n-1} + \lambda^{-1}\varrho(x_{n-1})y_0)\\
&= \sigma_{\varrho,\lambda^{-1}}(x) \cdot y,
\end{aligned} \tag{12}$$

so $\sigma_{\varrho,\lambda^{-1}}(x) \in C^{\perp}$; hence, $C^{\perp}$ is a skew θ_t-λ^{-1}-constacyclic code.

By Lemma 2, $C^{\perp}$ is a skew θ_t-λ^{-1}-constacyclic code of length n over $\mathfrak{R}_k$. By Theorem 1, $C_j^{\perp}$ is a skew θ_t-λ_j^{-1}-constacyclic code over $\mathbb{F}_q$ for $j = 1, 2, \cdots, k+1$. □

Theorem 3. *Let $C = \oplus_{j=1}^{k+1} \varsigma_j C_j$ be a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$, $\lambda = \lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1}$ is a unit in $\mathfrak{R}_k$ $ord(\varrho) \mid n$, $\varrho(\lambda) = \lambda$. Then, there exists a polynomial $\varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{k+1} g_{k+1}(x) \in \mathfrak{R}_k\,[x, \varrho]$ subject to $C = \langle \varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{k+1} g_{k+1}(x)\rangle$, where the right divisor of $x^n - \lambda$ is $\varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{k+1} g_{k+1}(x)$, the generator polynomial of skew θ_t-λ_j-constacyclic C_j is $g_j(x) \in \mathbb{F}_q[x, \theta_t]$, and $g_j(x)$ divides $x^n - \lambda_j$ on the right for $j = 1, 2, \cdots, k+1$.*

Proof. Let $C = \oplus_{j=1}^{k+1} \varsigma_j C_j$ be a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$. By Theorem 1, C_j is a skew θ_t-λ_j-constacyclic code of length n over $\mathbb{F}_q$ for $j = 1, 2, \cdots, k+1$.

Let $g_j(x)$ be the generator polynomial of C_j, then

$$C = \langle \varsigma_1 g_1(x), \varsigma_2 g_2(x), \cdots, \varsigma_{k+1} g_{k+1}(x)\rangle. \tag{13}$$

Let $C' = \langle \varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{k+1} g_{k+1}(x)\rangle$. Clearly, $C' \subseteq C$.

Because $\varsigma_j[(\varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{k+1} g_{k+1}(x)] = \varsigma_j g_j(x)$ *for* $j = 1, 2, \cdots, k+1$, so $C \subseteq C'$.

Hence, $C = C' = \langle \varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{k+1} g_{k+1}(x)\rangle$.

Because the right divisor of $x^n - \lambda_j$ is $g_j(x)$ for $j = 1, 2, \cdots, k+1$. Let $f_j(x) g_j(x) = x^n - \lambda_j$. Then, $[\varsigma_1 f_1(x) + \varsigma_2 f_2(x) + \cdots + \varsigma_{k+1} f_{k+1}(x)][\varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{k+1} g_{k+1}(x)] = x^n - (\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1}) = x^n - \lambda$.

So, the right divisor of $x^n - \lambda$ is $\varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{k+1} g_{k+1}(x)$. □

Corollary 1. *Let $C = \oplus_{j=1}^{k+1} \varsigma_j C_j$ be a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$, $\lambda = \lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1}$ is a unit in $\mathfrak{R}_k$, $\mathrm{ord}(\varrho) \mid n$, $\varrho(\lambda) = \lambda$. Then, $C^{\perp} = \langle \varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{k+1} f_{k+1}^*(x)\rangle$, $|C^{\perp}| = q^{\left(\sum_{j=1}^{k+1} \deg(g_j(x))\right)}$, where $f_j(x) g_j(x) = x^n - \lambda_j$, $f_j(x) = a_{0,j} + a_{1,j}x + \cdots + a_{n-r,j}x^{n-r} \in \mathbb{F}_q[x, \theta_t]$, $f_j^*(x) = a_{n-r,j} + \theta_t(a_{n-r-1,j})x + \cdots + \theta_t^{n-r}(a_{0,j})x^{n-r}$, and $f_j^*(x)$ is the generate polynomials of skew θ_t-λ_j^{-1}-constacyclic $C_j^{\perp}$ for $j = 1, 2, \cdots, k+1$.*

Proof. Let $C_j^{\perp} = \langle f_j^*(x)\rangle$ *for* $j = 1, 2, \cdots, k+1$, using Theorems 2 and 3, $C^{\perp} = \oplus_{j=1}^{k+1} \varsigma_j C_j^{\perp}$, then $|C^{\perp}| = \prod_{j=1}^{k+1} |C_j^{\perp}| = q^{\left(\sum_{j=1}^{k+1} \deg(g_j(x))\right)}$, and we can get that $C^{\perp} = \langle \varsigma_1 f_1^*(x), \varsigma_2 f_2^*(x), \cdots, \varsigma_{k+1} f_{k+1}^*(x)\rangle$. Let $\widetilde{D} = \langle \varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{k+1} f_{k+1}^*(x)\rangle$. Clearly, $\widetilde{D} \subseteq C^{\perp}$. Because $\varsigma_j[\varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{k+1} f_{k+1}^*(x)] = \varsigma_j f_j^*(x)$ *for* $j = 1, 2, \cdots, k+1$, so $C^{\perp} \subseteq \widetilde{D}$.

Therefore, $C^{\perp} = \widetilde{D} = \langle \varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{k+1} f_{k+1}^*(x)\rangle$. □

4. Quantum Codes from Skew ϱ-λ-Constacyclic Codes over $\mathfrak{R}_k$

Theorem 4. *Let $C = \oplus_{j=1}^{k+1} \varsigma_j C_j$ be a linear code of length n over $\mathfrak{R}_k$, with order $|C| = q^l$, and the minimum Gray distance of C is d_G. Then, $\phi_k(C)$ is a $[(k+1)n, l, d_G]$ linear code and $\phi_k(C)^{\perp} = \phi_k(C)^{\perp}$. If C is a self-dual code over $\mathfrak{R}_k$, then $\phi_k(C)$ is a self-dual code over $\mathbb{F}_q$.*

Proof. By the definition of ϕ_k, we can have that $\phi_k(C)$ is a $[(k+1)n, l, d_G]$ linear code.

Let $a = (a_0, a_1, \cdots, a_{n-1}) \in C, b = (b_0, b_1, \cdots, b_{n-1}) \in C^{\perp}$, $a_j = a_{1,j}\varsigma_1 + a_{2,j}\varsigma_2 + \cdots + a_{k+1,j}\varsigma_{k+1}, b_j = b_{1,j}\varsigma_1 + b_{2,j}\varsigma_2 + \cdots + b_{k+1,j}\varsigma_{k+1} \in \mathfrak{R}_k$, $j = 0, 1, 2, \cdots, n-1$, and $a^{(i)} = (a_{i,0}, a_{i,1}, \cdots, a_{i,n-1}), b^{(i)} = (b_{i,0}, b_{i,1}, \cdots, b_{i,n-1})$, $i = 1, 2, \ldots, k+1$.Then, $a \cdot b = \sum_{j=0}^{n-1} a_j b_j = \sum_{j=0}^{n-1} \sum_{i=1}^{k+1} a_{i,j} b_{i,j} \varsigma_i = \sum_{i=1}^{k+1} a^{(i)} b^{(i)^{\mathrm{T}}} \varsigma_i = 0$.So $a^{(i)} b^{(i)^{\mathrm{T}}} = 0$ and $i = 1, 2, \cdots, k+1$.Since $\phi_k(a) = (a^{(1)}, a^{(2)}, \cdots, a^{(k+1)})$ and $\phi_k(b) = (b^{(1)}, b^{(2)}, \cdots, b^{(k+1)})$,

$$\phi_k(a) \cdot \phi_k(b) = \phi_k(a)\phi_k(b)^{\mathrm{T}} = \sum_{i=1}^{k+1} a^{(i)} b^{(i)^{\mathrm{T}}} = 0. \tag{14}$$

So, we have $\phi_k(C^{\perp}) \subseteq \phi_k(C)^{\perp}$.

Because ϕ_k is bijective, $|C| = |\phi_k(C)|$. Then, $|\phi_k(C^{\perp})| = q^{(k+1)n}/|C| = q^{(k+1)n}/|\phi_k(C)| = |\phi_k(C)^{\perp}|$. We have $\phi_k(C^{\perp}) = \phi_k(C)^{\perp}$.

If C is a self-dual code, $C = C^{\perp}$ and then $\phi_k(C)^{\perp} = \phi_k(C^{\perp}) = \phi_k(C)$.

Therefore, $\phi_k(C)$ is a self-dual code over $\mathbb{F}_q$. □

Lemma 4. *Let C be a skew θ_t-λ-constacyclic code of length n over $\mathbb{F}_q$, whose generator polynomial is $g(x)$ and $ord(\theta_t) | n$. Then, C contains its dual code if and only if $x^n - \lambda$ is the right divisor of $f^*(x) f(x)$, where $\lambda = \pm 1$ and the generator polynomial of $C^{\perp}$ is $f^*(x)$.*

Proof. Let $C^{\perp} = \langle f^*(x)\rangle$, where $f(x) g(x) = (x^n - \lambda)$ and $\lambda = \pm 1$, C contains its dual code if and only if there exists $h(x) \in \mathbb{F}_q[x, \theta_t]$ subject to $f^*(x) = h(x) g(x)$, by Lemma 1, $f^*(x) f(x) = h(x) g(x) f(x) = h(x) f(x) g(x) = h(x)(x^n - \lambda)$ if and only if the right divisor of $f^*(x) f(x)$ is $x^n - \lambda$.

In the present section, we construct quantum codes from skew ϱ- λ-constacyclic over $\mathfrak{R}_k$ by using the CSS construction [1, 2]. □

Theorem 5. *(CSS Construction). Let $C = [n, k, d]q$ be a linear codes over $\mathbb{F}_q$, if $C^{\perp} \subseteq C$, then there exists a quantum code $[[n, 2k - n, d]]_q$.*

Theorem 6. *Let $C = \oplus_{j=1}^{k+1} \varsigma_j C_j$ be a skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$, $ord(\varrho) \mid n$, $\varrho(\lambda) = \lambda$, $\lambda = \lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1}$ is a unit in $\mathfrak{R}_k$. Then, $C^{\perp} \subseteq C$ if and only if the right divisor of $f_i^*(x) f_i(x)$ is $x^n - \lambda_i$, $\lambda_i = \pm 1$ and $i = 1, 2, \cdots, k+1$.*

Proof. Suppose the right divisor of $f_i^*(x)f_i(x)$ is $x^n - \lambda_i$, by Lemma 4, $C_i^{\perp} \subseteq C_i$, $i = 1, 2, \cdots, k+1$, then $\varsigma_i C_i^{\perp} \subseteq \varsigma_i C_i$, which implies $C^{\perp} = \oplus_{j=1}^{k+1} \varsigma_j C_j^{\perp} \subseteq \oplus_{j=1}^{k+1} \varsigma_j C_j = C$.

On the contrary, let $C^{\perp} \subseteq C$, then $C^{\perp} = \oplus_{j=1}^{k+1} \varsigma_j C_j^{\perp} \subseteq \oplus_{j=1}^{k+1} \varsigma_j C_j = C$. Hence, $C_i^{\perp} \subseteq C_i$. By Lemma 4, we have the right divisor of $f_i^*(x)f_i(x)$ as $x^n - \lambda_i$, $\lambda_i = \pm 1$, $i = 1, 2, \cdots, k+1$.

Using Lemma 4 and Theorem 6, we can get the following corollary. □

Corollary 2. *Let $C = \oplus_{j=1}^{k+1} \varsigma_j C_j$ be a skew ϱ- λ-constacyclic code of length n over $\mathfrak{R}_k$, where $ord(\varrho) \mid n$, $\varrho(\lambda) = \lambda$, and $\lambda = \lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1}$ is a unit in $\mathfrak{R}_k$. Then, $C^{\perp} \subseteq C$ if and only if $C_j^{\perp} \subseteq C_j$, $j = 1, 2, \cdots, k+1$.*

Theorem 7. *Let $C = \oplus_{j=1}^{k+1} \varsigma_j C_j$ be a skew ϱ- λ-constacyclic code of length n over $\mathfrak{R}_k$, where $ord(\varrho) \mid n$, $\varrho(\lambda) = \lambda$, and $\lambda = \lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{k+1}\varsigma_{k+1}$ is a unit in $\mathfrak{R}_k$. If C_j is a skew θ_t-λ_j-constacyclic code over $\mathbb{F}_q$ and $C_j^{\perp} \subseteq C_j$, where $\lambda_i = \pm 1$ and $j = 1, 2, \cdots, k+1$, then $\phi_k(C)^{\perp} \subseteq \phi_k(C)$ and there exists a quantum code $[[(k+1)n, 2l - (k+1)n, d_G]]_q$, where the minimum Gray weight of C is d_G and the dimension of $\phi_k(C)$ is l.*

Proof. Since C_j is a skew θ_t-λ_j-constacyclic code over $\mathbb{F}_q$ and $C_j^{\perp} \subseteq C_j$, $\lambda_j = \pm 1$, $j = 1, 2, \cdots, k+1$, using Corollary 2, and $C^{\perp} \subseteq C$. So, $\phi_k(C^{\perp}) \subseteq \phi_k(C)$, by Theorem 4 $\phi_k(C)^{\perp} = \phi_k(C^{\perp})$. Therefore, $\phi_k(C)^{\perp} \subseteq \phi_k(C)$, and by Theorem 4, $\phi_k(C) = [(k+1)n, l, d_G]$. Using Theorem 5, there exists a quantum code $[[(k+1)|n, 2l-(k+1)n, d_G]]_q$. □

Example 1. Let $n = 3$ and $\mathfrak{R}_2 = \mathbb{F}_{27}[u_1, u_2]/\langle u_1^2 = u_1, u_2^2 = u_2, u_1u_2 = u_2u_1 = 0\rangle$, $\varsigma_1 = u_1, \varsigma_2 = u_2, \varsigma_3 = 1 - u_1 - u_2$, $\forall a \in \mathbb{F}_{27}$, and $\theta_t\colon \mathbb{F}_{27} \longrightarrow \mathbb{F}_{27}$ is defined by $\theta_t(a) = a^3$, and $\forall (a_0 + a_1u_1 + a_2u_2) \in \mathfrak{R}_2, \varrho$ $(a_0 + a_1u_1 + a_2u_2) = ((a_0)^{\theta_t} + a_1)^{\theta_t}u_1 + (a_2)^{\theta_t}u_2$. Then, $|\varrho| = 3, ord(\varrho)|n$.

$$\begin{aligned} x^3 + 1 &= \left(\omega^{24} + \omega^7 x + \omega^8 x^2\right)\left(\omega^2 x + \omega^2\right) \\ &\in \mathbb{F}_{27}[x, \theta_t], x^3 - 1 = \left(\omega^{12} + \omega^{22}x + x^2\right)(x + \omega) \\ &\in \mathbb{F}_{27}[x, \theta_t]. \end{aligned} \tag{15}$$

Let C be a skew ϱ-$(\varsigma_1 + \varsigma_2 + (-1)\varsigma_3)$-constacyclic code of length 3 over $\mathfrak{R}_2$. Let $g(x) = \varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \varsigma_3 g_3(x)$, where $g_1(x) = g_2(x) = \omega^2 x + \omega^2$, $g_3(x) = x + \omega$. Then, $C_1 = \langle g_1(x)\rangle$ *and* $C_2 = \langle g_2(x)\rangle$ are skew negacyclic codes of length 3 over $\mathbb{F}_{27}$. $C_3 = \langle g_3(x)\rangle$ is a skew cyclic code of length 3 over $\mathbb{F}_{27}$. By Theorem 4, $\phi_2(C) = [9, 6, 3]_{27}$. Using Theorem 7, $C^{\perp} \subseteq C$. So, we can get a quantum code $[[9, 3, 3]]_{27}$ such that $n - k + 2 - 2d = 2$.

Example 2. Let $n = 8$ and $\mathfrak{R}_3 = \mathbb{F}_9[u_1, u_2, u_3]/ \langle u_i^2 = -u_i, u_iu_j = u_ju_i = 0\rangle, \varsigma_1 = -u_1, \varsigma_2 = -u_2, \varsigma_3 = -u_3$, $\varsigma_4 = 1 + u_1 + u_2 + u_3$, $\forall a \in \mathbb{F}_9$ and $\theta_t\colon \mathbb{F}_9 \longrightarrow \mathbb{F}_9$ is defined by $\theta_t(a) = a^3$, and $\forall (a_0 + a_1u_1 + a_2u_2 + a_3u_3) \in \mathfrak{R}_3$ and $\varrho(a_0 + a_1u_1 + a_2u_2 + a_3u_3) = (a_0)^{\theta_t} + (a_1)^{\theta_t}u_1 + (a_2)^{\theta_t}u_2 + (a_3)^{\theta_t}u_3$. Then, $|\varrho| = 2, ord(\varrho)|n$.

$$\begin{aligned} x^8 - 1 &= \left(\omega^2 x^5 + \omega^2 x^4 + \omega x^3 + \omega x^2 + 2x + 2\right) \\ &\quad \left(\omega^2 x^3 + \omega^2 x^2 + 2x + 1\right) \in \mathbb{F}_9[x, \theta_t], x^8 + 1 \\ &= \left(\omega^7 x^6 + \omega^3 x^5 + \omega^5 x^4 + \omega^7 x^3 + \omega^6 x^2 + \omega^7 x + 1\right) \\ &\quad \left(\omega x^2 + \omega^3 x + 1\right) \in \mathbb{F}_9[x, \theta_t]. \end{aligned} \tag{16}$$

Let C be a skew ϱ-$(\varsigma_1 + \varsigma_2 + (-1)\varsigma_3 + (-1)\varsigma_4)$-constacyclic code of length 8 over $\mathfrak{R}_3$. Let $g(x) = \varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \varsigma_3 g_3(x) + \varsigma_4 g_4(x)$, where $g_1(x) = g_2(x) = \omega^2 x^3 + \omega^2 x^2 + 2x + 1$, $g_3(x) = g_4(x) = \omega x^2 + \omega^3 x + 1$. Then, $C_1 = \langle g_1(x)\rangle$ and $C_2 = \langle g_2(x)\rangle$ are skew cyclic codes of length 8 over $\mathbb{F}_9$. $C_3 = \langle g_3(x)\rangle$ and $C_4 = \langle g_4(x)\rangle$ are skew negacyclic codes of length 8 over $\mathbb{F}_9$. By Theorem 4, $\phi_3(C) = [32, 22, 4]_9$. By Theorem 7, $C^{\perp} \subseteq C$. So, we can get a quantum code $[[32, 12, 4]]_9$.

Example 3. Let $n = 12$ and $\mathfrak{R}_3 = \mathbb{F}_9[u_1, u_2, u_3]/\langle u_i^2 = -u_i, u_iu_j = u_ju_i = 0\rangle, \varsigma_1 = -u_1, \varsigma_2 = -u_2, \varsigma_3 = -u_3, \varsigma_4 = 1 + u_1 + u_2 + u_3$, $\forall a \in \mathbb{F}_9$, and $\theta_t\colon \mathbb{F}_9 \longrightarrow \mathbb{F}_9$ is defined by $\theta_t(a) = a^3$, and $\forall (a_0 + a_1u_1 + a_2u_2 + a_3u_3) \in \mathfrak{R}_3$ and $\varrho(a_0 + a_1u_1 + a_2u_2 + a_3u_3) = (a_0)^{\theta_t} + (a_1)^{\theta_t}u_1 + (a_2)^{\theta_t}u_2 + (a_3)^{\theta_t}u_3$. Then, $|\varrho| = 2, ord(\varrho)|n$.

$$\begin{aligned} x^{12} - 1 &= \left(\omega^2 x^{10} + \omega^2 x^8 + \omega^2 x^6 + \omega^2 x^4 + \omega^2 x^2 + \omega^2\right) \\ &\quad \left(\omega^6 x^2 + \omega^2\right) \in \mathbb{F}_9[x, \theta_t], x^{12} + 1 \\ &= \left(x^{10} + \omega^6 x^8 + 2x^6 + \omega^2 x^4 + x^2 + \omega^6\right) \\ &\quad \left(x^2 + \omega^2\right) \in \mathbb{F}_9[x, \theta_t]. \end{aligned} \tag{17}$$

Let C be a skew ϱ-$(\varsigma_1 + (-1)\varsigma_2 + \varsigma_3 + (-1)\varsigma_4)$-constacyclic code of length 12 over $\mathfrak{R}_3$. Let $g(x) = \varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \varsigma_3 g_3(x) + \varsigma_4 g_4(x)$, where $g_1(x) = g_3(x) = \omega^6 x^2 + \omega^2$ and $g_2(x) = g_4(x) = x^2 + \omega^2$. Then, $C_1 = \langle g_1(x)\rangle$ and $C_3 = \langle g_3(x)\rangle$ are skew cyclic codes of length 12 over $\mathbb{F}_9$, and $C_2 = \langle g_2(x)\rangle$ and $C_4 = \langle g_4(x)\rangle$ are skew negacyclic codes of length 12 over $\mathbb{F}_9$. By Theorem 4, $\phi_3(C) = [48, 40, 3]_9$. By Theorem 7, $C^{\perp} \subseteq C$. So, we can get a quantum code $[[48, 32, 3]]_9$, which has larger dimension than $[[48, 24, 3]]_9$ in [21].

In Table 1, some new quantum codes are given from skew ϱ-λ constacyclic over $\mathfrak{R}_k$. Our quantum codes $[[18, 12, 3]]_{27}$, $[[9, 3, 3]]_{27}$, $[[18, 12, 3]]_{47}$, $[[18, 12, 3]]_{169}$, $[[28, 22, 3]]_{49}$ have the parameters such that $n - k - 2d + 2 = 2$. These codes are approached quantum MDS codes (satisfying quantum singleton bound $n - k - 2d + 2 = 0$). Moreover, our obtained quantum codes $[[48, 32, 3]]_9$, $[[40, 30, 3]]_{25}$, $[[56, 46, 3]]_{49}$ have larger dimensions than the quantum codes $[[48, 24, 3]]_9$, $[[40, 24, 3]]_{25}$, $[[56, 40, 3]]_{49}$ in [21].

Table 1: Quantum codes from skew ϱ-λ-constacyclic over $\mathfrak{R}_k$.

q	n	k	$(\lambda_1, \cdots, \lambda_{k+1})$	$<g_1(x), \cdots, g_{k+1}(x)>$	$\phi_k(C)$	$[[n, l, d]]_q$	$[[n', k', d']]_q$
49	6	2	$(1, -1, -1)$	$(31, \omega 1, \omega 1)$	$[18, 15, 3]$	$[[18, 12, 3]]_{49}$	$n - k - 2d + 2 = 0$
27	3	2	$(1, 1, -1)$	$(1\omega, 1\omega, \omega^2\omega^2)$	$[9, 6, 3]$	$[[9, 3, 3]]_{27}$	$n - k - 2d + 2 = 0$
49	6	2	$(1, -1, -1)$	$(31, \omega 1, \omega 1)$	$[18, 15, 3]$	$[[18, 12, 3]]_{49}$	$n - k - 2d + 2 = 0$
169	6	2	$(1, -1, 1)$	$(\omega^{16}1, \omega^{38}1, \omega^{16}1)$	$[18, 15, 3]$	$[[28, 22, 3]]_{49}$	$n - k - 2d + 2 = 0$
49	14	1	$(1, -1)$	$(1\omega^{22}1, \omega^3 1)$	$[28, 25, 3]$	$[[48, 32, 3]]_9$	$n - k - 2d + 2 = 0$
9	12	3	$(1, -1, 1, -1)$	$(\omega^6 0\omega^2, 10\omega^2, \omega^6 0\omega^2, 10\omega^2)$	$[48, 40, 3]$	$[[48, 32, 3]]_9$	$[[48, 32, 3]]_9$ [21]
25	10	3	$(1, 1, -1, -1)$	$(\omega^8\omega^4 1, \omega^4 1, \omega^{10}1, \omega^{10}1)$	$[40, 35, 3]$	$[[40, 30, 3]]_{25}$	$[[40, 24, 3]]_{25}$ [21]
49	14	3	$(1, -1, -1, 1)$	$(1\omega^{22}1, \omega^3 1, \omega^3 1, \omega^6 1)$	$[56, 51, 3]$	$[[56, 46, 3]]_{49}$	$[[56, 46, 3]]_{49}$ [21]

5. Conclusions

In this article, we construct quantum codes by studying the structure of skew ϱ-λ-constacyclic codes over a finite non-chain ring $\mathfrak{R}_k = \mathbb{F}_q[u_1, u_2, \cdots, u_k]/\langle u_i^2 = \alpha_i u_i, u_i u_j = u_j u_i = 0\rangle$, where $q = p^m$, p is an odd prime, and α_i is a unit over $\mathbb{F}_q$, $i, j = 1, 2, \cdots, k$. The major contributions are as follows: we study the structure of skew ϱ-λ-constacyclic code of length n over $\mathfrak{R}_k$ and give the necessary and sufficient conditions of dual-containing skew constacyclic codes. Our results will enrich the code source of quantum codes. Besides, we obtain some new quantum codes from skew ϱ-λ -constacyclic over $\mathfrak{R}_k$ by using the CSS construction. Our obtained quantum codes are approached quantum MDS codes or have larger dimensions than [21].

Data Availability

All data generated or analysed during this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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