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Retraction

Retracted: Hamacher Weighted Aggregation Operators Based on Picture Cubic Fuzzy Sets and Their Application to Group Decision-Making Problems

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This article has been retracted by Hindawi, as publisher, following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of systematic manipulation of the publication and peer-review process. We cannot, therefore, vouch for the reliability or integrity of this article.

Please note that this notice is intended solely to alert readers that the peer-review process of this article has been compromised.

Wiley and Hindawi regret that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] L. Yanhong, R. Ambrin, M. Ibrar, and M. Ali Khan, "Hamacher Weighted Aggregation Operators Based on Picture Cubic Fuzzy Sets and Their Application to Group Decision-Making Problems," *Security and Communication Networks*, vol. 2022, Article ID 1651017, 39 pages, 2022.

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Research Article

Hamacher Weighted Aggregation Operators Based on Picture Cubic Fuzzy Sets and Their Application to Group Decision-Making Problems

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In this study, an extended version of intuitionistic cubic fuzzy Hamacher weighted aggregation operators is the primary objective of the expected study. We establish new concept, picture cubic fuzzy set, and utilize this new concept for ranking in decision analysis. Picture cubic fuzzy Hamacher weighted averaging operator, picture cubic fuzzy Hamacher order weighted averaging, and picture cubic fuzzy Hamacher hybrid averaging operator are developed on the basis of picture cubic fuzzy sets. Some unique cases and few suitable properties of these proposed operators are also examined. In addition, based on these expected operators, we are designing a new multicriteria group decision-making framework. The proposed aggregation operators can be used in the performance evaluation of energy projects and security systems and devices. We give an illustrative example for the selection of small hydropower plants locations as an implementation and appropriateness of the proposed algorithm. Finally, we perform a comparative review of the designed algorithm with intuitionistic cubic fuzzy Hamacher weighted averaging operators to expose the new algorithm efficiency, feasibility, and goodness.

1. Introduction

Need for electricity globally is continuously increasing due to rapid growing of the demographic and comparison of growth in agricultural and global industries. Electricity is typically derived from two kinds of energy sources, such as nontraditional (renewable) and traditional (nonrenewable) energy sources. Nonrenewable or conventional sources of energy, such as flammable gas, oil, coal, and atomic separation, are normally used in the energy production, and a greater part of the exact areas of management limitation is represented. While renewable energy sources include hydropower, solar, biomass, hydroelectric, and air, the generate electricity from alternative sources of energy provides a smaller amount of the overall generation capacity, and this segment should also be unambiguously committed to an expected returns in the energy circumstance. Hydropower accounts with about 24% of the combined production of power, and in the all-out amounts of electricity produced annually, this was one of the prominent renewable energy sources [1]. Hydropower projects are divided into groups on the basis of power generation capacity, namely, large hydropower plant (LHPP) and small hydropower plant (SHPP). SHPP is typically the sort of run-of-the-stream or stream dependent and refers to hydroelectric plants with such a consumption level of less than 25 MW, while LHPP is almost always linked to a large dam built over a stream, and it refers to a plant with a productivity level greater than 25 MW.

Karimi Azari et al. [2] followed the method to order of preference by similarity to ideal solution (TOPSIS) to pick a fair risk management framework for development projects. Chen et al. [3] reported out research on the design of procurement systems China's development programs. The best range of SHPP construction projects is a significant task, as the parameters are often in disagreement between each other. The choice among the most favorable SHPP is therefore absolutely MCGDM issue. Many multicriteria group decision-making (MCGDM) issues (such as strategic financial management, and medical diagnosis and business) have been generated by aggregation operators [4, 5]. In 2018, Fahmi et al. [6] utilized some geometric operators under triangular cubic linguistic hesitant fuzzy environment and constructed a framework to group decision-making (GDM) complications. Weighted average rating algorithm, designed by [7], to solve group decision-making issues based on triangular cubic fuzzy hybrid aggregation. In [8], cubic fuzzy Einstein aggregation operators are introduced and its uses to decision-making (DM) issues. Triangular cubic linguistic hesitant fuzzy aggregation operators introduced by Aliya et al. [9], including its participation within group decision-making problem. Also, in 2018, Fahmi et al. [10] introduced planned values of aggregation operators based on cubic trapezoidal fuzzy value and their approach to the problems of MCDM. He [11] initiated typhoon disaster assessment focused on Dombi hesitant fuzzy data aggregation operators. In 2019, Jana et al. [12] initiated the notion of picture fuzzy Dombi aggregation operators and their approach to MCDM issues. Gagandeep and Harish [13] defined generalized cubic intuitionist fuzzy (CIF) aggregation operators through t-norm (TN) operations and their approach to GDM framework. Several linguistic intuitionist fuzzy aggregation operators and its uses to MAD issues are initiated by Peide and Peng [14]. Amiri et al. [15] designed a new fuzzy best-worst method technique for analyzing and choosing a supplier which is competitive in supply chain management. Utilizing Einstein t-norm and t-conorm and its approach to DM, generalize intuitionist fuzzy (IF) interactive geometric interaction operators [16]. Harish generalized the notion of IF multiplicative interactive geometric operators and its uses to MCDM [17]. Khan and Saleem [18] developed the concept of cubic aggregation operators. Liu [19] presented some interval-valued intuitionistic fuzzy (IVIF) Hamacher aggregation operators and applied it to the issue of MCGDM. Liang et al. [20] initiated the notion of GDM framework for multicriteria focused on generalized intuitionistic trapezoidal fuzzy prioritized aggregation operators. A few other IF Dombi Bonferroni mean operators including its connection to MAGDM are presented by Liu et al. [21]. Cubic hesitant fuzzy sets and respective approaches to DM under various parameters are defined in [22]. Only with the support of the new proposal of PF similarity, Haseeb and Singh present a novel framework recognized as the picture fuzzy inferior ratio method to solve MADM issues in the PF setting, which is focused on the very same concept as TOPSIS, taken into consideration both PIS and NIS similarities in [23]. Dombi operations are applied to neutrosophic cubic sets and proposed a neutrosophic cubic Dombi weighted arithmetic average operator and neutrosophic cubic Dombi weighted geometric average operator [24]. Ullah et al. [25] utilized the GRA method on the basis of picture hesitant fuzzy sets and discussed its application in GDM problems. Minxia and Huifeng defined transformation algorithm for a picture fuzzy value and trapezoidal fuzzy value and suggested a PF multiplication operation and an image fuzzy power operation on the basis of this procedure [26]. The creation of a modern hybrid model based on PFSs through linear assignment and its first cases involve for the design of sustainable transportation systems [27]. To determine the distinct requirements of the option between the DM methods, many aggregation operators (AOs), named PF Yager aggregation, under the PF data [28] have been developed. Two correlation coefficients for PFSs obtaining their values in $[-1, 1]$ have been introduced by Haseeb et al. while explaining the significance of the correlation more sufficiently [29].

In the field of DM environment, it is really tough to select real criterion values. In 1965, Zadeh coined his remarkable theory of fuzzy set (FS) to deal with uncertainty [30]. The notion of bi-parametric distance and similarity measures for PFSs was extended by Khan et al. [31]. The description of the PFS entropy measure has been developed by Thao [32]. They also analyzed such similarity measures caused by entropy measures at same period and then utilized it is applicable in MCDM issue regarding supplier selection. Focused on picture fuzzy numbers, Egrioglu et al. [33] model fuzzy time-series and a single-variable high-order PF time-series model are established. They also propose a better method of forecasting PF time series. Few operators were developed by Akram et al. [34], to aggregate CPF information, namely, complex picture fuzzy Hamacher weighted averaging, ordered weighted averaging, hybrid averaging and complex picture fuzzy Hamacher weighted geometric, ordered weighted geometric and hybrid geometric operators, gaining from standard Hamacher operations, and averaging geometric aggregation. Many PF geometric operators have been established by Chunyong et al. and its basic properties are examined [35]. In [36, 37], Xu and Yager initiated the concept of aggregation operators such as IF hybrid averaging (IFHA) operator, IF order weighted averaging (IFOWA) operator, IF weighted averaging (IFWA) operator, IF hybrid geometric (IFHG) operator, IF order weighted geometric (IFOWG) operator, and IF weighted geometric (IFWG) operator and utilized these proposed operators to MCGDM.

Hamacher operator (HO) [38] is an alloy of Einstein triangular cubic number, algebraic t-norm, and t-conorm [39]. Li studied Hamacher correlated averaging operator based on interval-valued IF (IVIF) information [40]. Hung introduced the concept of IF Hamacher aggregation (IFHA) operator and utilized this operator to MAGDM issues [41]. Garg [42] introduced IF Hamacher aggregation (IFHA) operators under depends on entropy weight and applied them to MCGDM problems.

Picture fuzzy Hamacher aggregation operator is presented by Jana and Pal [43], for assessment of enterprise performance. Xiao provided an order weighted Hamacher geometric operator [44], on the basis of IVIF environment. Hamacher and other aggregation operator functions are applied to the introduced MAGDM method [45,46]. Abdullah and Ashraf [47] established the concept of cubic picture fuzzy set and utilized this concept to evaluate the petroleum circulation center problem. Muneeza et al. [48] presented intuitionistic cubic fuzzy Hamacher weighted aggregation (ICFHWA) operators and applied them to the MCGDM method. In this study, we want to define picture cubic fuzzy set (PCFS) and some picture cubic fuzzy Hamacher weighted aggregation (PCFHWA) operator based on HOs [49]. PCFS has a ground-breaking ability to appear the unpredictable information of the real life problem. PCFS is a forceful ability to model the uncertain information of real world. From the literature study of HOs, there is a broad range of works matching to application of fuzzy aggregation algorithms in MAGDM issues; by means of this encouragement, we utilized PCFHWA operator, PCFHOWA operator, and PCFHHA operator based on PCF numbers, to obtain the best SHPP as a manufacture project for the developer.

Based on the internal characteristics of unmanned ground delivery vehicles (UGDVs), a multicriteria comprehensive evaluation system for UGDVs is constructed [50]. In [51], Zeng et al. proposed a multicriteria model based on a social network for assessing a digital reform under an intuitionistic fuzzy environment. Xie et al., in [52], introduced a novel progressively under the sampling method based on the density peaks sequence for imbalanced data. In [53], the concept of SFS and T-spherical fuzzy set (T-SFS) is introduced as a generalization of FS, IFS, and PFS. The principle of Bonferroni mean (BM) operators with interval-valued T-spherical fuzzy set (IVTSFS) to develop the principle of the intervalued T-spherical fuzzy (IVTSF) BM (IVTSFBM) operator, the IVTSF-weighted BM (IVTSFWBM) operator, the IVTSF geometric BM (IVTSFGBM) operator, and the IVTSF-weighted geometric BM (IVTSFWGBM) operator is introduced in [54]. Ullah et al. developed the concept of picture fuzzy Maclaurin symmetric mean (PFMSM) operators and investigated their validity [55].

The aim of this study is to propose a new concept called "picture cubic fuzzy set" (PCFS), as an extension of intuitionistic cubic fuzzy set (ICFS). The addition of a neutral membership degree to ICFS makes PCFS as generalized form of ICFS. The uniqueness of this new theory lies in the capability to attain the wider range with the help of degree of neutral membership, nonmembership, and membership. Hamacher aggregation operator is one such operator which is based on Hamacher t-norm and t-conorm. It is observed that the Hamacher aggregation operators of intuitionistic cubic fuzzy set have some limitations in their applicability. To overcome these limitations, some Hamacher aggregation operators based on picture cubic fuzzy numbers are introduced.

In 2020, Muneeza et al. [48] presented the concept of ICFS and utilized this concept in MADM problems. Since PFS is a generalized concept to IFS, so, we gave attention to bring the notion of PCFS and defined Hamacher aggregation operators on the bases of PCFS and utilized PCFHA operators for selection of SHPP. The proposed aggregation operators can be used in the performance evaluation of energy projects, security systems and devices, etc.

Based on to what is discussed with regard to the abovementioned studies, the contribution of this study is given as below:

(1) Propose an idea of PCF set and discuss its basic properties; also, few Hamacher operators for PCFS are established
(2) Propose PCF Hamacher weighted aggregation operators
(3) Propose new-MCGDM algorithm based on these planned operators
(4) Solve SHPP place choice on the basis of new algorithm
(5) Discuss the comparison analysis to describe the validity of the proposed new-MCGDM algorithm

The leftovers of this study are planned below. In Section 2, we shortly review of essential definitions and HCs on PFS. In Section 3, we first describe the idea of PCFS and discuss its different basic properties. Several Hamacher operators for PCFSs are established. In Section 4, a number of PCFHA operators are given, such as PCFHWA operator, PCFHOWA operator, and PCFHHA operator, and study some of their properties. In Section 5, we state a new-MCGDM algorithm on the basis of these planned operators. In Section 6, we utilize descriptive example of SHPP locations selection and conduct a comparison analysis. In Section 7, conclusion is presented.

2. Preliminaries

The study provides a brief overview of the basic concepts of PFS, as well as their operation and operators. We also go through several more well-known concepts that will be essential in the upcoming analysis.

2.1. Fuzzy Set (*FS*)

Definition 1 (see [30]). *A FS R in* $\check{T} \neq \phi$ *is expressed as*

$$R = \left\{ \langle \check{t}, \mu_R(\check{t}) \rangle | \check{t} \in \check{T} \right\}, \tag{1}$$

where $\mu_R(\check{t}) \in [0, 1]$ *is a membership degree of* $\check{t}$ *in R.*

Definition 2 (see [56]). *An interval-valued intuitionistic fuzzy set* (IVIFS)*R in* $\check{T} \neq \phi$ *is expressed as*

$$R = \left\{ \langle \check{t}, [\mu_R^-(\check{t}), \mu_R^+(\check{t})], [\nu_R^-(\check{t}), \nu_R^+(\check{t})] \rangle | \check{t} \in \check{T} \right\}, \tag{2}$$

where $[\mu_R^-(\check{t}), \mu_R^+(\check{t})]$ indicates the membership function and $[\nu_R^-(\check{t}), \nu_R^+(\check{t})]$ denotes the nonmembership function.

Definition 3 (see [57]). *A cubic fuzzy set* (CFS)*R in* $\check{T} \neq \phi$ *is expressed as*

$$R = \left\{ \langle \check{t}, [\mu_R^-(\check{t}), \mu_R^+(\check{t})], \nu_R(\check{t}) \rangle | \check{t} \in \check{T} \right\}, \tag{3}$$

where the first value is the IVF number, indicates the degree of membership, and the second value is a simple fuzzy number, indicates the nonmembership degree.

2.2. Hamacher Aggregation Operators on ICFS

Definition 4 (see [48]). *An ICFS R in $\breve{T} \neq \phi$ is expressed as*

$$R = \left\{\breve{t}, \langle a_R, b_R \rangle | \breve{t} \in \breve{T}\right\},$$

or,

$$R = \left\{\left(\breve{t}, \langle [a^-, a^+], \breve{\lambda} \rangle, \langle [b^-, b^+], \breve{\delta} \rangle\right) | \breve{t} \in \breve{T}\right\}, \tag{4}$$

where $\breve{\lambda}$ and $\breve{\delta}$ represent degree of membership and degree of nonmembership, respectively, $\langle [a^-, a^+], \breve{\lambda} \rangle$ denotes the exact degree of membership, and $\langle [b^-, b^+], \breve{\delta} \rangle$ denotes the exact degree of non-membership of ICFS. $[a^-, a^+]$ and $[b^-, b^+] \subset [0, 1]$ *and* $\breve{\lambda}\colon \breve{T} \longrightarrow [0, 1]$ and $\breve{\delta}\colon \breve{T} \longrightarrow [0, 1]$ *under restriction* $\mathrm{Sup}[a^-, a^+] + \mathrm{Sup}[b^-, b^+] \leq 1$ *and* $\breve{\lambda} + \breve{\delta} \leq 1$. *Besides, we obtain*

$$\breve{\pi}(R) = \left\{\langle [1, 1] - [[a^-, a^+] + [b^-, b^+]] \rangle, \langle 1 - (\breve{\lambda} + \breve{\delta}) \rangle\right\},$$

$$\breve{\pi}(R) = \left\{[1 - (a^- + b^-), 1 - (a^+ + b^+)], 1 - (\breve{\lambda} + \breve{\delta})\right\}, \tag{5}$$

is called index of ICFS or indeterminacy degree of $\breve{t} \in \breve{T}$ for R. For simplicity, an ICFN is represented by R, that is, $R = (\langle [a^-, a^+], \breve{\lambda} \rangle, \langle [b^-, b^+], \breve{\delta} \rangle)$.

Definition 5 (see [48]). *Let* $R_z = \langle a_{R_z}, b_{R_z} \rangle$ $(z = 1, 2, \ldots, g)$ *be a collection of ICFNs in* $\breve{T}$, *and suppose ICFHWA operator of dimension g is a function of ICFHWA*: $\Omega^g \longrightarrow \Omega$; *thus,*

$$\mathrm{ICFHWA}_{\varpi}\left(R_1, R_2, R_3, \ldots, R_g\right) = \oplus_{z=1}^{g} \varpi_z R_z,$$

$$= \left\{\begin{array}{c} \left(\left[\dfrac{\prod_{z=1}^{g}\left(1 + (\mathrm{d} - 1)a_z^-\right)^{\varpi_z} - \prod_{z=1}^{g}\left(1 - a_z^-\right)^{\varpi_z}}{\prod_{z=1}^{g}\left(1 + (\mathrm{d} - 1)a_z^-\right)^{\varpi_z} + (\mathrm{d} - 1)\prod_{z=1}^{g}\left(1 - a_z^-\right)^{\varpi_z}}, \dfrac{\prod_{z=1}^{g}\left(1 + (\mathrm{d} - 1)a_z^+\right)^{\varpi_z} - \prod_{z=1}^{g}\left(1 - a_z^+\right)^{\varpi_z}}{\prod_{z=1}^{g}\left(1 + (\mathrm{d} - 1)a_z^+\right)^{\varpi_z} + (\mathrm{d} - 1)\prod_{z=1}^{g}\left(1 - a_z^+\right)^{\varpi_z}}\right], \dfrac{\prod_{z=1}^{g}\left(1 + (\mathrm{d} - 1)\breve{\lambda}_z\right)^{\varpi_z} - \prod_{z=1}^{g}\left(1 - \breve{\lambda}_z\right)^{\varpi_z}}{\prod_{z=1}^{g}\left(1 + (\mathrm{d} - 1)\lambda_z\right)^{\varpi_z} + (\mathrm{d} - 1)\prod_{z=1}^{g}\left(1 - \breve{\lambda}_z\right)^{\varpi_z}}\right), \\ \left(\left[\dfrac{\mathrm{d}\prod_{z=1}^{g}\left(b_z^-\right)^{\varpi_z}}{\prod_{z=1}^{g}\left(1 + (\mathrm{d} - 1)\left(1 - b_z^-\right)\right)^{\varpi_z} + (\mathrm{d} - 1)\prod_{z=1}^{g}\left(b_z^-\right)^{\varpi_z}}, \dfrac{k\prod_{z=1}^{g}\left(b_z^+\right)^{\varpi_z}}{\prod_{z=1}^{g}\left(1 + (\mathrm{d} - 1)\left(1 - b_z^+\right)\right)^{\varpi_z} + (\mathrm{d} - 1)\prod_{z=1}^{g}\left(b_z^+\right)^{\varpi_z}}\right], \dfrac{\mathrm{d}\prod_{z=1}^{g}\left(\delta_z\right)^{\varpi_z}}{\prod_{z=1}^{g}\left(1 + (\mathrm{d} - 1)\left(1 - \breve{\delta}_z\right)\right)^{\varpi_z} + (\mathrm{d} - 1)\prod_{z=1}^{g}\left(\breve{\delta}_z\right)^{\varpi_z}}\right) \end{array}\right\}, \tag{6}$$

where $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_g)^T$ *are weight vector of* R_z $(z = 1, 2, 3, \ldots, g)$ *by* $\sum_{z=1}^{g} \varpi_z = 1$ *and* $\varpi_z \in [0, 1]$.

Definition 6 (see [48]). *Let* $R_z = \langle a_{R_z}, b_{R_z} \rangle$ $(z = 1, 2, \ldots, g)$ *be a collection of ICFNs in* $\breve{T}$, *and suppose ICFHOWA operator of dimension* g *is a function of* ICFHOWA: $\Omega^g \longrightarrow \Omega$; *thus,*

$$\text{ICFHOWA}_{\varpi}(R_1, R_2, R_3, \ldots, R_g) = \oplus_{z=1}^{g} \varpi_z R_{\sigma_{(z)}},$$

$$= \left\{ \begin{array}{l} \left(\left[\dfrac{\prod_{z=1}^{g} \left(1 + (\mathrm{d}-1) a^{-}_{\sigma_{(z)}}\right)^{\varpi_z} - \prod_{z=1}^{g} \left(1 - a^{-}_{\sigma_{(z)}}\right)^{\varpi_z}}{\prod_{z=1}^{g} \left(1 + (\mathrm{d}-1) a^{-}_{\sigma_{(z)}}\right)^{\varpi_z} + (\mathrm{d}-1) \prod_{z=1}^{g} \left(1 - a^{-}_{\sigma_{(z)}}\right)^{\varpi_z}}, \dfrac{\prod_{z=1}^{g} \left(1 + (\mathrm{d}-1) a^{+}_{\sigma_{(z)}}\right)^{\varpi_z} - \prod_{z=1}^{g} \left(1 - a^{+}_{\sigma_{(z)}}\right)^{\varpi_z}}{\prod_{z=1}^{g} \left(1 + (\mathrm{d}-1) a^{+}_{\sigma_{(z)}}\right)^{\varpi_z} + (\mathrm{d}-1) \prod_{z=1}^{g} \left(1 - a^{+}_{\sigma_{(z)}}\right)^{\varpi_z}} \right], \dfrac{\prod_{z=1}^{g} \left(1 + (\mathrm{d}-1) \breve{\lambda}_{\sigma_{(z)}}\right)^{\varpi_z} - \prod_{z=1}^{g} \left(1 - \breve{\lambda}_{\sigma_{(z)}}\right)^{\varpi_z}}{\prod_{z=1}^{g} \left(1 + (\mathrm{d}-1) \breve{\lambda}_{\sigma_{(z)}}\right)^{\varpi_z} + (\mathrm{d}-1) \prod_{z=1}^{g} \left(1 - \breve{\lambda}_{\sigma_{(z)}}\right)^{\varpi_z}} \right) \\ \left(\left[\dfrac{\mathrm{d} \prod_{z=1}^{g} \left(b^{-}_{\sigma_{(z)}}\right)^{\varpi_z}}{\prod_{z=1}^{g} \left(1 + (\mathrm{d}-1)\left(1 - b^{-}_{\sigma_{(z)}}\right)\right)^{\varpi_z} + (\mathrm{d}-1) \prod_{z=1}^{g} \left(b^{-}_{\sigma_{(z)}}\right)^{\varpi_z}}, \dfrac{k \prod_{z=1}^{g} \left(b^{+}_{\sigma_{(z)}}\right)^{\varpi_z}}{\prod_{z=1}^{g} \left(1 + (\mathrm{d}-1)\left(1 - b^{+}_{\sigma_{(z)}}\right)\right)^{\varpi_z} + (\mathrm{d}-1) \prod_{z=1}^{g} \left(b^{+}_{\sigma_{(z)}}\right)^{\varpi_z}} \right], \dfrac{\mathrm{d} \prod_{z=1}^{g} \left(\breve{\delta}_{\sigma_{(z)}}\right)^{\varpi_z}}{\prod_{z=1}^{g} \left(1 + (\mathrm{d}-1)\left(1 - \breve{\delta}_{\sigma_{(z)}}\right)\right)^{\varpi_z} + (\mathrm{d}-1) \prod_{z=1}^{g} \left(\breve{\delta}_{\sigma_{(z)}}\right)^{\varpi_z}} \right) \end{array} \right\}, \tag{7}$$

where $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_g)^T$ *are weight vector of* R_z $(z = 1, 2, \ldots, g)$ *by* $\sum_{z=1}^{g} \varpi_z = 1$ *and* $\varpi_z \in [0, 1]$. *For all z,* $R_{\sigma_{(z-1)}} \geq R_{\sigma_{(z)}}$ *and* $(\sigma_{(1)}, \sigma_{(2)}, \ldots, \sigma_{(g)})$ *are permutation of* $(1, 2, \ldots, g)$.

Definition 7 (see [48]). *Let* $R_z = \langle a_{R_z}, b_{R_z} \rangle$ $(z = 1, 2, \ldots, g)$ *be a collection of ICFNs in* $\breve{T}$, *and let ICFHWA operator of dimension g be a function of* ICFHHA: $\Omega^g \longrightarrow \Omega$, *with weight vector* $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_g)^T$ *which is the weight* *vector of* R_z $(z = 1, 2, \ldots, g)$ *with* $\sum_{z=1}^{g} \varpi_z = 1$ *and* $\varpi_z \in [0, 1]$ *and* $\varpi_z > 0$; *thus,*

$$\text{ICFHHA}_{w,v}(R_1, R_2, R_3, \ldots, R_z) = \mathring{A}_{z=1}^{g} v_z \tilde{R}_{\sigma_{(z)}}, \tag{8}$$

where $w = (w_1, w_2, \ldots, w_g)^T$ are weight of R_z with $\sum_{z=1}^{g} w_z = 1$ and $w_z \in [0, 1]$. Here, $\tilde{R}_{\sigma_{(z)}}$ is the zth largest of the weighted ICFNs $\tilde{R}_z$ and $\tilde{R}_{\sigma_{(z)}} = g w_z R_{\sigma_{(z)}} = (\langle[\tilde{a}^-_{\sigma_{(z)}}, \tilde{a}^+_{\sigma_{(z)}}], \tilde{\breve{\lambda}}_{\sigma_{(z)}}\rangle, \langle[\tilde{b}^-_{\sigma_{(z)}}, \tilde{b}^+_{\sigma_{(z)}}], \tilde{\breve{\delta}}_{\sigma_{(z)}}\rangle)$ $(z = 1, 2, 3, \ldots, g)$, where

$$\begin{aligned}
\tilde{a}^-_{\sigma_{(z)}} &= \left(1 + (d-1)a^-_{\sigma_{(z)}}\right)^{g w_z} - \frac{\left(1 - a^-_{\sigma_{(z)}}\right)^{g w_z}}{\left(1 + (d-1)a^-_{\sigma_{(z)}}\right)^{g w_p} + (d-1)\left(1 - a^-_{\sigma_{(z)}}\right)^{g w_z}}, \\
\tilde{a}^+_{\sigma_{(z)}} &= \left(1 + (d-1)a^+_{\sigma_{(z)}}\right)^{g w_z} - \frac{\left(1 - a^-_{\sigma_{(z)}}\right)^{g w_z}}{\left(1 + (d-1)a^+_{\sigma_{(z)}}\right)^{g w_p} + (d-1)\left(1 - a^+_{\sigma_{(z)}}\right)^{g w_z}}, \\
\tilde{\breve{\lambda}}_{\sigma_{(z)}} &= \frac{\left(1 + (d-1)\breve{\lambda}_{\sigma_{(z)}}\right)^{g w_z} - \left(1 - \breve{\lambda}_{\sigma_{(z)}}\right)^{g w_z}}{\left(1 + (d-1)\breve{\lambda}_{\sigma_{(z)}}\right)^{g w_z} + (d-1)\left(1 - \breve{\lambda}_{\sigma_{(z)}}\right)^{g w_z}}, \\
\tilde{b}^-_{\sigma_{(z)}} &= \frac{d\left(b^-_{\sigma_{(z)}}\right)^{g w_z}}{1 + (d-1)\left(1 - b^-_{\sigma_{(z)}}\right)^{g w_z} + (d-1)\left(b^-_{\sigma_{(z)}}\right)^{g w_z}}, \\
\tilde{b}^+_{\sigma_{(z)}} &= \frac{d\left(b^+_{\sigma_{(z)}}\right)^{g w_z}}{1 + (d-1)\left(1 - b^+_{\sigma_{(z)}}\right)^{g w_z} + (d-1)\left(b^+_{\sigma_{(z)}}\right)^{g w_z}}, \\
\tilde{\breve{\delta}}_{\sigma_{(z)}} &= \frac{d\left(\breve{\delta}_{\sigma_{(z)}}\right)^{g w_z}}{1 + (d-1)\left(1 - \breve{\delta}_{\sigma_{(z)}}\right)^{g w_z} + (d-1)\left(\breve{\delta}_{\sigma_{(z)}}\right)^{g w_z}},
\end{aligned} \tag{9}$$

where g is the balancing coefficient, which keeps the proper balance.

2.3. Picture Fuzzy set (PFS)

Definition 8 (see [58]). *A PFS R in* $\check{T} \neq \phi$ *is identified as*

$$R = \left\{\langle \check{t}, \mu_R(\check{t}), \eta_R(\check{t}), \nu_R(\check{t})\rangle | \check{t} \in \check{T}\right\}, \tag{10}$$

where $\mu_R(\check{t})$, $\eta_R(\check{t})$, and $\nu_R(\check{t})$ in unit closed interval ($[0, 1]$) are called positive-membership, neutral-membership, and negative-membership degrees respectively, under restriction $0 \leq \mu_R(\check{t}) + \eta_R(\check{t}) + \nu_R(\check{t}) \leq 1 \forall \check{t} \in \check{T}$. Furthermore, $\Psi_R = 1 - (\mu_R(\check{t}) + \eta_R(\check{t}) + \nu_R(\check{t}))$for all$\check{t} \in \check{T}$ is said to be the refusal-membership degree of the function. The pair (μ_R, η_R, ν_R) is called PF value (PFV) or PF number (PFN). Keep in mind that each one IFS can be expressed as

$$R = \left\{\langle \mu_R(\check{t}), 0, \nu_R(\check{t})\rangle | \check{t} \in \check{T}\right\}. \tag{11}$$

If we put $\eta_R(\check{t}) \neq 0$, in equation (2), then we get PFS.

2.4. Hamacher Operations (HOs)

Basically, t-operator are union and intersection operators in fuzzy set theory which are represent by t-conorm (T) and t-norm (T^*), respectively [59]. The generalized intersection and union of IFSs on the basis of t-conorm and t-norm were introduced by Deschrijver and Kerre [60]. Hamacher initiated HOs, so called as Hamacher sum ($\oplus$) and Hamacher product ($\otimes$), in 1978 [38], the t-conorm and t-norm examples, respectively. Hamacher T and Hamacher T^* are given in the following definition:

$$\begin{aligned}
T_H(s, t) &= s \otimes t = \frac{st}{d + (1 - d)(s + t - st)}, \\
T_H^*(s, t) &= s \oplus t = \frac{s + t - st - (1 - d)st}{1 - (1 - d)st}.
\end{aligned} \tag{12}$$

When $d = 1$, at that time, Hamacher T^* and T will change to the form:

$$T_H(s, t) = s \otimes t = st, \tag{13}$$

$$T_H^*(s, t) = s \oplus t = s + t - st, \tag{14}$$

which represent algebraic T and T^*. When $d=2$, at that time, Hamacher T^* and T will change to the form:

$$T_H(s,t) = s \otimes t = \frac{st}{1+(1-s)(1-t)}, \quad (15)$$

$$T_H^*(s,t) = s \oplus t = \frac{s+t}{1+st}, \quad (16)$$

called Einstein T and T^*, respectively.

2.5. Hamacher Operations (HOs) of PFS. Several Hamacher operations (HOs) on picture fuzzy values (PFVs) are given, which are introduced by Wei [49]. Suppose R_1 and R_2 be any two picture fuzzy set and $n>0$. Then, the Hamacher sum and Hamacher product of R_1 and R_2 are represented by $(R_1 \oplus R_2)$ and $(R_1 \otimes R_2)$ and defined by

(1) $(R_1 \oplus R_2) = (\mu_1 + \mu_2 - \mu_1\mu_2 - (1-\mathrm{d})\mu_1\mu_2/1-(1-\mathrm{d})\mu_1\mu_2, \eta_1\eta_2/\mathrm{d} + (1-\mathrm{d})(\eta_1+\eta_2-\eta_1\eta_2), \nu_1\nu_2/\mathrm{d} + (1-\mathrm{d})(\nu_1+\nu_2-\nu_1\nu_2))$

(2) $(R_1 \otimes R_2) = (\mu_1\mu_2/d + (1-d)(\mu_1+\mu_2-\mu_1\mu_2), \eta_1 + \eta_2 - \eta_1\eta_2 - (1-d)\eta_1\eta_2/1-(1-d)\eta_1\eta_2, \nu_1+\nu_2-\mu_1\nu_2 - (1-d)\nu_1\nu_2/1-(1-d)\nu_1\nu_2)$

(3) $nR_1 = \{((1+(d-1)\ \mu_1)^n - (1-\mu_1)^n/(1+(d-1)\mu_1)^n + (d-1)(1-\mu_1)^n), (d(\eta_1)^n/(1+(d-1)(1-\eta_1))^n + (d-1)(\eta_1)^n), (d(\nu_1)^n/(1+(d-1)(1-\nu_1))^n + (d-1)(\nu_1)^n)\}$

(4) $R_1^n = \{(d(\mu_1)^n/(1+(d-1)(1-\mu_1))^n + (d-1)(\mu_1)^n), ((1+(d-1)\eta_1)^n - (1-\eta_1)^n/(1+(d-1)\eta_1)^n + (d-1)(1-\eta_1)^n), ((1+(d-1)\nu_1)^n - (1-\nu_1)^n/(1+(d-1)\nu_1)^n + (d-1)(1-\nu_1)^n)\}$.

3. PCFS and Its Fundamental Relations and Operations

Definition 9. A picture cubic fuzzy set (PCFS) P in $\breve{T} \neq \phi$ is defined as

$$P = \left\{\breve{t}, \langle a_P, c_P, b_P \rangle | \breve{t} \in \breve{T}\right\}, \quad (17)$$

or

$$P = \left\{\left(\breve{t}, \langle [a^-, a^+], \breve{\lambda}\rangle, \langle [c^-, c^+], \breve{\psi}\rangle, \langle [b^-, b^+], \breve{\delta}\rangle\right) | \breve{t} \in \breve{T}\right\}, \quad (18)$$

where $\langle [a^-, a^+], \breve{\lambda}\rangle$ denotes the *exact degree of membership, $\langle [c^-, c^+], \breve{\psi}\rangle$ denotes the exact degree of neutral, and $\langle [b^-, b^+], \breve{\delta}\rangle$ denotes the exact degree of nonmembership of P. On this spot* $[a^-, a^+][c^-, c^+]$and$[b^-, b^+] \subset [0,1]$, $\breve{\lambda}$: $\breve{T} \longrightarrow [0,1]$, $\breve{\psi}$: $\breve{T} \longrightarrow [0,1]$, and$\delta$: $\breve{T} \longrightarrow [0,1]$ *subject to* $\mathrm{Sup}[a^-, a^+] + \mathrm{Sup}[c^-, c^+] + \mathrm{Sup}[b^-, b^+] \leq 1$ *and* $\breve{\lambda} + \breve{\psi} + \breve{\delta} \leq 1$. *Thus, we have*

$$\breve{\pi}(R) = \left\{\langle [1,1] - \left[[a^-, a^+] + [c^-, c^+] + [b^-, b^+]\right]\rangle, \langle 1 - (\breve{\lambda} + \breve{\psi} + \breve{\delta})\rangle\right\},$$
$$\breve{\pi}(R) = \left\{[1-(a^- + c^- + b^-), 1-(a^+ + c^+ + b^+)], 1-(\breve{\lambda} + \breve{\psi} + \breve{\delta})\right\}, \quad (19)$$

which is called index of PCFS or indeterminacy degree of $\breve{t} \in \breve{T}$ for PCFS. For simplicity, an PCFN is denoted by P, that is, $P = (\langle (a^-, a^+), \breve{\lambda}\rangle, \langle [c^-, c^+], \breve{\psi}\rangle, \langle [b^-, b^+], \breve{\delta}\rangle)$.

Example 1. Let us have $\breve{T} = \{\breve{t}_1, \breve{t}_2, \breve{t}_3, \breve{t}_4\}$ a set, and suppose P be a set given by

$$P = \left\{\begin{array}{c} (\breve{t}_1, ([0.1, 0.15], 0.4), ([0.25, 0.3], 0.1), ([0.4, 0.5], 0.3)), \\ (\breve{t}_2, ([0.1, 0.4], 0.2), ([0.35, 0.4], 0.6), ([0.1, 0.15], 0.2)), \\ (\breve{t}_3, ([0.2, 0.3], 0.1), ([0.1, 0.2], 0.35), ([0.2, 0.4], 0.3)), \\ (\breve{t}_4, ([0.1, 0.5], 0.4), ([0.15, 0.2], 0.3), ([0.2, 0.3], 0.3)) \end{array}\right\}. \quad (20)$$

Thus, P satisfies the definition of PCFS. Hence, P is PCFS.

Based on PCFN, we defined score function $S(P)$ and accuracy function $A(P)$.

Definition 10. Suppose $P = (\langle [a^-, a^+], \breve{\lambda}\rangle, \langle [c^-, c^+], \breve{\psi}\rangle, \langle [b^-, b^+], \breve{\delta}\rangle)$ *be a picture cubic fuzzy number. $S(P)$ is expressed as*

$$S(P) = \left(\frac{a^- + a^+ + \breve{\lambda} - \left(c^- + c^+ + \breve{\psi} + b^- + b^+ + \breve{\delta}\right)}{3}\right), \quad (21)$$

so as $S(P) \in [-1, 1]$. The function $S(P)$ computes the score of PCFN P.

Definition 11. Suppose $P = (\langle[a^-, a^+], \breve{\lambda}\rangle, \langle[c^-, c^+], \breve{\psi}\rangle, \langle[b^-, b^+], \breve{\delta}\rangle)$ *be a PCFN. The accuracy function* $A(P)$ *is expressed as*

$$A(P) = \left(\frac{a^- + a^+ + \breve{\lambda} + c^- + c^+ + \breve{\psi} + b^- + b^+ + \breve{\delta}}{3}\right), \quad (22)$$

where $A(P) \in [0, 1]$.

Definition 12. Suppose

$$\begin{aligned} P_1 &= \left(\langle[a_1^-, a_1^+], \breve{\lambda}_1\rangle, \langle[c_1^-, c_1^+], \breve{\psi}_1\rangle, \langle[b_1^-, b_1^+], \breve{\delta}_1\rangle\right), \\ P_2 &= \left(\langle[a_2^-, a_2^+], \breve{\lambda}_2\rangle, \langle[c_2^-, c_2^+], \breve{\psi}_2\rangle, \langle[b_2^-, b_2^+], \breve{\delta}_2\rangle\right), \end{aligned} \quad (23)$$

are two PCFNs, $S(P_1)$ and $S(P_2)$ are the score function of P_1 and P_2, and $A(P_1)$ and $A(P_2)$ are the accuracy function of P_1 and P_2, respectively. Then,

(1) If $S(P_1) < S(P_2) \Rightarrow P_1 < P_2$
(2) If $S(P_1) = S(P_2)$, $A(P_1) = A(P_2)$, then $P_1 = P_2$
(3) If $S(P_1) = S(P_2)$and $A(P_1) < A(P_2) \Rightarrow P_1 = P_2$

3.1. Hamacher Operations in PCFS. Let

$$\begin{aligned} P_1 &= \left\{\left(\langle[a_1^-, a_1^+], \breve{\lambda}_1\rangle, \langle[c_1^-, c_1^+], \breve{\psi}_1\rangle, \langle[b_1^-, b_1^+], \breve{\delta}_1\rangle\right)\right\}, \\ P_2 &= \left\{\left(\langle[a_2^-, a_2^+], \breve{\lambda}_2\rangle, \langle[c_2^-, c_2^+], \breve{\psi}_2\rangle, \langle[b_2^-, b_2^+], \breve{\delta}_2\rangle\right)\right\}, \end{aligned} \quad (24)$$

be two PCFN and $n > 0$; we introduce the below fundamental Hamacher operators in PCFS:

(1) $P_1 \oplus P_2 = \{ ([a_1^- + a_2^- - a_1^- a_2^- - (1-\mathrm{d}) a_1^- a_2^- /1 - (1-\mathrm{d}) a_1^- a_2^-, a_1^+ + a_2^+ - a_1^+ a_2^+ - (1-\mathrm{d}) a_1^+ a_2^+ /1 - (1-\mathrm{d}) a_1^+ a_2^+], \breve{\lambda}_1 + \breve{\lambda}_2 - \breve{\lambda}_1 \breve{\lambda}_2 - (1-\mathrm{d}) \breve{\lambda}_1 \breve{\lambda}_2 /1 - (1-\mathrm{d}) \breve{\lambda}_1 \breve{\lambda}_2),$
$\left(\begin{bmatrix} c_1^- c_2^- /\mathrm{d} + (1-\mathrm{d})(c_1^- + c_2^- - c_1^- c_2^-), \\ c_1^+ c_2^+ /\mathrm{d} + (1-\mathrm{d})(c_1^+ + c_2^+ - c_1^+ c_2^+) \end{bmatrix}, \atop \breve{\psi}_1 \breve{\psi}_2 /\mathrm{d} + (1-\mathrm{d})(\breve{\psi}_1 + \breve{\psi}_2 - \breve{\psi}_1 \breve{\psi}_2)\right)$
$\left(\begin{bmatrix} b_1^- b_2^- /\mathrm{d} + (1-\mathrm{d})(b_1^- + b_2^- - b_1^- b_2^-), \\ b_1^+ b_2^+ /\mathrm{d} + (1-\mathrm{d})(b_1^+ + b_2^+ - b_1^+ b_2^+) \end{bmatrix}, \atop \breve{\delta}_1 \breve{\delta}_2 /\mathrm{d} + (1-\mathrm{d})(\breve{\delta}_1 + \breve{\delta}_2 - \breve{\delta}_1 \breve{\delta}_2)\right), \}$

(2) $P_1 \otimes P_2 = \{ ([a_1^- a_2^- /d + (1-d) (a_1^- + a_2^- - a_1^- a_2^-), a_1^+ a_2^+ /d + (1-d)(a_1^+ + a_2^+ - a_1^+ a_2^+)], \breve{\lambda}_1 \breve{\lambda}_2 /d + (1-d)(\breve{\lambda}_1 + \breve{\lambda}_2 - \breve{\lambda}_1 \breve{\lambda}_2)), ([c_1^- + c_2^- - c_1^- c_2^- - (1-d) c_1^- c_2^- /1- (1-d) c_1^- c_2^-, c_1^+ + c_2^+ - c_1^+ c_2^+ - (1-d) c_1^+ c_2^+ /1- (1-d) c_1^+ c_2^+], \breve{\psi}_1 + \breve{\psi}_2 - \breve{\psi}_1 \breve{\psi}_2 - (1-d) \breve{\psi}_1 \breve{\psi}_2 /1 - (1-d) \breve{\psi}_1 \breve{\psi}_2), ([b_1^- + b_2^- - b_1^- b_2^- - (1-d) b_1^- b_2^- /1 - (1-d) b_1^- b_2^-, b_1^+ + b_2^+ - b_1^+ b_2^+ - (1-d) b_1^+ b_2^+ /1 - (1-d) b_1^+ b_2^+], \breve{\delta}_1 + \breve{\delta}_2 - \breve{\delta}_1 \breve{\delta}_2 - (1-d) \breve{\delta}_1 \breve{\delta}_2 /1 - (1-d) \breve{\delta}_1 \breve{\delta}_2)\}$

(3) $nP_1 = \{ ([(1 + (d- 1)a_1^-)^n - (1 - a_1^-)^n /(1 + (d-1)a_1^-)^n + (d-1)(1 - a_1^-)^n, (1 + (d-1)a_1^+)^n - (1- a_1^+)^n /(1 + (d-1)a_1^+)^n + (d-1)(1 - a_1^+)^n], (1 + (d-1)\breve{\lambda}_1)^n - (1 - \breve{\lambda}_1)^n /(1 + (d-1)\breve{\lambda}_1)^n + (d-1)(1 - \breve{\lambda}_1)^n) ([k(c_1^-)^n / (1 + (d-1)(1 - c_1^-))^n + (d-1)(c_1^-)^n, k(c_1^+)^n /(1 + (d-1)(1 - c_1^+))^n + (d-1)(c_1^+)^n], d(\breve{\psi}_1)^n /(1 + (d-1)(1 - \breve{\psi}_1))^n + (d-1)(\breve{\psi}_1)^n), ([d(b_1^-)^n /(1 + (1-d) (1 - b_1^-))^n + (1-d)(b_1^-)^n, d(b_1^+)^n /(1 + (1-d)(1 - b_1^+))^n + (1-d)(b_1^+)^n], d(\breve{\delta}_1)^n /(1 + (1-d)(1 - \breve{\delta}_1))^n + (1-d)(\breve{\delta}_1)^n)\}$

(4) $(P_1)^n = \{ ([d(a_1^-)^n / (1 + (d-1)(1 - a_1^-))^n + (d-1)(a_1^-)^n, d(a_1^+)^n /(1 + (d-1)(1 - a_1^+))^n + (d-1)(a_1^+)^n], d(\breve{\lambda}_1)^n /(1 + (d-1)(1 - \breve{\lambda}_1))^n + (d-1)(\breve{\lambda}_1)^n) ([(1 + (d-1)c_1^-)^n - (1 - c_1^-)^n /(1 + (d-1)c_1^-)^n + (d-1)(1 - c_1^-)^n, (1 + (d-1)c_1^+)^n - (1 - c_1^+)^n /(1 + (d-1)c_1^+)^n + (d-1)(1 - c_1^+)^n], (1+ (d-1)\breve{\psi}_1)^n - (1 - \breve{\psi}_1)^n /(1 + (d-1)\breve{\psi}_1)^n + (d-1)(1 - \breve{\psi}_1)^n), ([(1 + (d-1)b_1^-)^n - (1 - b_1^-)^n /(1 + (d-1)b_1^-)^n + (d-1)(1 - b_1^-)^n, (1 + (d-1)b_1^+)^n - (1 - b_1^+)^n /(1 + (d-1)b_1^+)^n + (d-1)(1 - b_1^+)^n], (1 + (d-1)\breve{\delta}_1)^n - (1 - \breve{\delta}_1)^n /(1 + (d-1)\breve{\delta}_1)^n + (d-1)(1 - \breve{\delta}_1)^n)\}$

Proposition 1. *Let*

$$\begin{aligned} P_1 &= \left\{\left(\langle[a_1^-, a_1^+], \breve{\lambda}_1\rangle, \langle[c_1^-, c_1^+], \breve{\psi}_1\rangle, \langle[b_1^-, b_1^+], \breve{\delta}_1\rangle\right)\right\}, \\ P_2 &= \left\{\left(\langle[a_2^-, a_2^+], \breve{\lambda}_2\rangle, \langle[c_1^-, c_1^+], \breve{\psi}_2\rangle, \langle[b_2^-, b_2^+], \breve{\delta}_2\rangle\right)\right\}, \end{aligned} \quad (25)$$

be two PCFN and $n_1, n_2 > 0$*; then, we have*

(1) $P_1 \oplus P_2 = P_2 \oplus P_1$
(2) $P_1 \otimes P_2 = P_2 \otimes P_1$
(3) $n(P_1 \oplus P_2) = nP_1 \oplus nP_2$
(4) $n(P_1 \otimes P_2) = nP_1 \otimes nP_2$
(5) $(P_1 \otimes P_2)^n = P_1^n \otimes P_2^n$
(6) $(P_1 \oplus P_2)^n = P_1^n \oplus P_2^n$
(7) $(P_1^{n_1})^{n_2} = (P_1)^{n_1 n_2}$.

4. Picture Cubic Fuzzy Hamacher Weighted Aggregation (PCFHWA) Operators

Here, we establish a number of PCFHA operators and describe few of its basic properties.

4.1. PCFHWA Operator

Definition 13. Suppose $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r}\rangle$ $(r = 1, 2, \ldots, g)$ *is a set of PCFVs in* $\breve{T}$*. A PCFHWA operator of dimension g is a function PCFHWA:* $\Omega^g \longrightarrow \Omega$*, such that*

$$\mathrm{PCFHWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) = \oplus_{r=1}^{g} \varpi_r P_r, \quad (26)$$

where Ω is the set of all PCFVs and $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_g)^T$ is the weight vector of P_r $(r = 1, 2, \ldots, g)$ with $\sum_{r=1}^{g} \varpi_r = 1$ and $\varpi_r \in [0, 1]$.

Under the Hamacher operational rules of the PCFNs, we get the theorem as given below.

Theorem 1. *Suppose that* $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r}\rangle$ $(r = 1, 2, \ldots, g)$ *is a set of PCFVs in* $\breve{T}$*; then, the aggregated value of them utilizing PCFHWA operator is again a PCFV and is given as*

$$\mathrm{PCFHWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right),$$
$$= \left\{ \begin{array}{l} \left(\left[\dfrac{\prod_{r=1}^{g}\left(1+(d-1)a_r^-\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-a_r^-\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^-\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(1-a_r^-\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{g}\left(1+(k-1)a_r^+\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-a_r^+\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^+\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(1-a_r^+\right)^{\varpi_r}} \right], \dfrac{\prod_{r=1}^{g}\left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-\breve{\lambda}_r\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(1-\breve{\lambda}_r\right)^{\varpi_r}} \right), \\ \left(\left[\dfrac{d\prod_{r=1}^{g}\left(c_r^-\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-c_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(c_r^-\right)^{\varpi_r}}, \dfrac{d\prod_{r=1}^{g}\left(c_r^+\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-c_r^+\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(c_r^+\right)^{\varpi_r}} \right], \dfrac{d\prod_{r=1}^{g}\left(\breve{\psi}_r\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-\breve{\psi}_r\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(\breve{\psi}_r\right)^{\varpi_r}} \right), \\ \left(\left[\dfrac{d\prod_{r=1}^{g}\left(b_r^-\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-b_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(b_r^-\right)^{\varpi_r}}, \dfrac{d\prod_{r=1}^{g}\left(b_r^+\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-b_r^+\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(b_r^+\right)^{\varpi_r}} \right], \dfrac{d\prod_{r=1}^{g}\left(\breve{\delta}_r\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-\breve{\delta}_r\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(\breve{\delta}_r\right)^{\varpi_r}} \right) \end{array} \right\}, \tag{27}$$

where $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_m)^T$ *is the weight vector of* $P_r\,(r = 1, 2, .., g)$ *with* $\sum_{r=1}^{g} \varpi_r = 1$ *and* $\varpi_r > 0$.

Proof. For proof, see Appendix A. □

4.2. Parameter d Effect on PCFHWA Operator. Subsequently, we suppose that two appropriate cases for the PCFHWA operator whenever $d = 1$ or 2.

Case (i): in case that $d = 1$, at that time, PCFHWA operator will change to PCFWA operator:

$$\mathrm{PCFWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) = \left\{\begin{array}{c}\left(\begin{bmatrix}1-\prod_{r=1}^{g}\left(1-a_r^{-}\right)^{\varpi_r}, \\ 1-\prod_{r=1}^{g}\left(1-a_r^{+}\right)^{\varpi_r}\end{bmatrix}, 1-\prod_{r=1}^{g}\left(1-\breve{\lambda}_r\right)^{\varpi_r}\right), \\ \left(\begin{bmatrix}\prod_{r=1}^{g}\left(c_r^{-}\right)^{\varpi_r}, \\ \prod_{r=1}^{g}\left(c_r^{+}\right)^{\varpi_r}\end{bmatrix}, \prod_{r=1}^{g}\left(\breve{\psi}_r\right)^{\varpi_r}\right), \\ \left(\begin{bmatrix}\prod_{r=1}^{g}\left(b_r^{-}\right)^{\varpi_r}, \\ \prod_{r=1}^{g}\left(b_r^{+}\right)^{\varpi_r}\end{bmatrix}, \prod_{r=1}^{g}\left(\breve{\delta}_r\right)^{\varpi_r}\right)\end{array}\right\}. \tag{28}$$

Case (ii): if $d = 2$, then the PCFHWA operator will change to picture cubic fuzzy Einstein weighted averaging (PCFEWA) operator:

$$\mathrm{PCFEWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) = \left\{\begin{array}{c}\left(\begin{bmatrix}\dfrac{\prod_{r=1}^{g}\left(1+a_r^{-}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+a_r^{-}\right)^{\varpi_r}+\prod_{r=1}^{g}\left(1-a_r^{-}\right)^{\varpi_r}}, \\ \dfrac{\prod_{r=1}^{g}\left(1+a_r^{+}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+a_r^{+}\right)^{\varpi_r}+\prod_{r=1}^{g}\left(1-a_r^{+}\right)^{\varpi_r}}\end{bmatrix}, \dfrac{\prod_{r=1}^{g}\left(1+\breve{\lambda}_r\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-\breve{\lambda}_r\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+\breve{\lambda}_r\right)^{\varpi_r}+\prod_{r=1}^{g}\left(1-\breve{\lambda}_r\right)^{\varpi_r}}\right), \\ \left(\begin{bmatrix}\dfrac{2\prod_{r=1}^{g}\left(c_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-c_r^{-}\right)^{\varpi_r}+\prod_{r=1}^{g}\left(c_r^{-}\right)^{\varpi_r}}, \\ \dfrac{2\prod_{r=1}^{g}\left(c_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-c_r^{+}\right)^{\varpi_r}+\prod_{r=1}^{g}\left(c_r^{+}\right)^{\varpi_r}}\end{bmatrix}, \dfrac{2\prod_{r=1}^{g}\left(\breve{\psi}_r\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-\breve{\psi}_r\right)^{\varpi_r}+\prod_{r=1}^{g}\left(\breve{\psi}_r\right)^{\varpi_r}}\right), \\ \left(\begin{bmatrix}\dfrac{2\prod_{r=1}^{g}\left(b_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-b_r^{-}\right)^{\varpi_r}+\prod_{r=1}^{g}\left(b_r^{-}\right)^{\varpi_r}}, \\ \dfrac{2\prod_{r=1}^{g}\left(b_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-b_r^{+}\right)^{\varpi_r}+\prod_{r=1}^{g}\left(b_r^{+}\right)^{\varpi_r}}\end{bmatrix}, \dfrac{2\prod_{r=1}^{g}\left(\breve{\delta}_r\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-\breve{\delta}_r\right)^{\varpi_r}+\prod_{r=1}^{g}\left(\breve{\delta}_r\right)^{\varpi_r}}\right)\end{array}\right\}. \tag{29}$$

Proposition 2. *Suppose that* $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r} \rangle$ $(r = 1, 2, \ldots, g)$ *is a set of PCFVs in* $\breve{T}$, $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_g)^T$ *are the weight vector of* P_r *with* $\sum_{r=1}^{g} \varpi_r = 1$, *and* ϖ_r *is the element of unit closed interval ([0,1]); then, the coming properties are initiated.*

Idempotency Property. If all $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r} \rangle$ $(r = 1, 2, \ldots, g)$ *are equal, that is,* $P_r = P$, *then*

$$\mathrm{PCFHWA}_{\varpi}\left(P_1, P_2, \ldots, P_g\right) = P. \tag{30}$$

Proof. For proof, see Appendix B.

Boundedness Property. For every ϖ,

$$P^{-} \leq \mathrm{PCFHWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) \leq P^{+}, \tag{31}$$

where

$$P^{+} = \left\{ \begin{array}{c} \left(\left[\min a^{-}, \max a^{+}\right], \max \breve{\lambda}\right), \\ \left(\left[\max c^{-}, \min c^{+}\right], \min \breve{\psi}\right), \\ \left(\left[\max b^{-}, \min b^{+}\right], \min \breve{\delta}\right) \end{array} \right\},$$
$$P^{-} = \left\{ \begin{array}{c} \left(\left[\min a^{-}, \max a^{+}\right], \min \breve{\lambda}\right), \\ \left(\left[\max c^{-}, \min c^{+}\right], \max \breve{\psi}\right), \\ \left[\max b^{-}, \min b^{+}\right], \min \breve{\delta} \end{array} \right\}. \tag{32}$$

Monotonicity property : let

$$P_r^{*} = \left\{ \left(\left\langle \left[a_r^{*^-}, a_r^{*^+}\right], \breve{\lambda}_r^{*} \right\rangle, \left\langle \left[c_r^{*^-}, c_r^{*^+}\right], \breve{\psi}_r^{*} \right\rangle, \left\langle \left[b_r^{*^-}, b_r^{*^+}\right], \breve{\delta}_r^{*} \right\rangle \right) \right\}, \tag{33}$$

where $(r = 1, 2, .., g)$ be a group of PCFNs if

$$\left[a_r^{-}, a_r^{+}\right] \leq \left[a_r^{*^-}, a_r^{*^+}\right], \breve{\lambda}_r \leq \breve{\lambda}_r^{*}, \left[c_r^{*^-}, c_r^{*^+}\right] \leq \left[c_r^{-}, c_r^{+}\right], \breve{\psi}_r^{*} \leq \breve{\psi}_r,$$
$$\left[b_r^{*^-}, b_r^{*^+}\right] \leq \left[b_r^{-}, b_r^{+}\right], \breve{\delta}_r^{*} \leq \breve{\delta}_r, \tag{34}$$

for all r. Then,

$$\mathrm{PCFHWA}_{\varpi}\left(P_1, P_2, \ldots, P_g\right) \leq \mathrm{PCFHWA}_{\varpi}\left(P_1^{*}, P_2^{*}, \ldots, P_g^{*}\right). \tag{35}$$

□

Proof. See Appendix C. □

4.3. PCFHOWA Operator

Definition 14. Let $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r} \rangle$ $(r = 1, 2, \ldots, g)$ *be a set of PCFVs in* $\breve{T}$. *A PCFHOWA operator of g dimension is a function PCFHOWA:* $\Omega^{g} \longrightarrow \Omega$, *such that*

$$\mathrm{PCFHOWA}_{\varpi}\left(P_1, P_2, \ldots, P_g\right) = \oplus_{r=1}^{g} \varpi_r P_{\sigma_{(r)}}. \tag{36}$$

For all $r P_{\sigma_{(r-1)}} \geq P_{\sigma_{(r)}}$, $(\sigma_{(1)}, \sigma_{(2)}, .., \sigma_{(g)})$ is a permutation of $(1, 2, .., g)$. Ω is the set of all PCFVs and $\varpi = (\varpi_1, \varpi_2, .., \varpi_g)^T$ is the weight vector of P_r $(r = 1, 2, .., g)$ with $\sum_{r=1}^{g} \varpi_r = 1$ and $\varpi_r \in [0, 1]$.

Hence, under the Hamacher operational rules of the PCFNs, we get the theorem as given below.

Theorem 2. *Suppose that* $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r} \rangle$ $(r = 1, 2, \ldots, g)$ *is a set of PCFVs in* $\breve{X}$; *then, the aggregated value of the utilizing PCFHOWA operator is also a PCFV and is written as*

$$
\begin{aligned}
&\mathrm{PCFHOWA}_{\varpi}\left(P_1, P_2, \ldots, P_g\right),\\
&=\left\{\begin{array}{l}
\left(\left[\dfrac{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)a^{-}_{\sigma_{(r)}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)a^{-}_{\sigma_{(r)}}\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(1-a^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)a^{+}_{\sigma_{(r)}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)a^{+}_{\sigma_{(r)}}\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(1-a^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}\right], \dfrac{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\breve{\lambda}_{\sigma_{(r)}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-\breve{\lambda}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\breve{\lambda}_{\sigma_{(r)}}\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(1-\breve{\lambda}_{\sigma_{(r)}}\right)^{\varpi_r}}\right),\\
\left(\left[\dfrac{\mathrm{d}\prod_{r=1}^{g}\left(c^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\left(1-c^{-}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(c^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \dfrac{\mathrm{d}\prod_{r=1}^{g}\left(c^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\left(1-c^{+}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(c^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}\right], \dfrac{\mathrm{d}\prod_{r=1}^{g}\left(\breve{\psi}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\left(1-\breve{\psi}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(\breve{\psi}_{\sigma_{(r)}}\right)^{\varpi_r}}\right),\\
\left(\left[\dfrac{\mathrm{d}\prod_{r=1}^{g}\left(b^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\left(1-b^{-}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(b^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \dfrac{\mathrm{d}\prod_{r=1}^{g}\left(b^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{m}\left(1+(\mathrm{d}-1)\left(1-b^{+}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(b^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}\right], \dfrac{\mathrm{d}\prod_{r=1}^{g}\left(\breve{\delta}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-\breve{\delta}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(\breve{\delta}_{\sigma_{(r)}}\right)^{\varpi_r}}\right)
\end{array}\right\}.
\end{aligned}
\tag{37}
$$

The weight vector of $P_r\,(r = 1, 2, .., g)$ is $\varpi = (\varpi_1, \varpi_2, .., \varpi_g)^T$ with $\sum_{r=1}^{g} \varpi_r = 1$ and $\varpi_r > 0$.

4.4. Parameter d Effect on PCFHOWA Operator. Subsequently, we suppose that two appropriate cases for the PCFHOWA operator when the parameter $d = 1$ or 2.

Case (i): whenever $d = 1$, at that time, the PCFHOWA operator will reduce to PCFOWA operator:

$$\mathrm{PCFOWA}_{\varpi}\left(P_1, P_2, \ldots, P_g\right) = \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} 1 - \prod_{r=1}^{g} \left(1 - a^{-}_{\sigma_{(r)}}\right)^{\varpi_r}, \\ 1 - \prod_{r=1}^{g} \left(1 - a^{+}_{\sigma_{(r)}}\right)^{\varpi_r} \end{array} \right], \\ 1 - \prod_{r=1}^{g} \left(1 - \breve{\lambda}_{\sigma_{(r)}}\right)^{\varpi_r} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \prod_{r=1}^{g} \left(c^{-}_{\sigma_{(r)}}\right)^{\varpi_r}, \\ \prod_{r=1}^{g} \left(c^{+}_{\sigma_{(r)}}\right)^{\varpi_r} \end{array} \right], \\ \prod_{r=1}^{g} \left(\breve{\psi}_{\sigma_{(r)}}\right)^{\varpi_r} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \prod_{r=1}^{g} \left(b^{-}_{\sigma_{(r)}}\right)^{\varpi_r}, \\ \prod_{r=1}^{g} \left(b^{+}_{\sigma_{(r)}}\right)^{\varpi_r} \end{array} \right], \\ \prod_{r=1}^{g} \left(\breve{\delta}_{\sigma_{(r)}}\right)^{\varpi_r} \end{array} \right) \end{array} \right\}. \tag{38}$$

Case (ii): if $d = 2$, then the PCFHOWA operator will change to PCF Einstein weighted averaging (PCFEWA) operator:

$$\text{PCFEOWA}_{\varpi}\left(P_1, P_2, \ldots, P_g\right) = \left\{ \begin{array}{c} \left(\left[\dfrac{\prod_{r=1}^{g}\left(1+a^{-}_{\sigma_{(r)}}\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-a^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+a^{-}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(1-a^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{g}\left(1+a^{+}_{\sigma_{(r)}}\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-a^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+a^{+}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(1-a^{+}_{\sigma_{(r)}}\right)^{\varpi_r}} \right], \dfrac{\prod_{r=1}^{g}\left(1+\breve{\lambda}_{\sigma_{(r)}}\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-\breve{\lambda}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+\breve{\lambda}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(1-\breve{\lambda}_{\sigma_{(r)}}\right)^{\varpi_r}} \right), \\ \left(\left[\dfrac{2\prod_{r=1}^{g}\left(c^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-c^{-}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(c^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \dfrac{2\prod_{r=1}^{g}\left(c^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-c^{+}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(c^{+}_{\sigma_{(r)}}\right)^{\varpi_r}} \right], \dfrac{2\prod_{r=1}^{g}\left(\breve{\psi}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-\breve{\psi}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(\breve{\psi}_{\sigma_{(r)}}\right)^{\varpi_r}} \right), \\ \left(\left[\dfrac{2\prod_{r=1}^{g}\left(b^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-b^{-}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(b^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \dfrac{2\prod_{r=1}^{g}\left(b^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-b^{+}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(b^{+}_{\sigma_{(r)}}\right)^{\varpi_r}} \right], \dfrac{2\prod_{r=1}^{g}\left(\breve{\delta}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-\breve{\delta}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(\breve{\delta}_{\sigma_{(r)}}\right)^{\varpi_r}} \right) \end{array} \right\}. \tag{39}$$

Proposition 3. *Suppose that* $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r} \rangle$ $(r = 1, 2, 3, \ldots, g)$ *be a set of PCFVs in* $\breve{T}$ *and* $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_g)^T$ *is the weight vector of* P_r *with* $\sum_{r=1}^{g} \varpi_r = 1$, *and* ϖ_r *is the element of unit closed interval* ([0, 1])*; then, the coming properties are initiated.*

Idempotency Property. If all $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r} \rangle$ $(r = 1, 2, 3, \ldots, g)$ *are equal, that is,* $P_r = P$*, then*

$$\text{PCFHOWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) = P. \quad (40)$$

Boundedness Property. For every ϖ,

$$P^- \leq \text{PCFHOWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) \leq P^+, \quad (41)$$

where

$$P^+ = \left\{ \begin{array}{l} \left(\left[\min a^-_{\sigma_{(r)}}, \max a^+_{\sigma_{(r)}}\right], \max\breve{\lambda}_{\sigma_{(r)}}\right), \\ \left(\left[\max c^-_{\sigma_{(r)}}, \min c^+_{\sigma_{(r)}}\right], \min\breve{\psi}_{\sigma_{(r)}}\right), \\ \left(\left[\max b^-_{\sigma_{(r)}}, \min b^+_{\sigma_{(r)}}\right], \min\breve{\delta}_{\sigma_{(r)}}\right) \end{array} \right\},$$
$$P^- = \left\{ \begin{array}{l} \left(\left[\min a^-_{\sigma_{(r)}}, \max a^+_{\sigma_{(r)}}\right], \min\breve{\lambda}_{\sigma_{(r)}}\right), \\ \left(\left[\max c^-_{\sigma_{(r)}}, \min c^+_{\sigma_{(r)}}\right], \max\breve{\psi}_{\sigma_{(r)}}\right), \\ \left(\left[\max b^-_{\sigma_{(r)}}, \min b^+_{\sigma_{(r)}}\right], \max\breve{\delta}_{\sigma_{(r)}}\right) \end{array} \right\}. \quad (42)$$

Monotonicity Property. Let

$$P^*_r = \left\{\left(\left\langle\left[a^{*-}_{\sigma_{(r)}}, a^{*+}_{\sigma_{(r)}}\right], \breve{\lambda}^*_{\sigma_{(r)}}\right\rangle, \left\langle\left[c^{*-}_{\sigma_{(r)}}, c^{*+}_{\sigma_{(r)}}\right], \breve{\psi}^*_{\sigma_{(r)}}\right\rangle, \left\langle\left[b^{*-}_{\sigma_{(r)}}, b^{*+}_{\sigma_{(r)}}\right], \breve{\delta}^*_{\sigma_{(r)}}\right\rangle\right)\right\}, \quad (43)$$

where $(r = 1, 2, 3, \ldots, g)$ *is a group of PCFNs if*

$$\begin{aligned} &\left[a^-_{\sigma_{(r)}}, a^+_{\sigma_{(r)}}\right] \leq \left[a^{*-}_{\sigma_{(r)}}, a^{*+}_{\sigma_{(r)}}\right], \breve{\lambda}_{\sigma_{(r)}} \leq \breve{\lambda}^*_{\sigma_{(r)}}, \\ &\left[c^{*-}_{\sigma_{(r)}}, c^{*+}_{\sigma_{(r)}}\right] \leq \left[c^-_{\sigma_{(r)}}, c^+_{\sigma_{(r)}}\right], \breve{\psi}^*_{\sigma_{(r)}} \leq \breve{\psi}_{\sigma_{(r)}}, \\ &\left[b^{*-}_{\sigma_{(r)}}, b^{*+}_{\sigma_{(r)}}\right] \leq \left[b^-_{\sigma_{(r)}}, b^+_{\sigma_{(r)}}\right], \breve{\delta}^*_{\sigma_{(r)}} \leq \breve{\delta}_{\sigma_{(r)}}. \end{aligned} \quad (44)$$

Then,

$$\begin{aligned} &\text{PCFHOWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) \\ &\leq \text{PCFOHWA}_{\varpi}\left(P^*_1, P^*_2, P^*_3 \ldots, P^*_g\right). \end{aligned} \quad (45)$$

4.5. PCFHHA Operator

Definition 15. Let $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r} \rangle$ $(r = 1, 2, 3, \ldots, g)$ *be a set of PCFVs in* $\breve{T}$, *and suppose PCFHHA operator of g dimension is a function of PCFHHA*: $\Omega^g \longrightarrow \Omega$, *having* $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_g)^T$ which *are the weight vector of* P_r $(r = 1, 2, \ldots, g)$ *with* $\sum_{r=1}^{g} \varpi_r = 1$ *and* ϖ_r *is the element of unit closed interval* ([0, 1]) *and* $\varpi_r > 0$ *such that*

$$\text{PCFHHA}_{w,\varpi}\left(P_1, P_2, \ldots, P_g\right) = \oplus_{r=1}^{g} \varpi_r \widetilde{P}_{\sigma_{(r)}}, \quad (46)$$

which is the weight of $P_r\, w = (w_1, w_2, .., w_g)^T$, *with* $\sum_{r=1}^{m} w_r = 1$, *and* w_r *is the element of unit closed interval* ([0, 1])*. Here,* $\widetilde{P}_{\sigma_{(r)}}$ *is the gth greatest of the weighted PCFNs* $\widetilde{P}_r$ *and* $\widetilde{P}_{\sigma_{(r)}} = g w_r P_{\sigma_{(r)}} = (\langle[\widetilde{a}^-_{\sigma_{(r)}}, \widetilde{a}^+_{\sigma_{(r)}}], \widetilde{\lambda}_{\sigma_{(r)}}\rangle, \langle[\widetilde{c}^-_{\sigma_{(r)}}, \widetilde{c}^+_{\sigma_{(r)}}], \widetilde{\psi}_{\sigma_{(r)}}\rangle, \langle[\widetilde{b}^-_{\sigma_{(r)}}, \widetilde{b}^+_{\sigma_{(r)}}], \widetilde{\delta}_{\sigma_{(r)}}\rangle)$ $(r = 1, 2, \ldots, g)$, *where*

$$\widetilde{a}^-_{\sigma_{(r)}} = \frac{\left(1 + (d-1)a^-_{\sigma_{(r)}}\right)^{gw_r} - \left(1 - a^-_{\sigma_{(r)}}\right)^{gw_r}}{\left(1 + (d-1)a^-_{\sigma_{(r)}}\right)^{gw_r} + (d-1)\left(1 - a^-_{\sigma_{(r)}}\right)^{gw_r}},$$

$$\widetilde{a}^+_{\sigma_{(r)}} = \frac{\left(1 + (d-1)a^+_{\sigma_{(r)}}\right)^{gw_r} - \left(1 - a^+_{\sigma_{(r)}}\right)^{gw_r}}{\left(1 + (d-1)a^+_{\sigma_{(r)}}\right)^{gw_r} + (g-1)\left(1 - a^+_{\sigma_{(r)}}\right)^{gw_r}},$$

$$\widetilde{\lambda}_{\sigma_{(r)}} = \frac{\left(1 + (d-1)\breve{\lambda}_{\sigma_{(r)}}\right)^{gw_r} - \left(1 - \breve{\lambda}_{\sigma_{(r)}}\right)^{gw_r}}{\left(1 + (d-1)\breve{\lambda}_{\sigma_{(r)}}\right)^{gw_r} + (d-1)\left(1 - \breve{\lambda}_{\sigma_{(r)}}\right)^{gw_r}},$$

$$\widetilde{c}^-_{\sigma_{(r)}} = \frac{d\left(c^-_{\sigma_{(r)}}\right)^{gw_r}}{1 + (d-1)\left(1 - c^-_{\sigma_{(r)}}\right)^{gw_r} + (d-1)\left(c^-_{\sigma_{(r)}}\right)^{gw_r}},$$

$$
\begin{aligned}
\tilde{c}^{+}_{\sigma_{(r)}} &= \frac{d\left(c^{+}_{\sigma_{(r)}}\right)^{gw_r}}{1+(d-1)\left(1-c^{+}_{\sigma_{(r)}}\right)^{gw_r}+(d-1)\left(c^{+}_{\sigma_{(r)}}\right)^{gw_r}},\\
\tilde{\tilde{\breve{\psi}}}_{\sigma_{(r)}} &= \frac{g\left(\breve{\psi}_{\sigma_{(r)}}\right)^{gw_r}}{1+(d-1)\left(1-\breve{\psi}_{\sigma_{(r)}}\right)^{gw_r}+(d-1)\left(\breve{\psi}_{\sigma_{(r)}}\right)^{gw_r}},\\
\tilde{b}^{-}_{\sigma_{(r)}} &= \frac{d\left(b^{-}_{\sigma_{(r)}}\right)^{gw_r}}{1+(d-1)\left(1-b^{-}_{\sigma_{(r)}}\right)^{gw_r}+(d-1)\left(b^{-}_{\sigma_{(r)}}\right)^{gw_r}},\\
\tilde{b}^{+}_{\sigma_{(r)}} &= \frac{d\left(b^{+}_{\sigma_{(r)}}\right)^{gw_r}}{1+(d-1)\left(1-b^{+}_{\sigma_{(r)}}\right)^{gw_r}+(d-1)\left(b^{+}_{\sigma_{(r)}}\right)^{gw_r}},\\
\tilde{\tilde{\breve{\psi}}}_{\sigma_{(r)}} &= \frac{d\left(\breve{\delta}_{\sigma_{(r)}}\right)^{gw_r}}{1+(d-1)\left(1-\breve{\delta}_{\sigma_{(r)}}\right)^{gw_r}+(d-1)\left(\breve{\delta}_{\sigma_{(r)}}\right)^{gw_r}}.
\end{aligned}
\tag{47}
$$

Here, the coefficient of balancing is m, which maintains the appropriate balance; particularly, whenever $\varpi = (1/g, 1/g, 1/g, \ldots, 1/g)^T$, then the PCFHWA and PCFHOWA operators are considered as an important case of PCFHHA operator.

Hence, under Hamacher operational rules of PCFNs, we get the theorem as given below.

Theorem 3. *Suppose that $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r} \rangle$ $(r = 1, 2, \ldots, g)$ is the set of PCFVs in $\breve{T}$; then, the aggregated value of them utilizing PCFHHA operator is also a PCFV and is given below:*

$$
\begin{aligned}
&PCFHHA_{\varpi}\left(P_1, P_2, .., P_g\right)\\
&= \left\{
\begin{array}{l}
\left(\left[\dfrac{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\widetilde{a}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-\widetilde{a}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\widetilde{a}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(1-\widetilde{a}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\widetilde{a}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-\widetilde{a}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\widetilde{a}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(1-\widetilde{a}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}\right], \dfrac{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\breve{\tilde{\lambda}}_{\sigma_{(r)}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-\breve{\tilde{\lambda}}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\breve{\tilde{\lambda}}_{\sigma_{(r)}}\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(1-\breve{\tilde{\lambda}}_{\sigma_{(r)}}\right)^{\varpi_r}}\right),\\
\left(\left[\dfrac{\mathrm{d}\prod_{r=1}^{g}\left(\widetilde{c}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\left(1-\widetilde{c}^{-}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+\prod_{r=1}^{g}\left(\widetilde{c}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}(\mathrm{d}-1)}, \dfrac{k\prod_{r=1}^{g}\left(\widetilde{c}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\left(1-\widetilde{c}^{+}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(\widetilde{c}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}\right], \dfrac{\mathrm{d}\prod_{r=1}^{g}\left(\breve{\tilde{\tilde{\psi}}}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\left(1-\breve{\tilde{\tilde{\psi}}}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(\breve{\tilde{\tilde{\psi}}}_{\sigma_{(r)}}\right)^{\varpi_r}}\right),\\
\left(\left[\dfrac{\mathrm{d}\prod_{r=1}^{g}\left(\widetilde{b}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\left(1-\widetilde{b}^{-}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(\widetilde{b}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \dfrac{\mathrm{d}\prod_{r=1}^{g}\left(\widetilde{b}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\left(1-\widetilde{b}^{+}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{m}\left(\widetilde{b}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}\right], \dfrac{\mathrm{d}\prod_{r=1}^{g}\left(\breve{\tilde{\tilde{\delta}}}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(\mathrm{d}-1)\left(1-\breve{\tilde{\tilde{\delta}}}_{\sigma_{(r)}}\right)\right)^{\varpi_r}+(\mathrm{d}-1)\prod_{r=1}^{g}\left(\breve{\tilde{\tilde{\delta}}}_{\sigma_{(r)}}\right)^{\varpi_r}}\right)
\end{array}
\right\}.
\end{aligned}
\tag{48}
$$

The weight vector of $P_r\,(r = 1, 2, \ldots, g)$ *are* $\varpi = (\varpi_1, \varpi_2, .., \varpi_g)^T$, *with* $\sum_{r=1}^{g} \varpi_r = 1$ *and* $\varpi_r > 0$.

4.6. Parameter d Effect on PCFHHA Operator. Subsequently, we suppose that two appropriate cases for the PCFHHA operator when $d = 1$ or 2.

Case (i): whenever $d = 1$, at that time, the PCFHHA operator will change to picture cubic fuzzy hybrid averaging (PCFHA) operator:

$$\mathrm{PCFHA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) = \left\{ \begin{array}{l} \left(\left[1 - \prod_{r=1}^{g} \left(1 - \tilde{a}^{-}_{\sigma_{(r)}} \right)^{\varpi_r}, 1 - \prod_{r=1}^{g} \left(1 - \tilde{a}^{+}_{\sigma_{(r)}} \right)^{\varpi_r} \right], 1 - \prod_{r=1}^{g} \left(1 - \tilde{\tilde{\lambda}}_{\sigma_{(r)}} \right)^{\varpi_r} \right), \\ \left(\left[\prod_{r=1}^{g} \left(\tilde{c}^{-}_{\sigma_{(r)}} \right)^{\varpi_r}, \prod_{r=1}^{g} \left(\tilde{c}^{+}_{\sigma_{(r)}} \right)^{\varpi_r} \right], \prod_{r=1}^{g} \left(\tilde{\tilde{\psi}}_{\sigma_{(r)}} \right)^{\varpi_r} \right), \\ \left(\left[\prod_{r=1}^{g} \left(\tilde{b}^{-}_{\sigma_{(r)}} \right)^{\varpi_r}, \prod_{r=1}^{g} \left(\tilde{b}^{+}_{\sigma_{(r)}} \right)^{\varpi_r} \right], \prod_{r=1}^{g} \left(\tilde{\tilde{\delta}}_{\sigma_{(r)}} \right)^{\varpi_r} \right) \end{array} \right\}. \tag{49}$$

Case (ii): whenever $d = 2$, at that time, the PCFHHA operator will change to PCF Einstein hybrid averaging (PCFEHA) operator:

$$\mathrm{PCFEHA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) = \left\{ \begin{array}{l} \left(\left[\frac{\prod_{r=1}^{g}\left(1+\tilde{a}^{-}_{\sigma_{(r)}}\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-\tilde{a}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+\tilde{a}^{-}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(1-\tilde{a}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \frac{\prod_{r=1}^{g}\left(1+\tilde{a}^{+}_{\sigma_{(r)}}\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-\tilde{a}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+\tilde{a}^{+}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(1-\tilde{a}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}} \right], \frac{\prod_{r=1}^{g}\left(1+\tilde{\tilde{\lambda}}_{\sigma_{(r)}}\right) - \prod_{r=1}^{g}\left(1-\tilde{\tilde{\lambda}}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+\tilde{\tilde{\lambda}}_{\sigma_{(r)}}\right) + \prod_{r=1}^{g}\left(1-\tilde{\tilde{\lambda}}_{\sigma_{(r)}}\right)} \right), \\ \left(\left[\frac{2\prod_{r=1}^{g}\left(\tilde{c}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-\tilde{c}^{-}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(\tilde{c}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \frac{2\prod_{r=1}^{g}\left(\tilde{c}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-\tilde{c}^{+}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(\tilde{c}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}} \right], \frac{2\prod_{r=1}^{g}\left(\tilde{\tilde{\psi}}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-\tilde{\tilde{\psi}}^{+}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(2-\tilde{\tilde{\psi}}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}} \right), \\ \left(\left[\frac{2\prod_{r=1}^{g}\left(\tilde{b}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-\tilde{b}^{-}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(\tilde{b}^{-}_{\sigma_{(r)}}\right)^{\varpi_r}}, \frac{2\prod_{r=1}^{g}\left(\tilde{b}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(2-\tilde{b}^{+}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(\tilde{b}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}} \right], \frac{2\prod_{r=1}^{g}\left(\tilde{\tilde{\delta}}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}}{\prod_{r=1}^{g} 2-\left(\tilde{\tilde{\delta}}^{+}_{\sigma_{(r)}}\right)^{\varpi_r} + \prod_{r=1}^{g}\left(\tilde{\tilde{\delta}}^{+}_{\sigma_{(r)}}\right)^{\varpi_r}} \right) \end{array} \right\}. \tag{50}$$

Proposition 4. *Suppose that* $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r}\rangle$ $(r = 1, 2, 3, \ldots, g)$ *is the set of PCFVs in* T *and* $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_g)^T$ *are the weight vector of* P_r *with* $\sum_{r=1}^{g} \varpi_r = 1$ *and* $\varpi_r \in [0, 1]$*; then, the coming properties are initiated.*

Idempotency Property. If all $P_r = \langle a_{P_r}, c_{P_r}, b_{P_r}\rangle$ $(r = 1, 2, 3, \ldots, g)$ *are equal, that is,* $P_r = P$*, then*

$$\text{PCFHHA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) = P. \tag{51}$$

Boundedness Property. For every ϖ,

$$P^- \leq \text{PCFHHA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) \leq P^+, \tag{52}$$

where

$$\begin{aligned} P^+ &= \left\{ \begin{array}{l} \left(\left[\min a^-_{\sigma_{(r)}}, \max a^+_{\sigma_{(r)}}\right], \max \breve{\lambda}_{\sigma_{(r)}}\right), \\ \left(\left[\max c^-_{\sigma_{(r)}}, \min c^+_{\sigma_{(r)}}\right], \min \breve{\psi}_{\sigma_{(r)}}\right), \\ \left(\left[\max b^-_{\sigma_{(r)}}, \min b^+_{\sigma_{(r)}}\right], \min \breve{\delta}_{\sigma_{(r)}}\right) \end{array} \right\}, \\ P^- &= \left\{ \begin{array}{l} \left(\left[\min a^-_{\sigma_{(r)}}, \max a^+_{\sigma_{(r)}}\right], \min \breve{\lambda}_{\sigma_{(r)}}\right), \\ \left(\left[\max c^-_{\sigma_{(r)}}, \min c^+_{\sigma_{(r)}}\right], \max \breve{\psi}_{\sigma_{(r)}}\right), \\ \left(\left[\max b^-_{\sigma_{(r)}}, \min b^+_{\sigma_{(r)}}\right], \max \breve{\delta}_{\sigma_{(r)}}\right) \end{array} \right\}. \end{aligned} \tag{53}$$

Monotonicity Property. Let

$$P^*_r = \left\{\left(\left\langle\left[\tilde{a}^{*^-}_{\sigma_{(r)}}, \tilde{a}^{*^+}_{\sigma_{(r)}}\right], \tilde{\breve{\lambda}}^*_{\sigma_{(r)}}\right\rangle, \left\langle\left[\tilde{c}^{*^-}_{\sigma_{(r)}}, \tilde{c}^{*^+}_{\sigma_{(r)}}\right], \tilde{\breve{\psi}}^*_{\sigma_{(r)}}\right\rangle, \left\langle\left[\tilde{b}^{*^-}_{\sigma_{(r)}}, \tilde{b}^{*^+}_{\sigma_{(r)}}\right], \tilde{\breve{\delta}}^*_{\sigma_{(r)}}\right\rangle\right)\right\}, \tag{54}$$

where $(r = 1, 2, 3, \ldots, g)$ *is a group of PCFNs if*

$$\begin{aligned} &\left[\tilde{a}^-_{\sigma_{(r)}}, \tilde{a}^+_{\sigma_{(r)}}\right] \leq \left[\tilde{a}^{*^-}_{\sigma_{(r)}}, \tilde{a}^{*^+}_{\sigma_{(r)}}\right], \tilde{\breve{\lambda}}_{\sigma_{(r)}} \leq \tilde{\breve{\lambda}}^*_{\sigma_{(r)}}, \\ &\left[\tilde{c}^{*^-}_{\sigma_{(r)}}, \tilde{c}^{*^+}_{\sigma_{(r)}}\right] \leq \left[\tilde{c}^-_{\sigma_{(r)}}, \tilde{c}^+_{\sigma_{(r)}}\right], \tilde{\breve{\psi}}^*_{\sigma_{(r)}} \leq \tilde{\breve{\psi}}_{\sigma_{(r)}}, \\ &\left[\tilde{b}^{*^-}_{\sigma_{(r)}}, \tilde{b}^{*^+}_{\sigma_{(r)}}\right] \leq \left[\tilde{b}^-_{\sigma_{(r)}}, \tilde{b}^+_{\sigma_{(r)}}\right], \tilde{\breve{\delta}}^*_{\sigma_{(r)}} \leq \tilde{\breve{\delta}}_{\sigma_{(r)}}. \end{aligned} \tag{55}$$

Then,

$$\text{PCFHHA}_{w,\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) \leq \text{PCFHHA}_{w,\varpi}\left(P^*_1, P^*_2, P^*_3, \ldots, P^*_g\right). \tag{56}$$

Theorem 4. *The PCFHWA operator is a special case of the PCFHHA operator.*

Proof. Suppose $\varpi = (1/g, 1/g, 1/g, \ldots, 1/g)^T$; then,

$$\begin{aligned} \text{PCFHHA}_{w,\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) &= \varpi_1 \tilde{P}_{\sigma_{(1)}} \oplus \varpi_2 \tilde{P}_{\sigma_{(2)}} \oplus, \ldots, \oplus \varpi_g \tilde{P}_{\sigma_{(g)}}, \\ &= \frac{1}{g}\left(\tilde{P}_{\sigma_{(1)}} \oplus \tilde{P}_{\sigma_{(2)}} \oplus, \ldots, \tilde{P}_{\sigma_{(g)}}\right) \\ &= \varpi_1 P_{\sigma} \oplus \varpi_2 P_{\sigma} \oplus, \ldots, \varpi_g P_{\sigma_{(g)}} \\ &= \text{PCFHWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right). \end{aligned} \tag{57}$$

Hence, it is proved. □

Theorem 5. *The PCFHOWA operator is a special case of the PCFHHA operator.*

Proof. Suppose $\varpi = (1/g, 1/g, 1/g, \ldots, 1/g)^T$, so $\tilde{P}_{\sigma_{(r)}} = P_{\sigma_{(r)}}$ $(r = 1, 2, 3, \ldots, g)$; thus,

$$\begin{aligned}\mathrm{PCFHHA}_{w,\varpi}\left(P_1,P_2,P_3,\ldots,P_g\right) &= \varpi_1\widetilde{P}_{\sigma_{(1)}}\oplus\varpi_2\widetilde{P}_{\sigma_{(2)}}\oplus,\ldots,\oplus\varpi_g\widetilde{P}_{\sigma_{(g)}}\\ &= \varpi_1 P_{\sigma_{(1)}}\oplus\varpi_2 P_{\sigma_{(2)}}\oplus,\ldots,\varpi_g P_{\sigma_{(g)}}\\ &= \mathrm{PCFHOWA}_{\varpi}\left(P_1,P_2,P_3,\ldots,P_g\right).\end{aligned}\tag{58}$$

Hence, the result is proved. □

5. MAGDM Methods on the Basis of Planned Operators

In this section, picture cubic fuzzy Hamacher weighted aggregation (PCFHWA) operators are utilized to MAGDM algorithm. Consider there are **u** alternatives $\widetilde{\varrho} = \{\widetilde{\varrho}_1, \widetilde{\varrho}_2, \ldots, \widetilde{\varrho}_u\}$ and **v** attributes $\widetilde{\Gamma} = \{\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \ldots, \widetilde{\Gamma}_v\}$ with weight vector $\varpi = \{\varpi_1, \varpi_2, \ldots, \varpi_v\}$ such that $\sum_{h=1}^{v}\varpi_h = 1$ and $\varpi_h \in [0,1]$. To evaluate the accomplishment of the alternatives $\widetilde{\varrho}_x \quad (x = 1,2,\ldots,u)$ based on attributes $\widetilde{\Gamma}_h$, the experts have to give only the statistics about the alternative $\widetilde{\varrho}_x$, fulfilling the attributes but also about the alternative $\widetilde{\varrho}_x$ and not fulfilling the attributes $\widetilde{\Gamma}_h \quad (h = 1,2,\ldots,v)$. Suppose that the rating of the alternatives $\widetilde{\varrho}_x$ based on the criteria $\widetilde{\Gamma}_h$ provided by the experts in the form of PCFNs in $\breve{T}$: $P_{xh} = \langle a_{xh}, b_{xh}, c_{xh}\rangle \quad (x = 1,2,\ldots,u; h = 1,2,\ldots,v)$. Suppose a_{xh} represent the degree of alternatives $\widetilde{\varrho}_x$ that satisfying the criteria $\widetilde{\Gamma}_h$, c_{xh} represents the degree of alternatives $\widetilde{\varrho}_x$ that are neutral, and b_{xh} represent the degree of alternatives $\widetilde{\varrho}_x$ that not satisfying the criteria $\widetilde{\Gamma}_h$, such that $a_{xh} = \langle [a^-_{xh}, a^+_{xh}], \breve{\lambda}_{xh}\rangle 1/2$, $c_{xh} = \langle [c^-_{xh}, c^+_{xh}], \breve{\psi}_{xh}\rangle$, and $b_{xh} = \langle [b^-_{xh}, b^+_{xh}], \delta_{xh}\rangle$, under the condition $[a^-_{xh}, a^+_{xh}], [c^-_{xh}, c^+_{xh}], [b^-_{xh}, b^+_{xh}] \subset [0,1]$ and $\breve{\lambda}_{xh}\colon \breve{T} \longrightarrow [0,1]$, $\breve{\psi}_{xh}\colon \breve{T} \longrightarrow [0,1]$, and $\breve{\delta}_{xh}\colon \breve{T} \longrightarrow [0,1]$, subject to $0 \le \sup[a^-_{xh}, a^+_{xh}] + \sup[c^-_{xh}, c^+_{xh}] + \sup[b^-_{xh}, b^+_{xh}] \le 1$ and $0 \le \breve{\lambda}_{xh} + \psi_{xh} + \delta_{xh} \le 1 \quad (x = 1,2,\ldots,u; h = 1,2,\ldots,v)$. Consequently, a MAGDM problem can be shortly given in a PCF decision matrix $M = (P_{xh})_{u\times v} = (\langle a_{xh}, c_{xh}, b_{xh}\rangle)_{u\times v}(x = 1,2,\ldots,u; h = 1,2,\ldots,v)$. Further steps which are involved in the MAGDM method are given below.

5.1. Main Algorithm

Start.

Step 1: construct PCF decision matrix $M = (P_{xh})_{u\times v} = (\langle a_{xh}, c_{xh}, b_{xh}\rangle)_{u\times v}(x = 1,2,\ldots,u;\ h = 1,2,\ldots,v)$. The attribute is usually classified as benefit and cost attributes. Of course, it is unnecessary of normalizing the rating values if all attributes be the identical types. The values of the cost attributes can be transformed into the values of benefit attributes by the following formula, when M contains the two cost and benefits attributes:

$$s_{xh} = \langle a_{xh}, b_{xh}, c_{xh}\rangle = \begin{cases} q_{xh} \text{ for benefit criteria,}\\ q^c_{xh} \text{ for cost criteria,}\end{cases}\tag{59}$$

where the complement of q_{xh} is q^c_{xh}. Consequently, we obtained a normalized PCF decision matrix denoted by M^u and is assigned as

$$M^u = s_{xh} = \left(\langle a_{xh}, b_{ij}, c_{xh}\rangle\right)_{u\times v}\quad (x = 1,2,\ldots,u; h = 1,2,\ldots,v).\tag{60}$$

Next, we will apply the PCFHWA, PCFHOWA, and PCFHHA operators to MCGDM, moreover consist of the coming steps.

Step 2: utilize planned operators to calculate PCFNs $Px \quad (x = 1,2,\ldots,u)$ for the alternatives $\widetilde{\varrho}_x$. This is the established operators to maintain the overall collective choice values $Px \quad (x = 1,2,\ldots,u)$ for the alternatives $\widetilde{\varrho}_x$, where $\varpi = \{\varpi_1, \varpi_2, \ldots, \varpi_v\}$ is the weight vector of the attributes.

Step 3: utilizing the score function formula of PCFNs, we calculate the score functions $S(P_x) \quad (x = 1,2,\ldots,u)$ of P_x to rank the alternatives $\widetilde{\varrho}_x$. When the two score functions $S(P_s)$ and $S(P_e)$ are equal, we need their accuracy function $A(P_s)$ and $S(P_e)$; then, we rank the alternative based on accuracy degree.

Step 4: rank the alternatives $\widetilde{\varrho}_x \quad (x = 1,2,\ldots,u)$ for choosing a suitable one.

End.

6. Application

Suppose that $\{\widetilde{\varrho}_1, \widetilde{\varrho}_2, \widetilde{\varrho}_3\}$ are three SHPPs from power plant category. The generation capacity of these three SHPPs is 25 MW, to be located at different geographical sites in Pakistan. Three decision makers have been appointed to select the most suitable plant for beginning the construction activity (adopted from [48]), on the basis of four criteria such that

$$\begin{aligned}&\widetilde{\Gamma}_1\colon \text{approachability},\\ &\widetilde{\Gamma}_2\colon \text{socioeconomic climate},\\ &\widetilde{\Gamma}_3\colon \text{constructability},\\ &\widetilde{\Gamma}_4\colon \text{technical feasibility}.\end{aligned}\tag{61}$$

Utilizing attributes' weighting vector $(0.35, 0.3, 0.21, 0.14)^T$ and the decision makers' weighting $(0.4, 0.35, 0.25)^T$, three experts present the information in the form of evaluation matrices, as shown in Table 1–3.

6.1. By PCFHWA Operator

Step 5: the experts' decision is given in Tables 1–3. On this spot, it is unnecessary of normalization because all the attributes are given as beneficial attributes.

Suppose $k = 2$ and $\varpi = (0.4, 0.35, 0.25)^T$ as a weight vector of experts; using PCFHWA operator, we have collected the information stated by the three experts in Tables 1

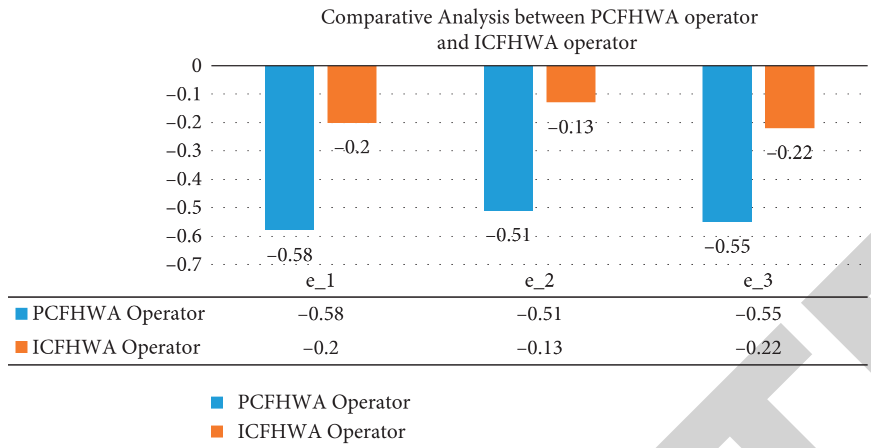

Figure 1: Comparative study between PCFHWA operator and ICFHWA operator and ranking of the alternatives.

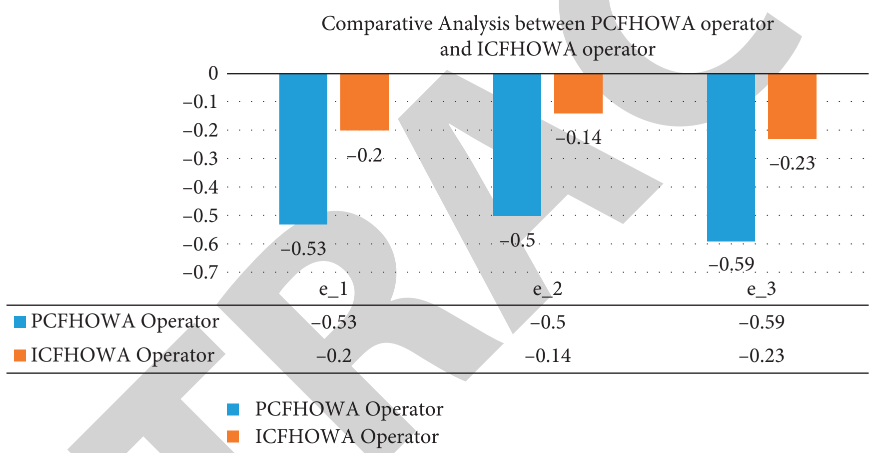

Figure 2: Comparative study between PCFHOWA operator and ICFHOWA operator and ranking of the alternatives.

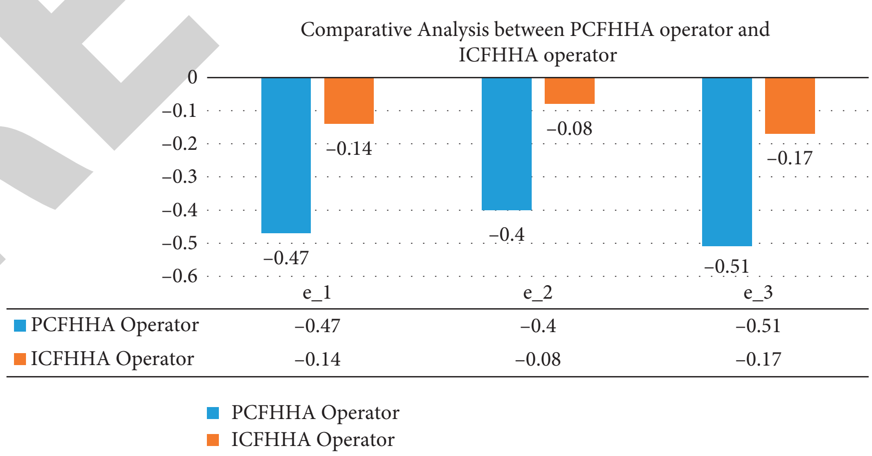

Figure 3: Comparative study between PCFHHA operator and ICFHHA operator and ranking of the alternatives.

Table 1: First experts' information.

Supp.	$\tilde{\Gamma}_1$	$\tilde{\Gamma}_2$	$\tilde{\Gamma}_3$	$\tilde{\Gamma}_4$
$\tilde{\varrho}_1$	$\begin{pmatrix} ([0.1, 0.12], 0.12), \\ ([0.1, 0.2], 0.5), \\ ([0.2, 0.3], 0.2) \end{pmatrix}$	$\left(\begin{pmatrix} ([0.3, 0.31], 0.1), \\ ([0.1, 0.3], 0.3), \\ ([0.1, 0.3], 0.1) \end{pmatrix}\right)$	$\begin{pmatrix} ([0.2, 0.21], 0.21), \\ ([0.1, 0.4], 0.2), \\ ([0.1, 0.2], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.14], 0.14), \\ ([0.1, 0.3], 0.3), \\ ([0.1, 0.2], 0.4) \end{pmatrix}$
$\tilde{\varrho}_2$	$\begin{pmatrix} ([0.2, 0.22], 0.21), \\ ([0.2, 0.21], 0.25), \\ ([0.1, 0.2], 0.15) \end{pmatrix}$	$\begin{pmatrix} ([0.3, 0.31], 0.31), \\ ([0.2, 0.3], 0.3), \\ ([0.1, 0.16], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.201], 0.21), \\ ([0.3, 0.4], 0.31), \\ ([0.1, 0.2], 0.22) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.201], 0.22), \\ ([0.1, 0.24], 0.3), \\ ([0.1, 0.12], 0.13) \end{pmatrix}$
$\tilde{\varrho}_3$	$\begin{pmatrix} ([0.1, 0.104], 0.14), \\ ([0.1, 0.4], 0.2), \\ ([0.1, 0.2], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.22], 0.21), \\ ([0.2, 0.3], 0.22), \\ ([0.2, 0.3], 0.31) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.102], 0.12), \\ ([0.1, 0.3], 0.1), \\ ([0.1, 0.2], 0.31) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.201], 0.21), \\ ([0.2, 0.3], 0.4), \\ ([0.1, 0.2], 0.4) \end{pmatrix}$

Table 2: Second experts' information.

Supp.	$\tilde{\Gamma}_1$	$\tilde{\Gamma}_2$	$\tilde{\Gamma}_3$	$\tilde{\Gamma}_4$
$\tilde{\varrho}_1$	$\begin{pmatrix} ([0.1, 0.13]0.13), \\ ([0.1, 0.2], 0.3), \\ ([0.1, 0.2], 0.1) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.11], 0.11), \\ ([0.2, 0.3], 0.4), \\ ([0.2, 0.3], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.11], 0.11), \\ ([0.1, 0.2], 0.4), \\ ([0.2, 0.3], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.12], 0.12), \\ ([0.1, 0.21], 0.2), \\ ([0.2, 0.4], 0.21) \end{pmatrix}$
$\tilde{\varrho}_2$	$\begin{pmatrix} ([0.2, 0.201], 0.201), \\ ([0.2, 0.25], 0.25), \\ ([0.1, 0.2], 0.21) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.201], 0.201), \\ ([0.1, 0.2], 0.2), \\ ([0.2, 0.22], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.22], 0.22), \\ ([0.3, 0.31], 0.32), \\ ([0.1, 0.2], 0.23) \end{pmatrix}$	$\begin{pmatrix} ([0.102, 0.23], 0.21), \\ ([0.2, 0.21], 0.24), \\ ([0.1, 0.11], 0.14) \end{pmatrix}$
$\tilde{\varrho}_3$	$\begin{pmatrix} ([0.1, 0.103], 0.12), \\ ([0.1, 0.3], 0.3), \\ ([0.1, 0.2], 0.5) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.103], 0.11), \\ ([0.1, 0.2], 0.3), \\ ([0.1, 0.2], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.103], 0.103), \\ ([0.1, 0.3], 0.31), \\ ([0.2, 0.3], 0.21) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.105], 0.102), \\ ([0.2, 0.3], 0.21), \\ ([0.1, 0.2], 0.3) \end{pmatrix}$

Table 3: Third experts' information.

Supp.	$\tilde{\Gamma}_1$	$\tilde{\Gamma}_2$	$\tilde{\Gamma}_3$	$\tilde{\Gamma}_4$
$\tilde{\varrho}_1$	$\begin{pmatrix} ([0.2, 0.22], 0.22), \\ ([0.3, 0.4], 0.1), \\ ([0.1, 0.2], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.21], 0.21), \\ ([0.1, 0.3], 0.1), \\ ([0.1, 0.2], 0.1) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.12], 0.12), \\ ([0.1, 0.2], 0.1), \\ ([0.2, 0.21], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.13], 0.13), \\ ([0.1, 0.2], 0.2), \\ ([0.1, 0.2], 0.21) \end{pmatrix}$
$\tilde{\varrho}_2$	$\begin{pmatrix} ([0.1, 0.101], 0.102), \\ ([0.1, 0.2], 0.15), \\ ([0.1, 0.5], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.101], 0.101), \\ ([0.2, 0.3], 0.25), \\ ([0.1, 0.15], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.2], 0.21), \\ ([0.1, 0.2], 0.23), \\ ([0.1, 0.2], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.2], 0.105), \\ ([0.2, 0.3], 0.3), \\ ([0.2, 0.21], 0.31) \end{pmatrix}$
$\tilde{\varrho}_3$	$\begin{pmatrix} ([0.1, 0.13], 0.11), \\ ([0.2, 0.4], 0.21), \\ ([0.1, 0.3], 0.21) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.105], 0.102), \\ ([0.1, 0.2], 0.3), \\ ([0.2, 0.3], 0.31) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.3], 0.201), \\ ([0.2, 0.4], 0.3), \\ ([0.2, 0.3], 0.22) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.104], 0.11), \\ ([0.1, 0.2], 0.15), \\ ([0.1, 0.3], 0.3) \end{pmatrix}$

to 3 on the basis of different importance of all the experts given in Table 4.

Step 6: let $k = 2$; again, using PCFHWA operator and utilizing aggregated report in Table 4 with weight vector $(0.35, 0.3, 0.21, 0.14^T)$, we obtain the collective PCFNs for alternatives $\{\tilde{\varrho}_1, \tilde{\varrho}_2, \tilde{\varrho}_3\}$:

$$\begin{aligned} \tilde{\varrho}_1 &= (([0.1, 0.16], 0.14), ([0.3, 0.4], 0.4), ([0.3, 0.4], 0.3)), \\ \tilde{\varrho}_2 &= (([0.18, 0.22], 0.21), ([0.33, 0.4], 0.4), ([0.3, 0.35], 0.36)), \\ \tilde{\varrho}_3 &= (([0.101, 0.2], 0.14), ([0.3, 0.4], 0.4), ([0.3, 0.4], 0.4)). \end{aligned} \tag{62}$$

Table 4: Collected information by PCFHWA operator.

Supp.	$\widetilde{\Gamma}_1$	$\widetilde{\Gamma}_2$	$\widetilde{\Gamma}_3$	$\widetilde{\Gamma}_4$
$\widetilde{\varrho}_1$	$\begin{pmatrix} ([0.1, 0.14], 0.14), \\ ([0.2, 0.3], 0.3), \\ ([0.2, 0.3], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.21], 0.13), \\ ([0.2, 0.4], 0.3), \\ ([0.2, 0.3], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.14], 0.14), \\ ([0.2, 0.3], 0.3), \\ ([0.2, 0.3], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.13], 0.13), \\ ([0.2, 0.3], 0.3), \\ ([0.2, 0.3], 0.3) \end{pmatrix}$
$\widetilde{\varrho}_2$	$\begin{pmatrix} ([0.18, 0.2], 0.18), \\ ([0.3, 0.31], 0.305), \\ ([0.2, 0.3], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.22], 0.23), \\ ([0.2, 0.3], 0.3), \\ ([0.2, 0.26], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.21], 0.21), \\ ([0.3, 0.4], 0.4), \\ ([0.2, 0.3], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.21], 0.19), \\ ([0.2, 0.3], 0.4), \\ ([0.2, 0.21], 0.2) \end{pmatrix}$
$\widetilde{\varrho}_3$	$\begin{pmatrix} ([0.1, 0.11], 0.13), \\ ([0.2, 0.4], 0.3), \\ ([0.2, 0.3], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.15], 0.15), \\ ([0.2, 0.3], 0.3), \\ ([0.2, 0.3], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.15], 0.13), \\ ([0.2, 0.4], 0.3), \\ ([0.2, 0.3], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.15], 0.15), \\ ([0.3, 0.4], 0.3), \\ ([0.2, 0.3], 0.3) \end{pmatrix}$

Abbreviation: PCFHWA, picture cubic fuzzy Hamacher weighted averaging

Step 7: utilizing score function formula of PCFNs, calculate $S(\widetilde{\varrho}_1), S(\widetilde{\varrho}_2)$, and $S(\widetilde{\varrho}_3)$:

$$\begin{aligned} S(\widetilde{\varrho}_1) &= -0.5667, \\ S(\widetilde{\varrho}_2) &= -0.5100, \\ S(\widetilde{\varrho}_3) &= -0.5527. \end{aligned} \tag{63}$$

Step 8: rank the alternatives:

$$\widetilde{\varrho}_2 > \widetilde{\varrho}_3 > \widetilde{\varrho}_1. \tag{64}$$

Rank all the alternatives to select the good alternative is $\widetilde{\varrho}_2$.

6.2. By PCFHOWA Operator

Step 9: the collected information stated by three experts with respect to different importance three experts is given in Table 4.

Step 10: let $k = 2$, and using PCFHOWA operator and utilizing the aggregated information in Table 4, with weight vector $(0.35, 0.3, 0.21, 0.14)^T$, we obtain the collective PCFNs for the alternatives $\{\widetilde{\varrho}_1, \widetilde{\varrho}_2, \widetilde{\varrho}_3\}$, which are given below:

$$\begin{aligned} \widetilde{\varrho}_{\sigma_{(1)}} &= (([0.14, 0.17], 0.14), ([0.29, 0.4], 0.38), ([0.29, 0.38], 0.31)), \\ \widetilde{\varrho}_{\sigma_{(2)}} &= (([0.17, 0.2], 0.21), ([0.31, 0.38], 0.41), ([0.29, 0.34], 0.35)), \\ \widetilde{\varrho}_{\sigma_{(3)}} &= (([0.101, 0.15], 0.14), ([0.3, 0.4], 0.38), ([0.29, 0.38], 0.41)). \end{aligned} \tag{65}$$

Step 11: utilizing score function formula of PCFNs, calculate $S(\widetilde{\varrho}_1), S(\widetilde{\varrho}_2)$, and $S(\widetilde{\varrho}_3)$:

$$\begin{aligned} S(\widetilde{\varrho}_1) &= -0.5333, \\ S(\widetilde{\varrho}_2) &= -0.5000, \\ S(\widetilde{\varrho}_3) &= -0.5897. \end{aligned} \tag{66}$$

Step 12: rank the alternatives:

$$\widetilde{\varrho}_2 > \widetilde{\varrho}_1 > \widetilde{\varrho}_3. \tag{67}$$

Rank all the alternatives to select the good alternative is $\widetilde{\varrho}_2$.

6.3. By PCFHHWA Operator.

The experts have presented their decision in Tables 1–3. Utilize the formula $\widetilde{P}_r = \text{uwP}_r = \langle ([\widetilde{a}_r^-, \widetilde{a}_r^+], \widetilde{\widetilde{\lambda}}_r), ([\widetilde{c}_r^-, \widetilde{c}_r^+], \widetilde{\widetilde{\psi}}_r), ([\widetilde{b}_r^-, \widetilde{b}_r^+], \widetilde{\widetilde{\delta}}_r)\rangle$ $(r = 1, 2, \ldots, u)$ to the information given in Tables 1–3, taking weight vector $w = (0.4, 0.35, 0.25)^T$.

The aggregated information by utilizing the PCFHHA operator with taking weight vector $(0.45, 0.3, 0.25)$, given by three experts w.r.t the varied importance of the experts, is presented in Table 5.

Step 14: again using PCFHHA operator, having $\varpi = (0.35, 0.3, 0.21, 0.14)$, as a weight vector, we get collective PCFNs of the alternatives as given below:

Table 6: Once more weight multiplied to Table 5.

Supp.	$\widetilde{\Gamma}_1$	$\widetilde{\Gamma}_2$	$\widetilde{\Gamma}_3$	$\widetilde{\Gamma}_4$
$\widetilde{\varrho}_1$	$\begin{pmatrix} ([0.14, 0.18], 0.18), \\ ([0.12, 0.2], 0.3), \\ ([0.14, 0.23], 0.15) \end{pmatrix}$	$\begin{pmatrix} ([0.24, 0.25], 0.13), \\ ([0.18, 0.36], 0.33), \\ ([0.18, 0.3], 0.18) \end{pmatrix}$	$\begin{pmatrix} ([0.16, 0.17], 0.17), \\ ([0.15, 0.35], 0.28), \\ ([0.2, 0.29], 0.31) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.11], 0.1), \\ ([0.24, 0.42], 0.4), \\ ([0.27, 0.41], 0.5) \end{pmatrix}$
$\widetilde{\varrho}_2$	$\begin{pmatrix} ([0.23, 0.24], 0.27), \\ ([0.17, 0.2], 0.22), \\ ([0.09, 0.21], 0.16) \end{pmatrix}$	$\begin{pmatrix} ([0.25, 0.25], 0.25), \\ ([0.22, 0.33], 0.32), \\ ([0.18, 0.24], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.19, 0.21], 0.22), \\ ([0.32, 0.38], 0.36), \\ ([0.15, 0.26], 0.28) \end{pmatrix}$	$\begin{pmatrix} ([0.09, 0.18], 0.16), \\ ([0.29, 0.4], 0.44), \\ ([0.26, 0.27], 0.31) \end{pmatrix}$
$\widetilde{\varrho}_3$	$\begin{pmatrix} ([0.13, 0.13], 0.18), \\ ([0.11, 0.33], 0.21), \\ ([0.09, 0.19], 0.24) \end{pmatrix}$	$\begin{pmatrix} ([0.16, 0.17], 0.17), \\ ([0.21, 0.31], 0.32), \\ ([0.22, 0.33], 0.36) \end{pmatrix}$	$\begin{pmatrix} ([0.12, 0.14], 0.13), \\ ([0.17, 0.36], 0.23), \\ ([0.2, 0.3], 0.32) \end{pmatrix}$	$\begin{pmatrix} ([0.13, 0.13], 0.14), \\ ([0.33, 0.44], 0.44), \\ ([0.24, 0.37], 0.4) \end{pmatrix}$

Table 5: Calculated experts' information by PCFHHA operator.

Supp.	$\widetilde{\Gamma}_1$	$\widetilde{\Gamma}_2$	$\widetilde{\Gamma}_3$	$\widetilde{\Gamma}_4$
$\widetilde{\varrho}_1$	$\begin{pmatrix} ([0.12, 0.15], 0.15), \\ ([0.18, 0.28], 0.38), \\ ([0.21, 0.31], 0.22) \end{pmatrix}$	$\begin{pmatrix} ([0.24, 0.25], 0.13), \\ ([0.18, 0.36], 0.33), \\ ([0.18, 0.3], 0.18) \end{pmatrix}$	$\begin{pmatrix} ([0.16, 0.17], 0.17), \\ ([0.15, 0.35], 0.28), \\ ([0.2, 0.29], 0.31) \end{pmatrix}$	$\begin{pmatrix} ([0.12, 0.14], 0.14), \\ ([0.15, 0.32], 0.31), \\ ([0.18, 0.31], 0.43) \end{pmatrix}$
$\widetilde{\varrho}_2$	$\begin{pmatrix} ([0.19, 0.2], 0.2), \\ ([0.24, 0.28], 0.3), \\ ([0.15, 0.29], 0.23) \end{pmatrix}$	$\begin{pmatrix} ([0.25, 0.25], 0.25), \\ ([0.22, 0.33], 0.32), \\ ([0.18, 0.24], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.19, 0.21], 0.22), \\ ([0.32, 0.38], 0.36), \\ ([0.15, 0.26], 0.28) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.22], 0.2), \\ ([0.2, 0.3], 0.34), \\ ([0.17, 0.18], 0.21) \end{pmatrix}$
$\widetilde{\varrho}_3$	$\begin{pmatrix} ([0.11, 0.11], 0.14), \\ ([0.17, 0.41], 0.29), \\ ([0.15, 0.27], 0.32) \end{pmatrix}$	$\begin{pmatrix} ([0.16, 0.17], 0.17), \\ ([0.21, 0.31], 0.32), \\ ([0.22, 0.33], 0.36) \end{pmatrix}$	$\begin{pmatrix} ([0.12, 0.14], 0.13), \\ ([0.17, 0.36], 0.23), \\ ([0.2, 0.3], 0.32) \end{pmatrix}$	$\begin{pmatrix} ([0.16, 0.16], 0.17), \\ ([0.24, 0.34], 0.34), \\ ([0.15, 0.27], 0.3) \end{pmatrix}$

Abbreviation: PCFHHA, picture cubic fuzzy Hamacher hybrid averaging

$$\begin{aligned} \widetilde{\varrho}_{\sigma_{(1)}} &= (([0.17, 0.19], 0.15), ([0.23, 0.37], 0.39), ([0.26, 0.37], 0.3)), \\ \widetilde{\varrho}_{\sigma_{(2)}} &= (([0.21, 0.24], 0.25), ([0.31, 0.37], 0.37), ([0.22, 0.32], 0.32)), \\ \widetilde{\varrho}_{\sigma_{(3)}} &= (([0.14, 0.14], 0.16), ([0.25, 0.41], 0.34), ([0.24, 0.35], 0.38)). \end{aligned} \tag{68}$$

Step 15: utilizing score function formula of PCFNs, calculate the score functions of $\widetilde{\varrho}_1, \widetilde{\varrho}_2$, and $\widetilde{\varrho}_3$:

$$\begin{aligned} S(\widetilde{\varrho}_1) &= -0.4700, \\ S(\widetilde{\varrho}_2) &= -0.4033, \\ S(\widetilde{\varrho}_3) &= -0.5100. \end{aligned} \tag{69}$$

Step 16: rank the alternatives:

$$\widetilde{\varrho}_2 > \widetilde{\varrho}_1 > \widetilde{\varrho}_3. \tag{70}$$

Step 13: utilize the formula $\widetilde{P}_r = uw\widetilde{P}_r = \langle ([\widetilde{a}_r^-, \widetilde{a}_r^+], \breve{\lambda}_r), ([\widetilde{c}_r^-, \widetilde{c}_r^+], \widetilde{\psi}_r), ([\widetilde{b}_r^-, \widetilde{b}_r^+], \widetilde{\delta}_r) \rangle$ $(r = 1, 2, \ldots, u)$ to the data given in Table 5, taking weight vector $w = (0.3, 0.25, 0.25, 0.2)^T$ of $\widetilde{\varrho}_r$; the result is presented in Table 6.

Rank all the alternatives to select the good alternative is $\widetilde{\Phi}_2$. By this ranking, we obtain $\widetilde{\varrho}_2$ is the best choice for construction company.

6.4. Comparative Analysis. To verify the superiority of our planned PCF Hamacher weighted aggregation operators, we conduct a comparative analysis of the planned operators with intuitionistic cubic fuzzy Hamacher aggregation operators [48]. The score values ($\widetilde{\varrho}_r$ $(r = 1, 2, 3)$) and rating order of the alternatives are compiled in Table 7.

From Table 7, we see that the most good alternative is $\widetilde{\varrho}_2$. Obviously, the picture cubic fuzzy Hamacher aggregation operators are more accurate and more stable. Because in any unit composed of membership, neutral and nonmembership functions are defined in PCFS. The membership function is represented as CFN is likewise neutral and nonmembership functions are represented in the same style, but each variable composed of membership and nonmembership functions is expressed in the intuitionist cubic fuzzy set. The function of membership is expressed by the CFN and the nonmembership function is defined in the same way. In case we suppose the discussed example, PCFS is the very modern set, so it is not desirable for existing intuitionistic cubic fuzzy Hamacher aggregation operators to solve the data involved

Table 7: Observation table.

)Operators.	$S(\tilde{\varrho}_1)$	$S(\tilde{\varrho}_2)$	$S(\tilde{\varrho}_3)$	Ranking order
ICFHWA operator [48]	−0.20	−0.13	−0.22	$\tilde{\varrho}_2 > \tilde{\varrho}_1 > \tilde{\varrho}_3$
ICFHOWA operator [48]	−0.20	−0.14	−0.23	$\tilde{\varrho}_2 > \tilde{\varrho}_1 > \tilde{\varrho}_3$
ICFHHA operator [48]	−0.14	−0.08	−0.17	$\tilde{\varrho}_2 > \tilde{\varrho}_1 > \tilde{\varrho}_3$
PCFHWA operator	−0.58	−0.51	−0.55	$\tilde{\varrho}_2 > \tilde{\varrho}_3 > \tilde{\varrho}_1$
PCFHOWA operator	−0.53	−0.50	−0.59	$\tilde{\varrho}_2 > \tilde{\varrho}_1 > \tilde{\varrho}_3$
PCFHHA operator	−0.47	−0.40	−0.51	$\tilde{\varrho}_2 > \tilde{\varrho}_1 > \tilde{\varrho}_3$

in the discussed numerical problem, which indicate the existing intuitionistic cubic fuzzy Hamacher aggregation operators have limited approach. So, our planned operators are most reliable for such types of problem.

Although picture fuzzy set theory has been trailed to handle MADM problems in various fields, some circumstances in real-world picture fuzzy sets are not appropriate. So, we employed picture cubic fuzzy sets in those cases, which are extension of intuitionistic fuzzy sets and picture fuzzy sets; therefore, PCFS is almost always superior than the IFS and PFS.

In some cases, the PFS is unable to handle an issue that the PCFS can; for example, if a DM provides information on positive, neutral, and negative degree of membership in an interval and as a single value, this type of issue is only handled with PCFS. PCFS, in particular, is better at dealing with ambiguous problems. We present a technique for making decisions in complicated real-life scenarios using picture cubic fuzzy information.

Preexisting structures such as fuzzy sets, intuitionistic fuzzy sets, cubic sets, and picture fuzzy sets cannot address the numerical example provided in this study. So, our suggested technique is a generalization of the existent structure of fuzzy sets in order to better deal with real-world decision-making challenges.

7. Discussion and Conclusion

In the literature, there have been several research related to Hamacher aggregation operators, but in 2020, Muneeza et al. introduced the notion of intuitionistic cubic fuzzy set (ICFS) and also introduced intuitionistic cubic fuzzy Hamacher weighted aggregation operators. ICFS is a new generalization of intuitionistic fuzzy set. We still have not seen, to date, any Hamacher weighted aggregation operators-based research for intelligence fusion under the picture cubic fuzzy set (PCFS) framework. Thus, it is essential to give attention to this subject. Inspired by ICFS, we concentrate our attention upon this concept in order to expand the Hamacher aggregation operators to aggregate picture cubic fuzzy data and their activity in multiple attribute decision-making problems.

The main purpose in this study is to present some Hamacher weighted aggregation operators under picture cubic fuzzy environment, i.e., PCFHWA operator, PCFHOWA operator, and PCFHHA operator, and applied them to MAGDM problems. Basically, SHPP locations' selection is MAGDMP problem, where the attributes' value is in the form of PCFNs. Firstly, we have established picture cubic fuzzy set, introduced score degree, and accuracy function formula for the comparison of PCFNs. The motivation of Hamacher operators also defined a number of PCF Hamacher aggregation operators. Furthermore, we discussed few properties of planned operators, such as monotonicity, boundedness, and idempotency, and demonstrated relationship among abovementioned planned operators. We have defined new MAGDM algorithm by utilizing these planned operators with PCFVs. To verify the superiority of these planned operators, we conduct a comparison analysis of these planned operators with intuitionistic cubic fuzzy Hamacher aggregation operators.

In future picture cubic fuzzy context, we may further study different types of averaging operators and apply them to realistic decision-making situations, taking advantage of the enhanced simulation capacity of picture cubic fuzzy sets.

Furthermore, numerous sectors, such as medical diagnostic, pattern recognition, weather forecasting, sustainable energy planning decision-making, robotics, and informatics can made significant methodological improvements of picture cubic fuzzy sets in future.

We expect that the convergence of these main climate-centric research fields will provide significant growth and opportunity to better comprehend our world.

Abbreviations

FS:	Fuzzy set
μ_R:	Membership degree for fuzzy set
IVIFS:	Interval-valued intuitionistic fuzzy set
$[\mu_R^-(\check{t}), \mu_R^+(\check{t})]$:	Membership function for IVIFS
$[\nu_R^-(\check{t}), \nu_R^+(\check{t})]$:	Nonmembership function for IVIFS
CFS:	Cubic fuzzy set
ICFS:	Intuitionistic cubic fuzzy set
$\breve{\underset{\smile}{\lambda}}$:	Degree of membership for ICFS
δ:	Degree of nonmembership for ICFS
$\langle [a^-, a^+], \breve{\underset{\smile}{\lambda}} \rangle$:	Exact degree of membership for ICFS
$\langle [b^-, b^+], \delta \rangle$:	Exact degree of nonmembership for ICFS
HOs:	Hamacher operations
PCFS:	Picture cubic fuzzy set
S:	Score function
A:	Accuracy function.

Appendix

A. Proof of Theorem 1

Proof. By the method of mathematical induction, the authors will prove this theorem.

(i) When $r = 1$ and $n = 1$, based *HOs* on PCFVs, we obtain the following result as

$$\mathrm{PCFHWA}_{\varpi}(P_1) = \left\{ \begin{array}{l} \left(\left[\dfrac{1 + (d-1)a^{-} - (1-a^{-})}{1 + (d-1)a^{-} + (d-1)(1-a^{-})}, \dfrac{1 + (d-1)a^{+} - (1-a^{+})}{1 + (d-1)a^{+} + (d-1)(1-a^{+})} \right], \dfrac{1 + (d-1)\breve{\lambda} - (1-\breve{\lambda})}{1 + (d-1)\breve{\lambda} + (d-1)(1-\breve{\lambda})} \right), \\ \left(\left[\dfrac{d(c^{-})}{1 + (d-1)(1-c^{-}) + (d-1)(c^{-})}, \dfrac{d(c^{+})}{1 + (d-1)(1-c^{+}) + (d-1)(c^{+})} \right], \dfrac{d(\breve{\psi})}{1 + (d-1)(1-\breve{\psi}) + (d-1)(\breve{\psi})} \right), \\ \left(\left[\dfrac{d(b^{-})}{1 + (d-1)(1-b^{-}) + (d-1)(b^{-})}, \dfrac{d(b^{+})}{1 + (d-1)(1-b^{+}) + (d-1)(b^{+})} \right], \dfrac{d(\breve{\delta})}{1 + (d-1)(1-\breve{\delta}) + (d-1)(\breve{\delta})} \right) \end{array} \right\}. \tag{A.1}$$

Hence, equation (14) is true, for $r = 1$.

(ii) Let equation (14) is true, for $g = n$; on the basis, from equation (14), we obtain

$$\mathrm{PCFHWA}_{\varpi}(P_1, P_2, P_3, \ldots, P_n)$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \dfrac{\prod_{r=1}^{n}(1+(d-1)a_r^-)^{\varpi_r} - \prod_{r=1}^{n}(1-a_r^-)^{\varpi_r}}{\prod_{r=1}^{n}(1+(d-1)a_r^-)^{\varpi_r} - \prod_{r=1}^{n}(1-a_r^-)^{\varpi_r}}, \\ \dfrac{\prod_{r=1}^{n}(1+(d-1)a_r^+)^{\varpi_r} - \prod_{r=1}^{n}(1-a_r^+)^{\varpi_r}}{\prod_{r=1}^{n}(1+(d-1)a_r^+)^{\varpi_r} - \prod_{r=1}^{n}(1-a_r^+)^{\varpi_r}} \end{array} \right], \\ \dfrac{\prod_{r=1}^{n}\left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{n}\left(1-\breve{\lambda}_r\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{n}\left(1-\breve{\lambda}_r\right)^{\varpi_r}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} d\dfrac{\prod_{r=1}^{n}(c_r^-)^{\varpi_r}}{\prod_{r=1}^{n}(1+(d-1)(1-c_r^-))^{\varpi_r} + (d-1)\prod_{r=1}^{n}(c_r^-)^{\varpi_r}}, \\ d\dfrac{\prod_{r=1}^{n}(c_r^+)^{\varpi_r}}{\prod_{r=1}^{n}(1+(d-1)(1-c_r^+))^{\varpi_r} + (d-1)\prod_{r=1}^{n}(c_r^+)^{\varpi_r}} \end{array} \right], \\ d\dfrac{\prod_{r=1}^{n}\left(\breve{\psi}_r\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)\left(1-\breve{\psi}_r\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n}\left(\breve{\psi}_r\right)^{\varpi_r}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \dfrac{\prod_{r=1}^{n}(b_r^-)^{\varpi_r}}{\prod_{r=1}^{n}(1+(d-1)(1-b_r^-))^{\varpi_r} + (d-1)\prod_{r=1}^{n}(b_r^-)^{\varpi_r}}, \\ \dfrac{\prod_{r=1}^{n}(b_r^+)^{\varpi_r}}{\prod_{r=1}^{n}(1+(d-1)(1-b_r^-))^{\varpi_r} + (d-1)\prod_{r=1}^{n}(b_r^+)^{\varpi_r}} \end{array} \right], \\ d\dfrac{\prod_{r=1}^{n}\left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)\left(\left(\breve{\delta}_r\right)\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n}\left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}} \end{array} \right) \end{array} \right\}. \tag{A.2}$$

Now, for $g = n + 1$, we have

$$\mathrm{PCFHWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_n, P_{n+1}\right) = \oplus_{r=1}^{n}\varpi_r P_r \oplus \varpi_{n+1} P_{n+1}$$

$$= \left\{ \begin{array}{c} \left(\left[\begin{array}{c} \dfrac{\prod_{r=1}^{n}\left(1+(d-1)a_r^{-}\right)^{\varpi_r} - \prod_{r=1}^{n}\left(1-a_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)a_r^{-}\right)^{\varpi_r} - \prod_{r=1}^{n}\left(1-a_r^{-}\right)^{\varpi_r}}, \\ \\ \dfrac{\prod_{r=1}^{n}\left(1+(d-1)a_r^{+}\right)^{\varpi_r} - \prod_{r=1}^{n}\left(1-a_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)a_r^{+}\right)^{\varpi_r} - \prod_{r=1}^{n}\left(1-a_r^{+}\right)^{\varpi_r}} \end{array} \right], \\ \\ \dfrac{\prod_{r=1}^{n}\left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{n}\left(1-\breve{\lambda}_r\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{n}\left(1-\breve{\lambda}_r\right)^{\varpi_r}} \right), \\ \\ \left(\left[\begin{array}{c} d\dfrac{\prod_{r=1}^{n}\left(c_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)\left(1-c_r^{-}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n}\left(c_r^{-}\right)^{\varpi_r}}, \\ \\ d\dfrac{\prod_{r=1}^{n}\left(c_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)\left(1-c_r^{+}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n}\left(c_r^{+}\right)^{\varpi_r}} \end{array} \right], \\ \\ d\dfrac{\prod_{r=1}^{n}\left(\breve{\psi}_r\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)\left(1-\breve{\psi}_r\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n}\left(\breve{\psi}_r\right)^{\varpi_r}} \right), \\ \\ \left(\left[\begin{array}{c} \dfrac{\prod_{r=1}^{n}\left(b_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)\left(1-b_r^{-}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n}\left(b_r^{-}\right)^{\varpi_r}}, \\ \\ \dfrac{\prod_{r=1}^{n}\left(b_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)\left(1-b_r^{-}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n}\left(b_r^{+}\right)^{\varpi_r}} \end{array} \right], \\ \\ d\dfrac{\prod_{r=1}^{n}\left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}}{\prod_{r=1}^{n}\left(1+(d-1)\left(\left(\breve{\delta}_r\right)\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n}\left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}} \end{array} \right) \right\},$$

$$
\oplus\left\{\begin{array}{c}
\left(\left[\dfrac{1+(d-1)a^{-}-(1-a^{-})}{1+(d-1)a^{-}+(d-1)(1-a^{-})},\ \dfrac{1+(d-1)a^{+}-(1-a^{+})}{1+(d-1)a^{+}+(d-1)(1-a^{+})}\right],\ \dfrac{1+(d-1)\breve{\lambda}-(1-\breve{\lambda})}{1+(d-1)\breve{\lambda}+(d-1)(1-\breve{\lambda})}\right) \\
\left(\left[\dfrac{d(c_{n+1}^{-})^{n+1}}{1+(d-1)(1-c_{n+1}^{-})^{n+1}+(d-1)(c_{n+1}^{-})^{n+1}},\ \dfrac{d(c_{n+1}^{+})^{n+1}}{1+(d-1)(1-(c_{n+1}^{+}))^{n+1}+(d-1)(c_{n+1}^{+})^{n+1}}\right],\ \dfrac{d\left(\breve{\psi}_{n+1}^{+}\right)^{n+1}}{1+(d-1)\left(1-\breve{\psi}_{n+1}^{+}\right)^{n+1}+(d-1)\left(\breve{\psi}_{n+1}^{+}\right)^{n+1}}\right), \\
\left(\left[\dfrac{d(b_{n+1}^{-})^{n+1}}{1+(d-1)(1-b_{n+1}^{-})^{n+1}+(d-1)(b_{n+1}^{-})^{n+1}},\ \dfrac{d(b_{n+1}^{+})^{n+1}}{1+(d-1)(1-b_{n+1}^{+})^{n+1}+(d-1)(b_{n+1}^{+})^{n+1}}\right],\ \dfrac{d\left(\breve{\delta}_{n+1}\right)^{n+1}}{1+(d-1)\left(1-\breve{\delta}_{n+1}\right)^{n+1}+(d-1)\left(\breve{\delta}_{n+1}\right)^{n+1}}\right)
\end{array}\right\}. \tag{A.3}
$$

$$\left\{\begin{array}{l} = \left(\left[\dfrac{\prod_{r=1}^{n+1}\left(1+(d-1)a_r^-\right)^{\varpi_r} - \prod_{r=1}^{n+1}\left(1-a_r^-\right)^{\varpi_r}}{\prod_{r=1}^{n+1}\left(1+(d-1)a_r^-\right)^{\varpi_r} - \prod_{r=1}^{n+1}\left(1-a_r^-\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{n+1}\left(1+(d-1)a_r^+\right)^{\varpi_r} - \prod_{r=1}^{n+1}\left(1-a_r^+\right)^{\varpi_r}}{\prod_{r=1}^{n+1}\left(1+(d-1)a_r^+\right)^{\varpi_r} - \prod_{r=1}^{n+1}\left(1-a_r^+\right)^{\varpi_r}} \right], \dfrac{\prod_{r=1}^{n+1}\left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{n+1}\left(1-\breve{\lambda}_r\right)^{\varpi_r}}{\prod_{r=1}^{n+1}\left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{n+1}\left(1-\breve{\lambda}_r\right)^{\varpi_r}} \right), \\ \left(\left[d\dfrac{\prod_{r=1}^{n+1}\left(c_r^-\right)^{\varpi_r}}{\prod_{r=1}^{n+1}\left(1+(d-1)\left(1-c_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n+1}\left(c_r^-\right)^{\varpi_r}}, d\dfrac{\prod_{r=1}^{n+1}\left(c_r^+\right)^{\varpi_r}}{\prod_{r=1}^{n+1}\left(1+(d-1)\left(1-c_r^+\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n+1}\left(c_r^+\right)^{\varpi_r}} \right], d\dfrac{\prod_{r=1}^{n+1}\left(\breve{\psi}_r\right)^{\varpi_r}}{\prod_{r=1}^{n+1}\left(1+(d-1)\left(1-\breve{\psi}_r\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n+1}\left(\breve{\psi}_r\right)^{\varpi_r}} \right), \\ \left(\left[\dfrac{\prod_{r=1}^{n+1}\left(b_r^-\right)^{\varpi_r}}{\prod_{r=1}^{n+1}\left(1+(d-1)\left(1-b_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n+1}\left(b_r^-\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{n+1}\left(b_r^+\right)^{\varpi_r}}{\prod_{r=1}^{n+1}\left(1+(d-1)\left(1-b_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n+1}\left(b_r^+\right)^{\varpi_r}} \right], d\dfrac{\prod_{r=1}^{n+1}\left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}}{\prod_{r=1}^{n+1}\left(1+(d-1)\left(\left(\breve{\delta}_r\right)\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{n+1}\left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}} \right) \end{array}\right\}. \tag{A.4}$$

Thus, equation (14) is true, for $r = n + 1$. By the method of mathematical induction, (i) to (ii), the authors conclude that equation (14) is true for each-and-every values of r. □

B. Proof of Proposition 2 (Idempotency)

Since, for all r, $P_r = P$, that is, $a_r^- = a^-$, $a_r^+ = a^+$, $c_r^- = c^-$, $c_r^+ = c^+$, $b_r^- = b^-$, $b_r^+ = b^+$, $\breve{\lambda}_r = \breve{\lambda}$, $\breve{\psi}_r = \breve{\psi}$, and $\breve{\delta}_r = \breve{\delta}$,

$$\mathrm{PCFHWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right)$$

$$= \left\{ \begin{array}{c} \left(\left[\dfrac{\prod_{r=1}^{g}\left(1+(d-1)a_r^{-}\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-a_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^{-}\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-a_r^{-}\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{g}\left(1+(d-1)a_r^{+}\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-a_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^{+}\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-a_r^{+}\right)^{\varpi_r}} \right], \dfrac{\prod_{r=1}^{g}\left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-\breve{\lambda}_r\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{g}\left(1-\breve{\lambda}_r\right)^{\varpi_r}} \right), \\ \left(\left[d\dfrac{\prod_{r=1}^{g}\left(c_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-c_r^{-}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(c_r^{-}\right)^{\varpi_r}}, d\dfrac{\prod_{r=1}^{g}\left(c_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-c_r^{+}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(c_r^{+}\right)^{\varpi_r}} \right], d\dfrac{\prod_{r=1}^{n+1}\left(\breve{\psi}_r\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-\breve{\psi}_r\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(\breve{\psi}_r\right)^{\varpi_r}} \right), \\ \left(\left[\dfrac{\prod_{r=1}^{g}\left(b_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-b_r^{-}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(b_r^{-}\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{n+1}\left(b_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(1-b_r^{-}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(b_r^{+}\right)^{\varpi_r}} \right], d\dfrac{\prod_{r=1}^{g}\left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)\left(\left(\breve{\delta}_r\right)\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g}\left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}} \right) \end{array} \right\}, \tag{B.1}$$

$$\begin{aligned}
&\mathrm{PCFHWA}_{\varpi}(P_1)\\
&=\left\{\begin{array}{l}
\left(\left[\begin{array}{l}\dfrac{1+(d-1)a^{-}-(1-a^{-})}{1+(d-1)a^{-}+(d-1)(1-a^{-})},\\[2ex] \dfrac{1+(d-1)a^{+}-(1-a^{+})}{1+(d-1)a^{+}+(d-1)(1-a^{+})}\end{array}\right],\ \dfrac{1+(d-1)\breve{\lambda}-(1-\breve{\lambda})}{1+(d-1)\breve{\lambda}+(d-1)(1-\breve{\lambda})}\right),\\[4ex]
\left(\left[\begin{array}{l}\dfrac{d(c^{-})}{1+(d-1)(1-c^{-})+(d-1)(c^{-})},\\[2ex] \dfrac{d(c^{+})}{1+(d-1)(1-c^{+})+(d-1)(c^{+})}\end{array}\right],\ \dfrac{d(\breve{\psi})}{1+(d-1)(1-\breve{\psi})+(d-1)(\breve{\psi})}\right),\\[4ex]
\left(\left[\begin{array}{l}\dfrac{d(b^{-})}{1+(d-1)(1-b^{-})+(d-1)(b^{-})},\\[2ex] \dfrac{d(b^{+})}{1+(d-1)(1-b^{+})+(d-1)(b^{+})}\end{array}\right],\ \dfrac{d(\breve{\delta})}{1+(d-1)(1-\breve{\delta})+(d-1)(\breve{\delta})}\right)
\end{array}\right\}.\\
&=\left\{\left([a^{-},a^{+}],\breve{\lambda}\right),\left([c^{-},c^{+}],\breve{\psi}\right),\left([b^{-},b^{+}],\breve{\delta}\right)\right\}\\
&=P
\end{aligned} \tag{B.2}$$

Hence, it is proved.

C. Proof of Proposition 2 (Monotonicity)

$$
\begin{aligned}
&\mathrm{PCFHWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) \\
&= \left\{ \begin{array}{l}
\left(\left[\dfrac{\prod_{r=1}^{g} \left(1+(d-1)a_r^-\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-a_r^-\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)a_r^-\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-a_r^-\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{g} \left(1+(d-1)a_r^+\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-a_r^+\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)a_r^+\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-a_r^+\right)^{\varpi_r}} \right], \dfrac{\prod_{r=1}^{g} \left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-\breve{\lambda}_r\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-\breve{\lambda}_r\right)^{\varpi_r}} \right), \\
\left(\left[d\dfrac{\prod_{r=1}^{g} \left(c_r^-\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-c_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(c_r^-\right)^{\varpi_r}}, d\dfrac{\prod_{r=1}^{g} \left(c_r^+\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-c_r^+\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(c_r^+\right)^{\varpi_r}} \right], d\dfrac{\prod_{r=1}^{n+1} \left(\breve{\psi}_r\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-\breve{\psi}_r\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(\breve{\psi}_r\right)^{\varpi_r}} \right), \\
\left(\left[\dfrac{\prod_{r=1}^{g} \left(b_r^-\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-b_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(b_r^-\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{n+1} \left(b_r^+\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-b_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(b_r^+\right)^{\varpi_r}} \right], d\dfrac{\prod_{r=1}^{g} \left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(\left(\breve{\delta}_r\right)\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}} \right)
\end{array} \right\},
\end{aligned}
\tag{C.1}
$$

as

$$
\begin{aligned}
&\frac{\prod_{r=1}^{g}\left(1+(d-1)a_r^{-}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^{-}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{-}\right)^{\varpi_r}},\\
&\geq\frac{\prod_{r=1}^{g}\left(1+(d-1)a_r^{*^{-}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{*^{-}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^{*^{-}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{*^{-}}\right)^{\varpi_r}},\\
&\quad\frac{\prod_{r=1}^{g}\left(1+(d-1)a_r^{*^{+}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{*^{+}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^{*^{+}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{*^{+}}\right)^{\varpi_r}},\\
&\quad\frac{\prod_{r=1}^{g}\left(1+(d-1)a_r^{+}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^{+}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{+}\right)^{\varpi_r}},\\
&\Rightarrow\left[\begin{array}{c}\dfrac{\prod_{r=1}^{g}\left(1+(d-1)a_r^{-}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{-}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^{-}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{-}\right)^{\varpi_r}},\\[2ex]\dfrac{\prod_{r=1}^{g}\left(1+(d-1)a_r^{+}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{+}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^{+}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{+}\right)^{\varpi_r}}\end{array}\right]\\
&\leq\left[\begin{array}{c}\dfrac{\prod_{r=1}^{g}\left(1+(d-1)a_r^{*^{-}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{*^{-}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^{*^{-}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{*^{-}}\right)^{\varpi_r}},\\[2ex]\dfrac{\prod_{r=1}^{g}\left(1+(d-1)a_r^{*^{+}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{*^{+}}\right)^{\varpi_r}}{\prod_{r=1}^{g}\left(1+(d-1)a_r^{*^{+}}\right)^{\varpi_r}-\prod_{r=1}^{g}\left(1-a_r^{*^{+}}\right)^{\varpi_r}}\end{array}\right].
\end{aligned}
\tag{C.2}
$$

Now,

$$
\Rightarrow \left[\frac{d \prod_{r=1}^{g} \left(c_r^{*^-}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - c_r^{*^-}\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(c_r^{*^-}\right)^{\varpi_r}}, \frac{d \prod_{r=1}^{g} \left(c_r^{*^+}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - c_r^{*^+}\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(c_r^{*^+}\right)^{\varpi_r}} \right]
$$

$$
\leq \left[\frac{d \prod_{r=1}^{g} \left(c_r^{*^-}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - c_r^{*^-}\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(c_r^{*^-}\right)^{\varpi_r}}, \frac{d \prod_{r=1}^{g} \left(c_r^{*^+}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - c_r^{*^+}\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(c_r^{*^+}\right)^{\varpi_r}} \right], \tag{C.3}
$$

$$
\Rightarrow \left[\frac{d \prod_{r=1}^{g} \left(b_r^{*^-}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - b_r^{*^-}\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(b_r^{*^-}\right)^{\varpi_r}}, \frac{d \prod_{r=1}^{g} \left(b_r^{*^+}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - b_r^{*^+}\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(b_r^{*+}\right)^{\varpi_r}} \right]
$$

$$
\leq \left[\frac{d \prod_{r=1}^{g} \left(b_r^{*^-}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - b_r^{*^-}\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(b_r^{*^-}\right)^{\varpi_r}}, \frac{d \prod_{r=1}^{g} \left(b_r^{*^+}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - b_r^{*^+}\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(b_r^{*+}\right)^{\varpi_r}} \right],
$$

$$
\begin{aligned}
&\breve{\lambda}_r \leq \breve{\lambda}_r^*, \\
&\quad \frac{\prod_{r=1}^{g} \left(1 + (d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1 - \breve{\lambda}_r\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\breve{\lambda}_r^*\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(1 - \breve{\lambda}_r^*\right)^{\varpi_r}} \\
&\quad \leq \frac{\prod_{r=1}^{g} \left(1 + (d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1 - \breve{\lambda}_r\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\breve{\lambda}_r^*\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(1 - \breve{\lambda}_r^*\right)^{\varpi_r}},
\end{aligned} \tag{C.4}
$$

and

$$
\begin{aligned}
&\breve{\psi}_r^* \leq \breve{\psi}_r, \\
&\frac{d \prod_{r=1}^{g} \left(\breve{\psi}_r\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - \breve{\psi}_r\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(\breve{\psi}_r\right)^{\varpi_r}} \\
&\leq \frac{d \prod_{r=1}^{g} \left(\breve{\psi}_r\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - \breve{\psi}_r\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(\breve{\psi}_r\right)^{\varpi_r}}.
\end{aligned} \tag{C.5}
$$

Also,

$$
\begin{aligned}
&\breve{\delta}_r^* \leq \breve{\delta}_r, \\
&d \frac{\prod_{r=1}^{g} \left(\breve{\delta}_r^*\right)^{\varpi_r}}{\prod_{r=1}^{g} 1 + (d-1)\left(1 - \breve{\delta}_r^*\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(\breve{\delta}_r^*\right)^{\varpi_r}}, \\
&\leq \frac{d \prod_{r=1}^{g} \left(\breve{\delta}_r\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1 + (d-1)\left(1 - \breve{\delta}_r\right)\right)^{\varpi_r} + (d-1) \prod_{r=1}^{g} \left(\breve{\delta}_r\right)^{\varpi_r}}
\end{aligned} \tag{C.6}
$$

Equations (15) to (24) imply

$$\mathrm{PCFHWA}_{\varpi}\left(P_1, P_2, P_3, \ldots, P_g\right) = \left\{ \begin{array}{l} \left(\left[\dfrac{\prod_{r=1}^{g} \left(1+(d-1)a_r^-\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-a_r^-\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)a_r^-\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-a_r^-\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{g} \left(1+(d-1)a_r^+\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-a_r^+\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)a_r^+\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-a_r^+\right)^{\varpi_r}} \right], \dfrac{\prod_{r=1}^{g} \left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-\breve{\lambda}_r\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\breve{\lambda}_r\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-\breve{\lambda}_r\right)^{\varpi_r}} \right), \\ \left(\left[d\dfrac{\prod_{r=1}^{g} \left(c_r^-\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-c_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(c_r^-\right)^{\varpi_r}}, d\dfrac{\prod_{r=1}^{g} \left(c_r^+\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-c_r^+\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(c_r^+\right)^{\varpi_r}} \right], d\dfrac{\prod_{r=1}^{n+1} \left(\breve{\psi}_r\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-\breve{\psi}_r\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(\breve{\psi}_r\right)^{\varpi_r}} \right), \\ \left(\left[\dfrac{\prod_{r=1}^{g} \left(b_r^-\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-b_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(b_r^-\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{n+1} \left(b_r^+\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-b_r^-\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(b_r^+\right)^{\varpi_r}} \right], d\dfrac{\prod_{r=1}^{g} \left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(\left(\breve{\delta}_r\right)\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(\left(\breve{\delta}_r\right)\right)^{\varpi_r}} \right) \end{array} \right\}, \tag{C.7}$$

$$
\leq \left\{ \begin{array}{l}
\left(\left[\dfrac{\prod_{r=1}^{g} \left(1+(d-1)a_r^{*^-}\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-a_r^{*^-}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)a_r^{*^-}\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(1-a_r^{*^-}\right)^{\varpi_r}}, \dfrac{\prod_{r=1}^{g} \left(1+(d-1)a_r^{*^+}\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-a_r^{*^+}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)a_r^{*^+}\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(1-a_r^{*^+}\right)^{\varpi_r}} \right], \dfrac{\prod_{r=1}^{g} \left(1+(d-1)\breve{\lambda}_r^{*}\right)^{\varpi_r} - \prod_{r=1}^{g} \left(1-\breve{\lambda}_r^{*}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\breve{\lambda}_r^{*}\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(1-\breve{\lambda}_r^{*}\right)^{\varpi_r}} \right), \\
\left(\left[\dfrac{d\prod_{r=1}^{g} \left(c_r^{*^-}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-c_r^{*^-}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(c_r^{*^-}\right)^{\varpi_r}}, \dfrac{d\prod_{r=1}^{g} \left(c_r^{*^+}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-c_r^{*^+}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(c_r^{*+}\right)^{\varpi_r}} \right], \dfrac{d\prod_{r=1}^{g} \left(\breve{\psi}_r^{*}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-\breve{\psi}_r^{*}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(\breve{\psi}_r^{*}\right)^{\varpi_r}} \right), \\
\left(\left[\dfrac{d\prod_{r=1}^{g} \left(b_r^{*^-}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-cb_r^{*^-}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(b_r^{*^-}\right)^{\varpi_r}}, \dfrac{d\prod_{r=1}^{g} \left(b_r^{*^+}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-cb_r^{*^+}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(b_r^{*^+}\right)^{\varpi_r}} \right], \dfrac{d\prod_{r=1}^{g} \left(\breve{\delta}_r^{*}\right)^{\varpi_r}}{\prod_{r=1}^{g} \left(1+(d-1)\left(1-\breve{\delta}_r^{*}\right)\right)^{\varpi_r} + (d-1)\prod_{r=1}^{g} \left(\breve{\delta}_r^{*}\right)^{\varpi_r}} \right)
\end{array} \right\}. \tag{C.8}
$$

The authors get our required inequality:

$$
\mathrm{PCFHWA}_{\varpi}\left(P_1, P_2, \ldots, P_g\right) \leq \mathrm{PCFHWA}_{\varpi}\left(P_1^*, P_2^*, \ldots, P_g^*\right) \tag{C.9}
$$

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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