Hindawi
Scientific Programming
Volume 2023, Article ID 1606642, 15 pages
https://doi.org/10.1155/2023/1606642

Research Article

A Distributionally Robust Fuzzy Optimization Method for Single-Period Inventory Management Problems

Zhaozhuang Guo, Yuefang Sun, Shengnan Tian, and Zikun Li

School of Liberal Arts and Sciences, North China Institute of Aerospace Engineering, Langfang 065000, Hebei, China

Correspondence should be addressed to Yuefang Sun; syuefang@163.com

Received 12 October 2022; Revised 28 October 2022; Accepted 25 January 2023; Published 18 February 2023

Academic Editor: Hongguang Ma

This paper investigates single-period inventory management problems with uncertain market demand, where the exact possibility distribution of demand is unavailable. In this condition, it is important to order a reliable quantity which can immunize against distribution uncertainty. To model this type of single-period inventory management problem, this paper characterizes the uncertain demand by generalized interval-valued possibility distributions. We present a novel concept about an uncertain distribution set to describe distribution perturbation characterization. First, we introduce a lambda selection of the interval-valued fuzzy variable, and the uncertain distribution set is a collection of all generalized possibility distributions of lambda selection variables. According to the uncertain distribution set, a new distributionally robust fuzzy optimization method is developed for single-period inventory management problems. Under mild assumptions, the robust counterpart of the proposed fuzzy single-period inventory management model is formulated, which is an optimization program with certain linear objectives and infinitely many integral constraints. We discuss the computational issue of integral constraints and reformulate equivalently the robust counterpart as three deterministic inventory submodels under generalized interval-valued trapezoidal possibility distributions. According to the characteristics of three submodels, a domain decomposition method is designed to find the robust optimal solution that can immunize against uncertainty in our single-period inventory management problem. Finally, some computational results demonstrate the efficiency of the proposed distributionally robust fuzzy optimization method.

1. Introduction

The single-period inventory management problem (newsvendor problem) is a classical problem in the literature on inventory management [1]. The essential characteristic of the problem is that only one period is relevant and that there is no chance to place any subsequent orders during the period. In our real life, many products have such characteristics like seasonal products, sports goods, and fashion items, so the single-period inventory management problem provides a very useful framework to make decision about the optimal order quantity. The single-period inventory management problem has been studied since the 18th century, and it has been widely used to analyze supply chains with perishable and fashionable products. Khouja [2] reviewed the extensive contributions to the single-period inventory problem. Qin et al. [1] reviewed additional contributions and extended the prior reviews by considering several specific extensions. Most of the extensions have been made in a probabilistic framework and focused on the case that market demand was assumed to be random and characterized by random variables. However, many products today have shorter and shorter life cycles due to rapid technology upgrades. That is to say, growing innovation rates and shorter product life cycles make the market demand extremely variable. In this case, decision makers do have not enough historical data to determine the exact probability distribution of demand. Under incomplete information about the probability distribution of demand, some interesting research studies have been documented in the literature [3–5]. The stochastic approach seems to be less conservative than the worst-case-oriented robust optimization approach. This is so if indeed the uncertain data are of a stochastic nature and if decision makers are able to provide the associated probability distribution.

However, the above two if's are too restrictive in some practical inventory management problems, in which the probability distribution of the market demand is unavailable due to the lack of the related historical data or information. At the same time, the demand in these situations can be approximately estimated based on the experts' experiences or subjective judgments. In this respect, there are some early applications of fuzzy set theory to inventory management problems in the literature [6–8]. Since then, there has been growing interest in the study of the single-period inventory model under fuzzy uncertainty [9–12]. In numerous uncertain inventory management problems, optimal order quantities often depend heavily on the distributions of uncertain market demands. The work mentioned above studied inventory management problems under the assumption that the exact possibility distributions of uncertain demands were available, which motivates us to address uncertain inventory management problems in a more advanced setting.

In this paper, we develop a new distributionally robust optimization method for the single-period inventory management problem, in which the uncertain demand is characterized by uncertain distribution sets. Compared with the existing literature, the novelties of this study include the following several aspects:

(i) When the distribution of uncertain market demand is only partially known, this paper presents a new method to model the uncertain demand by uncertain distribution sets.

(ii) Based on uncertain distribution sets, this paper investigates a new robust modeling framework for a single-period inventory management problem that incorporates the robust credibilistic optimization method and the risk-neutral criterion. Under mild assumptions, the robust counterpart of the original inventory management optimization model is formulated.

(iii) Under generalized interval-valued trapezoidal possibility distributions of uncertain market demand, theoretical analysis demonstrates that the robust counterpart of original inventory problems is equivalent to three submodels with different subregions. As a result, we design a domain decomposition method to find the robust optimal solution of our single-period inventory management problem.

(iv) The computational results demonstrate that the proposed distributionally robust optimization method can help the retailer order a reliable quantity to immunize against the uncertain demand in our single-period inventory management problem.

The rest of this paper is organized as follows: Section 2 gives an overview of related works. Section 3 first introduces generalized interval-valued trapezoidal fuzzy variables. Then, a new concept of an uncertain distribution set is proposed for a given parametric interval-valued possibility distribution. Section 4 develops a robust modeling framework for single-period inventory management problems, in which the possibility distributions of lambda selections vary in a given uncertain distribution set. Under mild assumptions, the robust counterpart of the original uncertain inventory problem is also formulated. Section 5 discusses the computational issue of infinitely many integral constraints and turns the robust counterpart problem into its equivalent parametric programming submodels. Section 6 designs a domain decomposition method to solve the obtained three deterministic parametric programming submodels. Section 7 performs some numerical experiments to demonstrate our new robust modeling idea. Section 8 gives our conclusions.

2. Literature Review

The single-period inventory management problem has been studied since the 18th century in the economic literature. Starting from the 1950s, the single-period inventory management problem has been extended to model a great variety of real-life problems. In the following, we classify the literature of single-period inventory management problems into three streams. The first stream is the random market demand with exact probability distributions. The second stream is the incomplete information about probability distributions of demand, and the third stream is the study of the single-period inventory model under fuzzy uncertainty.

2.1. Random Market Demand. The classical single-period inventory management problem was studied under random market demand, where the market demand was treated as an exogenous parameter. Based on the fact that the retailer may adjust the price in order to reduce or increase market demand, Lau and Lau [13] studied the extension of the classical single-period inventory management problem with a stochastic price-demand. For the simplest price-demand relationship, analytical solutions to the extended single-period inventory problem were obtained. For other cases, they developed numerical solution procedures. Based on the fact that market demand could be influenced by many marketing activities, such as advertising and sales calls, Kraiselburd et al. [14] studied the effect of marketing efforts on market demand, where the mean market demand was modeled as a nondecreasing function of the marketing effort. In addition, some marketing researchers believe that quantity stock has a positive impact on market demand. Balakrishnan et al. [15] generalized the single-period inventory problem to incorporate the stochastic and initial stock-level-dependent demand. In order to capture the effect of the stock level on demand, they considered a general random demand model via an inverse fractile function. Khouja [2] reviewed the extensive contributions to the single-period inventory management problem, such as different news-vendor pricing policies, discounting structures, and different states of information about demand. Qin et al. [1] reviewed additional contributions and extended the prior reviews by considering several specific extensions such as analyzing the impact of the price, marketing effort, and stocking quantity on market demand.

2.2. Incomplete Information about Probability Distribution of Demand. Some researchers considered the single-period inventory management problem where the market demand did not satisfy the assumption of having a specific probability distribution of demand. In this case, some interesting research studies have been documented in the literature. Scarf [16] addressed the single-period inventory management problem where only the mean and variance of market demand was known. He modeled the problem by maximizing the expected profit against the worst possible distribution of the demand. Gallego and Moon [17] presented a simpler proof of Scarf's ordering rule and provided four extensions to the distribution-free single-period inventory management problem. Hill [18] applied a Bayesian methodology to the single-period inventory management problem where random demands followed known distributions with unknown but fixed parameters. Ridder et al. [19] studied the effects of demand variability on the profit. That is, higher demand variability resulted in larger variance and smaller profit. When the distribution of demand had known support, mean, and variance, Kamburowski [3] studied the single-period inventory management problem and derived the closed form formulas for the worst-case and best-case order quantities. Qin and Shang [4] and Wang et al. [5] applied a robust optimization approach to stochastic inventory management problems. So far, the robust optimization approach [20] has become an important research direction.

2.3. Fuzzy Market Demand. In fuzzy decision systems, several researchers have studied the inventory management problem based on fuzzy set theory. Petrović et al. [8] proposed two fuzzy models for single-period inventory management problems and discussed the effects of changing the membership function shapes of fuzzy inventory data on the optimal order quantity. Ishii and Konno [21] considered the stochastic demand with fuzzy shortage cost in the single-period inventory management problem. Kao and Hsu [22] proposed a single-period inventory model to find the optimal order quantity of the newsboy, in which the demand was described by subjectively determined membership functions. Li et al. [23] considered a single-period fuzzy inventory model, in which the optimal order quantity was achieved through fuzzy ordering of fuzzy numbers with respect to their total integral values. Dutta et al. [24] formulated a single-period inventory model with reordering opportunities under fuzzy demand. Chen and Ho [11] considered the optimal inventory policy for the single-period inventory problem with quantity discount. Xu and Zhai [25] expressed the demand as an L-R type fuzzy number. Under perfect coordination and in contrast with the non-coordination case, they investigated the optimization of the vertically integrated two-stage supply chain. Yu and Jin [9] developed the return policy in a supply chain with symmetric channel information and asymmetric channel information, respectively. Yu et al. [10] proposed a single-period inventory model with fuzzy price-dependent demand and discussed the conditions to determine the optimal pricing and inventory decisions jointly so that the expected profit could be maximized. Sang [12] considered a supply chain model with two competitive manufacturers and a common retailer, where the parameters of demand function and manufacturing costs were treated as fuzzy variables. For a single-vendor multiretailer supply chain, Sadeghi et al. [26] developed a constrained vendor-managed inventory model with trapezoidal fuzzy demand. Based on credibility measures, Guo [27] proposed two optimization models where uncertain demands were characterized by discrete and continuous possibility distributions, respectively. The analytical expressions of the optimal order quantity were derived in the above cases. Under variable possibility distributions of uncertain demand, Guo et al. [28] studied a multiproduct single-period inventory management problem.

Most of the existing literature studied inventory management problems under the assumption that the exact membership function or possibility distribution of fuzzy variables was available. Based on a given uncertainty distribution set, Guo and Liu [29] studied a distributionally robust optimization method for the single-period inventory management problem, which motivates us to study inventory management problems from a new perspective.

3. Uncertain Distribution Set

Fuzzy possibility theory has been introduced in the literature [30, 31]. For a detailed overview of the relationship between interval-valued fuzzy variables and interval type-2 fuzzy variables, we refer to Pagola et al. [32] and Bustince et al. [33].

In order to characterize the perturbation of the possibility distribution in some practical inventory management problems, we first introduce the representation method for the interval-valued distribution of fuzzy variables [34]. If the secondary possibility distribution of a type-2 fuzzy variable ξ is the following subinterval,

$$\left[\frac{x-r_1}{r_2-r_1}-\theta_l\frac{x-r_1}{r_2-r_1},\frac{x-r_1}{r_2-r_1}+\theta_r\frac{r_2-x}{r_2-r_1}\right], \tag{1}$$

of [0,1] for $x\in[r_1,r_2]$, the subinterval $[1-\theta_l,1]$ of [0,1] for $x\in[r_2,r_3]$, and the following subinterval,

$$\left[\frac{r_4-x}{r_4-r_3}-\theta_l\frac{r_4-x}{r_4-r_3},\frac{r_4-x}{r_4-r_3}+\theta_r\frac{x-r_3}{r_4-r_3}\right], \tag{2}$$

of [0,1] for $x\in[r_3,r_4]$, then the fuzzy variable ξ is called a generalized parametric interval-valued trapezoidal fuzzy variable, where $\theta_l,\theta_r\in[0,1]$ are two parameters characterizing the degree of uncertainty that ξ takes on the value x. For simplicity, we denote the generalized parametric interval-valued trapezoidal fuzzy variable ξ by $\mathrm{Tra}(r_1,r_2,r_3,r_4;\theta)$ with $\theta=(\theta_l,\theta_r)$. The secondary possibility distribution is called the nominal possibility distribution of ξ when $\theta_l = \theta_r = 0$, and the fuzzy variable characterized by the nominal possibility distribution is denoted by ξ^n.

In order to address the fuzzy variable with the parametric interval-valued possibility distribution, we next introduce its selection variable, which is different from the existing literature [35, 36]. Suppose ξ is a generalized parametric interval-valued fuzzy variable with the secondary possibility distribution $\tilde{\mu}_\xi(x) = [\mu_{\xi^L}(x;\theta_l), \mu_{\xi^U}(x;\theta_r)]$. The fuzzy variable ξ^L is called the lower selection variable of ξ if ξ^L has the generalized parametric possibility distribution $\mu_{\xi^L}(x;\theta_l)$. The fuzzy variable ξ^U is called the upper selection variable of ξ if ξ^U has the generalized parametric possibility distribution $\mu_{\xi^U}(x;\theta_r)$. For any $\lambda \in [0,1]$, a fuzzy variable ξ^λ is called a lambda selection of ξ if ξ^λ has the following generalized parametric possibility distribution:

$$\mu_{\xi^\lambda}(x;\theta) = \lambda\mu_{\xi^U}(x;\theta_r) + (1-\lambda)\mu_{\xi^L}(x;\theta_l). \tag{3}$$

Theorem 1. *Let $\xi = Tra(r_1, r_2, r_3, r_4; \theta)$ be a generalized parametric interval-valued trapezoidal fuzzy variable and ξ^λ be its lambda selection. Then, for any real r, the credibility of $\{\xi^\lambda \le r\}$ is computed by*

$$Cr\{\xi^\lambda \le r\} = \begin{cases} 0, r \in (-\infty, r_1), \\ \frac{1}{2}\lambda\theta_r + \frac{(r-r_1)[1-(1-\lambda)\theta_l - \lambda\theta_r]}{2(r_2-r_1)}, r \in [r_1, r_2), \\ \frac{1}{2} - \frac{(1-\lambda)\theta_l}{2}, r \in [r_2, r_3), \\ 1-(1-\lambda)\theta_l, r \in [r_3, r_4), \\ -\frac{1}{2}\left\{\lambda\theta_r + \frac{(r_4-r)[1-(1-\lambda)\theta_l - \lambda\theta_r]}{r_4-r_3}\right\}, \\ 1-(1-\lambda)\theta_l, r \in [r_4, +\infty). \end{cases} \tag{4}$$

Proof. According to the definition of the generalized parametric interval-valued trapezoidal fuzzy variable, if $\xi = \mathrm{Tra}(r_1, r_2, r_3, r_4; \theta)$, then the generalized possibility distributions of the lower selection variable ξ^L and the upper selection variable ξ^U can be determined [34].

By the definition of ξ^λ, the generalized possibility distribution of ξ^λ is

$$\mu_{\xi^\lambda}(r;\theta) = \begin{cases} \lambda\theta_r + \frac{(r-r_1)[1-(1-\lambda)\theta_l - \lambda\theta_r]}{r_2-r_1}, & r \in [r_1, r_2], \\ 1-(1-\lambda)\theta_l, & r \in [r_2, r_3], \\ \lambda\theta_r + \frac{(r_4-r)[1-(1-\lambda)\theta_l - \lambda\theta_r]}{r_4-r_3}, & r \in [r_3, r_4]. \end{cases} \tag{5}$$

According to the definition of the credibility measure [37], the credibility of $\{\xi^\lambda \le r\}$ is computed by

$$\begin{aligned} \mathrm{Cr}\{\xi^\lambda \le r\} &= \frac{1}{2}\left(\sup_{t\in\Re}\mu_{\xi^\lambda}(t;\theta) + \sup_{t\le r}\mu_{\xi^\lambda}(t;\theta) - \sup_{t>r}\mu_{\xi^\lambda}(t;\theta)\right) \\ &= \begin{cases} 0, r \in (-\infty, r_1), \\ \frac{1}{2}\lambda\theta_r + \frac{(r-r_1)[1-(1-\lambda)\theta_l - \lambda\theta_r]}{2(r_2-r_1)}, r \in [r_1, r_2), \\ \frac{1}{2} - \frac{(1-\lambda)\theta_l}{2}, r \in [r_2, r_3), \\ 1-(1-\lambda)\theta_l, r \in [r_3, r_4), \\ -\frac{1}{2}\left\{\lambda\theta_r + \frac{(r_4-r)[1-(1-\lambda)\theta_l - \lambda\theta_r]}{r_4-r_3}\right\}, \\ 1-(1-\lambda)\theta_l, r \in [r_4 + \infty). \end{cases} \end{aligned} \tag{6}$$

The proof of the theorem is complete.

For a generalized parametric interval-valued trapezoidal fuzzy variable $\xi = \mathrm{Tra}(r_1, r_2, r_3, r_4; \theta)$, the generalized credibility $\mathrm{Cr}\{\xi^\lambda \le r\}$ with respect to r is plotted in Figure 1.

Given an interval-valued fuzzy variable ξ, we next define the uncertain distribution set to describe the distribution perturbation of the interval-valued possibility distribution $[\mu_{\xi^L}(x;\theta_l), \mu_{\xi^U}(x;\theta_r)]$.

Definition 1. Assume that ξ is a generalized parametric interval-valued fuzzy variable with the secondary possibility distribution $\tilde{\mu}_\xi(\mathrm{x}) = [\mu_{\xi^L}(\mathrm{x};\theta_l), \mu_{\xi^U}(\mathrm{x};\theta_r)]$. For any $\lambda \in [0,1]$, the generalized possibility distribution of the lambda selection ξ^λ is denoted as $\mu_{\xi^\lambda}(\mathrm{x};\theta)$. Then, the uncertain distribution set $\mathscr{U}$ of ξ is defined as a collection of all generalized possibility distributions $\mu_{\xi^\lambda}(\mathrm{x};\theta)$ of lambda selections ξ^λ, i.e.,

$$\begin{aligned} \mathscr{U} = \Big\{\mu_{\xi^\lambda}(x;\theta) \,\Big|\, \mu_{\xi^\lambda}(x;\theta) &= \lambda\mu_{\xi^U}(x;\theta_r) \\ &+ (1-\lambda)\mu_{\xi^L}(x;\theta_l), \lambda \in [0,1]\Big\}. \end{aligned} \tag{7}$$

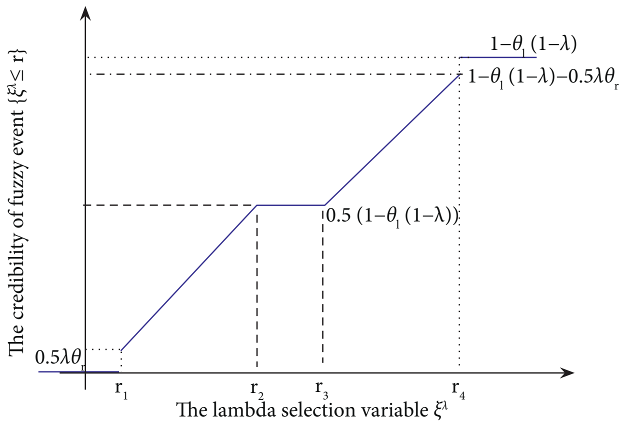

FIGURE 1: The credibility of the fuzzy event $\left\{\xi^\lambda \le r\right\}$.

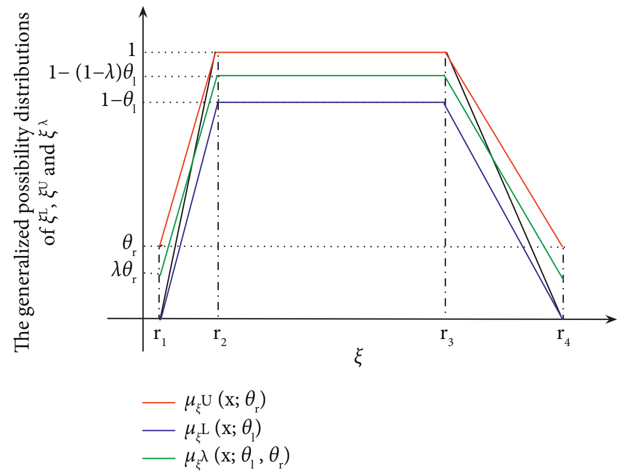

FIGURE 2: The generalized possibility distributions of ξ^L, ξ^U, and ξ^λ.

For a generalized parametric interval-valued trapezoidal fuzzy variable $\xi = \mathrm{Tra}(r_1, r_2, r_3, r_4; \theta)$, the generalized possibility distributions of ξ^L, ξ^U, and ξ^λ are plotted in Figure 2.

In the next section, we develop a robust modeling method for single-period inventory management problems under the given uncertain distribution set in (7).

4. Formulation of Robust Inventory Models

4.1. Problem Description and Notations. Consider a single-period inventory management problem consisting of one supplier and one retailer, where the supplier sells products to the retailer and the retailer faces the uncertain market demand. Suppose that products are sold only in one period and that the retailer has no chance to place a second order. After the supplier sets a price for his products, the retailer makes the decision to maximize his profit according to holding costs, goodwill costs for shortages, and his estimate on the demand. In practice, the difficulty faced by the retailer is to forecast the demand during the decision-making process. Because the uncertain market demand is nonnegative and bounded variable in the light of the actual conditions, the trapezoidal fuzzy demand variable is a common variable to characterize this demand. In order to build our distributionally robust optimization model, we next give some necessary notations and model parameters.

Fixed parameters are as follows.

c_r is the retailer's treatment cost of a unit product.

c is the total cost of a unit product and $c = c_r + w_s$.

s is the salvage value of a unit residual product.

g is the retailer's goodwill cost for unit unmet demand.

p is the retailer's sales price of a unit product.

w_s is the wholesale price of a unit product charged by suppliers.

The decision variable is as follows.

Q is the retailer's order quantity.

Uncertain parameters are as follows.

ξ is the uncertain market demand $\xi = \mathrm{Tra}(r_1, r_2, r_3, r_4; \theta)$.

ξ^λ is the lambda selection of the uncertain demand ξ.

In the following discussion, we assume $p > c > s$ to avoid trivial problems. Consider a lambda selection ξ^λ of a demand ξ, where $\mu_{\xi^\lambda}(x;\theta) \in \mathscr{U}$ is the generalized parametric possibility distribution of ξ^λ. When a retailer determines to order Q units of products, the sales volume, holding, and shortage quantity for the retailer are denoted as $\min(\xi^\lambda, Q)$, $\max(Q - \xi^\lambda, 0)$, and $\max(\xi^\lambda - Q, 0)$, respectively. Under this condition, the total profit for the retailer can be represented as

$$\pi\left(Q, \xi^\lambda\right) = p \min\left(\xi^\lambda, Q\right) + s \max\left(Q - \xi^\lambda, 0\right) - g \max\left(\xi^\lambda - Q, 0\right) - (w_s + c_r)Q. \tag{8}$$

The objective of a retailer is to maximize the total profit. When the demand ξ is known exactly in advance, the best decision is to order exactly quantity ξ. In the next subsection, we build a novel robust inventory optimization model, where the uncertain demand is characterized by a given uncertain distribution set. For a detailed overview of the robust optimization framework, we refer to Ben-Tal et al. [38], Bertsimas et al. [39], and Gorissen et al. [40].

4.2. Development of the Single-Period Inventory Model in the Fuzzy Decision System. In the fuzzy decision system, the single-period inventory problem has been studied in [8, 22, 23], in which the basic model is built as

$$\begin{aligned} &\max_{Q} \; I[\pi(Q,\xi)] \\ &\text{s.t. } r_1 \le Q \le r_4, \end{aligned} \tag{9}$$

where ξ is a fuzzy demand and I is some defuzzification method for the associated fuzzy profit $\pi(Q,\xi)$. The work mentioned above studied the single inventory management problem under the assumption that the exact membership function or possibility distribution of uncertain demand ξ was available.

In the present paper, we study the single-period inventory management problem from a new perspective. When the information of the uncertain demand ξ is partially known, we characterize it by the parametric interval-valued possibility distribution $\tilde{\mu}_\xi(x) = [\mu_{\xi^L}(x;\theta_l), \mu_{\xi^U}(x;\theta_r)]$. After that, we introduce the lambda selection ξ^{λ} of the interval-valued demand and define the uncertain distribution set as in (7).

Given the generalized possibility distribution $\mu_{\xi^{\lambda}}(x;\theta)$ of the lambda selection ξ^{λ}, the total profit $\pi(Q,\xi^{\lambda})$ for the retailer is represented by (8). Based on L–S integral [41], the mean profit of $\pi(Q,\xi^{\lambda})$ is computed by

$$\int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\left\{\xi^{\lambda} \le r\right\}, \quad (10)$$

where the measure is induced by the nondecreasing function $\mathrm{Cr}\left\{\xi^{\lambda} \le r\right\}$. It is obvious that the mean profit (10) depends on the generalized parametric possibility distribution $\mu_{\xi^{\lambda}}(x;\theta)$ of the lambda selection ξ^{λ} of the uncertain demand ξ.

Finally, we develop the following robust inventory optimization model for a single-period inventory management problem:

$$\left\{\max_{Q}\left\{\int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\left\{\xi^{\lambda} \le r\right\}:\ r_1 \le Q \le r_4\right\}\right\}_{\mu_{\xi^{\lambda}}(x;\theta)\in\mathcal{U}}, \quad (11)$$

which is a collection of optimization models.

$$\begin{aligned} &\max_{Q}\ \int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\left\{\xi^{\lambda} \le r\right\},\\ &\text{s.t. } r_1 \le Q \le r_4, \end{aligned} \quad (12)$$

where possibility distribution $\mu_{\xi^{\lambda}}(x;\theta)$ varies in $\mathcal{U}$.

It is evident that our robust inventory optimization model (11) is totally different from model (9). In addition, the solution concept to model (11) is different from that to model (9), which is addressed in details in the next subsection.

4.3. Robust Counterpart of the Proposed Inventory Optimization Model. In this subsection, we deal with the robust counterpart of distributionally robust single-period inventory problem (11). In contrast to fuzzy inventory optimization model (9), where the uncertain demand has a fixed membership function or possibility distribution, a collection of a single-period inventory optimization problem like (12) is not associated by itself with the concepts of feasible solutions, optimal solutions and optimal values. The answer to the question rests on some implicit assumptions on the underlying decision-making environment. In this paper, we focus on the environment with the following assumptions.

A1. The retailer's order quantity Q in problem (11) represents "here and now" decision; it should be assigned a specific numerical value as a result of solving the problem before the actual demand data ξ reveal themselves.

A2. The decision maker is fully responsible for the consequence of the decision to be made when and only when the parametric possibility distribution $\mu_{\xi^{\lambda}}(x;\theta)$ varies in the uncertain distribution set $\mathcal{U}$ specified by (7).

A3. The constraints in model (11) cannot be violated when the parametric possibility distribution $\mu_{\xi^{\lambda}}(x;\theta)$ varies in $\mathcal{U}$.

The three assumptions lead to the robust feasible solution to optimization problem (11). Considering all the generalized possibility distributions in $\mathcal{U}$, the objective function will be constructed with respect to the worst-case mean profit. Therefore, this leads to solving the distributionally robust optimization model. By **A1**, the meaningful feasible solution to the uncertain single-period inventory management problem should be a fixed variable; Based on the spirit of the worse-case-oriented assumptions, **A2** and **A3**, the robust mean value of the objective in (11) at a candidate order quantity Q is the smallest mean value $\int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\left\{\xi^{\lambda} \le r\right\}$ over all parametric possibility distribution $\mu_{\xi^{\lambda}}(x;\theta)$ from the uncertain distribution set $\mathcal{U}$, i.e.,

$$\inf_{\mu_{\xi^{\lambda}}(r;\theta)\in\mathcal{U}} \int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\left\{\xi^{\lambda} \le r\right\}. \quad (13)$$

The best robust feasible solution is the one that solves the following optimization problem:

$$\begin{aligned} &\max_{Q}\ \inf_{\mu_{\xi^{\lambda}}(x;\theta)\in\mathcal{U}} \int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\left\{\xi^{\lambda} \le r\right\},\\ &\text{s.t. } r_1 \le Q \le r_4. \end{aligned} \quad (14)$$

Problem (14) is called the robust counterpart of problem (11). We should seek the maximal robust mean value of the objective among all robust feasible solutions to the uncertain single-period inventory problem. If we introduce an extra variable t and reformulate problem (14) as an optimization problem with a certain objective, then robust counterpart (14) can be rewritten equivalently as the following optimization problem:

$$\begin{aligned} &\max_{t,Q}\ t\\ &\text{s.t.} \int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\left\{\xi^{\lambda} \le r\right\} \ge t, \forall \mu_{\xi^{\lambda}}(x;\theta) \in \mathcal{U},\\ &r_1 \le Q \le r_4. \end{aligned} \quad (15)$$

So far, we have obtained equivalent robust counterpart optimization problem (15) in variables Q and t, where the objective is not affected by uncertainty at all. The optimal solution and optimal value to robust counterpart (14) or (15) is called the robust optimal solution and the robust optimal value to problem (11), respectively. Note that robust counterpart (15) has a certain linear objective and infinitely many integral constraints, which are usually computationally intractable. In the next section, we discuss the equivalent deterministic programming model of problem (15) and design its solution algorithm.

5. The Equivalent Programming Models of the Robust Counterpart

5.1. The Analytical Representation of the Mean Profit. In order to solve robust counterpart (15), it is required to derive the analytical representation of the following mean profit:

$$\int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\{\xi^\lambda \le r\}, \tag{16}$$

where the generalized credibility $\mathrm{Cr}\{\xi \le r\}$ is computed by (4). We first give the following result about the analytical representation of the mean profit.

Theorem 2. *(Guo [27]). If ξ is a fuzzy demand variable with a finite expected value, then the analytical representation of the mean profit is*

$$\int_{[0,+\infty]} \pi(Q,r)\mathrm{d}Cr\{\xi \le r\} = (p+g-w_s-c_r)hQ - (p+g-s)\int_0^Q Cr\{\xi \le r\}\mathrm{d}r - g\mu, \tag{17}$$

where $h = \lim_{r \longrightarrow +\infty} Cr\{\xi \le r\}$ and $\mu = \int_{[0,+\infty]} r dCr\{\xi \le r\}$.

If ξ^λ is the lambda selection of $\mathrm{Tra}(r_1, r_2, r_3, r_4; \theta)$, then $\pi(Q, \xi^\lambda)$ is computed by

$$\int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\{\xi^\lambda \le r\} = (p+g-w_s-c_r)(1-(1-\lambda)\theta_l)Q - (p+g-s)\int_0^Q Cr\{\xi \le r\}dr - g\mu, \tag{18}$$

where $\mu = [1-(1-\lambda)\theta_l](r_1+r_2+r_3+r_4/4) + (1/4)\lambda\theta_r (r_1-r_2-r_3+r_4)$.

According to Theorem 1, one has

$$\int_{r_1}^{Q} \mathrm{Cr}\{\xi^\lambda \le r\}\mathrm{d}r = \begin{cases} \lambda\dfrac{\theta_r\left[-Q^2+2r_2Q-(2r_2-r_1)r_1\right]}{4(r_2-r_1)} + \dfrac{h(Q-r_1)^2}{4(r_2-r_1)}, & r_1 \le Q < r_2 \\ \lambda\dfrac{\theta_r(r_2-r_1)}{4} + \dfrac{h(2Q-r_1-r_2)}{4}, & r_2 \le Q < r_3 \\ \lambda\dfrac{\theta_r\left[-Q^2+2r_3Q+(r_2-r_1)(r_4-r_3)-r_3^2\right]}{4(r_4-r_3)} + \dfrac{h\left[Q^2+2(r_4-2r_3)Q\right]}{4(r_4-r_3)} + \dfrac{h\left[(r_1+r_2)(r_3-r_4)+r_3^2\right]}{4(r_4-r_3)}, & r_3 \le Q \le r_4. \end{cases} \tag{19}$$

By (18) and (19), when $r_1 \le Q < r_2$, the mean profit has the following analytical representation:

$$\int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\{\xi^{\lambda} \le r\} = \begin{cases} \lambda\left\{-\dfrac{g[\theta_l(r_1+r_2+r_3+r_4)+\theta_r(r_1-r_2-r_3+r_4)]}{4}\right. \\ -\dfrac{Q\{2\theta_l r_1(p+g+s-2c)-2r_2[2\theta_l(p+g-c)-\theta_r(p+g-s)]\}}{4(r_2-r_1)} \\ -\dfrac{r_1(p+g-s)[r_1(\theta_r+\theta_l)-2r_2\theta_r]}{4(r_2-r_1)} \\ \left.+\dfrac{Q^2(p+g-s)(\theta_r-\theta_l)}{4(r_2-r_1)}\right\} \\ +Q(1-\theta_l)\left[p+g-c+\dfrac{r_1(p+g-s)}{2(r_2-r_1)}\right] \\ -\dfrac{g(1-\theta_l)(r_1+r_2+r_3+r_4)}{4} \\ -\dfrac{(1-\theta_l)r_1^2(p+g-s)}{4(r_2-r_1)} \\ -\dfrac{Q^2(1-\theta_l)(p+g-s)}{4(r_2-r_1)}. \end{cases} \tag{20}$$

Similarly, when $r_2 \le Q < r_3$, the mean profit has the following analytical representation:

$$\int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\{\xi^{\lambda} \le r\} = \begin{cases} \lambda\left\{\dfrac{Q\theta_l(p+g+s-2c)}{2}+\dfrac{(p+g-s)[(r_1-r_2)\theta_r+(r_1+r_2)\theta_l]}{4}\right. \\ \left.-\dfrac{g[\theta_l(r_1+r_2+r_3+r_4)+\theta_r(r_1-r_2-r_3+r_4)]}{4}\right\} \\ +\dfrac{Q(1-\theta_l)(p+g+s-2c)}{2}+\dfrac{(p+g-s)(1-\theta_l)(r_1+r_2)}{4} \\ -\dfrac{g(1-\theta_l)(r_1+r_2+r_3+r_4)}{4}. \end{cases} \tag{21}$$

Finally, when $r_3 \le Q \le r_4$, the mean profit has the following analytical representation:

$$\int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\left\{\xi^{\lambda} \le r\right\} = \begin{cases} \lambda\left\{\dfrac{Q\{r_3[4\theta_l(c-s)-2\theta_r(p+g-s)]+2\theta_l r_4(p+g+s-2c)\}}{4(r_4-r_3)}\right. \\[2ex] -\dfrac{g[\theta_l(r_1+r_2+r_3+r_4)+\theta_r(r_1-r_2-r_3+r_4)]}{4} \\[2ex] +\dfrac{\theta_r(p+g-s)\left[r_3^2-(r_2-r_1)(r_4-r_3)\right]}{4(r_4-r_3)} \\[2ex] -\dfrac{\theta_l(p+g-s)\left[(r_1+r_2)(r_3-r_4)+r_3^2\right]}{4(r_4-r_3)} \\[2ex] \left.+\dfrac{Q^2(p+g-s)(\theta_r-\theta_l)}{4(r_4-r_3)}\right\} \\[2ex] +Q(1-\theta_l)\left[\dfrac{2r_3-r_4}{2(r_4-r_3)}(p+g-s)+p+g-c\right] \\[2ex] -\dfrac{(1-\theta_l)(p+g-s)\left[(r_1+r_2)(r_3-r_4)+r_3^2\right]}{4(r_4-r_3)} \\[2ex] -\dfrac{g(1-\theta_l)(r_1+r_2+r_3+r_4)}{4} \\[2ex] -\dfrac{Q^2(1-\theta_l)(p+g-s)}{4(r_4-r_3)}. \end{cases} \tag{22}$$

5.2. The Equivalent Parametric Programming Submodels of the Robust Counterpart. In order to solve robust counterpart problem (15), we introduce the following functions:

$$\begin{aligned} f_1(Q) = &-\frac{Q\{2\theta_l r_1(p+g+s-2c)-2r_2[2\theta_l(p+g-c)-\theta_r(p+g-s)]\}}{4(r_2-r_1)} \\ &-\frac{g[\theta_l(r_1+r_2+r_3+r_4)+\theta_r(r_1-r_2-r_3+r_4)]}{4} \\ &-\frac{r_1(p+g-s)[r_1(\theta_r+\theta_l)-2r_2\theta_r]}{4(r_2-r_1)}+\frac{Q^2(p+g-s)(\theta_r-\theta_l)}{4(r_2-r_1)}, \\ g_1(Q) = &-\frac{Q^2(1-\theta_l)(p+g-s)}{4(r_2-r_1)}-\frac{(1-\theta_l)r_1^2(p+g-s)}{4(r_2-r_1)} \\ &+Q(1-\theta_l)\left[p+g-c+\frac{r_1(p+g-s)}{2(r_2-r_1)}\right]-\frac{g(1-\theta_l)(r_1+r_2+r_3+r_4)}{4}, \end{aligned}$$

$$
\begin{aligned}
f_2(Q) &= \frac{Q\theta_l(p+g+s-2c)}{2} + \frac{(p+g-s)[(r_1-r_2)\theta_r+(r_1+r_2)\theta_l]}{4} \\
&\quad - \frac{g[\theta_l(r_1+r_2+r_3+r_4)+\theta_r(r_1-r_2-r_3+r_4)]}{4}, \\
g_2(Q) &= \frac{Q(1-\theta_l)(p+g+s-2c)}{2} + \frac{(p+g-s)(1-\theta_l)(r_1+r_2)}{4} - \frac{g(1-\theta_l)(r_1+r_2+r_3+r_4)}{4}, \\
f_3(Q) &= \frac{Q\{r_3[4\theta_l(c-s)-2\theta_r(p+g-s)]+2\theta_l r_4(p+g+s-2c)\}}{4(r_4-r_3)} - \frac{g[\theta_l(r_1+r_2+r_3+r_4)+\theta_r(r_1-r_2-r_3+r_4)]}{4} \\
&\quad + \frac{\theta_r(p+g-s)[r_3^2-(r_2-r_1)(r_4-r_3)]}{4(r_4-r_3)} - \frac{\theta_l(p+g-s)[(r_1+r_2)(r_3-r_4)+r_3^2]}{4(r_4-r_3)} + \frac{Q^2(p+g-s)(\theta_r-\theta_l)}{4(r_4-r_3)}, \\
g_3(Q) &= Q(1-\theta_l)\left[\frac{2r_3-r_4}{2(r_4-r_3)}(p+g-s)+p+g-c\right] - \frac{Q^2(1-\theta_l)(p+g-s)}{4(r_4-r_3)} - \frac{g(1-\theta_l)(r_1+r_2+r_3+r_4)}{4} \\
&\quad - \frac{(1-\theta_l)(p+g-s)[(r_1+r_2)(r_3-r_4)+r_3^2]}{4(r_4-r_3)}.
\end{aligned} \tag{23}
$$

Using the aforementioned notations, when $r_1 \le Q < r_2$, by the analytical representation of the mean profit, the robust mean value has the following equivalent representation:

$$
\inf_{\mu_{\xi^\lambda}(r;\theta)\in\mathcal{U}} \int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\{\xi^\lambda \le r\} = -\max\{-f_1(Q),0\} + g_1(Q). \tag{24}
$$

Similarly, when $r_2 \le Q < r_3$, by the analytical representation of the mean profit, the robust mean value has the following equivalent representation:

$$
\inf_{\mu_{\xi^\lambda}(r;\theta)\in\mathcal{U}} \int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\{\xi^\lambda \le r\} = -\max\{-f_2(Q),0\} + g_2(Q). \tag{25}
$$

Finally, when $r_3 \le Q \le r_4$, by the analytical representation of the mean profit, the robust mean value has the following equivalent representation:

$$
\inf_{\mu_{\xi^\lambda}(r;\theta)\in\mathcal{U}} \int_{[r_1,r_4]} \pi(Q,r)\mathrm{dCr}\{\xi^\lambda \le r\} = -\max\{-f_3(Q),0\} + g_3(Q). \tag{26}
$$

According to the above analytical representation of the robust mean value, we decompose the feasible region of robust counterpart (15) into three disjoint subregions according to the values of the decision Q.

When $r_1 \le Q < r_2$, robust counterpart (15) is equivalent to the quadratic programming submodel:

$$
\begin{aligned}
\max \quad & t_1 \\
\text{s.t.} \quad & g_1(Q) - u_1 \ge t_1, \\
& f_1(Q) + u_1 \ge 0, \\
& u_1 \ge 0, r_1 \le Q < r_2.
\end{aligned} \tag{27}
$$

Similarly, when $r_2 \le Q < r_3$, robust counterpart (15) is equivalent to the linear programming submodel:

$$\begin{aligned} \max \quad & t_2 \\ \text{s.t.} \quad & g_2(Q) - u_2 \geq t_2, \\ & f_2(Q) + u_2 \geq 0, \\ & u_2 \geq 0, r_2 \leq Q < r_3. \end{aligned} \tag{28}$$

Finally, when $r_3 \leq Q \leq r_4$, robust counterpart (15) is equivalent to the quadratic programming submodel:

$$\begin{aligned} \max \quad & t_3 \\ \text{s.t.} \quad & g_3(Q) - u_3 \geq t_3, \\ & f_3(Q) + u_3 \geq 0, \\ & u_3 \geq 0, r_3 \leq Q \leq r_4. \end{aligned} \tag{29}$$

6. Domain Decomposition Method

We have decomposed robust counterpart problem (15) into three deterministic parametric programming submodels (27)–(29). Therefore, the feasible region of problem (15) is decomposed into three disjoint subregions according to the values of the decision Q. The three subregions are just the feasible regions of submodels (27)–(29). From this observation, we know that the global optimal solution of problem (15) can be obtained by solving submodels (27)–(29). By comparing the objective values of the obtained local optimal solutions, we can find the global optimal solution. This solution procedure is called the domain decomposition method.

Given the values of distribution parameters θ_l and θ_r, the solution process described above is summarized as follows.

Step 1. Parametric programming submodels (27)–(29) are solved by using LINGO software, and the obtained local optimal solutions are denoted as $(Q_i, u_i, t_i), i = 1, 2, 3$.

Step 2. The local objective values $t_i = g_i(Q_i) - u_i$ are compared at (Q_i, u_i, t_i) for $i = 1, 2, 3$, and the global optimal solution is obtained by the following formula:

$$t_k = \max_{1 \leq i \leq 3} t_i. \tag{30}$$

Step 3. Q_k is returned as the global optimal solution to robust counterpart (15) with the optimal value $g_k(Q_k) - u_k$.

The obtained Q_k value is called the robust optimal solution to model (11).

7. Numerical Experiments

7.1. Problem Statement. In this section, we consider a practical fanner inventory management problem during the summer. The retailer needs to order fanners before a selling season. Based on the knowledge of the retailer, the number of fanner demands ξ during the sales cycle is between 300 and 600, but the exact distribution on the interval $[300, 600]$ is unavailable. To model this situation, we characterize the uncertain demand ξ of the fanner by the generalized parametric interval-valued trapezoidal variable $\text{Tra}(300, 450, 550, 600; \theta_l, \theta_r)$, where θ_l and θ_r represent the uncertainty degree of the market demand ξ in the interval $[300, 600]$. The unit wholesale price w_s of the fanner is set by the supplier, and the retailer orders the number of fanners based on holding costs, goodwill costs for shortages, and his estimate on demand during the summer. In this inventory management problem, the values of model parameters are set as follows. The unit wholesale price w_s is \$55, and the unit retail price p is \$130. The treatment cost c_r of the unit fanner is \$5, and the goodwill cost g for unit unmet demand is \$15. In order to avoid overstock, the retailer can hold a special clearance sale to sell all surplus fanners at the end of the sales cycle, and there is no initial inventory on hand. It is expected that any residual fanners could be sold at the price s = \$20.

7.2. Computational Results of the Robust Counterpart. In order to find the optimal order quantity of the fanner, we solve robust counterpart (15) with respect to the following uncertain distribution set:

$$\mathcal{U} = \left\{ \mu_{\xi^\lambda} \,\middle|\, \mu_{\xi^\lambda} \text{ is the possibility distribution of } \xi^\lambda \right\}, \tag{31}$$

where ξ^λ is the lambda selection of the uncertain demand $\xi = \text{Tra}(300, 450, 550, 600; \theta_l, \theta_r)$.

Based on our designed domain decomposition method, we employ LINGO software to solve three equivalent submodels (27)–(29), where the values of parameters are $r_1 = 300$, $r_2 = 450$, $r_3 = 550$, and $r_4 = 600$.

To identify the influence of distribution parameters θ_r and θ_l on solution results, we first set the values of parameter θ_l as 0.05, 0.08, and 0.1, respectively, and observe the relationship between the robust optimal order quantity and the parameter θ_r and the relationship between the robust optimal mean profit and the parameter θ_r. The computational results are plotted in Figures 3 and 4, respectively, from which we find that the robust optimal order quantity Q^* is increasing with respect to the parameter θ_r, while the robust optimal mean profit of the retailer is decreasing with respect to the parameter θ_r. To further illustrate the impact of distribution parameters θ_r and θ_l on the robust optimal order quantity, the computational results with various values of θ_r and fixed $\theta_l = 0.02$ are reported in Table 1.

We next set the values of θ_r as 0.05, 0.1, 0.2, and 0.3, respectively, and observe the relationship between the robust optimal order quantity the and parameter θ_l and the relationship between the robust optimal mean profit and the parameter θ_l. The computational results are plotted in Figures 5 and 6, respectively, from which we find that the robust optimal order quantity Q^* is decreasing with respect to the parameter θ_l, and the robust optimal mean profit of the retailer is also decreasing with respect to the parameter θ_l. To further illustrate the impact of distribution parameters θ_r and θ_l on the robust optimal order quantity, the computational results with various values of θ_l and fixed $\theta_r = 0.28$ are reported in Table 2.

So far, we have discussed the relationship between the robust optimal order quantity and distribution parameters θ_r and θ_l. By the definition of distribution parameters θ_r and

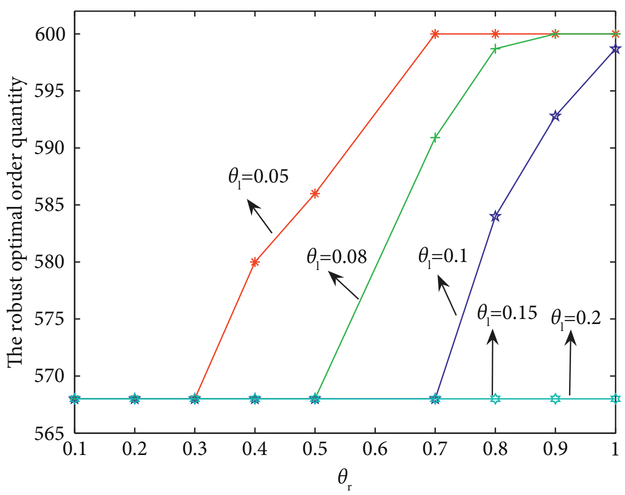

FIGURE 3: The relationship between the robust optimal order quantity and the parameter θ_r.

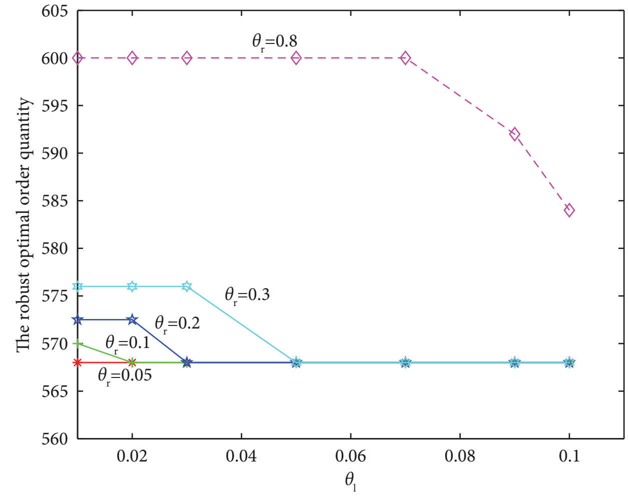

FIGURE 5: The relationship between the robust optimal order quantity and the parameter θ_l.

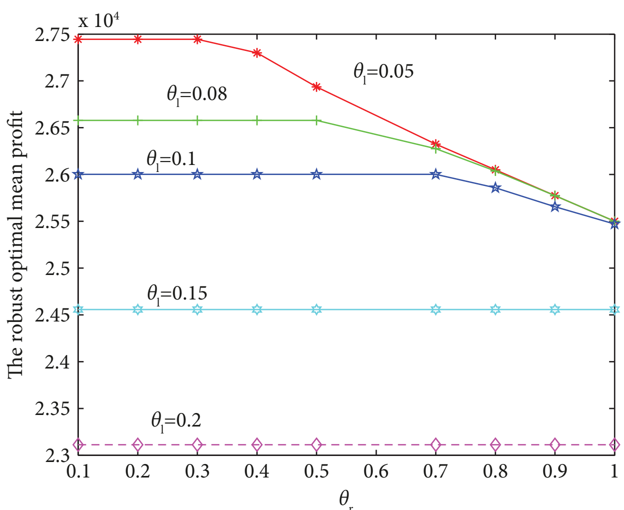

FIGURE 4: The relationship between the robust optimal mean profit and the parameter θ_r.

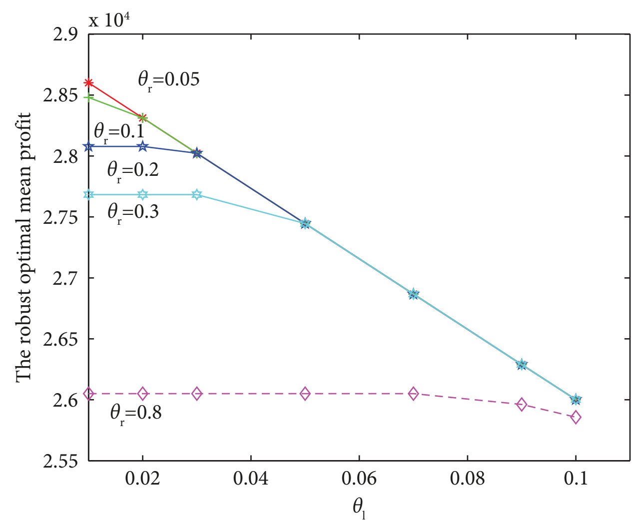

FIGURE 6: The relationship between the robust optimal mean profit and the parameter θ_l.

TABLE 1: The robust optimal order quantity and the mean profit with respect to θ_r.

θ_l	θ_r	Robust optimal order quantity	Robust optimal mean profit
0.02	0.15	571	28278.9
0.02	0.2	572	28078
0.02	0.22	573	27998.4
0.02	0.25	574	27879.4
0.02	0.28	575	27761.3
0.02	0.3	576	27683

TABLE 2: The robust optimal order quantity and the mean profit with respect to θ_l.

θ_l	θ_r	Robust optimal order quantity	Robust optimal mean profit
0.038	0.28	575	27761.2
0.0382	0.28	574	27760.8
0.0385	0.28	573	27759.5
0.0388	0.28	572	27757.2
0.039	0.28	571	27754.1
0.0395	0.28	570	27746.4

θ_l, the larger the distribution parameters, the larger the uncertainty degree of uncertain demand. The above computational results support our arguments. In the next section, we further compare the proposed distributionally robust optimization method with other optimization approaches to the single-period inventory management problem.

7.3. Comparison Study

7.3.1. Comparing with the Deterministic Optimization Method. We first compare our robust optimization method with the deterministic optimization method for the single-period inventory management problem, where the demand is known in advance. For the sake of comparison, we assume that the deterministic demand d is a mean value of 475 for the trapezoidal demand variable $\xi = \mathrm{Tra}(300, 450, 550, 600)$. In this situation, the optimization problem can be formulated as

$$\max_{r_1 \leq Q \leq r_4} \pi(Q), \tag{32}$$

where

$$\pi(Q) = \min\{(s - w_s - c_r)Q + 475(p - s), (p + g - w_s - c_r)Q - 475g\}. \tag{33}$$

Obviously, $\pi(Q)$ is a concave function with respect to Q. The total profit $\pi(Q)$ gets its maximum in $Q^* = d = 475$. That is, the optimal order quantity is 475 with a maximum profit of 33250. It is evident that the solution (475) is totally different from our robust optimal solutions reported in Tables 1 and 2. In fact, note that a value of 33250 is larger than all robust optimal values obtained in our numerical experiments, and the optimal solution (475) to model (32) is in the subregion [450,550]. Hence, there is no nonnegative number u_2 such that $(475, u_2, 33250)$ is a feasible solution to submodel (28), which implies $(475, u_2, 33250)$ is not feasible to robust counterpart problem (15).

7.3.2. Comparing with the Stochastic Optimization Method. We now compare our robust optimization method with the stochastic optimization method, in which the stochastic demand ξ follows a trapezoidal probability density function:

$$f(x) = \begin{cases} \dfrac{1}{30000}(x - 300), & 300 \leq x \leq 450, \\[2ex] \dfrac{1}{200}, & 450 \leq x \leq 550, \\[2ex] \dfrac{1}{10000}(600 - x), & 550 \leq x \leq 600. \end{cases} \tag{34}$$

According to the stochastic optimization method for the single-period inventory management problem [1], we know that the optimal order quantity is 511 with a maximum mean profit of 30007. In this case, the support of stochastic demand is [300, 600], which is the same as the support of uncertain demand in our single-period inventory management problem. However, compared with our robust optimal solutions and robust optimal values reported in Tables 1 and 2, the optimal solution (511) to the stochastic model is not a feasible solution to our robust counterpart problem (15), which can be explained in the same way as in the deterministic optimization method.

7.3.3. Comparing with the Fuzzy Optimization Method. For the sake of comparison, suppose uncertain demand ξ follows a trapezoidal possibility distribution (300, 450, 550, 600), which is the nominal possibility distribution of interval-valued possibility distributions used in our numerical example. In this case, we consider the following optimization model:

$$\max_{r_1 \leq Q \leq r_4} E[\pi(Q, \xi)], \tag{35}$$

where $E[\pi(Q, \xi)] = \int_{[r_1, r_4]} \pi(Q, r)\mathrm{dCr}\{\xi \leq r\}$. Model (35) corresponds to the case $\theta_r = \theta_l = 0$ in our robust counterpart problem (15). The optimal order quantity (568) to model (35) is called the nominal optimal order quantity, while a maximum mean profit of 28890 to model (35) is called the nominal maximal profit. The nominal maximal profit is larger than the robust optimal values obtained in Tables 1 and 2, which implies that the nominal optimal solution (568) in the subregion [550, 600] is not a feasible solution to our robust counterpart problem (15). The price of robustness is the reduction from its nominal optimal value to its robust optimal value. From the computational results reported in Tables 1 and 2, we observe that the price of robustness is increasing with respect to θ_r or θ_l, which can be explained easily by the definition of parameters θ_r and θ_l. It is noted that with the meaning of price of robustness, the obtained robust order quantity is the best uncertainty-immunized solution for our inventory management problem.

From the above comparison study, we obtain the following observations:

(i) Some of data like market demands in inventory management do not exist and usually are replaced with retailer's forecasting. Deterministic inventory optimization is based on the assumption that future demands can be forecasted exactly. In the case that future demands cannot be forecasted exactly, one should not adopt the resulting optimal solution of model (32) to order the optimal quantity.

(ii) The stochastic optimization method for inventory management problems is based on the assumption that future demands are random variables and that decision makers are able to point out the associated probability distribution. In the case that the probability distributions of future demands cannot be determined exactly, the resulting optimal solution of the stochastic model cannot be used to order optimal quantity.

(iii) In the conventional fuzzy optimization method for inventory management problems, small data perturbation in the possibility distribution is usually ignored. The inventory problem is solved as if the given nominal possibility distributions were exact. The comparison study demonstrates that there exists a real need of a technique which is able to detect cases when data perturbation in possibility distributions can heavily affect the quality of nominal solutions. Applying the proposed distributionally robust optimization method, the resulting robust optimal order quantity is the best uncertainty-immunized solution we can associate with our uncertain single-period inventory management problem.

8. Conclusions

In this paper, we studied the single-period inventory management problem from a new perspective. The major new results include the following several aspects.

First, when only partial information about the distribution of market demand is available, we characterized the uncertain demand by an uncertain distribution set, which is a collection of all generalized possibility distributions of lambda selections.

Second, based on the proposed uncertain distribution set, a new robust fuzzy optimization method was developed for single-period inventory management problems. Under mild assumptions, we built the robust counterpart of the original inventory management problem.

Third, we discussed the computational issue of the robust counterpart. Based on the structural characteristics of the three submodels, a domain decomposition method was designed to find the robust optimal solution that can immunize against uncertainty in inventory management problems.

Finally, some computational results were provided to demonstrate the primary benefit of using the proposed robust fuzzy optimization method.

This study limits the consideration to the generalized parametric trapezoidal fuzzy variables with bounded possibility distributions. Extension to considering other types of uncertain distribution sets with the robust optimization method is another interesting research direction.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the Social Science Foundation of Hebei Province (No. HB22TJ005) and the Doctoral Science Foundation of North China Institute of Aerospace Engineering (No. BKY-2018-23).

References

[1] Y. Qin, R. X. Wang, A. J. Vakharia, Y. W. Chen, and M. M. Seref, "The newsvendor problem: review and directions for future research," *European Journal of Operational Research*, vol. 213, no. 2, pp. 361–374, 2011.

[2] M. Khouja, "The single period (news-vendor) problem: literature review and suggestions for future research," *Omega*, vol. 27, no. 5, pp. 537–553, 1999.

[3] J. Kamburowski, "The distribution-free newsboy problem under the worst-case and best-case scenarios," *European Journal of Operational Research*, vol. 237, no. 1, pp. 106–112, 2014.

[4] R. Qiu and J. Shang, "Robust optimisation for risk-averse multi-period inventory decision with partial demand distribution information," *International Journal of Production Research*, vol. 52, no. 24, pp. 7472–7495, 2014.

[5] C. X. Wang, S. Webster, and S. D. Zhang, "Robust price-setting newsvendor model with interval market size and consumer willingness-to-pay," *International Journal of Production Economics*, vol. 154, pp. 100–112, 2014.

[6] J. Kacprzyk and P. Stanieski, "Long-term inventory policy-making through fuzzy decision-making models," *Fuzzy Sets and Systems*, vol. 8, no. 2, pp. 117–132, 1982.

[7] H. J. Zimmermann, *Fuzzy Sets Theory and its Applications*, Kluwer Academic Publishers, London, UK, 2 edition, 1985.

[8] D. Petrović, R. Petrović, and M. Vujošević, "Fuzzy models for the newsboy problem," *International Journal of Production Economics*, vol. 45, no. 1-3, pp. 435–441, 1996.

[9] Y. Yu and T. D. Jin, "The return policy model with fuzzy demands and asymmetric information," *Applied Soft Computing*, vol. 11, no. 2, pp. 1669–1678, 2011.

[10] Y. Yu, J. Zhu, and C. W. Wang, "A newsvendor model with fuzzy price-dependent demand," *Applied Mathematical Modelling*, vol. 37, no. 5, pp. 2644–2661, 2013.

[11] S. P. Chen and Y. H. Ho, "Optimal inventory policy for the fuzzy newsboy problem with quantity discounts," *Information Sciences*, vol. 228, pp. 75–89, 2013.

[12] S. J. Sang, "Price competition of manufacturers in supply chain under a fuzzy decision environment," *Fuzzy Optimization and Decision Making*, vol. 14, no. 3, pp. 335–363, 2015.

[13] A. H. L. Lau and H. S. Lau, "The newsboy problem with price-dependent demand distribution," *IIE Transactions*, vol. 20, no. 2, pp. 168–175, 1988.

[14] S. Kraiselburd, V. G. Narayanan, and A. Raman, "Contracting in a supply chain with stochastic demand and substitute products," *Production and Operations Management*, vol. 13, no. 1, pp. 46–62, 2009.

[15] A. Balakrishnan, M. S. Pangburn, and E. Stavrulaki, "Integrating the promotional and service roles of retail inventories," *Manufacturing and Service Operations Management*, vol. 10, no. 2, pp. 218–235, 2008.

[16] H. Scarf, "A min-max solution of an inventory problem," in *Studies in the Mathematical Theory of Inventory and Production*, K. Arrow, S. Karlin, and H. Scarf, Eds., pp. 201–209, Stanford University Press, Stanford, CA, USA, 1958.

[17] G. Gallego and I. Moon, "The distribution free newsboy problem: review and extensions," *Journal of the Operational Research Society*, vol. 44, no. 8, pp. 825–834, 1993.

[18] R. M. Hill, “Applying Bayesian methodology with a uniform prior to the single period inventory model,” *European Journal of Operational Research*, vol. 98, no. 3, pp. 555–562, 1997.

[19] A. Ridder, E. van der Laan, and M. Salomon, “How larger demand variability may lead to lower costs in the newsvendor problem,” *Operations Research*, vol. 46, no. 6, pp. 934–936, 1998.

[20] X. F. Xu, Z. R. Lin, X. Li, C. J. Shang, and Q. Shen, “Multi-objective robust optimisation model for MDVRPLS in refined oil distribution,” *International Journal of Production Research*, vol. 60, no. 22, pp. 6772–6792, 2021.

[21] H. Ishii and T. Konno, “A stochastic inventory problem with fuzzy shortage cost,” *European Journal of Operational Research*, vol. 106, no. 1, pp. 90–94, 1998.

[22] C. Kao and W. K. Hsu, “A single-period inventory model with fuzzy demand,” *Computers & Mathematics with Applications*, vol. 43, no. 6-7, pp. 841–848, 2002.

[23] L. Li, S. N. Kabadi, and K. P. K. Nair, “Fuzzy models for single-period inventory problem,” *Fuzzy Sets and Systems*, vol. 132, no. 3, pp. 273–289, 2002.

[24] P. Dutta, D. Chakraborty, and A. R. Roy, “An inventory model for single-period products with reordering opportunities under fuzzy demand,” *Computers & Mathematics with Applications*, vol. 53, no. 10, pp. 1502–1517, 2007.

[25] R. Xu and X. Zhai, “Analysis of supply chain coordination under fuzzy demand in a two-stage supply chain,” *Applied Mathematical Modelling*, vol. 34, no. 1, pp. 129–139, 2010.

[26] J. Sadeghi, S. M. Mousavi, and S. T. A. Niaki, “Optimizing an inventory model with fuzzy demand, backordering, and discount using a hybrid imperialist competitive algorithm,” *Applied Mathematical Modelling*, vol. 40, no. 15-16, pp. 7318–7335, 2016.

[27] Z. Z. Guo, “Optimal inventory policy for single-period inventory management problem under equivalent value criterion,” *Journal of Uncertain Systems*, vol. 10, no. 4, pp. 302–311, 2016.

[28] Z. Z. Guo, S. N. Tian, and Y. K. Liu, “A multi-product single-period inventory management problem under variable possibility distributions,” *Mathematical Problems in Engineering*, vol. 2017, Article ID 2159281, 14 pages, 2017.

[29] Z. Z. Guo and Y. K. Liu, “Modelling single-period inventory problem by distributionally robust fuzzy optimization method,” *Journal of Intelligent and Fuzzy Systems*, vol. 35, no. 1, pp. 1007–1019, 2018.

[30] Z. Q. Liu and Y. K. Liu, “Type-2 fuzzy variables and their arithmetic,” *Soft Computing*, vol. 14, no. 7, pp. 729–747, 2010.

[31] Y. Liu and Y. K. Liu, “The lambda selections of parametric interval-valued fuzzy variables and their numerical characteristics,” *Fuzzy Optimization and Decision Making*, vol. 15, no. 3, pp. 255–279, 2016.

[32] M. Pagola, C. Lopez-Molina, J. Fernandez, E. Barrenechea, and H. Bustince, “Interval type-2 fuzzy sets constructed from several membership functions. Application to the fuzzy thresholding algorithm,” *IEEE Transactions on Fuzzy Systems*, vol. 21, no. 2, pp. 230–244, 2013.

[33] H. Bustince, J. Fernandez, H. Hagras, F. Herrera, M. Pagola, and E. Barrenechea, “Interval type-2 fuzzy sets are generalization of interval-valued fuzzy sets: towards a wider view on their relationship,” *IEEE Transactions on Fuzzy Systems*, vol. 23, no. 5, pp. 1876–1882, 2015.

[34] Z. Z. Guo, Y. K. Liu, and Y. Liu, “Coordinating a three level supply chain under generalized parametric interval-valued distribution of uncertain demand,” *Journal of Ambient Intelligence and Humanized Computing*, vol. 8, no. 5, pp. 677–694, 2017.

[35] X. Bai and Y. K. Liu, “Semideviations of reduced fuzzy variables: a possibility approach,” *Fuzzy Optimization and Decision Making*, vol. 13, no. 2, pp. 173–196, 2014.

[36] X. Bai and Y. K. Liu, “CVaR reduced fuzzy variables and their second order moments,” *Iranian Journal of Fuzzy Systems*, vol. 12, no. 5, pp. 45–75, 2015.

[37] B. D. Liu and Y. K. Liu, “Expected value of fuzzy variable and fuzzy expected value models,” *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 4, pp. 445–450, 2002.

[38] A. Ben-Tal, L. E. Ghaoui, and A. Nemirovski, *Robust Optimization*, Princeton University Press, New Jersey, NJ, USA, 2009.

[39] D. Bertsimas, D. B. Brown, and C. Caramanis, “Theory and applications of robust optimization,” *Society of Indian Automobile Manufacturers Review*, vol. 53, no. 3, pp. 464–501, 2011.

[40] B. L. Gorissen, İ. Yanikoğlu, and D. Hertog, “A practical guide to robust optimization,” *Omega*, vol. 53, pp. 124–137, 2015.

[41] M. Carter and B. V. Brunt, *The Lebesgue-Stieltjes Integral*, Spinger-Verlag, New York, NY, USA, 2000.