Hindawi
Scientific Programming
Volume 2022, Article ID 6897405, 14 pages
https://doi.org/10.1155/2022/6897405

Research Article

Alpha Power Moment Exponential Model with Applications to Biomedical Science

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Received 7 January 2022; Revised 26 February 2022; Accepted 2 March 2022; Published 12 April 2022

Academic Editor: Sikandar Ali

The moment exponential distribution has recently been generalized by a number of authors. The two-parameter alpha power-transformed moment exponential (APTME) distribution is introduced. In terms of fit, the APTME distribution outperforms the moment exponential distribution. Exact expressions for ordinary moments, incomplete and conditional moments, the moment generating function, the cumulant generating function, and information measures are obtained for some APTME distribution features. To estimate the model parameters, six well-known frequentist techniques were applied. The behavior of the various estimators was investigated using a simulated exercise. To examine the practical significance of the APTME distribution, real-world datasets were used. In terms of performance, we show that the APTME distribution beats other models.

1. Introduction

Several univariate continuous distributions have been widely used for modelling lifetime data in environmental, engineering, financial, and biomedical sciences, among other fields. However, there is still a strong need for significant improvement of the classical distributions via various techniques for modelling various data lifetimes. In this regard, Mahdavi and Kundu [1] introduced the alpha power transformation (APT) method, which is based on adding a parameter to a family of distributions to improve their flexibility. The APT- G family's cumulative distribution function (CDF) is defined as follows:

$$F(x;\alpha) = \begin{cases} \dfrac{\alpha^{G(x)} - 1}{\alpha - 1}, & \alpha > 0, \alpha \neq 1, \\ G(x), & \alpha > 0, \alpha = 1. \end{cases} \quad (1)$$

The probability density function related to (1) is

$$f(x;\alpha) = \begin{bmatrix} \dfrac{\log\alpha}{\alpha - 1} g(x)\alpha^{G(x)} & \text{if } \alpha > 0, \alpha \neq 1 \\ g(x) & \text{if } \alpha = 1 \end{bmatrix}. \quad (2)$$

In literature, many probability distributions have been prepared by several researches using this approach; for instance, APT Weibull (APTW) distribution [2], APT inverse exponential [3], APT extended exponential distribution [4], APT Lindley model [5], APT inverse Lindley model [6], APT power Lindley [7], APT Kumaraswamy distribution [8], APT Pareto model [9], APT inverse Lomax [10], Marshall–Olkin APT Weibull distribution with different estimation methods based on Type-I and Type-II censored samples [11], and APT Lomax distribution [12], among others.

The moment exponential (ME) distribution (also known as the length biased exponential (LBE) distribution) is regarded as one of the most important univariate and parametric models. It is commonly used in the analysis of lifetime data as well as problems involving the modelling of failure processes. This distribution is also a flexible lifetime distribution model that may fit some failure datasets well. Dara and Ahmed [13] proposed the ME with the PDF and CDF files shown as follows:

$$g(x;v) = v^2 x e^{-x/v}; \quad x \geq 0, v > 0,$$
$$G(x;v) = 1 - \left(1 + \frac{x}{v} e^{-x/v}\right); \quad x \geq 0, v > 0, \tag{3}$$

where v is the scale parameter. Different values of the shape parameter lead to different shapes of the density function.

The following notable contributions to the associated literature are made by this study: As a generalization of the ME distribution, the APT moment exponential (APTME) distribution is presented; it is a more flexible model than the ME distribution. Two real-world datasets are used to assess the model's applicability. Maximum likelihood (M1) under complete and right-censored (RC) samples, least squares (M2), weighted least squares (M3), maximum product of spacing (M4), Cramer von Mises (M5), and Anderson Darling (M6) methods are used to estimate the unknown parameters of the APTME distribution. The APTME model's structural characteristics and parameter estimators are derived for a number of APTME models. The following is a presentation of the study's content. In Section 2, we present the APTME model's PDF, CDF, and the hazard rate function (HRF). Section 3 contains some structural properties. Section 4 discusses parameter estimation using the methods of M1, M2, M3, M4, M5, and M6. In Section 5, two applications to real-world datasets demonstrate the APTME model's empirical importance. Section 6 provides a comparison and precision of these methods versus a simulation study. Conclusions are provided at the end of the paper.

2. Description of the APTME Model

The APTME distribution, based on the APT method, is introduced in this section. The PDF, CDF, survival function, and the HRF of the APTME distribution are defined.

Definition 1. A random variable X is said to have APTME distribution, denoted by APTME (α, v), with shape parameter α and scale parameter v, if the PDF and CDF of X for $x \geq 0$ are given by

$$f(x;\alpha,v) = \begin{cases} \dfrac{x\alpha \log(\alpha)}{v^2(\alpha-1)} e^{-x/v} \alpha^{-(1+x/v)e^{-x/v}}, & \text{if } \alpha \neq 1, v, \alpha > 0, \\ \dfrac{x}{v^2} e^{-x/v}, & \text{if } \alpha = 1, v, \alpha > 0, \end{cases}$$
$$F(x;\alpha,v) = \begin{cases} \dfrac{\alpha^{1-(1+x/v)e^{-x/v}} - 1}{\alpha - 1}, & \text{if } \alpha \neq 1, v, \alpha > 0, \\ 1 - \left(1 + \dfrac{x}{v}\right) e^{-x/v}, & \text{if } \alpha = 1, v, \alpha > 0. \end{cases} \tag{4}$$

Some plots of the density (4) for different values of α and v are illustrated in Figure 1. They reveal that the PDF of X is quite flexible and can take asymmetric forms, among others.

Other important functions, such as the survival function and the HRF, can be used to highlight the APTME model in addition to the density function (4) and distribution function (4). These functions are especially important in analyzing the survival and are included in the definition below.

Definition 2. The survival function (SF) and the HRF of the APTME distribution are given, respectively, by

$$R(x;\alpha,v) = \begin{cases} \dfrac{\alpha - \alpha^{1-(1+x/v)e^{-x/v}}}{\alpha - 1}, & \text{if } \alpha \neq 1, v, \alpha > 0, \\ \left(1 + \dfrac{x}{v}\right) e^{-x/v}, & \text{if } \alpha = 1, v, \alpha > 0, \end{cases}$$
$$\delta(x;\alpha,v) = \begin{cases} \dfrac{x \log(\alpha) e^{-x/v} \alpha^{-(1+x/v)e^{-x/v}}}{\lambda^2\left(1 - \alpha^{(1+x/v)e^{-x/v}}\right)}, & \text{if } \alpha \neq 1, v, \alpha > 0, \\ \dfrac{x}{v^2(1 + x/v)} & \text{if } \alpha = 1, v, \alpha > 0. \end{cases} \tag{5}$$

Some plots of the HRF are displayed in Figure 2. An increasing or decreasing hazard rate is frequently used to model survival and failure time data. As seen from Figure 2, the HRF of the APTME takes an upside-down bathtub shape, i.e., it is increasing and then decreasing, with a single maximum.

Definition 3. The APTME quantile function, say $Q(u) = F^{-1}(u)$, is straightforward given by means of the inverse transformation method inverting (4) as follows:

$$\frac{\ln u(\alpha - 1) + 1}{\ln \alpha} - G(x_q) = 0, \tag{6}$$

where G (.) is the CDF of the ME distribution, $x_q = Q(u)$. Solving (6) numerically, we easily simulate data from the APTME. Quantities of interest are obtained from (6) by inserting appropriate values for q. Furthermore, Bowley's skewness that depends on quartiles is as follows:

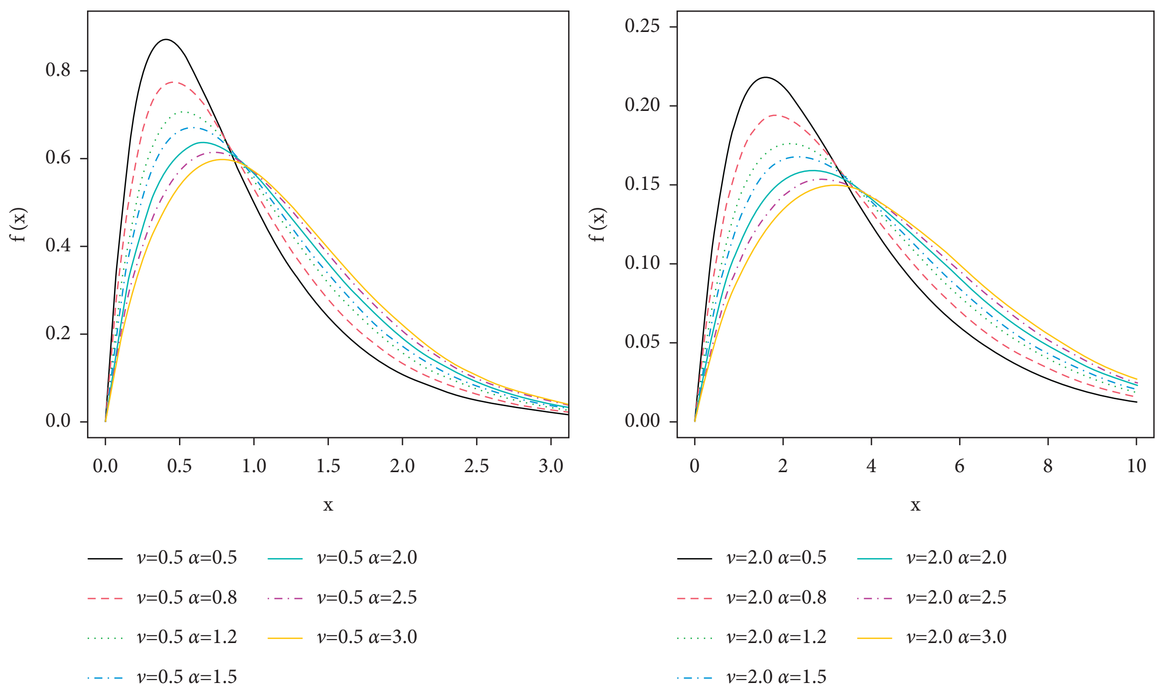

Figure 1: Plots of the APTME density function for some parameter values.

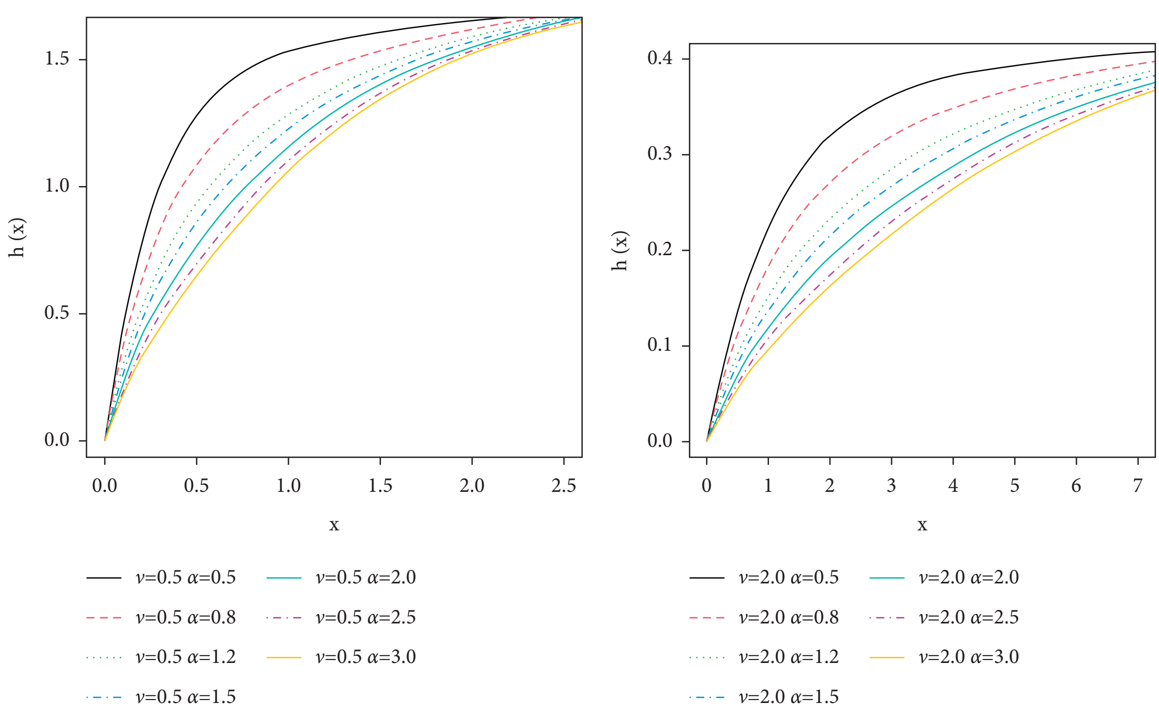

Figure 2: Plots of the APTME HRF for some parameter values.

$$SK_B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/2)}, \tag{7}$$

where Q (.) is the APITL quantile function. Moor's kurtosis is given as

$$KU_m = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}. \tag{8}$$

Skewness and kurtosis plots of the APITL model, based on quantile, are exhibited in Figure 3.

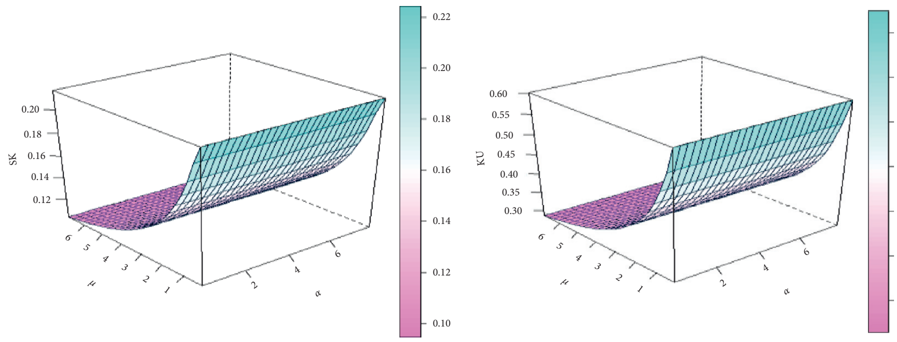

Figure 3: Skewness and kurtosis plots for APITL model.

3. Distributional Properties

In this section, we derive the expressions for some essential properties of the APTME model.

3.1. Important Expansion. Here, we obtain a simple expression for PDF (4) to aid the derivation of some structural properties for the APTME model.

The power series can be represented as follows:

$$\alpha^{w} = \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i}}{i!} w^{i}. \tag{9}$$

Hence, inserting (9) in PDF (4), then

$$f(x;\alpha,v) = \frac{x\alpha}{v^{2}(\alpha-1)} \sum_{i=0}^{\infty} \frac{(-1)^{i}(\log \alpha)^{i+1}}{i!}\left(1+\frac{x}{v}\right)^{i} e^{-(i+1)x/v}. \tag{10}$$

Using the binomial expansion in (10),

$$f(x;\alpha,v) = \frac{\alpha}{(\alpha-1)} \sum_{i=0}^{\infty}\sum_{j=0}^{i} \frac{(-1)^{i}(\log \alpha)^{i+1}}{v^{j+2} i!}\binom{i}{j} x^{j+1} e^{-(i+1)x/v}. \tag{11}$$

Hence, (11) is expressed as follows:

$$f(x;\alpha,v) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} w_{i,j} x^{j+1} e^{-(i+1)x/v} \tag{12}$$

$$\text{where } w_{i,j} = \frac{\alpha(-1)^{i}(\log \alpha)^{i+1}}{(\alpha-1)v^{j+2}(i-j)!j!}.$$

3.2. Ordinary and Incomplete Moments. If X has the PDF (12), then its kth moment can be obtained as follows:

$$\begin{aligned}\mu_k' &= \sum_{i=0}^{\infty}\sum_{j=0}^{i} w_{i,j}\int_0^{\infty} x^{k+j+1} e^{-(i+1)x/v}\,dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i} \frac{w_{i,j} v^{k+j+2}\Gamma(k+j+2)}{(i+1)^{k+j+2}},\end{aligned} \tag{13}$$

where $\Gamma(.)$ stands for the gamma function. The central moments (μ_k) of the APTME distribution can be obtained from

$$\mu_k = E(X-\mu_1')^{k} = \sum_{m=0}^{k} (-1)^{m}\binom{k}{m}(\mu_1')^{m}\mu_{k-m}'. \tag{14}$$

The skewness and kurtosis measures can be evaluated from the ordinary moments using the well-known relationships. The moment and cumulant generating functions are obtained as follows:

$$M_X(t) = \sum_{k,i=0}^{\infty}\sum_{j=0}^{i} \frac{w_{i,j} t^{k} \lambda^{k+j+2}\Gamma(k+j+2)}{k!(i+1)^{k+j+2}}, \tag{15}$$

$$\varpi_X(t) = \ln\left[\sum_{k,i=0}^{\infty}\sum_{j=0}^{i} \frac{w_{i,j} t^{k} \lambda^{k+j+2}\Gamma(k+j+2)}{k!(i+1)^{k+j+2}}\right]. \tag{16}$$

Numerical values of the $\mu_1', \mu_2', \mu_3', \mu_4'$, variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) of the APTEL distribution for certain values of parameters are obtained and recorded in Table 1.

Numerical outputs for the mean (μ_1'), variance (σ^2), skewness (S), kurtosis (K), and coefficient of variation (CoV) of the APTME model for some values of parameters are mentioned in Table 1.

Furthermore, the m^{th} upper incomplete (UI) moment, say $\eta_m(t)$ of the APTME distribution, is given by

Table 1: Results of μ_1', σ^2, S, K, and CoV for the APTME model at $v = 0.5$

$\alpha\downarrow$	$\mu_1'\downarrow$	$\sigma^2\downarrow$	$S\uparrow$	$K\uparrow$	CoV$\uparrow$
0.5	0.874	0.431	1.612	1.612	0.751
0.8	0.959	0.479	1.475	1.475	0.722
1.2	1.034	0.517	1.367	1.367	0.695
1.5	1.077	0.536	1.312	1.312	0.680
2.0	1.133	0.559	1.246	1.246	0.660
2.5	1.176	0.575	1.199	1.199	0.645
3.0	1.211	0.587	1.163	1.163	0.632

$$\eta_m(t) = \int_t^\infty x^m f(x)\mathrm{d}x = \sum_{i=0}^{\infty}\sum_{j=0}^{i} w_{i,j}\int_t^\infty x^{m+j+1}e^{-(i+1)x/v}\mathrm{d}x = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{w_{i,j}v^{m+j+2}\Gamma(m+j+2,((i+1)t/v))}{(i+1)^{m+j+2}}, \tag{17}$$

where $\Gamma(m,t) = \int_t^\infty x^{m-1}e^{-x}\mathrm{d}x$ is the UI gamma function. Similarly, the m^{th} lower incomplete (LI) moment is given by

$$\phi_m(t) = \int_0^t x^m f(x)\mathrm{d}x = \sum_{i=0}^{\infty}\sum_{j=0}^{i} w_{i,j}\int_0^t x^{m+j+1}e^{-(i+1)x/v}\mathrm{d}x = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{w_{i,j}v^{m+j+2}\gamma(m+j+2,((i+1)t/v))}{(i+1)^{m+j+2}}, \tag{18}$$

where $\gamma(m,t) = \int_0^t x^{m-1}e^{-x}\mathrm{d}x$ is the LI gamma function.

3.3. Information Measures. The Rényi entropy (RE), presented by Rényi [14], is defined by

$$\mathrm{I}_R(\varepsilon) = (1-\varepsilon)^{-1}\log\left(\int_0^\infty (f(x))^\varepsilon \mathrm{d}x\right), \quad \varepsilon \neq 1,\ \varepsilon > 0. \tag{19}$$

Substituting (4) in (19), we have

$$\mathrm{I}_R(\varepsilon) = (1-\varepsilon)^{-1}\log\left(\int_0^\infty \frac{x^\varepsilon\alpha^\varepsilon(\log(\alpha))^\varepsilon}{v^{2\varepsilon}(\alpha-1)^\varepsilon}e^{-\varepsilon x/v}\alpha^{-\varepsilon(1+x/v)e^{-x/v}}\mathrm{d}x\right). \tag{20}$$

Using expansion (9) in (20), we get

$$\mathrm{I}_R(\varepsilon) = (1-\varepsilon)^{-1} \log\left(\int_0^\infty \sum_{i=0}^{\infty}\frac{(-\varepsilon)^i x^\varepsilon\alpha^\varepsilon(\log(\alpha))^{\varepsilon+i}}{i!\varepsilon^{2\gamma}(\alpha-1)^\varepsilon}\left(1+\frac{x}{v}\right)^i e^{-(i+1)\varepsilon x/v}\mathrm{d}x\right). \tag{21}$$

Employing the binomial expansion in (21), we have

$$\mathrm{I}_R(\varepsilon) = (1-\varepsilon)^{-1}\log\left(\int_0^\infty \frac{\alpha^\varepsilon}{(\alpha-1)^\varepsilon}\sum_{i=0}^{\infty}\sum_{j=1}^{i}\frac{(-\gamma)^i(\log(\alpha))^{\varepsilon+i}}{v^{2\varepsilon+j}i!}\binom{i}{j}\int_0^\infty x^{\varepsilon+j}e^{-(i+1)\varepsilon x/v}\mathrm{d}x\right) = (1-\varepsilon)^{-1}\log\left[\sum_{i=0}^{\infty}\sum_{j=1}^{i}\frac{(-\varepsilon)^i(\log(\alpha))^{\varepsilon+i}\Gamma(\varepsilon+j+1)}{v^{\varepsilon-1}i!((i+1)\varepsilon)^{\gamma+j+1}}\binom{i}{j}\right]. \tag{22}$$

The Havrda and Charvat entropy (HCE) measure (see [15]) is defined by

$$\mathrm{HC}_R(\varepsilon) = \frac{1}{2^{1-\varepsilon}-1}\left[\left(\int_0^\infty f^\varepsilon(x)\mathrm{d}x\right)^{1/\varepsilon} - 1\right], \quad \varepsilon \neq 1,\ \varepsilon > 0. \tag{23}$$

Hence, the HCE of the APTME distribution is given by

$$\mathrm{HC}_R(\varepsilon) = \frac{1}{2^{1-\varepsilon}-1}\left[\frac{\alpha^\varepsilon}{(\alpha-1)^\varepsilon}\sum_{i=0}^{\infty}\sum_{j=1}^{i}\frac{(-\varepsilon)^i(\log(\alpha))^{\varepsilon+i}\Gamma(\varepsilon+j+1)}{v^{\varepsilon-1}i!((i+1)\varepsilon)^{\varepsilon+j+1}}\binom{i}{j} - 1\right]. \tag{24}$$

The Arimoto entropy (AE) measure (see Arimoto [16]) of the APTME is defined by

$$A_R(\varepsilon) = \frac{\varepsilon}{1-\varepsilon}\left[\left(\int_0^{\infty} f^{\varepsilon}(x)\mathrm{d}x\right)^{1/\varepsilon} - 1\right], \quad \varepsilon \neq 1,\ \varepsilon > 0. \quad (25)$$

Hence, the AE of the APTME distribution is given by

$$A_R(\varepsilon) = \frac{\varepsilon}{1-\varepsilon}\left[\left(\frac{\alpha^{\varepsilon}}{(\alpha-1)^{\varepsilon}}\sum_{i=0}^{\infty}\sum_{j=1}^{i}\frac{(-\varepsilon)^i(\log(\alpha))^{\varepsilon+i}\Gamma(\varepsilon+j+1)}{\upsilon^{\varepsilon-1}i!((i+1)\varepsilon)^{\varepsilon+j+1}}\binom{i}{j}\right)^{1/\varepsilon} - 1\right]. \quad (26)$$

The Tsallis entropy (TE) measure (see [17]) is defined by

$$T_R(\varepsilon) = \frac{1}{\varepsilon-1}\left[1 - \int_0^{\infty} f^{\varepsilon}(x)\mathrm{d}x\right], \quad \varepsilon \neq 1,\ \varepsilon > 0. \quad (27)$$

Hence, the TE of the APTME distribution is obtained as follows:

$$T_R(\varepsilon) = \frac{1}{\varepsilon-1}\left[1 - \frac{\alpha^{\varepsilon}}{(\alpha-1)^{\varepsilon}}\sum_{i=0}^{\infty}\sum_{j=1}^{i}\frac{(-\varepsilon)^i(\log(\alpha))^{\varepsilon+i}\Gamma(\varepsilon+j+1)}{\upsilon^{\varepsilon-1}i!((i+1)\varepsilon)^{\varepsilon+j+1}}\binom{i}{j}\right]. \quad (28)$$

Some of the numerical values of the RE, HCE, AE, and TE for some selected values of parameters are given in Tables 2–7.

As the value of α and ε increases, the measures values of the RE, HCE, AE, and TE also increase. For the same value of ε and ε, the measures values of the RE, HCE, AE, and TE increase with α. As the value of α and ε increases, for fixed value of ε, the measures values of the RE, HCE, AE, and TE increase.

4. Parameter Estimation

This section deals with the parameter estimation for the APTME distribution based on six frequentist estimation procedures under complete and RC samples.

4.1. M1 Estimators under the Complete Sample. Let X_1, X_2, ..., X_n be the observed values from the APTME with parameters α and υ; then, from (4), we can write the log-likelihood function (LLFu) under the complete sample as follows:

$$\ln \ell = n \ln \alpha + n \ln\left(\frac{\ln \alpha}{\alpha-1}\right) + \sum_{i=1}^{n}\ln x_i - 2\ln \upsilon - \sum_{i=1}^{n}\frac{x_i}{\upsilon} - \ln \alpha \sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\upsilon}\right)e^{-x_i/\upsilon}\right]. \quad (29)$$

The partial derivatives of $\ln \ell$ for α and υ are given by

$$\frac{\partial \ln \ell}{\partial \alpha} = \frac{n}{\alpha} + \frac{n[\alpha - 1 - \alpha\ln\alpha]}{\alpha(\alpha-1)\ln\alpha} - \frac{1}{\alpha}\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\upsilon}\right)e^{-x_i/\upsilon}\right], \quad (30)$$

and

$$\frac{\partial \ln \ell}{\partial \upsilon} = \sum_{i=1}^{n}\frac{x_i}{\upsilon^2} - \frac{2}{\upsilon} + \frac{\ln\alpha}{\upsilon^2}\left\{\sum_{i=1}^{n}e^{-x_i/\upsilon}\left[\left(1+\frac{x_i}{\upsilon}\right)+x_i\right]\right\}. \quad (31)$$

The nonlinear equations $\partial \ln \ell/\partial\alpha = 0$ and $\partial \ln \ell/\partial\upsilon = 0$ are solved numerically; we obtain the ML estimators of α and υ.

4.2. M1 Estimators under the RC Sample. Let $X_1, X_2, \ldots, X_r$ be a RC sample of size r observed from the lifetime testing experiment on n items whose lifetime has the PDF for APTME. The LLFu of the RC sample is

Table 2: Numerical values of the RE, HCE, AE, and TE for the APTME model at $\upsilon = 0.5$ and $\varepsilon = 0.5$.

$\alpha\downarrow$	RE$\downarrow$	HCE$\downarrow$	AE$\downarrow$	TE$\downarrow$
0.5	1.061	5.779	10.516	4.787
0.8	1.088	6.031	11.238	4.997
1.2	1.108	6.232	11.826	5.163
1.5	1.118	6.334	12.132	5.247
2.0	1.130	6.457	12.503	5.349
2.5	1.139	6.545	12.773	5.422
3.0	1.146	6.613	12.981	5.478

Table 3: Numerical values of the RE, HCE, AE, and TE for the APTME model at $\upsilon = 0.5$ and $\varepsilon = 1.5$.

$\alpha\downarrow$	RE$\downarrow$	HCE$\downarrow$	AE$\downarrow$	TE$\downarrow$
0.5	0.877	2.171	1.470	1.272
0.8	0.917	2.226	1.516	1.304
1.2	0.948	2.267	1.550	1.328
1.5	0.963	2.287	1.567	1.340
2.0	0.981	2.310	1.587	1.353
2.5	0.993	2.326	1.600	1.362
3.0	1.002	2.337	1.610	1.369

Table 4: Numerical values of the RE, HCE, AE, and TE for the APTME model at $\upsilon = 0.5$ and $\varepsilon = 2.5$.

$\alpha\downarrow$	RE$\downarrow$	HCE$\downarrow$	AE$\downarrow$	TE$\downarrow$
0.5	0.819	1.455	1.129	0.627
0.8	0.863	1.468	1.160	0.633
1.2	0.897	1.477	1.184	0.637
1.5	0.914	1.481	1.195	0.638
2.0	0.933	1.485	1.208	0.640
2.5	0.947	1.488	1.216	0.641
3.0	0.957	1.490	1.222	0.642

Table 5: Numerical values of the RE, HCE, AE, and TE for the APTME model at $\upsilon = 2$ and $\varepsilon = 0.5$.

$\alpha\downarrow$	RE$\downarrow$	HCE$\downarrow$	AE$\downarrow$	TE$\downarrow$
0.5	0.459	1.682	1.879	1.394
0.8	0.486	1.809	2.060	1.498
1.2	0.506	1.909	2.206	1.581
1.5	0.516	1.960	2.283	1.624
2.0	0.528	2.022	2.376	1.675
2.5	0.537	2.066	2.443	1.711
3.0	0.543	2.099	2.495	1.739

Table 6: Numerical values of the RE, HCE, AE, and TE for the APTME model at $\upsilon = 2$ and $\varepsilon = 1.5$.

$\alpha\downarrow$	RE$\downarrow$	HCE$\downarrow$	AE$\downarrow$	TE$\downarrow$
0.5	0.275	0.928	0.572	0.543
0.8	0.315	1.038	0.644	0.608
1.2	0.345	1.120	0.699	0.656
1.5	0.361	1.160	0.726	0.680
2.0	0.379	1.206	0.756	0.706
2.5	0.391	1.237	0.777	0.725
3.0	0.400	1.260	0.793	0.738

Table 7: Numerical values of the RE, HCE, AE, and TE for the APTME model at $v = 2$ and $\varepsilon = 2.5$.

$\alpha\downarrow$	RE↓	HCE↓	AE↓	TE↓
0.5	0.217	0.815	0.431	0.351
0.8	0.260	0.918	0.504	0.396
1.2	0.295	0.988	0.557	0.426
1.5	0.312	1.020	0.583	0.439
2.0	0.331	1.054	0.612	0.454
2.5	0.345	1.077	0.632	0.464
3.0	0.355	1.093	0.646	0.471

$$\ln \ell = \ln\left(\frac{n!}{(n-r)!}\right) + r \ln \alpha + r \ln\left(\frac{\ln \alpha}{\alpha - 1}\right) + \sum_{i=1}^{r} \ln x_i - 2 \ln v - \sum_{i=1}^{r} \frac{x_i}{v} - \ln \alpha \sum_{i=1}^{r}\left[\left(1 + \frac{x_i}{v}\right) e^{-x_i/v}\right] + (n-r)\ln\left(\alpha - \alpha^{1-\left(1+x_r/v\right)e^{-x_r/v}}\right) - (n-r)\ln(\alpha - 1). \tag{32}$$

The partial derivatives of $\ln \ell$ for α and v are given by

$$\frac{\partial \ln \ell}{\partial \alpha} = \frac{n}{\alpha} + \frac{n[\alpha - 1 - \alpha \ln \alpha]}{\alpha(\alpha - 1)\ln \alpha} - \frac{1}{\alpha}\sum_{i=1}^{n}\left[\left(1 + \frac{x_i}{v}\right)e^{-x_i/v}\right] - \frac{(n-r)}{(\alpha - 1)} + (n-r)\frac{\left(1 - \left(1 - \left(1 + x_r/v\right)e^{-x_r/v}\right)\alpha^{-\left(1+x_r/v\right)e^{-x_r/v}}\right)}{\alpha - \alpha^{1-\left(1+x_r/v\right)e^{-x_r/v}}}, \tag{33}$$

and

$$\frac{\partial \ln \ell}{\partial v} = \sum_{i=1}^{n}\frac{x_i}{v^2} - \frac{2}{v} + \frac{\ln \alpha}{v^2}\left\{\sum_{i=1}^{n} e^{-x_i/v}\left[\left(1 + \frac{x_i}{v}\right) + x_i\right]\right\} - (n-r)\ln \alpha \frac{\alpha^{1-\left(1+x_r/v\right)e^{-x_r/v}}\left(x_r^2/v^3\right)e^{-x_r/v}}{\alpha - \alpha^{1-\left(1+x_r/v\right)e^{x_r/v}}}. \tag{34}$$

The nonlinear equations $\partial \ln \ell/\partial\alpha = 0$ and $\partial \ln \ell/\partial v = 0$ are solved numerically, and we obtain the ML estimators of α and v.

4.3. The M2 and M3 Estimators. Consider a random sample of size n from the APTME distribution, and let $x_{(1)} < x_{(2)} < \ldots < x_{(n)}$ be the associated order random observations; then, the M2 estimators of α and v are obtained by minimizing the following:

$$\mathrm{M2}(\alpha, v) = \sum_{i=1}^{n}\left[(\alpha - 1)^{-1}\left(\alpha^{1-\left(1+x_{(i)}/v\right)e^{-x_{(i)}/v}} - 1\right) - \frac{i}{n+1}\right]^2, \tag{35}$$

with respect to α and v.

The M3 estimator of α and v is derived by minimizing the following:

$$\mathrm{M3}(\alpha, v) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[(\alpha - 1)^{-1}\left(\alpha^{1-\left(1+x_{(i)}/v\right)e^{-x_{(i)}/v}} - 1\right) - \frac{i}{n+1}\right]^2, \tag{36}$$

with respect to α and v.

4.4. The M5 and M6 Estimators. The M5 method is regarded as a class of minimum distance estimators based on the difference between the estimate of the CDF and the empirical CDF. The M5 estimator of the APTME parameters is derived by minimizing the following:

$$M5(\alpha, v) = \frac{1}{12n} + \sum_{i=1}^{n}\left[(\alpha-1)^{-1}\left(\alpha^{1-\left(1+x_{(i)}/v\right)e^{-x_{(i)}/v}}-1\right)-\frac{2i-1}{2n}\right]^2, \tag{37}$$

with respect to α and v.

The M5 estimator of the APTME distribution parameters is obtained by minimizing the following:

$$M6(\alpha, v) = -n-\frac{1}{n}+\sum_{i=1}^{n}(2i-1)\left[(\alpha-1)^{-1}\left(\alpha^{1-x_{(i)}/ve^{-x_{(i)}/v}}-1\right)-\frac{2i-1}{2n}\right]^2, \tag{38}$$

with respect to α and v.

4.5. The M4 Estimators. For estimating the population parameters of continuous distributions, the M4 method is a powerful alternative to the M1 method. Let $D_i(\alpha, v) = F(x_{(i)}|\alpha, v) - F(x_{(i-1)}|\alpha, v), i = 1, 2, \ldots, n+1$, be the uniform spacings of a random sample drawn from the APTME distribution, $F(x_{(0)}|\alpha, v) = 0, F(x_{(n+1)}|\alpha, v) = 1$ and $\sum_{i=1}^{n+1} D_i(\alpha, v) = 1$.

The M6 estimator is obtained by maximizing the geometric mean (GEOM) of the spacings:

$$GEOM(\alpha, v) = \left\{\prod_{i=1}^{n+1} D_i(\alpha, v)\right\}^{1/n+1}. \tag{39}$$

The M6 estimator of α and v can be obtained by maximizing the logarithm of the GEOM of sample spacing's (39).

5. Applications to Real Data

In this section, we present two applications to real-world datasets to demonstrate the empirical significance of the APTME distribution. The APTME model is compared to the Marshall–Olkin E (MOE), beta exponential (BE), ME, and E models. Data I shows the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli (see Bjerkedal [18]). Data II, which consists of 20 patients, refers to the lifetimes of patients who received an analgesic and their relief times (in minutes) (see Gross [19]). Table 8 shows some descriptive statistics for both datasets.

The M1 estimates (M1Es) of the parameters and their standard errors (SEs) for all models are computed for both datasets (see Tables 9 and 10). We use some model selections like Akaike information criterion (B1), Bayesian information criterion (B2), consistent AIC (B3), and Hannan–Quinn information criterion (B4). In addition, we calculate the Kolmogorov–Smirnov (B5) along with their *P*–values (B6), Anderson–Darling (B7), and Cram`er–von Mises (B8) test statistics. The values of the criteria measures and the values of the test statistics are listed in Tables 11 and 12 for both datasets.

Plots of the estimated PDFs, CDFs, SF, and probability–probability (PP) of the APTME model for the two considered data are displayed in Figures 4 and 5. According to the numerical values provided in Tables 11 and 12 along with Figures 4 and 5, the APTME model is much better than the above-mentioned extensions of the exponential model; thus, the APTME model is a good alternative to these models in both datasets.

6. Simulation Study

In this portion, a simulation analysis evaluates the output of six different estimates of the APTME parameters. We chose different parameter values as $\alpha = 0.15, 0.75, 1.5, 3$ and $v = 0.6, 1.6, 3, 5$. 10000 random sample of sizes $n = 50$, 100, 150 are generated from the APTME distribution using (6). The average values of all estimates (mean) and their corresponding mean square error (MSE) are calculated. The results of the different estimates are evaluated via their mean and MSE. Simulation outputs are obtained via the R program. Tables 13–16 list the mean and MSE for M1, M2, M3, M4, M5, and M5 estimate values.

6.1. Final Thoughts on the Simulation Results. For the same value of α and increased values of v, the MSE of both the parameters for the APTME distribution increases in most cases except when $\alpha = 0.15$:

(1) For a fixed value of α and v, the mean values of the estimated parameters tend to the true values with increased sample size. Also, the MSE of the estimated parameters decreases with increased sample size.

(2) In most cases, the M2 method was the best method for estimating the parameter referring to the mean values of the parameter estimates and MSE as in Tables 13–16.

Table 8: Descriptive statistics for both datasets.

	n	μ_1'	σ^2	S	K
Data I	72	1.768	1.055	1.371	2.225
Data II	20	1.9	0.471	1.862	4.185

Table 9: M1Es and SEs for data I.

Models	M1Es and SEs
APTME (α, v)	12.636, 0.591 (1.1979), (0.07)
MOE (α, β)	8.778, 1.379 (3.555), (0.193)
BE (a, b, β)	0.8073, 3.4612, 1.3311 (0.6961), (1.0032), (0.8551)
ME (β)	0.9252 (0.0768)

Table 10: M1Es and SEs for data II.

Models	M1Es and SEs
APTME (α, v)	2158, 0.445 (31.87), (0.043)
MOE (α, β)	54.474, 2.316 (35.582), (0.374)
BE (a, b, β)	81.6333, 0.5421, 3.5142 (120.4104), (0.3272), (1.4101)
ME (β)	0.9502 (0.1501)

Table 11: Criteria measures and goodness of fit for data I.

Models	B1	B2	B3	B4	B7	B8	B5	B6
APTME	192.67	192.38	192.84	194.48	0.76	0.13	0.088	0.63
MOE	210.36	214.92	210.53	212.16	1.18	0.17	0.10	0.43
BE	207.38	214.22	207.73	210.08	0.98	0.15	0.11	0.34
ME	210.40	212.68	210.45	211.30	1.52	0.25	0.14	0.13

Table 12: Criteria measures and goodness of fit for data II.

Models	B1	B2	B3	B4	B7	B8	B5	B6
APTME	39.37	37.98	40.08	39.76	0.55	0.08	0.143	0.81
MOE	43.51	45.51	44.22	43.90	0.80	0.14	0.18	0.55
BE	43.48	46.45	44.98	44.02	0.70	0.12	0.16	0.80
ME	54.32	55.31	54.54	54.50	2.76	0.53	0.32	0.07

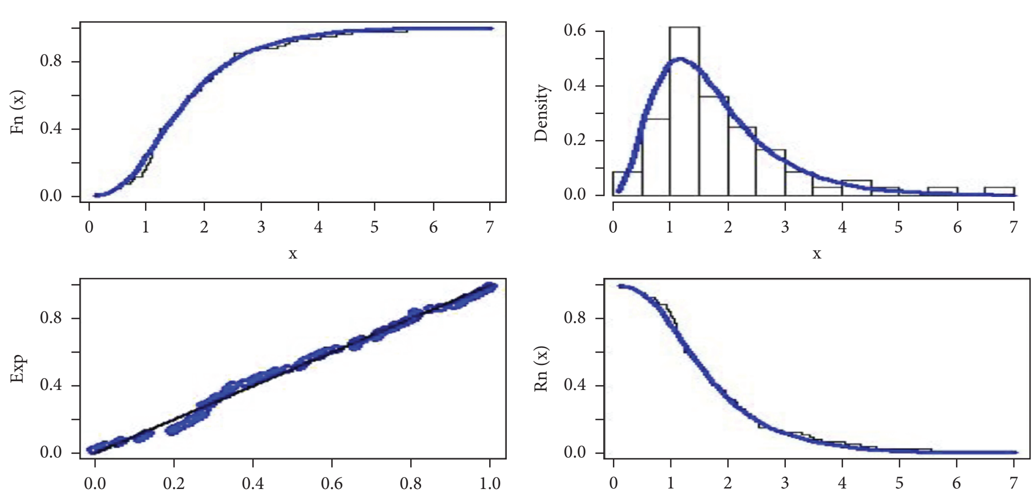

Figure 4: Estimated CDF, PDF, PP, and SF plots for Data I.

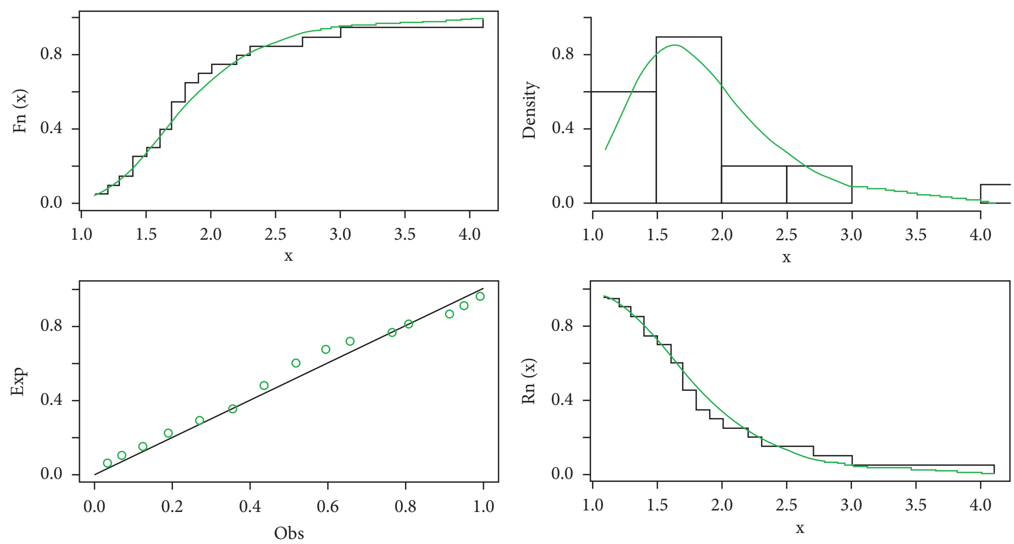

Figure 5: Estimated CDF, PDF, SF, and PP plots for Data II.

Table 13: Av and MSE of the M1, M2, M3, M4, M5, and M6 estimates for the APTME distribution when $\alpha = 0.15$.

			M1		M2		M3		M4		M5		M6	
υ	n		Av	MSE	Av	MSE	Av	MSE	Av	MSE	Av	MSE	Av	MSE
0.6	50	α	0.2032	1.1518	0.1545	0.0070	0.1721	0.0257	0.2400	0.0658	0.1483	0.0085	0.1624	0.0186
		υ	0.6749	0.0411	0.6225	0.0091	0.6315	0.0151	0.6000	0.0361	0.6325	0.0107	0.6396	0.0158
	100	α	0.1810	0.8744	0.1685	0.0096	0.1859	0.0255	0.1500	0.0385	0.1613	0.0102	0.1721	0.0197
		υ	0.6702	0.0369	0.6128	0.0070	0.6162	0.0102	0.6000	0.0284	0.6214	0.0083	0.6268	0.0116
	150	α	0.1628	0.7868	0.1480	0.0029	0.1648	0.0526	0.1500	0.0280	0.1447	0.0030	0.1478	0.0055
		υ	0.6677	0.0326	0.6154	0.0039	0.6192	0.0067	0.6000	0.0247	0.6190	0.0041	0.6248	0.0072
1.6	50	α	0.2460	1.7617	0.2169	0.0513	0.3473	1.1966	0.3900	0.1323	0.2338	0.0792	0.2708	0.1568
		υ	1.7078	0.2979	1.5710	0.0270	1.5735	0.0540	1.6000	0.1478	1.5729	0.0313	1.5813	0.0530
	100	α	0.2028	0.9746	0.1710	0.0121	0.2181	0.1018	0.1500	0.0697	0.1711	0.0153	0.1835	0.0334
		υ	1.7038	0.2696	1.5922	0.0136	1.5915	0.0327	1.6000	0.1137	1.5981	0.0141	1.6069	0.0255
	150	α	0.1970	0.7986	0.1632	0.0052	0.2215	0.0850	0.1500	0.0514	0.1614	0.0052	0.1811	0.0178
		υ	1.6767	0.2642	1.5963	0.0104	1.5739	0.0324	1.6000	0.0885	1.6012	0.0108	1.5974	0.0266
3	50	α	0.2285	0.7354	0.1970	0.0368	0.3130	0.3772	0.6000	0.0923	0.2083	0.0543	0.2751	0.2062
		υ	3.0982	0.1133	2.9667	0.0338	2.9384	0.1406	3.0000	0.2787	2.9622	0.0415	2.9406	0.1149
	100	α	0.1835	0.7308	0.1801	0.0241	0.2354	0.1616	0.1500	0.0410	0.1817	0.0298	0.1995	0.0652
		υ	3.1017	0.1103	2.9728	0.0243	2.9642	0.0821	3.0000	0.1803	2.9779	0.0285	2.9802	0.0530
	150	α	0.1836	0.7225	0.1598	0.0040	0.1988	0.0625	0.1500	0.0292	0.1626	0.0088	0.1692	0.0174
		υ	3.0558	0.1091	2.9927	0.0098	2.9676	0.0645	3.0000	0.1436	2.9928	0.0129	2.9954	0.0306
5	50	α	0.1841	0.7192	0.1814	0.0198	0.2902	0.4398	0.9000	0.0293	0.1881	0.0308	0.2410	0.1802
		υ	5.1062	0.0385	4.9737	0.0306	4.9191	0.2244	5.0000	0.2984	4.9668	0.0356	4.9367	0.1639
	100	α	0.1585	0.7023	0.1723	0.0192	0.2281	0.1801	0.1500	0.0124	0.1727	0.0233	0.1947	0.0801
		υ	5.1186	0.0382	4.9732	0.0293	4.9429	0.1505	5.0000	0.1751	4.9784	0.0330	4.9755	0.0920
	150	α	0.1572	0.6723	0.1584	0.0040	0.1854	0.0441	0.1500	0.0070	0.1574	0.0034	0.1633	0.0111
		υ	5.0703	0.0394	4.9935	0.0106	4.9611	0.0988	5.0000	0.1205	4.9962	0.0090	4.9967	0.0322

(3) All estimates perform very well and provide very small MSE and the average (Av) values of all estimates tend to the true value of the parameters.

(4) The differences between all estimates' values are very small, referring to the MSE values and the average value of the estimated parameters.

Table 14: Av and MSE of the M1, M2, M3, M4, M5, and M6 estimates for the APTME distribution when $\alpha = 0.75$.

			M1		M2		M3		M4		M5		M6	
υ	N		Av	MSE	Av	MSE	Av	MSE	Av	MSE	Av	MSE	Av	MSE
0.6	50	α	0.6048	1.8540	0.7164	0.0501	0.7582	0.2131	0.6042	0.2494	0.7657	0.0530	0.7613	0.1332
		υ	0.7266	0.0084	0.6190	0.0063	0.6340	0.0118	0.7272	0.0569	0.6084	0.0057	0.6202	0.0077
	100	α	0.5835	0.5012	0.7325	0.0781	0.7518	0.1149	0.5830	0.1882	0.7686	0.0734	0.7587	0.3026
		υ	0.7054	0.0051	0.6154	0.0041	0.6184	0.0059	0.7056	0.0398	0.6075	0.0036	0.6230	0.0073
	150	α	0.6224	0.3268	0.7120	0.0627	0.7330	0.0484	0.6228	0.1340	0.7371	0.0616	0.7362	0.0715
		υ	0.6708	0.0038	0.6202	0.0045	0.6129	0.0038	0.6707	0.0233	0.6143	0.0040	0.6170	0.0056
1.6	50	α	0.7181	3.1035	0.8197	0.3032	0.9729	0.7892	0.7175	0.4372	0.9351	0.3729	0.9374	0.5236
		υ	1.8711	0.0695	1.6799	0.0802	1.6818	0.1160	1.8721	0.2823	1.6235	0.0703	1.6663	0.1012
	100	α	0.6623	1.0816	0.8183	0.2913	0.8602	0.3958	0.6618	0.2863	0.9052	0.3539	0.8906	0.4263
		υ	1.8261	0.0416	1.6686	0.0679	1.6796	0.0926	1.8266	0.1934	1.6291	0.0615	1.6587	0.0777
	150	α	0.6839	0.5367	0.8136	0.2271	0.8315	0.2679	0.6844	0.2063	0.8630	0.2511	0.8690	0.3038
		υ	1.7557	0.0304	1.6485	0.0495	1.6481	0.0656	1.7554	0.1180	1.6237	0.0462	1.6332	0.0533
3	50	α	0.7372	3.6222	1.0184	0.7520	1.1178	1.7233	0.7368	0.3908	1.1969	1.0144	1.2293	1.5815
		υ	3.3658	0.1946	3.0547	0.2617	3.0988	0.3728	3.3668	0.5858	2.9415	0.2589	3.0392	0.3643
	100	α	0.6856	1.1972	0.8630	0.3485	0.9304	0.5941	0.6854	0.2436	0.9476	0.4131	0.9483	0.5652
		υ	3.2965	0.1291	3.0824	0.1912	3.1012	0.2700	3.2970	0.3816	3.0135	0.1796	3.0727	0.2446
	150	α	0.7041	0.5830	0.1750	0.3193	0.1906	0.5080	0.1409	0.1710	0.1882	0.3721	0.1859	0.4139
		υ	3.1985	0.0993	3.0549	0.1669	3.0324	0.1883	3.1981	0.2267	3.0030	0.1613	3.0309	0.1774
5	50	α	0.7510	4.6122	1.0280	0.7823	1.4226	3.8892	0.7509	0.2992	1.2002	1.0843	1.2619	1.7012
		υ	5.3849	0.3246	5.0404	0.5619	5.0388	0.9725	5.3860	0.8192	4.8767	0.5816	4.9881	0.8161
	100	α	0.7085	1.0436	1.0073	0.6754	1.0838	1.0552	0.7084	0.1751	1.1182	0.8640	1.0805	1.0174
		υ	5.2980	0.1674	4.9921	0.4418	5.0240	0.6186	5.2985	0.4945	4.8874	0.4669	5.0269	0.6399
	150	α	0.7233	0.4938	0.9493	0.4527	0.9540	0.4803	0.7237	0.1195	1.0272	0.5564	0.9880	0.5618
		υ	5.1933	0.1347	4.9968	0.3704	5.0271	0.4701	5.1929	0.2876	4.9158	0.3836	4.9950	0.4493

Table 15: Av and MSE of the M1, M2, M3, M4, M5, and M6 estimates for the APTME distribution when $\alpha = 1.5$.

			M1		M2		M3		M4		M5		M6	
υ	N		Av	MSE	Av	MSE	Av	MSE	Av	MSE	Av	MSE	Av	MSE
0.6	50	α	1.2106	1.5626	1.4770	0.0749	1.4934	0.6657	1.2104	0.4538	1.5198	0.0534	1.4703	0.0983
		υ	0.6855	0.0056	0.6090	0.0052	0.6167	0.0070	0.6856	0.0363	0.6029	0.0043	0.6093	0.0041
	100	α	1.2929	0.5603	1.4781	0.0858	1.4958	0.1499	1.2928	0.2580	1.5202	0.0968	1.4649	0.1484
		υ	0.6452	0.0033	0.6052	0.0020	0.6075	0.0041	0.6452	0.0136	0.6014	0.0022	0.6103	0.0030
	150	α	1.2998	0.5189	1.3689	0.0606	1.4023	0.1150	1.2998	0.2390	1.4463	0.0912	1.5122	0.1322
		υ	0.6347	0.0027	0.6147	0.0020	0.6093	0.0037	0.6347	0.0091	0.6042	0.0021	0.5971	0.0018
1.6	50	α	1.2391	5.3878	1.4102	0.4715	1.6032	1.4707	1.2389	0.7879	1.5609	0.4682	1.5528	0.7795
		υ	1.8483	0.0508	1.7009	0.0766	1.7250	0.1873	1.8486	0.2322	1.6501	0.0589	1.6886	0.0881
	100	α	1.2593	1.8709	1.4411	0.4921	1.5210	0.6896	1.2592	0.5310	1.5484	0.4984	1.5288	0.6993
		υ	1.7627	0.0316	1.6841	0.0587	1.6822	0.0862	1.7628	0.1139	1.6491	0.0473	1.6777	0.0670
	150	α	1.3455	1.1303	1.4375	0.4295	1.4900	0.4466	1.3462	0.3431	1.5192	0.4462	1.5074	0.4833
		υ	1.6928	0.0223	1.6754	0.0477	1.6581	0.0441	1.6927	0.0531	1.6507	0.0409	1.6569	0.0436
3	50	α	1.2576	6.1258	1.6137	1.2024	1.7072	1.8448	1.2537	0.6778	1.8612	1.4515	1.9327	2.6429
		υ	3.3416	0.1812	3.2020	0.3373	3.2308	0.5049	3.3421	0.4630	3.0853	0.2784	3.1394	0.3612
	100	α	1.2956	2.7107	1.4794	0.6361	1.6817	1.3977	1.2956	0.4571	1.5946	0.6630	1.6715	1.1470
		υ	3.2282	0.1170	3.1514	0.1980	3.1358	0.2519	3.2284	0.2324	3.0877	0.1676	3.1190	0.2359
	150	α	1.3687	1.4668	1.5273	0.5614	1.6485	0.8587	1.3694	0.2979	1.6223	0.5959	1.6283	0.7126
		υ	3.1356	0.0810	3.1104	0.1582	3.0929	0.2283	3.1354	0.1172	3.0608	0.1362	3.0747	0.1559
5	50	α	1.3466	10.4321	1.6601	1.2018	2.0295	3.2231	1.3429	0.6160	1.9123	1.4978	1.9950	2.5070
		υ	5.3950	0.3421	5.2437	0.4675	5.2415	0.7448	5.4000	0.5036	5.0554	0.3938	5.1491	0.5533
	100	α	1.4195	2.8337	1.7902	1.1344	1.9804	1.9234	1.4193	0.3723	1.9565	1.3381	1.9935	1.8771
		υ	5.2716	0.1928	5.1289	0.2960	5.0966	0.3620	5.2722	0.2560	5.0160	0.2649	5.0699	0.3345
	150	α	1.3193	1.1638	1.4628	0.6838	1.5486	1.0308	1.3209	0.2744	1.5698	0.7528	1.5823	0.9621
		υ	5.2172	0.1413	5.2296	0.2888	5.1910	0.2903	5.2165	0.1693	5.1431	0.2588	5.1646	0.2781

Table 16: Av and MSE of the M1, M2, M3, M4, M5, and M6 estimates for the APTME distribution when $\alpha = 3$.

			M1		M2		M3		M4		M5		M6	
υ	N		Av	MSE	Av	MSE	Av	MSE	Av	MSE	Av	MSE	Av	MSE
0.6	50	α	2.9099	0.9809	2.9957	0.0311	3.0399	0.3817	2.9090	0.1044	3.0270	0.0318	3.0015	0.0646
		υ	0.6126	0.0049	0.6026	0.0027	0.6025	0.0024	0.6129	0.0029	0.6003	0.0027	0.6022	0.0026
	100	α	2.9605	0.8900	2.9993	0.0486	3.0609	0.1085	2.9607	0.0433	3.0339	0.0719	3.0689	0.2824
		υ	0.6123	0.0032	0.6079	0.0017	0.6052	0.0016	0.6124	0.0016	0.6062	0.0016	0.6061	0.0016
	150	α	2.9245	0.5139	2.9476	0.0519	2.9769	0.0142	2.9251	0.0392	2.9444	0.0653	2.9716	0.0378
		υ	0.6048	0.0017	0.5998	0.0015	0.5984	0.0013	0.6049	0.0012	0.5998	0.0016	0.5988	0.0013
1.6	50	α	2.5586	1.5278	2.8392	0.4697	3.1926	2.7864	2.5515	0.8890	3.0707	0.4422	3.0195	0.7371
		υ	1.7564	0.0502	1.6672	0.0513	1.7147	0.1316	1.7580	0.1255	1.6296	0.0404	1.6576	0.0593
	100	α	2.7558	0.9622	3.0447	0.4276	3.1584	0.6448	2.7564	0.5074	3.2199	0.4828	3.1518	0.7072
		υ	1.6958	0.0307	1.6453	0.0262	1.6393	0.0278	1.6959	0.0514	1.6242	0.0242	1.6500	0.0389
	150	α	2.6262	0.8403	2.6344	0.6055	2.8201	0.9647	2.6291	0.3904	2.8021	0.6490	2.8228	0.8308
		υ	1.6711	0.0235	1.6856	0.0418	1.6750	0.0876	1.6709	0.0330	1.6607	0.0341	1.6655	0.0380
3	50	α	2.6165	1.9217	3.0350	1.3823	3.2313	2.2369	2.6103	0.7960	3.4402	1.5505	3.6285	2.8923
		υ	3.2280	0.1847	3.1872	0.2892	3.2046	0.3810	3.2306	0.2727	3.0787	0.2254	3.1215	0.3118
	100	α	2.7940	1.3673	3.0318	0.7471	3.5966	2.9520	2.7946	0.4538	3.2363	0.7678	3.5113	1.6164
		υ	3.1493	0.1180	3.1321	0.1478	3.1060	0.2745	3.1496	0.1233	3.0770	0.1198	3.0751	0.1621
	150	α	2.6549	0.8893	2.7707	0.7512	2.9463	1.0842	2.6577	0.3530	2.9214	0.7604	2.9101	0.8797
		υ	3.1122	0.0861	3.1347	0.1335	3.1094	0.1336	3.1120	0.0816	3.0921	0.1139	3.1066	0.1274
5	50	α	2.6974	2.1614	3.1053	1.3390	3.9134	4.7655	2.6923	0.6588	3.5058	1.5608	3.7362	3.2498
		υ	5.2727	0.5328	5.2375	0.5959	5.2421	1.0101	5.2762	0.4448	5.0753	0.4855	5.1782	0.8214
	100	α	2.8530	1.5956	3.4017	1.4936	3.6933	2.2939	2.8537	0.3783	3.6983	1.7368	3.7865	2.4773
		υ	5.1918	0.3348	5.1547	0.4236	5.1240	0.5141	5.1923	0.2177	5.0514	0.3747	5.0830	0.4428
	150	α	2.7016	1.3973	2.9477	1.2908	3.1818	1.8412	2.7042	0.2956	3.1482	1.3983	3.1963	1.7347
		υ	5.1445	0.2385	5.2257	0.4438	5.1721	0.4776	5.1442	0.1509	5.1449	0.3928	5.1506	0.4010

7. Conclusions

We developed a generalized version of the moment exponential as a member of the alpha power-transformed family. The alpha power-converted moment exponential distribution has two parameters. We investigated some of its most significant statistical features. The model parameters are estimated using six different estimation methods. The estimation methods given are M1, M2, M3, M4, M5, and M6. A simulation research was carried out in order to assess the accuracy and behavior of several estimators. According to the numerical results, even with small sample numbers, the estimates offer desirable qualities such as minimal biases and variances. Finally, we demonstrate empirically that the APTME distribution better matches a real dataset than alternative models.

Data Availability

If you would like to get the numerical dataset used to conduct the study reported in the publication, please contact the appropriate author.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

This work was funded by the Researchers Supporting Project (RSP2022R464), King Saud University, Riyadh, Saudi Arabia.

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