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Retraction

Retracted: More General Form of Interval-Valued Fuzzy Ideals of BCK/BCI-Algebras

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] G. Muhiuddin, D. Al-Kadi, and A. Mahboob, "More General Form of Interval-Valued Fuzzy Ideals of BCK/BCI-Algebras," *Security and Communication Networks*, vol. 2021, Article ID 9930467, 10 pages, 2021.

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Research Article

More General Form of Interval-Valued Fuzzy Ideals of BCK/BCI-Algebras

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The concepts of interval-valued $(\in, \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy subalgebras, interval-valued $(\in, \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy ideals, and interval-valued $(\in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}), \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy ideals are introduced, and related properties are studied. Many examples are given in support of these new notions. Furthermore, interval-valued $(\in, \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy commutative ideals are defined, and some important properties are discussed. For a BCK-algebra X, it is proved that every interval-valued $(\in, \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy commutative ideal of BCK-algebra X is an interval-valued $(\in, \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy ideal of X, but the converse need not be true, in general, and then a counterexample is constructed.

1. Introduction

As an extension of fuzzy sets, Zadeh defined fuzzy sets with an interval-valued membership function proposing the concept of interval-valued fuzzy sets. This concept has been studied from various points of view in different algebraic structures as BCK-algebras and some of its generalization (see, for example, [1–5]), groups (see for example, [6–10]), and rings (see, for example, [11–13]). Moreover, as novel approaches in decision-making, theoretical models were introduced based on (fuzzy) soft sets in [14–19]. In BCK/BCI-algebras and other related algebraic structures, different kinds of related concepts were investigated in various ways (see, for example, [20–33]). Jun [34] studied interval-valued fuzzy ideals in BCI-algebras. Zhan et al. [35, 36] studied $(\in, \in\vee q)$-fuzzy ideals of BCI-algebras. The concept of "quasi-coincidence" of an interval-valued fuzzy point together with "belongingness" within an interval-valued fuzzy set was used in the studies made by Ma et al. in [37, 38] where they discussed properties of some types of $(\in, \in\vee q)$-interval-valued fuzzy ideals of BCI-algebras.

It is natural to introduce the general form of the existing interval-valued fuzzy ideals of BCK/BCI-algebras. For this purpose, we first recall in Section 2 some elementary notions used in the sequel. Then, in Section 3, we introduce the concepts: interval-valued $(\in, \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy subalgebras, interval-valued $(\in, \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy ideals, interval-valued $(\in, \in\vee q)$-fuzzy ideals, and interval-valued $(\in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}), \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy ideals, and related properties are studied. In Section 4, interval-valued $(\in, \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy commutative ideals are introduced, some properties are studied, and their relation with interval-valued $(\in, \in\vee(\widetilde{\kappa}^*, q_{\widetilde{\kappa}}))$-fuzzy ideals is investigated.

2. Preliminaries

An algebra $X = (X; *, 0)$ of type $(2, 0)$ is a BCI-algebra if for all $\upsilon, p, \hbar \in X$,

(1) $((\upsilon * p) * (\upsilon * \hbar)) * (\hbar * p) = 0$

(2) $(\upsilon * (\upsilon * p)) * p = 0$

(3) $\upsilon * \upsilon = 0$

(4) $\upsilon * p = 0$ and $p * \upsilon = 0 \Rightarrow \upsilon = p$

If X satisfies (1)–(4) and (5) $0 * \upsilon = 0$, then X is a BCK-algebra.

Any BCK/BCI-algebra X satisfies

(1) $\upsilon * 0 = \upsilon$

(2) $(\upsilon * p) * \hbar = (\upsilon * \hbar) * p$

From now on, let X be a BCK/BCI-algebra unless otherwise specified.

We define a partially ordered set $(X, \leq)$, where $\upsilon \leq \kappa \Leftrightarrow \upsilon * p = 0$.

A nonempty subset P of X is said to be a subalgebra of X if $\hbar * \upsilon \in P$ for all $\hbar, \upsilon \in P$.

A nonempty subset I of X is said to be an ideal of X if

(I_1) $0 \in I$

(I_2) $\forall \hbar, \upsilon \in I, \hbar * \upsilon \in I$, and $\upsilon \in I \Rightarrow \hbar \in I$

By an interval number $\tilde{a}$, we mean an interval, denoted by $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all interval numbers is denoted by $D[0,1]$. In whatever follows, the interval $[a, a]$ is identified by the number $a \in [0,1]$. For the interval numbers $\tilde{a}_i = [a_i^-, a_i^+]$ and $\tilde{b}_i = [b_i^-, b_i^+] \in D[0,1]$, $i \in I$, we define

$$\begin{gathered}
\min\{\tilde{a}_i, \tilde{b}_i\} = [\min(a_i^-, b_i^-), \min(a_i^+, b_i^+)],\\
\max\{\tilde{a}_i, \tilde{b}_i\} = [\max(a_i^-, b_i^-), \max(a_i^+, b_i^+)],\\
\tilde{a}_1 \leq \tilde{a}_2 \Leftrightarrow a_1^- \leq a_2^- \text{ and } a_1^+ \leq a_2^+,\\
\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow a_1^- = a_2^- \text{ and } a_1^+ = a_2^+,\\
\kappa\tilde{a} = [\kappa a^-, \kappa a^+], \quad \text{whenever } 0 \leq \kappa \leq 1,\\
\widetilde{\kappa^*} = [\kappa^{*-}, \kappa^{*+}], \quad \text{whenever } 0 < \kappa^* \leq 1.
\end{gathered} \tag{1}$$

A mapping $\widetilde{\mathscr{U}}: X \longrightarrow D[0,1]$ is called an interval-valued fuzzy subset (briefly, IVFS) of X, where $\widetilde{\mathscr{U}}(x) = [\widetilde{\mathscr{U}}^-(x), \widetilde{\mathscr{U}}^+(x)]$ for all $x \in X$, $\widetilde{\mathscr{U}}^-$ and $\widetilde{\mathscr{U}}^+$ are fuzzy sets of X with $\widetilde{\mathscr{U}}^-(x) \leq \widetilde{\mathscr{U}}^+(x)$ for all $x \in X$.

3. Interval-Valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}}))$-Fuzzy Ideals

Definition 1. Let $a \in X$ and $\tilde{\ell} \in D(0,1]$. An interval-valued ordered fuzzy point (briefly, IVOFP) $a_{\tilde{\ell}}$ of X is defined as:

$$a_{\tilde{\ell}}(x) = \begin{cases} \tilde{\ell}, & \text{if } x \in (a],\\ [0,0], & \text{if } x \notin (a], \end{cases} \tag{2}$$

for all $x \in X$.

Clearly, $a_{\tilde{\ell}}$ is an IVFS of X. For any IVFS $\widetilde{\mathscr{U}}$ of X, we denote $a_{\tilde{\ell}} \subseteq \widetilde{\mathscr{U}}$ as $a_{\tilde{\ell}} \in \widetilde{\mathscr{U}}$ in the sequel. So, $a_{\tilde{\ell}} \in \widetilde{\mathscr{U}} \Leftrightarrow \widetilde{\mathscr{U}}(a) \geq \tilde{\ell}$.

Definition 2. Let $\hbar_{\tilde{l}}$ be an IVOFP of X and $\widetilde{\kappa^*} \in D(0,1]$. Then, $\hbar_{\tilde{l}}$ is called $(\widetilde{\kappa^*}, q)$-quasi-coincident with an IVFS $\widetilde{\mathscr{U}}$ of X, represented as $\hbar_{\tilde{l}}(\widetilde{\kappa^*}, q)\widetilde{\mathscr{U}}$, if $\widetilde{\mathscr{U}}(\hbar) + \tilde{l} > \widetilde{\kappa^*}$.

Assume $[0,0] \leq \tilde{\kappa} < \widetilde{\kappa^*} \leq [1,1]$. For an IVOFP $\hbar_{\tilde{\ell}}$, we write

(1) $\hbar_{\tilde{\ell}}(\widetilde{\kappa^*}, q_{\tilde{\kappa}})\widetilde{\mathscr{U}}$ if $\widetilde{\mathscr{U}}(\hbar) + \tilde{\ell} + \tilde{\kappa} > \widetilde{\kappa^*}$

(2) $\hbar_{\tilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}})\widetilde{\mathscr{U}}$ if $\hbar_{\tilde{\ell}} \in \widetilde{\mathscr{U}}$ or $\hbar_{\tilde{\ell}}(\widetilde{\kappa^*}, q_{\tilde{\kappa}})\widetilde{\mathscr{U}}$

(3) $\hbar_{\tilde{\ell}}\overline{\alpha}\widetilde{\mathscr{U}}$ if $\hbar_{\tilde{\ell}}\alpha\widetilde{\mathscr{U}}$ does not hold for $\alpha \in \{(\widetilde{\kappa^*}, q_{\tilde{\kappa}}), \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}})\}$

Definition 3. An IVFS $\widetilde{\mathscr{U}}$ of X is called an interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}}))$-fuzzy subalgebra (in short, IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}}))$-FS) of X if $\hbar_{\tilde{\ell}} \in \widetilde{\mathscr{U}}$ and $\upsilon_{\tilde{j}} \in \widetilde{\mathscr{U}}$ imply $(\hbar * \upsilon)_{\min\{\tilde{\ell}, \tilde{j}\}} \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}})\widetilde{\mathscr{U}}$ for all $\tilde{\ell}, \tilde{j} \in D(0,1]$ and $\hbar, \upsilon \in X$.

Theorem 1. *An IVFS $\widetilde{\mathscr{U}}$ of X is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}}))$-FS of $X \Leftrightarrow$*

$$\widetilde{\mathscr{U}}(\hbar * \upsilon) \geq \min\left\{\widetilde{\mathscr{U}}(\hbar), \widetilde{\mathscr{U}}(\upsilon), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}, \tag{3}$$

for all $\hbar, \upsilon \in X$.

Proof. $(\Rightarrow)$ On the contrary, suppose that $\widetilde{\mathscr{U}}(\hbar * \upsilon) < \min\{\widetilde{\mathscr{U}}(\hbar), \widetilde{\mathscr{U}}(\upsilon), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$, for some $\hbar, \upsilon \in X$. Choose $\omega \in D(0,1]$ such that

$$\widetilde{\mathscr{U}}(\hbar * \upsilon) < \omega \leq \min\left\{\widetilde{\mathscr{U}}(\hbar), \widetilde{\mathscr{U}}(\upsilon), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}. \tag{4}$$

Then, $\hbar_\omega \in \widetilde{\mathscr{U}}$ and $\upsilon_\omega \in \widetilde{\mathscr{U}}$, but $(\hbar * \upsilon)_\omega \overline{\in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}})}\widetilde{\mathscr{U}}$, which is impossible. Hence, $\widetilde{\mathscr{U}}(\hbar * \upsilon) \geq \min\{\widetilde{\mathscr{U}}(\hbar), \widetilde{\mathscr{U}}(\upsilon), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$.

$(\Leftarrow)$ Assume that $\widetilde{\mathscr{U}}(\hbar * \upsilon) \geq \min\{\widetilde{\mathscr{U}}(\hbar), \widetilde{\mathscr{U}}(\upsilon), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$ for all $\hbar, \upsilon \in X$. Let $\hbar_{\tilde{\ell}} \in \widetilde{\mathscr{U}}$ and $\upsilon_{\tilde{j}} \in \widetilde{\mathscr{U}}$ for all $\tilde{\ell}, \tilde{j} \in D(0,1]$. Then, $\widetilde{\mathscr{U}}(\hbar) \geq \tilde{\ell}$ and $\widetilde{\mathscr{U}}(\upsilon) \geq \tilde{j}$. So, $\widetilde{\mathscr{U}}(\hbar * \upsilon) \geq \min\{\widetilde{\mathscr{U}}(\hbar), \widetilde{\mathscr{U}}(\upsilon), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\} \geq \min\{\tilde{\ell}, \tilde{j}, [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$. If $\min\{\tilde{\ell}, \tilde{j}\} \leq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then $\widetilde{\mathscr{U}}(\hbar * \upsilon) \geq \min\{\tilde{\ell}, \tilde{j}\}$ implies that $(\hbar * \upsilon)_{\min\{\tilde{\ell}, \tilde{j}\}} \in \widetilde{\mathscr{U}}$. If $\min\{\tilde{\ell}, \tilde{j}\} > [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then $\widetilde{\mathscr{U}}(\hbar * \upsilon) \geq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$. So, $\widetilde{\mathscr{U}}(\hbar * \upsilon) + \min\{\tilde{\ell}, \tilde{j}\} > [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)] + [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)] = [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$ implies that $(\hbar * \upsilon)_{\min\{\tilde{\ell}, \tilde{j}\}}(\widetilde{\kappa^*}, q_{\tilde{\kappa}})\widetilde{\mathscr{U}}$. Hence, $(\hbar * \upsilon)_{\min\{\tilde{\ell}, \tilde{j}\}} \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}})\widetilde{\mathscr{U}}$. Therefore, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}}))$-FS of X. □

Definition 4. An IVFS $\widetilde{\mathscr{U}}$ of X is called an interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}}))$-fuzzy ideal (in short, IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}}))$-FI) of X if

(1) $\hbar_{\tilde{\ell}} \in \widetilde{\mathscr{U}}$ implies $0_{\tilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}})\widetilde{\mathscr{U}}$

(2) $(\hbar * \upsilon)_{\tilde{\ell}} \in \widetilde{\mathscr{U}}$ and $\upsilon_{\tilde{j}} \in \widetilde{\mathscr{U}}$ imply $\hbar_{\min\{\tilde{\ell}, \tilde{j}\}} \in\vee(\widetilde{\kappa^*}, q_{\tilde{\kappa}})\widetilde{\mathscr{U}}$

for all $\hbar, \upsilon \in X$ and $\tilde{\ell}, \tilde{j} \in D(0,1]$.

Example 1. Consider a BCI-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ as defined in Table 1.

Define $\widetilde{\mathscr{U}}: X \longrightarrow D[0,1]$ by

$$\widetilde{\mathscr{U}}(\hbar) = \begin{cases} [0.9, 1], & \text{if } \hbar = 0,\\ [0.3, 0.4], & \text{if } \hbar \in \{1, 2, 3\}. \end{cases} \tag{5}$$

Table 1: Cayley Table of the binary operation*.

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Choose $\widetilde{\kappa^*} = [0.2, 0.3]$ and $\widetilde{\kappa} = [0.1, 0.2]$. Then, with direct computation, we find that $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X.

Definition 5. An IVFS $\widetilde{\mathscr{U}}$ of X is called an interval-valued $(\in, \in\vee q)$-fuzzy ideal (in short, IV $(\in, \in\vee q)$-FI) of X if

(1) $\hbar_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ implies $0_{\widetilde{\ell}} \in\vee q\widetilde{\mathscr{U}}$

(2) $(\hbar * \upsilon)_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ and $\upsilon_{\widetilde{j}} \in \widetilde{\mathscr{U}}$ imply $\hbar_{\min\{\widetilde{\ell},\widetilde{j}\}} \in\vee q\widetilde{\mathscr{U}}$

for all $\hbar, \upsilon \in X$ and $\widetilde{\ell}, \widetilde{j} \in D(0, 1]$.

Theorem 2. *In X, every IV $(\in, \in\vee q)$-FI is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI.*

Proof. Let $\widetilde{\mathscr{U}}$ be any IV $(\in, \in\vee q)$-FI of X. Take any $\hbar_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ for $\hbar \in X$ and $\widetilde{\ell} \in D(0, 1]$. Then, by hypothesis, $0_{\widetilde{\ell}} \in\vee q\widetilde{\mathscr{U}}$. It follows that $\widetilde{\mathscr{U}}(0) \geq \widetilde{\ell}$ or $\widetilde{\mathscr{U}}(0) + \widetilde{\ell} \geq [1, 1]$, and so, $\widetilde{\mathscr{U}}(0) \geq \widetilde{\ell}$ or $\widetilde{\mathscr{U}}(0) + \widetilde{\kappa} + \widetilde{\ell} \geq \widetilde{\kappa^*}$. Therefore, $0_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. Next, let $(\hbar * \upsilon)_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ and $\upsilon_{\widetilde{j}} \in \widetilde{\mathscr{U}}$. So, $\hbar_{\min\{\widetilde{\ell},\widetilde{j}\}} \in\vee q\widetilde{\mathscr{U}}$ implies $\widetilde{\mathscr{U}}(\hbar) \geq \min\{\widetilde{\ell}, \widetilde{j}\}$ or $\widetilde{\mathscr{U}}(\hbar) + \min\{\widetilde{\ell}, \widetilde{j}\} > [1, 1]$. Therefore, $\widetilde{\mathscr{U}}(\hbar) \geq \min\{\widetilde{\ell}, \widetilde{j}\}$ or $\widetilde{\mathscr{U}}(\hbar) + \widetilde{\kappa} + \min\{\widetilde{\ell}, \widetilde{j}\} > \widetilde{\kappa^*}$. Thus, $\hbar_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. Hence, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X. □

Example 2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ as defined in Table 2.

Define $\widetilde{\mathscr{U}}\colon X \longrightarrow D[0, 1]$ by

$$\widetilde{\mathscr{U}}(\hbar) = \begin{cases} [0.4, 0.5], & \text{if } \hbar = 0, \\ [0.2, 0.3], & \text{if } \hbar \in \{1, 2\}, \\ [0.5, 0.6], & \text{if } \hbar = 3, \\ [0.1, 0.2], & \text{if } \hbar = 4. \end{cases} \tag{6}$$

Choose $\widetilde{\kappa^*} = [0.2, 0.3]$ and $\widetilde{\kappa} = [0.1, 0.2]$. Then, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X but is not an IV $(\in, \in\vee q)$-FI of X as $2_{\widetilde{\ell}=[0.6,0.6]} = (4 * 2)_{\widetilde{\ell}=[0.6,0.6]} \in \widetilde{\mathscr{U}}$ and $2_{\widetilde{j}=[0.6,0.6]} \in \widetilde{\mathscr{U}}$ but $4_{\min\{\widetilde{\ell},\widetilde{j}\}=[0.6,0.6]} \overline{\in\vee q}\widetilde{\mathscr{U}}$.

Definition 6. An IVFS $\widetilde{\mathscr{U}}$ of X is called an interval-valued $(\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}), \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy ideal (in short, IV $(\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}), \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI) of X if

(1) $\hbar_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$ implies $0_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$

(2) $(\hbar * \upsilon)_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$ and $\upsilon_{\widetilde{j}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$ imply $\hbar_{\min\{\widetilde{\ell},\widetilde{j}\}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$

for all $\hbar, \upsilon \in X$ and $\widetilde{\ell}, \widetilde{j} \in D(0, 1]$.

Theorem 3. *In X, every IV $(\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}), \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI.*

Table 2: Cayley Table of the binary operation*.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

Proof. Let $\widetilde{\mathscr{U}}$ be any IV $(\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}), \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X. Take any $\hbar_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ for $\hbar \in X$ and $\widetilde{\ell} \in D(0, 1]$. Then, $\hbar_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. So, by hypothesis, $0_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. Suppose that $(\hbar * \ell)_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ and $\ell_{\widetilde{j}} \in \widetilde{\mathscr{U}}$. Then, $(\hbar * \ell)_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$ and $\ell_{\widetilde{j}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. Therefore, by hypothesis, $\hbar_{\min\{\widetilde{\ell},\widetilde{j}\}} \in \vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. Hence, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X. □

Example 3. Consider a BCK-algebra of Example 2. Define $\widetilde{\mathscr{U}}\colon X \longrightarrow D[0, 1]$ by

$$\widetilde{\mathscr{U}}(\hbar) = \begin{cases} [0.4, 0.5], & \text{if } \hbar = 0, \\ [0.6, 0.7], & \text{if } \hbar \in \{1, 3\}, \\ [0.1, 0.2], & \text{if } \hbar = \{2, 4\}. \end{cases} \tag{7}$$

Choose $\widetilde{\kappa} = [0, 0]$ and $\widetilde{\kappa^*} = [0.7, 0.9]$. Then, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X but is not an IV $(\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}), \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X as $2_{\widetilde{\ell}=[0.95,0.95]} = (2 * 1)_{\widetilde{\ell}=[0.95,0.95]} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$ and $1_{\widetilde{j}=[0.5,0.6]} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$ but $2_{\min\{\widetilde{\ell},\widetilde{j}\}=[0.5,0.6]} \overline{\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})}\widetilde{\mathscr{U}}$.

Lemma 1. *Let $\widetilde{\mathscr{U}}$ be an IVFS of X. Then, $\hbar_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ implies $0_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}} \Leftrightarrow \forall \hbar \in X, \widetilde{\mathscr{U}}(0) \geq \min\{\widetilde{\mathscr{U}}(\hbar), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$.*

Proof. ($\Rightarrow$) On the contrary, suppose that, for some $\hbar \in X$, $\widetilde{\mathscr{U}}(0) < \min\{\widetilde{\mathscr{U}}(\hbar), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$. Take $\widetilde{\ell} \in D(0, ((\kappa^* - k)/2)]$ such that

$$\widetilde{\mathscr{U}}(0) < \widetilde{\ell} \leq \min\left\{\widetilde{\mathscr{U}}(\hbar), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}. \tag{8}$$

Then, $\hbar_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$, but $0_{\widetilde{\ell}} \overline{\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})}\widetilde{\mathscr{U}}$, a contradiction. Hence, $\widetilde{\mathscr{U}}(0) \geq \min\{\widetilde{\mathscr{U}}(\hbar), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$.

($\Leftarrow$) Let $\hbar \in X$ such that $\hbar_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$. Then, $\widetilde{\mathscr{U}}(\hbar) \geq \widetilde{\ell}$. So,

$$\begin{aligned} \widetilde{\mathscr{U}}(0) &\geq \min\left\{\widetilde{\mathscr{U}}(\hbar), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\} \\ &\geq \min\left\{\widetilde{\ell}, \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}. \end{aligned} \tag{9}$$

Now, if $\widetilde{\ell} \leq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then $\widetilde{\mathscr{U}}(0) \geq \widetilde{\ell}$. Therefore, $0_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$. On the contrary, if $\widetilde{\ell} > [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then $\widetilde{\mathscr{U}}(0) \geq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$. So, $\widetilde{\mathscr{U}}(0) + \widetilde{\ell} > [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)] + [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)] = [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$. This implies that $0_{\widetilde{\ell}}(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. Hence, $0_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. □

Lemma 2. *Let $\widetilde{\mathcal{U}}$ be an IVFS of X. Then, $(\hbar * v)_{\widetilde{\ell}} \in \widetilde{\mathcal{U}}$ and $v_{\widetilde{j}} \in \widetilde{\mathcal{U}}$ imply $(\hbar)_{\min\{\widetilde{\ell}, \widetilde{j}\}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathcal{U}} \Leftrightarrow \widetilde{\mathcal{U}}(\hbar) \geq \min \{\widetilde{\mathcal{U}}(\hbar * v), \widetilde{\mathcal{U}}(v), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$.*

Proof. ($\Rightarrow$) On the contrary, suppose that $\widetilde{\mathcal{U}}(\hbar) < \min \{\widetilde{\mathcal{U}}(\hbar * v), \widetilde{\mathcal{U}}(v), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$ for some $\hbar, v \in X$. Choose $\widetilde{\ell} \in D(0, ((\kappa^* - k)/2)]$ such that $\widetilde{\mathcal{U}}(\hbar) < \widetilde{\ell} \leq \min\{\widetilde{\mathcal{U}}(\hbar * v), \widetilde{\mathcal{U}}(v), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$. Then, $(\hbar * v)_{\widetilde{\ell}} \in \widetilde{\mathcal{U}}$ and $v_{\widetilde{\ell}} \in \widetilde{\mathcal{U}}$, but $\hbar_{\widetilde{\ell}} \overline{\in\vee}(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathcal{U}}$, which is not possible. Thus, we have shown that

$$\widetilde{\mathcal{U}}(\hbar) \geq \min\left\{\widetilde{\mathcal{U}}(\hbar * v), \widetilde{\mathcal{U}}(v), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}. \tag{10}$$

($\Leftarrow$) Let $(\hbar * v)_{\widetilde{\ell}} \in \widetilde{\mathcal{U}}$ and $v_{\widetilde{j}} \in \widetilde{\mathcal{U}}$ for all $\widetilde{\ell}, \widetilde{j} \in D(0, 1]$. Then, $\widetilde{\mathcal{U}}(\hbar * v) \geq \widetilde{\ell}$ and $\widetilde{\mathcal{U}}(v) \geq \widetilde{j}$. Thus,

$$\begin{aligned} \widetilde{\mathcal{U}}(\hbar) &\geq \min\left\{\widetilde{\mathcal{U}}(\hbar * v), \widetilde{\mathcal{U}}(v), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\} \\ &\geq \min\left\{\widetilde{\ell}, \widetilde{j}, \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}. \end{aligned} \tag{11}$$

Now, if $\min\{\widetilde{\ell}, \widetilde{j}\} \leq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then $\widetilde{\mathcal{U}}(\hbar) \geq \min\{\widetilde{\ell}, \widetilde{j}\}$ and $(\hbar)_{\min\{\widetilde{\ell}, \widetilde{j}\}} \in \widetilde{\mathcal{U}}$; otherwise, i.e., when $\min\{\widetilde{\ell}, \widetilde{j}\} > [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then $\widetilde{\mathcal{U}}(\hbar) \geq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$. So, we have

$$\widetilde{\mathcal{U}}(\hbar) + \min\{\widetilde{\ell}, \widetilde{j}\} > \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right] + \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right] = [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]. \tag{12}$$

This implies that $\hbar_{\min\{\widetilde{\ell}, \widetilde{j}\}} (\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathcal{U}}$. Hence, $\hbar_{\min\{\widetilde{\ell}, \widetilde{j}\}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathcal{U}}$, as required.

By combining Lemmas 1 and 2, we have the following theorem. □

Theorem 4. *An IVFS $\widetilde{\mathcal{U}}$ of X is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of $X \Leftrightarrow$*

(1) $\widetilde{\mathcal{U}}(0) \geq \min\{\widetilde{\mathcal{U}}(\hbar), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$ and

(2) $\widetilde{\mathcal{U}}(\hbar) \geq \min\{\widetilde{\mathcal{U}}(\hbar * v), \widetilde{\mathcal{U}}(v), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$

for all $\hbar, v \in X$.

Lemma 3. *Let $\widetilde{\mathcal{U}}$ be an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X such that $\hbar \leq v$. Then, $\widetilde{\mathcal{U}}(\hbar) \geq \min\{\widetilde{\mathcal{U}}(v), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$.*

Proof. Let $\hbar \leq v$ for $\hbar, v \in X$. Then, $\hbar * v = 0$. By hypothesis, we have

$$\begin{aligned} \widetilde{\mathcal{U}}(\hbar) &\geq \min\left\{\widetilde{\mathcal{U}}(\hbar * v), \mathcal{U}(v), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\} \\ &= \min\left\{\widetilde{\mathcal{U}}(0), \mathcal{U}(v), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\} \\ &\geq \min\left\{\widetilde{\mathcal{U}}(v), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}. \end{aligned} \tag{13}$$

□

Lemma 4. *Let $\widetilde{\mathcal{U}}$ be an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X. Then, for any $\hbar, v, z \in X$, $\hbar * v \leq z \Rightarrow \widetilde{\mathcal{U}}(\hbar) \geq \min\{\widetilde{\mathcal{U}}(v), \widetilde{\mathcal{U}}(z), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$.*

Proof. Suppose that $\hbar * v \leq z$ for $\hbar, v, z \in X$. Then, we have

$$\begin{aligned} \widetilde{\mathcal{U}}(\hbar) &\geq \min\left\{\widetilde{\mathcal{U}}(\hbar * v), \mathcal{U}(v), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\} \\ &\geq \min\left\{\min\left\{\widetilde{\mathcal{U}}(z), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}, \widetilde{\mathcal{U}}(v), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\} \\ &= \min\left\{\widetilde{\mathcal{U}}(v), \mathcal{U}(z), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}. \end{aligned} \tag{14}$$

□

Theorem 5. *Every IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of BCK-algebra X is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FS of X.*

Proof. Let $\widetilde{\mathcal{U}}$ be an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X and $v, \hbar \in X$. As $v * \hbar \leq v$ in X, by Lemma 3, we have

$$\widetilde{\mathscr{U}}(v * \hbar) \geq \min\left\{\widetilde{\mathscr{U}}(v), \left[\frac{\kappa^{*-} - \kappa^{-}}{2}, \frac{\kappa^{*+} - \kappa^{+}}{2}\right]\right\}. \tag{15}$$

Since $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X, we have

$$\begin{aligned}\widetilde{\mathscr{U}}(v * \hbar) &\geq \min\left\{\widetilde{\mathscr{U}}(v), \left[\frac{\kappa^{*-} - \kappa^{-}}{2}, \frac{\kappa^{*+} - \kappa^{+}}{2}\right]\right\}\\ &\geq \min\left\{\min\left\{\widetilde{\mathscr{U}}(v * \hbar), \mathscr{U}(\hbar), \left[\frac{\kappa^{*-} - \kappa^{-}}{2}, \frac{\kappa^{*+} - \kappa^{+}}{2}\right]\right\}, \left[\frac{\kappa^{*-} - \kappa^{-}}{2}, \frac{\kappa^{*+} - \kappa^{+}}{2}\right]\right\}\\ &\geq \min\left\{\widetilde{\mathscr{U}}(v), \widetilde{\mathscr{U}}(\hbar), \left[\frac{\kappa^{*-} - \kappa^{-}}{2}, \frac{\kappa^{*+} - \kappa^{+}}{2}\right]\right\}.\end{aligned} \tag{16}$$

Hence, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FS of X. □

Example 4. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ as defined in Table 3.

Consider the IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FS $\widetilde{\mathscr{U}}$ of X, where $\widetilde{\mathscr{U}}: X \longrightarrow D[0, 1]$ is defined by

$$\widetilde{\mathscr{U}}(\hbar) = \begin{cases} [0.3, 0.4], & \text{if } \hbar = 0,\\ [0.1, 0.2], & \text{if } \hbar \in \{1, 2\},\\ [0.2, 0.3], & \text{if } \hbar = 3.\end{cases} \tag{17}$$

Choose $\widetilde{\kappa^*} = [0.8, 0.9]$ and $\widetilde{\kappa} = [0.1, 0.2]$. Then, $\widetilde{\mathscr{U}}$ is not an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X as $0_{\widetilde{\ell}=[0.3,0.3]} = (1 * 3)_{\widetilde{\ell}=[0.3,0.3]} \in \widetilde{\mathscr{U}}$ and $3_{\widetilde{j}=[0.2,0.2]} \in \widetilde{\mathscr{U}}$ but $1_{\min\{\widetilde{\ell}, \widetilde{j}\}=[0.2,0.2]} \overline{\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})}\widetilde{\mathscr{U}}$.

Theorem 6. *Let $\widetilde{\mathscr{U}}$ be an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FS of X. Then, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI $\Leftrightarrow$ for all $v, \hbar, z \in X$ such that $v * \hbar \leq z$ implies $\widetilde{\mathscr{U}}(v) \geq \min\{\widetilde{\mathscr{U}}(\hbar), \widetilde{\mathscr{U}}(z), [((\kappa^{*-} - \kappa^{-})/2), ((\kappa^{*+} - \kappa^{+})/2)]\}$.*

Proof. $(\Rightarrow)$ It follows from Lemma 4.

$(\Leftarrow)$ Let $\widetilde{\mathscr{U}}$ be an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FS such that for all $v, \hbar, z \in X$ with $v * \hbar \leq z$ imply $\widetilde{\mathscr{U}}(v) \geq \min\{\widetilde{\mathscr{U}}(\hbar), \widetilde{\mathscr{U}}(z), [((\kappa^{*-} - \kappa^{-})/2), ((\kappa^{*+} - \kappa^{+})/2)]\}$. As $v * (v * \hbar) \leq \hbar$, by hypothesis,

$$\widetilde{\mathscr{U}}(v) \geq \min\left\{\widetilde{\mathscr{U}}(v * \hbar), \widetilde{\mathscr{U}}(\hbar), \left[\frac{\kappa^{*-} - \kappa^{-}}{2}, \frac{\kappa^{*+} - \kappa^{+}}{2}\right]\right\}. \tag{18}$$

Hence, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X. □

Theorem 7. *Let $\widetilde{\mathscr{U}}$ be an IVFS of X. Then, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of $X \Leftrightarrow$ the set $\widetilde{\mathscr{U}}_{\widetilde{\ell}} (\neq \varnothing)$ is an ideal of X for each $\widetilde{\ell} \in D(0, ((\kappa^* - \kappa)/2)]$.*

Proof. $(\Rightarrow)$ Let $\widetilde{\ell} \in D(0, ((\kappa^* - \kappa)/2)]$ such that $\widetilde{\mathscr{U}}_{\widetilde{\ell}} \neq \varnothing$. By Theorem 4, we have

$$\widetilde{\mathscr{U}}(0) \geq \min\left\{\widetilde{\mathscr{U}}(\hbar), \left[\frac{\kappa^{*-} - \kappa^{-}}{2}, \frac{\kappa^{*+} - \kappa^{+}}{2}\right]\right\}, \tag{19}$$

with $\hbar \in \widetilde{\mathscr{U}}_{\widetilde{\ell}}$. It follows that $\widetilde{\mathscr{U}}(0) \geq \min\{\widetilde{\ell}, [((\kappa^{*-} - \kappa^{-})/2), ((\kappa^{*+} - \kappa^{+})/2)]\} = \widetilde{\ell}$. Therefore, $0 \in \widetilde{\mathscr{U}}_{\widetilde{\ell}}$.

Next, suppose that $\hbar * v \in \widetilde{\mathscr{U}}_{\widetilde{\ell}}$ and $v \in \widetilde{\mathscr{U}}_{\widetilde{\ell}}$. Then, $\widetilde{\mathscr{U}}(\hbar * v) \geq \widetilde{\ell}$ and $\widetilde{\mathscr{U}}(v) \geq \widetilde{\ell}$. Again, by Theorem 4, we have

$$\begin{aligned}\widetilde{\mathscr{U}}(\hbar) &\geq \min\left\{\widetilde{\mathscr{U}}(\hbar * v), \mathscr{U}(v), \left[\frac{\kappa^{*-} - \kappa^{-}}{2}, \frac{\kappa^{*+} - \kappa^{+}}{2}\right]\right\}\\ &\geq \min\left\{\widetilde{\ell}, \left[\frac{\kappa^{*-} - \kappa^{-}}{2}, \frac{\kappa^{*+} - \kappa^{+}}{2}\right]\right\} = \widetilde{\ell}.\end{aligned} \tag{20}$$

Therefore, $\hbar \in \widetilde{\mathscr{U}}_{\widetilde{\ell}}$. Hence, $\widetilde{\mathscr{U}}_{\widetilde{\ell}}$ is an ideal of X.

$(\Leftarrow)$ Suppose that $\widetilde{\mathscr{U}}_{\widetilde{\ell}}$ is an ideal of X for all $\widetilde{\ell} \in D(0, ((\kappa^* - k)/2)]$. If $\widetilde{\mathscr{U}}(0) < \min\{\widetilde{\mathscr{U}}(\hbar), [((\kappa^{*-} - \kappa^{-})/2), ((\kappa^{*+} - \kappa^{+})/2)]\}$ for some $\hbar \in X$, then $\exists \widetilde{\ell} \in D(0, ((\kappa^* - k)/2)]$ such that $\widetilde{\mathscr{U}}(0) < \widetilde{\ell} \leq \min\{\widetilde{\mathscr{U}}(\hbar), [((\kappa^{*-} - \kappa^{-})/2), ((\kappa^{*+} - \kappa^{+})/2)]\}$. It follows that $\hbar \in \widetilde{\mathscr{U}}_{\widetilde{\ell}}$ but $0 \notin \widetilde{\mathscr{U}}_{\widetilde{\ell}}$, a contradiction. Therefore, $\widetilde{\mathscr{U}}(0) \geq \min\{\widetilde{\mathscr{U}}(\hbar), [((\kappa^{*-} - \kappa^{-})/2), ((\kappa^{*+} - \kappa^{+})/2)]\}$. Also, if $\widetilde{\mathscr{U}}(\hbar) < \min\{\widetilde{\mathscr{U}}(\hbar * v), \widetilde{\mathscr{U}}(v), [((\kappa^{*-} - \kappa^{-})/ 2), ((\kappa^{*+} - \kappa^{+})/2)]\}$ for some $\hbar, v \in X$, then $\exists \widetilde{\ell} \in D(0, ((\kappa^* - k)/2)]$ such that

$$\widetilde{\mathscr{U}}(\hbar) < \widetilde{\ell} \leq \min\left\{\widetilde{\mathscr{U}}(\hbar * v), \widetilde{\mathscr{U}}(v), \left[\frac{\kappa^{*-} - \kappa^{-}}{2}, \frac{\kappa^{*+} - \kappa^{+}}{2}\right]\right\}. \tag{21}$$

It follows that $\hbar * v \in \widetilde{\mathscr{U}}_{\widetilde{\ell}}$ and $v \in \widetilde{\mathscr{U}}_{\widetilde{\ell}}$ but $\hbar \notin \widetilde{\mathscr{U}}_{\widetilde{\ell}}$, which is again a contradiction. Therefore, $\widetilde{\mathscr{U}}(\hbar) \geq \min\{\widetilde{\mathscr{U}}(\hbar * v), \widetilde{\mathscr{U}}(v), [((\kappa^{*-} - \kappa^{-})/2), ((\kappa^{*+} - \kappa^{+})/2)]\}$. Hence, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X. □

Definition 7. Let $\widetilde{\mathscr{U}}$ be an IVFS of X. The set

$$[\widetilde{\mathscr{U}}]_{\widetilde{\ell}} = \left\{\hbar \in X \mid \hbar_{\widetilde{\ell}} \in\vee\left(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}\right)\widetilde{\mathscr{U}}\right\}, \quad \text{where } \widetilde{\ell} \in D(0, 1], \tag{22}$$

is called an $(\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-level subset of $\widetilde{\mathscr{U}}$.

Theorem 8. *Let $\widetilde{\mathscr{U}}$ be an IVFS of X. Then, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of $X \Leftrightarrow$ the $(\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-level subset $[\widetilde{\mathscr{U}}]_{\widetilde{\ell}}$ of $\widetilde{\mathscr{U}}$ is an ideal of X for each $\widetilde{\ell} \in D(0, 1]$.*

Table 3: Cayley Table of the binary operation*.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Proof. ($\Rightarrow$) Suppose $\widetilde{\mathcal{U}}$ is an interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X. Take any $\hbar \in [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$. Then, $\hbar_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{k}})\widetilde{\mathcal{U}}$. So, $\widetilde{\mathcal{U}}(\hbar) \geq \widetilde{\ell}$ or $\widetilde{\mathcal{U}}(\hbar) + \widetilde{\ell} > [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$. Now, by Theorem 4, we have $\widetilde{\mathcal{U}}(0) \geq \min\{\widetilde{\mathcal{U}}(\hbar), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$. Thus, $\widetilde{\mathcal{U}}(0) \geq \min\{\widetilde{\ell}, [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$ when $\widetilde{\mathcal{U}}(\hbar) \geq \widetilde{\ell}$. If $\widetilde{\ell} > [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then $\widetilde{\mathcal{U}}(0) \geq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$ implies $0 \in [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$. Also, if $\widetilde{\ell} \leq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then $\widetilde{\mathcal{U}}(0) \geq \widetilde{\ell}$ implies $0 \in [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$. Similarly, $0 \in [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$ when $\widetilde{\mathcal{U}}(\hbar) + \widetilde{\ell} > [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$.

Next, take any $\hbar * \upsilon \in [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$ and $\upsilon \in [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$. Then, $(\hbar * \upsilon)_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{k}})\widetilde{\mathcal{U}}$ and $\upsilon_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{k}})\widetilde{\mathcal{U}}$, i.e., either $\widetilde{\mathcal{U}}(\hbar * \upsilon) \geq \widetilde{\ell}$ or $\widetilde{\mathcal{U}}(\hbar * \upsilon) + \widetilde{\ell} > [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$ and either $\widetilde{\mathcal{U}}(\upsilon) \geq \widetilde{\ell}$ or $\widetilde{\mathcal{U}}(\upsilon) + \widetilde{\ell} > [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$. By assumption, $\widetilde{\mathcal{U}}(\hbar) \geq \min\{\widetilde{\mathcal{U}}(\hbar * \upsilon), \widetilde{\mathcal{U}}(\upsilon), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$. Thus, the following cases arise. □

Case 1. Let $\widetilde{\mathcal{U}}(\hbar * \upsilon) \geq \widetilde{\ell}$ and $\widetilde{\mathcal{U}}(\upsilon) \geq \widetilde{\ell}$. If $\widetilde{\ell} > [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then

$$\begin{aligned}\widetilde{\mathcal{U}}(\hbar) &\geq \min\left\{\widetilde{\mathcal{U}}(\hbar * \upsilon), \widetilde{\mathcal{U}}(\upsilon), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}\\ &\geq \min\left\{\widetilde{\ell}, \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}\\ &= \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right],\end{aligned} \tag{23}$$

and so, $\hbar_{\widetilde{\ell}}(\widetilde{\kappa^*}, q_{\widetilde{k}})\widetilde{\mathcal{U}}$. If $\widetilde{\ell} \leq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then

$$\begin{aligned}\widetilde{\mathcal{U}}(\hbar) &\geq \min\left\{\widetilde{\mathcal{U}}(\hbar * \upsilon), \widetilde{\mathcal{U}}(\upsilon), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}\\ &\geq \min\left\{\widetilde{\ell}, \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}\\ &= \widetilde{\ell}.\end{aligned} \tag{24}$$

So, $\hbar_{\widetilde{\ell}} \in \widetilde{\mathcal{U}}$. Hence, $\hbar_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{k}})\widetilde{\mathcal{U}}$.

Case 2. Let $\widetilde{\mathcal{U}}(\hbar * \upsilon) \geq \widetilde{\ell}$ and $\widetilde{\mathcal{U}}(\upsilon) + \widetilde{\ell} \geq [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$. If $\widetilde{\ell} > [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then

$$\begin{aligned}\widetilde{\mathcal{U}}(\hbar) &\geq \min\left\{\widetilde{\mathcal{U}}(\hbar * \upsilon), \widetilde{\mathcal{U}}(\upsilon), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}\\ &\geq \min\{\widetilde{\ell}, ([\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+] - \widetilde{\ell}),\\ &\quad \cdot \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}\\ &= [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+] - \widetilde{\ell},\end{aligned} \tag{25}$$

i.e., $\widetilde{\mathcal{U}}(\hbar) + \widetilde{\ell} > [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$, and thus, $\hbar_{\widetilde{\ell}}(\widetilde{\kappa^*}, q_{\widetilde{k}})\widetilde{\mathcal{U}}$. If $\widetilde{\ell} \leq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then

$$\begin{aligned}\widetilde{\mathcal{U}}(\hbar) &\geq \min\left\{\widetilde{\mathcal{U}}(\hbar * \upsilon), \widetilde{\mathcal{U}}(\upsilon), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}\\ &\geq \min\{\widetilde{\ell}, ([\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+] - \widetilde{\ell}),\\ &\quad \cdot \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\} = \widetilde{\ell},\end{aligned} \tag{26}$$

and so, $\hbar_{\widetilde{\ell}} \in \widetilde{\mathcal{U}}$. Hence, $\hbar_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{k}})\widetilde{\mathcal{U}}$.

Similarly, in other two cases, i.e., when $\widetilde{\mathcal{U}}(\hbar * \upsilon)) + \widetilde{\ell} > [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$, $\widetilde{\mathcal{U}}(\upsilon) \geq \widetilde{\ell}$, and $\widetilde{\mathcal{U}}(\hbar * \upsilon) + \widetilde{\ell} > [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$, $\widetilde{\mathcal{U}}(\upsilon) + \widetilde{\ell} > [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]$, we have $\hbar_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{k}})\widetilde{\mathcal{U}}$. Hence, in each case, $\hbar_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{k}})\widetilde{\mathcal{U}}$, and thus, $\hbar \in [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$.

($\Leftarrow$) Let $[\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$ be an ideal of X for all $\widetilde{\ell} \in D(0,1]$. On the contrary, let

$$\widetilde{\mathcal{U}}(0) < \min\left\{\widetilde{\mathcal{U}}(\hbar), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}, \tag{27}$$

with $\hbar \in X$. Then, $\exists \widetilde{\ell} \in D(0,1]$ such that $\widetilde{\mathcal{U}}(0) < \widetilde{\ell} \leq \min\{\widetilde{\mathcal{U}}(\hbar), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$. It follows that $\hbar \in [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$, but $0 \notin [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$, which is not possible. Therefore,

$$\widetilde{\mathcal{U}}(0) \geq \min\left\{\widetilde{\mathcal{U}}(\hbar), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}. \tag{28}$$

Also, if $\widetilde{\mathcal{U}}(\hbar) < \min\{\widetilde{\mathcal{U}}(\hbar * \upsilon), \widetilde{\mathcal{U}}(\upsilon), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$ for some $\hbar, \upsilon \in X$, then $\exists \widetilde{\ell} \in D(0,1]$ such that

$$\widetilde{\mathcal{U}}(\hbar) < \widetilde{\ell} \leq \min\left\{\widetilde{\mathcal{U}}(\hbar * \upsilon), \widetilde{\mathcal{U}}(\upsilon), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}. \tag{29}$$

It follows that $\hbar * \upsilon \in [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$ and $\upsilon \in [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$ but $\hbar \notin [\widetilde{\mathcal{U}}]_{\widetilde{\ell}}$, which is again a contradiction. Therefore, $\widetilde{\mathcal{U}}(\hbar) \geq \min\{\widetilde{\mathcal{U}}(\hbar * \upsilon), \widetilde{\mathcal{U}}(\upsilon), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$. Hence, $\widetilde{\mathcal{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X.

4. Interval-Valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-Fuzzy Commutative Ideals

Throughout the following sections, X denotes BCK-algebras unless stated otherwise.

Definition 8. Let X be a BCK-algebra. An IVFS $\widetilde{\mathscr{U}}$ is called an interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy commutative ideal (in short, IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FCI) if for all $v, p, \hbar \in X$:

(1) $\hbar_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ implies $0_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$

(2) $((v * p) * \hbar)_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ and $\hbar_{\widetilde{j}} \in \widetilde{\mathscr{U}}$ imply $(v * (p * (p * v)))_{\min\{\widetilde{\ell}, \widetilde{j}\}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$

Example 5. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ as defined in Table 4.

Define $\widetilde{\mathscr{U}}: X \longrightarrow D[0, 1]$ by

$$\widetilde{\mathscr{U}}(\hbar) = \begin{cases} [0.5, 0.6], & \text{if } \hbar = 0, \\ [0.3, 0.4], & \text{if } \hbar \in \{1, 2\}, \\ [0.1, 0.2], & \text{if } \hbar = 3. \end{cases} \tag{30}$$

Choose $\widetilde{\kappa^*} = [0.2, 0.3]$ and $\widetilde{\kappa} = [0.1, 0.2]$. Then, it is easy to see that $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FCI of X.

Theorem 9. *An IVFS $\widetilde{\mathscr{U}}$ of X is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FCI of $X \Leftrightarrow$*

(1) $\widetilde{\mathscr{U}}(0) \geq \min\{\widetilde{\mathscr{U}}(\hbar), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$ and

(2) $\widetilde{\mathscr{U}}(v * (p * (p * v))) \geq \min\{\widetilde{\mathscr{U}}((v * p) * \hbar), \widetilde{\mathscr{U}}(\hbar), [((\kappa^{*-} - \kappa^-)/ 2), ((\kappa^{*+} - \kappa^+)/2)]\}$

for all $v, p, \hbar \in X$.

Table 4: Cayley Table of the binary operation $*$.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Proof. $(\Rightarrow)$ Condition (1) follows from Lemma 1. To show that (2) holds in X, suppose, on the contrary, that (2) does not hold in X, so we have

$$\widetilde{\mathscr{U}}(v * (p * (p * v))) < \min\left\{\widetilde{\mathscr{U}}((v * p) * \hbar), \widetilde{\mathscr{U}}(\hbar), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}, \tag{31}$$

for some $v, p, \hbar \in X$. Choose $\widetilde{\ell} \in D[0, 1]$ such that $\widetilde{\mathscr{U}}(v * (p * (p * v))) < \widetilde{\ell} \leq \min\{\widetilde{\mathscr{U}}((v * p) * \hbar), \widetilde{\mathscr{U}}(\hbar), [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]\}$. Then, $((v * p) * \hbar)_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ and $\hbar_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$, but $(v * (p * (p * v)))_{\widetilde{\ell}} \overline{\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})}\widetilde{\mathscr{U}}$, which is not possible. Thus,

$$\widetilde{\mathscr{U}}(v * (p * (p * v))) \geq \min\left\{\widetilde{\mathscr{U}}((v * p) * \hbar), \widetilde{\mathscr{U}}(\hbar), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}, \tag{32}$$

for each $v, p, \hbar \in X$.

$(\Leftarrow)$ Suppose that conditions (1) and (2) hold in X. It follows from condition (1) and Lemma 1 that $\hbar_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ implies $0_{\widetilde{\ell}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. Let $((v * p) * \hbar)_{\widetilde{\ell}} \in \widetilde{\mathscr{U}}$ and $\hbar_{\widetilde{j}} \in \widetilde{\mathscr{U}}$ for all $\widetilde{\ell}, \widetilde{j} \in D(0, 1]$. Then, $\widetilde{\mathscr{U}}((v * p) * \hbar) \geq \widetilde{\ell}$ and $\widetilde{\mathscr{U}}(\hbar) \geq \widetilde{j}$. Thus,

$$\begin{aligned} \widetilde{\mathscr{U}}(v * (p * (p * v))) &\geq \min\left\{\widetilde{\mathscr{U}}((v * p) * \hbar), \widetilde{\mathscr{U}}(\hbar), \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\} \\ &\geq \min\left\{\widetilde{\ell}, \widetilde{j}, \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right]\right\}. \end{aligned} \tag{33}$$

Now, if $\min\{\widetilde{\ell}, \widetilde{j}\} \leq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then $\widetilde{\mathscr{U}}(v * (p * (p * v))) \geq \min\{\widetilde{\ell}, \widetilde{j}\}$ implies $(v * (p * (p * v)))_{\min\{\widetilde{\ell}, \widetilde{j}\}} \in \widetilde{\mathscr{U}}$; otherwise, i.e., when $\min\{\widetilde{\ell}, \widetilde{j}\} > [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$, then $\widetilde{\mathscr{U}}(v * (p * (p * v))) \geq [((\kappa^{*-} - \kappa^-)/2), ((\kappa^{*+} - \kappa^+)/2)]$. So, we have

$$\begin{aligned} &\widetilde{\mathscr{U}}(v * (p * (p * v))) + \min\{\widetilde{\ell}, \widetilde{j}\} \\ &\geq \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right] + \left[\frac{\kappa^{*-} - \kappa^-}{2}, \frac{\kappa^{*+} - \kappa^+}{2}\right] \\ &= [\kappa^{*-} - \kappa^-, \kappa^{*+} - \kappa^+]. \end{aligned} \tag{34}$$

This implies that $(v * (p * (p * v)))_{\min\{\widetilde{\ell}, \widetilde{j}\}}(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. Therefore, $(v * (p * (p * v)))_{\min\{\widetilde{\ell}, \widetilde{j}\}} \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})\widetilde{\mathscr{U}}$. Hence, $\widetilde{\mathscr{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FCI of X. □

Theorem 10. *Every IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FCI of BCK-algebra X is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X.*

Proof. Let $\widetilde{\mathscr{U}}$ be any IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FCI of X and $v, \hbar \in X$. Then, we have

Table 5: Cayley Table of the binary operation*.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	3	2	0

$$\begin{aligned}\widetilde{\mathcal{U}}(v) &= \widetilde{\mathcal{U}}(v*(0*(0*v)))\\ &\geq \min\left\{\widetilde{\mathcal{U}}((v*0)*\hbar), \widetilde{\mathcal{U}}(\hbar), \left[\frac{\kappa^{*-}-\kappa^{-}}{2}, \frac{\kappa^{*+}-\kappa^{+}}{2}\right]\right\}\\ &= \min\left\{\widetilde{\mathcal{U}}((v*\hbar), \widetilde{\mathcal{U}}(\hbar), \left[\frac{\kappa^{*-}-\kappa^{-}}{2}, \frac{\kappa^{*+}-\kappa^{+}}{2}\right]\right\}.\end{aligned} \tag{35}$$

Hence, $\widetilde{\mathcal{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X. □

Example 6. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ as defined in Table 5.

Choose $\widetilde{\kappa^*} = [0.7, 0.8]$ and $\widetilde{\kappa} = [0.2, 0.3]$. Then, $\widetilde{\mathcal{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of X but is not an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FCI of X as $(0)_{\widetilde{\ell}=[0.2,0.2]} = (2*(3*(3*2)))_{\widetilde{\ell}=[0.2,0.2]} \in \widetilde{\mathcal{U}}$ and $(0)_{\widetilde{j}=[0.3,0.3]} \in \widetilde{\mathcal{U}}$ but $(2*(3*(3*2)))_{\min\{\widetilde{\ell},\widetilde{j}\}=[0.2,0.2]} = (2)_{\widetilde{\ell}=[0.2,0.2]} \overline{\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}})}\widetilde{\mathcal{U}}$.

$$\widetilde{\mathcal{U}}(\hbar) = \begin{cases} [0.5, 0.6], & \text{if } \hbar = 0,\\ [0.4, 0.5], & \text{if } \hbar = 1,\\ [0.0, 0.1], & \text{if } \hbar \in \{2, 3, 4\}.\end{cases} \tag{36}$$

Theorem 11. *Let $\widetilde{\mathcal{U}}$ be an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FI of BCK-algebra X. Then, $\widetilde{\mathcal{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FCI of $X \Leftrightarrow$ for all $v, p \in X$,*

$$\widetilde{\mathcal{U}}(v*(p*(p*v))) \geq \min\left\{\widetilde{\mathcal{U}}(v*p), \left[\frac{\kappa^{*-}-\kappa^{-}}{2}, \frac{\kappa^{*+}-\kappa^{+}}{2}\right]\right\}. \tag{37}$$

Proof. $(\Rightarrow)$ Let $\widetilde{\mathcal{U}}$ be an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FCI of X. Then, for all $v, p, \hbar \in X$, we have

$$\begin{aligned}&\widetilde{\mathcal{U}}(v*(p*(p*v)))\\ &\quad\geq \min\left\{\widetilde{\mathcal{U}}((v*p)*\hbar), \widetilde{\mathcal{U}}(\hbar), \left[\frac{\kappa^{*-}-\kappa^{-}}{2}, \frac{\kappa^{*+}-\kappa^{+}}{2}\right]\right\}.\end{aligned} \tag{38}$$

By taking $\hbar = 0$, we have

$$\begin{aligned}\widetilde{\mathcal{U}}(v*(p*(p*v))) &\geq \min\left\{\widetilde{\mathcal{U}}((v*p)*0), \widetilde{\mathcal{U}}(0), \left[\frac{\kappa^{*-}-\kappa^{-}}{2}, \frac{\kappa^{*+}-\kappa^{+}}{2}\right]\right\}\\ &= \min\left\{\widetilde{\mathcal{U}}(v*p), \left[\frac{\kappa^{*-}-\kappa^{-}}{2}, \frac{\kappa^{*+}-\kappa^{+}}{2}\right]\right\}.\end{aligned} \tag{39}$$

$(\Leftarrow)$ Let $v, p, \hbar \in X$. By assumption, we have

$$\widetilde{\mathcal{U}}(v*p) \geq \min\left\{\widetilde{\mathcal{U}}((v*p)*\hbar), \widetilde{\mathcal{U}}(\hbar), \left[\frac{\kappa^{*-}-\kappa^{-}}{2}, \frac{\kappa^{*+}-\kappa^{+}}{2}\right]\right\}. \tag{40}$$

By assumption and equation (40), we have

$$\begin{aligned}\widetilde{\mathcal{U}}(v*(p*(p*v))) &\geq \min\left\{\widetilde{\mathcal{U}}(v*p), \left[\frac{\kappa^{*-}-\kappa^{-}}{2}, \frac{\kappa^{*+}-\kappa^{+}}{2}\right]\right\}\\ &\geq \min\left\{\widetilde{\mathcal{U}}((v*p)*\hbar), \widetilde{\mathcal{U}}(\hbar), \left[\frac{\kappa^{*-}-\kappa^{-}}{2}, \frac{\kappa^{*+}-\kappa^{+}}{2}\right]\right\}.\end{aligned} \tag{41}$$

Hence, $\widetilde{\mathcal{U}}$ is an IV $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-FCI of X. □

5. Conclusion

The aim of this paper is to present a general form of interval-valued fuzzy ideals of BCK/BCI-algebras. In fact, we introduced the concepts of interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy (subalgebras) ideals and interval-valued $(\in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}), \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy ideals of BCK/BCI-algebras. In addition, interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy commutative ideals were defined, and some essential properties were discussed. Moreover, the relationship between $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy ideals and interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy commutative ideals is considered. In this present study, we conclude the following cases:

(1) If we take $\widetilde{\kappa^*} = [1, 1]$ and $\widetilde{\kappa} = [0, 0]$, then interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy subalgebras and interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy ideals reduce to the concepts of interval-valued $(\in, \in\vee q)$-fuzzy subalgebras and interval-valued $(\in, \in\vee q)$-fuzzy ideals of X as introduced in [34]

(2) If we take $\widetilde{\kappa^*} = [1, 1]$ and $\widetilde{\kappa} = \widetilde{\kappa}$, then interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy subalgebras and interval-valued $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy ideals reduce to the concepts of interval-valued $(\in, \in\vee q_{\widetilde{\kappa}})$-fuzzy subalgebras and interval-valued $(\in, \in\vee q_{\widetilde{\kappa}})$-fuzzy ideals of X as introduced in [37]

Consequently, the notions introduced in this paper, i.e., $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy subalgebras and $(\in, \in\vee(\widetilde{\kappa^*}, q_{\widetilde{\kappa}}))$-fuzzy ideals, are more general than $(\in, \in\vee q_{\widetilde{\kappa}})$-fuzzy subalgebras and $(\in, \in\vee q_{\widetilde{\kappa}})$-fuzzy ideals. In future work, one may extend these concepts to various algebraic structures such as rings, hemirings, LA-semigroups, semi-hypergroups, semi-hyperrings, BL-algebras, MTL-algebras, $R0$-algebras, MV-algebras, EQ-algebras, and lattice implication algebras.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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