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Retraction

Retracted: Intuitionistic Fuzzy Double Controlled Metric Spaces and Related Results

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] M. Farheen, K. Ahmed, K. Javed, V. Parvaneh, F. U. Din, and U. Ishtiaq, "Intuitionistic Fuzzy Double Controlled Metric Spaces and Related Results," *Security and Communication Networks*, vol. 2022, Article ID 6254055, 15 pages, 2022.

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Research Article

Intuitionistic Fuzzy Double Controlled Metric Spaces and Related Results

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In this article, we introduce the concept of intuitionistic fuzzy double controlled metric spaces that generalizes the concept of intuitionistic fuzzy b-metric spaces. For this purpose, two noncomparable functions are used in triangle inequalities. We generalize the concepts of the Banach contraction principle and fuzzy contractive mappings by giving authentic proof of these mappings in the sense of intuitionistic fuzzy double controlled metric spaces. To validate the superiority of these results, examples are imparted. A possible application to solving integral equations is also set forth towards the end of this work to support the proposed results.

1. Introduction

The concept of metric spaces and the Banach contraction principle are the backbone of the field of fixed point theory. Since the axiomatic interpretations of metric space, it has attracted researchers due to its spaciousness. So far, different developments in metric space have appeared in the literature, either by improving contraction conditions or by relaxing the axioms of metric space.

Zadeh [1] was the first to put forward the concept of fuzzy sets and this idea has deeply influenced many scientific fields since its inception. Using the concepts of probabilistic metric space and fuzzy sets, fuzzy metric space was introduced in [2]. Afterward, the utility of FMS appeared in applied sciences such as fixed-point theory, image and signal processing, medical imaging, and decision-making. This concept succeeded in shifting a lot of mathematical structures within itself. In this continuation, Kramosil and Michalek [3] initiated the notion of fuzzy metric spaces. Khalil et al. [4] generalized this concept by introducing fuzzy b-metric-like spaces. Fuzzy metric space only discusses membership functions, so for dealing with membership and nonmembership functions, the notion of intuitionistic fuzzy metric spaces introduced by Park [5] and this concept was generalized into intuitionistic fuzzy b-metric spaces by Konwar [6]. In this connectedness, many important results appeared in the literature, such as fixed point theorems on intuitionistic fuzzy metric space [7], fixed point theorems for a generalized intuitionistic fuzzy contraction in intuitionistic fuzzy metric spaces [8], extension of fixed point results in intuitionistic fuzzy b-metric spaces [6], fixed points in intuitionistic fuzzy metric spaces [4], fuzzy fixed point [9], and some more work in generalized metric space in [10], ordered defined in fuzzy b-metric [11], partial metric defining the relation in [12], orthogonal neutrosophic metric space [13], and orthogonal partial metric space [14]. More details related to generalized metric spaces can be seen in [15].

Recently, Saleem et al. [10] introduced the notion of fuzzy double controlled metric spaces and generalized the Banach contraction principle. This result was generalized in different spaces and different structures were attained using this topic. One may recall some basic results related to this topic, such as controlled fuzzy metric spaces and some related fixed point results in [16], controlled metric type spaces, and the related contraction principle in [17] and orthogonally controlled metric type spaces in [18], and very recently, the new aspects of metric spaces in [19].

In this article, we aim to generalize the concept of intuitionistic fuzzy b-metric spaces and introduce the concept of intuitionistic fuzzy double controlled metric spaces. Some nontrivial examples are given and an application to solving integral equations is also imparted in this work. Table 1 of abbreviations of notions will be used throughout this study.

1.1. Preliminaries. First, we define some necessary definitions that are helpful for readers to understand the main results.

Definition 1 (see [2]). A binary operation $*: [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called a CTN if

(1) $\pi * \mu = \mu * \pi$, $(\forall)\pi, \mu \in [0, 1]$;

(2) $*$ is continuous;

(3) $\pi * 1 = \pi$, $(\forall)\pi \in [0, 1]$;

(4) $(\pi * \mu) * \rho = \pi * (\mu * \rho)$, $(\forall)\pi, \mu, \rho \in [0, 1]$;

(5) $\pi \leq \rho$ and $\mu \leq d$, with $\pi, \mu, \rho, d \in [0, 1]$, then $\pi * \mu \leq \rho * d$.

Example 1 (see [20]). Some fundamental examples of CTNs are $\pi * \mu = \pi \cdot \mu, \pi * \mu = \min\{\pi, \mu\}$ and $\pi * \mu = \max\{\pi + \mu - 1, 0\}$.

Definition 2 (see [2]). A binary operation $\bigcirc: [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called a CTCN if it meets the following assertions:

(1) $\pi \bigcirc \mu = \mu \bigcirc \pi$, or all $\pi, \mu \in [0, 1]$

(2) $\bigcirc$ is continuous;

(3) $\pi \bigcirc 0 = 0$, for all $\pi \in [0, 1]$;

(4) $(\pi \bigcirc \mu) \bigcirc \rho = \pi \bigcirc (\mu \bigcirc \rho)$, for all $\pi, \mu, \rho \in [0, 1]$;

(5) If $\pi \leq \rho$ and $\mu \leq d$, with $\pi, \mu, \rho, d \in [0, 1]$, then $\pi \bigcirc \mu \leq \rho \bigcirc d$.

Example 2 (see [2]). $\pi \bigcirc \mu = \max\{\pi, \mu\}$ and $\pi \bigcirc \mu = \min\{\pi + \mu, 1\}$ are examples of CTCNs.

Definition 3 (see [21]). Let functions $\phi, \eta: \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, \infty)$ be noncomparable. Let $\partial: \mathfrak{B} \times \mathfrak{B} \longrightarrow [0, \infty)$ be fulfilling:

(a) $\partial(x, y) = 0$ if $x = y$;

(b) $\partial(x, y) = \partial(y, x)$;

(c) $\partial(x, y) \leq \phi(x, z)\partial(x, z) + \eta(z, y)\partial(z, y)$, for all $x, y, z \in \mathfrak{B}$. Then, ∂ is called a double controlled metric and $(\mathfrak{B}, \partial)$ is called a double controlled metric space.

Definition 4 (see [21]). Let $\mathfrak{B} \neq \varnothing$ and $\phi, \eta: \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, \infty)$ be given noncomparable functions, and $*$ is a CTN and P be a FS on $\mathfrak{B} \times \mathfrak{B} \times (0, \infty)$ which is called fuzzy double controlled metric on $\mathfrak{B}$; if for all $x, y, z \in \mathfrak{B}$, the below circumstances are fulfilling:

(I) $P(x, y, 0) = 0$;

(II) $P(x, y, t) = 1$ for all $t > 0$, if and only if $x = y$;

(III) $P(x, y, t) = P(y, x, t)$;

(IV) $P(x, z, t + s) \geq P(x, y, t/\phi(x, y)) * P(y, z, s/\eta(y, z))$;

(V) $P(x, y, \cdot): (0, \infty) \longrightarrow [0, 1]$ is left continuous.

Then, $(\mathfrak{B}, P, Q, *)$ is named a FDCMS.

Definition 5 (see [6]). Take $\mathfrak{B} \neq \varnothing$. Let $*$ be a CTN, $\bigcirc$ be a CTCN, $b \geq 1$, and P, Q be FSs on $\mathfrak{B} \times \mathfrak{B} \times (0, \infty)$, if the following for all $x, y \in \mathfrak{B}$ and $s, t > 0$:

(I) $P(x, y, t) + Q(x, y, t) \leq 1$;

(II) $P(x, y, t) > 0$;

(III) $P(x, y, t) = 1 \Leftrightarrow x = y$;

(IV) $P(x, y, t) = P(y, x, t)$;

(V) $P(x, z, b(t + s)) \geq P(x, y, t) * P(y, z, s)$;

(VI) $P(x, y, \cdot)$ is a nondecreasing function of $\mathbb{R}^+$ and $\lim_{t \longrightarrow \infty} P(x, y, t) = 1$;

(VII) $Q(x, y, t) > 0$;

(VIII) $Q(x, y, t) = 0 \Leftrightarrow x = y$;

(IX) $Q(x, y, t) = Q(y, x, t)$;

(X) $Q(x, z, b(t + s)) \leq Q(x, y, t) \bigcirc Q(y, z, s)$;

(XI) $Q(x, y, \cdot)$ is a nonincreasing function of $\mathbb{R}^+$ and $\lim_{t \longrightarrow \infty} Q(x, y, t) = 0$, then $(\mathfrak{B}, P, Q, *, \bigcirc)$ is an IFBMS.

2. Main Results

In this section, we introduce the concept of IFDCMSs and prove some fixed point (FP) results.

Definition 6. Let $\mathfrak{B} \neq \varnothing$ and $\phi, \eta: \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, \infty)$ be given noncomparable functions, and $*$ is a CTN and $\bigcirc$ is a CTCN. P, Q are FSs on $\mathfrak{B} \times \mathfrak{B} \times (0, \infty)$ which are named intuitionistic fuzzy double controlled metrics on $\mathfrak{B}$; if for all $x, y, z \in \mathfrak{B}$, the below circumstances are fulfilling:

(I) $P(x, y, t) + Q(x, y, t) \leq 1$

(II) $P(x, y, t) > 0$;

(III) $P(x, y, t) = 1$ for all $t > 0$, if and only if $x = y$;

(IV) $P(x, y, t) = P(y, x, t)$;

(V) $P(x, z, t + s) \geq P(x, y, t/\phi(x, y)) * P(y, z, s/\eta(y, z))$;

(VI) $P(x, y, \cdot): (0, \infty) \longrightarrow [0, 1]$ is left continuous;

Table 1: Abbreviations.

Abbreviations	Meanings
FSs	Fuzzy sets
FMSs	Fuzzy metric spaces
IFBMSs	Intuitionistic fuzzy b-metric spaces
FDCMSs	Fuzzy double controlled metric spaces
IFDCMSs	Intuitionistic fuzzy double controlled metric spaces
CTN	Continuous triangle norm
CTCN	Continuous triangle conorm

(VII) $Q(x, y, t) < 1$;

(VIII) $Q(x, y, t) = 0$ for all $t > 0$, if and only if $x = y$;

(IX) $Q(x, y, t) = Q(y, x, t)$;

(X) $Q(x, z, t + s) \leq Q(x, y, t/\phi(x, y)) \bigcirc Q(y, z, s/\eta(y, z))$;

(XI) $Q(x, y, \cdot)\colon (0, \infty) \longrightarrow [0, 1]$ is left continuous.

Then, $(\mathfrak{B}, \mathrm{P}, Q, *, \bigcirc)$ is called an IFDCMS.

Remark 1. If we take $\phi(x, y) = \eta(y, z) = b \geq 1$, then IFDCMS becomes an IFBMS.

Example 3. Let $\mathfrak{B} = \{1, 2, 3\}$ and $\phi, \eta\colon \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, \infty)$ be two noncomparable functions given by $\phi(x, y) = x + y + 1$ and $\eta(x, y) = x^2 + y^2 - 1$. Define $\mathrm{P}, Q\colon \mathfrak{B} \times \mathfrak{B} \times (0, \infty) \longrightarrow [0, 1]$ as

$$\mathrm{P}(x, y, t) = \begin{cases} 1, & \text{if} \quad x = y, \\ \dfrac{t}{t + \max\{x, y\}}, & \text{if} \quad \text{otherwise}. \end{cases}$$
$$Q(x, y, t) = \begin{cases} 1, & \text{if} \quad x = y, \\ \dfrac{t}{t + \max\{x, y\}}, & \text{if} \quad \text{otherwise}. \end{cases} \tag{1}$$

Then, $(\mathfrak{B}, \mathrm{P}, Q, *, \bigcirc)$ is an IFDCMS with CTN $\pi * \mu = \pi\mu$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Proof. Conditions (i)–(iv), (vi)–(ix), and (xi) are easy to examine Here, we prove (v) and (x).

Let $x = 1$, $y = 2$, and $z = 3$. Then,

$$\begin{aligned} \mathrm{P}(1, 3, t + s) &= \frac{t + s}{t + s + \max\{1, 3\}} \\ &= \frac{t + s}{t + s + 3}. \end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned} \mathrm{P}\left(1, 2, \frac{t}{\phi(1, 2)}\right) &= \frac{t/\phi(1, 2)}{\phi(1, 2) + \max\{1, 2\}} \\ &= \frac{t/4}{t/4 + 2} = \frac{t}{t + 8}, \\ \mathrm{P}\left(2, 3, \frac{s}{\eta(2, 3)}\right) &= \frac{s/\eta(2, 3)}{s/\eta(2, 3) + \max\{2, 3\}} \\ &= \frac{s/12}{s/12 + 3} = \frac{s}{s + 36}. \end{aligned} \tag{3}$$

That is,

$$\frac{t + s}{t + s + 3} \geq \frac{t}{t + 8} \cdot \frac{s}{s + 36}. \tag{4}$$

Then, it satisfies for all $t, s > 0$. Hence,

$$\mathrm{P}(x, z, t + s) \geq \mathrm{P}\left(x, y, \frac{t}{\phi(x, y)}\right) * \mathrm{P}\left(y, z, \frac{s}{\phi(y, z)}\right). \tag{5}$$

Now,

$$\begin{aligned} Q(1, 3, t + s) &= \frac{\max\{1, 3\}}{t + s + \max\{1, 3\}} \\ &= \frac{3}{t + s + 3}. \end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned} Q\left(1, 2, \frac{t}{\eta(1, 2)}\right) &= \frac{\max\{1, 2\}}{t/\eta(1, 2) + \max\{1, 2\}} \\ &= \frac{2}{t/4 + 2} = \frac{8}{t + 8}, \\ Q\left(2, 3, \frac{s}{\eta(2, 3)}\right) &= \frac{\max\{2, 3\}}{s/\eta(2, 3) + \max\{2, 3\}} \\ &= \frac{3}{s/12 + 3} = \frac{36}{s + 36}. \end{aligned} \tag{7}$$

That is,

$$\frac{3}{t + s + 3} \leq \max\left\{\frac{8}{t + 8}, \frac{36}{s + 36}\right\}. \tag{8}$$

Then, it satisfies for all $t, s > 0$. Hence,

$$Q(x, z, t + s) \leq Q\left(x, y, \frac{t}{\phi(x, y)}\right) \bigcirc Q\left(y, z, \frac{s}{\phi(y, z)}\right). \tag{9}$$

On the same lines, one can examine all other cases. Hence, $(\mathfrak{B}, \mathrm{P}, Q, *, \bigcirc)$ is an IFDCMS. □

Remark 2. The above example also satisfied for CTN $\pi * \mu = \min\{\pi, \mu\}$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Example 4. Let $\mathfrak{B} = (0, \infty)$ and $\phi, \eta\colon \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, \infty)$ be two noncomparable functions given by $\phi(x, y) = x + y + 1$ and $\eta(x, y) = x^2 + y^2 - 1$.

Define $P, Q\colon \mathfrak{B} \times \mathfrak{B} \times (0, \infty) \longrightarrow [0, 1]$ as

$$P(x, y, t) = \frac{t}{t + |x - y|^2}, \quad Q(x, y, t) = \frac{|x - y|^2}{t + |x - y|^2}. \tag{10}$$

Then, $(\mathfrak{B}, P, Q, *, \bigcirc)$ is an IFDCMS with CTN $\pi * \mu = \pi\mu$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Remark 3. The above example also holds for

$$\phi(x, y) = \begin{cases} 1, & \text{if } x = y, \\ \dfrac{1 + \max\{x, y\}}{\min\{x, y\}}, & \text{if } x \neq y, \end{cases} \quad \eta(x, y) = \begin{cases} 1, & \text{if } x = y, \\ \dfrac{1 + \max\{x^2, y^2\}}{\min\{x^2, y^2\}}, & \text{if } x \neq y. \end{cases} \tag{11}$$

Remark 4. The above example is also satisfied for CTN $\pi * \mu = \min\{\pi, \mu\}$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Example 5. Let $\mathfrak{B} = \{1, 2, 3\}$ and $\phi, \eta\colon \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, \infty)$ be two noncomparable functions given by $\phi(x, y) = x + y + 1$ and $\eta(x, y) = x^2 + y^2 - 1$. Define $P, Q\colon \mathfrak{B} \times \mathfrak{B} \times (0, \infty) \longrightarrow [0, 1]$ as

$$P(x, y, t) = \frac{t + \min\{x, y\}}{t + \max\{x, y\}}, \quad Q(x, y, t) = 1 - \frac{t + \min\{x, y\}}{t + \max\{x, y\}}. \tag{12}$$

Then, $(\mathfrak{B}, P, Q, *, \bigcirc)$ is an IFDCMS with CTN $\pi * \mu = \pi\mu$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Proof. It is easy to examine the line of the above example. □

Remark 5. The above example is not IFDCMS if we take CTN $\pi * \mu = \min\{\pi, \mu\}$, CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$, and $x = 1, y = 2, z = 3, t = 0.02, s = 0.03$ with $\phi(x, y) = x + y + 1$ and $\eta(x, y) = x^2 + y^2 - 1$.

Proposition 1. *Let $\mathfrak{B} = [0, 1]$ and $\phi, \eta\colon \mathfrak{B} \times \mathfrak{B} \longrightarrow [0, 1]$ be two noncomparable functions given by $\phi(x, y) = 2(x + y + 1)$and $\eta(x, y) = 2(x^2 + y^2 + 1)$. Define Q, P as*

$$P(x, y, t^n) = e^{-\left((x-y)^2/t^n\right)}, \quad Q(x, y, t^n) = 1 - e^{-\left((x-y)^2/t^n\right)} \text{ for all } x, y \in \mathfrak{B}, t > 0. \tag{13}$$

Then, $(\mathfrak{B}, P, Q, *, \bigcirc)$ is an IFDCMS with CTN $\pi * \mu = \pi\mu$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Remark 6. The above proposition is also satisfied for CTN $\pi * \mu = \min\{\pi, \mu\}$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Proposition 2. *Let $\mathfrak{B} = [0, 1]$ and $\phi, \eta\colon \mathfrak{B} \times \mathfrak{B} \longrightarrow [0, 1]$ be two noncomparable functions given by $\phi(x, y) = 2(x + y + 1)$and $\eta(x, y) = 2(x^2 + y^2 + 1)$. Define Q, P as*

$$P(x, y, t^n) = \left[e^{-\left((x-y)^2/t^n\right)}\right]^{-1}, \quad Q(x, y, t^n) = 1 - \left[e^{-\left((x-y)^2/t^n\right)}\right]^{-1} \quad \text{for all } x, y \in \mathfrak{B}, t > 0. \tag{14}$$

Then, $(\mathfrak{B}, P, Q, *, \bigcirc)$ is an IFDCMS with CTN $\pi * \mu = \pi\mu$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Remark 7. The above proposition is also satisfied for CTN $\pi * \mu = \min\{\pi, \mu\}$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Definition 7. Let $(\mathfrak{B}, P, Q, *, \bigcirc)$ be an IFDCMS. Then, we define an open ball $B(x, r, t)$ with center x, radius r, $0 < r < 1$, and $t > 0$ as follows:

$$B(x, r, t) = \{y \in \mathfrak{B}\colon P(x, y, t) > 1 - r, Q(x, y, t) < r\}. \tag{15}$$

Remark 8. An IFDCMS $(\mathfrak{B}, P, Q, *, \bigcirc)$ needs not to be a Hausdorff.

Proof. Let $\mathfrak{B} = \{1, 2, 3\}$ and $\phi, \eta\colon \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, \infty)$ be two noncomparable functions given by $\phi(x, y) = x + y + 1$ and $\eta(x, y) = x^2 + y^2 - 1$. Define $P, Q\colon \mathfrak{B} \times \mathfrak{B} \times (0, \infty) \longrightarrow [0, 1]$ as

$$P(x, y, t) = \frac{t + \min\{x, y\}}{t + \max\{x, y\}}, \quad Q(x, y, t) = 1 - \frac{t + \min\{x, y\}}{t + \max\{x, y\}}. \tag{16}$$

Then, $(\mathfrak{B}, P, Q, *, \bigcirc)$ is an IFDCMS with CTN $\pi * \mu = \pi\mu$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$. Consider the open ball $B(1, 0.4, 6)$ centered at 1, with radius $r = 0.4$ and $t = 6$. Then,

$$B(1, 0.4, 6) = \{y \in \mathfrak{B}\colon P(1, y, 6) > 0.6, Q(1, y, 6) < 0.4\}. \tag{17}$$

Now,

$$
\begin{aligned}
\mathrm{P}(1,2,6) &= \frac{1+6}{2+6} \\
&= \frac{7}{8} = 0.875, \\
Q(1,2,6) &= 1 - \frac{1+6}{2+6} \\
&= 1 - \frac{7}{8} = 1 - 0.875 \\
&= 0.125, \\
\mathrm{P}(1,3,6) &= \frac{1+6}{3+6} \\
&= \frac{7}{9} = 0.777, \\
Q(1,3,6) &= 1 - \frac{1+6}{3+6} \\
&= 1 - \frac{7}{9} = 1 - 0.777 \\
&= 0.223.
\end{aligned} \tag{18}
$$

Thus, $B(1, 0.4, 6) = \{2, 3\}$. Now, consider the open ball $B(2, 0.6, 12)$ with radius $r = 0.6$, centered at 2, and $t = 12$. Then,

$$
B(2, 0.6, 12) = \{y \in \mathfrak{B}\colon \mathrm{P}(2, y, 12) > 0.4, Q(2, y, 12) < 0.6\}. \tag{19}
$$

Now,

$$
\begin{aligned}
\mathrm{P}(2,1,12) &= \frac{1+12}{2+12} \\
&= \frac{13}{14} = 0.928, \\
Q(2,1,12) &= 1 - \frac{1+12}{2+12} \\
&= 1 - \frac{13}{14} \\
&= 0.071, \\
\mathrm{P}(2,3,12) &= \frac{2+12}{3+12} \\
&= \frac{14}{15} = 0.933, \\
Q(2,3,12) &= 1 - \frac{2+12}{3+12} \\
&= 1 - \frac{14}{15} = 1 - 0.933 \\
&= 0.066.
\end{aligned} \tag{20}
$$

Thus, $B(2, 0.6, 12) = \{1, 3\}$. Now,

$$
\begin{aligned}
B(1, 0.4, 6) \cap B(2, 0.6, 12) &= \{2, 3\} \cap \{1, 3\}, \\
&= \{3\} \neq .
\end{aligned} \tag{21}
$$

Hence, an IFDCMS is not necessarily Hausdorff. □

Definition 8. Let $(\mathfrak{B}, \mathrm{P}, Q, *, \bigcirc)$ be an IFDCMS and $\{x_n\}$ be a sequence in $\mathfrak{B}$. Then, $\{x_n\}$ is named to be

(a) Convergent, if there exists $x \in \mathfrak{B}$ such that

$$
\begin{aligned}
&\lim_{n\longrightarrow\infty} \mathrm{P}(x_n, x, t) = 1, \\
&\lim_{n\longrightarrow\infty} Q(x_n, x, t) = 0, \text{for all } t > 0.
\end{aligned} \tag{22}
$$

(b) A Cauchy sequence (CS), if and only if for each $\mu > 0, t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$
\begin{aligned}
&\mathrm{P}\left(x_n, x_{n+q}, t\right) \geq 1 - \mu, \\
&Q\left(x_n, x_{n+q}, t\right) \leq \mu, \text{ for all } n, m \geq n_0.
\end{aligned} \tag{23}
$$

If every Cauchy sequence is convergent in $\mathfrak{B}$, then $(\mathfrak{B}, \mathrm{P}, Q, *, \bigcirc)$ is called a complete IFDCMS.

Lemma 1. *Let $\{x_n\}$ be a Cauchy sequence in IFDCMS $(\mathfrak{B}, \mathrm{P}, Q, *, \bigcirc)$ such that $x_n \neq x_m$ whenever $m, n \in \mathbb{N}$ with $n \neq m$. Then, the sequence $\{x_n\}$ can converge to at most one limit point.*

Proof. Ccontrarily, assume that $x_n \longrightarrow x$ and $x_n \longrightarrow y$, for $x \neq y$. Then, $\lim_{n\longrightarrow\infty} \mathrm{P}(x_n, x, t) = 1, \lim_{n\longrightarrow\infty} Q(x_n, x, t) = 0$ and $\lim_{n\longrightarrow\infty} \mathrm{P}(x_n, y, t) = 1, \lim_{n\longrightarrow\infty} Q(x_n, y, t) = 0$, for all $t > 0$. Therefore,

$$
\begin{aligned}
\mathrm{P}(x, y, t) &\geq \mathrm{P}\left(x, x_n, \frac{t}{2\phi(x, x_n)}\right) * \mathrm{P}\left(x_n, y, \frac{t}{2\eta(x_n, y)}\right) \\
&\longrightarrow 1 * 1, \text{as } n \longrightarrow \infty, \\
Q(x, y, t) &\leq Q\left(x, x_n, \frac{t}{2\phi(x, x_n)}\right) \bigcirc Q\left(x_n, y, \frac{t}{2\eta(x_n, y)}\right) \\
&\longrightarrow 0 \bigcirc 0, \text{as } n \longrightarrow \infty.
\end{aligned} \tag{24}
$$

That is, $\mathrm{P}(x, y, t) \geq 1 * 1 = 1$ and $Q(x, y, t) \leq 0 \bigcirc 0 = 0$. Hence, $x = y$, that is, the sequence converges to at most one limit point. □

Lemma 2. *Let $(\mathfrak{B}, \mathrm{P}, Q, *, \bigcirc)$ be an IFDCMS. If for some $0 < \theta < 1$ and for any $x, y \in \mathfrak{B}, t > 0$,*

$$
\begin{aligned}
\mathrm{P}(x, y, t) &\geq \mathrm{P}\left(x, y, \frac{t}{\theta}\right), \\
Q(x, y, t) &\leq Q\left(x, y, \frac{t}{\theta}\right).
\end{aligned} \tag{25}
$$

then $x = y$.

Proof. (1) implies that

$$\begin{aligned} &\mathrm{P}(x, y, t) \geq \mathrm{P}\left(x, y, \frac{t}{\theta^n}\right),\\ &Q(x, y, t) \leq Q\left(x, y, \frac{t}{\theta^n}\right), \quad n \in \mathbb{N}, t > 0. \end{aligned} \tag{26}$$

Now,

$$\begin{aligned} &\mathrm{P}(x, y, t) \geq \lim_{n \longrightarrow \infty} \mathrm{P}\left(x, y, \frac{t}{\theta^n}\right) = 1,\\ &Q(x, y, t) \leq \lim_{n \longrightarrow \infty} Q\left(x, y, \frac{t}{\theta^n}\right) = 0, \quad t > 0. \end{aligned} \tag{27}$$

By (iii) and (viii), then $x = y$.

At this time, we prove the intuitionistic fuzzy controlled Banach contraction result. □

Theorem 1. *Suppose that* $(\mathfrak{B}, \mathrm{P}, Q, *, \bigcirc)$ *is a complete IFDCMS in the company of* $\phi, \eta\colon \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, 1/\theta)$ *with* $0 < \theta < 1$ *and suppose that*

$$\lim_{t \longrightarrow \infty} \mathrm{P}(x, y, t) = 1 \text{ and } \lim_{t \longrightarrow \infty} Q(x, y, t) = 0. \tag{28}$$

for all $x, y \in \mathfrak{B}$, *and* $t > 0$. *Let* $\Psi\colon \mathfrak{B} \longrightarrow \mathfrak{B}$ *be a mapping satisfying for all* $x, y \in \mathfrak{B}$ *and* $t > 0$. *Then,* Ψ *has a unique FP.*

Proof. Let x_0 be a random integer of $\mathfrak{B}$ and describe a sequence x_n by $x_n = \Psi^n x_0 = \Psi x_{n-1}$, $n \in \mathbb{N}$. By using (2) for all $t > 0$, we have

$$\begin{aligned} \mathrm{P}(x_n, x_{n+1}, \theta t) &= \mathrm{P}(\Psi x_{n-1}, \Psi x_n, \theta t) \geq \mathrm{P}(x_{n-1}, x_n, t) \geq \mathrm{P}\left(x_{n-2}, x_{n-1}, \frac{t}{\theta}\right)\\ &\geq \mathrm{P}\left(x_{n-3}, x_{n-2}, \frac{t}{\theta^2}\right) \geq \cdots \geq \mathrm{P}\left(x_0, x_1, \frac{t}{\theta^{n-1}}\right),\\ Q(x_n, x_{n+1}, \theta t) &= Q(\Psi x_{n-1}, \Psi x_n, \theta t) \leq Q(x_{n-1}, x_n, t) \leq Q\left(x_{n-2}, x_{n-1}, \frac{t}{\theta}\right),\\ &\leq Q\left(x_{n-3}, x_{n-2}, \frac{t}{\theta^2}\right) \leq \cdots \leq Q\left(x_0, x_1, \frac{t}{\theta^{n-1}}\right). \end{aligned} \tag{29}$$

We obtain

$$\begin{aligned} &\mathrm{P}(x_n, x_{n+1}, \theta t) \geq \mathrm{P}\left(x_0, x_1, \frac{t}{\theta^{n-1}}\right),\\ &Q(x_n, x_{n+1}, \theta t) \leq Q\left(x_0, x_1, \frac{t}{\theta^{n-1}}\right). \end{aligned} \tag{30}$$

For any $q \in \mathbb{N}$, using (v) and (x), we deduce

$$\begin{aligned} \mathrm{P}(x_n, x_{n+q}, t) &\geq \mathrm{P}\left(x_n, x_{n+1}, \frac{t}{2(\phi(x_n, x_{n+1}))}\right) * \mathrm{P}\left(x_{n+1}, x_{n+q}, \frac{t}{2(\eta(x_{n+1}, x_{n+q}))}\right)\\ &\geq \mathrm{P}\left(x_n, x_{n+1}, \frac{t}{2(\phi(x_n, x_{n+1}))}\right) * \mathrm{P}\left(x_{n+1}, x_{n+2}, \frac{t}{(2)^2(\eta(x_{n+1}, x_{n+q})\phi(x_{n+1}, x_{n+2}))}\right)\\ &\quad * \mathrm{P}_{\phi}\left(x_{n+2}, x_{n+q}, \frac{t}{(2)^2(\eta(x_{n+1}, x_{n+q})\eta(x_{n+2}, x_{n+q}))}\right)\\ &\geq \mathrm{P}\left(x_n, x_{n+1}, \frac{t}{2(\phi(x_n, x_{n+1}))}\right) * \mathrm{P}\left(x_{n+1}, x_{n+2}, \frac{t}{(2)^2(\eta(x_{n+1}, x_{n+q})\phi(x_{n+1}, x_{n+2}))}\right)\\ &\quad * \mathrm{P}\left(x_{n+2}, x_{n+3}, \frac{t}{(2)^3(\eta(x_{n+1}, x_{n+q})\eta(x_{n+2}, x_{n+q})\phi(x_{n+2}, x_{n+3}))}\right)\\ &\quad * \mathrm{P}\left(x_{n+3}, x_{n+q}, \frac{t}{(2)^3(\eta(x_{n+1}, x_{n+q})\eta(x_{n+2}, x_{n+q})\eta(x_{n+3}, x_{n+q}))}\right) \end{aligned}$$

$$
\begin{aligned}
&\geq \mathrm{P}\left(x_n, x_{n+1}, \frac{t}{2(\phi(x_n, x_{n+1}))}\right) * \mathrm{P}\left(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi(x_{n+1}, x_{n+2})\right)}\right)\\
&* \mathrm{P}\left(x_{n+2}, x_{n+3}, \frac{t}{(2)^3\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\phi(x_{n+2}, x_{n+3})\right)}\right)\\
&\quad * \mathrm{P}\left(x_{n+3}, x_{n+4}, \frac{t}{(2)^4\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\phi(x_{n+3}, x_{n+4})\right)}\right) * \cdots *\\
&\quad \mathrm{P}\left(x_{n+q-2}, x_{n+q-1}, \frac{t}{(2)^{q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\cdots\eta\left(x_{n+q-2}, x_{n+q}\right)\phi\left(x_{n+q-2}, x_{n+q-1}\right)\right)}\right)\\
&\quad * \mathrm{P}\left(x_{n+q-1}, x_{n+q}, \frac{t}{(2)^{q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\cdots\left(x_{n+q-1}, x_{n+q}\right)\right)}\right),\\
&\mathrm{Q}\left(x_n, x_{n+q}, t\right) \leq \mathrm{Q}\left(x_n, x_{n+1}, \frac{t}{2(\phi(x_n, x_{n+1}))}\right)\mathrm{Q}\left(x_{n+1}, x_{n+q}, \frac{t}{2\left(\eta\left(x_{n+1}, x_{n+q}\right)\right)}\right),\\
&\leq \mathrm{Q}\left(x_n, x_{n+1}, \frac{t}{2(\phi(x_n, x_{n+1}))}\right)\bigcirc\mathrm{Q}\left(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi(x_{n+1}, x_{n+2})\right)}\right),\\
&\quad \bigcirc\mathrm{Q}\left(x_{n+2}, x_{n+q}, \frac{t}{(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\right)}\right),\\
&\leq \mathrm{Q}\left(x_n, x_{n+1}, \frac{t}{2(\phi(x_n, x_{n+1}))}\right)\bigcirc\mathrm{Q}\left(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi(x_{n+1}, x_{n+2})\right)}\right),\\
&\bigcirc\mathrm{Q}\left(x_{n+2}, x_{n+3}, \frac{t}{(2)^3\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\phi(x_{n+2}, x_{n+3})\right)}\right),\\
&\bigcirc\mathrm{Q}\left(x_{n+3}, x_{n+q}, \frac{t}{(2)^3\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\right)}\right),\\
&\leq \mathrm{Q}\left(x_n, x_{n+1}, \frac{t}{2(\phi(x_n, x_{n+1}))}\right)\bigcirc\mathrm{Q}\left(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi(x_{n+1}, x_{n+2})\right)}\right),\\
&\quad \bigcirc\mathrm{Q}\left(x_{n+2}, x_{n+3}, \frac{t}{(2)^3\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\phi(x_{n+2}, x_{n+3})\right)}\right),\\
&\quad \bigcirc\mathrm{Q}\left(x_{n+3}, x_{n+4}, \frac{t}{(2)^4\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\phi(x_{n+3}, x_{n+4})\right)}\right)\bigcirc\cdots\bigcirc,\\
&\quad \mathrm{Q}\left(x_{n+q-2}, x_{n+q-1}, \frac{t}{(2)^{q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\cdots\eta\left(x_{n+q-2}, x_{n+q}\right)\phi\left(x_{n+q-2}, x_{n+q-1}\right)\right)}\right),\\
&\quad \bigcirc\mathrm{Q}\left(x_{n+q-1}, x_{n+q}, \frac{t}{(2)^{q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\cdots\eta\left(x_{n+q-1}, x_{n+q}\right)\right)}\right).
\end{aligned}
\tag{31}
$$

Using (4) in the above inequalities, we deduce

$$\begin{aligned}
&\geq \mathrm{P}\left(x_0, x_1, \frac{t}{2(\theta)^{n-1}(\phi(x_n, x_{n+1}))}\right) * \mathrm{P}\left(x_0, x_1, \frac{t}{(2)^2(\theta)^n\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi(x_{n+1}, x_{n+2})\right)}\right),\\
&* \mathrm{P}\left(x_0, x_1, \frac{t}{(2)^3(\theta)^{n+1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\phi(x_{n+2}, x_{n+3})\right)}\right),\\
&* \mathrm{P}\left(x_0, x_1, \frac{t}{(2)^4(\theta)^{n+2}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\phi(x_{n+3}, x_{n+4})\right)}\right) * \cdots *,\\
&\mathrm{P}\left(x_0, x_1, \frac{t}{(2)^{q-1}(\theta)^{n+q-2}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\cdots\eta\left(x_{n+q-2}, x_{n+q}\right)\phi\left(x_{n+q-2}, x_{n+q-1}\right)\right)}\right),\\
&* \mathrm{P}\left(x_0, x_1, \frac{t}{(2)^{q-1}(\theta)^{n+q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\cdots\eta\left(x_{n+q-1}, x_{n+q}\right)\right)}\right),\\
&\leq \mathrm{Q}\left(x_0, x_1, \frac{t}{2(\theta)^{n-1}(\phi(x_n, x_{n+1}))}\right)\mathrm{Q}\left(x_0, x_1, \frac{t}{(2)^2(\theta)^n\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi(x_{n+1}, x_{n+2})\right)}\right),\\
&\bigcirc \mathrm{Q}\left(x_0, x_1, \frac{t}{(2)^3(\theta)^{n+1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\phi(x_{n+2}, x_{n+3})\right)}\right),\\
&\bigcirc \mathrm{Q}\left(x_0, x_1, \frac{t}{(2)^4(\theta)^{n+2}\left(\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\phi(x_{n+3}, x_{n+4})\right)}\right),\\
&\bigcirc \cdots \bigcirc\\
&\mathrm{Q}\left(x_0, x_1, \frac{t}{(2)^{q-1}(\theta)^{n+q-2}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\cdots\eta\left(x_{n+q-2}, x_{n+q}\right)\phi\left(x_{n+q-2}, x_{n+q-1}\right)\right)}\right),\\
&\bigcirc \mathrm{Q}\left(x_0, x_1, \frac{t}{(2)^{q-1}(\theta)^{n+q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\cdots\eta\left(x_{n+q-1}, x_{n+q}\right)\right)}\right).
\end{aligned} \tag{32}$$

Using (2), if $n \longrightarrow \infty$, we deduce

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \mathrm{P}\left(x_n, x_{n+q}, t\right) = 1 * 1 * \cdots * 1 = 1,\\
&\lim_{n\longrightarrow\infty} \mathrm{Q}\left(x_n, x_{n+q}, t\right) = 0\bigcirc 0\bigcirc \cdots \bigcirc 0 = 0,
\end{aligned} \tag{33}$$

i.e., $\{x_n\}$ is a CS. Since $(\mathfrak{B}, \mathrm{P}, Q, *, \bigcirc)$ is a complete IFDCMS, there exists

$$\lim_{n\longrightarrow\infty} x_n = x. \tag{34}$$

Now, we investigate that x is a FP of Ψ. Using (v), (x), and (2), we obtain

$$\begin{aligned}
\mathrm{P}(x, \Psi x, t) &\geq \mathrm{P}\left(x, x_{n+1}, \frac{t}{2(\phi(x, x_{n+1}))}\right) * \mathrm{P}\left(x_{n+1}, \Psi x, \frac{t}{2(\eta(x_{n+1}, \Psi x))}\right)\\
&\geq \mathrm{P}\left(x, x_{n+1}, \frac{t}{2(\phi(x, x_{n+1}))}\right) * \mathrm{P}\left(\Psi x_n, \Psi x, \frac{t}{2(\eta(x_{n+1}, \Psi x))}\right)\\
&\geq \mathrm{P}\left(x, x_{n+1}, \frac{t}{2(\phi(x, x_{n+1}))}\right) * \mathrm{P}\left(x_n, x, \frac{t}{2\theta(\eta(x_{n+1}, \Psi x))}\right) \longrightarrow 1 * 1 = 1,
\end{aligned} \tag{35}$$

as $n \longrightarrow \infty$, and

$$\begin{aligned} Q(x,\Psi x,t) &\leq Q\left(x,x_{n+1},\frac{t}{2(\phi(x,x_{n+1}))}\right)\bigcirc Q\left(x_{n+1},\Psi x,\frac{t}{2(\eta(x_{n+1},\Psi x))}\right)\\ &\leq Q\left(x,x_{n+1},\frac{t}{2(\phi(x,x_{n+1}))}\right)\bigcirc Q\left(\Psi x_n,\Psi x,\frac{t}{2(\eta(x_{n+1},\Psi x))}\right)\\ &\leq Q\left(x,x_{n+1},\frac{t}{2(\phi(x,x_{n+1}))}\right)\bigcirc Q\left(x_n,x,\frac{t}{2\theta(\eta(x_{n+1},\Psi x))}\right)\longrightarrow 0\bigcirc 0=0, \end{aligned} \tag{36}$$

as $n \longrightarrow \infty$. This implies that $\Psi x = x$, a FP. Now, we show the uniqueness. Suppose that $\Psi\rho = \rho$ for some $\rho \in \mathfrak{B}$. Then,

$$\begin{aligned} &1 \geq P(\rho,x,t) = P(\Psi\rho,\Psi x,t) \geq P\left(\rho,x,\frac{t}{\theta}\right) = P\left(\Psi\rho,\Psi x,\frac{t}{\theta}\right)\\ &\geq P\left(\rho,x,\frac{t}{\theta^2}\right) \geq \cdots \geq P\left(\rho,x,\frac{t}{\theta^n}\right) \longrightarrow 1 \text{ as } n \longrightarrow \infty\\ &0 \leq Q(\rho,x,t) = Q(\Psi\rho,\Psi x,t) \leq Q\left(\rho,x,\frac{t}{\theta}\right) = Q\left(\Psi\rho,\Psi x,\frac{t}{\theta}\right)\\ &\leq Q\left(\rho,x,\frac{t}{\theta^2}\right) \leq \cdots \leq Q\left(\rho,x,\frac{t}{\theta^n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{37}$$

By using (iii)and (viii), then $x = \rho$. □

Corollary 1. *Suppose that $(\mathfrak{B}, P, Q, *, \bigcirc)$ is a complete IFDCMS in the company of $\phi, \eta: \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, 1/\theta)$ with $0 < \theta < 1$ and suppose that*

$$\begin{aligned} &\lim_{t\longrightarrow\infty} P(x,y,t) = 1,\\ &\lim_{t\longrightarrow\infty} Q(x,y,t) = 0, \end{aligned} \tag{38}$$

for all $x, y \in \mathfrak{B}$ and $t > 0$. Let $\Psi: \mathfrak{B} \longrightarrow \mathfrak{B}$ be a mapping satisfying

$$\begin{aligned} &P(\Psi x,\Psi y,\theta t) \geq \min\{P(x,y,t),P(x,\Psi x,t),P(y,\Psi y,t)\},\\ &Q(\Psi x,\Psi y,\theta t) \leq \min\{Q(x,y,t),Q(x,\Psi x,t),Q(y,\Psi y,t)\}, \end{aligned} \tag{39}$$

for all $x, y \in \mathfrak{B}$, and $t > 0$. Then, Ψ has a unique FP.

Proof. It is easy to prove by using Theorem 1 and Lemma 2. □

Definition 9. Let $(\mathfrak{B}, P, Q, *, \bigcirc)$ be an IFDCMS. A map $\Psi: \mathfrak{B} \longrightarrow \mathfrak{B}$ is a D-controlled intuitionistic fuzzy contraction if there exists $0 < \theta < 1$, such that

$$\begin{aligned} &\frac{1}{P_\phi(\Psi x,\Psi y,t)} - 1 \leq \theta\left[\frac{1}{P_\phi(x,y,t)} - 1\right],\\ &Q_\phi(\Psi x,\Psi y,t) \leq \theta Q_\phi(x,y,t), \end{aligned} \tag{40}$$

for all $x, y \in \mathfrak{B}$ and $t > 0$.

Now, we prove a theorem for D-controlled intuitionistic fuzzy contractions.

Theorem 2. *Let $(\mathfrak{B}, P, Q, *, \bigcirc)$ be a complete IFDCMS with $\phi, \eta: \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, \infty)$ and suppose that*

$$\begin{aligned} &\lim_{t\longrightarrow\infty} P(x,y,t) = 1,\\ &\lim_{t\longrightarrow\infty} Q(x,y,t) = 0, \end{aligned} \tag{41}$$

for all $x, y \in \mathfrak{B}$ and $t > 0$. Let $\Psi: \mathfrak{B} \longrightarrow \mathfrak{B}$ be a D-controlled intuitionistic fuzzy contraction. Further, suppose that for an arbitrary $x_0 \in \mathfrak{B}$, and $n, q \in \mathbb{N}$, $x_n = \Psi^n x_0 = \Psi x_{n-1}$. Then, Ψ has a unique FP.

Proof. Let x_0 be a random integer of $\mathfrak{B}$ and describe a sequence x_n by $x_n = \Psi^n x_0 = \Psi x_{n-1}$, $n \in \mathbb{N}$. By using (5) and (6) for all $t > 0$, $n > q$, we have

$$\begin{aligned} \frac{1}{P(x_n,x_{n+1},t)} - 1 &= \frac{1}{P(\Psi x_{n-1},x_n,t)} - 1\\ &\leq \theta\left[\frac{1}{P(x_{n-1},x_n,t)} - 1\right] = \frac{\theta}{P(x_{n-1},x_n,t)} - \theta\\ &\Rightarrow \frac{1}{P(x_n,x_{n+1},t)} \leq \frac{\theta}{P(x_{n-1},x_n,t)} + (1-\theta)\\ &\leq \frac{\theta^2}{P(x_{n-2},x_{n-1},t)} + \theta(1-\theta) + (1-\theta). \end{aligned} \tag{42}$$

Continuing in this way, we get

$$\begin{aligned} \frac{1}{P(x_n,x_{n+1},t)} &\leq \frac{\theta^n}{P(x_0,x_1,t)} + \theta^{n-1}(1-\theta) + \theta^{n-2}(1-\theta) + \cdots + \theta(1-\theta) + (1-\theta)\\ &\leq \frac{\theta^n}{P(x_0,x_1,t)} + \left(\theta^{n-1} + \theta^{n-2} + \cdots + 1\right)(1-\theta) \leq \frac{\theta^n}{P(x_0,x_1,t)} + \left(1-\theta^n\right). \end{aligned} \tag{43}$$

We obtain

$$\frac{1}{\theta^n/\mathrm{P}(x_0,x_1,t)+(1-\theta^n)}\le \mathrm{P}(x_n,x_{n+1},t),$$
$$\begin{aligned}\mathrm{Q}(x_n,x_{n+1},t)=\mathrm{Q}(\Psi x_{n-1},x_n,t)\le\theta\mathrm{Q}(x_{n-1},x_n,t)\\ =\mathrm{Q}(\Psi x_{n-2},x_{n-1},t),\\ \le\theta^2\mathrm{Q}(x_{n-2},x_{n-1},t)\le\cdots\le\theta^n\mathrm{Q}(x_0,x_1,t),\end{aligned}\tag{44}$$

for any $q\in\mathbb{N}$Using (v) and (x), we deduce

$$\begin{aligned}
\mathrm{P}(x_n,x_{n+q},t)&\ge\mathrm{P}\left(x_n,x_{n+1},\frac{t}{2(\phi(x_n,x_{n+1}))}\right)*\mathrm{P}\left(x_{n+1},x_{n+q},\frac{t}{2(\eta(x_{n+1},x_{n+q}))}\right),\\
&\ge\mathrm{P}\left(x_n,x_{n+1},\frac{t}{2(\phi(x_n,x_{n+1}))}\right)*\mathrm{P}\left(x_{n+1},x_{n+2},\frac{t}{(2)^2(\eta(x_{n+1},x_{n+q})\phi(x_{n+1},x_{n+2}))}\right),\\
&*\mathrm{P}\left(x_{n+2},x_{n+q},\frac{t}{(2)^2(\eta(x_{n+1},x_{n+q})\eta(x_{n+2},x_{n+q}))}\right),\\
&\ge\mathrm{P}\left(x_n,x_{n+1},\frac{t}{2(\phi(x_n,x_{n+1}))}\right)*\mathrm{P}\left(x_{n+1},x_{n+2},\frac{t}{(2)^2(\eta(x_{n+1},x_{n+q})\phi(x_{n+1},x_{n+2}))}\right),\\
&*\mathrm{P}\left(x_{n+2},x_{n+3},\frac{t}{(2)^3(\eta(x_{n+1},x_{n+q})\eta(x_{n+2},x_{n+q})\phi(x_{n+2},x_{n+3}))}\right),\\
&*\mathrm{P}\left(x_{n+3},x_{n+q},\frac{t}{(2)^3(\eta(x_{n+1},x_{n+q})\eta(x_{n+2},x_{n+q})\eta(x_{n+3},x_{n+q}))}\right),\\
&\ge\mathrm{P}\left(x_n,x_{n+1},\frac{t}{2(\phi(x_n,x_{n+1}))}\right)*\mathrm{P}\left(x_{n+1},x_{n+2},\frac{t}{(2)^2(\eta(x_{n+1},x_{n+q})\phi(x_{n+1},x_{n+2}))}\right),\\
&*\mathrm{P}\left(x_{n+2},x_{n+3},\frac{t}{(2)^3(\eta(x_{n+1},x_{n+q})\eta(x_{n+2},x_{n+q})\phi(x_{n+2},x_{n+3}))}\right),\\
&*\mathrm{P}\left(x_{n+3},x_{n+4},\frac{t}{(2)^4(\eta(x_{n+1},x_{n+q})\eta(x_{n+2},x_{n+q})\eta(x_{n+3},x_{n+q})\phi(x_{n+3},x_{n+4}))}\right)*\cdots*,\\
&\mathrm{P}\left(x_{n+q-2},x_{n+q-1},\frac{t}{(2)^{q-1}(\eta(x_{n+1},x_{n+q})\eta(x_{n+2},x_{n+q})\cdots\eta(x_{n+q-2},x_{n+q})\phi(x_{n+q-2},x_{n+q-1}))}\right),\\
&*\mathrm{P}\left(x_{n+q-1},x_{n+q},\frac{t}{(2)^{q-1}(\eta(x_{n+1},x_{n+q})\eta(x_{n+2},x_{n+q})\eta(x_{n+3},x_{n+q})\cdots\eta(x_{n+q-1},x_{n+q}))}\right),\\
\mathrm{Q}(x_n,x_{n+q},t)&\le\mathrm{Q}\left(x_n,x_{n+1},\frac{t}{2(\phi(x_n,x_{n+1}))}\right)\circ\mathrm{Q}\left(x_{n+1},x_{n+q},\frac{t}{2(\eta(x_{n+1},x_{n+q}))}\right),
\end{aligned}$$

$$
\begin{aligned}
&\leq Q\left(x_n, x_{n+1}, \frac{t}{2\left(\phi\left(x_n, x_{n+1}\right)\right)}\right) \bigcirc Q\left(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi\left(x_{n+1}, x_{n+2}\right)\right)}\right),\\
&\bigcirc Q\left(x_{n+2}, x_{n+q}, \frac{t}{(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\right)}\right),\\
&\leq Q\left(x_n, x_{n+1}, \frac{t}{2\left(\phi\left(x_n, x_{n+1}\right)\right)}\right) \bigcirc Q\left(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi\left(x_{n+1}, x_{n+2}\right)\right)}\right),\\
&\bigcirc Q\left(x_{n+2}, x_{n+3}, \frac{t}{(2)^3\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\phi\left(x_{n+2}, x_{n+3}\right)\right)}\right),\\
&\bigcirc Q\left(x_{n+3}, x_{n+q}, \frac{t}{(2)^3\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\right)}\right),\\
&\leq Q\left(x_n, x_{n+1}, \frac{t}{2\left(\phi\left(x_n, x_{n+1}\right)\right)}\right) \bigcirc Q\left(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi\left(x_{n+1}, x_{n+2}\right)\right)}\right),\\
&\bigcirc Q\left(x_{n+2}, x_{n+3}, \frac{t}{(2)^3\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\phi\left(x_{n+2}, x_{n+3}\right)\right)}\right),\\
&\bigcirc Q\left(x_{n+3}, x_{n+4}, \frac{t}{(2)^4\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\phi\left(x_{n+3}, x_{n+4}\right)\right)}\right) \bigcirc \cdots \bigcirc,\\
&Q\left(x_{n+q-2}, x_{n+q-1}, \frac{t}{(2)^{q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\cdots\eta\left(x_{n+q-2}, x_{n+q}\right)\phi\left(x_{n+q-2}, x_{n+q-1}\right)\right)}\right),\\
&\bigcirc Q\left(x_{n+q-1}, x_{n+q}, \frac{t}{(2)^{q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\cdots\eta\left(x_{n+q-1}, x_{n+q}\right)\right)}\right),\\
&P\left(x_n, x_{n+q}, t\right) \geq \frac{1}{\theta^n/P\left(x_0, x_1, t/2\left(\phi\left(x_n, x_{n+1}\right)\right)\right) + \left(1-\theta^n\right)} * \frac{1}{\theta^{n+1}/P\left(x_0, x_1, t/(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi\left(x_{n+1}, x_{n+2}\right)\right)\right) + \left(1-\theta^{n+1}\right)},\\
&* \frac{1}{\theta^{n+2}/P\left(x_0, x_1, t/(2)^3\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\phi\left(x_{n+2}, x_{n+3}\right)\right)\right) + \left(1-\theta^{n+2}\right)} * \cdots *,\\
&\frac{1}{\theta^{n+q-2}/P\left(x_0, x_1, t/(2)^{q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\cdots\eta\left(x_{n+q-2}, x_{n+q}\right)\phi\left(x_{n+q-2}, x_{n+q-1}\right)\right)\right) + \left(1-\theta^{n+q-2}\right)},\\
&* \frac{1}{\theta^{n+q-1}/P\left(x_0, x_1, t/(2)^{q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\cdots\eta\left(x_{n+q-1}, x_{n+q}\right)\right)\right) + \left(1-\theta^{n+q-1}\right)},\\
&Q\left(x_n, x_{n+q}, t\right) \leq \theta^n Q\left(x_0, x_1, \frac{t}{2\left(\phi\left(x_n, x_{n+1}\right)\right)}\right) \bigcirc \theta^{n+1} Q\left(x_0, x_1, \frac{t}{(2)^2\left(\eta\left(x_{n+1}, x_{n+q}\right)\phi\left(x_{n+1}, x_{n+2}\right)\right)}\right),\\
&\bigcirc \theta^{n+2} Q\left(x_0, x_1, \frac{t}{(2)^3\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\phi\left(x_{n+2}, x_{n+3}\right)\right)}\right) \bigcirc \cdots \bigcirc,\\
&\theta^{n+q-2} Q\left(x_0, x_1, \frac{t}{(2)^{q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\cdots\eta\left(x_{n+q-2}, x_{n+q}\right)\phi\left(x_{n+q-2}, x_{n+q-1}\right)\right)}\right),\\
&\bigcirc \theta^{n+q-1} Q\left(x_0, x_1, \frac{t}{(2)^{q-1}\left(\eta\left(x_{n+1}, x_{n+q}\right)\eta\left(x_{n+2}, x_{n+q}\right)\eta\left(x_{n+3}, x_{n+q}\right)\cdots\eta\left(x_{n+q-1}, x_{n+q}\right)\right)}\right).
\end{aligned}
\tag{45}
$$

Therefore,

$$\lim_{n\longrightarrow\infty} P\left(x_n, x_{n+q}, t\right) = 1 * 1 * \cdots * 1 = 1,$$
$$\lim_{n\longrightarrow\infty} Q\left(x_n, x_{n+q}, t\right) = 0\bigcirc 0\bigcirc \cdots \bigcirc 0 = 0, \tag{46}$$

i.e., $\{x_n\}$ is a CS. Since $(\mathfrak{B}, P, Q, *, \bigcirc)$ is a complete IFDCMS, there exists

$$\lim_{n\longrightarrow\infty} x_n = x. \tag{47}$$

Now, we investigate that x is a FP of Ψ. Using (v) and (x), we obtain

$$\frac{1}{P(\Psi x_n, \Psi x, t)} - 1 \leq \theta\left[\frac{1}{P(x_n, x, t)} - 1\right] = \frac{\theta}{P(x_n, x, t)} - \theta$$
$$\Rightarrow \frac{1}{\theta/P(x_n, x, t) + (1-\theta)} \leq P(\Psi x_n, \Psi x, t). \tag{48}$$

Using the above inequality, we obtain

$$\begin{aligned} P(x, \Psi x, t) &\geq P\left(x, x_{n+1}, \frac{t}{2ф(x, x_{n+1})}\right) * P\left(x_{n+1}, \Psi x, \frac{t}{2\eta(x_{n+1}, \Psi x)}\right) \\ &\geq P\left(x, x_{n+1}, \frac{t}{2ф(x, x_{n+1})}\right) * P\left(\Psi x_n, \Psi x, \frac{t}{2\eta(x_{n+1}, \Psi x)}\right) \\ &\geq P\left(x_n, x_{n+1}, \frac{t}{(2ф(x, x_{n+1}))}\right) * \frac{1}{\theta/P(x_n, x, t/2\eta(x_{n+1}, \Psi x)) + (1-\theta)} \longrightarrow 1 * 1 = 1, \end{aligned} \tag{49}$$

as $n \longrightarrow \infty$, and

$$\begin{aligned} Q(x, \Psi x, t) &\leq Q\left(x, x_{n+1}, \frac{t}{2ф(x, x_{n+1})}\right) \bigcirc Q\left(x_{n+1}, \Psi x, \frac{t}{2\eta(x_{n+1}, \Psi x)}\right) \\ &\leq Q\left(x, x_{n+1}, \frac{t}{2ф(x, x_{n+1})}\right) \bigcirc Q\left(\Psi x_n, \Psi x, \frac{t}{2\eta(x_{n+1}, \Psi x)}\right) \\ &\leq Q\left(x_n, x_{n+1}, \frac{t}{2ф(x, x_{n+1})}\right) \bigcirc \theta Q\left(x_n, x, \frac{t}{2\eta(x_{n+1}, \Psi x)}\right) \longrightarrow 0\bigcirc 0 = 0 \ as\ n \longrightarrow \infty. \end{aligned} \tag{50}$$

This implies that $\Psi x = x$, a FP. Now, we show the uniqueness. Suppose that $\Psi\rho = \rho$ for some $\rho \in \mathfrak{B}$. Then,

$$\begin{aligned} \frac{1}{P(x, \rho, t)} - 1 &= \frac{1}{P(\Psi x, \Psi\rho, t)} - 1 \\ &\leq \theta\left[\frac{1}{P(x, \rho, t)} - 1\right] < \frac{1}{P(x, \rho, t)} - 1, \end{aligned} \tag{51}$$

a contradiction, and

$$Q(x, \rho, t) = Q(\Psi x, \Psi\rho, t) \leq \theta Q(x, \rho, t) < Q(x, \rho, t), \tag{52}$$

a contradiction. Therefore, we must have $P(x, \rho, t) = 1$ and $Q(x, \rho, t) = 0$, hence $x = \rho$. □

Example 6. Let $\mathfrak{B} = [0, 1]$ and $ф, \eta\colon \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, \infty)$ be two noncomparable functions given by

$$ф(x, y) = \begin{cases} 1, & \text{if} \quad x = y, \\ \dfrac{1 + \max\{x, y\}}{\min\{x, y\}}, & \text{if} \quad x \neq y \neq 0. \end{cases}$$
$$\eta(x, y) = \begin{cases} 1, & \text{if} \quad x = y, \\ \dfrac{1 + \max\{x^2, y^2\}}{\min\{x^2, y^2\}}, & \text{if} \quad x \neq y. \end{cases} \tag{53}$$

Define $P, Q\colon \mathfrak{B} \times \mathfrak{B} \times (0, \infty) \longrightarrow [0, 1]$ as

$$P(x, y, t) = \frac{t}{t + |x - y|^2},$$
$$Q(x, y, t) = \frac{|x - y|^2}{t + |x - y|^2}. \tag{54}$$

Then, $(\mathfrak{B}, \mathrm{P}, \mathrm{Q}, *, \bigcirc)$ is a complete IFDCMS with CTN $\pi * \mu = \pi\mu$ and CTCN $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Define $\Psi\colon \mathfrak{B} \longrightarrow \mathfrak{B}$ by $\Psi(x) = 1 - 2^{-x}/3$ and take $\theta \in [1/2, 1)$. Then,

$$
\begin{aligned}
\mathrm{P}(\Psi x, \Psi y, \theta t) &= \mathrm{P}\left(\frac{1-2^{-x}}{3}, \frac{1-2^{-y}}{3}, \theta t\right)\\
&= \frac{\theta t}{\theta t + \left|1 - 2^{-x}/3 - 1 - 2^{-y}/3\right|^2}\\
&= \frac{\theta t}{\theta t + \left|2^{-x} - 2^{-y}\right|^2/9}\\
&\geq \frac{\theta t}{\theta t + |x - y|^2/9} = \frac{9\theta t}{9\theta t + |x - y|^2} \geq \frac{t}{t + |x - y|^2} = \mathrm{P}(x, y, t),\\
\mathrm{Q}(\Psi x, \Psi y, \theta t) &= \mathrm{Q}\left(\frac{1-2^{-x}}{3}, \frac{1-2^{-y}}{3}, \theta t\right)\\
&= \frac{\left|1 - 2^{-x}/3 - 1 - 2^{-y}/3\right|^2}{\theta t + \left|1 - 2^{-x}/3 - 1 - 2^{-y}/3\right|^2}\\
&= \frac{\left|2^{-x} - 2^{-y}\right|^2/9}{\theta t + \left|2^{-x} - 2^{-y}\right|^2/9}\\
&= \frac{\left|2^{-x} - 2^{-y}\right|^2}{9\theta t + \left|2^{-x} - 2^{-y}\right|^2} \leq \frac{|x - y|^2}{9\theta t + |x - y|^2} \leq \frac{|x - y|^2}{t + |x - y|^2} = \mathrm{Q}(x, y, t).
\end{aligned}
\tag{55}
$$

Hence, all circumstances of Theorem 1 are fulfilled and 0 is a unique fixed point for Ψ.

3. Application to Fuzzy Fredholm Integral Equation

Let $\mathfrak{B} = C([e, g], \mathbb{R})$ be the set of the entire continuous functions so that their domain is real values and defined on $[e, g]$.

Now, we consider the following fuzzy integral equation:

$$
x(l) = f(j) + \delta \int_e^g F(l, j) x(l) \mathrm{d}j, \quad \text{for } j \in [e, g], \tag{56}
$$

where $\delta > 0$, $f(j)$ is a fuzzy function of $j\colon j \in [e, g]$ and $F \in \mathfrak{B}$

Define P and Q by

$$
\begin{aligned}
\mathrm{P}(x(l), y(l), t) &= \sup_{l \in [e,g]} \frac{t}{t + |x(l) - y(l)|^2} \quad \text{for all } x, y \in \mathfrak{B} \text{ and } t > 0,\\
\mathrm{P}(x(l), y(l), t) &= 1 - \sup_{l \in [e,g]} \frac{t}{t + |x(l) - y(l)|^2} \quad \text{for all } x, y \in \mathfrak{B} \text{ and } t > 0,
\end{aligned}
\tag{57}
$$

with CTN and CTCN defined by $\pi * \mu = \pi.\mu$ and $\pi \bigcirc \mu = \max\{\pi, \mu\}$.

Define $\phi, \eta\colon \mathfrak{B} \times \mathfrak{B} \longrightarrow [1, \infty)$ as

$$\phi(x,y) = \begin{cases} 1, & \text{if } \quad x = y, \\ \dfrac{1+\max\{x,y\}}{\min\{x,y\}}, & \text{if } \quad x \neq y \neq 0. \end{cases}$$

$$\eta(x,y) = \begin{cases} 1, & \text{if } \quad x = y, \\ \dfrac{1+\max\{x^2,y^2\}}{\min\{x^2,y^2\}}, & \text{if } \quad x \neq y. \end{cases} \tag{58}$$

Then, $(\mathfrak{B}, \mathrm{P}, \mathrm{P}, *, \bigcirc)$ is a complete IFDCMS. Assume that

$$|F(l,j)x(l) - F(l,j)y(l)| \leq |x(l) - y(l)|, \tag{59}$$

for $x, y \in \mathfrak{B}$, $\theta \in (0,1)$ and for all $l, j \in [e,g]$.

Also, consider $(\delta\pi \int_e^g \mathrm{d}j)^2 \leq \theta < 1$. Then, the fuzzy integral equation in equation (56) has a unique solution.

Proof: . Define $\Psi\colon \mathfrak{B} \longrightarrow \mathfrak{B}$ by

$$\Psi x(l) = f(j) + \delta \int_e^g F(l,j)e(l)\mathrm{d}j, \quad \text{for all } j \in [e,g]. \tag{60}$$

Scrutinize that survival of an FP of the operator Ψ has come to the survival of solution of the fuzzy integral equation.

Now, for all $x, y \in \mathfrak{B}$, we obtain

$$\begin{aligned}
\mathrm{P}(\Psi x(l), \Psi y(l), \theta t) &= \sup_{l\in[e,g]} \frac{\theta t}{\theta t + |\Psi x(l) - \Psi y(l)|^2} \\
&= \sup_{l\in[e,g]} \frac{\theta t}{\theta t + \left|f(j) + \delta\int_e^g F(l,j)e(l)\mathrm{d}j - f(j) - \delta\int_e^g F(l,j)e(l)\mathrm{d}j\right|^2} \\
&= \sup_{l\in[e,g]} \frac{\theta t}{\theta t + \left|\delta\int_e^g F(l,j)e(l)\mathrm{d}j - \delta\int_e^g F(l,j)e(l)\mathrm{d}j\right|^2} \\
&= \sup_{l\in[e,g]} \frac{\theta t}{\theta t + |F(l,j)x(l) - F(l,j)y(l)|^2\left(\delta\int_e^g \mathrm{d}j\right)^2} \\
&\geq \sup_{l\in[e,g]} \frac{t}{t + |x(l) - y(l)|^2} \\
&\geq \mathrm{P}(x(l), y(l), t) \\
Q(\Psi x(l), \Psi y(l), \theta t) &= 1 - \sup_{l\in[e,g]} \frac{\theta t}{\theta t + |\Psi x(l) - \Psi y(l)|^2} \\
&= 1 - \sup_{l\in[e,g]} \frac{\theta t}{\theta t + \left|f(j) + \delta\int_e^g F(l,j)e(l)\mathrm{d}j - f(j) - \delta\int_e^g F(l,j)e(l)\mathrm{d}j\right|^2} \\
&= 1 - \sup_{l\in[e,g]} \frac{\theta t}{\theta t + \left|\delta\int_e^g F(l,j)e(l)\mathrm{d}j - \delta\int_e^g F(l,j)e(l)\mathrm{d}j\right|^2} \\
&= 1 - \sup_{l\in[e,g]} \frac{\theta t}{\theta t + |F(l,j)x(l) - F(l,j)y(l)|^2\left(\delta\int_e^g \mathrm{d}j\right)^2} \\
&\leq 1 - \sup_{l\in[e,g]} \frac{t}{t + |x(l) - y(l)|^2} \\
&\leq Q(x(l), y(l), t).
\end{aligned} \tag{61}$$

Therefore, all circumstances of Theorem 1 are fulfilled. Hence, operator Ψ has a single FP. This implies that fuzzy integral (56) has a single solution. □

4. Conclusion

Herein, we introduced the notion of intuitionistic fuzzy double controlled metric spaces and some new types of fixed point theorems in this new setting. Moreover, we provided a nontrivial example to demonstrate the viability of the proposed methods. We have supplemented this work with an application that demonstrates how the built method outperforms those found in the literature. Since our structure is more general than the class of fuzzy intuitionistic and double controlled metric spaces, our results and notions expand and generalize many previously published results [6, 22, 23].

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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