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Research Article

A Compound Class of Unit Burr XII Model: Theory, Estimation, Fuzzy, and Application

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The current research offers an enhanced three-parameter lifetime model that combines the unit Burr XII distribution with a power series distribution. The novel class of distribution is named the unit Burr XII power series (UBXIIPS). This compounding technique allows for the production of flexible distributions with strong physical meanings in domains such as biology and engineering. The UBXIIPS class includes the unit Burr XII Poisson (UBXIIP) distribution, the unit Burr XII binomial distribution, the unit Burr XII geometric distribution, and the unit Burr XII negative binomial distribution. The statistical properties of the class include formulas for the density and cumulative distribution functions, and limiting behaviour, moments and incomplete moments, entropy measures, and quantile function are provided. For estimating population parameters and fuzzy reliability for the UBXIIP model, maximum likelihood and Bayesian approaches are studied by the Metropolis–Hastings algorithm. For maximum likelihood estimators, the length of asymptotic confidence intervals is specified, whereas, for Bayesian estimators, the length of credible confidence intervals is assigned. A simulation investigation of the UBXIIP model was established to evaluate the performance of suggested estimates. In addition, the UBXIIP distribution is explored using real-world data. The UBXIIP distribution appears to offer some benefits in understanding lifetime data when compared to unit Weibull, beta, Kumaraswamy, Kumaraswamy Kumaraswamy, Marshall-Olkin Kumaraswamy, and Topp–Leone Weibull Lomax distributions.

1. Introduction

In recent years, the development of unit distributions has risen rapidly. These distributions focus on modelling a wide variety of occurrences using data with values ranging from 0 to 1, such as proportions, probabilities, and percentages. The design of parametric, semiparametric, and regression models for the analysis of such data is also in high demand in applied disciplines. The majority of today's unit distributions are created by appropriately modifying older distributions. Our attention has been drawn to the recently presented unit Burr XII (UBXII) created by Korkmaz and Chesneau [1]. The cumulative distribution function (CDF) and probability density function (PDF) of the UBXII distribution are specified by

$$G(t) = \left(1 + (-\ln t)^{\beta}\right)^{-\varphi}, \beta, \varphi > 0, 0 < t < 1, \quad (1)$$

and

$$g(t) = \beta\varphi t^{-1}(-\ln t)^{\beta-1}\left(1 + (-\ln t)^{\beta}\right)^{-\varphi-1}, \beta, \varphi > 0, 0 < t < 1, \quad (2)$$

where $\beta > 0$ and $\varphi > 0$ are shape parameters. They investigated some of the UBXII distribution's characteristics and developed a regression model for it.

Due to theoretical considerations, practical applications, or both, academics have recently grown more interested in the design of novel univariate distributions, which are widely utilised in statistics and related disciplines. Compounding is a useful strategy for creating new distributions by combining certain useful lifetime with truncated discrete distributions. The basic idea behind creating these compounding distributions is that the lifetime of a system with K components and a positive continuous random variable, T_i, that denotes the lifetime of the i^{th} component, can be represented by a nonnegative random variable, $T = \min(T_1, \ldots, T_K)$ if the components are in a series, or $T = \max(T_1, \ldots, T_K)$ if the components are in parallel. The continuous random variables T_i are considered to be independent of K in both contexts.

Various compound classes have been provided by mixing continuous distributions with power series (PS) distribution, for example, Weibull-PS and extended Weibull-PS (Morais and Barreto-Souza [2] and Silva et al. [3]), generalized exponential PS (Mahmoudi and Jafari [4]), complementary exponential PS (Flores et al. [5]), the Burr XII-PS (Silva and Corderio [6]), Gompertz PS (Jafari and Tahmasebi [7]), generalized modified Weibull-PS (Bagheri et al. [8]), exponential Pareto PS (Elbatal et al. [9]), exponentiated power Lindley-PS (Alizadeh et al. [10]), generalized inverse Lindley-PS (Alkarni, [11]), Burr–Weibull PS (Oluyede et al. [12]), odd log-logistic PS (Goldoust et al. [13]), new generalized Lindley-Weibull class (Makubate et al. [14]), inverse gamma PS (Rivera et al. [15]), and inverted exponentiated Lomax PS (Hassan et al. [16]) among others.

The unit Burr XII power series (UXIIPS) class is suggested in the current study by mixing the UBXII and PS distributions in a device made up of parallel components. As special examples, this class also includes a number of compound lifetime models. Additionally, it offers us the flexibility to simulate many different behavioural forms of lifetime data using any compound lifetime. There are several hazard rate shapes present in this class of distributions. In addition to the quantile function, some moment measurements, the mean residual life, and uncertainty measures are other distributional features that we offer. The parameters and fuzzy reliability of the UBXII Poisson (UBXIIP) distribution, a specific example from the proposed class, are estimated using Bayesian and non-Bayesian techniques. A numerical simulation experiment is run to assess the accuracy of the estimated values. A real data set is used to explain the utility of the UBXIIP distribution.

The followings are the main physical justifications and significance of the distribution's UBXII class:

(i) To approximate the time to the last failure of a system made up of components linked in parallel.

(ii) To construct and generate distributions with symmetric, left-skewed, right-skewed, and U shapes.

(iii) To define special models that have a variety of hazard rate functions, including monotonic and nonmonotonic shapes.

(iv) To consistently provide better fits than other generated distributions having the same or greater number of parameters.

(v) To discuss the Bayesian and non-Bayesian estimators of parameters and fuzzy reliability for one special model from the provided class.

(vi) To assess the accuracy of the produced estimators, a numerical simulation investigation is undertaken.

(vii) To build heavy-tailed distributions for modelling diverse real data sets used in numerous domains, such as business, environmental research, medicinal studies, demographics, and industrial reliability.

(viii) A data study demonstrated the UBXIIP distribution's superiority over a few other well-known models.

This article is structured as follows: Section 2 provides the PDF of the UBXIIPS class as well as gives associated models. In Section 3, we deduce certain structural features of the UBXIIPS class. The Bayesian and non-Bayesian estimators of the UBXIIP distribution parameters and fuzzy reliability are explained in Section 4. Numerical examination and real data analysis are discussed, respectively, in Sections 5 and 6. Section 7 concludes the article.

2. Construction of the New Class

The new class of UBXIIPS is defined as follows. Given K, supposed that $T_1, \ldots, T_K$ be identically independent distributed random variables having the UBXII distribution (1) with shape parameters $\beta > 0$ and $\varphi > 0$, where K is a discrete random variable having a PS (truncated at zero) distribution. The probability mass function of K is specified by

$$P(K = k) = \frac{a_k \eta^k}{D(\eta)}, k = 1, 2, 3 \cdots, \tag{3}$$

where a_k depends only on k, $\eta > 0$ is the scale parameter, and $D(\eta) = \sum_{k=1}^{\infty} a_k \eta^k D'(\eta)$, and $D''(\eta)$ denote the first and second derivatives of $D(\eta)$, respectively. We provide certain PS distributions (truncated at zero) defined by (3) in Table 1, including the Poisson, logarithmic, geometric, and binomial distributions.

Let $T = \max\{T_i\}_{i=1}^{K}$, the conditional CDF of $T|K$ is given by

$$F_{T|K=k}(t) = [G(t)]^k = \left(1 + (-\ln t)^{\beta}\right)^{-\varphi k}. \tag{4}$$

Hence, $T|K$ is the UBXII distribution with parameters β and $k\varphi$, so we obtain

$$P(T \le t; K = k) = \frac{a_k \eta^k}{D(\eta)}\left(1 + (-\ln t)^{\beta}\right)^{-\varphi k}, 0 < t < 1, k \ge 1. \tag{5}$$

So, the marginal CDF of (5) is given by

Table 1: Basic quantities of some PS distributions.

Distributions	Poisson	Logarithm	Geometric	Binomial
a_k	$(k!)^{-1}$	$(k)^{-1}$	1	$\binom{m}{k}$
$D(\eta)$	$e^{\eta}-1$	$-\ln(1-\eta)$	$\eta(1-\eta)^{-1}$	$(1+\eta)^{m}-1$
$D'(.)$	e^{η}	$(1-\eta)^{-1}$	$(1-\eta)^{-2}$	$m(1+\eta)^{m-1}$
$D''(.)$	e^{η}	$(1-\eta)^{-2}$	$2(1-\eta)^{-3}$	$m(m-1)/(\eta+1)^{2-m}$
$(D(\eta))^{-1}$	$\ln(\eta+1)$	$1-e^{-\eta}$	$\eta(1+\eta)^{-1}$	$(\eta-1)^{1/m}-1$
η	$\eta\in(0,\infty)$	$\eta\in(0,1)$	$\eta\in(0,1)$	$\eta\in(0,\infty)$

$$F(t;\Theta)=\sum_{k=1}^{\infty}\frac{a_k}{D(\eta)}\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}\right)^{k}=\frac{1}{D(\eta)}D\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}\right),0<t<1. \quad (6)$$

A random variable with CDF (6) has UBXIIPS class with parameters $\Theta\equiv(\beta,\varphi,\eta)$ shall be denoted by T~UBXIIPS (Θ). The PDF of the UBXIIPS class corresponding to (6) is

$$f(t;\Theta)=\frac{\beta\varphi\eta t^{-1}(-\ln t)^{\beta-1}\left(1+(-\ln t)^{\beta}\right)^{-\varphi-1}D'\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}\right)}{D(\eta)},0<t<1. \quad (7)$$

The survival function and hazard rate function of the UBXIIPS class are represented by

$$\overline{F}(t;\Theta)=1-\frac{1}{D(\eta)}D\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}\right),0<t<1,$$

$$h(t;\Theta)=\frac{\beta\varphi\eta t^{-1}(-\ln t)^{\beta-1}\left(1+(-\ln t)^{\beta}\right)^{-\varphi-1}D'\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}\right)}{D(\eta)-D\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}\right)}. \quad (8)$$

Some special submodels are listed as follows, based on PDF (7) and Table 1:

(i) For $D(\eta)=e^{\eta}-1$, we obtain the PDF of the UBXIIP distribution as follows:

$$f_1(t;\Theta)=\frac{\beta\varphi\eta t^{-1}(-\ln t)^{\beta-1}\left(1+(-\ln t)^{\beta}\right)^{-\varphi-1}e^{\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}}}{\left(e^{\eta}-1\right)},0<t<1,\beta,\varphi,\eta>0. \quad (9)$$

(ii) For $D(\eta)=-\ln(1-\eta)$, we obtain the PDF of the UBXII logarithmic distribution as follows:

$$f_2(t;\Theta)=\frac{\beta\varphi\eta t^{-1}(-\ln t)^{\beta-1}\left(1+(-\ln t)^{\beta}\right)^{-\varphi-1}}{-\ln(1-\eta)\left(1-\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}\right)},0<t,\eta<1,\beta,\varphi>0. \quad (10)$$

(iii) For $D(\eta)=\eta(1-\eta)^{-1}$, we obtain the PDF of the UBXII geometric distribution as follows:

$$f_3(t;\Theta)=\frac{\beta\varphi(1-\eta)(-\ln t)^{\beta-1}\left(1+(-\ln t)^{\beta}\right)^{-\varphi-1}}{t\left(1-\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}\right)^{2}},0<t,\eta<1,\ \beta,\varphi>0. \quad (11)$$

(iv) For $D(\eta)=(1-\eta)^{m}-1$, we get UBXII binomial distribution as follows:

$$f_4(t;\Theta)=\frac{\beta\varphi\eta t^{-1}m(-\ln t)^{\beta-1}\left(1+(-\ln t)^{\beta}\right)^{-\varphi-1}\left(1+\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}\right)^{m-1}}{(1+\eta)^{m}-1},0<t<1,\beta,\varphi,\eta>0. \quad (12)$$

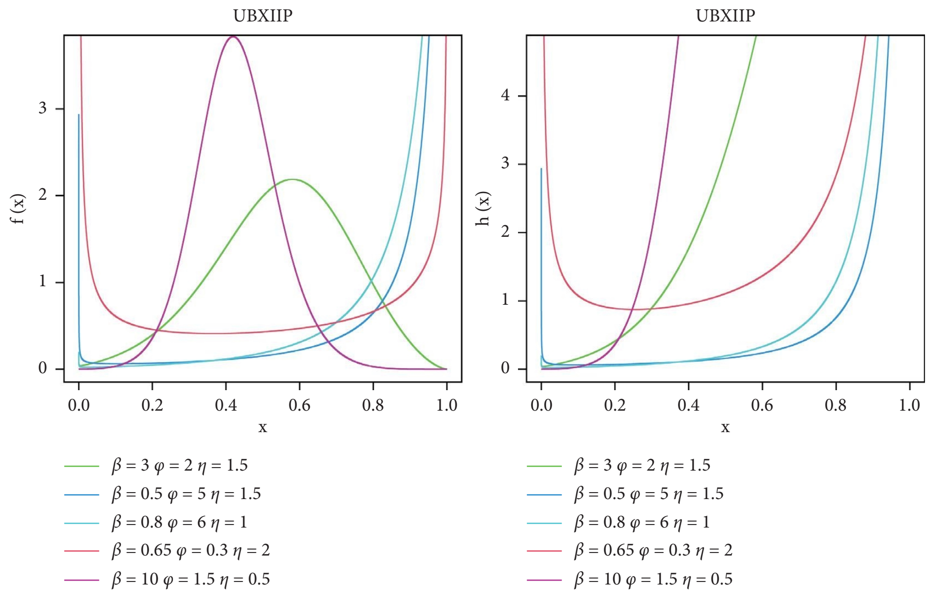

FIGURE 1: UBXIIP density and hazard functions for different values.

Figure 1 illustrates graphs of the UBXIIP density for various parameter values. These graphs demonstrate the new distribution's reliability and modality. The UBXIIP density is unimodal or bell-shaped, as seen in Figure 1. For a given set of parameters, it is left-skewed and reversed J-shaped. The hazard rate function can be decreasing, increasing, bath-tub, and J-shaped.

Proposition 1. *The UBXIIPS density function (7) can be explained as an infinite mixture of UBXII densities with parameters* $(\beta, \varphi k)$.

Proof. Using $D'(\eta) = \sum_{k=1}^{\infty} k a_k \eta^{k-1}$ in PDF (7), then it can be reformed as follows:

$$f(t;\Theta) = \sum_{k=1}^{\infty} \frac{k a_k \beta\phi\eta^k t^{-1}(-\ln t)^{\beta-1}\left(1+(-\ln t)^{\beta}\right)^{-\phi k-1}}{D(\eta)} = \sum_{k=1}^{\infty} P(K=k)g(t;\phi k,\beta), \tag{13}$$

where $g(t;\phi k,\beta)$ is the UBXII density function with parameters $(\beta,\phi k)$ and $P(K=k)$ defined in (3). □

Proposition 2. *The UBXII distribution is the limiting distribution of the UBXIIPS class of distributions when* $\eta \longrightarrow 0^+$.

Proof. The limiting distribution of (6), for $\eta \longrightarrow 0^+$, is determined, using L'Hospital's rule, as follows:

$$\begin{aligned}\lim_{\eta\longrightarrow 0^+} F(t;\Theta) &= \lim_{\eta\longrightarrow 0^+} \frac{\sum_{k=1}^{\infty} a_k\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\phi}\right)^k}{\sum_{k=1}^{\infty} a_k\eta^k} = \lim_{\eta\longrightarrow 0^+} \frac{\left(1+(-\ln t)^{\beta}\right)^{-\phi}\left[1+a_1^{-1}\sum_{k=2}^{\infty} k a_k\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\phi}\right)^{k-1}\right]}{1+a_1^{-1}\sum_{k=2}^{\infty} k a_k\eta^{k-1}}\\ &= \left(1+(-\ln t)^{\beta}\right)^{-\phi},\end{aligned} \tag{14}$$

which is CDF of UBXII distribution. □

3. General Properties

This section covers some statistical characteristics of the UBXIIPS class. Furthermore, these measures are focused on a one model, specifically the UBXIIP distribution.

3.1. Moments Measures. The moments of a probability distribution are essential tools in any statistical analysis. The r^{th} moment of T has the UBXIIPS class is presented from (13) as follows:

$$\mu_r' = \sum_{k=1}^{\infty} P(K=k)k\beta\phi \int_0^1 t^{r-1} (-\ln t)^{\beta-1}\left(1+(-\ln t)^{\beta}\right)^{-\phi k-1} \mathrm{d}t. \tag{15}$$

Using binomial expansion in (15) and let $z=-\ln t$, then μ_r' of UBXIIPS class has the form

$$\begin{aligned}\mu_r' &= \sum_{u=0}^{\infty}\sum_{k=1}^{\infty} (-1)^u P(K=k)\binom{\varphi k+u}{u} k\beta\varphi \int_0^1 t^{r-1} (-\ln t)^{\beta(u+1)-1}\mathrm{d}t \\ &= \sum_{u=0}^{\infty}\sum_{k=1}^{\infty} (-1)^u P(K=k)\binom{\varphi k+u}{u} k\beta\varphi \int_0^{\infty} e^{-zr} z^{\beta(u+1)-1}\mathrm{d}z = \frac{\Delta_{u,k}\Gamma(\beta(u+1))}{(r)^{\beta(u+1)}},\end{aligned} \tag{16}$$

where $\Delta_{u,k}=\sum_{u=0}^{\infty}\sum_{k=1}^{\infty}(-1)^u P(K=k)\binom{\phi k+u}{u}k\beta\phi$, and $\Gamma(.)$ is the gamma function (GF). The r^{th} central moment (μ_r) of the UBXIIPS class is given by

$$\mu_r = E\left(T-\mu_1'\right)^r = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\left(\mu_1'\right)^j\mu_{r-j}'. \tag{17}$$

Based on (17) and by using well-known relationships, we can determine some measures such as variance (σ^2), skewness (α_3), and kurtosis (α_4). In particular, numerical measurements such as $\mu_1', \mu_2', \mu_3', \mu_4', \sigma^2, \alpha_3$, and α_4 of the UBXIIP distribution are displayed in Table 2, for various parameter values, including (a) $(\phi=1.3, \eta=0.5, \beta=1.5)$, (b) $(\phi=1.3, \eta=1, \beta=1.5)$, (c) $(\phi=1.3, \eta=2, \beta=1.5)$, (d) $\equiv (\phi=1.3, \eta=0.5, \beta=0.5)$, (e) $(\phi=1.3, \eta=1, \beta=0.5)$, and (f) $(\phi=1.3, \eta=2, \beta=0.5)$.

As seen from Table 2, as the value of η increases, values of $\mu_1', \mu_2', \mu_3', \mu_4', \alpha_4$ increase while the value of α_3 decreases expect for set (d). The UBXIIP distribution is negatively skewed and platykurtic.

Furthermore, the r^{th} incomplete moment of the UBXIIPS class is derived as follows:

$$\mathfrak{I}_r(x) = \sum_{k=1}^{\infty} P(K=k)k\beta\phi \int_0^x t^{r-1} (-\ln t)^{\beta-1}\left(1+(-\ln t)^{\beta}\right)^{-\phi k-1} \mathrm{d}t. \tag{18}$$

Using binomial expansion and let $z=-\ln t$, then $\mathfrak{I}_r(x)$ of UBXIIPS class is

$$\mathfrak{I}_r(x) = \Delta_{u,k}\int_{\ln(1/x)}^{\infty} e^{-zr} z^{\beta(u+1)-1}\mathrm{d}z = \frac{\Delta_{u,k}\Gamma(\beta(u+1), r\ln(1/x))}{(r)^{\beta(u+1)}}, \tag{19}$$

where $\Gamma(.,.,v)$, is the upper incomplete GF. Additionally, the Lorenz (LZ) and Bonferroni (BI) curves of UBXIIP distribution are derived using the following expressions: $\mathrm{LZ}(x)=\mathfrak{I}_1(x)/\mu_1'$ and $\mathrm{BI}(x)=\mathfrak{I}_1(x)/\mu_1' F(x;\Theta)$.

Moreover, the r^{th} moment and incomplete moment of the UBXIIP distribution are computed from (16) and (19), respectively, by putting $P(K=k)=e^{-\eta}\eta^k/k!(1-e^{-\eta})$, $k=1,2,\cdots$.

Another important function is the mean residual life (MRL) function. The MRL function is defined as an item's expected life after t years. It is a conditional notion that tells you how long you may expect the object to last.

Table 2: Moment values for the UBXIIP distribution.

μ_r'	(a)	(b)	(c)	(d)	(e)	(f)
μ_1'	0.488	0.527	0.598	0.578	0.632	0.729
μ_2'	0.313	0.350	0.422	0.494	0.549	0.653
μ_3'	0.223	0.255	0.319	0.444	0.500	0.605
μ_4'	0.169	0.196	0.252	0.410	0.465	0.570
σ^2	0.075	0.073	0.064	0.160	0.150	0.122
α_3	−0.142	−0.301	−0.601	−0.398	−0.637	−1.144
α_4	1.939	2.057	2.499	1.466	1.743	2.757

Consequently, we obtain the n^{th} moment of the residual life of T via the general following formula:

$$\Phi_n(y) = E((T-y)\,|\,T>y) = \frac{1}{\overline{F}(y;\Theta)}\sum_{k=1}^{\infty} P(K=k)k\beta\varphi \int_y^1 (t-y)^n t^{-1} (-\ln t)^{\beta-1}\left(1+(-\ln t)^{\beta}\right)^{-\varphi k-1}\mathrm{d}t. \tag{20}$$

Using binomial series more than one time, then (20) is represented as follows:

$$\begin{aligned}\Phi_n(z) &= \frac{1}{\overline{F}(y;\Theta)}\sum_{j=0}^{n}\binom{n}{j}(-y)^{n-j}\Delta_{k,u}\int_y^1 t^{n-1}(-\ln t)^{\beta(u+1)-1}\mathrm{d}t \\ &= \frac{1}{\overline{F}(y;\Theta)}\sum_{j=0}^{n}\binom{n}{j}\frac{(-y)^{n-j}\Delta_{k,u}\gamma(\beta(u+1);n\ln(1/y))}{n^{\beta(u+1)}},\end{aligned} \tag{21}$$

where $\gamma(.,.,v)$, is the lower incomplete GF. Put $n=1$ in (21), we can calculate the MRL of UBXIIP distribution.

3.2. Quantile Function. The quantile function (QF) is defined as $Q(u)=F^{-1}(u)$ for any CDF. Consequently, the QF of the UBXIIPS class, based on (6), is calculated as follows:

$$t_u = Q(u) = \exp -\left\{\left[\frac{1}{\eta}D^{-1}(uD(\eta))\right]^{-1/\phi} - 1\right\}^{1/\beta}, 0<u<1. \tag{22}$$

For $u=0.25$, $u=0.5$, and $u=0.75$ in (22), we can determine the first, median, and third quantiles of the UBXIIPS class. More precisely, the QF of the UBXIIP distribution is derived from (22) by letting $D(\eta)=e^{\eta}-1$, and $D^{-1}(\eta)=\ln(1+\eta)$ as follows:

$$t_u = Q(u) = \exp -\left[\left\{\frac{1}{\eta}\left(\ln\left[u\left(e^{\eta}-1\right)+1\right]\right)\right\}^{-1/\phi} - 1\right]^{1/\beta}, 0<u<1. \tag{23}$$

Equation (23) can be used to generate UBXIIP random variates.

3.3. Uncertainty Measures. We look at certain information measures including Rényi (Ré) entropy and τ−entropy. The entropy quantifies the data's uncertainty; the higher the entropy number, the greater the data's uncertainty. The Ré entropy of T has UBXIIPS class is defined by

$$\Xi(\tau) = (1-\tau)^{-1}\log\left(\int_0^1 (f(t))^{\tau}\mathrm{d}t\right). \tag{24}$$

Expressions for different entropy measures of the UBXIIPS class are obtained. From (7), we deduce $(f(t;\Theta))^{\tau}$ as follows:

$$(f(t;\Theta))^{\tau} = \left(\beta\varphi\eta t^{-1}\right)^{\tau}(-\ln t)^{\tau(\beta-1)}\left(1+(-\ln t)^{\beta}\right)^{-\tau(\varphi+1)}\left\{\sum_{k=1}^{\infty}\frac{ka_k}{D(\eta)}\left[\eta\left(1+(-\ln t)^{\beta}\right)^{-\varphi}\right]^{k-1}\right\}^{\tau}. \tag{25}$$

But

$$\left\{\sum_{k=1}^{\infty} k a_k \left[\eta\left(1+(-\ln t)^{\beta}\right)^{-\phi}\right]^{k-1}\right\}^{\tau} = a_1^{\tau}\left(\sum_{m=0}^{\infty}\omega_m\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\phi}\right)^{m}\right)^{\tau},$$
$$\omega_m = \frac{a_{m+1}}{a_1}(m+1), m = 1, 2, \cdots. \tag{26}$$

Use the relation $(\sum_{m=0}^{\infty} h_m w^m)^{\tau} = \sum_{m=0}^{\infty} d_{\tau,m} w^m$ (Gradshteyn and Ryzhik [17]) in (26) yields

$$\left(\sum_{m=0}^{\infty}\omega_m\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\phi}\right)^{m}\right)^{\tau} = \sum_{m=0}^{\infty} d_{\tau,m}\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\phi}\right)^{m}. \tag{27}$$

Therefore, (26) will look like this:

$$\left\{\sum_{k=1}^{\infty} k a_k \left[\eta\left(1+(-\ln t)^{\beta}\right)^{-\phi}\right]^{k-1}\right\}^{\tau} = a_1^{\tau}\sum_{m=0}^{\infty} d_{\tau,m}\left(\eta\left(1+(-\ln t)^{\beta}\right)^{-\phi}\right)^{m}, \tag{28}$$

where, for $\ell \geq 1, d_{\tau,\ell} = \ell^{-1}\sum_{m=1}^{\ell}[m(\tau+1)-\ell]\omega_m d_{\tau,\ell-m}$ and $d_{\tau,0} = 1$. Therefore, (25) can be written as

$$(f(t;\Theta))^{\tau} = \sum_{m,p=0}^{\infty}\frac{d_{\tau,m}\eta^{m}(-1)^{p}a_1^{\tau}}{(D(\eta)^{\tau})}\binom{\tau(\varphi+1)+\varphi m+p}{p}\left(\beta\varphi\eta t^{-1}\right)^{\tau}(-\ln t)^{\tau(\beta-1)+\beta p}. \tag{29}$$

Put (29) in (24), we acquire Ré entropy of the UBXIIPS class as follows:

$$\Xi(\tau) = (1-\tau)^{-1}\log\left\{\sum_{m,p=0}^{\infty}\frac{d_{\tau,m}\eta^{m}(-1)^{p}(\beta\varphi\eta)\tau a_1^{\tau}}{(D(\eta))^{\tau}}\binom{\tau(\varphi+1)+\varphi m+p}{p}\frac{\Gamma(\tau(\beta-1)+\beta p+1)}{(1-\tau)^{\tau(\beta-1)+\beta p+1}}\right\}. \tag{30}$$

The $\tau-$ entropy of T has UBXIIPS class is defined by

$$\Lambda(\tau) = \frac{1}{(\tau-1)}\left(1-\int_0^1 (f(t))^{\tau}\mathrm{d}t\right), \tau \neq 1, \tau > 0. \tag{31}$$

Hence, $\tau-$ entropy of UBXIIPS class is obtained as follows:

$$\Lambda(\tau) = \frac{1}{(\tau-1)}\left(1-\left\{\sum_{m,p=0}^{\infty}\frac{d_{\tau,m}\eta^{m}(-1)^{p}(\beta\varphi\eta)\tau a_1^{\tau}}{(D(\eta))^{\tau}}\binom{\tau(\varphi+1)+\varphi m+p}{p}\frac{\Gamma(\tau(\beta-1)+\beta p+1)}{(1-\tau)^{\tau(\beta-1)+\beta p+1}}\right\}\right). \tag{32}$$

Putting $D(\eta) = e^{\eta-1}$ in (30) and (32), we get the Ré entropy and $\tau-$ entropy of the UBXIIP distribution.

4. Parameter Estimation of the UBXIIP Model

The parameter and reliability estimators of the UBXIIP distribution based on ML, and Bayesian estimation methods are discussed in this section.

4.1. ML Method. Let $T_1,\ldots,T_n$ be the observed values from the UBXIIP distribution with parameters $\beta, \phi,$ and η. The likelihood function, say $L(t|\Theta)$ of the UBXIIP distribution is expressed as:

$$L(t|\Theta) = \frac{\eta^n \varphi^n \beta^n}{(e^\eta - 1)^n} \prod_{i=1}^{n} \frac{1}{t_i} [-\ln(t_i)]^{\beta-1} (\zeta_i)^{-\varphi-1} e^{\eta(\zeta_i)^{-\varphi}}, \quad (33)$$

where $\zeta_i = 1 + [-\ln(t_i)]^\beta$. Then, the log-likelihood function, say ℓ, of the UBXIIP distribution is given as follows:

$$\ell = n[\ln(\beta) + \ln(\phi) + \ln(\eta)] - n\ln(e^\eta - 1) - \sum_{i=1}^{n} \ln(t_i) - (\phi+1)\sum_{i=1}^{n} \ln(\zeta_i) + (\beta-1)\sum_{i=1}^{n} \ln[-\ln(t_i)] + \eta \sum_{i=1}^{n} (\zeta_i)^{-\phi}. \quad (34)$$

Therefore, the ML equations are given by

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} - (\phi+1)\sum_{i=1}^{n} \frac{[-\ln(t_i)]^\beta \ln[-\ln(t_i)](\zeta_i)^{-1}}{\zeta_i} + \sum_{i=1}^{n} \ln[-\ln(t_i)] - \phi\eta \sum_{i=1}^{n} (\zeta_i)^{-\phi-1}[-\ln(t_i)]^\beta \ln[-\ln(t_i)],\\
\frac{\partial \ell}{\partial \phi} &= \frac{n}{\phi} - \sum_{i=1}^{n} \ln(\zeta_i) - \eta \sum_{i=1}^{n} (\zeta_i)^{-\phi} \ln(\zeta_i),\\
\frac{\partial \ell}{\partial \eta} &= \frac{n}{\eta} + \sum_{i=1}^{n} (\zeta_i)^{-\phi}.
\end{aligned} \quad (35)$$

Solving the nonlinear equations $\partial\ell/\partial\beta = 0, \partial\ell/\partial\phi = 0,$ and $\partial\ell/\partial\eta = 0,$ numerically using optimization algorism as conjugate-gradient optimization, we get the ML estimators of $\beta, \phi,$ and η.

To construct confidence intervals (CIs) for $\beta, \phi,$ and η we need to compute the asymptotic variance-covariance matrix which obtained by inverting the Fisher information matrix $I(\beta, \phi, \eta)$, in which elements are negatives of expected values of the second partial derivatives of ℓ. The elements of the sample information matrix for the ML method will be

$$I(\beta,\phi,\eta) = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \beta^2} & & \\ \frac{\partial^2 \ell}{\partial \phi \partial \beta} & \frac{\partial^2 \ell}{\partial \phi^2} & \\ \frac{\partial^2 \ell}{\partial \eta \partial \beta} & \frac{\partial^2 \ell}{\partial \eta \partial \phi} & \frac{\partial^2 \ell}{\partial \eta^2} \end{bmatrix}. \quad (36)$$

Under some regularity conditions, $(\hat{\beta}, \hat{\phi}, \hat{\eta})$ is approximately normal with mean (β, ϕ, η) and covariance matrix $I^{-1}(\beta, \phi, \eta)$. Practically, we estimate $I^{-1}(\beta, \phi, \eta)$ by $I^{-1}(\hat{\beta}, \hat{\phi}, \hat{\eta})$, then

$$I^{-1}(\beta,\phi,\eta) = \begin{bmatrix} \operatorname{var}(\hat{\beta}) & & \\ \operatorname{cov}(\hat{\phi},\hat{\beta}) & \operatorname{var}(\hat{\phi}) & \\ \operatorname{cov}(\hat{\eta},\hat{\beta}) & \operatorname{cov}(\hat{\eta},\hat{\phi}) & \operatorname{var}(\hat{\eta}) \end{bmatrix}. \quad (37)$$

Now, the approximate CIs (ACIs) for $\beta, \phi,$ and η can be obtained as follows:

$$\begin{aligned} &\hat{\beta} \pm z_{1-q/2}.\sqrt{\operatorname{var}(\hat{\beta})}, \hat{\phi} \pm z_{1-q/2}.\sqrt{\operatorname{var}(\hat{\phi})},\\ &\hat{\eta} \pm z_{1-q/2}.\sqrt{\operatorname{var}(\hat{\eta})}, \end{aligned} \quad (38)$$

where z_q is the 100 $q-th$ percentile of a standard normal distribution.

4.2. Bayesian Method. Here, we get the Bayesian estimator of the UBXIIP parameters. The Bayesian estimator is regarded as symmetric (squared error loss function (SELF)), which is defined as follows:

$$L(\tilde{\beta}, \beta) = E(\tilde{\beta} - \beta)^2, L(\tilde{\phi}, \phi) = E(\tilde{\phi} - \phi)^2, L(\tilde{\eta}, \eta) = E(\tilde{\eta} - \eta)^2. \quad (39)$$

Assuming that the prior distribution of $\beta, \phi,$ and η, denoted by $\pi(\beta), \pi(\phi), \pi(\eta),$ has an independent gamma distribution. The joint gamma prior density of $\beta, \phi,$ and η can be written as follows:

$$\pi(\beta,\phi,\eta) \propto \beta^{a_1-1}e^{-b_1\beta}\phi^{a_2-1}e^{-b_2\phi}\eta^{a_3-1}e^{-b_3\eta}; a_i, b_i > 0, i = 1,2,3. \tag{40}$$

In Bayesian estimation, the value of the hyperparameter is crucial. If the appropriate prior information for $\beta, \phi,$ and η is provided, $a_i, b_i; i = 1,2,3$, then the joint prior distribution (40) is proportional to the likelihood function (33). As a result, if one does not have previous knowledge of the unknown parameters, it is preferable to employ ML estimation rather than Bayesian estimation because the latter is computationally expensive. We used ML information (estimators and variance) as well as prior information (mean and variance) and solve these functions using the moments method or any iterative approach to obtain the hyperparameter values $\beta, \phi,$ and η, which are referred to as prior values. From (33) and (40), the joint posterior of the UBXIIP distribution with parameters $\beta, \phi,$ and η is

$$\pi\left(\beta,\phi,\eta \,\middle|\, t\right) \propto \frac{\eta^{n+a_3-1}\phi^{n+a_2-1}\beta^{n+a_1-1}}{\left(e^{\eta}-1\right)^n}e^{-b_1\beta-b_2\phi}\prod_{i=1}^{n}\left[-\ln\left(t_i\right)\right]^{\beta-1}\left(1+\left[-\ln\left(t_i\right)\right]^{\beta}\right)^{-\phi-1}e^{-\eta\left\{b_3-\left(1+\left[-\ln\left(t_i\right)\right]^{\beta}\right)^{-\phi}\right\}}. \tag{41}$$

Bayesian estimators may be produced using the Markov chain Monte Carlo (MCMC) technique. The Gibbs sampling, as well as the more generic Metropolis within Gibbs samplers, is important MCMC techniques. The Metropolis–Hastings (MH) algorithm and the Gibbs sampling are two well-known applications of the MCMC approach. The conditional posterior densities of $\beta, \phi,$ and η are produced as follows:

$$\begin{aligned}
\pi\left(\beta|\phi,\eta,t\right) &\propto \beta^{n+a_1-1}e^{-b_1\beta}\prod_{i=1}^{n}\left[-\ln\left(t_i\right)\right]^{\beta-1}\left(1+\left[-\ln\left(t_i\right)\right]^{\beta}\right)^{-\phi-1}e^{\left(1+\left[-\ln\left(t_i\right)\right]^{\beta}\right)^{-\phi}},\\
\pi\left(\phi|\beta,\eta,t\right) &\propto \phi^{n+a_2-1}e^{-b_2\phi}\prod_{i=1}^{n}\left(1+\left[-\ln\left(t_i\right)\right]^{\beta}\right)^{-\phi-1}e^{\left(1+\left[-\ln\left(t_i\right)\right]^{\beta}\right)^{-\phi}},\\
\pi\left(\eta|\beta,\phi,t\right) &\propto \frac{\eta^{n+a_3-1}}{\left(e^{\eta}-1\right)^n}e^{-\eta\left\{b_3-\sum_{i=1}^{n}\left(1+\left[-\ln\left(t_i\right)\right]^{\beta}\right)^{-\phi}\right\}}.
\end{aligned} \tag{42}$$

The Bayesian estimators are obtained based on SELF. It is clear that samples of $\beta, \phi,$ and η can be easily generated by using their conditional posterior distributions which are obtained above based on MH algorithm. Furthermore, using the samples generated from the suggested MH method, one may create the highest posterior density (HPD) credible intervals for $\beta, \phi,$ and η of the UBXIIP distribution following the MH proposed by Chen and Shao [18].

With two endpoints from the MCMC sample outputs, the lower is 2.5%, and the upper is 97.5% percentiles, respectively, and a 95% HPD interval can be generated. The credible intervals of $\beta, \phi,$ and η are calculated as follows:

(1) Arrange as $\beta^{[1]} < \beta^{[2]} < \cdots < \beta^{[H]}, \phi^{[1]} < \phi^{[2]} < \cdots < \phi^{[H]}$ and $\eta^{[1]} < \eta^{[2]} < \cdots < \eta^{[H]}$, where H is the length of MCMC generated

(2) The 95% symmetric credible intervals of $\beta, \phi,$ and η become $(\beta^{[H25/1000]}, \beta^{[H975/1000]})$, $(\phi^{[H25/1000]}, \phi^{[H975/1000]})$, and $(\eta^{[H25/1000]}, \eta^{[H975/1000]})$

4.3. Fuzzy Reliability. Let X be a continuous random variable that represents a system's failure time. The fuzzy reliability can be calculated using the fuzzy probability formula proposed by Chen and Pham [19]

$$R(\alpha) = P(T > x) = \int_x^{\infty}\mu(t)f(t)\mathrm{d}t, 0 \le x \le t \le \infty, \tag{43}$$

where $\mu(t)$ is a membership function that describes the degree to which each element of a given universe belongs to a fuzzy set. Now, assume that $\mu(t)$ is

$$\mu(\mathrm{t}) = \begin{cases} 0, & t \le x_1, \\ \dfrac{t-x_1}{x_2-x_1}, & 0 \le x_1 \le t \le x_2, \quad \text{where } 0 \le x_1 \le x_2 \\ 1, & t \ge x_2. \end{cases} \tag{44}$$

For $\mu(t)$ by the computational method of the function of fuzzy numbers, the lifetime $x(\alpha)$ can be obtained correspond to a certain value of $\alpha -$ cut where $0 \le \alpha \le 1$ proposed by Chen and Pham [19]

$$\mu(t) = \alpha \longrightarrow \frac{t - x_1}{x_2 - x_1} = \alpha, \text{then} \begin{cases} \mu(t) \le x, & \alpha = 0, \\ \mu(t) = x_1 + \alpha(x_2 - x_1), & 0 < \alpha < 1, \\ \mu(t) \ge x_2, & \alpha = 1. \end{cases} \quad (45)$$

Thus, for all $\alpha -$ cut values, fuzzy reliability values can be calculated as follows:

$$R_F(\alpha) = \begin{cases} \int_{x_1}^{x_1} f(t)\mathrm{d}t = 0, & \alpha = 0, \\ \int_{t_1}^{x(\alpha)=x_1+\alpha(x_2-x_1)} f(t)\mathrm{d}t = 0, & 0 < \alpha < 1, \\ \int_{x_1}^{x_2} f(t)\mathrm{d}t & \alpha = 1. \end{cases} \quad (46)$$

The reliability estimator via ML method based on nonfuzzy and fuzzy is as follows:

$$\begin{aligned} \widehat{R}(x_1) &= 1 - \frac{1}{e^{\widehat{\eta}} - 1}\left[e^{\widehat{\eta}\left(1+\left(-\ln x_1\right)^{\widehat{\beta}}\right)^{-\widehat{\phi}}} - 1\right], \\ \widehat{R}(x_2) &= 1 - \frac{1}{e^{\widehat{\eta}} - 1}\left[e^{\widehat{\eta}\left(1+\left(-\ln x_2\right)^{\widehat{\beta}}\right)^{-\widehat{\phi}}} - 1\right], \\ \widehat{R}(x) &= 1 - \frac{1}{e^{\widehat{\eta}} - 1}\left[e^{\widehat{\eta}\left(1+\left(-\ln x\right)^{\widehat{\beta}}\right)^{-\widehat{\phi}}} - 1\right], \\ \widehat{R}_F(\alpha) &= \widehat{R}(\mathrm{x}_1) - \widehat{R}(x_1 + \alpha(x_2 - x_1)). \end{aligned} \quad (47)$$

The reliability estimator via the Bayesian method based on the nonfuzzy and fuzzy parts of the UBXIIP distribution is as follows:

$$\begin{aligned} \widetilde{R}(x_i) &= \int_0^\infty \int_0^\infty \int_0^\infty \left(1 - \frac{1}{e^{\eta} - 1}\left[e^{\eta\left(1+\left(-\ln x_i\right)^{\beta}\right)^{-\phi}} - 1\right]\right)\pi\left(\beta, \phi, \eta \middle| t\right)\mathrm{d}\phi\, \mathrm{d}\beta\, \mathrm{d}\eta, i = 1, 2, \\ \widetilde{R}(x) &= \int_0^\infty \int_0^\infty \int_0^\infty \overline{F}(x; \Theta)\pi\left(\beta, \phi, \eta \middle| t\right)\mathrm{d}\phi\, \mathrm{d}\beta\, \mathrm{d}\eta, \\ \widetilde{R}_F(\alpha) &= \widetilde{R}(x_1) - \widetilde{R}(x_1 + \alpha(x_2 - x_1)). \end{aligned} \quad (48)$$

5. Simulation Study

The following algorithm is used to obtain the likelihood and Bayesian estimation of parameters for the UBXIIP model and reliability with tradition and fuzzy with different α cuts, and their properties are studied using the mean squared errors (MSE), average absolute bias (AAB), and length of CIs. Our simulation algorithm begins with the creation of all simulation controls. At this stage, we must proceed in the following order:

Step 1: Assume various values for the UBXIIP distribution's parameters vector, where $\eta = 0.5, 1, 2, \beta = 0.5, 1.5$, and $\phi = 1.3$.

Step 2: Select the appropriate strength sample size $n = 50$, and 150.

Step 3: Generate the sample random values of UBXIIP distribution by using quantile function in (23).

Step 4: Solve differential equations of ML method, to obtain the estimators of the parameters for UBXIIP distribution, we calculate $\hat{\beta}, \hat{\phi}, \hat{\eta}$. Also, we calculate reliability based on nonfuzzy and fuzzy.

Step 5: Estimate hyperparameter values as

$$a_i = \left[\frac{1}{H}\sum_{j=1}^{H}\widehat{\Theta}^j\right]^2 \left[\frac{1}{H-1}\sum_{j=1}^{H}\left(\widehat{\Theta}_i^j - \frac{1}{H}\sum_{j=0}^{H}\widehat{\Theta}_i^j\right)^2\right]^{-1} \& b_i = \left[\frac{1}{H}\sum_{j=1}^{H}\widehat{\Theta}_i^j\right]\left[\frac{1}{H-1}\sum_{j=1}^{H}\left(\widehat{\Theta}_i^j - \frac{1}{H}\sum_{j=0}^{H}\widehat{\Theta}_i^j\right)^2\right]^{-1}, i = 1, 2, 3, \widehat{\Theta} \equiv (\widehat{\beta}, \widehat{\phi}, \widehat{\eta}). \quad (49)$$

Step 6: Generate posterior distribution by MCMC techniques of the Bayesian method as H = 10000, to obtain the estimates of the parameters for UBXIIP distribution, we calculate $\widetilde{\beta}, \widetilde{\phi}$, and $\widetilde{\eta}$. Also, we calculate reliability based on nonfuzzy and fuzzy.

Step 7: Compute AAB, MSE, and length of ACI (LACI) for ML estimate and length of credible CI (LCCI) for Bayesian estimates where the level of CIs is 95%. The measures AAB, MSE, and length of CI are defined by: AAB = Mean $|\hat{\Theta} - \Theta|$, MSE = Mean $(\hat{\Theta} - \Theta)^2$, and length of CI = Upper CI - Lower CI.

The following observations about the behavior of estimates are listed below, as shown in Tables 3–4.

(i) The suggested estimates of β, ϕ, and η, are pretty satisfactory in terms of lowest AAB, MSE, and LACI.

(ii) The accuracy of the estimates improves with n.

(iii) In terms of AAB, MSE, and LACI, Bayesian MCMC estimates utilizing gamma informative priors outperform frequentist estimates since they include prior knowledge.

(iv) The credible CI surpasses the asymptotic CI in terms of the smallest CI length due to the gamma prior knowledge.

(v) As η decreases, the accuracy of the estimates improves.

(vi) As β decreases, the accuracy of the estimates improves.

(vii) MSE of fuzzy reliability values has smaller values than MSE of nonfuzzy reliability values

(viii) As $\alpha -$ cut increases, then fuzzy reliability values improve, as well as MSE increases

6. Data Application

This data set is constituted by the total milk production from the first birth of 107 cows of the SINDI race. The data can be found in Cordeiro and Birto [20] and analyzed by Muhammad et al. [21]. Concretely, the data set is 0.4365, 0.4260, 0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, 0.2747.

The ML estimate evaluates all of the competing model's parameters. The Bayesian method is also used to analyze the parameters of the UBXIIP model. The Akaike information criterion (AIC), Bayesian IC (BIC), consistent AIC (CAIC), and Hannan–Quinn IC (HQIC) are all used to compare fitted models, and they all use the estimated log-likelihood as the principal constituent. The Kolmogorov-Smirnov (KSV), Anderson-Darling value (ADV), and Cramér-von Mises value (CvMV) goodness-of-fit statistics are also taken into account. As is customary, the model with the lowest value of these measures better represents the data than the other models. We compute the ML and Bayesian parameter estimators of the UBXIIP model using the standard error (SE). Kumaraswamy (K), beta, Kumaraswamy Kumaraswamy (KK) distribution (El-Sherpieny and Ahmed [22]), Marshall-Olkin Kumaraswamy (MOK) distribution (George and Thobias [23]), unit Weibull (UW) distribution (Mazucheli et al. [24]), and Topp-Leone Weibull-Lomax (TLWL) distribution (Jamal et al. [25]) are used as competing models. Table 5 shows the numerical values of the statistical measures that were considered for each model. The findings reveal that the UBXIIP model fits the data better than the other models.

Table 6 shows different estimates using ML and Bayesian estimation methods. Also, it contains reliability with tradition and fuzzy with different α-cut. We conclude from this table that Bayesian estimates perform better than ML estimates.

Figure 2 gives the plots of the histogram with the fitted PDF, empirical CDF with the fitted CDF, and P-P plots of UBXIIP model. From this figure, we conclude that the UBXIIP distribution is suitable for this data set.

Figure 3 shows convergence plots of MCMC for parameter estimates of the UBXIIP distribution. It illustrates the iterations obtained using the MH algorithm and the Gibbs sampling technique for each parameter. Also, it includes the posterior PDFs of each parameter based on the

Table 3: ML and Bayesian estimations of parameters and reliability based on nonfuzzy and fuzzy: I.

		$\beta = 1.5, \phi = 1.3$		Parameter			Nonfuzzy			Fuzzy		
η	n			β	ϕ	η	$R(t_1)$	$R(t)$	$R(t_2)$	$\alpha = 0.25$	$\alpha = 0.55$	$\alpha = 0.9$
0.5	50	ML	AAB	0.0315	0.5563	1.0294	0.0121	0.0030	0.0007	0.0037	0.0023	0.0079
			MSE	0.2332	0.5967	2.4594	0.0008	0.0046	0.0070	0.0004	0.0017	0.0058
			LACI	1.8889	2.1017	4.6401	0.0922	0.2657	0.3274	0.0771	0.1632	0.2892
		Bayesian	AAB	0.0589	0.0884	0.0658	0.0131	0.0242	0.0206	0.0082	0.0120	0.0094
			MSE	0.0274	0.0623	0.0763	0.0006	0.0035	0.0039	0.0003	0.0010	0.0020
			LCCI	0.5927	0.8852	0.9128	0.0823	0.2153	0.2354	0.0577	0.1155	0.1712
	150	ML	AAB	0.0042	0.3571	0.6636	0.0170	0.0060	0.0017	0.0026	0.0032	0.0153
			MSE	0.0661	0.2163	0.7759	0.0005	0.0017	0.0025	0.0001	0.0006	0.0021
			LACI	1.0085	1.1686	2.2720	0.0550	0.1584	0.1945	0.0441	0.0963	0.1706
		Bayesian	AAB	0.0480	0.0695	0.0120	0.0128	0.0127	0.0064	0.0051	0.0044	0.0034
			MSE	0.0232	0.0289	0.0514	0.0004	0.0014	0.0018	0.0001	0.0004	0.0013
			LCCI	0.5617	0.6001	0.7586	0.0557	0.1403	0.1673	0.0364	0.0794	0.1383
1	50	ML	AAB	0.0470	0.7050	1.2235	0.0157	0.0084	0.0022	0.0055	0.0017	0.0133
			MSE	0.2524	0.7991	3.0997	0.0007	0.0043	0.0069	0.0004	0.0016	0.0054
			LACI	1.9617	2.1556	4.9653	0.0796	0.2549	0.3260	0.0737	0.1560	0.2821
		Bayesian	AAB	0.0674	0.1034	0.0456	0.0109	0.0221	0.0189	0.0075	0.0115	0.0098
			MSE	0.0310	0.0751	0.0872	0.0005	0.0034	0.0041	0.0003	0.0010	0.0022
			LCCI	0.6211	0.9587	1.0448	0.0695	0.2097	0.2425	0.0543	0.1127	0.1821
	150	ML	AAB	0.0448	0.5815	1.0702	0.0153	0.0064	0.0005	0.0027	0.0014	0.0128
			MSE	0.0930	0.5231	1.7384	0.0004	0.0016	0.0026	0.0001	0.0005	0.0020
			LACI	1.1688	1.6869	3.0202	0.0470	0.1525	0.1982	0.0409	0.0900	0.1692
		Bayesian	AAB	0.0635	0.0782	0.0098	0.0103	0.0136	0.0083	0.0052	0.0061	0.0006
			MSE	0.0296	0.0337	0.0659	0.0003	0.0015	0.0019	0.0001	0.0004	0.0012
			LCCI	0.5937	0.6331	0.9222	0.0482	0.1403	0.1680	0.0365	0.0792	0.1342
2	50	ML	AAB	0.0465	0.8958	1.4360	0.0126	0.0072	0.0088	0.0063	0.0046	0.0137
			MSE	0.3169	1.2278	3.6640	0.0003	0.0033	0.0070	0.0003	0.0011	0.0048
			LACI	2.2003	2.5577	4.9638	0.0507	0.2219	0.3268	0.0595	0.1314	0.2674
		Bayesian	AAB	0.0699	0.1272	0.0335	0.0066	0.0188	0.0179	0.0058	0.0106	0.0120
			MSE	0.0310	0.0945	0.0944	0.0002	0.0025	0.0037	0.0002	0.0007	0.0020
			LCCI	0.6255	1.0813	1.1544	0.0435	0.1778	0.2240	0.0420	0.0933	0.1676
	150	ML	AAB	0.0335	0.8512	1.4411	0.0123	0.0071	0.0008	0.0048	0.0026	0.0086
			MSE	0.1257	0.9823	3.0928	0.0002	0.0012	0.0024	0.0001	0.0004	0.0017
			LACI	1.3841	1.9909	3.9532	0.0285	0.1305	0.1930	0.0336	0.0761	0.1586
		Bayesian	AAB	0.0608	0.0954	0.0003	0.0062	0.0142	0.0106	0.0049	0.0079	0.0058
			MSE	0.0305	0.0412	0.0657	0.0001	0.0012	0.0018	0.0001	0.0003	0.0011
			LCCI	0.6253	0.6868	0.9670	0.0328	0.1191	0.1594	0.0284	0.0633	0.1238

Table 4: ML and Bayesian estimations of parameters and reliability based on nonfuzzy and fuzzy: II.

		$\beta = 0.5, \phi = 1.3$		Parameter			Nonfuzzy			Fuzzy		
η	n			β	ϕ	η	$R(t_1)$	$R(t)$	$R(t_2)$	$\alpha = 0.25$	$\alpha = 0.55$	$\alpha = 0.9$
0.5	50	ML	AAB	0.1580	0.7828	1.1471	0.0899	0.0394	0.0208	0.0140	0.0336	0.0605
			MSE	0.0417	0.6823	1.5926	0.0094	0.0061	0.0061	0.0004	0.0020	0.0058
			LACI	0.5074	1.0342	2.0632	0.1405	0.2634	0.2949	0.0600	0.1175	0.1824
		Bayesian	AAB	0.2302	0.2097	0.0966	0.0861	0.0504	0.0349	0.0084	0.0224	0.0439
			MSE	0.0693	0.1201	0.0884	0.0085	0.0060	0.0056	0.0002	0.0011	0.0035
			LCCI	0.4788	1.0885	0.9865	0.1293	0.2310	0.2544	0.0477	0.0960	0.1522
	150	ML	AAB	0.1255	0.7586	0.9303	0.0896	0.0354	0.0204	0.0135	0.0320	0.0574
			MSE	0.0191	0.6320	1.2799	0.0084	0.0032	0.0025	0.0003	0.0013	0.0040
			LACI	0.2262	0.6996	1.2413	0.0789	0.1528	0.1721	0.0351	0.0689	0.1065
		Bayesian	AAB	0.2142	0.1840	0.0268	0.0833	0.0407	0.0234	0.0082	0.0218	0.0415
			MSE	0.0668	0.0642	0.0573	0.0074	0.0031	0.0024	0.0002	0.0010	0.0034
			LCCI	0.3499	0.6730	0.8217	0.0824	0.1518	0.1709	0.0328	0.0648	0.1041

Table 4: Continued.

	$\beta = 0.5, \phi = 1.3$			Parameter			Nonfuzzy			Fuzzy		
1	50	ML	AAB	0.1733	0.9256	1.2260	0.0845	0.0467	0.0293	0.0081	0.0231	0.0470
			MSE	0.0635	0.9712	2.0088	0.0082	0.0065	0.0066	0.0003	0.0015	0.0048
			LACI	0.7173	1.3270	2.7892	0.1282	0.2580	0.2976	0.0601	0.1214	0.1982
		Bayesian	AAB	0.2291	0.2502	0.0699	0.0757	0.0506	0.0370	0.0042	0.0143	0.0322
			MSE	0.0724	0.1463	0.0974	0.0066	0.0055	0.0053	0.0001	0.0008	0.0026
			LCCI	0.5122	1.0624	1.1128	0.1178	0.2166	0.2480	0.0443	0.0912	0.1494
	150	ML	AAB	0.1292	0.9068	1.0418	0.0840	0.0419	0.0234	0.0077	0.0215	0.0428
			MSE	0.0238	0.9070	1.9314	0.0074	0.0038	0.0029	0.0001	0.0007	0.0026
			LACI	0.3298	0.8845	2.1541	0.0697	0.1434	0.1647	0.0330	0.0663	0.1059
		Bayesian	AAB	0.2144	0.2122	0.0411	0.0748	0.0441	0.0288	0.0036	0.0138	0.0319
			MSE	0.0690	0.0782	0.0674	0.0060	0.0032	0.0025	0.0001	0.0006	0.0022
			LCCI	0.3660	0.7166	0.9822	0.0751	0.1395	0.1624	0.0298	0.0618	0.1031
2	50	ML	AAB	0.3160	1.3071	1.4667	0.0761	0.0536	0.0330	0.0010	0.0085	0.0329
			MSE	0.1959	1.9614	3.1583	0.0061	0.0053	0.0052	0.0001	0.0008	0.0035
			LACI	1.2153	1.9728	3.9364	0.0693	0.1945	0.2522	0.0470	0.1033	0.1926
		Bayesian	AAB	0.2566	0.3206	0.0656	0.0572	0.0480	0.0375	0.0018	0.0024	0.0145
			MSE	0.0953	0.1974	0.0953	0.0037	0.0039	0.0039	0.0001	0.0003	0.0014
			LCCI	0.6091	1.1230	1.1810	0.0738	0.1544	0.1901	0.0316	0.0704	0.1273
	150	ML	AAB	0.2353	1.2388	1.3620	0.0757	0.0509	0.0329	0.0010	0.0064	0.0249
			MSE	0.0707	1.9101	2.9109	0.0058	0.0043	0.0033	0.0001	0.0003	0.0014
			LACI	0.4857	1.1502	2.0969	0.0417	0.1170	0.1473	0.0278	0.0609	0.1092
		Bayesian	AAB	0.2480	0.2895	0.0627	0.0509	0.0472	0.0352	0.0011	0.0024	0.0138
			MSE	0.0916	0.1275	0.0795	0.0035	0.0030	0.0025	0.0000	0.0002	0.0010
			LCCI	0.5293	0.8067	1.0771	0.0512	0.1053	0.1310	0.0228	0.0523	0.0994

Table 5: Estimates, SE, goodness-of-fit test by using tests and different criteria measures.

		Estimates	SE	KSV	PVKS	AIC	BIC	CAIC	HQIC	CvMV	ADV
UBXIIP	β	2.7988	0.2304	0.0413	0.9932	−51.5156	−43.4972	−51.2826	−48.2650	0.0249	0.2088
	ϕ	0.9189	0.3633								
	η	2.0920	1.2821								
UW	α	0.9846	0.1015	0.1206	0.0890	−29.8423	−24.4966	−29.7269	−27.6752	0.3963	2.4244
	β	1.5619	0.1064								
K	α	2.1949	0.2224	0.0763	0.5622	−46.7894	−41.4437	−46.6740	−44.6223	0.1561	1.0090
	β	3.4366	0.5821								
Beta	α	2.4125	0.3145	0.0910	0.3384	−43.5545	−38.2088	−43.4391	−41.3874	0.2083	1.3263
	β	2.8297	0.3744								
KK	α	0.3781	0.2337	0.0761	0.5654	−45.9247	−35.2334	−45.5326	−41.5906	0.1120	0.7195
	β	5.1808	3.9516								
	θ	3.8954	9.2215								
	λ	1.4834	2.8696								
TLWL	α	9.6340	0.0248	0.0858	0.4099	−43.8721	−33.1808	−43.4800	−39.5380	0.1353	0.8556
	β	10.8776	0.0248								
	θ	0.2493	0.0260								
	λ	0.3225	0.0043								
MOK	α	11.5237	14.9262	0.0455	0.9796	−51.4916	−42.8973	−50.9853	−48.1289	0.0294	0.2015
	β	1.0163	0.5328								
	θ	3.8608	0.5487								

Table 6: ML and Bayesian estimation of parameters and reliability with traditional and fuzzy.

		ML	SE	Bayesian	SE
Parameters	β	2.7988	0.2304	2.7796	0.2300
	ϕ	0.9189	0.3633	0.9794	0.2860
	η	2.0920	1.2821	2.0102	0.9747
Traditional	$R(t_1)$	0.9821		0.9843	
	$R(t_2)$	0.4581		0.4708	
	$R(t)$	0.0634		0.0671	
R fuzzy	0.25	0.0481		0.0455	
	0.55	0.1661		0.1596	
	0.9	0.4300		0.4194	

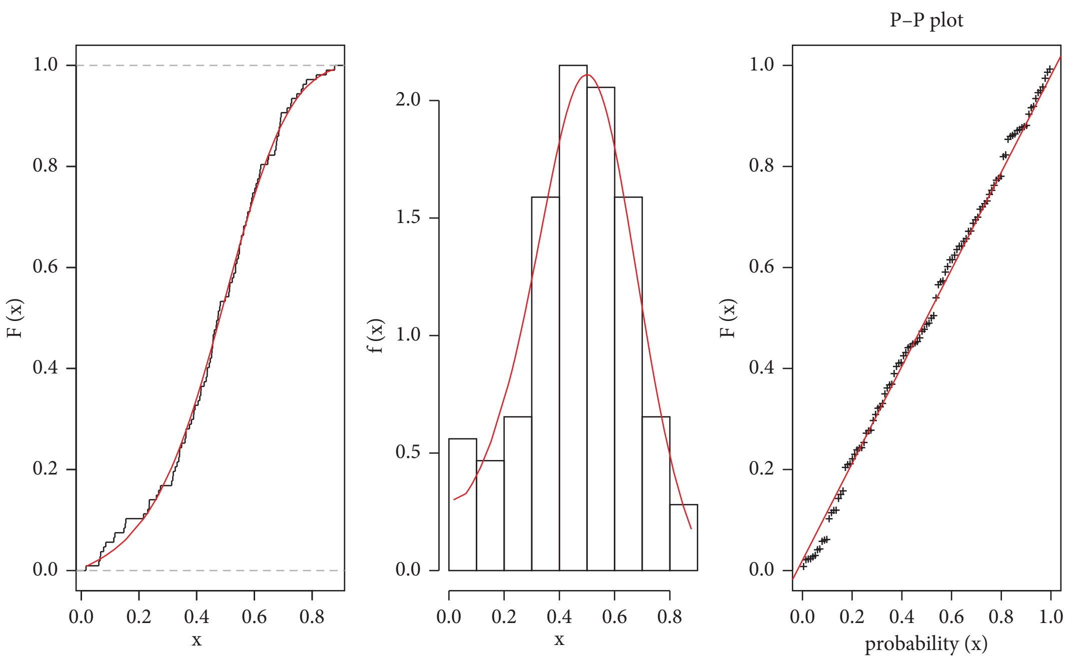

Figure 2: The cumulative function and empirical CDF, histogram, and P–P plots for UBXIIP distribution.

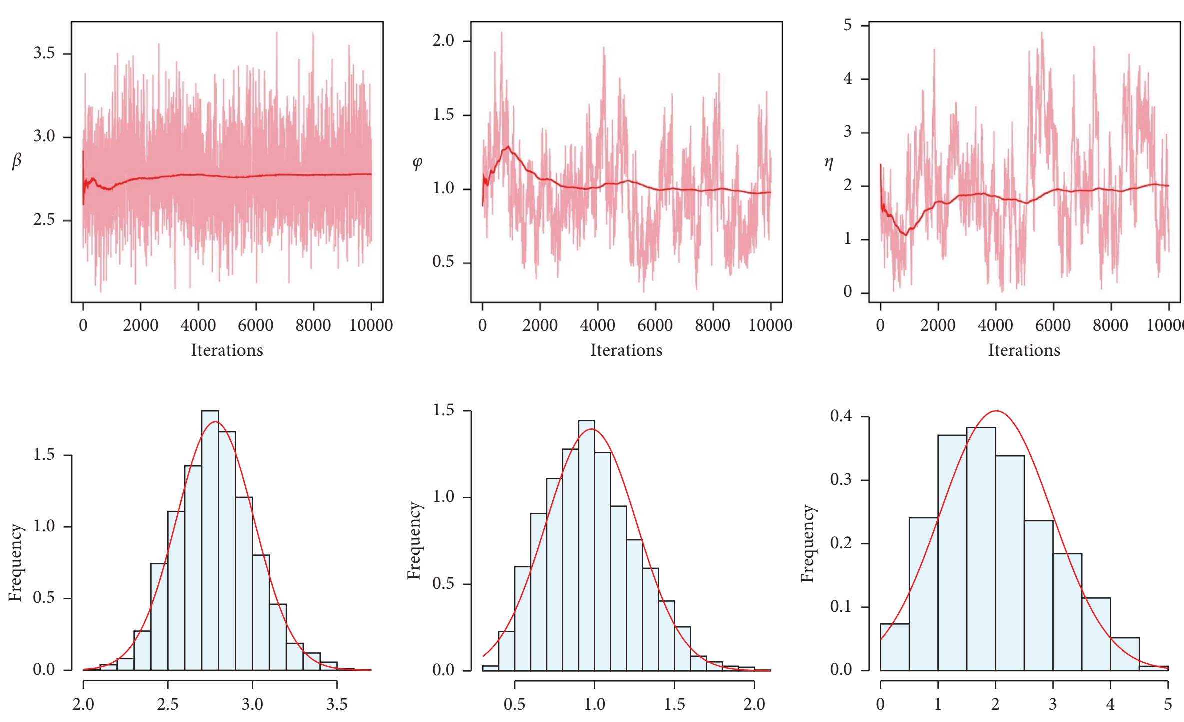

Figure 3: Iterations with convergence lines and plot of posterior of MCMC results.

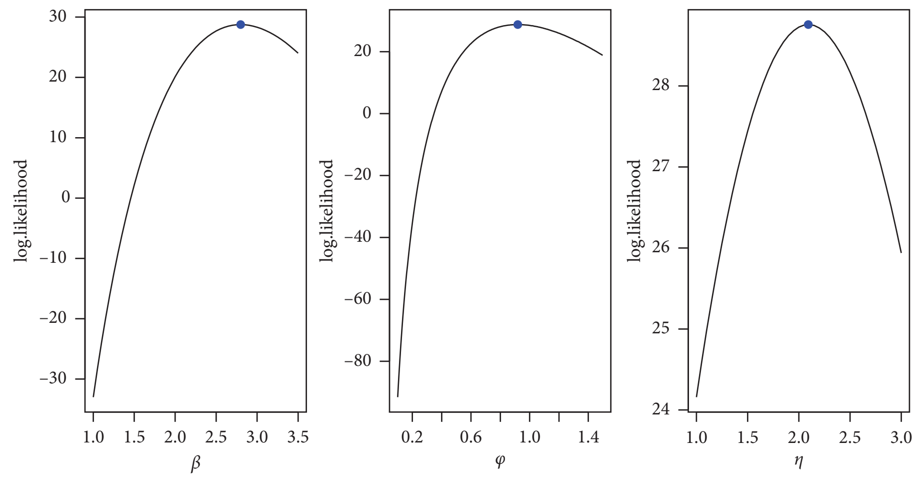

Figure 4: Profile likelihood for the three parameters for UBXIIP distribution.

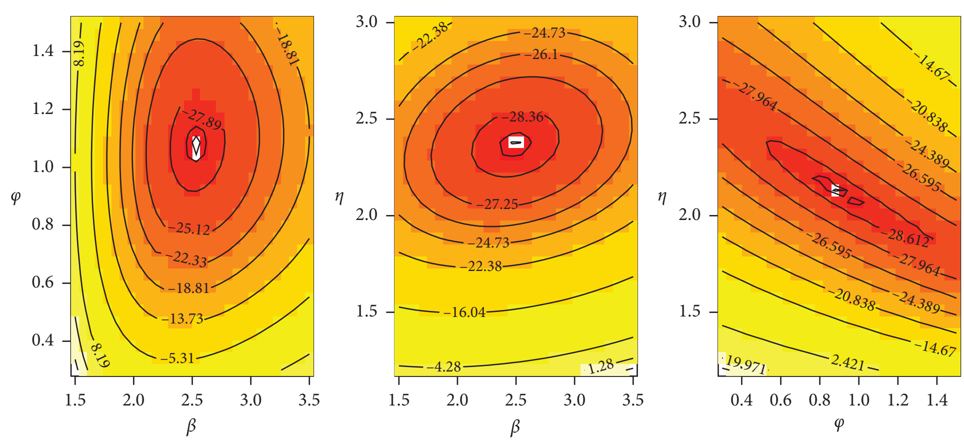

Figure 5: Contour plots for log-likelihood.

iterations obtained, to demonstrate the performance of the Bayesian estimates.

The log-likelihood function is maximized by the estimated parameters (Figure 4). The roots were found to always point to the global maximum rather than the local maximum. We were able to corroborate our findings by presenting the log-likelihood function. The estimate is at its maximum location along the curve, as indicated by the blue dot.

We sketched the log-likelihood by contour plot for each parameter for the fraction of total milk production data in Figure 5 by fixing two parameters and modifying the rest. The fraction of total milk production data set performs admirably, as shown in Figure 5, because the three roots of the parameters are global maximums.

7. Concluding Remarks

The unit Burr XII power series class is a four-parameter lifetime distribution that we introduced. The UBXII and PS distributions were combined to produce the proposed class. The hazard rate function of the UBXII distribution can take on a number of forms, including the bathtub and J-shaped. Moments and incomplete moments, mean residual life, information measures, and quantile function are among the statistical features of the UBXII distribution that we have derived. Some new submodels of the class are presented. The classical and Bayesian methods are used to obtain the estimators of parameters as well as reliability estimators via fuzzy and nonfuzzy. Furthermore, the length of ACI and the length of Bayesian credible intervals are determined. For the Bayesian method, we use the MCMC techniques to obtain the proposed estimators. To assess the effectiveness of the provided estimates, a simulation analysis of the UBXIIP model was created. Due to gamma previous information, the credible interval is less than the asymptotic CI in terms of CI length. According to simulation research findings, Bayesian estimates are preferred over non-Bayesian estimates since they have a higher degree of accuracy. Compared to nonfuzzy reliability estimates, the MSE of fuzzy reliability estimates has smaller values. In comparison to the asymptotic CI, the credible interval also has a shorter length. Application to one real data set showed that the UBXIIP distribution performs well than some other distributions. Additionally, results from real data and MCMC plots for parameter estimates of the UBXIIP distribution support the conclusions of the simulation research. The study of stress-strength reliability in light of the ranked set sampling (RSS) technique has recently captured the interest of multiple writers due to its application in a variety of fields. Therefore, in our upcoming work, we want to address the problem of stress-strength estimation for a certain distribution in the class using the RSS method [26–28], and also, for applications of lifetime data [29–31].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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