

Retraction

Retracted: Fixed Point of Rational Contractions and Its Application for Secure Dynamic Routing in Wireless Sensor Networks

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

- [1] A. Asif, E. Savas, H. AlSalman, M. Arshad, A. Gumaei, and A. Rehman, "Fixed Point of Rational Contractions and Its Application for Secure Dynamic Routing in Wireless Sensor Networks," *Security and Communication Networks*, vol. 2021, Article ID 9939546, 10 pages, 2021.

Research Article

Fixed Point of Rational Contractions and Its Application for Secure Dynamic Routing in Wireless Sensor Networks

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Security is one of the major concerns for data communication over wireless sensor networks (WSNs). Dynamic routing algorithms can provide small similarity paths of data delivery between two consecutive transmitted packets, improving data security without adding extra information or control messages. This article illustrates the iteration of the fixed point (FP) of rational contractions and generalized Banach contractions (BC) in the setting of F -metric space (F-MS). It also describes an FP of the said mappings, while restricting the imposition of the contraction only to a subset of the F-MS, the closed ball, rather than executing it on the entire F-MS. The results have been verified and supported by concise examples. Further, the application of the functional equation proved results with randomization is given to find a solution for secure dynamic routing of data transmission in WSNs. The application is a tool to analyze and model a network structure in which sensors can be deployed with high security and low risk in a greater region (sensor field), thus boosting the accuracy.

1. Introduction and Preliminaries

The idea of metric space is extended and generalized by many authors in different ways (see [1–8]). Getting motivation from Wardowski's F -contraction, an interesting generalization was put forwarded by Jleli [8], which he named as F -metric space (F-MS). He proved fixed point results of Banach contractions (BC) in the frame of F-MS and defined the topological properties in the new given setting. Following this conception, various authors (see [8–13]) furthered this idea by presenting their fixed point (FP) models in F-MS, and a new and more discussion was initiated on the topic.

In continuation to this, our paper focuses on the advancement of the said F-MS, and we prove certain FP and typical FP results in F-MS. Also, following the concept of the closed ball in generalized metric spaces (for discussion on closed ball, see [13–17]), we will present both FP and typical

FP results for rational contractions and Banach contractions, which are imposed only on a subset that is closed ball for the whole F-MS. With the help of examples, the difference between locally contractive and globally contractive mappings and their outcomes will be explained.

Section 1 of this manuscript defines the basic notions and definitions that will be used or referred to in our paper. Section 2 of the article is dedicated to the Banach as well rational contractions and its investigation for FP in F-MS, while Section 3 of the paper revises the same results for the structure of the closed ball. Section 4 presents the application of our results. The role of the functional equation in solving many real-world problems is inevitable. Besides the application discussed in the abstract, the functional equation approach may be executed to construct the algorithm that finds the solution point in no more than steps equal to $N-1$ for a network having N nodes (see [18]). Also, the functional equations enable multistage decision-making and

lead to water security sustainability evaluation (see [19]). Therefore, the importance and contribution of our work are not limited to what we have discussed in this Section 4 of our article. It can apply to stochastic processes, economics, classical mechanics, dynamic programming, computer graphics, game theory, neural networks, artificial intelligence, fuzzy set theory, decision theory, digital image processing, multivalued logic, and many other fields. At last, the work done in this paper is concluded in the Conclusion section. Following are related and relevant to the main study of our work.

Definition 1 (see [8]). Suppose G is a family of mappings $g: (0, +\infty) \rightarrow R$ with the following constraints:

- (F1) g be a nondecreasing mapping, i.e., $0 < q < s \implies g(q) \leq g(s)$
- (F2) For every sequence $u_n \subset (0, +\infty)$, we have

$$\lim_{n \rightarrow \infty} u_n = 0 \iff \lim_{n \rightarrow \infty} g(u_n) = -\infty. \quad (1)$$

Definition 2 (see [8]). Suppose a set $E \neq \emptyset$ and a map $d: E \times E \rightarrow [0, \infty)$. as well as suppose there is some $(g, \sigma) \in tGn \times q[0, +\infty)$ in which

Distance (d1) $(q, s) \in E \times E$, $d(q, s) = 0 \iff q = s$ and $(q, s) \in E \times E$, $d(q, s) = d(s, q)$

Distance (d2) $(q, s) \in E \times E$, $d(q, s) = d(s, q)$

Distance (d3) For each $(q, s) \in E \times E$, and for all $N \ni \iota N \iota \geq 2$, as well as for each $(u_i)_{i=1}^n \subset E$ with $(u_1, u_N) = (q, s)$, we have

$$(q, s) > 0 \implies g(d(q, s)) \leq g\left(\sum_{i=1}^{N-1} d(u_i, u_{i+1})\right) + \sigma. \quad (2)$$

Then, (E, d) is called an F-metric space (F-MS)

Example 1 (see [8]). Let $E = N$ and $d: E \times E \rightarrow (0, \infty)$ be given as

$$d(q, s) = \begin{cases} (q-s)^2 & \text{if } (q, s) \in [0, 3] \times [0, 3], \\ |q-s| & \text{if } (q, s) \notin [0, 3] \times [0, 3]. \end{cases} \quad (3)$$

For each $(q, s) \in E \times E$, then d is an F-metric space (F-MS).

Example 2 (see [8]). Assuming $E = N$ and distance $(d): E \times E \rightarrow (0, \infty)$ is given as

$$d(q, s) = \begin{cases} 0, & \text{if } q = s, \\ e^{|q-s|}, & \text{if } q \neq s. \end{cases} \quad (4)$$

For each $(q, s) \in E \times E$, then d is an F-metric space (F-MS).

Definition 3 (see [8]). Let $(q_n) \in E$. If

- (i) $\lim_{n \rightarrow \infty} d(q_n, q) = 0$ for some $q \in E$, at this point (q_n) will be F-convergent to q

- (ii) $\lim_{n, m \rightarrow \infty} d(q_n, q_m) = 0$, then the sequence (q_n) F-Cauchy
- (iii) (E, d) is said to be F-complete if for every $(q_n) \subset E \implies (q_n)$

Definition 4 (see [8]). Let (E, d) an F-metric space (F-MS). A set $O \subset E$ is an F-metric open (F-MO) if and only if for each $q \in O$, there is a number $\sigma > 0$ provided that $B(q, \sigma) \subset O$, and

$$B(q, \sigma) = \{s \in E: d(q, s) < \sigma\}. \quad (5)$$

While a subset C from the set E is an F-metric closed (F-MC) whenever E or C is an F-open.

Definition 5 (see [8]). Let $\phi \neq B \subset E$, and distance (d) be an F-metric space (F-MS); then, the subsequent rules are equivalent:

- (i) B is an F-MC
- (ii) For every $(q_n) \subset B$, we have $\lim_{n \rightarrow \infty} d(q_n, q) = 0, q \in E \implies q \in B$. (6)

For convenience, we write only E rather than (E, d) for an F-MS. For a complete F-MS, we will write CF-MS.

Theorem 1 (see [8]). Let h be a self-mapping on a CF-MS E and $k \in (0, 1)$ such that

$$d(h(q), h(s)) \leq k d(q, s), (q, s) \in E \times E. \quad (7)$$

Then, $h q^* = q^*$ for a unique $q^* \in E$. Further, for any $q_0 \in E$, the sequence $(q_n) \subset E$ defined by $q_{n+1} = h(q_n)$, $n \in N$, is F-convergent to q^* .

Theorem 2 (see [8]). Let (E, d) is a complete MS and h is a self-mapping on E such that

$$d(h(q), h(s)) \leq \alpha d(q, s) + \beta d(q, h(s)) + \gamma d(q, h(s)). \quad (8)$$

For all q and $s \in E$, where α, β , and γ are nonnegative numbers satisfying $\alpha + \beta/1 - \gamma < 1$, then h has a unique fixed point in E .

Lemma 1 (see [8]). Let $(B(W), \|\cdot\|)$ be a Banach space and distance (d) is stated by

$$d(J, h) = \|J - h\| = \max_{q \in W} |J(q) - h(q)|, J, h \in B(W). \quad (9)$$

Then, $(B(W), \|\cdot\|)$ will be an F-MS.

2. Fixed Points of Generalized Contractions in F-MS

This portion of the article illustrates FP and typical FP results of rational type and Banach type contraction in the setting of F-MS. The results have been explained with the help of

concise examples, and some essential corollaries are developed.

Theorem 3. Assume $(g, \alpha) \in \mathcal{G} \times (0, \infty)$ and (A, d) is a CF-MS, and suppose $S, T: A \rightarrow A$ be self-mappings such that

$$d(Ss, Ty) \leq \lambda d(s, y). \quad (10)$$

For $\lambda \in [0, \infty)$ and for all $(s, y) \in A \times A$, then there exists a single typical FP of the mappings S and T in A .

Proof. Pick an arbitrary element s_0 and iterate a sequence (s_n) as

$$\begin{aligned} Ss_{2x} &= s_{2x+1}, \\ Ts_{2x+1} &= s_{2x+2}, \\ x &= 0, 1, 2, \dots \end{aligned} \quad (11)$$

From (12) and (13), we obtain

$$d(s_{2x+1}, s_{2x+2}) = d(Ss_{2x}, Ts_{2x+1}) \leq \lambda d(s_{2x}, s_{2x+1}). \quad (12)$$

This implies

$$d(s_{2x+1}, s_{2x+2}) < \lambda d(s_{2x}, s_{2x+1}). \quad (13)$$

Similarly,

$$d(s_{2x}, s_{2x+1}) = d(Ss_{2x-1}, Ts_{2x}) \leq \lambda d(s_{2x-1}, s_{2x}), \quad (14)$$

i.e.,

$$d(s_{2x}, s_{2x+1}) \leq \lambda d(s_{2x-1}, s_{2x}). \quad (15)$$

Generalizing this as follows:

$$d(s_n, s_{n+1}) < \lambda d(s_{n-1}, s_n) \text{ for all } n \in \mathbb{N}, \quad (16)$$

which yields

$$\begin{aligned} d(s_n, s_{n+1}) &< \lambda d(s_{n-1}, s_n) < \lambda^2 d(s_{n-2}, s_{n-1}) \\ &< \dots < \lambda^n d(s_0, s_1), n \in \mathbb{N}. \end{aligned} \quad (17)$$

Using (16), we can write

$$\begin{aligned} \sum_{k=n}^{m-1} d(s_k, s_{k+1}) &= d(s_n, s_{n+1}) + d(s_{n+1}, s_{n+2}) + \dots + d(s_{m-1}, s_m) \\ &< \lambda^n [1 + \lambda + \lambda^2 + \dots + \lambda^{m-n-1}] d(s_0, s_1) \\ &\leq \frac{\lambda^n}{1 - \lambda} d(s_0, s_1), \quad m > n. \end{aligned} \quad (18)$$

i.e.,

$$\sum_{k=n}^{m-1} d(s_k, s_{k+1}) \leq \frac{\lambda^n}{1 - \lambda} d(s_0, s_1), \quad m > n. \quad (19)$$

Since $\lim_{n \rightarrow \infty} \lambda^n / (1 - \lambda) d(s_0, s_1) = 0$, for any $\delta > 0$, there are some $n' \in \mathbb{N}$ in which

$$0 < \frac{\lambda^n}{1 - \lambda} d(s_0, s_1) < \delta, \quad (20)$$

$$n \geq n'.$$

Further, suppose $(g, \alpha) \in tGn \times q[0, \infty)$ meets (d3) and $\epsilon > 0$ is fixed. By (F2), there exists some $\delta > 0$ where

$$0 < l < \delta \implies g(l) < g(\epsilon) - \alpha. \quad (21)$$

By (20) and (21), we write

$$\begin{aligned} g\left(\sum_{k=n}^{m-1} d(s_k, s_{k+1})\right) &\leq g\left(\frac{\lambda^n}{1 - \lambda} d(s_0, s_1)\right) \\ &< g(\epsilon) - \alpha, m > n \geq n'. \end{aligned} \quad (22)$$

By (d3) and the above equation, we can get

$$d(s_n, s_m) > 0, m > n \geq n' \implies g(d(s_n, s_m)) < g(\epsilon). \quad (23)$$

This shows

$$d(s_n, s_m) < \epsilon, m > n \geq n'. \quad (24)$$

Thus, it proved that the sequence (s_n) is F -Cauchy in A . Now, as (A, d) is F -complete, so there must be some element c^* in A and (s_n) converges to c^* i.e.,

$$\lim_{n \rightarrow \infty} d(s_n, c^*) = 0. \quad (25)$$

Next, we show that c^* is the FP of S . For this,

$$d(Sc^*, s_{2x+2}) = d(Sc^*, Ts_{2x+1}) \leq \lambda d(c^*, s_{2x+1}). \quad (26)$$

If limit $x \rightarrow \infty$, we obtain

$$d(Sc^*, c^*) \leq \lambda d(c^*, c^*). \quad (27)$$

Thus, $d(Sc^*, c^*) = 0$, i.e., $Sc^* = c^*$.

Similarly, we obtain $Tc^* = c^*$. Hence, $Tc^* = Sc^* = c^*$.

Uniqueness: let c^{**} is also a typical FP of S and T , and $c^* \neq c^{**}$. Then,

$$d(c^*, c^{**}) = d(Sc^*, Tc^{**}) \leq \lambda d(c^*, c^{**}), \quad (28)$$

which is a contradiction. Thus, $c^* = c^{**}$.

Taking $S = T$, the following outcome is achieved. \square

Corollary 1. Suppose $(g, \alpha) \in tGn \times q[0, \infty)$ and (A, d) is an F -complete F -MS and $S: A \rightarrow A$ is a self-mapping such that

$$d(Ss, Sy) \leq \lambda d(s, y). \quad (29)$$

For $\lambda \in (0, \infty)$ and each $(s, y) \in A \times A$, then there exists a single FP of the mappings S in A .

Theorem 4. Assume $(g, \alpha) \in tGn \times q[0, \infty)$, (A, d) is a CF-MS, and $S, T: A \rightarrow A$ are self-maps such that

$$d(Ss, Ty) \leq \alpha d(s, y) + \beta \frac{(p + d(s, Ss)) \cdot d(y, Ty)}{(p + d(s, y))} + \gamma [d(s, Ss) + d(y, Ty)]. \quad (30)$$

For α, β, γ , and $p \in (0, \infty)$ such that $(\alpha + \gamma)/(1 - \beta - \gamma) < 1$ and for all $(s, y) \in A \times A$, then there exists a single typical FP of the mappings S and T in A .

$$\begin{aligned} d(s_{2x+1}, s_{2x+2}) &= d(Ss_{2x}, Ts_{2x+1}) \leq \alpha d(s_{2x}, s_{2x+1}) + \beta \frac{(p + d(s_{2x}, Ss_{2x})) \cdot d(s_{2x+1}, Ts_{2x+1})}{(p + d(s_{2x}, s_{2x+1}))} + \gamma [d(s_{2x}, Ss_{2x}) + d(s_{2x+1}, Ts_{2x+1})] \\ &= \alpha d(s_{2x}, s_{2x+1}) + \beta \frac{(p + d(s_{2x}, s_{2x+1})) \cdot d(s_{2x+1}, s_{2x+2})}{(p + d(s_{2x}, s_{2x+1}))} + \gamma [d(s_{2x}, s_{2x+1}) + d(s_{2x+1}, s_{2x+2})] \\ &= \alpha d(s_{2x}, s_{2x+1}) + \beta d(s_{2x+1}, s_{2x+2}) + \gamma [d(s_{2x}, s_{2x+1}) + d(s_{2x+1}, s_{2x+2})] \\ &= (\alpha + \gamma) d(s_{2x}, s_{2x+1}) + (\beta + \gamma) d(s_{2x+1}, s_{2x+2}), \end{aligned} \quad (32)$$

or

$$(1 - \beta - \gamma) d(s_{2x+1}, s_{2x+2}) \leq (\alpha + \gamma) d(s_{2x+1}, s_{2x+2}), \quad (33)$$

which implies that

$$\begin{aligned} d(s_{2x+1}, s_{2x+2}) &\leq \frac{(\alpha + \gamma)}{(1 - \beta - \gamma)} d(s_{2x}, s_{2x+1}) \\ &= \lambda d(s_{2x}, s_{2x+1}) \quad \text{say } \frac{(\alpha + \gamma)}{(1 - \beta - \gamma)} = \lambda. \end{aligned} \quad (34)$$

Similarly,

$$d(s_{2x}, s_{2x+1}) \leq \lambda d(s_{2x-1}, s_{2x}). \quad (35)$$

Continuing the same way, we obtain

$$d(s_n, s_{n+1}) < \lambda d(s_{n-1}, s_n). \quad (36)$$

For all $n \in \mathbb{N}$, it yields

$$\begin{aligned} d(s_n, s_{n+1}) &< \lambda d(s_{n-1}, s_n) < \lambda^2 d(s_{n-2}, s_{n-1}) \\ &< \dots < \lambda^n d(s_0, s_1), \quad n \in \mathbb{N}. \end{aligned} \quad (37)$$

Using (37), we can write

$$\begin{aligned} \sum_{k=n}^{m-1} d(s_k, s_{k+1}) &= d(s_n, s_{n+1}) + d(s_{n+1}, s_{n+2}) + \dots + d(s_{m-1}, s_m) \\ &< \lambda^n [1 + \lambda + \lambda^2 + \dots + \lambda^{m-n-1}] d(s_0, s_1) \\ &\leq \frac{\lambda^n}{1 - \lambda} d(s_0, s_1), \quad m > n. \end{aligned} \quad (38)$$

i.e.,

Proof. Choose an arbitrary element s_0 and iterate a sequence (s_n) as

$$\begin{aligned} Ss_{2x} &= s_{2x+1}, \\ Ts_{2x+1} &= s_{2x+2}; \quad x = 0, 1, 2, \dots \end{aligned} \quad (31)$$

From (30) and (31), we obtain

$$\sum_{k=n}^{m-1} d(s_k, s_{k+1}) \leq \frac{\lambda^n}{1 - \lambda} d(s_0, s_1), \quad m > n. \quad (39)$$

Since $\lim_{n \rightarrow \infty} \lambda^n / (1 - \lambda) d(s_0, s_1) = 0$, for any $\delta > 0$, there exists some $n' \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1 - \lambda} d(s_0, s_1) < \delta \quad n \geq n'. \quad (40)$$

Further, suppose $(g, \alpha) \in \mathcal{F} \times [0, \infty)$ satisfies (d3), and $\epsilon > 0$ is fixed. By (F2), there is some $\delta > 0$ such that

$$0 < l < \delta \implies g(l) < g(\epsilon) - \alpha. \quad (41)$$

By (40) and (41), we write

$$g\left(\sum_{k=n}^{m-1} d(s_k, s_{k+1})\right) \leq g\left(\frac{\lambda^n}{1 - \lambda} d(s_0, s_1)\right) \quad (42)$$

$$< g(\epsilon) - \alpha m > n \geq n'.$$

By (d3) and the above equation, we obtain

$$d(s_n, s_m) > 0, \quad m > n > n' \implies g(d(s_n, s_m)) < g(\epsilon). \quad (43)$$

This shows

$$d(s_n, s_m) < \epsilon, \quad m > n \geq n'. \quad (44)$$

Thus, it proved that the sequence (s_n) is F -Cauchy in A . Now, as (A, d) is F -complete, so there must be some element c^* in A and (s_n) converges to c^* i.e.,

$$\lim_{n \rightarrow \infty} d(s_n, c^*) = 0. \quad (45)$$

Next, we show that c^* is the FP of S :

$$\begin{aligned}
d(Sc^*, s_{2x+2}) &= d(Sc^*, Ts_{2x+1}) \leq \alpha d(c^*, s_{2x+1}) + \beta \frac{(p + d(c^*, Sc^*)) \cdot d(s_{2x+1}, Ts_{2x+1})}{(p + d(c^*, s_{2x+1}))} + \gamma [d(c^*, Sc^*) + d(s_{2x+1}, Ts_{2x+1})] \\
&= \alpha d(c^*, s_{2x+1}) + \beta \frac{(p + d(c^*, s_{2x+1})) \cdot d(s_{2x+1}, s_{2x+2})}{(p + d(c^*, s_{2x+1}))} + \gamma [d(c^*, s_{2x+1}) + d(s_{2x+1}, s_{2x+2})].
\end{aligned} \tag{46}$$

Applying limit $n \rightarrow \infty$, we obtain

$$d(Sc^*, c^*) \leq \alpha d(c^*, c^*). \tag{47}$$

Hence, $d(Sc^*, c^*) = 0$, i.e., $Sc^* = c^*$.

Similarly, we obtain $Tc^* = c^*$. Hence, $Tc^* = Sc^* = c^*$.
Uniqueness: if there is another distinct typical FP c^{**} and T , then

$$\begin{aligned}
d(c^*, c^{**}) &= d(Sc^*, Tc^{**}) \leq \alpha d(c^*, c^{**}) + \beta \frac{(p + d(c^*, Sc^*)) \cdot d(c^{**}, Tc^{**})}{(p + d(c^*, c^{**}))} + \gamma [d(c^*, Sc^*) + d(c^{**}, Tc^{**})] \\
&= \alpha d(c^*, c^{**}) + \beta \frac{(p + d(c^*, c^*)) \cdot d(c^{**}, c^{**})}{(p + d(c^*, c^{**}))} + \gamma [d(c^*, c^*) + d(c^{**}, c^{**})].
\end{aligned} \tag{48}$$

Thus, $(1 - \alpha)d(c^*, c^{**}) \leq d(c^*, c^{**})$, and this implies that $d(c^*, c^{**}) = 0$, i.e., $c^* = c^{**}$. \square

One can easily verify that d is an F -metric and g satisfies (F1) – (F2). Suppose $x \neq k$, $\alpha = e^{-1/2}$, and $\beta = \gamma = 0$; then,

Example 3. Suppose $A = A_x \{6x + 2/3, x \in N\}$, $d(A_x, A_k) = \begin{cases} 0 & \text{if } A_x = A_k \\ e^{|A_x - A_k|} & \text{if } A_x \neq A_k \end{cases}$, $g(A_x) = -1/\sqrt{A_x}$, and $S, T: A \rightarrow A$ are defined by

$$\begin{aligned}
TA_x &= \begin{cases} A_1, & \text{if } x = 1, 2 \\ A_{x-1}, & \text{if } x > 2, \end{cases} \\
SA_x &= \begin{cases} A_1, & \text{if } x = 1, \\ A_2, & \text{if } x = 2, \\ A_{x-2}, & \text{if } x > 4. \end{cases}
\end{aligned} \tag{49}$$

$$\begin{aligned}
d(SA_x, TA_k) &= e^{|A_{x-2} - A_{k-1}|} = e^{|2(x-k)-2|} < e^{-\frac{1}{2}} \cdot e^{|2(x-k)|} = \alpha d(A_x, A_k) \\
&< \alpha d(A_x, A_k) + \beta \frac{(p + d(A_x, SA_x)) \cdot d(A_k, TA_k)}{(p + d(A_x, A_k))} + \gamma [d(A_k, SA_k) + d(A_x, TA_x)].
\end{aligned} \tag{50}$$

Hence, inequality (30) holds. Clearly, A_1 is the only typical FP of S and T .

3. Fixed Points of Rational Type Contractions (RTC) on Metric Closed (F-MC) Ball

In this section of the manuscript, we explore similar results in the domain of an metric closed (F-MC) ball rather than in the whole F -MS. We will show that the FP of the above

contraction can be iterated even if the contractive condition is imposed only on the metric closed (F-MC) ball. The example in this section gives a comparative analysis of the results in this section to those in the previous section.

Definition 6. Let (A, d) be a CF-MS and $S, T: A \rightarrow A$ be self-mappings. Suppose for α, β, γ , and $p \in (0, \infty)$ with $(\alpha + \gamma)/(1 - \beta - \gamma) < 1$; then, the map T is called an RTC on $B(s_0, \mu) \subseteq A$ if

$$d(Ss, Ty) \leq \alpha d(s, y) + \beta \frac{(p + d(s, Ss)).d(y, Ty)}{(p + d(s, y))} + \gamma [d(s, Ss) + d(y, Ty)], \quad (51)$$

for each $(s, y) \in B(s_0, \mu) \subseteq A$.

Theorem 5. Assume $(g, \alpha) \in \mathcal{G} \times [0, \infty)$, (A, d) is a CF-MS, and T be an RTC on $B(s_0, \mu)$. Suppose for some $s_0 \in A$ and $\mu > 0$, and the next conditions can hold:

- (a) $B(s_0, \mu)$ is an F-MC
- (b) $d(s_0, s_1) \leq (1 - \lambda)\mu$, for $s_1 \in A$ and $\lambda = (\alpha + \gamma)/(1 - \beta - \gamma)$
- (c) There exist $0 < \epsilon < \mu$ such as $g((1 - \lambda^{k+1})\mu) \leq g(\epsilon) - \alpha$, where $k \in \mathbb{N}$

Then, there exists a single typical FP of the mappings S and T in $B(s_0, \mu)$.

Proof: Choose an arbitrary element s_0 and iterate a sequence (s_n) as

$$\begin{aligned} T(s_{2x}) &= s_{2x+1}, \\ S(s_{2x+1}) &= s_{2x+2}, \\ x &= 0, 1, 2, \dots \end{aligned} \quad (52)$$

Before proceeding to our proof, first we show that s_n is in $B(s_0, \mu)$ for every $n \in \mathbb{N}$. We do this by mathematical induction. Using (b), we write

$$d(s_0, s_1) < \mu \quad (53)$$

Hence, $s_1 \in B(s_0, \mu)$. Let $s_2, \dots, s_k \in B(s_0, \mu)$ for some $k \in \mathbb{N}$. Next, if $s_{2x+1} \leq s_k$, then by (51), we can write

$$\begin{aligned} d(s_{2x}, s_{2x+1}) &= d(Ss_{2x-1}, Ts_{2x}) \leq \alpha d(s_{2x-1}, s_{2x}) + \beta \frac{(p + d(s_{2x-1}, Ss_{2x-1})).d(s_{2x}, Ts_{2x})}{(p + d(s_{2x-1}, s_{2x}))} + \gamma [d(s_{2x-1}, Ss_{2x-1}) + d(s_{2x}, Ts_{2x})] \\ &= \alpha d(s_{2x-1}, s_{2x}) + \beta \frac{(p + d(s_{2x-1}, s_{2x})).d(s_{2x}, s_{2x+1})}{(p + d(s_{2x-1}, s_{2x}))} + \gamma [d(s_{2x-1}, s_{2x}) + d(s_{2x}, s_{2x+1})] \\ &= \alpha d(s_{2x-1}, s_{2x}) + \beta d(s_{2x}, s_{2x+1}) + \gamma [d(s_{2x-1}, s_{2x}) + d(s_{2x}, s_{2x+1})] \\ &= (\alpha + \gamma)d(s_{2x-1}, s_{2x}) + (\beta + \gamma)d(s_{2x}, s_{2x+1}), \end{aligned} \quad (54)$$

or

$$(1 - \beta - \gamma)d(s_{2x}, s_{2x+1}) \leq (\alpha + \gamma)d(s_{2x-1}, s_{2x}), \quad (55)$$

which implies that

$$\begin{aligned} d(s_{2x}, s_{2x+1}) &\leq \frac{(\alpha + \gamma)}{(1 - \beta - \gamma)} d(s_{2x-1}, s_{2x}) \\ &= \lambda d(s_{2x-1}, s_{2x}) \text{ say } \frac{(\alpha + \gamma)}{(1 - \beta - \gamma)} = \lambda. \end{aligned} \quad (56)$$

Similarly,

$$d(s_{2x-1}, s_{2x}) \leq \lambda d(s_{2x-2}, s_{2x-1}). \quad (57)$$

Similarly, if $s_{2x} \leq s_k$,

$$\begin{aligned} d(s_{2x-1}, s_{2x}) &< \frac{(\alpha + \gamma)}{(1 - \beta - \gamma)} d(s_{2x-2}, s_{2x-1}) \\ &= \lambda d(s_{2x-2}, s_{2x-1}). \end{aligned} \quad (58)$$

Therefore, from inequality (55) and (56), we write

$$d(s_{2x}, s_{2x+1}) < \lambda d(s_{2x-1}, s_{2x}) < \dots < \lambda^{2x} d(s_0, s_1), \quad (59)$$

and

$$d(s_{2x-1}, s_{2x}) < \lambda d(s_{2x-2}, s_{2x-1}) < \dots < \lambda^{2x-1} d(s_0, s_1). \quad (60)$$

From (59) and (60), we write

$$d(s_k, s_{k+1}) \leq \lambda^k d(s_0, s_1) \text{ for some } k \in \mathbb{N}. \quad (61)$$

Now, using (61), we have

$$\begin{aligned} g(d(s_0, s_{k+1})) &\leq g\left(\sum_{i=1}^{k+1} d(s_{i-1}, s_i)\right) + \alpha = g(d(s_0, s_1) + \dots + d(s_k, s_{k+1})) + \alpha \\ &\leq g\left[\left(1 + \lambda + \lambda^2 + \dots + \lambda^k\right)d(s_0, s_1)\right] + \alpha = g\left[\frac{1 - \lambda^{k+1}}{1 - \lambda} d(s_0, s_1)\right] + \alpha. \end{aligned} \quad (62)$$

By (b) and (c), we can write

$$g(d(s_0, s_{k+1})) \leq g((1 - \lambda^{k+1})\mu) + \alpha \leq g(\epsilon) < g(\mu). \quad (63)$$

Hence, by (F1), we deduce that

$$s_{k+1} \in B(s_0, \mu). \quad (64)$$

Thus, $s_n \in B(s_0, \mu)$ for all $n \in \mathbb{N}$. By (30), we have the following equations:

$$\begin{aligned} d(s_{2i+1}, s_{2i+2}) &= d(Ss_{2i}, Ts_{2i+1}) \leq \alpha d(s_{2i}, s_{2i+1}) + \beta \frac{(p + d(s_{2i}, Ss_{2i})) \cdot d(s_{2i+1}, Ts_{2i+1})}{(p + d(s_{2i}, s_{2i+1}))} + \gamma [d(s_{2i}, Ss_{2i}) + d(s_{2i+1}, Ts_{2i+1})] \\ &= \alpha d(s_{2i}, s_{2i+1}) + \beta \frac{(p + d(s_{2i}, s_{2i+1})) \cdot d(s_{2i+1}, s_{2i+2})}{(p + d(s_{2i}, s_{2i+1}))} + \gamma [d(s_{2i}, s_{2i+1}) + d(s_{2i+1}, s_{2i+2})]. \end{aligned} \quad (65)$$

Now, proceeding in a similar way as in Theorem 4 and using (a), we find that (s_n) converges to some c^* in $B(s_0, \mu)$. Also, we prove c^* as the single common FP of S and T by following the method of Theorem 4.

Put $\alpha = 0$ in Theorem 5, and the following results are given. \square

Corollary 2. Assume $(g, \alpha) \in \mathcal{G} \times [0, \infty)$, (A, d) is an F -complete F -MS, $S, T: A \rightarrow A$ are self-mappings, and $\beta/(1 - \alpha - \beta)$ with $\alpha, \beta, p \in (0, \infty)$. Let, for $s_0 \in A$ and $\mu > 0$, the following conditions hold:

- (i) $B(s_0, \mu) \subseteq A$ is F -closed
- (ii) $d(Ss, Ty) \leq \alpha \cdot (p + d(s, Ss)) \cdot d(y, Ty) / (p + d(s, y)) + \beta [d(s, Ss) + d(y, Ty)]$, for all s and $y \in B(s_0, \mu)$
- (iii) $d(s_0, s_1) \leq (1 - \lambda)\mu$, for $s_1 \in A$ and $\lambda = \beta/(1 - \alpha - \beta)$
- (iv) There exists $0 < \epsilon < \mu$ such as $g((1 - \lambda^{k+1})\mu) \leq g(\epsilon) - \alpha$, where $k \in \mathbb{N}$.

Then, there exists a single typical FP of the mappings S and T in $B(s_0, \mu)$.

If we take $S = T$ in Theorem 5, the following results are developed.

Corollary 3. Let $(g, \alpha) \in \mathcal{G} \times [0, \infty)$, (A, d) is an F -complete F -MS, and $T: A \rightarrow A$ is a self-mapping, and assume that $(\alpha + \gamma)/(1 - \beta - \gamma)$ with α, β, γ , and $p \in (0, \infty)$. Assume that, for $s_0 \in A$ and $\mu > 0$, the following conditions hold:

- (i) $B(s_0, \mu) \subseteq A$ is F -closed

$$(ii) \ d(Ts, Ty) \leq \alpha d(s, y) + \beta \cdot (p + d(s, Ts)) \cdot d(y, Ty) / (p + d(s, y)) + \gamma [d(s, Ts) + d(y, Ty)], \text{ for every } s \text{ and } y \in B(s_0, \mu)$$

$$(iii) \ d(s_0, s_1) \leq (1 - \lambda)\mu, \text{ for } s_1 \in A \text{ and } \lambda = (\alpha + \gamma) / (1 - \beta - \gamma)$$

$$(iv) \ \text{There exist } 0 < \epsilon < \mu \text{ such as } g((1 - \lambda^{k+1})\mu) \leq g(\epsilon) - \alpha, \text{ where } k \in \mathbb{N}$$

Then, there is a single FP of the mapping T in $B(s_0, \mu)$.

Example 4. Let $A = [0, \infty)$ and $g(s) = -1/\sqrt{s}$. Define $T: A \rightarrow A$ by

$$Ts = \begin{cases} \frac{s}{3}, & \text{if } s \in [0, 2], \\ s^3, & \text{if } s \in (2, \infty), \end{cases} \quad (66)$$

and define $d\beta y$:

$$d(s, y) = \begin{cases} (s - y)^2, & \text{if } (s, y) \in [0, 2] \times [0, 2], \\ |s - y|, & \text{if } (s, y) \notin [0, 2] \times [0, 2]. \end{cases} \quad (67)$$

Observe that d is an F -metric space (F -MS) and the mapping g fulfills (F1)-(F2). Choose $s_0 = \mu = 1$; then, $B(s_0, \mu) = [0, 2]$. Notice that $B(s_0, \mu)$ is \mathcal{F} -closed, so condition (a) of Corollary 2 is fulfilled. Next, if $\alpha = 3/4$ and $\beta = \gamma = 0$, then $\lambda = \alpha$ and

$$d(s_0, s_1) = d(s_0, Ts_0) = \left(\frac{1}{2} - \frac{1}{6}\right)^2 = \frac{1}{9} < \left(1 - \frac{3}{4}\right) \cdot 1 = (1 - \lambda)\mu. \quad (68)$$

Thus, condition (b) is obeyed. Further, as the function g is increasing and $\lambda < 1$, therefore, for every $k \in \mathbb{N}$, we can

locate some $\epsilon < \mu$ and $\alpha \in [0, \infty)$ such that $g((1 - \lambda^{k+1})\mu) = g(\epsilon) - \alpha$ is satisfied. i.e., condition (c) is obeyed.

Now, when $(s, y) \in B(s_0, \mu) \times B(s_0, \mu)$, then

$$\begin{aligned} d(Ts, Ty) &= \left(\frac{s}{3} - \frac{y}{3}\right)^2 = \frac{1}{9}(s - y)^2 < \frac{3}{4}(s - y)^2 = \alpha d(s, y) \\ &= \alpha d(s, y) + \beta \cdot \frac{(p + d(s, Ts)) \cdot d(y, Ty)}{(p + d(s, y))} + \gamma [d(s, Ts) + d(y, Ty)]. \end{aligned} \quad (69)$$

Hence, (d) holds for all $(s, y) \in B(s_0, \mu) \times B(s_0, \mu)$. But, if $(s, y) \notin B(s_0, \mu) \times B(s_0, \mu)$ e.g., $s = 3$ and $y = 4$, then

$$d(Ts, Ty) = |3^3 - 4^3| > \frac{3}{4}|3 - 4| = (\alpha d(s, y)) = (\alpha d(s, y)) + \beta \cdot \frac{(p + d(s, Ts)) \cdot d(y, Ty)}{(p + d(s, y))} + \gamma [d(s, Ts) + d(y, Ty)]. \quad (70)$$

Hence, it is now verified that the condition (b) holds true only for and not on the whole space A . Finally, $0 \in B(s_0, \mu)$ is the FP of T .

the problem of DP in the structure of a function equation as follows:

$$p(s) = \max_{t \in T} \{F(s, t) + f_1(s, t, p(\eta(s, t)))\} \text{ for } s \in S. \quad (71)$$

4. Application

This section assures a solution for dynamic programming (DP) by using the fixed point technique. Thus, in turn, it renders a solution to dynamic routing with randomization for secure wireless sensor networks.

A DP is two states process: a state space (SS) and a decision space (DC). One can further divide SS into (a) initial state, (b) transitional state, and (c) action state. On the contrary, DS comprises the procedure and steps adopted to iterate the solution of the particular problem. Such algorithms are mainly used in computer programming and optimization.

There used to be hundreds or thousands of sensor nodes in a wireless sensor. They sometimes communicate among themselves and sometimes directly to a base station. If there are more sensors in a network, they can sense a large region with more considerable accuracy, minimizing the risk. The sensor nodes are often dispersed on a broader sensor field, where they share the quality data/information.

All such sensor nodes tend to gather and route the data back to the base station or to among themselves. Sending the traffic of a network can be done by the process of routing. For a high communication capacity, efficient tools are needed to select a network that can respond sharply to changes in the communication link. For this purpose, many algorithms have already been developed for routing protocols in a network.

Bellman [20], in 1958, using the approach of functional equations, designed an algorithm that converges in $N - 1$ (or lesser) iterations to perform dynamic routing with randomization in a sensor network of N nodes and supporting its security in terms of data transmission. One can formulate

$$x(s) = \max_{t \in T} \{F(s, t) + f_2(s, t, p(\eta(s, t)))\} \text{ for } s \in S, \quad (72)$$

where Y and Z are Banach spaces such as $S \subseteq Y$ and $T \subseteq Z$, and

$$\begin{aligned} \eta: S \times T &\longrightarrow S, \\ F: S \times T &\longrightarrow R, \\ f_1, f_2: S \times T \times R &\longrightarrow R. \end{aligned} \quad (73)$$

Suppose S and T are the DS and SS, respectively. We tend to investigate a similar solution point occurrence for both (71) and (72). Let us represent the family of real-valued bounded mappings on S by $W(S)$. Suppose an arbitrary element $j \in W(S)$, and $\|j\| = \max_{s \in S} |j(s)|$. Then, $(W(S), \|\cdot\|)$ is Banach space endowed with d defined as

$$d(j, k) = \max_{s \in S} |j(s) - k(s)|. \quad (74)$$

Suppose the below speculations hold:

(C1): F, f_1 , and f_2 are bounded.

(C2): For $s \in S$ and $j \in W(S)$, define $Q, Z: W(S) \longrightarrow W(S)$ by

$$\begin{aligned} Qj(u) &= \max_{t \in T} \{F(s, t) + f_1(s, t, j(\eta(s, t)))\} \text{ for } s \in S, \\ Zj(u) &= \max_{t \in T} \{F(s, t) + f_2(s, t, j(\eta(s, t)))\} \text{ for } s \in S. \end{aligned} \quad (75)$$

Observe that the mappings F, f_1 , and f_2 are all bounded.

(C3): For $(s, t) \in S \times T$, j and $k \in W(S)$ and $l \in S$, and we write

$$|f_1(s, t, j(l)) - f_1(s, t, k(l))| \leq M(j, k), \quad (76)$$

where

$$M(j, k) = \alpha d(j, k) + \beta \frac{(p + d(j, Qj)) \cdot d(k, Zk)}{(p + d(j, k))} + \gamma (d(j, Qj) + d(k, Zk)), \quad (77)$$

for α, β , and $\gamma \in [0, \infty)$ and $\alpha + 2\beta + 2\gamma < 1$.

Now, we develop the following theorem.

Theorem 6. *Suppose conditions (C1)-(C3) hold; then, at most one identical bounded solution for both (71) and (72).*

$$\begin{aligned} Qj_1(s) - Zj_2(s) &< f_1(s, t_1, j_1(\eta(s, t_1))) - f_1(s, t_1, j_2(\eta(s, t_1))) + \omega \\ &\leq |f_1(s, t_1, j_1(\eta(s, t_1))) - f_1(s, t_1, j_2(\eta(s, t_1)))| + \omega \leq M(j_1(s), j_2(s)) + \omega. \end{aligned} \quad (82)$$

Also, from (78) and (79), we get

$$Zj_2(s) - Qj_1(s) < M(j_1(s), j_2(s)) + \omega. \quad (83)$$

Merging the above two inequalities, we write

$$|Qj_1(s) - Zj_2(s)| < M(j_1(s), j_2(s)) + \omega, \quad (84)$$

for all $\omega > 0$. Thus,

$$d(Qj_1(s), Zj_2(s)) \leq M(j_1(s), j_2(s)), \quad (85)$$

i.e.,

$$d(Sj_1, Tj_2) \leq M(j_1, j_2), \quad (86)$$

for every $s \in S$. All the requirements of Theorem 4 are fulfilled. Therefore, using Theorem 4, S and T have a unique bounded and standard solution for equations (70) and (71). \square

5. Conclusion

This research work has highlighted the essentialness and usefulness of the newly-introduced F-MS by establishing interesting FP theorems of some contractions. It is obtained that the FP and typical FP of a contractive mapping is beneficial even if the contraction condition is not imposed on the whole F-MS and is shrunk only to a closed ball inside it. The two different examples are indeed a comparative analysis of the outcome of assessing the contraction locally and globally. Few practical corollaries have been developed from the proven results. We will extend this idea to the frame of fuzzy metric space and picture fuzzy metric space in the future. The effects will be investigated in the mentioned setting, and its application in multistage optimization will be discussed. Finally, there is a discussion on the applications

Proof: By Lemma 1, it is evident that $(W(S), d)$ is an F -complete F -MS. d is defined by (74), and from (C1), we deduce that S and T are self-mappings on $W(S)$. Let ω be an arbitrary positive number and j_1 and $j_2 \in W(S)$. Take $s \in S$ and t_1 and $t_2 \in T$ such as

$$Qj_x < F(s, t_x) + f_1(s, t_x, j_x(\eta(s, t_x))) + \omega, \quad (78)$$

$$Zj_x < F(s, t_x) + f_2(s, t_x, j_x(\eta(s, t_x))) + \omega, \quad (79)$$

and

$$Qj_1 \geq F(s, t_2) + f_1(s, t_2, j_1(\eta(s, t_2))), \quad (80)$$

$$Zj_2 \geq F(s, t_1) + f_1(s, t_1, j_2(\eta(s, t_1))). \quad (81)$$

Then, using (78) and (81), we obtain

side for development and establishing a single solution of the functional equation, which leads to a dynamic routing with randomization and improving data security tasks in a wireless sensor network.

Data Availability

Data sharing is not applicable to this article as no data set were generated during this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The authors have equally contributed to this research work.

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