# and Abstract Applied Analysis

Special Issue
Ulam's Type Stability

Guest Editors
Janusz Brzdęk, Nicole Brillouët-Belluot, Krzysztof Ciepliński, and Bing Xu

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# Editorial Board

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*Editorial*

# Ulam's Type Stability

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The original stability problem was posed by S. M. Ulam in 1940 and concerned approximate homomorphisms. The pursuit of solutions to this problem, but also to its generalizations and modifications for various classes of (difference, functional, differential, and integral) equations and inequalities, is an expanding area of research and has led to the development of what is now quite often called *Ulam's type stability theory* or *the Hyers-Ulam stability theory*. This theory has been the subject of many papers as well as talks presented at various conferences, especially at the series of ICFEI conferences (International Conference on Functional Equations and Inequalities) organized by the Department of Mathematics of the Pedagogical University in Cracow (Poland) since 1984.

This special issue on *Ulam's type stability* is focused on the recent achievements in that type of stability for various objects. It contains 16 articles (a survey and 15 regular research papers) which have been written by 29 authors from 11 countries.

As usual, most of the authors use in their investigations direct and fixed point methods. Some open problems are also formulated.

The issue covers a wide variety of problems for different classes of functional equations both in a single variable and in several variables. Their stability is traditionally investigated in classical Banach spaces, but also in complete (probabilistic) metric spaces, complete probabilistic quasimetric spaces, $n$-Banach spaces, $(\beta, p)$-Banach spaces, and fuzzy Banach spaces.

Several papers deal with the stability of several kinds of derivations, and, thus, derivations in Riesz algebras, $(m, n)_{(\sigma,\tau)}$-derivations in normed algebras, cubic $*$-derivations in Banach $*$-algebras, and some higher ring derivations in intuitionistic fuzzy Banach algebras are studied.

The issue contains a few papers on the phenomenon of superstability, an article on the stability of a functional inequality in $p$-Banach spaces, and a paper on the Cauchy fractional differential equation in the unit disk.

Moreover, general solutions of two conditional quadratic functional equations of Pexider type and the structure of the set of all regular points and the set of all irregular points for a Brouwer homeomorphism which is embeddable in a flow are also considered.

Finally, the survey presents some selected recent developments (results and methods) in the theory of Ulam's type stability. In particular, some aspects of stability and nonstability of functional equations in a single variable, the effect ″stability implies completeness,″ some methods of proofs applied in that theory (the Forti method and the methods of fixed points), stability in non-Archimedean spaces, selected results on functional congruences, the notion of hyperstability, and stability of composite functional equations (e.g., of the Gołąb-Schinzel equation and its generalizations) are discussed there.

We believe that this volume will have some influence on the further research in that area of mathematics.

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*Review Article*

# On Some Recent Developments in Ulam's Type Stability

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We present a survey of some selected recent developments (results and methods) in the theory of Ulam's type stability. In particular we provide some information on hyperstability and the fixed point methods.

## 1. Introduction

The theory of Ulam's type stability (also quite often connected, e.g., with the names of Bourgin, Găvruţa, Ger, Hyers, and Rassias) is a very popular subject of investigations at the moment. In this expository paper we do not give an introduction to it or an ample historical background; for this we refer to [1–11]. Here we only want to attract the readers attention to some selected topics by presenting some new results and methods in several areas of the theory, which have not been treated at all or only marginally in those publications and which are somehow connected to the research interests of the authors of this paper. Also the number of references is significantly limited (otherwise the list of references would be the major part of the paper) and is only somehow representative (but certainly not fully) to the subjects discussed in this survey.

First we present a brief historical background for the stability of the Cauchy equation. Next we discuss some aspects of stability and nonstability of functional equations in single variable, some methods of proofs applied in that theory (the Forti method and the methods of fixed points), stability in non-Archmedean spaces, selected results on functional congruences, stability of composite type functional equations (in particular of the Gołąb-Schinzel equation and its generalizations), and finally the notion of hyperstability. We end the paper with remarks also on some other miscellaneous issues.

## 2. Some Classical Results Concerning the Cauchy Equation

Throughout this paper $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote, as usual, the sets of positive integers, integers, reals, and complex numbers, respectively. Moreover, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For the beginning let us mention that the first known result on stability of functional equations is due to Pólya and Szegő [12] and reads as follows.

*For every real sequence $(a_n)_{n\in\mathbb{N}}$ with*

$$\sup_{n,m\in\mathbb{N}} |a_{n+m} - a_n - a_m| \le 1, \tag{2.1}$$

*there is a real number $\omega$ such that*

$$\sup_{n\in\mathbb{N}} |a_n - \omega n| \le 1. \tag{2.2}$$

*Moreover,*

$$\omega = \lim_{n\to\infty} \frac{a_n}{n}. \tag{2.3}$$

But the main motivation for study of that subject is due to Ulam (cf. [13]), who in 1940 in his talk at the University of Wisconsin presented some unsolved problems and among them was the following question.

*Let $G_1$ be a group and $(G_2, d)$ a metric group. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies*

$$d(f(xy), f(x)f(y)) < \delta, \quad x, y \in G_1, \tag{2.4}$$

*then a homomorphism $T : G_1 \to G_2$ exists with*

$$d(f(x), T(x)) < \varepsilon, \quad x \in G_1? \tag{2.5}$$

In 1941 Hyers [14] published the following answer to it.

Let $X$ and $Y$ be Banach spaces and $\varepsilon > 0$. Then for every $g : X \to Y$ with

$$\sup_{x,y\in X} \|g(x+y) - g(x) - g(y)\| \le \varepsilon, \tag{2.6}$$

there exists a unique function $f : X \to Y$ such that

$$\sup_{x\in X} \|g(x) - f(x)\| \le \varepsilon, \tag{2.7}$$

$$f(x+y) = f(x) + f(y), \quad x, y \in X. \tag{2.8}$$

We can describe that latter result saying that *the Cauchy functional equation* (2.8) *is Hyers-Ulam stable (or has the Hyers-Ulam stability) in the class of functions* $Y^X$. For examples of various possible definitions of stability for functional equations and some discussions on them we refer to [9].

The result of Hyers was extended by Aoki [15] (for $0 < p < 1$; see also [16–18]), Gajda [19] (for $p > 1$), and Rassias [20] (for $p < 0$; see also [21, p. 326] and [22]), in the following way.

**Theorem 2.1.** *Let $E_1$ and $E_2$ be two normed spaces, let $E_2$ be complete, $c \geq 0$, and let $p \neq 1$ be a real number. Let $f : E_1 \to E_2$ be an operator such that*

$$\|f(x+y) - f(x) - f(y)\| \leq c\left(\|x\|^p + \|y\|^p\right), \quad x, y \in E_1 \setminus \{0\}. \tag{2.9}$$

*Then there exists a unique additive operator $T : E_1 \to E_2$ with*

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{\left|1 - 2^{p-1}\right|}, \quad x \in E_1 \setminus \{0\}. \tag{2.10}$$

A further generalization was suggested by Bourgin [22] (see also [2, 6–8, 23]), without a proof, and next rediscovered and improved many years later by Găvruţa [24]. Below, we present the Găvruţa type result in a bit generalized form (on the restricted domain), which can be easily derived from [25, Theorem 1].

**Corollary 2.2.** *Let $X$ be a linear space over a field with $2 \neq 0$ and let $Y$ be a Banach space. Let $V \subset X$ be nonempty, $\varphi : V^2 \to \mathbb{R}$, and $f : V \to Y$ satisfy*

$$\|g(x+y) - g(x) - g(y)\| \leq \varphi(x, y), \quad x, y \in V, \ x + y \in V. \tag{2.11}$$

*Suppose that there is $\varepsilon \in \{-1, 1\}$ such that $2^\varepsilon V \subset V$ and*

$$H(x) := \sum_{i=0}^{\infty} 2^{-i\varepsilon} \varphi\left(2^{i\varepsilon} x, 2^{i\varepsilon} x\right) < \infty, \quad x \in V, \tag{2.12}$$

$$\liminf_{n \to \infty} \left|2^{-n\varepsilon} \varphi\left(2^{n\varepsilon} x, 2^{n\varepsilon} y\right)\right| = 0, \quad x, y \in V. \tag{2.13}$$

*Then there exists a unique $F : V \to Y$ such that*

$$\begin{gathered} F(x+y) = F(x) + F(y), \quad x, y \in V, \ x + y \in V, \\ \|F(x) - f(x)\| \leq H_0(x), \quad x \in V, \end{gathered} \tag{2.14}$$

*where*

$$H_0(x) := \begin{cases} 2^{-1} H(x), & \textit{if } \varepsilon = 1, \\ H\left(2^{-1} x\right), & \textit{if } \varepsilon = -1. \end{cases} \tag{2.15}$$

Corollary 2.2 generalizes several already classical results on stability of (2.8). In fact, if we take $\varepsilon = -1$ and

$$\varphi(x,y) := L_1\|x\|^p + L_2\|y\|^q + L_3\|x\|^r\|y\|^s, \quad x,y \in V \tag{2.16}$$

with some $L_1, L_2, L_3 \in \mathbb{R}_+$, $p,q \in (1,\infty)$, and $r,s \in \mathbb{R}$ with $r+s > 1$, then $H_0$ has the form

$$H_0(x) = \frac{L_1\|x\|^p}{2^p-2} + \frac{L_2\|y\|^q}{2^q-2} + \frac{L_3\|y\|^{r+s}}{2^{r+s}-2}, \quad x \in V. \tag{2.17}$$

On the other hand, if $\varepsilon = 1$, $V \subset X \setminus \{0\}$ and

$$\varphi(x,y) := \delta + L_1\|x\|^p + L_2\|y\|^q + L_3\|x\|^r\|y\|^s, \quad x,y \in V, \tag{2.18}$$

with some $\delta, L_1, L_2, L_3 \in \mathbb{R}_+$, $q,r \in (-\infty,1)$, and $r,s \in \mathbb{R}$ with $r+s < 1$, then

$$H_0(x) = \delta + \frac{L_1\|x\|^p}{2-2^p} + \frac{L_2\|y\|^q}{2-2^q} + \frac{L_3\|y\|^{r+s}}{2-2^{r+s}}, \quad x \in V. \tag{2.19}$$

It is easily seen that, in this way, with $V = X$ and $L_1 = L_2 = L_3 = 0$ we get the result of Hyers [14], with $V = X$, $p = q$, $L_1 = L_2$ and $\delta = L_3 = 0$ we obtain Theorem 2.1, with $V = X$ and $\delta = L_1 = L_2 = 0$ we have the results of Rassias [26, 27].

*Remark 2.3.* Actually, as it is easily seen in the proof of [25, Theorem 1], it is enough to assume in Corollary 2.2 that $(X,+)$ is a commutative semigroup that is uniquely divisible by 2 (i.e., for each $x \in X$ there exists a unique $y \in X$ with $x = y + y$.)

For recent results on stability of some conditional versions of the Cauchy functional equation (2.8) we refer to, for example, [28–31].

## 3. Stability of the Linear Functional Equation in Single Variable

In this section $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, $X$ stands for a Banach space over $\mathbb{K}$, $S$ is a nonempty set, $F : S \to X$, $m \in \mathbb{N}$, $f_1,\dots,f_m : S \to S$, and $a_1,\dots,a_m : S \to \mathbb{K}$, unless explicitly stated otherwise.

The functional equation

$$\varphi(x) = \sum_{i=1}^{m} a_i(x)\varphi(f_i(x)) + F(x), \tag{3.1}$$

for $\varphi : S \to X$, is known as the linear functional equation of order $m$. For some information on it we refer to [32, 33] and the references therein.

A simply particular case of functional equation (3.1), with $S \in \{\mathbb{N}_0,\mathbb{Z}\}$, is the difference equation:

$$y_n = \sum_{i=1}^{m} a_i(n)y_{n+i} + b_n, \quad n \in S, \tag{3.2}$$

for sequences $(y_n)_{n\in S}$ in $X$, where $(b_n)_{n\in S}$ is a fixed sequence in $X$, namely, (3.1) becomes difference equation (3.2) with

$$f_i(n) = n + i, \qquad y_n := \varphi(n) = \varphi(f_1(0)), \quad b_n := F(n), \quad n \in S. \tag{3.3}$$

There are only few results on stability of (3.1), and actually only of some particular cases of it. For example, [34, Corollary 4] (cf. [34, Remark 5]) yields the following stability result.

**Corollary 3.1.** *Assume that*

$$q(x) := \sum_{i=1}^{m} |a_i(x)| < 1, \quad x \in S, \tag{3.4}$$

*and* $\varepsilon : S \longrightarrow \mathbb{R}_+$ *are such that*

$$\varepsilon(f_i(x)) \le \varepsilon(x), \qquad q(f_i(x)) \le q(x), \quad x \in S,\ i = 1, \dots, m, \tag{3.5}$$

*(e.g.,* $\varepsilon$ *and* $q$ *are constant). If a function* $\varphi : S \longrightarrow X$ *satisfies the inequality*

$$\left\| \varphi(x) - \sum_{i=1}^{m} a_i(x)\varphi(f_i(x)) - F(x) \right\| \le \varepsilon(x), \quad x \in S, \tag{3.6}$$

*then there exists a unique solution* $\psi : S \longrightarrow X$ *to* (3.1) *with*

$$\|\varphi(x) - \psi(x)\| \le \frac{\varepsilon(x)}{1 - q(x)}, \quad x \in S. \tag{3.7}$$

The assumption (3.4) seems to be quite restrictive. So far we only know that it can be avoided for some special cases of (3.1). For instance, this is the case when each function $a_i$ is constant, $a_m$ is nonzero, and $f_i = f^i$ for $i = 1, \dots, m$ (with some function $f : S \to S$), where as usual, for each $p \in \mathbb{N}_0$, $f^p$ denotes the $p$th iterate of $f$, that is,

$$f^0(x) = x, \qquad f^{p+1}(x) = f(f^p(x)), \quad p \in \mathbb{N}_0,\ x \in S. \tag{3.8}$$

Then (3.1) can be written in the following form

$$\varphi(f^m(x)) = \sum_{i=0}^{m-1} d_i \varphi\left(f^i(x)\right) + F(x), \tag{3.9}$$

with some $d_0, \dots, d_{m-1} \in \mathbb{K}$, and [35, Theorem 2] implies the following stability result.

**Theorem 3.2.** *Let $\delta \in \mathbb{R}_+$, $d_0, \ldots, d_{m-1} \in \mathbb{K}$, $\varphi_s : S \to X$ satisfy*

$$\left\| \varphi_s(f^m(x)) - \sum_{j=0}^{m-1} d_j \varphi_s\left(f^j(x)\right) - F(x) \right\| \leq \delta, \quad x \in S, \tag{3.10}$$

*and $r_1, \ldots, r_m \in \mathbb{C}$ denote the roots of the characteristic equation*

$$r^m - \sum_{j=0}^{m-1} d_j r^j = 0. \tag{3.11}$$

*Assume that one of the following three conditions is valid.*

1° $|r_j| > 1$ *for* $j = 1, \ldots, m$.

2° $|r_j| \in (1, \infty) \cup \{0\}$ *for* $j = 1, \ldots, m$ *and* $f$ *is injective.*

3° $|r_j| \neq 1$ *for* $j = 1, \ldots, m$ *and* $f$ *is bijective.*

*Then there is a solution $\varphi : S \to X$ of* (3.9) *with*

$$\|\varphi_s(x) - \varphi(x)\| \leq \frac{\delta}{|1 - |r_1|| \cdot \ldots \cdot |1 - |r_m||}, \quad x \in S. \tag{3.12}$$

*Moreover, in the case where 1° or 3° holds, $\varphi$ is the unique solution of* (3.9) *such that*

$$\sup_{x \in S} \|\varphi_s(x) - \varphi(x)\| < \infty. \tag{3.13}$$

The following example (see [35, Example 1]) shows that the statement of Theorem 3.2 need not to be valid in the general situation if $|r_j| = 1$ for some $j \in \{1, \ldots, m\}$.

*Example 3.3.* Fix $\delta > 0$. Let $S = X = \mathbb{K}$ and let the functions $f$ and $\varphi_s$ be given by

$$f(x) = x + 1, \quad \varphi_s(x) := \frac{\delta}{2} x^2, \quad x \in \mathbb{K}. \tag{3.14}$$

Then it is easily seen that

$$\begin{aligned} &\left|\varphi_s\left(f^2(x)\right) - 2\varphi_s(f(x)) + \varphi_s(x)\right| \\ &\quad = \left|\frac{\delta}{2}(x+2)^2 - \delta(x+1)^2 + \frac{\delta}{2}x^2\right| = \delta, \quad x \in \mathbb{K}. \end{aligned} \tag{3.15}$$

Suppose that $\varphi : \mathbb{K} \to \mathbb{K}$ is a solution of

$$\varphi\left(f^2(x)\right) = 2\varphi(f(x)) - \varphi(x). \tag{3.16}$$

Clearly,

$$r^2 - 2r + 1 = 0 \tag{3.17}$$

is the characteristic equation of (3.16) with the roots $r_1 = r_2 = 1$. Let

$$\psi(x) := \varphi(x+1) - \varphi(x), \qquad x \in \mathbb{K}. \tag{3.18}$$

Then it is easily seen that $\psi(x+1) = \psi(x)$ for $x \in \mathbb{K}$, whence by a simple induction on $n \in \mathbb{N}$ we get

$$\varphi(n) = \varphi(0) + n\psi(0), \quad n \in \mathbb{N}. \tag{3.19}$$

Consequently

$$\lim_{n\to\infty} \left|\varphi_s(n) - \varphi(n)\right| = \lim_{n\to\infty} \left|\frac{\delta}{2}n^2 - \varphi(0) - n\psi(0)\right| = \infty, \tag{3.20}$$

which means that

$$\sup_{x\in\mathbb{K}} \left|\varphi_s(x) - \varphi(x)\right| = \infty. \tag{3.21}$$

Thus we have shown that the statement of Theorem 3.2 is not valid in this case.

Estimation (3.12) is not optimal at least in some cases; for details we refer to [36, Remark 1.5, and Theorem 3.1] (see also [37]).

For some investigations of stability of the functional equation

$$\varphi(f^m(x)) = \sum_{j=1}^{m} a_j(x)\varphi\left(f^{m-j}(x)\right) + F(x), \tag{3.22}$$

with $m > 1$, we refer to [38] (note that the equation is a special case of (3.1) and a generalization of (3.9)). Here we only present one simplified result from there.

To this end we need a hypothesis concerning the roots of the equations

$$z^m - \sum_{j=1}^{m} a_j(x) z^{m-j} = 0, \tag{3.23}$$

with $x \in S$, which reads as follows.

($\mathscr{H}$) Functions $r_1, \ldots, r_m : S \to \mathbb{C}$ satisfy the condition

$$\prod_{i=1}^{m} (z - r_i(x)) = z^m - \sum_{j=1}^{m} a_j(x) z^{m-j}, \quad x \in S, \ z \in \mathbb{C}. \tag{3.24}$$

Hypothesis $(\mathscr{H})$ means that, for every $x \in S$, $r_1(x), \ldots, r_m(x) \in \mathbb{C}$ are the complex roots of (3.23). Clearly, the functions $r_1, \ldots, r_m$ are not unique, but for every $x \in S$ the sequence

$$(r_1(x), \ldots, r_m(x)) \tag{3.25}$$

is uniquely determined up to a permutation. Moreover, $0 \notin a_m(S)$ if and only if $0 \notin r_j(S)$ for each $j = 1, \ldots, m$ (see [38, Remark 1]).

We say that a function $g : S \to X$ is $f$-invariant provided

$$g(f(x)) = g(x), \quad x \in S. \tag{3.26}$$

Now we are in a position to present a result that can de deduced from [38, Theorem 1].

**Theorem 3.4.** *Let $\varepsilon_0 : S \to \mathbb{R}_+$, let $(\mathscr{H})$ be valid, and let $r_j$ be $f$-invariant for $j > 1$.*

*Assume that $0 \notin a_m(S)$ and $\varphi_s : S \to X$ fulfills the inequality*

$$\left\| \varphi_s(f^m(x)) - \sum_{j=1}^{m} a_j(x) \varphi_s\left(f^{m-j}(x)\right) - F(x) \right\| \le \varepsilon_0(x), \quad x \in S. \tag{3.27}$$

*Further, suppose that*

$$\begin{gathered} \varepsilon_1(x) := \sum_{k=0}^{\infty} \frac{\varepsilon_0(f^k(x))}{\prod_{p=0}^{k} \left| r_1(f^p(x)) \right|} < \infty, \quad x \in S, \\ \varepsilon_j(x) := \sum_{k=0}^{\infty} \frac{\varepsilon_{j-1}(f^k(x))}{\left| r_j(x) \right|^{k+1}} < \infty, \quad x \in S, \ j > 1. \end{gathered} \tag{3.28}$$

*Then* (3.22) *has a solution $\varphi : S \to X$ with*

$$\left\| \varphi_s(x) - \varphi(x) \right\| \le \varepsilon_m(x), \quad x \in S. \tag{3.29}$$

As it folows from [38, Remark 8], the form of $\varphi$ in Theorem 3.4 can be explicitly described in some recurrent way.

Some further results on stability of (3.9), particular cases of it and some other similar equations in single variable can be found in [1, 35, 39–51]. For instance, it has been shown in [34, 52, 53] that stability of numerous functional equations of this kind is a direct consequence of some fixed point results. We deal with that issue in the section on the fixed point methods.

At the end of this part we would like to suggest some terminology that might be useful in the investigation of stability also for some other equations (as before, $B^D$ denotes the class of functions mapping a nonempty set $D$ into a nonempty set $B$). Moreover, that terminology could be somehow helpful in clarification of the notion of nonstability, which is very briefly discussed in the next section.

*Definition 3.5.* Let $\mathcal{C} \subset \mathbb{R}_+^S$ be nonempty and let $\mathcal{T}$ be an operator mapping $\mathcal{C}$ into $\mathbb{R}_+^S$. We say that (3.1) is $\mathcal{T}$-stable (with uniqueness, resp.) provided for every $\varepsilon \in \mathcal{C}$ and $\varphi : S \to X$ with

$$\left\| \varphi_s(x) - \sum_{i=1}^{m} a_i(x) \varphi_s(f_i(x)) - F(x) \right\| \le \varepsilon(x), \quad x \in S \tag{3.30}$$

there exists a (unique, resp.) solution $\widetilde{\varphi} : S \to X$ of (3.1) such that

$$\| \varphi(x) - \widetilde{\varphi}(x) \| \le \mathcal{T}\varepsilon(x), \quad x \in S. \tag{3.31}$$

In connection with the original statement of Ulam's problem we might think of yet another definition that seems to be quite natural and useful sometimes.

*Definition 3.6.* Let $\varepsilon : S \to \mathbb{R}_+$ and $L \in \mathbb{R}_+$. We say that functional equation (3.1) is $(\varepsilon, L)$-stable (with uniqueness, resp.,) provided for every function $\varphi : S \to X$ satisfying (3.30), there exists a (unique, resp.,) solution $\widetilde{\varphi} : S \to X$ to (3.1) such that

$$\| \varphi(x) - \widetilde{\varphi}(x) \| \le L\varepsilon(x), \quad x \in S. \tag{3.32}$$

Given $a : S \to \mathbb{R}_+ \setminus \{0\}$, for each $\phi : S \to \mathbb{R}_+$ we write

$$\begin{gathered} \mathcal{A}_a^f \phi(x) := \sum_{j=0}^{\infty} \frac{\phi(f^j(x))}{\prod_{k=0}^{j} |a(f^k(x))|}, \quad x \in S, \\ \mathfrak{D} := \left\{ \varepsilon : S \to \mathbb{R}_+^0 : \mathcal{A}_a^f \varepsilon(x) < \infty, x \in S \right\}. \end{gathered} \tag{3.33}$$

Then $\mathcal{A}_a^f$ is an operator mapping $\mathfrak{D}$ into $\mathbb{R}^S$ and, according to Theorem 3.4, the functional equation

$$\psi(f(x)) = a(x)\psi(x) + F(x), \quad x \in S \tag{3.34}$$

(i.e., (3.22) with $m = 1$) is $\mathcal{A}_a^f$-stable with uniqueness (cf. [48, Theorem 2.1]).

Further, note that for every $\varepsilon \in \mathbb{R}_+^S$ with

$$\begin{gathered} \varepsilon(f(t)) \le \varepsilon(t), \quad t \in S, \\ s := \inf_{t \in S} |a(t)| > 1, \end{gathered} \tag{3.35}$$

we have

$$\mathcal{A}_a^f \varepsilon(x) \le \sum_{j=0}^{\infty} \frac{\varepsilon(x)}{s^k} = \frac{\varepsilon(x)}{s-1}, \quad x \in S, \tag{3.36}$$

and consequently (3.34) is $(\varepsilon, L)$-stable with

$$L := \frac{1}{s-1}. \tag{3.37}$$

## 4. Nonstability

There are only few outcomes of which we could say that they concern nonstability of functional equation. The first well-known one is due to Gajda [19] and answers a question raised by Rassias [54]. Namely, he gave an example of a function showing that a result analogous to that described in Theorem 2.1 cannot be obtained for $p = 1$ (for further such examples see [21]; cf. also, e.g., [55, 56]).

In general it is not easy to define the notion of nonstability precisely, mostly because at the moment there are several notions of stability in use (see [9, 57]). For instance, we could understand nonstability as in Example 3.3. The other possibility is to refer to Definitions 3.5 and 3.6 and define $\mathfrak{T}$-nonstability and $(\varepsilon, L)$-nonstability, respectively. Finally, if there does not exist an $L \in \mathbb{R}_+$ such that the equation is $(\varepsilon, L)$-stable, then we could say that it is $\varepsilon$-nonstable.

For some further propositions of such definitions and preliminary results on nonstability we refer to [58–62]. As an example we present below the result from [60, Theorem 1] concerning nonstability of the difference equation

$$x_{n+1} = \overline{a}_n x_n + b_n, \quad n \in \mathbb{N}_0, \tag{4.1}$$

where $(x_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are sequences in X and $(\overline{a}_n)_{n\geq 0}$ is a sequence in $\mathbb{K}$.

**Theorem 4.1.** *Let $(\varepsilon_n)_{n\geq 0}$ be a sequence of positive real numbers, $(b_n)_{n\geq 0}$ a sequence in X, and $(\overline{a}_n)_{n\geq 0}$ a sequence in $\mathbb{K}$ with the property*

$$\lim_{n\to\infty} \frac{\varepsilon_n |\overline{a}_{n+1}|}{\varepsilon_{n+1}} = 1. \tag{4.2}$$

*Then there exists a sequence $(y_n)_{n\geq 0}$ in X with*

$$\left\| y_{n+1} - \overline{a}_n y_n - b_n \right\| \leq \varepsilon_n, \quad n \in \mathbb{N}_0, \tag{4.3}$$

*such that, for every sequence $(x_n)_{n\geq 0}$ in X satisfying recurrence* (4.1)*,*

$$\sup_{n\in\mathbb{N}} \frac{\left\| x_n - y_n \right\|}{\varepsilon_{n-1}} = \infty. \tag{4.4}$$

The issue of nonstability seems to be a new promising area for research.

## 5. Stability and Completeness

It is well known that the completeness of the target space is of great importance in the theory of Hyers-Ulam stability of functional equations; we could observe this fact for the stability of the Cauchy equation in the second section.

In [63], Forti and Schwaiger proved that if $X$ is a commutative group containing an element of infinite order, $Y$ is a normed space, and the Cauchy functional equation is Hyers-Ulam stable in the class $Y^X$, then the space $Y$ has to be complete (let us also mention here that Moszner [64] showed that all four assumptions are essential to get the completeness of $Y$).

The above-described effect, *stability implies completeness*, was recently proved for some other equations (see [65–68]). Here we present only one result of this kind. It concerns the quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{5.1}$$

and comes from [67].

**Theorem 5.1.** *Let $X$ be a finitely generated free commutative group and $Y$ be a normed space. If* (5.1) *is Hyers-Ulam stable in the class $Y^X$, then the space $Y$ is complete.*

## 6. The Method of Forti

As Forti [43] (see also, e.g., [69]) has clearly demonstrated, the stability of functional equations in single variable, in particular of the form:

$$\Psi \circ F \circ a = F \tag{6.1}$$

plays a basic role in many investigations of the stability of functional equations in several variables. Some examples presenting that method can be found in [25, 70, 71] (see also [72]). Here we give only one such example that corrects [70, Corollary 3.2], which unfortunately has been published with some details confused. The main tool is the following theorem (see [70, Theorem 2.1]; cf. [43]).

**Theorem 6.1.** *Assume that $(Y,d)$ is a complete metric space, $K$ is a nonempty set, $f : K \to Y$, $\Psi : Y \to Y$, $a : K \to K$, $h : K \to \mathbb{R}_+$, $\lambda \in \mathbb{R}_+$,*

$$\begin{gathered} d(\Psi \circ f \circ a(x), f(x)) \le h(x), \quad x \in K, \\ d(\Psi(x), \Psi(y)) \le \lambda d(x,y), \quad x,y \in Y, \\ H(x) := \sum_{i=0}^{\infty} \lambda^i h\left(a^i(x)\right) < \infty, \quad x \in K. \end{gathered} \tag{6.2}$$

*Then, for every $x \in K$, the limit*

$$F(x) := \lim_{n\to\infty} \Psi^n \circ f \circ a^n(x) \tag{6.3}$$

*exists and* $F : K \to Y$ *is the unique function such that* (6.1) *holds and*

$$d\big(f(x), F(x)\big) \le H(x), \quad x \in K. \tag{6.4}$$

The next corollary presents the corrected version of [70, Corollary 3.2] and its proof. Let us make some preparations for it.

First, let us recall that a groupoid $(G, +)$ (i.e., a nonempty set $G$ endowed with a binary operation $+ : G^2 \to G$) is uniquely divisible by 2 provided, for each $x \in X$, there is a unique $y \in X$ with $x = 2y := y + y$; such $y$ we denote by $(1/2)x$. Next, we use the notion:

$$2^0 x := x, \qquad 2^n x = 2\left(2^{n-1} x\right), \tag{6.5}$$

and (only if the groupoid is uniquely divisible by 2)

$$2^{-n} x = \frac{1}{2}\left(2^{-n+1} x\right), \tag{6.6}$$

for every $x \in G$, $n \in \mathbb{N}$.

A groupoid $(G, +)$ is square symmetric provided the operation $+$ is square symmetric, that is, $2(x + y) = 2x + 2y$ for $x, y \in G$; it is easy to show by induction that, for each $n \in \mathbb{N}$ (for all $n \in \mathbb{Z}$, if the groupoid is uniquely divisible by 2), we have

$$2^n\left(x + y\right) = 2^n x + 2^n y, \quad x, y \in G. \tag{6.7}$$

Clearly every commutative semigroup is a square symmetric groupoid. Next, let $X$ be a linear space over a field $\mathbb{K}$, $a, b \in \mathbb{K}$, $z \in X$, and define a binary operation $* : X^2 \to X$ by

$$x * y := ax + by + z, \quad x, y \in X. \tag{6.8}$$

Then it is easy to check that $(X, *)$ provides a simple example of a square symmetric groupoid.

The square symmetric groupoids have been already considered in several papers investigating the stability of some functional equations (see, e.g., [73–79]). For a description of square symmetric operations we refer to [80].

Finally, we say that $(G, +, d)$ is a complete metric groupoid provided $(G, +)$ is a groupoid, $(G, d)$ is a complete metric space, and the operation $+ : G^2 \to G$ is continuous, in both variables simultaneously, with respect to the metric $d$.

Now we are in a position to present the mentioned above corrected version of [70, Corollary 3.2].

**Corollary 6.2.** *Let $(X,+)$ and $(Y,+)$ be square symmetric groupoids, $(Y,+)$ be uniquely divisible by 2, $(Y,+,d)$ be a complete metric groupoid, $K \subset X$, $2K \subset K$ (i.e., $2a \in K$ for $a \in K$), and $\chi : X^2 \to \mathbb{R}_+$. Suppose that there exist $\xi, \eta \in \mathbb{R}_+$ such that $\xi\eta < 1$,*

$$d\left(\frac{1}{2}x, \frac{1}{2}y\right) \le \xi d(x,y), \quad x,y \in Y,$$
$$\chi(2x,2y) \le \eta\chi(x,y), \quad x,y \in K, \tag{6.9}$$

*and $\varphi : K \to Y$ satisfies*

$$d(\varphi(x+y), \varphi(x)+\varphi(y)) \le \chi(x,y), \quad x,y \in K,\ x+y \in K. \tag{6.10}$$

*Then there is a unique function $F : K \to Y$ with*

$$F(x+y) = F(x) + F(y), \quad x,y \in K,\ x+y \in K, \tag{6.11}$$

$$d(\varphi(x), F(x)) \le \frac{\xi}{1-\xi\eta}\chi(x,x), \quad x \in K. \tag{6.12}$$

*Proof.* From (6.10), with $x = y$, we obtain $d(\varphi(2x), 2\varphi(x)) \le \chi(x,x)$ for $x \in K$, which yields

$$d\left(\frac{1}{2}\varphi(2x), \varphi(x)\right) \le \xi d(\varphi(2x), 2\varphi(x)) \le \xi\chi(x,x), \quad x \in K. \tag{6.13}$$

Hence, by Theorem 6.1 (with $\lambda = \xi$, $f = \varphi$, $\Psi(z) = (1/2)z$, $h(x) = \xi\chi(x,x)$, and $a(x) = 2x$) the limit

$$F(x) := \lim_{n\to\infty} 2^{-n}\varphi(2^n x) \tag{6.14}$$

exists for every $x \in K$ and

$$d(\varphi(x), F(x)) \le \xi\chi(x,x)\sum_{i=0}^{\infty}(\xi\eta)^i \le \frac{\xi}{1-\xi\eta}\chi(x,x), \quad x \in K. \tag{6.15}$$

Next, by (6.7) and (6.10), for every $x, y \in K$ with $x+y \in K$, we have

$$d(2^{-n}\varphi(2^n(x+y)), 2^{-n}\varphi(2^n x) + 2^{-n}\varphi(2^n y)) \le (\xi\eta)^n\chi(x,y), \tag{6.16}$$

for $n \in \mathbb{N}$, whence letting $n \to \infty$ we deduce that $F$ is a solution of (6.11).

It remains to show the uniqueness of $F$. So suppose that $G : K \to Y$,

$$d(\varphi(x), G(x)) \le \frac{\xi}{1-\xi\eta}\chi(x,x), \quad x \in K, \tag{6.17}$$

$$G(x+y) = G(x) + G(y), \quad x,y \in K,\ x+y \in K. \tag{6.18}$$

Then

$$d(F(x),G(x)) \leq d\big(F(x),\varphi(x)\big) + d\big(\varphi(x),G(x)\big) \leq \frac{2\xi}{1-\xi\eta}\chi(x,x), \quad x \in K, \tag{6.19}$$

and by induction it is easy to show that (6.11) and (6.18) yield $F(2^n x) = 2^n F(x)$ and $G(2^n x) = 2^n G(x)$ for every $x \in K$ and $n \in \mathbb{N}$. Hence, for each $x \in K$,

$$\begin{aligned} d(F(x),G(x)) &= d\big(2^{-n}F(2^n x), 2^{-n}G(2^n x)\big) \\ &\leq \xi^n \chi(2^n x, 2^n x) \leq (\xi\eta)^n \chi(x,x). \end{aligned} \tag{6.20}$$

Since $\xi\eta < 1$, letting $n \to \infty$ we get $F = G$. □

## 7. The Fixed Point Methods

Apart from the classical method applied by Hyers and its modification proposed by Forti (see also [72]), the fixed point methods seem to be the most popular at the moment in the investigations of the stability of functional equations, both in single and several variables. Although the fixed point method was used for the first time by Baker [39] who applied a variant of Banach's fixed point theorem to obtain the Hyers-Ulam stability of the functional equation

$$f(t) = F\big(t, f(\varphi(t))\big), \tag{7.1}$$

most authors follow Radu's approach (see [81], where a new proof of Theorem 2.1 for $p \in \mathbb{R}_+ \setminus \{1\}$ was given) and make use of a theorem of Diaz and Margolis. Here we only present one of the recent results obtained in this way.

Let us recall that a mapping $f : V^n \to W$, where $V$ is a commutative group, $W$ is a linear space, and $n$ is a positive integer, is called *multiquadratic* if it is quadratic in each variable. Similarly we define *multiadditive* and *multi-Jensen* mappings. Some basic facts on multiadditive functions can be found for instance in [82] (where their application to the representation of polynomial functions is also presented), whereas for the general form of multi-Jensen mappings and their connection with generalized polynomials we refer to [83].

The stability of multiadditive, multi-Jensen, and multiquadratic mappings was recently investigated in [68, 84–93]. In particular, in [88] Radu's approach was applied to the proof of the following theorem.

**Theorem 7.1.** *Let $W$ be a Banach space and for every $i \in \{1,\dots,n\}$, let $\varphi_i : V^{n+1} \to \mathbb{R}_+$ be a mapping such that*

$$\begin{aligned}
&\lim_{j\to\infty} \frac{1}{4^j}\varphi_i\left(2^j x_1, x_2,\dots,x_{n+1}\right) = \cdots \\
&\qquad = \lim_{j\to\infty}\frac{1}{4^j}\varphi_i\left(x_1,\dots,x_{i-2},2^j x_{i-1},x_i,\dots,x_{n+1}\right) \\
&\qquad = \lim_{j\to\infty}\frac{1}{4^j}\varphi_i\left(x_1,\dots,x_{i-1},2^j x_i,2^j x_{i+1},x_{i+2},\dots,x_{n+1}\right) \\
&\qquad = \lim_{j\to\infty}\frac{1}{4^j}\varphi_i\left(x_1,\dots,x_{i+1},2^j x_{i+2},x_{i+3},\dots,x_{n+1}\right) = \cdots \\
&\qquad = \lim_{j\to\infty}\frac{1}{4^j}\varphi_i\left(x_1,\dots,x_n,2^j x_{n+1}\right) = 0, \quad (x_1,\dots,x_{n+1}) \in V^{n+1},
\end{aligned} \tag{7.2}$$

$$\begin{aligned}
&\varphi_i(x_1,\dots,x_{i-1},2x_i,2x_i,x_{i+1},\dots,x_n) \\
&\quad \le 4L_i\varphi_i(x_1,\dots,x_i,x_i,x_{i+1},\dots,x_n), \quad (x_1,\dots,x_n)\in V^n,
\end{aligned} \tag{7.3}$$

*for an $L_i \in (0,1)$. If $f : V^n \to W$ is a mapping satisfying, for any $i \in \{1,\dots,n\}$,*

$$f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n) = 0, \quad (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) \in V^{n-1}, \tag{7.4}$$

$$\begin{aligned}
&\left\| f(x_1,\dots,x_{i-1},x_i+x_i',x_{i+1},\dots,x_n) + f(x_1,\dots,x_{i-1},x_i-x_i',x_{i+1},\dots,x_n) \right. \\
&\quad \left. - 2f(x_1,\dots,x_n) - 2f(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_n)\right\| \\
&\quad \le \varphi_i(x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n), \quad (x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n)\in V^{n+1},
\end{aligned} \tag{7.5}$$

*then for every $i \in \{1,\dots,n\}$ there exists a unique multiquadratic mapping $F_i : V^n \to W$ such that*

$$\begin{aligned}
\left\| f(x_1,\dots,x_n) - F_i(x_1,\dots,x_n)\right\| \le \frac{1}{4-4L_i}\varphi_i(x_1,\dots,x_i,x_i,x_{i+1},\dots,x_n), \\
(x_1,\dots,x_n)\in V^n.
\end{aligned} \tag{7.6}$$

Baker's idea (to prove his result it is enough to define suitable (complete) metric space and (contractive) operator, which form follows from the considered equation (in this case $T(a)(t) := F(t,a(\varphi(t)))$), and apply the (Banach) fixed point theorem) was used by several mathematicians, who applied other fixed point theorems to extend and generalize Baker's result. Now, we present some of these recent outcomes.

To formulate the first of them, let us recall that a mapping $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* if it is nondecreasing and

$$\lim_{n\to\infty} \gamma^n(t) = 0, \quad t \in (0,\infty). \tag{7.7}$$

In [94], Matkowski's fixed point theorem was applied to the proof of the following generalization of Baker's result.

**Theorem 7.2.** *Let $S$ be a nonempty set, let $(X,d)$ be a complete metric space, $\varphi : S \to S$, and $F : S \times X \to X$. Assume also that*

$$d(F(t,u), F(t,v)) \le \gamma(d(u,v)), \quad t \in S,\ u,v \in X, \tag{7.8}$$

*where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, and let $g : S \to X$, $\delta > 0$ be such that*

$$d(g(t), F(t, g(\varphi(t)))) \le \delta, \quad t \in S. \tag{7.9}$$

*Then there is a unique function $f : S \to X$ satisfying* (7.1) *and*

$$\rho(f,g) := \sup\{d(f(t), g(t)), t \in S\} < \infty. \tag{7.10}$$

*Moreover, $\rho(f,g) - \gamma(\rho(f,g)) \le \delta$.*

On the other hand, in [95], Baker's idea and a variant of Ćirić's fixed point theorem were used to obtain the following result concerning the stability of (7.1).

**Theorem 7.3.** *Let $S$ be a nonempty set, let $(X,d)$ be a complete metric space, $\varphi : S \to S$, and $F : S \times X \to X$ and*

$$\begin{aligned} d(F(t,x), F(t,y)) \le{} & \alpha_1(x,y) d(x,y) + \alpha_2(x,y) d(x, F(t,x)) \\ & + \alpha_3(x,y) d(y, F(t,y)) + \alpha_4(x,y) d(x, F(t,y)) \\ & + \alpha_5(x,y) d(y, F(t,x)), \quad t \in S,\ x,y \in X, \end{aligned} \tag{7.11}$$

*where $\alpha_1, \dots, \alpha_5 : X \times X \to \mathbb{R}_+$ satisfy*

$$\sum_{i=1}^{5} \alpha_i(x,y) \le \lambda, \tag{7.12}$$

*for all $x, y \in X$ and a $\lambda \in [0,1)$. If $g : S \to X$, $\delta > 0$, and* (7.9) *holds, then there is a unique function $f : S \to X$ satisfying* (7.1) *and*

$$d(f(t), g(t)) \le \frac{(2+\lambda)\delta}{2(1-\lambda)}, \quad t \in S. \tag{7.13}$$

A consequence of Theorem 7.3 is the following result on the stability of the linear functional equation of order 1.

**Corollary 7.4.** *Let $S$ be a nonempty set, let $E$ be a real or complex Banach space, $\varphi : S \to S$, $\alpha : S \to E$, $B : S \to \mathfrak{L}(E)$ (here $\mathfrak{L}(E)$ denotes the Banach algebra of all bounded linear operators on $E$), $\lambda \in [0,1)$, and*

$$\|B(t)\| \leq \lambda, \quad t \in S. \tag{7.14}$$

*If $g : S \to E$, $\delta > 0$, and*

$$\left\| g(t) - \left(\alpha(t) + B(t)\left(g(\varphi(t))\right)\right) \right\| \leq \delta, \quad t \in S, \tag{7.15}$$

*then there exists a unique function $f : S \to E$ satisfying*

$$f(t) = \alpha(t) + B(t)\left(f\left(\varphi(t)\right)\right), \quad t \in S, \tag{7.16}$$

*and the condition*

$$\left\| f(t) - g(t) \right\| \leq \frac{\delta}{1-\lambda}, \quad t \in S. \tag{7.17}$$

In [96], Miheţ gave one more generalization of Baker's result. In order to do this he proved a fixed point alternative and used it in the proof of this generalization. To formulate Miheţ's theorem, let us recall that a mapping $\gamma : [0,\infty] \to [0,\infty]$ is called a *generalized strict comparison function* if it is nondecreasing, $\gamma(\infty) = \infty$,

$$\begin{gathered} \lim_{n\to\infty} \gamma^n(t) = 0, \quad t \in (0,\infty), \\ \lim_{t\to\infty} \left(t - \gamma(t)\right) = \infty. \end{gathered} \tag{7.18}$$

**Theorem 7.5.** *Let $S$ be a nonempty set, let $(X,d)$ be a complete metric space, $\varphi : S \to S$, and $F : S \times X \to X$. Assume also that*

$$d(F(t,u), F(t,v)) \leq \gamma(d(u,v)), \quad t \in S,\ u,v \in X, \tag{7.19}$$

*where $\gamma : [0,\infty] \to [0,\infty]$ is a generalized strict comparison function and let $g : S \to X$, $\delta > 0$ be such that (7.9) holds. Then there is a unique function $f : S \to X$ satisfying (7.1) and*

$$d\left(f(t), g(t)\right) \leq \sup\left\{s > 0 : s - \gamma(s) \leq \delta\right\}, \quad t \in S. \tag{7.20}$$

A somewhat different fixed point approach to the Hyers-Ulam stability of functional equations, in which the stability results are simple consequences of some new fixed point theorems, can be found in [34, 52, 53, 97].

Given a nonempty set $S$ and a metric space $(X, d)$, we define $\Delta : (X^S)^2 \to \mathbb{R}_+{}^S$ by

$$\Delta(\xi, \mu)(t) := d(\xi(t), \mu(t)), \quad \xi, \mu \in X^S, \ t \in S. \tag{7.21}$$

Now, we are in a position to present the following fixed point theorem from [34].

**Theorem 7.6.** *Let $S$ be a nonempty set, let $(X, d)$ be a complete metric space, $k \in \mathbb{N}$, $f_1, \dots, f_k : S \to S$, $L_1, \dots, L_k : S \to \mathbb{R}_+$, and let $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$ be given by*

$$(\Lambda\delta)(t) := \sum_{i=1}^{k} L_i(t)\delta(f_i(t)), \quad \delta \in \mathbb{R}_+^S, \ t \in S. \tag{7.22}$$

*If $\mathcal{T} : X^S \to X^S$ is an operator satisfying the inequality*

$$\Delta(\mathcal{T}\xi, \mathcal{T}\mu)(t) \le \Lambda(\Delta(\xi, \mu))(t), \quad \xi, \mu \in X^S, \ t \in S \tag{7.23}$$

*and functions $\varepsilon : S \to \mathbb{R}_+$ and $g : S \to X$ are such that*

$$\Delta(\mathcal{T}g, g)(t) \le \varepsilon(t), \quad t \in S, \tag{7.24}$$

$$\sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(t) =: \sigma(t) < \infty, \quad t \in S, \tag{7.25}$$

*then for every $t \in S$ the limit*

$$\lim_{n \to \infty} (\mathcal{T}^n g)(t) =: f(t) \tag{7.26}$$

*exists and the function $f : S \to X$, defined in this way, is a unique fixed point of $\mathcal{T}$ with*

$$\Delta(g, f)(t) \le \sigma(t), \quad t \in S. \tag{7.27}$$

A consequence of Theorem 7.6 is the following result on the stability of a quite wide class of functional equations in a single variable.

**Corollary 7.7.** *Let $S$ be a nonempty set, let $(X, d)$ be a complete metric space, $k \in \mathbb{N}$, $f_1, \dots, f_k : S \to S$, $L_1, \dots, L_k : S \to \mathbb{R}_+$, and let a function $\Phi : S \times X^k \to X$ satisfy the inequality*

$$d(\Phi(t, y_1, \dots, y_k), \Phi(t, z_1, \dots, z_k)) \le \sum_{i=1}^{k} L_i(t) d(y_i, z_i), \tag{7.28}$$

*for any $(y_1, \dots, y_k), (z_1, \dots, z_k) \in X^k$ and $t \in S$, and $\mathcal{T} : X^S \to X^S$ be an operator defined by*

$$(\mathcal{T}\varphi)(t) := \Phi(t, \varphi(f_1(t)), \dots, \varphi(f_k(t))), \quad \varphi \in X^S, \ t \in S. \tag{7.29}$$

*Assume also that* $\Lambda$ *is given by* (7.22) *and functions* $g : S \to X$ *and* $\varepsilon : S \to \mathbb{R}_+$ *are such that*

$$d\big(g(t), \Phi\big(t, g\big(f_1(t)\big), \ldots, g\big(f_k(t)\big)\big)\big) \le \varepsilon(t), \quad t \in S \tag{7.30}$$

*and* (7.25) *holds. Then for every* $t \in S$ *limit* (7.26) *exists and the function* $f : S \to X$ *is a unique solution of the functional equation*

$$\Phi\big(t, f\big(f_1(t)\big), \ldots, f\big(f_k(t)\big)\big) = f(t), \quad t \in S \tag{7.31}$$

*satisfying inequality* (7.27).

Let us also mention here that very recently Cădariu et al. [97] improved the above two outcomes considering, instead of that given by (7.22), a more general operator $\Lambda$.

Next, following [53], we deal with the case of non-Archimedean metric spaces. In order to do this, we introduce some notations and definitions.

Let $S$ be a nonempty set. For any $\delta_1, \delta_2 \in \mathbb{R}_+^S$ we write $\delta_1 \le \delta_2$ provided

$$\delta_1(t) \le \delta_2(t), \quad t \in S, \tag{7.32}$$

and we say that an operator $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$ is *nondecreasing* if it satisfies the condition

$$\Lambda\delta_1 \le \Lambda\delta_2, \quad \delta_1, \delta_2 \in \mathbb{R}_+^S,\ \delta_1 \le \delta_2. \tag{7.33}$$

Moreover, given a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^S$, we write $\lim_{n\to\infty} g_n = 0$ provided

$$\lim_{n\to\infty} g_n(t) = 0, \quad t \in S. \tag{7.34}$$

We will also use the following hypothesis concerning operators $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$:

($\mathcal{C}$) $\lim_{n\to\infty} \Lambda\delta_n = 0$ for every sequence $(\delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^S$ with $\lim_{n\to\infty}\delta_n = 0$.

Finally, recall that a metric $d$ on a nonempty set $X$ is called *non-Archimedean* (or an *ultrametric*) provided

$$d(x, z) \le \max\{d(x, y), d(y, z)\}, \quad x, y, z \in X. \tag{7.35}$$

We can now formulate the following fixed point theorem.

**Theorem 7.8.** *Let* $S$ *be a nonempty set, let* $(X, d)$ *be a complete non-Archimedean metric space, and let* $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$ *be a nondecreasing operator satisfying hypothesis* ($\mathcal{C}$). *If* $\mathcal{T} : X^S \to X^S$ *is an operator satisfying inequality* (7.23) *and functions* $\varepsilon : S \to \mathbb{R}_+$ *and* $g : S \to X$ *are such that*

$$\lim_{n\to\infty} \Lambda^n \varepsilon = 0, \tag{7.36}$$

*and* (7.24) *holds, then for every $t \in S$ limit* (7.26) *exists and the function $f : S \to X$, defined in this way, is a fixed point of $\mathcal{T}$ with*

$$\Delta(g, f)(t) \leq \sup_{n \in \mathbb{N}_0} (\Lambda^n \varepsilon)(t) =: \sigma(t), \quad t \in S. \tag{7.37}$$

*If, moreover,*

$$(\Lambda\sigma)(t) \leq \sup_{n \in \mathbb{N}_0} \left(\Lambda^{n+1} \varepsilon\right)(t), \quad t \in S, \tag{7.38}$$

*then $f$ is the unique fixed point of $\mathcal{T}$ satisfying* (7.37).

An immediate consequence of Theorem 7.8 is the following result on the stability of (7.31) in complete non-Archimedean metric spaces.

**Corollary 7.9.** *Let $S$ be a nonempty set, $(X, d)$ be a complete non-Archimedean metric space, $k \in \mathbb{N}$, $f_1, \ldots, f_k : S \to S$, $L_1, \ldots, L_k : S \to \mathbb{R}_+$, and a function $\Phi : S \times X^k \to X$ satisfy the inequality*

$$d(\Phi(t, y_1, \ldots, y_k), \Phi(t, z_1, \ldots, z_k)) \leq \max_{i \in \{1, \ldots, k\}} L_i(t) d(y_i, z_i), \tag{7.39}$$

*for any $(y_1, \ldots, y_k), (z_1, \ldots, z_k) \in X^k$ and $t \in S$, and $\mathcal{T} : X^S \to X^S$ be an operator defined by* (7.29). *Assume also that $\Lambda$ is given by*

$$(\Lambda\delta)(t) := \max_{i \in \{1, \ldots, k\}} L_i(t) \delta(f_i(t)), \quad \delta \in \mathbb{R}_+^S, \ t \in S, \tag{7.40}$$

*and functions $g : S \to X$ and $\varepsilon : S \to \mathbb{R}_+$ are such that* (7.30) *and* (7.36) *hold. Then for every $t \in S$ limit* (7.26) *exists and the function $f : S \to X$ is a solution of functional equation* (7.31) *satisfying inequality* (7.37).

Given nonempty sets $S, Z$ and functions $\varphi : S \to S$, $F : S \times Z \to Z$, we define an operator $\mathcal{L}_\varphi^F : Z^S \to Z^S$ by

$$\mathcal{L}_\varphi^F(g)(t) := F(t, g(\varphi(t))), \quad g \in Z^S, \ t \in S, \tag{7.41}$$

and we say that $\mathcal{U} : Z^S \to Z^S$ is an *operator of substitution* provided $\mathcal{U} = \mathcal{L}_\psi^G$ with some $\psi : S \to S$ and $G : S \times Z \to Z$. Moreover, if $G(t, \cdot)$ is continuous for each $t \in S$ (with respect to a topology in $Z$), then we say that $\mathcal{U}$ is *continuous*.

The following fixed point theorem was proved in [52].

**Theorem 7.10.** *Let $S$ be a nonempty set, let $(X, d)$ be a complete metric space, $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$, $\mathcal{T} : X^S \to X^S$, $\varphi : S \to S$, and*

$$\Delta(\mathcal{T}\alpha, \mathcal{T}\beta)(t) \leq \Lambda(t, \Delta(\alpha \circ \varphi, \beta \circ \varphi)(t)), \quad \alpha, \beta \in X^S, \ t \in S. \tag{7.42}$$

*Assume also that for every* $t \in S$, $\Lambda_t := \Lambda(t,\cdot)$ *is nondecreasing,* $\varepsilon : S \to \mathbb{R}_+$, $g : S \to X$,

$$\sum_{n=0}^{\infty}\left(\left(\mathcal{L}_{\varphi}^{\Lambda}\right)^{n}\varepsilon\right)(t) =: \sigma(t) < \infty, \quad t \in S, \tag{7.43}$$

*and* (7.24) *holds. Then for every* $t \in S$ *limit* (7.26) *exists and inequality* (7.27) *is satisfied. Moreover, the following two statements are true.*

(i) *If* $\mathcal{T}$ *is a continuous operator of substitution or* $\Lambda_t$ *is continuous at* 0 *for each* $t \in S$*, then* $f$ *is a fixed point of* $\mathcal{T}$.

(ii) *If* $\Lambda_t$ *is subadditive (that is,*

$$\Lambda_t(a+b) \le \Lambda_t(a) + \Lambda_t(b), \tag{7.44}$$

*for all* $a, b \in \mathbb{R}_+$*) for each* $t \in S$*, then* $\mathcal{T}$ *has at most one fixed point* $f \in X^S$ *such that*

$$\Delta(g,f)(t) \le M\sigma(t), \quad t \in S, \tag{7.45}$$

*for a positive integer* $M$.

Theorem 7.10 with $\mathcal{T} = \mathcal{L}_{\varphi}^{F}$ immediately gives the following generalization of Baker's result.

**Corollary 7.11.** *Let* $S$ *be a nonempty set, let* $(X,d)$ *be a complete metric space,* $F : S \times X \to X$, $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$*, and*

$$d(F(t,x),F(t,y)) \le \Lambda(t,d(x,y)), \quad t \in S, \quad x,y \in X. \tag{7.46}$$

*Assume also that* $\varphi : S \to S$, $\varepsilon : S \to \mathbb{R}_+$, (7.43) *holds,* $g : S \to X$*, for every* $t \in S$, $\Lambda_t := \Lambda(t,\cdot)$ *is nondecreasing,* $F(t,\cdot)$ *is continuous, and*

$$d(g(t),F(t,g(\varphi(t)))) \le \varepsilon(t), \quad t \in S. \tag{7.47}$$

*Then for every* $t \in S$ *the limit*

$$f(t) := \lim_{n\to\infty}\left(\mathcal{L}_{\varphi}^{F}\right)^{n}(g)(t) \tag{7.48}$$

*exists,* (7.27) *holds and* $f$ *is a solution of* (7.1)*. Moreover, if for every* $t \in S$, $\Lambda_t$ *is subadditive and* $M \in \mathbb{N}$*, then* $f : S \to X$ *is the unique solution of* (7.1) *fulfilling* (7.45).

Let us finally mention that the fixed point method is also a useful tool for proving the Hyers-Ulam stability of differential (see [98, 99]) and integral equations (see for instance [100–102]). Some further details and information on the connections between the fixed point theory and the Hyers-Ulam stability can be found in [103].

## 8. Stability in Non-Archimedean Spaces

Let us recall that a *non-Archimedean valuation* in a field $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ with

$$\begin{gathered} |r| = 0, \quad \text{iff } r = 0, \\ |rs| = |r||s|, \quad r, s \in \mathbb{K}, \\ |r+s| \le \max\{|r|, |s|\}, \quad r, s \in \mathbb{K}. \end{gathered} \tag{8.1}$$

A field endowed with a non-Archimedean valuation is said to be *non-Archimedean*. Let $X$ be a linear space over a field $\mathbb{K}$ with a non-Archimedean valuation that is nontrivial (i.e., we additionally assume that there is an $r_0 \in \mathbb{K}$ such that $0 \ne |r_0| \ne 1$). A function $\|\cdot\| : X \to \mathbb{R}_+$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

$$\begin{gathered} \|x\| = 0, \quad \text{iff } x = 0, \\ \|rx\| = |r|\|x\|, \quad r \in \mathbb{K},\ x \in X, \\ \|x+y\| \le \max\{\|x\|, \|y\|\}, \quad x, y \in X. \end{gathered} \tag{8.2}$$

If $\|\cdot\| : X \to \mathbb{R}_+$ is a non-Archimedean norm in $X$, then the pair $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

If $(X, \|\cdot\|)$ is a non-Archimedean normed space, then it is easily seen that the function $d_X : X^2 \to \mathbb{R}_+$, given by $d_X(x, y) := \|x - y\|$, is a non-Archimedean metric on $X$. Therefore non-Archimedean normed spaces are special cases of metric spaces. The most important examples of non-Archimedean normed spaces are the $p$-adic numbers $\mathbb{Q}_p$ (here $p$ is any prime number), which have gained the interest of physicists because of their connections with some problems coming from quantum physics, $p$-adic strings, and superstrings (see, for instance, [104]).

In [105], correcting the mistakes in the proof given by the second author in 1968, Arriola and Beyer showed that the Cauchy functional equation is Hyers-Ulam stable in $\mathbb{R}^{\mathbb{Q}_p}$. Schwaiger [106] did the same in the class of functions from a commutative group which is uniquely divisible by $p$ to a Banach space over $\mathbb{Q}_p$. In 2007, Moslehian and Rassias [107] proved the generalized Hyers-Ulam stability of the Cauchy equation in a more general setting, namely, in complete non-Archimedean normed spaces. After their results a lot of papers (see, for instance, [87–89, 93] and the references given there) on the stability of other equations in such spaces have been published. Here we present only one example of these outcomes which is a generalization of the result of Moslehian and Rassias and was obtained in [87] (cf. also Theorem 7 in [106]).

**Theorem 8.1.** *Let $V$ be a commutative semigroup and $W$ be a complete non-Archimedean normed space over a non-Archimedean field of characteristic different from 2. Assume also that $n \in \mathbb{N}$ and for every $i \in \{1, \ldots, n\}$, $\varphi_i : V^{n+1} \to \mathbb{R}_+$ is a mapping such that for each $(x_1, \ldots, x_{n+1}) \in V^{n+1}$,*

$$\begin{aligned}
\lim_{j\to\infty} \frac{1}{|2|^j}\varphi_i\Big(2^j x_1, x_2, \ldots, x_{n+1}\Big) &= \ldots \\
&= \lim_{j\to\infty} \frac{1}{|2|^j}\varphi_i\Big(x_1, \ldots, x_{i-2}, 2^j x_{i-1}, x_i, \ldots, x_{n+1}\Big) \\
&= \lim_{j\to\infty} \frac{1}{|2|^j}\varphi_i\Big(x_1, \ldots, x_{i-1}, 2^j x_i, 2^j x_{i+1}, x_{i+2}, \ldots, x_{n+1}\Big) \\
&= \lim_{j\to\infty} \frac{1}{|2|^j}\varphi_i\Big(x_1, \ldots, x_{i+1}, 2^j x_{i+2}, x_{i+3}, \ldots, x_{n+1}\Big) = \ldots \\
&= \lim_{j\to\infty} \frac{1}{|2|^j}\varphi_i\Big(x_1, \ldots, x_n, 2^j x_{n+1}\Big) = 0,
\end{aligned} \tag{8.3}$$

*and the limit*

$$\lim_{k\to\infty} \max\left\{ \frac{1}{|2|^j}\varphi_i\Big(x_1, \ldots, x_{i-1}, 2^j x_i, 2^j x_i, x_{i+1}, \ldots, x_n\Big) : 0 \le j < k \right\}, \tag{8.4}$$

*denoted by $\widetilde{\varphi}_i(x_1, \ldots, x_n)$, exists. If $f : V^n \to W$ is a function satisfying*

$$\begin{aligned}
&\big\| f(x_1, \ldots, x_{i-1}, x_i + x_i', x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n) \\
&\quad - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)\big\| \\
&\le \varphi_i(x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n), \\
&\quad (x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n) \in V^{n+1}, \quad i \in \{1, \ldots, n\},
\end{aligned} \tag{8.5}$$

*then for every $i \in \{1, \ldots, n\}$ there exists a multiadditive mapping $F_i : V^n \to W$ for which*

$$\big\| f(x_1, \ldots, x_n) - F_i(x_1, \ldots, x_n)\big\| \le \frac{1}{|2|}\widetilde{\varphi}_i(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in V^n. \tag{8.6}$$

*For every $i \in \{1, \ldots, n\}$ the function $F_i$ is given by*

$$F_i(x_1, \ldots, x_n) := \lim_{j\to\infty} \frac{1}{2^j} f\Big(x_1, \ldots, x_{i-1}, 2^j x_i, x_{i+1}, \ldots, x_n\Big), \quad (x_1, \ldots, x_n) \in V^n. \tag{8.7}$$

*If, moreover,*

$$\lim_{l\to\infty}\lim_{k\to\infty}\max\left\{\frac{1}{|2|^j}\varphi_i\left(x_1,\ldots,x_{i-1},2^jx_i,2^jx_i,x_{i+1},\ldots,x_n\right):l\le j<k+l\right\}=0,\qquad i\in\{1,\ldots,n\},(x_1,\ldots,x_n)\in V^n, \tag{8.8}$$

*then for every* $i\in\{1,\ldots,n\}$, $F_i$ *is the unique multiadditive mapping satisfying condition* (8.6).

It seems that [53] was the first paper where the Hyers-Ulam stability was considered in the most general setting, namely, in complete non-Archimedean metric spaces. One of its results (Corollary 7.9) was mentioned in Section 6; the others, which can be also derived from Theorem 7.8, read as follows (from now on $X$ denotes a nonempty set and $(Y,d)$ stands for a complete non-Archimedean metric space).

**Corollary 8.2.** *Suppose that* $(Y,*)$ *is a groupoid and*

$$d(x*z,y*z)=d(x,y),\quad x,y,z\in Y. \tag{8.9}$$

*Let* $k,m\in\mathbb{N}$, $L_1,\ldots,L_k:X\to\mathbb{R}_+$, $G:X\times Y^m\to Y$, $f_1,\ldots,f_k,g_1,\ldots,g_m:X\to X$, *and* $\Phi:X\times Y^k\to Y$ *satisfy inequality* (7.39) *for any* $(y_1,\ldots,y_k),(z_1,\ldots,z_k)\in Y^k$ *and* $t\in X$. *Assume also that functions* $\varphi,\mu_1,\ldots,\mu_m:X\to Y$, *and* $\varepsilon:X\to\mathbb{R}_+$ *are such that*

$$d(\varphi(x),\Phi(x,\varphi(f_1(x)),\ldots,\varphi(f_k(x)))*G(x,\mu_1(g_1(x)),\ldots,\mu_m(g_m(x))))\le\varepsilon(x),\quad x\in X \tag{8.10}$$

*and* (7.36) *holds with* $\Lambda$ *given by* (7.40). *Then the limit* $\lim_{n\to\infty}(\mathcal{T}_0{}^n\varphi)(x)=:\psi(x)$ *exists for every* $x\in X$, *where* $\mathcal{T}_0:Y^X\to Y^X$ *is defined by*

$$(\mathcal{T}_0\xi)(x):=\Phi(x,\xi(f_1(x)),\ldots,\xi(f_k(x)))*G(x,\mu_1(g_1(x)),\ldots,\mu_m(g_m(x))), \tag{8.11}$$

*and the functions* $\mu_1,\ldots,\mu_m$, *and* $\psi:X\to Y$ *fulfil*

$$\psi(x)=\Phi(x,\psi(f_1(x)),\ldots,\psi(f_k(x)))*G(x,\mu_1(g_1(x)),\ldots,\mu_m(g_m(x))),\qquad d(\varphi(x),\psi(x))\le\sup_{n\in\mathbb{N}_0}(\Lambda^n\varepsilon)(x),\quad x\in X. \tag{8.12}$$

**Corollary 8.3.** *Suppose that* $(Y,+)$ *is a commutative group and* $d$ *is invariant (i.e.,* $d(x+z,y+z)=d(x,y)$ *for* $x,y,z\in Y$*). Let* $k\in\mathbb{N}$, $\varphi_1,\ldots,\varphi_k:X\to Y$, $\Phi_1,\ldots,\Phi_k:X\times Y\to Y$, *and* $\varepsilon:X\to\mathbb{R}_+$ *satisfy*

$$d\left(\sum_{i=1}^k\varphi_i(x),\sum_{i=1}^k\Phi_i(x,\varphi_i(x))\right)\le\varepsilon(x),\quad x\in X. \tag{8.13}$$

*Assume also that there is a number* $j \in \{1, \ldots, k\}$ *such that*

$$d\left(\Phi_j(x, y), \Phi_j(x, z)\right) \leq L_j(x) d(y, z), \quad x \in X, \ y, z \in Y \tag{8.14}$$

*with a function* $L_j : X \to [0, 1)$. *Then the limit* $\lim_{n\to\infty}(\mathcal{T}^n \varphi_j)(x) =: \psi(x)$ *exists for every* $x \in X$, *where* $\mathcal{T} : Y^X \to Y^X$ *is given by*

$$(\mathcal{T}\varphi)(x) := \Phi_j(x, \varphi(x)) + \sum_{i=1, i\neq j}^{k} \Phi_i(x, \varphi_i(x)) - \sum_{i=1, i\neq j}^{k} \varphi_i(x), \tag{8.15}$$

*and the function* $\psi : X \to Y$, *defined in this way, is the unique solution of the functional equation*

$$\Phi_j(x, \psi(x)) + \sum_{i=1, i\neq j}^{k} \Phi_i(x, \varphi_i(x)) = \psi(x) + \sum_{i=1, i\neq j}^{k} \varphi_i(x), \tag{8.16}$$

*such that* $d(\varphi_j(x), \psi(x)) \leq \varepsilon(x)$ *for* $x \in X$.

**Corollary 8.4.** *Let* $(X, *)$ *be a groupoid,* $k \in \mathbb{N}$, $d_1, \ldots, d_k \in X$, $c \in \mathbb{R}_+$, $\varphi : X \to Y$, $L_1, \ldots, L_k : X \to \mathbb{R}_+$, *a function* $\Phi : X \times Y^k \to Y$ *satisfy inequality* (7.39) *for any* $(y_1, \ldots, y_k), (z_1, \ldots, z_k) \in Y^k$ *and* $t \in X$, *and* $\mathcal{T} : Y^X \to Y^X$ *be an operator defined by*

$$(\mathcal{T}\varphi)(x) := \Phi(x, \varphi(x * d_1), \ldots, \varphi(x * d_k)), \quad \varphi \in Y^X, \ x \in X. \tag{8.17}$$

*Assume also that a function* $\sigma : X \to \mathbb{R}_+$ *is such that*

$$\begin{gathered} q := \sup_{x\in X}\left(\max_{i\in\{1,\ldots,k\}} L_i(x)\sigma(d_i)\right) < 1, \\ \sigma(x * y) \leq \sigma(x)\sigma(y), \qquad x, y \in X, \\ d(\varphi(x), \Phi(x, \varphi(x * d_1), \ldots, \varphi(x * d_k))) \leq c\ \sigma(x), \quad x \in X. \end{gathered} \tag{8.18}$$

*Then there exists a function* $\psi : X \to Y$ *such that*

$$\begin{gathered} \psi(x) = \Phi(x, \psi(x * d_1), \ldots, \psi(x * d_k)), \quad x \in X, \\ d(\varphi(x), \psi(x)) \leq c\sigma(x), \quad x \in X. \end{gathered} \tag{8.19}$$

## 9. Functional Congruences

In this section $Y$ denotes a real Banach space, $K$ stands for a subgroup of the group $(Y, +)$, and $E$ is a real linear space, unless explicitly stated otherwise. We write

$$D_1 + TD_2 := \{x + ty : x \in D_1, y \in D_2, t \in T\}, \tag{9.1}$$

for $T \subset \mathbb{R}$ and $D_1, D_2 \subset E$.

Baron et al. [108] have started the study of conditions on a convex set $C \subset Y$ and a function $h : E \to Y$ with

$$h(x+y) - h(x) - h(y) \in K + C, \quad \text{for } x, y \in E, \tag{9.2}$$

which guarantee that there exists an additive function $A : E \to Y$ (i.e., $A(x+y) = A(x)+A(y)$ for $x, y \in E$) such that

$$h(x) - A(x) \in K + C, \quad \text{for } x \in E, \tag{9.3}$$

or, in other words, that $h$ can be represented in the form

$$h = A + \gamma + \kappa, \tag{9.4}$$

with some $\gamma : E \to C$, $\kappa : E \to K$. That is a continuation and an extension of some earlier investigations in [109–111]. Here we present some examples of results from [112] (see also [113, 114]), which generalize those in [108].

They correspond simultaneously to the classical Ulam's problem of stability for the Cauchy equation (with $K = \{0\}$) and to the subjects considered, for example, in [115–128], where functions satisfying (9.2) with $C = \{0\}$ (mainly on restricted domains), have been investigated. The latter issue appears naturally in connection with descriptions of subgroups of some groups (see [129]) and some representations of characters (see, e.g., [109, 115, 116, 122–125]).

It is proved in [112, Example 1] that without any additional assumptions on $h$, the mentioned above decomposition of $h$ is not possible in general.

In what follows we say that two nonempty sets $D_1, D_2 \subset Y$ are separated provided

$$\inf \left\{ \|x - y\| : x \in D_1, y \in D_2 \right\} > 0. \tag{9.5}$$

In the rest of this section $C$ stands for a nonempty closed, symmetric (i.e., $-x \in C$ for each $x \in C$), and convex subset of $Y$. The next theorem (see [112, Theorem 10]) involves the notion of Christensen measurability and we refer to [130] (cf. [131, 132]) for the details concerning it.

**Theorem 9.1.** *Suppose that $E$ is a Polish real linear space, $h : E \to Y$ is Christensen measurable,* (9.2) *holds, and one of the following three conditions is valid.*

(i) *The sets $4C$ and $K \setminus \{0\}$ are separated and $Y$ is separable.*

(ii) *The sets $10C$ and $K \setminus \{0\}$ are separated, $K$ is countable, and $C$ is bounded.*

(iii) *The sets $(10 + \varepsilon)C$ and $K \setminus \{0\}$ are separated for some $\varepsilon \in (0, \infty)$ and $K$ is countable.*

*Then there exists an additive function $A : E \to Y$ satisfying* (9.3).

*Moreover, in the case where $C$ is bounded, $A$ is unique and continuous.*

*Remark 9.2.* There arises a natural question to what extent each of assumptions (i)–(iii) in Theorem 9.1, but also in Theorems 9.3 and 9.4, can be weakened (if at all)?

Certainly, the boundedness of $C$ in Theorem 9.1 is necessary for the uniqueness and continuity of $A$ as it follows from [112, Remark 4]. It is also the case for the uniqueness, linearity, and continuity of $A$ in Theorems 9.3 and 9.4.

For the next theorem we need the notion of Baire property. Let us recall that $h : E \to Y$ has the Baire property provided, for every open set $V \subset Y$, the set $h^{-1}(V)$ has the Baire property, that is, there are an open set $U \subset E$ and sets $T_1, T_2 \subset E$ of the first category, with

$$h^{-1}(V) = (U \cup T_1) \setminus T_2. \tag{9.6}$$

Let us yet recall that a topology in a real linear space $Z$ is called semilinear provided the mapping

$$\mathbb{R} \times Z \times Z \ni (\alpha, x, y) \longrightarrow \alpha x + y \in Z \tag{9.7}$$

is separately continuous with respect to each variable (see, e.g., [133]). A real linear space $Z$ endowed with a semilinear topology is called a semilinear topological space.

Now we are in a position to present [112, Theorem 13].

**Theorem 9.3.** *Suppose that $E$ is a real semilinear topological space of the second category of Baire (in itself), one of conditions (i)–(iii) of Theorem 9.1 is valid, and $h : E \to Y$ fulfills* (9.2) *and has the Baire property. Then there exists an additive function $A : E \to Y$ such that* (9.3) *holds.*

*Moreover, in the case where $C$ is bounded, $A$ is unique and linear; in the case where $C$ is bounded and $E$ is a linear topological space, $A$ is unique and continuous.*

For our last theorem (see [112, Theorem 15]), let us recall that $f$, mapping a topological space $X$ into $Y$, is universally measurable provided, for every open set $U \subset Y$, the set $f^{-1}(U)$ is universally measurable, that is, it is in the universal completion of the Borel field in $E$ (see e.g., [131, 132]); $f$ is Borel provided, for every Borel set $D \subset Y$, the set $f^{-1}(D)$ is Borel in $X$.

**Theorem 9.4.** *Let $E$ be endowed with a topology such that the mapping*

$$\mathbb{R} \ni t \longrightarrow tx \in E \tag{9.8}$$

*is Borel for every $x \in E$, one of conditions (i)–(iii) of Theorem 9.1 be valid, and $h : E \to Y$ fulfill* (9.2) *and be universally measurable. Then there exists an additive function $A : E \to Y$ such that* (9.3) *holds.*

*Moreover, if $C$ is bounded, then $A$ is unique and linear; if $C$ is bounded and the topology in $E$ is linear and metrizable with a complete metric, then $A$ is unique and continuous.*

## 10. Hyperstability

In this part, $X$ and $Y$ are normed spaces, $U \subset X$ is nonempty, and $\varphi : U^2 \to \mathbb{R}_+$. We say that the following conditional Cauchy functional equation

$$f(x+y) = f(x) + f(y), \quad x, y \in U, \ x+y \in U \tag{10.1}$$

is $\varphi$-hyperstable in the class of functions $f : U \to Y$ provided each $f : U \to Y$ satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y), \quad x,y \in U, \ x+y \in \mathrm{U}, \tag{10.2}$$

must fulfil (10.1).

According to our best knowledge, the first hyperstability result was published in [134] (for the constant function $\varphi$) and concerned the ring homomorphisms. However, the term *hyperstability* has been used for the first time probably in [135].

Now we present two very elementary hyperstability results for (10.1). The first one is a simple consequence of Corollary 2.2.

**Corollary 10.1.** *Let $L$ and $p \neq 1$ be fixed positive real numbers, $2U = U$, $C : U \to X$, and $C(2x) = 2C(x)$ for $x \in U$. Assume that $f : U \to Y$ satisfies* (10.2) *with $\varphi : U^2 \to \mathbb{R}$ given by*

$$\varphi(x,y) = L\|C(x) - C(y)\|^p, \quad x,y \in U. \tag{10.3}$$

*Then $f$ is a solution to* (10.1).

*Proof.* It is easily seen that condition (2.13) is valid with $\varepsilon = 1$ for $p < 1$ and with $\varepsilon = -1$ for $p > 1$. Hence it is enough to use Corollary 2.2. □

We have as well the following.

**Proposition 10.2.** *Let* $X > 2$ *and let $g : X \to Y$. Suppose that there exist functions $\eta, \mu : \mathbb{R} \to \mathbb{R}$ with $\mu(0) = 0$ and*

$$\|g(x+y) - g(x) - g(y)\| \leq \mu(\eta(\|x\|) - \eta(\|y\|)), \quad x,y \in X. \tag{10.4}$$

*Then $g$ is additive.*

*Proof.* Inequality (10.4) yields

$$g(x+y) = g(x) + g(y), \quad x,y \in X, \quad \|x\| = \|y\|. \tag{10.5}$$

Hence, by [136, Theorem 3.1], $g$ is additive. □

Below we provide two simple examples of applications of those hyperstability results; they correspond to the investigations in [137–149] concerning the inhomogeneous Cauchy equation and the cocycle equation.

**Corollary 10.3.** *Let $G : U^2 \to Y$ be such that $G(x_0, y_0) \neq 0$ for some $x_0, y_0 \in U$ with $x_0 + y_0 \in U$. Assume that one of the following two conditions is valid.*

(a) $2U = U$ *and there exist $C : U \to X$ and positive reals $L$ and $p \neq 1$ with*

$$\begin{gathered} C(2x) = 2C(x), \quad x \in U, \\ \|G(x,y)\| \leq L\|C(x) - C(y)\|^p, \quad x,y \in U. \end{gathered} \tag{10.6}$$

(b) $U = X$, $X > 2$ *and there are functions* $\eta, \mu : \mathbb{R} \to \mathbb{R}$ *with* $\mu(0) = 0$, *and*

$$\|G(x,y)\| \le \mu(\eta(\|x\|) - \eta(\|y\|)), \quad x, y \in X. \tag{10.7}$$

*Then the conditional functional equation*

$$g(x+y) = g(x) + g(y) + G(x,y), \quad x, y \in U,\ x+y \in U \tag{10.8}$$

*has no solutions in the class of functions* $g : U \to Y$.

*Proof.* Let $g : U \to Y$ be a solution to (10.8). Then

$$\|g(x+y) - g(x) - g(y)\| \le \|G(x,y)\|, \quad x, y \in U,\ x+y \in U. \tag{10.9}$$

Hence, by Corollary 10.1 (if (a) holds) and Proposition 10.2 (if (b) holds), $g$ is a solution to (10.1). This means that $G(x_0, y_0) = 0$, which is a contradiction. □

**Corollary 10.4.** *Let* $U = X$, *and* $G : X^2 \to Y$ *be a symmetric (i.e.,* $G(x,y) = G(y,x)$ *for* $x, y \in X$*) solution to the cocycle functional equation*

$$G(x,y) + G(x+y,z) = G(x,y+z) + G(y,z), \quad x, y, z \in X. \tag{10.10}$$

*Assume that one of conditions (a) and (b) holds. Then* $G(x,y) = 0$, *for* $x, y \in X$.

*Proof.* $G$ is coboundary (see [146] or [149]), that is, there is $g : X \to Y$ with $G(x,y) = g(x+y) - g(x) - g(y)$ for $x, y \in X$. Clearly $g$ is a solution to (10.8). Hence Corollary 10.3 implies the statement. □

For some further (more involved) examples of hyperstability results, concerning also some other functional equations, we refer to [150–153]. The issue of hyperstability seems to be a very promising subject to study within the theory of Ulam's type stability.

## 11. Stability of Composite Functional Equations

The problem of studying the stability of the composite functional equations was raised by Ger in 2000 (at the 38th International Symposium on Functional Equations) and in particular it concerned the Hyers-Ulam stability of the Gołąb-Schinzel equation

$$f(x + f(x)y) = f(x)f(y), \tag{11.1}$$

for the information concerning that equation and generalizations of it we refer to the survey paper [154].

In 2005, Chudziak [155] answered this question and proved that in the class of continuous real functions equation (11.1) is superstable. More precisely, he showed that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\left| f(x + f(x)y) - f(x)f(y) \right| \le \varepsilon, \quad x, y \in \mathbb{R}, \tag{11.2}$$

with a positive real number $\varepsilon$, then either $f$ is bounded or it is a solution of (11.1).

In [156], Chudziak and Tabor generalized this result. Namely, they proved that if $\mathbb{K}$ is a subfield of $\mathbb{C}$, $X$ is a vector space over $\mathbb{K}$ and $f : X \to \mathbb{K}$, is a function satisfying the inequality

$$\left| f(x + f(x)y) - f(x)f(y) \right| \le \varepsilon, \quad x, y \in X \tag{11.3}$$

and such that the limit

$$\lim_{t \to 0} f(tx) \tag{11.4}$$

exists (not necessarily finite) for every $x \in X \setminus f^{-1}(0)$, then either $f$ is bounded or it is a solution of (11.1) on $X$. Therefore, (11.1) is superstable also in this class of functions.

Later on, in [157, 158], the same results have been proved for the generalized Gołąb-Schinzel equation

$$f(x + f(x)^n y) = \lambda f(x)f(y), \tag{11.5}$$

where $n$ is a positive integer and $\lambda$ is a nonzero complex number. If $\lambda \in \mathbb{R}$, then functional equation (11.5) is superstable in the class of continuous real functions. If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\lambda \in \mathbb{K} \setminus \{0\}$, and $X$ is a vector space over $\mathbb{K}$, then (11.5) is superstable in the class of functions $f : X \to \mathbb{K}$ such that the limit (11.4) (not necessarily finite) exists for every $x \in X \setminus f^{-1}(0)$.

It is known (see [159]) that the phenomenon of superstability is caused by the fact that we mix two operations. Namely, on the right-hand side of (11.1) we have a product, but in (11.2) we measure the distance between the two sides of (11.1) using the difference. Therefore, it is more natural to measure the difference between 1 and the quotients of the sides of (11.1). In [159] it has been proved that for the exponential equation this approach leads to the traditional stability. The result is different in the case of the Gołąb-Schinzel equation.

In [160] it is proved that if $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0 and satisfies the following two inequalities

$$\begin{gathered} \left| \frac{f(x)f(y)}{f(x + f(x)y)} - 1 \right| \le \varepsilon, \quad \text{whenever } f(x + f(x)y) \ne 0, \\ \left| \frac{f(x + f(x)y)}{f(x)f(y)} - 1 \right| \le \varepsilon, \quad \text{whenever } f(x)f(y) \ne 0 \end{gathered} \tag{11.6}$$

for a given $\varepsilon \in (0, 1)$, then either $f$ is close to 1 or it is a solution of (11.1). Therefore, with this definition of (quotient) stability, the Gołąb-Schinzel equation is also superstable in the class of real functions that are continuous at 0. This approach to stability, using quotients, is now called *the stability in the sense of* Ger.

Chudziak generalized this result in [161] where he proved that if $X$ is a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\lambda \in \mathbb{K} \setminus \{0\}$, $n$ is a positive integer and $f : X \to \mathbb{K}$ is such that $X \setminus f^{-1}(0)$ admits an algebraically interior point (i.e., a point $a$ such that, for every $x \in X \setminus \{0\}$, there exists $r_x > 0$ such that $a + sx \subset X \setminus f^{-1}(0)$ for $s \in \mathbb{K}$ with $|s| \leq r_x$) and $f$ satisfies the following two inequalities

$$\left|\frac{\lambda f(x) f(y)}{f(x + f(x)^n y)} - 1\right| \leq \varepsilon, \quad \text{whenever } f(x + f(x)^n y) \neq 0,$$
$$\left|\frac{f(x + f(x)^n y)}{\lambda f(x) f(y)} - 1\right| \leq \varepsilon, \quad \text{whenever } f(x) f(y) \neq 0, \tag{11.7}$$

for a given $\varepsilon \in (0,1)$, then either $f$ is bounded or it is a solution of (11.5). Thus, in the class of functions $f : X \to \mathbb{K}$ such that $X \setminus f^{-1}(0)$ admits an algebraically interior point, (11.5) is superstable in the sense of Ger.

In [162] those results were extended to a class of functional equations which includes (11.1), (11.4), and the exponential equation. Consider, namely, the functional equation

$$f(x + M(f(x))y) = \lambda f(x) f(y), \tag{11.8}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ and $M : \mathbb{R} \to \mathbb{R}$ is a continuous nonzero multiplicative function. It turns out (see [162]) that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the inequality

$$\frac{1}{\varepsilon_1 + 1} \leq \left|\frac{f(x + M(f(x))y)}{\lambda f(x) f(y)}\right| \leq \varepsilon_2 + 1, \quad \text{whenever } f(x + M(f(x))y) f(x) f(y) \neq 0, \tag{11.9}$$

then either $f$ is a solution of the functional equation

$$f(x + M(f(x))y) f(x) f(y) = 0, \tag{11.10}$$

or the following three conditions are valid.

(i) If $M$ is odd, then either $f$ is bounded or it is a solution of (11.8) with $\lambda = 1$.

(ii) If $M$ is even and $M(\mathbb{R}) \neq \{1\}$, then either $f$ is bounded or it is a solution of (11.8) with some $\lambda \in \{1, -1\}$.

(iii) If $M(\mathbb{R}) = \{1\}$, then there exists a unique $\alpha \in \mathbb{R}$ such that

$$\left|\frac{\lambda f(x)}{e^{\alpha x}}\right| \in \left[\frac{1}{\varepsilon_1 + 1}, \varepsilon_2 + 1\right], \quad x \in \mathbb{R}. \tag{11.11}$$

(For some results on (11.10) see [163]).

In [164], the stability in the sense of Ger of (11.8) was studied in the following more general setting.

**Theorem 11.1.** *Let $X$ be a real linear space and let $M$ be multiplicative and continuous at a point $x_0 \in \mathbb{R}$. Assume also that $f : X \to \mathbb{R}$ with $f(X) \neq \{0\}$ satisfies the inequalities*

$$\left|\frac{f(x+M(f(x))y)}{\lambda f(x)f(y)} - 1\right| \leq \varepsilon_1, \quad \text{whenever } f(x)f(y) \neq 0,$$
$$\left|\frac{\lambda f(x)f(y)}{f(x+M(f(x))y)} - 1\right| \leq \varepsilon_2, \quad \text{whenever } (x+M(f(x))y) \neq 0, \tag{11.12}$$

*for some $\varepsilon_1, \varepsilon_2 \in (0,1)$. If $M(\mathbb{R}) \subset \{-1,0,1\}$, then there exists a unique function $g : X \to \mathbb{R}$ with $g^{-1}(0) = f^{-1}(0)$ satisfying* (11.8) *and*

$$\left|\frac{f(x)}{g(x)}\right| \in \left[\frac{1}{\varepsilon_1+1}, \varepsilon_2+1\right], \quad x \in X \setminus g^{-1}(0). \tag{11.13}$$

*If $M(\mathbb{R}) \not\subset \{-1,0,1\}$ and the set $X \setminus f^{-1}(0)$ has an algebraically interior point, then either $f$ is bounded or it is solution of* (11.8) *with $\lambda$ replaced by sign($\lambda$).*

In view of the above result, some questions arise. Can we obtain analogous results in the complex case? Are the assumptions on $M$ and the set $X \setminus f^{-1}(0)$ necessary?

The results related to the stability of composite functional equations which have been obtained up to now and which have been described previously concern essentially the Gołąb-Schinzel type functional equations. A few other equations have been investigated in [165–167]. For instance, another very important example of composite functional equations is the translation equation

$$F(t, F(s,x)) = F(s+t, x), \tag{11.14}$$

(see [168–171] for more information on it and its applications) and its stability has been studied in [172–177].

It would be interesting to study also the stability of other composite type functional equations such as the Baxter functional equation [178] and the Ebanks functional equation [179].

## 12. Miscellaneous

At the end of this survey we would like to attract the attention of the readers to the results and new techniques of proving the stability results in [77, 180–186]; those techniques involve the methods of multivalued function.

A new approach to the stability of functional equations has been proposed by Paneah (see, e.g., [187]) with some critique of the notions that have been commonly accepted so far. Another method, using the concept of shadowing, was presented in [188] and recently applied in [79, 189–192].

An approach to stability in the ring of formal power series is suggested in [173].

Stability of some conditional versions of the Cauchy equation has been studied in [193–197], for example, of the following Mikusiński functional equation

$$f(x+y)\big(f(x+y)-f(x)-f(y)\big)=0. \tag{12.1}$$

For some connections between Ulam's type stability and the number theory see [198–200].

For some recent results on stability of derivations in rings and algebras see, for example, [201, 202] and the references therein.

Stability for ODE and PDE has been studied, for example, in [98, 99, 203–225], for stability results for some integral equations see [100–102, 226].

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*Research Article*

# Probabilistic (Quasi)metric Versions for a Stability Result of Baker

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By using the fixed point method, we obtain a version of a stability result of Baker in probabilistic metric and quasimetric spaces under triangular norms of Hadžić type. As an application, we prove a theorem regarding the stability of the additive Cauchy functional equation in random normed spaces.

## 1. Introduction

The use of the fixed point theory in the study of Ulam-Hyers stability was initiated by Baker in the paper [1]. Baker used the classical Banach fixed point theorem to prove the stability of the nonlinear functional equation

$$f(x) = \Phi(x, f(\eta(x))). \tag{1.1}$$

His result reads as follows.

**Theorem 1.1** (see [1, Theorem 2]). *Suppose $S$ is a nonempty set, $(X, d)$ is a complete metric space, $\eta : S \to S$, $\Phi : S \times X \to X$, $\lambda \in [0, 1)$, and*

$$d(\Phi(u, x), \Phi(u, y)) \leq \lambda d(x, y), \quad \forall u \in S, \ x, y \in X. \tag{1.2}$$

*Also, suppose that* $f : S \to X$, $\delta > 0$, *and*

$$d(f(u), \Phi(u, f(\eta(u)))) \leq \delta, \quad \forall u \in S. \tag{1.3}$$

*Then there exists a unique mapping* $g : S \to X$ *such that*

$$\begin{aligned} g(u) &= \Phi(u, g(\eta(u))), \quad \forall u \in S, \\ d(f(u), g(u)) &\leq \frac{\delta}{1-\lambda}, \quad \forall u \in S. \end{aligned} \tag{1.4}$$

Starting with the papers [2, 3], the fixed point method has become a fundamental tool in the study of Ulam-Hyers stability. In the probabilistic and fuzzy setting, this approach was first used in the papers [4, 5] for the case of random and fuzzy normed spaces endowed with the strongest triangular norm $T_M$. In fact, by identifying a suitable deterministic metric, the stability problem in such spaces was reduced to a fixed point theorem in generalized metric spaces. This idea was adopted by many authors, see for example, [6–11]. It is worth noting that, in applying this method, the fact that the triangular norm is $T_M$ is essential.

In this paper we study the stability of (1.1) when the unknown $f$ takes values in a probabilistic (quasi-) metric space endowed with a triangular norm of Hadžić type. To this end, we employ the fixed point theory in probabilistic metric spaces, rather than that in metric spaces.

## 2. Hyers-Ulam Stability of the Equation $f(x) = \Phi(x, f(\eta(x)))$ in Probabilistic Metric Spaces

In this section, we study the stability of the equation $f(x) = \Phi(x, f(\eta(x)))$, where the unknown function $f$ is a mapping from a nonempty set $S$ to a probabilistic metric space $(X, F, T)$, and $\Phi : S \times X \to X$ and $\eta : S \to S$ are given mappings.

We assume that the reader is familiar with the basic concepts of the theory of probabilistic metric spaces. As usual, $\Delta_+$ denotes the space of all functions $F : \mathbb{R} \to [0,1]$, such that $F$ is left-continuous and nondecreasing on $\mathbb{R}$, $F(0) = 0$, and $D_+$ denotes the subspace of $\Delta_+$ consisting of functions $F$ with $\lim_{t\to\infty} F(t) = 1$. Here we adopt the terminology from [12], hence the probabilistic metric takes values in $\Delta_+$.

We recall some facts from the fixed point theory in probabilistic metric spaces.

*Definition 2.1.* A $t$-norm $T$ is said to be of $H$-type [13] if the family of its iterates $\{T^n\}_{n\in\mathbb{N}}$, given by $T^0(x) = 1$, and $T^n(x) = T(T^{n-1}(x), x)$ for all $n \geq 1$, is equicontinuous at $x = 1$.

A trivial example of a $t$-norm of $H$-type is the $t$-norm $T_M$, $T_M(a, b) = \text{Min}\{a, \text{b}\}$, but there exist $t$-norms of $H$-type different from Min [14].

The theorem below provides a characterization of continuous $t$-norms of $H$-type.

**Proposition 2.2** (see [15]). (i) *Suppose that there exists a strictly increasing sequence* $(b_n)_{n\in\mathbb{N}}$ *in* $[0, 1)$ *such that* $\lim_{n\to\infty} b_n = 1$ *and* $T(b_n, b_n) = b_n$. *Then T is of H-type.*

(ii) *Conversely, if T is continuous and of H-type, then there exists a sequence as in* (i).

*Definition 2.3* (see [16]). Let $(X, F, T)$ be a probabilistic metric space. A mapping $f : X \to X$ is said to be a Sehgal contraction (or $B$-contraction) if the following relation holds:

$$F_{f(p)f(q)}(kt) \geq F_{pq}(t), \quad (p, q \in X,\ t > 0). \tag{2.1}$$

**Theorem 2.4** (see [17]). *Let $(X, F, T)$ be a complete probabilistic metric space with $T$ of Hadžić-type and $f : X \to X$ be a $B$-contraction. Then $f$ has a fixed point if and only if there is $p \in X$ such that $F_{pf(p)} \in D_+$. If $F_{pf(p)} \in D_+$, then $p^* := \lim_{n\to\infty} f^n(p)$ is the unique fixed point of $f$ in the set $Y = \{q \in X : F_{pq} \in D_+\}$.*

The following lemma completes Theorem 2.4 with an estimation relation, in the case $T = T_M$.

**Lemma 2.5** (see [18]). *Let $(X, F, T_M)$ be a complete probabilistic metric space and $f : X \to X$ be a $k - B$ contraction. Suppose that $F_{pf(p)} \in D_+$ and let $p^* = \lim_{n\to\infty} f^n(p)$. Then*

$$F_{pp^*}(t+0) \geq F_{pf(p)}((1-k)t), \quad \forall t > 0. \tag{2.2}$$

This lemma can be extended to the case of probabilistic metric spaces under a continuous $t$-norm of $H$-type.

**Lemma 2.6.** *Let $(X, F, T)$ be a complete probabilistic metric space, with $T$ a continuous $t$-norm of $H$-type and $(b_n)_n$ be a strictly increasing sequence of idempotents of $T$. Suppose $f : X \to X$ is a $B$-contraction with Lipschitz constant $k \in (0,1)$. If there exists $p \in X$ such that $F_{pf(p)} \in D_+$, then $p^* = \lim_{n\to\infty} f^n(p)$ is the unique fixed point of $f$ in the set*

$$\{q \in X : F_{pq} \in D_+\}. \tag{2.3}$$

*Moreover, if $t > 0$ is so that $F_{pf(p)}((1-k)t) \geq b_n$, then $F_{pp^*}(t+0) \geq b_n$.*

*Proof.* We have to prove only the last part of the theorem. We show by induction on $m$ that $F_{pf(p)}((1-k)s) \geq b_n$ implies $F_{pf^m(p)}(s) \geq b_n$, for all $m \geq 1$.

The case $m = 1$ is obvious. Now, suppose that $F_{pf^m(p)}(s) \geq b_n$. Then

$$\begin{aligned} F_{pf^{m+1}(p)}(s) &\geq T\left(F_{pf(p)}((1-k)s), F_{f(p)f^{m+1}(p)}(ks)\right) \\ &\geq T\left(F_{pf(p)}((1-k)s), F_{pf^m(p)}(s)\right) \\ &\geq T(b_n, b_n) = b_n. \end{aligned} \tag{2.4}$$

Let $t > 0$ be such that $F_{pf(p)}((1-k)t) \geq b_n$, and let $s > 0$. Then

$$F_{pp^*}(t+s) \geq T\left(F_{pf^m(p)}(t), F_{f^m(p)p^*}(s)\right) \geq T\left(b_n, F_{f^m(p)p^*}(s)\right), \tag{2.5}$$

for all $m \geq 1$. Since $(f^m(p))$ converges to $p^*$, $F_{f^m(p)p^*}(s)$ goes to 1 as $m$ tends to infinity, so

$$F_{pp^*}(t+s) \geq T(b_n, 1) = b_n. \tag{2.6}$$

By taking $s \to 0$ we obtain

$$F_{pp^*}(t+0) \geq b_n. \tag{2.7}$$

□

In order to state our first stability result, we define an appropriate concept of approximate solution for the functional equation (1.1).

*Definition 2.7.* A probabilistic uniform approximate solution of (1.1) is a function $f : S \to X$ with the property that

$$\lim_{t\to\infty} F_{f(u)\Phi(u,f(\eta(u)))}(t) = 1 \tag{2.8}$$

uniformly on $S$.

*Example 2.8.* Let $(X,d)$ be a metric space, and let $F : X \times X \to D_+$ be defined by

$$F_{xy}(t) = \frac{t}{t+d(x,y)} \quad (x,y \in X,\ t \geq 0). \tag{2.9}$$

Then $(X,F,T_M)$ is a probabilistic metric space (the induced probabilistic metric space). One can easily verify that $f$ is a probabilistic uniform approximate solution of (1.1) if and only if it satisfies relation (1.3), thus being an approximate solution in the sense of Theorem 1.1.

**Theorem 2.9.** *Let $S$ be a nonempty set, $(X,F,T)$ be a complete probabilistic metric space, with $T$ a continuous t-norm of $H$-type, and $(b_n)_n$ be a strictly increasing sequence of idempotents of $T$. Suppose $\Phi : S \times X \to X$ is a mapping for which there exists $k \in (0,1)$ with*

$$F_{\Phi(u,x)\Phi(u,y)}(kt) \geq F_{xy}(t), \tag{2.10}$$

*for all $u \in S$, $x,y \in X$ and $t > 0$.*

*If $f : S \to X$ is a probabilistic uniform approximate solution of (1.1), then there exists a function $a : S \to X$ which is an exact solution of (1.1), with the property that, if $t > 0$ is such that*

$$F_{f(u)\Phi(u,f(\eta(u)))}(t) > b_n, \quad \forall u \in S, \tag{2.11}$$

*then*

$$F_{f(u)a(u)}\left(\frac{t}{1-k}+0\right) \geq b_n, \quad \forall u \in S. \tag{2.12}$$

*Proof.* Denote by $Y$ the set of all mappings $g : S \to X$, and let $J : Y \to Y$ be Baker's operator, given by $J(g)(u) = \Phi(u, g(\eta(u)))$ for all $g \in Y,\ u \in S$. We define the distribution function $\mathcal{F}_{gh}$ by

$$\mathcal{F}_{gh}(t) = \sup_{s<t} \inf_{u\in S} F_{g(u)h(u)}(s), \tag{2.13}$$

for all $g, h \in Y$.

The assumptions on the space $(X, F, T)$ ensure that $(Y, \mathcal{F}, T)$ is a complete probabilistic metric space. Also,

$$\begin{aligned} \mathcal{F}_{J(g)J(h)}(kt) &= \sup_{s<kt} \inf_{u\in S} F_{J(g)(u)J(h)(u)}(s) = \sup_{s<t} \inf_{u\in S} F_{J(g)(u)J(h)(u)}(ks) \\ &\geq \sup_{s<t} \inf_{u\in S} F_{g(\eta(u))h(\eta(u))}(s) \geq \mathcal{F}_{gh}(t), \end{aligned} \tag{2.14}$$

that is, $J$ is a Sehgal contraction on $(Y, \mathcal{F}, T)$.

Moreover, the relation $\lim_{t\to\infty} F_{f(u)\Phi(u,f(\eta(u)))}(t) = 1$, uniformly on $X$ implies

$$\mathcal{F}_{fJ(f)} \in D_+. \tag{2.15}$$

Now we can apply Lemma 2.6 to obtain a fixed point of $J$, that is a mapping $a : S \to X$ which is a solution of (1.1), with $a(u) = \lim_{n\to\infty} J^n f(u)$ for all $u \in S$.

Next, let $t > 0$ be such that $F_{f(u)\Phi(u,f(\eta(u)))}(t) > b_n$ for all $u \in S$. Then, from the left continuity of $F$, it follows that $F_{f(u)\Phi(u,f(\eta(u)))}(s_0) > b_n (u \in S)$, for some $s_0 \in (0, t)$. Therefore $\inf_{u\in S} F_{f(u)\Phi(u,f(\eta(u)))}(s_0) \geq b_n$ , so $\mathcal{F}_{fJ(f)}(t) \geq b_n$. By Lemma 2.6, $\mathcal{F}_{fa}(t/(1-k)+0) \geq b_n$, whence we conclude that the estimation (2.12) holds. □

*Remark 2.10.* The result of Baker [1] can be obtained as a particular case of Theorem 2.9, by considering in this theorem the induced probabilistic metric space (see Example 2.8).

From Theorem 2.9 one can derive a stability result for the Cauchy additive functional equation

$$f(x + y) = f(x) + f(y) \tag{2.16}$$

in random normed spaces.

Recall (see [12]) that a random normed space ($RN$-space) is a triple $(X, \nu, T)$, where $X$ is a real linear space, $\nu$ is a mapping from $X$ to $D_+$, and $T$ is a $t$-norm, satisfying the following conditions ($\nu(x)$ will be denoted by $\nu_x$):

(i) $\nu_x(t) = 1$ for all $t > 0$ iff $x = \theta$, the null vector of $X$;

(ii) $\nu_{\alpha x}(t) = \nu_x(t/|\alpha|)$, for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and all $x \in X$;

(iii) $\nu_{x+y}(t + s) \geq T(\nu_x(t), \nu_y(s))$, for all $x, y \in X$ and all $t, s > 0$.

*Definition 2.11.* A probabilistic uniform approximate solution of (2.16) is a function $f : S \to X$ with the property that

$$\lim_{t \to \infty} \nu_{f(u+v)-f(u)-f(v)}(t) = 1 \tag{2.17}$$

uniformly on $S \times S$.

**Theorem 2.12.** *Let $S$ be a real linear space, $(X, \nu, T)$ be a complete RN-space with $T$—a continuous t-norm of $H$-type, and $(b_n)_n$ be a strictly increasing sequence of idempotents of $T$.*

*If $f : S \to X$ is a probabilistic uniform approximate solution of* (2.16)*, then there exists a mapping $a : S \to X$ which is an exact solution of* (2.16)*, with the property that, if $t > 0$ is such that*

$$\nu_{f(u)-f(2u)/2}(t) > b_n, \quad \forall u \in S, \tag{2.18}$$

*then*

$$\nu_{f(u)-a(u)}(2t + 0) \geq b_n, \quad \forall u \in S. \tag{2.19}$$

*Proof.* We apply Theorem 2.9 for $\Phi : S \times X \to X$, $\Phi(u, x) = x/2$, and $\eta : S \to S$, $\eta(u) = 2u$ in the probabilistic metric space $(X, F, T)$ with $F$ defined by

$$F_{xy}(t) = \nu_{x-y}(t) \tag{2.20}$$

for all $x, y \in X$, $t > 0$. Note that $F$ satisfies (2.10) for $k = 1/2$, since

$$F_{\Phi(u,x)\Phi(u,y)}\left(\frac{t}{2}\right) = F_{(x/2)(y/2)}\left(\frac{t}{2}\right) = \nu_{(1/2)(x-y)}\left(\frac{t}{2}\right) = \nu_{x-y}(t) = F_{xy}(t), \tag{2.21}$$

for all $u \in S$, $x, y \in X$ and $t > 0$.

It is easy to see that $f$ is a probabilistic uniform approximate solution of (1.1), so there exists an exact solution of (1.1), that is, a mapping $a : S \to X$ satisfying $a(u) = (1/2)a(2u)$ for all $u \in S$. The estimation (2.19) can be immediately derived from the corresponding one in Theorem 2.9.

It remains to show that $a$ is additive. This follows from the fact that $a(u) = \lim_{n\to\infty}(1/2^n)f(2^n u)$, for all $u \in S$, and $f$ is a probabilistic uniform approximate solution of (2.16). Namely, for all $t > 0$,

$$\begin{aligned}\nu_{a(u+v)-a(u)-a(v)}(t) &\geq T\left(\nu_{a(u+v)-f(2^n(u+v))/2^n}\left(\frac{t}{4}\right), \nu_{a(u)-f(2^n u)/2^n}\left(\frac{t}{4}\right),\right.\\ &\qquad \left.\nu_{a(v)-f(2^n v)/2^n}\left(\frac{t}{4}\right), \nu_{f(2^n(u+v))/2^n-f(2^n u)/2^n-f(2^n v)/2^n}\left(\frac{t}{4}\right)\right)\\ &\geq T\left(\nu_{a(u+v)-f(2^n(u+v))/2^n}\left(\frac{t}{4}\right), \nu_{a(u)-f(2^n u)/2^n}\left(\frac{t}{4}\right),\right.\\ &\qquad \left.\nu_{a(v)-f(2^n v)/2^n}\left(\frac{t}{4}\right), \nu_{f(2^n(u+v))-f(2^n u)-f(2^n v)}\left(\frac{2^n t}{4}\right)\right) \xrightarrow{n\to\infty} 1,\end{aligned} \tag{2.22}$$

implying $a(u+v) = a(u) + a(v)$ for all $u, v \in S$. □

## 3. Hyers-Ulam Stability of the Equation $f(x) = \Phi(x, f(\eta(x)))$ in Probabilistic Quasimetric Spaces

The defining feature of quasimetric structures is the absence of symmetry. This allows one to consider different notions of convergence and completeness. We state the terminology and notations, following [19] (also see [20]).

*Definition 3.1.* A probabilistic quasimetric space is a triple $(X, P, T)$, where $X$ is a nonempty set, $T$ is a $t$-norm, and $P : X \times X \to \Delta_+$ is a mapping satisfying

(i) $P_{xy} = P_{yx} = \varepsilon_0$ if and only if $x = y$;

(ii) $P_{xy}(t+s) \geq T(P_{xz}(t), P_{zy}(s))$, for all $x, y, z \in X$, for all $t, s > 0$.

We note that if $P$ verifies the symmetry assumption $P_{xy} = P_{yx}$, for all $x, y \in X$, then $(X, P, T)$ is a probabilistic metric space.

If $(X, P, T)$ is a probabilistic quasimetric space, then the mapping $Q : X^2 \to \Delta_+$ defined by $Q_{xy} = P_{yx}$ for all $x, y \in X$ is called the conjugate probabilistic quasimetric of $P$.

*Definition 3.2.* Let $(X, P, T)$ be a probabilistic quasimetric space. A sequence $(x_n)_n$ in $X$ is said to be:

(i) right $K$-Cauchy (left $K$-Cauchy) if, for each $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists $k \in \mathcal{N}$ so that, for all $m \geq n \geq k$, $P_{x_n x_m}(\varepsilon) > 1 - \lambda$ ($Q_{x_n x_m}(\varepsilon) > 1 - \lambda$ resp.);

(ii) $P$-convergent ($Q$-convergent) to $x \in X$ if, for each $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists $k \in \mathcal{N}$ so that $P_{x x_n}(\varepsilon) > 1 - \lambda$ ($Q_{x x_n}(\varepsilon) > 1 - \lambda$), for all $n \geq k$.

*Definition 3.3.* Let $A \in \{\text{right } K, \text{left } K\}$ and $B \in \{P, Q\}$. The space $(X, P, T)$ is $(A\text{–}B)$ complete if every $A$-Cauchy sequence is $B$ convergent.

*Definition 3.4.* The probabilistic quasimetric space $(X, P, T)$ has the *L-US* (*R-US*) property if every $P$-($Q$-) convergent sequence has a unique limit.

The following lemma is a quasimetric analogue of Lemma 2.6.

**Lemma 3.5.** *Let $(X, P, T)$ be a (right $K-Q$)-complete probabilistic quasimetric space with the R-US property, where $T$ is a continuous t-norm of H-type. Let $(b_n)_n$ be a strictly increasing sequence of idempotents of $T$.*

*Suppose $f : X \to X$ is a Sehgal contraction with Lipschitz constant $k \in (0,1)$, and $p$ is an element of X such that $P_{pf(p)} \in D_+$. Then $p^* := \lim_{n\to\infty} f^n(p)$ is a fixed point of $f$ and if $t > 0$ is so that $P_{pf(p)}((1-k)t) \geq b_n$, then $P_{pp^*}(t+0) \geq b_n$.*

*Proof.* We proceed in the classical manner to show that the sequence of iterates $(f^n(p))_n$ is right $K$-Cauchy, therefore it is $Q$-convergent to $p^* \in X$. The fact that $p^*$ is a fixed point of f is a consequence of the *R-US* property of the space $X$. Next, as in the proof of Lemma 2.6 we show by induction on $m$ that $P_{pf(p)}((1-k)s) \geq b_n$ implies $P_{pf^m(p)}(s) \geq b_n$, for all $m \geq 1$.

Let $t > 0$ be such that $P_{pf(p)}((1-k)t) \geq b_n$, and let $s > 0$. Then

$$P_{pp^*}(t+s) \geq T\left(P_{pf^m(p)}(t), P_{f^m(p)p^*}(s)\right) \geq T\left(b_n, P_{f^m(p)p^*}(s)\right), \tag{3.1}$$

for all $m \geq 1$. Since $(f^m(p))$ is $Q$-convergent to $p^*$, $P_{f^m(p)p^*}(s)$ goes to 1 as $m$ tends to infinity, so

$$P_{pp^*}(t+s) \geq T(b_n, 1) = b_n. \tag{3.2}$$

By taking $s \to 0$ we obtain

$$P_{pp^*}(t+0) \geq b_n. \tag{3.3}$$

□

The probabilistic quasimetric version of Baker's theorem can be stated as follows.

**Theorem 3.6.** *Let $S$ be a nonempty set, $(X, P, T)$ be a (right $K-Q$)-complete probabilistic quasimetric space with the R-US property, with $T$ a continuous t-norm of H-type, and $(b_n)_n$ be a strictly increasing sequence of idempotents of $T$. Suppose $\Phi : S \times X \to X$ is a mapping for which there exists $k \in (0,1)$ with*

$$P_{\Phi(u,x)\Phi(u,y)}(kt) \geq P_{xy}(t), \tag{3.4}$$

*for all $u \in S$, $x, y \in X$ and $t > 0$.*

*If $f : S \to X$ is a probabilistic uniform approximate solution of (1.1), then there exists a function $a : S \to X$ which is an exact solution of (1.1), with the property that, if $t > 0$ is such that*

$$P_{f(u)\Phi(u,f(\eta(u)))}(t) > b_n, \quad \forall u \in S, \tag{3.5}$$

*then*

$$P_{f(u)a(u)}\left(\frac{t}{1-k} + 0\right) \geq b_n, \quad \forall u \in S. \tag{3.6}$$

*Proof.* We only sketch the proof, as it is very similar to that of Theorem 2.9.

As in the mentioned proof, denote by $Y$ the set of all mappings $g : S \to X$, and define the distribution function $F_{gh}$ by

$$F_{gh}(t) = \sup_{s<t} \inf_{u\in S} P_{g(u)h(u)}(s), \tag{3.7}$$

for all $g, h \in Y$ and Baker's operator $J : Y \to Y$, $J(g)(u) = \Phi(u, g(\eta(u)))$ for all $g \in Y$, $u \in S$.

The assumptions on the space $(X, P, T)$ ensure that $(Y, F, T)$ is a (right $K-Q$)-complete probabilistic quasimetric space with the $R$-$US$ property and that $J$ is a Sehgal contraction on $(Y, F, T)$, and the relation $\lim_{t\to\infty} P_{f(u)\Phi(u,f(\eta(u)))}(t) = 1$, uniformly on $X$ implies

$$F_{fJ(f)} \in D_{+}. \tag{3.8}$$

We can now apply Lemma 3.5 to obtain a mapping $a : S \to X$ which is a solution of (1.1), with $a(u) = \lim_{n\to\infty} J^n f(u)$ for all $u \in S$.

The estimation (3.6) follows by using the left continuity of $P$, as in the proof of Theorem 2.9. □

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*Research Article*

# Approximate Riesz Algebra-Valued Derivations

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Let $F$ be a Riesz algebra with an extended norm $\|\cdot\|_u$ such that $(F, \|\cdot\|_u)$ is complete. Also, let $\|\cdot\|_v$ be another extended norm in $F$ weaker than $\|\cdot\|_u$ such that whenever (a) $x_n \to x$ and $x_n \cdot y \to z$ in $\|\cdot\|_v$, then $z = x \cdot y$; (b) $y_n \to y$ and $x \cdot y_n \to z$ in $\|\cdot\|_v$, then $z = x \cdot y$. Let $\varepsilon$ and $\delta >$ be two nonnegative real numbers. Assume that a map $f : F \to F$ satisfies $\|f(x+y) - f(x) - f(y)\|_u \leq \varepsilon$ and $\|f(x \cdot y) - x \cdot f(y) - f(x) \cdot y\|_v \leq \delta$ for all $x, y \in F$. In this paper, we prove that there exists a unique derivation $d : F \to F$ such that $\|f(x) - d(x)\|_u \leq \varepsilon$, $(x \in F)$. Moreover, $x \cdot (f(y) - d(y)) = 0$ for all $x, y \in F$.

## 1. Introduction

Let $E$ and $E'$ be Banach spaces and let $\delta > 0$. A function $f : E \to E'$ is called $\delta$-additive if $\|f(x+y) - f(x) - f(y)\| < \delta$ for all $x, y \in E$. The well-known problem of stability of functional equation $f(x+y) = f(x) + f(y)$ started with the following question of Ulam [1]. Does there exist for each $\varepsilon > 0$, a $\delta > 0$ such that, to each $\delta$-additive function $f$ of $E$ into $E'$ there corresponds an additive function $l$ of $E$ into $E'$ satisfying the inequality $\|f(x) - l(x)\| \leq \varepsilon$ for each $x \in E$? In 1941, Hyers [2] answered this question in the affirmative way and showed that $\delta$ may be taken equal to $\varepsilon$. The answer of Hyers is presented in a great number of articles and books. For the theory of the stability of functional equations see Hyers et al [3].

Let $F$ be an algebra. A mapping $d : F \to F$ is called a derivation if and only if it satisfies the following functional equations:

$$d(a+b) = d(a) + d(b), \tag{1.1}$$

$$d(ab) = ad(b) + d(a)b, \tag{1.2}$$

for all $a, b \in F$.

The stability of derivations was first studied by Jun and Park [4]. Further, approximate derivations were investigated by a number of mathematicians (see, e.g., [5–7]).

The aim of the present paper is to examine the stability problem of derivations for Riesz algebras with extended norms.

## 2. Preliminaries

A vector space $F$ with a partial order $\leq$ satisfying the following two conditions:

(1) $x \leq y \Rightarrow \alpha x + z \leq \alpha y + z$ for all $z \in F$ and $0 \leq \alpha \in \mathbb{R}$,

(2) for all $x, y \in F$, the supremum $x \vee y$ and infimum $x \wedge y$ exist in $F$ (hence, the modulus $|x| := x \vee (-x)$ exists for each $x \in F$),

is called a Riesz space or vector lattice. Typical examples of Riesz spaces are provided by the function spaces. $C(K)$ the spaces of real valued continuous functions on a topological space $K$, $l_p$ real valued absolutely summable sequences, $c$ the spaces of real valued convergent sequences, and $c_0$ the spaces of real valued sequences converging to zero are natural examples of Riesz spaces under the pointwise ordering. A Riesz space $F$ is called Archimedean if $0 \leq u, v \in F$ and $nu \leq v$ for each $n \in \mathbb{N}$ imply $u = 0$. A subset $S$ in a Riesz space $F$ is said to be solid if it follows from $|u| \leq |v|$ in $F$ and $v \in S$ that $u \in S$. A solid linear subspace of a Riesz space $F$ is called an ideal. Every subset $D$ of a Riesz space $F$ is included in a smallest ideal $F_D$, called ideal generated *by* $D$. A principal ideal of a Riesz space $F$ is any ideal generated by a singleton $\{u\}$. This ideal will be denoted by $I_u$. It is easy to see that

$$I_u = \{v \in F : \lambda \geq 0 \text{ such that } |v| \leq \lambda |u|\}. \tag{2.1}$$

Let $F$ be a Riesz space and $0 \leq u \in F$. Firstly, we give the following definition.

*Definition 2.1.* (1) The sequence $(x_n)$ in $F$ is said to be $u$-uniformly convergent to the element $x \in F$ whenever, for every $\varepsilon > 0$, there exists $n_0$ such that $|x_{n_0+k} - x| \leq \varepsilon u$ holds for each $k$.

(2) The sequence $(x_n)$ in $F$ is said to be relatively uniformly convergent to $x$ whenever $x_n$ converges $u$-uniformly to $x \in F$ for some $0 \leq u \in F$.

When dealing with relative uniform convergence in an Archimedean Riesz space $F$, it is natural to associate with every positive element $u \in F$ an extended norm $\|\cdot\|_u$ in $F$ by the following formula:

$$\|x\|_u = \inf\{\lambda \geq 0 : |x| \leq \lambda u\} \quad (x \in F). \tag{2.2}$$

Note that $\|x\|_u < \infty$ if and only if $x \in I_u$. Also $|x| \leq \delta u$ if and only if $\|x\|_u \leq \delta$.

A Banach lattice is a vector lattice $F$ that is simultaneously a Banach space whose norm is monotone in the following sense.
For all $x, y \in F$, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. Hence, $\|x\| = \||x|\|$ for all $x \in F$.

The sequence $(x_n)$ in $(F, \|\cdot\|_u)$ is called an extended $u$-normed Cauchy sequence, if for every $\varepsilon > 0$ there exists $k$ such that $\|x_{n+k} - x_{m+k}\|_u < \varepsilon$ for all $m, n$. If every extended $u$-normed Cauchy sequence is convergent in $F$, then $F$ is called an extended $u$-normed Banach lattice.

A Riesz space $F$ is called a Riesz algebra or a lattice ordered algebra if there exists an associative multiplication in $F$ with the usual algebra properties such that $0 \leq u \cdot v$ for all $0 \leq u, v \in F$.

For more detailed information about Riesz spaces, the reader can consult the book *Riesz Spaces* by Luxemburg and Zaanen [8]. In the sequel, all the Riesz spaces are assumed to be Archimedean.

## 3. Main Result

Recently, Polat [9] generalized the Hyers' result [2] to Riesz spaces with extended norms and proved the following.

**Theorem 3.1.** *Let $E$ be a linear space and $F$ a Riesz space equipped with an extended norm $\|\cdot\|_u$ such that the space $(F, \|\cdot\|_u)$ is complete. If, for some $\delta > 0$, a map $f : E \to (F, \|\cdot\|_u)$ is $\delta$-additive, then limit $l(x) = \lim_{n\to\infty} f(2^n x)/2^n$ exists for each $x \in E$. $l(x)$ is the unique additive function satisfying the inequality $\|f(x) - l(x)\|_u \leq \delta$ for all $x \in E$.*

By using Theorem 3.1, we give the main result of the paper as follows.

**Theorem 3.2.** *Let $F$ be a Riesz algebra with an extended norm $\|\cdot\|_u$ such that $(F, \|\cdot\|_u)$ is complete. Also, let $\|\cdot\|_v$ be another extended norm in $F$ weaker than $\|\cdot\|_u$ such that whenever*

(a) *$x_n \to x$ and $x_n \cdot y \to z$ in $\|\cdot\|_v$, then $z = x \cdot y$;*

(b) *$y_n \to y$ and $x \cdot y_n \to z$ in $\|\cdot\|_v$, then $z = x \cdot y$.*

*Let $\varepsilon$ and $\delta$ be two nonnegative real numbers. Assume that a map $f : F \to F$ satisfies*

$$\left\|f(x+y) - f(x) - f(y)\right\|_u \leq \varepsilon, \tag{3.1}$$

$$\left\|f(x \cdot y) - x \cdot f(y) - f(x) \cdot y\right\|_v \leq \delta, \tag{3.2}$$

*for all $x, y \in F$. Then, there exists a unique derivation $d : F \to F$ such that $\|f(x) - d(x)\|_u \leq \varepsilon$, $(x \in F)$. Moreover, $x \cdot (f(y) - d(y)) = 0$ for all $x, y \in F$.*

*Proof.* By Condition (3.1), Theorem 3.1 shows that there exists a unique additive function $d : F \to F$ such that

$$\left\|f(x) - d(x)\right\|_u \leq \varepsilon, \tag{3.3}$$

for each $x \in F$. It is enough to show that $d$ satisfies Condition (1.2). The inequality (3.3) implies that

$$\left\|f(nx) - d(nx)\right\|_u \leq \varepsilon \quad (x \in F, n \in \mathbb{N}). \tag{3.4}$$

By the additivity of $d$, we then have

$$\left\|\frac{1}{n} f(nx) - d(x)\right\|_u \leq \frac{1}{n}\varepsilon \quad (x \in F, n \in \mathbb{N}), \tag{3.5}$$

which means that

$$d(x) = \lim_{n\to\infty} \frac{1}{n} f(nx), \quad (x \in F), \tag{3.6}$$

with respect to $\|\cdot\|_u$ norm and so is with respect to $\|\cdot\|_v$ norm. Condition (3.2) implies that the function $r : F \times F \to F$ defined by $r(x,y) = f(x\cdot y) - x\cdot f(y) - f(x)\cdot y$ is bounded. Hence

$$\lim_{n\to\infty} \frac{1}{n} r(nx,y) = 0, \quad (x,y \in F), \tag{3.7}$$

with respect to $\|\cdot\|_v$ norm. Applying (3.6) and (3.7), we have

$$d(x\cdot y) = x\cdot f(y) + d(x)\cdot y, \quad (x,y \in F). \tag{3.8}$$

Indeed, we have the following with respect to $\|\cdot\|_v$ norm,

$$\begin{aligned} d(x\cdot y) &= \lim_{n\to\infty} \frac{1}{n} f(n(x\cdot y)) = \lim_{n\to\infty} \frac{1}{n} f((nx)\cdot y) \\ &= \lim_{n\to\infty} \frac{1}{n}(nx\cdot f(y) + f(nx)\cdot y + r(nx,y)) \\ &= \lim_{n\to\infty}\left( x\cdot f(y) + \frac{f(nx)}{n}\cdot y + \frac{r(nx,y)}{n}\right) \\ &= x\cdot f(y) + d(x)\cdot y, \quad (x,y\in F). \end{aligned} \tag{3.9}$$

Let $x, y \in F$ and $n \in \mathbb{N}$ be fixed. Then using (3.8) and additivity of $d$, we have

$$\begin{aligned} x\cdot f(ny) + nd(x)\cdot y = x\cdot f(ny) + d(x)\cdot ny &= d(x\cdot ny) \\ = d(nx\cdot y) &= nx\cdot f(y) + d(nx)\cdot y \\ &= nx\cdot f(y) + nd(x)\cdot y. \end{aligned} \tag{3.10}$$

Therefore,

$$x\cdot f(y) = x\cdot \frac{f(ny)}{n}, \quad (x,y\in F,\ n\in\mathbb{N}). \tag{3.11}$$

Sending $n$ to infinity, by (3.6), we see that

$$x\cdot f(y) = x\cdot d(y), \quad (x,y\in F). \tag{3.12}$$

Combining this formula with (3.8), we have that $d$ satisfies (1.2) which is the desired result. Moreover, the last formula yields $x\cdot(f(y) - d(y)) = 0$ for all $x, y \in F$. □

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*Research Article*

# On the Structure of Brouwer Homeomorphisms Embeddable in a Flow

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We present two theorems describing the structure of the set of all regular points and the set of all irregular points for a Brouwer homeomorphism which is embeddable in a flow. The theorems are counterparts of structure theorems proved by Homma and Terasaka. To obtain our results, we use properties of the codivergence relation.

## 1. Introduction

Throughout the paper, $f$ will denote a *Brouwer homeomorphism*, that is, orientation preserving homeomorphism of the plane onto itself which has no fixed points.

For any sequence of subsets $(A_n)_{n\in\mathbb{Z}_+}$ of the plane, we define *limes superior* $\limsup_{n\to\infty} A_n$ as the set of all points $p \in \mathbb{R}^2$ such that any neighbourhood of $p$ has common points with infinitely many elements of the sequence $(A_n)_{n\in\mathbb{N}}$. For any subset $B$ of the plane, we define the *positive limit set* $\omega_f(B)$ as the limes superior of the sequence of its iterates $(f^n(B))_{n\in\mathbb{N}}$ and *negative limit set* $\alpha_f(B)$ as the limes superior of the sequence $(f^{-n}(B))_{n\in\mathbb{N}}$. Under the assumption that $B$ is compact, Nakayama [1] proved that

$$\begin{aligned}
\omega_f(B) = \Big\{ q \in \mathbb{R}^2 :\ & \text{there exist sequences } (p_j)_{j\in\mathbb{N}} \text{ and } (n_j)_{j\in\mathbb{N}} \\
& \text{such that } p_j \in B, n_j \in \mathbb{N}, n_j \longrightarrow +\infty, f^{n_j}(p_j) \longrightarrow q \text{ as } j \longrightarrow +\infty \Big\}, \\
\alpha_f(B) = \Big\{ q \in \mathbb{R}^2 :\ & \text{there exist sequences } (p_j)_{j\in\mathbb{N}} \text{ and } (n_j)_{j\in\mathbb{N}} \\
& \text{such that } p_j \in B, n_j \in \mathbb{N}, n_j \longrightarrow +\infty, f^{-n_j}(p_j) \longrightarrow q \text{ as } j \longrightarrow +\infty \Big\}.
\end{aligned} \tag{1.1}$$

A point $p$ is called *positively irregular* if $\omega_f(B) \neq \emptyset$ for each Jordan domain $B$ containing $p$ in its interior, and *negatively irregular* if $\alpha_f(B) \neq \emptyset$ for each Jordan domain $B$ containing $p$ in its interior, where by a Jordan domain we mean the union of a Jordan curve $J$ and the Jordan region determined by $J$ (i.e., the bounded component of $\mathbb{R}^2 \setminus J$). A point which is not positively irregular is said to be *positively regular*. Similarly, a point which is not negatively irregular is called *negatively regular*. A point which is positively or negatively irregular is called *irregular*, otherwise it is *regular*.

We say that a set $A \subset \mathbb{R}^2$ is *invariant* if $f(A) = A$. An invariant region $M$ is said to be *parallelizable* if there exists a homeomorphism $\varphi : M \to \mathbb{R}^2$ such that

$$f|_M = \varphi^{-1} \circ T \circ \varphi, \tag{1.2}$$

where $T$ is given by the formula $T(t,s) = (t+1, s)$. On account of the Brouwer Translation Theorem, for each $p \in \mathbb{R}^2$, there exists a parallelizable region $M$ containing $p$ (see [2]). This implies that a Brouwer homeomorphism looks locally like a translation. However, its global behaviour may be very complicated (cf. [3, 4]).

For any $p \in \mathbb{R}^2$, one can construct an arc $K$ with endpoints $p$ and $f(p)$ such that $f(K) \cap K = \{f(p)\}$ (see [5]). Such an arc is called a *translation arc*. The Brouwer Lemma says that if $K$ is a translation arc, then $\bigcup_{n\in\mathbb{Z}} f^n(K)$ is a homeomorphic image of a straight line (see [2]). The set $\bigcup_{n\in\mathbb{Z}} f^n(K)$ is called a *translation line*. A translation line needs not be a topological line, where by a *topological line* we mean a closed set which is a homeomorphic image of a straight line.

Homma and Terasaka [6] proved two theorems describing the structure of a Brouwer homeomorphism. The theorems can be formulated in the following way.

**Theorem 1.1** (see [6], First Structure Theorem). *Let $f$ be a Brouwer homeomorphism. Then, the plane is divided into at most three kinds of pairwise disjoint sets: $\{O_i : i \in I\}$, where $I = \mathbb{N}$ or $I = \{1, \dots, n\}$ for a positive integer $n$, $\{O'_i : i \in \mathbb{N}\}$ and $F$. The sets $\{O_i : i \in I\}$ and $\{O'_i : i \in \mathbb{N}\}$ are the components of the set of all regular points such that each $O_i$ is a parallelizable unbounded simply connected region, and each $O'_i$ is a simply connected region satisfying the condition $O'_i \cap f^n(O'_i) = \emptyset$ for $n \in \mathbb{Z} \setminus \{0\}$. The set $F$ is invariant, closed, and consists of all irregular points.*

**Theorem 1.2** (see [6], Second Structure Theorem). *Let $f$ be a Brouwer homeomorphism. Then, the plane is divided into at most three kinds of pairwise disjoint sets: $\{O_i : i \in I\}$, where $I = \mathbb{N}$ or $I = \{1, \dots, n\}$ for a positive integer $n$, $\{O'_i : i \in \mathbb{N}\}$ and $F$. The sets $\{O_i : i \in I\}$ and $\{O'_i : i \in \mathbb{N}\}$ are the components of the set of all negatively regular points such that each $O_i$ is an invariant unbounded simply connected region and can be filled with a family of translation lines which are closed sets in $O_i$, and each $O'_i$ is a simply connected region satisfying the condition $O'_i \cap f^n(O'_i) = \emptyset$ for $n \in \mathbb{Z} \setminus \{0\}$. The set $F$ is invariant, closed, and consists of all negatively irregular points.*

The set $F$ occurring in the theorems above is the union of sets called *singular lines* and their cluster set. Homma and Terasaka [6] showed many properties describing mutual relationships among singular lines. Moreover, they proved that the set of all singular lines is at most countable. But the set $F$ occurring in the theorems above can also contain the cluster points of singular lines which do not belong to any singular line. Thus, to obtain the complete description of the set $F$, the study of the set of these cluster points is needed. In the case of an arbitrary Brouwer homeomorphism, the problem is still open.

In this paper, we prove the counterparts of the structure theorems under the assumption that $f$ is embeddable in a flow. By a *flow*, we mean a group of homeomorphisms of the plane onto itself $\{f^t : t \in \mathbb{R}\}$ under the operation of composition which satisfies the following conditions:

(1) the function $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \phi(x,t) = f^t(x)$ is continuous,

(2) $f^0(x) = x$ for $x \in \mathbb{R}^2$,

(3) $f^t(f^s(x)) = f^{t+s}(x)$ for $x \in \mathbb{R}^2$, $t, s \in \mathbb{R}$.

We say that $f$ is *embeddable in a flow* if there exists a flow $\{f^t : t \in \mathbb{R}\}$ such that $f = f^1$.

## 2. Codivergence Relation

In this section, we characterize the sets of regular and irregular points of any Brouwer homeomorphism embeddable in a flow using the codivergence relation defined by Andrea [7].

For any Brouwer homeomorphism $f$, the *codivergence relation* is defined in the following way:

$$p \sim q \quad \text{if } p = q \text{ or } p \text{ and } q \text{ are endpoints of some arc } K \text{ for which } f^n(K) \longrightarrow \infty \text{ as } n \longrightarrow \pm\infty. \tag{2.1}$$

By an *arc* $K$ with endpoints $p$ and $q$, we mean the image of a homeomorphism $c : [0,1] \to c([0,1])$ satisfying conditions $c(0) = p$, $c(1) = q$, where the topology on $c([0,1])$ is induced by the topology of $\mathbb{R}^2$.

It turns out that the relation defined above is an equivalence relation and under the assumption that $f$ is embeddable in a flow each equivalence class of the relation is an invariant simply connected set (see [7, 8]).

**Proposition 2.1.** *Let $f$ be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then, the set of all regular points is equal to the union of the interiors of all equivalence classes of the codivergence relation.*

*Proof.* First we prove that every point $p$ belonging to the interior of an equivalence class $G_0$ is regular. By the definition of the interior, there exists a Jordan curve $J$ contained in $G_0$ such that the point $p$ belongs to the Jordan region $U$ whose boundary is equal to $J$. In the proof of the main theorem of [8], it has been showed that for every Jordan domain $B$ contained in an equivalence class which does not consist of just one orbit we have $f^n(B) \to \infty$ as $n \to \pm\infty$. Thus, $\omega_f(\mathrm{cl}\, U) = \emptyset$ and $\alpha_f(\mathrm{cl}\, U) = \emptyset$.

Conversely, if a point $p$ is regular, then there exists a Jordan region $U$ containing $p$ such that $f^n(\mathrm{cl}\, U) \to \infty$ as $n \to \pm\infty$. Since $U$ is arcwise connected, for each $q \in U \setminus \{p\}$ there exists an arc $K$ with endpoints $p$, $q$ contained in $U$. Hence, $K$ satisfies the condition $f^n(K) \to \infty$ as $n \to \pm\infty$. Thus, each point of the Jordan region $U$ belongs to the same equivalence class as $p$. Consequently, $p$ belongs to the interior of this equivalence class.

From the proposition above, we obtain immediately the following. □

**Corollary 2.2.** *Let $f$ be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then, the set of all irregular points is equal to the union of the boundaries of all equivalence classes of the codivergence relation.*

## 3. Structure of the Set of Regular Points

In this section, we show an application of properties of the codivergence relation to describe the set of all regular points for a Brouwer homeomorphism $f$ which is embeddable in a flow.

**Proposition 3.1.** *Let $f$ be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Let $p$ be a regular point. Then, each point of the trajectory $C_p = \{f^t(p) : t \in \mathbb{R}\}$ is a regular point.*

*Proof.* Let $p$ be a regular point. Denote by $G_0$ the equivalence class which contains $p$. By Proposition 2.1, we have $p \in \operatorname{int} G_0$. Hence, the trajectory $C_p$ is contained in $\operatorname{int} G_0$, since the interior of each equivalence class is invariant under any element of the flow $\{f^t : t \in \mathbb{R}\}$ (see [9]). Using Proposition 2.1 once again, we obtain that each element of the trajectory is a regular point.

In Theorem 1.1 describing the structure of any Brouwer homeomorphism, there are three types of sets: $O_i$, $O'_i$, and $F$. Under the assumption that a Brouwer homeomorphism is embeddable in a flow, we only have two types of sets: $O_i$ and $F$. However, sets of type $O'_i$ cannot occur. □

**Theorem 3.2.** *Let $f$ be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then, the plane is divided into at most two kinds of pairwise disjoint sets: $\{O_i : i \in I\}$, where $I = \mathbb{N}$ or $I = \{1, \dots, n\}$ for a positive integer $n$, and $F$. The sets $\{O_i : i \in I\}$ are the components of the set of all regular points such that each $O_i$ is a parallelizable unbounded simply connected region. The set $F$ is closed and consists of all irregular points.*

*Proof.* Suppose, on the contrary, that there exists a family of simply connected regions $\{O'_i : i \in \mathbb{N}\}$ occurring in Theorem 1.1. Let us fix a point $p \in O'_i$ for some $i \in \mathbb{N}$. Then, by Theorem 1.1, $p$ is a regular point and there exists a $j \in \mathbb{N}$, $j \neq i$ such that $f(p) \in O'_j$.

By Proposition 3.1, each point of the trajectory $C_p$ is regular. In particular, all points belonging to the arc with endpoints $p$ and $f(p)$ contained in this trajectory are regular. On the other hand, the arc $K$ has to contain an irregular point, since $p$ and $f(p)$ belong to different components $O'_i$ and $O'_j$ of the sets of all regular points.

At the end of this section, let us note that the invariance of the set of all irregular points (and the set of all regular points) under each element of a flow $\{f^t : t \in \mathbb{R}\}$ such that $f = f^1$ can also be obtained from the relation $f = f^{-t} \circ f \circ f^t$ (see [10]). □

## 4. Structure of the Set of Irregular Points

In this section, we proceed to study the structure of the set $F$ of all irregular points for a Brouwer homeomorphism $f$ which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$.

For any irregular point $p$, the set $P^+$ is defined as the intersection of all $\omega_f(B)$ and the set $P^-$ as the intersection of all $\alpha_f(B) \neq \emptyset$, where $B$ is a Jordan domain containing $p$ in its interior. An irregular point $p$ is *strongly positively irregular* if $P^+ \neq \emptyset$, otherwise it is *weakly positively irregular*. Similarly, $p$ is *strongly negatively irregular* if $P^- \neq \emptyset$, otherwise it is *weakly* *negatively irregular*. We say that $p$ is *strongly irregular* if it is strongly positively irregular or strongly negatively irregular. Otherwise, an irregular point $p$ is said to be *weakly irregular*.

Nakayama [10] has proved that for any Brouwer homeomorphism the subset of $F$ consisting of all strongly irregular points has no interior points. In the case where $f$ is embeddable in a flow, the set $F$ is the union of a family of invariant topological lines, since the boundary of each equivalence class is the union of trajectories of the flow $\{f^t : t \in \mathbb{R}\}$ (see [9]). But some of these trajectories are not singular lines in the sense of Homma and Terasaka. The union of all singular lines is equal to the set of all strongly irregular points, and, moreover, the cluster points of singular lines which do not belong to any singular line are weakly irregular points (see [6]).

In the description of the set $F$, the notion of the first prolongational limit set can be used. For any point $p$, we define *the first prolongational limit set* of $p$ as $J(p) = J^+(p) \cup J^-(p)$, where

$$\begin{aligned} J^+(p) &:= \Big\{q \in \mathbb{R}^2 : \text{ there exist sequences } (p_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}} \\ &\qquad \text{such that } p_n \longrightarrow p, t_n \longrightarrow +\infty, f^{t_n}(p_n) \longrightarrow q \text{ as } n \longrightarrow +\infty\Big\}, \\ J^-(p) &:= \Big\{q \in \mathbb{R}^2 : \text{ there exist sequences } (p_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}} \\ &\qquad \text{such that } p_n \longrightarrow p, t_n \longrightarrow -\infty, f^{t_n}(p_n) \longrightarrow q \text{ as } n \longrightarrow +\infty\Big\}. \end{aligned} \tag{4.1}$$

For an $H \subset \mathbb{R}^2$, we put

$$J(H) = \bigcup_{p\in H} J(p) \tag{4.2}$$

(see [11]). From the definition above, we obtain that

$$p \in J^+(q) \Longleftrightarrow q \in J^-(p) \tag{4.3}$$

for all $p, q \in \mathbb{R}^2$. Hence,

$$J(p) \neq \emptyset \Longleftrightarrow p \in J\left(\mathbb{R}^2\right). \tag{4.4}$$

**Proposition 4.1.** *Let $f$ be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Let $p$ be a strongly irregular point. Then, $J(p) \neq \emptyset$.*

*Proof.* Without loss of generality, we assume that $P^+ \neq \emptyset$. We will show that $P^+ \subset J^+(p)$. Let $q \in P^+$. For every positive integer $n$, we denote by $C_n$ the ball with centre $p$ and radius $1/n$ and by $D_n$ the ball with centre $q$ and radius $1/n$. Fix an $n \in \mathbb{N}$. Then, $q \in \omega_f(C_n)$. By the definition of $\omega_f(C_n)$, there exist sequences $(p_j)_{j\in\mathbb{N}}$ and $(m_j)_{j\in\mathbb{N}}$ such that $p_j \in C_n$, $m_j \in \mathbb{N}$, $m_j \to +\infty$, $f^{m_j}(p_j) \to q$ as $j \to +\infty$. Hence, there exists an $i \in \mathbb{N}$ such that $m_i > n$ and

$f^{m_i}(p_i) \in D_n$. Put $q_n = p_i$ and $t_n = m_i$. Thus, we constructed sequences $(q_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ such that

$$q_n \in C_n, \quad t_n > n, \quad f^{t_n}(q_n) \in D_n \tag{4.5}$$

for every $n \in \mathbb{N}$. Hence, $q_n \to p$, $t_n \to +\infty$ and $f^{t_n}(q_n) \to q$ as $n \to +\infty$. Consequently, $q \in J^+(p)$. □

From the proposition above, we obtain the following.

**Corollary 4.2.** *Let $f$ be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then, the set of all irregular points is equal to the closure of the first prolongational limit set of the plane.*

*Proof.* By Proposition 4.1, if $p$ is a strongly irregular point, then $p \in J(\mathbb{R}^2)$. If $p$ is a weakly irregular point, then it belongs to the closure of the set of all strongly irregular points (see [6]). Consequently, $p \in \mathrm{cl}\ J(\mathbb{R}^2)$. The closure of the first prolongational limit set of the plane cannot contain any regular point, since for each $p$ belonging to the interior of an equivalence class we have $p \notin J(\mathbb{R}^2)$ (see [12]).

Using the main theorem of [13], we replace the regions $O_i$ occurring in Theorem 3.2 by larger parallelizable unbounded simply connected regions $U_i$ such that the union of all these regions $U_i$ contains the set of all weakly irregular points. A strongly irregular point can belong either to a region $U_i$ or to the set $F$. Moreover, for every singular line contained in the boundary of a region $U_i$, there can exist at most one singular line contained in the region (see [14]). Therefore, the counterpart of the Second Structure Theorem can be stated in the following way. □

**Theorem 4.3.** *Let $f$ be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then, the plane is divided into at most two kinds of pairwise disjoint sets: $\{U_i : i \in I\}$, where $I = \mathbb{N}$ or $I = \{1, \ldots, n\}$ for a positive integer $n$, and $F$. The sets $\{U_i : i \in I\}$ are parallelizable unbounded simply connected regions. The set $F$ is closed, contained in $J(\mathbb{R}^2)$, and is the union of at most countable family of trajectories of the flow. Each of these trajectories is contained in the boundary of an region $U_i$.*

Using a decomposition described in the theorem above, we can obtain generalizations of results concerning Reeb homeomorphisms given by Béguin and Le Roux in [15].

## 5. Final Remarks

Let us consider the one-point compactification of a plane into the sphere $S^2$. Then, we can extend any Brouwer homeomorphism $f$ to a homeomorphism of the sphere by putting $f(\infty) = \infty$. Let us assume that $f$ is embeddable in a flow. Then, all trajectories are closed sets on the plane, since for all $p \in \mathbb{R}^2$ we have $f^t(p) \to \infty$ as $t \to \pm\infty$ (see [7]). Since the closure of each trajectory contains the stationary point $\infty$ of the flow, the phase portrait of the flow restricted to a Jordan region $U$ containing $\infty$ is divided into sectors (see [16], pages 161–174).

The index of $\infty$ is equal to

$$1 + \frac{n_e - n_h}{2}, \tag{5.1}$$

where $n_e$ is the number of elliptic sectors and $n_h$ is the number of hyperbolic sectors (the expression gives an integer, since the difference of the number of elliptic sectors and the number of hyperbolic sectors is even). Applying the Lefschetz-Hopf Theorem to our case, we obtain that the index of the stationary point $\infty$ equals 2, since the Euler characteristic of the sphere equals 2. In the case where $f$ is a translation, there are two elliptic sectors and two parabolic sectors. In the case where $f$ is a Reeb homeomorphism, there are three elliptic sectors, one hyperbolic sector and four parabolic sectors.

If a Jordan domain $B$ is contained in an elliptic sector of $U$, then $f^n(B)$ is contained in this sector for each $n \in \mathbb{Z}$. However, this property does not hold for parabolic and hyperbolic sectors. In the case where $f$ is a translation, for each Jordan region $U$ containing $\infty$ and each Jordan domain $B$ contained in one of the parabolic sectors, there exists an $n \in \mathbb{N}$ such that $f^n(B)$ is not contained in $U$. Thus even in case $f$ is a translation, the fixed point $\infty$ is not stable in the sense of the following definition: an invariant set $C$ is called *Lyapunov stable* if for any Jordan domain $U$ containing $C$ there is a Jordan domain $V$ containing $C$ such that $f^n(V) \subset U$ for all $n \in \mathbb{N}$ (see, e.g., [17]).

For a subset $D$ of the set of all homeomorphisms of a metric space $M$ equipped with the topology of uniform convergence on compact subsets, we say that $f \in D$ is *structurally stable* if there exists a neighborhood $U$ of $f$ in $D$ such that each $g \in U$ is topologically conjugate to $f$. If $M = \mathbb{R}^2$ and $D$ is the set of all Brouwer homeomorphisms, then there are no $f \in D$ which are structurally stable. Moreover, each of the topological conjugacy classes is dense in $D$ (see [18]).

Le Roux [19] gave a classification of the topological conjugacy classes of flows whose orbits are leaves of a given Reeb foliation of the plane. It could be interesting to study the structural stability of flows of Brouwer homeomorphisms. A flow $\{f^t : t \in \mathbb{R}\}$ is said to be *structurally stable* if for any flow $\{g^t : t \in \mathbb{R}\}$ in a neighbourhood of $\{f^t : t \in \mathbb{R}\}$ there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ that sends the orbits of $\{f^t : t \in \mathbb{R}\}$ to the orbits of $\{g^t : t \in \mathbb{R}\}$ preserving the orientation of the orbits. This means that the phase portraits of the flows are homeomorphic.

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*Research Article*

# Generalized Stability of Euler-Lagrange Quadratic Functional Equation

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The main goal of this paper is the investigation of the general solution and the generalized Hyers-Ulam stability theorem of the following Euler-Lagrange type quadratic functional equation $f(ax+by)+af(x-by)=(a+1)b^2f(y)+a(a+1)f(x)$, in $(\beta,p)$-Banach space, where $a,b$ are fixed rational numbers such that $a\neq-1,0$ and $b\neq 0$.

## 1. Introduction

In 1940, Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $G$ be a group and let $G'$ be a metric group with metric $\rho(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if $f:G\to G'$ satisfies $\rho(f(xy),f(x)f(y))<\delta$ for all $x,y\in G$, then a homomorphism $h:G\to G'$ exists with $\rho(f(x),h(x))<\epsilon$ for all $x\in G$?

In 1941, the first result concerning the stability of functional equations was presented by Hyers [2]. He has answered the question of Ulam for the case where $G_1$ and $G_2$ are Banach spaces.

Let $E_1$ and $E_2$ be real vector spaces. A function $f:E_1\to E_2$ is called a quadratic function if and only if $f$ is a solution function of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

It is well known that a function $f$ between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $B$ such that $f(x)=B(x,x)$ for all $x$, where the mapping $B$ is given by $B(x,y) = (1/4)(f(x+y) - f(x-y))$. See [3, 4] for the details. The Hyers-Ulam stability of the quadratic functional equation (1.1) was first proved by Skof [5] for functions $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [6] demonstrated that Skof's theorem is also valid if $E_1$ is replaced by an Abelian group $G$. Assume that a function $f : G \to E$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta, \tag{1.2}$$

for some $\delta \geq 0$ and for all $x, y \in G$. Then there exists a unique quadratic function $Q : G \to E$ such that

$$\|f(x) - Q(x)\| \leq \frac{\delta}{2}, \tag{1.3}$$

for all $x \in G$. Czerwik [7] proved the Hyers-Ulam-Rassias stability of quadratic functional equation (1.1). Let $E_1$ and $E_2$ be a real normed space and a real Banach space, respectively, and let $p \neq 2$ be a positive constant. If a function $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p), \tag{1.4}$$

for some $\epsilon > 0$ and for all $x, y \in E_1$, then there exists a unique quadratic function $q : E_1 \to E_2$ such that

$$\|f(x) - q(x)\| \leq \frac{2\epsilon}{|4 - 2^p|}\|x\|^p, \tag{1.5}$$

for all $x \in E_1$. Furthermore, according to the theorem of Borelli and Forti [8], we know the following generalization of stability theorem for quadratic functional equation. Let $G$ be an Abelian group and $E$ a Banach space, and let $f : G \to E$ be a mapping with $f(0) = 0$ satisfying the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varphi(x,y), \tag{1.6}$$

for all $x, y \in G$. Assume that one of the series

$$\Phi(x,y) := \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{1}{2^{2(k+1)}} \varphi(2^k x, 2^k y) < \infty, \\ \displaystyle\sum_{k=0}^{\infty} 2^{2k} \varphi\left(\frac{x}{2^{(k+1)}}, \frac{y}{2^{(k+1)}}\right) < \infty, \end{cases} \tag{1.7}$$

then there exists a unique quadratic function $Q : G \to E$ such that

$$\|f(x) - Q(x)\| \leq \Phi(x,x), \tag{1.8}$$

for all $x \in G$. During the last three decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability of several functional equations, and there are many interesting results concerning this problem [9–13].

The notion of quasi-$\beta$-normed space was introduced by Rassias and Kim in [14]. This notion is a generalization of that of quasi-normed space. We consider some basic concepts concerning quasi-$\beta$-normed space. We fix a real number $\beta$ with $0 < \beta \le 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let $X$ be a linear space over $\mathbb{K}$. A quasi-$\beta$-norm $\|\cdot\|$ is a real-valued function on $X$ satisfying the following:

(1) $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\lambda x\| = |\lambda|^{\beta}\|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$,

(3) there is a constant $K \ge 1$ such that $\|x + y\| \le K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-$\beta$-normed space if $\|\cdot\|$ is a quasi-$\beta$-norm on $X$. The smallest possible $K$ is called the modulus of concavity of $\|\cdot\|$. A quasi-$\beta$-Banach space is a complete quasi-$\beta$-normed space. A quasi-$\beta$-norm $\|\cdot\|$ is called a $(\beta, p)$-norm $(0 < p \le 1)$ if

$$\|x + y\|^p \le \|x\|^p + \|y\|^p, \tag{1.9}$$

for all $x, y \in X$. In this case, the quasi-$\beta$-Banach space is called a $(\beta, p)$-Banach space. We observe that if $x_1, x_2, \ldots, x_n$ are nonnegative real numbers, then

$$\left(\sum_{i=1}^{n} x_i\right)^p \le \sum_{i=1}^{n} x_i^p, \tag{1.10}$$

where $0 < p \le 1$ [15].

J. M. Rassias investigated the stability of Ulam for the Euler-Lagrange functional equation

$$f(ax + by) + f(bx - ay) = \left(a^2 + b^2\right)[f(x) + f(y)] \tag{1.11}$$

in the paper of [16]. Gordji and Khodaei investigated the generalized Hyers-Ulam stability of other Euler-Lagrange quadratic functional equations [17]. Jun et al. [18] introduced a new quadratic Euler-Lagrange functional equation

$$f(ax + y) + af(x - y) = (a + 1)f(y) + a(a + 1)f(x), \tag{1.12}$$

for any fixed $a \in \mathbb{Z}$ with $a \ne 0, -1$, which was a modified and instrumental equation for [19], and solved the generalized stability of (1.12). Now, we improve the functional equation (1.12) to the following functional equations:

$$f(ax + by) + af(x - by) = (a + 1)f(by) + a(a + 1)f(x), \tag{1.13}$$

$$f(ax + by) + af(x - by) = (a + 1)b^2 f(y) + a(a + 1)f(x), \tag{1.14}$$

for any fixed rational numbers $a, b \in \mathbb{Q}$ with $a \neq 0, -1$ and $b \neq 0$, which are generalized versions of (1.12). In this paper, we establish the general solution of (1.13) and (1.14) and then prove the generalized Hyers-Ulam stability of (1.13) and (1.14). We remark that there are some interesting papers concerning the stability of functional equations in quasi-Banach spaces [15, 20–23] and quasi-$\beta$-normed spaces [14, 24, 25].

## 2. General Solution of (1.13) and (1.14)

First, we present the general solution of (1.14) in the class of all functions between vector spaces.

**Lemma 2.1.** *Let $X$ and $Y$ be vector spaces over $\mathbb{K}$. Then a mapping $f : X \to Y$ is a solution of the functional equation* (1.12) *for any fixed rational number $a \in \mathbb{Q}$ with $a \neq 0, -1$ if and only if $f$ is quadratic.*

*Proof.* See the same proof in [18]. □

**Lemma 2.2.** *Let $X$ and $Y$ be vector spaces over $\mathbb{K}$. Then a mapping $f : X \to Y$ is a solution of the functional equation* (1.13) *if and only if $f$ is quadratic.*

*Proof.* We assume that a mapping $f : X \to Y$ satisfies the functional equation (1.13). Letting $by = u$ in (1.13), then (1.13) is equivalent to (1.12). Then by Lemma 2.1, $f$ is quadratic. Conversely, if $f$ is quadratic, then it is obvious that $f$ satisfies (1.13). □

**Theorem 2.3.** *Let $X$ and $Y$ be vector spaces over $\mathbb{K}$. Then a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation* (1.14) *if and only if $f$ is quadratic. In this case, $f(ax) = a^2 f(x)$ and $f(bx) = b^2 f(x)$ hold for all $x \in X$.*

*Proof.* We assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation (1.14). Then replacing $y$ in (1.14) by 0, we also get the equality $f(ax) = a^2 f(x)$ for all $x \in X$. Now, we decompose $f$ into the even part and the odd part by setting

$$f_e(x) = \frac{1}{2}\big(f(x) + f(-x)\big), \qquad f_o(x) = \frac{1}{2}\big(f(x) - f(-x)\big), \tag{2.1}$$

for all $x \in X$. Then $f_e$ and $f_o$ satisfy the functional equation (1.14). Therefore, we may assume without loss of generality that $f$ is even and satisfies (1.14) for all $x, y \in X$. If we replace $x$ in (1.14) by 0, then we get

$$f(by) + af(-by) = (a+1)b^2 f(y), \tag{2.2}$$

for all $y \in X$. From this equality, we have $f(by) = b^2 f(y)$ for all $y \in X$. Therefore, (1.14) implies (1.13) for all $x, y \in X$. By Lemma 2.2, $f$ is quadratic.

Now, we assume that $f$ is odd and satisfies (1.14) for all $x, y \in X$. For the case $a = 1$, we have

$$f(x + by) + f(x - by) = 2b^2 f(y) + 2f(x), \tag{2.3}$$

for all $x, y \in X$. Setting $x$ by 0 in (2.3), one obtains $f \equiv 0$. Let $a \neq 1$. Replacing $x$ by 0 in (1.14), we have

$$(1-a)f(by) = (a+1)b^2 f(y), \tag{2.4}$$

for all $y \in X$. From (1.14) and (2.4), we get

$$f(ax+by) + af(x-by) = (1-a)f(by) + a(a+1)f(x), \tag{2.5}$$

for all $x, y \in X$. Putting $by = u$ in (2.5), then we obtain

$$f(ax+u) + af(x-u) = (1-a)f(u) + a(a+1)f(x), \tag{2.6}$$

for all $x, u \in X$. Replacing $u$ by $au$ in (2.6), we get

$$f(ax+au) + af(x-au) = (1-a)f(au) + a(a+1)f(x), \tag{2.7}$$

for all $x, u \in X$. Since $f(ax) = a^2 f(x)$, (2.7) yields

$$af(x+u) + f(x-au) = (1-a)af(u) + (a+1)f(x), \tag{2.8}$$

for all $x, u \in X$. Interchanging $x$ and $u$ in (2.8), we have by oddness of $f$

$$-f(ax-u) + af(x+u) = (1-a)af(x) + (a+1)f(u), \tag{2.9}$$

for all $x, u \in X$. Replacing $u$ by $-u$ in (2.6), we get

$$f(ax-u) + af(x+u) = -(1-a)f(u) + a(a+1)f(x), \tag{2.10}$$

for all $x, u \in X$. Adding (2.9) and (2.10) side by side, this leads to

$$f(x+u) = f(x) + f(u), \tag{2.11}$$

for all $x, u \in X$. Therefore, $f$ is additive and so $f(ax) = af(x)$ for all $x \in X$ and for any odd function satisfying (1.14). Using the equality $f(ax) = a^2 f(x)$, we obtain $f(x) = 0$ for all $x \in X$. Therefore, $f(x) = f_e(x) + f_o(x)$ is a quadratic mapping, as desired.

Conversely, if $f$ is quadratic, then it is obvious that $f$ satisfies (1.14). □

We note that $f(0) = 0$ if $a + b^2 \neq 1$ and $f$ satisfies (1.14).

## 3. Generalized Stability of (1.14) for $a \neq 1$

For convenience, we use the following abbreviation: for any fixed rational numbers $a$ and $b$ with $a \neq -1, 0, 1$ and $b \neq 0$,

$$D_f(x,y) := f(ax+by) + af(x-by) - (a+1)b^2 f(y) - a(a+1)f(x), \tag{3.1}$$

for all $x, y \in X$, which is called the approximate remainder of the functional equation (1.14) and acts as a perturbation of the equation.

From now on, let $X$ be a vector space, and let $Y$ be a $(\beta, p)$-Banach space unless we give any specific reference. We will investigate the generalized Hyers-Ulam stability problem for the functional equation (1.14). Thus, we find some conditions such that there exists a true quadratic function near an approximate solution of (1.14).

**Theorem 3.1.** *Let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\Phi(x) := \sum_{n=0}^{\infty} \frac{1}{|a|^{2\beta n p}} (\varphi(a^n x, 0))^p < \infty, \tag{3.2}$$

$$\lim_{n\to\infty} \frac{1}{|a|^{2\beta n}} \varphi(a^n x, a^n y) = 0, \tag{3.3}$$

*for all $x, y \in X$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\|D_f(x,y)\|_Y \leq \varphi(x,y), \tag{3.4}$$

*for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying*

$$\|f(x) - Q(x)\|_Y \leq \frac{1}{|a|^{2\beta}} [\Phi(x)]^{1/p}, \tag{3.5}$$

*for all $x \in X$. The function $Q$ is given by*

$$Q(x) = \lim_{k\to\infty} \frac{1}{a^{2k}} f\left(a^k x\right), \tag{3.6}$$

*for all $x \in X$.*

*Proof.* Letting $y$ by 0 in (3.4), we get

$$\left\| f(ax) - a^2 f(x) \right\|_Y \leq \varphi(x,0), \tag{3.7}$$

for all $x \in X$. Multiplying both sides by $1/|a|^{2\beta}$ in (3.7), we have

$$\left\| \frac{1}{a^2} f(ax) - f(x) \right\|_Y \leq \frac{1}{|a|^{2\beta}} \varphi(x,0), \tag{3.8}$$

for all $x \in X$. Replacing $x$ by $a^n x$ and multiplying both sides by $1/|a|^{2n\beta}$ in (3.8), we have

$$\left\| \frac{1}{a^{2(n+1)}} f\left(a^{n+1}x\right) - \frac{1}{a^{2n}} f(a^n x) \right\|_Y \le \frac{1}{|a|^{2\beta(n+1)}} \varphi(a^n x, 0), \tag{3.9}$$

for all $x \in X$. Next we show that the sequence $\{(1/a^{2n}) f(a^n x)\}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}$, $m > n \ge 0$, and $x \in X$, it follows from (3.9) that

$$\begin{aligned} \left\| \frac{1}{a^{2(m+1)}} f\left(a^{m+1}x\right) - \frac{1}{a^{2n}} f(a^n x) \right\|_Y^p &= \left\| \sum_{i=n}^{m} \frac{1}{a^{2(i+1)}} f\left(a^{i+1}x\right) - \frac{1}{a^{2i}} f\left(a^i x\right) \right\|_Y^p \\ &\le \sum_{i=n}^{m} \left\| \frac{1}{a^{2(i+1)}} f\left(a^{i+1}x\right) - \frac{1}{a^{2i}} f\left(a^i x\right) \right\|_Y^p \\ &\le \sum_{i=n}^{m} \frac{1}{|a|^{2\beta p(i+1)}} \left(\varphi\left(a^i x, 0\right)\right)^p \\ &= \frac{1}{|a|^{2\beta p}} \sum_{i=n}^{m} \frac{1}{|a|^{2\beta p i}} \left(\varphi\left(a^i x, 0\right)\right)^p, \end{aligned} \tag{3.10}$$

for all $x \in X$. It follows from (3.2) and (3.10) that the sequence $\{(1/a^{2n}) f(a^n x)\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is a $(\beta, p)$-Banach space, the sequence $\{(1/a^{2n}) f(a^n x)\}$ converges for all $x \in X$. Therefore, we can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n \to \infty} \frac{1}{a^{2n}} f(a^n x), \tag{3.11}$$

for all $x \in X$. Taking $m \to \infty$ and $n = 0$ in (3.10), we have

$$\|Q(x) - f(x)\|_Y^p \le \frac{1}{|a|^{2\beta p}} \sum_{i=0}^{\infty} \frac{1}{|a|^{2\beta p i}} \left(\varphi\left(a^i x, 0\right)\right)^p = \frac{1}{|a|^{2\beta p}} \Phi(x), \tag{3.12}$$

for all $x \in X$. Therefore,

$$\|Q(x) - f(x)\|_Y \le \frac{1}{|a|^{2\beta}} [\Phi(x)]^{1/p}, \tag{3.13}$$

for all $x \in X$, that is, the mapping $Q$ satisfies (3.5). It follows from (3.3) and (3.4) that

$$\begin{aligned} \|D_Q(x, y)\|_Y &= \lim_{n \to \infty} \left\| \frac{1}{a^{2n}} D_f(a^n x, a^n y) \right\|_Y \\ &= \lim_{n \to \infty} \frac{1}{|a|^{2\beta n}} \|D_f(a^n x, a^n y)\|_Y \\ &\le \lim_{n \to \infty} \frac{1}{|a|^{2\beta n}} \varphi(a^n x, a^n y) = 0, \end{aligned} \tag{3.14}$$

for all $x, y \in X$. Therefore, $Q$ satisfies (1.14), and so the function $Q$ is quadratic.

To prove the uniqueness of the quadratic function $Q$, let us assume that there exists a quadratic function $Q' : X \to Y$ satisfying the inequality (3.5). Then we have

$$\begin{aligned}
\|Q(x) - Q'(x)\|_Y^p &= \left\| \frac{1}{a^{2n}} Q(a^n x) - \frac{1}{a^{2n}} Q'(a^n x) \right\|_Y^p \\
&= \frac{1}{a^{2n\beta p}} \|Q(a^n x) - Q'(a^n x)\|_Y^p \\
&\le \frac{1}{a^{2n\beta p}} \left( \|Q(a^n x) - f(a^n x)\|_Y^p + \|Q'(a^n x) - f(a^n x)\|_Y^p \right) \\
&\le \frac{1}{|a|^{2n\beta p}} \frac{2}{|a|^{2\beta p}} \Phi(a^n x) \\
&= \frac{2}{|a|^{2\beta p(n+1)}} \sum_{i=0}^{\infty} \frac{1}{|a|^{2\beta p i}} \left( \varphi\left( a^{i+n} x, 0 \right) \right)^p \\
&= \frac{2}{|a|^{2\beta p}} \sum_{i=n}^{\infty} \frac{1}{|a|^{2\beta p i}} \left( \varphi\left( a^{i} x, 0 \right) \right)^p,
\end{aligned} \tag{3.15}$$

for all $x \in X$ and $n \in \mathbb{N}$. Therefore, letting $n \to \infty$, one has $Q(x) - Q'(x) = 0$ for all $x \in X$, completing the proof of uniqueness. □

In the following corollary, we get a stability result of (1.14).

**Corollary 3.2.** *Let $X$ be a quasi-$\alpha$-normed space for fixed real number $\alpha$ with $0 < \alpha \le 1$. Let $\theta_1, \theta_2, \theta_3, \alpha_1, \alpha_2, \gamma_1, \gamma_2$ be positive reals such that either (1) $|a| > 1$, $(\alpha_1 + \alpha_2)\alpha < 2\beta$, and $\gamma_i \alpha < 2\beta$ or (2) $|a| < 1$, $(\alpha_1 + \alpha_2)\alpha > 2\beta$, and $\gamma_i \alpha > 2\beta$, for $i = 1, 2$. Assume that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_f(x, y)\|_Y \le \theta_1 \|x\|^{\alpha_1} \|y\|^{\alpha_2} + \theta_2 \|x\|^{\gamma_1} + \theta_3 \|y\|^{\gamma_2}, \tag{3.16}$$

*for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality*

$$\|f(x) - Q(x)\|_Y \le \frac{\theta_2 \|x\|^{\gamma_1}}{\left( |a|^{2\beta p} - |a|^{\gamma_1 \alpha p} \right)^{1/p}}, \tag{3.17}$$

*for all $x \in X$. The function $Q$ is given by*

$$Q(x) = \lim_{k \to \infty} \frac{f(a^k x)}{a^{2k}}, \tag{3.18}$$

*for all $x \in X$.*

*Proof.* Let $\varphi(x,y) = \theta_1\|x\|^{\alpha_1}\|y\|^{\alpha_2} + \theta_2\|x\|^{\gamma_1} + \theta_3\|y\|^{\gamma_2}$. Then

$$\begin{aligned}\Phi(x) &= \sum_{n=0}^{\infty}\frac{1}{|a|^{2\beta np}}\left(\varphi(a^n x,0)\right)^p = \sum_{n=0}^{\infty}\frac{1}{|a|^{2\beta np}}\theta_2^p\|a^n x\|^{\gamma_1 p} \\ &= \theta_2^p\|x\|^{\gamma_1 p}\sum_{n=0}^{\infty}|a|^{(\gamma_1\alpha-2\beta)np} < \infty,\end{aligned} \tag{3.19}$$

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{|a|^{2\beta n}}\varphi(a^n x, a^n y) &= \lim_{n\to\infty}\frac{1}{|a|^{2\beta n}}\left[\theta_1\left(\|a^n x\|^{\alpha_1}\left\|a^n y\right\|^{\alpha_2}\right) + \theta_2\|a^n x\|^{\gamma_1} + \theta_3\left\|a^n y\right\|^{\gamma_2}\right] \\ &= \theta_1\left(\|x\|^{\alpha_1}\left\|y\right\|^{\alpha_2}\right)\lim_{n\to\infty}|a|^{((\alpha_1+\alpha_2)\alpha-2\beta)n} + \theta_2\|x\|^{\gamma_1}\lim_{n\to\infty}|a|^{(\gamma_1\alpha-2\beta)n} \\ &\quad + \theta_3\left\|y\right\|^{\gamma_2}\lim_{n\to\infty}|a|^{(\gamma_2\alpha-2\beta)n} = 0.\end{aligned} \tag{3.20}$$

By Theorem 3.1, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}\left\|f(x) - Q(x)\right\|_Y &\le \frac{1}{|a|^{2\beta}}[\Phi(x)]^{1/p} \\ &= \frac{\theta_2\|x\|^{\gamma_1}}{|a|^{2\beta}}\left(\sum_{n=0}^{\infty}|a|^{(\gamma_1\alpha-2\beta)np}\right)^{1/p} \\ &= \frac{\theta_2\|x\|^{\gamma_1}}{\left(|a|^{2\beta p} - |a|^{\gamma_1\alpha p}\right)^{1/p}},\end{aligned} \tag{3.21}$$

for all $x \in X$. □

**Theorem 3.3.** *Let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\Psi(x) := \sum_{n=0}^{\infty}|a|^{2\beta np}\left(\varphi\left(\frac{x}{a^{n+1}},0\right)\right)^p < \infty, \tag{3.22}$$

$$\lim_{n\to\infty}|a|^{2\beta n}\varphi\left(\frac{x}{a^n},\frac{y}{a^n}\right) = 0, \tag{3.23}$$

*for all $x, y \in X$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\left\|D_f(x,y)\right\|_Y \le \varphi(x,y), \tag{3.24}$$

*for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying*

$$\left\|f(x) - Q(x)\right\|_Y \le [\Psi(x)]^{1/p}, \tag{3.25}$$

*for all $x \in X$. The function $Q$ is given by*

$$Q(x) = \lim_{k \to \infty} a^{2k} f\left(\frac{x}{a^k}\right), \tag{3.26}$$

*for all $x \in X$.*

*Proof.* Letting $y$ by 0 in (3.24), we get

$$\left\| f(ax) - a^2 f(x) \right\|_Y \leq \varphi(x, 0), \tag{3.27}$$

for all $x \in X$. Replacing $x$ by $x/a$ in (3.27), we have

$$\left\| f(x) - a^2 f\left(\frac{x}{a}\right) \right\|_Y \leq \varphi\left(\frac{x}{a}, 0\right), \tag{3.28}$$

for all $x \in X$. Replacing $x$ by $x/a^n$ and multiplying both sides by $|a|^{2\beta n}$ in (3.28), we have

$$\left\| a^{2n} f\left(\frac{x}{a^n}\right) - a^{2(n+1)} f\left(\frac{x}{a^{n+1}}\right) \right\|_Y \leq |a|^{2\beta n} \varphi\left(\frac{x}{a^{n+1}}, 0\right), \tag{3.29}$$

for all $x \in X$. Next we show that the sequence $\{a^{2n} f(x/a^n)\}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}$, $m > n \geq 0$, and $x \in X$, it follows from (3.29) that

$$\begin{aligned} \left\| a^{2n} f\left(\frac{x}{a^n}\right) - a^{2(m+1)} f\left(\frac{x}{a^{m+1}}\right) \right\|_Y^p &= \left\| \sum_{i=n}^{m} a^{2i} f\left(\frac{x}{a^i}\right) - a^{2(i+1)} f\left(\frac{x}{a^{i+1}}\right) \right\|_Y^p \\ &\leq \sum_{i=n}^{m} \left\| a^{2i} f\left(\frac{x}{a^i}\right) - a^{2(i+1)} f\left(\frac{x}{a^{i+1}}\right) \right\|_Y^p \\ &\leq \sum_{i=n}^{m} |a|^{2\beta p i} \left( \varphi\left(\frac{x}{a^{i+1}}, 0\right) \right)^p. \end{aligned} \tag{3.30}$$

It follows from (3.22) and (3.30) that the sequence $\{a^{2n} f(x/a^n)\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is a $(\beta, p)$-Banach space, the sequence $\{a^{2n} f(x/a^n)\}$ converges for all $x \in X$. Therefore, we can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n \to \infty} a^{2n} f\left(\frac{x}{a^n}\right), \tag{3.31}$$

for all $x \in X$. The rest of the proof is similar to the corresponding proof of Theorem 3.1. □

**Corollary 3.4.** *Let $X$ be a quasi-$\alpha$-normed space for fixed real number $\alpha$ with $0 < \alpha \leq 1$. Let $\theta_1, \theta_2, \theta_3, \alpha_1, \alpha_2, \gamma_1, \gamma_2$ be positive reals such that either (1) $|a| > 1$, $(\alpha_1 + \alpha_2)\alpha > 2\beta$, and $\gamma_i \alpha > 2\beta$ or*

*(2) $|a| < 1$, $(\alpha_1 + \alpha_2)\alpha < 2\beta$, and $\gamma_i\alpha < 2\beta$, for $i = 1, 2$. Assume that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_f(x,y)\|_Y \le \theta_1\|x\|^{\alpha_1}\|y\|^{\alpha_2} + \theta_2\|x\|^{\gamma_1} + \theta_3\|y\|^{\gamma_2}, \tag{3.32}$$

*for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality*

$$\|f(x) - Q(x)\|_Y \le \frac{\theta_2\|x\|^{\gamma_1}}{\left(|a|^{\gamma_1\alpha p} - |a|^{2\beta p}\right)^{1/p}}, \tag{3.33}$$

*for all $x \in X$. The function $Q$ is given by*

$$Q(x) = \lim_{k\to\infty} a^{2k} f\left(\frac{x}{a^k}\right), \tag{3.34}$$

*for all $x \in X$.*

*Proof.* Let $\varphi(x,y) = \theta_1\|x\|^{\alpha_1}\|y\|^{\alpha_2} + \theta_2\|x\|^{\gamma_1} + \theta_3\|y\|^{\gamma_2}$. Then $\varphi$ satisfies the conditions (3.22) and (3.23). Applying Theorem 3.3, we obtain the results, as desired. □

## 4. Generalized Stability of (1.13)

For convenience, we use the following abbreviation: for any fixed rational numbers $a$ and $b$ with $a \ne -1, 0$ and $b \ne 0$,

$$E_f(x,y) := f(ax + by) + af(x - by) - (a+1)f(by) - a(a+1)f(x), \tag{4.1}$$

for all $x, y \in X$, which is called the approximate remainder of the functional equation (1.13) and acts as a perturbation of the equation.

We will investigate the generalized Hyers-Ulam stability problem for the functional equation (1.13).

**Theorem 4.1.** *Let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\Phi(x) := \sum_{n=0}^{\infty} \frac{1}{|a+1|^{2\beta np}} \left(\varphi\left((a+1)^n x, \frac{(a+1)^n x}{b}\right)\right)^p < \infty, \tag{4.2}$$

$$\lim_{n\to\infty} \frac{1}{|a+1|^{2\beta n}} \varphi\left((a+1)^n x, (a+1)^n y\right) = 0, \tag{4.3}$$

*for all $x, y \in X$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\|E_f(x,y)\|_Y \le \varphi(x,y), \tag{4.4}$$

*for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying*

$$\|f(x) - Q(x)\|_Y \le \frac{1}{|a+1|^{2\beta}}[\Phi(x)]^{1/p}, \tag{4.5}$$

*for all $x \in X$. The function $Q$ is given by*

$$Q(x) = \lim_{k\to\infty} \frac{1}{(a+1)^{2k}} f\left((a+1)^k x\right), \tag{4.6}$$

*for all $x \in X$.*

*Proof.* Replacing $x$ by $by$ in (4.4), we get

$$\left\|f((a+1)by) - (a+1)^2 f(by)\right\|_Y \le \varphi(by, y), \tag{4.7}$$

for all $y \in X$. Letting $by$ be $x$ in (4.7), we have

$$\left\|f((a+1)x) - (a+1)^2 f(x)\right\|_Y \le \varphi\left(x, \frac{x}{b}\right), \tag{4.8}$$

for all $x \in X$. Multiplying both sides by $1/|a+1|^{2\beta}$ in (4.8), we have

$$\left\|\frac{1}{(a+1)^2} f((a+1)x) - f(x)\right\|_Y \le \frac{1}{|a+1|^{2\beta}}\varphi\left(x, \frac{x}{b}\right), \tag{4.9}$$

for all $x \in X$. Replacing $x$ by $(a+1)^i x$ and multiplying both sides by $1/|a+1|^{2i\beta}$ in (4.9), we have

$$\left\|\frac{1}{(a+1)^{2(i+1)}} f\left((a+1)^{i+1}x\right) - \frac{1}{(a+1)^{2i}} f\left((a+1)^i x\right)\right\|_Y \le \frac{1}{|a+1|^{2\beta(i+1)}}\varphi\left((a+1)^i x, \frac{(a+1)^i x}{b}\right), \tag{4.10}$$

for all $x \in X$. Next we show that the sequence $\{(1/(a+1)^{2n}) f((a+1)^n x)\}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}$, $m > n \ge 0$, and $x \in X$, it follows from (4.10) that

$$\begin{aligned}
&\left\|\frac{1}{(a+1)^{2(m+1)}} f\left((a+1)^{m+1}x\right) - \frac{1}{(a+1)^{2n}} f((a+1)^n x)\right\|_Y^p \\
&\quad = \left\|\sum_{i=n}^{m} \frac{1}{(a+1)^{2(i+1)}} f\left((a+1)^{i+1}x\right) - \frac{1}{(a+1)^{2i}} f\left((a+1)^i x\right)\right\|_Y^p \\
&\quad \le \sum_{i=n}^{m} \left\|\frac{1}{(a+1)^{2(i+1)}} f\left((a+1)^{i+1}x\right) - \frac{1}{(a+1)^{2i}} f\left((a+1)^i x\right)\right\|_Y^p
\end{aligned}$$

$$\leq \sum_{i=n}^{m} \frac{1}{|a+1|^{2\beta p(i+1)}} \left( \varphi\left( (a+1)^i x, \frac{(a+1)^i x}{b} \right) \right)^p$$
$$= \frac{1}{|a+1|^{2\beta p}} \sum_{i=n}^{m} \frac{1}{|a+1|^{2\beta p i}} \left( \varphi\left( (a+1)^i x, \frac{(a+1)^i x}{b} \right) \right)^p, \tag{4.11}$$

for all $x \in X$. It follows from (4.2) and (4.11) that the sequence $\{f((a+1)^n x)/(a+1)^{2n}\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is a $(\beta, p)$-Banach space, the sequence $\{f((a+1)^n x)/(a+1)^{2n}\}$ converges for all $x \in X$. Therefore, we can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \frac{1}{(a+1)^{2n}} f((a+1)^n x), \tag{4.12}$$

for all $x \in X$. The rest of the proof is similar to the corresponding proof of Theorem 3.1. □

In the following corollary, we get a stability result of (1.13).

**Corollary 4.2.** *Let $X$ be a quasi-$\alpha$-normed space for fixed real number $\alpha$ with $0 < \alpha \leq 1$. Let $\theta_1, \theta_2, \theta_3$, $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ be positive reals such that either (1) $|a+1| > 1$, $(\alpha_1 + \alpha_2)\alpha < 2\beta$, and $\gamma_i \alpha < 2\beta$ or (2) $|a+1| < 1$, $(\alpha_1 + \alpha_2)\alpha > 2\beta$, and $\gamma_i \alpha > 2\beta$, for $i = 1, 2$. Assume that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\left\| E_f(x,y) \right\|_Y \leq \theta_1 \|x\|^{\alpha_1} \|y\|^{\alpha_2} + \theta_2 \|x\|^{\gamma_1} + \theta_3 \|y\|^{\gamma_2}, \tag{4.13}$$

*for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality*

$$\|f(x) - Q(x)\|_Y \leq \left\{ \frac{\theta_1^p \|x\|^{(\alpha_1+\alpha_2)p}}{|b|^{\alpha\alpha_2 p}\left(|a+1|^{2\beta p} - |a+1|^{(\alpha_1+\alpha_2)\alpha p}\right)} + \frac{\theta_2^p \|x\|^{\gamma_1 p}}{|a+1|^{2\beta p} - |a+1|^{\gamma_1 \alpha p}} + \frac{\theta_3^p \|x\|^{\gamma_2 p}}{|b|^{\gamma_2 \alpha p}\left(|a+1|^{2\beta p} - |a+1|^{\gamma_2 \alpha p}\right)} \right\}^{1/p}, \tag{4.14}$$

*for all $x \in X$. The function $Q$ is given by*

$$Q(x) = \lim_{k\to\infty} \frac{1}{(a+1)^{2k}} f\left((a+1)^k x\right), \tag{4.15}$$

*for all $x \in X$.*

*Proof.* Let $\varphi(x,y) = \theta_1\|x\|^{\alpha_1}\|y\|^{\alpha_2} + \theta_2\|x\|^{\gamma_1} + \theta_3\|y\|^{\gamma_2}$. Then $\varphi$ satisfies the conditions (4.2) and (4.3). By Theorem 4.1, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}
\|f(x) - Q(x)\|_Y &\le \frac{1}{|a+1|^{2\beta}}\left[\sum_{n=0}^{\infty}\frac{1}{|a+1|^{2\beta np}}\left(\varphi\left((a+1)^n x, \frac{(a+1)^n x}{b}\right)\right)^p\right]^{1/p} \\
&\le \left\{\frac{\theta_1^p\|x\|^{(\alpha_1+\alpha_2)p}}{|b|^{\alpha\alpha_2 p}\left(|a+1|^{2\beta p} - |a+1|^{(\alpha_1+\alpha_2)\alpha p}\right)}\right. \\
&\qquad \left. + \frac{\theta_2^p\|x\|^{\gamma_1 p}}{|a+1|^{2\beta p} - |a+1|^{\gamma_1\alpha p}} + \frac{\theta_3^p\|x\|^{\gamma_2 p}}{|b|^{\gamma_2\alpha p}\left(|a+1|^{2\beta p} - |a+1|^{\gamma_2\alpha p}\right)}\right\}^{1/p},
\end{aligned} \tag{4.16}$$

for all $x \in X$. □

**Theorem 4.3.** *Let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\begin{aligned}
\Psi(x) &:= \sum_{n=0}^{\infty}|a+1|^{2\beta np}\left(\varphi\left(\frac{x}{(a+1)^{n+1}}, \frac{x}{(a+1)^{n+1}b}\right)\right)^p < \infty, \\
&\lim_{n\to\infty}|a+1|^{2\beta n}\varphi\left(\frac{x}{(a+1)^n}, \frac{y}{(a+1)^n}\right) = 0,
\end{aligned} \tag{4.17}$$

*for all $x, y \in X$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\|E_f(x,y)\|_Y \le \varphi(x,y), \tag{4.18}$$

*for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying*

$$\|f(x) - Q(x)\|_Y \le [\Psi(x)]^{1/p}, \tag{4.19}$$

*for all $x \in X$. The function $Q$ is given by*

$$Q(x) = \lim_{k\to\infty}(a+1)^{2k} f\left(\frac{x}{(a+1)^k}\right), \tag{4.20}$$

*for all $x \in X$.*

*Proof.* Replacing $x$ by $x/(a+1)$ in (4.8), we have

$$\left\|f(x) - (a+1)^2 f\left(\frac{x}{a+1}\right)\right\|_Y \le \varphi\left(\frac{x}{a+1}, \frac{x}{(a+1)b}\right), \tag{4.21}$$

for all $x \in X$. The rest of the proof is similar to the corresponding proof of Theorem 3.3. □

**Corollary 4.4.** *Let X be a quasi-$\alpha$-normed space for fixed real number $\alpha$ with $0 < \alpha \leq 1$. Let $\theta_1, \theta_2, \theta_3, \alpha_1, \alpha_2, \gamma_1, \gamma_2$ be positive reals such that either (1) $|a+1| > 1$ and $(\alpha_1+\alpha_2)\alpha > 2\beta$, $\gamma_i\alpha > 2\beta$ or (2) $|a+1| < 1$ and $(\alpha_1+\alpha_2)\alpha < 2\beta$, $\gamma_i\alpha < 2\beta$, for $i = 1, 2$. Assume that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\left\|E_f(x,y)\right\|_Y \leq \theta_1\|x\|^{\alpha_1}\|y\|^{\alpha_2} + \theta_2\|x\|^{\gamma_1} + \theta_3\|y\|^{\gamma_2}, \tag{4.22}$$

*for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality*

$$\left\|f(x) - Q(x)\right\|_Y \leq \left\{\frac{\theta_1^p\|x\|^{(\alpha_1+\alpha_2)p}}{|b|^{\alpha\alpha_2 p}\left(|a+1|^{(\alpha_1+\alpha_2)\alpha p} - |a+1|^{2\beta p}\right)} + \frac{\theta_2^p\|x\|^{\gamma_1 p}}{|a+1|^{\gamma_1\alpha p} - |a+1|^{2\beta p}} + \frac{\theta_3^p\|x\|^{\gamma_2 p}}{|b|^{\alpha\gamma_2 p}\left(|a+1|^{\gamma_2\alpha p} - |a+1|^{2\beta p}\right)}\right\}^{1/p}, \tag{4.23}$$

*for all $x \in X$. The function $Q$ is given by*

$$Q(x) = \lim_{k\to\infty}(a+1)^{2k} f\left(\frac{x}{(a+1)^k}\right), \tag{4.24}$$

*for all $x \in X$.*

*Proof.* Let $\varphi(x,y) = \theta_1\|x\|^{\alpha_1}\|y\|^{\alpha_2} + \theta_2\|x\|^{\gamma_1} + \theta_3\|y\|^{\gamma_2}$. Then $\varphi$ satisfies the conditions (4.17). Applying Theorem 4.3, we obtain the results, as desired. □


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*Research Article*

# Hyers-Ulam Stability of Jensen Functional Inequality in $p$-Banach Spaces

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We prove the Hyers-Ulam stability of the following Jensen functional inequality $\|f((x-y)/n+z)+f((y-z)/n+x)+f((z-x)/n+y)\| \le \|f((x+y+z)\|$ in $p$-Banach spaces for any fixed nonzero integer $n$.

## 1. Introduction

The stability problem of equations originated from a question of Ulam [1] concerning the stability of group homomorphisms.

We are given a group $G_1$ and a metric group $G_2$ with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a number $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

In 1941, Hyers [2] considered the case of approximately additive mappings between Banach spaces and proved the following result.

Suppose that $E_1$ and $E_2$ are Banach spaces and $f : E_1 \to E_2$ satisfies the following condition: if there is a number $\epsilon \ge 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon \tag{1.1}$$

for all $x, y \in E_1$, then the limit $h(x) = \lim_{n\to\infty} f(2^n x)/2^n$ exists for all $x \in E_1$ and there exists a unique additive mapping $h : E_1 \to E_2$ such that

$$\|f(x) - h(x)\| \le \epsilon. \tag{1.2}$$

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x \in E_1$, then the mapping $h$ is $\mathbb{R}$-linear.

The method which was provided by Hyers, and which produces the additive mapping $h$, is called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers' theorem was generalized by Aoki [3] and Bourgin [4] for additive mappings by considering an unbounded Cauchy difference. In 1978, Rassias [5] also provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded. Let $E_1$ and $E_2$ be two Banach spaces and let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x$. Assume that there exist $\epsilon > 0$ and $0 \le p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon\left(\|x\|^p + \|y\|^p\right), \quad \forall x, y \in E_1. \tag{1.3}$$

Then, there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.4}$$

for all $x \in E_1$. A generalized result of Rassias' theorem was obtained by Găvruţa in [6] and Jung in [7]. In 1990, Rassias [8] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. In 1991, Gajda [9], following the same approach as in [5], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [9], as well as by Rassias and Šemrl [10], that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [9], as well as of Rassias and Šemrl [10], have stimulated several mathematicians to invent new approximately additive or approximately linear mappings.

We recall some basic facts concerning quasinormed spaces and some preliminary results. Let $X$ be a real linear space. A quasinorm is a real-valued function on $X$ satisfying the following:

(1) $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $M \ge 1$ such that $\|x + y\| \le M(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasinormed space if $\|\cdot\|$ is a quasinorm on $X$ [11, 12]. The smallest possible $M$ is called the modulus of concavity of $\|\cdot\|$. A quasi-Banach space is a complete quasinormed space.

A quasinorm $\|\cdot\|$ is called a $p$-norm $(0 < p \le 1)$ if

$$\|x + y\|^p \le \|x\|^p + \|y\|^p \tag{1.5}$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a $p$-Banach space.

Given a $p$-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on $X$. By the Aoki-Rolewicz theorem [12], each quasinorm is equivalent to some $p$-norm (see also [11]). Since it is much easier to work with $p$-norms, henceforth, we restrict our attention mainly to $p$-norms. We observe that if $x_1, x_2, \ldots, x_n$ are nonnegative real numbers, then

$$\left(\sum_{i=1}^{n} x_i\right)^p \le \sum_{i=1}^{n} x_i^p, \tag{1.6}$$

where $0 < p \le 1$.

In 2009, Moslehian and Najati [13] introduced the Hyers-Ulam stability of the additive functional inequality:

$$\left\| f\left(\frac{x-y}{2}+z\right)+f\left(\frac{y-z}{2}+x\right)+f\left(\frac{z-x}{2}+y\right)\right\| \le \|f(x+y+z)\|, \tag{1.7}$$

and then have investigated the general solution and the Hyers-Ulam stability problem for the functional inequality. The stability problems of several functional equations in quasi-normed spaces and several functional inequalities have been investigated by a number of authors and there are many interesting results concerning the stability of various functional inequalities [14–17].

In this paper, we consider a modified and general Jensen functional inequality:

$$\left\| f\left(\frac{x-y}{n}+z\right)+f\left(\frac{y-z}{n}+x\right)+f\left(\frac{z-x}{n}+y\right)\right\| \le \|f(x+y+z)\| \tag{1.8}$$

for any fixed nonzero integer $n$. First of all, it is easy to see that a function $f$ satisfies the inequality (1.8) if and only if $f(x)$ is additive. Thus the inequality (1.8) may be called the Jensen functional inequality and the general solution of inequality (1.8) may be called the Jensen function. In the sequel, we investigate the generalized Hyers-Ulam stability of (1.8) in $p$-Banach spaces for any fixed nonzero integer $n$ by using the techniques of [14, 15].

## 2. Generalized Hyers-Ulam Stability

First, we present the general solution of the inequality (1.8).

**Lemma 2.1.** *Let both X and Y be real vector spaces. A function $f : X \to Y$ satisfies* (1.8) *for all $x, y, z \in X$ if and only if $f$ is additive.*

*Proof.* Letting $x = y = z = 0$ in (1.8), we have $f(0) = 0$. Putting $y = -(n+1)x/2$ and $z = (n-1)x/2$ in (1.8), we get

$$\left\| f\left(\frac{(n^2+3)x}{2n}\right)+f\left(\frac{-(n^2+3)x}{2n}\right)\right\| \le \|f(0)\| \tag{2.1}$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$. Replacing $z$ by $-x-y$ in (1.8), we obtain

$$\left\| f\left(\frac{(1-n)x-(n+1)y}{n}\right)+f\left(\frac{(n+1)x+2y}{n}\right)+f\left(\frac{-2x+(n-1)y}{n}\right)\right\| \le \|f(0)\|, \tag{2.2}$$

that is,

$$f((1-n)x-(n+1)y)+f((n+1)x+2y)+f(-2x+(n-1)y) = 0 \tag{2.3}$$

for all $x, y \in X$. Putting $u = (n+1)x + 2y$ and $v = -2x + (n-1)y$ in (2.3), we get by oddness of $f$,

$$f(u+v) = f(u) + f(v) \tag{2.4}$$

for all $u, v \in X$. So $f$ is additive.

The proof of the converse is trivial. □

From now on, assume that $X$ is a quasinormed space with quasinorm $\|\cdot\|$ and that $Y$ is a $p$-Banach space with $p$-norm $\|\cdot\|$. Let $M$ be the modulus of concavity of $\|\cdot\|$ in $Y$.

Before taking up the main subject, given a mapping $f : X \to Y$, we define the difference operator $Df : X^3 \to Y$ by

$$Df(x,y,z) := \left\| f\left(\frac{x-y}{n} + z\right) + f\left(\frac{y-z}{n} + x\right) + f\left(\frac{z-x}{n} + y\right) \right\| - \|f(x+y+z)\| \tag{2.5}$$

for all $x, y, z \in X$ and for any fixed nonzero integer $n$.

**Theorem 2.2.** *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$Df(x,y,z) \le \varphi(x,y,z) \tag{2.6}$$

*for all $x, y, z \in X$ and the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies*

$$\Phi(x,y,z) := \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \varphi\left(2^i x, 2^i y, 2^i z\right)^p < \infty \tag{2.7}$$

*for all $x, y, z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ defined by $h(x) = \lim_{k\to\infty}(1/2^k) f(2^k x)$ such that*

$$\begin{aligned} \|f(x) - h(x)\| \le \frac{M}{2}\Bigg[ &\Phi\left(\frac{n(n-3)x}{n^2+3}, \frac{n(n+3)x}{n^2+3}, \frac{-2n^2 x}{n^2+3}\right) \\ &+\Phi\left(\frac{-2n(n+1)x}{n^2+3}, \frac{2n(n-1)x}{n^2+3}, \frac{4nx}{n^2+3}\right)\Bigg]^{1/p} \end{aligned} \tag{2.8}$$

*for all $x \in X$.*

*Proof.* Replacing $z$ by $-x-y$ in (2.6), we obtain

$$\begin{aligned} &\left\| f\left(\frac{(1-n)x - (n+1)y}{n}\right) + f\left(\frac{(n+1)x + 2y}{n}\right) + f\left(\frac{-2x + (n-1)y}{n}\right) \right\| \\ &\quad \le \varphi(x, y, -x-y) \end{aligned} \tag{2.9}$$

for all $x, y \in X$. Letting $x = (n-3)x/(n^2+3)$ and $y = (n+3)x/(n^2+3)$ in (2.9), we get

$$\left\| f\left(-\frac{2x}{n}\right) + 2f\left(\frac{x}{n}\right) \right\| \le \varphi\left(\frac{(n-3)x}{n^2+3}, \frac{(n+3)x}{n^2+3}, \frac{-2nx}{n^2+3}\right) \tag{2.10}$$

for all $x \in X$. Putting $x = -(n+1)z/2$ and $y = (n-1)z/2$ in (2.6), we have

$$\left\| f\left(\frac{-(n^2+3)z}{2n}\right) + f\left(\frac{(n^2+3)z}{2n}\right) \right\| \le \varphi\left(\frac{-(n+1)z}{2}, \frac{(n-1)z}{2}, z\right) \tag{2.11}$$

for all $z \in X$. Replacing $z$ by $4x/(n^2+3)$ in (2.11), we obtain

$$\left\| f\left(-\frac{2x}{n}\right) + f\left(\frac{2x}{n}\right) \right\| \le \varphi\left(\frac{-2(n+1)x}{n^2+3}, \frac{2(n-1)x}{n^2+3}, \frac{4x}{n^2+3}\right) \tag{2.12}$$

for all $x \in X$. It follows from (2.10) and (2.12) that

$$\begin{aligned} \left\| f\left(\frac{2x}{n}\right) - 2f\left(\frac{2x}{n}\right) \right\| &\le M\left[\left\| f\left(-\frac{2x}{n}\right) + 2f\left(\frac{x}{n}\right) \right\| + \left\| f\left(-\frac{2x}{n}\right) + f\left(\frac{2x}{n}\right) \right\|\right] \\ &\le M\left[\varphi\left(\frac{(n-3)x}{n^2+3}, \frac{(n+3)x}{n^2+3}, \frac{-2nx}{n^2+3}\right)\right. \\ &\quad \left. +\varphi\left(\frac{-2(n+1)x}{n^2+3}, \frac{2(n-1)x}{n^2+3}, \frac{4x}{n^2+3}\right)\right] \end{aligned} \tag{2.13}$$

for all $x \in X$. If we replace $x$ by $nx$ in (2.13), then we get that

$$\begin{aligned} \| f(2x) - 2f(x) \| \le M&\left[\varphi\left(\frac{n(n-3)x}{n^2+3}, \frac{n(n+3)x}{n^2+3}, \frac{-2n^2x}{n^2+3}\right)\right. \\ &\left. +\varphi\left(\frac{-2n(n+1)x}{n^2+3}, \frac{2n(n-1)x}{n^2+3}, \frac{4nx}{n^2+3}\right)\right]. \end{aligned} \tag{2.14}$$

It follows from (2.14) that

$$\begin{aligned}\left\|\frac{f(2^l x)}{2^l}-\frac{f(2^m x)}{2^m}\right\|^p &\le \sum_{i=l}^{m-1}\left\|\frac{1}{2^i}f\left(2^i x\right)-\frac{1}{2^{i+1}}f\left(2^{i+1}x\right)\right\|^p\\ &= \sum_{i=l}^{m-1}\frac{1}{2^{ip}}\left\|f\left(2^i x\right)-\frac{1}{2}f\left(2^{i+1}x\right)\right\|^p\\ &\le \frac{M^p}{2^p}\sum_{i=l}^{m-1}\frac{1}{2^{ip}}\left[\varphi\left(\frac{n(n-3)2^i x}{n^2+3},\frac{n(n+3)2^i x}{n^2+3},\frac{(-2n^2)2^i x}{n^2+3}\right)^p\right.\\ &\qquad\left.+\varphi\left(\frac{-2n(n+1)2^i x}{n^2+3},\frac{2n(n-1)2^i x}{n^2+3},\frac{(4n)2^i x}{n^2+3}\right)^p\right]\end{aligned} \tag{2.15}$$

for all nonnegative integers $m$ and $l$ with $m > l \ge 0$ and $x \in X$. Since the right-hand side of (2.15) tends to zero as $l \to \infty$, by the convergence of the series (2.7), we obtain that the sequence $\{f(2^m x)/2^m\}$ is Cauchy for all $x \in X$. Because of the fact that $Y$ is complete, it follows that the sequence $\{f(2^m x)/2^m\}$ converges in $Y$. Therefore, we can define a mapping $h : X \to Y$ as

$$h(x) = \lim_{m\to\infty}\frac{f(2^m x)}{2^m}, \quad x \in X. \tag{2.16}$$

Moreover, letting $l = 0$ and taking $m \to \infty$ in (2.15), we get

$$\begin{aligned}\|f(x)-h(x)\| \le \frac{M}{2}&\left[\Phi\left(\frac{n(n-3)x}{n^2+3},\frac{n(n+3)x}{n^2+3},\frac{-2n^2x}{n^2+3}\right)\right.\\ &\left.+\Phi\left(\frac{-2n(n+1)x}{n^2+3},\frac{2n(n-1)x}{n^2+3},\frac{4nx}{n^2+3}\right)\right]^{1/p}\end{aligned} \tag{2.17}$$

for all $x \in X$.

It follows from (2.6) and (2.7) that

$$\begin{aligned}&\left\|h\left(\frac{x-y}{n}+z\right)+h\left(\frac{y-z}{n}+x\right)+h\left(\frac{z-x}{n}+y\right)\right\|^p\\ &\quad= \lim_{m\to\infty}\left\|\frac{1}{2^m}\left\{f\left(2^m\left(\frac{x-y}{n}+z\right)\right)+f\left(2^m\left(\frac{y-z}{n}+x\right)\right)+f\left(2^m\left(\frac{z-x}{n}+y\right)\right)\right\}\right\|^p\\ &\quad\le \lim_{m\to\infty}\left\{\left\|\frac{1}{2^m}f(2^m(x+y+z))\right\|^p+\frac{1}{2^{mp}}\varphi(2^m x,2^m y,2^m z)^p\right\}\\ &\quad= \|h(x+y+z)\|^p\end{aligned} \tag{2.18}$$

for all $x, y, z \in X$. So the mapping $h$ is additive.

Next, let $h' : X \to Y$ be another additive mapping satisfying (2.8). Then, we have

$$\begin{aligned}
&\|h(x) - h'(x)\|^p \\
&\quad = \left\| \frac{1}{2^k} h\left(2^k x\right) - \frac{1}{2^k} h'\left(2^k x\right) \right\|^p \\
&\quad \le \frac{1}{2^{kp}} \left( \left\| h\left(2^k x\right) - f\left(2^k x\right) \right\|^p + \left\| f\left(2^k x\right) - h'\left(2^k x\right) \right\|^p \right) \\
&\quad \le \sum_{i=0}^{\infty} \frac{2M^p}{2^{(i+k+1)p}} \left[ \varphi\left( \frac{n(n-3)2^{i+k}x}{n^2+3}, \frac{n(n+3)2^{i+k}x}{n^2+3}, \frac{(-2n^2)2^{i+k}x}{n^2+3} \right)^p \right. \\
&\qquad\qquad \left. + \varphi\left( \frac{-2n(n+1)2^{i+k}x}{n^2+3}, \frac{2n(n-1)2^{i+k}x}{n^2+3}, \frac{(4n)2^{i+k}x}{n^2+3} \right)^p \right] \\
&\quad = \sum_{i=k}^{\infty} \frac{2M^p}{2^{(i+1)p}} \left[ \varphi\left( \frac{n(n-3)2^{i}x}{n^2+3}, \frac{n(n+3)2^{i}x}{n^2+3}, \frac{(-2n^2)2^{i}x}{n^2+3} \right)^p \right. \\
&\qquad\qquad \left. + \varphi\left( \frac{-2n(n+1)2^{i}x}{n^2+3}, \frac{2n(n-1)2^{i}x}{n^2+3}, \frac{(4n)2^{i}x}{n^2+3} \right)^p \right]
\end{aligned} \tag{2.19}$$

for all $k \in \mathbb{N}$ and all $x \in X$. Taking the limit as $k \to \infty$, we conclude that

$$h(x) = h'(x) \tag{2.20}$$

for all $x \in X$. This completes the proof. □

If we put $\varphi(x, y, z) := \theta(\|x\|^{r_1}\|y\|^{r_2}\|z\|^{r_3})$ and $\varphi(x, y, z) := \theta_1\|x\|^{r_1} + \theta_2\|y\|^{r_2} + \theta_3\|z\|^{r_3}$ in the following corollaries, respectively, then we lead to the desired results.

**Corollary 2.3.** *Let $r_i > 0$ for $i = 1, 2, 3$ with $\sum_{i=1}^{3} r_i < 1$ and $\theta \ge 0$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality*

$$Df(x, y, z) \le \theta\left(\|x\|^{r_1}\|y\|^{r_2}\|z\|^{r_3}\right) \tag{2.21}$$

*for all $x, y, z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that*

$$\begin{aligned}
\|f(x) - h(x)\| \le \frac{M\theta\|x\|^r}{\sqrt[p]{2^p - 2^{rp}}} &\left( \left| \frac{n(n-3)}{n^2+3} \right|^{r_1 p} \left| \frac{n(n+3)}{n^2+3} \right|^{r_2 p} \left| \frac{2n^2}{n^2+3} \right|^{r_3 p} \right. \\
&\left. + \left| \frac{2n(n+1)}{n^2+1} \right|^{r_1 p} \left| \frac{2n(n-3)}{n^2+3} \right|^{r_2 p} \left| \frac{4n}{n^2+3} \right|^{r_3 p} \right)^{1/p}
\end{aligned} \tag{2.22}$$

*for all $x \in X$, where $r = \sum_{i=1}^{3} r_i$.*

**Corollary 2.4.** *Let $0 < r_i < 1$ and $\theta_i \geq 0$ for $i = 1,2,3$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality*

$$Df(x,y,z) \leq \theta_1 \|x\|^{r_1} + \theta_2 \|y\|^{r_2} + \theta_3 \|z\|^{r_3} \tag{2.23}$$

*for all $x, y, z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that*

$$\begin{aligned} \|f(x) - h(x)\| \leq M\Bigg[&\left(\left|\frac{n(n-3)}{n^2+3}\right|^{r_1 p} + \left|\frac{2n(n+1)}{n^2+3}\right|^{r_1 p}\right)\frac{\theta_1^p \|x\|^{r_1 p}}{2^p - 2^{r_1 p}} \\ &+\left(\left|\frac{n(n+3)}{n^2+3}\right|^{r_2 p} + \left|\frac{2n(n-1)}{n^2+3}\right|^{r_2 p}\right)\frac{\theta_2^p \|x\|^{r_2 p}}{2^p - 2^{r_2 p}} \\ &+\left(\left|\frac{2n^2}{n^2+3}\right|^{r_3 p} + \left|\frac{4n}{n^2+3}\right|^{r_3 p}\right)\frac{\theta_3^p \|x\|^{r_3 p}}{2^p - 2^{r_3 p}}\Bigg]^{1/p} \end{aligned} \tag{2.24}$$

*for all $x \in X$.*

**Theorem 2.5.** *Suppose that a mapping $f : X \to Y$ satisfies the functional inequality*

$$Df(x,y,z) \leq \varphi(x,y,z) \tag{2.25}$$

*for all $x, y, z \in X$, and the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies*

$$\Phi(x,y,z) := \sum_{i=0}^{\infty} 2^{ip} \varphi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1} y}, \frac{z}{2^{i+1}}\right)^p < \infty \tag{2.26}$$

*for all $x, y, z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ defined by $h(x) = \lim_{k\to\infty} 2^k f(x/2^k)$ such that*

$$\begin{aligned} \|f(x) - h(x)\| \leq M\Bigg[&\Phi\left(\frac{n(n-3)x}{n^2+3}, \frac{n(n+3)x}{n^2+3}, \frac{-2n^2 x}{n^2+3}\right) \\ &+\Phi\left(\frac{-2n(n+1)x}{n^2+3}, \frac{2n(n-1)x}{n^2+3}, \frac{4nx}{n^2+3}\right)\Bigg]^{1/p} \end{aligned} \tag{2.27}$$

*for all $x \in X$.*

*Proof.* We note that $f(0) = 0$ since $\varphi(0,0,0) = 0$ by the convergence of (2.26). Now, if we replace $x$ by $x/2$ in (2.14),

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq M\left[\varphi\left(\frac{n(n-3)x}{2(n^2+3)}, \frac{n(n+3)x}{2(n^2+3)}, \frac{-n^2x}{(n^2+3)}\right) + \varphi\left(\frac{-n(n+1)x}{n^2+3}, \frac{n(n-1)x}{n^2+3}, \frac{2nx}{n^2+3}\right)\right] \tag{2.28}$$

for all $x \in X$. Then, it follows from the last inequality that

$$\left\| f(x) - 2^m f\left(\frac{x}{2^m}\right) \right\|^p \leq M^p \sum_{i=0}^{m-1} 2^{ip} \left[\varphi\left(\frac{n(n-3)x}{2^{i+1}(n^2+3)}, \frac{n(n+3)x}{2^{i+1}(n^2+3)}, \frac{-2n^2x}{2^{i+1}(n^2+3)}\right)^p + \varphi\left(\frac{-2n(n+1)x}{2^{i+i}(n^2+3)}, \frac{2n(n-1)x}{2^{i+1}(n^2+3)}, \frac{4nx}{2^{i+1}(n^2+3)}\right)^p\right] \tag{2.29}$$

for all nonnegative integer $m$ and all $x \in X$. The remaining proof is similar to the corresponding part of Theorem 2.2. This completes the proof. □

If we put $\varphi(x,y,z) := \theta(\|x\|^{r_1}\|y\|^{r_2}\|z\|^{r_3})$ and $\varphi(x,y,z) := \theta_1\|x\|^{r_1} + \theta_2\|y\|^{r_2} + \theta_3\|z\|^{r_3}$ in the following corollaries, respectively, then we lead to the desired results.

**Corollary 2.6.** *Let $r_i > 0$ for $i = 1,2,3$ with $\sum_{i=1}^{3} r_i > 1$ and $\theta \geq 0$. If a mapping $f : X \to Y$ satisfies the following functional inequality*

$$Df(x,y,z) \leq \theta(\|x\|^{r_1}\|y\|^{r_2}\|z\|^{r_3}) \tag{2.30}$$

*for all $x, y, z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{M\theta\|x\|^r}{\sqrt[p]{2^{rp} - 2^p}}\left(\left|\frac{n(n-3)}{n^2+3}\right|^{r_1p}\left|\frac{n(n+3)}{n^2+3}\right|^{r_2p}\left|\frac{2n^2}{n^2+3}\right|^{r_3p} + \left|\frac{2n(n+1)}{n^2+1}\right|^{r_1p}\left|\frac{2n(n-3)}{n^2+3}\right|^{r_2p}\left|\frac{4n}{n^2+3}\right|^{r_3p}\right)^{1/p} \tag{2.31}$$

*for all $x \in X$, where $r = \sum_{i=1}^{3} r_i$.*

**Corollary 2.7.** *Let $r_i > 1$ and $\theta_i \geq 0$ for $i = 1,2,3$. If a mapping $f : X \to Y$ satisfies the following functional inequality*

$$Df(x,y,z) \leq \theta_1 \|x\|^{r_1} + \theta_2 \|y\|^{r_2} + \theta_3 \|z\|^{r_3} \tag{2.32}$$

*for all $x, y, z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that*

$$\begin{aligned}\|f(x) - h(x)\| \leq M\Bigg[&\left(\left|\frac{n(n-3)}{n^2+3}\right|^{r_1p} + \left|\frac{2n(n+1)}{n^2+3}\right|^{r_1p}\right)\frac{\theta_1^p\|x\|^{r_1p}}{2^{r_1p-2^p}} \\ &+\left(\left|\frac{n(n+3)}{n^2+3}\right|^{r_2p} + \left|\frac{2n(n-1)}{n^2+3}\right|^{r_2p}\right)\frac{\theta_2^p\|x\|^{r_2p}}{2^{r_2p}-2^p} \\ &+\left(\left|\frac{2n^2}{n^2+3}\right|^{r_3p} + \left|\frac{4n}{n^2+3}\right|^{r_3p}\right)\frac{\theta_3^p\|x\|^{r_3p}}{2^{r_3p}-2^p}\Bigg]^{1/p}\end{aligned} \tag{2.33}$$

*for all $x \in X$.*

The following is a simple example that the additive functional inequality $Df(x,y,z) \leq \theta(\|x\| + \|y\| + \|z\|)$ is not stable for the singular case $r_1, r_2, r_3 = 1$ in Corollaries 2.4 and 2.7.

*Example 2.8.* Fix $\theta \geq 0$ and put $\mu := \theta/8$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} \mu & \text{for } x \in [1,\infty), \\ \mu x & \text{for } x \in (-1,1), \\ -\mu & \text{for } x \in (-\infty,-1], \end{cases} \tag{2.34}$$

and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{i=0}^{\infty} \frac{\phi(2^i x)}{2^i}, \quad \forall x \in \mathbb{R}, \tag{2.35}$$

which can be found in [9]. It follows from the same argument as in the example of [9] that $f$ satisfies the functional inequality

$$\begin{aligned}&\left|\left|f\left(\frac{x-y}{n}+z\right) + f\left(\frac{y-z}{n}+x\right) + f\left(\frac{z-x}{n}+y\right)\right| - |f(x+y+z)|\right| \\ &\quad \leq 8\mu(|x| + |y| + |z|)\end{aligned} \tag{2.36}$$

for all $x, y, z \in \mathbb{R}$. In fact, if $x = y = z = 0$, then (2.36) is trivially fulfilled. Next, if $0 < |x| + |y| + |z| < 1$, then there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{2^N} \leq |x| + |y| + |z| < \frac{1}{2^{N-1}}, \tag{2.37}$$

which implies that

$$2^i\left(\frac{x-y}{n}+z\right), 2^i\left(\frac{y-z}{n}+x\right), 2^i\left(\frac{z-x}{n}+y\right), 2^i(x+y+z) \in (-1,1), \quad \forall i \in \{0, \dots, N-1\}. \tag{2.38}$$

Thus, we see that

$$\phi\left(2^i\left(\frac{x-y}{n}+z\right)\right) + \phi\left(2^i\left(\frac{y-z}{n}+x\right)\right) + \phi\left(2^i\left(\frac{z-x}{n}+y\right)\right) - \phi\left(2^i(x+y+z)\right) = 0 \tag{2.39}$$

for all $i \in \{0, \dots, N-1\}$. As a result, we infer that

$$\begin{aligned} &\frac{\left|f(((x-y)/n)+z) + f(((y-z)/n)+x) + f(((z-x)/n)+y) - f(x+y+z)\right|}{|x|+|y|+|z|} \\ &\leq \sum_{i=N}^{\infty} \frac{\left|\phi(2^i(((x-y)/n)+z)) + \phi(2^i(((y-z)/n)+x)) + \phi(2^i(((z-x)/n)+y)) - \phi(2^i(x+y+z))\right|}{2^i(|x|+|y|+|z|)} \\ &\leq 8\mu \end{aligned} \tag{2.40}$$

for all $x, y, z \in \mathbb{R}$. Finally, if $|x| + |y| + |z| \geq 1$, then one has by use of boundedness of $f$

$$\frac{\left|f(((x-y)/n)+z) + f(((y-z)/n)+x) + f(((z-x)/n)+y) - f(x+y+z)\right|}{|x|+|y|+|z|} \leq 8\mu \tag{2.41}$$

for all $x, y, z \in \mathbb{R}$. Therefore, $f$ satisfies the functional inequality (2.36) and so

$$Df(x, y, z) \leq 8\mu(|x| + |y| + |z|) \tag{2.42}$$

for all $x, y, z \in \mathbb{R}$. However, there do not exist an additive function $T : \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$ such that

$$\left|f(x) - T(x)\right| \leq c|x| \quad \forall x \in \mathbb{R}. \tag{2.43}$$

*Remark 2.9.* The stability problem on the singular case $r = 1$ in Corollaries 2.3 and 2.6 is not easy and it remains with us unsolved for providing a counterexample on the singular case $r = 1$.

## 3. Alternative Generalized Hyers-Ulam Stability of (1.8)

From now on, we investigate the generalized Hyers-Ulam stability of the functional inequality (1.8) using the contractive property of perturbing term of the inequality (1.8).

**Theorem 3.1.** *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$Df(x,y,z) \leq \varphi(x,y,z) \tag{3.1}$$

*for all $x, y, z \in X$ and there exists a constant $L$ with $0 < L < 1$ for which the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies*

$$\varphi(2x, 2y, 2z) \leq 2L\varphi(x,y,z) \tag{3.2}$$

*for all $x, y, z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ given by $h(x) = \lim_{k\to\infty}(1/2^k)f(2^k x)$ such that*

$$\begin{aligned}\|f(x) - h(x)\| \leq \frac{M}{2\sqrt[p]{1-L^p}}\Bigg[&\varphi\left(\frac{n(n-3)x}{n^2+3}, \frac{n(n+3)x}{n^2+3}, \frac{-2n^2x}{n^2+3}\right)^p \\ &+\varphi\left(\frac{-2n(n+1)x}{n^2+3}, \frac{2n(n-1)x}{n^2+3}, \frac{4nx}{n^2+3}\right)^p\Bigg]^{1/p}\end{aligned} \tag{3.3}$$

*for all $x \in X$.*

*Proof.* It follows from (2.15) and (3.2) that

$$\begin{aligned}\left\|\frac{f(2^l x)}{2^l} - \frac{f(2^m x)}{2^m}\right\|^p &\leq \frac{M^p}{2^p}\sum_{i=l}^{m-1}\frac{1}{2^{ip}}\Bigg[\varphi\left(\frac{n(n-3)2^i x}{n^2+3}, \frac{n(n+3)2^i x}{n^2+3}, \frac{(-2n^2)2^i x}{n^2+3}\right)^p \\ &\qquad+\varphi\left(\frac{-2n(n+1)2^i x}{n^2+3}, \frac{2n(n-1)2^i x}{n^2+3}, \frac{(4n)2^i x}{n^2+3}\right)^p\Bigg] \\ &\leq \frac{M^p}{2^p}\sum_{i=l}^{m-1}L^{ip}\Bigg[\varphi\left(\frac{n(n-3)x}{n^2+3}, \frac{n(n+3)x}{n^2+3}, \frac{(-2n^2)x}{n^2+3}\right)^p \\ &\qquad+\varphi\left(\frac{-2n(n+1)x}{n^2+3}, \frac{2n(n-1)x}{n^2+3}, \frac{(4n)x}{n^2+3}\right)^p\Bigg]\end{aligned} \tag{3.4}$$

for all nonnegative integers $m$ and $l$ with $m > l \geq 0$ and $x \in X$. Since the sequence $\{f(2^m x)/2^m\}$ is Cauchy for all $x \in X$, we can define a mapping $h : X \to Y$ by

$$h(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^m}, \quad x \in X. \tag{3.5}$$

Moreover, letting $l = 0$ and $m \to \infty$ in the last inequality yields the approximation (3.3).

The remaining proof is similar to the corresponding part of Theorem 2.2. This completes the proof. □

**Corollary 3.2.** *Let $\xi : [0,\infty) \to [0,\infty)$ be a nontrivial function satisfying*

$$\xi(2t) \leq \xi(2)\xi(t), \quad (t \geq 0),\ 0 < \xi(2) < 2. \tag{3.6}$$

*If $f : X \to Y$ with $f(0) = 0$ is a mapping satisfying the following functional inequality*

$$Df(x,y,z) \leq \theta\{\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|)\} \tag{3.7}$$

*for all $x,y,z \in X$ and for some $\theta \geq 0$, then there exists a unique additive mapping $h : X \to Y$ such that*

$$\begin{aligned}
&\|f(x) - h(x)\| \\
&\quad \leq \frac{M\theta}{\sqrt[p]{2^p - \xi(2)^p}} \left[\xi\left(\left|\frac{n(n-3)}{n^2+3}\right|\|x\|\right)^p + \xi\left(\left|\frac{n(n+3)}{n^2+3}\right|\|x\|\right)^p + \xi\left(\left|\frac{2n^2}{n^2+3}\right|\|x\|\right)^p\right. \\
&\qquad \left. +\xi\left(\left|\frac{2n(n+1)}{n^2+3}\right|\|x\|\right)^p + \xi\left(\left|\frac{2n(n-1)}{n^2+3}\right|\|x\|\right)^p + \xi\left(\left|\frac{4n}{n^2+3}\right|\|x\|\right)^p\right]^{1/p}
\end{aligned} \tag{3.8}$$

*for all $x \in X$.*

*Proof.* Letting $\varphi(x,y,z) = \theta\{\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|)\}$ and applying Theorem 3.1 with $L := \xi(2)/2$, we obtain the desired result. □

**Theorem 3.3.** *Suppose that a mapping $f : X \to Y$ satisfies the functional inequality*

$$Df(x,y,z) \leq \varphi(x,y,z) \tag{3.9}$$

*for all $x,y,z \in X$ and there exists a constant $L$ with $0 < L < 1$ for which the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies*

$$\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right) \leq \frac{L}{2}\varphi(x,y,z) \tag{3.10}$$

*for all* $x, y, z \in X$*. Then, there exists a unique additive mapping* $h : X \to Y$ *defined by* $h(x) = \lim_{k\to\infty} 2^k f(x/2^k)$ *such that*

$$\|f(x) - h(x)\| \leq \frac{ML}{2\sqrt[p]{1-L^p}}\left[\varphi\left(\frac{n(n-3)x}{n^2+3}, \frac{n(n+3)x}{n^2+3}, \frac{-2n^2x}{n^2+3}\right)^p + \varphi\left(\frac{-2n(n+1)x}{n^2+3}, \frac{2n(n-1)x}{n^2+3}, \frac{4nx}{n^2+3}\right)^p\right]^{1/p} \tag{3.11}$$

*for all* $x \in X$*.*

*Proof.* We observe that $f(0) = 0$ because $\varphi(0,0,0) = 0$, which follows from the condition $\varphi(0,0,0) \leq L/2\ \varphi(0,0,0)$. It follows from (2.29) and (3.10) that

$$\begin{aligned}\left\|f(x) - 2^m f\left(\frac{x}{2^m}\right)\right\|^p &\leq M^{\mathrm{p}} \sum_{i=0}^{m-1} 2^{ip}\left[\varphi\left(\frac{n(n-3)x}{2^{i+1}(n^2+3)}, \frac{n(n+3)x}{2^{i+1}(n^2+3)}, \frac{-2n^2x}{2^{i+1}(n^2+3)}\right)^p \right.\\ &\qquad \left. + \varphi\left(\frac{-2n(n+1)x}{2^{i+i}(n^2+3)}, \frac{2n(n-1)x}{2^{i+1}(n^2+3)}, \frac{4nx}{2^{i+1}(n^2+3)}\right)^p\right]\\ &\leq \frac{M^p}{2^p}\sum_{i=0}^{m-1} L^{(i+1)p}\left[\varphi\left(\frac{n(n-3)x}{(n^2+3)}, \frac{n(n+2)x}{(n^2+3)}, \frac{-2n^2x}{(n^2+3)}\right)^p\right.\\ &\qquad \left. + \varphi\left(\frac{-2n(n+1)x}{(n^2+3)}, \frac{2n(n-1)x}{(n^2+3)}, \frac{4nx}{(n^2+3)}\right)^p\right]\end{aligned} \tag{3.12}$$

for all nonnegative integer $m$ and all $x \in X$.

The remaining proof is similar to the corresponding part of Theorem 2.2. This completes the proof. □

**Corollary 3.4.** *Let* $\xi : [0,\infty) \to [0,\infty)$ *be a nontrivial function satisfying*

$$\xi\left(\frac{t}{2}\right) \leq \xi\left(\frac{1}{2}\right)\xi(t), \quad (t \geq 0),\ 0 < \xi\left(\frac{1}{2}\right) < \frac{1}{2}. \tag{3.13}$$

*If* $f : X \to Y$ *is a mapping satisfying the following functional inequality*

$$Df(x,y,z) \leq \theta\{\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|)\} \tag{3.14}$$

*for all $x, y, z \in X$ and for some $\theta \geq 0$, then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{M\theta\xi(1/2)}{\sqrt[p]{1 - 2^p\xi(1/2)^p}} \left[ \xi\left( \left| \frac{n(n-3)}{n^2+3} \right| \|x\| \right)^p + \xi\left( \left| \frac{n(n+3)}{n^2+3} \right| \|x\| \right)^p + \xi\left( \left| \frac{2n^2}{n^2+3} \right| \|x\| \right)^p + \xi\left( \left| \frac{2n(n+1)}{n^2+3} \right| \|x\| \right)^p + \xi\left( \left| \frac{2n(n-1)}{n^2+3} \right| \|x\| \right)^p + \xi\left( \left| \frac{4n}{n^2+3} \right| \|x\| \right)^p \right]^{1/p} \tag{3.15}$$

*for all $x \in X$.*

*Proof.* Letting $\varphi(x, y, z) = \theta\{\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|)\}$ and applying Theorem 3.3 with $L := 2\xi(1/2)$, we lead to the approximation. □

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*Research Article*

# General Solutions of Two Quadratic Functional Equations of Pexider Type on Orthogonal Vectors

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Based on the studies on the Hyers-Ulam stability and the orthogonal stability of some Pexider-quadratic functional equations, in this paper we find the general solutions of two quadratic functional equations of Pexider type. Both equations are studied in restricted domains: the first equation is studied on the restricted domain of the orthogonal vectors in the sense of Rätz, and the second equation is considered on the orthogonal vectors in the inner product spaces with the usual orthogonality.

## 1. Introduction

Stability problems for some functional equations have been extensively investigated by several authors, and in particular one of the most important functional equation studied in this topic is the quadratic functional equation,

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

(Skof [1], Cholewa [2], Czerwik [3], Rassias [4], among others).

Recently, many articles have been devoted to the study of the stability or orthogonal stability of quadratic functional equations of Pexider type on the restricted domain of orthogonal vectors in the sense of Rätz.

We remind the definition of orthogonality space (see [5]). The pair $(X,\perp)$ is called orthogonality space in the sense of Rätz if $X$ is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on $X$ with the following properties:

(i) $x \perp 0$, $0 \perp x$ for all $x \in X$,

(ii) if $x, y \in X - \{0\}$, $x \perp y$, then the vectors are linearly independent,

(iii) if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in R$,

(iv) let $P$ be a 2-dimensional subspace of $X$. If $x \in P$ then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$.

An example of orthogonality in the sense of Rätz is the ordinary orthogonality on an inner product space $(H, (\cdot,\cdot))$ given by $\perp y \Leftrightarrow (x, y) = 0$.

In the class of real functionals $f, g, h$ defined on an orthogonality space in the sense of Rätz, $f, g, h : (X, \perp) \to R$, a first version of the quadratic equation of Pexider type is

$$f(x+y) + f(x-y) = 2g(x) + 2h(y) \tag{1.2}$$

and its relative conditional form is

$$x \perp y \Longrightarrow f(x+y) + f(x-y) = 2g(x) + 2h(y). \tag{1.3}$$

Although the Hyers-Ulam stability of the conditional quadratic functional equation (1.3) has been studied by Moslehian [6], we do not know the characterization of the solutions of the conditional equation (1.3).

In the same class of functions, $f, g, h, k : (X, \perp) \to R$, another version of the quadratic functional equation of Pexider type is

$$f(x+y) + g(x-y) = h(x) + k(y), \tag{1.4}$$

and its relative conditional form is

$$x \perp y \Longrightarrow f(x+y) + g(x-y) = h(x) + k(y). \tag{1.5}$$

Equation (1.4) has been solved by Ebanks et al. [7]; its stability has been studied, among others, by Jung and Sahoo [8] and Yang [9] and its orthogonal stability has been studied by Mirzavaziri and Moslehian [10], but also in this case we do not know the general solutions of (1.5).

Based on those studies, we intend to consider the above-mentioned functional equations (1.3) and (1.5) on the restricted domain of orthogonal vectors in order to present the characterization of their general solutions.

Throughout the paper, the orthogonality $\perp$ in the sense of Rätz is assumed to be symmetric.

## 2. The Conditional Equation $x \perp y \Rightarrow f(x+y) + f(x-y) = 2g(x) + 2h(y)$ in Orthogonality Spaces in the Sense of Ratz

In the class of real functionals $f, g, h$ defined on an orthogonality space in the sense of Rätz, $f, g, h : (X, \perp) \to R$, let us consider the conditional equation (1.3).

We describe its solutions first assuming that $f$ is an odd functional, then an even functional, finally, using the decomposition of the functionals $f, g, h$ into their even and odd parts, we describe the general solutions.

**Theorem 2.1.** *Let $f, g, h : (X, \perp) \to R$ be real functionals satisfying* (1.3).

*If $f$ is an odd functional, then the solutions of* (1.3) *are given by*

$$\begin{aligned} f(x) &= A(x), \\ g(x) &= A(x) + g(0), \\ h(x) &= h(0), \end{aligned} \tag{2.1}$$

*where $A : (X, \perp) \to R$ is an additive function, that is, $A$ is solution of $A(x+y) = A(x) + A(y)$ for all $(x, y) \in X^2$.*

*If $f$ is an even functional, then the solutions of* (1.3) *are given by*

$$\begin{aligned} f(x) &= Q(x) + f(0), \\ g(x) &= Q(x) + g(0), \\ h(x) &= Q(x) + h(0), \end{aligned} \tag{2.2}$$

*where $Q : (X, \perp) \to R$ is an orthogonally quadratic function, that is, solution of $Q(x+y)+Q(x-y) = 2Q(x) + 2Q(y)$ for $x \perp y$.*

*Proof.* Let us first consider $f$ an odd functional. Letting $x = 0$ and $y = 0$ in (1.3), by $f(0) = 0$ for the oddness of $f$, we obtain

$$g(0) + h(0) = 0. \tag{2.3}$$

Now, putting $(x, 0)$ in place of $(x, y)$ in (1.3), we have $f(x) = g(x) + h(0)$, then putting again $(0, x)$ in place of $(x, y)$ we get $g(0) + h(x) = 0$ for all $x \in X$, since $f$ is odd. The first equation gives

$$g(x) = f(x) + g(0) \tag{2.4}$$

from (2.3), and the last equation proves that

$$h(x) = h(0) \tag{2.5}$$

using (2.3) again.

From the above results, (1.3) may be rewritten in the following way: $f(x+y)+f(x-y) = 2f(x)$ for all $x \perp y$. Hence by Lemma 3.1, [6], we have $f(x) - f(0) = A(x)$ where $A : X \to R$ is an orthogonally additive functional. But since $f(0) = 0$ and from [5, Theorem 5], we deduce that $A$ is everywhere additive.

Consider now $f$ an even functional. Substituting in (1.3) $(0,0)$ in place of $(x,y)$, we obtain

$$g(0)+h(0)=f(0). \tag{2.6}$$

Now writing (1.3) with $(x,y)$ replaced, respectively, first by $(x,0)$, then by $(0,y)$, we get

$$f(x)=g(x)+h(0), \tag{2.7}$$

$$f(y)=g(0)+h(y), \tag{2.8}$$

for all $x,y\in X$, since $f$ is even. From (1.3), using (2.7), (2.8), and (2.6), we obtain

$$f(x+y)+f(x-y)=2f(x)+2f(y)-2f(0). \tag{2.9}$$

Hence, setting $Q(t)=f(t)-f(0)$, we infer $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$ for $x\perp y$, that is, $Q$ is an orthogonally quadratic functional. So, $f(x)=Q(x)+f(0)$, and from (2.7), using (2.6), $g(x)=Q(x)+f(0)-h(0)=Q(x)+g(0)$, and from (2.8), $h(x)=Q(x)+f(0)-g(0)=Q(x)+h(0)$.

The theorem is so proved. □

**Lemma 2.2.** *Let $f,g,h:(X,\perp)\to R$ be real functionals satisfying* (1.3).

*Then both the even parts and the odd parts of $f,g,h$, namely, $f_e,g_e,h_e$ and $f_o,g_o,h_o$, satisfy* (1.3).

*Proof.* Denoting by $f_e,g_e,h_e$ and $f_o,g_o,h_o$ the even and odd parts, respectively, of $f,g,h$, we have from (1.3)

$$f_e(x+y)+f_o(x+y)+f_e(x-y)+f_o(x-y)=2g_e(x)+2g_o(x)+2h_e(y)+2h_o(y),\quad \text{for } x\perp y. \tag{2.10}$$

From the homogeneity of the orthogonality relation (property (iii)), we have $x\perp y\Rightarrow -x\perp -y$, so that, by (1.3), choosing $-x,-y$, we get

$$f_e(x+y)-f_o(x+y)+f_e(x-y)-f_o(x-y)=2g_e(x)-2g_o(x)+2h_e(y)-2h_o(y),\quad \text{for } x\perp y. \tag{2.11}$$

Adding and then subtracting (2.10) and (2.11), we easily prove the lemma.

From Lemma 2.2 and Theorem 2.1, we may easily prove the following theorem. □

**Theorem 2.3.** *The general solution $f,g,h:(X,\perp)\to R$ of the functional equation* (1.3) *is given by*

$$\begin{aligned} f(x)&=A(x)+Q(x)+f(0),\\ g(x)&=A(x)+Q(x)+g(0),\\ h(x)&=Q(x)+h(0), \end{aligned} \tag{2.12}$$

*where* $A : (X, \perp) \to R$ *is an additive function and* $Q : (X, \perp) \to R$ *is an orthogonally quadratic function.*

In the case of an inner product space $(H, (\cdot, \cdot))$ $(\dim H > 2)$ which is a particular orthogonality space in the sense of Rätz, with the ordinary orthogonality given by $\perp y \Leftrightarrow (x, y) = 0$, we have the characterization of the orthogonally quadratic mappings from [11, Theorem 2]. Hence we have the following corollary.

**Corollary 2.4.** *Let* $H$ *be an inner product space with* $\dim\ H > 2$ *and* $f, g, h : (H, (\cdot, \cdot)) \to R$*. The general solution of the functional equation* (1.3) *is given by*

$$\begin{aligned} f(x) &= A(x) + Q(x) + f(0), \\ g(x) &= A(x) + Q(x) + g(0), \\ h(x) &= Q(x) + h(0), \end{aligned} \tag{2.13}$$

*where* $A : (H, (\cdot, \cdot)) \to R$ *is an additive function and* $Q : (H, (\cdot, \cdot)) \to R$ *is a quadratic function.*

## 3. The Conditional Equation $x \perp y \Rightarrow f(x + y) + g(x - y) = h(x) + k(y)$ in Inner Product Spaces

Consider now $H$ an inner product space with $\dim H > 2$ and the usual orthogonality given by $\perp y \Leftrightarrow (x, y) = 0$. In the class of real functionals $f, g, h, k$ defined on $H$, we consider the conditional equation (1.5).

First prove the following lemma.

**Lemma 3.1.** *Let* $f, g, h, k : H \to R$ *be solutions of* (1.5)*; then*

$$h(x) = A(x) + Q(x) + h(0), \tag{3.1}$$

*where* $A : H \to R$ *is an additive function and* $Q : H \to R$ *is a quadratic function.*

*Proof.* Replacing in (1.5) $(x, y)$ by $(0, 0)$, then by $(x, 0)$ and finally by $(0, y)$, we obtain

(i) $f(0) + g(0) = h(0) + k(0)$,

(ii) $f(x) + g(x) = h(x) + k(0)$,

(iii) $f(y) + g(-y) = h(0) + k(y)$.

Hence (1.5) may be rewritten as

$$f(x + y) + g(x - y) = f(x) + f(y) + g(x) + g(-y) - f(0) - g(0). \tag{3.2}$$

So that, setting $F(t) = f(t) - f(0)$ and $G(t) = g(t) - g(0)$, we infer

$$F(x + y) + G(x - y) = F(x) + F(y) + G(x) + G(-y). \tag{3.3}$$

Now, substituting $-y$ in (3.3) in place of $y$, we have

$$F(x-y)+G(x+y)=F(x)+F(-y)+G(x)+G(y). \tag{3.4}$$

Adding (3.3) and (3.4), we get

$$F(x+y)+F(x-y)+G(x+y)+G(x-y)=2F(x)+F(y)+F(-y)+2G(x)+G(y)+G(-y). \tag{3.5}$$

So, defining the functional $S : H \to R$ by

$$S(t)=F(t)+G(t), \tag{3.6}$$

the above equation becomes

$$x \perp y \Longrightarrow S(x+y)+S(x-y)=2S(x)+S(y)+S(-y). \tag{3.7}$$

From [11, Theorem 3], we have

$$S(x)=A(x)+Q(x), \tag{3.8}$$

where $A : H \to R$ is an additive function and $Q : H \to R$ is a quadratic function. From (3.6), we have, $F(x)+G(x)=A(x)+Q(x)$, that is, $f(x)-f(0)+g(x)-g(0)=A(x)+Q(x)$. Using (ii) and (i), the left-hand side of the above equation may be written in the following way: $h(x)+k(0)-f(0)-g(0)=h(x)+k(0)-h(0)-k(0)=h(x)-h(0)$; hence we get $h(x)=A(x)+Q(x)+h(0)$. The theorem is so proved. □

Our aim is now to characterize the general solutions of (1.5): this is obtained using the decomposition of the functionals $f, g, h, k$ into their even and odd parts. Using the same approach of Lemma 2.2, we easily prove the following lemma.

**Lemma 3.2.** *Let $f, g, h, k : H \to R$ be real functionals satisfying* (1.5).

*Then both the even parts and the odd parts of $f, g, h, k$, namely, $f_e, g_e, h_e, k_e$ and $f_o, g_o, h_o, k_o$, satisfy* (1.5), *that is,*

$$x \perp y \Longrightarrow f_o(x+y)+g_o(x-y)=h_o(x)+k_o(y), \tag{3.9}$$

$$x \perp y \Longrightarrow f_e(x+y)+g_e(x-y)=h_e(x)+k_e(y). \tag{3.10}$$

Now consider (3.9): the characterization of its solutions is given by the following theorem.

**Theorem 3.3.** *Let $f_o, g_o, h_o, k_o : H \to R$ be real odd functionals satisfying* (3.9)*; then the solutions of* (3.9) *are given by*

$$f_o(x) = \frac{A(x)+B(x)}{2},$$
$$g_o(x) = \frac{A(x)-B(x)}{2}, \tag{3.11}$$
$$h_o(x) = A(x),$$
$$k_o(x) = B(x),$$

*where $A : H \to R$ and $B : H \to R$ are additive functions.*

*Proof.* Substituting in (3.9) first $(0,x)$, then $(x,0)$ in place of $(x,y)$, and by $h_o(0) = 0$ and $k_o(0) = 0$ by the oddness of the functions, we obtain

$$f_o(x) - g_o(x) = k_o(x),$$
$$f_o(x) + g_o(x) = h_o(x). \tag{3.12}$$

Adding and then subtracting the above equations, we get

$$2f_o(x) = h_o(x) + k_o(x),$$
$$2g_o(x) = h_o(x) - k_o(x). \tag{3.13}$$

By (3.1), $h_o(x) = A(x)$, hence from the above equations,

$$2f_o(x) = A(x) + k_o(x), \tag{3.14}$$

$$2g_o(x) = A(x) - k_o(x). \tag{3.15}$$

Consider now $x, y \in H$ with $x \perp y$. Writing (3.14) with $x+y$ instead of $x$ and (3.15) with $x-y$ instead of $x$, we get

$$2f_o(x+y) = A(x+y) + k_o(x+y),$$
$$2g_o(x-y) = A(x-y) - k_o(x-y). \tag{3.16}$$

Adding the above equations, from (3.9), the additivity of $A$ and $h_o(x) = A(x)$, we obtain

$$k_0(x+y) - k_0(x-y) = 2k_0(y) \tag{3.17}$$

for $x \perp y$. By the symmetry of the orthogonality relation, we get, changing $x$ and $y$ and from the oddness of the function,

$$k_0(x+y) + k_0(x-y) = 2k_0(x), \tag{3.18}$$

hence $k_0(x+y) = k_0(x) + k_0(y)$ for $x \perp y$. By [5, Theorem 5], $k_0$ is an additive function; consequently, there exists an additive function $B : H \to R$ such that $k_0(x) = B(x)$ for all $x \in H$. Now (3.14) and (3.15) give $f_o(x) = (A(x)+B(x))/2$ and $g_o(x) = (A(x)-B(x))/2$, so the theorem is proved. □

Finally, consider equation (3.10): the characterization of its solutions is given by the following theorem.

**Theorem 3.4.** *Let $f_e, g_e, h_e, k_e : H \to R$ be real even functionals satisfying* (3.10)*; then there exist a quadratic function $Q : H \to R$ and a function $\varphi : [0,\infty) \to R$ such that*

$$\begin{aligned} f_e(x) &= \frac{Q(x)+\varphi(\|x\|)+h_e(0)+k_e(0)}{2}, \\ g_e(x) &= \frac{Q(x)-\varphi(\|x\|)+h_e(0)+k_e(0)}{2}, \\ h_e(x) &= Q(x)+h_e(0), \\ k_e(x) &= Q(x)+k_e(0). \end{aligned} \tag{3.19}$$

*Proof.* From Lemma 3.1, we first notice that

$$h_e(x) = Q(x)+h_e(0). \tag{3.20}$$

Substituting now in (3.10) first $(x,0)$ then $(0,x)$ instead of $(x,y)$, we obtain, respectively

$$\begin{aligned} f_e(x)+g_e(x) &= h_e(x)+k_e(0), \\ f_e(x)+g_e(x) &= h_e(0)+k_e(x). \end{aligned} \tag{3.21}$$

Consequently, by subtraction and from (3.20), we have

$$k_e(x) = Q(x)+k_e(0). \tag{3.22}$$

Substitution of (3.20) and (3.22) in (3.10) gives

$$f_e(x+y)+g_e(x-y) = Q(x)+Q(y)+h_e(0)+k_e(0). \tag{3.23}$$

Then, we substitute $-y$ in place of $y$ in (3.23) and have

$$f_e(x-y)+g_e(x+y) = Q(x)+Q(y)+h_e(0)+k_e(0) \tag{3.24}$$

for all $x \perp y$. Hence, for $y=0$ in (3.24), we obtain

$$f_e(x)+g_e(x) = Q(x)+h_e(0)+k_e(0). \tag{3.25}$$

Subtracting now (3.23) and (3.24), we get $f_e(x+y)+g_e(x-y)-f_e(x-y)-g_e(x+y)=0$ for all $x \perp y$. Consider $u, v \in H$ with $\|u\| = \|v\|$: it follows that $(u+v)/2 \perp (u-v)/2$, hence in the above equation we may replace $x, y$ with $(u+v)/2$, $(u-v)/2$, respectively. We obtain $f_e(u)+g_e(v)-f_e(v)-g_e(u)=0$, that is, $f_e(u)-g_e(u)=f_e(v)-g_e(v)$ for all $u, v \in H$ with $\|u\| = \|v\|$. Thus the function $f_e(t)-g_e(t)$ is constant on each sphere with center 0, and $\varphi : [0,\infty) \to R$ is well defined by

$$\varphi(\|x\|) = f_e(x) - g_e(x). \tag{3.26}$$

Hence (3.25) and (3.26) lead to

$$\begin{aligned} f_e(x) &= \frac{Q(x)+\varphi(\|x\|)+h_e(0)+k_e(0)}{2}, \\ g_e(x) &= \frac{Q(x)-\varphi(\|x\|)+h_e(0)+k_e(0)}{2}, \end{aligned} \tag{3.27}$$

which finishes the proof. □

Finally, the general solution of (1.5) is characterized by the following theorem.

**Theorem 3.5.** *Let $f, g, h, k : H \to R$ be real functionals satisfying* (1.5)*; then there exist additive functions, $B : H \to R$, a quadratic function $Q : H \to R$, and a function $\varphi : [0,\infty) \to R$ such that*

$$\begin{aligned} f(x) &= \frac{A(x)+B(x)+Q(x)+\varphi(\|x\|)+h(0)+k(0)}{2}, \\ g(x) &= \frac{A(x)-B(x)+Q(x)-\varphi(\|x\|)+h(0)+k(0)}{2}, \\ h(x) &= A(x)+Q(x)+h(0), \\ k(x) &= B(x)+Q(x)+k(0). \end{aligned} \tag{3.28}$$

*Conversely, the above functionals satisfy* (1.5).

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*Research Article*

# Fixed Points and Generalized Hyers-Ulam Stability

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In this paper we prove a fixed-point theorem for a class of operators with suitable properties, in very general conditions. Also, we show that some recent fixed-points results in Brzdęk et al., (2011) and Brzdęk and Ciepliński (2011) can be obtained directly from our theorem. Moreover, an affirmative answer to the open problem of Brzdęk and Ciepliński (2011) is given. Several corollaries, obtained directly from our main result, show that this is a useful tool for proving properties of generalized Hyers-Ulam stability for some functional equations in a single variable.

## 1. Introduction

The study of functional equations stability originated from a question of Ulam [1], concerning the stability of group homomorphisms. In 1941 Hyers [2] gave an affirmative answer to the question of Ulam for Cauchy equation in Banach spaces. The Hyers result was generalized by Aoki [3] for additive mappings and independently by Rassias [4] for linear mappings, by considering the unbounded Cauchy differences. A further generalization was obtained by Găvruţa [5] in 1994, by replacing the Cauchy differences by a control mapping $\varphi$, in the spirit of Rassias approach. See also [6] for more generalizations. We mention that the proofs of the results in the above mentioned papers used the direct method (of Hyers): the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution. We refer the reader to the expository papers [7, 8] and to the books [9–11] (see also the papers [12–17], for supplementary details).

On the other hand, in 1991 Baker [18] used the Banach fixed-point theorem to give Hyers-Ulam stability results for a nonlinear functional equation. In 2003, Radu [19] proposed a new method, successively developed in [20], to obtaining the existence of the exact solutions and the error estimations, based on the fixed-point alternative. Concerning the stability of some functionals equations in a single variable, we mention the articles of Cădariu and Radu [21], of Miheţ [22], which applied the Luxemburg-Jung fixed-point theorem in generalized metric spaces, as well as the paper of Găvruţa [23] which used the Matkowski's fixed-point theorem. Also, Găvruţa introduced a new method in [24], called the *weighted space method*, for the generalized Hyers-Ulam stability (see, also [25]). It is worth noting that two fixed-point alternatives together with the error estimations for generalized contractions of type Bianchini-Grandolfi and Matkowski are pointed out by Cădariu and Radu, and then used as fundamental tools for proving stability of Cauchy functional equation in $\beta$-normed spaces [26], as well as of the monomial functional equation [27]. We also mention the new survey of Ciepliński [28], where some applications of different fixed-point theorems to the theory of the Hyers-Ulam stability of functional equations are presented.

Very recently, Brzdęk et al. proved in [29] a fixed-point theorem for (not necessarily) linear operators and they used it to obtain Hyers-Ulam stability results for a class of functional equations in a single variable. A fixed-point result of the same type was proved by Brzdęk and Ciepliński [30], in complete non-Archimedean metric spaces as well as in complete metric spaces. Also, they formulated an open problem concerning the uniqueness of the fixed point of the operator $\mathcal{T}$, which will be defined in the next section.

Our principal purpose is to obtain a fixed point theorem for a class of operators with suitable properties, in very general conditions. After that, we will show that some recent results in [29, 30] can be obtained as particular cases of our theorem. Moreover, by using our outcome, we will give an affirmative answer to the open problem of Brzdęk and Ciepliński, posed in the end of the paper [30]. Finally, we will show that main Theorem 2.2 is an efficient tool for proving generalized Hyers-Ulam stability results of several functional equations in a single variable.

## 2. Results

We consider a nonempty set $X$, a complete metric space $(Y, d)$, and the mappings $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ and $\mathcal{T} : Y^X \to Y^X$. We recall that $Y^X$ is the space of all mappings from X into Y.

*Definition 2.1.* One says that $\mathcal{T}$ is $\Lambda$-contractive if for $u, v : X \to Y$ and $\delta \in \mathbb{R}_+^X$ with

$$d(u(t), v(t)) \leq \delta(t), \quad \forall t \in X, \tag{2.1}$$

it follows

$$d((\mathcal{T}u)(t), (\mathcal{T}v)(t)) \leq (\Lambda\delta)(t), \quad \forall t \in X. \tag{2.2}$$

In the following, we assume that $\Lambda$ satisfies the condition:

$(C_1)$ for every sequence $(\delta_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{R}_+^X$ and every $t \in X$,

$$\lim_{n\to\infty} \delta_n(t) = 0 \Longrightarrow \lim_{n\to\infty} (\Lambda\delta_n)(t) = 0. \tag{2.3}$$

Also, we suppose that $\varepsilon \in \mathbb{R}_+^X$ is a given function such that

$(C_2)$

$$\varepsilon^*(t) := \sum_{k=0}^{\infty}\left(\Lambda^k \varepsilon\right)(t) < \infty, \quad t \in X. \tag{2.4}$$

**Theorem 2.2.** *One supposes that the operator $\mathcal{T}$ is $\Lambda$-contractive and the conditions $(C_1)$ and $(C_2)$ hold. One considers a mapping $f \in Y^X$ such that*

$$d((\mathcal{T}f)(t), f(t)) \leq \varepsilon(t), \quad \forall t \in X. \tag{2.5}$$

*Then, for every $t \in X$, the limit*

$$g(t) := \lim_{n \to \infty} (\mathcal{T}^n f)(t) \tag{2.6}$$

*exists and the mapping $g$ is the unique fixed point of $\mathcal{T}$ with the property*

$$d((\mathcal{T}^m f)(t), g(t)) \leq \sum_{k=m}^{\infty}\left(\Lambda^k \varepsilon\right)(t), \quad t \in X,\ m \in \mathbb{N} = \{0, 1, 2, \ldots\}. \tag{2.7}$$

*Moreover, if one has*

$(C_3)$

$$\lim_{n \to \infty} (\Lambda^n \varepsilon^*)(t) = 0, \quad \forall t \in X, \tag{2.8}$$

*then $g$ is the unique fixed point of $\mathcal{T}$ with the property*

$$d(f(t), g(t)) \leq \varepsilon^*(t), \quad \forall t \in X. \tag{2.9}$$

*Proof.* We have

$$d\left(\left(\mathcal{T}^{n+1} f\right)(t), (\mathcal{T}^n f)(t)\right) \leq (\Lambda^n \varepsilon)(t), \quad t \in X. \tag{2.10}$$

Indeed, for $n = 0$, the relation (2.10) is (2.5).

We suppose that (2.10) holds. Since $\mathcal{T}$ is $\Lambda$-contractive, we have

$$d\left(\left(\mathcal{T}^{n+2} f\right)(t), \left(\mathcal{T}^{n+1} f\right)(t)\right) \leq (\Lambda(\Lambda^n \varepsilon))(t), \quad t \in X. \tag{2.11}$$

By using the triangle inequality and (2.10), we obtain, for $n > m$

$$d((\mathcal{T}^n f)(t), (\mathcal{T}^m f)(t)) \leq \sum_{k=m}^{n-1}\left(\Lambda^k \varepsilon\right)(t), \quad t \in X. \tag{2.12}$$

Hence the sequence $\{\mathcal{T}^n f(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $(Y, d)$ is complete, it results that there exists $g \in Y^X$ defined by

$$g(t) := \lim_{n\to\infty} (\mathcal{T}^n f)(t). \tag{2.13}$$

Then, in view of (2.12), we get (2.7).

Now, we prove that $g$ is a fixed point for the operator $\mathcal{T}$. To this end, we show that $\mathcal{T}$ is a pointwise continuous. Indeed, if $h_m(t) \xrightarrow[m\to\infty]{} h(t)$, $t \in X$, then

$$|h_m, h|(t) := d(h_m(t), h(t)) \xrightarrow[m\to\infty]{} 0, \quad t \in X. \tag{2.14}$$

By using condition $(C_1)$ we have $(\Lambda|h_m, h|)(t) \xrightarrow[m\to\infty]{} 0, t \in X$. But

$$d((\mathcal{T}h_m)(t), (\mathcal{T}h)(t)) \leq (\Lambda|h_m, h|)(t), \tag{2.15}$$

so it follows that $d((\mathcal{T}h_m)(t), (\mathcal{T}h)(t)) \xrightarrow[m\to\infty]{} 0$.

Since $\mathcal{T}$ is a pointwise continuous, we obtain $(\mathcal{T}(\mathcal{T}^n f))(t) \xrightarrow[n\to\infty]{} (\mathcal{T}g)(t)$. Hence $g(t) = (\mathcal{T}g)(t)$ for $t \in X$.

It is easy to prove that $g$ is the unique point of $\mathcal{T}$, which satisfies (2.7): for $n \to \infty$ in (2.12), it results

$$d(g(t), (\mathcal{T}^m f)(t)) \leq \sum_{k=m}^{\infty} \left(\Lambda^k \varepsilon\right)(t), \quad t \in X. \tag{2.16}$$

If $g_1$ is another fixed point of $\mathcal{T}$ such that (2.7) holds, then we have

$$d(g_1(t), (\mathcal{T}^m f)(t)) \leq \sum_{k=m}^{\infty} \left(\Lambda^k \varepsilon\right)(t), \quad t \in X. \tag{2.17}$$

Hence

$$d(g_1(t), g(t)) \leq 2\sum_{k=m}^{\infty} \left(\Lambda^k \varepsilon\right)(t), \quad t \in X, \tag{2.18}$$

so letting $m \to \infty$ we obtain $d(g_1(t), g(t)) = 0$ for $t \in X$. Thus $g_1 = g$.

To prove the last part of the theorem, we take $m = 0$ in (2.7) and we obtain (2.9). Moreover, if $(C_3)$ holds and $g_2$ is another fixed point of $\mathcal{T}$ such that (2.9) is satisfied, then we have

$$d((\mathcal{T}^n g_2)(t), (\mathcal{T}^n f)(t)) \leq (\Lambda^n \varepsilon^*)(t), \quad t \in X, \tag{2.19}$$

hence

$$d\big(g_2(t), (\mathcal{T}^n f)(t)\big) \le (\Lambda^n \varepsilon^*)(t), \quad t \in X. \tag{2.20}$$

Letting $n \to \infty$, we obtain $d(g_2(t), g(t)) = 0$, for $t \in X$, so $g = g_2$. □

**Corollary 2.3.** *Let $X$ be a nonempty set, $(Y, d)$ a complete metric space, and let $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ be a nondecreasing operator satisfying the hypothesis $(C_1)$.*

*If $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality*

$$d\big((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)\big) \le \Lambda\big(d\big(\xi(x), \mu(x)\big)\big), \quad \xi, \mu \in Y^X,\ x \in X, \tag{2.21}$$

*and the functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ are such that*

$$\begin{gathered} d\big((\mathcal{T}\varphi)(x), \varphi(x)\big) \le \varepsilon(x), \quad x \in X, \\ \varepsilon^*(x) := \sum_{k=0}^{\infty} \Big(\Lambda^k \varepsilon\Big)(x) < \infty, \quad x \in X, \end{gathered} \tag{2.22}$$

*then, for every $x \in X$, the limit*

$$\psi(x) := \lim_{n \to \infty} (\mathcal{T}^n \varphi)(x) \tag{2.23}$$

*exists and the function $\psi \in Y^X$, defined in this way, is a fixed point of $\mathcal{T}$, with*

$$d\big(\varphi(x), \psi(x)\big) \le \varepsilon^*(x), \quad x \in X. \tag{2.24}$$

*Moreover, if the condition $(C_3)$ holds, then the mapping $\psi$ is the unique fixed point of $\mathcal{T}$ with the property*

$$d\big(\varphi(x), \psi(x)\big) \le \varepsilon^*(x), \quad x \in X. \tag{2.25}$$

*Proof.* To apply Theorem 2.2 it is sufficient to show that the operator $\mathcal{T}$ from the above corollary is $\Lambda$-contractive, in the sense of the Definition 2.1. To this end, let us suppose that $\xi$, $\mu \in Y^X$, $\delta \in \mathbb{R}_+^X$ and

$$d\big(\xi(x), \mu(x)\big) \le \delta(x), \quad x \in X. \tag{2.26}$$

By using (2.21) and the non-decreasing property of $\Lambda$, we obtain that

$$d\big((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)\big) \le \Lambda\big(d\big(\xi(x), \mu(x)\big)\big) \le \Lambda(\delta(x)), \quad x \in X. \tag{2.27}$$

Hence $\mathcal{T}$ is $\Lambda$-contractive. The uniqueness follows from Theorem 2.2. □

The results of Corollary 2.3 (except for the uniqueness of $\psi$) have been proved recently by Brzdęk and Ciepliński [30]. Actually, the authors have stated there an open question concerning the uniqueness of $\psi$.

Another recent result proved in [29], by Brzdęk et al., can be obtained from Theorem 2.2.

**Corollary 2.4** (Corollary [see [29], Theorem 1]). *Let $X$ be a nonempty set, $(Y, d)$ a complete metric space, $f_1, \ldots, f_s : X \to X$, and let $L_1, \ldots, L_s : X \to \mathbb{R}_+$ be given maps. Let $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ be a linear operator defined by*

$$(\Lambda\delta)(x) := \sum_{i=1}^{s} L_i(x)\delta(f_i(x)) \tag{2.28}$$

*for $\delta : X \to \mathbb{R}_+$ and $x \in X$. If $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality*

$$d((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)) \leq \sum_{i=1}^{s} L_i(x) d(\xi(f_i(x)), \mu(f_i(x))), \quad \xi, \mu \in Y^X, \ x \in X, \tag{2.29}$$

*and the functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ are such that*

$$\begin{aligned} d((\mathcal{T}\varphi)(x), \varphi(x)) &\leq \varepsilon(x), \quad x \in X, \\ \varepsilon^*(x) := \sum_{k=0}^{\infty} \left(\Lambda^k \varepsilon\right)(x) &< \infty, \quad x \in X, \end{aligned} \tag{2.30}$$

*then, for every $x \in X$, the limit*

$$\psi(x) := \lim_{n \to \infty} (\mathcal{T}^n \varphi)(x) \tag{2.31}$$

*exists and the function $\psi \in Y^X$ so defined is a unique fixed point of $\mathcal{T}$, with*

$$d(\varphi(x), \psi(x)) \leq \varepsilon^*(x), \quad x \in X. \tag{2.32}$$

*Proof.* We apply Theorem 2.2. Therefore, it is necessary to prove that the operator $\mathcal{T}$, defined in (2.28), is $\Lambda$-contractive. To this end, let us suppose that $\xi, \mu \in Y^X$, $\delta \in \mathbb{R}_+^X$ and

$$d(\xi(x), \mu(x)) \leq \delta(x), \quad \forall x \in X. \tag{2.33}$$

By using (2.28) and (2.29), we obtain that

$$\begin{aligned} d\big((\mathcal{T}\xi)(x),(\mathcal{T}\mu)(x)\big) &\le \sum_{i=1}^{s} L_i(x) d\big(\xi(f_i(x)),\mu(f_i(x))\big) \\ &\le \sum_{i=1}^{s} L_i(x)\delta\big(f_i(x)\big) \\ &= \Lambda(\delta(x)), \quad x \in X, \end{aligned} \tag{2.34}$$

so $\mathcal{T}$ is $\Lambda$-contractive.

On the other hand, from definition of $\Lambda$, it results immediately that the relation $(C_1)$ holds.

The uniqueness of $\psi$ results also from Theorem 2.2. To this end, we prove that the linear operator $\Lambda$ satisfy the hypotheses $(C_3)$:

$$\begin{aligned} \Lambda^n(\varepsilon^*(x)) &= \Lambda^n\left(\sum_{k=0}^{\infty}\left(\Lambda^k\varepsilon\right)(x)\right) \\ &= \sum_{k=0}^{\infty}\left(\Lambda^{n+k}\varepsilon\right)(x) = \sum_{m=n}^{\infty}(\Lambda^m\varepsilon)(x). \end{aligned} \tag{2.35}$$

Thus

$$\lim_{n\to\infty}\Lambda^n(\varepsilon^*(x)) = 0, \quad x \in X. \tag{2.36}$$

□

The following result of generalized Hyers-Ulam stability for the functional equation:

$$\Theta\big(x,\varphi(f_1(x)),\dots,\varphi(f_s(x))\big) = \varphi(x), \quad x \in X, \tag{2.37}$$

can be also derived from Theorem 2.2. (The unknown mapping is $\varphi$; the others are given functions.)

**Corollary 2.5.** *Let $X$ be a nonempty set, let $(Y,d)$ be a complete metric space, and let the operators $\Theta : X \times Y^s \to Y$ and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$. We suppose that $\Theta$ is $\Lambda$-contractive, the conditions $(C_1)$ and $(C_2)$ hold, and let one consider a function $\varphi \in Y^X$ such that*

$$d\big(\varphi(x),\Theta\big(x,\varphi(f_1(x)),\dots,\varphi(f_s(x))\big)\big) \le \varepsilon(x), \quad x \in X, \tag{2.38}$$

*for the given mappings $f_1,\dots,f_s : X \to X$. Then, for every $x \in X$, the limit*

$$\psi(x) := \lim_{n\to\infty}(\mathcal{T}^n\varphi)(x), \tag{2.39}$$

*where* $(\mathcal{T}\varphi)(x) = \Theta(x, \varphi(f_1(x)), \ldots, \varphi(f_s(x)))$, *exists and the function* $\psi \in Y^X$, *above defined, is the unique solution of the functional equation* (2.37) *with property*

$$d\left(\left(\mathcal{T}^m\varphi\right)(x), \psi(x)\right) \leq \sum_{k=m}^{\infty}\left(\Lambda^k\varepsilon\right)(x), \quad x \in X,\ m \in \mathbb{N} = \{0,1,2,\ldots\}. \tag{2.40}$$

*Moreover, if one has*

$$\lim_{n\to\infty}(\Lambda^n\varepsilon^*)(x) = 0, \quad \forall x \in X, \tag{2.41}$$

*then* $\psi$ *is the unique solution of* (2.37), *with the property*

$$d\left(\psi(x), \varphi(x)\right) \leq \varepsilon^*(x), \quad \forall x \in X. \tag{2.42}$$

*Remark 2.6.* It is easy to see that if we take in the above result

$$(\Lambda\delta)(x) := \sum_{i=1}^{s} L_i(x)\delta\left(f_i(x)\right), \quad \forall x \in X \tag{2.43}$$

for the given mappings $L_1, \ldots, L_s : X \to \mathbb{R}_+$ and $\delta : X \to \mathbb{R}_+$, we obtain the Corollary 3 in [29].

From Theorem 2.2 we obtain the following fixed-point result.

**Corollary 2.7.** *Let* $(Y, d)$ *be a metric space and let* $c : [0,\infty) \to [0,\infty)$ *be a function, with the property: for every sequence* $\varepsilon_n \in [0,\infty)$, *with* $\lim_{n\to\infty}\varepsilon_n = 0 \Rightarrow \lim_{n\to\infty}c(\varepsilon_n) = 0$. *Let one consider an operator* $T : Y \to Y$ *such that, for* $u, v \in Y$ *and* $\lambda \geq 0$, *with* $d(u,v) \leq \lambda$, *it follows* $d(Tu, Tv) \leq c(\lambda)$. *Moreover, let* $\varepsilon > 0$ *and* $f \in Y$ *be such that*

$$\varepsilon^* = \sum_{n=0}^{\infty} c^n(\varepsilon) < \infty \tag{2.44}$$

*and* $d(Tf, f) \leq \varepsilon$. *Then there exists*

$$g := \lim_{n\to\infty} T^n f, \tag{2.45}$$

*which is the unique fixed point of* $T$, *with*

$$d(T^m f, g) \leq \sum_{k=m}^{\infty} c^k(\varepsilon), \quad \forall m \in \mathbb{N} = \{0,1,2,\ldots\}. \tag{2.46}$$

*Moreover, if*

$$\lim_{n\to\infty} c^n(\varepsilon^*) = 0 \tag{2.47}$$

*holds, then $g$ is the unique fixed point of $T$, with the property $d(f,g) \leq \varepsilon^*$.*

*Proof.* The result follows immediately from Theorem 2.2 by taking $X$ to be the set with a single element. □

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*Research Article*

# Approximate Cubic $*$-Derivations on Banach $*$-Algebras

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We study the stability of cubic $*$-derivations on Banach $*$-algebras. We also prove the superstability of cubic $*$-derivations on a Banach $*$-algebra $A$, which is left approximately unital.

## 1. Introduction

In [1], Ulam proposed the stability problems for functional equations concerning the stability of group homomorphisms. In fact, a functional equation is called *stable* if any approximately solution to the functional equation is near a true solution of that functional equation and is *superstable* if every approximate solution is an exact solution to it. In [2], Hyers considered the case of approximate additive mappings in Banach spaces and satisfying the well-known weak Hyers inequality controlled by a positive constant. Bourgin [3] was the second author to treat this problem for additive mappings (see also [4]). In [5], Rassias provided a generalization of Hyers Theorem, which allows the Cauchy difference to be unbounded. Găvruţa then generalized the Rassias' result in [6] for the unbounded Cauchy difference. Subsequently, various approaches to the problem have been studied by a number of authors (see, e.g., [7–11]).

Recall that a Banach $*$-algebra is a Banach algebra (complete normed algebra) which has an isometric involution. For a locally compact group $G$, the algebraic group algebra $L^1(G)$ is a Banach $*$-algebra. The bounded operators on Hilbert space $\mathscr{H}$ is also a Banach $*$-algebra. In general, all $C^*$-algebras are Banach $*$-algebra. A left- (right-) bounded approximate identity for a normed algebra $\mathcal{A}$ is a bounded net $(e_j)_j$ in $\mathcal{A}$ such that $\lim_j e_j a = a$ ($\lim_j a e_j = a$) for each $a \in \mathcal{A}$. A bounded approximate identity for $\mathcal{A}$ is a bounded net $(e_j)_j$, which is both a left- and a right-bounded approximate identity. Every group algebra and every $C^*$-algebra has a bounded approximate identity.

The stability of functional equations of $*$-derivations and of quadratic $*$-derivations with the Cauchy functional equation and the Jensen functional equation on Banach $*$-algebras is investigated in [12]. The author also proved the superstability of $*$-derivations and of quadratic $*$-derivations on $C^*$-algebras.

In 2003, Cădariu and Radu employed the fixed point method to the investigation of the Jensen functional equation. They presented a short and a simple proof (different from the "*direct method*," initiated by Hyers in 1941) for the Cauchy functional equation [13] and for the quadratic functional equation [14] (see also [15–18]).

The functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.1}$$

which is called cubic functional equation. In addition, every solution of functional equation (1.1) is said to be a *cubic mapping*. It is easy to check that function $f(x) = ax^3$ is a solution of (1.1).

In [19], Bodaghi et al. proved the generalized Hyers-Ulam stability and the superstability for the functional equation (1.1) by using the alternative fixed point (Theorem 3.1) under certain conditions on Banach algebras. Also, the stability and the superstability of homomorphisms on $C^*$-algebras by using the same fixed point method was proved in [20]. The generalized Hyers-Ulam-Rassias stability of $*$-homomorphisms between unital $C^*$-algebras associated with the Trif functional equation and of linear $*$-derivations on unital $C^*$-algebras has earlier been proved by Park and Hou in [21].

In this paper, we prove the stability and the superstability of cubic $*$-derivations on Banach $*$-algebras. We also show that these functional equations, under some mild conditions, are superstable. We also establish the stability and the superstability of cubic $*$-derivations on a Banach $*$-algebra with a left-bounded approximate identity.

## 2. Stability of Cubic $*$-Derivation

Throughout this paper, we assume that $A$ is a Banach $*$-algebra. A mapping $D : A \to A$ is a cubic derivation if $D$ is a cubic homogeneous mapping, that is, $D$ is cubic and $D(\mu a) = \mu^3 D(a)$ for all $a \in A$ and $\mu \in \mathbb{C}$, and $D(ab) = D(a)b^3 + a^3 D(b)$ for all $a, b \in A$. In addition, if $D$ satisfies in condition $D(a^*) = D(a)^*$ for all $a \in A$, then it is called the cubic $*$-derivation. An example of cubic derivations on Banach algebras is given in [22].

Let $\mu \in \mathbb{C}$. For the given mapping $f : A \to A$, we consider

$$\begin{gathered} \mathfrak{D}_\mu f(a,b) := f(2\mu a + \mu b) + f(2\mu a - \mu b) - 2\mu^3 f(a+b) - 2\mu^3 f(a-b) - 12\mu^3 f(a), \\ \mathfrak{D} f(a,b) = f(ab) - f(a)b^3 - a^3 f(b) \end{gathered} \tag{2.1}$$

for all $a, b \in A$.

**Theorem 2.1.** *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A^5 \to [0,\infty)$ such that*

$$\widetilde{\varphi}(a,b,x,y,z) := \sum_{k=0}^{\infty} \frac{1}{8^k} \varphi\left(2^k a, 2^k b, 2^k x, 2^k y, 2^k z\right) < \infty, \tag{2.2}$$

$$\left\| \mathfrak{D}_\mu f(a,b) \right\| \leq \varphi(a,b,0,0,0), \tag{2.3}$$

$$\left\| \mathfrak{D} f(x,y) + f(z^*) - f(z)^* \right\| \leq \varphi(0,0,x,y,z), \tag{2.4}$$

*for all $\mu \in \mathbb{T}^1_{1/n_0} = \{e^{i\theta} : 0 \leq \theta \leq 2\pi/n_0\}$ and all $a,b,x,y,z \in A$ in which $n_0 \in \mathbb{N}$. Also, if for each fixed $a \in A$ the mapping $t \mapsto f(ta)$ from $\mathbb{R}$ to $A$ is continuous, then there exists a unique cubic $*$-derivation $D$ on $A$ satisfying*

$$\|f(a) - D(a)\| \leq \frac{1}{16} \widetilde{\varphi}(a), \quad (a \in A), \tag{2.5}$$

*in which $\widetilde{\varphi}(a) = \widetilde{\varphi}(a,0,0,0,0)$.*

*Proof.* Putting $b = 0$ and $\mu = 1$ in (2.3), we have

$$\left\| \frac{1}{8} f(2a) - f(a) \right\| \leq \frac{1}{16} \varphi(a) \tag{2.6}$$

for all $a \in A$ in which $\varphi(a) = \varphi(a,0,0,0,0)$. We can use induction to show that

$$\left\| \frac{f(2^n a)}{8^n} - \frac{f(2^m a)}{8^m} \right\| \leq \frac{1}{16} \sum_{k=m}^{n-1} \frac{\varphi(2^k a)}{8^k} \tag{2.7}$$

for all $a \in A$ and $n > m \geq 0$. On the other hand,

$$\left\| \frac{f(2^n a)}{8^n} - f(a) \right\| \leq \frac{1}{16} \sum_{k=0}^{n-1} \frac{\varphi(2^k a)}{8^k} \tag{2.8}$$

for all $a \in A$ and $n > 0$. It follows from (2.2) and (2.7) that the sequence $\{f(2^n a)/8^n\}$ is a Cauchy sequence. Since $A$ is a Banach algebra, this sequence converges to the map $D$, that is,

$$\lim_{n \to \infty} \frac{f(2^n a)}{8^n} = D(a). \tag{2.9}$$

Thus the inequalities (2.2) and (2.8) show that (2.5) holds. Substituting $a, b$ by $2^n a$, $2^n b$, respectively, in (2.3), we get

$$\left\| \mathfrak{D}_\mu D(a,b) \right\| = \lim_{n \to \infty} \frac{1}{8^n} \left\| \mathfrak{D}_\mu f(2^n a, 2^n b) \right\| \leq \lim_{n \to \infty} \frac{\varphi(2^n a, 2^n b, 0,0,0)}{8^n} = 0 \tag{2.10}$$

for all $a, b \in A$ and $\mu \in \mathbb{T}^1_{1/n_0}$. Since $\mathfrak{D}_1 D(a,b) = 0$, the mapping $D$ is cubic. The equality $\mathfrak{D}_\mu D(a,0) = 0$ implies that $D(\mu a) = \mu^3 D(a)$ for all $a \in A$ and $\mu \in \mathbb{T}^1_{1/n_0}$. Now, let $\mu \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ such that $\mu = e^{i\theta}$ in which $0 \le \theta < 2\pi$. We set $\mu_1 = e^{i\theta/n_0}$, thus $\mu_1$ belongs to $\mathbb{T}^1_{1/n_0}$ and $D(\mu a) = D(\mu_1^{n_0} a) = \mu_1^{3n_0} D(a) = \mu^3 D(a)$ for all $a \in A$. Now, suppose that $\mathcal{F}$ is any continuous linear functional on $A$ and $a$ is a fixed element in $A$. Define the mapping $g : \mathbb{R} \to \mathbb{R}$ via $g(\mu) = \mathcal{F}[D(\mu a)]$ for each $\mu \in \mathbb{R}$. Obviously, $g$ is a cubic function. Under the hypothesis that $f(ta)$ is continuous in $t \in \mathbb{R}$ for each fixed $a \in A$, the function $g$ is the pointwise limit of the sequence of measurable functions $\{g_n\}$ in which $g_n(\mu) = \mathcal{F}(2^n \mu a)/8^n$, $n \in \mathbb{N}$, $\mu \in \mathbb{R}$. Hence, $g$ is a continuous function and has the form $g(\mu) = \mu^3 g(1)$ for all $\mu \in \mathbb{R}$. Therefore,

$$\mathcal{F}[D(\mu a)] = g(\mu) = \mu^3 g(1) = \mu^3 \mathcal{F}[D(a)] = \mathcal{F}\left[\mu^3 D(a)\right]. \tag{2.11}$$

Since $\mathcal{F}$ is an arbitrary continuous linear functional on $A$, $D(\mu a) = \mu^3 D(a)$ for all $\mu \in \mathbb{R}$ and $a \in A$. Thus

$$D(\mu a) = D\left(\frac{\mu}{|\mu|}|\mu| a\right) = \frac{\mu^3}{|\mu|^3} D(|\mu| a) = \frac{\mu^3}{|\mu|^3} |\mu|^3 D(a) = \mu^3 D(a) \tag{2.12}$$

for all $a \in A$ and $\mu \in \mathbb{C}$ $(\mu \neq 0)$. Therefore, $D$ is a cubic homogeneous. If we replace $x, y$ by $2^n x$, $2^n y$, respectively, and put $z = 0$ in (2.4), we have

$$\frac{1}{8^{2n}} \|\mathfrak{D} f(2^n x, 2^n y)\| \le \frac{\varphi(0,0,2^n x, 2^n y, 0)}{8^{2n}} \le \frac{\varphi(0,0,2^n x, 2^n y, 0)}{8^n} \tag{2.13}$$

for all $x, y \in A$. Taking the limit as $n$ tends to infinity, we get $\mathfrak{D} D(x,y) = 0$, for all $x, y \in A$. Putting $x = y = 0$ and substituting $z$ by $2^n z$ in (2.4) and then dividing the both sides of the obtained inequality by $8^n$, then we get

$$\left\| \frac{f(2^n z^*)}{8^n} - \frac{f(2^n z)^*}{8^n} \right\| \le \frac{\varphi(0,0,0,0,2^n z)}{8^n} \tag{2.14}$$

for all $z \in A$. Passing to the limit as $n \to \infty$ in (2.14), we get $D(z^*) = D(z)^*$ for all $z \in A$. This shows that $D$ is a cubic $*$-derivation.

Now, let $D' : A \to A$ be another cubic $*$-derivation satisfying (2.5). Then we have

$$\begin{aligned} \|D(a) - D'(a)\| &= \frac{1}{8^n} \|D(2^n a) - D'(2^n a)\| \\ &\le \frac{1}{8^n} \left(\|D(2^n a) - f(2^n a)\| + \|f(2^n a) - D'(2^n a)\|\right) \\ &\le \frac{1}{8^{n+1}} \widetilde{\varphi}(2^n a) = \frac{1}{8} \sum_{k=n}^{\infty} \frac{1}{8^k} \varphi\left(2^k a\right), \end{aligned} \tag{2.15}$$

which tends to zero as $n \to \infty$ for all $a \in$ A. So we can conclude that $D(a) = D'(a)$ for all $a \in A$. This proves the uniqueness of $D$. □

We have the following theorem, which is analogous to Theorem 2.1. Since the proof is similar, it is omitted.

**Theorem 2.2.** *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A^5 \to [0,\infty)$ satisfying* (2.3), (2.4), *and*

$$\widetilde{\varphi}(a,b,x,y,z) := \sum_{k=1}^{\infty} 8^k \varphi\left(2^{-k}a, 2^{-k}b, 2^{-k}x, 2^{-k}y, 2^{-k}z\right) < \infty \tag{2.16}$$

*for all $a,b,x,y,z \in A$. Also, if for each fixed $a \in A$ the mappings $t \mapsto f(ta)$ from $\mathbb{R}$ to $A$ is continuous, then there exists a unique cubic $*$-derivation $D$ on $A$ satisfying*

$$\|f(a) - D(a)\| \le \frac{1}{16}\widetilde{\varphi}(a), \quad (a \in A), \tag{2.17}$$

*where $\widetilde{\varphi}(a) = \widetilde{\varphi}(a,0,0,0,0)$.*

**Corollary 2.3.** *Let $\theta, r$ be positive real numbers with $r \neq 3$, and let $f : A \to A$ be a mapping with $f(0) = 0$ such that*

$$\begin{gathered} \|\mathfrak{D}_\mu f(a,b)\| \le \theta(\|a\|^r + \|b\|^r), \\ \|\mathfrak{D} f(x,y) + f(z^*) - f(z)^*\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r), \end{gathered} \tag{2.18}$$

*for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $a,b,x,y,z \in A$. Then there exists a unique cubic $*$-derivation $D$ on $A$ satisfying*

$$\|f(a) - D(a)\| \le \frac{\theta\|a\|^r}{|16 - 2^{r+1}|}, \tag{2.19}$$

*for all $a \in A$.*

*Proof.* We can obtain the result from Theorem 2.1 and Theorem 2.2 by taking

$$\varphi(a,b,x,y,z) = \theta(\|a\|^r + \|b\|^r + \|x\|^r + \|y\|^r + \|z\|^r) \tag{2.20}$$

for all $a,b,x,y,z \in A$. □

In the next theorem, we investigate the superstability of cubic $*$-derivations of Banach $*$-algebras with a left-bounded approximate identity.

**Theorem 2.4.** *Suppose that $A$ is a Banach $*$-algebra with a left-bounded approximate identity and $s \in \{-1, 1\}$. Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A \times A \to [0, \infty)$ such that*

$$\lim_{n\to\infty} n^{-3s}\varphi(n^s a, b) = \lim_{n\to\infty} n^{-3s}\varphi(a, n^s b) = 0, \tag{2.21}$$

$$\left\| a^3 f(b) - f(a)b^3 \right\| \le \varphi(a, b), \tag{2.22}$$

$$\left\| f(c)(ab)^3 - c^3\left[f(a)b^3 + a^3 f(b)\right] \right\| \le \varphi(c, ab), \tag{2.23}$$

$$\left\| a^3 f(b^*) - f(a)\left(b^3\right)^* \right\| \le \varphi(a, b) \tag{2.24}$$

*for all $a, b, c \in A$. Then $f$ is a cubic $*$-derivation on $A$.*

*Proof.* First, we show that $f$ is cubic. For each $a, b, c \in A$, we have

$$\begin{aligned}
&\left\| c^3 [f(2a+b) + f(2a-b) - 2f(a+b) - 2f(a-b) - 12f(a)] \right\| \\
&\quad = n^{-3s}\left\| n^{3s}c^3 f(2a+b) + n^{3s}c^3 f(2a-b) - 2n^{3s}c^3 f(a+b) - 2n^{3s}c^3 f(a-b) - 12n^{3s}c^3 f(a) \right\| \\
&\quad \le n^{-3s}\Big[\left\| n^{3s}c^3 f(2a+b) - f\left(n^{3s}c^3\right)(2a+b)^3 \right\| + \left\| n^{3s}c^3 f(2a-b) - f\left(n^{3s}c^3\right)(2a-b)^3 \right\| \\
&\qquad\qquad + 2\left\| n^{3s}c^3 f(a+b) - f\left(n^{3s}c^3\right)(a+b)^3 \right\| \\
&\qquad\qquad + 2\left\| n^{3s}c^3 f(a-b) - f\left(n^{3s}c^3\right)(a-b)^3 \right\| \\
&\qquad\qquad + 12\left\| n^{3s}c^3 f(a) - f\left(n^{3s}c^3\right)a^3 \right\|\Big] \\
&\quad \le n^{-3s}\left[\varphi(n^s c, 2a+b) + \varphi(n^s c, 2a-b) + 2\varphi(n^s c, a+b) + 2\varphi(n^s c, a-b) + 12\varphi(n^s c, a)\right].
\end{aligned} \tag{2.25}$$

Taking the limit from the right side as $n$ tends to infinity and using (2.21), we get

$$c^3 [f(2a+b) + f(2a-b) - 2f(a+b) - 2f(a-b) - 12f(a)] = 0 \tag{2.26}$$

for all $a, b, c \in A$. If $(e_j)$ is a left-bounded approximate identity for $A$, then so is $(e_j^3)$. Now, it follows from (2.26) that $f$ is cubic. For being cubic homogeneous of $f$, we have

$$\begin{aligned}
\left\| n^{3s}b^3\left[f(\mu a) - \mu^3 f(a)\right] \right\| &\le \left\| n^{3s}b^3 f(\mu a) - f(n^s b)(\mu a)^3 \right\| \\
&\quad + \left\| (\mu a)^3 f(n^s b) - n^{3s}(\mu b)^3 f(a) \right\| \\
&\le \varphi(n^s b, \mu a) + |\mu|^3 \varphi(n^s b, a).
\end{aligned} \tag{2.27}$$

Thus $\|b^3[f(\mu a) - \mu^3 f(a)]\| \le n^{-3s}\psi(n^s b, \mu a) + n^{-3s}|\mu|^3\psi(n^s b, a)$. By the same reasoning as in the above, $f$ is cubic homogeneous. For each $a, b, c \in A$, we have

$$\begin{aligned}\left\|c^3\left[f(ab) - f(a)b^3 - a^3 f(b)\right]\right\| &= n^{-3s}\left\|n^{3s}c^3\left[f(ab) - f(a)b^3 - a^3 f(b)\right]\right\| \\ &\le n^{-3s}\left\|n^{3s}c^3 f(ab) - f(n^s c)(ab)^3\right\| \\ &\quad + n^{-3s}\left\|f(n^s c)(ab)^3 - n^{3s}c^3 f(a)b^3 - n^{3s}c^3 a^3 f(b)\right\| \\ &\le 2n^{-3s}\psi(n^s c, ab).\end{aligned} \tag{2.28}$$

The above inequality and (2.21), (2.22), and (2.23) show that $f(ab) = f(a)b^3 + a^3 f(b)$ for all $a, b \in A$. Finally, we have

$$\begin{aligned}\left\|b^3[f(a^*) - f(a)^*]\right\| &= n^{-3s}\left\|n^{3s}b^3 f(a^*) - n^{3s}b^3 f(a)^*\right\| \\ &\le n^{-3s}\left\|n^{3s}b^3 f(a^*) - f(n^s b)\left(a^3\right)^*\right\| \\ &\quad + n^{-3s}\left\|f(n^s b)\left(a^3\right)^* - n^{3s}b^3 f(a)^*\right\| \\ &\le n^{-3s}\psi(n^s b, a^*) + n^{-3s}\psi(n^s b, a)\end{aligned} \tag{2.29}$$

for all $a, b \in A$. Note that in the last inequality we have used (2.22) and (2.24). This completes the proof. □

**Corollary 2.5.** *Let $r, \delta$ be the nonnegative real numbers with $r \ne 3$, and let $A$ be a Banach $*$-algebra with a left bounded approximate identity. Suppose that $f : A \to A$ is a mapping satisfying*

$$\begin{gathered}\left\|a^3 f(b) - f(a)b^3\right\| \le \delta\left(\|a\|^r\|b\|^r\right), \\ \left\|f(c)(ab)^3 - c^3\left[f(a)b^3 + a^3 f(b)\right]\right\| \le \delta\left(\|ab\|^r\|c\|^r\right), \\ \left\|a^3 f(b^*) - f(a)\left(b^3\right)^*\right\| \le \delta\left(\|a\|^r\|b\|^r\right)\end{gathered} \tag{2.30}$$

*for all all $a, b, c \in A$. Then $f$ is a cubic $*$-derivation on $A$.*

*Proof.* Using Theorem 2.4 with $\psi(a, b) = \delta(\|a\|^r\|b\|^r)$, we get the desired result. □

## 3. A Fixed Point Approach

Before proceeding to the main results in this section, we bring the upcoming theorem, which is useful to our purpose (For an extension of the result see [23]).

**Theorem 3.1** (The fixed point alternative [24]). *Let $(\Omega, d)$ be a complete generalized metric space and $\mathcal{T} : \Omega \to \Omega$ a mapping with Lipschitz constant $L < 1$. Then, for each element $\alpha \in \Omega$, either $d(\mathcal{T}^n\alpha, \mathcal{T}^{n+1}\alpha) = \infty$ for all $n \ge 0$, or there exists a natural number $n_0$ such that:*

(i) *$d(\mathcal{T}^n\alpha, \mathcal{T}^{n+1}\alpha) < \infty$ for all $n \ge n_0$;*

(ii) *the sequence* $\{\mathfrak{T}^n\alpha\}$ *is convergent to a fixed point* $\beta^*$ *of* $\mathfrak{T}$;

(iii) $\beta^*$ *is the unique fixed point of* $\mathfrak{T}$ *in the set* $\Lambda = \{\beta \in \Omega : d(\mathfrak{T}^{n_0}\alpha, \beta) < \infty\}$;

(iv) $d(\beta, \beta^*) \leq 1/(1-L)d(\beta, \mathfrak{T}\beta)$ *for all* $\beta \in \Lambda$.

**Theorem 3.2.** *Let* $f : A \to A$ *be a continuous mapping with* $f(0) = 0$, *and let* $\varphi : A^4 \to [0,\infty)$ *be a continuous function such that*

$$\left\|\mathfrak{D}_\mu f(a,b) + \mathfrak{D} f(c,d)\right\| \leq \varphi(a,b,c,d), \tag{3.1}$$

$$\left\|f(a^*) - f(a)^*\right\| \leq \varphi(a,a,a,a) \tag{3.2}$$

*for all* $\mu \in \mathbb{T}^1_{1/n_0}$ *and all* $a,b,c,d \in A$. *If there exists a constant* $k \in (0,1)$ *such that*

$$\varphi(2a,2b,2c,2d) \leq 8k\varphi(a,b,c,d) \tag{3.3}$$

*for all* $a,b,c,d \in A$, *then there exists a unique cubic* $*$*-derivation* $D$ *on* $A$ *satisfying*

$$\left\|f(a) - D(a)\right\| \leq \frac{1}{16(1-k)}\widetilde{\varphi}(a) \quad (a \in A), \tag{3.4}$$

*in which* $\widetilde{\varphi}(a) = \varphi(a,0,0,0)$.

*Proof.* First, we wish to provide the conditions of Theorem 3.1. We consider the set

$$\Omega = \left\{g : A \longrightarrow A \mid g(0) = 0\right\} \tag{3.5}$$

and define the mapping $d$ on $\Omega \times \Omega$ as follows:

$$d(g_1, g_2) := \inf\left\{C \in (0,\infty) : \left\|g_1(a) - g_2(a)\right\| \leq C\widetilde{\varphi}(a),\ (\forall a \in A)\right\} \tag{3.6}$$

if there exist such constant $C$ and $d(g_1, g_2) = \infty$, otherwise. It is easy to check that $d(g,g) = 0$ and $d(g_1,g_2) = d(g_2,g_1)$, for all $g, g_1, g_2 \in \Omega$. For each $g_1,\ g_2,\ g_3 \in \Omega$, we have

$$\begin{aligned}
&\inf\left\{C \in (0,\infty) : \left\|g_1(a) - g_2(a)\right\| \leq C\widetilde{\varphi}(a)\ \forall a \in A\right\}\\
&\quad\leq \inf\left\{C \in (0,\infty) : \left\|g_1(a) - g_3(a)\right\| \leq C\widetilde{\varphi}(a)\ \forall a \in A\right\}\\
&\qquad+ \inf\left\{C \in (0,\infty) : \left\|g_3(a) - g_2(a)\right\| \leq C\widetilde{\varphi}(a)\ \forall a \in A\right\}.
\end{aligned} \tag{3.7}$$

Hence $d(g_1,g_2) \leq d(g_1,g_3) + d(g_3,g_2)$. If $d(g_1,g_2) = 0$, then for every fixed $a_0 \in A$, we have $\|g_1(a_0) - g_2(a_0)\| \leq C\widetilde{\varphi}(a_0)$ for all $C > 0$. This implies $g_1 = g_2$. Let $\{g_n\}$ be a $d$-Cauchy sequence in $\Omega$. Then $d(g_m, g_n) \to 0$, and thus $\|g_m(a) - g_n(a)\| \to 0$ for all $a \in A$. Since $A$ is complete, then there exists $g \in \Omega$ such that $g_n \xrightarrow{d} g$ in $\Omega$. Therefore, $d$ is a generalized metric on $\Omega$ and the metric space $(\Omega, d)$ is complete. Now, we define the mapping $\mathcal{T} : \Omega \to \Omega$ by

$$\mathcal{T}g(a) = \frac{1}{8}g(2a), \quad (a \in A). \tag{3.8}$$

If $g_1, g_2 \in \Omega$ such that $d(g_1, g_2) < C$, by definition of $d$ and $\mathcal{T}$, we have

$$\left\|\frac{1}{8}g_1(2a) - \frac{1}{8}g_2(2a)\right\| \leq \frac{1}{8}C\varphi(2a, 0, 0, 0) \tag{3.9}$$

for all $a \in A$. By using (3.3), we get

$$\left\|\frac{1}{8}g_1(2a) - \frac{1}{8}g_2(2a)\right\| \leq Ck\varphi(a, 0, 0, 0) \tag{3.10}$$

for all $a \in A$. The above inequality shows that $d(\mathcal{T}g_1, \mathcal{T}g_2) \leq kd(g_1, g_2)$ for all $g_1, g_2 \in \Omega$. Hence, $\mathcal{T}$ is a strictly contractive mapping on $\Omega$ with a Lipschitz constant $k$. To achieve inequality (3.4), we prove that $d(\mathcal{T}f, f) < \infty$. Putting $b = c = d = 0$ and $\mu = 1$ in (3.1), we obtain

$$\|2f(2a) - 16f(a)\| \leq \tilde{\varphi}(a) \tag{3.11}$$

for all $a \in A$. Hence

$$\left\|\frac{1}{8}f(2a) - f(a)\right\| \leq \frac{1}{16}\tilde{\varphi}(a) \tag{3.12}$$

for all $a \in A$. We conclude from (3.12) that $d(\mathcal{T}f, f) \leq 1/16$. It follows from Theorem 3.1 that $d(\mathcal{T}^n g, \mathcal{T}^{n+1} g) < \infty$ for all $n \geq 0$, and thus in this theorem we have $n_0 = 0$. Therefore, the parts (iii) and (iv) of Theorem 3.1 hold on the whole $\Omega$. Hence there exists a unique mapping $D : A \to A$ such that $D$ is a fixed point of $\mathcal{T}$ and that $\mathcal{T}^n f \to D$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{f(2^n a)}{8^n} = D(a) \tag{3.13}$$

for all $a \in A$, hence

$$d(f, D) \leq \frac{1}{1-k}d(\mathcal{T}f, f) \leq \frac{1}{16(1-k)}. \tag{3.14}$$

The above equalities show that (3.4) is true for all $a \in A$. It follows from (3.3) that

$$\lim_{n\to\infty} \frac{\varphi(2^n a, 2^n b, 2^n c, 2^n d)}{8^n} = 0. \tag{3.15}$$

Putting $c = d = 0$ and substituting $a, b$ by $2^n a,\ 2^n b$, respectively, in (3.1), we get

$$\frac{1}{8^n}\left\|\mathfrak{D}_\mu f(2^n a, 2^n b)\right\| \leq \frac{\varphi(2^n a, 2^n b, 0, 0)}{8^n}. \tag{3.16}$$

Taking the limit as $n$ tend to infinity, we obtain $\mathfrak{D}_\mu D(a,b) = 0$ for all $a, b \in A$ and all $\mu \in \mathbb{T}^1_{1/n_0}$. Similar to the proof of Theorem 2.1, we have $D(\mu a) = \mu^3 D(a)$ for all $a \in A$ and $\mu \in \mathbb{T}^1$. Since $\mathfrak{D}_1 D(a,b) = 0$, we can show that $D(ra) = r^3 D(a)$ for any rational number $r$. The continuity of $f$ and $\varphi$ imply that $D(\mu a) = \mu^3 D(a)$, for all $a \in A$ and $\mu \in \mathbb{R}$. Hence $D(\mu a) = \mu^3 D(a)$, for all $a \in A$ and $\mu \in \mathbb{C}$ $(\mu \neq 0)$. Therefore, $D$ is a cubic homogeneous. If we put $a = b = 0$ and replace $c, d$ by $2^n c, 2^n d$, respectively, in (3.1), we have

$$\frac{1}{8^{2n}}\left\|\mathfrak{D} f(2^n c, 2^n d)\right\| \leq \frac{\varphi(0,0,2^n c, 2^n d)}{8^{2n}} \leq \frac{\varphi(0,0,2^n c, 2^n d)}{8^n} \tag{3.17}$$

for all $c, d \in A$. By letting $n \to \infty$ in the preceding inequality, we find $\mathfrak{D} D(c,d) = 0$ for all $c, d \in A$. Substituting $a$ by $2^n a$ in (3.2) and then dividing the both sides of the obtained inequality by $8^n$, we get

$$\left\|\frac{f(2^n a^*)}{8^n} - \frac{f(2^n a)^*}{8^n}\right\| \leq \frac{\varphi(2^n a, 2^n a, 2^n a, 2^n a)}{8^n} \tag{3.18}$$

for all $a \in A$. Passing to the limit as $n \to \infty$ in (3.18) and applying (3.13), we conclude that $D(a^*) = D(a)^*$ for all $a \in A$. This shows that $D$ is a unique cubic $*$-derivation. □

**Corollary 3.3.** *Let $\theta, r$ be positive real numbers with $r < 3$, and let $f : A \to A$ be a mapping with $f(0) = 0$ such that*

$$\begin{gathered} \left\|\mathfrak{D}_\mu f(a,b) + \mathfrak{D} f(c,d)\right\| \leq \theta\left(\|a\|^r + \|b\|^r + \|c\|^r + \|d\|^r\right), \\ \left\|f(a^*) - f(a)^*\right\| \leq 4\theta\|a\|^r \end{gathered} \tag{3.19}$$

*for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $a, b, c, d \in A$. Then there exists a unique cubic $*$-derivation $D$ on $A$ satisfying*

$$\left\|f(a) - D(a)\right\| \leq \frac{\theta}{2(8 - 2^r)}\|a\|^r \tag{3.20}$$

*for all $a \in A$.*

*Proof.* The result follows from Theorem 3.2 by letting

$$\varphi(a, b, c, d) = \theta(\|a\|^r + \|b\|^r + \|c\|^r + \|d\|^r). \tag{3.21}$$

□

In the following corollary, we show the superstability for cubic $*$-derivations.

**Corollary 3.4.** *Let* $r_j$ $(1 \le j \le 4)$ $\theta$ *be nonnegative real numbers with* $0 < \sum_{j=1}^{4} r_j \neq 3$*, and let* $f : A \to A$ *be a mapping such that*

$$\left\| \mathfrak{D}_\mu f(a,b) + \mathfrak{D} f(c,d) \right\| \le \theta\left( \|a\|^{r_1} \|b\|^{r_2} \|c\|^{r_3} \|d\|^{r_4} \right), \tag{3.22}$$

$$\left\| f(a^*) - f(a)^* \right\| \le \theta \|a\|^{\sum_{j=1}^{4} r_j} \tag{3.23}$$

*for all* $\mu \in \mathbb{T}^1_{1/n_0}$ *and all* $a,\ b,\ c,\ d \in A$*. Then* $f$ *is a cubic* $*$*-derivation on* $A$*.*

*Proof.* Putting $a = b = c = d = 0$ in (3.22), we get $f(0) = 0$. Now, if we put $b = c = d = 0$, $\mu = 1$ in (3.22), then we have $f(2a) = 8f(a)$ for all $a \in A$. It is easy to see by induction that $f(2^n a) = 8^n f(a)$, and thus $f(a) = f(2^n a)/8^n$ for all $a \in A$ and $n \in \mathbb{N}$. It follows from Theorem 3.2 that $f$ is a cubic mapping. Now, by putting $\varphi(a,b,c,d) = \theta(\|a\|^{r_1}\|b\|^{r_2}\|c\|^{r_3}\|d\|^{r_4})$ in Theorem 3.2, we can obtain the desired result. □

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*Research Article*

# Higher Ring Derivation and Intuitionistic Fuzzy Stability

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We take account of the stability of higher ring derivation in intuitionistic fuzzy Banach algebra associated to the Jensen type functional equation. In addition, we deal with the superstability of higher ring derivation in intuitionistic fuzzy Banach algebra with unit.

## 1. Introduction and Preliminaries

The stability problem of functional equations has originally been formulated by Ulam [1]: *under what condition does there exist a homomorphism near an approximate homomorphism?* Hyers [2] answered the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [3] and for approximately linear mappings was presented by Rassias [4] by considering an unbounded Cauchy difference. The paper work of Rassias [4] has had a lot of influence in the development of what is called the *generalized Hyers-Ulam stability* of functional equations. Since then, more generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings have been investigated (e.g., [5–7]). In particular, Badora [8] gave a generalization of the Bourgin's result [9], and he also dealt with the stability and the Bourgin-type superstability of derivations in [10]. Recently, fuzzy version is discussed in [11, 12]. Quite recently, the intuitionistic fuzzy stability problem for Jensen functional equation and cubic functional equation is considered in [13–15], respectively, while the idea of intuitionistic fuzzy normed space was introduced in [16], and there are some recent and important results which are directly related to the central theme of this paper, that is, intuitionistic fuzziness (see e.g., [17–20]).

In this paper, we establish the stability of higher ring derivation in intuitionistic fuzzy Banach algebra associated to the Jensen type functional equation $lf(x + y/l) = f(x) + f(y)$. Moreover, we consider the superstability of higher ring derivation in intuitionistic fuzzy Banach algebra with unit.

We now recall some notations and basic definitions used in this paper.

*Definition 1.1* (see [5]). Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over the real or complex field $\mathbb{F}$. Let $\mathbb{N}$ be the set of the natural numbers. From $m \in \mathbb{N} \cup \{0\}$, a sequence $H = \{h_0, h_1, \ldots, h_m\}$ (resp., $H = \{h_0, h_1, \ldots, h_k, \ldots\}$) of additive operators from $\mathcal{A}$ into $\mathcal{B}$ is called a *higher ring derivation* of rank $m$ (resp., infinite rank) if the functional equation $h_k(xy) = \sum_{i=0}^{k} h_i(x)h_{k-i}(y)$ holds for each $k = 0, 1, \ldots, m$ (resp., $k = 0, 1, \ldots$) and for all $x, y \in \mathcal{A}$. A higher ring derivation $H$ of additive operators on $\mathcal{A}$, particularly, is called *strong* if $h_0$ is an identity operator.

Of course, a higher ring derivation of rank 0 from $\mathcal{A}$ into $\mathcal{B}$ (resp., a strong higher ring derivation of rank 1 on $\mathcal{A}$) is a ring homomorphism (resp., a ring derivation). Note that a higher ring derivation is a generalization of both a ring homomorphism and a ring derivation.

*Definition 1.2.* A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:

(1) $*$ is associative and commutative, (2) $*$ is continuous, (3) $a * 1 = a$ for all $a \in [0,1]$, and (4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

*Definition 1.3.* A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:

(1) $\diamond$ is associative and commutative, (2) $\diamond$ is continuous, (3) $a \diamond 0 = a$ for all $a \in [0,1]$, and (4) $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

Using the notions of continuous $t$-norm and $t$-conorm, Saadati and Park [16] have recently introduced the concept of intuitionistic fuzzy normed space as follows.

*Definition 1.4.* The five-tuple $(\mathcal{X}, \mu, \nu, *, \diamond)$ is said to be an *intuitionistic fuzzy normed space* if $\mathcal{X}$ is a vector space, $*$ is a continuous $t$-norm, $\diamond$ is a continuous $t$-conorm, and $\mu, \nu$ are fuzzy sets on $\mathcal{X} \times (0, \infty)$ satisfying the following conditions. For every $x, y \in \mathcal{X}$ and $s, t > 0$, (1) $\mu(x,t) + \nu(x,t) \le 1$, (2) $\mu(x,t) > 0$, (3) $\mu(x,t) = 1$ if and only if $x = 0$, (4) $\mu(\alpha x, t) = \mu(x, t/|\alpha|)$ for each $\alpha \neq 0$, (5) $\mu(x,t) * \mu(y,s) \le \mu(x+y, t+s)$, (6) $\mu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous, (7) $\lim_{t\to\infty}\mu(x,t) = 1$ and $\lim_{t\to 0}\mu(x,t) = 0$, (8) $\nu(x,t) < 1$, (9) $\nu(x,t) = 0$ if and only if $x = 0$, (10) $\nu(\alpha x, t) = \nu(x, t/|\alpha|)$ for each $\alpha \neq 0$, (11) $\nu(x,t) \diamond \mu(y,s) \ge \nu(x+y, t+s)$, (12) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous, (13) $\lim_{t\to\infty}\nu(x,t) = 0$ and $\lim_{t\to 0}\nu(x,t) = 1$.

In this case, $(\mu, \nu)$ is called an intuitionistic fuzzy norm.

*Example 1.5.* Let $(\mathcal{X}, \|\cdot\|)$ be a normed space, $a * b = ab$, and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in \mathcal{X}$ and every $t > 0$, consider

$$\mu(x,t) = \begin{cases} 1, & \text{if } t > \|x\|, \\ 0, & \text{if } t \le \|x\|, \end{cases} \qquad \nu(x,t) = \begin{cases} 0, & \text{if } t > \|x\|, \\ 1, & \text{if } t \le \|x\|. \end{cases} \tag{1.1}$$

Then $(\mathcal{X}, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy normed space.

*Example 1.6.* Let $(\mathcal{X}, \|\cdot\|)$ be a normed space, $a * b = ab$, and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in \mathcal{X}$ and every $t > 0$ and $k = 1, 2$, consider

$$\mu(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & \text{if } t > 0, \\ 0, & \text{if } t \le 0, \end{cases} \qquad \nu(x,t) = \begin{cases} \dfrac{k\|x\|}{t+k\|x\|}, & \text{if } t > 0, \\ 0, & \text{if } t \le 0. \end{cases} \tag{1.2}$$

Then $(\mathcal{X}, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy normed space.

*Definition 1.7* (see [21]). The five-tuple $(\mathcal{X}, \mu, \nu, *, \diamond)$ is said to be an *intuitionistic fuzzy normed algebra* if $\mathcal{X}$ is an algebra, $*$ is a continuous $t$-norm, $\diamond$ is a continuous $t$-conorm, and $\mu, \nu$ are fuzzy sets on $\mathcal{X} \times (0, \infty)$ satisfying the conditions (1)–(13) of the Definition 1.4. Furthermore, for every $x, y \in \mathcal{X}$ and $s, t > 0$, (14) $\max\{\mu(x,t), \mu(y,s)\} \le \mu(xy, t+s)$, (15) $\min\{\nu(x,t), \nu(y,s)\} \ge \nu(xy, t+s)$.

For an intuitionistic fuzzy normed algebra $(\mathcal{X}, \mu, \nu, *, \diamond)$, we further assume that (16) $a * a = a$ and $a \diamond a = a$ for all $a \in [0,1]$.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [16]. Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space or intuitionistic fuzzy normed algebra. A sequence $x = \{x_k\}$ is said to be *intuitionistic fuzzy convergent* to $L \in \mathcal{X}$ if $\lim_{k\to\infty}\mu(x_k - L, t) = 1$ and $\lim_{k\to\infty}\nu(x_k - L, t) = 0$ for all $t > 0$. In this case, we write $(\mu, \nu) - \lim_{k\to\infty} x_k = L$ or $x_k \overset{IF}{\to} L$ as $k \to \infty$. A sequence $x = \{x_k\}$ in $(\mathcal{X}, \mu, \nu, *, \diamond)$ is said to be *intuitionistic fuzzy Cauchy sequence* if $\lim_{k\to\infty}\mu(x_{k+p} - x_k, t) = 1$ and $\lim_{k\to\infty}\nu(x_{k+p} - x_k, t) = 0$ for all $t > 0$ and $p = 1, 2, \ldots$. An intuitionistic fuzzy normed space (resp., intuitionistic fuzzy normed algebra) $(\mathcal{X}, \mu, \nu, *, \diamond)$ is said to be *complete* if every intuitionistic fuzzy Cauchy sequence in $(\mathcal{X}, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent in $(\mathcal{X}, \mu, \nu, *, \diamond)$. A complete intuitionistic fuzzy normed space (resp., intuitionistic fuzzy normed algebra) is also called an *intuitionistic fuzzy Banach space* (resp., *intuitionistic fuzzy Banach algebra*).

## 2. Stability of Higher Ring Derivation in Intuitionistic Fuzzy Banach Algebra

As a matter of convenience in this paper, we use the following abbreviation:

$$\prod_{j=0}^{n} a_j := a_1 * a_2 * \cdots * a_n, \qquad \prod_{j=0}^{\infty} a_j := a_1 * a_2 * \cdots . \tag{2.1}$$

In addition,

$$\coprod_{j=0}^{n} a_j := a_1 \diamond a_2 \diamond \cdots \diamond a_n, \qquad \coprod_{j=0}^{\infty} a_j := a_1 \diamond a_2 \diamond \cdots . \tag{2.2}$$

We begin with a generalized Hyers-Ulam theorem in intuitionistic fuzzy Banach space for the Jensen type functional equation. The following result is also the generalization of the theorem introduced in [13].

**Theorem 2.1.** *Let $\mathcal{A}$ be a vector space, and let $f$ be a mapping from $\mathcal{A}$ to an intuitionistic fuzzy Banach space $(\mathcal{B}, \mu, \nu, *, \diamond)$ with $f(0) = 0$. Suppose that $\varphi$ is a function from $\mathcal{A}$ to an intuitionistic fuzzy normed space $(C, \mu', \nu', *, \diamond)$ such that*

$$\mu\left(lf\left(\frac{x+y}{l}\right) - f(x) - f(y), t+s\right) \geq \mu'(\varphi(x), t) * \mu'(\varphi(y), s), \tag{2.3}$$

$$\nu\left(lf\left(\frac{x+y}{l}\right) - f(x) - f(y), t+s\right) \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(y), s) \tag{2.4}$$

*for all $x, y \in \mathcal{A} \setminus \{0\}$, $t > 0$ and $s > 0$. If $l > 1$ is a fixed integer, and $\varphi((l+1)x) = \alpha\varphi(x)$ for some real number $\alpha$ with $0 < |\alpha| < l+1$, then there exists a unique additive mapping $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{L}(x) := (\mu, \nu) - \lim_{n\to\infty}(f((l+1)^n x)/(l+1)^n)$,*

$$\begin{aligned}\mu(\mathcal{L}(x) - f(x), t) &\geq \prod_{j=0}^{\infty} M\left(x, \frac{((l+1)-\alpha)t}{2(l+1)}\right),\\ \nu(\mathcal{L}(x) - f(x), t) &\leq \coprod_{j=0}^{\infty} N\left(x, \frac{((l+1)-\alpha)t}{2(l+1)}\right)\end{aligned} \tag{2.5}$$

*for all $x \in \mathcal{A}$ and $t > 0$, where*

$$\begin{aligned}M(x,t) &:= \mu'\left(\varphi(x), \frac{l+1}{4}t\right) * \mu'\left(\varphi(-x), \frac{l+1}{4}t\right) * \mu'\left(\varphi(-x), \frac{l+1}{4}t\right) * \mu'\left(\varphi((l+1)x), \frac{l+1}{4}t\right),\\ N(x,t) &:= \nu'\left(\varphi(x), \frac{l+1}{4}t\right) \diamond \nu'\left(\varphi(-x), \frac{l+1}{4}t\right) \diamond \nu'\left(\varphi(-x), \frac{l+1}{4}t\right) \diamond \nu'\left(\varphi((l+1)x), \frac{l+1}{4}t\right).\end{aligned} \tag{2.6}$$

*Proof.* Without loss of generality, we assume that $0 < \alpha < l+1$. From (2.3) and (2.4), we get

$$\begin{aligned}\mu(f(x) + f(-x), lt) &\geq \mu'\left(\varphi(x), \frac{l}{2}t\right) * \mu'\left(\varphi(-x), \frac{l}{2}t\right),\\ \nu(f(x) + f(-x), lt) &\leq \nu'\left(\varphi(x), \frac{l}{2}t\right) \diamond \left(\varphi(-x), \frac{l}{2}t\right)\end{aligned} \tag{2.7}$$

for all $x \in \mathcal{A}$ and $t > 0$. Again, by (2.3) and (2.4), we obtain

$$\begin{aligned}\mu(lf(x) - f(-x) - f((l+1)x), lt) &\geq \mu'\left(\varphi(-x), \frac{l}{2}t\right) * \mu'\left(\varphi((l+1)x), \frac{l}{2}t\right),\\ \nu(lf(x) - f(-x) - f((l+1)x), lt) &\leq \nu'\left(\varphi(-x), \frac{l}{2}t\right) \diamond \nu'\left(\varphi((l+1)x), \frac{l}{2}t\right)\end{aligned} \tag{2.8}$$

for all $x \in \mathcal{A}$ and $t > 0$. Combining (2.7) and (2.8), we arrive at

$$\begin{aligned}
\mu((l+1)f(x) - f((l+1)x), 2lt) &\geq \mu'\left(\varphi(x), \frac{l}{2}t\right) * \mu'\left(\varphi(-x), \frac{l}{2}t\right) * \mu'\left(\varphi(-x), \frac{l}{2}t\right) \\
&\quad * \mu'\left(\varphi((l+1)x), \frac{l}{2}t\right), \\
\nu((l+1)f(x) - f((l+1)x), 2lt) &\leq \nu'\left(\varphi(x), \frac{l}{2}t\right) \diamond \nu'\left(\varphi(-x), \frac{l}{2}t\right) \diamond \nu'\left(\varphi(-x), \frac{l}{2}t\right) \\
&\quad \diamond \nu'\left(\varphi((l+1)x), \frac{l}{2}t\right),
\end{aligned} \tag{2.9}$$

for all $x \in \mathcal{A}$ and $t > 0$. This implies that

$$\begin{aligned}
\mu\left(f(x) - \frac{f((l+1)x)}{(l+1)}, t\right) &\geq \mu'\left(\varphi(x), \frac{l+1}{4}t\right) * \mu'\left(\varphi(-x), \frac{l+1}{4}t\right) * \mu'\left(\varphi(-x), \frac{l+1}{4}t\right) \\
&\quad * \mu'\left(\varphi((l+1)x), \frac{l+1}{4}t\right), \\
\nu\left(f(x) - \frac{f((l+1)x)}{(l+1)}, t\right) &\leq \nu'\left(\varphi(x), \frac{l+1}{4}t\right) \diamond \nu'\left(\varphi(-x), \frac{l+1}{4}t\right) \diamond \nu'\left(\varphi(-x), \frac{l+1}{4}t\right) \\
&\quad \diamond \nu'\left(\varphi((l+1)x), \frac{l+1}{4}t\right),
\end{aligned} \tag{2.10}$$

for all $x \in \mathcal{A}$ and $t > 0$. Now we define

$$\begin{aligned}
M(x,t) &:= \mu'\left(\varphi(x), \frac{l+1}{4}t\right) * \mu'\left(\varphi(-x), \frac{l+1}{4}t\right) * \mu'\left(\varphi(-x), \frac{l+1}{4}t\right) * \mu'\left(\varphi((l+1)x), \frac{l+1}{4}t\right), \\
N(x,t) &:= \nu'\left(\varphi(x), \frac{l+1}{4}t\right) \diamond \nu'\left(\varphi(-x), \frac{l+1}{4}t\right) \diamond \nu'\left(\varphi(-x), \frac{l+1}{4}t\right) \diamond \nu'\left(\varphi((l+1)x), \frac{l+1}{4}t\right),
\end{aligned} \tag{2.11}$$

for all $x \in \mathcal{A}$ and $t > 0$. Then we have by assumption

$$M((l+1)x, t) = M\left(x, \frac{t}{\alpha}\right), \qquad N((l+1)x, t) = N\left(x, \frac{t}{\alpha}\right), \tag{2.12}$$

for all $x \in \mathcal{A}$ and $t > 0$. Using (2.10) and (2.12), we get

$$\begin{aligned}
\mu\left(\frac{f((l+1)^n x)}{(l+1)^n} - \frac{f\left((l+1)^{n+1}x\right)}{(l+1)^{n+1}}, \frac{\alpha^n t}{(l+1)^n}\right) &= \mu\left(f((l+1)^n x) - \frac{f\left((l+1)^{n+1}x\right)}{l+1}, \alpha^n t\right) \\
&\geq M((l+1)^n x, \alpha^n t) = M(x,t),
\end{aligned}$$

$$\nu\left(\frac{f\left((l+1)^n x\right)}{(l+1)^n}-\frac{f\left((l+1)^{n+1} x\right)}{(l+1)^{n+1}}, \frac{\alpha^n t}{(l+1)^n}\right)=\nu\left(f\left((l+1)^n x\right)-\frac{f\left((l+1)^{n+1} x\right)}{l+1}, \alpha^n t\right)$$
$$\leq N\left((l+1)^n x, \alpha^n t\right)=N(x, t), \tag{2.13}$$

for all $x \in \mathcal{A}$ and $t>0$. Therefore, for all $n>m$, we have

$$\begin{aligned}
&\mu\left(\frac{f\left((l+1)^m x\right)}{(l+1)^m}-\frac{f\left((l+1)^n x\right)}{(l+1)^n}, \sum_{j=m}^{n-1} \frac{\alpha^j t}{(l+1)^j}\right) \\
&\quad=\mu\left(\sum_{j=m}^{n-1}\left[\frac{f\left((l+1)^j x\right)}{(l+1)^j}-\frac{f\left((l+1)^{j+1} x\right)}{(l+1)^{j+1}}\right], \sum_{j=m}^{n-1} \frac{\alpha^j t}{(l+1)^j}\right) \\
&\quad\geq \prod_{j=m}^{n-1} \mu\left(\frac{f\left((l+1)^j x\right)}{(l+1)^j}-\frac{f\left((l+1)^{j+1} x\right)}{(l+1)^{j+1}}, \frac{\alpha^j t}{(l+1)^j}\right) \geq \prod_{j=m}^{n-1} M(x, t), \\
&\nu\left(\frac{f\left((l+1)^m x\right)}{(l+1)^m}-\frac{f\left((l+1)^n x\right)}{(l+1)^n}, \sum_{j=m}^{n-1} \frac{\alpha^j t}{(l+1)^j}\right) \\
&\quad=\nu\left(\sum_{j=m}^{n-1}\left[\frac{f\left((l+1)^j x\right)}{(l+1)^j}-\frac{f\left((l+1)^{j+1} x\right)}{(l+1)^{j+1}}\right], \sum_{j=m}^{n-1} \frac{\alpha^j t}{(l+1)^j}\right) \\
&\quad\leq \coprod_{j=m}^{n-1} \nu\left(\frac{f\left((l+1)^j x\right)}{(l+1)^j}-\frac{f\left((l+1)^{j+1} x\right)}{(l+1)^{j+1}}, \frac{\alpha^j t}{(l+1)^j}\right) \leq \coprod_{j=m}^{n-1} N(x, t),
\end{aligned} \tag{2.14}$$

for all $x \in \mathcal{A}$ and $t>0$. Let $\varepsilon>0$ and $\delta>0$ be given. Since $\lim_{t \to \infty} \prod_{j=m}^{n-1} M(x, t)=1$ and $\lim_{t \to \infty} \coprod_{j=m}^{n-1} N(x, t)=0$, there exists some $t_0$ such that $\prod_{j=m}^{n-1} M(x, t_0)>1-\varepsilon$, $\coprod_{j=m}^{n-1} N(x, t_0)<\varepsilon$. Since $\sum_{j=0}^{\infty}(\alpha^j t/(l+1)^j)<\infty$, there exists a positive integer $n_0$ such that $\sum_{j=m}^{n-1}(\alpha^j t/(l+1)^j)<\delta$ for all $n>m \geq n_0$.

Then

$$\mu\left(\frac{f\left((l+1)^m x\right)}{(l+1)^m}-\frac{f\left((l+1)^n x\right)}{(l+1)^n}, \delta\right) \geq \mu\left(\frac{f\left((l+1)^m x\right)}{(l+1)^m}-\frac{f\left((l+1)^n x\right)}{(l+1)^n}, \sum_{j=m}^{n-1} \frac{\alpha^j t_0}{(l+1)^j}\right)$$
$$\geq \prod_{j=m}^{n-1} M(x, t_0)>1-\varepsilon,$$

$$\nu\left(\frac{f((l+1)^m x)}{(l+1)^m}-\frac{f((l+1)^n x)}{(l+1)^n},\delta\right)\le \nu\left(\frac{f((l+1)^m x)}{(l+1)^m}-\frac{f((l+1)^n x)}{(l+1)^n},\sum_{j=m}^{n-1}\frac{\alpha^j t_0}{(l+1)^j}\right)$$
$$\le \coprod_{j=m}^{n-1} N(x,t_0)<\varepsilon. \tag{2.15}$$

This shows that $\{(f((l+1)^n x))/((l+1)^n)\}$ is a Cauchy sequence in $(\mathcal{B},\mu',\nu',*,\diamond)$. Since $\mathcal{B}$ is complete, we can define a mapping $\mathcal{L}$ by $\mathcal{L}(x) := (\mu,\nu)-\lim_{n\to\infty}(f((l+1)^n x)/(l+1)^n)$ for all $x\in\mathcal{A}$. Moreover, if we let $m=0$ in (2.14), then we get

$$\mu\left(\frac{f((l+1)^n x)}{(l+1)^n}-f(x),\sum_{j=0}^{n-1}\frac{\alpha^j t}{(l+1)^j}\right)\ge \prod_{j=0}^{n-1} M(x,t),$$
$$\nu\left(\frac{f((l+1)^n x)}{(l+1)^n}-f(x),\sum_{j=0}^{n-1}\frac{\alpha^j t}{(l+1)^j}\right)\le \coprod_{j=0}^{n-1} N(x,t), \tag{2.16}$$

for all $x\in\mathcal{A}$ and $t>0$. Therefore, we find that

$$\mu\left(\frac{f((l+1)^n x)}{(l+1)^n}-f(x),t\right)\ge \prod_{j=0}^{n-1} M\left(x,\frac{t}{\sum_{j=0}^{n-1}\left(\alpha^j/(l+1)^j\right)}\right),$$
$$\nu\left(\frac{f((l+1)^n x)}{(l+1)^n}-f(x),t\right)\le \coprod_{j=0}^{n-1} N\left(x,\frac{t}{\sum_{j=0}^{n-1}\left(\alpha^j/(l+1)^j\right)}\right). \tag{2.17}$$

Next, we will show that $\mathcal{L}$ is additive mapping. Note that

$$\mu\left(l\mathcal{L}\left(\frac{x+y}{l}\right)-\mathcal{L}(x)-\mathcal{L}(y),t\right)\ge \mu\left(l\mathcal{L}\left(\frac{x+y}{l}\right)-\frac{lf(((l+1)^n(x+y))/l)}{(l+1)^n},\frac{t}{4}\right)$$
$$*\,\mu\left(\frac{f((l+1)^n x)}{(l+1)^n}-\mathcal{L}(x),\frac{t}{4}\right)*\mu\left(\frac{f((l+1)^n y)}{(l+1)^n}-\mathcal{L}(y),\frac{t}{4}\right)$$
$$*\,\mu\left(\frac{lf(((l+1)^n(x+y))/(l))}{(l+1)^n}-\frac{f((l+1)^n x)}{(l+1)^n}-\frac{f((l+1)^n y)}{(l+1)^n},\frac{t}{4}\right),$$

$$\nu\left(l\mathcal{L}\left(\frac{x+y}{l}\right)-\mathcal{L}(x)-\mathcal{L}(y),t\right)\leq\nu\left(l\mathcal{L}\left(\frac{x+y}{l}\right)-\frac{lf(((l+1)^n(x+y))/l)}{(l+1)^n},\frac{t}{4}\right)$$
$$\diamond\,\nu\left(\frac{f((l+1)^n x)}{(l+1)^n}-\mathcal{L}(x),\frac{t}{4}\right)\diamond\,\nu\left(\frac{f((l+1)^n y)}{(l+1)^n}-\mathcal{L}(y),\frac{t}{4}\right)$$
$$\diamond\,\nu\left(\frac{lf(((l+1)^n(x+y))/(l))}{(l+1)^n}-\frac{f((l+1)^n x)}{(l+1)^n}-\frac{f((l+1)^n y)}{(l+1)^n},\frac{t}{4}\right). \tag{2.18}$$

On the other hand, (2.3) and (2.4) give the following:

$$\begin{aligned}
&\mu\left(\frac{lf(((l+1)^n(x+y))/l)}{(l+1)^n}-\frac{f((l+1)^n x)}{(l+1)^n}-\frac{f((l+1)^n y)}{(l+1)^n},\frac{t}{4}\right)\\
&\quad\geq\mu'\left(\varphi(x),\left(\frac{l+1}{\alpha}\right)^n\frac{t}{8}\right)*\mu'\left(\varphi(y),\left(\frac{l+1}{\alpha}\right)^n\frac{t}{8}\right),\\
&\nu\left(\frac{lf(((l+1)^n(x+y))/l)}{(l+1)^n}-\frac{f((l+1)^n x)}{(l+1)^n}-\frac{f((l+1)^n y)}{(l+1)^n},\frac{t}{4}\right)\\
&\quad\leq\nu'\left(\varphi(x),\left(\frac{l+1}{\alpha}\right)^n\frac{t}{8}\right)\diamond\,\nu'\left(\varphi(y),\left(\frac{l+1}{\alpha}\right)^n\frac{t}{8}\right).
\end{aligned}\tag{2.19}$$

Letting $n\to\infty$ in (2.18) and (2.19), we yield

$$\mu\left(l\mathcal{L}\left(\frac{x+y}{l}\right)-\mathcal{L}(x)-\mathcal{L}(y),t\right)=1,\qquad\nu\left(l\mathcal{L}\left(\frac{x+y}{l}\right)-\mathcal{L}(x)-\mathcal{L}(y),t\right)=0. \tag{2.20}$$

So we see that $\mathcal{L}$ is additive mapping.

Now, we approximate the difference between $f$ and $\mathcal{L}$ in an intuitionistic fuzzy sense. By (2.17), we get

$$\begin{aligned}
\mu(\mathcal{L}(x)-f(x),t)&\geq\mu\left(\mathcal{L}(x)-\frac{f((l+1)^n x)}{(l+1)^n},\frac{t}{2}\right)*\mu\left(\frac{f((l+1)^n x)}{(l+1)^n}-f(x),\frac{t}{2}\right)\\
&\geq\prod_{j=0}^{\infty}M\left(x,\frac{((l+1)-\alpha)t}{2(l+1)}\right),\\
\nu(\mathcal{L}(x)-f(x),t)&\leq\nu\left(\mathcal{L}(x)-\frac{f((l+1)^n x)}{(l+1)^n},\frac{t}{2}\right)\diamond\,\nu\left(\frac{f((l+1)^n x)}{(l+1)^n}-f(x),\frac{t}{2}\right)\\
&\leq\coprod_{j=0}^{\infty}N\left(x,\frac{((l+1)-\alpha)t}{2(l+1)}\right),
\end{aligned}\tag{2.21}$$

for all $x\in\mathcal{A}$ and $t>0$ and sufficiently large $n$.

In order to prove the uniqueness of $\mathcal{L}$, we assume that $T$ is another additive mapping from $\mathcal{A}$ to $\mathcal{B}$, which satisfies the inequality (2.5). Then

$$\begin{aligned}\mu(\mathcal{L}(x)-T(x),t) &\geq \mu\left(\mathcal{L}(x)-f(x),\frac{t}{2}\right)*\mu\left(T(x)-f(x),\frac{t}{2}\right)\\ &\geq \prod_{j=0}^{\infty} M\left(x,\frac{((l+1)-\alpha)t}{4(l+1)}\right),\\ \nu(\mathcal{L}(x)-T(x),t) &\leq \nu\left(\mathcal{L}(x)-f(x),\frac{t}{2}\right)\diamond\nu\left(T(x)-f(x),\frac{t}{2}\right)\\ &\leq \coprod_{j=0}^{\infty} N\left(x,\frac{((l+1)-\alpha)t}{4(l+1)}\right),\end{aligned}\tag{2.22}$$

for all $x\in\mathcal{A}$ and $t>0$. Therefore, due to the additivity of $\mathcal{L}$ and $T$, we obtain that

$$\begin{aligned}\mu(\mathcal{L}(x)-T(x),t) &= \mu\left(\mathcal{L}\left((l+1)^n x\right)-T\left((l+1)^n x\right),(l+1)^n t\right)\\ &\geq \prod_{j=0}^{\infty} M\left(x,\left(\frac{l+1}{\alpha}\right)^n\frac{((l+1)-\alpha)t}{4(l+1)}\right),\\ \nu(\mathcal{L}(x)-T(x),t) &= \nu\left(\mathcal{L}\left((l+1)^n x\right)-T\left((l+1)^n x\right),(l+1)^n t\right)\\ &\leq \coprod_{j=0}^{\infty} M\left(x,\left(\frac{l+1}{\alpha}\right)^n\frac{((l+1)-\alpha)t}{4(l+1)}\right).\end{aligned}\tag{2.23}$$

Since $0<\alpha<l+1,\ \lim_{n\to\infty}((l+1)/\alpha)^n=\infty$, and we get

$$\lim_{n\to\infty} M\left(x,\left(\frac{l+1}{\alpha}\right)^n\frac{((l+1)-\alpha)t}{4(l+1)}\right)=1,\qquad \lim_{n\to\infty} N\left(x,\left(\frac{l+1}{\alpha}\right)^n\frac{((l+1)-\alpha)t}{4(l+1)}\right)=0,\tag{2.24}$$

that is, $\mu(\mathcal{L}(x)-T(x),t)=1$ and $\nu(\mathcal{L}(x)-T(x),t)=0$ for all $x\in\mathcal{A},\ t>0$. So $\mathcal{L}=T$, which completes the proof. □

In particular, we can prove the preceding result for the case when $\alpha>l+1$. In this case, the mapping $\mathcal{L}(x):=(\mu,\nu)-\lim_{n\to\infty}(l+1)^n f((l+1)^{-n}x)$. We now establish a generalized Hyers-Ulam stability in intuitionistic fuzzy Banach algebra for the higher ring derivation.

**Theorem 2.2.** *Let $\mathcal{A}$ be an algebra, and let $F=\{f_0,f_1,\ldots,f_k,\ldots\}$ be a sequence of mappings from $\mathcal{A}$ to an intuitionistic fuzzy Banach algebra $(\mathcal{B},\mu,\nu,*,\diamond)$ with $f_k(0)=0$ for each $k=0,1,\ldots$. Suppose* *that* $\varphi$ *is a function from* $\mathcal{A}$ *to an intuitionistic fuzzy normed algebra* $(C, \mu', \nu', *, \diamond)$ *such that for each* $k = 0, 1, \ldots,$

$$\mu\left(lf_k\left(\frac{x+y}{l}\right) - f_k(x) - f_k(y), t+s\right) \geq \mu'(\varphi(x), t) * \mu'(\varphi(y), s),$$
$$\nu\left(lf_k\left(\frac{x+y}{l}\right) - f_k(x) - f_k(y), t+s\right) \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(y), s) \tag{2.25}$$

*for all* $x, y \in \mathcal{A} \setminus \{0\}, t > 0$ *and* $s > 0$*, and that* $\Phi$ *is a function from* $\mathcal{A}$ *to an intuitionistic fuzzy normed space* $(D, \mu'', \nu'', *, \diamond)$ *such that for each* $k = 0, 1, \ldots,$

$$\mu\left(f_k(xy) - \sum_{i=0}^{k} f_i(x) f_{k-i}(y), t+s\right) \geq \max\{\mu''(\Phi(x), t), \mu''(\Phi(y), s)\},$$
$$\nu\left(f_k(xy) - \sum_{i=0}^{k} f_i(x) f_{k-i}(y), t+s\right) \leq \min\{\nu''(\Phi(x), t), \nu''(\Phi(y), s)\} \tag{2.26}$$

*for all* $x, y \in \mathcal{A}, t > 0$*, and* $s > 0$*. If* $l > 1$ *is a fixed integer,* $\varphi((l+1)x) = \alpha\varphi(x)$*, and* $\Phi((l+1)x) = \beta\Phi(x)$ *for some real numbers* $\alpha$ *and* $\beta$ *with* $0 < |\alpha| < l+1$ *and* $0 < |\beta| < l+1$*, then there exists a unique higher ring derivation* $H = \{\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_k, \ldots\}$ *of any rank such that for each* $k = 0, 1, \ldots,$

$$\mu(\mathcal{L}_k(x) - f_k(x), t) \geq M\left(x, \frac{((l+1)-\alpha)t}{2(l+1)}\right),$$
$$\nu(\mathcal{L}_k(x) - f_k(x), t) \leq N\left(x, \frac{((l+1)-\alpha)t}{2(l+1)}\right), \tag{2.27}$$

*for all* $x \in \mathcal{A}$ *and* $t > 0$*. In this case,*

$$M(x,t) := \mu'\left(\varphi(x), \frac{l+1}{4}t\right) * \mu'\left(\varphi(-x), \frac{l+1}{4}t\right) * \mu'\left(\varphi((l+1)x), \frac{l+1}{4}t\right),$$
$$N(x,t) := \nu'\left(\varphi(x), \frac{l+1}{4}t\right) \diamond \nu'\left(\varphi(-x), \frac{l+1}{4}t\right) \diamond \nu'\left(\varphi((l+1)x), \frac{l+1}{4}t\right). \tag{2.28}$$

*Moreover, the identity*

$$\sum_{i=0}^{k} \mathcal{L}_i(y)\{\mathcal{L}_{k-i}(y) - f_{k-i}(y)\} = 0 \tag{2.29}$$

*holds for each* $k = 0, 1, \ldots$ *and all* $x, y \in \mathcal{A}$*.*

*Proof.* It follows by Theorem 2.1 that for each $k = 0, 1, \ldots$ and all $x \in \mathcal{A}$, there exists a unique additive mapping $\mathcal{L}_k : \mathcal{A} \to \mathcal{B}$ given by

$$\mathcal{L}_k(x) := (\mu, \nu) - \lim_{n\to\infty} \frac{f_k((l+1)^n x)}{(l+1)^n}, \tag{2.30}$$

satisfying (2.27) since $(C, \mu', \nu', *, \diamond)$ is an intuitionistic fuzzy normed algebra.

Without loss of generality, we suppose that $0 < \beta < l + 1$. Now, we need to prove that the sequence $H = \{\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_k, \ldots\}$ satisfies the identity $\mathcal{L}_k(xy) = \sum_{i=0}^{k} \mathcal{L}_i(x)\mathcal{L}_{k-i}(y)$ for each $k = 0, 1, \ldots$ and all $x \in \mathcal{A}$. It is observed that for each $k = 0, 1, \ldots$,

$$\begin{aligned}
&\mu\left(\mathcal{L}_k(xy) - \sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y), t\right)\\
&\quad\geq \mu\left(\mathcal{L}_k(xy) - \frac{f_k((l+1)^n xy)}{(l+1)^n}, \frac{t}{3}\right) * \mu\left(\frac{f_k((l+1)^n xy)}{(l+1)^n} - \sum_{i=0}^{k}\frac{f_i((l+1)^n x)}{(l+1)^n} f_{k-i}(y), \frac{t}{3}\right)\\
&\qquad * \mu\left(\sum_{i=0}^{k}\frac{f_i((l+1)^n x)}{(l+1)^n} f_{k-i}(y) - \sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y), \frac{t}{3}\right),\\
&\nu\left(\mathcal{L}_k(xy) - \sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y), t\right)\\
&\quad\leq \nu\left(\mathcal{L}_k(xy) - \frac{f_k((l+1)^n xy)}{(l+1)^n}, \frac{t}{3}\right) \diamond \nu\left(\frac{f_k((l+1)^n xy)}{(l+1)^n} - \sum_{i=0}^{k}\frac{f_i((l+1)^n x)}{(l+1)^n} f_{k-i}(y), \frac{t}{3}\right)\\
&\qquad \diamond \nu\left(\sum_{i=0}^{k}\frac{f_i((l+1)^n x)}{(l+1)^n} f_{k-i}(y) - \sum_{i=0}^{k}\mathcal{L}_{\mathrm{i}}(x)f_{k-i}(y), \frac{t}{3}\right)
\end{aligned} \tag{2.31}$$

for all $x, y \in \mathcal{A}$ and $t > 0$. On account of (2.26), we see that for each $k = 0, 1, \ldots$,

$$\begin{aligned}
&\mu\left(\frac{f_k((l+1)^n x\cdot y)}{(l+1)^n} - \sum_{i=0}^{k}\frac{f_i((l+1)^n x)}{(l+1)^n} f_{k-i}(y), \frac{t}{3}\right)\\
&\quad= \mu\left(f_k((l+1)^n x\cdot y) - \sum_{i=0}^{k} f_i((l+1)^n x) f_{k-i}(y), \frac{(l+1)^n t}{3}\right)\\
&\quad\geq \max\left\{\mu''\left(\Phi(x), \left(\frac{l+1}{\beta}\right)^n \frac{t}{6}\right), \mu''\left(\Phi(y), \frac{(l+1)^n t}{6}\right)\right\},
\end{aligned}$$

$$\begin{aligned}
&\nu\left(\frac{f_k((l+1)^n x\cdot y)}{(l+1)^n}-\sum_{i=0}^{k}\frac{f_i((l+1)^n x)}{(l+1)^n}f_{k-i}(y),\frac{(l+1)^n t}{3}\right)\\
&\quad=\nu\left(f_k((l+1)^n x\cdot y)-\sum_{i=0}^{k}f_i((l+1)^n x)f_{k-i}(y),\frac{(l+1)^n t}{3}\right)\\
&\quad\le\min\left\{\mu''\left(\Phi(x),\left(\frac{l+1}{\beta}\right)^n\frac{t}{6}\right),\nu''\left(\Phi(y),\frac{(l+1)^n t}{6}\right)\right\},
\end{aligned}\tag{2.32}$$

for all $x,y\in\mathcal{A}$ and $t>0$. Due to additivity of $\mathcal{L}_k$, for each $k=0,1,\ldots$,

$$\begin{aligned}
&\mu\left(\sum_{i=0}^{k}\frac{f_i((l+1)^n x)}{(l+1)^n}f_{k-i}(y)-\sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y),\frac{t}{3}\right)\\
&\quad\ge\prod_{i=0}^{k}\mu\left(f_i((l+1)^n x)f_{k-i}(y)-(l+1)^n\mathcal{L}_i(x)f_{k-i}(y),\frac{(l+1)^n t}{3(k+1)}\right),\\
&\nu\left(\sum_{i=0}^{k}\frac{f_i((l+1)^n x)}{(l+1)^n}f_{k-i}(y)-\sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y),\frac{t}{3}\right)\\
&\quad\le\coprod_{i=0}^{k}\nu\left(f_i((l+1)^n x)f_{k-i}(y)-(l+1)^n\mathcal{L}_i(x)f_{k-i}(y),\frac{(l+1)^n t}{3(k+1)}\right)
\end{aligned}\tag{2.33}$$

for all $x,y\in\mathcal{A}$ and $t>0$. In addition, we feel that

$$\begin{aligned}
&\mu\left(f_i((l+1)^n x)f_{k-i}(y)-(l+1)^n\mathcal{L}_i(x)f_{k-i}(y),\frac{(l+1)^n t}{3(k+1)}\right)\\
&\quad\ge\max\left\{\mu\left(f_i((l+1)^n x)-(l+1)^n\mathcal{L}_i(x),\frac{(l+1)^n t}{6(k+1)}\right),\mu\left(f_{k-i}(y),\frac{(l+1)^n t}{6(k+1)}\right)\right\},\\
&\nu\left(f_i((l+1)^n x)f_{k-i}(y)-(l+1)^n\mathcal{L}_i(x)f_{k-i}(y),\frac{(l+1)^n t}{3(k+1)}\right)\\
&\quad\le\min\left\{\nu\left(f_i((l+1)^n x)-(l+1)^n\mathcal{L}_i(x),\frac{(l+1)^n t}{6(k+1)}\right),\nu\left(f_{k-i}(y),\frac{(l+1)^n t}{6(k+1)}\right)\right\}.
\end{aligned}\tag{2.34}$$

Letting $n\to\infty$ in (2.31), (2.32), (2.33), and (2.34), we get $\mu(\mathcal{L}_k(xy)-\sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y),t)=1$ and $\nu(\mathcal{L}_k(xy)-\sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y),t)=0$. This implies that

$$\mathcal{L}_k(xy)=\sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y),\tag{2.35}$$

for each $k=0,1,\ldots$ and all $x,y\in\mathcal{A}$.

Using additivity of $\mathcal{L}_k$ and (2.35), we find that

$$(l+1)^n \sum_{i=0}^{k} \mathcal{L}_i(x) f_{k-i}(y) = \mathcal{L}_k\left((l+1)^n x \cdot y\right) = \mathcal{L}_k\left(x \cdot (l+1)^n y\right) = \sum_{i=0}^{k} \mathcal{L}(x) f\left((l+1)^n y\right). \tag{2.36}$$

So we obtain $\sum_{i=0}^{k} \mathcal{L}_i(x) f_{k-i}(y) = \sum_{i=0}^{k} \mathcal{L}_i(x)(f_{k-i}((l+1)^n y)/(l+1)^n)$. Hence for each $k = 0, 1, \ldots,$

$$\begin{aligned} &\mu\left(\sum_{i=0}^{k} \mathcal{L}_i(x) f_{k-i}(y) - \sum_{i=0}^{k} \mathcal{L}_i(x) \frac{f_{k-i}\left((l+1)^n y\right)}{(l+1)^n}, t\right) = 1, \\ &\nu\left(\sum_{i=0}^{k} \mathcal{L}_i(x) f_{k-i}(y) - \sum_{i=0}^{k} \mathcal{L}_i(x) \frac{f_{k-i}\left((l+1)^n y\right)}{(l+1)^n}, t\right) = 0, \end{aligned} \tag{2.37}$$

for all $x, y \in \mathcal{A}$ and $t > 0$. This relation yields that for each $k = 0, 1, \ldots,$

$$\begin{aligned} &\mu\left(\sum_{i=0}^{k} \mathcal{L}_i(x) \mathcal{L}_{k-i}(y) - \sum_{i=0}^{k} \mathcal{L}_i(x) f_{k-i}(y), t\right) \\ &\quad \geq \mu\left(\sum_{i=0}^{k} \mathcal{L}_i(x) \mathcal{L}_{k-i}(y) - \sum_{i=0}^{k} \mathcal{L}_i(x) \frac{f_{k-i}\left((l+1)^n y\right)}{(l+1)^n}, \frac{t}{2}\right) \\ &\qquad * \mu\left(\sum_{i=0}^{k} \mathcal{L}_i(x) \frac{f_{k-i}\left((l+1)^n y\right)}{(l+1)^n} - \sum_{i=0}^{k} \mathcal{L}_i(x) f_{k-i}(y), \frac{t}{2}\right) \\ &\quad \geq \prod_{i=0}^{k} \mu\left(\mathcal{L}_i(x) \mathcal{L}_{k-i}(y) - \mathcal{L}_i(x) \frac{f_{k-i}\left((l+1)^n y\right)}{(l+1)^n}, \frac{t}{2(k+1)}\right), \end{aligned} \tag{2.38}$$

$$\begin{aligned} &\nu\left(\sum_{i=0}^{k} \mathcal{L}_i(x) \mathcal{L}_{k-i}(y) - \sum_{i=0}^{k} \mathcal{L}_i(x) f_{k-i}(y), t\right) \\ &\quad \leq \nu\left(\sum_{i=0}^{k} \mathcal{L}_i(x) \mathcal{L}_{k-i}(y) - \sum_{i=0}^{k} \mathcal{L}_i(x) \frac{f_{k-i}\left((l+1)^n y\right)}{(l+1)^n}, \frac{t}{2}\right) \\ &\qquad \diamond \nu\left(\sum_{i=0}^{k} \mathcal{L}_i(x) \frac{f_{k-i}\left((l+1)^n y\right)}{(l+1)^n} - \sum_{i=0}^{k} \mathcal{L}_i(x) f_{k-i}(y), \frac{t}{2}\right) \\ &\quad \leq \coprod_{i=0}^{k} \nu\left(\mathcal{L}_i(x) \mathcal{L}_{k-i}(y) - \mathcal{L}_i(x) \frac{f_{k-i}\left((l+1)^n y\right)}{(l+1)^n}, \frac{t}{2(k+1)}\right), \end{aligned} \tag{2.39}$$

for all $x, y \in \mathcal{A}$ and $t > 0$. On the other hand, we see that

$$\begin{aligned}
&\mu\left(\mathcal{L}_i(x)\mathcal{L}_{k-i}(y) - \mathcal{L}_i(x)\frac{f_{k-i}((l+1)^n y)}{(l+1)^n}, \frac{t}{2(k+1)}\right) \\
&\quad \geq \max\left\{\mu\left(\mathcal{L}_i(x), \frac{(l+1)^n t}{4(k+1)}\right), \mu\left(\mathcal{L}_{k-i}(y) - \frac{f_{k-i}((l+1)^n y)}{(l+1)^n}, \frac{t}{4(k+1)}\right)\right\}, \\
&\mu\left(\mathcal{L}_i(x)\mathcal{L}_{k-i}(y) - \mathcal{L}_i(x)\frac{f_{k-i}((l+1)^n y)}{(l+1)^n}, \frac{t}{2(k+1)}\right) \\
&\quad \leq \min\left\{\nu\left(\mathcal{L}_i(x), \frac{(l+1)^n t}{4(k+1)}\right), \nu\left(\mathcal{L}_{k-i}(y) - \frac{f_{k-i}((l+1)^n y)}{(l+1)^n}, \frac{t}{4(k+1)}\right)\right\}.
\end{aligned} \tag{2.40}$$

Sending $n \to \infty$ in (2.38) and (2.40), we have that for each $k = 0, 1, \ldots,$

$$\begin{aligned}
&\mu\left(\sum_{i=0}^{k}\mathcal{L}_i(x)\mathcal{L}_{k-i}(y) - \sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y), t\right) = 1, \\
&\nu\left(\sum_{i=0}^{k}\mathcal{L}_i(x)\mathcal{L}_{k-i}(y) - \sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y), t\right) = 0,
\end{aligned} \tag{2.41}$$

for all $x, y \in \mathcal{A}$ and $t > 0$. Thus, we conclude that

$$\sum_{i=0}^{k}\mathcal{L}_i(x)\mathcal{L}_{k-i}(y) = \sum_{i=0}^{k}\mathcal{L}_i(x)f_{k-i}(y), \tag{2.42}$$

for each $k = 0, 1, \ldots$ and all $x, y \in \mathcal{A}$.

Therefore, by combining (2.35) and (2.42), we get the required result, which completes the proof. □

As a consequence of Theorem 2.2, we get the following superstability.

**Corollary 2.3.** *Let $(\mathcal{B}, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy Banach algebra with unit, and let a sequence of operators $F = \{f_0, f_1, \ldots, f_k, \ldots\}$ on $\mathcal{A}$ satisfy $f_k(0) = 0$ for each $k = 0, 1, \ldots$, where $f_0$ is an identity operator. Suppose that $\varphi$ is a function from $\mathcal{A}$ to an intuitionistic fuzzy normed algebra $(C, \mu', \nu', *, \diamond)$ satisfying* (2.25) *and (2.14) and that $\Phi$ is a function from $\mathcal{A}$ to an intuitionistic fuzzy normed space $(D, \mu'', \nu'', *, \diamond)$ satisfying* (2.26). *If $l > 1$ is a fixed integer, $\varphi((l+1)x) = \alpha\varphi(x)$, and $\Phi((l+1)x) = \beta\Phi(x)$ for some real numbers $\alpha$ and $\beta$ with $0 < |\alpha| < l+1$ and $0 < |\beta| < l+1$, then $F$ is a strong higher ring derivation on $\mathcal{A}$.*

*Proof.* According to (2.30), we have $\mathcal{L}_0(x) = x$ for all $x \in \mathcal{A}$, and so $\mathcal{L}_0(=f_0)$ is an identity operator on $\mathcal{A}$. By induction, we get the conclusion. If $k = 1$, then it follows from (2.29) that $f_1(\mathrm{x}) = \mathcal{L}_1(x)$ holds for all $x \in \mathcal{A}$ since $\mathcal{A}$ contains the unit element. Let us assume that $f_m(x) = \mathcal{L}_m(x)$ is valid for all $x \in \mathcal{A}$ and $m < k$. Then (2.29) implies that $x\{\mathcal{L}_m(y) - f_m(y)\} = 0$ for all $x, y \in \mathcal{A}$. Since $\mathcal{A}$ has the unit element, $f_k(y) = \mathcal{L}_k(y)$ for all $x \in \mathcal{A}$. Hence we conclude that $f_k(y) = \mathcal{L}_k(y)$ for each $k = 0, 1, 2, \ldots$ and all $x \in \mathcal{A}$. So this tells us that $F$ is a higher ring derivation of any rank from $\mathcal{A}$ and $\mathcal{B}$. The proof of the corollary is complete. □

We remark that we can prove the preceding result for the case when $\alpha > l + 1$ and $\beta > l + 1$.

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*Research Article*

# The Hyers-Ulam-Rassias Stability of $(m,n)_{(\sigma,\tau)}$-Derivations on Normed Algebras

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We study the Hyers-Ulam-Rassias stability of $(m,n)_{(\sigma,\tau)}$-derivations on normed algebras.

## 1. Introduction

A classical question in the theory of functional equations is as follows. *Under what conditions is it true that a mapping which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$*? This problem was formulated by Ulam in 1940 (see [1, 2]). He investigated the stability of group homomorphisms. *Let $(\mathcal{G}_1,\circ)$ be a group, and let $(\mathcal{G}_2,*,\delta)$ be a metric group with a metric $\delta(\cdot,\cdot)$. Suppose that $f:\mathcal{G}_1\to\mathcal{G}_2$ is a map and $\epsilon>0$ a fixed scalar. Does there exists $\lambda>0$such that if $f$ satisfies the inequality*

$$\delta(f(x\circ y),f(x)*f(y))\le\lambda \tag{1.1}$$

*for all $x,y\in\mathcal{G}_1$, then there exists a group homomorphism $F:\mathcal{G}_1\to\mathcal{G}_2$ with the property*

$$\delta(f(x),F(x))\le\epsilon \tag{1.2}$$

*for all $x\in\mathcal{G}_1$?*

One year later, Ulam's problem was affirmatively solved by Hyers [3] for the Cauchy functional equation $f(x+y)=f(x)+f(y)$.: *Let $\mathcal{X}_1$ be a normed space, $\mathcal{X}_2$ a Banach space, and $\epsilon>0$ a fixed scalar. Suppose that $f:\mathcal{X}_1\to\mathcal{X}_2$ is a map with the property*

$$\|f(x+y)-f(x)-f(y)\|<\epsilon \tag{1.3}$$

*for all* $x, y \in \mathcal{X}_1$. *Then there exists a unique additive mapping* $F : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ *such that*

$$\| f(x) - F(x) \| < \epsilon \tag{1.4}$$

*for all* $x \in \mathcal{X}_1$. This gave rise to the stability theory of functional equations.

The famous Hyers stability result has been generalized in the stability of additive mappings involving a sum of powers of norms by Aoki [4] which allowed the Cauchy difference to be unbounded. In 1978, Rassias [5] proved the stability of linear mappings in the following way. *Let* $\mathcal{X}_1$ *be a real normed space and* $\mathcal{X}_2$ *a real Banach space. If there exist scalars* $\epsilon \geq 0$ *and* $0 \leq p < 1$ *such that*

$$\| f(x+y) - f(x) - f(y) \| \leq \epsilon \left( \|x\|^p + \|y\|^p \right) \tag{1.5}$$

*for all* $x, y \in \mathcal{X}_1$, *then there exists a unique additive mapping* $F : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ *with the property*

$$\| f(x) - F(y) \| \leq \frac{2\epsilon}{2 - 2^p} \|x\|^p \tag{1.6}$$

*for all* $x \in \mathcal{X}_1$. *Moreover, if the map* $r \mapsto f(rx)$ *is continuous on* $\mathbb{R}$ *for each* $x \in \mathcal{X}_1$, *then* $F$ *is linear.* This result has provided a lot of influence in the development of what we now call the Hyers-Ulam-Rassias stability of functional equations.

Later, Găvruţa [6] generalized the Rassias' theorem as follows: *Let* $(\mathcal{G}, +)$ *be an Abelian group and* $\mathcal{X}$ *a Banach space. Suppose that the so-called admissible control function* $\varphi : \mathcal{G} \times \mathcal{G} \longrightarrow [0, \infty)$ *satisfies*

$$\sum_{k=0}^{\infty} \frac{\varphi(2^k x, 2^k y)}{2^{k+1}} < \infty \tag{1.7}$$

*for all* $x, y \in \mathcal{G}$. If $f : \mathcal{G} \longrightarrow \mathcal{X}$ *is a mapping with the property*

$$\| f(x+y) - f(x) - f(y) \| \leq \varphi(x, y) \tag{1.8}$$

*for all* $x, y \in \mathcal{G}$, *then there exists a unique additive mapping* $F : \mathcal{G} \longrightarrow \mathcal{X}$ *such that*

$$\| f(x) - F(x) \| \leq \sum_{k=0}^{\infty} \frac{\varphi(2^k x, 2^k y)}{2^{k+1}} \tag{1.9}$$

*for all* $x \in \mathcal{G}$.

In the last few decades, various approaches to the problem have been introduced by several authors. Moreover, it is surprising that in some cases the *approximate mapping* is actually a *true mapping*. In such cases we call the equation $\mathcal{E}$ superstable. For the history and various aspects of this theory we refer the reader to monographs [7–9].

As we are aware, the stability of derivations was first investigated by Jun and Park [10]. During the past few years, approximate derivations were studied by a number of mathematicians (see [11–18] and references therein).

Moslehian [19] studied the stability of $(\sigma,\tau)$-derivations and generalized some results obtained in [18]. He also established the generalized Hyers-Ulam-Rassias stability of $(\sigma,\tau)$-derivations on normed algebras into Banach bimodules. This motivated us to investigate approximate $(m,n)_{(\sigma,\tau)}$-derivations on normed algebras. The aim of this paper is to study the stability of $(m,n)_{(\sigma,\tau)}$-derivations and to generalize some results given in [19].

## 2. Preliminaries

Throughout, $\mathcal{A}$ will be a normed algebra and $\mathcal{M}$ a Banach $\mathcal{A}$-bimodule. Let $\sigma$ and $\tau$ be two linear operators on $\mathcal{A}$. An additive mapping $d:\mathcal{A}\to\mathcal{M}$ is called an $(\sigma,\tau)$-derivation if

$$d(xy)=d(x)\sigma(y)+\tau(x)d(y) \tag{2.1}$$

holds for all $x,y\in\mathcal{A}$. Ordinary derivations from $\mathcal{A}$ to $\mathcal{M}$ and maps defined by $x\mapsto a\sigma(x)-\tau(x)a$, where $a\in\mathcal{A}$ is a fixed element and $\sigma,\tau$ are endomorphisms on $\mathcal{A}$, are natural examples of $(\sigma,\tau)$-derivations on $\mathcal{A}$. Moreover, if $\psi$ is an endomorphism on $\mathcal{A}$, then $\psi$ is a $((1/2)\psi,(1/2)\psi)$-derivation on $\mathcal{A}$. We refer the reader to [20], where further information about $(\sigma,\tau)$-derivations can be found.

In [19] Moslehian studied stability of $(\sigma,\tau)$-derivations. The natural question here is, whether the analogue results hold true for $(m,n)_{(\sigma,\tau)}$-derivations. Theorem 3.1 answers this question in the affirmative.

Let $m$ and $n$ be nonnegative integers with $m+n\neq 0$. An additive mapping $d:\mathcal{A}\to\mathcal{M}$ is called a $(m,n)_{(\sigma,\tau)}$-derivation if

$$(m+n)d(xy)=2md(x)\sigma(y)+2n\tau(x)d(y) \tag{2.2}$$

holds for all $x,y\in\mathcal{A}$. Clearly, $(m,n)_{(\sigma,\tau)}$-derivations are one of the natural generalizations of $(\sigma,\tau)$-derivations (the case $m=n$). If $\sigma,\tau=id$, where $id$ denotes the identity map on $\mathcal{A}$, and an additive mapping $d:\mathcal{A}\to\mathcal{M}$ satisfies (2.2), then $d$ is called a $(m,n)$-derivation. In the last few decades a lot of work has been done on the field of $(m,n)$-derivations on rings and algebras (see, e.g, [21–25]). This motivated us to study the Hyers-Ulam-Rassias stability of functional inequalities associated with $(m,n)_{(\sigma,\tau)}$-derivations.

In the following, we will assume that $m$ and $n$ are nonnegative integers with $m+n\neq 0$. We will use the same symbol $\|\cdot\|$ in order to represent the norms on a normed algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule $\mathcal{M}$. For a given (admissible control) function $\varphi:\mathcal{A}\times\mathcal{A}\to[0,\infty)$ we will use the following abbreviation:

$$\phi(x,y):=\sum_{k=0}^{\infty}\frac{\varphi(2^kx,2^ky)}{2^{k+1}},\quad x,y\in\mathcal{A}. \tag{2.3}$$

Let us start with one well-known lemma.

**Lemma 2.1** (see [6]). *Suppose that a function $\varphi:\mathcal{A}\times\mathcal{A}\to[0,\infty)$ satisfies $\phi(x,y)<\infty$, $x,y\in\mathcal{A}$. If $f:\mathcal{A}\to\mathcal{M}$ is a mapping with*

$$\|f(x+y)-f(x)-f(y)\|\leq\varphi(x,y) \tag{2.4}$$

*for all $x, y \in \mathcal{A}$, then there exists a unique additive mapping $F : \mathcal{A} \to \mathcal{M}$ such that*

$$\|f(x) - F(x)\| \leq \phi(x, x) \tag{2.5}$$

*for all $x \in \mathcal{A}$.*

We say that an additive mapping $f : \mathcal{A} \to \mathcal{M}$ is $\mathbb{C}$-linear if $f(\lambda x) = \lambda f(x)$ for all $x \in \mathcal{A}$ and all scalars $\lambda \in \mathbb{C}$. In the following, $\Lambda$ will denote the set of all complex units, that is,

$$\Lambda = \{\lambda \in C : |\lambda| = 1\}. \tag{2.6}$$

For a given additive mapping $f : \mathcal{A} \to \mathcal{M}$, Park [26] obtained the next result.

**Lemma 2.2.** *If $f(\lambda x) = \lambda f(x)$ for all $x \in \mathcal{A}$ and all $\lambda \in \Lambda$, then $f$ is $\mathbb{C}$-linear.*

## 3. The Results

Our first result is a generalization of [19, Theorem 2.1] (the case $m = n$). We use the direct method to construct a unique $\mathbb{C}$-linear mapping from an approximate one and prove that this mapping is an appropriate $(m, n)_{(\sigma,\tau)}$-derivation on $\mathcal{A}$. This method was first devised by Hyers [3]. The idea is taken from [19].

**Theorem 3.1.** *Let $d : \mathcal{A} \to \mathcal{M}$ and $f, g : \mathcal{A} \to \mathcal{A}$ be mappings with $d(0) = f(0) = g(0) = 0$. Suppose that there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that $\phi(x, y) < \infty$ for all $x, y \in \mathcal{A}$ and*

$$\|d(\lambda x + \lambda y) - \lambda d(x) - \lambda d(y)\| \leq \varphi(x, y), \tag{3.1}$$

$$\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y)\| \leq \varphi(x, y), \tag{3.2}$$

$$\|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| \leq \varphi(x, y), \tag{3.3}$$

$$\|(m + n)d(xy) - 2md(x)f(y) - 2ng(x)d(y)\| \leq \varphi(x, y) \tag{3.4}$$

*for all $x, y \in \mathcal{A}$ and $\lambda \in \Lambda$. Then there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\|f(x) - \sigma(x)\| \leq \phi(x, x), \qquad \|g(x) - \tau(x)\| \leq \phi(x, x) \tag{3.5}$$

*for all $x \in \mathcal{A}$, and a unique $\mathbb{C}$-linear $(m, n)_{(\sigma,\tau)}$-derivation $D : \mathcal{A} \to \mathcal{M}$ such that*

$$\|d(x) - D(x)\| \leq \phi(x, x) \tag{3.6}$$

*for all $x \in \mathcal{A}$.*

*Proof.* Taking $\lambda = 1$ in (3.1) and using Lemma 2.1, it follows that there exists a unique additive mapping $D : \mathcal{A} \to \mathcal{M}$ such that $\|d(x) - D(x)\| \le \phi(x,x)$ holds for all $x \in \mathcal{A}$. More precisely, using the induction, it is easy to see that

$$\left\| \frac{d(2^l x)}{2^l} - d(x) \right\| \le \sum_{k=0}^{l-1} \frac{\varphi(2^k x, 2^k x)}{2^{k+1}}, \tag{3.7}$$

$$\left\| \frac{d(2^p x)}{2^p} - \frac{d(2^q x)}{2^q} \right\| \le \sum_{k=q}^{p-1} \frac{\varphi(2^k x, 2^k x)}{2^{k+1}} \tag{3.8}$$

for all $x \in \mathcal{A}$, all positive integers $l$, and all $0 \le q < p$. According to the assumptions on $\phi(x,y)$, it follows that the sequence $\{d(2^k x)/2^k\}_{k=0}^{\infty}$ is Cauchy. Thus, by the completeness of $\mathcal{M}$, this sequence is convergent and we can define a map $D : \mathcal{A} \to \mathcal{M}$ as

$$D(x) := \lim_{k\to\infty} \frac{d(2^k x)}{2^k}, \quad x \in \mathcal{A}. \tag{3.9}$$

Using (3.1), we get

$$\begin{aligned} &\|D(\lambda x + \lambda y) - \lambda D(x) - \lambda D(y)\| \\ &\quad = \lim_{k\to\infty} 2^{-k} \left\| d\left(\lambda 2^k x + \lambda 2^k y\right) - \lambda d\left(2^k x\right) - \lambda d\left(2^k y\right) \right\| \\ &\quad \le \lim_{k\to\infty} 2^{-k} \varphi\left(2^k x, 2^k y\right) = 0. \end{aligned} \tag{3.10}$$

This yields that

$$D(\lambda x + \lambda y) = \lambda D(x) + \lambda D(y) \tag{3.11}$$

for all $x, y \in \mathcal{A}$ and $\lambda \in \Lambda$. Using Lemma 2.2, it follows that the map $D$ is $\mathbb{C}$-linear. Moreover, according to inequality (3.7), we have

$$\|d(x) - D(x)\| = \lim_{k\to\infty} \left\| d(x) - \frac{d(2^k x)}{2^k} \right\| \le \phi(x,x) \tag{3.12}$$

for all $x \in \mathcal{A}$.

Next, we have to show the uniqueness of $D$. So, suppose that there exists another $\mathbb{C}$-linear mapping $\widetilde{D} : \mathcal{A} \to \mathcal{M}$ such that $\|d(x) - \widetilde{D}(x)\| \leq \phi(x, x)$ for all $x \in \mathcal{A}$. Then

$$\begin{aligned}
\left\|D(x) - \widetilde{D}(x)\right\| &= \lim_{k\to\infty} 2^{-k}\left\|d\left(2^k x\right) - \widetilde{D}\left(2^k x\right)\right\| \\
&\leq \lim_{k\to\infty} 2^{-k}\phi\left(2^k x, 2^k x\right) \\
&= \lim_{k\to\infty} 2^{-k}\sum_{j=0}^{\infty}\frac{\varphi(2^{j+k}x, 2^{j+k}x)}{2^{j+1}} \\
&= \lim_{k\to\infty}\sum_{j=k}^{\infty}\frac{\varphi(2^j x, 2^j x)}{2^{j+1}} = 0.
\end{aligned} \tag{3.13}$$

Therefore, $D(x) = \widetilde{D}(x)$ for all $x \in \mathcal{A}$, as desired.

Similarly we can show that there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned}
\sigma(x) &:= \lim_{k\to\infty}\frac{f(2^k x)}{2^k}, \quad x \in \mathcal{A}, \\
\tau(x) &:= \lim_{k\to\infty}\frac{g(2^k x)}{2^k}, \quad x \in \mathcal{A}.
\end{aligned} \tag{3.14}$$

Furthermore,

$$\|f(x) - \sigma(x)\| \leq \phi(x, x), \qquad \|g(x) - \tau(x)\| \leq \phi(x, x) \tag{3.15}$$

for all $x \in \mathcal{A}$.

It remains to prove that $D$ is an $(m, n)_{(\sigma,\tau)}$-derivation. Writing $2^k x$ in the place of $x$ and $2^k y$ in the place of $y$ in (3.4), we obtain

$$\left\|(m+n)d\left(4^k xy\right) - 2md\left(2^k x\right)f\left(2^k y\right) - 2ng\left(2^k x\right)d\left(2^k y\right)\right\| \leq \varphi\left(2^k x, 2^k y\right). \tag{3.16}$$

This yields that

$$\begin{aligned}
&\|((m+n)D(xy) - 2mD(x)\sigma(y) - 2n\tau(x)D(y)\| \\
&\quad = \lim_{k\to\infty} 4^{-k}\left\|(m+n)d\left(4^k xy\right) - 2md\left(2^k x\right)f\left(2^k y\right) - 2ng\left(2^k x\right)d\left(2^k y\right)\right\| \\
&\quad \leq \lim_{k\to\infty} 4^{-k}\varphi\left(2^k x, 2^k y\right) = 0
\end{aligned} \tag{3.17}$$

for all $x, y \in \mathcal{A}$. Thus, mappings $D$ and $\sigma, \tau$ satisfy (2.2). The proof is completed. □

*Remark 3.2.* If there exists $x_0 \in \mathcal{A}$ such that $d$ and the map $x \mapsto \phi(x,x)$ are continuous at point $x_0$, then $D$ is continuous on $\mathcal{A}$. Namely, if $D$ was not continuous, then there would exist an integer $C$ and a sequence $\{x_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} x_k = 0$ and $\|D(x_k)\| > 1/C$, $k \geq 0$. Let $t > C(2\phi(x_0,x_0)+1)$. Then

$$\lim_{k\to\infty} d(tx_k + x_0) = d(x_0) \tag{3.18}$$

since $d$ is continuous at point $x_0$. Thus, there exists an integer $k_0$ such that for every $k > k_0$ we have

$$\|d(tx_k + x_0) - d(x_0)\| < 1. \tag{3.19}$$

Therefore,

$$\begin{aligned} 2\phi(x_0,x_0) + 1 < \frac{t}{C} &< \|D(tx_k)\| = \|D(tx_k + x_0) - D(x_0)\| \\ &\leq \|D(tx_k + x_0) - d(tx_k + x_0)\| + \|d(tx_k + x_0) - d(x_0)\| + \|d(x_0) - D(x_0)\| \\ &< \phi(tx_k + x_0, tx_k + x_0) + 1 + \phi(x_0,x_0) \end{aligned} \tag{3.20}$$

for every $k > k_0$. Letting $k \to \infty$ and using the continuity of the map $x \mapsto \phi(x,x)$ at point $x_0$, we get a contradiction.

Let $\epsilon \geq 0$ and $0 \leq p < 1$. Applying Theorem 3.1 for the case

$$\varphi(x,y) := \epsilon\left(\|x\|^p + \|y\|^p\right), \quad x,y \in \mathcal{A}. \tag{3.21}$$

**Corollary 3.3.** *Let $d : \mathcal{A} \to \mathcal{M}$ and $f, g : \mathcal{A} \to \mathcal{A}$ be mappings with $d(0) = f(0) = g(0) = 0$. Suppose that* (3.1), (3.2), (3.3), *and* (3.4) *hold true for all $x, y \in \mathcal{A}$ and $\lambda \in \Lambda$, where a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ is defined as above. Then there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\|f(x) - \sigma(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p, \quad \|g(x) - \tau(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{3.22}$$

*for all $x \in \mathcal{A}$ and a unique $\mathbb{C}$-linear $(m,n)_{(\sigma,\tau)}$-derivation $D : \mathcal{A} \to \mathcal{M}$ such that*

$$\|d(x) - D(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{3.23}$$

*Proof.* Note that $\phi(x,y) < \infty$ for all $x, y \in \mathcal{A}$ and

$$\phi(x,y) = \frac{\epsilon}{2-2^p}\left(\|x\|^p + \|y\|^p\right), \quad x,y \in \mathcal{A}. \tag{3.24}$$

□

*Remark 3.4.* Recall that we can actually take any map $\varphi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ in the form

$$\varphi(x,y) := \nu + \epsilon\left(\|x\|^p + \|y\|^p\right), \quad x,y \in \mathcal{A}, \tag{3.25}$$

where $\nu \geq 0$. In this case we have

$$\phi(x,y) = \nu + \frac{\epsilon\left(\|x\|^p + \|y\|^p\right)}{(2-2^p)}, \quad x,y \in \mathcal{A}. \tag{3.26}$$

Before stating our next result, let us write one well-known lemma about the continuity of measurable functions (see, e.g., [27]).

**Lemma 3.5.** *If a measurable function $\psi : \mathbb{R} \to \mathbb{R}$ satisfies $\psi(r_1+r_2) = \psi(r_1)+\psi(r_2)$ for all $r_1, r_2 \in \mathbb{R}$, then $\psi$ is continuous.*

Now we are in the position to state a result for normed algebras $\mathcal{A}$ which are spanned by a subset $\mathcal{S}$ of $\mathcal{A}$. For example, $\mathcal{A}$ can be a $C^*$-algebra spanned by the unitary group of $\mathcal{A}$ or the positive part of $\mathcal{A}$

**Theorem 3.6.** *Let $\mathcal{A}$ be a normed algebra which is spanned by a subset $\mathcal{S}$ of $\mathcal{A}$ and $d : \mathcal{A} \to \mathcal{M}$, $f,g : \mathcal{A} \to \mathcal{A}$ mappings with $d(0) = f(0) = g(0) = 0$. Suppose that there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ such that $\phi(x,y) < \infty$ for all $x,y \in \mathcal{A}$ and* (3.1), (3.2), (3.3) *holds true for all $x,y \in \mathcal{A}$ and $\lambda = 1, i$. Moreover, suppose that* (3.4) *holds true for all $x,y \in \mathcal{S}$. If for all $x \in \mathcal{A}$ the functions $r \mapsto d(rx)$, $r \mapsto f(rx)$, and $r \mapsto g(rx)$ are continuous on $\mathbb{R}$, then there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\left\|f(x) - \sigma(x)\right\| \leq \phi(x,x), \qquad \left\|g(x) - \tau(x)\right\| \leq \phi(x,x) \tag{3.27}$$

*for all $x \in \mathcal{A}$ and a unique $\mathbb{C}$-linear $(m,n)_{(\sigma,\tau)}$-derivation $D : \mathcal{A} \to \mathcal{M}$ such that*

$$\|d(x) - D(x)\| \leq \phi(x,x) \tag{3.28}$$

*for all $x \in \mathcal{A}$.*

We will give just a sketch of the proof since most of the steps are the same as in the proof of Theorem 3.1.

*Proof.* As in the proof of Theorem 3.1, we can show that there exists a unique additive mapping $D : \mathcal{A} \to \mathcal{M}$ defined by $D(x) := \lim_{k\to\infty}(d(2^k x)/2^k)$, $x \in \mathcal{A}$. Moreover, $\|d(x) - D(x)\| \leq \phi(x,x)$ for all $x \in \mathcal{A}$.

Writing $y = 0$, $\lambda = i$ in (3.1), we get

$$\|d(ix) - id(x)\| \leq \varphi(x,0). \tag{3.29}$$

Therefore,

$$\|D(ix) - iD(x)\| = \lim_{k\to\infty} 2^{-k}\left\|d\left(2^k ix\right) - id\left(2^k x\right)\right\| \leq \lim_{k\to\infty} 2^{-k}\varphi\left(2^k x, 0\right) = 0. \tag{3.30}$$

This yields that

$$D(ix) = iD(x) \tag{3.31}$$

for all $x \in \mathcal{A}$. In the next step we will show that $D$ is $\mathbb{R}$-linear, that is,

$$D(rx) = rD(x) \tag{3.32}$$

for all $x \in \mathcal{A}$ and all $r \in \mathbb{R}$.

Since $D$ is additive, we have $D(qx) = qx$ for every $x \in \mathcal{A}$ and all rational numbers $q$. Let us fix elements $x_0 \in \mathcal{A}$ and $\rho \in \mathcal{M}^*$, where $\mathcal{M}^*$ denotes the dual space of $\mathcal{M}$. Then we can define a function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(r) = \rho(D(rx_0)), \quad r \in \mathbb{R}. \tag{3.33}$$

Firstly, we would like to prove that $\psi$ is continuous. Recall that

$$\psi(r_1 + r_2) = \rho(D((r_1 + r_2)x_0)) = \rho(D(r_1x_0)) + \rho(D(r_2x_0)) = \psi(r_1) + \psi(r_2) \tag{3.34}$$

for all $r_1, r_2 \in \mathbb{R}$. Furthermore,

$$\psi(r) = \lim_{k\to\infty} \rho\left(\frac{d(2^k r x_0)}{2^k}\right) \tag{3.35}$$

for all $r \in \mathbb{R}$. Set

$$\psi_k(r) = \rho\left(\frac{d(2^k r x_0)}{2^k}\right), \quad k \ge 0. \tag{3.36}$$

Obviously, $\{\psi_k\}_{k=0}^{\infty}$ is a sequence of continuous functions and $\psi$ is its pointwise limit. This yields that $\psi$ is a Borel function and, by Lemma 3.5 it is continuous. Therefore, we have $\psi(r) = r\psi(1)$ for all $r \in \mathbb{R}$. This implies $D(rx_0) = rD(x_0)$. Since $x_0$ was an arbitrary element from $\mathcal{A}$, we proved that $D$ is $\mathbb{R}$-linear.

Now, let $\lambda \in \mathbb{C}$. Then $\lambda = r_1 + ir_2$ for some real numbers $r_1, r_2$. Using (3.31), we have

$$D(\lambda x) = D((r_1 + ir_2)x) = D(r_1x) + D(ir_2x) = r_1D(x) + ir_2D(x) = \lambda D(x) \tag{3.37}$$

for all $x \in \mathcal{A}$. This means that $D$ is $\mathbb{C}$-linear.

Similarly we can show that there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ satisfying

$$\|f(x) - \sigma(x)\| \le \phi(x, x), \qquad \|g(x) - \tau(x)\| \le \phi(x, x) \tag{3.38}$$

for all $x \in \mathcal{A}$. Moreover, (2.2) holds true for all $x, y \in \mathcal{S}$. Since $\mathcal{A}$ is linearly generated by $\mathcal{S}$, we conclude that $D$ is an $(m,n)_{(\sigma,\tau)}$-derivation on $\mathcal{A}$. The proof is completed. □

*Remark 3.7.* As above, we can apply Theorem 3.6 for the case

$$\varphi(x,y) := \nu + \epsilon\left(\|x\|^p + \|y\|^p\right), \quad x, y \in \mathcal{A}, \tag{3.39}$$

where $\nu, \epsilon \geq 0$ and $0 \leq p < 1$.

*Remark 3.8.* If $\epsilon \geq 0$ and $0 \leq p < 1/2$, then we can use in Theorem 3.1 as well as in Theorem 3.6 a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ given by

$$\varphi(x,y) := \epsilon\|x\|^p\|y\|^p, \quad x, y \in \mathcal{A}. \tag{3.40}$$

In this case

$$\phi(x,y) = \frac{\epsilon}{2-4^p}\|x\|^p\|y\|^p, \quad x, y \in \mathcal{A}. \tag{3.41}$$

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*Research Article*

# Ulam-Hyers Stability for Cauchy Fractional Differential Equation in the Unit Disk

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We prove the Ulam-Hyers stability of Cauchy fractional differential equations in the unit disk for the linear and non-linear cases. The fractional operators are taken in sense of Srivastava-Owa operators.

## 1. Introduction

A classical problem in the theory of functional equations is that if a function $f$ approximately satisfies functional equation $\mathcal{E}$, when does there exists an exact solution of $\mathcal{E}$ which $f$ approximates. In 1940, Ulam [1, 2] imposed the question of the stability of the Cauchy equation, and in 1941, Hyers solved it [3]. In 1978, Rassias [4] provided a generalization of Hyers, theorem by proving the existence of unique linear mappings near approximate additive mappings. The problem has been considered for many different types of spaces (see [5–7]). The Ulam-Hyers stability of differential equations has been investigated by Alsina and Ger [8] and generalized by Jung [9–11]. Recently, Li and Shen [12] have investigated the Ulam-Hyers stability of the linear differential equations of second order, Abdollahpour and Najati [13] have studied the Ulam-Hyers stability of the linear differential equations of third order, and Lungu and Popa have imposed the Ulam-Hyers stability of a first-order partial differential equation [14].

The analysis on stability of fractional differential equations is more complicated than the classical differential equations, since fractional derivatives are nonlocal and have weakly singular kernels. Recently, Li and Zhang [15] provided an overview on the stability results of the fractional differential equations. Particularly, Li et al. [16] devoted to study the Mittag-Leffler stability and the Lyapunov's methods, Deng [17] derived sufficient conditions for the local asymptotical stability of nonlinear fractional differential equations, and Li et al. studied the stability of fractional-order nonlinear dynamic systems using the Lyapunov direct method and generalized Mittag-Leffler stability [18]. Furthermore, there are few works on the Ulam stability of fractional differential equations, which maybe provide a new way for the researchers to investigate the stability of fractional differential equations from different perspectives. First the Ulam stability and data dependence for fractional differential equations with Caputo derivative have been posed by Wang et al. [19] and Ibrahim [20] with Riemann-Liouville derivative in complex domain. Moreover, Wang et al. [21–24] considered and established the Ulam stability for various types of fractional differential equation. Finally, the author generalized the Ulam-Hyers stability for fractional differential equation including infinite power series [25, 26].

In this work, we continue our study by imposing the Ulam-Hyers stability for the Cauchy fractional differential equations in complex domain. The operators are taken in sense of the Srivastava-Owa fractional derivative and integral.

## 2. Fractional Calculus

The theory of fractional calculus has found interesting applications in the theory of analytic functions. The classical definitions of fractional operators and their generalizations have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates [27], distortion inequalities [28], and convolution structures for various subclasses of analytic functions and the works in the research monographs. In [29], Srivastava and Owa gave definitions for fractional operators (derivative and integral) in the complex $z$-plane $\mathbb{C}$ as follows.

*Definition 2.1.* The fractional derivative of order $\alpha$ is defined, for a function $f(z)$, by

$$D_z^{\alpha} f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} d\zeta, \quad 0 \leq \alpha < 1, \tag{2.1}$$

where the function $f(z)$ is analytic in simply connected region of the complex $z$-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

*Definition 2.2.* The fractional integral of order $\alpha$ is defined, for a function $f(z)$, by

$$I_z^{\alpha} f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta, \quad \alpha > 0, \tag{2.2}$$

where the function $f(z)$ is analytic in simply connected region of the complex $z$-plane $(\mathbb{C})$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

*Remark 2.3.* From Definitions 2.1 and 2.2, we have

$$D_z^{\alpha} z^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \quad \mu > -1,\ 0 \le \alpha < 1,$$
$$I_z^{\alpha} z^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \quad \mu > -1,\ \alpha > 0. \tag{2.3}$$

We need the following preliminaries in the sequel.

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}$ denote the space of all analytic functions on $U$. Also for $a \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\mathcal{H}[a, m]$ be the subspace of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \cdots, \quad z \in U. \tag{2.4}$$

Let $\mathcal{A}$ be the class of functions $f$, analytic in $U$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is called univalent ($\mathcal{S}$) if it is one-one in $U$.

**Lemma 2.4** (see [28]). *Let the function $f(z)$ be in the class $\mathcal{S}$. Then*

$$\left|D_z^{\alpha} f(z)\right| \le \frac{r^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{1+rt}{(1-t)^{\alpha}(1-rt)^3} dt \quad (r = |z|,\ z \in U,\ 0 < \alpha < 1). \tag{2.5}$$

**Lemma 2.5** (see [28]). *Let the function $f(z)$ be in the class $\mathcal{S}$. Then*

$$\left|D_z^{\alpha+1} f(z)\right| \le \frac{r^{-\alpha}}{\Gamma(1-\alpha)} (rF(2,1;1-\alpha;r))' \quad (r = |z|,\ z \in U \setminus \{0\},\ 0 < \alpha < 1). \tag{2.6}$$

## 3. Ulam-Hyers Stability for Fractional Problems

In this section, we will study the Ulam-Hyers stability for two different types of fractional Cauchy problems involving the differential operator in Definition 2.1. The first initial value problem is

$$D_z^{\alpha} u(z) = \rho(z) u(z), \quad (u(0) = 0,\ z \in U,\ 0 < \alpha < 1), \tag{3.1}$$

where $u(z), \rho(z) \in \mathcal{H}[U, \mathbb{C}]$ (the space of analytic function on the unit disk). While the second problem is

$$D_z^{\alpha} u(z) = f(z, u(z)), \quad (u(0) = 0,\ z \in U,\ 0 < \alpha < 1), \tag{3.2}$$

where $f : U \times \mathbb{C} \to \mathbb{C}$ is analytic in $z \in U$. Finally, we consider the problem

$$D_z^{1+\alpha} u(z) = f(z, u(z)), \quad (u(z_0) = c,\ z_0 \in U \setminus \{0\},\ 0 < \alpha < 1), \tag{3.3}$$

where $u(z) \in \mathcal{H}[U, \mathbb{C}]$ and $f : U \times \mathbb{C} \to \mathbb{C}$ is analytic in $z \in U$.

*Definition 3.1.* Problem (3.1) has the Ulam-Hyers stability if there exists a positive constant $K$ with the following property: for every $\epsilon > 0$, $u \in \mathcal{H}[U, \mathbb{C}]$, if

$$\left| D_z^{\alpha} u(z) - \rho(z) u(z) \right| < \epsilon, \tag{3.4}$$

then there exists some $v \in \mathcal{H}[U, \mathbb{C}]$ satisfying

$$D_z^{\alpha} v(z) = \rho(z) v(z) \tag{3.5}$$

with $v(0) = 0$ such that

$$|u(z) - v(z)| < K\epsilon. \tag{3.6}$$

*Definition 3.2.* Problem (3.2) has the Ulam-Hyers stability if there exists a positive constant $K$ with the following property: for every $\epsilon > 0$, $u \in \mathcal{H}[U, \mathbb{C}]$, if

$$\left| D_z^{\alpha} u(z) - f(z, u(z)) \right| < \epsilon, \tag{3.7}$$

then there exists some $v \in \mathcal{H}[U, \mathbb{C}]$ satisfying

$$D_z^{\alpha} v(z) = f(z, v(z)) \tag{3.8}$$

with $v(0) = 0$ such that

$$|u(z) - v(z)| < K\epsilon. \tag{3.9}$$

*Definition 3.3.* Problem (3.3) has the Ulam-Hyers stability if there exists a positive constant $K$ with the following property: for every $\epsilon > 0$, $u \in \mathcal{H}[U, \mathbb{C}]$, if

$$\left| D_z^{1+\alpha} u(z) - f(z, u(z)) \right| < \epsilon, \tag{3.10}$$

then there exists some $v \in \mathcal{H}[U, \mathbb{C}]$ satisfying

$$D_z^{1+\alpha} v(z) = f(z, v(z)) \tag{3.11}$$

with $v(z_0) = c$, $z_0 \in U \setminus \{0\}$ such that

$$|u(z) - v(z)| < K\epsilon. \tag{3.12}$$

We start with the following result.

**Theorem 3.4.** *Let $u \in \mathcal{S}$, such that*

$$\max|u(z)| \leq \frac{h_\alpha}{2}, \quad \forall z \in U, \tag{3.13}$$

*where*

$$h_\alpha = \frac{r^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{1+rt}{(1-t)^\alpha (1-rt)^3} dt. \tag{3.14}$$

*If* $\max|\rho(z)| < 1$, *then problem* (3.1) *has the Ulam-Hyers stability.*

*Proof.* For every $\epsilon > 0$, $u \in \mathcal{S}$, we let

$$\left|D_z^\alpha u(z) - \rho(z)u(z)\right| < \epsilon \tag{3.15}$$

with $u(0) = 0$. In view of Lemma 2.4, we obtain

$$\max|D_z^\alpha u(z)| = h_\alpha \quad (\text{sharp case}), \tag{3.16}$$

consequently, we have

$$\begin{aligned} \max|u(z)| &\leq \max\left|D_z^\alpha u(z) - \rho(z)u(z)\right| + \max\left|\rho(z)\right| \max|u(z)| \\ &\leq \epsilon + \max\left|\rho(z)\right| \max|u(z)|; \end{aligned} \tag{3.17}$$

hence we impose that

$$\max|u(z)| \leq \frac{\epsilon}{1 - \max\left|\rho(z)\right|} := K\epsilon. \tag{3.18}$$

Obviously, $v(z) = 0$ is a solution of the problem (3.1) and yields

$$|u(z)| \leq K\epsilon. \tag{3.19}$$

Hence (3.1) has the Ulam-Hyers stability. □

**Corollary 3.5.** *Let $u \in \mathcal{H}[\mathbb{D}, \mathbb{C}]$, where $\mathbb{D} \subset \mathbb{C}$ is a convex domain, satisfying one of the following conditions:*

(1) $\Re\{u'(z)\} > 0$, $z \in U$,

(2) $\Re\{zu'(z)/u(z)\} > 0$, $z \in U$,

(3) $\Re\{1 + zu''(z)/u'(z)\} > 0$, $z \in U$.

*If* $\max|u(z)| \leq h_\alpha/2$ *and* $\max|\rho(z)| < 1$, *then problem* (3.1) *has the Ulam-Hyers stability.*

*Proof.* Assume that $u \in \mathscr{H}[D, \mathbb{C}]$ satisfying one of the conditions (1)–(3), then $u$ is a univalent function in the unit disk; that is, $u \in \mathscr{A}$ (see [30]). Thus, in view of Theorem 3.4, problem (3.1) has the Ulam-Hyers stability. □

*Remark 3.6.* A function $f \in \mathscr{A}$ is called bounded turning function if it satisfies the following inequality:

$$\Re\{f'(z)\} > 0 \quad (z \in U). \tag{3.20}$$

A function $f \in \mathscr{A}$ is called star-like if it satisfies the following inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in U). \tag{3.21}$$

A function $f \in \mathscr{A}$ is called convex if it satisfies the following inequality

$$\Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > 0 \quad (z \in U). \tag{3.22}$$

These subclasses of analytic functions in the unit disk play an important role in the theory of geometric function (see [30]).

Next, we consider the Ulam-Hyers stability for the nonlinear problems (3.2) and (3.3).

**Theorem 3.7.** *Let $u \in \mathcal{S}$, such that* $\max|u(z)| \le h_\alpha/2$, *where*

$$h_\alpha = \frac{r^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{1+rt}{(1-t)^\alpha (1-rt)^3} dt. \tag{3.23}$$

*If*

$$\max\left|f(z, u(z))\right| \le M \max|u(z)|, \quad M \in (0,1), \tag{3.24}$$

*then problem* (3.2) *has the Ulam-Hyers stability.*

*Proof.* For every $\epsilon > 0$, $u \in \mathcal{S}$, we let

$$\left|D_z^\alpha u(z) - f(z, u(z))\right| < \epsilon \tag{3.25}$$

with $u(0) = 0$. In view of Lemma 2.4, it implies that

$$\max|D_z^\alpha u(z)| = h_\alpha \quad (\text{sharp case}); \tag{3.26}$$

therefore, we pose

$$\begin{aligned} \max|u(z)| &\le \max\left|D_z^\alpha u(z) - f(z,u(z))\right| + \max\left|f(z,u(z))\right| \\ &\le \epsilon + \max\left|f(z,u(z))\right| \\ &\le \epsilon + M\max|u(z)|; \end{aligned} \tag{3.27}$$

that is,

$$\max|u(z)| \le \frac{\epsilon}{1-M} := K\epsilon. \tag{3.28}$$

It is clear that

$$v(0) = I_z^\alpha f(z,v(z))\big|_{z=0} = 0 \tag{3.29}$$

yields

$$|u(z)| \le K\epsilon. \tag{3.30}$$

Hence (3.2) has the Ulam-Hyers stability. □

Now by applying Lemma 2.5, we study the Ulam-Hyers stability for the nonlinear problems (3.3).

**Theorem 3.8.** *Let* $u \in \mathcal{S}$, *such that* $\max|u(z)| \le g_\alpha/2$, *where*

$$\begin{gathered} g_\alpha = \frac{r^{-\alpha}}{\Gamma(1-\alpha)} \quad (rF(2,1;1-\alpha;r))', \\ \left|f(z,u(z)) - f(z,v(z))\right| \le L|u(z)-v(z)|. \end{gathered} \tag{3.31}$$

*If* $L \in (0,1)$, *then problem* (3.3) *has the Ulam-Hyers stability.*

*Proof.* Since $f$ is a contraction mapping, then the Banach fixed-point theorem implies that problem (3.3) has a unique solution. For every $\epsilon > 0$, $u \in \mathcal{S}$, we let

$$\left|D_z^{1+\alpha} u(z) - f(z,u(z))\right| < \epsilon \tag{3.32}$$

with $u(z_0) = c$, $z_0 \in U \setminus \{0\}$. In view of Lemma 2.5, we impose

$$\max\left|D_z^{1+\alpha} u(z)\right| = g_\alpha \quad \text{(sharp case)}, \tag{3.33}$$

and consequently we have

$$\begin{aligned}
&\max|u(z)-v(z)| \\
&\quad\le \max|D_z^\alpha(u(z)-v(z))| \\
&\quad\le \left|D_z^\alpha u(z)-D_z^\alpha v(z)-f(z,u(z))+f(z,v(z))\right| + \max\left|f(z,u(z))-f(z,v(z))\right| \\
&\quad\le \epsilon + L\max|u(z)-v(z)|;
\end{aligned} \tag{3.34}$$

hence we receive

$$\max|u(z)-v(z)| \le \frac{\epsilon}{1-L} := K\epsilon. \tag{3.35}$$

It is clear that $v(z_0)=c$ for some $z_0 \in U \setminus \{0\}$ yields

$$|u(z)-v(z)| \le K\epsilon. \tag{3.36}$$

Thus (3.3) has the Ulam-Hyers stability. □

## 4. Conclusion

From above, the Ulam-Hyers stability is considered for different types of fractional Cauchy problems in the unit disk and in the puncture unit disk. We have observed that the problems (3.1) and (3.2) have the Ulam-Hyers stability when $\alpha \in (0,1)$ and $u \in \mathcal{S}$ (univalent solution). While the Ulam-Hyers stability for higher-order fractional Cauchy problem of the form (3.3) is studied in Theorem 3.8, for $z \in U \setminus \{0\}$ and $u \in \mathcal{S}$. This leads to a set of questions:

(1) Is the fractional Cauchy problem (linear and nonlinear) has the Ulam-Hyers stability for all $u \in \mathcal{H}[U,\mathbb{C}]$? (under what conditions).

(2) Is the higher-order fractional Cauchy problem has the Ulam-Hyers stability for all $u \in \mathcal{H}[U,\mathbb{C}]$? (under what conditions). More specific,

(3) does the higher-order fractional Cauchy problem of the form

$$D_z^{m+\alpha}u(z) = f(z,u(z)) \quad (u \in \mathcal{H}[U,\mathbb{C}],\ m=2,3,\ldots) \tag{4.1}$$

have the Ulam-Hyers stability?

(4) If we extend our study to complex Banach space, does the last problem have the Ulam-Hyers stability?

(5) If the study in complex Banach space, does the problem

$$D^m u(z) = f(z,u(z)), \qquad D := \frac{d}{dz} \tag{4.2}$$

have the Ulam-Hyers stability?

More generalization

(6) If the study in complex Banach space, does the problem

$$D^m u(z) = f\left(z, u(z), D^{m-1}u(z)\right), \quad m = 2, 3, \ldots, \tag{4.3}$$

have the Ulam-Hyers stability?

Another special case

(7) If the study in complex Banach space, does the problem

$$D^m u(z) = f\left(z, zD^{m-1}u(z)\right), \quad m = 2, 3, \ldots, \tag{4.4}$$

have the Ulam-Hyers stability?

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*Research Article*
# On the Stability Problem in Fuzzy Banach Space

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We investigate the generalized Ulam-Hyers stability of the Cauchy functional equation and pose two open problems in fuzzy Banach space.

## 1. Introduction and Preliminaries

In 1940, Ulam [1] asked the first question on the stability problem. In 1941, Hyers [2] solved the problem of Ulam. This result was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an *unbounded Cauchy difference*.

**Theorem 1.1** (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality:*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

*for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $0 \leq p < 1$. Then, the limit $L(x) = \lim_{n\to\infty}(1/2^n) f(2^n x)$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.2}$$

*for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then $L$ is linear.*

In 1990, Th. M. Rassias [5] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6], as well as by Th. M. Rassias and Šemrl [7], that one cannot prove a Th. M. Rassias type theorem when $p = 1$. Găvruţa [8] proved that the function $f(x) = x \ln|x|$, if $x \neq 0$ and $f(0) = 0$ satisfies (1.1) with $\epsilon = p = 1$ but

$$\sup_{x\neq 0}\frac{\left|f(x)-A(x)\right|}{|x|} \geq \sup_{n\in\mathbb{N}}\frac{|n\ln n - A(n)|}{n} = \sup_{n\in\mathbb{N}}|\ln n - A(1)| = \infty \tag{1.3}$$

for any additive function $A : \mathbb{R} \to \mathbb{R}$. J. M. Rassias [9] replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^{p_1}\|y\|^{p_2}$ for $p_1, p_2 \in \mathbb{R}$ with $p_1 + p_2 \neq 1$ (see also [10, 11]) and has obtained the following theorem.

**Theorem 1.2.** *Let $X$ be a real normed linear space and $Y$ a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p = p_1 + p_2 \neq 1$ such that $f$ satisfies the inequality:*

$$\left\|f(x+y) - f(x) - f(y)\right\| \leq \theta\|x\|^{p_1}\left\|y\right\|^{p_2} \tag{1.4}$$

*for all $x, y \in X$. Then, there exists a unique additive mapping $L : X \to Y$ satisfying*

$$\left\|f(x) - L(x)\right\| \leq \frac{\theta}{|2^p - 2|}\|x\|^p \tag{1.5}$$

*for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $L$ is an $\mathbb{R}$-linear mapping.*

In the case $p = 1$, we do not have stability [12]. In 1994, a further generalization of Th. M. Rassias' Theorem was obtained by Găvruţa [13], in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Isac and Th. M. Rassias [14] replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^{p_1} + \|y\|^{p_2}$ in Theorem 1.1 and solved stability problem when $p_2 \leq p_1 < 1$ or $1 < p_2 \leq p_1$, also they asked the question whether such a theorem can be proved for $p_2 < 1 < p_1$. Găvruţa [8] gave a negative answer to this question. Isac and Th. M. Rassias [15] applied the Ulam-Hyers-Rassias stability theory to prove fixed point theorems and study some new applications in nonlinear analysis. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Ulam-Hyers stability to a number of functional equations and mappings (see [16–40]). We also refer the readers to the books of Czerwik [41] and Hyers et al. [42].

Th. M. Rassias [43] has obtained the following theorem and posed a problem.

**Theorem 1.3.** *Let $E_1$ and $E_2$ be two Banach spaces, and let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in $t$ for each fixed $x$. Assume that there exist $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\left\|f(x+y) - f(x) - f(y)\right\| \leq \theta\left(\|x\|^p + \left\|y\right\|^p\right) \tag{1.6}$$

*for all* $x, y \in X$. *Let* $k$ *be a positive integer* $k > 2$. *Then, there exists a unique linear mapping* $T : E_1 \to E_2$ *such that*

$$\|f(x) - T(x)\| \leq \frac{k\theta}{k - k^p}\|x\|^p s(k,p) \tag{1.7}$$

*for all* $x \in X$, *where*

$$s(k,p) = 1 + \frac{1}{k}\sum_{m=2}^{k-1} m^p. \tag{1.8}$$

*Th. M. Rassias Problem*

What is the best possible value of $k$ in Theorem 1.3?

Găvruţa et al. have given a generalization of [13] and have answered to Th. M. Rassias problem [44].

In [45], J. M. Rassias et al. have investigated the generalized Ulam-Hyers "product-sum" stability of functional equations and have obtained the following theorem.

**Theorem 1.4** (see [45]). *Let* $f : E \to F$ *be a mapping which satisfies the inequality*

$$\begin{aligned} &\left\| f(mx+y) + f(mx-y) - 2f(x+y) - 2f(x-y) - 2\left(m^2-2\right)f(x) + 2f(y)\right\|_F \\ &\quad \leq \epsilon\left(\|x\|_E^p\|y\|_E^p + \|x\|_E^{2p} + \|y\|_E^{2p}\right) \end{aligned} \tag{1.9}$$

*for all* $x, y \in E$ *with* $x \perp y$, *where* $\epsilon$ *and* $p$ *are constants with* $\epsilon, p > 0$ *and either* $m > 1, p < 1$ *or* $m < 1$, $p > 1$ *with* $m \neq 0$, $m \neq \pm 1, m \neq \sqrt{\pm 2}$, *and* $-1 \neq |m|^{p-1} < 1$. *Then, the limit* $\lim_{n\to\infty} m^{-2n} f(m^n x)$ *exists for all* $x \in E$ *and* $Q : E \to F$ *is the unique orthogonally Euler-Lagrange quadratic mapping such that*

$$\|f(x) - Q(x)\|_F \leq \frac{\epsilon}{2\left|m^2 - m^{2p}\right|}\|x\|_E^{2p} \tag{1.10}$$

*for all* $x \in E$.

Note that the mixed "product-sum" function was introduced by J. M. Rassias in 2008-2009 [46–48].

We recall some basic facts concerning fuzzy normed space.

Let $X$ be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ (so-called fuzzy subset) is said to be a fuzzy norm on $X$ if for all $x, y \in X$ and all $c, t \in \mathbb{R}$,

$(N1)$ $N(x,c) = 0$ for $c \leq 0$;

$(N2)$ $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;

$(N3)$ $N(cx,t) = N(x, t/|c|)$ if $c \neq 0$;

$(N4)$ $N(x+y, t) \geq \min\{N(x,t), N(y,t)\}$;

$(N5)$ $N(x, \cdot)$ is a nondecreasing function of $\mathbb{R}$ and

$$\lim_{t \to \infty} N(x, t) = 1. \tag{1.11}$$

The pair $(X, N)$ is called a fuzzy normed linear space. The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [49–51].

Let $(X, N)$ be a fuzzy normed space and let $\{x_n\}$ be a sequence in $X$. Then, $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in a fuzzy normed space $(X, N)$ is called Cauchy if, for each $\epsilon > 0$ and $\delta > 0$, one can find some $n_0$ such that

$$N(x_m - x_n, \delta) > 1 - \epsilon \tag{1.12}$$

for all $n, m \geq n_0$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy. If, in a fuzzy-normed space, each Cauchy sequence is convergent, then the fuzzy-norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Stability of Cauchy, Jensen, quadratic, and cubic function equation in fuzzy normed spaces have first been investigated in [50–53].

In this paper, we give a generalization of the results from [13] and pose two open problems in fuzzy Banach space. For convenience, we use the following abbreviation for a given mapping $f$:

$$Df(x, y) =: f(x + y) - f(x) - f(y). \tag{1.13}$$

## 2. Stability of the Cauchy Functional Equation

Hereafter, unless otherwise stated, we will assume that $X$ is real vector space, $(Y, N)$ is a complete fuzzy norm space and $k$ is a fixed integer greater than 1.

**Theorem 2.1.** *Let $(Z, N')$ be a fuzzy normed space and $\varphi : X \times X \to Z$ be a mapping such that, $\varphi(kx, ky) = \alpha\varphi(x, y)$ for some $\alpha$ with $0 < \alpha < k$. Suppose that $f : X \to Y$ be mapping such that*

$$N(Df(x, y), t) \geq N'(\varphi(x, y), t) \tag{2.1}$$

*for all $x, y \in X$ and all positive real number $t$. Then, there is a unique additive mapping $T_k : X \to Y$ such that $T_k(x) = \lim_{n\to\infty} f(k^n x)/k^n$ and*

$$N(T_k(x) - f(x), t) \geq M_k(x, (k - \alpha)t), \tag{2.2}$$

*where $M_k(x, t) := \min\{N'(\varphi(x, ix), t) : 1 \leq i < k\}$.*

*Proof.* By induction on $k$, we show that

$$N(f(kx) - kf(x), t) \geq M_k(x, t) := \min\{N'(\varphi(x, ix), t) : 1 \leq i < k\} \tag{2.3}$$

for all $x \in X$ and all positive real number $t$. Letting $y = x$ in (2.1), we get

$$N(f(2x) - 2f(x), t) \geq N'(\varphi(x,x), t). \tag{2.4}$$

So we get (2.3) for $k = 2$.

Assume that (2.3) holds for $k$ with $k > 2$. Letting $y = kx$ in (2.1), we get

$$N(f((k+1)x) - f(x) - f(kx), t) \geq N'(\varphi(x,kx), t). \tag{2.5}$$

for all $x \in X$. By using (2.3) and (2.5), we get (2.3) for $k+1$ and this completes the induction argument. Replacing $x$ by $k^n x$ in (2.3), we get

$$N\left(f\left(k^{n+1}x\right) - kf(k^n x), t\right) \geq M_k(k^n x, t). \tag{2.6}$$

Thus

$$N\left(\frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(k^n x)}{k^n}, \frac{t}{k^{n+1}}\right) \geq M_k\left(x, \frac{t}{\alpha^n}\right) \tag{2.7}$$

for all $x \in X$ and all positive real number $t$. Hence,

$$\begin{aligned}
&N\left(\frac{1}{k^{n+1}}f\left(k^{n+1}x\right) - \frac{1}{k^m}f(k^m x), \sum_{i=m}^{n}\frac{\alpha^i}{k^{i+1}}t\right)\\
&\quad\geq N\left(\sum_{i=m}^{n}\frac{1}{k^{i+1}}f\left(k^{i+1}x\right) - \frac{1}{k^i}f\left(k^i x\right), \sum_{i=m}^{n}\frac{\alpha^i}{k^{i+1}}t\right)\\
&\quad\geq \min\bigcup_{i=m}^{n}\left\{N\left(\frac{1}{k^{i+1}}f\left(k^{i+1}x\right) - \frac{1}{k^i}f\left(k^i x\right), \frac{\alpha^i}{k^{i+1}}t\right)\right\}\\
&\quad\geq M_k(x,t).
\end{aligned} \tag{2.8}$$

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} M_k(x,t) = 1$, there is some $t_0 > 0$ such that $M_k(x,t_0) > 1-\epsilon$. Since $\sum_{n=0}^{\infty}(\alpha^n/k^n)t_0 < \infty$, there is some $n_0 \in N$ such that $\sum_{i=m}^{n}(\alpha^i/k^i)t_0 < k\delta$ for all $n > m \geq n_0$. It follows that

$$\begin{aligned}
&N\left(\frac{1}{k^{n+1}}f\left(k^{n+1}x\right) - \frac{1}{k^m}f(k^m x), \delta\right)\\
&\quad\geq N\left(\frac{1}{k^{n+1}}f\left(k^{n+1}x\right) - \frac{1}{k^m}f(k^m x), \sum_{i=m}^{n}\frac{\alpha^i}{k^{i+1}}t_0\right)\\
&\quad\geq M_k(x,t_0) > 1-\epsilon
\end{aligned} \tag{2.9}$$

for all $x \in X$ and all nonnegative integers $n$ and $m$ with $n > m \geq n_0$. Therefore, the sequence $\{(1/k^n)f(k^n x)\}$ is a Cauchy sequence in $(Y,N)$ for all $x \in X$. Since $(Y,N)$ is complete, the sequence $\{(1/k^n)f(k^n x)\}$ converges in $Y$ for all $x \in X$. So one can define the mapping $T_k : X \to Y$ by

$$T_k(x) := \lim_{n\to\infty} \frac{1}{k^n} f(k^n x) \tag{2.10}$$

for all $x \in X$. Now, we show that $T_k$ is an additive mapping. It follows from (2.1) and (2.10) that

$$\begin{aligned} N(DT_k(x,y),t) &= \lim_{n\to\infty} N\left(\frac{Df(k^n x, k^n y)}{k^n}, t\right) \\ &\geq \lim_{n\to\infty} N'\left(\frac{\varphi(k^n x, k^n y)}{k^n}, t\right) \\ &= \lim_{n\to\infty} N'\left(\varphi(x,y), \frac{k^n}{\alpha^n} t\right) \\ &= 1 \end{aligned} \tag{2.11}$$

for all $x, y \in X$ and all positive real number $t$. Therefore, the mapping $T_k$ is additive. Moreover, if we put $m = 0$ in (2.8), we observe that

$$N\left(\frac{1}{k^{n+1}} f\left(k^{n+1}x\right) - f(x), \sum_{i=0}^{n} \frac{\alpha^i}{k^{i+1}} t\right) \geq M_k(x,t). \tag{2.12}$$

Therefore,

$$N\left(\frac{1}{k^{n+1}} f\left(k^{n+1}x\right) - f(x), t\right) \geq M_k\left(x, \frac{t}{\sum_{i=0}^{n}(\alpha^i/k^{i+1})}\right). \tag{2.13}$$

It follows from (2.13), for large enough $n$, that

$$\begin{aligned} N(T_k(x) - f(x), t) &\geq \min\left\{N\left(\frac{f(k^{n+1}x)}{k^{n+1}} - f(x), t\right), N\left(T_k(x) - \frac{f(k^{n+1}x)}{k^{n+1}}, t\right)\right\} \\ &\geq M_k\left(x, \frac{t}{\sum_{i=0}^{n}(\alpha^i/k^{i+1})}\right) \\ &\geq M_k(x, (k-\alpha)t). \end{aligned} \tag{2.14}$$

Now, we show that $T_k$ is unique. Let $T'$ be another additive mapping from $X$ into $Y$, which satisfies the required inequality. Then, for each $x \in X$ and $t > 0$, we have

$$\begin{aligned} N(T_k(x) - T'(x), t) &\geq \min\{N(T_k(x) - f(x), t), N(f(x) - T'(x), t)\} \\ &\geq M_k(x, (k-\alpha)t). \end{aligned} \tag{2.15}$$

So,

$$\begin{aligned} N(T_k(x) - T'(x), t) &= N\left(\frac{T_k(k^n x)}{k^n} - \frac{T'(k^n x)}{k^n}, t\right) \\ &= N(T_k(k^n x) - T'(k^n x), k^n t) \\ &\geq M_k(k^n x, (k-\alpha)k^n t) \\ &\geq M_k\left(x, (k-\alpha)\frac{k^n}{\alpha^n} t\right). \end{aligned} \tag{2.16}$$

Hence, the right-hand side of the above inequality tends to 1 as $n \to \infty$. It follows that $T_k(x) = T'(x)$ for all $x \in X$. □

**Theorem 2.2.** *Let $(Z, N')$ be a fuzzy normed space and, $\Phi : X \times X \to Z$ be a mapping such that $\Phi(k^{-1}x, k^{-1}y) = \alpha^{-1}\Phi(x, y)$ for some $\alpha$ with $\alpha > k$. Suppose that $f : X \to Y$ be mapping such that*

$$N(Df(x, y), t) \geq N'(\Phi(x, y), t) \tag{2.17}$$

*for all $x, y \in X$ and all positive real number $t$. Then, there is a unique additive mapping $T_k : X \to Y$ such that $T_k(x) = \lim_{n\to\infty} k^n f(x/k^n)$ and*

$$N(T_k(x) - f(x), t) \geq M_k(x, (\alpha - k)t), \tag{2.18}$$

*where $M_k(x, t) := \min\{N'(\Phi(x, ix), t) : 1 \leq i < k\}$.*

*Proof.* Similarly to the proof of Theorem 2.1, we have

$$N(f(kx) - kf(x), t) \geq M_k(x, t) \tag{2.19}$$

for all $x \in X$ and all positive real number $t$. Replacing $x$ by $x/k^{n+1}$ in (2.19), we get

$$N\left(f\left(\frac{x}{k^n}\right) - kf\left(\frac{x}{k^{n+1}}\right), t\right) \geq M_k\left(\frac{x}{k^{n+1}}, t\right). \tag{2.20}$$

Thus,

$$N\left(k^n f\left(\frac{x}{k^n}\right) - k^{n+1} f\left(\frac{x}{k^{n+1}}\right), k^n t\right) \geq M_k\left(x, \alpha^{n+1} t\right) \tag{2.21}$$

for all $x \in X$ and all positive real number $t$. Hence,

$$
\begin{aligned}
N\left(k^{n+1} f\left(\frac{x}{k^{n+1}}\right)-k^m f\left(\frac{x}{k^m}\right), \sum_{i=m}^n \frac{k^i}{\alpha^{i+1}} t\right) & \geq N\left(\sum_{i=m}^n k^{i+1} f\left(\frac{x}{k^{i+1}}\right)-k^i f\left(\frac{x}{k^i}\right), \sum_{i=m}^n \frac{k^i}{\alpha^{i+1}} t\right) \\
& \geq \min \bigcup_{i=m}^n\left\{N\left(k^{i+1} f\left(\frac{x}{k^{i+1}}\right)-k^i f\left(\frac{x}{k^i}\right), \frac{k^i}{\alpha^{i+1}} t\right)\right\} \\
& \geq M_k(x, t).
\end{aligned}
\tag{2.22}
$$

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} M_k(x,t) = 1$, there is some $t_0 > 0$ such that $M_k(x, t_0) > 1-\epsilon$. Since $\sum_{n=0}^{\infty}(k^n/\alpha^n)t_0 < \infty$, there is some $n_0 \in N$ such that $\sum_{i=m}^{n}(k^i/\alpha^i)t_0 < \alpha\delta$ for all $n > m \geq n_0$. It follows from (2.22) that

$$
\begin{aligned}
N\left(k^{n+1} f\left(\frac{x}{k^{n+1}}\right)-k^m f\left(\frac{x}{k^m}\right), \delta\right) & \geq N\left(k^{n+1} f\left(\frac{x}{k^{n+1}}\right)-k^m f\left(\frac{x}{k^m}\right), \sum_{i=m}^n \frac{k^i}{\alpha^{i+1}} t_0\right) \\
& \geq M_k(x, t_0) > 1-\epsilon
\end{aligned}
\tag{2.23}
$$

for all $x \in X$ and all nonnegative integers $n$ and $m$ with $n > m \geq n_0$. Therefore, the sequence $\{k^n f(x/k^n)\}$ is a Cauchy sequence in $(Y, N)$ for all $x \in X$. Since $(Y, N)$ is complete, the sequence $\{k^n f(x/k^n)\}$ converges in $Y$ for all $x \in X$. So one can define the mapping $T_k : X \to Y$ by

$$
T_k(x) := \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right) \tag{2.24}
$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 2.1 □

**Theorem 2.3.** *Let $X$ be a normed space, let $(Z, N')$ be a fuzzy normed space, and let $\psi : [0,\infty) \to [0,\infty)$ be a function such that*

(1) $\psi(ts) = \psi(t)\psi(s)$,

(2) $\psi(t) < t$ *for all* $t > 1$.

*Suppose that a mapping $f : X \to Y$ satisfies the inequality:*

$$
N(Df(x,y), t) \geq N'((\psi(\|x\|) + \psi(\|y\|))z_0, t) \tag{2.25}
$$

*for all $x, y \in X$ and all positive real number $t$, where $z_0$ is a fixed vector of $Z$. Then, there exists a unique additive mapping $T_k : X \to Y$ satisfying $T_k(x) := \lim_{n\to\infty}(f(k^n x)/k^n)$ and*

$$
N(T_k(x) - f(x), t) \geq N'\left(\psi(\|x\|)z_0, \frac{k-\psi(k)}{\sigma_k(\psi)} t\right) \tag{2.26}
$$

*for all $x \in X$, where $\sigma_k(\psi) = \max\{1 + \psi(i) : 1 \leq i < k\}$. Moreover, $T_k = T_2$ for all $k \geq 2$.*

*Proof.* Let

$$\varphi(x,y) = (\psi(\|x\|) + \psi(\|y\|))z_0 \tag{2.27}$$

for all $x, y \in X$. So,

$$\varphi(kx,ky) = \psi(k)\varphi(x,y). \tag{2.28}$$

where $\psi(k) < k$. By using Theorem 2.1, we can get (2.26). Now, we show that $T_k = T_2$. It follows from (1) that $\psi(k^n) = (\psi(k))^n$. Replacing $x$ by $2^n x$ in (2.26), we get

$$N(T_k(2^n x) - f(2^n x), t) \geq N'\left(\psi(\|2^n x\|)z_0, \frac{k-\psi(k)}{\sigma_k(\psi)}t\right) \tag{2.29}$$

for all $x \in X$. So we have

$$N\left(T_k(x) - \frac{f(2^n x)}{2^n}, t\right) \geq N'\left(\psi(\|x\|)z_0, \frac{k-\psi(k)}{\sigma_k(\psi)\psi(2^n)}2^n t\right) \tag{2.30}$$

Using (2) and passing the limit $n \to \infty$ in (2.30), we get $T_k = T_2$. □

**Theorem 2.4.** *Let $X$ be a normed space, let $(Z, N')$ be a fuzzy normed space, and let $\psi : [0,\infty) \to [0,\infty)$ be a function such that*

(1) $\psi(ts) = \psi(t)\psi(s)$,

(2) $\psi(t) > t$ *for all* $t > 1$.

*Suppose that a mapping $f : X \to Y$ satisfies the inequality:*

$$N(Df(x,y),t) \geq N'((\psi(\|x\|) + \psi(\|y\|))z_0, t) \tag{2.31}$$

*for all $x, y \in X$ and all positive real number $t$, where $z_0$ is a fixed vector of $Z$. Then, there exists a unique additive mapping $T_k : X \to Y$ satisfying $T_k(x) := \lim_{n\to\infty} k^n f(x/k^n)$ and*

$$N(T_k(x) - f(x), t) \geq N'\left(\psi(\|x\|)z_0, \frac{\psi(k)-k}{\sigma_k(\psi)}t\right) \tag{2.32}$$

*for all $x \in X$, where*

$$\sigma_k(\psi) = \max\{1 + \psi(i) : 1 \leq i < k\}. \tag{2.33}$$

*Moreover, $T_k = T_2$ for all $k \geq 2$.*

*Proof.* Let

$$\Phi(x,y) = (\psi(\|x\|) + \psi(\|y\|))z_0 \tag{2.34}$$

for all $x, y \in X$. So, we have

$$\Phi\left(k^{-1}x, k^{-1}y\right) = \psi\left(k^{-1}\right)\Phi(x, y), \tag{2.35}$$

where $\psi(k^{-1}) = \psi(k)^{-1} < k^{-1}$. It follows from (1) that $\psi(k^{-n}) = (\psi(k))^{-n}$. By using Theorem 2.2, we can get (2.32). Now, we show that $T_k = T_2$. Replacing $x$ by $x/2^n$ in (2.32), we get

$$N\left(T_k\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right), t\right) \geq N'\left(\psi\left(\left\|\left(\frac{x}{2^n}\right)\right\|\right)z_0, \frac{\psi(k) - k}{\sigma_k(\psi)}t\right). \tag{2.36}$$

for all $x \in X$. So we have

$$N\left(T_k(x) - 2^n f\left(\frac{x}{2^n}\right), t\right) \geq N'\left(\psi(\|x\|)z_0, \frac{\psi(k) - k}{2^n\sigma_k(\psi)\psi(2^{-n})}t\right). \tag{2.37}$$

Using (2) and passing the limit $n \to \infty$ in (2.37), we get $T_k = T_2$. □

**Theorem 2.5.** *Let $X$ be a normed space, let $p$ be a nonnegative real number such that $p \neq 1$, and let $H : [0,\infty) \times [0,\infty) \to [0,\infty)$ be a homogeneous function of degree $p$. Suppose that $(Z, N')$ be a fuzzy normed space and let $f : X \to Y$ be mapping such that*

$$N(Df(x,y), t) \geq N'(H(\|x\|, \|y\|)z_0, t) \tag{2.38}$$

*for all $x, y \in X$ and all positive real number $t$, where $z_0$ is a fixed vector of $Z$. Then, there exists a unique additive mapping $T_k : X \to Y$ such that*

$$N(T_k(x) - f(x), t) \geq M_k(x, |k^p - k|t), \tag{2.39}$$

*where $M_k(x,t) := \min\{N'(\|x\|^p H(1,i)z_0, t) : 1 \leq i < k\}$.*

*Proof.* The proof follows from Theorems 2.1 and 2.2. □

For the particular cases $H(x,y) = \theta(x^p + y^p)$, $H(x,y) = x^r y^s$, $H(x,y) = x^r y^s + x^{r+s} + y^{r+s} (r+s = p)$, and $H(x,y) = \min\{x^p, y^p\}$, we have the following corollaries.

**Corollary 2.6.** *Let $X$ be a normed space, let $p$ be a nonnegative real number such that $p \neq 1$. Suppose that $(Z, N')$ be a fuzzy normed space and $f : X \to Y$ be mapping such that*

$$N(Df(x,y), t) \geq N'((\|x\|^p + \|y\|^p)\theta, t) \tag{2.40}$$

*for all $x, y \in X$ and all positive real number $t$, where $\theta$ is a fixed vector of $Z$. Then, there exists a unique additive mapping $T_k : X \to Y$ such that*

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p\theta, \frac{|k^p - k|}{1 + (k-1)^p}t\right). \tag{2.41}$$

**Corollary 2.7.** *Let $X$ be a normed space, $r, s$ be non-negative real numbers such that $p := r + s \neq 1$. Suppose that $(Z, N')$ be a fuzzy normed space and $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'(\|x\|^r \|y\|^s \theta, t) \tag{2.42}$$

*for all $x, y \in X$ and all positive real number $t$, where $\theta$ is a fixed vector of $Z$. Then there exists a unique additive mapping $T_k : X \to Y$ such that*

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p \theta, \frac{|k^p - k|}{(k-1)^s} t\right). \tag{2.43}$$

**Corollary 2.8.** *Let $X$ be a normed space, and let $r, s$ be nonnegative real numbers such that $p := r + s \neq 1$. Suppose that $(Z, N')$ be a fuzzy normed space and let $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'\left(\theta\|x\|^r \|y\|^s + \theta\|x\|^{r+s} + \theta\|y\|^{r+s}, t\right) \tag{2.44}$$

*for all $x, y \in X$ and all positive real number $t$, where $\theta$ is a fixed vector of $Z$. Then, there exists a unique additive mapping $T_k : X \to Y$ such that*

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p \theta, \frac{|k^p - k|}{(k-1)^s + (k-1)^p + 1} t\right). \tag{2.45}$$

**Corollary 2.9.** *Let $X$ be a normed space, let $p$ be a nonnegative real number such that $p \neq 1$. Suppose that $(Z, N')$ be a fuzzy normed space and let $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'(\min\{\|x\|^p, \|y\|^p\}\theta, t) \tag{2.46}$$

*for all $x, y \in X$ and all positive real number $t$, where $\theta$ is a fixed vector of $Z$. Then, there exists a unique additive mapping $T_k : X \to Y$ such that*

$$N(T_k(x) - f(x), t) \geq N'(\|x\|^p \theta, |k^p - k| t). \tag{2.47}$$

*Problem 1.* Whether Theorem 2.5 and/or such Corollaries can be proved for $p = 1$?

*Problem 2.* What is the best possible value of $k$ in Corollaries 2.6 and 2.7?

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*Research Article*

# A Fixed Point Approach to the Stability of a Cauchy-Jensen Functional Equation

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We find out the general solution of a generalized Cauchy-Jensen functional equation and prove its stability. In fact, we investigate the existence of a Cauchy-Jensen mapping related to the generalized Cauchy-Jensen functional equation and prove its uniqueness. In the last section of this paper, we treat a fixed point approach to the stability of the Cauchy-Jensen functional equation.

## 1. Introduction

In 1940, Ulam [1] gave a wide-range talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of homomorphisms.

Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by Hyers [2] under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1978, Rassias [3] gave a generalization of Hyers's result. Many authors investigated solutions or stability of various functional equations (see [4–7]).

Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

In this paper, let $X$ and $Y$ be two real vector spaces.

*Definition 1.1.* A mapping $f : X \times X \to Y$ is called a *Cauchy-Jensen mapping* if $f$ satisfies the system of equations:

$$\begin{aligned} f(x+y,z) &= f(x,z)+f(y,z), \\ 2f\left(x,\frac{y+z}{2}\right) &= f(x,y)+f(x,z). \end{aligned} \tag{1.1}$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $f(x,y) := axy+bx$ is a solution of (1.1).

For a mappings $f : X \times X \to Y$, consider the functional equation:

$$nf\left(\sum_{i=1}^{n}x_i, \frac{1}{n}\sum_{j=1}^{n}y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}f(x_i,y_j), \tag{1.2}$$

where $n$ is a fixed integer greater than 1. In 2006, the authors [8] solved the functional equation:

$$2f\left(x+y,\frac{z+w}{2}\right) = f(x,z)+f(x,w)+f(y,z)+f(y,w), \tag{1.3}$$

which is a special case of (1.2) for $n = 2$.

In this paper, we find out the general solution and we prove the generalized Hyers-Ulam stability of the functional equation (1.2).

## 2. General Solution of (1.2)

The following lemma ia a well-known fact (see, e.g., [6]).

**Lemma 2.1.** *A mapping $g : X \to Y$ satisfies Jensen's functional equation:*

$$2g\left(\frac{y+z}{2}\right) = g(y)+g(z) \tag{2.1}$$

*for all $y, z \in X$ if and only if it satisfies the generalized Jensen's functional equation:*

$$ng\left(\frac{y_1+\cdots+y_n}{n}\right) = g(y_1)+\cdots+g(y_n) \tag{2.2}$$

*for all $y_1,\ldots,y_n \in X$.*

**Theorem 2.2.** *A mapping $f : X \times X \to Y$ satisfies* (1.1) *if and only if it satisfies* (1.2).

*Proof.* If $f$ satisfies (1.1), then we get

$$nf\left(\sum_{i=1}^{n} x_i, \frac{1}{n}\sum_{j=1}^{n} y_j\right) = n\sum_{i=1}^{n} f\left(x_i, \frac{1}{n}\sum_{j=1}^{n} y_j\right), \tag{2.3}$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Hence, we obtain that $f$ satisfies (1.2) by Lemma 2.1.

Conversely, assume that $f$ satisfies (1.2). Letting $x_1 = \cdots = x_n = 0$ and $y_1 = \cdots = y_n = z$ in (1.2), we get $f(0, z) = 0$ for all $z \in X$. Putting $x_1 = x, x_2 = y, x_3 = \cdots = x_n = 0$, and $y_1 = \cdots = y_n = z$ in (1.2), we have

$$f(x + y, z) = f(x, z) + f(y, z) \tag{2.4}$$

for all $x, y, z \in X$. Setting $x_1 = x$ and $x_2 = \cdots = x_n = 0$ in (1.2), we obtain that

$$nf\left(x, \frac{1}{n}\sum_{j=1}^{n} y_j\right) = \sum_{j=1}^{n} f(x, y_j) \tag{2.5}$$

for all $x, y_1, \ldots, y_n \in X$. By Lemma 2.1, we see that

$$2f\left(x, \frac{y + z}{2}\right) = f(x, y) + f(x, z), \tag{2.6}$$

for all $x, y, z \in X$. □

## 3. Stability of (1.3) Using the Alternative of Fixed Point

In this section, let $Y$ be a real Banach space. We investigate the stability of functional equation (1.3) using the alternative of fixed point. Before proceeding the proof, we will state the theorem which is the alternative of fixed point.

**Theorem 3.1** (The alternative of fixed point [9]). *Suppose that one is given a complete generalized metric space $(\Omega, d)$ and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant $L$. Then, for each given $x \in \Omega$, either*

$$d\left(T^n x, T^{n+1} x\right) = \infty \quad \forall n \geq 0, \tag{3.1}$$

*Or there exists a positive integer $n_0$ such that*

(i) $d(T^n x, T^{n+1} x) < \infty$ *for all $n \geq n_0$;*

(ii) *the sequence $(T^n x)$ is convergent to a fixed point $y^*$ of $T$;*

(iii) *$y^*$ is the unique fixed point of $T$ in the set $\Delta = \{y \in \Omega \mid d(T^{n_0} x, y) < \infty\}$;*

(iv) $d(y, y^*) \leq 1/(1 - L) d(y, Ty)$ *for all $y \in \Delta$.*

*From now on, let $\Omega$ be the set of all mappings $g : X \times X \to Y$ satisfying $g(0, 0) = 0$.*

**Lemma 3.2.** *Let $\psi : X \times X \to [0,\infty)$ be a function. Consider the generalized metric $d$ on $\Omega$ given by*

$$d(g,h) = d_\psi(g,h) := \inf S_\psi(g,h), \tag{3.2}$$

*where $S_\psi(g,h) := \{K \in [0,\infty] \mid \|g(x,y) - h(x,y)\| \leq K\psi(x,y) \text{ forall } x,y \in X\}$ for all $g,h \in \Omega$. Then, $(\Omega,d)$ is complete.*

*Proof.* Let $\{g_n\}$ be a Cauchy sequence in $(\Omega,d)$. Then, given $\varepsilon > 0$, there exists $N$ such that $d(g_n,g_k) < \varepsilon$ if $n,k \geq N$. Let $n,k \geq N$. Since $d(g_n,g_k) = \inf S_\psi(g_n,g_k) < \varepsilon$, there exists $K \in [0,\varepsilon)$ such that

$$\|g_n(x,y) - g_k(x,y)\| \leq K\psi(x,y) \leq \varepsilon\psi(x,y) \tag{3.3}$$

for all $x,y \in X$. So, for each $x,y \in X$, $\{g_n(x,y)\}$ is a Cauchy sequence in $Y$. Since $Y$ is complete, for each $x,y \in X$, there exists $g(x,y) \in Y$ such that $g_n(x,y) \to g(x,y)$ as $n \to \infty$. So $g(0,0) = \lim_{n\to\infty} g_n(0,0) = 0$. Thus, we have $g \in \Omega$. Taking the limit as $k \to \infty$ in (3.3), we obtain that

$$\begin{aligned} n \geq N &\Longrightarrow \|g_n(x,y) - g(x,y)\| \leq \varepsilon\psi(x,y), \quad \forall x,y \in X \\ &\Longrightarrow \varepsilon \in S_\psi(g_n,g) \\ &\Longrightarrow d(g_n,g) = \inf S_\psi(g_n,g) \leq \varepsilon. \end{aligned} \tag{3.4}$$

Hence, $g_n \to g \in \Omega$ as $n \to \infty$. □

Using an idea of Cădariu and Radu (see [10] and also [4] where applications of different fixed point theorems to the theory of the Hyers-Ulam stability can be found), we will prove the generalized Hyers-Ulam stability of (1.3).

**Theorem 3.3.** *Let $L \in (0,1)$ and $\varphi$ satisfy*

$$\varphi(x,y,z,w) \leq 6L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{3},\frac{w}{3}\right) \tag{3.5}$$

*for all $x,y,z,w \in X$. Suppose that a mapping $f : X \times X \to Y$ fulfils $f(0,0) = 0$ and the functional inequality:*

$$\left\|2f\left(x+y,\frac{z+w}{2}\right) - f(x,z) - f(x,w) - f(y,z) - f(y,w)\right\| \leq \varphi(x,y,z,w) \tag{3.6}$$

*for all $x,y,z,w \in X$. Then, there exists a unique mapping $F : X \times X \to Y$ satisfying (1.3) such that*

$$\|f(x,y) - F(x,y)\| \leq \frac{L}{1-L}\psi(x,y), \tag{3.7}$$

*where* $\psi : X \times X \to [0,\infty)$ *is a function given by*

$$\psi(x,y) := \varphi(x,x,y,-y) + 2\varphi(x,x,-y,y) + \varphi(x,x,y,y) + \varphi(x,x,-y,3y) + \frac{1}{2}\varphi(x,x,3y,3y) \tag{3.8}$$

*for all* $x, y \in X$.

*Proof.* By a similar method to the proof of Theorem 2.3 in [11], we have the inequality:

$$\begin{aligned}(\|6f(x,y) - f(2x,3y)\|) \leq \varphi(x,x,y,-y) + 2\varphi(x,x,-y,y) \\ + \varphi(x,x,y,y) + \varphi(x,x,-y,3y) + \frac{1}{2}\varphi(x,x,3y,3y)\end{aligned} \tag{3.9}$$

for all $x, y \in X$. By (3.5), we get

$$\|6f(x,y) - f(2x,3y)\| \leq \psi(x,y) \leq 6L\psi\left(\frac{x}{2}, \frac{y}{3}\right) \tag{3.10}$$

for all $x, y \in X$. Consider the generalized metric $d$ on $\Omega$ given by

$$d(g,h) = d_\psi(g,h) := \inf S_\psi(g,h) \tag{3.11}$$

for all $g, h \in \Omega$. Then, we obtain

$$d(f,Tf) \leq L < \infty. \tag{3.12}$$

By Lemma 3.2, the generalized metric space $(\Omega, d)$ is complete. Now, we define a mapping $T : \Omega \to \Omega$ by

$$Tg(x,y) := \frac{1}{6}g(2x,3y) \tag{3.13}$$

for all $g \in \Omega$ and all $x, y \in X$. Observe that, for all $g, h \in \Omega$,

$$\begin{aligned}&K' \in S_\psi(g,h), \quad K' < K \\ &\Longrightarrow \|g(x,y) - h(x,y)\| \leq K'\psi(x,y) \leq K\psi(x,y) \quad \forall x, y \in X \\ &\Longrightarrow K \in S_\psi(g,h).\end{aligned} \tag{3.14}$$

Let $g, h \in \Omega$, $K \in [0,\infty]$ and $d(g,h) < K$. Then, there is a $K' \in S_\psi(g,h)$ such that $K' < K$. By the above observation, we gain $K \in S_\psi(g,h)$. So, we get $\|g(x,y) - h(x,y)\| \le K\psi(x,y)$ for all $x, y \in X$. Thus, we have

$$\left\| \frac{1}{6} g(2x,3y) - \frac{1}{6} h(2x,3y) \right\| \le \frac{1}{6} K\psi(2x,3y) \tag{3.15}$$

for all $x, y \in X$. By (3.5), we obtain that

$$\left\| \frac{1}{6} g(2x,3y) - \frac{1}{6} h(2x,3y) \right\| \le LK\psi(x,y) \tag{3.16}$$

for all $x, y \in X$. Hence, $d(Tg, Th) \le LK$. Therefore, we obtain that

$$d(Tg, Th) \le Ld(g,h) \tag{3.17}$$

for all $g, h \in \Omega$, that is, $T$ is a strictly contractive mapping of $\Omega$ with Lipschitz constant $L$. Applying the alternative of fixed point, we see that there exists a fixed point $F$ of $T$ in $\Omega$ such that

$$F(x,y) = \lim_{n\to\infty} \frac{1}{6^n} f(2^n x, 3^n y) \tag{3.18}$$

for all $x, y \in X$. Replacing $x, y, z, w$ by $2^n x, 2^n y, 3^n z, 3^n w$ in (3.6), respectively, and dividing by $4^n$, we have

$$\begin{aligned}
&\|F(x+y, z-w) + F(x-y, z+w) - 2F(x,z) - 2F(y,w)\| \\
&= \lim_{n\to\infty} \frac{1}{6^n} \| f(2^n(x+y), 3^n(z-w)) + f(2^n(x-y), 3^n(z+w)) \\
&\qquad -2f(2^n x, 3^n z) - 2f(2^n y, 3^n w) \| \\
&\le \lim_{n\to\infty} \frac{1}{6^n} \varphi(2^n x, 2^n y, 3^n z, 3^n w)
\end{aligned} \tag{3.19}$$

for all $x, y, z, w \in X$. By (3.5), the mapping $F$ satisfies (1.3). By (3.5) and (3.10), we obtain that

$$\begin{aligned}
\left\| T^n f(x,y) - T^{n+1} f(x,y) \right\| &= \frac{1}{6^n} \left\| f(2^n x, 3^n y) - \frac{1}{6} f\left(2^{n+1} x, 3^{n+1} y\right) \right\| \\
&\le \frac{L}{6^n} \psi\left(2^{n-1} x, 3^{n-1} y\right) \le \cdots \le \frac{L}{6^n} (6L)^{n-1} \psi(x,y) \\
&= \frac{L^n}{6} \psi(x,y)
\end{aligned} \tag{3.20}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$, that is, $d(T^n f, T^{n+1} f) \leq L^n/6 < \infty$ for all $n \in \mathbb{N}$. By the fixed point alternative, there exists a natural number $n_0$ such that the mapping $F$ is the unique fixed point of $T$ in the set $\Delta = \{g \in \Omega \mid d(T^{n_0} f, g) < \infty\}$. So, we have $d(T^{n_0} f, F) < \infty$. Since

$$d(f, T^{n_0} f) \leq d(f, Tf) + d\left(Tf, T^2 f\right) + \cdots + d\left(T^{n_0-1} f, T^{n_0} f\right) < \infty, \tag{3.21}$$

we get $f \in \Delta$. Thus, we have $d(f, F) \leq d(f, T^{m_0} f) + d(T^{m_0} f, F) < \infty$. Hence, we obtain

$$\|f(x, y) - F(x, y)\| \leq K\varphi(x, y) \tag{3.22}$$

for all $x, y \in X$ and a $K \in [0, \infty)$. Again, using the fixed point alternative, we have

$$d(f, F) \leq \frac{1}{1 - L} d(f, Tf). \tag{3.23}$$

By (3.12), we may conclude that

$$d(f, F) \leq \frac{L}{1 - L}, \tag{3.24}$$

which implies inequality (3.7). □

**Theorem 3.4.** $L \in (0, 1)$ *and* $\varphi$ *satisfy*

$$\varphi(x, y, z, w) \leq \frac{L}{6} \varphi(2x, 2y, 3z, 3w) \tag{3.25}$$

*for all* $x, y, z, w \in X$. *Suppose that a mapping* $f : X \times X \to Y$ *fulfils* $f(0, 0) = 0$ *and the functional inequality* (3.6). *Then, there exists a unique mapping* $F : X \times X \to Y$ *satisfying* (1.3) *such that*

$$\|f(x, y) - F(x, y)\| \leq \frac{1}{1 - L} \psi(x, y), \tag{3.26}$$

*where* $\psi : X \times X \to [0, \infty)$ *is a function given by*

$$\begin{aligned} \psi(x, y) := \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{3}, -\frac{y}{3}\right) + 2\varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{y}{3}, \frac{y}{3}\right) + \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{3}, \frac{y}{3}\right) \\ + \varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{y}{3}, y\right) + \frac{1}{2}\varphi\left(\frac{x}{2}, \frac{x}{2}, y, y\right) \end{aligned} \tag{3.27}$$

*for all* $x, y \in X$.

*Proof.* By a similar method to the proof of Theorem 2.3 in [11], we have the inequality

$$\|6f(x,y) - f(2x,3y)\| \le \varphi(x,x,y,-y) + 2\varphi(x,x,-y,y) + \varphi(x,x,y,y) + \varphi(x,x,-y,3y) + \frac{1}{2}\varphi(x,x,3y,3y) \tag{3.28}$$

for all $x, y \in X$. So, we get

$$\left\| f(x,y) - 6f\left(\frac{x}{2}, \frac{y}{3}\right) \right\| \le \psi(x,y) \tag{3.29}$$

for all $x, y \in X$. Consider the generalized metric $d$ on $\Omega$ given by

$$d(g,h) = d_\psi(g,h) := \inf S_\psi(g,h) \tag{3.30}$$

for all $g, h \in \Omega$. Then, we obtain

$$d(f, Tf) \le 1 < \infty. \tag{3.31}$$

By Lemma 3.2, the generalized metric space $(\Omega, d)$ is complete. Now, we define a mapping $T : \Omega \to \Omega$ by

$$Tg(x,y) := 6g\left(\frac{x}{2}, \frac{y}{3}\right) \tag{3.32}$$

for all $g \in \Omega$ and all $x, y \in X$. By the same argument as in the proof of Theorem 2.3 in [11], $T$ is a strictly contractive mapping of $\Omega$ with Lipschitz constant $L$. Applying the alternative of fixed point, we see that there exists a fixed point $F$ of $T$ in $\Omega$ such that

$$F(x,y) = \lim_{n\to\infty} 6^n f\left(\frac{x}{2^n}, \frac{y}{3^n}\right) \tag{3.33}$$

for all $x, y \in X$. Replacing $x, y, z, w$ by $x/2^n, y/2^n, z/3^n, w/3^n$ in (3.6), respectively, and multiplying by $6^n$, we have

$$\begin{aligned} &\|F(x+y, z-w) + F(x-y, z+w) - 2F(x,z) - 2F(y,w)\| \\ &\quad = \lim_{n\to\infty} 6^n \left\| f\left(\frac{x+y}{2^n}, \frac{z-w}{3^n}\right) + f\left(\frac{x-y}{2^n}, \frac{z+w}{3^n}\right) - 2f\left(\frac{x}{2^n}, \frac{z}{3^n}\right) - 2f\left(\frac{y}{2^n}, \frac{w}{3^n}\right) \right\| \\ &\quad \le \lim_{n\to\infty} 6^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{3^n}, \frac{w}{3^n}\right) \end{aligned} \tag{3.34}$$

for all $x, y, z, w \in X$. By (3.25), the mapping $F$ satisfies (1.3). By (3.25), we obtain that

$$\begin{aligned}
&\|T^n f(x,y) - T^{n+1} f(x,y) \\
&\quad = 6^n \left\| f\left(\frac{x}{2^n}, \frac{y}{3^n}\right) - 6f\left(\frac{x}{2^{n+1}}, \frac{y}{3^{n+1}}\right) \right\| \\
&\quad \le 6^n \psi\left(\frac{x}{2^n}, \frac{y}{3^n}\right) \le 6^{n-1} L\psi\left(\frac{x}{2^{n-1}}, \frac{y}{3^{n-1}}\right) \le 6^{n-2} L^2 \psi\left(\frac{x}{2^{n-2}}, \frac{y}{3^{n-2}}\right) \le \cdots \le L^n \psi(x,y)
\end{aligned} \tag{3.35}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$, that is, $d(T^n f, T^{n+1} f) \le L^n < \infty$ for all $n \in \mathbb{N}$. By the same reasoning as in the proof of Theorem 2.3 in [11], we have

$$d(f,F) \le \frac{1}{1-L} d(f,Tf). \tag{3.36}$$

By (3.31), we may conclude that

$$d(f,F) \le \frac{1}{1-L}, \tag{3.37}$$

which implies inequality (3.26). □

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*Research Article*

# On the Hyers-Ulam Stability of a General Mixed Additive and Cubic Functional Equation in *n*-Banach Spaces

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The objective of the present paper is to determine the generalized Hyers-Ulam stability of the mixed additive-cubic functional equation in $n$-Banach spaces by the direct method. In addition, we show under some suitable conditions that an approximately mixed additive-cubic function can be approximated by a mixed additive and cubic mapping.

## 1. Introduction and Preliminaries

A basic question in the theory of functional equations is as follows: when is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation?

If the problem accepts a unique solution, we say the equation is stable (see [1]). The study of stability problems for functional equations is related to a question of Ulam [2] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [3]. The result of Hyers was generalized by Aoki [4] for approximate additive mappings and by Rassias [5] for approximate linear mappings by allowing the Cauchy difference operator $\mathrm{CD}f(x,y) = f(x+y) - [f(x) + f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a generalization of Rassias' theorem was obtained by Găvruţa [6], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x,y)$. On the other hand, several further interesting discussions, modifications, extensions, and generalizations of the original problem of Ulam have been proposed (see, e.g. [7–12] and the references therein).

Recently, Park [9] investigated the approximate additive mappings, approximate Jensen mappings, and approximate quadratic mappings in 2-Banach spaces and proved the generalized Hyers-Ulam stability of the Cauchy functional equation, the Jensen functional equation, and the quadratic functional equation in 2-Banach spaces. This is the first result for the stability problem of functional equations in 2-Banach spaces.

In [11, 12], we introduced the following mixed additive-cubic functional equation for fixed integers $k$ with $k \neq 0, \pm 1$:

$$f(kx+y)+f(kx-y)=kf(x+y)+kf(x-y)+2f(kx)-2kf(x), \tag{1.1}$$

with $f(0)=0$, and investigated the generalized Hyers-Ulam stability of (1.1) in quasi-Banach spaces and non-Archimedean fuzzy normed spaces, respectively.

In this paper, we investigate, approximate mixed additive-cubic mappings in $n$-Banach spaces. That is, we prove the generalized Hyers-Ulam stability of a general mixed additive-cubic equation (1.1) in $n$-Banach spaces by the direct method.

The concept of 2-normed spaces was initially developed by Gähler [13, 14] in the middle of 1960s, while that of $n$-normed spaces can be found in [15, 16]. Since then, many others have studied this concept and obtained various results; see for instance [15, 17–19].

We recall some basic facts concerning $n$-normed spaces and some preliminary results.

*Definition 1.1.* Let $n \in \mathbb{N}$, and let $X$ be a real linear space with $\dim X \geq n$ and $\|\cdot,\ldots,\cdot\| : X^n \to \mathbb{R}$ a function satisfying the following properties:

(N1) $\|x_1,x_2,\ldots,x_n\|=0$ if and only if $x_1,x_2,\ldots,x_n$ are linearly dependent,

(N2) $\|x_1,x_2,\ldots,x_n\|$ is invariant under permutation,

(N3) $\|\alpha x_1,x_2,\ldots,x_n\|=|\alpha|\|x_1,x_2,\ldots,x_n\|$,

(N4) $\|x+y,x_2,\ldots,x_n\| \leq \|x,x_2,\ldots,x_n\|+\|y,x_2,\ldots,x_n\|$

for all $\alpha \in \mathbb{R}$ and $x,y,x_1,x_2,\ldots,x_n \in$ X. Then the function $\|\cdot,\ldots,\cdot\|$ is called an $n$-norm on $X$ and the pair $(X,\|\cdot,\ldots,\cdot\|)$ is called an $n$-normed space.

*Example 1.2.* For $x_1,x_2,\ldots,x_n \in \mathbb{R}^n$, the Euclidean $n$-norm $\|x_1,x_2,\ldots,x_n\|_E$ is defined by

$$\|x_1,x_2,\ldots,x_n\|_E=\left|\det(x_{ij})\right|=\text{ abs }\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn}\end{vmatrix}\right), \tag{1.2}$$

where $x_i=(x_{i1},\ldots,x_{in}) \in \mathbb{R}^n$ for each $i=1,2,\ldots,n$.

*Example 1.3.* The standard $n$-norm on $X$, a real inner product space of dimension $\dim X \geq n$, is as follows:

$$\|x_1,x_2,\ldots,x_n\|_S=\begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_1,x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,x_1\rangle & \cdots & \langle x_n,x_n\rangle\end{vmatrix}^{1/2}, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on X. If $X=\mathbb{R}^n$, then this $n$-norm is exactly the same as the Euclidean $n$-norm $\|x_1,x_2,\ldots,x_n\|_E$ mentioned earlier. For $n=1$, this $n$-norm is the usual norm $\|x_1\|=\langle x_1,x_1\rangle^{1/2}$.

*Definition 1.4.* A sequence $\{x_k\}$ in an $n$-normed space $X$ is said to converge to some $x \in X$ in the $n$-norm if

$$\lim_{k\to\infty} \|x_k - x, y_2, \ldots, y_n\| = 0, \tag{1.4}$$

for every $y_2, \ldots, y_n \in X$.

*Definition 1.5.* A sequence $\{x_k\}$ in an $n$-normed space $X$ is said to be a Cauchy sequence with respect to the $n$-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y_2, \ldots, y_n\| = 0, \tag{1.5}$$

for every $y_2, \ldots, y_n \in X$. If every Cauchy sequence in $X$ converges to some $x \in X$, then $X$ is said to be complete with respect to the $n$-norm. Any complete $n$-normed space is said to be an $n$-Banach space.

Now we state the following results as lemma (see [9] for the details).

**Lemma 1.6.** *Let $X$ be an $n$-normed space. Then,*

(1) *For $x_i \in X (i = 1, \ldots, n)$ and $\gamma$, a real number,*

$$\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\| = \|x_1, \ldots, x_i, \ldots, x_j + \gamma x_i, \ldots, x_n\| \tag{1.6}$$

*for all $1 \leq i \neq j \leq n$,*

(2) $|\|x, y_2, \ldots, y_n\| - \|y, y_2, \ldots, y_n\|| \leq \|x - y, y_2, \ldots, y_n\|$ *for all $x, y, y_2, \ldots, y_n \in X$,*

(3) *if $\|x, y_2, \ldots, y_n\| = 0$ for all $y_2, \ldots, y_n \in X$, then $x = 0$,*

(4) *for a convergent sequence $\{x_j\}$ in $X$,*

$$\lim_{j\to\infty} \|x_j, y_2, \ldots, y_n\| = \left\| \lim_{j\to\infty} x_j, y_2, \ldots, y_n \right\| \tag{1.7}$$

*for all $y_2, \ldots, y_n \in X$.*

## 2. Approximate Mixed Additive-Cubic Mappings

In this section, we investigate the generalized Hyers-Ulam stability of the generalized mixed additive-cubic functional equation in $n$-Banach spaces. Let $X$ be a linear space and $Y$ an $n$-Banach space. For convenience, we use the following abbreviation for a given mapping $f : X \to Y$:

$$Df(x,y) := f(kx+y) + f(kx-y) - kf(x+y) - kf(x-y) - 2f(kx) + 2kf(x) \tag{2.1}$$

for all $x, y \in X$.

**Theorem 2.1.** *Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that*

$$\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left(2^j x, 2^j y, u_2, \dots, u_n\right) < \infty, \tag{2.2}$$

$$\left\| Df(x, y), u_2, \dots, u_n \right\|_Y \le \varphi(x, y, u_2, \dots, u_n) \tag{2.3}$$

*for all $x, y, u_2, \dots, u_n \in X$. Then, there is a unique additive mapping $A : X \to Y$ such that*

$$\left\| f(2x) - 8f(x) - A(x), u_2, \dots, u_n \right\|_Y \le \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \dots, u_n\right) \tag{2.4}$$

*for all $x, u_2, \dots, u_n \in X$, where*

$$\begin{aligned}
&\widetilde{\varphi}(x, u_2, \dots, u_n) \\
&\quad := \frac{1}{|k^3 - k|} \Bigg\{ (|k| + 1) \left[ \varphi(x, (2k+1)x, u_2, \dots, u_n) + \varphi(x, (2k-1)x, u_2, \dots, u_n) \right] \\
&\qquad + \varphi(3x, x, u_2, \dots, u_n) + \left(8k^2 + 1\right) \varphi(x, x, u_2, \dots, u_n) + \varphi(x, 3kx, u_2, \dots, u_n) \\
&\qquad + \varphi(x, kx, u_2, \dots, u_n) + k^2 \varphi(2x, 2x, u_2, \dots, u_n) + \varphi(2x, 2kx, u_2, \dots, u_n) \\
&\qquad + 2\varphi(x, (k+1)x, u_2, \dots, u_n) + 2\varphi(x, (k-1)x, u_2, \dots, u_n) + 2\varphi(2x, x, u_2, \dots, u_n) \\
&\qquad + 2\varphi(2x, kx, u_2, \dots, u_n) + 8\varphi\left(\frac{x}{2}, \frac{kx}{2}, u_2, \dots, u_n\right) \\
&\qquad + 8|k| \varphi\left(\frac{x}{2}, \frac{(2k-1)x}{2}, u_2, \dots, u_n\right) + 8|k| \varphi\left(\frac{x}{2}, \frac{(2k+1)x}{2}, u_2, \dots, u_n\right) \\
&\qquad + 8\varphi\left(\frac{x}{2}, \frac{3kx}{2}, u_2, \dots, u_n\right) + \frac{|k| + 1}{|k - 1|} \varphi(0, (k+1)x, u_2, \dots, u_n) \\
&\qquad + \frac{8k^2 + 1}{|k - 1|} \varphi(0, (k-1)x, u_2, \dots, u_n) + \frac{2}{|k - 1|} \varphi(0, x, u_2, \dots, u_n) \\
&\qquad + \frac{|k|}{|k - 1|} \varphi(0, (3k-1)x, u_2, \dots, u_n) + \frac{k^2}{|k - 1|} \varphi(0, 2(k-1)x, u_2, \dots, u_n) \\
&\qquad + \frac{k^2 + |k| - 1}{|k - 1|} \varphi(0, 2kx, u_2, \dots, u_n) \\
&\qquad + \frac{8|k|}{|k - 1|} \varphi\left(0, \frac{(3k-1)x}{2}, u_2, \dots, u_n\right) + \frac{8|k|}{|k - 1|} \varphi\left(0, \frac{(k+1)x}{2}, u_2, \dots, u_n\right) \\
&\qquad + \frac{8k^2 + 2|k| - 8}{|k - 1|} \varphi(0, kx, u_2, \dots, u_n) \Bigg\}.
\end{aligned} \tag{2.5}$$

*Proof.* Letting $x = 0$ in (2.3), we get

$$\|f(y) + f(-y), u_2, \ldots, u_n\|_Y \leq \frac{1}{|k-1|}\varphi(0, y, u_2, \ldots, u_n) \tag{2.6}$$

for all $y, u_2, \ldots, u_n \in X$. Putting $y = x$ in (2.3), we have

$$\|f((k+1)x) + f((k-1)x) - kf(2x) - 2f(kx) + 2kf(x), u_2, \ldots, u_n\|_Y \leq \varphi(x, x, u_2, \ldots, u_n) \tag{2.7}$$

for all $x, u_2, \ldots, u_n \in X$. Thus

$$\begin{aligned} &\|f(2(k+1)x) + f(2(k-1)x) - kf(4x) - 2f(2kx) + 2kf(2x), u_2, \ldots, u_n\|_Y \\ &\quad\leq \varphi(2x, 2x, u_2, \ldots, u_n) \end{aligned} \tag{2.8}$$

for all $x, u_2, \ldots, u_n \in X$. Letting $y = kx$ in (2.3), we get

$$\|f(2kx) - kf((k+1)x) - kf(-(k-1)x) - 2f(kx) + 2kf(x), u_2, \ldots, u_n\|_Y \leq \varphi(x, kx, u_2, \ldots, u_n) \tag{2.9}$$

for all $x, u_2, \ldots, u_n \in X$. Letting $y = (k+1)x$ in (2.3), we have

$$\begin{aligned} &\|f((2k+1)x) + f(-x) - kf((k+2)x) - kf(-kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n\|_Y \\ &\quad\leq \varphi(x, (k+1)x, u_2, \ldots, u_n) \end{aligned} \tag{2.10}$$

for all $x, u_2, \ldots, u_n \in X$. Letting $y = (k-1)x$ in (2.3), we have

$$\begin{aligned} &\|f((2k-1)x) - (k+2)f(kx) - kf(-(k-2)x) + (2k+1)f(x), u_2, \ldots, u_n\|_Y \\ &\quad\leq \varphi(x, (k-1)x, u_2, \ldots, u_n) \end{aligned} \tag{2.11}$$

for all $x, u_2, \ldots, u_n \in X$. Replacing $x$ and $y$ by $2x$ and $x$ in (2.3), respectively, we get

$$\begin{aligned} &\|f((2k+1)x) + f((2k-1)x) - 2f(2kx) - kf(3x) + 2kf(2x) - kf(x), u_2, \ldots, u_n\|_Y \\ &\quad\leq \varphi(2x, x, u_2, \ldots, u_n) \end{aligned} \tag{2.12}$$

for all $x, u_2, \ldots, u_n \in X$. Replacing $x$ and $y$ by $3x$ and $x$ in (2.3), respectively, we get

$$\begin{aligned} &\|f((3k+1)x) + f((3k-1)x) - 2f(3kx) - kf(4x) - kf(2x) + 2kf(3x), u_2, \ldots, u_n\|_Y \\ &\quad\leq \varphi(3x, x, u_2, \ldots, u_n) \end{aligned} \tag{2.13}$$

for all $x, u_2, \ldots, u_n \in X$. Replacing $x$ and $y$ by $2x$ and $kx$ in (2.3), respectively, we have

$$\begin{aligned}&\left\| f(3kx) + f(kx) - kf((k+2)x) - kf(-(k-2)x) - 2f(2kx) + 2kf(2x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(2x, kx, u_2, \ldots, u_n)\end{aligned} \tag{2.14}$$

for all $x, u_2, \ldots, u_n \in X$. Setting $y = (2k+1)x$ in (2.3), we have

$$\begin{aligned}&\left\| f((3k+1)x) + f(-(k+1)x) - kf(2(k+1)x) - kf(-2kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(x, (2k+1)x, u_2, \ldots, u_n)\end{aligned} \tag{2.15}$$

for all $x, u_2, \ldots, u_n \in X$. Letting $y = (2k-1)x$ in (2.3), we have

$$\begin{aligned}&\left\| f((3k-1)x) + f(-(k-1)x) - kf(-2(k-1)x) - kf(2kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(x, (2k-1)x, u_2, \ldots, u_n)\end{aligned} \tag{2.16}$$

for all $x, u_2, \ldots, u_n \in X$. Letting $y = 3kx$ in (2.3), we have

$$\begin{aligned}&\left\| f(4\mathrm{k}x) + f(-2kx) - kf((3k+1)x) - kf(-(3k-1)x) - 2f(kx) + 2kf(x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(x, 3kx, u_2, \ldots, u_n)\end{aligned} \tag{2.17}$$

for all $x, u_2, \ldots, u_n \in X$. By (2.6), (2.7), (2.13), (2.15), and (2.16), we get

$$\begin{aligned}&\left\| kf(2(k+1)x) + kf(-2(k-1)x) + 6f(kx) - 2f(3kx) - kf(4x) + 2kf(3x) - 6kf(x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(x, (2k+1)x, u_2, \ldots, u_n) + \varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(3x, x, u_2, \ldots, u_n) \\ &\qquad + \varphi(x, x, u_2, \ldots, u_n) + \frac{1}{|k-1|}\varphi(0, (k+1)x, u_2, \ldots, u_n) \\ &\qquad + \frac{1}{|k-1|}\varphi(0, (k-1)x, u_2, \ldots, u_n) + \frac{|k|}{|k-1|}\varphi(0, 2kx, u_2, \ldots, u_n)\end{aligned} \tag{2.18}$$

for all $x, u_2, \ldots, u_n \in X$. By (2.6), (2.10), and (2.11), we have

$$\begin{aligned}&\left\| f((2k+1)x) + f((2k-1)x) - kf((k+2)x) - kf(-(k-2)x) - 4f(kx) + 4kf(x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(x, (k+1)x, u_2, \ldots, u_n) + \varphi(x, (k-1)x, u_2, \ldots, u_n) + \frac{1}{|k-1|}\varphi(0, x, u_2, \ldots, u_n) \\ &\qquad + \left|\frac{k}{k-1}\right|\varphi(0, kx, u_2, \ldots, u_n)\end{aligned} \tag{2.19}$$

for all $x, u_2, \ldots, u_n \in X$. It follows from (2.12) and (2.19) that

$$\begin{aligned}
&\left\| kf((k+2)x) + kf(-(k-2)x) - 2f(2kx) + 4f(kx) - kf(3x) + 2kf(2x) - 5kf(x), u_2, \ldots, u_n \right\|_Y \\
&\quad \le \varphi(x, (k+1)x, u_2, \ldots, u_n) + \varphi(x, (k-1)x, u_2, \ldots, u_n) + \varphi(2x, x, u_2, \ldots, u_n) \\
&\qquad + \frac{1}{|k-1|}\varphi(0, x, u_2, \ldots, u_n) + \left|\frac{k}{k-1}\right|\varphi(0, kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.20}$$

for all $x, u_2, \ldots, u_n \in X$. By (2.14) and (2.20), we have

$$\begin{aligned}
&\left\| f(3kx) - 4f(2kx) + 5f(kx) - kf(3x) + 4kf(2x) - 5kf(x), u_2, \ldots, u_n \right\|_Y \\
&\quad \le \varphi(x, (k+1)x, u_2, \ldots, u_n) + \varphi(x, (k-1)x, u_2, \ldots, u_n) + \varphi(2x, x, u_2, \ldots, u_n) \\
&\qquad + \varphi(2x, kx, u_2, \ldots, u_n) + \frac{1}{|k-1|}\varphi(0, x, u_2, \ldots, u_n) + \left|\frac{k}{k-1}\right|\varphi(0, kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.21}$$

for all $x, u_2, \ldots, u_n \in X$. By (2.6), (2.15), (2.16), and (2.17), we have

$$\begin{aligned}
&\Big\| kf(-(k+1)x) - kf(-(k-1)x) - k^2 f(2(k+1)x) + k^2 f(-2(k-1)x) \\
&\qquad + k^2 f(2kx) - \left(k^2 - 1\right) f(-2kx) + f(4kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n \Big\|_Y \\
&\quad \le |k|\varphi(x, (2k+1)x, u_2, \ldots, u_n) + |k|\varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
&\qquad + \left|\frac{k}{k-1}\right|\varphi(0, (3k-1)x, u_2, \ldots, u_n)
\end{aligned} \tag{2.22}$$

for all $x, u_2, \ldots, u_n \in X$. It follows from (2.6), (2.8), (2.9), and (2.22) that

$$\begin{aligned}
&\left\| f(4kx) - 2f(2kx) - k^3 f(4x) + 2k^3 f(2x), u_2, \ldots, u_n \right\|_Y \\
&\quad \le |k|\varphi(x, (2k+1)x, u_2, \ldots, u_n) + |k|\varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
&\qquad + \varphi(x, kx, u_2, \ldots, u_n) + k^2\varphi(2x, 2x, u_2, \ldots, u_n) + \left|\frac{k}{k-1}\right|\varphi(0, (3k-1)x, u_2, \ldots, u_n) \\
&\qquad + \left|\frac{k}{k-1}\right|\varphi(0, (k+1)x, u_2, \ldots, u_n) + \frac{k^2}{|k-1|}\varphi(0, 2(k-1)x, u_2, \ldots, u_n) \\
&\qquad + \frac{k^2-1}{|k-1|}\varphi(0, 2kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.23}$$

for all $x, u_2, \ldots, u_n \in X$. Hence,

$$\begin{aligned}
&\left\|f(2kx) - 2f(kx) - k^3 f(2x) + 2k^3 f(x), u_2, \ldots, u_n\right\|_Y \\
&\quad \leq |k|\varphi\left(\frac{x}{2}, \frac{(2k+1)x}{2}, u_2, \ldots, u_n\right) + |k|\varphi\left(\frac{x}{2}, \frac{(2k-1)x}{2}, u_2, \ldots, u_n\right) + \varphi\left(\frac{x}{2}, \frac{3kx}{2}, u_2, \ldots, u_n\right) \\
&\qquad + \varphi\left(\frac{x}{2}, \frac{kx}{2}, u_2, \ldots, u_n\right) + k^2\varphi(x, x, u_2, \ldots, u_n) + \left|\frac{k}{k-1}\right|\varphi\left(0, \frac{(3k-1)x}{2}, u_2, \ldots, u_n\right) \\
&\qquad + \left|\frac{k}{k-1}\right|\varphi\left(0, \frac{(k+1)x}{2}, u_2, \ldots, u_n\right) + \frac{k^2}{|k-1|}\varphi(0, (k-1)x, u_2, \ldots, u_n) \\
&\qquad + \frac{k^2-1}{|k-1|}\varphi(0, kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.24}$$

for all $x, u_2, \ldots, u_n \in X$. By (2.9), we have

$$\begin{aligned}
&\|f(4kx) - kf(2(k+1)x) - kf(-2(k-1)x) - 2f(2kx) + 2kf(2x), u_2, \ldots, u_n\|_Y \\
&\quad \leq \varphi(2x, 2kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.25}$$

for all $x, u_2, \ldots, u_n \in X$. From (2.23) and (2.25), we have

$$\begin{aligned}
&\left\|kf(2(k+1)x) + kf(-2(k-1)x) - k^3 f(4x) + \left(2k^3 - 2k\right) f(2x)\right\|_Y \\
&\quad \leq |k|\varphi(x, (2k+1)x, u_2, \ldots, u_n) + |k|\varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
&\qquad + \varphi(x, kx, u_2, \ldots, u_n) + k^2\varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) \\
&\qquad + \left|\frac{k}{k-1}\right|\varphi(0, (3k-1)x, u_2, \ldots, u_n) + \left|\frac{k}{k-1}\right|\varphi(0, (k+1)x, u_2, \ldots, u_n) \\
&\qquad + \frac{k^2}{|k-1|}\varphi(0, 2(k-1)x, u_2, \ldots, u_n) + \frac{k^2-1}{|k-1|}\varphi(0, 2kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.26}$$

for all $x, u_2, \ldots, u_n \in X$. Also, from (2.18) and (2.26), we get

$$\begin{aligned}
&\left\|2f(3kx) - 6f(kx) + \left(k - k^3\right) f(4x) - 2kf(3x) + \left(2k^3 - 2k\right) f(2x) + 6kf(x), u_2, \ldots, u_n\right\|_Y \\
&\quad \leq (|k|+1)\left[\varphi(x, (2k+1)x, u_2, \ldots, u_n) + \varphi(x, (2k-1)x, u_2, \ldots, u_n)\right] + \varphi(3x, x, u_2, \ldots, u_n) \\
&\qquad + \varphi(x, x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) + \varphi(x, kx, u_2, \ldots, u_n) \\
&\qquad + k^2\varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) + \frac{|k|+1}{|k-1|}\varphi(0, (k+1)x, u_2, \ldots, u_n)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|k-1|}\varphi(0,(k-1)x,u_2,\ldots,u_n)+\frac{k^2+|k|-1}{|k-1|}\varphi(0,2kx,u_2,\ldots,u_n)\\
&+\left|\frac{k}{k-1}\right|\varphi(0,(3k-1)x,u_2,\ldots,u_n)+\frac{k^2}{|k-1|}\varphi(0,2(k-1)x,u_2,\ldots,u_n)
\end{aligned}
\tag{2.27}
$$

for all $x,u_2,\ldots,u_n\in X$.

On the other hand, it follows from (2.21) and (2.27) that

$$
\begin{aligned}
&\left\|8f(2kx)-16f(kx)+\left(k-k^3\right)f(4x)+\left(2k^3-10k\right)f(2x)+16kf(x),u_2,\ldots,u_n\right\|_Y\\
&\quad\le(|k|+1)\left[\varphi(x,(2k+1)x,u_2,\ldots,u_n)+\varphi(x,(2k-1)x,u_2,\ldots,u_n)\right]+\varphi(3x,x,u_2,\ldots,u_n)\\
&\qquad+\varphi(x,x,u_2,\ldots,u_n)+\varphi(x,3kx,u_2,\ldots,u_n)+\varphi(x,kx,u_2,\ldots,u_n)\\
&\qquad+k^2\varphi(2x,2x,u_2,\ldots,u_n)+\varphi(2x,2kx,u_2,\ldots,u_n)+2\varphi(x,(k+1)x,u_2,\ldots,u_n)\\
&\qquad+2\varphi(x,(k-1)x,u_2,\ldots,u_n)+2\varphi(2x,x,u_2,\ldots,u_n)+2\varphi(2x,kx,u_2,\ldots,u_n)\\
&\qquad+\frac{2}{|k-1|}\varphi(0,x,u_2,\ldots,u_n)+\frac{2|k|}{|k-1|}\varphi(0,kx,u_2,\ldots,u_n)+\frac{|k|+1}{|k-1|}\varphi(0,(k+1)x,u_2,\ldots,u_n)\\
&\qquad+\frac{1}{|k-1|}\varphi(0,(k-1)x,u_2,\ldots,u_n)+\frac{k^2+|k|-1}{|k-1|}\varphi(0,2kx,u_2,\ldots,u_n)\\
&\qquad+\left|\frac{k}{k-1}\right|\varphi(0,(3k-1)x,u_2,\ldots,u_n)+\frac{k^2}{|k-1|}\varphi(0,2(k-1)x,u_2,\ldots,u_n)
\end{aligned}
\tag{2.28}
$$

for all $x,u_2,\ldots,u_n\in X$. Therefore by (2.24) and (2.28), we get

$$
\begin{aligned}
&\left\|f(4x)-10f(2x)+16f(x),u_2,\ldots,u_n\right\|_Y\\
&\quad\le\frac{1}{|k^3-k|}\\
&\qquad\times\Big\{(|k|+1)\left[\varphi(x,(2k+1)x,u_2,\ldots,u_n)+\varphi(x,(2k-1)x,u_2,\ldots,u_n)\right]\\
&\qquad\quad+\varphi(3x,x,u_2,\ldots,u_n)+\left(8k^2+1\right)\varphi(x,x,u_2,\ldots,u_n)+\varphi(x,3kx,u_2,\ldots,u_n)\\
&\qquad\quad+\varphi(x,kx,u_2,\ldots,u_n)+k^2\varphi(2x,2x,u_2,\ldots,u_n)+\varphi(2x,2kx,u_2,\ldots,u_n)\\
&\qquad\quad+2\varphi(x,(k+1)x,u_2,\ldots,u_n)+2\varphi(x,(k-1)x,u_2,\ldots,u_n)+2\varphi(2x,x,u_2,\ldots,u_n)\\
&\qquad\quad+2\varphi(2x,kx,u_2,\ldots,u_n)+8\varphi\left(\frac{x}{2},\frac{kx}{2},u_2,\ldots,u_n\right)+8|k|\varphi\left(\frac{x}{2},\frac{(2k-1)x}{2},u_2,\ldots,u_n\right)\\
&\qquad\quad+8|k|\varphi\left(\frac{x}{2},\frac{(2k+1)x}{2},u_2,\ldots,u_n\right)+8\varphi\left(\frac{x}{2},\frac{3kx}{2},u_2,\ldots,u_n\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{|k|+1}{|k-1|}\varphi(0,(k+1)x,u_2,\dots,u_n) + \frac{8k^2+1}{|k-1|}\varphi(0,(k-1)x,u_2,\dots,u_n)\\
&\quad + \frac{2}{|k-1|}\varphi(0,x,u_2,\dots,u_n) + \left|\frac{k}{k-1}\right|\varphi(0,(3k-1)x,u_2,\dots,u_n)\\
&\quad + \frac{k^2}{|k-1|}\varphi(0,2(k-1)x,u_2,\dots,u_n) + \frac{k^2+|k|-1}{|k-1|}\varphi(0,2kx,u_2,\dots,u_n)\\
&\quad + \frac{8|k|}{|k-1|}\varphi\left(0,\frac{(3k-1)x}{2},u_2,\dots,u_n\right)\\
&\quad + \frac{8|k|}{|k-1|}\varphi\left(0,\frac{(k+1)x}{2},u_2,\dots,u_n\right) + \frac{8k^2+2|k|-8}{|k-1|}\varphi(0,kx,u_2,\dots,u_n)\Bigg\}\\
&:= \widetilde{\varphi}(x,u_2,\dots,u_n)
\end{aligned}
\tag{2.29}
$$

for all $x, u_2, \dots, u_n \in X$.

Now, let $g : X \to Y$ be the mapping defined by $g(x) := f(2x) - 8f(x)$ for all $x, u_2, \dots, u_n \in X$. Then, (2.29) means that

$$\left\| f(4x) - 10f(2x) + 16f(x), u_2, \dots, u_n \right\|_Y \le \widetilde{\varphi}(x, u_2, \dots, u_n) \tag{2.30}$$

for all $x, u_2, \dots, u_n \in X$. Also, we get

$$\left\| g(2x) - 2g(x), u_2, \dots, u_n \right\|_Y \le \widetilde{\varphi}(x, u_2, \dots, u_n) \tag{2.31}$$

for all $x \in X$. Replacing $x$ by $2^j x$ in (2.31) and dividing both sides of (2.31) by $2^{j+1}$, we get

$$\left\| \frac{1}{2^j} g(2^j x) - \frac{1}{2^{j+1}} g(2^{j+1}x), u_2, \dots, u_n \right\|_Y \le \frac{1}{2^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \dots, u_n\right) \tag{2.32}$$

for all $x, u_2, \dots, u_n \in X$ and all integers $j \ge 0$. For all integers $l, m$ with $0 \le l < m$, we have

$$
\begin{aligned}
\left\| \frac{1}{2^l} g\left(2^l x\right) - \frac{1}{2^m} g(2^m x), u_2, \dots, u_n \right\|_Y &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} g\left(2^j x\right) - \frac{1}{2^{j+1}} g\left(2^{j+1} x\right), u_2, \dots, u_n \right\|_Y\\
&\le \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \dots, u_n\right)
\end{aligned}
\tag{2.33}
$$

for all $x, u_2, \dots, u_n \in X$. So, we get

$$\lim_{l,m\to\infty} \left\| \frac{1}{2^l} g(2^l x) - \frac{1}{2^m} g(2^m x), u_2, \dots, u_n \right\|_Y = 0 \tag{2.34}$$

for all $x, u_2, \ldots, u_n \in X$. This shows that the sequence $\{(1/2^j)g(2^j x)\}$ is a Cauchy sequence in $Y$. Since $Y$ is an $n$-Banach space, the sequence $\{(1/2^j)g(2^j x)\}$ converges. So, we can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{j \to \infty} \frac{1}{2^j} g\left(2^j x\right) \tag{2.35}$$

for all $x \in X$. Putting $l = 0$, then passing the limit $m \to \infty$ in (2.33), and using Lemma 1.6(4), we get

$$\|g(x) - A(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.36}$$

for all $x, u_2, \ldots, u_n \in X$.

Now we show that $A$ is additive. By Lemma 1.6, (2.2), (2.32), and (2.35), we have

$$\begin{aligned} \|A(2x) - 2A(x), u_2, \ldots, u_n\|_Y &= \lim_{j \to \infty} \left\| \frac{1}{2^j} g\left(2^{j+1} x\right) - \frac{1}{2^{j-1}} g\left(2^j x\right), u_2, \ldots, u_n \right\|_Y \\ &= 2 \lim_{j \to \infty} \left\| \frac{1}{2^{j+1}} g\left(2^{j+1} x\right) - \frac{1}{2^j} g\left(2^j x\right), u_2, \ldots, u_n \right\|_Y \\ &\leq \lim_{j \to \infty} \frac{1}{2^j} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) = 0 \end{aligned} \tag{2.37}$$

for all $x, u_2, \ldots, u_n \in X$. By Lemma 1.6(3), $A(2x) = 2A(x)$ for all $x \in X$. Also, by Lemma 1.6(4), (2.2), (2.3), and (2.35), we get

$$\begin{aligned} &\|DA(x, y), u_2, \ldots, u_n\|_Y \\ &\quad = \lim_{j \to \infty} \frac{1}{2^j} \left\| Dg\left(2^j x, 2^j y\right), u_2, \ldots, u_n \right\|_Y \\ &\quad = \lim_{j \to \infty} \frac{1}{2^j} \left\| Df\left(2^{j+1} x, 2^{j+1} y\right) - 8Df\left(2^j x, 2^j y\right), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \lim_{j \to \infty} \frac{1}{2^j} \left[ \left\| Df\left(2^{j+1} x, 2^{j+1} y\right), u_2, \ldots, u_n \right\|_Y + 8 \left\| Df\left(2^j x, 2^j y\right), u_2, \ldots, u_n \right\|_Y \right] \\ &\quad \leq \lim_{j \to \infty} \frac{1}{2^j} \left[ \varphi\left(2^{j+1} x, 2^{j+1} y, u_2, \ldots, u_n\right) + 8\varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) \right] = 0 \end{aligned} \tag{2.38}$$

for all $x, y, u_2, \ldots, u_n \in X$. By Lemma 1.6(3), $DA(x, y) = 0$ for all $x, y \in X$. Hence, the mapping $A$ satisfies (1.1). By [11, Lemma 2.3], the mapping $x \to A(2x) - 8A(x)$ is additive. Therefore, $A(2x) = 2A(x)$ implies that the mapping $A$ is additive.

To prove the uniqueness of $A$, let $B : X \to Y$ be another additive mapping satisfying (2.4). Fix $x \in X$. Clearly, $A(2^l x) = 2^l A(x)$ and $B(2^l x) = 2^l B(x)$ for all $l \in \mathbb{N}$. It follows from (2.4) that

$$\begin{aligned} \|A(x) - B(x), u_2, \ldots, u_n\|_Y &= \left\| \frac{A(2^l x)}{2^l} - \frac{B(2^l x)}{2^l}, u_2, \ldots, u_n \right\|_Y \\ &\le \frac{1}{2^l}\left[\left\| f\left(2^{l+1}x\right) - 8f\left(2^l x\right) - A\left(2^l x\right), u_2, \ldots, u_n \right\|_Y \right. \\ &\qquad \left. + \left\| B\left(2^l x\right) - f\left(2^{l+1}x\right) + 8f\left(2^l x\right), u_2, \ldots, u_n \right\|_Y \right] \\ &\le \frac{1}{2^l}\sum_{j=0}^{\infty}\frac{1}{2^j}\tilde{\varphi}\left(2^{j+l}x, u_2, \ldots, u_n\right) \\ &\le \sum_{j=0}^{\infty}\frac{1}{2^{j+l}}\tilde{\varphi}\left(2^{j+l}x, u_2, \ldots, u_n\right) = \sum_{j=l}^{\infty}\frac{1}{2^j}\tilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \end{aligned} \tag{2.39}$$

for all $x, u_2, \ldots, u_n \in X$, and $l \in \mathbb{N}$. By (2.2), we see that the right-hand side of the above inequality tends to 0 as $l \to \infty$. Therefore, $\|A(x) - B(x), u_2, \ldots, u_n\|_Y = 0$ for all $u_2, \ldots, u_n \in X$. By Lemma 1.6, we can conclude that $A(x) = B(x)$ for all $x \in X$. So, $A = B$. This proves the uniqueness of $A$. □

**Theorem 2.2.** *Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that*

$$\begin{gathered} \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, u_2, \ldots, u_n\right) < \infty, \\ \left\| Df(x, y), u_2, \ldots, u_n \right\|_Y \le \varphi(x, y, u_2, \ldots, u_n) \end{gathered} \tag{2.40}$$

*for all $x, y, u_2, \ldots, u_n \in X$. Then, there is a unique additive mapping $A : X \to Y$ such that*

$$\left\| f(2x) - 8f(x) - A(x), u_2, \ldots, u_n \right\|_Y \le \sum_{j=1}^{\infty} 2^{j-1}\tilde{\varphi}\left(\frac{x}{2^j}, u_2, \ldots, u_n\right) \tag{2.41}$$

*for all $x, u_2, \ldots, u_n \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.*

*Proof.* The proof is similar to the proof of Theorem 2.1. □

**Corollary 2.3.** *Let $X$ be a normed space and $Y$ an $n$-Banach space. Let $\theta \in [0, \infty), p, r_2, \ldots, r_n \in (0, \infty)$ such that $p \neq 1$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\left\| Df(x, y), u_2, \ldots, u_n \right\|_Y \le \theta\left(\|x\|_X^p + \|y\|_X^p\right)\|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n} \tag{2.42}$$

*for all* $x, y, u_2, \ldots, u_n \in \mathrm{X}$. *Then, there exists a unique additive mapping* $A : \mathrm{X} \to Y$ *such that*

$$\left\|f(2x) - 8f(x) - A(x), u_2, \ldots, u_n\right\|_Y \leq \frac{\theta\epsilon\|x\|_{\mathrm{X}}^p \|u_2\|_{\mathrm{X}}^{r_2} \cdots \|u_n\|_{\mathrm{X}}^{r_n}}{|(2-2^p)(k^3-k)|} \tag{2.43}$$

*for all* $x, u_2, \ldots, u_n \in \mathrm{X}$, *where*

$$\begin{aligned}\epsilon = {} & \left(1 + |k| + 2^{3-p}|k|\right)\left[(2k+1)^p + (2k-1)^p\right] + 2|k| + 13 + 3^p + 3|k|^p + 16k^2 + 3^p|k|^p + 2^{p+1}k^2 \\ & + 2^p\left(5 + |k|^p\right) + 2|k+1|^p + 2|k-1|^p + 2^{3-p}\left(2 + |k| + |k|^p + 3^p|k|^p\right) + \frac{(|k|+1)|k+1|^p}{|k-1|} \\ & + \frac{2^{3-p}|k|}{|k-1|}|k+1|^p + \left(1 + 8k^2 + 2^pk^2\right)|k-1|^{p-1} + \frac{2^p|k|^p\left(k^2 + |k| - 1\right)}{|k-1|} \\ & + \frac{2}{|k-1|} + \frac{|k|\left(2^{3-p}+1\right)}{|k-1|}|3k-1|^p + \frac{8k^2 + 2|k| - 8}{|k-1|}|k|^p.\end{aligned} \tag{2.44}$$

*Proof.* Define $\varphi(x, y) = \theta(\|x\|_{\mathrm{X}}^p + \|y\|_{\mathrm{X}}^p)\|u_2\|_{\mathrm{X}}^{r_2} \cdots \|u_n\|_{\mathrm{X}}^{r_n}$ for all $x, y, u_2, \ldots, u_n \in \mathrm{X}$, and apply Theorems 2.1 and 2.2. □

The following example shows that the assumption $p \neq 1$ cannot be omitted in Corollary 2.3.

*Example 2.4.* Let $\mathrm{X} = \mathbb{C}$ be a linear space over $\mathbb{R}$. Define $\|\cdot,\cdot\| : \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ by $\|x_1, x_2\| = |a_1b_2 - a_2b_1|$, where $x_j = a_j + b_j i \in \mathbb{C}$, $a_j, b_j \in \mathbb{R}$, $j = 1, 2$ ($i = \sqrt{-1}$ is the imaginary unit). Then, $(\mathrm{X}, \|\cdot,\cdot\|)$ is a 2-normed linear space.

Let $\phi : \mathbb{C} \to \mathbb{C}$ defined by

$$\phi(x) = \begin{cases} x, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \geq 1. \end{cases} \tag{2.45}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x) = \sum_{m=0}^{\infty} \alpha^{-m}\phi(\alpha^m x) \tag{2.46}$$

for all $x \in \mathbb{C}$, where $\alpha > |k|$. Then, $f$ satisfies the functional inequality

$$\|Df(x, y), u\| \leq \frac{4\alpha^2(|k|+1)}{\alpha - 1}(|x| + |y|)|u| \tag{2.47}$$

for all $x, y, u \in \mathbb{C}$, but there do not exist an additive mapping $A : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $\|f(x) - A(x), u\| \leq d\ |x||u|$ for all $x, u \in \mathbb{C}$.

It is clear that $|f(x)| \le \alpha/(\alpha-1)$ for all $x \in \mathbb{C}$. If $|x|+|y|=0$ or $|x|+|y| \ge 1/\alpha$ for all $x, y \in \mathbb{C}$, then the inequality (2.47) holds. Now suppose that $0<|x|+|y|<1/\alpha$. Then, there exists an integer $n \ge 1$ such that

$$\frac{1}{\alpha^{n+1}} \le |x|+|y| < \frac{1}{\alpha^{n}}. \tag{2.48}$$

Hence, $\alpha^m|kx \pm y|<1, \alpha^m|x \pm y|<1, \alpha^m|x|<1$ for all $m=0,1,\ldots,n-1$. From the definition of $f$ and (2.48), we obtain that

$$\begin{aligned}
&\|Df(x,y),u\| \\
&\quad= \left\|\sum_{m=n}^{\infty}\alpha^{-m}\phi(\alpha^m(kx+y)) + \sum_{m=n}^{\infty}\alpha^{-m}\phi(\alpha^m(kx-y)) \ - k\sum_{m=n}^{\infty}\alpha^{-m}\phi(\alpha^m(x+y))\right. \\
&\qquad \left. -k\sum_{m=n}^{\infty}\alpha^{-m}\phi(\alpha^m(x-y)) \ - 2\sum_{m=n}^{\infty}\alpha^{-m}\phi(\alpha^m kx) + 2k\sum_{m=n}^{\infty}\alpha^{-m}\phi(\alpha^m x), u\right\| \\
&\quad\le \frac{4\alpha^2(|k|+1)}{\alpha-1}(|x|+|y|)|u|.
\end{aligned} \tag{2.49}$$

Therefore, $f$ satisfies (2.47). Now, we claim that the functional equation (1.1) is not stable for $p=1$ in Corollary 2.3. Suppose on the contrary that there exist an additive mapping $A : \mathbb{C} \to \mathbb{C}$ and a constant $d>0$ such that $\|f(x)-A(x),u\| \le d\ |x||u|$ for all $x,u \in \mathbb{C}$. Then, there exists a constant $c \in \mathbb{C}$ such that $A(x)=cx$ for all rational numbers $x$. So, we obtain that

$$\|f(x),u\| \le (d+|c|)\ |x||u| \tag{2.50}$$

for all rational numbers $x$ and all $u \in \mathbb{C}$. Let $s \in \mathbb{N}$ with $s+1>d+|c|$. If $x$ is a rational number in $(0,\alpha^{-s})$ and $u=bi$ $(b \in \mathbb{R})$, then $\alpha^m x \in (0,1)$ for all $m=0,1,\ldots,s$, and we get

$$\|f(x),u\| = \left\|\sum_{m=0}^{\infty}\frac{\phi(\alpha^m x)}{\alpha^m},u\right\| \ge \sum_{m=0}^{s}\frac{\phi(\alpha^m x)}{\alpha^m}|b| = (s+1)x|b| > (d+|c|)x|b| = (d+|c|)|x||u|, \tag{2.51}$$

which contradicts (2.50).

**Theorem 2.5.** *Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0)=0$ for which there is a function $\varphi : X^{n+1} \to [0,\infty)$ such that*

$$\sum_{j=0}^{\infty}\frac{1}{8^j}\varphi\left(2^j x, 2^j y, u_2,\ldots,u_n\right) < \infty, \tag{2.52}$$

$$\|Df(x,y),u_2,\ldots,u_n\|_Y \le \varphi(x,y,u_2,\ldots,u_n) \tag{2.53}$$

*for all $x, y, u_2, \ldots, u_n \in X$. Then, there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \tilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.54}$$

*for all $x, u_2, \ldots, u_n \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.*

*Proof.* As in the proof of Theorem 2.1, we have

$$\|f(4x) - 10f(2x) + 16f(x), u_2, \ldots, u_n\|_Y \leq \tilde{\varphi}(x, u_2, \ldots, u_n) \tag{2.55}$$

for all $x \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Now, let $h : X \to Y$ be the mapping defined by $h(x) := f(2x) - 2f(x)$. By (2.55), we have

$$\|h(2x) - 8h(x), u_2, \ldots, u_n\|_Y \leq \tilde{\varphi}(x, u_2, \ldots, u_n) \tag{2.56}$$

for all $x \in X$. Replacing $x$ by $2^j x$ in (2.56) and dividing both sides of (2.56) by $8^{j+1}$, we get

$$\left\|\frac{1}{8^j} h(2^j x) - \frac{1}{8^{j+1}} h(2^{j+1} x), u_2, \ldots, u_n\right\|_Y \leq \frac{1}{8^{j+1}} \tilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.57}$$

for all $x, u_2, \ldots, u_n \in X$ and all integers $j \geq 0$. For all integers $l, m$ with $0 \leq l < m$, we have

$$\begin{aligned} \left\|\frac{1}{8^l} h(2^l x) - \frac{1}{8^m} h(2^m x), u_2, \ldots, u_n\right\|_Y &\leq \sum_{j=l}^{m-1} \left\|\frac{1}{8^j} h(2^j x) - \frac{1}{8^{j+1}} h(2^{j+1} x), u_2, \ldots, u_n\right\|_Y \\ &\leq \sum_{j=l}^{m-1} \frac{1}{8^{j+1}} \tilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \end{aligned} \tag{2.58}$$

for all $x, u_2, \ldots, u_n \in X$. So, we get

$$\lim_{l,m \to \infty} \left\|\frac{1}{8^l} h(2^l x) - \frac{1}{8^m} h(2^m x), u_2, \ldots, u_n\right\|_Y = 0 \tag{2.59}$$

for all $x, u_2, \ldots, u_n \in X$. This shows that the sequence $\{(1/8^j) h(2^j x)\}$ is a Cauchy sequence in $Y$. Since $Y$ is an $n$-Banach space, the sequence $\{(1/8^j) h(2^j x)\}$ converges. So, we can define a mapping $C : X \to Y$ by

$$C(x) := \lim_{j \to \infty} \frac{1}{8^j} h\left(2^j x\right) \tag{2.60}$$

for all $x \in X$. Putting $l = 0$, then passing the limit $m \to \infty$ in (2.58), and using Lemma 1.6(4), we get

$$\|h(x) - C(x), u_2, \ldots, u_n\|_Y \le \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.61}$$

for all $x, u_2, \ldots, u_n \in X$.

Now we show that C is cubic. By Lemma 1.6, (2.52), (2.58), and (2.60), we have

$$\begin{aligned}
\|C(2x) - 8C(x), u_2, \ldots, u_n\|_Y &= \lim_{j\to\infty} \left\| \frac{1}{8^j} h(2^{j+1}x) - \frac{1}{8^{j-1}} h(2^j x), u_2, \ldots, u_n \right\|_Y \\
&= 8 \lim_{j\to\infty} \left\| \frac{1}{8^{j+1}} h(2^{j+1}x) - \frac{1}{8^j} h(2^j x), u_2, \ldots, u_n \right\|_Y \\
&\le \lim_{j\to\infty} \frac{1}{8^j} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) = 0
\end{aligned} \tag{2.62}$$

for all $x, u_2, \ldots, u_n \in X$. By Lemma 1.6(3), $C(2x) = 8C(x)$ for all $x \in X$. Also, by Lemma 1.6(4), (2.52), (2.53), and (2.60), we get

$$\begin{aligned}
&\left\| DC(x, y), u_2, \ldots, u_n \right\|_Y \\
&\quad = \lim_{j\to\infty} \frac{1}{8^j} \left\| Dh(2^j x, 2^j y), u_2, \ldots, u_n \right\|_Y \\
&\quad = \lim_{j\to\infty} \frac{1}{8^j} \left\| Df(2^{j+1}x, 2^{j+1}y) - 2Df(2^j x, 2^j y), u_2, \ldots, u_n \right\|_Y \\
&\quad \le \lim_{j\to\infty} \frac{1}{8^j} \left[ \left\| Df(2^{j+1}x, 2^{j+1}y), u_2, \ldots, u_n \right\|_Y + 2\left\| Df(2^j x, 2^j y), u_2, \ldots, u_n \right\|_Y \right] \\
&\quad \le \lim_{j\to\infty} \frac{1}{8^j} \left[ \varphi\left(2^{j+1}x, 2^{j+1}y, u_2, \ldots, u_n\right) + 2\varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) \right] = 0
\end{aligned} \tag{2.63}$$

for all $x, y, u_2, \ldots, u_n \in X$. By Lemma 1.6(3), $DC(x, y) = 0$ for all $x, y \in X$. Hence the mapping $C$ satisfies (1.1). By [11, Lemma 2.3], the mapping $x \to C(2x) - 2C(x)$ is cubic. Therefore, $C(2x) = 8C(x)$ implies that the mapping $C$ is cubic.

To prove the uniqueness of $C$, let $S : X \to Y$ be another cubic mapping satisfying (2.54). Fix $x \in X$. Clearly, $C(2^l x) = 8^l A(x)$ and $S(2^l x) = 8^l S(x)$ for all $l \in \mathbb{N}$. It follows from (2.54) that

$$\begin{aligned}
\|C(x) - S(x), u_2, \ldots, u_n\|_Y &= \left\| \frac{C(2^l x)}{8^l} - \frac{S(2^l x)}{8^l}, u_2, \ldots, u_n \right\|_Y \\
&\leq \frac{1}{8^l}\left[ \left\| f(2^{l+1}x) - 2f(2^l x) - C(2^l x), u_2, \ldots, u_n \right\|_Y \right. \\
&\quad \left. + \left\| S(2^l x) - f(2^{l+1}x) + 2f(2^l x), u_2, \ldots, u_n \right\|_Y \right] \\
&\leq \frac{1}{8^l}\sum_{j=0}^{\infty} \frac{1}{8^j}\widetilde{\varphi}\left(2^{j+l}x, u_2, \ldots, u_n\right) \\
&\leq \sum_{j=0}^{\infty} \frac{1}{8^{j+l}}\widetilde{\varphi}\left(2^{j+l}x, u_2, \ldots, u_n\right) = \sum_{j=l}^{\infty} \frac{1}{8^j}\widetilde{\varphi}\left(2^{j}x, u_2, \ldots, u_n\right)
\end{aligned} \tag{2.64}$$

for all $x, u_2, \ldots, u_n \in X$, and $l \in \mathbb{N}$. By (2.52), we see that the right-hand side of the above inequality tends to 0 as $l \to \infty$. Therefore, $\|C(x) - S(x), u_2, \ldots, u_n\|_Y = 0$ for all $u_2, \ldots, u_n \in X$. By Lemma 1.6, we can conclude that $C(x) = S(x)$ for all $x \in X$. So $C = S$. This proves the uniqueness of $C$. □

**Theorem 2.6.** *Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that*

$$\begin{gathered}
\sum_{j=1}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, u_2, \ldots, u_n\right) < \infty, \\
\left\| Df(x, y), u_2, \ldots, u_n \right\|_Y \leq \varphi(x, y, u_2, \ldots, u_n)
\end{gathered} \tag{2.65}$$

*for all $x, y, u_2, \ldots, u_n \in X$. Then, there is a unique cubic mapping $C : X \to Y$ such that*

$$\left\| f(2x) - 2f(x) - C(x), u_2, \ldots, u_n \right\|_Y \leq \sum_{j=1}^{\infty} 8^{j-1}\widetilde{\varphi}\left(\frac{x}{2^j}, u_2, \ldots, u_n\right) \tag{2.66}$$

*for all $x, u_2, \ldots, u_n \in X$, where $\widetilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.*

*Proof.* The proof is similar to the proof of Theorem 2.5. □

**Corollary 2.7.** *Let $X$ be a normed space and $Y$ an $n$-Banach space. Let $\theta \in [0, \infty), p, r_2, \ldots, r_n \in (0, \infty)$ such that $p \neq 3$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\left\| Df(x, y), u_2, \ldots, u_n \right\|_Y \leq \theta\left(\|x\|_X^p + \|y\|_X^p\right)\|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n} \tag{2.67}$$

*for all $x, y, u_2, \ldots, u_n \in X$. Then, there exists a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x), u_2, \ldots, u_n\|_Y \leq \frac{\theta\epsilon\|x\|_X^p\|u_2\|_X^{r_2}\cdots\|u_n\|_X^{r_n}}{|(8-2^p)(k^3-k)|} \tag{2.68}$$

*for all $x, u_2, \ldots, u_n \in X$, where $\epsilon$ is defined as in Corollary 2.3.*

*Proof.* Define $\varphi(x,y) = \theta(\|x\|_X^p + \|y\|_X^p)\|u_2\|_X^{r_2}\cdots\|u_n\|_X^{r_n}$ for all $x, y, u_2, \ldots, u_n \in X$, and apply Theorems 2.5 and 2.6. □

The following example shows that the the generalized Hyers-Ulam stability problem for the case of $p = 3$ was excluded in Corollary 2.7.

*Example 2.8.* Let $X = \mathbb{C}$ be a linear space over $\mathbb{R}$, and let $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ be defined as in Example 2.4. Then, $(X, \|\cdot,\cdot\|)$ is a 2-normed linear space.

Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x^3, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \geq 1. \end{cases} \tag{2.69}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x) = \sum_{m=0}^{\infty} \alpha^{-3m}\phi(\alpha^m x) \tag{2.70}$$

for all $x \in \mathbb{C}$, where $\alpha > |k|$. Then, $f$ satisfies the functional inequality

$$\|Df(x,y), u\| \leq \frac{4\alpha^6(|k|+1)}{\alpha^3 - 1}\left(|x|^3 + |y|^3\right)|u| \tag{2.71}$$

for all $x, y, u \in \mathbb{C}$, but there do not exist a cubic mapping $C : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $\|f(x) - C(x), u\| \leq d\,|x|^3|u|$ for all $x, u \in \mathbb{C}$.

It is clear that $|f(x)| \leq \alpha^3/(\alpha^3 - 1)$ for all $x \in \mathbb{C}$. If $|x|^3 + |y|^3 = 0$ or $|x|^3 + |y|^3 \geq 1/\alpha^3$ for all $x, y \in \mathbb{C}$, then inequality (2.71) holds. Now suppose that $0 < |x|^3 + |y|^3 < 1/\alpha^3$. Then, there exists an integer $n \geq 1$ such that

$$\frac{1}{\alpha^{3(n+1)}} \leq |x|^3 + |y|^3 < \frac{1}{\alpha^{3n}}. \tag{2.72}$$

Hence, $\alpha^m|kx \pm y| < 1, \alpha^m|x \pm y| < 1, \alpha^m|x| < 1$ for all $m = 0, 1, \ldots, n-1$. From the definition of $f$ and (2.72), we obtain that

$$\begin{aligned}
\|Df(x,y),u\| &= \Bigg\| \sum_{m=n}^{\infty} \alpha^{-3m}\phi(\alpha^m(kx+y)) + \sum_{m=n}^{\infty} \alpha^{-3m}\phi(\alpha^m(kx-y)) - k\sum_{m=n}^{\infty} \alpha^{-3m}\phi(\alpha^m(x+y)) \\
&\quad -k\sum_{m=n}^{\infty} \alpha^{-3m}\phi(\alpha^m(x-y)) - 2\sum_{m=n}^{\infty} \alpha^{-3m}\phi(\alpha^m kx) + 2k\sum_{m=n}^{\infty} \alpha^{-3m}\phi(\alpha^m x), u \Bigg\| \\
&\leq \frac{4\alpha^6(|k|+1)}{\alpha^3 - 1}\left(|x|^3 + |y|^3\right)|u|.
\end{aligned} \tag{2.73}$$

Therefore, $f$ satisfies (2.71). Now, we claim that the functional equation (1.1) is not stable for $p = 3$ in Corollary 2.7. Suppose on the contrary that there exist a cubic mapping $C : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $\|f(x) - C(x), u\| \leq d\ |x|^3|u|$ for all $x, u \in \mathbb{C}$. Then, there exists a constant $\beta \in \mathbb{C}$ such that $C(x) = \beta x^3$ for all rational numbers $x$. So, we obtain that

$$\|f(x), u\| \leq (d + |\beta|)|x|^3|u| \tag{2.74}$$

for all rational numbers $x$ and all $u \in \mathbb{C}$. Let $s \in \mathbb{N}$ with $s+1 > d+|\beta|$. If $x$ is a rational number in $(0, \alpha^{-s})$ and $u = bi$ $(b \in \mathbb{R})$, then $\alpha^m x \in (0,1)$ for all $m = 0, 1, \ldots, s$, and we get

$$\begin{aligned}
\|f(x), u\| &= \left\| \sum_{m=0}^{\infty} \frac{\phi(\alpha^m x)}{\alpha^{3m}}, u \right\| \geq \sum_{m=0}^{s} \frac{\phi(\alpha^m x)}{\alpha^{3m}}|b| \\
&= (s+1)x^3|b| > (d + |\beta|)x^3|b| = (d + |\beta|)|x|^3|u|,
\end{aligned} \tag{2.75}$$

which contradicts (2.74).

**Theorem 2.9.** *Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that*

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) < \infty, \tag{2.76}$$

$$\|Df(x,y), u_2, \ldots, u_n\|_Y \leq \varphi(x, y, u_2, \ldots, u_n) \tag{2.77}$$

*for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that*

$$\|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y \leq \frac{1}{6}\sum_{j=0}^{\infty}\left(\frac{1}{2^{j+1}} + \frac{1}{8^{j+1}}\right)\widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.78}$$

*for all $x, u_2, \ldots, u_n \in X$, where $\widetilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.*

*Proof.* By Theorems 2.1 and 2.5, there exist an additive mapping $A' : X \to Y$ and a cubic mapping $C' : X \to Y$ such that

$$\begin{aligned}
\left\| f(2x) - 8f(x) - A'(x), u_2, \ldots, u_n \right\|_Y &\le \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right), \\
\left\| f(2x) - 2f(x) - C'(x), u_2, \ldots, u_n \right\|_Y &\le \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right)
\end{aligned} \tag{2.79}$$

for all $x, u_2, \ldots, u_n \in X$. Hence,

$$\left\| f(x) + \frac{1}{6}A'(x) - \frac{1}{6}C'(x), u_2, \ldots, u_n \right\|_Y \le \frac{1}{6} \sum_{j=0}^{\infty} \left( \frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.80}$$

for all $x \in X$. So, we obtain (2.78) by letting $A(x) = -(1/6)A'(x)$ and $C(x) = (1/6)C'(x)$ for all $x \in X$.

To prove the uniqueness of $A$ and $C$, let $A'', C'' : X \to Y$ be another additive and cubic mapping satisfying (2.78). Fix $x \in X$. Let $A_1 = A - A''$ and $C_1 = C - C''$. So,

$$\begin{aligned}
&\|A_1(x) + C_1(x), u_2, \ldots, u_n\|_Y \\
&\quad \le \left\| f(x) - A(x) - C(x), u_2, \ldots, u_n \right\|_Y + \left\| f(x) - A''(x) - C''(x), u_2, \ldots, u_n \right\|_Y \\
&\quad \le \frac{1}{3} \sum_{j=0}^{\infty} \left( \frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right)
\end{aligned} \tag{2.81}$$

for all $x, u_2, \ldots, u_n \in X$. Then (2.76) implies that

$$\lim_{n \to \infty} \frac{1}{8^n} \|A_1(2^n x) + C_1(2^n x), u_2, \ldots, u_n\|_Y = 0 \tag{2.82}$$

for all $x, u_2, \ldots, u_n \in X$. Thus, $C_1 = 0$. So, it follows from (2.81) that

$$\|A_1(x), u_2, \ldots, u_n\|_Y \le \frac{1}{3} \sum_{j=0}^{\infty} \left( \frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.83}$$

for all $u_2, \ldots, u_n \in X$. Therefore, $A_1 = 0$. □

Similarly to Theorem 2.9, one can prove the following result.

**Theorem 2.10.** *Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0,\infty)$ such that*

$$\sum_{j=0}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, u_2, \ldots, u_n\right) < \infty,$$
$$\|Df(x,y), u_2, \ldots, u_n\|_Y \le \varphi(x, y, u_2, \ldots, u_n) \tag{2.84}$$

*for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that*

$$\|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y \le \frac{1}{6}\sum_{j=1}^{\infty}\left(2^{j-1} + 8^{j-1}\right)\widetilde{\varphi}\left(\frac{x}{2^j}, u_2, \ldots, u_n\right) \tag{2.85}$$

*for all $x, u_2, \ldots, u_n \in X$, where $\widetilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.*

*Proof.* The proof is similar to the proof of Theorem 2.9 and the result follows from Theorems 2.2 and 2.6. □

**Theorem 2.11.** *Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0,\infty)$ such that*

$$\sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, u_2, \ldots, u_n\right) < \infty, \quad \sum_{j=0}^{\infty} \frac{1}{8^j}\varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) < \infty,$$
$$\|Df(x,y), u_2, \ldots, u_n\|_Y \le \varphi(x, y, u_2, \ldots, u_n) \tag{2.86}$$

*for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that*

$$\begin{aligned} &\|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y \\ &\quad \le \frac{1}{6}\left[\sum_{j=1}^{\infty} 2^{j-1}\widetilde{\varphi}\left(\frac{x}{2^j}, u_2, \ldots, u_n\right) + \sum_{j=0}^{\infty}\frac{1}{8^{j+1}}\widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right)\right] \end{aligned} \tag{2.87}$$

*for all $x, u_2, \ldots, u_n \in X$, where $\widetilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.*

*Proof.* The proof is similar to the proof of Theorem 2.9 and the result follows from Theorems 2.2 and 2.5. □

**Corollary 2.12.** *Let $X$ be a normed space and $Y$ an $n$-Banach space. Let $\theta \in [0,\infty), r_2, \ldots, r_n \in (0,\infty)$, $p \in (0,1) \cup (1,3) \cup (3,\infty)$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\|Df(x,y), u_2, \ldots, u_n\|_Y \le \theta\left(\|x\|_X^p + \|y\|_X^p\right)\|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n} \tag{2.88}$$

*for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that*

$$\left\| f(x) - A(x) - C(x), u_2, \ldots, u_n \right\|_Y \leq \frac{1}{6|k^3 - k|}\left(\frac{1}{|2 - 2^p|} + \frac{1}{|8 - 2^p|}\right)\theta\epsilon \|x\|_X^p \|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n} \tag{2.89}$$

*for all $x, u_2, \ldots, u_n \in X$, where $\epsilon$ is defined as in Corollary 2.3.*

*Proof.* Define $\varphi(x, y) = \theta(\|x\|_X^p + \|y\|_X^p)\|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n}$ for all $x, y, u_2, \ldots, u_n \in X$, and apply Theorems 2.9–2.11. □

*Remark 2.13.* The generalized Hyers-Ulam stability problem for the cases of $p = 1$ and $p = 3$ was excluded in Corollary 2.12 (see Examples 2.4 and 2.8).

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*Research Article*

# Ulam Stability of a Quartic Functional Equation

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The oldest quartic functional equation was introduced by J. M. Rassias in (1999), and then was employed by other authors. The functional equation $f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$ is called a *quartic functional equation*, all of its solution is said to be a *quartic function*. In the current paper, the Hyers-Ulam stability and the superstability for quartic functional equations are established by using the fixed-point alternative theorem.

## 1. Introduction

We say a functional equation $\digamma$ is *stable* if any function $f$ satisfying the equation $\digamma$ approximately is near to true solution of $\digamma$. Moreover, a functional equation $\digamma$ is *superstable* if any function $f$ satisfying the equation $\digamma$ approximately is a true solution of $\digamma$ (see [1] for another notion of the superstability which may be called *superstability modulo the bounded functions*).

The stability problem for functional equations originated from a question by Ulam [2] in 1940, concerning the stability of group homomorphisms: let $(G_1,\cdot)$ be a group, and let $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist $\delta>0$ such that, if a mapping $h: G_1 \to G_2$ satisfies the inequality $d(h(s\cdot t),h(s)*h(t))<\delta$ for all $s,t\in G_1$, then there exists a homomorphism $H: G_1\to G_2$ with $d(h(s),H(s))<\epsilon$ for all $s\in G_1$? In other words, under what condition a functional equation is stable? In the following year, Hyers [3] gave a partial affirmative answer to the question of Ulam for Banach spaces. In 1978, the generalized Hyers' theorem was independently rediscovered by Th. M. Rassias [4] by obtaining a unique linear mapping under certain continuity assumption.

The functional equations

$$f(x+y)+f(x-y)=2f(x)+2f(y),$$
$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

are called *quadratic* and *cubic* functional equations, respectively. During the last decades, several stability problems for functional equations especially the quadratic and cubic and their generalized have been extensively investigated by many mathematicians (for instances, [5–9]).

In [10], Lee et al. considered the following quartic functional equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to check that for every $a \in \mathbb{R}$, the function $f(x) = ax^4$ is a solution of the above functional equation. They solved (1.2) and in fact showed that a function $f : \mathcal{X} \to \mathcal{Y}$ whenever $\mathcal{X}$ and $\mathcal{Y}$ are real vector spaces is quadratic if and only if there exists a symmetric biquadratic function $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that $f(x) = F(x,x)$ for all $x \in \mathcal{X}$. They also proved the stability of (1.2). Zhou Xu et al. in [11] used the fixed-point alternative (Theorem 2.1 of the current paper) to establish Hyers-Ulam-Rassias stability of the general mixed additive-cubic functional equation, where functions map a linear space into a complete quasifuzzy $p$-normed space. The generalized Hyers-Ulam stability of a general mixed AQCQ-functional in multi-Banach spaces is also proved by using the mentioned theorem in [12].

Recently, Bodaghi et al. in [13, 14] investigated the stability and the superstability of quadratic and cubic functional equations by a fixed-point method and applied this method to prove the stability of (quadratic, cubic) multipliers on Banach algebras.

In this paper we prove the generalized Hyers-Ulam stability and the superstability for quartic functional equation (1.2) by using the alternative fixed point (Theorem 2.1) under certain conditions.

## 2. Main Results

Throughout this paper, assume that $\mathcal{X}$ is a normed vector space and $\mathcal{Y}$ is a Banach space. For a given mapping $f : \mathcal{X} \to \mathcal{Y}$, we consider

$$Df(x,y) := f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y), \tag{2.1}$$

for all $x, y \in \mathcal{X}$.

To achieve our aim, we need the following known fixed-point theorem which has been proved in [15].

**Theorem 2.1.** *Suppose that $(\Delta, d)$ is a complete generalized metric space, and let $\mathcal{J} : \Delta \to \Delta$ be a strictly contractive mapping with Lipschitz constant $L < 1$, Then for each element $g \in \Delta$, either $d(\mathcal{J}^n g, \mathcal{J}^{n+1} g) = \infty$ for all $n \geq 0$, or there exists a natural number $n_0$ such that*

(i) $d(\mathcal{J}^n g, \mathcal{J}^{n+1} g) < \infty$, *for all $n \geq n_0$,*

(ii) *the sequence $\{\mathcal{J}^n g\}$ is convergent to a fixed-point $g^*$ of $\mathcal{J}$,*

(iii) $g^*$ *is the unique fixed point of* $\mathcal{J}$ *in the set*

$$\Omega = \{g \in \Delta : d(\mathcal{J}^{n_0} g, g) < \infty\}; \tag{2.2}$$

(iv) $d(g, g^*) \leq (1/(1-L)) d(g, \mathcal{J} g)$, *for all* $g \in \Omega$.

**Theorem 2.2.** *Assume that* $\phi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *is a function satisfying*

$$\|Df(x, y)\| \leq \phi(x, y), \tag{2.3}$$

*for all* $x, y \in \mathcal{X}$. *Let a mapping* $f : \mathcal{X} \to \mathcal{Y}$ *satisfy* $f(0) = 0$. *If there exists* $K \in (0, 1)$ *such that*

$$\phi(x, y) \leq 2^4 K \phi\left(\frac{x}{2}, \frac{y}{2}\right), \tag{2.4}$$

*for all* $x, y \in \mathcal{X}$, *then there exists a unique quartic mapping* $Q : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{1}{32(1-K)} \phi(x, 0), \tag{2.5}$$

*for all* $x \in \mathcal{X}$.

*Proof.* By recurrence method, we can conclude from (2.4) that $\phi(2^n x, 2^n y)/2^{4n} \leq K^n \phi(x, y)$ for all $x, y \in \mathcal{X}$. Passing to the limit, we get

$$\lim_{n \to \infty} \frac{\phi(2^n x, 2^n y)}{2^{4n}} = 0, \tag{2.6}$$

for all $x, y \in \mathcal{X}$. Here, we intend to build the conditions of Theorem 2.1 and so consider the set $\Delta := \{h : \mathcal{X} \to \mathcal{Y} \mid h(0) = 0\}$ and the mapping $d$ defined on $\Delta \times \Delta$ by

$$d(g, h) := \inf\{C \in (0, \infty) : \|g(x) - h(x)\| \leq C\phi(x, 0)\ \forall x \in \mathcal{X}\} \tag{2.7}$$

if there exists such constant $C$, and $d(g, h) = \infty$ otherwise. It is easy to see that $d(h, h) = 0$ and $d(g, h) = d(h, g)$, for all $g, h \in \Delta$. For each $g, h, p \in \Delta$, we have

$$\begin{aligned}
&\inf\{C \in (0, \infty) : \|g(x) - h(x)\| \leq C\phi(x, 0)\ \forall x \in \mathcal{X}\} \\
&\quad \leq \inf\{C \in (0, \infty) : \|g(x) - p(x)\| \leq C\phi(x, 0)\ \forall x \in \mathcal{X}\} \\
&\qquad + \inf\{C \in (0, \infty) : \|p(x) - h(x)\| \leq C\phi(x, 0)\ \forall x \in \mathcal{X}\}.
\end{aligned} \tag{2.8}$$

Hence, $d(g, h) \leq d(g, p) + d(p, h)$. Now if $d(g, h) = 0$, then for every fixed $x_0 \in \mathcal{X}$, we have $\|g(x_0) - h(x_0)\| \leq C\phi(x_0, 0)$, for all $C > 0$. This implies $g = h$. Let $\{h_n\}$ be a $d$-Cauchy sequence in $\Delta$, then $d(h_m, h_n) \to 0$, and thus $\|h_m(x) - h_n(x)\| \to 0$, for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, then there exists $h \in \Delta$ such that $h_n \xrightarrow{d} h$ in $\Delta$. Therefore, $d$ is a generalized metric on $\Delta$, and the metric space $(\Delta, d)$ is complete. Now, we define the mapping $\mathcal{J} : \Delta \to \Delta$ by

$$\mathcal{J}g(x) = \frac{1}{2^4}g(2x), \quad (x \in \mathcal{X}). \tag{2.9}$$

Fix a $C \in (0,\infty)$ and take $g, h \in \Delta$ such that $d(g,h) < C$. The definitions of $d$ and $\mathcal{J}$ show that

$$\left\| \frac{1}{2^4}g(2x) - \frac{1}{2^4}h(2x) \right\| \le \frac{1}{2^4}C\phi(2x,0), \tag{2.10}$$

for all $x \in \mathcal{X}$. By using (2.4), we have

$$\left\| \frac{1}{2^4}g(2x) - \frac{1}{2^4}h(2x) \right\| \le CK\phi(x,0), \tag{2.11}$$

for all $x \in \mathcal{X}$. It follows from the above inequality that $d(\mathcal{J}g, \mathcal{J}h) \le Kd(g,h)$, for all $g, h \in \Delta$. Hence, $\mathcal{J}$ is a strictly contractive mapping on $\Delta$ with a Lipschitz constant $K$. Putting $y = 0$ in (2.3) and dividing both sides of the resulting inequality by 32, we have

$$\left\| f(x) - \frac{1}{16}f(2x) \right\| \le \frac{1}{32}\phi(x,0), \tag{2.12}$$

for all $x \in$ X. Thus, $d(f, \mathcal{J}f) \le 1/32 < \infty$. Note that by Theorem 2.1, $d(\mathcal{J}^n g, \mathcal{J}^{n+1}g) < \infty$, for all $n \ge 0$. Thus, we get $n_0 = 0$ in this theorem, so (iii) and (iv) of Theorem 2.1 are true on the whole $\Delta$. However, the sequence $\{\mathcal{J}^n f\}$ converges to a unique fixed-point $Q : \mathcal{X} \to \mathcal{Y}$ in the set $\{g \in \Delta; d(f,g) < \infty\}$, that is,

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{4n}}, \tag{2.13}$$

for all $x \in$ X. By the part (iv) of Theorem 2.1, we have

$$d(f,Q) \le \frac{d(f,\mathcal{J}f)}{1-K} \le \frac{1}{32(1-K)}. \tag{2.14}$$

From (2.14), we observe that the inequality (2.5) holds for all $x \in \mathcal{X}$. Substituting $x, y$ by $2^n x, 2^n y$ in (2.3), respectively, and applying (2.6) and (2.13), we have

$$\|DQ(x,y)\| = \lim_{n\to\infty} \frac{1}{2^{4n}} \|Df(2^n x, 2^n y)\| \le \lim_{n\to\infty} \frac{1}{2^{4n}}\phi(2^n x, 2^n y) = 0, \tag{2.15}$$

for all $x \in \mathcal{X}$. Therefore, $Q$ is a quartic mapping which is unique by part (iii) of Theorem 2.1. □

**Corollary 2.3.** *Let $p, \theta$ be nonnegative real numbers such that $p < 4$, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping (with $f(0) = 0$ when $p = 0$) satisfying*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p), \tag{2.16}$$

*for all $x, y \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{32 - 2^{p+1}}\|x\|^p, \tag{2.17}$$

*for all $x \in \mathcal{X}$.*

*Proof.* The result follows from Theorem 2.2 by using $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$. □

Now, we establish the superstability of quartic mapping on Banach spaces under some conditions.

**Corollary 2.4.** *Let $p, q, \theta$ be nonnegative real numbers such that $p + q \in (0, 4)$. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\|Df(x,y)\| \leq \theta\|x\|^p\|y\|^q, \tag{2.18}$$

*for all $x, y \in \mathcal{X}$, then $f$ is a quartic mapping on $\mathcal{X}$.*

*Proof.* Letting $\phi(x,y) = \theta\|x\|^p\|y\|^q$ in Theorem 2.2, we have

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{2^{4n}} = 0, \tag{2.19}$$

which shows (2.6) holds for $\phi$. Putting $x = y = 0$ in (2.18), we get $f(0) = 0$. Furthermore, if we put $y = 0$ in (2.18), then we have $f(2x) = 2^4 f(x)$, for all $x \in \mathcal{X}$. It is easy to see that by induction, we have $f(2^n x) = 2^{4n} f(x)$, and so $f(x) = f(2^n x)/2^{4n}$, for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Now, it follows from Theorem 2.2 that $f$ is a quartic mapping. □

Let $\theta$ and $p$ be positive real numbers. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies

$$\|Df(x,y)\| \leq \theta\|y\|^p, \tag{2.20}$$

for all $x, y \in \mathcal{X}$, then by considering $\phi(x,y) = \theta\|y\|^p$ in Theorem 2.2, the mapping $f$ is again a quartic mapping on $\mathcal{X}$.

The following result is proved in [16, Theorem 1].

**Theorem 2.5.** *Let $\mathcal{X}$ be a linear space, and let $\mathcal{Y}$ be a Banach space. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that*

$$\begin{gathered} \tilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 2^{-4k} \varphi\left(2^k x, 2^k y\right) < \infty, \\ \| Df(x, y) \| \le \delta + \varphi(x, y) \end{gathered} \tag{2.21}$$

*for all $x, y \in \mathcal{X}$, where $\delta \ge 0$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that*

$$\left\| f(x) - Q(x) + \frac{1}{5} f(0) \right\| \le \frac{1}{30} \delta + \frac{1}{32} \tilde{\varphi}(x, 0) \tag{2.22}$$

*for all $x \in \mathcal{X}$.*

One should note that in the above theorem, $f(0)$ is not necessarily zero, but in the following result, we assume that $f(0) = 0$ and also consider the case $\delta = 0$. By these hypotheses and by applying Theorem 2.1, we obtain the specific result which is a way to prove the superstability of a quartic functional equation.

**Theorem 2.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$, and let $\psi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function satisfying*

$$\lim_{n \to \infty} 2^{4n} \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \tag{2.23}$$

$$\| Df(x, y) \| \le \psi(x, y), \tag{2.24}$$

*for all $x, y \in \mathcal{X}$. If there exists $L \in (0, 1)$ such that*

$$\psi(x, 0) \le 2^{-4} L \psi(2x, 0), \tag{2.25}$$

*for all $x \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that*

$$\| f(x) - Q(x) \| \le \frac{L}{32(1 - L)} \psi(x, 0), \tag{2.26}$$

*for all $x \in \mathcal{X}$.*

*Proof.* We take the set $\Omega := \{ g : \mathcal{X} \to \mathcal{Y} \mid g(0) = 0 \}$ and consider the generalized metric on $\Omega$,

$$d(g_1, g_2) := \inf\{ C \in (0, \infty) : \| g_1(x) - g_2(x) \| \le C \psi(x, 0) \ \forall x \in \mathcal{X} \}, \tag{2.27}$$

if there exists such a constant $C$, and $d(g_1, g_2) = \infty$ otherwise. It follows from the proof of Theorem 2.2 that the metric space $(\Omega, d)$ is complete (see the proof of Theorem 2.2).

We will show that the mapping $\mathcal{J} : \Omega \to \Omega$ defined by $\mathcal{J}g(x) = 2^4 g(x/2)(x \in \mathcal{X})$ is strictly contractive. Fix a $C \in (0, \infty)$ and take $g_1, g_2 \in \Omega$ such that $d(g_1, g_2) < C$, then we have

$$\left\|2^4 g_1\left(\frac{x}{2}\right) - 2^4 g_2\left(\frac{x}{2}\right)\right\| \leq 2^4 C\varphi\left(\frac{x}{2}, 0\right), \tag{2.28}$$

for all $x \in \mathcal{X}$. By using (2.25), we obtain

$$\left\|2^4 g_1\left(\frac{x}{2}\right) - 2^4 g_2\left(\frac{x}{2}\right)\right\| \leq CL\varphi(x, 0), \tag{2.29}$$

for all $x \in \mathcal{X}$. It follows from the last inequality that $d(\mathcal{J}g_1, \mathcal{J}g_2) \leq Ld(g_1, g_2)$, for all $g_1, g_2 \in \Omega$. Hence, $\mathcal{J}$ is a strictly contractive mapping on $\Omega$ with a Lipschitz constant $L$. By putting $y = 0$, replacing $x$ by $x/2$ in (2.24) and using (2.25), and then dividing both sides of the resulting inequality by 2, we have

$$\left\|2^4 f\left(\frac{x}{2}\right) - f(x)\right\| \leq \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right) \leq 2^{-5} L\varphi(x, 0), \tag{2.30}$$

for all $x \in \mathcal{X}$. Hence, $d(f, \mathcal{J}f) \leq 2^{-5}L < \infty$. By applying the fixed-point alternative Theorem 2.1, there exists a unique mapping $Q : \mathcal{X} \to \mathcal{Y}$ in the set $\Omega_1 = \{g \in \Omega; d(f, g) < \infty\}$ such that

$$Q(x) = \lim_{n\to\infty} 2^{4n} f\left(\frac{x}{2^n}\right), \tag{2.31}$$

for all $x \in \mathcal{X}$. Again Theorem 2.1 shows that

$$d(f, Q) \leq \frac{d(f, \mathcal{J}f)}{1 - L} \leq \frac{2^{-5}L}{1 - L}. \tag{2.32}$$

Hence, inequality (2.32) implies (2.26). Replacing $x, y$ by $2^n x, 2^n y$ in (2.24), respectively, and using (2.23) and (2.31), we conclude that

$$\begin{aligned} \|DQ(x, y)\| &= \lim_{n\to\infty} 2^{4n} \left\|Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 2^{4n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \end{aligned} \tag{2.33}$$

for all $x \in \mathcal{X}$. Therefore, $Q$ is a quartic mapping. □

**Corollary 2.7.** *Let $p$ and $\lambda$ be nonnegative real numbers such that $p > 4$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying*

$$\|Df(x, y)\| \leq \lambda(\|x\|^p + \|y\|^p), \tag{2.34}$$

*for all $x, y \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - Q(x)\| \leq \frac{\lambda}{2(2^p - 2^4)}\|x\|^p \tag{2.35}$$

*for all $x \in \mathcal{X}$.*

*Proof.* It is enough to let $\varphi(x, y) = \lambda(\|x\|^p + \|y\|^p)$ in Theorem 2.6. □

**Corollary 2.8.** *Let $p, q, \lambda$ be nonnegative real numbers such that $p + q \in (4, \infty)$. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\|Df(x, y)\| \leq \lambda\|x\|^p\|y\|^q \tag{2.36}$$

*for all $x, y \in \mathcal{X}$. Then $f$ is a quartic mapping on $\mathcal{X}$.*

*Proof.* Putting $\varphi(x, y) = \theta\|x\|^p\|y\|^q$ in Theorem 2.6, we have

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^{4n}} = 0, \tag{2.37}$$

and thus, (2.6) holds. If we put $x = y = 0$ in (2.36), then we get $f(0) = 0$. Again, letting $y = 0$ in (2.36), we conclude that $f(x) = 2^4 f(x/2)$, and thus, $f(x) = 2^{4n} f(x/2^n)$, for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Now, we can obtain the desired result by Theorem 2.6.

From Corollaries 2.4 and 2.8 we deduce the following result. □

**Corollary 2.9.** *Let $p, q,$ and $\lambda$ be nonnegative real numbers such that $p + q > 0$ and $p + q \neq 4$. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies* (2.36), *for all $x, y \in \mathcal{X}$ then $f$ is a quartic mapping on $\mathcal{X}$.*

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