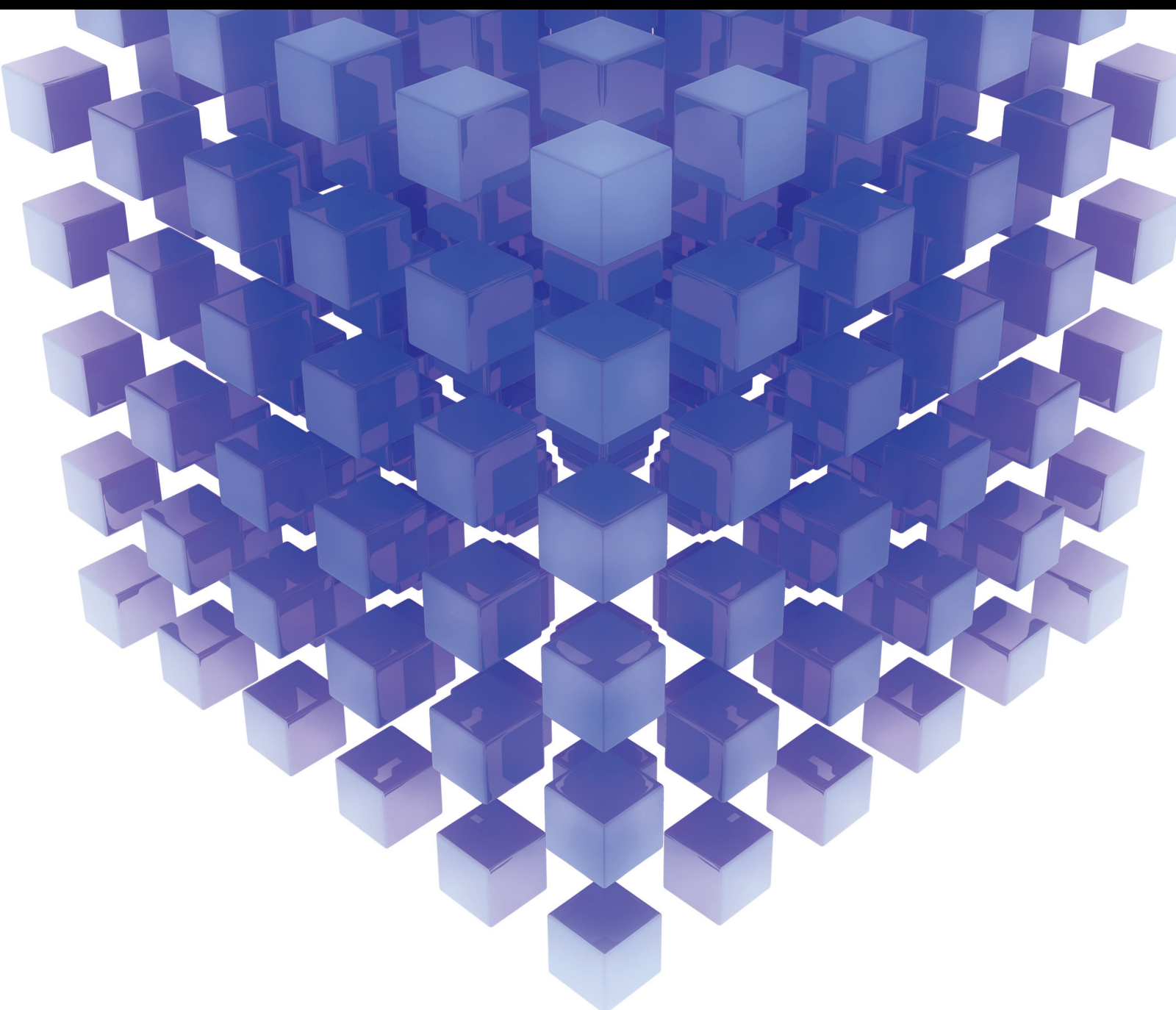


Fuzzy Applications in Engineering and Risk Management

Lead Guest Editor: Naeem Jan

Guest Editors: Ewa Rak, Lazim Abdullah, and László T. Kóczy





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Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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In addition, our investigation has also shown that one or more of the following human-subject reporting requirements has not been met in this article: ethical approval by an Institutional Review Board (IRB) committee or equivalent, patient/participant consent to participate, and/or agreement to publish patient/participant details (where relevant).

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Retraction

Retracted: Optimization of LR-Type Fully Bipolar Fuzzy Linear Programming Problems

Mathematical Problems in Engineering

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Retraction

Retracted: Neutrosophic Soft Quad Structures Concerning Neutrosophic Soft Points

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Retraction

Retracted: Estimating Lane Change Duration for Overtaking in Nonlane-Based Driving Behavior by Local Linear Model Trees (LOLIMOT)

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Retraction

Retracted: Analysis of the Shortest Path in Spherical Fuzzy Networks Using the Novel Dijkstra Algorithm

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Retraction

Retracted: A Public-Participation-Based Mixed Multiattribute Decision-Making Approach for Major Public Affairs

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Retraction

Retracted: A New Method for the Aggregate Proportion Calculation and Gradation Optimization of Asphalt-Treated Base (ATB-25)

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Retraction

Retracted: Spherical Cubic Fuzzy Extended TOPSIS Method and Its Application in Multicriteria Decision-Making

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Retraction

Retracted: Some Improved Correlation Coefficients for q-Rung Orthopair Fuzzy Sets and Their Applications in Cluster Analysis

Mathematical Problems in Engineering

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Retraction

Retracted: A Solution of Fredholm Integral Inclusions via Suzuki-Type Fuzzy Contractions

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Retraction

Retracted: Some Topological Approaches for Generalized Rough Sets via Ideals

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Retraction

Retracted: Global Dynamics of Sixth-Order Fuzzy Difference Equation

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Retraction

Retracted: Arab and Malay Students' Attitudes toward Statistics and Their Learning Styles: A Rasch Measurement Approach

Mathematical Problems in Engineering

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In addition, our investigation has also shown that one or more of the following human-subject reporting requirements has not been met in this article: ethical approval by an Institutional Review Board (IRB) committee or equivalent, patient/participant consent to participate, and/or agreement to publish patient/participant details (where relevant).

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Research Article

Graphical Analysis of q-Rung Orthopair Fuzzy Information with Application

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The q-rung orthopair fuzzy graph (q-ROFG) is an expansion of the intuitionistic fuzzy graph (IFG) and Pythagorean fuzzy graph (PFG); q-rung orthopair fuzzy model is an influential model for describing vagueness and uncertainty as a comparison to an intuitionistic fuzzy model and Pythagorean fuzzy model. The research aims to illustrate the notion of the graph of q-rung orthopair fuzzy sets (q-ROFSs). Furthermore, in this article, we examine the ideas of domination theory (DT) and double domination theory (DDT) in q-ROFGs. Additionally, the structure of q-ROFG is developed and its associated concept is presented through the assistance of instructive instances. Furthermore, the DT of q-ROFGs is established, as are cardinality, power, and completeness on dominance in a q-ROFG and bipartite q-ROFG, and double domination set (DDS), as well as some results, is investigated in the concept of q-ROFGs. A political campaign is simulated using the proposed structure as an application, and the impact of double dominance (DD) on political campaigns is investigated. Finally, a comparison is given between the proposed study and actual studies, as well as the advantages of working in the q-ROFG scenario.

1. Introduction

In 1965, Zadeh [1] initiated the notion of the fuzzy set (FS). The FS theory has been shown to be a useful tool for defining scenarios with uncertain or imprecise data. FS is an expansion of the crisp set. The intuitionistic fuzzy set (IFS) was first proposed by Atanassov [2]; here we can define the level of membership (LM) and the level of nonmembership (LNM) of an object. The IFS is an extension of FS which can be used in instances where FS is unsuitable. Yager [3] proposed the notion of Pythagorean fuzzy sets (PFSs) in 2013. PFS is an expansion of FS and IFS. As a consequence, it is better able to express and handle fuzzy data in practice and scientific study. The notion of q-ROFSs was also established by Yager [4], q-ROFS is an extension of IFS and PFS, and the major feature of q-ROFS is that the LM and LNM have a broader uncertain space. A q-ROFS is additionally more

useful and efficient than IFS or a PFS to describe uncertainty in different decision-making issues.

Kaufman [5] and Rosenfeld [6] proposed the notion of fuzzy graphs (FGs) as the generalization of crisp graphs. The vertices and edges of an FG are fuzzy numbers; therefore, a variety of investigators have applied the idea of FGs to real-world problems; for example, Sameena and Sunitha [7] utilized FGs in fuzzy neural networks, Sunitha and Mathew [8] investigated FG theory, and Yeh and Bang [9] used FGs in clustering analysis and cognitive and decision-making processes. IFG is expanded by Parvathi and Karunambigai [10], which is an extension of the FG. As previously stated, IFS is more beneficial than FS, and the same is true for IFGs and FGs. Talebi et al. [11] worked on new idea of IFG with application. Rangasamy et al. [12] worked out in a network the intuitionistic fuzzy shortest hyperpath. The IFSs and IFGs were extensively explored and presented by Akram and

Al-Shehrie [13]. Bozhenyuk et al. [14] worked on the dominance set in IFG.

Graph theory has been applied to a variety of fields. The DT and its uses are very prominent in graph theory. Sigaretta initiated the concept of total dominance on Certain Graphical Functions [15]. Different forms of dominance were developed by Karunambigai et al. [16] in IFG. DT is extremely useful in FGs and IFGs and has several applications. Somasundaram and Somasundaram [17] initiated the concept of DT of FGs, while Parvathi and Thamizhendhi [18] introduced the domination in IFGs. Additionally, [19, 20] addressed some features of strong domination in FGs. Borzooei and Rashmanlou [21] compiled and described the ideas of FG domination and their applications. Manjusha and Sunitha [22] presented some concepts in FGs; Zhang et al. [23] discussed how dynamic dominance develops in fuzzy causal systems. Shubatah [24] explored the notions of dominance and total domination in FGs. Jan et al. [25] investigated interval-valued PFGs decision-making and shortest path issues. Pachamuthu and Praveenkumar [26] introduced the idea of dominance in IFGs of second type. The graphs for n^{th} type IFGs and their applications were presented by Davvaz et al. [27].

The q-ROFG theory, which is based on q-ROFSs, was introduced by Habib and Farooq [28]; q-ROFGs are a generalization of IFGs and PFGs. FG simply addresses the level of membership, whereas IFGs, PFGs, and q-ROFGs describe both levels of membership and the level of non-memberships. Yin et al. [29] worked on product operations on q-ROFGs. Zeng et al. [30] expanded the concepts of q-rung orthopair fuzzy weight induced logarithm. Wan et al. [31] worked on the weight average LINMAP group decision-making based on q-rung orthopair fuzzy triangular. Peng et al. [32] worked on the q-rung orthopair fuzzy decision-making system which was developed for implementing mobile edge caching method preferences. Peng and Luo [33] introduced a study that collects a total of 80 publications related to q-ROFS in Web of Science for in-depth analysis. Roughly important results regarding the country, annual trends level, journal level, institutional level, and research landscape and highly cited papers are illustrated and generated. They summarized eighteen research challenges or future directions for the q-ROFS theory. Therefore, the motivation is given by the need to examine the ideas of domination theory (DT) and double domination theory (DDT) in q-ROFGs.

This article is arranged as follows. In Section 1, we give a quick overview of some basic definitions. In Section 2, we look through several essential terminologies and ideas associated with IFGs theory and q-ROFGs. In Section 3, we extend the notion of q-ROFGs, q-rung orthopair fuzzy subgraph (q-ROFSG), and the component of edge relationship, bridge, and cut-vertices. In Section 4, we clearly illustrate the notion of dominance in a q-rung orthopair fuzzy environment. Section 5 presents the idea of q-ROFG double dominance and related terminology. In part 6, we discuss an innovative use of double domination in a candidate's political campaign in a constituency. In part 7, we

conduct a comparative analysis of the proposed methodologies with previous work dominance. Eventually, in part 8, we provide a summary of the study.

2. Preliminaries

In this portion, we will go over various essential ideas associated with the theory of FGs, IFGs, intuitionistic fuzzy subgraph (IFSG), and q-ROFSs. In this part, we will also examine the DT of IFGs.

Definition 1 (see [5]). A pair $F = (\check{C}, \check{D})$ is said to be FG if

- (i) $\check{C} = \{c_1, c_2, c_3, \dots, c_i\}$ is the set of vertices, and $\xi_{\check{C}}: \check{C} \rightarrow [0, 1]$ represents the LM of c_i in \check{C} such that $0 \leq \xi_{\check{C}}(c_i) \leq 1$.
- (ii) $\check{D} = \{d_1, d_2, d_3, \dots, d_i\}$ is the set of edges, and $\check{D} \subseteq \check{C} \times \check{C}$. The mapping $\xi_{\check{D}}: \check{C} \times \check{C} \rightarrow [0, 1]$ represents the LM of d_i , where $\xi_{\check{D}}(c_i, c_j) \leq \min\{\xi_{\check{C}}(c_i), \xi_{\check{C}}(c_j)\}$ for $i, j \in \mathbb{N}$. The condition of $0 \leq \xi_{\check{D}}(c_i, c_j) \leq 1$ is satisfied.

Definition 2 (see [10]). A pair $F = (\check{C}, \check{D})$ is said to be IFG if

- (i) $\check{C} = \{c_1, c_2, c_3, c_4, \dots, c_i\}$ is the set of vertices, and $\xi_{\check{C}}: \check{C} \rightarrow [0, 1]$ and $\psi_{\check{C}}: \check{C} \rightarrow [0, 1]$ show the LM and LNM of c_i in \check{C} , respectively, such that

$$0 \leq \xi_{\check{C}}(c_i) + \psi_{\check{C}}(c_i) \leq 1. \quad (1)$$
- (ii) $\check{D} = \{d_1, d_2, d_3, \dots, d_i\}$ is the set of edges, and $\check{D} \subseteq \check{C} \times \check{C}$. The mappings $\xi_{\check{D}}: \check{C} \times \check{C} \rightarrow [0, 1]$ and $\psi_{\check{D}}: \check{C} \times \check{C} \rightarrow [0, 1]$ are known as the LM and the LNM of d_i , respectively, where $\xi_{\check{D}}(c_i, c_j) \leq \min\{\xi_{\check{C}}(c_i), \xi_{\check{C}}(c_j)\}$ and $\psi_{\check{D}}(c_i, c_j) \leq \max\{\psi_{\check{C}}(c_i), \psi_{\check{C}}(c_j)\}$ for $i, j \in \mathbb{N}$.

$$0 \leq \xi_{\check{D}}(c_i, c_j) + \psi_{\check{D}}(c_i, c_j) \leq 1. \quad (2)$$

Example 1. Let $F = (\check{C}, \check{D})$ be an IFG in Figure 1, let $\check{C} = \{c_1, c_2, c_3, c_4\}$ be the collection of vertices, and let $\check{D} = \{c_1c_2, c_3c_4, c_2c_4, c_1c_3\}$ be the collection of edges.

Definition 3 (see [10]). A pair $F = (\check{C}, \check{D})$ is called IFG; then an IFSG is of the form $F' = (\check{C}', \check{D}')$ if $\check{C}' \subseteq \check{C}$ and $\check{D}' \subseteq \check{D}$, where $\xi_{\check{C}'}(c_i) \leq \xi_{\check{C}}(c_i)$, $\xi_{\check{D}'}(d_i) \leq \xi_{\check{D}}(d_i)$, $\psi_{\check{C}'}(c_i) \geq \psi_{\check{C}}(c_i)$, and $\psi_{\check{D}'}(d_i) \geq \psi_{\check{D}}(d_i)$, for $i \in \mathbb{N}$.

Definition 4 (see [18]). Let $F = (\check{C}, \check{D})$ be an IFG; then, for any $c_1, c_2 \in \check{D}$, c_1 is said to dominate c_2 in F if $\xi_{\check{D}}(c_1, c_2) = \min\{\xi_{\check{C}}(c_1), \xi_{\check{C}}(c_2)\}$ and $\psi_{\check{D}}(c_1, c_2) = \max\{\psi_{\check{C}}(c_1), \psi_{\check{C}}(c_2)\}$. For $\check{C}' \subseteq \check{C}$, if, $\forall c' \notin \check{C}', \exists c \in \check{C} \ni c$ dominates c' , then \check{C} is called the DS in F . The dominant number (DN) of an IFG F is the fuzzy cardinality of DS with the smallest fuzzy cardinality in F . The DN of F is symbolized as $\varphi(F)$.

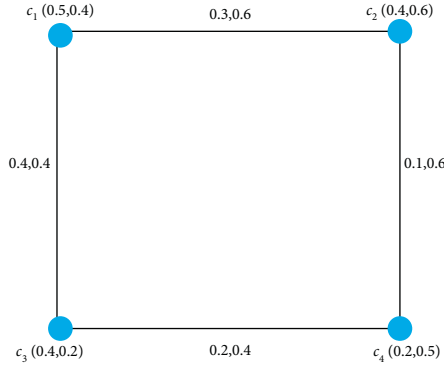


FIGURE 1: IFG.

Remark 1

- (i) For any $c_1, c_2 \in \check{C}$, if c_1 dominates c_2 , then c_2 dominates c_1 in F ; it displays the symmetry of domination
- (ii) If $\xi_{\check{D}}(c_1, c_2) = \min\{\xi_{\check{C}}(c_1), \xi_{\check{C}}(c_2)\}$ and $\psi_{\check{D}}(c_1, c_2) = \max\{\psi_{\check{C}}(c_1), \psi_{\check{C}}(c_2)\}$, for all $c_1, c_2 \in \check{C}$, then clearly the only DS in F is \check{C}

Definition 5 (see [18]). The strong edge of the edge (c_1, c_2) in an IFG is of the form $\xi_{\check{D}}(c_1, c_2) \geq \xi_{\check{D}}^{\infty}(c_1, c_2)$, where $\psi_{\check{D}}(c_1, c_2) \geq \psi_{\check{D}}^{\infty}(c_1, c_2)$.

Definition 6 (see [18]). If there is no proper dominating subset, the DS S of an IFG is known as minimum DS.

Definition 7 (see [18]). If there is a strong independent edge between a couple of vertices in an IFG F , they are considered independent.

Definition 8 (see [4]). Let \check{C} be a q-ROFS. \check{C} in E is defined as $\check{C} = \{e, \xi(e), \psi(e) | e \in E\}$, where $\xi(e): \rightarrow [0, 1]$ shows the LM and $\psi(e): \rightarrow [0, 1]$ denotes the LNM and the following condition is satisfied:

$$0 \leq \xi_{\check{C}}^n(e) + \psi_{\check{C}}^n(e) \leq 1 \quad \forall e \in E. \quad (3)$$

The double (ξ, ψ) represents the q-rung orthopair fuzzy number (q-ROFN).

3. q-Rung Orthopair Fuzzy Graphs

In this part, we develop the idea of q-ROFG, q-rung orthopair fuzzy subgraph (q-ROFSG), and the relationship between edges, bridge, and cut-vertices. With the help of examples, these ideas are demonstrated.

Definition 9 (see [28]). A pair $F = (\check{C}, \check{D})$ is said to be q-ROFG if we have the following:

- (i) $\check{C} = \{c_1, c_2, c_3, \dots, c_i\}$ is the collection of vertices, and $\xi_{\check{C}}: \check{C} \rightarrow [0, 1]$ and $\psi_{\check{C}}: \check{C} \rightarrow [0, 1]$ show the LM and LNM of c_i in \check{C} such that

$$0 \leq \xi_{\check{C}}^n(c_i) + \psi_{\check{C}}^n(c_i) \leq 1. \quad (4)$$

- (ii) $\check{D} = \{d_1, d_2, d_3, \dots, d_i\}$ is the set of edges, and $\check{D} \subseteq \check{C} \times \check{C}$. The mappings $\xi_{\check{D}}: \check{C} \times \check{C} \rightarrow [0, 1]$ and $\psi_{\check{D}}: \check{C} \times \check{C} \rightarrow [0, 1]$ indicate the LM and LNM of d_i , correspondingly, where $\xi_{\check{D}}(c_i, c_j) \leq \min\{\xi_{\check{C}}(c_i), \xi_{\check{C}}(c_j)\}$ and $\psi_{\check{D}}(c_i, c_j) \leq \max\{\psi_{\check{C}}(c_i), \psi_{\check{C}}(c_j)\}$ for $i, j \in \mathbb{N}$.

$$0 \leq \xi_{\check{D}}^n(c_i, c_j) + \psi_{\check{D}}^n(c_i, c_j) \leq 1. \quad (5)$$

Example 2. In Figure 2, $F = (\check{C}, \check{D})$ is a q-ROFG, $\check{C} = \{c_1, c_2, c_3, c_4\}$ is a collection of vertices, and $\check{D} = \{c_1c_2, c_2c_3, c_3c_4, c_4c_1\}$ is a collection of edges.

Figure 2 is simply q-ROFG for $n = 4$.

Remark 2. IFGs and PFGs are q-ROFGs but the inverse is not true.

Definition 10. Let $F = (\check{C}, \check{D})$ be a q-ROFG; then q-ROFSG is of the form $F' = (\check{C}', \check{D}')$ such that $\check{C}' \subseteq \check{C}$ and $\check{D}' \subseteq \check{D}$. $\xi_{\check{C}'}$ and $\xi_{\check{D}'}$ represent the LM and $\psi_{\check{C}'}$, $\psi_{\check{D}'}$ represent the LNM, such that $\xi_{\check{C}'}^n(c_i) \leq \xi_{\check{C}}^n(c_i)$, $\xi_{\check{D}'}^n(d_i) \leq \xi_{\check{D}}^n(d_i)$, $\psi_{\check{C}'}^n(c_i) \geq \psi_{\check{C}}^n(c_i)$, and $\psi_{\check{D}'}^n(d_i) \geq \psi_{\check{D}}^n(d_i)$, for $i \in \mathbb{N}$.

Definition 11. (c_i, c_j) is an edge in a q-ROFG $F = (\check{C}, \check{D})$. If removing this edge reduces the power of connectivity between pairs of vertices in F , it is called a bridge.

Example 3. Let $F = (\check{C}, \check{D})$ be a q-ROFG in Figure 3, and $\check{C} = \{c_1, c_2, c_3, c_4\}$ is a set of vertices and $\check{D} = \{c_1c_2, c_2c_3, c_3c_4, c_4c_1\}$ is a set of edges.

In Figure 3, (c_3, c_4) is a bridge.

Definition 12. The combination of two edge relationships $(d_{ij}, \xi_{\check{D}_{ij}}, \psi_{\check{D}_{ij}})$ and $(d_{jk}, \xi_{\check{D}_{jk}}, \psi_{\check{D}_{jk}})$ in a q-ROFG F , represented by $d_{ij} \circ d_{jk}$, is of the following form: $(d_{ik}, \xi_{\check{D}_{ik}}, \psi_{\check{D}_{ik}})$; here, $\xi_{\check{D}_{ik}} = \max\{\xi_{\check{D}_{ij}}, \xi_{\check{D}_{jk}}\}$ and $\psi_{\check{D}_{ik}} = \min\{\psi_{\check{D}_{ij}}, \psi_{\check{D}_{jk}}\}$, $\forall c_i, c_k \in \check{C}$.

Definition 13. If d_{ij} is an edge connection of q-ROFG F , then the power of d_{ij} is defined as

$$\begin{aligned} d_{ij}^1 &= d_{ij} = (d_{ij}, \xi_{\check{D}_{ij}}, \psi_{\check{D}_{ij}}), \\ d_{ij}^2 &= d_{ij} \circ d_{ij} = (d_{ij}, \xi_{\check{D}_{ij}}^2, \psi_{\check{D}_{ij}}^2), \\ d_{ij}^3 &= d_{ij} \circ d_{ij} \circ d_{ij} = (d_{ij}, \xi_{\check{D}_{ij}}^3, \psi_{\check{D}_{ij}}^3), \end{aligned} \quad (6)$$

and so on.

$$d_{ij}^{\infty} = (d_{ij}, \xi_{\check{D}_{ij}}^{\infty}, \psi_{\check{D}_{ij}}^{\infty}). \quad (7)$$

Here, $\xi_{\check{D}_{ij}}^{\infty} = \max_{y=1,2,\dots,n} \{\xi_{\check{D}_{ij}}^y\}$ and $\psi_{\check{D}_{ij}}^{\infty} = \min_{y=1,2,\dots,n} \{\psi_{\check{D}_{ij}}^y\}$ are the LM strength and the LNM strength of connecting between two vertices of c_i and c_j .

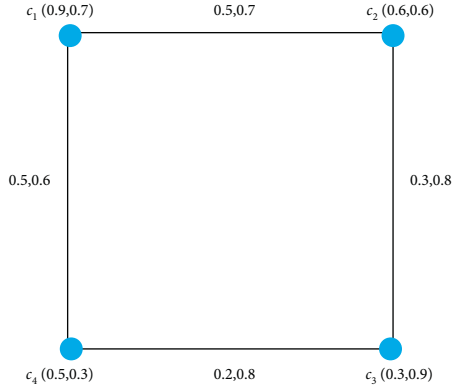


FIGURE 2: The q-ROFG.

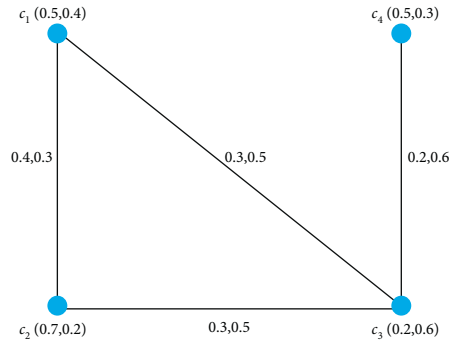


FIGURE 3: The q-ROFG for bridge.

We have

$$d_{ij}^0 = \begin{cases} 0, & \text{if } c_i \neq c_j \text{ and } (c_i, \xi_{\check{C}i}, \psi_{\check{C}i}) \\ (c_i, \xi_{\check{C}i}, \psi_{\check{C}i}) & \text{if } c_i \text{ is equal to } c_j \end{cases}. \quad (8)$$

Theorem 1. If $d_{ij} = (c_i, c_j)$ is an edge in F , then, for any two vertices in q-ROFG $F = (\check{C}, \check{D})$, the following statements are equipped:

- (i) d_{ij} is a bridge
- (ii) $\xi_{\check{D}ij}^{*\infty} < \xi_{\check{D}ij}$, and $\psi_{\check{D}ij}^{*\infty} > \psi_{\check{D}ij}$
- (iii) d_{ij} , there is no cycle's edge

Proof: We will do it as follows: $(ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$. Let $\xi_{\check{D}ij}^{*\infty} < \xi_{\check{D}ij}$ and $\psi_{\check{D}ij}^{*\infty} > \psi_{\check{D}ij}$. To demonstrate that (i) is accurate, we suppose that the opposite is true. After that,

$$\xi_{\check{D}ij}^{*\infty} = \xi_{\check{D}ij}^{\infty} \geq \xi_{\check{D}ij}, \text{ and } \psi_{\check{D}ij}^{*\infty} = \psi_{\check{D}ij}^{\infty} \leq \psi_{\check{D}ij} \quad (9)$$

$\Rightarrow \xi_{\check{D}ij}^{*\infty} \geq \xi_{\check{D}ij}$, and $\psi_{\check{D}ij}^{*\infty} \leq \psi_{\check{D}ij}$, resulting in a contradiction, so (i) is true.

$(i) \Rightarrow (iii)$. Now, we let d_{ij} be a cycle's edge; then, containing edge d_{ij} to any path can be converted onto a path that does not contain (c_i, c_j) . A cycle is a route from c_i to $c_j = d_{ij}$ which is not a bridge, since it is a contradiction. Thus, d_{ij} is not a cycle's edge.

$(iii) \Rightarrow (ii)$. The conclusion is simple. \square

Definition 14. c_i is a vertex in a q-ROFG $F = (\check{C}, \check{D})$; if removing this vertex reduces the power of connectivity between some vertices in F , it is called a cut-vertex.

Example 4. Suppose that $F = (\check{C}, \check{D})$ is a q-ROFG in Figure 4, and $\check{C} = \{c_1, c_2, c_3, c_4, c_5, \dots, c_i\}$ is a set of vertices and $\check{D} = \{c_1c_2, c_2c_4, c_4c_5, c_4c_3, c_3c_1\}$ is a set of edges.

In Figure 4, $n = 2$, and c_3 is a cut-vertex.

4. Domination in q-Rung Orthopair Fuzzy Graph

In this part, for q-ROFGs, we introduced the ideas of cardinality, power, and completeness. Additionally, we define the domination in q-ROFG and prove it with some results. First, we define vertex cardinality (VC) and edge cardinality (EC).

Definition 15. Let $F = (\check{C}, \check{D})$ be a q-ROFG. The VC of \check{C} is defined and represented by

$$|\check{C}| = \sum_{c_i \in \check{C}} \frac{1 + \xi_{\check{C}}(c_i) - \psi_{\check{C}}(c_i)}{2}, \quad \forall c_i \in \check{C}, \quad (10)$$

where $(\xi_{\check{C}}(c_i), \psi_{\check{C}}(c_i))$ shows the level of membership and the level of nonmembership.

Definition 16. Let $F = (\check{C}, \check{D})$ be a q-ROFG. Then, the EC of \check{D} is represented and defined by

$$|\check{D}| = \sum_{c_i, c_j \in \check{D}} \frac{1 + \xi_{\check{D}}(c_i, c_j) - \psi_{\check{D}}(c_i, c_j)}{2}, \quad \forall c_i, c_j \in \check{D}, \quad (11)$$

where $\xi_{\check{D}}(c_i, c_j), \psi_{\check{D}}(c_i, c_j)$ show the level of membership and the level of nonmembership.

Definition 17. The cardinality of a q-ROFG $F = (\check{C}, \check{D})$ is defined and represented by

$$|F| = \left| \sum_{c_i \in \check{C}} \frac{1 + \xi_{\check{C}}(c_i) - \psi_{\check{C}}(c_i)}{2} + \sum_{c_i, c_j \in \check{D}} \frac{1 + \xi_{\check{D}}(c_i, c_j) - \psi_{\check{D}}(c_i, c_j)}{2} \right|. \quad (12)$$

Remark 3. In a q-ROFG, the order of q-ROFG means the collection of vertices, and it is indicated by $\mathcal{O}(F)$. The size of a q-ROFG is defined as the set of edges in the q-ROFG F , and it is indicated by $S(F)$.

Definition 18. For $F = (\check{C}, \check{D})$ is a q-ROFG, the degree of vertex c_i is defined as the sum of the weight of SE occurrence to c_i , and it is represented by $k_F(c_i)$. The least degree of F is $\delta(F) = \min\{k_F(c_i) : c_i \in \check{C}\}$, and the maximum degree of F is $\Delta(F) = \max\{k_F(c_i) : c_i \in \check{C}\}$.

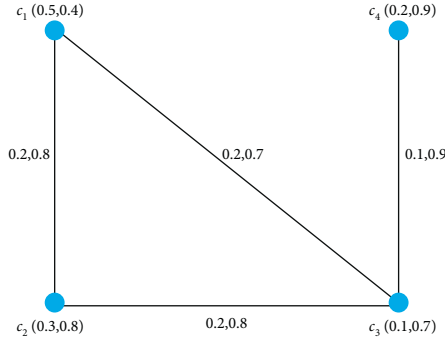


FIGURE 4: The q-ROFG for cut-vertex.

Definition 19. In a q-ROFG, a couple of vertices c_i and c_j are considered to be neighbors, if one or more of the following attributes is accurate:

- (i) $\xi_D(c_i, c_j) > 0, \psi_D(c_i, c_j) > 0$
- (ii) $\xi_D(c_i, c_j) = 0, \psi_D(c_i, c_j) > 0$
- (iii) $\xi_D(c_i, c_j) > 0, \psi_D(c_i, c_j) = 0, c_i, c_j \in \check{C}$

Definition 20. In a q-ROFG, a path is a collection of various vertices such that c_1, c_2, \dots, c_n . Then, one of the following requirements must be completed:

- (i) $\xi_D(c_i, c_j) > 0, \psi_D(c_i, c_j) > 0$
- (ii) $\xi_D(c_i, c_j) = 0, \psi_D(c_i, c_j) > 0$
- (iii) $\xi_D(c_i, c_j) > 0, \psi_D(c_i, c_j) = 0, c_i, c_j \in \check{C}, \text{ for } i, j \in \mathbb{N}$

The length of a path is $P = c_1, c_2, \dots, c_{n+1} (n > 0)$.

Remark 4. If a path connects a couple of vertices, they are said to be connected.

Definition 21. If c_i and c_j are a couple of vertices in q-ROFG F associated using the path, then the path's strength is defined as $(\min_{i,j} \xi_{Dij}, \max_{i,j} \psi_{Dij})$, where $\min \xi_{Dij}$ is the ξ -strength of the weakest edge and $\max_{i,j} \psi_{Dij}$ is the ψ -strength of the strongest edge.

Definition 22. If $c_i, c_j \in \check{C} \subseteq F$, then ξ -strength of connection between c_i and c_j is $\xi_D^{\infty}(c_i, c_j) = \sup\{\xi_D^x(c_i, c_j): x = 1, 2, \dots, n\}$, and ψ -strength of connectivity between c_i and c_j is $\psi_D^{\infty}(c_i, c_j) = \inf\{\psi_D^x(c_i, c_j): x = 1, 2, \dots, n\}$; if c and h are associated by ways of the path of length x ; then, $\xi_D^x(c, h)$ is defined as

$$\sup\{\xi_D(c, h_1) \min \xi_D(h_1, h_2) \min \xi_D(h_2, h_3) \dots \min \xi_D(h_{x-1}, h): c, h, h_2, \dots, h_{x-1}, h \in \check{C}\} \quad (13)$$

$$\psi_D^x(c, h) \text{ is defined as } \inf\{\psi_D(c, h_1) \max \psi_D(h_1, h_2) \max \psi_D(h_2, h_3) \dots \max \psi_D(h_{x-1}, h): c, h_1, h_2, \dots, h_{x-1}, h \in \check{C}\}. \quad (14)$$

Definition 23. A pair $F = (\check{C}, \check{D})$ is said to be a q-ROFG; then the complete q-ROFG is defined as $\xi_{Dij} = \min\{\xi_{Ci}, \psi_{Cj}\}$ and $\psi_{Dij} = \max\{\xi_{Ci}, \psi_{Cj}\}$ for every $c_i, c_j \in \check{C}$.

Definition 24. For a q-ROFG $F = (\check{C}, \check{D})$, an edge (c, h) is said to be an SE if $\xi_D(c, h) \geq \xi_D^{\infty}(c, h)$, and $\psi_D(c, h) \geq \psi_D^{\infty}(c, h)$. If there is an SE in $c, h \in \check{C}$, we say that c dominates h . A node's neighbor is denoted by $N(c) = \{h \in \check{C}: (c, h) \text{ is a strong edge}\}$. $\check{S} \subseteq \check{C}$ is assumed to be DS in F if, $\forall h \in \check{C} - \check{S}, c \in \check{S}$ such that c dominates h . If no proper subset of a DS occurs, it is called a minimal DS \check{S} , and the subset of \check{S} is DS cardinality. The lowest cardinality among all minimal DS is said to be the lower-dominant number (LDN) of F and is represented by $k_L(F)$. The upper dominant number (UDN) is defined as the maximum cardinality among all DS, and it is represented by $K_U(F)$. If there is no edge connecting the vertices, they are autonomous. A subset $\check{S} \subseteq \check{C}$ is an autonomous set of F , if $\xi_D(c, h) < \xi_D^{\infty}(c, h)$, and $\psi_D(c, h) < \psi_D^{\infty}(c, h)$, where $c, h \in \check{S}$. If the set $\check{S} \cup \{c\}$ is not independent for every vertex $h \in \check{C} - \check{S}$, the independent set \check{S} of F is assumed to be maximally independent. If each vertex $c \in \check{C} - K$, the set $\check{S} \cup \{c\}$ is not independent. For a minimum DS of F , if $\check{C} - K$ contains a DS K^{-1} of F , then K^{-1} is said to be the inverse of DS of F concerning K . Inverse dominating number $M^{-1}(F)$ of F is the cardinality of a minimum inverse DS of F . If a vertex $c \in \check{C}$ has merely one strong neighbor in F , it is known as an end-vertex.

5. Double Domination in q-Rung Orthopair Fuzzy Graph

In this part, we analyze the DDT of q-ROFGs. The idea of a DDS is described and illustrated with instances. The cardinality of DDS is investigated and the idea of minimal DD and maximal cardinality is studied.

Definition 25. Let $F = (\check{C}, \check{D})$ be a q-ROFG and $K \subseteq \check{C}$, where K is said to be DDS of F if each vertex in $K - \check{C}$ has minimum couple of vertices dominant in K . The double domination number (DDN) of F is described as the minimum fuzzy cardinality of all DDS of F and it is represented by $M_{kk}(F)$.

Example 5. Let $F = (\check{C}, \check{D})$ be a q-ROFG in Figure 5, and $C = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ is a collection of vertices and $\check{D} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ is a set of edges.

In Figure 5, we find out the DDN, where $n = 3$ d_1, d_2, d_3 , and d_4 are strong edges (SE) and the minimum dominating set $\{c_1, c_2, c_3, c_4, c_5\}$ is a DDS of F . So $\check{C} - K = \{c_6\}$ and the DDN is $M_{kk}(F) = 3.1$

Theorem 2 shows the presence requirements for double domination and is presented below.

Theorem 2. If the vertex in $\check{C} - K$ has at least two strong neighbors in a q-ROFG F , then DDS occurs in F .

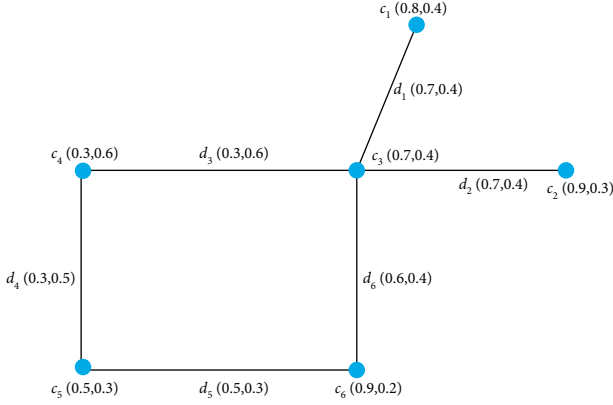


FIGURE 5: Graph with double domination number.

Proof. Let K be a DDS. If a vertex $c \in \check{C} - K$ includes only one strong neighbor, while further vertices $\check{C} - K$ have a minimum of two strong neighbors, then for all $c \in \check{C} - K$, there is a vertex $\check{h} \in K$ such that K is DS. This is an inconsistency, and so our assumption is incorrect. Therefore, each vertex in $\check{C} - K$ should hold a minimum of two strong neighbors. \square

Example 6. In Figure 6, $F = (\check{C}, \check{D})$ is a q-ROFG, where $\check{C} = \{c_1, c_2, c_3, c_4\}$ is a combination of vertices and $\check{D} = \{d_1, d_2, d_3, d_4, d_5\}$ is a combination of edges.

Here $n = 3$, d_1, d_2 , and d_5 are SEs and $K = \{c_1, c_3, c_4\}$. So $U - K = \{c_2\}$, and hence c_2 possesses at least a couple of strong neighbors seen in Figure 6.

Theorem 3 demonstrates the cardinality relation.

Theorem 3. If F is a q-ROFG and K is a DDS in F , then $|K| \geq |\check{C} - K|$.

Proof. By DD of K , each vertex \check{h} in $\check{C} - K$ requires at least two vertices in K and each \check{h} 's neighbor will appear in \check{D} . Furthermore, assume that the vertex is strong, and further dominating sets can be produced and neighboring vertices of \check{h} will appear in \check{D} . Hence, $|K| \geq |\check{C} - K|$. \square

Example 7. Suppose that $F = (\check{C}, \check{D})$ is a q-ROFG in Figure 7, and $\check{C} = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ is a set of vertices and $\check{D} = \{d_1, d_2, d_3, d_4, d_5\}$ is a collection of edges.

In Figure 7, $n = 4$ and all the edges are SEs and $K = \{c_1, c_3, c_5, c_6\}$. So $\check{C} - K = \{c_2, c_4\}$; therefore, $|K| = 4$ and $|\check{C} - K| = 2$, and thus $|K| \geq |\check{C} - K|$.

Theorem 4 examines the required conditions for minimizing double domination.

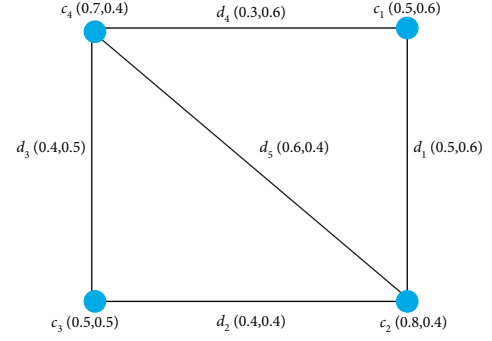
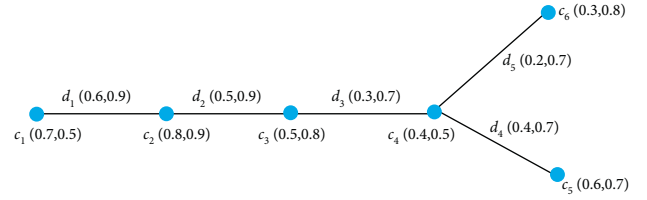
FIGURE 6: At least two neighbors in $\check{C} - K$.

FIGURE 7: Graph with cardinality.

Theorem 4. The DDS K is a minimal “iff” any two vertices $\{\check{h}, \omega\} \in K$; then at least one of its statements is correct.

- (i) There exists a vertex $c \in \check{C} - K$, such that $N(\check{C}) \cap K \neq \{\check{h}, \omega\}$
- (ii) $\check{C} - K$ is isolated

Proof. For minimal DS in q-ROFG F , suppose that $\check{h}, \omega \in K$ such that \check{h}, ω do not fulfill conditions (1) and (2); suppose that $K' = K - \{\check{h}, \omega\}$ is a DDS fulfilling properties (1) and (2). Hence $\check{C} - K'$ is isolated and there is the assumption that $\check{h}, \omega \in K$. This supports our statement.

Conversely, for each \check{h}, ω in DDS K , at least one of qualities (1) and (2) is true. Suppose on the contrary that K is not a minimum DDS. Then, $\check{h}, \omega \in K$ s.t. $K - \{\check{h}, \omega\}$ is DDS. As a consequence, \check{h}, ω are adjacent to at least one node in $K - \{\check{h}, \omega\}$, implying that \check{h}, ω are weak neighbors to all nodes in K . Hence, a node $c \in \check{C} - K$ s.t. $N(c) \cap K \neq \{\check{h}, \omega\}$ which is an inconsistency. Thus, K is minimal DDS.

Theorem 5 can be used to establish a relationship between double domination, greatest degree, and least degree correspondingly. \square

Theorem 5. Let $F = (\check{C}, \check{D})$ be a q-ROFG. If K is the minimal DDS, then the following property holds:

$$\check{W}(K) \leq \delta(F) + 2. \quad (15)$$

$$(i) \ddot{W}(K) \geq \Delta(F) - 1$$

Here $\ddot{W}(K)$, $\Delta(F)$, and $\delta(F)$ show the weight of DDS, greatest degree, and the least degree of F correspondingly.

Proof. Suppose that K is a minimal DDS.

$$\begin{aligned} \ddot{W}(K) &= \sum_{1 \leq i \leq n} \min \left(\sum [k_{\xi}(c_i)], \sum \max [k_{\psi}(c_i)] \right) \\ &\geq \left(\min \left[\sum [k_{\xi}(c_i)] \right], \min \left[\sum [k_{\psi}(c_i)] \right] \right) \\ &= (\delta_{\xi}(F), \delta_{\psi}(F)) \\ &= \delta(F) \\ &\leq \delta(F) + 2 \\ \ddot{W}(K) &= \left(\sum \min [k_{\xi}(u_i)], \sum \max [k_{\psi}(u_i)] \right) \\ &\leq \left(\max \left[\sum [k_{\xi}(u_i)] \right], \max \left[\sum [k_{\psi}(u_i)] \right] \right) \\ &= (\Delta_{\xi}(F), \Delta_{\psi}(F)) \\ &= \Delta(F) \\ &\geq \Delta(F) - 1. \end{aligned} \quad (16) \quad \square$$

Theorem 6. Suppose that $F = (\check{C}, \check{D})$ is a q -ROFG with only end vertex. Then DDS K does not exist.

Proof. For a q -ROFG F containing only end vertices, assume that $K \subseteq \check{C}$. As F is only ended vertex, for every $c \in \check{C} - K \exists c \in K$ such that K is DS.

Furthermore, not any $\check{C} - K$ vertex is dominated by a minimum of two vertices. As a result, there is not any DDS K . \square

Example 8. In Figure 8, $F = (\check{C}, \check{D})$ is a q -ROFG, where $\check{C} = \{c_1, c_2, c_3, c_4, c_5\}$ is a combination of vertices and $\check{D} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ is a combination of edges.

In Figure 8, $n = 2$, d_1 and d_3 are the SEs, and there is not any DDS. So, the other SEs are required to be d_2 and d_4 .

Theorem 7. For any q -ROFG $F = (\check{C}, \check{D})$, $M_{kk}(F) \geq \mathcal{O}(F)/\Delta_{\psi}(F) + 1$, where $\Delta_{\psi}(F)$ is the greatest ψ -degree of F and $\mathcal{O}(F)$ represents the order of q -ROF.

Proof. Suppose that K is a DDS of q -ROFG F with $|K| = M_{kk}(F)$. While each vertex in $\check{C} - K$ is nearby to a similar vertex in K , we have

$$\begin{aligned} |\check{C} - K| &\leq \sum_{i=1}^n k(c_i) \leq M_{kk}(F) \cdot \Delta_{\xi}(F), \\ \mathcal{O}(F) - M_{kk}(F) &\leq M_{kk}(F) \Delta_{\psi}(F), \\ \mathcal{O}(F) &\leq M_{kk}(F) \Delta_{\psi}(F) + M_{kk}(F), \\ &\leq M_{kk}(F) (\Delta_{\psi}(F) + 1). \end{aligned} \quad (17)$$

This implies that

$$M_{kk}(F) \geq \frac{\mathcal{O}(F)}{\Delta_{\psi}(F) + 1}. \quad (18)$$

Hence the proof is completed. Theorem 8 illustrates the presence of cut-vertices in DDS. \square

Theorem 8. If F is a q -ROFG with cut-vertex, then DDS K will have a minimum of one cut-vertex.

Proof. For a q -ROFG F with cut-vertex, it is assumed that no cut-vertex occurs in DDS K . Consider a cut-vertex $c \in \check{C} - K$. While cut-vertices are usually the last vertices, as a result, c only has one strong neighbor, which is located in K . Furthermore, K is not a DDS, since c in K is not dominated by two or more vertices. This is in opposition to our supposition. Consequently, a cut-vertex c should occur in the DDS K . The proof is now complete. \square

Example 9. Let $F = (\check{C}, \check{D})$ be a q -ROFG in Figure 9, and $\check{C} = \{c_1, c_2, c_3, c_4\}$ is a collection of vertices and $\check{D} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ is a collection of edges.

For $n = 2$, note that d_1, d_2, d_4, d_5 are the SEs and $K = \{c_1, c_3, c_4\}$; after that $\check{C} - K = \{c_2\}$; thus c_2 is a cut-vertex that can be seen in Figure 9.

Theorem 9. Let $F = (\check{C}, \check{D})$ be a q -ROFG, and $M^{-1}(F) \leq M_{kk}(F) \leq |C|$, where $M^{-1}(F)$ is a double dominating number.

Proof. Suppose that F is a q -ROFG. By Theorem 3, inverse domination set $K^{-1} \subseteq \check{C} - K$, $|K| \geq |\check{C} - K|$, implying that the DDN is higher than the inverse dominating number; that is, $M^{-1}(F) \leq M_{kk}(F)$, and DDS does not have all the vertices of F . This means that at least one of the vertices h must be in $\check{C} - K$. Thus $F - \{h\}$ provides the double domination number. Obviously $M_{kk}(F) \leq |\check{C}|$. Hence proved. \square

Theorem 10. For F being a q -ROFG, DDS K in F is independent but not in \bar{F} .

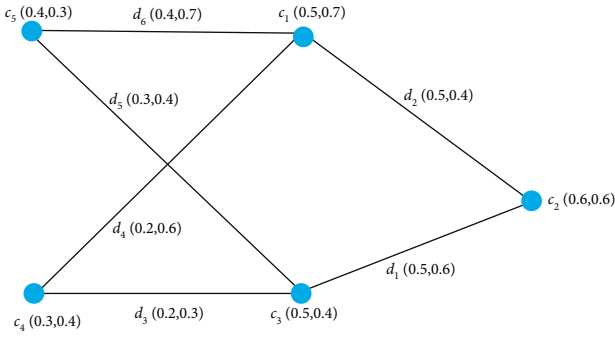
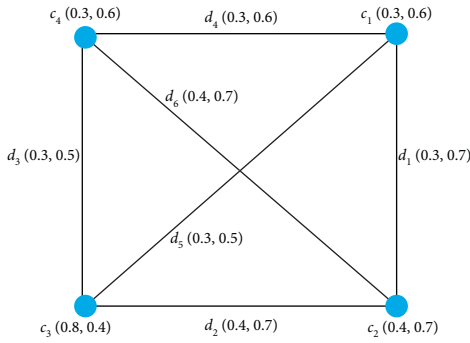
FIGURE 8: Graph F with only end nodes.

FIGURE 9: Graph of a DDS with at least one cut-vertex.

Proof. In q-ROFG with an independent DDS, it is assumed that \bar{F} is the complement of F , and $\bar{\bar{C}} = \check{C}$.

$\bar{\xi}_{\check{C}_{ij}} = \min(\bar{\xi}_{\check{C}_i}, \bar{\xi}_{\check{C}_j}) - \xi_{\check{C}_{ij}}$ and $\bar{\psi}_{\check{D}_{ij}} = \max(\bar{\psi}_{\check{D}_i}, \bar{\psi}_{\check{D}_j}) - \psi_{\check{D}_{ij}}$. Here the only difference is in the values of the edges in F . This indicates that neighboring vertices in \bar{F} contain significant neighbors having distinct DDS in \bar{F} . Thus, in F , the similar DDS K is not independent in \bar{F} . The proof is complete now. \square

6. Application

In political races, a lot of the time a politician's target is to achieve as many supporters as feasible in the shortest possible time frame. The idea of a q-rung orthopair DS could be quite useful for this purpose. In general, when a region has a considerable lot of electorates, each elector in that area has registered a node of q-ROFG. By doing so, a large amount of data could be readily managed. Because every node symbolizes an elector, it is possible that two electorates are quite familiar with one another, such as close colleagues or relatives, or that they have just known one another for a short time, or that they are complete strangers. Such relationships between two electorates are indicated by an edge between two vectors. Every edge will display the voting power of two people who are linked to it. If two electorates do not know each other, there is not any edge between them. This is well understood if a political leader contacts a specific

elector and secures his support; then, with the assistance of this candidate, he gets accessibility to each of his close friends or colleagues and so there is no need to communicate to each of them separately.

Employing the DT of q-ROFGs, the political leader just wants to convene the member of the DSs of q-ROFGs. Therefore all elements of DS will persuade more members to support the same political leader; thus the majority of electorates can support a certain political leader.

Example 10. Consider the q-ROFG in Figure 10, where we considered a precinct of 7 electorates, showing 7 vertices of q-ROFG, and used edges to discuss their connections. The q-ROFNs are used to represent the values of vertices and edges. $\check{C} = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$ is the total number of vertices and $\{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}\}$ is the total number of edges. The minimum DS in F is certain by $K = \{c_1, c_3\}$; thus, $K - \check{C} = \{c_2, c_4, c_5, c_6, c_7\}$.

Now the issue can be dealt with by utilizing the q-rung orthopair DS. In order to build q-ROFG, the vertices are designed to denote the electorate in the area. These vertices are associated via edges if the allocated electorates have any relationship. Every relationship between two electorates is given a fuzzy value based on the quality of their interaction. When two people have no relationships, they are considered disconnected.

Utilizing the domination in q-ROFG, in the graph, there is a minimum DS, as well as the political leader only assembling the participants of that group. Therefore, all DS participants now can demand votes from all non-DS members. Even if the political leader is unable to meet with all of the electorates, his party will receive a majority of the votes. In Figure 10, the political leader only desires to meet the electorate c_1, c_3 for winning an election. Algorithm 1 gives the steps to construct the q-ROFG.

7. Comparative Study and Advantages

The dominant idea of q-ROFG will be discussed in this part, which is more flexible than the dominant ideas of FG and IFG. The q-ROFG is an important tool for representing uncertainty and fuzziness. It can be used for enhancing decision-makers capability over orthopairs and their choices. In Example 10, a q-ROFG is discussed wherein the vertices and edges are in the form of q-ROFNs. In this scenario, a q-ROFN is preferable to fuzzy numbers and intuitionistic fuzzy numbers for representing doubtful conditions. Consider the IFG in Figure 11. This graph can be understood as an FG with neutral and a level of non-membership equivalent to zero. Nonetheless, if we look at the q-ROFG shown in Figures 11 and 12, it cannot be addressed by using ideas of FG and IFG, whereas a q-ROFG may or may not be considered FG or IFG.

Furthermore, FG and IFG are unable to deal with the issue raised previously, since these frameworks are bound to certain types of grades. On the other side, if we try and replicate this research in the domain of q-rung orthopair fuzzy environment, we will find that it fails owing to structural restrictions. On the other hand, if we try to do this

- (1) In the q-ROFNs configuration put all the values, including, LM and LNM all the vertices
- (2) Using q-ROFGs, describe all of the edges
- (3) Draw a contrast of all the edge's values
- (4) Construct the power for all edges
- (5) Outcome

ALGORITHM 1: The main steps to construct the q-ROFG.

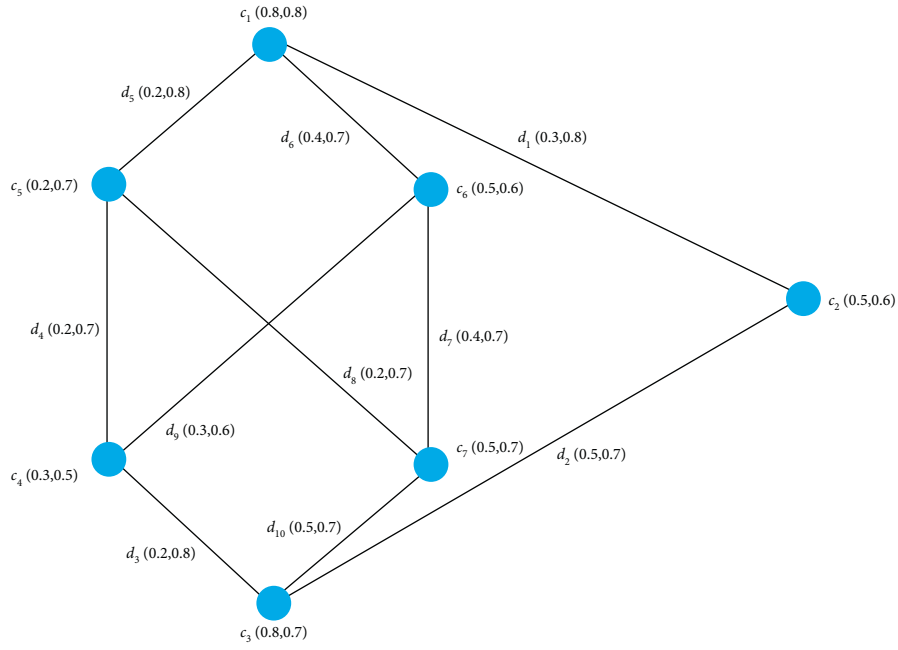


FIGURE 10: The q-rung orthopair fuzzy graphs for analysis.

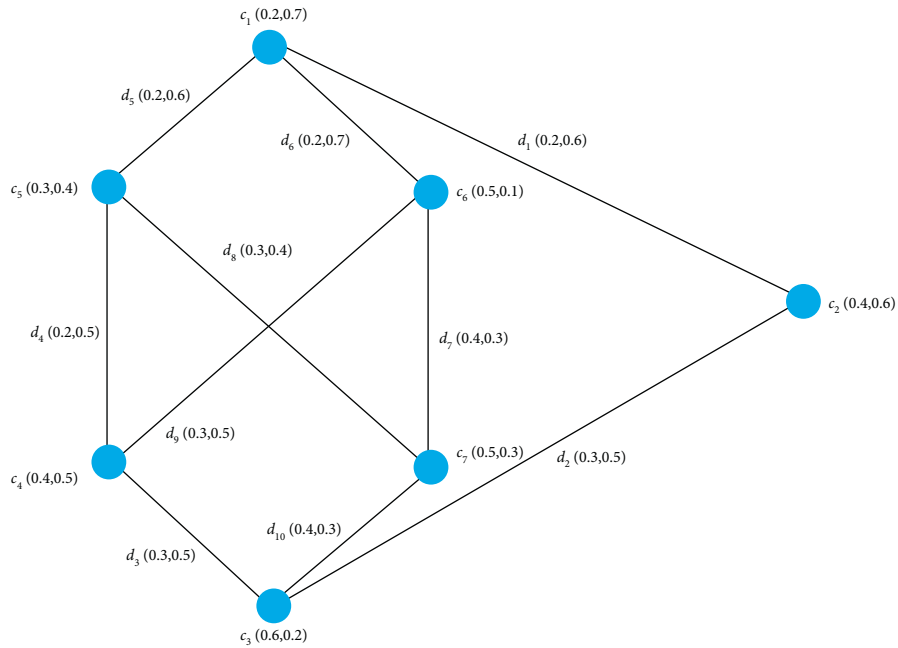


FIGURE 11: IFG for analysis.

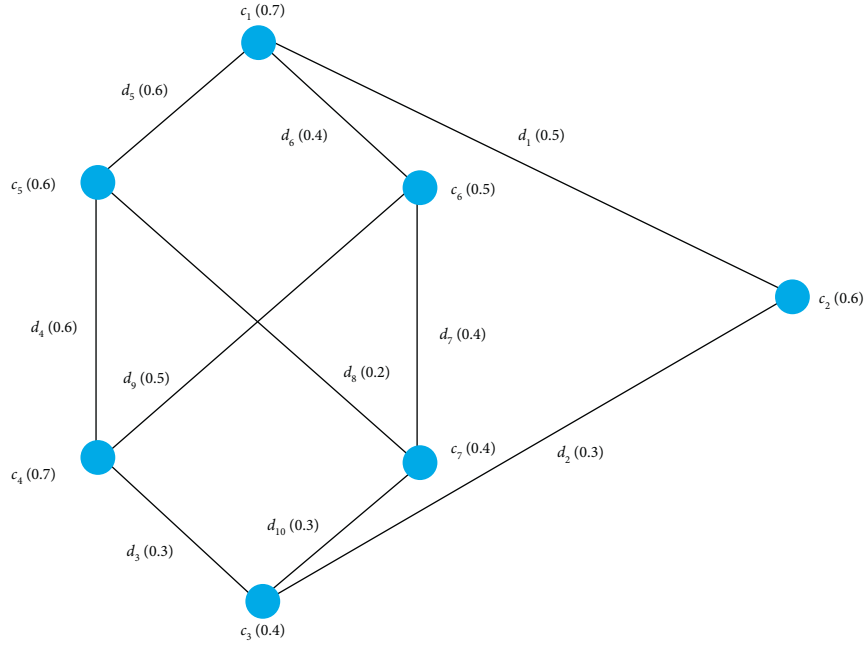


FIGURE 12: FG for analysis.

TABLE 1: q-rung orthopair fuzzy information when dealing with fuzzy graphs and intuitionistic fuzzy networks.

Object	Levels	Sum of levels	Explanations	Sum of squares of levels	Explanations
c_1	(0.8, 0.8)	1.6	False	1.28	False
c_2	(0.5, 0.6)	1.1	False	0.61	True
c_3	(0.8, 0.7)	1.5	False	1.13	False
c_4	(0.3, 0.5)	0.8	True	0.34	True
c_5	(0.2, 0.7)	0.9	True	0.53	True
c_6	(0.5, 0.6)	1.1	False	0.61	True
c_7	(0.5, 0.7)	1.2	False	0.74	True
d_{10}	(0.3, 0.8)	1.1	False	0.73	True
d_2	(0.5, 0.7)	1.2	False	0.74	True
d_3	(0.2, 0.8)	1	True	0.68	True
d_4	(0.3, 0.7)	1	True	0.58	True
d_5	(0.2, 0.8)	0.8	True	0.68	True
d_6	(0.4, 0.7)	1.1	False	0.65	True
d_7	(0.4, 0.7)	1.1	False	0.65	True
d_8	(0.2, 0.9)	1.1	False	0.85	True
d_9	(0.3, 0.6)	0.9	True	0.65	True
d_{10}	(0.5, 0.7)	1.2	False	0.74	True

research in the framework of fuzzy information or intuitionistic fuzzy information, we will most likely fail due to structural restrictions. Table 1 provides a thorough examination of the topic.

8. Conclusion

A new concept of q-ROFGs is suggested in this paper. This innovative notion enables us to extend all notions such as FGs and IFGs. Moreover, we conceptualized and modified the concept of double domination theory which elaborates all the currently accessible ideas in graph theory for diverse

structures. Basic operations such as cardinality, order, strength, and completeness on dominance, bipartite q-ROFG, and DDS are introduced and demonstrated. This research proposed the ideas of dominant and DDSs, as well as a study of associated terms based on examples. In addition, associated terminologies of q-ROFGs have been defined based on their attributes. The DT is demonstrated in conditions of an election campaign study. A political leader seeks to contact as many electors as possible in a short period. The idea of q-rung orthopair domination could be very useful in this particular skill scenario. The relative examination revealed that the planned framework is

innovative and aids us in resolving the deficiencies of already available concepts to deal with a situation where other instruments fail to work. We intend to investigate the expansion and fuse of the q-ROFS graph forms in relation to certain application scenes in future study.

Data Availability

Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of the research article.

Acknowledgments

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Retraction

Retracted: Risk Assessment and Prediction of Rainstorm and Flood Disaster Based on Henan Province, China

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] G. Deng, H. Chen, and S. Wang, "Risk Assessment and Prediction of Rainstorm and Flood Disaster Based on Henan Province, China," *Mathematical Problems in Engineering*, vol. 2022, Article ID 5310920, 17 pages, 2022.

Research Article

Risk Assessment and Prediction of Rainstorm and Flood Disaster Based on Henan Province, China

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To reasonably evaluate and predict the loss of rainstorm and flood disaster, this study is based on the rainfall data and rainstorm and flood disaster data of 18 cities in Henan Province from 2010 to 2020, using GIS technology and weighted comprehensive evaluation method to analyze the risk of rainstorm and flood disaster factors in various regions. The four risk factors of hazard risk, hazard-pregnant environment sensitivity, hazard-bearing body vulnerability, and disaster resilience were analyzed in compartment analysis. At the same time, a new rainstorm and flood disaster prediction model was constructed in combination with the hybrid PSO-SVR algorithm. The research results show that there are many rivers in Henan Province, the terrain tends to be higher in the west and lower in the east, and most areas are low plains, making most cities in Henan Province at a moderate risk level. For the more developed cities such as Zhengzhou, Luoyang, and Nanyang, the hazard risk, sensitivity, vulnerability, and disaster resistance are high, and they are prone to heavy rains and floods. For the economically underdeveloped, the terrain is high or hills, such as Sanmenxia City; Xinyang City and other places have low hazard risk and are not prone to rainstorms and floods. By constructing a hybrid PSO-SVR model, selecting two representative cities of Zhengzhou and Luoyang, and predicting the daily rainfall, the number of disasters, and the direct economic loss, the calculated RMSE and MAPE values are both less than GA-SVR, the traditional SVR, and BPNN models, which have verified the superiority of the model proposed in this study and the practical value it brings. To further verify the prediction accuracy of the hybrid model, the average value of RMSE and MAPE of other 16 cities are calculated, and the result is still smaller than other three models, and the study can provide some decision-making references for the urban rainstorm and flood management.

1. Introduction

The rainstorm disaster risk assessment is based on some obtained disaster evaluation index values, which is a comprehensive evaluation of the risk of hazard factors, sensitivity of hazard-pregnant environment, vulnerability of hazard-bearing body, and disaster resilience in the study area, providing reference for regional rainstorm risk management, flood prevention, and disaster mitigation planning [1–4]. It also provides a reasonable reference for long-term social and economic development planning [5–11]. With the deepening of high-intensity human engineering activities on a global scale, the global climate warming trend is becoming more and more significant, and extreme weather events occur frequently and cause a series of disasters. According to the global risk report released by World Economic Forum (WEF), extreme

weather emergencies from 2017 to 2020 have ranked first in terms of the probability of occurrence of the top ten global risks for four consecutive years. Affected by factors such as global temperature rise, sea level rise, and surface subsidence in some areas, the frequent and widespread problems of storm and flood disasters have become increasingly prominent [12–15]. Rainstorm and flood disasters have become one of the most common and serious natural disasters in many large cities in the world. In recent years, cities in the social and economic development centers of developing countries have suffered heavy rain disasters [16–18]. In this context, it is particularly important to conduct a scientific and reasonable risk assessment of storms and floods.

Zandvoort and Vlist [19] used a multilayer safety analysis method to evaluate rainstorm disasters to design a more robust and effective avoidance plan to improve the disaster

resistance in the area and reduce post-disaster losses. Otari et al. [20] constructed a model of the relationship between hazards in Georgia and used the geographic information system (GIS) analysis to obtain a zoning map of storm and flood risk. Benito et al. [21] integrated geological conditions, water conservancy construction, and historical statistical disaster records based on long-term extreme storm and flood risk assessment and constructed a multidisciplinary storm disaster assessment model. Alfa et al. [22] developed a storm flood risk assessment system including height and slope based on the Ofu River Basin in Nigeria. Weerasinghe et al. [23] used indicators such as hazard risk and carrier vulnerability to assess the flood risk level and constructed a rainstorm risk assessment system in the western provinces of Sri Lanka. Mahootchi and Golmohammadi [24] extended the two-stage stochastic optimization mathematical model to reduce the losses caused by the rainstorm disaster and further control the emergency cost and developed a rainstorm disaster model with higher matching degree. Alsubaie et al. [25] constructed a systematic disaster response planning platform based on Supervisory Control and Data Acquisition (SCADA).

With the rapid development of China's urbanization process, the frequent extreme rainstorms in major cities in China have caused huge losses to the society and economy [26–28]. Zhang and Cai [29] relied on fuzzy mathematics theory and used quantitative analysis to measure the ambiguity of risk factors and then constructed a fuzzy comprehensive index evaluation system. Peng et al. [30] simulated different causes of the degree of flood risk in the Maozhou River Basin under the dangerous situation of disaster factors, which are based on the volume submergence algorithm. However, the above methods cannot evaluate composite systems with multiple uncertainties. In recent years, with the rapid development of artificial intelligence technology, many scholars tend to apply intelligent algorithms to various evaluation tasks. Feng and Chen [31] used support vector machines (SVM) for the first time to establish a regression inference rainfall forecast model in the Sichuan Basin. The test results show that the SVM inference model has good forecasting capabilities. Xie et al. [32] established a support vector regression (SVR) rainfall prediction model based on the high-dimensional nonlinearity and periodicity of rainfall data, using seasonal autoregressive input feature selection method and grid parameter optimization method. In addition, some scholars have used algorithms such as backpropagation neural network (BPNN) and extreme gradient boosting (XGBoost) in the rainstorm disaster assessment model [33–36], but due to the uncertainty of model parameter settings and research, the incompleteness of the construction of the index system in the process caused the accuracy of model evaluation to be greatly reduced.

In summary, based on disaster risk theory, comprehensive consideration of the natural environment, and social economic conditions, to further improve the rainstorm and flood disaster risk assessment indicators and methods [37–40], this research will first use the logistic binary regression analysis to affect the occurrence of rainstorm and flood disaster. Then, with the help of the weighted

comprehensive evaluation method and GIS-related technology, the compartment analysis of rainstorm and flood disaster in Henan Province was carried out. Finally, the particle swarm optimization (PSO) algorithm was combined with SVR to establish a hybrid PSO-SVR rainstorm and flood disaster model [41, 42], which is expected to provide a scientific basis for the comprehensive evaluation of storm and flood disasters.

2. Research Data and Index System Construction

2.1. Overview of Study Area. Henan Province is located in the middle and lower reaches of the Yellow River in east-central China. It is bounded by $31^{\circ}23'$ – $36^{\circ}22'$ north latitude and $110^{\circ}21'$ – $116^{\circ}39'$ east longitude. As for the neighboring provinces, Henan borders Anhui and Shandong to the east, Hebei and Shanxi to the north, Shanxi to the west, and Hubei to the south. Of the total area of approximately 167,000 km², there are 74,000 km² of mountains and hills (44.3% of the province's total area) and 93,000 km² of plains and basins (55.7% of the province's total area). The terrain of Henan is high in the west and low in the east, with the altitude difference between the highest point and the lowest point as high as 2390.6 meters. Plains are widely distributed in the province, among which the central, eastern, and northern plains are formed by the alluvial accumulation of the Yellow River, Huaihe River, and Haihe River, also known as the Huanghuaihai Plain [43]. Due to its location in the warm temperate zone and subtropical zone, it belongs to a humid-semihumid monsoon climate with large seasonal differences. The annual average temperature is between 12°C and 16°C, and the annual average precipitation is about 500 mm–900 mm. There are many mountains in the south and west. Especially, the precipitation of Dabie Mountains can reach more than 1100 mm. Annual precipitation is concentrated in summer, often accompanied by torrential rain disasters [44]. This paper lists the historical rainstorm process information in Henan Province (see Table 1). Data come from the China Meteorological Administration.

According to data about July 2021 released by the Henan Meteorological Bureau, rainstorm occurred in many areas in Henan. Within the 13 days from July 10 to July 23, there were 11 days of rainfall in Zhengzhou. It should be noted that the rainfall from 16:00 to 17:00 on July 20 reached 201.9 mm per hour, breaking through the historical extreme value in mainland China. The rainstorm and flood disaster has caused great damage to people's lives in Henan province, causing 302 deaths and 50 missing. Among them, 292 people were killed and 47 people were missing in Zhengzhou City; 7 people were killed and 3 people were missing in Xinxiang City; 2 people were killed in Pingdingshan City; 1 person was killed in Luohe City. In view of this, the construction of rainstorm and flood disaster risk assessment model and the implementation of comprehensive evaluation of the disaster in this region can provide strong technical support for the research on storm flood management and risk warning services.

TABLE 1: Historical rainstorm and precipitation process information in Henan Province.

Time (year/ month/day)	Main distribution area of rainfall (city)	Maximum accumulated rainfall (mm)	Maximum daily rainfall (mm) (month/day)	The average rainfall of the province (mm)
1963/8/2-8/8	Xinxiang	699	253 (8/3)	199
1975/8/4-8/8	Zhumadian	1631	755 (8/7)	163
1982/7/28-8/5	Luoyang	666	265 (8/1)	169
1996/7/28-8/6	Xinyang Zhumadian	438	249 (8/3)	135
2005/6/25-26	Kaifeng Zhumadian	496	160 (6/26)	107
2007/8/2-8/3	Zhengzhou Nanyang	374	87 (8/2)	132
2008/7/22-7/23	Nanyang	542	229 (7/22)	142
2010/7/18-7/20	Zhengzhou Luoyang	596	182 (7/18)	150
2012/8/19-8/20	Zhengzhou	295	78 (8/20)	114
2016/7/18-7/20	Anyang	732	703 (7/19)	80
2021/7/16-7/20	Zhengzhou	617	362 (7/20)	88

2.2. Construction of Indices System. The evaluation of rainstorm and flood disaster is a comprehensive evaluation process based on the selection of reasonable indicators. Therefore, the reliability of the evaluation results is inseparable from the correctness of the index selection. In this research, based on the theory of disaster risk system, referring to existing research results and considering the availability of data for rainstorms and floods, index system of rainstorm and flood disaster loss prediction is constructed from the aspects of hazard factors, hazard-pregnant environment, hazard-bearing body, disaster resilience, and disaster loss (see Table 2).

2.2.1. Hazard Factors. The distribution of rainstorms directly affects the possibility of flooding and the degree of loss of disasters. According to relevant studies, the duration of the rainstorm and the maximum rainfall in a short time are closely related to the occurrence of rainstorm and flood disaster. Based on this, considering the availability of data, this research selects the bulletin data issued by the urban meteorological stations corresponding to 18 cities in Henan Province as the benchmark and selects indicators such as process accumulated rainfall, continuous rainfall days, and accumulated rainfall in 12 h and 24 h.

2.2.2. Hazard-Pregnant Environment. The occurrence of rainstorm and flood disaster is closely related to the terrain. Generally speaking, the greater the terrain undulation, the less likely the occurrence of flood disasters; the smaller the terrain undulation, the greater the possibility of flood disasters. The degree of vegetation coverage directly affects the water conservation capacity to a certain extent. At the same time, the density of the river network in the study area can indirectly reflect the relative magnitude of the risk of rainstorms and floods to a certain extent. Therefore, places with high river network density are more likely to encounter flood disasters.

2.2.3. Hazard-Bearing Body. It mainly refers to the object of the impact of rainstorms and floods. At the same time, the spatial distribution of population, roads, and houses are inseparable from the extent of flood damage. On the one

hand, the GDP of per person of city's residents in the area reflects the development status of social and economic construction. On the other hand, it can also reflect the residents' ability to withstand rainstorm and flood disaster.

2.2.4. Disaster Resilience. It refers to the ability of a region to defend against rainstorms and floods. This article mainly considers the regional GDP and the average output value. On the one hand, it not only reflects the local fiscal revenue, but more importantly, it reflects the comprehensive flood resilience capability of the region in response to rainstorms and floods. Third, from the perspective of drainage capacity, the capacity of the drainage system directly affects the occurrence of flood disasters.

2.2.5. Rainstorm and Flood Disaster Loss. Based on the reliability and availability of disaster loss data caused by historical rainstorms and floods, this article selects accumulated rainfall in 24 h (daily rainfall), number of rainstorm and flood disasters, and direct economic loss as the target variables for the hybrid PSO-SVR model prediction.

2.3. Data Sources. This paper adopts a total of 2,700 pieces of data selected from various regions in Henan Province from 2010 to 2020.

2.3.1. Hazard Factor Data. Relevant data are collected from China Meteorological Network, China Meteorological Administration, Henan Meteorological Administration, and Henan Meteorological Observation Center, as well as the data of 120 meteorological stations corresponding to 18 cities in Henan Province. Considering that June to August is the rainy season every year, the data of 15 days with rainfall greater than 60 mm in each city from June to August are selected as the data of the current year for analysis and processing.

2.3.2. Hazard-Pregnant Environment and Hazard-Bearing Body Data. Data of the digital elevation model (DEM) come from geographic space data cloud (ASTER G-DEM 30M resolution digital elevation data), and data of

TABLE 2: Indices system of rainstorm and flood disaster loss prediction.

Variable attributes	Disaster elements	Primary indices	Secondary indices	Reference source
Characteristic variable	Hazard factors	Rainfall data	Process accumulated rainfall (J1: mm)	Gong et al. [45], Li et al. [46], Huang et al. [47]
			Continuous rainfall days (J2: h)	
			Accumulated rainfall in 12 h (J3: mm)	
			Accumulated rainfall in 24 h (J4: mm)	
	Hazard-pregnant environment	Natural environment	Digital elevation model (DEM) (Z1: m)	Wang and Deng [48], Hu et al. [49]
			Relief amplitude (Z2: m)	
			Vegetation coverage (Z3: %)	
			Density of the river network (Z4: m ² /m ²)	
Characteristic variable	Hazard-bearing body	Social economic data	Density of population (S1: thousand people/km ²)	Gong et al. [50], Pan et al. [51], and Li et al. [46]
			GDP of per person (S2: people/ten thousand yuan)	
			Proportion of garden green space area (S3: %)	
			Proportion of house area (S4: %)	
	Disaster resilience	Disaster resilience and construction	GDP of every city (K1: ten million yuan)	Zhang et al. [52] and Xu et al. [53]
			GDP of per square kilometer (K2: ten million yuan/km ²)	
			Length of urban drainage pipeline (K3: ten thousand km)	
Target variable	Rainstorm and flood disaster loss	Disaster loss situation	Number of rainstorm and flood disasters (Y1: time)	Zhang et al. [54] and Wang et al. [55]
			Direct economic loss (Y2: hundred million yuan)	

topographic undulation (select the DEM standard deviation within the city to which you belong) and river network density come from the National Basic Geographic Information Center. Considering the uncertainty of vegetation coverage, vegetation coverage is selected as the evaluation standard of water conservation capacity. Vegetation coverage data come from “China Forestry Statistical Yearbook” of the database in China Forestry and Economic and Social Development Statistics. The carrier data come from the “China Statistical Yearbook” of 2010–2020 National Bureau of Statistics and “Henan Statistical Yearbook” of Henan Statistics Bureau.

2.3.3. Disaster Resilience and Rainstorm and Flood Disaster Loss. Relevant data come from “China Statistical Yearbook” of 2010–2020 National Bureau of Statistics, “National Water Regime Annual Report” of Ministry of Water Resources of the People’s Republic of China, and the “Statistical Bulletin” of the Ministry of Civil Affairs of the People’s Republic of China. To facilitate the calculation, the disaster loss caused by heavy rains and floods in each year is selected as the calculation data. Considering the availability of data released by meteorological stations, elevation distribution maps of meteorological stations in various regions of Henan Province are shown in Figure 1.

3. Risk Compartment Analysis of Rainstorm and Flood Disaster

After the construction of the indices system, the logistic regression model is used for correlation analysis to reduce

the impact of indices data on the error of the hybrid PSO-SVR prediction model. Since the model is a regression analysis for the binary variable of dependent variable, the data of each evaluation index are used as independent variables in the evaluation of flood disaster. In addition, the occurrence of rainstorm and flood disasters is represented by 0 (the number of disasters is 0) and 1 (the number of disasters is not 0) as the dependent variables of two classifications [56]. In this research, risk compartment analysis of rainstorm and flood disaster includes four parts of analysis. The risk analysis of hazard factors is mainly analyzing the influencing factors of risk sources. The sensitivity analysis of hazard-pregnant environment reflects the impact of natural geographical environment on rainstorm and flood disaster. The vulnerability analysis of hazard-bearing body mainly analyzes the influence of different rainstorm and flood intensity disasters for the distribution of population and the condition of regional economic and infrastructure. The analysis of disaster resilience reflects the level of the ability of disaster prevention and recovery [57–63]. This paper firstly presents the results of regression analysis (see Table 3).

According to the significance at the 95% level, the significance value greater than 0.05 indicates that the correlation of variables is low. Therefore, the impact of Z1, S2, and K2 will not be considered in this paper.

3.1. Risk Analysis of Hazard Factors. The disasters caused by rainstorm are mainly manifested as strong rains and high intensity. The rainstorm and flood disaster occurs when a short period of accumulated rainfall is too large to drain water. In this paper, J1, J3, J4, and J2 are selected to

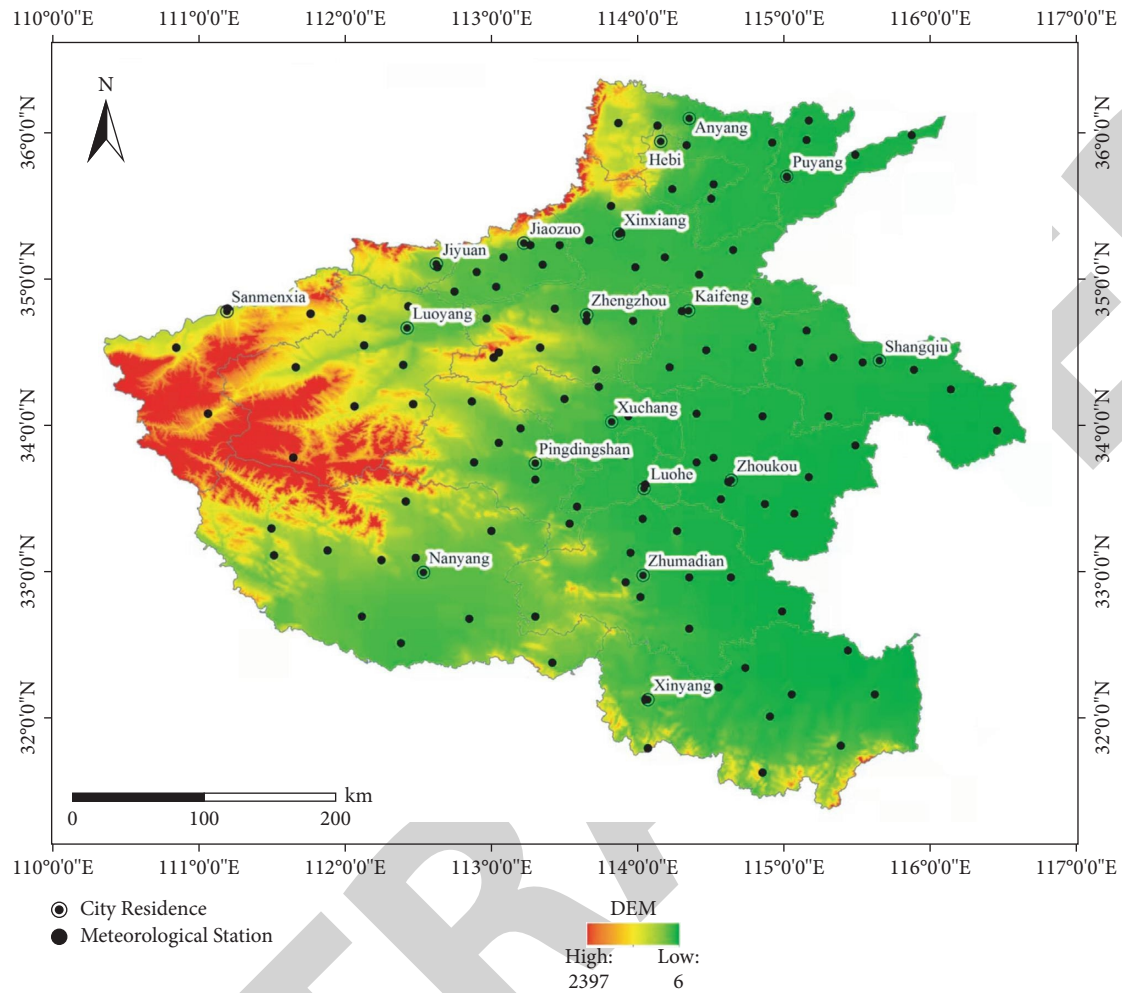


FIGURE 1: Spatial distribution of automatic meteorological observation stations in Henan Province.

TABLE 3: Results of logistic correlation regression analysis.

Indices	B	S.E.	Wald	df	Significance level	Exp (B)	Confidence interval 95% exp (B)	
							Lower limit	Upper limit
J_1	5.216	2.767	2.314	1	0.000	1.355	0.627	2.326
J_2	4.507	1.936	2.132	1	0.000	5.236	3.022	7.658
J_3	2.064	0.216	18.125	1	0.024	0.235	0.029	2.157
J_4	1.672	0.652	2.921	1	0.036	1.229	0.158	2.192
Z_1	-0.056	0.108	0.725	1	0.213	0.002	—	—
Z_2	-1.249	8.254	0.452	1	0.000	0.025	0.012	0.704
Z_3	-2.353	0.832	19.831	1	0.000	0.037	0.005	0.236
Z_4	0.237	1.602	1.374	1	0.035	1.368	0.529	2.871
S_1	1.203	0.362	0.257	1	0.016	0.765	0.233	1.986
S_2	0.075	1.206	0.034	1	0.107	0.013	—	—
S_3	-1.215	3.371	1.562	1	0.047	1.103	0.716	4.965
S_4	2.127	0.762	5.261	1	0.022	0.125	0.062	2.185
K_1	1.862	0.935	2.317	1	0.028	0.757	0.214	2.166
K_2	-0.051	0.236	0.965	1	0.154	0.006	—	—
K_3	-1.239	7.362	1.322	1	0.012	1.667	0.971	3.492
Constant term	3.653	0.368	197.532	1	0.000	0.042	—	—

Note. B represents the logistic regression coefficient; $\text{Exp}(B)$ is the odds ratio, which reflects the influence of each evaluation factor on the risk; SE is the standard error, which indicates the average error of the estimated value; Wald is a statistic used to test whether the independent variable has an impact on the dependent variable; df is the degree of freedom; “—” means no data.

represent rainfall intensity and rainfall frequency, respectively. To facilitate the assessment of hazard factors, accumulated rainfall in 24 h at 95th, 90th, 80th, 70th, and 60th percentiles is used as the critical disaster-causing rainfall for rainfall classification, respectively. The specific grading standards are as follows. The rainfall in the digits of 60%–70% is level 1. The rainfall in the digits of 70%–80% is level 2. The rainfall in the digits of 80%–90% is level 3. The rainfall in the digits of 90%–95% is level 4. The rainfall in the digits above 95% is level 5. The higher the classification level, the greater the effect of inducing flood formation. Therefore, the weights for levels 1 to 5 are set to 1/15, 2/15, 3/15, 4/15, and 5/15. According to the criterion of critical rainfall, the number of rainstorm intensity occurrences in each city within 15 days is counted. In addition, the sum of products of weights of precipitation intensity and frequency of different grades after normalization is calculated. The hazard factors that characterize each city is assigned to the GIS as the attribute value for rasterization. With the GIS built in the natural segment point classification method, the hazard factors are divided into lowest risk area, lower risk area, moderate risk area, higher risk area, and highest risk area.

3.2. Sensitivity Analysis of Hazard-Pregnant Environment. Based on the analysis of the formation mechanism of rainstorm and flood, the hazard-pregnant environment mainly considers factors such as relief amplitude, vegetation coverage, and density of the river network. The greater the difference in terrain undulation is, the less likely it is to cause flood disasters. The greater the vegetation coverage, the greater the water conservation capacity and thus the lower the probability of flooding. The higher the density of the river network, the closer it is to the water source and the higher the risk of flooding is. Based on the research experience, the relief amplitude level 1 to 5 standards are set as 1.5, 1.2, 0.85, 0.5, and 0.2. The vegetation coverage level 1 to 5 standards are set as 0.8, 0.7, 0.6, 0.45, and 0.2. The density of river network level 1 to 5 standards is set as 0.055, 0.04, 0.03, 0.02, and 0.01. After standardizing these factors, the corresponding weight index is calculated according to the degree of influence of each factor on the rainstorm and flood. Based on GIS, the hazard-pregnancy environment sensitivity can be divided into lowest sensitivity area, lower sensitivity area, moderate sensitivity area, higher sensitivity area, and highest sensitivity area.

3.3. Vulnerability Analysis of Hazard-Bearing Body and Analysis of Disaster Resilience. The degree of risk caused by rainstorm and flood disaster is related to the hazard-bearing body that bears the rainstorm and flood disaster. The greater the population density in the area, the greater the number of people affected by the disaster. To a certain extent, the regional GDP of per person reflects the individual's bearing strength after the disaster. Garden green space has the function of blocking and weakening rainstorm and flood; the larger the proportion of garden green

space area, the greater the ability to resist flood. The large proportion of house area will increase the possibility of rainstorm and flood disaster. From the results of logistic correlation regression analysis, the corresponding significance value of S2 is greater than 0.05, which indicates the level of S2 was not significant. Thus, this article selects S1, S3, and S4 as the evaluation indices of vulnerability. Because the relative bearing capacity of each region in the province to rainstorm and flood disaster is different, the weight of each region should be considered when calculating the vulnerability of the hazard-bearing body. Therefore, the vulnerability index of the hazard-bearing body will be obtained according to the weighted comprehensive evaluation method, and the vulnerability of the hazard-bearing body is divided into lowest vulnerability area, lower vulnerability area, moderate vulnerability area, higher vulnerability area, and highest vulnerability area by using GIS.

For all regions, the ability to withstand disasters can be measured from K1 and K3 during rainstorm. Due to the gap in economic development and social infrastructure among regions, the weighted comprehensive evaluation method is still used to calculate the disaster resilience index, and the disaster resilience capacity is divided into lowest disaster resilience area, lower disaster resilience area, moderate disaster resilience area, higher disaster resilience area, and highest disaster resilience area.

3.4. Compartment Analysis of Disaster Factors. Using the weighted comprehensive evaluation method and GIS-related technologies, combined with the data standardization formula (1), and the weighted comprehensive evaluation method formula (2), calculate the weight of relevant factors and establish a risk evaluation set for the four factors related to rainstorm and flood disaster compartment analysis, as shown in Table 4. The compartment analysis of rainstorm disaster-causing risk of hazard factors, sensitivity of hazard-pregnant environment, vulnerability of hazard-bearing body, and disaster resilience in various regions of Henan Province is shown:

$$Y_{ij} = \frac{X_{ij} - X_{\min_i}}{X_{\max_i} - X_{\min_i}}, \quad (1)$$

where Y_{ij} represents the standardized value of the i th index in the j th area, X_{ij} represents the i th index value in the j th area, and X_{\max_i} and X_{\min_i} represent the minimum and maximum values of the i th index:

$$V = \sum_{i=1}^n W_i Y_i, \quad (2)$$

where V represents the value of the evaluation factor, W_i represents the weight of index i , Y_i is the standardized value of the i th index, and n represents the number of evaluation indexes.

It can be seen from the compartment analysis map of disaster risk elements of rainstorm and flood in Henan Province (Figure 2); the risk has certain regional differences:

TABLE 4: Risk assessment criteria for rainstorm and flood disaster.

Disaster risk elements (weight)	Indices	Weight	Risk assessment criteria				
			First level (lowest)	Second level (lower)	Third level (moderate)	Forth level (higher)	Fifth level (highest)
Hazard factors (0.48)	J1	0.16	150	200	250	350	450
	J2	0.24	40	45	50	55	60
	J3	0.18	30	50	70	100	150
	J4	0.42	50	100	150	200	250
Hazard-pregnant environment (0.23)	Z2	0.51	1.50	1.20	0.85	0.50	0.20
	Z3	0.35	0.80	0.70	0.60	0.45	0.20
	Z4	0.14	0.055	0.041	0.03	0.02	0.01
Hazard-bearing body (0.17)	S1	0.39	0.5	1.0	2.0	3.0	4.0
	S3	0.36	0.75	0.65	0.55	0.45	0.35
	S4	0.25	0.4	0.45	0.5	0.55	0.60
Disaster resilience (0.12)	K1	0.78	5.5	4.5	3.5	2.5	1.5
	K3	0.22	4.0	3.0	2.5	2.0	1.0

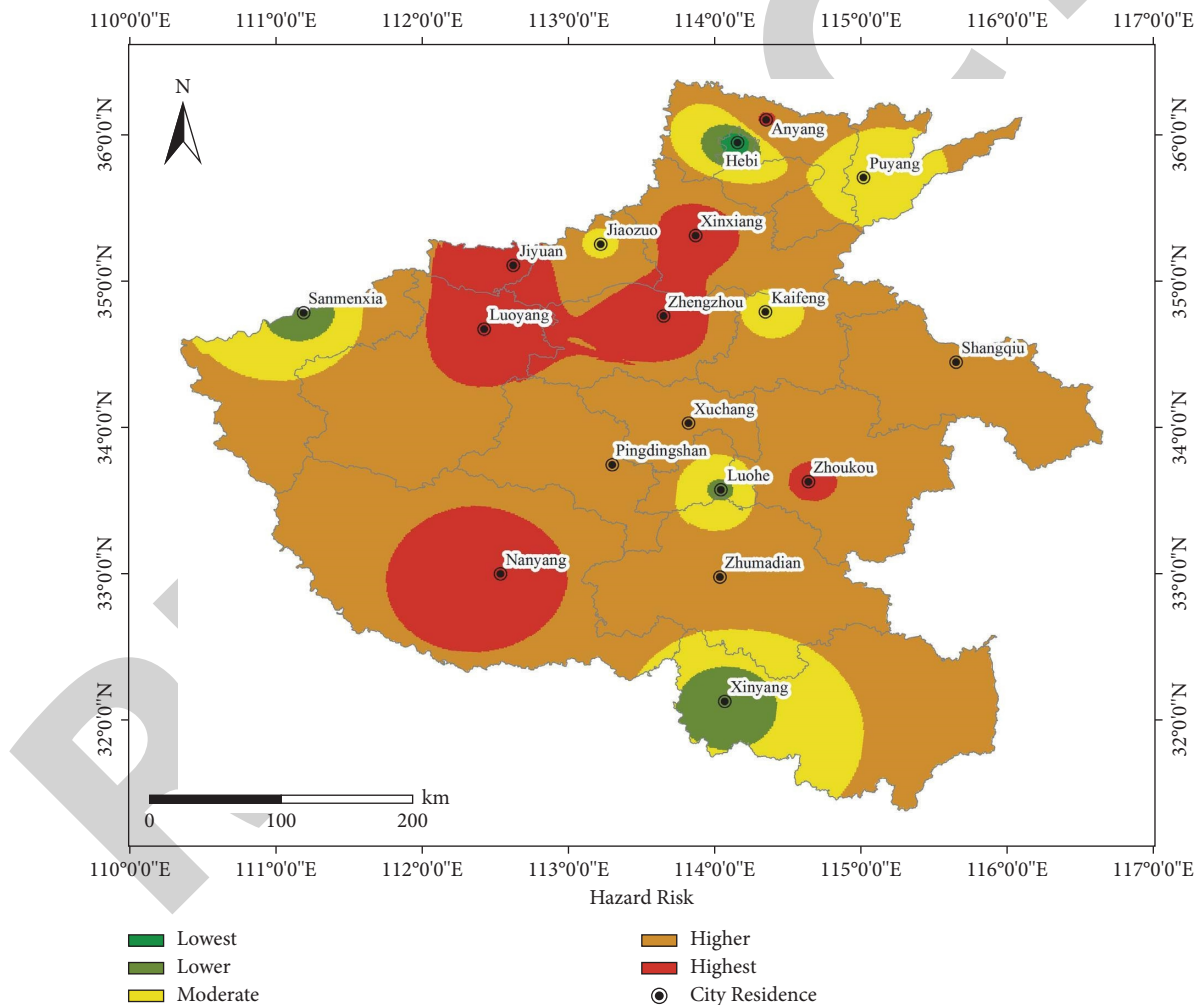


FIGURE 2: Assessment map of hazard risk of rainstorm and flood disaster in Henan Province.

generally, most areas of Henan are prone to rainstorm and flood disaster due to the geographical location of most plains and the influence of the Yellow River and Huaihe River basins. Among them, Zhengzhou, Luoyang, Jiyuan, and Xinxian in the northwest of Henan Province, Nanyang in

the southwest, and Zhoukou in the Middle East are highest risk areas of rainstorm and flood disaster. Anyang, Shangqiu, Xuchang, Pingdingshan, and Zhumadian are higher risk areas of rainstorm and flood disaster. Puyang, Jiaozuo, and Kaifeng are moderate risk areas of rainstorm and flood

disaster. Sanmenxia and Luohe are lower risk areas of rainstorm and flood disaster. Xinyang is the most southern city in Henan Province, located in the upper reaches of the Huaihe River, bordering Hubei Province and Anhui Province. Xinyang is located in the transition area from subtropical zone to warm temperate zone, with obvious seasonal climate and abundant rainfall, but Xinyang is high in the South and low in the north; it is a ladder landform with alternating hills and rivers and diverse forms, and its forest vegetation is dense, which is not easy to cause rainstorm and flood disasters. Thus, it shows lower risk of rainstorm and flood disaster. Hebi has a warm temperate climate with dry climate and little rain, and its geographical location is high in the West and low in the East. The climate and geographical conditions make the risk of rainstorm and flood disaster in this area the lowest.

As can be seen from the compartment analysis map of sensitivity of rainstorm and flood disaster in Henan Province (Figure 3), Zhengzhou, as the intersection of hilly land and plain flood, is a part of the plain in North China, and the terrain is relatively flat. Moreover, the Yellow River and the Huaihe River pass through the region. Among them, the Yellow River system has many river networks such as the main stream of the Yellow River, Yiluo River, and Sishui River. Therefore, Zhengzhou has become a highest sensitivity environment prone to rainstorm and flood disaster in Henan Province. Luoyang, Xinxiang, Anyang, and Nanyang in the southwest are higher sensitivity areas of rainstorm and flood disaster. Puyang, Kaifeng, Xuchang, Pingdingshan, Zhumadian, and Xinyang are moderate sensitivity areas of rainstorm and flood disaster. Hebi, Jiaozuo, Jiyuan, Shangqiu, Zhoukou, and Luohe are lower sensitivity areas of rainstorm and flood disaster. Sanmenxia is the lowest risk area of rainstorm and flood disaster. The landforms of Sanmenxia are mainly mountains, hills, and plateau, and its altitude is between 300 and 1500 meters. The unique geomorphic characteristics make Sanmenxia the lower sensitivity area of rainstorm and flood disaster.

It can be seen from the compartment analysis map of the vulnerability of the rainstorm and flood disaster in Henan Province (Figure 4) that Zhengzhou, Luoyang, and Nanyang are the highest vulnerability cities with high population density and large proportion of garden green space area. Xinxiang, Jiaozuo, Xuchang, and Zhoukou are higher vulnerability areas of rainstorm and flood disaster. Anyang, Kaifeng, Shangqiu, Zhumadian, and Shangqiu are moderate vulnerability areas of rainstorm and flood disaster. Puyang, Sanmenxia, and Pingdingshan are lower vulnerability areas of rainstorm and flood disaster. Due to the influence of the density of population, the proportion of garden green space area and the proportion of house area, the vulnerability of rainstorm, and flood disaster in Hebi, Jiyuan, and Luohe are lowest.

In addition, it can be seen from the compartment analysis map of the disaster resilience of the rainstorm and flood disaster in Henan Province (Figure 5) that cities with larger GDP in regions such as Zhengzhou and Luoyang have highest disaster resilience. Nanyang, Xinxiang, Xuchang, and Zhoukou are higher disaster resilience areas of

rainstorm and flood disaster. Jiaozuo and Xinyang are moderate disaster resilience areas of rainstorm and flood disaster. Anyang, Kaifeng, Shangqiu, Pingdingshan, and Zhumadian are lower disaster resilience areas of rainstorm and flood disaster. Due to the limitations of economic and backward infrastructure construction, Hebi, Puyang, Jiyuan, and Luohe are lowest disaster resilience areas of rainstorm and flood disaster. At the same time, Sanmenxia, a mountainous area with higher altitudes, the northeastern region where economic development is relatively backward, has the lowest resilience.

4. Unequal Weight Clustering Hybrid PSO-SVR Algorithm

The unequal weight clustering hybrid PSO-SVR algorithm is an integrated machine learning algorithm. The main idea is to perform stepwise regression and dimensionality reduction processing on the data first and then perform clustering after unequal weight processing. Small sample data can better reflect the superiority of the PSO-SVR algorithm.

4.1. SVR Algorithm Principle. SVM refers to a common discriminated method. In adherence to the SRM principle, it shows unique advantages in handling small samples and high-dimensional feature space problems. SVM is first used to solve the model recognition problems, but recently, it has also been applied to address nonlinear regression estimation problems through introducing the insensitive loss function ε . When being used to tackle the regression problems, SVM is referred to as support vector regression (SVR), and the main thinking of SVR is to map the dataset $x_i (i = 1, \dots, n)$ to a high-dimensional feature space through nonlinear function. The specific relation involved can be expressed as

$$f(x) = \omega^T \phi(x) + b, \quad (3)$$

where $f(x)$ is the output value, ω and b are the coefficients, and $\phi(x)$ is the nonlinear mapping function which can convert the input value to the high-dimensional feature space. The regulatory value of ω and b is indicated by

$$\begin{aligned} \text{Min}_{\omega, b, \xi_i^*, \xi_i} R_\varepsilon(\omega, \xi_i^*, \xi_i) &= \frac{1}{2} \omega^T \omega + C \sum_{i=1}^n (\xi_i^* + \xi_i), \\ \begin{cases} y_i - \omega^T \phi(x_i) - b \leq \varepsilon + \xi_i^*, & i = 1, 2, 3, \dots, n, \\ -y_i + \omega^T \phi(x_i) - b \leq \varepsilon + \xi_i, & i = 1, 2, 3, \dots, n, \\ \xi_i^*, \xi_i \geq 0, & i = 1, 2, 3, \dots, n, \end{cases} \end{aligned} \quad (4)$$

where $R_\varepsilon()$ is the empirical risk [64, 65], C is the regularization parameter, ξ_i^* is the error greater than ε , and ξ_i is the error less than $-\varepsilon$. The above function denotes a quadratic optimization problem which can be converted to the dual problem. Given this, the final equation of SVR is:

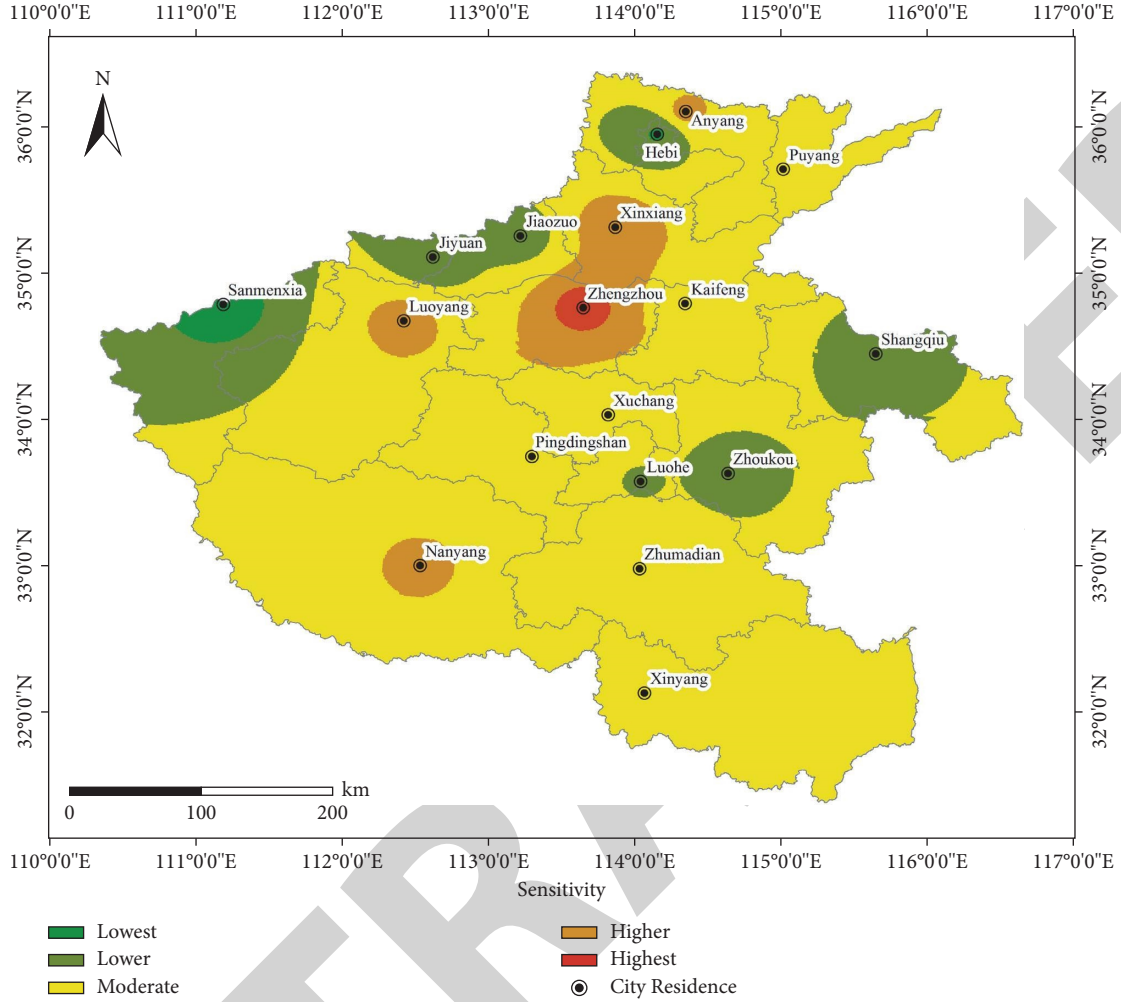


FIGURE 3: Assessment map of sensitivity of rainstorm and flood disaster in Henan Province.

$$f(x) = \sum_{i=1}^n (\beta_i^* - \beta_i) K(x_i, x_j) + b, \quad (5)$$

where β_i^* and β_i are the Lagrangian coefficients, and $K(x_i, x_j)$ is the kernel function of SVR which stands for the inner product of two vectors. The kernel function of vectors x_i and x_j can be defined as

$$K(x_i, x_j) = \phi(x_i)\phi(x_j). \quad (6)$$

There are several types of kernel function in the existing research, including both linear kernel function and Gaussian kernel function. Gaussian kernel function, as one of the most commonly used kernel functions, is also referred to as the radial basis function (RBF). This function is able to map the data to infinite dimensions and has relatively lower computational complexity. Therefore, the research uses RBF as the kernel function of SVR, and the function can be defined as

$$K(x_i, x_j) = \exp\left(-\frac{\|x_i - x_j\|^2}{2\gamma^2}\right), \quad (7)$$

where γ is the Gaussian parameter. SVR parameter combination is the key to realize high-precision prediction. Accordingly, the paper adopts the PSO algorithm to determine parameters including γ .

4.2. PSO Algorithm Principle. PSO represents a population computing technology developed on the basis of iteration optimization. Its first step is to initialize a group of particles. Then, the rate and location of these particles in the following iteration can be updated by tracking two extremums (single extremum P_{ibest} and global extremum P_{gbest}). When these two extremums are found, the PSO algorithm will be taken to recognize the rate and distance of each particle.

Suppose there are m particles in the d -dimensional search space. The i_{th} particle is indicated by $x_i = (x_{i1}, \dots, x_{id})$, where $i = 1, \dots, m$. In another word, the position of the i_{th} particle is x_i . The rate of the i_{th} particle is also a vector, expressed as $v_i = (v_{i1}, \dots, v_{id})$. The optimal location of the particle is $p_i = (p_{i1}, \dots, p_{id})$, and the optimal location of the whole population is $p_g = (p_{g1}, \dots, p_{gd})$. The standard PSO algorithm updates existing PSO algorithm, which is defined as

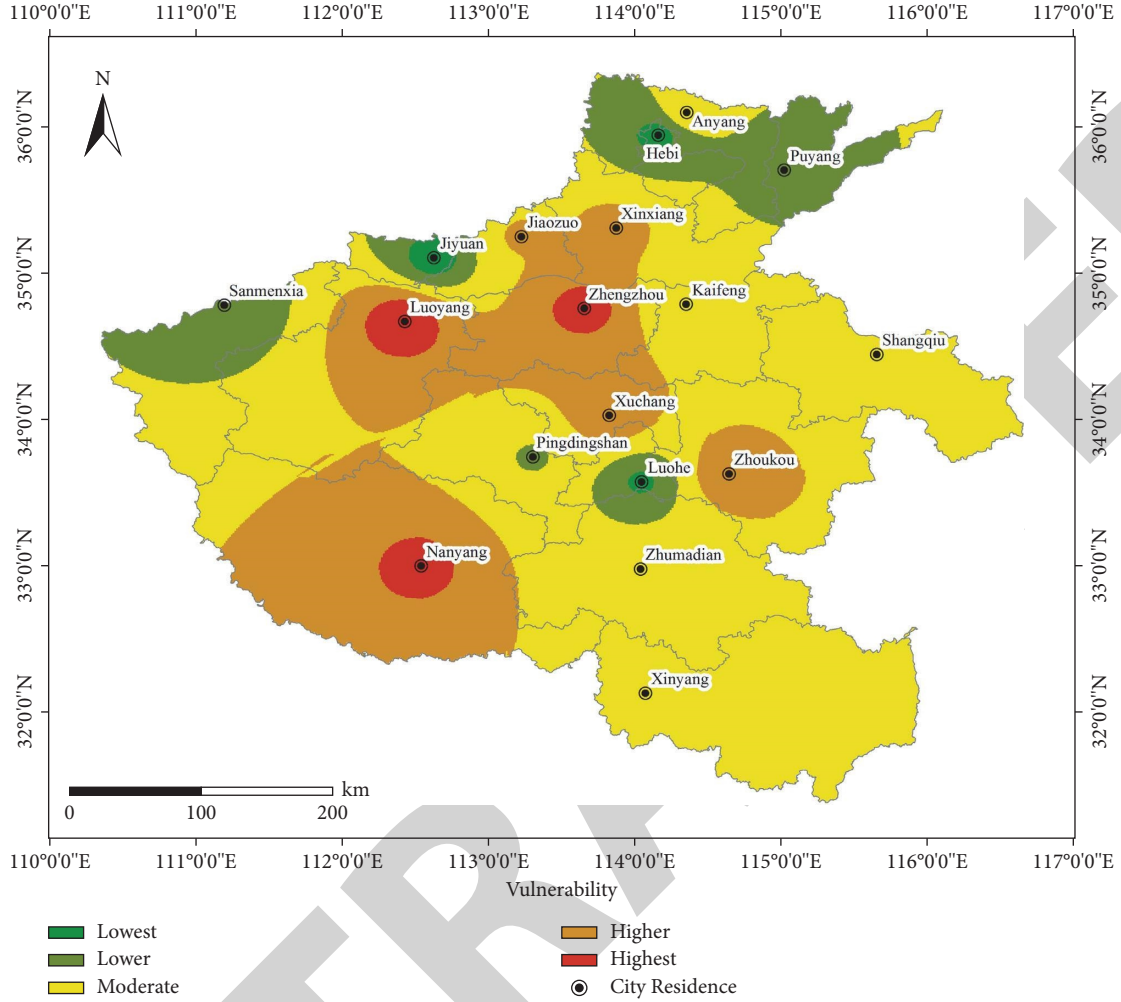


FIGURE 4: Assessment map of vulnerability of rainstorm and flood disaster in Henan Province.

$$\begin{aligned}
 v_{i,k+1}^d &= \bar{\omega} v_{i,k}^d + c_1 r_1 (P_{i,k}^d - x_{i,k}^d) + c_2 r_2 (P_{g,k}^d - x_{i,k}^d), x_{i,k+1}^d \\
 &= x_{i,k}^d + v_{i,k+1}^d.
 \end{aligned} \quad (8)$$

4.3. Unequal Weight Clustering Hybrid PSO-SVR Model. The model is mainly based on the PSO-SVR algorithm and mainly includes the following steps.

4.3.1. Generating Data Matrix. Before performing stepwise regression and dimensionality reduction, it is assumed that there exist c sample data and n independent variables (variable data after feature processing) in the experimental data. The set can be expressed as $X = (X_1, \dots, X_n)$, the dependent variable in this study is denoted by Y , and the model is written as follows:

$$Y = \beta_0 + \beta_i X_i + \varepsilon, \quad i = 1, \dots, n. \quad (9)$$

By calculating the regression coefficient of X_i , the F -test statistic values of the corresponding coefficients are (F'_1, \dots, F'_n) with F'_{τ_1} being the maximum value. Under a

given significance level, $\alpha = 0.05$, the corresponding critical value is $F^{(1)}$. When $F'_{\tau_1} \geq F^{(1)}$, X_{τ_1} is added to the regression model and S_1 is represented by the selected variable index set.

4.3.2. Establish a Binary Regression Model. Establish a binary regression model of dependent variable Y and a subset of independent variables $\{X_{\tau_1}, X_1\}, \dots, \{X_{\tau_1}, X_n\}$. There are a total of $n - 1$ subsets. The regression coefficient and the corresponding F -test statistic are calculated as F''_k ($k \notin S_1$). F''_{τ_2} refers to the maximum value. With a given significance level $\alpha = 0.05$, the corresponding critical value is $F^{(2)}$. When $F''_{\tau_2} \geq F^{(2)}$, then X_{τ_2} is added to the regression model. Otherwise, the variable introduction process is ended.

4.3.3. Repeat the Above Operation. Repeat the operation of Step 2 to obtain the final required equation model of this research as follows:

$$Y_{\tau} = \sum_{i=1}^k \beta_i X_{\tau_i}, \quad k \leq n. \quad (10)$$

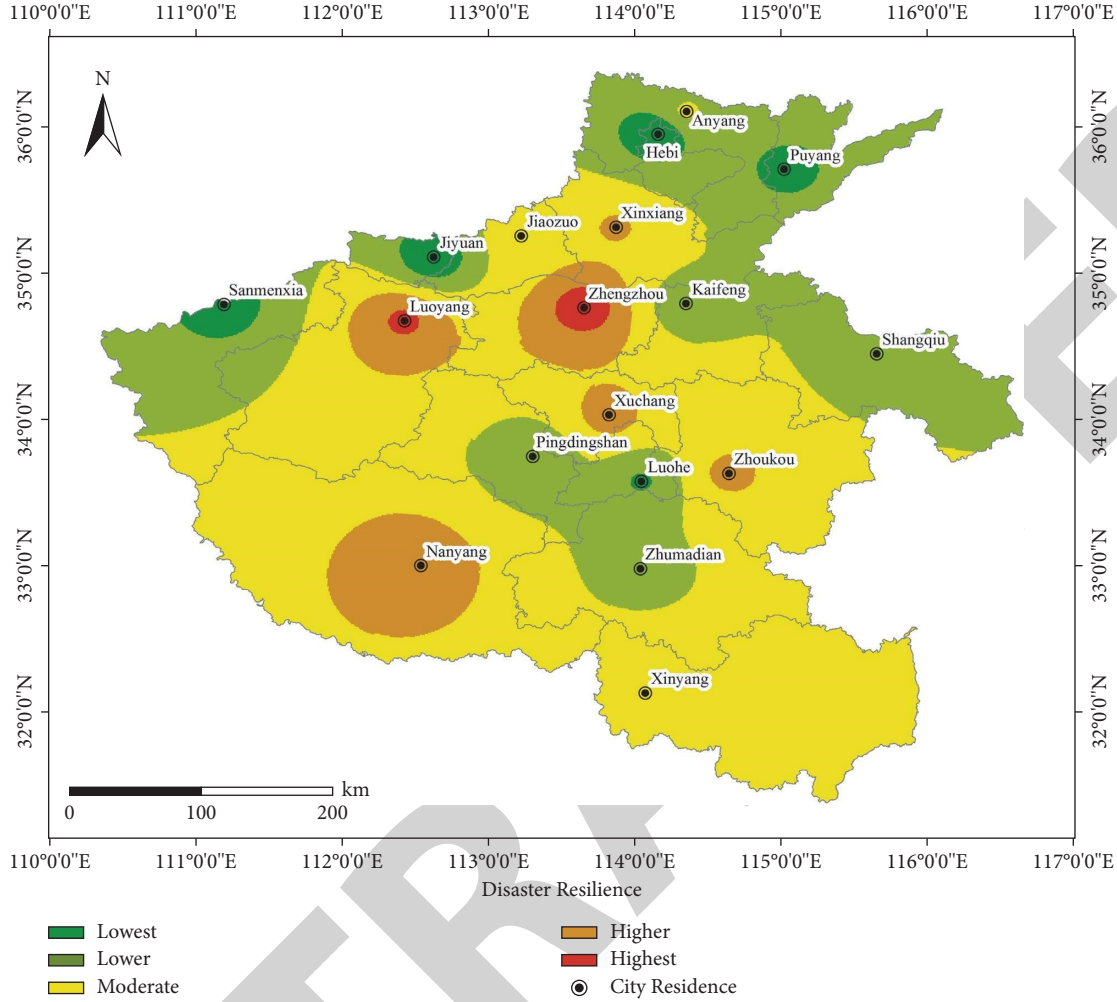


FIGURE 5: Assessment map of disaster resilience of rainstorm and flood disaster in Henan Province.

4.3.4. Select Centroids. Select l centroids, multiply the feature-encoded original data by the corresponding coefficient β_i , and input the new dataset Y_τ into the K -means clustering algorithm to obtain l datasets. These are $u_1, \dots, u_l \in R^n$, $Y_{\tau_i} \in R^n, i = 1, \dots, c$. The Euclidean distance from each sample in the dataset Y_τ to the centroid u_j is calculated, the centroid of the collection is continuously updated, and it was classified into l collections. The specific calculation formula is written as follows:

$$S_i = \arg \min \left\| Y_{\tau_i} - \frac{\sum_{i=1}^c Y_{\tau_i} |S_i = j|}{\sum_{i=1}^c 1 |S_i = j|} \right\| \quad S_i = j = 1, \dots, l. \quad (11)$$

4.3.5. Model Test. Take 10% of each set S_i as the test set, and finally, input it into the PSO-SVR model to obtain $S(S_i, P(c, \gamma, \epsilon))$ as the predicted result.

5. Result Analysis and Discussion

5.1. Evaluation Methods. The logistic binary regression built in SPSS 26.0 is used to preliminarily screen the relevant

factors affecting the occurrence of rainstorm and flood disaster, and then, the compartment analysis map of the four factors clearly shows the characteristics of each region. To further predict the rainstorm disasters, this research selects the accumulated rainfall in 24 h (daily rainfall), the number of rainstorm and flood disasters (selecting the daily rainfall exceeding 60 mm per month), and direct economic loss (average monthly rainstorm and flood disaster economic loss) in each region of Henan Province from 2010 to 2020 which are used as target variables to participate in the construction of the hybrid PSO-SVR model. The data from 2010 to 2019 are selected as the training samples, and the data from 2020 are used to verify the prediction accuracy of the model. To more objectively analyze the accuracy of the hybrid PSO-SVR model in rainstorm and flood disaster prediction, the SVR model without parameter optimization algorithm and the GA-SVR model and artificial neural network BPNN model are constructed from the same experimental samples for comparative verification.

The SVR model without parameter optimization algorithm is selected for comparison, which is mainly used to highlight the impact of parameter optimization on the prediction results. The GA-SVR model is chosen to compare

and highlight that PSO is more applicable to this model than GA algorithm due to its better optimization parameter effects. The BPNN model in artificial neural network (ANN) is chosen mainly because the model can still guarantee the sound prediction effects through establishing a relatively stable generalized regression neural network (GRNN) via radial basis neurons and linear neurons even when the model has limited experimental data. The research chooses the root mean square error (RMSE) and mean absolute percentage error (MAPE) to test the proposed hybrid model:

$$\text{RMSE} = \frac{1}{n} \sqrt{\sum_{i=1}^n (x_i^* - x_i)^2}. \quad (12)$$

RMSE is the mean square root of the square sum of the errors of the corresponding points of the predicted result and the real value. The smaller the value of RMSE is, the better the accuracy of the prediction model is:

$$\text{MAPE} = \sum_{i=1}^n \left| \frac{x_i^* - x_i}{x_i} \right| \times \frac{100}{n}. \quad (13)$$

MAPE is often used as a statistical index to measure the accuracy of prediction. The value of MAPE is smaller, the accuracy of the prediction model is better and the deviation from the real value is smaller, where n represents the number of time instances and x_i^* and x_i represent the predicted results and the real value in formulas (12) and (13).

5.2. Result Analysis. According to the compartment analysis of rainstorm and flood disaster, to explain the result, this research selects the data of Zhengzhou City and Luoyang City in Henan Province for result analysis. The performance comparison of each model is shown (see Table 5), and the prediction results of each model are shown in Figure 6. Note: the RMSE and MAPE calculate the average daily rainfall in 15 days.

The hybrid PSO-SVR model is run by Python compiler, and the RMSE and MAPE of each model are calculated (see Table 5). It can be seen that the four models perform well in the prediction of daily rainfall, with RMSE controlled within 24 and MAPE controlled within 31%. Among them, the hybrid PSO-SVR model performs best, with RMSE lower than 10 and MAPE lower than 15%. The effect of the GA-SVR model with parameter optimization is lower than that of the hybrid PSO-SVR model, followed by the BPNN model and the original SVR model, respectively. At the same time, the prediction of hybrid PSO-SVR for the number of disasters and disaster economic loss is still more accurate than other models.

Collecting the real values and the predicted results and using the gray correlation analysis method [66] to calculate the correlation between the two values, the calculated average correlations of the hybrid PSO-SVR, GA-SVR, SVR, and BPNN models to the predicted results and real values of the target variables are 0.696, 0.667, 0.639, and 0.625; the average correlations are greater than 0.625, respectively. It can be seen that the predicted results of the hybrid PSO-SVR is closest to the real value, which shows that the model has

better learning ability and generalization ability for the data of rainstorm and flood disaster.

At the same time, it can be seen from the predicted trend graph in Figure 6 that, for daily rainfall (Figures 6(a) and 6(b)), number of disasters (Figures 6(c) and 6(d)), and direct economic loss (Figures 6(e) and 6(f)), the prediction effect of hybrid PSO-SVR is better than other algorithm models; especially, at peak values of rainfall, the fitting effect is the best of all. It is further verified that the model proposed in this research has a strong learning ability to a certain extent on the complicated problem of rainstorm and flood disaster, and it also shows the better generalization ability of SVR to deal with small samples of high-dimensional data.

To explain and verify the practicability of the hybrid model proposed in this study, the hybrid PSO-SVR model is predicted for other 16 cities in Henan Province. The average value of RMSE and MAPE corresponding to the prediction results is shown in Figure 7.

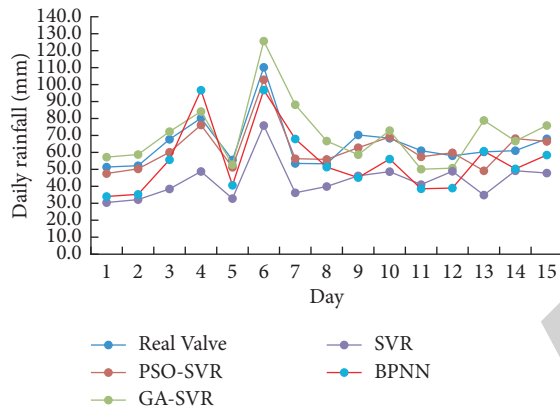
The average RMSE of 16 cities is less than 18 (Figure 7(a)), and the average MAPE of 16 cities is less than 21% (Figure 7(b)). The experimental results show that the hybrid PSO-SVR model is better than the other three models. When there are many characteristic dimensions of experimental data, it will affect the accuracy of prediction and increase the computational complexity. Firstly, regression analysis is selected to reduce the dimension of rainstorm and flood disaster index system in the research, so as to obtain a better combination of variables and reduce the complexity of data processing and the time required for prediction; By using the PSO optimization algorithm, this research achieves the automatic selection of parameters and overcomes the premature convergence problem of SVR. When the experimental data are complex, the processing efficiency and performance of GA algorithm are not as good as PSO algorithm. The comparison of RMSE and MAPE shows that the PSO algorithm is more suitable to optimize the parameters of SVR than GA algorithm. Due to the average daily rainfall, the number of disasters and disaster economic loss belong to small sample data; the performance of SVR prediction is better than BP neural network. Therefore, it is worthwhile to use the hybrid PSO-SVR model to predict the data of rainstorm and flood disaster.

Through the evaluation and analysis of rainstorm and flood disasters in Henan Province, a reasonable and effective evaluation mechanism is found (Figure 8). To a certain extent, we hope that the mechanism can help the government in disaster prevention and reduction. We will continue to conduct in-depth research and explore and establish a better evaluation mechanism in the future. Combined with historical disaster information, relevant analysis, and disaster risk zoning, this research puts forward the following suggestions:

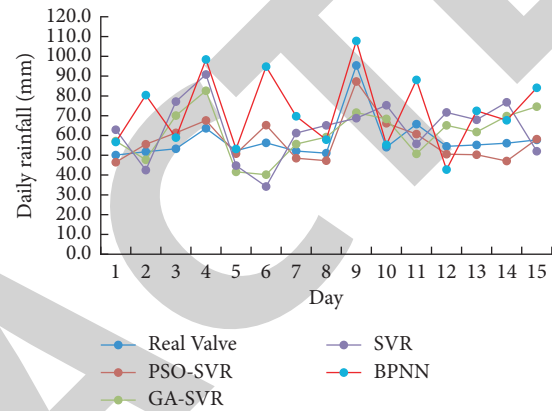
- (1) For cities with backward economic development and low population density, we should increase the construction of infrastructure, so as to improve the resistance and response ability to rainstorm and flood disasters.

TABLE 5: Model performance comparison.

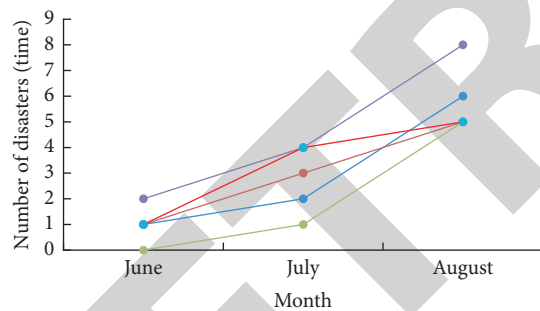
Target variable	PSO-SVR (RMSE/MAPE)		GA-SVR (RMSE/MAPE)		SVR (RMSE/MAPE)		BPNN (RMSE/MAPE)	
	Zhengzhou	Luoyang	Zhengzhou	Luoyang	Zhengzhou	Luoyang	Zhengzhou	Luoyang
Average daily rainfall	7.25	8.16	9.18	11.58	18.96	23.75	14.38	21.73
	11.36%	14.95%	17.78%	18.82%	30.25%	30.80%	20.18%	27.68%
Number of disasters	8.57	7.62	12.19	16.15	55.23	61.95	41.20	26.35
	14.58%	13.11%	15.77%	20.39%	38.33%	43.81%	36.59%	33.76%
Disaster economic loss	9.61	10.36	13.31	15.77	35.61	52.73	30.84	29.97
	18.65%	16.74%	20.36%	23.18%	41.93%	46.52%	29.71%	32.63%



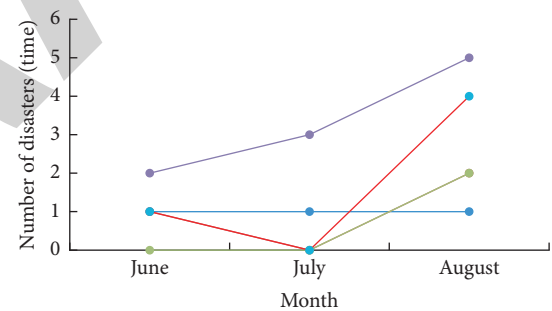
(a)



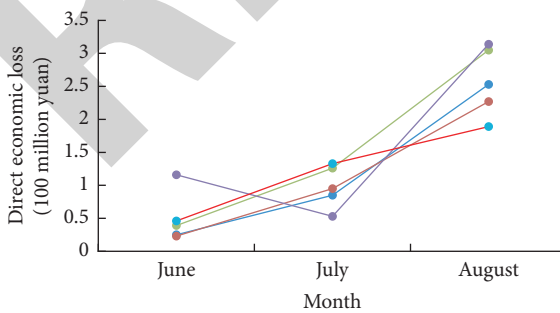
(b)



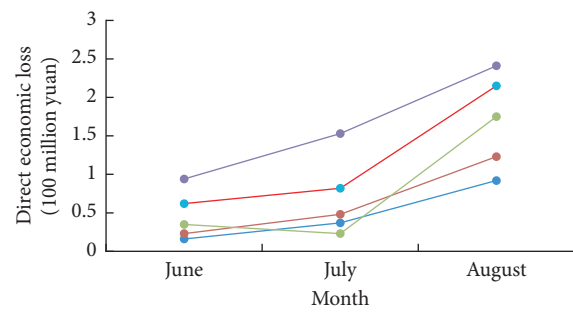
(c)



(d)



(e)



(f)

FIGURE 6: Assessment comparison of prediction results of various indicators in Zhengzhou City and Luoyang City in Henan Province: (a) and (b) rainstorm forecast; (c) and (d) number of disasters; (e) and (f) direct economic loss.

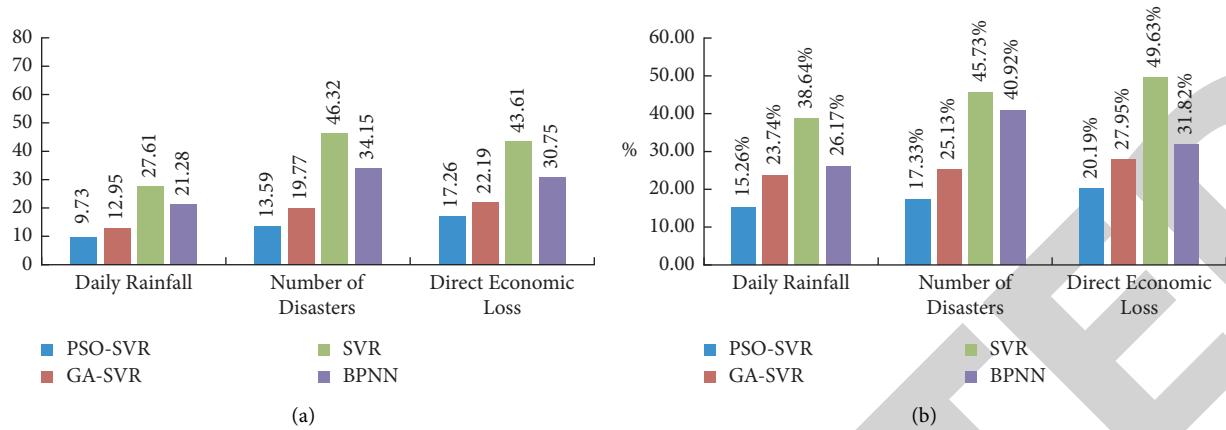


FIGURE 7: Index average values of 16 cities in Henan Province under different models: (a) RMSE. (b) MAPE.

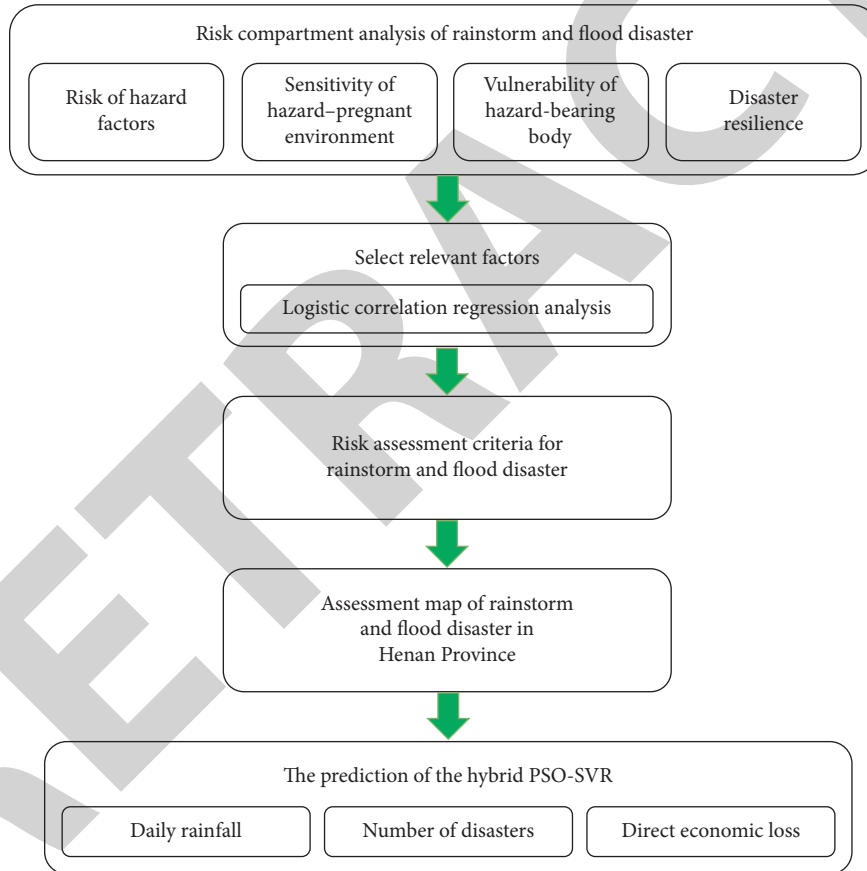


FIGURE 8: Risk evaluation mechanism of rainstorm and flood disaster.

- (2) For cities with developed economy and concentrated population density, we should strengthen the cultivation of residents' awareness of disaster resistance, so as to improve residents' ability to face rainstorm and flood disasters. At the same time, the government should increase the area of garden green space, expand the area of urban vegetation, and continue to strengthen the construction of drainage capacity, so as to well resist the invasion of rainstorm and flood disasters.

With the rapid development of global climate change and urbanization, more and more cities are suffering from extreme rainstorm and flood disasters, which has caused huge losses to people's lives and social and economic construction. Therefore, it is very important to carry out risk assessment and prediction of rainstorm and flood disaster, which will help to improve the ability of regional emergency prediction, reduce losses caused by rainstorm and flood disasters. Then, to expand the applicable scope of the model, the experiments of the hybrid PSO-SVR model will be tested

in more cities, and better improvements will be made in the continuous experimental process, to strive to provide more accurate analysis and assessment and disaster loss prediction.

6. Conclusions

Based on the hybrid PSO-SVR machine learning algorithm, this article constructs a rainstorm and flood disaster assessment and prediction model in Henan Province. First, based on the existing relevant research, considering the availability of data, establish a reasonable index system and initially screen the relevant factors that affect the occurrence of rainstorms and floods through logistic binary regression; Then, using GIS technology and a weighted comprehensive evaluation method to analyze the various regions in Henan Province, and the hazard factors, hazard-pregnant environment, hazard-bearing body, and disaster resilience were analyzed. Research and analysis showed that the risk of hazard factors in most parts of Henan Province from June to August was high. There are many rivers distributed, and most areas are lower plains, which are prone to rainstorm and flood disaster. For economically developed areas, due to the influence of geographical location, the sensitivity of the hazard-pregnant environment is high; at the same time, the vulnerability and resilience are also high. Through the regional analysis of hazard factors, the compartment analysis characteristics of rainstorm and flood in various regions of Henan Province are clearly demonstrated. Finally, to solve the complex disaster loss prediction situation involving nonlinear multidimensional factors, a hybrid PSO-SVR rainstorm and flood disaster model was constructed. The research results show that the hybrid PSO-SVR rainstorm and flood disaster prediction model is better than the GA-SVR, SVR, and BPNN models. The four models involved in the experiment have a better prediction of daily rainfall than the number of disasters and direct economic loss. Hybrid PSO-SVR has the best fitting effect for high peaks values in rainstorm and flood disaster prediction involving complex multidimensional factors.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Semantic Role Labeling Integrated with Multilevel Linguistic Cues and Bi-LSTM-CRF

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] F. Wan, Y. Yang, D. Zhu et al., "Semantic Role Labeling Integrated with Multilevel Linguistic Cues and Bi-LSTM-CRF," *Mathematical Problems in Engineering*, vol. 2022, Article ID 6300530, 8 pages, 2022.

Research Article

Semantic Role Labeling Integrated with Multilevel Linguistic Cues and Bi-LSTM-CRF

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Chinese Semantic Role Labeling (SRL) is the core technology of semantic understanding. In the field of Chinese information processing, where statistical machine learning is still the mainstream, the traditional labeling methods rely heavily on the parsing degree of syntax and semantics of sentences. Therefore, the labeling precision is limited and cannot meet the current needs. This paper adopts the model based on a bidirectional long short-term memory network combined with the Conditional Random Field (Bi-LSTM-CRF). In the feature processing stage, pooling technology is combined with sampling and selecting multifeature vector groups to improve the performance of the sequence labeling model. Lexical, syntactic, and other multilevel linguistic features are integrated into the training to realize in-depth improvement of the original labeling model. Through several groups of experiments, the precision of model annotation in this paper has been significantly improved combined with linguistic-assisted analysis, which proves that it can optimize the annotation performance of the model by integrating relevant linguistic features into the model based on Bi-LSTM-CRF and sampling and extracting multifeature groups; the evaluation of F increases to 82.18 percent.

1. Introduction

In natural language processing (NLP), semantic role labeling (SRL) is one of the important techniques of semantic analysis. The purpose is to label all semantic roles related to predicates in sentences. SRL belongs to shallow semantic analysis. Compared with deep semantic analysis, SRL has the characteristics of simple labeling, clear structure, and easy display. It has a wide range of practical value prospects in many application fields such as question answering (QA) system, information extraction (IE), machine translation (MT), etc. What is more, it can promote the research of deep semantic analysis and text understanding. The main research methods for SRL include CRF, support vector machine (SVM), and other linear machine learning methods; research direction includes named entity recognition (NER), part-of-speech (POS) for SRL, and so on.

According to the existing research on SRL, taking the Bi-LSTM-CRF model as the basic model, this paper studies the

improvement method of Chinese SRL, selectively expands the existing basic features, and adds multilevel linguistic features for comparative analysis and the improved precision is demonstrated in experiments.

2. Relevant Research

SRL is a practical scheme in current semantic analysis and processing. With the emergence of statistical machine learning methods in the field of NLP, many large-scale corpus resources with semantic information have been established, which greatly accelerates the practical pace of the development of SRL methods based on feature learning. In the research of English SRL based on machine learning, Pradhan et al. [1, 2] applied the machine learning method of SVM to SRL and achieved good results. Blunsom [3] introduced a more advanced machine learning model-maximum entropy Markov model into this field and achieved better labeling results. Cohn and Blunsom [4]

successfully applied CRF to SRL for the first time. With the rise of artificial intelligence (AI) in the past two years, deep learning methods have been applied to this field. Ronan and Jason [5] applied deep neural networks to frame SRL. This method slows down the manual intervention of traditional machine learning methods to deal with complex features and achieves ideal labeling results. Subsequently, multilayer neural networks of deep learning method also began to be introduced into this field. Socher et al. [6] used the combination of tree structure encoders and neural network units for classification, and Yin and Schutze [7] used a multilayer CNN network model for semantic classification. While these methods have achieved good performance, due to the increase in the number of network layers, the model does not depict linguistic phenomena well. The appearance of the LSTM model can not only effectively solve problems such as gradient disappearance but also consider the dependency relationship between contexts. Therefore, Zhou and Xu [8] used long short-term memory (LSTM) model to label semantic roles and added a small number of lexical features in the model training process, which has good experimental results. Zhen et al. [9] used the Bi-LSTM model, which exceeded the best results known at that time without introducing other resources. Jiang et al. [10] focused on syntactic path information and used Bi-LSTM to model, which improves the performance of the system. At present, the dropout punishment mechanism is widely used.

In Chinese SRL task, the sequence labeling model has made remarkable progress in the application of SRL. In CoNLL 2004, SRL was established as the theme for the first time and was carried out on the basis of shallow syntactic analysis theory. Wang's [11] research on SRL was based on a neural network with an optimized output layer. Although the experimental effect is still far from that of traditional machine learning annotation, this research provided a reference for the application of deep learning algorithm in this field. Therefore, the team tried again to apply the bidirectional cyclic neural network algorithm in 2015. The method avoids a large number of complex feature extraction and can make better use of the information in the annotation sequence. In order to solve the problems of poor information transmission caused by multilayer neural network and gradient explosion caused by too many network layers, Wang et al. [12] proposed setting up a "straight ladder unit" with information connection inside the multilayer LSTM model unit. The labeling information can be quickly transmitted between different layers. Li et al. [13] constructed a lightweight single-layer RNN model by external memory cells. The lightweight model has the advantages of simple training, high labeling efficiency, but its precision is close to that of the multilevel network model. Yang [14] introduced word distribution representation and dropout punishment mechanism into the neural network model, which greatly alleviated the problem of overfitting of the neural network model and significantly improved the labeling performance of the system. Significant progress has also been made in rule-based SRL [15–18]. This study also draws lessons from the model construction methods of relevant works [19–22].

Generally speaking, due to the limited corpus resources of Chinese SRL available for training and because some differences brought by Chinese itself are different from those of English (for example, the target verbs in Chinese are not easy to determine and Chinese SRL need to process word segmentation but English do not; the basic modules of Chinese automatic analysis, such as word segmentation, POS, dependency syntactic analysis [23], and other restrictions, etc.), the development of Chinese SRL is rather tortuous, so there is still much field for improvement in the Chinese SRL.

3. Model Construction

The main idea of the SRL model is to manually mark various semantic roles such as agent, subject, result, and mode in a certain scale corpus. The deep learning method is used for data training from the labeled large-scale corpus, and the probability rules of various semantic roles in different sentences are extracted to estimate and label each semantic role in the new corpus with the greatest probability. For the annotation model, role recognition and role classification are the core steps. Therefore, this paper uses bidirectional long short-term memory (Bi-LSTM) algorithm to solve these problems, and our end-to-end model can obtain features of role recognition and role classification from the embedding layer and further improves the sequence annotation performance of the original model by adding various linguistic features such as lexical and syntax.

3.1. Theoretical Methods. This method adopts the labeling strategy of word sequence labeling and uses a neural network classifier to identify and label various semantic roles in sentences at the same time. In the postprocessing stage, the pooling layer on CNN is used to sample the features and eliminate redundant feature information. After predicting all the matching semantic roles, simple postprocessing rules are adopted to identify the semantic role components that cannot be matched, and the semantic role with the highest prediction probability is retained.

In the selection of the annotation model, this paper mainly considers the current mainstream sequence annotation model based on deep learning: Bi-LSTM model. LSTM is an improved model based on the recurrent neural network (RNN) which has a strong nonlinear fitting ability. During model training, examples are mapped through complex nonlinear transformations in high-order and high-latitude heterogeneous spaces to obtain a low-dimensional sequence model. Compared with the traditional machine learning model, there is no flexibility to add custom features. When the semantic roles in a complex corpus are not completely separable, the labeling performance is poor, and the relevant information between elements in the sequence cannot be fully considered. Due to the design characteristics of LSTM, it can greatly improve the shortcomings of traditional machine learning methods and better take into account the sequence relationship of elements before and after the sequence, which is very suitable for modeling

complex nonlinear sequence data, such as text data. Therefore, the Bi-LSTM model would be adopted for achieving greater improvement in labeling performance.

3.2. Labeling Model. We use Bi-LSTM-CRF model for SRL task. Bi-LSTM is an improved model based on recurrent neural network (RNN), which consists of forward LSTM and backward LSTM. Due to the design characteristics, the model has a super nonlinear fitting ability and can realize automatic feature extraction and bidirectional encoding of context information, which can solve the long-distance dependence in sentences and is very suitable for processing time-series data, such as text data. However, the results obtained by Bi-LSTM contain a lot of useless information. After Bi-LSTM, the label transition probability matrix is introduced as a constraint condition, and CRF is used to fuse the global label information to obtain the optimal label sequence, which can improve the performance of the model. In this model, average pooling in CNN is used to sample the input features of the word embedding layer in the data preprocessing stage. The purpose is to sample and extract multifeature groups, eliminate redundant features, and complete word vector adjustment. This model is mainly composed of an input layer, word vector layer, average pooling layer, Bi-LSTM layer, and CRF layer architecture. The main architecture of the model is shown in Figure 1.

3.2.1. Pretreatment Layer. The model sends a sequence after the input layer; each word in the sequence is mapped into a corresponding word vector through the preprocessing layer and sent to the Bi-LSTM layer. Assuming that the input sentence A contains n word, $A = \{x_1, x_2, \dots, x_n\}$, x_i represents the i word in the input sentence. The word vector matrix E_w is used to obtain the word vector. v^w represents the vocabulary size. By (1), a word x_i can be converted into a word vector e_i :

$$e_i = E_w v^i, \quad (1)$$

where v^i is the absolute value distance of the vector v^i . After the above preprocessing, the initial sequence fragment sentences will enter the Bi-LSTM layer network in the form of word vectors.

3.2.2. Bi-LSTM Layer. The basic idea of the Bi-LSTM layer is to synchronously add a training sentence to forward and backward cyclic RNN and the units trained by the two RNNs points to a Max pooling layer interface at the same time. This two-way structure can provide a max-pooling layer with sufficient context-related information for each word in the input sentence. The network framework of its LSTM layer is shown in Figure 2.

Among them, the current input word corresponding to the time sequence t is X_t , the cell state is C_t , and the hidden layer state is h_t . The training process can be understood as processing the new element information in the current time sequence state through forgetting and memory units, retaining and transmitting the information with larger

influence factors to the cells in the next time sequence state, filtering the information with smaller influence factors, and outputting the hidden layer state h_t in the time sequence state. After iteration, in turn, the hidden layer timing state $\{h_0, h_1, \dots, h_{n-1}\}$ corresponding to the sentence sequence can be obtained. "AretanA" is function of arctangent.

Bi-LSTM is a two-way combination of the forward pass and backward pass, which takes into account the common context information in semantic role annotation tasks. The hidden layer state of the i word is shown as follows:

$$h_i = \vec{h}_i \oplus \overleftarrow{h}_i. \quad (2)$$

3.2.3. Posttreatment Layer. In the process of SRL, the location information of features inside the sentence sequence is very important. For example, the agent is generally located at the beginning of the sentence, the subject is located at the end of the sentence, and the core component is between them. The location information of these features is very important for role recognition and classification. However, the advantages of max-pooling technology are as follows. (1) The positions of the main features in the annotation sequence can still keep the feature position information and rotation unchanged after model training. (2) It can be used to reduce network parameters, model complexity, and iteration times in neural network model training. (3) When the features are pooled, the number of parameters of each filter and the number of neurons corresponding to fixed feature vectors can be significantly reduced. Therefore, this model introduces Max pooling technology in the post-processing layer.

3.2.4. CRF Layer. In SRL, there is a strong connection between the labels of adjacent words. For example, in the labeling system in this paper (Table 1), the label I_ARG0 can only be B_ARG0 or I_ARG0 before, while the label B_ARG0 can only be I_ARG0, O, or B_X after, and the rest of the labels are illegal. It is unreasonable to directly select the label with the highest score corresponding to each word output by the Bi-LSTM layer as the optimal label. The CRF layer introduces the label transition probability matrix, which can learn the label constraints of adjacent words from the training data and improve the performance of the label model. In addition, this paper adopts the Viterbi algorithm to infer the optimal label sequence.

4. Experiments

At present, the LSTM model has achieved good results in text information processing tasks of sequence labeling. This method not only overcomes the conflict between the sentence vector representation and the original sentence semantics caused by the original convolution neural network model, not considering the word order relationship in the sentence sequence, but also significantly improves the memory ability of the initial part of elements in long sentence patterns.

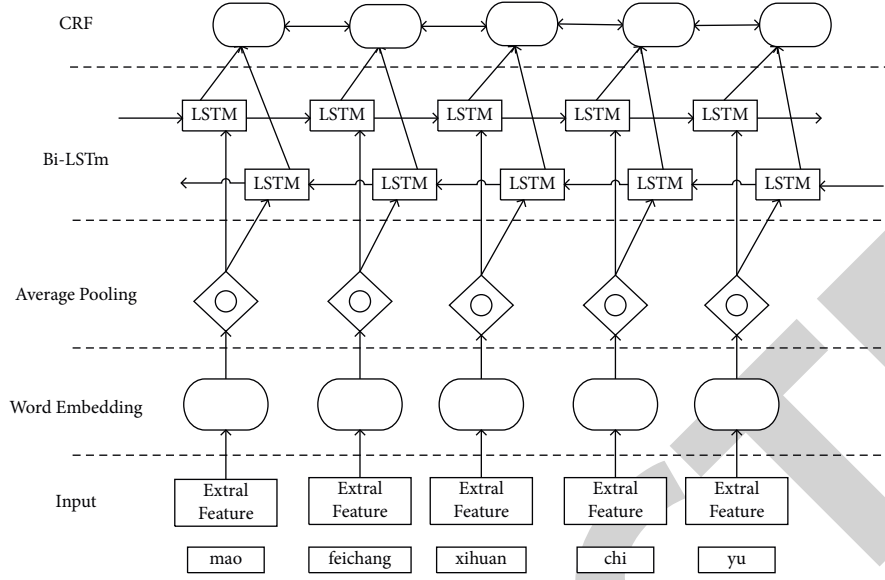


FIGURE 1: Model architecture of SRL.

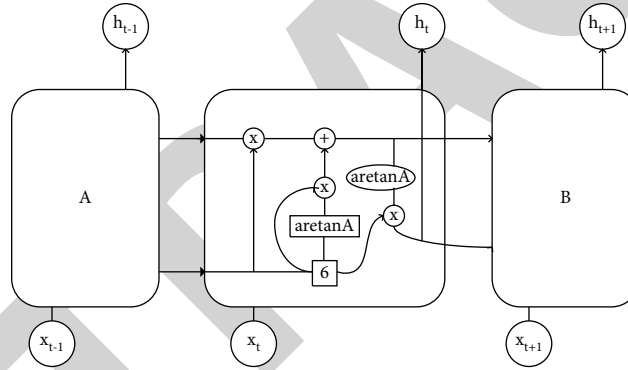


FIGURE 2: Network framework at the LSTM layer.

TABLE 1: Comparison of annotation results of whether CRF is added or not.

Sentence	Qunain	Shixian	Jinchukou	Zongzhi	a	1,090	Meiyuan
Correct label	B-ARG0	I-ARG0	I-ARG0	I-ARG0	el	B-ARG1	I-ARG1
Forecast label without CRF	ARGM-LOC	I-ARG0	I-ARG0	I-ARG0	el	B-ARG1	I-ARG1
Forecast label add CRF	B-ARG0	I-ARG0	I-ARG0	I-ARG0	el	B-ARG1	I-ARG1

This paper uses public CPB corpus for experiments and Bi-LSTM-CRF for training. Based on this initial model, several groups of new features are added to the corpus, average pooling is integrated to sample and select the formed feature vectors, and the basic model is gradually optimized and trained. Finally, the model is evaluated, and relevant experimental conclusions are obtained through analysis and comparison.

4.1. Corpus Selection. In terms of Chinese SRL corpus, due to the lack of large-scale training corpus in different fields, there is no good breakthrough in various types of SRL methods in domain adaptation. Therefore, we only consider the labeling problem in a single field. In the experiment, the

SRL system adopted Chinese PropBank (CPB). This system divides the semantic roles into two categories (Table 2). (1) The core semantic roles ARG0 ~ ARG5. ARG0 denotes the agent of the action, ARG1 denotes the subject of the action, and ARG2 ~ ARG5 have different semantic meanings according to different predicates. (2) Additional semantic roles. 13 subtypes such as location, reason, time, etc., are marked as ARG-X. For example, the location is marked as ARGM-LOC.

The corpus constructed in this experiment uses CPB partial data sets with a comprehensive semantic description and moderate granularity. After screening and statistics, there are 18,000 sentences in the training set, 1200 sentences in the development set, and 2,000 sentences in the test set.

TABLE 2: Format of basic training corpus.

Sentence	Zhongguo	Youse	Jinshu	Hangye	Baochi	Shiwu	Nian	Chixu	Zengzhang
Labeling	ARG-1	ARG-1	ARG-1	ARG-1	Rel	ARG-2	ARG-2	ARG-2	ARG-2

There are 18,418 words in the statistics. The first 13,000 words are selected as the vocabulary, and all words not included are replaced by_UNK. Table 2 shows the labeling schema.

In addition, this paper adopts precision, recall, and F1 values as evaluation indexes of argument recognition performance in SRL. The evaluation index equation in this paper is as follows, P is expressed as precision, R is expressed as recall, and $F1$ is equal to $2 * P * R / (P + R)$:

$$P = \frac{T}{W}, \quad (3)$$

$$R = \frac{T}{V}.$$

T is the number of proposition arguments correctly recognized by the system, W is the number of all proposition arguments recognized by the system, and V is the number of all proposition arguments in the standard mark.

4.2. Model Parameters. Bi-LSTM model is used to extract the feature attributes of sentence sequences, and the hidden layer of Bi-LSTM is output and pooled to the maximum extent to obtain all possible labeling results Y^* . Y^* is input into the discrimination function of the postprocessing layer, and the labeling result Y with the maximum probability is output after discrimination. The experimental flow of the labeling model is shown in Figure 3.

The model hyperparameters are shown in Table 3. The model is trained using the hyperparameter settings in Table 3.

4.3. Experimental Comparison. Chinese sentences include many linguistic cues, for example, POS, dependency syntax, and sentence framework; we consider these features can do help for our model. So experiments below are also based on these features.

Experiment 1: on the basis of the Bi-LSTM-CRF model, test and compare the performance improvement effect of integrating POS and argument into the basic corpus. The test results are shown in Table 4.

Analyzing the test results and comparing the evaluation of the precision, recall, and F1 value of the two models, it is shown that the performance of the model integrated with lexical features is generally better than that of the model trained with the basic corpus.

Experiment 2: on the basis of adding two groups of features such as POS in Experiment 1, dependency syntactic features are added to the training corpus. That is, the distance from the current word to the predicate and the dependency relationship are added to the training corpus. The test compares the performance improvement effect of integrating dependent syntactic features (the details of

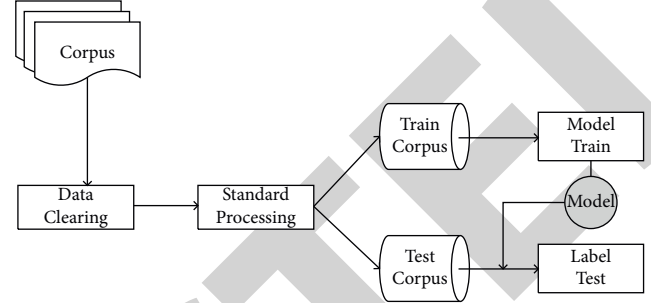


FIGURE 3: Experimental flow chart based on the Bi-LSTM model.

TABLE 3: Hyperparameter setting table.

Hyperparameter	Settings
Word vector dimension	150
Each feature vector dimension	15
Hidden layer dimension	128
Dropout	0.5
Learning rate	0.001

TABLE 4: Comparison of labeling results incorporating POS features.

	Precision (%)	Recall (%)	F1 (%)
Incorporate new feature groups	78.09	74.28	77.74
Control group	78.39	74.03	77.30

dependent syntactic features are added to the table, the sentence threshold discrimination table, and the labeling number is wrong) into the corpus of Experiment 1, and the results are shown in Table 5.

Analysis of the results in Table 4 shows that the model that integrates dependency features on the basis of adding part-of-speech features greatly improves the labeling performance, which is better than the model that only adds two groups of features such as part-of-speech.

Experiment 3: based on the analysis of Experiment 2, it is found that there is a big gap in the prediction results of sentences of different lengths. Among them, the labeling error rate of core components in long sentences is higher, while the labeling error rate of non-core components in short sentences is higher. Therefore, the experiment assumes to add sentence pattern discrimination features to the first three groups of feature corpus to explore whether to further improve the performance of the model. The discrimination method is to add a column of sentence pattern discrimination features to the corpus, and the threshold setting of sentence pattern features is shown in Table 6.

According to statistics, various sentence patterns in the corpus are shown in Figure 4. It can be seen that super-long

TABLE 5: Comparison of annotation results incorporating dependency features.

	Precision (%)	Recall (%)	F1 (%)
Incorporate new feature groups	80.83	75.30	78.44
Control group	78.09	74.29	77.74

TABLE 6: Sentence threshold discrimination.

Sentence type	Number of elements/ X
Short sentence	$X \leq 10$
Middle sentence	$X \leq 15$
Long sentence	$X \leq 20$
Super-long sentence	$X > 20$

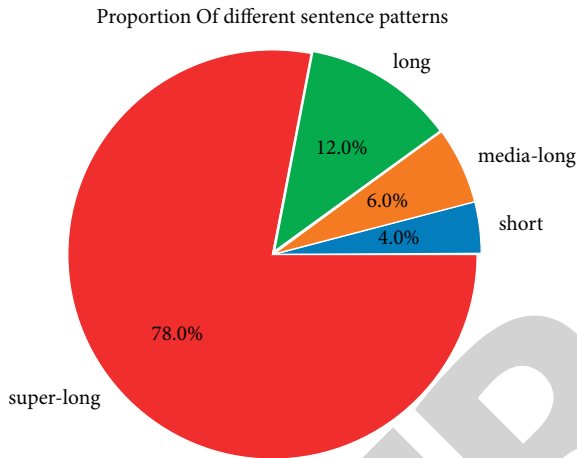


FIGURE 4: Proportion of different sentence patterns.

sentences account for the vast majority of the training corpus, followed by long sentences and short sentences.

After comparative tests, the results are shown in Table 7.

The analysis of the experimental results shows that the labeling precision of the non-core components of short sentences and the core components of long sentences with similar semantics and consistent sentence patterns in the training corpus has been improved to a certain extent.

Experiment 4: through the above experiments, it is found that integrating part-of-speech features, dependency features, and sentence features into the basic corpus can improve the performance of the model. However, with the increase of integrated features, the performance improvement effect of the model slows down. Therefore, the experiment assumes further feature extraction and sampling to explore whether the performance improvement effect brought by multilevel cue features to the model can be better released. The experimental test results are shown in Table 8.

The corpus used in this experiment is a multifeature training corpus that integrates POS, whether it belongs to arguments, dependent syntactic features, and sentence patterns. The above two models are trained and tested, respectively. Through the analysis of Table 8, it can be seen that the Bi-LSTM-CRF model integrated with the average pooling method is superior to the Bi-LSTM-CRF model

TABLE 7: Comparison of labeling results incorporating sentence pattern features.

	Precision (%)	Recall (%)	F1 (%)
Incorporate feature set	81.44	75.17	78.98
Control group	80.83	75.30	78.44

TABLE 8: Comparison of annotation results of two models.

	Precision (%)	Recall (%)	F1 (%)
Bi-LSTM-CRF + pooling	82.18	76.52	79.41
Bi-LSTM-CRF	81.44	75.17	78.69

without feature extraction and sampling in three evaluation indexes.

The comparison between the previous SRL results and the experimental results finally obtained in this paper is shown in Table 9.

4.4. Experimental Analysis. This experiment is an improvement attempt to the deep learning Chinese SRL model. The constructed corpus contains not only a large number of sentences with verbal predicates but also a large number of sentences with nominal predicates, which is close to the real language environment. Experiments show that different kinds of CNN models have a great influence on the annotation precision, and it is very important to select the appropriate annotation model. The Bi-LSTM model with Max pooling technology not only overcomes the conflict between the sentence vector representation and the original sentence semantics caused by the traditional CNN model not only considering the word order relationship in the sentence sequence but also significantly improves the memory ability of the initial part of elements in long sentence patterns. In order to verify the hypothesis of a correlation between different part-of-speech granularity and labeling precision, three groups of control experiments of coarse particle size, fine particle size, and coarse-fine particle size were carried out. The experimental results show that different particle size of a POS has a certain influence on the labeling precision, and the fine particle size training model has better labeling results than the coarse particle size training model. However, when the coarse and fine particle size training model is tried to combine, the labeling performance of the model does not increase but decreases. Compared with the training logs of the model, it is found that, with the complexity of POS granularity, the number of semantic role labels increases exponentially. The increase in the number of labels will be directly reflected in the increase in the number of features, while the excessive number of features will lead to the deterioration of the convergence speed of the model. The analysis of test results shows the following. (1) A large number of redundant or irrelevant features appear in the training model generation process. (2) Due to the difference in the size and granularity of POS, the model has different meanings when labeling named entities such as person names and organization names. For linear sequence classification annotation, the more detailed the

TABLE 9: Comparison of final experimental results with previous results.

Experimental team	Precision (%)	Models
Literature [3]	79.25	Maximum entropy Markov
Literature [8]	81.07	End-to-end
Ours	82.18	Bi-LSTM-CRF

features, the better the model performs. Too many features will easily lead to information redundancy, increase the system burden, and drag down the overall annotation precision of the model. The error analysis of the labeling results shows that, under the condition of similar sentence patterns, the error rate of non-core components in short sentences and core components in long sentences is relatively high, and it is conjectured that the labeling precision of sentence patterns and similar semantic sequences can be improved by adding long and short sentence labels. Experiment 3 verified this conjecture. By adding sentence pattern threshold features, the labeling precision of the model for the parts with a higher error rate of long and short sentences has been improved to different degrees.

Experiments show that each new feature will have different degrees of influence on the experimental results. In addition, when the model makes an annotated prediction, it may produce some unexpected prediction results (such as multiple core components, transcending boundaries, depending on edge crossing). It is the focus of our next task to solve these problems.

5. Conclusion and Next Work

Semantic roles are widely used because of their conciseness, clarity, and easy labeling. SRL, as a research key point connecting syntax and semantic layer, has high research value in NPL applications. In the construction of Chinese SRL model based on Bi-LSTM-CRF, this paper attempts to integrate multilevel linguistic features into the training corpus. During model training, use average pooling to sample and extract multifeature vector groups in the word vector processing stage; with these processes, the model can reduce the training difficulty and better release the potential of multicue features. The experimental results show that, by adding new features, it is confirmed that the annotation performance of the model can be improved to a certain extent. In addition, in the feature vector processing stage, average pooling integrated with CNN can further improve the labeling precision. In the future research work, we will focus on integrating high-order features that can embody structure into the model, making detailed role discrimination rules, and introducing semantic similarity calculation, so that the model can identify semantic role targets faster and more accurately. Next, we will add phrase syntactic features to our model and try to find some influence and achieve some improvement or not.

Data Availability

The data used to support the findings of this study are included within the article.

Disclosure

An earlier version of this paper has been presented at a conference in IEEEExplore at <https://ieeexplore.ieee.org/document/9362502>.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Fucheng Wan contributed to design of paper ideas and paper writing. Yimin Yang sorted out the current situation of research at home and abroad. Dengyun Zhu contributed to the extraction of linguistic features and the study of linguistic ontology. Hongzhi Yu* participated in paper writing and experimental methods. Ao Zhu contributed to construction of the underlying model of SRL. Guoyi Che carried out experiments test and analysis. Ning Ma contributed to revision of the paper and format adjustment.

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Retraction

Retracted: Neutrosophic Soft Quad Structures Concerning Neutrosophic Soft Points

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Neutrosophic Soft Quad Structures Concerning Neutrosophic Soft Points

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In this article, a new concept of neutrosophic soft quad-topological structures is introduced. Generalised neutrosophic soft $*_b$ open sets in neutrosophic soft quad-topological structures concerning soft points of the space are introduced. Different results are addressed in neutrosophic soft quad-topological structure on the basis of these new neutrosophic soft $*_b$ open sets. Neutrosophic soft separation axioms and other separation axioms are addressed in neutrosophic soft quad-topological structures. The engagement of these axioms is switched over with different results with respect to soft points. Neutrosophic soft topological properties of some results are also addressed in neutrosophic soft quad-topological structure. To secure the results, examples are constructed. The nonvalidity of some results are justified with examples. The understanding of some complicated problems is secured with simple techniques.

1. Introduction

Zadeh [1] introduced fuzzy set theory. It addresses vagueness and incomplete data used in various fields of science. Atanassov [2] addressed the short comings in fuzzy set theory with cultured way and opened a door to a new idea with new title intuitionistic fuzzy set theory. Molodtsov [3] inaugurated the concept of soft set theory to address the uncertainty. Soft set theory hugged with many applications in many fields, like smoothness of function, Riemann integration, measurement theory, and game theory. Molodtsov [4] continuously work on soft set theory. Alshehri et al. [5] applied the concept of soft set theory to K-algebra and traced some characteristics of Abelian soft K-algebras. In addition, they introduced the notion of intersection K-algebras and addressed some of their properties. Maji et al. [6] initiated first practical application of soft sets in decision-making problems. Feng et al. [7] for the first time considered

the combination of soft sets, fuzzy sets, and rough sets. Using soft sets as the granulation structures, Feng et al. [8] defined soft approximation spaces, soft rough approximations, and soft rough sets, which are generalizations of Pawlak's rough set model based on soft sets. It has been proved that, in some cases, Feng's soft rough set model could provide better approximations than classical rough sets.

Xu et al. [9] unveiled the concept of vague soft set theory which is an extension to soft set theory. Huang et al. [10] deeply studied [9] and pointed out some incorrect results. They verified the incorrect results with examples and gave some more new definitions. Chang Wang and Yaya Li [11] initiated the concept of vague soft topological structures with title topological structure of vague soft sets. The authors discussed the basic concepts related to vague soft topological space and studied the results in vague soft topology. The soft set to the hyper-soft set was generalized by Smarandache [12]. In addition to this, the author introduced the hybrids of

crisp, fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic hyper-soft set. Bera and Mahapatra [13] introduced the concept of neutrosophic soft topology and discussed some fundamental results. Ozturk in [14] introduced new concepts in neutrosophic soft topological spaces. These concepts are boundary, dense set, and neutrosophic soft basis. In addition, the concept of soft subspace on neutrosophic soft topological spaces. Some interesting results are addressed with respect to soft points. Some complicated results are secured with best examples.

Yolcu et al. redefined neutrosophic soft mapping and studied the images and inverse images of neutrosophic soft sets. The authors continued to trace the basic operations and other related properties of neutrosophic soft mapping. The authors beautifully applied neutrosophic soft mapping to application in decision-making problems.

Ozturk et al. [14] are pioneers of new operations on neutrosophic soft sets. Ozturk et al. [15] unveiled the concept of neutrosophic soft mapping, neutrosophic soft open mapping, and neutrosophic soft homeomorphism on the basis of operation defined in [14]. Some results are secured with best understandable examples. Gunduz et al. [16] introduced new concepts of neutrosophic soft sets. They defined new separation axioms in neutrosophic soft topological spaces with respect to soft points. In continuation, the relationship among these neutrosophic soft axioms has been addressed. Interior and closer of neutrosophic soft sets are also addressed. On the basis of these concepts, some other structures are also discussed. Most of the difficult results are secured with best examples.

AL-Nafee [17] introduced new family of neutrosophic soft sets. The author defined new operations on the neutrosophic set. These operations are union and intersection. On the basis of these new operations, the author defined neutrosophic soft topological space. AL-Nafee et al. [18] continued their work and extended the neutrosophic soft topological space to neutrosophic soft bitopological space on the basis of operations defined in [18]. The authors regenerated all the fundamental results of NSBTS on these basic operations. Dadas and Demiralp [19] inaugurated NSBTS on the basis of the operations defined in [13]. The authors introduced pairwise neutrosophic soft (closed) sets in NSBTS. These references [13–19] became source of motivation for my new research.

In our study, we have worked with the operations given in references [14–16] which are entirely different from references [13, 17]. In Section 2, some basic recipes are inaugurated. Originality begins in Section 3. In this section, new concepts of neutrosophic soft quad-topological spaces are addressed with examples. Some results, union and intersection are also studied in neutrosophic soft quad-topological spaces. In Section 4, some important definitions of generalized neutrosophic soft open sets in neutrosophic soft quad topological spaces are introduced. These definitions are semiopen, preopen, and $*_b$ open sets, respectively. These definitions became source of generation of different neutrosophical soft separation axioms and neutrosophical soft other separation axioms in neutrosophic soft quad topological spaces with respect to soft points of the second space.

Neutrosophical soft quad homeomorphism which is a safe carriage for different structure from one space to another is defined. Soft closer attachment with neutrosophical soft separation axioms and neutrosophical soft other separation axioms in neutrosophic soft quad topological spaces with respect to soft points of the second space are addressed. In Section 5, more main results are addressed. Among neutrosophical soft separation axioms, Hausdorff space is considered to be the most important separation axioms. Important things should be given serious attention. So, this section has almost been engaged with the study of Hausdorff space. Through neutrosophical soft function, the characteristics of one space can be migrated to another space if the neutrosophical soft function is satisfying conditions of neutrosophical soft bijections and bicontinuousness. A soft function satisfying these conditions is known as neutrosophical soft homeomorphism. Neutrosophical soft homeomorphism is giving birth to neutrosophical soft topological property. Some neutrosophical soft topological properties of Hausdorff space are addressed with respect to soft points. Sequentially, compactness and countably compact in neutrosophic soft quad topological spaces with respect to soft points of the second space are addressed. Engagement of Hausdorff space with closed sets in neutrosophic soft quad topological spaces is addressed. In Section 6, some concluding remarks and future work are given.

2. Fundamental Concepts

In this section, fundamental concepts are addressed. These fundamental concepts are neutrosophic set, soft set, neutrosophic soft set, neutrosophic soft complement, neutrosophic soft subsets, neutrosophic soft union, neutrosophic soft intersection, null neutrosophic soft, neutrosophic soft absolute set, neutrosophic soft points, and neutrosophic soft topological spaces.

Definition 1 (see [20]). A neutrosophic set (NS) symbolized by A on the key set \mathfrak{Z} is defined as: $A = \{\mathfrak{E}, \mathbb{T}_A \mathfrak{E}, \mathbb{I}_A \mathfrak{E},$

$$\mathbb{F}_A \mathfrak{E}: \mathfrak{E} \in \mathfrak{Z}\}, \text{ where } \begin{bmatrix} T: \mathfrak{Z}(0^-, 1^+) \\ I: \mathfrak{Z}(0^-, 1^+), \\ F: \mathfrak{Z}(0^-, 1^+) \\ 0^- \circ \mathbb{T}_A \langle \mathfrak{E} \rangle + \mathbb{I} \langle \mathfrak{E} \rangle \\ + \mathbb{F}_A \langle \mathfrak{E} \rangle \circ 3^+ \end{bmatrix},$$

Definition 2 (see [3]). Let \mathfrak{Z} be the key set, ∂ be a set of parameters, and $\mathcal{L}(\mathfrak{Z})$ symbolizes the power set of \mathfrak{Z} . A pair $\tilde{\eta}, \partial$ is referred to as a soft set (SS) over \mathfrak{Z} , where $\tilde{\eta}$ is a map given by $\tilde{\eta}: \partial \longrightarrow \mathcal{L}(\mathfrak{Z})$.

Definition 3 (see [21, 22]). Let \mathfrak{Z} be the key set and ∂ set of parameters. Let $\mathcal{L}(\mathfrak{Z})$ signifies power set of all neutrosophic sets on \mathfrak{Z} . Then, a neutrosophic soft set $\tilde{\eta}, \partial$ over \mathfrak{Z} is a set defined by a set valued function $\tilde{\eta}$ representing a mapping $\tilde{\eta}: \partial \longrightarrow \mathcal{L}(\mathfrak{Z})$, where $\tilde{\eta}$ is called approximate function of the neutrosophic soft set $\langle \tilde{\eta}, \partial \rangle$. It can be written as a set of

ordered pairs: $\tilde{\eta}, \partial = \{(\mathfrak{s}, \mathfrak{E}, \mathbb{T}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{I}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{F}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) : \mathfrak{E} \in \mathfrak{Q}) : \mathfrak{s} \in \partial\}$.

Definition 4 (see [13]). Let $\tilde{\eta}, \partial$ be a neutrosophic soft set over key set \mathfrak{Q} , the complement of $\tilde{\eta}, \partial$ is signified $\tilde{\eta}, \partial^c$ and is defined as follows:

$\tilde{\eta}, \partial^c = \{(\mathfrak{s}, \mathfrak{E}, \mathbb{T}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}), 1 - \mathbb{I}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{F}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) : \mathfrak{E} \in \mathfrak{Q}) : \mathfrak{s} \in \partial\}$. It is clear that

$$(\langle \tilde{\eta}, \partial^c \rangle)^c = \tilde{\eta}, \partial. \quad (1)$$

Definition 5 (see [23, 24]). Let $\langle \tilde{\eta}, \partial \rangle$ and $\langle \tilde{\rho}, \partial \rangle$ are two neutrosophic soft set over key set \mathfrak{Q} . Then, $\langle \tilde{\eta}, \partial \rangle$ is said to be neutrosophic soft subset of $\langle \tilde{\rho}, \partial \rangle$ if $\mathbb{T}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) \leq \mathbb{T}_{\tilde{\rho}(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{I}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) \leq \mathbb{I}_{\tilde{\rho}(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{F}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) \geq \mathbb{F}_{\tilde{\rho}(\mathfrak{s})}^-(\mathfrak{E}), \forall \mathfrak{s} \in \partial$, and $\forall \mathfrak{E} \in \mathfrak{Q}$. It is denoted by $\langle \tilde{\eta}, \partial \rangle \subseteq \langle \tilde{\rho}, \partial \rangle$. Neutrosophic soft $\langle \tilde{\eta}, \partial \rangle$ is said to be neutrosophic soft equal to $\langle \tilde{\rho}, \partial \rangle$, if $\langle \tilde{\eta}, \partial \rangle$ is neutrosophic soft subset of $\langle \tilde{\rho}, \partial \rangle$ and $\langle \tilde{\rho}, \partial \rangle$ neutrosophic soft subset of $\langle \tilde{\eta}, \partial \rangle$. It is denoted by $\langle \tilde{\eta}, \partial \rangle = \langle \tilde{\rho}, \partial \rangle$.

Definition 6 (see [16]). Let $\langle \tilde{\eta}_1, \partial \rangle, \langle \tilde{\eta}_2, \partial \rangle$ be two neutrosophic soft subsets over key set \mathfrak{Q} so that $\langle \tilde{\eta}_1, \partial \rangle \neq \langle \tilde{\eta}_2, \partial \rangle$, then their union is denoted by $\langle \tilde{\eta}_1, \partial \rangle \cup \langle \tilde{\eta}_2, \partial \rangle = \langle \tilde{\eta}_3, \partial \rangle$ and is defined as $\langle \tilde{\eta}_3, \partial \rangle = \{(\mathfrak{s}, \mathfrak{E}, \mathbb{T}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{I}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{F}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) : \mathfrak{E} \in \mathfrak{Q}) : \mathfrak{s} \in \partial\}$, where

$$\begin{cases} \mathbb{T}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = \max \left[\mathbb{T}_{\tilde{\mathfrak{f}}_1(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{T}_{\tilde{\mathfrak{f}}_2(\mathfrak{s})}^-(\mathfrak{E}) \right] \\ \mathbb{I}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = \max \left[\mathbb{I}_{\tilde{\mathfrak{f}}_1(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{I}_{\tilde{\mathfrak{f}}_2(\mathfrak{s})}^-(\mathfrak{E}) \right] \\ \mathbb{F}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = \min \left[\mathbb{F}_{\tilde{\mathfrak{f}}_1(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{F}_{\tilde{\mathfrak{f}}_2(\mathfrak{s})}^-(\mathfrak{E}) \right] \end{cases} \quad (2)$$

Definition 7 (see [16]). Let $\langle \tilde{\eta}_1, \partial \rangle$ and $\langle \tilde{\eta}_2, \partial \rangle$ be two neutrosophic soft subsets over key set \mathfrak{Q} such that $\langle \tilde{\eta}_1, \partial \rangle \neq \langle \tilde{\eta}_2, \partial \rangle$, then their intersection is defined as $\langle \tilde{\eta}_1, \partial \rangle \cap \langle \tilde{\eta}_2, \partial \rangle = \langle \tilde{\eta}_3, \partial \rangle$ is given as $[\langle \tilde{\eta}_3, \partial \rangle = \{(\mathfrak{s}, \mathfrak{E}, \mathbb{T}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{I}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{F}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) : \mathfrak{E} \in \mathfrak{Q}) : \mathfrak{s} \in \partial\}]$ where

$$\begin{cases} \mathbb{T}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = \min \left[\mathbb{T}_{\tilde{\mathfrak{f}}_1(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{T}_{\tilde{\mathfrak{f}}_2(\mathfrak{s})}^-(\mathfrak{E}) \right] \\ \mathbb{I}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = \min \left[\mathbb{I}_{\tilde{\mathfrak{f}}_1(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{I}_{\tilde{\mathfrak{f}}_2(\mathfrak{s})}^-(\mathfrak{E}) \right] \\ \mathbb{F}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = \max \left[\mathbb{F}_{\tilde{\mathfrak{f}}_1(\mathfrak{s})}^-(\mathfrak{E}), \mathbb{F}_{\tilde{\mathfrak{f}}_2(\mathfrak{s})}^-(\mathfrak{E}) \right] \end{cases} \quad (3)$$

Definition 8 (see [14]). Let $\tilde{\eta}, \partial$ be a neutrosophic soft set over a key set \mathfrak{Q} , then it is said to be a null neutrosophic soft set if $\mathbb{T}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = 0, \mathbb{I}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = 0, \mathbb{F}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = 1; \forall \mathfrak{s} \in \partial, \forall \mathfrak{E} \in \mathfrak{Q}$.

It is signified as $0_{(\mathfrak{Q}, \partial)}$.

Definition 9 (see [14]). Let $\tilde{\eta}, \partial$ be a neutrosophic soft set over a key set \mathfrak{Q} , then it is said to be an absolute soft set if $\mathbb{T}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = 1, \mathbb{I}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = 1, \mathbb{F}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}) = 0; \forall \mathfrak{s} \in \partial$ and for all $\mathfrak{E} \in \mathfrak{Q}$.

It is signified as $1_{(\mathfrak{Q}, \partial)}$; clearly, $0_{(\mathfrak{Q}, \partial)}^c = 1_{(\mathfrak{Q}, \partial)}, 1_{(\mathfrak{Q}, \partial)}^c = 0_{(\mathfrak{Q}, \partial)}$.

Definition 10 (see [14]). Let $NSS(\tilde{\mathfrak{Q}}, \partial)$ be the family of all neutrosophic soft sets and $\tau \subseteq NSS(\tilde{\mathfrak{Q}}, \partial)$, then τ is said to be a neutrosophic soft topology on $\tilde{\mathfrak{Q}}$ if (1): $0_{(\mathfrak{Q}, \partial)}, 1_{(\mathfrak{Q}, \partial)} \in \tau$, (2): the union of any number of neutrosophic soft sets in $\tau \in \tau$, (3): the intersection of a finite number of neutrosophic soft sets in $\tau \in \tau$; then, $(\tilde{\mathfrak{Q}}, \tau, \partial)$ is said to be an NSTS over $\tilde{\mathfrak{Q}}$.

Definition 11 (see [14]). Let NS be the family of all neutrosophic sets over $\tilde{\mathfrak{Q}}, \mathfrak{E} \in \tilde{\mathfrak{Q}}$, then neutrosophic set $\mathfrak{E}_{(\gamma_1, \gamma_2, \gamma_3)}^{(y)}$ is neutrosophic point, for $0 < \gamma_1, \gamma_2, \gamma_3 \leq 1$ and is defined as follows:

$$\mathfrak{E}_{(\gamma_1, \gamma_2, \gamma_3)}^{(y)} = \begin{cases} (\gamma_1, \gamma_2, \gamma_3) & \text{if } y = \mathfrak{E} \\ (0, 0, 1) & \text{if } y \neq \mathfrak{E} \end{cases} \quad (4)$$

Definition 12 (see [14]). Let $NSS(\tilde{\mathfrak{Q}}, \partial)$ be the family of all neutrosophic soft sets over KS \mathfrak{Q} . Then, neutrosophic soft set $(\mathfrak{E}_{(\gamma_1, \gamma_2, \gamma_3)}^{(y)})^{\mathfrak{s}}$ is called a neutrosophic soft point if for every $\mathfrak{E} \in \tilde{\mathfrak{Q}}, 0 < \gamma_1, \gamma_2, \gamma_3 \leq 1, \mathfrak{s} \in \partial$ and is defined as follows:

$$\mathfrak{E}^{\mathfrak{s}} = \begin{cases} (\gamma_1, \gamma_2, \gamma_3) & \text{if } \mathfrak{s} = \mathfrak{s}, \text{ then } y = \mathfrak{E} \\ (0, 0, 1) & \text{if } \mathfrak{s} \neq \mathfrak{s}, \text{ then } y \neq \mathfrak{E} \end{cases} \quad (5)$$

Definition 13 (see [14]). Let $\tilde{\eta}, \partial$ (be a neutrosophic soft sets over KS \mathfrak{Q} , then $\mathfrak{E}^{\mathfrak{s}} \in NSS \tilde{\eta}, \partial$) if

$$\gamma_1 \leq \mathbb{T}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}), \gamma_2 \leq \mathbb{I}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}), \gamma_3 \geq \mathbb{F}_{\tilde{\mathfrak{f}}(\mathfrak{s})}^-(\mathfrak{E}). \quad (6)$$

Definition 14 (see [14]). Let $(\tilde{\mathfrak{Q}}, \tau, \partial)$ be a soft neutrosophic topological space over \mathfrak{Q} . Let $\tilde{\eta}, \partial$ be a neutrosophic soft set. Then, $\tilde{\eta}, \partial$ is called a neutrosophic soft neighborhood of the neutrosophic soft point $\mathfrak{E}^{\mathfrak{s}}$, if there exists a neutrosophic soft open set (\mathcal{U}, ∂) such that $\mathfrak{E}^{\mathfrak{s}} \in (\mathcal{U}, \partial) \subseteq \tilde{\eta}, \partial$.

3. Characterisation of Neutrosophic Soft Quad-Topological Structures

In this section, the concept of neutrosophic soft quad topological space (NSQTS) is defined. Furthermore, new types of open and closed sets have been introduced in neutrosophic soft quad topological spaces.

3.1. Definition. If $(\mathfrak{Q}, \tau_1, \partial), (\mathfrak{Q}, \tau_2, \partial), (\mathfrak{Q}, \tau_3, \partial)$, and $(\mathfrak{Q}, \tau_4, \partial)$ are four neutrosophic soft topological space (NSTS), then $(\mathfrak{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is called neutrosophic soft quad topological space. A neutrosophic soft subset $\langle \tilde{\eta}, \partial \rangle$ is said to be neutrosophic soft quad open in $(\mathfrak{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ if there exists a neutrosophic soft open set $\langle \tilde{\eta}_1, \partial \rangle$ in τ_1 , neutrosophic soft open set $\langle \tilde{\eta}_2, \partial \rangle$ in τ_2 , neutrosophic soft open set $\langle \tilde{\eta}_3, \partial \rangle$ in τ_3 and neutrosophic soft open $\langle \tilde{\eta}_4, \partial \rangle$ in τ_4 such that $\langle \tilde{\eta}, \partial \rangle = \langle \tilde{\eta}_1, \partial \rangle \cup \langle \tilde{\eta}_2, \partial \rangle \cup \langle \tilde{\eta}_3, \partial \rangle \cup \langle \tilde{\eta}_4, \partial \rangle$.

3.2. *Example.* Let $\mathfrak{X} = \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3\}$, $\partial = \{\mathfrak{s}_1, \mathfrak{s}_2\}$ and $\tau_1 = \{0_{(\mathfrak{X}, \partial)}, 1_{(\mathfrak{X}, \partial)}, (p_1, \partial), (p_2, \partial)\}$, $\tau_2 = \{0_{(\mathfrak{X}, \partial)}, 1_{(\mathfrak{X}, \partial)}, (\mathbb{J}_1, \partial), (\mathbb{J}_2, \partial)\}$, $\tau_3 = \{0_{(\mathfrak{X}, \partial)}, 1_{(\mathfrak{X}, \partial)}, (\mathfrak{Q}_1, \partial), (\mathfrak{Q}_2, \partial)\}$, and $\tau_4 = \{0_{(\mathfrak{X}, \partial)}, 1_{(\mathfrak{X}, \partial)}\}$ where $(p_1, \partial), (p_2, \partial), (\mathbb{J}_1, \partial), (\mathbb{J}_2, \partial)$ and $(\mathfrak{Q}_1, \partial), (\mathfrak{Q}_2, \partial)$ being NSSs as follows:

$$f_{(p_1, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.2 \times 10^{-1}, 0.3 \times 10^{-1}, 0.8 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.2 \times 10^{-4}, 0.3 \times 10^{-4}, 0.8 \times 10^{-4} \rangle, \\ \langle \mathfrak{E}_3, 0.2 \times 10^{-1}, 0.4 \times 10^{-1}, 0.3 \times 10^{-1} \rangle \end{bmatrix} \quad (7)$$

$$f_{(p_1, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.3 \times 10^{-1}, 0.2 \times 10^{-1}, 0.6 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.1 \times 10^{-1}, 0.5 \times 10^{-1}, 0.5 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.4 \times 10^{-1}, 0.3 \times 10^{-1}, 0.4 \times 10^{-1} \rangle \end{bmatrix} \quad (8)$$

$$f_{(p_2, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.4 \times 10^{-1}, 0.3 \times 10^{-1}, 0.6 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.4 \times 10^{-1}, 0.5 \times 10^{-1}, 0.3 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.3 \times 10^{-1}, 0.5 \times 10^{-1}, 0.2 \times 10^{-1} \rangle \end{bmatrix} \quad (9)$$

$$f_{(p_2, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.3 \times 10^{-1}, 0.4 \times 10^{-1}, 0.5 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.2 \times 10^{-1}, 0.6 \times 10^{-1}, 0.4 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.4 \times 10^{-1}, 0.6 \times 10^{-1}, 0.3 \times 10^{-1} \rangle \end{bmatrix} \quad (10)$$

$$f_{(\mathbb{J}_1, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.5 \times 10^{-1}, 0.4 \times 10^{-1}, 0.4 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.6 \times 10^{-1}, 0.6 \times 10^{-1}, 0.2 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.4 \times 10^{-1}, 0.6 \times 10^{-1}, 0.1 \times 10^{-1} \rangle \end{bmatrix} \quad (11)$$

$$f_{(\mathbb{J}_1, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.4 \times 10^{-1}, 0.6 \times 10^{-1}, 0.3 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.3 \times 10^{-1}, 0.7 \times 10^{-1}, 0.3 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.5 \times 10^{-1}, 0.7 \times 10^{-1}, 0.1 \times 10^{-1} \rangle \end{bmatrix} \quad (12)$$

$$\mathbb{J}f_{(\mathbb{J}_2, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.1 \times 10^{-1}, 0.2 \times 10^{-1}, 0.7 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.3 \times 10^{-1}, 0.3 \times 10^{-1}, 0.3 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.1 \times 10^{-1}, 0.2 \times 10^{-1}, 0.2 \times 10^{-1} \rangle \end{bmatrix} \quad (13)$$

$$f_{(\mathbb{J}_2, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.1 \times 10^{-1}, 0.2 \times 10^{-1}, 0.7 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.3 \times 10^{-1}, 0.3 \times 10^{-1}, 0.3 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.1 \times 10^{-1}, 0.2 \times 10^{-1}, 0.2 \times 10^{-1} \rangle \end{bmatrix} \quad (14)$$

$$f_{(\mathfrak{Q}_1, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.6 \times 10^{-1}, 0.5 \times 10^{-1}, 0.2 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.4 \times 10^{-1}, 0.5 \times 10^{-1}, 0.2 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.4 \times 10^{-1}, 0.5 \times 10^{-1}, 0.1 \times 10^{-1} \rangle \end{bmatrix} \quad (15)$$

$$f_{(\mathfrak{Q}_1, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.3 \times 10^{-1}, 0.4 \times 10^{-1}, 0.4 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.3 \times 10^{-1}, 0.6 \times 10^{-1}, 0.2 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.4 \times 10^{-1}, 0.6 \times 10^{-1}, 0.1 \times 10^{-1} \rangle \end{bmatrix} \quad (16)$$

$$f_{(\mathfrak{Q}_2, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.5 \times 10^{-1}, 0.4 \times 10^{-1}, 0.2 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.4 \times 10^{-1}, 0.5 \times 10^{-1}, 0.2 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.3 \times 10^{-1}, 0.5 \times 10^{-1}, 0.1 \times 10^{-1} \rangle \end{bmatrix} \quad (17)$$

$$f_{(\mathfrak{Q}_2, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.3 \times 10^{-1}, 0.4 \times 10^{-1}, 0.5 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.2 \times 10^{-1}, 0.6 \times 10^{-1}, 0.2 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.4 \times 10^{-1}, 0.5 \times 10^{-1}, 0.1 \times 10^{-1} \rangle \end{bmatrix} \quad (18)$$

Therefore, τ_1, τ_2, τ_3 , and τ_4 are NSQTS on \mathfrak{X} and so $(\mathfrak{X}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is a NSQTS.

3.3. *Theorem.* Let $(\mathfrak{X}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS. Then, $\tau_1 \widetilde{\cap} \tau_2 \widetilde{\cap} \tau_3 \widetilde{\cap} \tau_4$ is a NSQTS on \mathfrak{X} .

Proof. For this, we have to verify all the three conditions of neutrosophic soft quad topological space. Conditions (1) and (3) are obvious; for condition (2), let $\{(p_i, \partial); i \in I\} \in \tau_1 \widetilde{\cap} \tau_2 \widetilde{\cap} \tau_3 \widetilde{\cap} \tau_4$, then $(p_i, \partial) \in \tau_1, (p_i, \partial) \in \tau_2, (p_i, \partial) \in \tau_3$, and $(p_i, \partial) \in \tau_4$. As τ_1, τ_3 , and τ_4 are NSTS on \mathfrak{X} , then $\bigcup_i (p_i, \partial) \in \tau_1, \bigcup_i (p_i, \partial) \in \tau_2, \bigcup_i (p_i, \partial) \in \tau_3$, and $\bigcup_i (p_i, \partial) \in \tau_4$. Therefore, $\bigcup_i (p_i, \partial) \in \tau_1 \widetilde{\cap} \tau_2 \widetilde{\cap} \tau_3 \widetilde{\cap} \tau_4$. \square

Remark 1. Let $(\mathfrak{X}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS, then $\tau_1 \widetilde{\cup} \tau_2 \widetilde{\cup} \tau_3 \widetilde{\cup} \tau_4$ need not be a NSQTS on \mathfrak{X} .

3.4. *Example.* Let $\mathfrak{X} = \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3\}$, $\partial = \{\mathfrak{s}_1, \mathfrak{s}_2\}$ and $\tau_1 = \{0_{(\mathfrak{X}, \partial)}, 1_{(\mathfrak{X}, \partial)}, (p_1, \partial), (p_2, \partial), (p_3, \partial)\}$, $\tau_2 = \{0_{(\mathfrak{X}, \partial)}, 1_{(\mathfrak{X}, \partial)}, (\mathbb{J}_1, \partial), (\mathbb{J}_2, \partial)\}$, $\tau_3 = \{0_{(\mathfrak{X}, \partial)}, 1_{(\mathfrak{X}, \partial)}, (\mathfrak{Q}_1, \partial), (\mathfrak{Q}_2, \partial)\}$, and $\tau_4 = \{0_{(\mathfrak{X}, \partial)}, 1_{(\mathfrak{X}, \partial)}\}$, where $(p_1, \partial), (p_2, \partial), (\mathbb{J}_1, \partial), (\mathbb{J}_2, \partial)$ and $(\mathfrak{Q}_1, \partial), (\mathfrak{Q}_2, \partial)$ being NSSs as follows:

$$f_{(p_1, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.2 \times 10^{-1}, 0.3 \times 10^{-1}, 0.8 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.4 \times 10^{-1}, 0.4 \times 10^{-1}, 0.4 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.2 \times 10^{-1}, 0.4 \times 10^{-1}, 0.3 \times 10^{-1} \rangle \end{bmatrix} \quad (19)$$

$$f_{(p_1, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 0.3 \times 10^{-1}, 0.2 \times 10^{-1}, 0.6 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 0.1 \times 10^{-1}, 0.5 \times 10^{-1}, 0.5 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 0.4 \times 10^{-1}, 0.3 \times 10^{-1}, 0.5 \times 10^{-1} \rangle \end{bmatrix} \quad (20)$$

$$f_{(p_2, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 04 \times 10^{-1}, 03 \times 10^{-1}, 06 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 04 \times 10^{-1}, 05 \times 10^{-1}, 03 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 03 \times 10^{-1}, 05 \times 10^{-1}, 02 \times 10^{-1} \rangle \end{bmatrix} \quad (21)$$

$$f_{(p_2, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 03 \times 10^{-1}, 04 \times 10^{-1}, 05 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 02 \times 10^{-1}, 06 \times 10^{-1}, 04 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 04 \times 10^{-1}, 06 \times 10^{-1}, 03 \times 10^{-1} \rangle \end{bmatrix} \quad (22)$$

$$f_{(p_3, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 05 \times 10^{-1}, 04 \times 10^{-1}, 04 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 06 \times 10^{-1}, 06 \times 10^{-1}, 02 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 04 \times 10^{-1}, 06 \times 10^{-1}, 00 \times 10^{-1} \rangle \end{bmatrix} \quad (23)$$

$$f_{(p_3, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 04 \times 10^{-1}, 06 \times 10^{-1}, 03 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 03 \times 10^{-1}, 07 \times 10^{-1}, 03 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 05 \times 10^{-1}, 07 \times 10^{-1}, 01 \times 10^{-1} \rangle \end{bmatrix} \quad (24)$$

$$f_{(\mathbb{J}_1, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 05 \times 10^{-1}, 04 \times 10^{-1}, 04 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 06 \times 10^{-1}, 06 \times 10^{-1}, 02 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 04 \times 10^{-1}, 06 \times 10^{-1}, 01 \times 10^{-1} \rangle \end{bmatrix} \quad (25)$$

$$f_{(\mathbb{J}_1, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 04 \times 10^{-1}, 06 \times 10^{-1}, 03 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 03 \times 10^{-1}, 07 \times 10^{-1}, 03 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 05 \times 10^{-1}, 07 \times 10^{-1}, 01 \times 10^{-1} \rangle \end{bmatrix} \quad (26)$$

$$f_{(\mathbb{J}_2, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 01 \times 10^{-1}, 02 \times 10^{-1}, 07 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 03 \times 10^{-1}, 03 \times 10^{-1}, 03 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 01 \times 10^{-1}, 02 \times 10^{-1}, 02 \times 10^{-1} \rangle \end{bmatrix} \quad (27)$$

$$f_{(\mathbb{J}_2, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 01 \times 10^{-1}, 02 \times 10^{-1}, 07 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 03 \times 10^{-1}, 03 \times 10^{-1}, 03 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 01 \times 10^{-1}, 02 \times 10^{-1}, 02 \times 10^{-1} \rangle \end{bmatrix} \quad (28)$$

$$f_{(\mathfrak{Q}_1, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 06 \times 10^{-1}, 05 \times 10^{-1}, 02 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 04 \times 10^{-1}, 05 \times 10^{-1}, 02 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 04 \times 10^{-1}, 05 \times 10^{-1}, 01 \times 10^{-1} \rangle \end{bmatrix} \quad (29)$$

$$f_{(\mathfrak{Q}_1, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 03 \times 10^{-1}, 04 \times 10^{-1}, 04 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 03 \times 10^{-1}, 06 \times 10^{-1}, 02 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 04 \times 10^{-1}, 06 \times 10^{-1}, 01 \times 10^{-1} \rangle \end{bmatrix} \quad (30)$$

$$f_{(\mathfrak{Q}_2, \partial)}(\mathfrak{s}_1) = \begin{bmatrix} \langle \mathfrak{E}_1, 03 \times 10^{-1}, 04 \times 10^{-1}, 04 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 03 \times 10^{-1}, 06 \times 10^{-1}, 02 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 04 \times 10^{-1}, 06 \times 10^{-1}, 01 \times 10^{-1} \rangle \end{bmatrix} \quad (31)$$

$$f_{(\mathfrak{Q}_2, \partial)}(\mathfrak{s}_2) = \begin{bmatrix} \langle \mathfrak{E}_1, 04 \times 10^{-1}, 05 \times 10^{-1}, 02 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_2, 04 \times 10^{-1}, 05 \times 10^{-1}, 02 \times 10^{-1} \rangle, \\ \langle \mathfrak{E}_3, 03 \times 10^{-1}, 05 \times 10^{-1}, 01 \times 10^{-1} \rangle \end{bmatrix} \quad (32)$$

Here, $\tau_1 \tilde{\cup} \tau_2 \tilde{\cup} \tau_3 \tilde{\cup} \tau_4 = \{0_{(\mathfrak{Q}, \partial)}, 1_{(\mathfrak{Q}, \partial)}, (p_1, \partial), (p_2, \partial), (p_3, \partial), (\mathbb{J}_1, \partial), (\mathbb{J}_2, \partial), (\mathfrak{Q}_1, \partial), (\mathfrak{Q}_2, \partial)\}$ is not a NSQTS on \mathfrak{Q} .

3.5. Definition. Let $(\mathfrak{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS. Then, an NSS

$$(\mathbb{J}, \partial) = \{\mathfrak{s}, \{\langle \mathfrak{E}, T_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E}), I_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E}), F_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E}) \rangle\} : \mathfrak{E} \in \mathfrak{Q}, \mathfrak{s} \in \partial\}, \quad (33)$$

is called as a pairwise NSQ open set (PNSOS) if there exists a NSOS (\mathbb{J}_1, ∂) in τ_1 , NSOS (\mathbb{J}_2, ∂) in τ_2 , NSOS (\mathbb{J}_3, ∂) in τ_3 and NSOS (\mathbb{J}_4, ∂) in τ_4 such that for all $\mathfrak{E} \in \mathfrak{Q}$ such that

$$\begin{aligned} (\mathbb{J}, \partial) &= (\mathbb{J}_1, \partial) \tilde{\cup} (\mathbb{J}_2, \partial) \tilde{\cup} (\mathbb{J}_3, \partial) \tilde{\cup} (\mathbb{J}_4, \partial) \\ &= \left\{ \left(\mathfrak{s}, \left\{ \begin{array}{l} \langle \mathfrak{E}, \max\{T_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E}), T_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E})\}, \\ \max\{I_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E}), I_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E})\}, \min\{F_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E}), F_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E})\} \rangle \end{array} \right\} : \mathfrak{s} \in \partial \right\}. \end{aligned} \quad (34)$$

3.6. Definition. Let $(\mathfrak{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS. Then, an NSS

$$(\mathbb{J}, \partial) = \left\{ \left(\mathfrak{s}, \{\langle \mathfrak{E}, T_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E}), I_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E}), F_{\mathbb{J}(\mathfrak{s})}(\mathfrak{E}) \rangle\} \right) : \mathfrak{E} \in \mathfrak{Q}, \mathfrak{s} \in \partial \right\} \quad (35)$$

is called as a PNSOS if there exist a NSOS (\mathbb{J}_1, ∂) in τ_1 and a NSOS (\mathbb{J}_2, ∂) in τ_2 , NSOS (\mathbb{J}_3, ∂) in τ_3 , and NSOS (\mathbb{J}_4, ∂) in τ_4 such that, for all $\mathbb{E} \in \mathfrak{L}$,

$$(\mathbb{J}, \partial) = (\mathbb{J}_1, \partial) \widetilde{\cup} (\mathbb{J}_2, \partial) \widetilde{\cup} (\mathbb{J}_3, \partial) \widetilde{\cup} (\mathbb{J}_4, \partial) = \left\{ \left(\mathfrak{s}, \left\{ \begin{array}{l} < \mathbb{E}, \max\{T_{\mathbb{J}(\mathfrak{s})}(\mathbb{E}), T_{\mathbb{J}(\mathfrak{s})}(\mathbb{E})\}, \\ \max\{I_{\mathbb{J}(\mathfrak{s})}(\mathbb{E}), I_{\mathbb{J}(\mathfrak{s})}(\mathbb{E})\}, \min\{F_{\mathbb{J}(\mathfrak{s})}(\mathbb{E}), F_{\mathbb{J}(\mathfrak{s})}(\mathbb{E})\} > \end{array} \right\} \right) : \mathfrak{s} \in \partial \right\}. \quad (36)$$

The set of all pairwise neutrosophic soft quad open sets in $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is denoted by PNSO $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$.

3.7. Definition. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS. Then, a NSS

$$(\mathbb{J}, \partial) = \left\{ \left(\mathfrak{s}, \left\{ \langle \mathbb{E}, T_{\mathbb{J}(\mathfrak{s})}(\mathbb{E}), I_{\mathbb{J}(\mathfrak{s})}(\mathbb{E}), F_{\mathbb{J}(\mathfrak{s})}(\mathbb{E}) \rangle \right\} \right) : \mathbb{E} \in \mathfrak{L}, \mathfrak{s} \in \partial \right\}, \quad (37)$$

is called as a pairwise neutrosophic soft quad closed set (PNSC) if $(\mathbb{J}, \partial)^c$ is a PNSO. It is clear that (\mathbb{J}, ∂) is a PNSC set if there exists a NSOC (\mathbb{J}_1, ∂) in τ_1 , NSOC (\mathbb{J}_2, ∂) in τ_2 ,

NSOC (\mathbb{J}_3, ∂) in τ_3 , and NSOC (\mathbb{J}_4, ∂) in τ_4 such that, for all $\mathbb{E} \in \mathfrak{L}$,

$$(\mathbb{J}, \partial) = (\mathbb{J}_1, \partial) \widetilde{\cup} (\mathbb{J}_2, \partial) \widetilde{\cup} (\mathbb{J}_3, \partial) \widetilde{\cup} (\mathbb{J}_4, \partial) = \left\{ \left(\mathfrak{s}, \left\{ \begin{array}{l} < \mathbb{E}, \max\{T_{\mathbb{J}(\mathfrak{s})}(\mathbb{E}), T_{\mathbb{J}(\mathfrak{s})}(\mathbb{E})\}, \\ \max\{I_{\mathbb{J}(\mathfrak{s})}(\mathbb{E}), I_{\mathbb{J}(\mathfrak{s})}(\mathbb{E})\}, \min\{F_{\mathbb{J}(\mathfrak{s})}(\mathbb{E}), F_{\mathbb{J}(\mathfrak{s})}(\mathbb{E})\} > \end{array} \right\} \right) : \mathfrak{s} \in \partial \right\}. \quad (38)$$

The set of all PNSC in $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is denoted by PNSC $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$.

3.8. Theorem. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSBTS. In this case,

- (1) $0_{(\mathfrak{L}, \partial)}, 1_{(\mathfrak{L}, \partial)} \in \text{PNSO}(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$
- (2) If $\{(\mathbb{J}_i, \partial) | i \in I\} \in \text{PNSO}(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$, then $\bigcup_{i \in I} (\mathbb{J}_i, \partial) \in \text{PNSO}(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$
- (3) If $\{(\mathbb{U}_i, \partial) | i \in I\} \in \text{PNSC}(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ then $\bigcup_{i \in I} (\mathbb{U}_i, \partial) \in \text{PNSC}(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$

Proof

- (1) Since $0_{(\mathfrak{L}, \partial)} \cup 0_{(\mathfrak{L}, \partial)} = 0_{(\mathfrak{L}, \partial)}$ and $1_{(\mathfrak{L}, \partial)} \cup 1_{(\mathfrak{L}, \partial)} = 1_{(\mathfrak{L}, \partial)}$, then $0_{(\mathfrak{L}, \partial)}$ and $1_{(\mathfrak{L}, \partial)}$ are PNSO.
- (2) Since $(\mathbb{J}_i, \partial) \in \text{PNSO}(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$, there exist $(\mathbb{J}_1^1, \partial) \in \tau_1$, $(\mathbb{J}_1^2, \partial) \in \tau_2$, $(\mathbb{J}_1^3, \partial) \in \tau_3$, and $(\mathbb{J}_1^4, \partial) \in \tau_4$ such that $(\mathbb{J}_i, \partial) = (\mathbb{J}_i^1, \partial) \widetilde{\cup} (\mathbb{J}_i^2, \partial) \widetilde{\cup} (\mathbb{J}_i^3, \partial) \widetilde{\cup} (\mathbb{J}_i^4, \partial)$ for all $i \in I$. Then,

$$\bigcup_{i \in I} (\mathbb{J}_i, \partial) = \bigcup_{i \in I} ((\mathbb{J}_i, \partial) = (\mathbb{J}_i^1, \partial) \widetilde{\cup} (\mathbb{J}_i^2, \partial) \widetilde{\cup} (\mathbb{J}_i^3, \partial) \widetilde{\cup} (\mathbb{J}_i^4, \partial)) = \left(\bigcup_{i \in I} (\mathbb{J}_i^1, \partial) \right) \widetilde{\cup} \left(\bigcup_{i \in I} (\mathbb{J}_i^2, \partial) \right) \widetilde{\cup} \left(\bigcup_{i \in I} (\mathbb{J}_i^3, \partial) \right) \widetilde{\cup} \left(\bigcup_{i \in I} (\mathbb{J}_i^4, \partial) \right). \quad (39)$$

As τ_1, τ_2, τ_3 and τ_4 are NSTS on $U_{i \in I} (\mathbb{J}_i^1, \partial) \in \tau_1$, $U_{i \in I} (\mathbb{J}_i^2, \partial) \in \tau_2$, $U_{i \in I} (\mathbb{J}_i^3, \partial) \in \tau_3$, and $U_{i \in I} (\mathbb{J}_i^4, \partial) \in \tau_4$. Therefore, $\bigcup_{i \in I} (\mathbb{J}_i, \partial) \in \text{PNSO}(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$.

- (3) Since $(\mathbb{U}_i, \partial) \in \text{PNSC}(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$, there exist $(\mathbb{U}_i^1, \partial)^c \in \tau_1$, $(\mathbb{U}_i^2, \partial)^c \in \tau_2$, $(\mathbb{U}_i^3, \partial)^c \in \tau_3$, and $(\mathbb{U}_i^4, \partial)^c \in \tau_4$ such that $(\mathbb{U}_i, \partial) = (\mathbb{U}_i^1, \partial) \widetilde{\cap} (\mathbb{U}_i^2, \partial) \widetilde{\cap} (\mathbb{U}_i^3, \partial) \widetilde{\cap} (\mathbb{U}_i^4, \partial)$, for all $i \in I$. Then,

$$\begin{aligned} \bigcup_{i \in I} (\mathbb{U}_i, \partial) &= \bigcap_{i \in I} ((\mathbb{U}_i^1, \partial) \widetilde{\cap} (\mathbb{U}_i^2, \partial) \widetilde{\cap} (\mathbb{U}_i^3, \partial) \widetilde{\cap} (\mathbb{U}_i^4, \partial)) \\ &= \left(\bigcap_{i \in I} (\mathbb{U}_i^1, \partial) \right) \widetilde{\cap} \left(\bigcap_{i \in I} (\mathbb{U}_i^2, \partial) \right) \widetilde{\cap} \left(\bigcap_{i \in I} (\mathbb{U}_i^3, \partial) \right) \widetilde{\cap} \left(\bigcap_{i \in I} (\mathbb{U}_i^4, \partial) \right). \end{aligned} \quad (40)$$

Then, $\bigcap_{i \in I} (\mathbb{U}_i, \partial) \in \text{PNSC}(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ as $(\bigcap_{i \in I} (\mathbb{U}_i^1, \partial))^c \in \tau_1$, $(\bigcap_{i \in I} (\mathbb{U}_i^2, \partial))^c \in \tau_2$, $(\bigcap_{i \in I} (\mathbb{U}_i^3, \partial))^c \in \tau_3$, and $(\bigcap_{i \in I} (\mathbb{U}_i^4, \partial))^c \in \tau_4$. \square

3.9. Definition. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS, $(\mathbb{J}, \partial) \in \text{NSS}(\mathfrak{L})$. The PNS closure of (\mathbb{J}, ∂) , denoted by $\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial)$, is the intersection of all PNSC containing (\mathbb{J}, ∂) , i.e.,

$$\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial) = \{(\omega, \partial) \in \text{PNSC}(\mathfrak{L}) \mid (\mathbb{J}, \partial) \in (\omega, \partial)\}. \quad (41)$$

It is clear that $\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial)$ is the smallest PNSCS containing (\mathbb{J}, ∂) .

3.10. Theorem. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS and $(\mathbb{J}, \partial), (\mathfrak{Q}, \partial) \in \text{NSS}(\mathfrak{L})$. Then,

- (1) $\text{cl}_p^{\text{NSS}}(0_{(\mathfrak{L}, \partial)}) = 0_{(\mathfrak{L}, \partial)}$ and $\text{cl}_p^{\text{NSS}}(1_{(\mathfrak{L}, \partial)}) = 1_{(\mathfrak{L}, \partial)}$
- (2) $(\mathbb{J}, \partial) \in \text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial)$
- (3) (\mathbb{J}, ∂) is a PNSCS if $\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial) = (\mathbb{J}, \partial)$
- (4) $\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial) \in \text{cl}_p^{\text{NSS}}(\mathfrak{Q}, \partial)$ if $(\mathbb{J}, \partial) \in (\mathfrak{Q}, \partial)$
- (5) $\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial) \cup \text{cl}_p^{\text{NSS}}(\mathfrak{Q}, \partial) \in \text{cl}_p^{\text{NSS}}((\mathbb{J}, \partial) \cup (\mathfrak{Q}, \partial))$
- (6) $\text{cl}_p^{\text{NSS}}(\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial)) = \text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial)$, i.e., $\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial)$ is a PNSCS

Proof. It is obvious. \square

3.11. Theorem. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS, $(\mathbb{J}, \partial) \in \text{NSS}(\mathfrak{L})$. Then, $\mathbb{E}^{\mathbb{J}} \in \text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial)$ if and only if for all $U_{\mathbb{E}^{\mathbb{J}}} \in \tau_{1234}$, where $U_{\mathbb{E}^{\mathbb{J}}}$ any is PNSOS contains $\mathbb{E}^{\mathbb{J}}$ and $\tau_{1234}(\mathbb{E}^{\mathbb{J}})$ is the family of all PNSOS contains $\mathbb{E}^{\mathbb{J}}$, $U_{\mathbb{E}^{\mathbb{J}}} \cap (\mathbb{J}, \partial) \neq 0_{(\mathfrak{L}, \partial)}$.

Proof. Let $\mathbb{E}^{\mathbb{J}} \in \text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial)$ and suppose that there exists $U_{\mathbb{E}^{\mathbb{J}}} \in \tau_{1234}(\mathbb{E}^{\mathbb{J}})$ such that $U_{\mathbb{E}^{\mathbb{J}}} \cap (\mathbb{J}, \partial) = 0_{(\mathfrak{L}, \partial)}$. Then, $(\mathbb{J}, \partial) \in (U_{\mathbb{E}^{\mathbb{J}}})^c$. Thus, $\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial) \in \text{cl}_p^{\text{NSS}}(U_{\mathbb{E}^{\mathbb{J}}})^c = (U_{\mathbb{E}^{\mathbb{J}}})^c$ which implies $\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial) \cap U_{\mathbb{E}^{\mathbb{J}}} = 0_{(\mathfrak{L}, \partial)}$, a contradiction.

Conversely, that $\mathbb{E}^{\mathbb{J}} \notin \text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial)$, then $\mathbb{E}^{\mathbb{J}} \in (\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial))^c \in \tau_{1234}(\mathbb{E}^{\mathbb{J}})$. Therefore, by hypothesis $(\text{cl}_p^{\text{NSS}}(\mathbb{J}, \partial))^c \cap (\mathbb{J}, \partial) \neq 0_{(\mathfrak{L}, \partial)}$, a contradiction. \square

4. Exhibition of Some Definitions and Main Results

In this section, some important definitions of generalized neutrosophic soft open sets in neutrosophic soft quad topological spaces are introduced which pave the way to our new results. These definitions are semiopen, preopen, and $*_b$ open sets, respectively. These definitions became source of generation of different neutrosophical soft separation axioms and neutrosophical soft other separation axioms in neutrosophic soft quad topological spaces with respect to soft points of the second space. Neutrosophical soft quad homeomorphism which is a safe carriage for different structure from one space to another is defined. Soft closer attachment with neutrosophical soft separation axioms and neutrosophical soft other separation axioms in neutrosophic soft quad topological spaces with respect to soft points of the second space are addressed.

Definition 15. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS over \mathfrak{L} , $(\tilde{\eta}, \partial)$ be a NS set over \mathfrak{L} . Then, $(\tilde{\eta}, \partial)$ is

- (1) Neutrosophic soft quad semiopen if $(\tilde{\eta}, \partial) \in \text{VScl}(\text{VSint}(\tilde{\eta}, \partial))$
- (2) : Neutrosophic soft quad preopen if $(\tilde{\eta}, \partial) \in \text{VSint}(\text{VScl}(\tilde{\eta}, \partial))$
- (3) : Neutrosophic soft quad $*_b$ open if $(\tilde{\eta}, \partial) \in \text{VScl}(\text{VSint}(\tilde{\eta}, \partial)) \cup \text{VSint}(\text{VScl}(\tilde{\eta}, \partial))$ and neutrosophic soft quad $*_b$ close if $(\tilde{\eta}, \partial) \supseteq \text{VScl}(\text{VSint}(\tilde{\eta}, \partial)) \cap \text{VSint}(\text{VScl}(\tilde{\eta}, \partial))$

Definition 16. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS over \mathfrak{L} , $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \neq \mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3)$ are NSQ points if there exist NSQ $*_b$ -open sets $(\tilde{\eta}, \partial)$ and (\mathbb{U}, ∂) such that $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \in (\tilde{\eta}, \partial)$, $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \cap (\mathbb{U}, \partial) = 0_{(\mathfrak{L}, \partial)}$ or $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{U}, \partial)$, $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) \cap (\tilde{\eta}, \partial) = 0_{(\mathfrak{L}, \partial)}$. Then, $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is called a NSQ $*_b$ space.

Definition 17. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS over \mathfrak{L} , $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \neq \mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3)$ are NSQ points. If there exist NSQ $*_b$ -open sets $(\tilde{\eta}, \partial)$, (\mathbb{U}, ∂) such that $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \in (\tilde{\eta}, \partial)$, $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \cap (\mathbb{U}, \partial) = 0_{(\tilde{\eta}, \partial)}$, $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{U}, \partial)$, $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) \cap (\tilde{\eta}, \partial) = 0_{(\tilde{\eta}, \partial)}$, then $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is called a NSQ $*_b$ space.

Definition 18. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ NSQTS over \mathfrak{L} , $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \neq \mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3)$ are NSQ points. If there exists NSQ $*_b$ open set $(\tilde{\eta}, \partial, nUq, h\partial)$ such that $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \in \tilde{\eta}, \partial$, $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{U}, \partial)$ and $(\tilde{\eta}, \partial) \cap (\mathbb{U}, \partial) = 0_{(\tilde{\eta}, \partial)}$, then $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \theta)$ is called a NSQ $*_b$ space.

Definition 19. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \theta)$ and $(\tilde{\mathfrak{Y}}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ be two NSQTS. A NS function $\tilde{\eta}, \partial: (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial) \rightarrow (\tilde{\mathfrak{Y}}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ is said to NSQ homeomorphism if (i) $(\tilde{\eta}, \partial)$ is NSQ bijective; (ii) $(\tilde{\eta}, \partial)$ is NSQ continuous; (iii) $((\tilde{\eta}, \partial))^{-1}$ is NSQ continuous or $(\tilde{\eta}, \partial)$ is NSO or $(\tilde{\eta}, \partial)$ is NSC.

Theorem 1. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \theta)$ be a NSTQS over the father set \mathfrak{L} , then is NSQ $*_b$ space if and only if for distinct NS points $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3)$ and $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3)$, there exists an NSQ $*_b$ -open set $(\tilde{\eta}, \partial)$ containing but not $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3)$ such that $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) \notin (\tilde{\eta}, \partial)$.

Proof. Let $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \neq \mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3)$ be two NS points in NSQ $*_b$ space, then there exists disjoint NSQ $*_b$ open sets $(\tilde{\eta}, \partial)$, (\mathbb{U}, ∂) such that $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \in (\tilde{\eta}, \partial)$ and $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{U}, \partial)$, since $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \cap \mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) = 0_{(\mathfrak{L}, \partial)}$ and $(\tilde{\eta}, \partial) \cap (\mathbb{U}, \partial) = 0_{(\mathfrak{L}, \partial)}$. $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) \notin (\tilde{\eta}, \partial) \Rightarrow \mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) \notin (\tilde{\eta}, \partial)$. Next suppose that, $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3) \neq \mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3)$ then there exists a NSQ $*_b$ open set $(\tilde{\eta}, \partial)$ containing $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3)$ but not $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3)$ such that $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3) \notin \tilde{\eta}, \partial^c$ that is and $(\tilde{\eta}, \partial)^c$ are mutually exclusive NSQ $*_b$ open sets supposing $\mathbb{E}_1(\gamma_1, \gamma_2, \gamma_3)$ and $\mathbb{E}_2(\gamma_1, \gamma_2, \gamma_3)$ in turn. \square

Theorem 2. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS. Then, $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \theta)$ is NSQ $*_b$ space if every NSQ point $\xi_1(\gamma_1, \gamma_2, \gamma_3) \in \tilde{\eta}, \partial \in (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ if there exists an NSQ $*_b$ open set (\mathfrak{U}, ∂) such that $\xi_1(\gamma_1, \gamma_2, \gamma_3) \in (\mathfrak{U}, \partial) \subseteq \overline{(\mathfrak{U}, \partial)} \subseteq \tilde{\eta}, \partial$, then $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \theta)$ an NSQ $*_b$ space.

Proof. Suppose $\xi_1(\gamma_1, \gamma_2, \gamma_3) \cap \xi_2(\gamma_1, \gamma_2, \gamma_3) = 0_{(\mathfrak{L}, \partial)}$. Since $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is NSQ $*_b$ space. $\xi_1(\gamma_1, \gamma_2, \gamma_3)$ and $\xi_2(\gamma_1, \gamma_2, \gamma_3)$ are NSQ $*_b$ closed sets in $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$; then, $\xi_1(\gamma_1, \gamma_2, \gamma_3) \in \xi_2(\gamma_1, \gamma_2, \gamma_3) \in (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$. Thus, there exists a NSQ $*_b$ open set $(\mathfrak{U}, \partial) \in (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ such that $\xi_1(\gamma_1, \gamma_2, \gamma_3) \in (\mathfrak{U}, \partial) \subseteq \overline{(\mathfrak{U}, \partial)} \subseteq (\xi_2(\gamma_1, \gamma_2, \gamma_3))^c$. So, we have $\xi_2(\gamma_1, \gamma_2, \gamma_3) \in (\mathfrak{U}, \partial)$ and $(\mathfrak{U}, \partial) \cap (\mathfrak{U}, \partial)^c = 0_{(\mathfrak{L}, \partial)}$, that is, $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is a NSQ $*_{b2}$ space. \square

Definition 20. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS. $\tilde{\eta}, \partial$ be a NSQ $*_b$ closed set and $\xi_1(\gamma_1, \gamma_2, \gamma_3) \cap \tilde{\eta}, \partial = 0_{(\mathfrak{L}, \partial)}$. If there exists NSQ $*_b$ open sets $(\mathfrak{U}_1, \partial)$ and $(\mathfrak{U}_2, \partial)$ such that $\xi_1(\gamma_1, \gamma_2, \gamma_3) \in (\mathfrak{U}_1, \partial), \tilde{\eta}, \partial \in (\mathfrak{U}_2, \partial)$ and $\xi_1(\gamma_1, \gamma_2, \gamma_3) \cap (\mathfrak{U}_1, \partial) = 0_{(\mathfrak{L}, \partial)}$, then $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is called a NSQ $*_b$ -regular space. $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is said to be NSQ $*_{b3}$ space, if is both a NSQ regular and NSQ $*_b$ space.

Theorem 3. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQTS. $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is NSQ $*_{b3}$ space iff for every $\xi_1(\gamma_1, \gamma_2, \gamma_3) \in \tilde{\eta}, \partial$, that is, $(\mathfrak{U}, \partial) \in ((\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial))$ such that $\xi_1(\gamma_1, \gamma_2, \gamma_3) \in (\mathfrak{U}, \partial) \subseteq \overline{(\mathfrak{U}, \partial)} \subseteq \tilde{\eta}, \partial$.

Proof. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is NSQ $*_{b3}$ space and $\xi_1(\gamma_1, \gamma_2, \gamma_3) \in (\eta, \partial) \in (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$. Since $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is NSQ $*_{b3}$ space for the NSQ point $\xi_1(\gamma_1, \gamma_2, \gamma_3)$ and NSQ $*_b$ closed set $\tilde{\eta}, \partial^c$, there exists $(\mathfrak{U}_1, \partial)$ and $(\mathfrak{U}_2, \partial)$ such that $\xi_1(\gamma_1, \gamma_2, \gamma_3) \in (\mathfrak{U}_1, \partial), (\eta, \partial)^c \in (\mathfrak{U}_2, \partial)$ and $(\mathfrak{U}_1, \partial) \cap (\mathfrak{U}_2, \partial) = 0_{(\mathfrak{L}, \partial)}$. Then, we have $\xi_1^s \in (\mathfrak{U}_1, \partial) \subseteq (\mathfrak{U}_2, \partial)^c \subseteq \tilde{\eta}, \partial$. Since $(\mathfrak{U}_2, \partial)^c$ NSQ $*_b$ closed set. $(\mathfrak{U}_1, \partial) \subseteq (\mathfrak{U}_2, \partial)^c$. Conversely, let $\xi_1^s \cap (h, \partial) = 0_{(\mathfrak{L}, \partial)}$ and (h, ∂) be a NSQ $*_b$ closed set. $\xi_1^s \in (h, \partial)^c$ and we have $\xi_1^s \in (\mathfrak{U}, \partial) \subseteq \overline{(\mathfrak{U}, \partial)} \subseteq (h, \partial)^c$. Thus, $\xi_1^s \in (\mathfrak{U}, \partial), (h, \partial) \subseteq \overline{(\mathfrak{U}, \partial)}$ and $(\mathfrak{U}, \partial) \cap (\mathfrak{U}, \partial)^c = 0_{(\mathfrak{L}, \partial)}$. So, $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is NSQ $*_{b3}$ space. \square

5. Characterization of More Results Concerning New Definition

In this section, more main results are addressed. Among neutrosophical soft quad separation axioms, Hausdorff space is considered to be the most important separation axioms. Important things should be given serious attention. So, this section has almost been engaged with the study of Hausdorff space. Through neutrosophical soft quad function, the characteristics of one space can be migrated to another space if the neutrosophical soft quad function is satisfying conditions of neutrosophical soft quad bijections and bicontinuousness. A soft quad function satisfying these conditions is known as neutrosophical soft quad homeomorphism. Neutrosophical soft quad homeomorphism is giving birth to neutrosophical soft quad topological property. Some neutrosophical soft quad topological properties

of Hausdorff space are addressed with respect to soft points. Sequentially compactness and countably compact in neutrosophical soft quad topological spaces with respect to soft points of the second space are addressed. Engagement of Hausdorff space with closed sets in neutrosophical soft quad topological spaces is addressed.

Theorem 4. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be NSQTS such that it is NSQ $*_b$ Hausdorff space and $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ be another NSQTS such that let $f, \theta: (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial) \rightarrow (\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ be NSQ homeomorphism. Then, $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ is also of the same behavior of NSQ $*_b$ Hausdorffness.

Proof. Let $\sqcup_1(\gamma_1, \gamma_2, \gamma_3), \sqcup_2(\gamma_1, \gamma_2, \gamma_3) \in \tilde{Y}$ such that $\sqcup_1(\gamma_1, \gamma_2, \gamma_3) \neq \sqcup_2(\gamma_1, \gamma_2, \gamma_3)$. Since $\langle f, \partial \rangle: (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial) \rightarrow (\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ be NSQ homeomorphism. So, there exists $\xi_1(\gamma_1, \gamma_2, \gamma_3), \xi_2(\gamma_1, \gamma_2, \gamma_3) \in \mathfrak{L}$ such that $\xi_1(\gamma_1, \gamma_2, \gamma_3) \neq \xi_2(\gamma_1, \gamma_2, \gamma_3)$. Since $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is NSQ $*_b$ Hausdorff space, so there exist (h, ∂) and (\mathfrak{U}, ∂) in NSQ $*_b$ Hausdorff space such that $\xi_1(\gamma_1, \gamma_2, \gamma_3) \in (h, \partial)$ and $\xi_2(\gamma_1, \gamma_2, \gamma_3) \in (\mathfrak{U}, \partial)$ such that $\xi_1(\gamma_1, \gamma_2, \gamma_3) \cap (\mathfrak{U}, \partial) = 0_{(\mathfrak{L}, \partial)}$, $\xi_2(\gamma_1, \gamma_2, \gamma_3) \cap (h, \partial) = 0_{(\mathfrak{L}, \partial)}$. Hence, $f(\xi_1(\gamma_1, \gamma_2, \gamma_3)) \in f((h, \partial)), f(\xi_2(\gamma_1, \gamma_2, \gamma_3)) \in f((\mathfrak{U}, \partial))$. Since $\langle f, \partial \rangle: (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial) \rightarrow (\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ is NSQ homeomorphism, so f, θ is NSQ $*_b$ open. Also, $f((h, \partial)) \cap f((\mathfrak{U}, \partial)) = f((h, \partial) \cap (\mathfrak{U}, \partial)) = f(0_{(\mathfrak{L}, \partial)}) = 0_{(\tilde{Y}, \partial)}$. This proves $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ is also of characteristics of NSQ $*_b$ Hausdorffness. \square

Theorem 5. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be NSQTS and $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ be another NSQTS which satisfies one more condition of NSQ $*_b$ Hausdorffness. Let $f, \theta: (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial) \rightarrow (\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ a NSQ function such that it is NSQ homeomorphism, then $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is also of characteristics of NSQ $*_b$ Hausdorffness.

Proof. Suppose $\xi_1^{s'}, \xi_2^{s'} \in \tilde{\mathfrak{L}}$ such that $\xi_1^{s'} \neq \xi_2^{s'}$. Since f, ∂ NSQ homeomorphism, so $f\xi_1^{s'} \neq f\xi_2^{s'}$. Let $\sqcup_1^s, \sqcup_2^s \in \tilde{Y}$ such that $\sqcup_1^s \neq \sqcup_2^s$.

Let $f(\xi_1^{s'}) = \sqcup_1^s, f(\xi_2^{s'}) = \sqcup_2^s \Rightarrow f \Rightarrow \xi_1(\gamma_1, \gamma_2, \gamma_3) = f^{-1}(\sqcup_1^s)$ and $\xi_2(\gamma_1, \gamma_2, \gamma_3) = f^{-1}(\sqcup_2^s)$

Let $\sqcup_1^s, \sqcup_2^s \in \tilde{Y}$ such that $\sqcup_1^s \neq \sqcup_2^s$. Since $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ is NSQ $*_b$ Hausdorff space. There definitely exists NSQ $*_b$ open sets $\langle h_1, \theta \rangle$ and $\langle h_2, \partial \rangle$ such that $\sqcup_1^s \in h_1, \partial$ and $\sqcup_2^s \in h_2, \partial$ with $h_1, \partial \neq h_2, \partial$.

$f^{-1}(\sqcup_1^s), f^{-1}(\sqcup_2^s)$ are NSQ $*_b$ open in $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ Now, $f^{-1}(\sqcup_1^s) \cap f^{-1}(\sqcup_2^s) = f^{-1}(\sqcup_1^s \cap \sqcup_2^s) \Rightarrow f^{-1}(0_{(\tilde{Y}, \partial)}) = 0_{(\mathfrak{L}, \partial)}$. So is NSQTS is $*_b$ Hausdorff space. \square

Theorem 6. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be NSQTS and $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ be another NSQTS. Let $f, \partial: (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \theta) \rightarrow (\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ be a NSQ mapping. Let $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ is NSQ $*_b$ Hausdorff space, then it is guaranteed that $\mathcal{M}, \theta = \{(\xi_1(\gamma_1, \gamma_2, \gamma_3), \xi_2(\gamma_1, \gamma_2, \gamma_3)): f$

$(\mathcal{E}_1(\gamma_1, \gamma_2, \gamma_3)) = f(\mathcal{E}_2(\gamma_1, \gamma_2, \gamma_3))$ is a NSQ^*_b closed subset of $\tilde{\mathcal{Q}} \times \tilde{\mathcal{Y}}$.

Proof. Given that $(\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \theta)$ be NSQTS and $(\tilde{\mathcal{Y}}, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \partial)$ be another NSQTS. Let $(\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \theta) \longrightarrow (\tilde{\mathcal{Y}}, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \partial)$ be a NSQ mapping such that it is NSQ continuous mapping $(\tilde{\mathcal{Y}}, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \partial)$ is NSQ^*_b Hausdorff space. Then, we will prove that \mathcal{M}, ∂ is NSQ close subset of $\tilde{\mathcal{Q}} \times \tilde{\mathcal{Q}}$. Equivalently, we will prove that \mathcal{M}, ∂^c is NSQ^*_b an open subset of $\tilde{\mathcal{Q}} \times \tilde{\mathcal{Q}}$. Let $(\mathcal{E}_1^s, \mathcal{E}_2^s) \in \mathcal{M}, \partial^c$. Then, $\mathcal{E}_1^s \neq \mathcal{E}_2^s \Rightarrow f(\mathcal{E}_1^s) > f(\mathcal{E}_2^s)$ or $< f(\mathcal{E}_1^s) < f(\mathcal{E}_2^s)$ accordingly. Since, $(\tilde{\mathcal{Y}}, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \partial)$ is NSQ^*_b Hausdorff space. Certainly, $f(\mathcal{E}_1^s \neq \mathcal{E}_2^s), f(\mathcal{E}_2^s)$ are NS points of $\tilde{\mathcal{Y}}$, so there exists NSQ^*_b open sets $\langle \mathcal{G}, \partial \rangle, \langle h, \partial \rangle \in \langle \tilde{\mathcal{Y}} \rangle$ such that $f(\mathcal{E}_1^s) \in \mathcal{G}, \partial, f(\mathcal{E}_2^s) \in h, \partial$ provided $\langle \mathcal{G}, \partial \rangle \cap \langle h, \partial \rangle = \emptyset_{(\mathcal{Q}, \partial)}$. Since, $\langle f, \partial \rangle$ is soft continuous, $f^{-1}(\langle \mathcal{G}, \partial \rangle) \times f^{-1}(\langle h, \partial \rangle)$ are NSQ^*_b open sets in $\tilde{\mathcal{Q}}$. So, $f^{-1}(\langle \mathcal{G}, \partial \rangle) \times f^{-1}(\langle h, \partial \rangle)$ is NSQ^*_b open in $\tilde{\mathcal{Q}} \times \tilde{\mathcal{Q}}$. Now, we show that $f^{-1}(\langle \mathcal{G}, \partial \rangle) \times f^{-1}(\langle h, \partial \rangle)$ For this, let $(\mathcal{E}_1(\gamma_1, \gamma_2, \gamma_3), \mathcal{E}_2(\gamma_1, \gamma_2, \gamma_3)) \in \Rightarrow f^{-1}(\mathcal{G}, \partial \times f^{-1}f, \partial) \longrightarrow \mathcal{E}_1(\gamma_1, \gamma_2, \gamma_3) \in f^{-1}(\mathcal{G}, \partial)$ and $\mathcal{E}_2(\gamma_1, \gamma_2, \gamma_3) \in f^{-1}(\langle h, \partial \rangle) \Rightarrow f(\mathcal{E}_1^s) = \mathcal{G}, \partial$ and $(\mathcal{E}_2^s) = \langle h, \partial \rangle$. Since $\mathcal{G}, \partial \cap h, \partial = \emptyset_{(\pi, \partial)} \Rightarrow \mathcal{E}_1^s \neq \mathcal{E}_2^s \Rightarrow \mathcal{E}_1^s \in \mathcal{M}, \partial^c \Rightarrow f^{-1}(\langle \mathcal{G}, \partial \rangle) \times f^{-1}(h, \partial) \subseteq \mathcal{M}, \partial^c$. Thus, $f^{-1}(\langle \mathcal{G}, \partial \rangle) \times f^{-1}(\langle h, \partial \rangle)$ is NSQ^*_b open in $\tilde{\mathcal{Q}} \times \tilde{\mathcal{Q}}$. So, this implies that $(\mathcal{E}_1^s, \mathcal{E}_2^s)$ is NSQ interior point of \mathcal{M}, ∂^c . So, every point of \mathcal{M}, ∂^c is NSQ interior point of \mathcal{M}, ∂^c . So \mathcal{M}, ∂^c is NSQ^*_b open in $\tilde{\mathcal{Q}} \times \tilde{\mathcal{Q}}$. \square

Theorem 7. Let $(\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be NSQTS and $(\tilde{\mathcal{Y}}, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \partial)$ be another NSQTS. Let $\langle h, \partial \rangle: (\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial) \longrightarrow (\tilde{\mathcal{Y}}, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \partial)$ be an NSQ^*_b open mapping such that it is onto. If the soft set $\mathcal{M}, \partial = \{(\mathcal{E}_1^s, \mathcal{E}_2^s): f(\mathcal{E}_1^s) = f(\mathcal{E}_2^s)\}$ is NSQ^*_b closed in $\tilde{\mathcal{Q}} \times \tilde{\mathcal{Q}}$, then $\tilde{\mathcal{Y}}$ is NSQ^*_b Hausdorff space.

Proof. Suppose $f(\mathcal{E}_1^s), f(\mathcal{E}_2^s)$ be two NSQ points of $\tilde{\mathcal{Y}}$ such $f(\mathcal{E}_1^s) \neq f(\mathcal{E}_2^s) \longrightarrow f(\mathcal{E}_1^s) > f(\mathcal{E}_2^s)$ or $< \mathcal{E}_1^s < f(\mathcal{E}_2^s)$. Then, $(\mathcal{E}_1(\gamma_1, \gamma_2, \gamma_3), \mathcal{E}_2(\gamma_1, \gamma_2, \gamma_3)) \notin \mathcal{M}, \partial$, that is, $(\mathcal{E}_1(\gamma_1, \gamma_2, \gamma_3), \mathcal{E}_2(\gamma_1, \gamma_2, \gamma_3)) \in \mathcal{M}, \partial^c$. Since \mathcal{M}, ∂^c NSQ^*_b open in $\tilde{\mathcal{Q}} \times \tilde{\mathcal{Q}}$, there exist NSQ^*_b open sets \mathcal{G}, ∂ and $\langle h, \partial \rangle$ in $\tilde{\mathcal{Y}}$ such that $(\mathcal{E}_1^s, \mathcal{E}_2^s) \in \langle \mathcal{G}, \partial \rangle \times \langle h, \partial \rangle \in \mathcal{M}, \partial^c$. Then, since NSQ^*_b is open, $\langle \mathcal{G}, \partial \rangle, f(h, \partial)$ are NSQ^*_b open in $\tilde{\mathcal{Y}}$ containing $f(\mathcal{E}_1^s), f(\mathcal{E}_2^s)$, and $\langle \mathcal{G}, \partial \rangle$ as $\langle h, \partial \rangle$ disjoint. It follows that $\tilde{\mathcal{Y}}$ is NSQ^*_b Hausdorff space. \square

Theorem 8. Let $(\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ be a NSQ second countable space and let $\langle h, \partial \rangle$ be NSQ uncountable subset of $(\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$, then at least one point of $\langle h, \partial \rangle$ is a NSQ limit point of $\langle h, \partial \rangle$.

Proof. Let $\mathfrak{B} = \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \dots, \mathcal{B}_n: n \in \mathbb{N}$ for $(\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$.

Let, if possible, no point of $\langle f, \partial \rangle$ be a NSQ limit point of $\langle f, \partial \rangle$. Then, for each $\mathcal{E}_1^s \in \langle f, \partial \rangle$, there exists NSQ^*_b open set $\rho, \partial_{(\mathcal{E}_1^s)}$ such that $\mathcal{E}_1^s \in \rho, \partial_{(\mathcal{E}_1^s)}$ and $\rho, \partial_{(\mathcal{E}_1^s)} \cap f, \partial = \{\mathcal{E}_1^s\}$. Since \mathfrak{B} is soft base, there exists $\mathcal{B}_n \in \mathfrak{B}$ such that $(\mathcal{E}_1^s) \in \mathcal{B}_n \in \rho, \partial_{(\mathcal{E}_1^s)}$. Therefore, $\mathcal{B}_n \cap f, \partial \in \rho, \partial_{(\mathcal{E}_1^s)} \cap f, \partial = \{\mathcal{E}_1^s\}$. Moreover, if \mathcal{E}_1^s and \mathcal{E}_2^s be any two NSQ points such that $\mathcal{E}_1^s \neq \mathcal{E}_2^s$ which means either $\mathcal{E}_1^s > \mathcal{E}_2^s$ or $\mathcal{E}_1^s < \mathcal{E}_2^s$, then there exists \mathcal{B}_n and \mathcal{B}_m in \mathfrak{B} such that $\mathcal{B}_n \cap f, \partial = \{\mathcal{E}_1^s\}$ and $\mathcal{B}_m \cap f, \partial = \{\mathcal{E}_2^s\}$. Now, $\mathcal{E}_1^s \neq \mathcal{E}_2^s$ which guarantees that $\{\mathcal{E}_1^s\} \neq \{\mathcal{E}_2^s\}$ which implies that $\mathcal{B}_n \cap f, \partial \neq \mathcal{B}_m \cap f, \partial$ which implies $\mathcal{B}_n \neq \mathcal{B}_m$. Thus, there exists a one to one NSQ correspondence of $\langle f, \partial \rangle$ on to $\{\mathcal{B}_n: (\mathcal{E}_1^s) \in f, \partial\}$. Now, $\langle f, \partial \rangle$ being NSQ uncountable, it follows that $\{\mathcal{B}_n: \mathcal{E}_1^s \in \langle f, \partial \rangle\}$ is NSQ uncountable. This is a contradiction. \square

Theorem 9. Let $(\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ and $(\tilde{\mathcal{Y}}, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \partial)$ be two NSQTS and suppose \mathfrak{f}, ∂ be a NSQ continuous function such that $\mathfrak{f}, \partial: (\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial) \longrightarrow (\tilde{\mathcal{Y}}, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \partial)$ is NSQ continuous function and let $\mathcal{L}, \partial \in (\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ supposes the B.V.P. Then, safely, $\mathfrak{f}(\mathcal{L}, \partial)$ has the B.V.P.

Proof. Suppose \mathcal{L}, ∂ be an infinite NSQ subset of \mathfrak{f}, ∂ , so that \mathcal{L}, ∂ contains an enumerable NS set $(\mathcal{E}_1^s)_n: n \in \mathbb{N}$, then there exists enumerable NSQ set $(\mathcal{E}_2^s)_n: n \in \mathbb{N} \in \mathcal{L}, \partial$ such that $\mathfrak{f}((\mathcal{E}_2^s)_n) = (\mathcal{E}_1^s)_n$. \mathcal{L}, ∂ has B.V.P implying that every infinite soft subset \mathcal{L}, ∂ supposes NSQ accumulation point belonging to \mathcal{L}, ∂ , and this implies that $(\mathcal{E}_2^s)_n: n \in \mathbb{N}$ has NSQ limit point, say, $(\mathcal{E}_2^s)_0 \in \mathcal{L}, \partial$ implies that the limit of NSQ sequence $(\mathcal{E}_2^s)_n: n \in \mathbb{N}$ is $(\mathcal{E}_2^s)_0 \in \mathcal{L}, \partial \longrightarrow (\mathcal{E}_2^s)_n \longrightarrow (\mathcal{E}_2^s)_0 \in \mathcal{L}, \partial$. \mathfrak{f} is NSQ continuous. Furthermore, $(\mathcal{E}_2^s)_n (\mathcal{E}_2^s)_0 \in \mathcal{L}, \partial \longrightarrow \mathfrak{f}((\mathcal{E}_2(\gamma_1, \gamma_2, \gamma_3))_n) \mathfrak{f}((\mathcal{E}_2^s)_0) \in \mathfrak{f}(\mathcal{L}, \partial) \longrightarrow (\mathcal{E}_1^s)_n \mathfrak{f}(\mathcal{E}_2^s)_0 \in \mathfrak{f}(\mathcal{L}, \partial)$ implies that NSQ limit of a NSQ sequence $(\mathcal{E}_1^s)_n: n \in \mathbb{N}$ is $\mathfrak{f}(\mathcal{E}_2^s)_0 \in \mathfrak{f}(\mathcal{L}, \partial)$ implying that NSQ limit of a NSQ sequence $(\mathcal{E}_1^s)_n: n \in \mathbb{N}$ is $\mathfrak{f}(\mathcal{E}_2^s)_0 \in \mathfrak{f}, \partial(\mathcal{L}, \partial)$. Finally, we have shown that there exists infinite soft subset $(\mathcal{E}_1^s)_n: n \in \mathbb{N}$ of $\mathfrak{f}(\mathcal{L}, \partial)$ containing a limit point $\mathfrak{f}((\mathcal{E}_2^s)_0) \in \mathfrak{f}(\mathcal{L}, \partial)$; this guarantees that $\mathfrak{f}(\mathcal{L}, \partial)$ has B.V.P. \square

Theorem 10. Let $(\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ NSQTS and let $\widetilde{\mathcal{E}_1^s}_n$ be a NSQ sequence in $(\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ such that it converges to NSQ a point $(\mathcal{E}_1^s)_0$, then the NSQ set $\langle f, \partial \rangle$ consisting of NSQ points $(\mathcal{E}_1^s)_{n_0}$ and $(\mathcal{E}_1^s)_n (n = 1, 2, 3, \dots)$ is soft NSQ compact.

Proof. Given $(\mathcal{Q}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ NSQTS and let $(\widetilde{\mathcal{E}_1^s})_n$ be an NSQ sequence in $(\tilde{\mathcal{Q}}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ such that it converges to a point $(\mathcal{E}_1^s)_{n_0}$, that is $(\mathcal{E}_1^s)_n \longrightarrow (\mathcal{E}_1^s)_{n_0} \in \tilde{\mathcal{Q}}$. Let $\langle g, \partial \rangle = \left[\begin{array}{c} (\mathcal{E}_1^s)_1, (\mathcal{E}_1^s)_2, (\mathcal{E}_1^s)_3, \\ (\mathcal{E}_1^s)_4, (\mathcal{E}_1^s)_5, \\ (\mathcal{E}_1^s)_6, (\mathcal{E}_1^s)_7, \dots \end{array} \right]$. Let $\{\mathcal{G}, \partial_\alpha: \alpha \in \Delta\}$ be NSQ^*_b open covering of $\langle g, \partial \rangle$ so that $\langle g, \partial \rangle \in \bigcup \{\mathcal{G}, \partial_\alpha:$

$\alpha \in \Delta$, $(\xi_1^s)_{n_0} \in \langle g, \partial \rangle$ implies that there exists $\alpha_0 \in \Delta$ such that $(\xi_1^s)_{n_0} \in \mathfrak{S}, \partial_{\alpha_0}$. By NSQ convergence, $(\xi_1^s)_{n_0} \in \mathfrak{S}, \partial_{\alpha_0} \in (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ implies that there exists $n_0 \in \mathbb{N}$ s.t. $n \geq n_0$ and $(\xi_1^s)_n \in \mathfrak{S}, \partial_{\alpha_0}$. Evidently, $\mathfrak{S}, \partial_{\alpha_0}$ contains the NSQ points $(\xi_1^s)_{n_0}, (\xi_1^s)_{n_0+1}, (\xi_1^s)_{n_0+2}, (\xi_1^s)_{n_0+3}, (\xi_1^s), \dots, (\xi_1^s)_{n_0+n}, \dots$. Look carefully at the points and train them in a way as $(\xi_1^s)_1, (\xi_1^s)_2, (\xi_1^s)_3, (\xi_1^s)_4, \dots, (\xi_1^s)_n$ generating a finite soft set. Let $1 \leq n_{0-1}$. Then, $(\xi_1^{(y_1, y_2, y_3)})_i \in \langle g, \partial \rangle$. For this $i, (\xi_1^s)_i \in \langle g, \partial \rangle$. Hence, there exists $\alpha_i \in \Delta$ such that $(\xi_1^s)_i \in \mathfrak{S}, \partial_{\alpha_i}$. Evidently, $\langle g, \partial \rangle \in \bigcup_{i=0}^{n_{0-1}} \mathfrak{S}, \partial_{\alpha_i}$. This shows that $\{\mathfrak{S}, \partial_{\alpha_i} : 0 \leq n_{0-1}\}$ is NSQ*_b open cover of $\langle g, \partial \rangle$. Thus, an arbitrary NSQ*_b open cover $\{\mathfrak{S}, \partial_{\alpha} : \alpha \in \Delta\}$ of $\langle g, \partial \rangle$ is reducible to a finite NSQ subcover $\{\mathfrak{S}, \partial_{\alpha_i} : i = 0, 1, 2, 3, \dots, n_{0-1}\}$, and it follows that $\langle g, \partial \rangle$ is NSQ*_b compact. \square

Theorem 11. If $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ NSQTS such that it has the characteristics of NSQ*_b sequentially compactness. Then, $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is NSQ*_b countably compact.

Proof. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ NSQTS and let ρ, ∂ be finite

soft subset of $(L, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$. Let $\left[\begin{array}{c} (\xi_1^s)_1, (\xi_1^s)_2, (\xi_1^s)_3, \\ (\xi_1^s)_4, (\xi_1^s)_5, \\ (\xi_1^s)_6, (\xi_1^s)_7, \dots \end{array} \right]$

be an NSQ sequence of NSQ points of ρ, ∂ . Then, ρ, ∂ being finite, at least one of the elements in ρ, ∂ say $(\xi_1^s)_0$ must be duplicated an infinite number of times in the NSQ sequence.

Hence, $\left[\begin{array}{c} (\xi_1^s)_0, (\xi_1^s)_0, \\ (\xi_1^s)_0, (\xi_1^s)_0, (\xi_1^s)_0, \\ (\xi_1^s)_0, (\xi_1^s)_0, \dots \end{array} \right]$ is soft NSQ subsequence of

$(\xi_1^s)_n$ such that it is NSQ constant sequence and repeatedly constructed by single soft number $(\xi_1^s)_0$, and we know that a soft constant sequence converges on its self. So, it converges to $(\xi_1^s)_0$ which belongs to ρ, ∂ . Hence, ρ, ∂ is soft sequentially NSQ*_b compact. \square

Theorem 12. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ NSQTS and $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ be another NSQTS. Let $\langle f, \partial \rangle$ be a soft continuous mapping of an NSQ sequentially compact NSQ*_b space $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ into $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$, then $\langle f, \partial \rangle (\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ is NSQ*_b sequentially compact.

Proof. Given $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ NSQTS and $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ be another NSQTS. Let $\langle f, \partial \rangle$ be a NSQ continuous mapping of a NSB sequentially compact space $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ into $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$, then we have to prove $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ NSQ sequentially. For this, we

proceed as follows. Let $\left[\begin{array}{c} (\xi_2^s)_1, (\xi_2^s)_2, \\ (\xi_2^s)_3, (\xi_2^s)_4, \dots \\ (\xi_2^s)_7, \dots, (\xi_2^s)_n, \dots \end{array} \right]$ be a NSQ

sequence of NSQ points. Then, for each $n \in \mathbb{N}$, there exists $(\xi_1^s)_1, (\xi_1^s)_2, (\xi_1^s)_4, (\xi_1^s)_5, \dots, (\xi_1^s)_7, \dots, (\xi_1^s)_n, \dots \in (L,$

$\tau_1, \tau_2, \tau_3, \tau_4, \partial)$. Thus, we obtain an NSQ sequence

$\left[\begin{array}{c} (\xi_1^s)_1, (\xi_1^s)_2, \\ (\xi_1^s)_3, (\xi_1^s)_4, \dots \\ (\xi_1^s)_5, (\xi_1^s)_6, \\ (\xi_1^s)_7, \dots, (\xi_1^s)_n, \dots \end{array} \right]$ in $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$. But $(\mathfrak{L}, \tau_1, \tau_2,$

$\tau_3, \tau_4, \partial)$ being soft sequentially NSQ*_b compact, there is an NSQ subsequence $(\xi_1^s)_{n_i}$ of

$(\xi_1^s)_n$ such that $(\xi_1^s)_{n_i} \rightarrow (\xi_1^s) \in (\tilde{\mathfrak{L}}, \tau_1, \tau_2, \tau_3, \partial)$. So, by NSQ*_b continuity of $f, (\xi_1^s)_{n_i} \rightarrow (\xi_1^s)$ implies that $f((\xi_1^s)_{n_i}) \rightarrow f((\xi_1^s)) \in f(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$. Thus,

$f((\xi_2^{(y_1, y_2, y_3)})_{n_i})$ is a soft subsequence of

$\left[\begin{array}{c} (\xi_2^s)_1, (\xi_2^s)_2, \\ (\xi_2^s)_3, (\xi_2^s)_5, \\ (\xi_2^s)_5, (\xi_2^s)_6, \\ (\xi_2^s)_7, \dots, (\xi_2^s)_n, \dots \end{array} \right]$ converges to $f(\kappa_1)$ in $(\tilde{\mathfrak{L}}, \tau_1,$

$\tau_2, \tau_3, \tau_4, \partial)$. Hence, $((\tilde{\mathfrak{L}}, \tau_1, \tau_2, \tau_3, \tau_4, \partial))$ NSQ*_b sequentially compact. \square

Theorem 13. Let $(\tilde{\mathfrak{L}}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ NSQTS and suppose $\langle f, \partial \rangle, \langle g, \partial \rangle$ be two NSQ continuous function on an NSQTS $(\tilde{\mathfrak{L}}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ in to an NSQTS $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ which is NSQ*_b Hausdorff. Then, soft set $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)((\xi_1^s))\}$ is NSQ*_b closed of $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$.

Proof. Let if $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)((\xi_1^s))\}$ is an NSQ set of function. If $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)((\xi_1^s))\} = 0_{(\mathfrak{L}, \partial)}$, it is clearly NSQ*_b open, and therefore, $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)((\xi_1^s))\}$ is NSQ*_b closed, that is, nothing is proved in this case. Let us consider the case when $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)((\xi_1^s))\}$ and let $\rho \in \{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)((\xi_1^s))\}$. Then, ρ does not belong $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)(k_1)\}$. Result in $(f)(\rho) \neq (g)(\rho)$. Now, $(\tilde{Y}, \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2, \tilde{\mathfrak{F}}_3, \tilde{\mathfrak{F}}_4, \partial)$ being NSQ*_b Hausdorff space, so there exists NSQ*_b open sets $(f, \partial), (g, \partial)$ of $(f)(\rho)$ and $(g)(\rho)$, respectively, such that (f, ∂) and (g, ∂) such that these NSQ sets such that the possibility of one rules out the possibility of other. By soft continuity of $(f, \partial), (g, \partial), (f, \partial)^{-1}$ as well as is NSQ*_b open nhd of ρ , and therefore, $(f, \partial)^{-1} \cap (g, \partial)^{-1}$ is contained in $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)((\xi_1^s))\}$ for, $\{(\xi_1^s) \in ((f, \partial)^{-1} \cap (g, \partial)^{-1}) (g)((\xi_1^{(y_1, y_2, y_3)})) \in (g, \partial)\}$ and $(g)(f)((\xi_1^s)) \neq (g)((\xi_1^s))$ because (f, ∂) and (g, ∂) are mutually exclusive. This implies that κ_1 does not belong to $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (h)((\xi_1^s))\}$. Therefore, this shows that $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)((\xi_1^s))\}$ is nhd of each of its points. So, $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)((\xi_1^s))\}$ NSQ*_b open, and hence $\{(\xi_1^s) \in \tilde{\mathfrak{L}} : (f)((\xi_1^s)) = (g)((\xi_1^s))\}$ is NSQ*_b closed. \square

Theorem 14. Let $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ NSQTS such that it is $NSQ*_b$ Hausdorff space and let (f) be soft continuous function of $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$ into itself. Then, the NSQ set of fixed points under (g) is a $NSQ*_b$ closed set.

Proof. Let $\mathfrak{S} = \{(g)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\}$. If $\mathfrak{S}^c = 0_{(\mathfrak{L}, \partial)}$, Then, $NSQ*_b$ is open, and therefore, $\mathfrak{S} = \{(g)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\}$ closed. So, let $\mathfrak{S} = \{(g)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\}$ and let $\mathfrak{E}_2^{\mathfrak{g}'} \in \{(g)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\}^c$. Then, $\mathfrak{E}_2^{\mathfrak{g}'}$ does not belong to $\{(g)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\}$, and therefore, $(g)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})$. Now, $\mathfrak{E}_2^{\mathfrak{g}'}$, $(g)(\mathfrak{E}_2^{\mathfrak{g}'})$ being two distinct points of the $NSQ*_b$ Hausdorff space $(\mathfrak{L}, \tau_1, \tau_2, \tau_3, \tau_4, \partial)$, so there exist $NSQ*_b$ open sets (g, ∂) , (\mathfrak{H}, ∂) such that $\mathfrak{E}_2^{\mathfrak{g}'} \in (g, \partial)$, $(h)(\mathfrak{E}_2^{\mathfrak{g}'}) \in (\mathfrak{H}, \partial)$ and (g, ∂) , (\mathfrak{H}, ∂) are disjoint. Also, by the NSQ continuity, $NSQ*_b$ is an open set containing $\mathfrak{E}_2^{\mathfrak{g}'}$, we pretend that $(g, \partial) \cap (f)^{-1}(\mathfrak{H}, \partial) \in \{(g)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\}^c$. Since $\mu \in (g, \partial) \cap (f)^{-1}(\mathfrak{H}, \partial) \Rightarrow \mu \in (g, \partial)$, $\mu \in (h)^{-1} \Rightarrow \mu \in (g, \partial)$, $(h)(\mu) \in (\mathfrak{H}, \partial) \Rightarrow \mu \neq (g)(\mu)$. As $(g, \partial) \cap (\mathfrak{H}, \partial) = 0_{(\mathfrak{L}, \partial)}$ implies that μ does not belong to $\{(f)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\} \Rightarrow \mu \in \{(g)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\}^c$. Therefore, $\mathfrak{E}_1^{\mathfrak{g}} \in (g, \partial) \cap (g)^{-1}(\mathfrak{H}, 0_{(\mathfrak{L}, \partial)}) \in \{(f)(\kappa_{1 \vee_1, \vee_2, \vee_3}) = (\kappa_{1 \vee_1, \vee_2, \vee_3})\}^c$. Thus, $\{(f)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\}^c$ is the NSQ neighborhood of each of its points. So, $\{(g)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\}^c$ is $NSQ*_b$ open, and hence, $\{(g)((\mathfrak{E}_1^{\mathfrak{g}})) = (\mathfrak{E}_1^{\mathfrak{g}})\}$ is $NSQ*_b$ closed. \square

6. Conclusion

Neutrosophic soft topology (NST) is extension of vague soft topology (VST) and VST is extension of fuzzy soft topology (FST). VST gives two type of informations. One is true, and second one is false. It does not give informations about the indeterminacy (doubtful) case. NST is dominant over VST because it supposes all the three informations that is true, false, and indeterminacy at the same time. NST has narrow domain as compared to neutrosophic soft bitopology (NSBT). Some problems are very hard to discuss in NST, so need of NSBT is felt. Still NSTS is unable to afford problems of large number of domain. So, extension is needed to neutrosophic soft tritopology (NSTT) and neutrosophic soft quad-topology (NSQT). In our work, we regenerated some structures in NSQTS with new definition that is $*_b$ open sets relative to soft points. We worked with the operations given in references [14–16] which are entirely different from references [13, 17]. In future, we will develop neutrosophic soft penta-topology and neutrosophic soft hexa-topology relative to soft points of the space under more generalized neutrosophic soft open sets. We will try to develop neutrosophic soft separation axioms and their engagement with each other. In addition to this, neutrosophic soft other separation axioms will also be given special attention. After doing this, on the basis of reference [12], we will try to build the same structures with respect to hyper-soft sets and then with respect to plithogenic hyper-soft sets.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

Authors' Contributions

All authors read and approved the final manuscript.

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Retraction

Retracted: Arab and Malay Students' Attitudes toward Statistics and Their Learning Styles: A Rasch Measurement Approach

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

In addition, our investigation has also shown that one or more of the following human-subject reporting requirements has not been met in this article: ethical approval by an Institutional Review Board (IRB) committee or equivalent, patient/participant consent to participate, and/or agreement to publish patient/participant details (where relevant).

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] A. Fayomi, Z. Mahmud, A. Algarni, and A. M. Almarashi, "Arab and Malay Students' Attitudes toward Statistics and Their Learning Styles: A Rasch Measurement Approach," *Mathematical Problems in Engineering*, vol. 2022, Article ID 4144254, 10 pages, 2022.

Research Article

Arab and Malay Students' Attitudes toward Statistics and Their Learning Styles: A Rasch Measurement Approach

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Students' learning of statistics has been studied from a variety of angles, and this study is no different. The main purpose is to compare the Malay and Arab students' attitudes toward learning statistics and their learning styles in understanding statistics. A survey questionnaire and face-to-face interview techniques were used to elicit information from 150 students based on the cohort. They were asked about how they learn to solve statistical problems based on Kolb's four learning cycles: feeling, reflective observation, thinking, and doing. Attitude responses were numerically recorded based on a five-point Likert scale, while preference for learning styles was recorded as 1 (Do and Feel) or 0 (Watch and Think). Both attitude and learning style data were combined and subjected to Rasch analysis. Results show that a majority of the Arab and Malay students have moderate to high positive attitude toward learning statistics. Generally, students from both cultures are classified as the "Accommodating" type with a preference for doing and feeling from the experience of doing statistical problems. Arab students are classified as the "Assimilating" type with a preference for thinking, reflecting, and learning from observation, while Malay students are classified as the "Converging" type with a preference for thinking and doing statistical exercises.

1. Introduction

Statistics researchers and educators have performed several studies on students' attitudes toward statistics and their influence on their statistical learning [1–3]. Despite the focus on the relevance of statistics, students continue to struggle with understanding them. The study in [4] said that the targeted outcomes of basic statistics courses should include students' competence and conviction in their capacity to master statistical skills and use them in a real-world situation.

But while knowing the importance of being able to statistically think, many students in the social sciences, arts, and management sciences cannot exhibit this ability. Some people are hesitant to learn statistics because they are afraid of numbers and formulae, even if they are aware of the relevance and necessity of its usage later [5]. Student awareness of their statistical thinking handicap [6] or the

degree of their statistical literacy is key to ensuring that they are aware of their statistical thinking level and understanding. This knowledge would allow them to develop the necessary statistical abilities, tools, and information.

In higher education, the potential instructional information regarding students' learning styles has been highlighted. The study in [7] found that "there are obvious connections between how an individual conceptualizes learning, the methods through which the individual strives to learn, and the consequences of the individual's attempts to learn" [8]. Learning style choices have an influence on how students respond to educational programs or curricula in terms of comprehending their goals and objectives. According to [9], students may be better equipped to adapt to diverse settings if they are aware of their learning styles. The study in [10] separated learning styles into three interconnected components: information processing, instructional preferences, and learning methods. When it

comes to learning approaches, the work in [11] argues that pupils who are familiar with a range of tactics are more likely to pick the right one.

Your cultural background might also influence your learning style. An individual's cultural background may also be influenced based on learning styles. According to [12], learning methods related to diverse cultural backgrounds have a role in maximizing academic success; high achievers report using more strategies than lesser achievers [13], but these strategies may vary among students [14]. The study in [15] found that persons in nations with high levels of gender equality, in-group collectivism, and institutional collectivism are likely to have a more abstract learning style. This is confirmed further by [16–18], who demonstrated that learners' beliefs about their chances of success are influenced by the learning techniques they employ.

Even considerable research has been conducted to explore cultural variations across nations [19], and cultural differences in thinking and learning statistics have received less attention. This study will look into how students from the Arab and Malay cultures understand statistical concepts. Kolb's learning type model, which is based on four learning cycles, namely, feeling-concrete experience (CE), watching-reflective observation (RO), thinking-abstract conceptualization (AC), and doing-active experimentation (AE), will be used to assist learning style identification (AE). This study investigates students of Arab and Malay cultures' attitudes about studying statistics and learning methods.

The main purpose is to compare the Malay and Arab students' attitudes toward learning statistics and their learning styles in understanding statistics.

The following are the objectives of the study:

- (1) To describe whether students of Arab and Malay cultures differ in their perceived attitude toward learning statistics
- (2) To describe whether students of Arab and Malay cultures show similar or different learning styles in learning statistics
- (3) To compare certain attitudes of Arab and Malay students based on item characteristic curves

2. Attitude toward Statistics

Egyptian instructors enrolled in an introductory statistics course utilized the Survey of Attitudes Toward Statistics (SATS) to examine the relationship between attitudes toward statistics, anxiety, mathematical talent, and statistics achievement. The instructors' statistical success was evaluated using ten open-ended questions that included descriptive and inferential statistics. According to the findings of the study, instructors in Egypt have a minor beneficial influence on attitudes toward statistics on the statistical success [20]. Another study [21] concentrated on four major components to define instructors' views about statistics: affect, cognitive competency, value, and challenge. Based on the SATS, the study looked at 367 preservice teachers at the Faculty of Education, University of Lleida, Spain, and their attitudes about statistics, anxiety, mathematical ability, and

statistics success. The SATS questionnaire, which consisted of 28 items on a five-point Likert scale (from 1 "Strongly Disagree" to 5 "Strongly Agree"), was utilized in the study. According to the findings of the survey, participants regard statistics as little challenging, with small respect for the utility and significance of the statistics. Their personal and professional lives are also heavily influenced by statistics. They also have a positive attitude toward statistics education.

There were significant differences between current and future elementary school teachers' perceptions of statistics [22]. Instructors were evaluated based on their gender, a number of prior statistics courses, specialization (the topic that the teacher teaches), and teaching experience using the EAEE scale instrument. EAEE is a mixture of the Statistics Attitude Survey (SAS) [23], the Attitudes Toward Statistics (ATS) [24], and the Spanish scale [25], with 25 items that include both positive and negative items to prevent acquiescence bias. The EAEE instrument includes a five-point Likert scale ranging from 1-"Strongly Disagree," through 3-"Neutral," to 5-"Strongly Agree," on which respondents must indicate their degree of agreement or disagreement with the items.

The study indicated that older instructors exclude statistical subjects from their instruction as compared to younger teachers because they find the subject difficult to teach. Furthermore, instructors who have a negative attitude toward statistics do not use statistics in their professional activities.

In this study, the SATS scales were modified in the survey instrument to assess students of Arab and Malay cultures' attitudes toward statistics learning. Many studies on how students learn statistics have been conducted in a variety of domains and cognitive elements of learning [26, 27]. Students start learning processes with varying goals and preferences, which allow them to attain satisfying learning results via several learning routes [28].

Understanding statistical principles is the foundation for understanding statistics. Numerous studies on statistical reasoning (e.g., on variation and sample distributions) have shed light on how students learn to use statistical reasoning [28–31]. Statistical literacy, statistical reasoning, and statistical thinking are increasingly recognized as three separate but related cognitive processes in statistics education research today. It is possible to have a certain level of literacy, logic, and reasoning even before formal statistics schooling [32].

Reference [33] used statistical reasoning assessment (SRA) to evaluate students' reasoning ability on probability subjects. Statistics concept inventory (SCI), on the other hand, was created to test engineering students' statistics comprehension [34]. In their Assessment Resource Tools for Improving Statistical Thinking (ARTIST) initiative, [35] designed an online test called Comprehensive Assessment of Outcomes in Statistics (CAOS). In basic statistics classes, CAOS aims to test students' understanding of statistical concepts.

This study, on the other hand, took a different approach by incorporating Kolb's learning model into statistics to identify students' learning styles based on four learning

cycles, in which Kolb relates abstract conceptualization (thinking) to concrete experience (feeling) and reflective observation (watching) to active experimentation (doing). Figure 1 depicts one such case. Individuals' learning styles may be determined using the two-by-two matrix view. For example, a person with a dominant learning style of "doing" rather than "watching" the task and "feeling" rather than "thinking" about the experience will have a learning style that combines and represents those processes, namely, an "Accommodating" learning style.

3. Methods

A survey was conducted on a total of 150 undergraduate students who enrolled in a statistics subject at a public university in Selangor, Malaysia, and a public university in Jeddah, Saudi Arabia. The students were from various backgrounds of studies and were taught by statistics lecturers in the respective countries. The investigation only focused on a selected cohort and measured the students' attitude and their styles in learning statistics at their respective universities. They were interviewed after the tenth week of learning the subject.

4. Instruments

In this investigation, two types of equipment were used. The first tool was a questionnaire on people's attitudes about statistics learning. It comprised of 28 items that assessed students' perceived attitudes toward learning probability concepts on a five-point Likert scale ranging from 1 ("Strongly Disagree") to 5 ("Strongly Agree"). The attitude questions about statistics were modified from the Student's Attitude Towards Learning Statistics questionnaire [36] and the Survey of Attitudes Toward Statistics (SATS) scale [37]. Based on the students' learning environment, the original conceptions of SATS and the Student's Attitude Towards Learning Statistics questionnaire were updated. Figure 2 depicts the tool used to assess students' attitudes about learning statistics.

As for the second instrument, it was based on Kolb's learning style model, which was used to evaluate students' learning styles in statistics classes (Figure 3).

Figure 3 shows Kolb's learning model chart that was used in this study, based on the student's responses to Kolb's learning chart and the following questions:

- (i) How do you begin a task?
- (ii) How have you emotionally reacted to the experience?

The replies of the students were tallied and classified based on four potential combinations, namely, watch-think, watch-feel, do-think, and do-feel [38]. As stated in Table 1, the combination would be cross-tabulated depending on each student's response.

A majority of the students (47%) prefer to think through while learning a new topic in the class and then learn by doing the exercise or solving the problems later. About 30% initially prefer to watch and think through the lessons

instead of doing it. Based on Kolb's learning model, those students were categorized as having "converging" and "assimilating" characteristics, respectively.

5. Rasch Measurement Model

All data gathered from the instruments were subjected to Rasch measurement using Winsteps 3.90.2. Rasch rating scale and Rasch dichotomous models were used in the analysis of data gathered from the questionnaire. A person's logit score may, therefore, be used to assess a person's ability, and an item's logit score can be used to evaluate the difficulty of an item. Because a person's competence is defined by the proportion of correct answers and an item's difficulty is determined by the proportion of incorrect answers, both estimations are connected and may be mathematically expressed, i.e., the Rasch dichotomous measurement model, as follows:

$$x_{ni} = \left(\frac{1}{B_n}, D_i \right) = \frac{e^{B_n - D_i}}{1 + e^{B_n - D_i}}. \quad (1)$$

A Rasch model analysis determines how well the data match the Rasch model. It considers two parameters: the complexity of the test item and the person's aptitude. By examining the gap between the two parameters, it is believed that these parameters are interdependent. A probabilistic technique is used to achieve the separation, in which a person's raw test result is transformed into a success-to-failure ratio and subsequently into logarithmic chances that the individual would properly answer the items. A logit scale is used to depict this, and it may be displayed on a single scale ruler.

The model expresses the likelihood of getting the right response (1 rather than 0) as a function of the amount of the difference between the person's ability (B_n) and the item's difficulty (D_i) (i). The Rasch model is used to compute a person's skills and item difficulties and then plot the person's abilities and item difficulties on the same scale. The model states that the likelihood of a person succeeding on a particular item is an exponential function of the gap between that person's aptitude and the difficulty of the item [39].

The Rasch model gives two sorts of indicators to assist researchers in determining whether there is sufficient item spread and sufficient spreadability among individuals. The person dependability index reflects the likelihood of replicability of the person ordering if the sample individuals were given a different set of items measuring the same construct [40]. It also necessitates a wide enough range of skills throughout the sample. People with more of the trait of interest should be given a higher score [41]. In analyzing data quality, infit and outfit mean square fit statistics are used to see whether answers deviate from what the Rasch model predicts for each item and person. A substantial number of unexpected replies are indicated by high mean square fit statistics. High person mean square scores suggest that test takers who filled in replies at random had exceptional knowledge gaps. Item infit

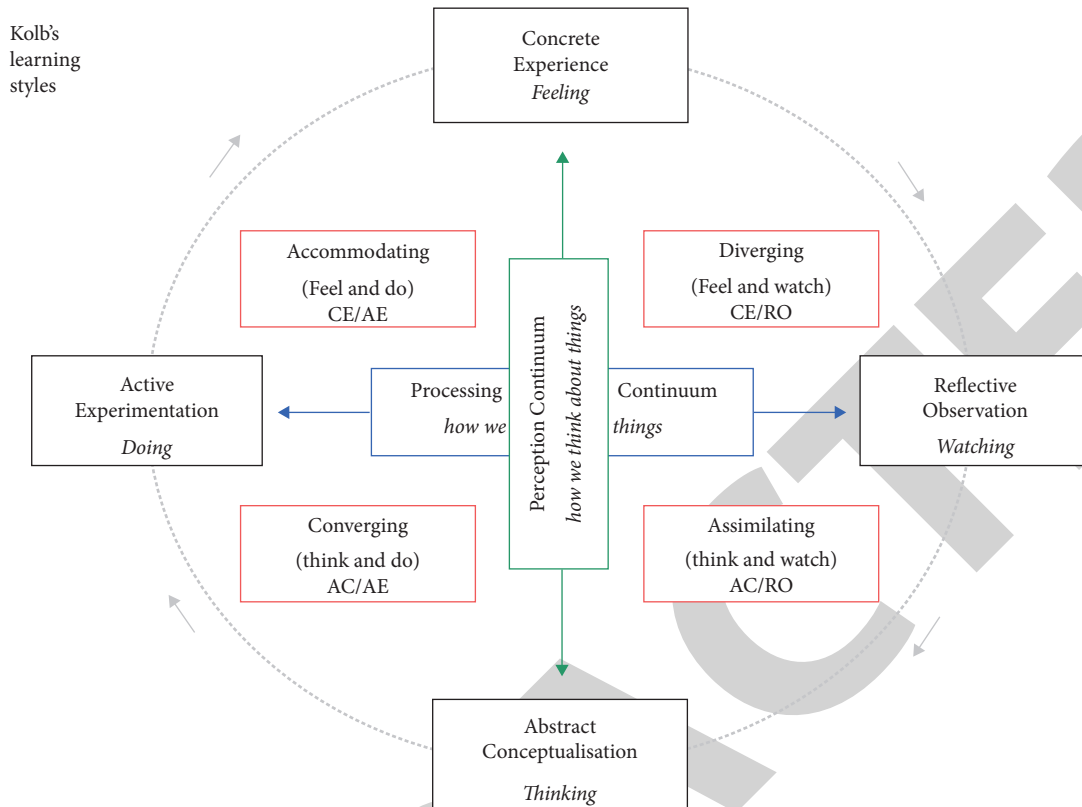


FIGURE 1: Kolb's learning style model.

mean square values between 1.5 and 2.0 are deemed unproductive for measurement, whereas values more than 2.0 are regarded as deteriorating [40].

The Rasch measurement seeks to achieve generalizability while avoiding bias due to characteristics such as gender and race. A Rasch analysis calculates the anticipated value of an item for people of the same skill level. These projected values may then be compared to observed values for various groups of people (for example, men and women, or different groups of classes).

5.1. Wright Map. The Wright map displays the distribution of person measure (left side of logit ruler) and item measure (right side of logit ruler) based on the polytomous Rasch model. When a higher person logit is calibrated against a lower item logit, the probability that the person has a positive attitude toward learning statistics is at least 0.5. The probability is less than 0.5 if a lower person logit is calibrated against a higher item logit.

6. Data Structure in Rasch Measurement

Data structure for Rasch analysis includes data preparation in Excel format as in formatted text (space delimited). It was then transformed to Winsteps 3.90.2 __.prn file that was used to execute the necessary Rasch outputs. Extract of the Excel data structure of the questionnaire items and __.prn file are shown in Figures 4 and 5, respectively. __.prn file in

Figure 5 is the base Winsteps program used to execute the necessary Winsteps outputs.

7. Results and Discussion

7.1. Objective 1 and Objective 2. The Wright map in Figure 6 displays the attitude responses and learning styles of Arab students. The person mean logit at 0.1 is slightly above and very close to the item mean logit at 0.0. This indicates that Arab students display a high positive attitude toward learning statistics but at varying probability levels. In terms of their learning styles, a majority of the students (84%) prefer to learn statistics by feeling and doing the problems. In other words, the probability of the students preferring to solve statistical problems by doing and understanding is between 0.46 and 0.76, respectively. On the other hand, only about 12% of the students prefer to watch and think about how other people (lecturer and fellow students) solve the problem, with a probability of between 0.24 and 0.55, respectively. Clearly, a majority of Arab students prefer to learn statistics by doing and solving problems.

Figure 7 shows the probability that the Arab students perceive themselves as confident (C13r), do not experience anxiety (C9R), and know the direction of the subject (C14r) is at least 0.85. A probability of at least 0.72 is observed for students who do not experience stress in learning statistics (C12r), do not find difficulties in learning statistics (C16r), and do not make lots of errors in statistical calculations (C15r). The probability that statistics is useful for solving

		Strongly Disagree (1)	Disagree (2)	Neutral (3)	Agree (4)	Strongly Agree (5)
1.	Learning statistics is exciting.					
2	I like to learn statistic topics.					
3	I never get tired of learning statistics.					
4	I do not feel bored when learning statistics.					
5	Statistics is useful in solving real life problems.					
6	Statistics is useful in my life.					
7	Statistics is relevant to my profession.					
8	Statistics is useful for making important decisions.					
9r	I have anxiety while learning statistics.					
10	Learning statistics is easy for me.					
11	I enjoy learning statistics.					
12r	I am under stress while learning statistics.					
13r	I am not confident while learning statistics.					
14r	I have no idea of what's going on in the statistics topic.					
15r	I tend to make lots of errors in statistics calculation.					
16r	I find it is difficult to understand statistics concepts.					
17	Statistics involves massive computation.					

FIGURE 2: Perceived attitude toward learning statistic items.

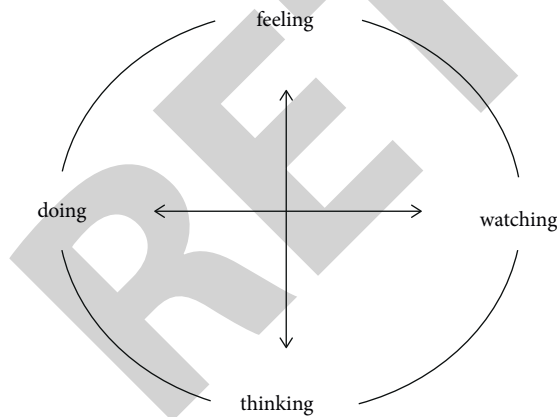


FIGURE 3: Instrument to gauge students' learning styles in statistics.

real-life problems (C5), useful in their lives (C6), relevant to their profession (C7), and enjoyable (C11) is at least 0.65, while the probability that the students perceive learning statistics as exciting (C1) and useful for making important decisions (C8) is at least 0.65.

A summary of the 20 things that were measured is shown in Table 2.

TABLE 1: Characteristics of students' learning styles based on Kolb's learning model.

		Do you prefer to think or feel?	
		Think	Feel
Do you prefer to watch or do?	Watch	45 (30%)	20 (13%)
	Do	70 (47%)	15 (10%)

Table 2 shows item reliability of 0.85, which indicates that the items are replicable for measuring the same attitude traits over a suitable range of Arab students' abilities. The mean infit and outfit for item mean square are both 1.00 logit, with z-scores of 0.2 and 0.1, respectively. The item mean logit is set to 0, and the item separation index is 2.41. This implies that things are divided into two difficulty levels.

It is shown in Figure 8 that the polytomous Rasch model is used to create a Wright map, where the person and item measurements are spread along the logit ruler. When a higher person logit is calibrated against a lower item logit, the probability that the person has a positive attitude toward learning statistics is at least 0.5. The probability is less than 0.5 if a lower person logit is calibrated against a higher item logit. The map shows that the person mean logit at 0.3 is

Program	Gender	C1	C2	C3	C4	C5	C6	C7	C8	C9r	C10	C11	C12r	C13r	C14r	C15r	C16r	C17	Watch	Do	Think	Feel
S	M	5	4	4	4	3	4	5	4	5	3	5	4	5	4	5	4	3	1	0	0	1
IS	M	3	3	4	3	4	4	4	4	5	3	3	5	5	4	3	5	2	1	0	0	1
EG	M	3	4	5	3	4	5	3	5	5	2	4	5	5	5	5	4	1	0	1	0	1
EG	M	4	5	4	2	4	4	4	4	5	3	5	4	5	4	4	5	2	1	0	1	0
B	M	5	4	4	3	4	5	5	5	4	2	4	3	4	4	3	4	1	0	1	1	0
M	M	4	5	3	4	4	5	3	5	4	4	5	4	4	3	4	3	1	0	1	1	0
M	M	5	4	4	4	4	4	5	4	4	3	5	5	4	4	4	5	1	0	1	0	1
P	M	5	5	3	4	4	4	4	4	4	3	2	4	4	3	5	3	4	1	1	0	1
EG	M	4	5	4	5	5	3	5	3	4	3	4	5	4	4	3	3	2	1	0	0	1
H	M	4	4	3	4	5	3	5	3	4	4	3	4	4	5	5	4	1	0	1	1	0
P	M	3	4	4	5	5	5	4	5	4	4	3	4	4	3	5	4	1	0	1	1	0
B	M	3	3	4	5	5	3	4	3	5	4	5	3	5	4	5	5	1	1	0	1	0
A	M	5	3	3	4	4	3	4	3	3	5	3	5	3	4	5	4	1	1	0	1	0
IS	M	3	5	4	4	5	4	3	4	5	4	3	4	5	4	4	5	2	0	1	1	0
P	M	3	3	4	3	3	4	3	4	4	3	4	3	4	5	4	4	2	1	0	0	1
B	M	4	3	4	3	3	4	4	4	3	3	4	4	3	4	3	5	3	0	1	1	0
H	M	4	4	4	5	3	4	4	4	4	3	4	3	4	5	3	4	2	0	1	1	0
EG	M	4	4	4	3	3	4	4	4	5	4	4	4	5	4	4	5	1	0	1	1	0
M	M	4	4	2	3	5	4	3	4	4	5	4	4	4	4	4	4	2	1	0	1	0
M	M	4	4	3	4	4	4	4	4	5	4	4	4	5	5	4	4	3	1	0	1	0
EG	M	4	4	3	4	4	4	4	4	4	3	4	4	4	4	5	3	0	1	1	1	0
MD	M	4	4	3	4	5	5	4	5	5	4	4	3	5	5	4	5	3	0	1	0	1
A	M	4	4	3	4	4	4	5	4	4	4	5	4	4	4	3	5	2	0	1	0	1
MD	M	5	4	3	4	5	3	4	2	5	3	4	4	5	3	4	5	2	0	1	0	1

FIGURE 4: Excel data structure as in formatted text.

```

File Edit Format View Help
&INST
  TITLE = "Learning Styles and Attitude Toward Statistics"
  PERSON = Person ; persons are ...
  ITEM = Item ; items are ...
  ITEM1 = 11 ; column of response to first item in data record
  NI = 21 ; number of items
  NAME1 = 1 ; column of first character of person identifying label
NAMELEN = 10 ; length of person label
  XWIDE = 1 ; number of columns per item response
  CODES = "012345 " ; valid codes in data file
  UIMEAN = 0 ; item mean for local origin
  USCALE = 1 ; user scaling for logits
  UDECIM = 2 ; reported decimal places for user scaling
ISGROUPS=*
1-17 L ; Attitude Likert items
18-21 D ; Competency Dichotomy items
*
linelength=123;
&END

```

FIGURE 5: Winsteps 3.90.2 __.prn file for a data structure.

slightly above the item mean logit at 0.0. This indicates that Malay students have a moderately positive attitude toward learning statistics.

A majority of the Malay students (89%) perceive that statistics is useful in solving real-life problems (C5) and that it is useful for making important decisions (C8). About 75% perceive that statistics is useful in their lives (C6), but at the same time they have anxiety while learning statistics; 71% are not under stress while learning statistics (C12r), and they do not find it difficult to understand statistical concepts (C16r). On the contrary, at least 75% often feel tired and not excited when learning statistics.

In terms of learning styles in statistics, a majority of the Malay students prefer to feel the experience of doing the statistical problems (Feel and Do), which categorizes them under the "Accommodating" type of learners, compared to thinking and watching what other people do (Think and

Watch), which would have categorized them under the "Assimilating" type of learners.

A perusal of the Wright map in Figure 6 shows that the Malay students' attitude toward learning statistics is moderately distributed across the map between -0.5 and $+1.5$ logit. The distribution of attitude responses shows that the students are fairly spread out in terms of their attitude, depending on the items that they responded to. A majority of the students feel that learning statistics is tiring, less exciting, and sometimes boring. In spite of these feelings, a majority acknowledge that learning statistics is useful in solving real-life problems and that it is useful for making important decisions.

Table 3 shows item reliability of 0.91, which indicates that the items are replicable for measuring the same attitude traits over a suitable range of the students' abilities. The mean infit for item mean square is 1.00 logit, and the mean

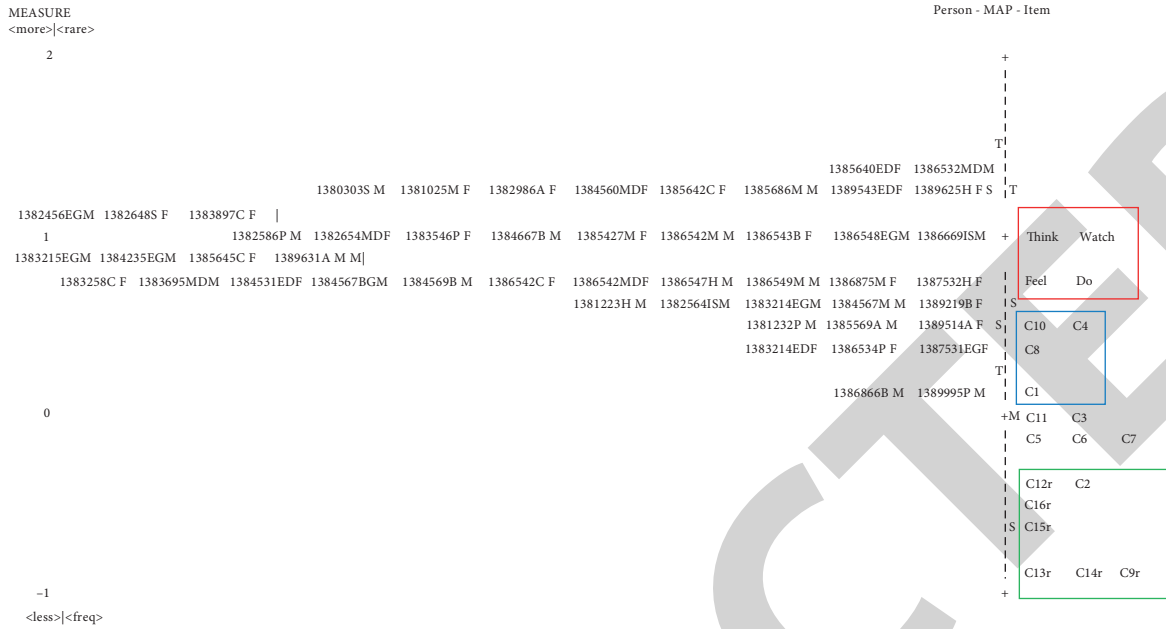


FIGURE 6: Wright map on the distribution of attitude toward statistics and learning styles of Arab students.

Item	MEASURE	PERSON	C14R	C13R	C9R	C15R	C16R	C12R	C2	C6	C7	C5	C11	C3	C1	C8	C10	C4
PERSON	MEASURE		-0.93	-0.88	-0.88	-0.80	-0.56	-0.34	-0.34	-0.18	-0.14	-0.14	-0.06	0.02	0.14	0.07	0.53	0.53
22	1.41		91.21	90.80	90.80	88.18	87.76	85.20	85.20	83.06	82.49	82.49	81.31	80.06	78.07	73.11	70.68	70.68
49	1.41		91.21	90.80	90.80	88.18	87.76	85.20	85.20	83.06	82.49	82.49	81.31	80.06	78.07	73.11	70.68	70.68
1	1.28	90.11	89.66	89.66	86.76	86.29	83.48	83.48	81.15	80.53	80.53	79.25	77.90	75.77	70.47	67.92	67.92	67.92
7	1.28	90.11	89.66	89.66	86.76	86.29	83.48	83.48	81.15	80.53	80.53	79.25	77.90	75.77	70.47	67.92	67.92	67.92
38	1.28	90.11	89.66	89.66	86.76	86.29	83.48	83.48	81.15	80.53	80.53	79.25	77.90	75.77	70.47	67.92	67.92	67.92
39	1.28	90.11	89.66	89.66	86.76	86.29	83.48	83.48	81.15	80.53	80.53	79.25	77.90	75.77	70.47	67.92	67.92	67.92
46	1.28	90.11	89.66	89.66	86.76	86.29	83.48	83.48	81.15	80.53	80.53	79.25	77.90	75.77	70.47	67.92	67.92	67.92
47	1.28	90.11	89.66	89.66	86.76	86.29	83.48	83.48	81.15	80.53	80.53	79.25	77.90	75.77	70.47	67.92	67.92	67.92
48	1.28	90.11	89.66	89.66	86.76	86.29	83.48	83.48	81.15	80.53	80.53	79.25	77.90	75.77	70.47	67.92	67.92	67.92
42	1.24	89.75	89.28	89.28	86.29	85.81	82.92	82.92	80.53	79.90	79.90	78.58	77.21	75.03	69.64	67.04	67.04	67.04
3	1.16	88.99	88.49	88.49	85.32	84.81	81.76	81.76	79.25	78.58	78.58	77.21	75.77	73.50	67.92	65.25	65.25	65.25
32	1.16	88.99	88.49	88.49	85.32	84.81	81.76	81.76	79.25	78.58	78.58	77.21	75.77	73.50	67.92	65.25	65.25	65.25
41	1.16	88.99	88.49	88.49	85.32	84.81	81.76	81.76	79.25	78.58	78.58	77.21	75.77	73.50	67.92	65.25	65.25	65.25
4	1.04	87.76	87.21	87.21	83.75	83.20	79.90	79.90	77.21	76.49	76.49	75.03	73.50	71.09	65.25	62.48	62.48	62.48
11	1.04	87.76	87.21	87.21	83.75	83.20	79.90	79.90	77.21	76.49	76.49	75.03	73.50	71.09	65.25	62.48	62.48	62.48
12	1.04	87.76	87.21	87.21	83.75	83.20	79.90	79.90	77.21	76.49	76.49	75.03	73.50	71.09	65.25	62.48	62.48	62.48
14	1.04	87.76	87.21	87.21	83.75	83.20	79.90	79.90	77.21	76.49	76.49	75.03	73.50	71.09	65.25	62.48	62.48	62.48
20	1.04	87.76	87.21	87.21	83.75	83.20	79.90	79.90	77.21	76.49	76.49	75.03	73.50	71.09	65.25	62.48	62.48	62.48
26	1.04	87.76	87.21	87.21	83.75	83.20	79.90	79.90	77.21	76.49	76.49	75.03	73.50	71.09	65.25	62.48	62.48	62.48
31	1.04	87.76	87.21	87.21	83.75	83.20	79.90	79.90	77.21	76.49	76.49	75.03	73.50	71.09	65.25	62.48	62.48	62.48
40	1.04	87.76	87.21	87.21	83.75	83.20	79.90	79.90	77.21	76.49	76.49	75.03	73.50	71.09	65.25	62.48	62.48	62.48
44	1.04	87.76	87.21	87.21	83.75	83.20	79.90	79.90	77.21	76.49	76.49	75.03	73.50	71.09	65.25	62.48	62.48	62.48
18	0.92	86.41	85.81	85.81	82.05	81.46	77.90	77.90	75.03	74.27	74.27	72.71	71.09	68.57	62.48	59.63	59.63	59.63
23	0.92	86.41	85.81	85.81	82.05	81.46	77.90	77.90	75.03	74.27	74.27	72.71	71.09	68.57	62.48	59.63	59.63	59.63

FIGURE 7: Attitude probability scores for Arab students' sample.

TABLE 2: Summary statistics of Arab pupils' assessed attitude items.

	Total score	Count	Measure	Model SE	Infit MNSQ	InfitZSTD	OutfitMNSQ	Infit ZSTD
Mean	166.0	49.9	0.00	0.22	1.00	0.2	1.00	0.1
SD	71.2	0.3	0.60	0.03	0.13	0.9	0.13	0.9
Max.	218.0	50.0	1.03	0.29	1.28	1.8	1.27	1.8
Min.	23.0	49.0	-0.93	0.20	0.77	-1.4	0.76	-1.4
Real MSE		0.23	True SD	0.55	Separation	2.41	Reliability	0.85
Model RMSE		0.22	True SD	0.55	Separation	2.47	Reliability	0.86

outfit is 1.01 logit, with both infit and outfit z-scores at -0.1 . The item mean logit is set at 0, and the separation index for item is 3.25. This indicates that the items are separated into three levels of the item's difficulty.

7.2. Objective 3. To compare certain attitude of the students based on item characteristic curves, the following

results are described. A summary of 20 measured items is shown in Table 3.

Technically, the item characteristic curve (ICC) is used to describe the distribution of the response pattern of the students toward the items based on the logit and item scores. In Figure 9, the expected and empirical item characteristic curves are constructed to observe the attitude of Arab students from different attitude items. A

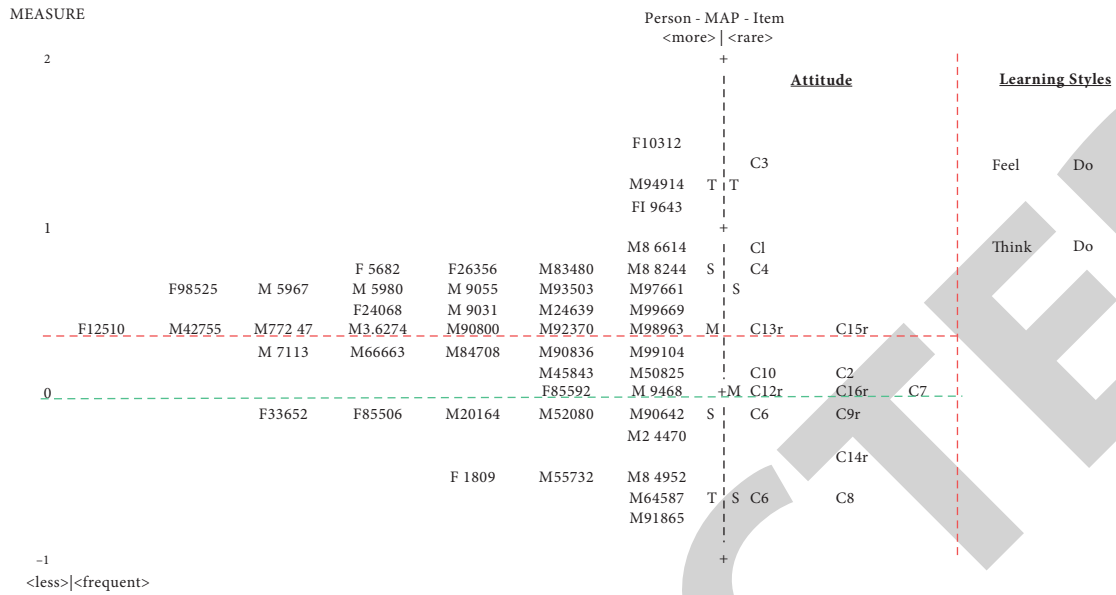


FIGURE 8: Wright map on the distribution of attitude toward statistics and learning styles of Malay students.

TABLE 3: Summary statistics of measured attitude items of Malay students.

	Total score	Count	Measure	Model SE	Infit MNSQ	Infit ZSTD	Outfit MNSQ	Infit ZSTD
Mean	149.6	45.0	0.00	0.17	1.00	-0.1	1.01	-0.1
SD	22.2	0.0	0.61	0.01	0.31	1.4	0.32	1.5
Max.	184.0	45.0	1.40	0.20	2.04	3.7	2.00	3.6
Min.	96.0	45.0	-1.04	0.16	0.58	-2.2	0.52	-2.6
Real MSE		0.18	True SD	0.59	Separation	3.25	Reliability	0.91
Model RMSE		0.17	True SD	0.59	Separation	3.48	Reliability	0.92

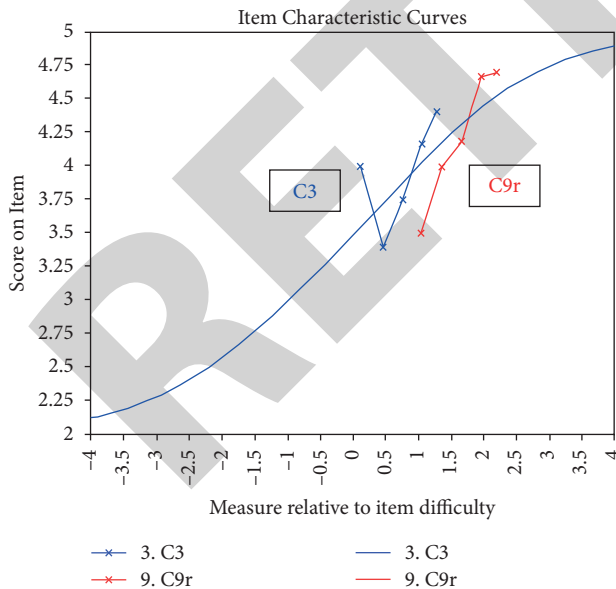


FIGURE 9: Attitude logit scores for Arab sample.

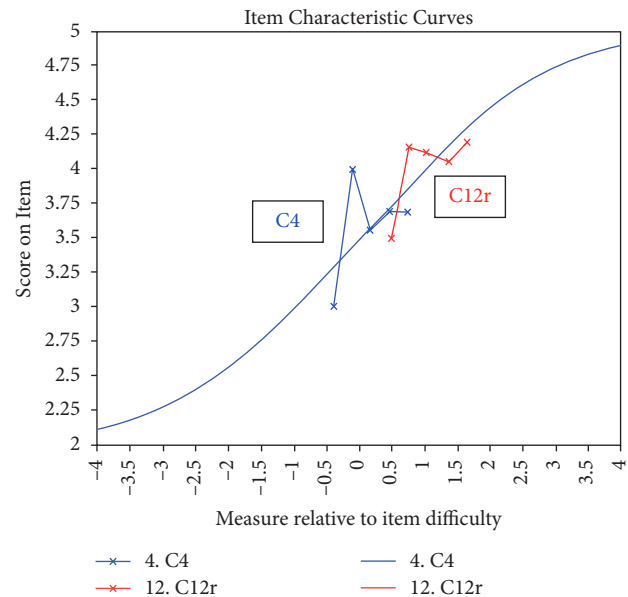


FIGURE 10: Wright map on the distribution of attitude toward statistics and learning styles of Malay students.

cursory look at the ICC shows that item C9r (“I do not have anxiety while learning statistics”) is easier to endorse compared to item C3 (“I never get tired of learning statistics”) because the response distribution for item C9r is

located above the response distribution for item C3. A comparison of items C3 and C9r shows that there are more students who get tired of learning statistics

compared to those who experience anxiety while learning statistics.

In Figure 10, a comparison of two items shows that item C12r (“I am under stress while learning statistics”) is easier to endorse compared to item C4 (“I do not feel bored when learning statistics”) because the response distribution for item C12r is located above the response distribution for item C4. This indicates that students are under stress while learning statistics; however, they do not feel bored learning about statistics.

8. Conclusions

Research studies on students’ attitudes toward statistics and their learning styles in statistics have been investigated from various perspectives. This study used a psychometric approach based on the Rasch models to explore both Malay and Arab students’ attitudes toward statistics and their learning styles. The connections between the students’ attitudes and their learning styles and the fact that they are from different cultural backgrounds have shown some similarities and differences in the students’ attitude toward statistics and their learning patterns in statistics. A comparison of the item reliability index shows that Malay students have a slightly higher item reliability index compared to Arab students, which indicates that the attitude items are more agreeable to the Malay students compared to the Arab students, given their range of abilities. Generally, students from both the Arab and Malay cultures are classified as the “Accommodating” type with a preference for doing (active experimentation) and feeling from the experience of doing/solving statistical exercises. However, Arab students are also classified as the “Assimilating” type with a preference for thinking, reflecting, and learning from observation, while Malay students are classified as the “Converging” type with a preference for thinking (abstract conceptualization) and doing statistical exercises (active experimentation).

Data Availability

Numerical dataset used to conduct the study reported in the publication is available from corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

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Retraction

Retracted: Asymptotically Effective Method to Explore Euler Path in a Graph

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

- [1] M. Fahad, S. Ali, M. Khan, M. Husnain, Z. Shafi, and A. Samad, "Asymptotically Effective Method to Explore Euler Path in a Graph," *Mathematical Problems in Engineering*, vol. 2021, Article ID 8018373, 7 pages, 2021.

Research Article

Asymptotically Effective Method to Explore Euler Path in a Graph

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Euler path is one of the most interesting and widely discussed topics in graph theory. An Euler path (or Euler trail) is a path that visits every edge of a graph exactly once. Similarly, an Euler circuit (or Euler cycle) is an Euler trail that starts and ends on the same node of a graph. A graph having Euler path is called Euler graph. While tracing Euler graph, one may halt at arbitrary nodes while some of its edges left unvisited. In this paper, we have proposed some precautionary steps that should be considered in exploring a deadlock-free Euler path, i.e., without being halted at any node. Simulation results show that our proposed approach improves the process of exploring the Euler path in an undirected connected graph without interruption. Furthermore, our proposed algorithm is complete for all types of undirected Eulerian graphs. The paper concludes with the proofs of the correctness of proposed algorithm and its computation complexity.

1. Introduction

Graph is one of the discrete structures that consists of nodes (or nodes) and edges that connect these nodes. In order to avoid confusion, we will use the term node in the rest of paper. Generally, the graphs may be either directed or undirected [1]. Mathematically a graph can be defined as $G = (V, E)$ which consists of V , a nonempty set of nodes and E , and a set of edges [2, 3]. Each edge has either two nodes v and w , associated with it, called its endpoints. The edges are unordered pairs of nodes in undirected graphs, also represented as (v, w) . In directed graphs, the edges are ordered pairs (v, w) of nodes, where v is the tail and w is the head of the edge. A path in a graph is a finite or infinite sequence of edges which connect a sequence of nodes which, by most definitions, are all distinct from one another [2, 3]. These paths can be used in designing the framework of a number of graph-related issues. For example, the graph model can be used in determining whether a communication link is established among different computers in a network or not.

Furthermore, a number of issues such as planning efficient routes in mail and postage delivery, diagnosis of transportation system, etc., can be efficiently planned using paths in graphs. It is noteworthy that all graphs discussed in this paper are assumed to be undirected. Furthermore, it is also supposed that these graphs have a finite number of nodes and edges.

Among various types of paths in graph theory, Euler path is a special path that visits every edge of connected graph only once [4, 5]. In a connected undirected graph, there are no unreachable nodes and all the edges are bidirectional. A connected undirected graph will have an Euler path, but not an Euler circuit if and only if it has exactly two nodes of odd degree. The degree of node a is defined as the number of edges connected to that node. For the existence of the Euler path, it is necessary that exactly two nodes in a graph must have odd degree [4, 5]. A graph having the Euler path is called as the Euler graph. It is observed that while tracing the Euler graph, one may halt at any arbitrary node with some nodes (and its associated edges) remain unreachable.

Some noteworthy classic work reported in [6, 7] concluded that the algorithm mentioned in Fleury's algorithm [8–10] is an elegant but inefficient algorithm, since, at the end of the algorithm, there are no edges left, and the sequence from which the edges were chosen forms of an Eulerian cycle if the graph has no vertices of odd degree. Similarly, in [10], the proposed algorithm may also be implemented with the Deque. Because it is only possible to get stuck when the Deque represents a closed tour, one should rotate the Deque by removing edges from the tail and adding them to the head until unstuck, and then continue until all edges are accounted for. These issues were addressed in our proposed approach.

In computers, there are a number of ways to represent graphs. One of these ways is to use adjacency lists. An adjacency list represents a graph as an array of linked list. The index of the array represents a node and each element in its linked list represents the other nodes that form an edge with the node [1, 2]. Figure 1 shows a simple graph G and its representation in adjacency list.

In this paper, we have presented an approach in exploring deadlock-free Euler paths in Euler graphs by revising adjacency list. The rest of this paper is organized as follows. Section 2 introduces the problem statement and traditional algorithm. In Section 3, the proposed approach is demonstrated in detail in comparison with conventional Euler path algorithm. Section 4 gives the proof of correctness of our proposed approach. Complexity of the proposed approach is given in Section 1. Some real-life applications of Euler path are discussed in Section 6. Concluding remarks are given in Section 7.

2. Problem Statement

Algorithm 1 depicts the construction of the Euler path. While tracing the Euler graph according to Algorithm 1, one may halt (or stall) at any arbitrary node and fail to visit the remaining unvisited edges of the graph thus leading to an incomplete traversal of the Euler path. This situation of halting may result in exponential increase of the computation cost even if a single node with its associated edges is added in the Euler graph.

Figure 2 shows Euler graph G and its two incomplete Euler paths halted at node a (Figure 2(b)) and b (Figure 2(c)), respectively.

The number on the edges shows the order of traverse. The order of traversal of the graph shown in Figure 2(b) is $b \rightarrow a \rightarrow c \rightarrow d \rightarrow a$ resulting to halting situation at node a , while the edges (c, e) , (d, e) , and (b, d) remain unvisited. Similarly, in Figure 2(c), the order of traverse is $a \rightarrow c \rightarrow b \rightarrow d \rightarrow a \rightarrow b$ that also freeze at node a with (c, e) , (c, d) , and (d, e) left unvisited.

In other Euler graph H shown in Figure 3(a), two of many incomplete Euler paths halted at nodes d and b , respectively, are shown in Figures 3(b) and 3(c). The order of traversal of the graph of Figure 3(b) is $b \rightarrow d \rightarrow c \rightarrow f \rightarrow g \rightarrow e \rightarrow d$ and thus results in halting at node d . The same scenario exists in the faulty traversal shown in Figure 3(c).

The *bridge* edge, as mentioned in Algorithm 1, is defined as an edge that when removed increases the number of connected components. The problem in faulty-Euler path lies when we accidentally visit the *bridge* edge. The procedure of finding the *bridge* edge by classical algorithm (Tarjan's bridge-finding algorithm) [13] is itself a complicated task for strong connected Euler graphs because it finds out the bridge edge(s) by extracting spanning forest from the graph. This process incurs space and time complexity $O(V \times (V + E))$ as computed by [1, 2]. Furthermore, the process becomes more complex when it tries to compute the size of multiplex computer network (represented by a graph) using conventional algorithm since, for every edge, another module is called in order to identify that the edge is *bridge* or not. Furthermore, the addition of extranode and its associated edges to an Euler graph (in a way that the resulting graph is also Euler) makes the graph too complex to explore the Euler path without being halted. In order to address this issue, we have proposed an informal solution to find the Euler path in any Euler graph without being halted, as discussed in Section 3.

3. Our Proposed Approach

Our proposed algorithm for finding deadlock-free Euler path in some Euler bidirectional connected graph is depicted in Algorithm 2. We applied our proposed algorithm on a number of Euler graphs. Figures 4 and 5 show the simulation results on Euler graphs of Figures 2 and 3. The results showed that our proposed algorithm outperformed the conventional approach. The first column of the table depicts the edge currently being visited. The edges to be visited are shown in column "Remaining edges." The last column shows the trace of our proposed approach to explore deadlock-free Euler path.

Still another classical example will clearly explain the performance of our proposed approach and help in exploring a deadlock-free Euler path. Consider a simple graph M (see Figure 6) having two odd-degree nodes c and d . If we explore the Euler path from any node other than c or d , we will be stuck with some edges remained unvisited. For example, the path $e \rightarrow f \rightarrow d \rightarrow c \rightarrow b \rightarrow a \rightarrow c$ will result to halt at node c with the edge (d, e) remain unvisited. If we apply our modified Euler algorithm, we can have deadlock-free path. Some of the many deadlock free Euler paths explored by our proposed algorithms are $d \rightarrow e \rightarrow f \rightarrow d \rightarrow c \rightarrow a \rightarrow b \rightarrow c$ and $c \rightarrow a \rightarrow b \rightarrow c \rightarrow d \rightarrow f \rightarrow e \rightarrow d$. It is noteworthy that both the paths start from one odd-degree node and ends on other odd-degree node.

Our proposed algorithm search for the odd-degree node first of an Euler graph from its revised adjacency list (see Figure 1) and then visit its adjacent even-degree nodes one by one prior to visiting the other odd-degree node. In other words, as long as there exist edges connected to even-degree nodes and traverse these edges prior to the other odd-degree node in order to avoid any deadlock until there is no choice.

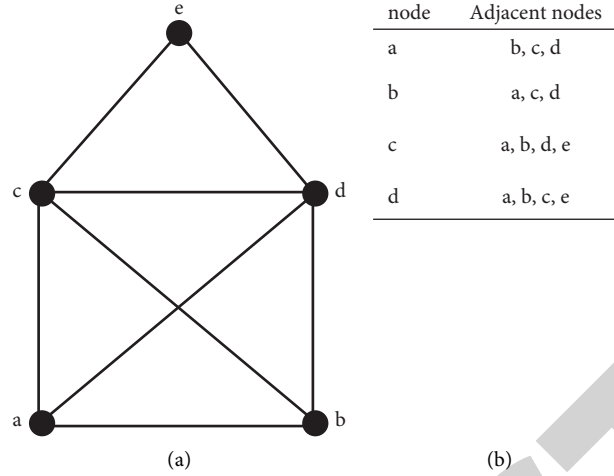


FIGURE 1: A graph G (a) and its adjacency list (b).

EulerPathInConnectedGraph (G: connected undirected graph with exactly two nodes of odd degree)

- (1) Make sure the graph G has 2 odd nodes.
- (2) Start at any one of the two odd nodes.
- (3) Traverse the edges one by one and then add in the array EP.
- (4) While traversing a **bridge** or a **nonbridge** edge, select the nonbridge edge.
- (5) Stop when you run out of edges.
return EP

ALGORITHM 1: Euler path [11, 12].

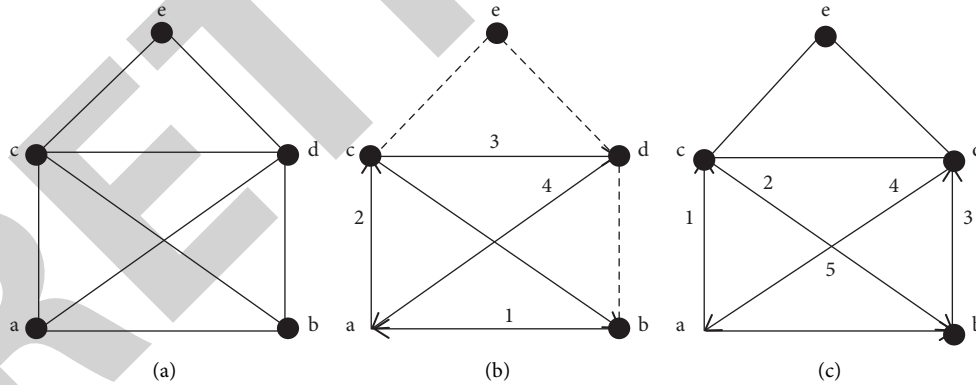


FIGURE 2: (a) Simple graph G. (b) A faulty-Euler path. (c) Other faulty-Euler path.

4. Proof of Correctness of Our Proposed Approach

Does our proposed approach always produce deadlock-free Euler path? The answer is yes. Let us prove by induction that each of the subgraph G_i , of a Euler graph G, where $i = 1, 2, \dots, n$, that is generated by removing the visited edges one by one by our proposed approach (Algorithm 2), leads to a deadlock-free Euler path. It immediately concludes that the last subgraph G_n (having

only one edge with two nodes) will have at least one odd-degree node. The induction phase is trivial since G_1 consists of a single edge connecting an odd-degree and even-degree node and also part of Euler path of the graph G. While solving the inductive step, we assume (for the sake of computation) that G_{i-1} is among one of the deadlock-free Euler paths. We need to prove that G_i , generated from G_{i-1} by our proposed approach, is leading towards a successful deadlock-free Euler path. We tried to prove this assertion by the contradictory rule by assuming

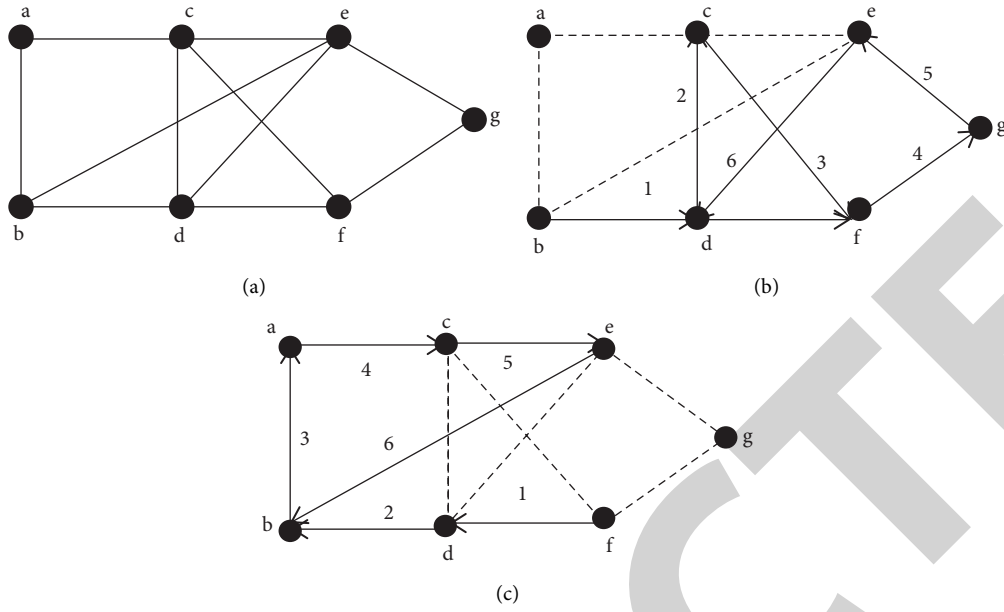


FIGURE 3: (a) Simple graph H (b) A faulty Euler path. (c) Other faulty Euler path.

Starting edge of graph	Remaining edges	Illustration
(b, d)	(d, e), (e, c), (a, c), (c, d), (a, d), (a, b), (b, c)	 G1
(d, e)	(e, c), (c, a), (c, d), (a, d), (a, b), (c, b)	 G2
(e, c)	(c, a), (c, d), (a, d), (a, b), (c, b)	 G3
(c, d)	(c, a), (a, d), (a, b), (c, b)	 G4
(a, d)	(a, c), (a, b), (b, c)	 G5
(a, b)	(b, c), (a, c)	 G6
(b, c)	(a, c)	 G7
(a, c)	---	 G8

FIGURE 4: Application of our proposed algorithm. The parenthesized labels of a node in the middle column indicate the nearest tree edge top to be selected; selected edges are shown in bold. The subscript in G_i depicts the order of edge visited.

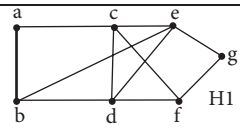
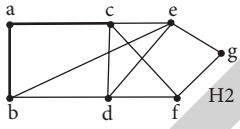
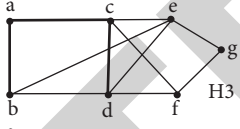
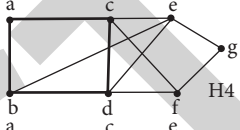
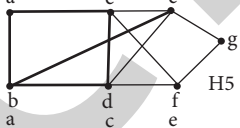
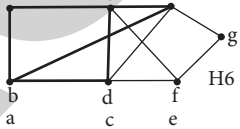
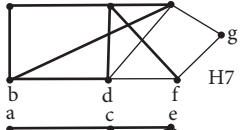
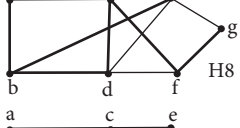
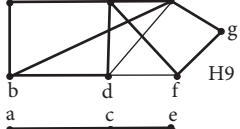
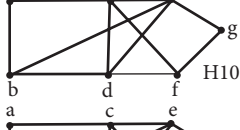
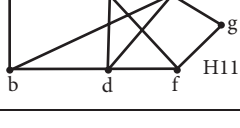
Starting edge of graph	Remaining edges	Illustration
(a, b)	(a, c), (c, e), (e, g), (f, g), (d, f), (b, d), (b, e), (c, d), (c, f), (d, e)	 H1
(a, c)	(c, e), (e, g), (f, g), (d, f), (b, d), (b, e), (c, d), (c, f), (d, e)	 H2
(c, d)	(c, e), (e, g), (f, g), (d, f), (b, d), (b, e), (c, f), (d, e)	 H3
(b, d)	(c, e), (e, g), (f, g), (d, f), (b, e), (c, f), (d, e)	 H4
(b, e)	(c, e), (e, g), (f, g), (d, f), (c, f), (d, e)	 H5
(c, e)	(e, g), (f, g), (d, f), (c, f), (d, e)	 H6
(c, f)	(e, g), (f, g), (d, f), (d, e)	 H7
(f, g)	(e, g), (d, f), (d, e)	 H8
(e, g)	(d, f), (d, e)	 H9
(d, e)	(d, f)	 H10
(d, f)	—	 H11

FIGURE 5: Application of our proposed algorithm. The parenthesized labels of a node in the middle column indicate the nearest tree node to be selected; selected edges are shown in bold. The subscript in H_i depicts the order of edge visited.

that no Euler path of any undirected graph may have G_i . Let $e = (v, u)$ be the edge from a specific node of G_{i-1} to a node that is not in G_{i-1} . This assertion is used by our proposed method to expand G_{i-1} to G_i . This assertion initiated our proposed approach by the assumption that e cannot belong to deadlock-free Euler path including G . As a result, by adding e to G , a cyclic path is constituted (Figure 7). Furthermore, the edge $e = (v, u)$, in this cyclic

path, must have the other edge (v', w') that is connecting a node v' of G_{i-1} to a node u' that is not in G_{i-1} (it is possible that v' coincides with v or u' coincides with u but not both). By removing the edge (v', w') from this cyclic path, the resultant graph is with two disconnected components since the only edge e is confirming the deadlock-free Euler path. Hence, this Euler path of G_{i-1} leads a deadlock-free Euler path which is clearly

EulerPathRevised (G : undirected graph with exactly two nodes of odd degree)

- (1) Make sure the graph G has 2-odd nodes.
- (2) Start at any one of the two odd nodes v from the *Degree* column of new adjacency list.
 - (a) Follow the edges to even-degree neighboring node w of node v .
 - (b) Add the visited edge in the list EP and remove it from Graph G .
- (3) **If** there is no choice then select the other odd-degree node
- (4) **else** repeat step **a** and **b**.
- (5) Stop when you run out of edges.
return EP

ALGORITHM 2: Our proposed algorithm for Euler path.

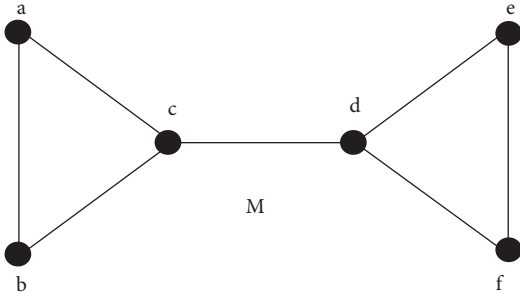


FIGURE 6: Classical Euler graph M.

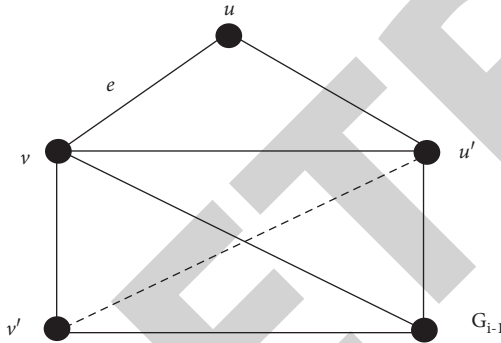


FIGURE 7: Correctness proof of our proposed approach (dashed edge is critical edge since its removal will produce the graph in two disconnected components).

contradicting the assumption that no deadlock-free Euler path will exist in graph G_i . This completes the correctness proof of our proposed approach.

5. Complexity of Our Proposed Approach

While computing the complexity of our proposed algorithm, we considered the revised adjacency list (see Figure 1). We start the Euler tour by selecting one of the odd-degree nodes from the revised adjacency list. This selection can be done in $O(1)$ time as the adjacency list is sorted in the descending order according to the degree of each node. After selecting one odd-degree node, we then select its neighboring even-degree node from the revised adjacency list that will take

$O(e-1)$ because remaining edges are now $e-1$. This process will work in an iterative way until we have no choice other than visiting the other odd-degree node or all the edges are visited. It is pertinent to mention that the numbers of edges are equally important as the number of nodes since every edge has two end points and visiting one edge is the same as visiting the two nodes. Now, putting all together, we have the initial constant work done on first odd-degree node plus the work done on each edge on a Eulerian graph of n nodes, described above

$$2^{(n+1)/2} \pi^{-0.5} e^{(-n^2/2)+n(n-2)(n+1)/2} 1 + O(n^{-0.5+e}) \text{ for } n \rightarrow \infty, \quad (1)$$

where n is odd.

A detail discussion of the complexity of the proposed algorithm for Euler path is given in [14]. The proof of correctness of the complexity is given below.

5.1. Proof of Correctness. This equation is derived from the classic work on the Euler path and circuits reported in [14, 15]. Since $n \rightarrow \infty$, without loss of generality, we assume $e < 0.5$; then, the statement of the second part of equation (1) can be considered as random regular circular tournament on n nodes on average $2^{(n+1)/2} e^{0.5} n^{n-2} (1 + O(n^{-0.5} + e))$ is considered or both the directed (and directed) spanning trees rooted at the n th node. It is noteworthy that it is $e^{0.5}$ more than the average for a random Euler tournament that needs not to be a regular tree.

It is noteworthy that these computations have several favorable interpretations in applied statistics models. Assume having an Eulerian graph G having start node v ; the random walk may proceed to each adjacent edge that might be used at most once.

6. Applications of Euler Path

In real world, we come across many applications that ask for a path that traverses each street in a neighborhood, each road in a transportation network, or each link in a large computer network exactly once [1, 2, 12]. Among the other areas where Euler path are applied is in the layout of circuits, in network

Research Article

A Novel Method for Navigational Risk Assessment in Wind Farm Waters Based on the Fuzzy Inference System

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Offshore wind power is an effective way to solve the energy crisis problem and achieve sustainable economic development. Aiming at the problems that the navigational risk of ships in the waters of offshore wind farms is difficult to quantify due to complex factors, this paper proposes a method of navigational risk assessment in the waters of offshore wind farms based on a fuzzy inference system. Firstly, through the analysis of the factors affecting the navigation system of wind farm waters, it is found that the navigational risk is affected by natural factors and navigational environment factors. Then, the visibility, the number of traffic flows, the number of encounter areas, and the distance between the sailing route and the wind farm are extracted to evaluate the risk of natural factors and the risk of the sailing environment in the navigation system of the wind farm waters, respectively. Considering the mutual influence of the factors, the fuzzy inference rules of navigational risk influence are established according to the expert experience, and a method of navigational risk assessment based on the fuzzy inference system in offshore wind farm waters is developed. In order to verify the effectiveness of the proposed method, a comprehensive evaluation of the navigational risk of wind farm waters in Changle offshore sea of Fujian Province is carried out, and the evaluation results are consistent with the actual situation. The proposed method has important theoretical significance for the navigational safety supervision of offshore wind farm waters.

1. Introduction

With the continuous development of the economy and society, offshore wind farms for wind power generation are of great significance to alleviate the energy crisis and improve the environment. China is rich in offshore wind energy resources. By the end of 2019, China's grid-connected wind power has reached 210 million kilowatts, an increase of 10.9% over the previous year [1]. In the future, China's offshore wind farms will enter a period of accelerated development, but the construction of offshore wind farms is bound to have a certain impact on maritime traffic, especially the navigational safety of ships. Therefore, it is particularly important to evaluate and quantify the navigational risk of offshore wind farms, which has an essential theoretical significance for improving maritime traffic safety and optimizing shipping routes.

At present, the research on the navigational risk of ships in the waters of offshore wind farms mainly starts from the collision problem between the route and the wind farm and studies the collision avoidance method between ships and wind turbines. Xue et al. [2] studied the collision avoidance algorithm for ships based on software A*. The key point is to find an optimal path, but the influence of wind, wave, and current on the safe navigation of ships is not considered in the modelling. Pu et al. [3] established a safety distance model and collision warning model between the ship and the wind farm and proposed a collision warning method based on an electronic fence to visually display the risk value of the ship collision with the wind turbine. Florian and Eike [4] proposed different anticollision measures for different situations by analyzing the risk of different types of ships hitting wind turbines. Many scholars analyze the

relationship between waterways and wind farms and constantly improve the safe distance between ships and wind farms. Based on the collision probability model of ships and wind farms, Nie et al. [5] proposed a method to define the safe distance between wind farms and waterways based on the collision probability of ships and wind farms and obtained the conclusion that the safe distance between wind farms and routes is related to the average speed of ships, load tons, and the boundary length of wind farms. Wang et al. [6] built a calculation model for the safe distance between the offshore wind farms and the waterways based on the improved out-of-control drift model by comprehensively considering the factors such as ships, fans, wind, and flow and obtained the safe distance between different types of ships and wind turbines in the out-of-control state. Andrew and Edward [7] compared and analyzed the changes of ship traffic around five offshore wind farms before and after they were built and found that developers and regulators all regard 1 nautical mile (n mile) as the safe passage distance for offshore wind farms.

In order to determine the locations of the wind farm or put forward suggestions for the safety of navigation, scholars conducted many studies of navigational risk assessments in offshore wind farm waters. Yun [8] proposed combining fuzzy comprehensive evaluation and the analytic hierarchy method to determine the best scheme of wind farm site selection. Wang [9] analyzed the navigational risk of wind farms from three aspects, nature, traffic, and wind farms, and proposed corresponding safety measures. Chen et al. [10] made a comparative analysis of various navigational risk assessment models and, based on the shortcomings of the NRA model, designed a navigational risk assessment framework that could provide a prediction of ship collision probability, and consequences, acceptability, and operability are yet to be verified. In the risk assessment, many scholars use the method of fuzzy inference. Ozturk et al. [11] presented three new parameters of distance, area, and speed and proposed a novel methodology based on machine learning and fuzzy inference to assess the risk of collision in the port approach; the proposed NCR assessment methodology gives accurate and reasonable risk degree in accordance with the navigation environment. Bukhari et al. [12] studied a dynamic method to calculate DCPA, TCPA, and azimuth directly by using the VTS radar input and proposed an intelligent real-time multiship collision risk assessment system based on the fuzzy inference system, which improved the process of manual risk assessment.

Based on the above analysis, most of the research studies on the navigational risk of ships in the waters of offshore wind farms are based on the site selection of wind farms and the collision avoidance between ships and wind turbines and other obstacles. They do not consider the collision avoidance between different traffic flows and ships nor carry out quantitative analysis on the overall navigational risk of ships in the waters of offshore wind farms. Compared with existing risk assessment methods such as AHP and fuzzy comprehensive evaluation [13], fuzzy inference system (FIS) can implement multiple complex factors as the input or

single factor as the output of the nonlinear mapping relationship, and it has been widely used in many fields such as the study of ocean detector performance [14], solar irradiance prediction [15], and flood prediction [16]. The quantification of the navigational risk of wind farm waters is a complex problem under multiple factors. The main contributions of this paper are as follows:

- (1) Through analyzing the factors affecting the navigational risk of wind farm waters, a hierarchical evaluation structure for navigational risk in offshore wind farm waters is built
- (2) The multifactor navigational risk assessment in the hierarchical evaluation structure has been implemented using a fuzzy inference system, which can solve the problem that the influence factors of navigational risk in the waters of offshore wind farms are difficult to be quantified

Based on the analysis of factors affecting ship navigation in wind farm waters, a fuzzy inference model is established by extracting factors such as the number of traffic flows, the number of meeting areas, and the distance between the sailing route and the wind farm. The proposed navigational risk assessment method in offshore wind farm water based on the FIS can take the influencing factors into account directly and give a visual perception of the assessment results. It is an improvement compared with the existing research. Analyzing and quantifying the navigational risk of wind farm waters under the influence of multiple factors can provide certain theoretical support for improving the supervision of wind farm waters' navigation and optimizing the sea route.

2. Influencing Factors of Navigational Risk in Offshore Wind Farm Waters

The "risk" refers to a specific danger, and risk assessment is to determine the possibility and severity of the accident caused by the risk and make a comprehensive evaluation [17]. The risk in this paper refers to the probability that the ship sailing in the waters of the wind farm may cause losses due to various uncertain factors. Through the model constructed in this paper, this probability can be expressed with accurate numbers so that the ship pilots and water traffic management departments have a more intuitive perception of the risk degree of ship navigation.

To objectively and accurately evaluate the overall navigational risk of offshore wind farm waters, this paper selects several indicators from two aspects of natural conditions and navigational environment to build the navigational risk assessment model for ships in offshore wind farm waters, as shown in Figure 1.

2.1. Natural Conditions. The influence of natural conditions on ship navigation is mainly due to the influence of visibility, wind, flow, and other factors on navigational safety.

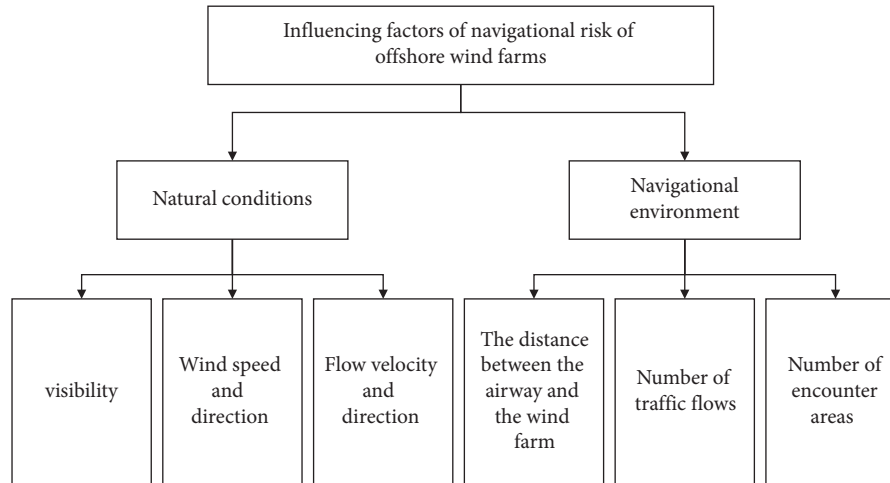


FIGURE 1: Hierarchical structure for offshore wind farm navigational waters' risk evaluation.

2.1.1. Visibility. Visibility is the maximum distance at which a person with normal vision can distinguish an object from the background. Too low visibility will greatly shorten the ship operator's response time and visual range, greatly increasing the risk of the ship operators. In 2018, there was more severe weather such as strong winds and fog in China. Under the condition of poor visibility, a total of 19 collision accidents occurred, and a total of 30 people were killed or missing, increasing 46.2% and 50.0%, respectively, year on year [18]. It can be seen that good visibility is an essential factor to ensure the safety of normal navigation.

2.1.2. Wind Speed and Direction. The influence of wind on ships' navigation can be divided into two aspects: the speed of the wind and the direction of the wind. Ships' navigation is more or less affected by wind direction, especially in gale weather, which may cause ships to yaw, increase, or stall to a certain extent. In addition to the speed of the wind, the direction is also an essential factor that cannot be ignored. In particular, the lateral wind pressure will cause obvious lateral deviation of the ship and even the risk of rollover, which will seriously affect the normal driving of the driver. In 2018, 16 self-sinking accidents occurred under the influence of magnitude of seven storms and waves, with a total of 53 deaths and missing people, increasing by 23.3% and 48.3%, respectively, year on year [18]. These data are enough to show that severe weather conditions such as strong winds and fog bring great threats to life safety and bring serious challenges to the maritime authorities' on-site supervision and emergency response.

2.1.3. Flow Velocity and Direction. Water flow is the most basic factor to ensure the normal sailing of ships. Many collision and grounding accidents are inseparable from the velocity and direction of water flow. When the traffic flow direction is in the same straight line as the ship's course, the current will only change the ship's speed and affect the difficulty of ship operation. When there is an angle between the direction of the flow and the course of the ship, the

course of the ship will be affected, resulting in lateral drift, which brings unpredictable risk to the ship's navigation. Especially when sailing in narrow waterways or special waters, the flow velocity and direction of water are the factors that should be paid attention to ensure the safety of ship driving (the velocity in this paper refers to the ratio of the flow velocity and the ship speed).

2.2. Navigational Environment

2.2.1. Horizontal Distance between the Route and Wind Farms. When the ship sails into the waters of the wind farm, the whole wind farm as a navigational obstacle will pose a potential threat to the ship. The ship will inevitably be affected by uncertain factors such as wind, wave, and flow in the actual sailing process, which will lead to deviation from the scheduled course and change the scheduled speed. The most serious consequence is that the ship cannot sail autonomously and has lateral drift due to mechanical failure during this period. Therefore, the sailing route near the wind farm must keep a certain safe distance from the wind turbine, that is, the transverse drift distance in Figure 2, which provides enough time for emergency braking of out of control of the ship to avoid collision accidents.

2.2.2. Number of Traffic Flows. The concept of traffic flow was first put forward in the study of road traffic engineering and applied to marine traffic engineering, where it is the total number of ships sailing along a certain channel. The difference is that the road traffic flow focuses on improving traffic efficiency, while the maritime traffic flow focuses on improving traffic safety [19], which makes the number of traffic flows as an essential evaluation index for assessing the risk of ship operation in the navigational environment.

2.2.3. Number of Encounter Areas. When a ship travels in the sea, it will inevitably meet with other ships. To avoid danger, the ship must take a series of measures. This kind of

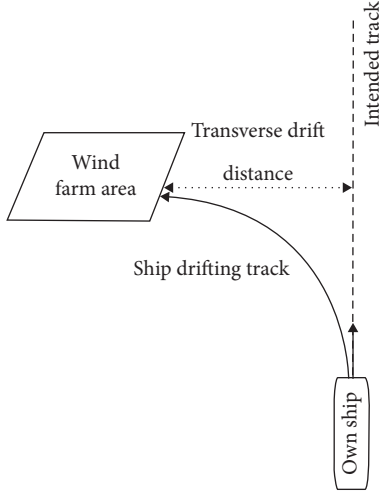


FIGURE 2: The position relationship between ship drifting with wind farms.

meeting of ships sailing on the sea and taking action is called “encounter.” The “encounter” generally includes head on, crossing, and overtaking (this paper mainly studies the cases of head on and crossing situations), and the number of ship encounters and space-time distribution can be used to evaluate the marine traffic risk degree.

3. Construction of the Risk Assessment Model for Offshore Wind Farm Waters

Fuzzy theory was proposed by Professor Zedeh, an automatic control expert from the University of California at Berkeley in the United States, in 1965. At present, the fuzzy theory has been successfully applied in the fields of automatic control, data processing, decision analysis, and pattern recognition [20]. Fuzzy inference is essentially a computational process that maps a given input space to a specific output space utilizing logic. Fuzzy inference system (FIS) is a computational structure based on the concept of fuzzy set theory, fuzzy if-then rules, and fuzzy logic inference. From the perspective of function, the FIS is mainly composed of four parts: fuzzification, fuzzy rule base, fuzzy reasoning method, and defuzzification [21]. The specific structure is shown in Figure 3.

3.1. Fuzzification Method. There are four common fuzzification methods, and the appropriate method can be chosen to construct a membership function according to the actual situation [21].

- (1) Fuzzy single-value method: let x^* be the exact measured value and \tilde{A}^* be the fuzzy set converted by the fuzzy single-value method; then,

$$\mu_{\tilde{A}^*}(x) = \begin{cases} 1, & x = x^*, \\ 0, & x \neq x^*. \end{cases} \quad (1)$$

- (2) Triangle membership function method: let x^* be a given exact quantity and \tilde{A}^* be the result of

fuzzification; then, the triangle membership function can generally be written as

$$\mu_{\tilde{A}}(x) = \begin{cases} \left(\frac{x-a}{b-a}\right), & a \leq x \leq b, \\ \left(\frac{c-x}{c-b}\right), & b \leq x \leq c, \\ 0, & \text{others.} \end{cases} \quad (2)$$

When the parameters $a=2$, $b=5$, and $c=8$, the membership function curve of the triangle distribution is shown in Figure 4.

- (3) Gauss membership function method: let x^* be a given exact quantity and \tilde{A}^* be the result of fuzzification; then, the Gaussian membership function can generally be written as follows:

$$\mu_{\tilde{A}}(x) = e^{-((x-x^*)^2/2\sigma^2)}. \quad (3)$$

In this formula, the parameter $\sigma > 0$ determines the width of the Gaussian function, and x^* is the center point of the Gaussian function. The Gaussian curve when $x^* = 2$ and $\sigma = 5$ is shown in Figure 5.

- (4) Trapezoidal membership function method: the shape of the membership function curve of the intermediate trapezoid distribution is determined by four parameters a , b , c , and d , and its expression is as follows:

$$f(x; a, b, c, d) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ \frac{d-x}{d-c}, & c \leq x \leq d, \\ 0, & d \leq x. \end{cases} \quad (4)$$

When the parameters $a=2$, $b=4$, $c=6$, and $d=8$, the membership function curve of the intermediate trapezoid distribution is shown in Figure 6.

3.2. Basic Form of Fuzzy Rules. The fuzzy rule base can be formulated according to the experience of experts. In this module, the extracted influencing factors can be combined to formulate risk degree rules for different combinations. On the deterministic domain X and Y , the one-dimensional fuzzy rules are expressed as follows:

$$\text{if } x \text{ is } \tilde{A}, \text{ then } y \text{ is } \tilde{B}. \quad (5)$$

The multidimensional fuzzy rules are expressed as follows:

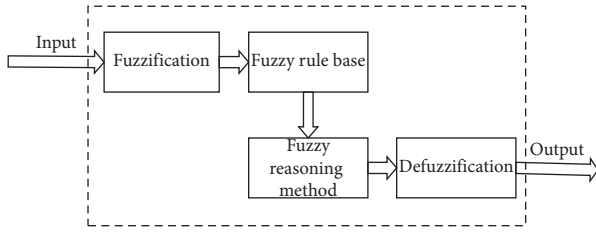


FIGURE 3: Fuzzy inference system structure diagram.

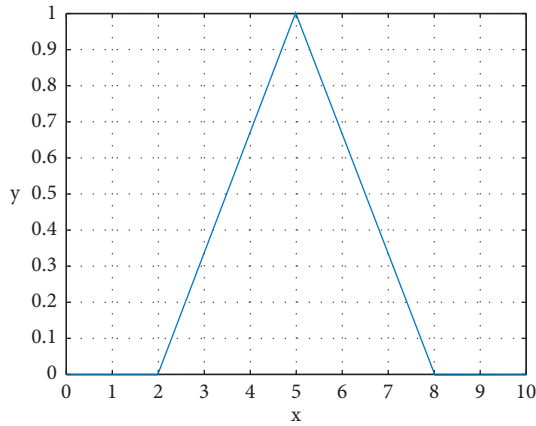


FIGURE 4: Triangle membership function method fuzzification process.

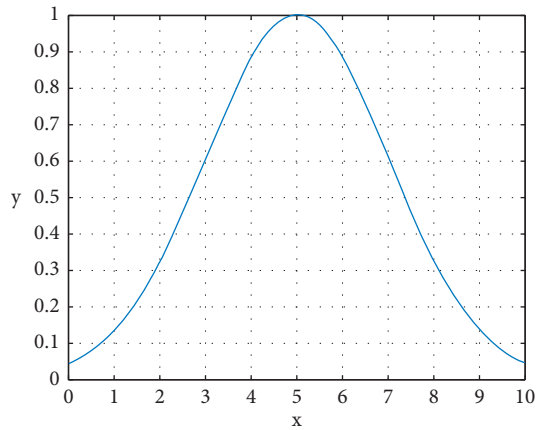


FIGURE 5: Gaussian membership function method.

if x_1 is \tilde{A}_1 and x_2 is \tilde{A}_2 and ... and x_n is \tilde{A}_n , then y is \tilde{B} . (6)

Among them, $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n$ are fuzzy sets on the domain X , and B is a fuzzy set on Y .

3.3. Integrity and Monotonicity of Fuzzy Rules. The set of fuzzy rules is strict and rigorous work. an incomplete or not monotonous fuzzy rule base will make the evaluation results in deviation. Usually, it is hard to get a complete and monotonous fuzzy rule base only from the expert experience; existing research has many ways to address this

question, such as the notion of a monotone fuzzy partition [22]. In this paper, genetic algorithm (GA) and approximate analogical reasoning schema (AARS) are combined to reduce the expert experience to be collected and meet the monotonicity:

Stage 1: GA is used to search for a small set of fuzzy rules gathered from expert experience

Stage 2: the remaining fuzzy rules are approximated by the similarity-based AARS [23]

3.4. Construction of the Risk Assessment Model. In order to quantitatively analyze the navigational risk of offshore wind farm waters, this paper extracts 6 influencing factors closely related to the overall navigational risk as input indexes and designs the model of navigational risk assessment in offshore wind farm waters based on the fuzzy inference system, as shown in Figures 7 and 8. After two layers of fuzzy inference, this model can obtain a quantitative navigational risk of ships in the waters of offshore wind farms.

Taking the second layer of fuzzy inference as an example [24], its fuzzy inference system is shown in Figure 9.

4. Model Validation

4.1. Introduction to Changle Offshore C Wind Farm. In this paper, the area C of the offshore wind farm waters in Changle, Fujian Province, is taken as an example to verify the validity of the model. The project is located in the eastern sea area of Changle of Fujian Province, the south bank of the Minjiang River estuary, and the waters on the northeast side of Pingtan Island. The site center is about 40 km away from the coastline of Changle, and the theoretical water depth is 41–47 m. There are a number of recommended coastal routes in this place [25], and since sailing ships do not strictly follow the recommended routes, a number of customary routes are formed, resulting in a high navigational density of ships and relatively complex traffic conditions.

4.1.1. Natural Conditions. This sea area is rich in wind energy resources and affected by the “narrow tube effect” of the Taiwan Sea, the annual average wind speed can reach 8 m/s, and the annual average flow velocity is recommended to be 0.5. According to statistics, the visibility in this sea area is good, and the visibility level can be set as 7. Although the situation in which the wind direction and flow direction are abscissa is of great risk to the safe navigation of the ship, it can be simplified in the modelling process due to the less frequent occurrence.

4.1.2. Navigational Environment. To study the actual distribution of ship traffic flow in the offshore waters of Changle, the AIS thermal map is shown in Figure 10. It can be seen that there are a total of 11 customary routes near the wind farm, namely, the number of traffic flows is 11, and the number of encounter areas is 6. The nearest route to this project is the connecting route of Songxia Port, which is 0.8 n mile.

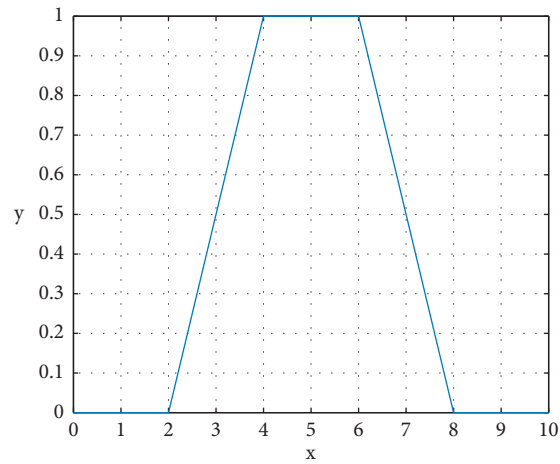


FIGURE 6: Trapezoidal Gaussian membership function.

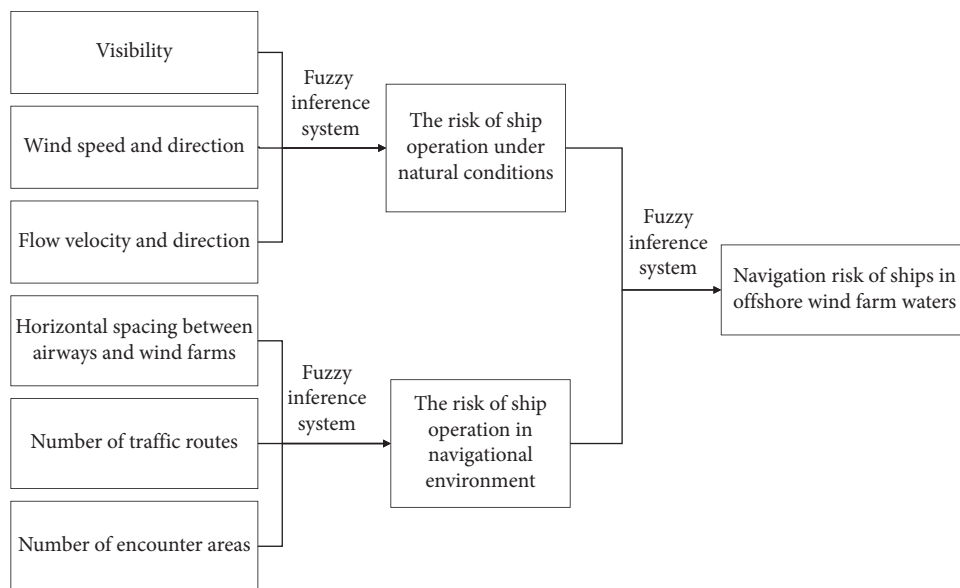


FIGURE 7: Assessment process of navigational risk in offshore wind farm waters.

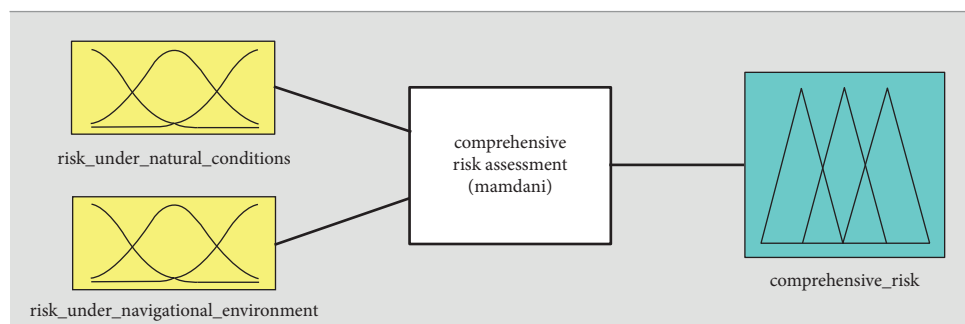


FIGURE 8: Navigational risk evaluation system for ships in the waters of offshore wind farms.

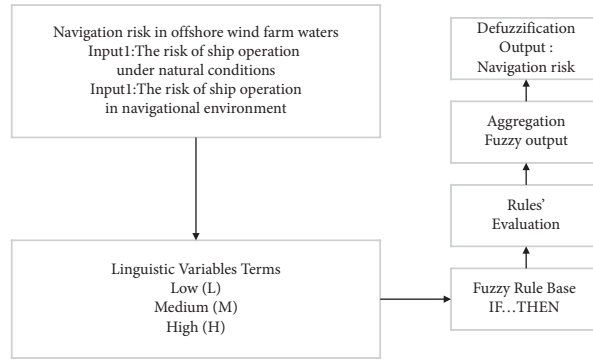


FIGURE 9: Fuzzy logic controller in navigational risk evaluation for ships in offshore wind farm waters.

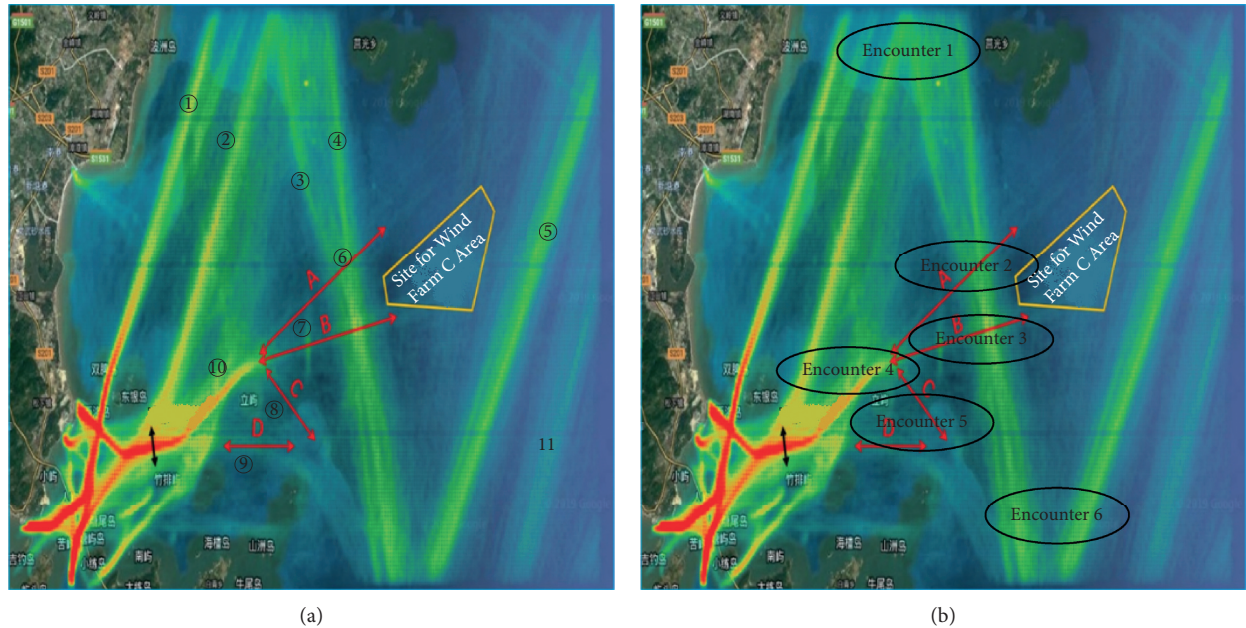


FIGURE 10: Thermal chart of ship trajectories in wind farm waters of area C. (a) Number of traffic routes. (b) Number of encounter areas.

4.2. Model Validation Results

4.2.1. The Risk of Ship Operation under Natural Conditions. According to the expert experience and actual influence, the fuzzy inference rule is established. It can be intuitive to see the relationship between various factors and ship operation risk under natural conditions, as shown in Figure 11. Then, the selected parameter values are brought into the risk model of sailing under natural conditions, and the exact value of the risk of sailing under natural conditions as shown in Figure 12 is “3,” which belongs to “medium risk.”

4.2.2. The Risk of Ship Operation in the Navigational Environment. According to the impact analysis of the number of ship encounter areas, number of traffic flows, and horizontal space between sailing routes and wind farms, we conclude that when the number of encounter areas and traffic flows increases, the navigational risk of wind farms increases, while the horizontal space between sailing routes and wind farms increases,

and the risk decreases. After consulting the ship officers and maritime experts, we build the membership function of each of two influencing factors, as shown in Figure 13.

When determining the safe distance between sailing routes and wind farms, scholars often refer to relevant international conventions, laws and regulations, industry regulations, or other research results. Considering comprehensively, the minimum safe distance between the sailing route and the wind farm is set as 2800 m [6] in this paper, and this is used as the critical value to define the risk under the influence of the distance between the route and the wind farm. The sailing conditions of Changle offshore sea are input into this fuzzy inference system, and the precise value of navigational risk under natural conditions is obtained, as shown in Figure 14; the value of risk is “5,” which belongs to “high risk.”

4.2.3. Comprehensive Risk Assessment. The comprehensive FIS is established based on the comprehensive consideration of the influence of natural condition sailing risk and

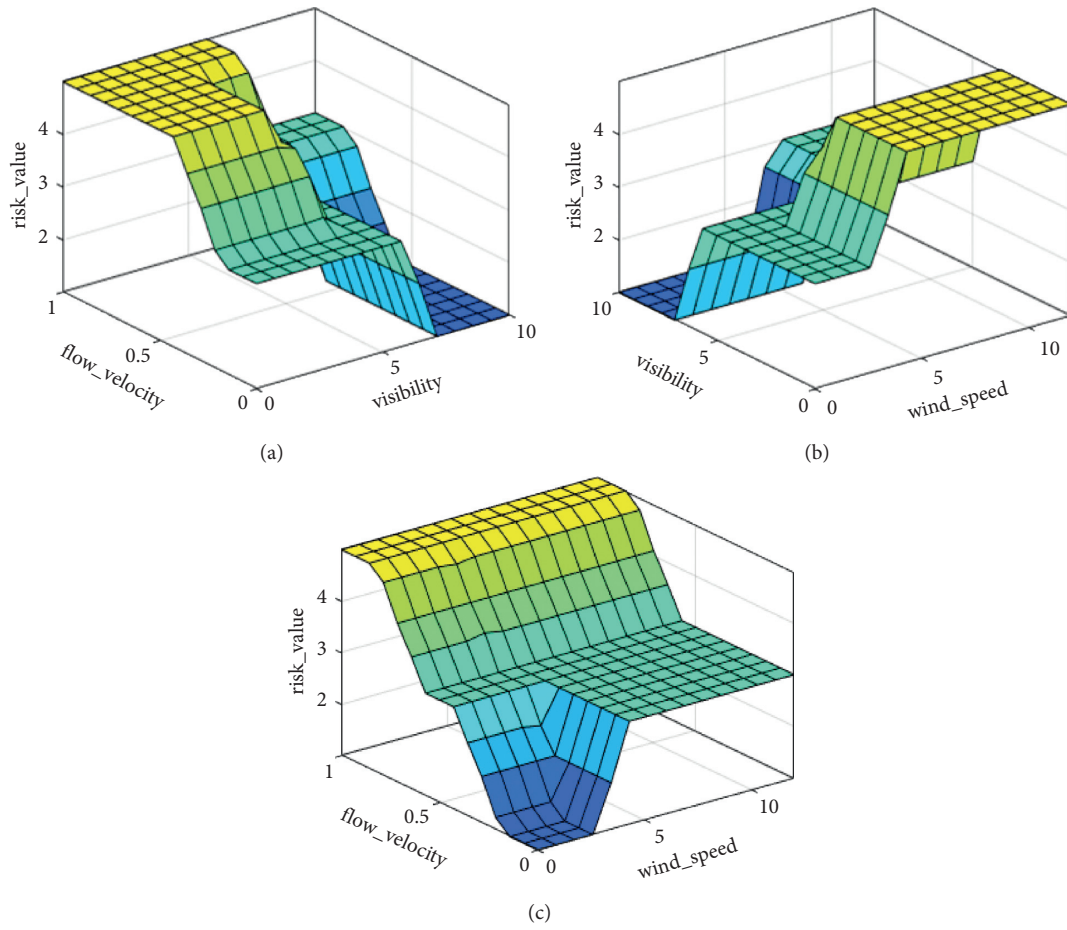


FIGURE 11: Natural conditions' risk membership function surface. (a) The relation surface between visibilities. (b) The relation surface between wind speed and visibility. (c) The relation surface between wind speed and flow visibility.

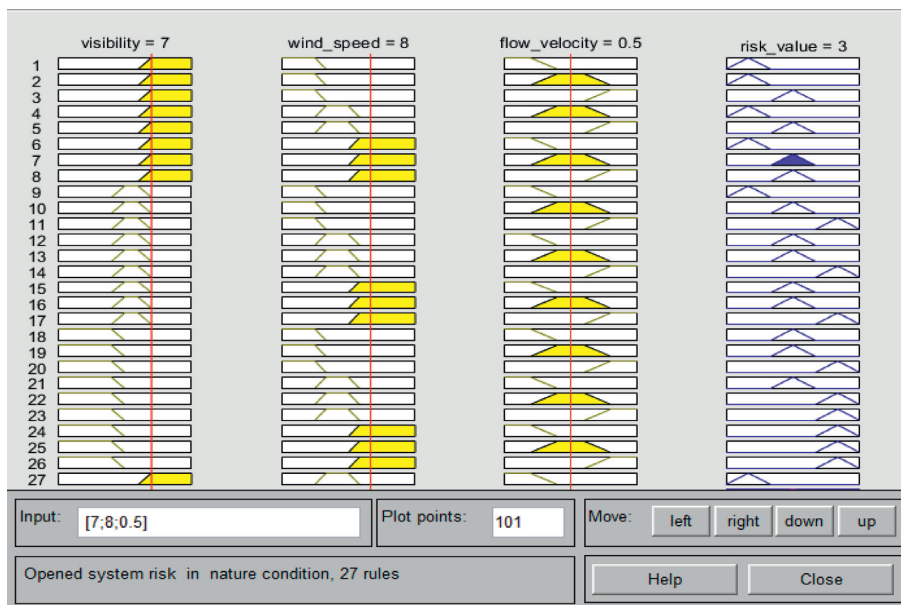


FIGURE 12: Navigation risk under natural conditions.

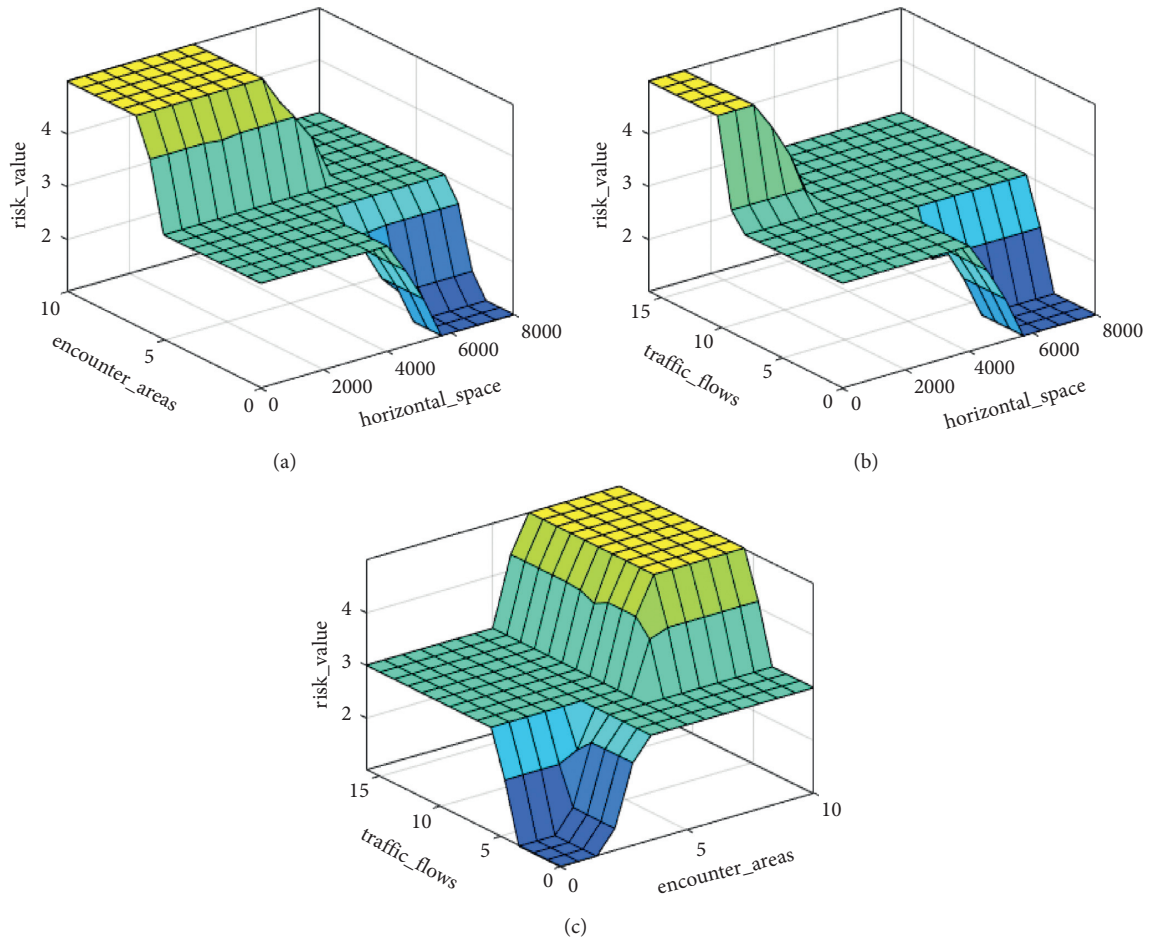


FIGURE 13: Navigational environment risk membership function surface. (a) The relation surface between the horizontal space and encounter area. (b) The relation surface between the horizontal space and traffic flow. (c) The relation surface between encounter area and traffic flow.

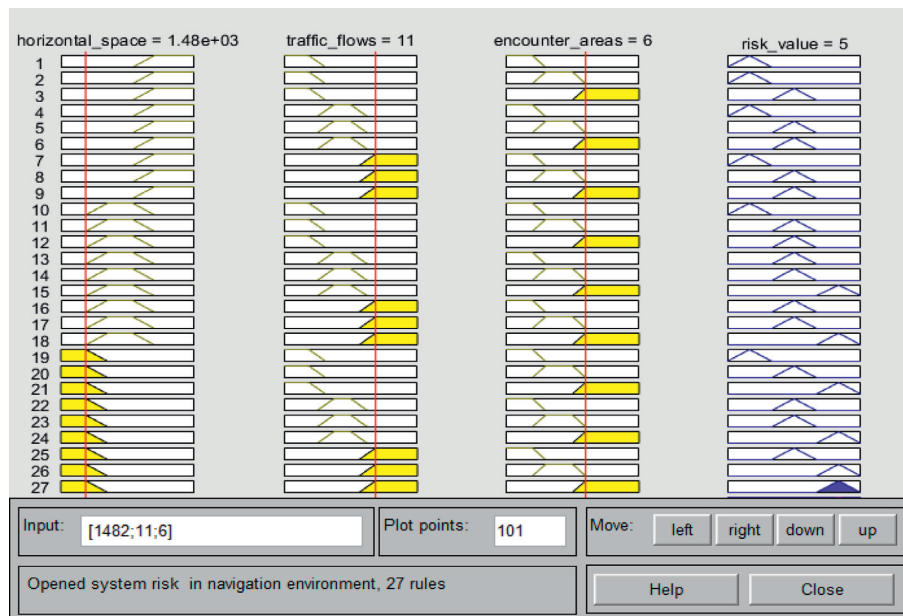


FIGURE 14: Navigational risk under the navigational environment.

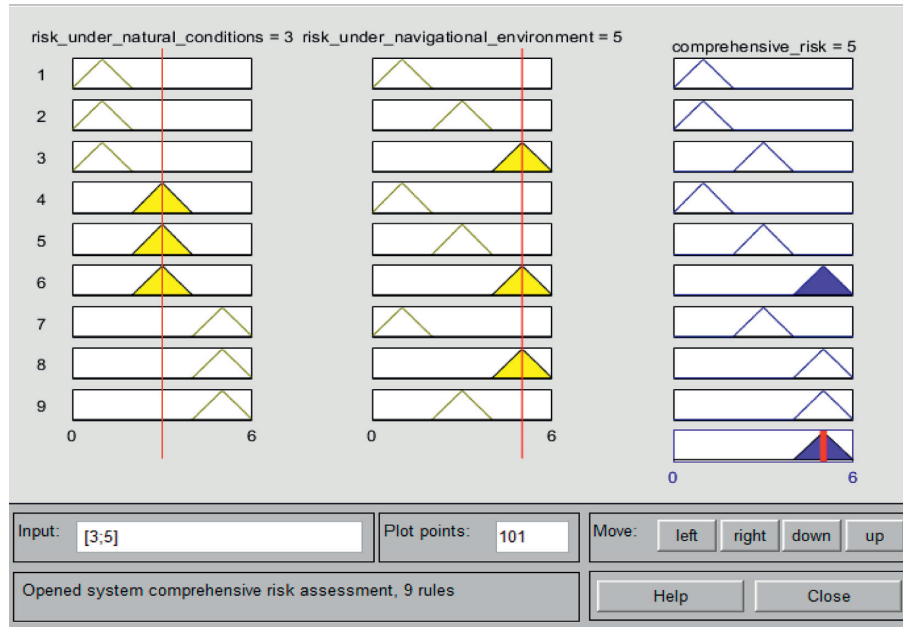


FIGURE 15: Original comprehensive navigational risk assessment.

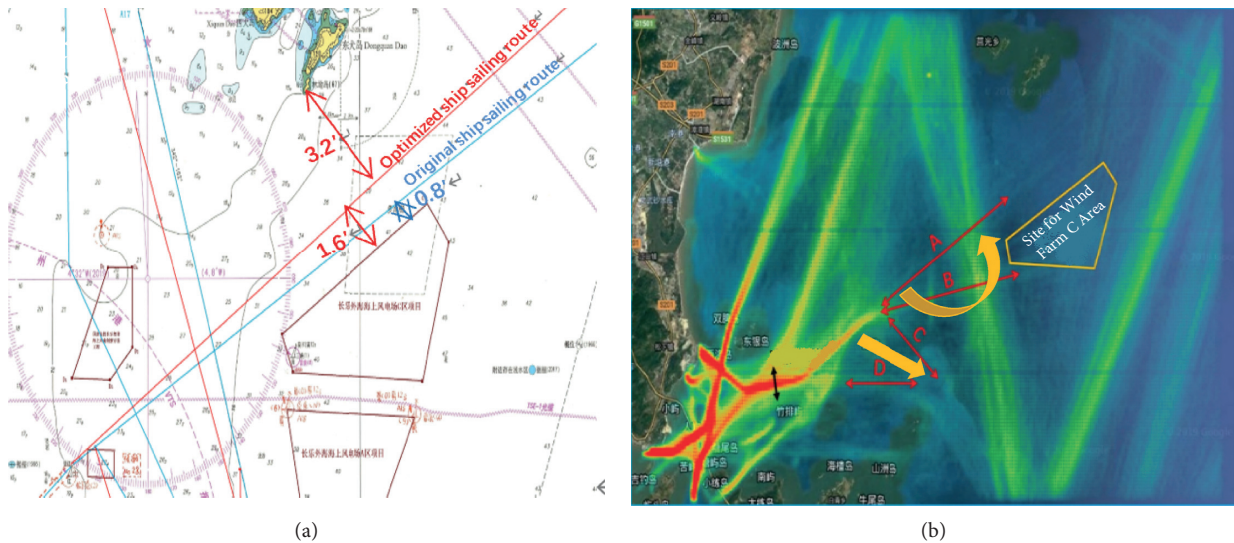


FIGURE 16: Optimized route distribution diagram. (a) Distance optimization between the airway and the wind farm. (b) Traffic flow number optimization.

navigational environment sailing risk on the overall navigational risk, and the obtained natural condition sailing risk “3” and navigational environment sailing risk “5” are input into the comprehensive fuzzy inference system, and the navigational risk of ships in the whole Changle offshore wind farm area of Fujian is finally obtained as “5.” These are “high-risk” navigational waters, as shown in Figure 15.

4.3. Route Optimization. The construction of the project of Changle Offshore Wind Farm in zone C has affected the navigation of ships in the area, especially the routes from Songxia Port to the main channel, and increased the

navigational risk of ships. It can be intuitively seen from the above model verification results that the navigational risk level of the waters near the project is as high as “5,” so it is necessary to optimize the nearby route.

Through investigation and analysis, as shown in Figure 16, the distance between Matsushita Port area’s connecting route and the wind farm can be adjusted to 1.6 n miles. At the same time, airway B can be merged into airway A, and airways C and D can be integrated into a traffic flow.

After the sailing route optimization, the number of traffic flows is reduced to 9, the number of encounter areas is reduced to 4, and the distance between the airways and the wind farm is increased to 1.6 n miles. The optimized data are

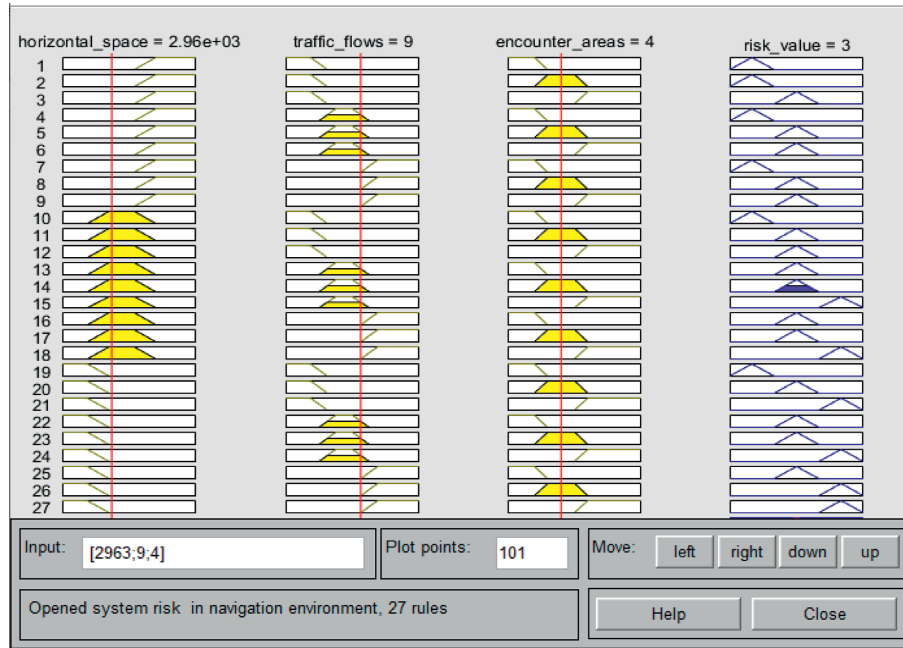


FIGURE 17: Navigational environment risk after sailing route optimization.

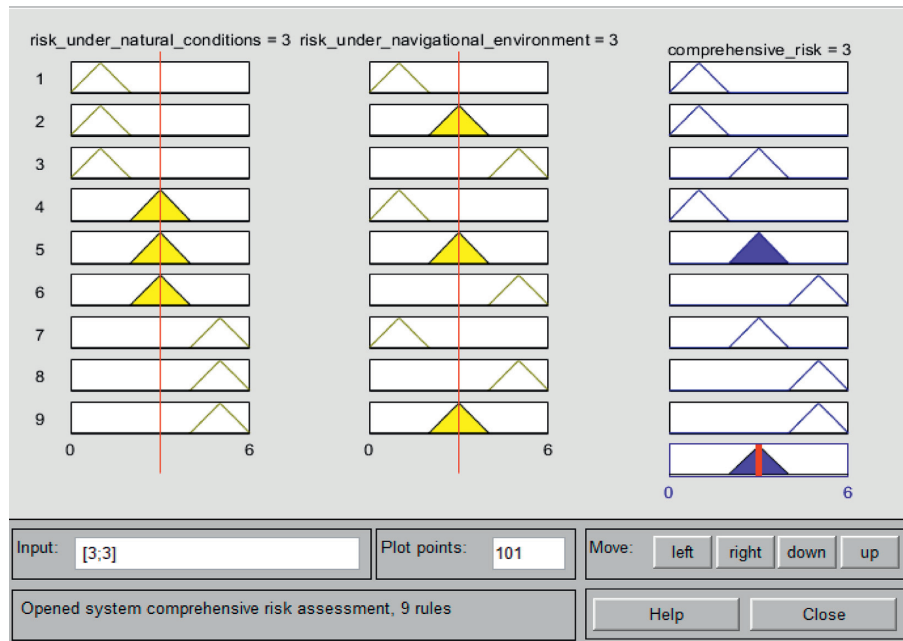


FIGURE 18: Comprehensive navigational risk assessment after route optimization.

resubstituted into the navigational risk model and the comprehensive risk assessment model, as shown in Figures 17 and 18; we find that the navigational risk of the ships in this area is reduced to “3,” belonging to “medium risk” navigational risk.

The above analysis shows that the evaluation method studied in this paper is feasible, and the fuzzy inference system can effectively calculate the navigational risk in the waters of offshore wind farms, and the calculated results meet the actual situation, thus verifying the rationality of the model.

5. Discussion

In order to verify the reliability of the model in this paper, other risk assessment methods, such as analytic hierarchy process (AHP) and fuzzy comprehensive evaluation, have been compared; we also implemented a comparative case study of the AHP in the same wind farm waters, and the results are shown in Figure 19. The blue decagon in the figure represents the risk value of each factor before optimization, and each corner represents the risk value of a factor. Similarly, red represents the risk value after optimization.

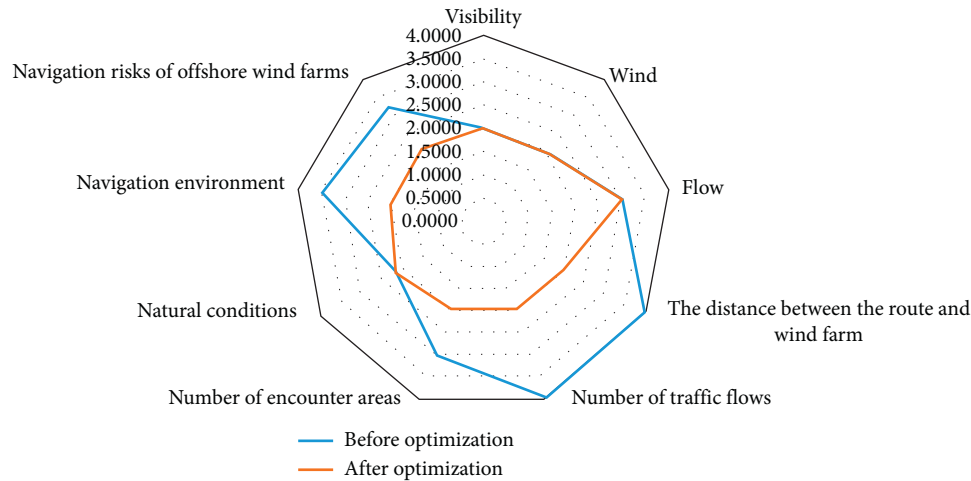


FIGURE 19: Evaluation of each risk index before and after route optimization using AHP and fuzzy comprehensive evaluation.

The research results showed that the AHP-based method has consistent results with the method proposed in this paper, and the evaluation value of each index obtained by the AHP and fuzzy comprehensive evaluation method has been improved before and after optimization. However, the AHP-based method depends on the subjective factors of people, the evaluation value obtained is easy to change, the calculation is complicated, and every scene needs to be recalculated. After the comparison, the FIS-based method proposed in this paper is more reliable and convenient for practical use.

6. Conclusions

In order to quantitatively evaluate the impact of the construction of offshore wind farms on the maritime navigational risk, this paper proposes a method of navigational risk assessment for offshore wind farms based on the fuzzy inference system. We analyze the influence of natural condition factors and navigational environment factors on the navigational environment. By extracting visibility, the number of traffic flows, the number of meeting areas, and distance between sailing routes and wind farms, the risk of natural condition and navigational environment operation in the navigational system of ships in the wind farm area is, respectively, evaluated. Considering the mutual influence of several factors, combined with the expert experience and established fuzzy inference rules, then finally the framework of navigational risk assessment based on the fuzzy inference system in offshore wind farm waters is formed. Taking the actual scene of wind farm water area C of the Changle coast of Fujian Province as an example, the proposed method is verified. The experimental results showed that the proposed method can quantitatively evaluate the navigational risk in the wind farm waters, which provides a theoretical basis for the optimization of sea routes and the supervision of ships' navigational risk. Compared with the existing research, the proposed navigational risk assessment method in offshore wind farm water based on the FIS can take all the influencing factors into account directly and give a visual perception of

the assessment results. In this paper, the natural conditions and navigational environment and their subfactors are considered as the main factors affecting the risk of wind farm waters for fuzzy inference system modelling. Actually, the factors of human and machine can raise the accident of ship's sailing in the waters of offshore wind farms, such as misjudgement, miscommunication between officers on the ship, and malfunction of navigation equipment of ships; we will take the above factors into account in the following research in the future.

Data Availability

The data presented in this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

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Research Article

The Adoption of Geographic Information Systems in the Public Sector of Saudi Arabia: A Conceptual Model

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The development of fuzzy sets in geographic information systems (GIS) arose out of the need to handle uncertainty and the ability of soft computing technology to support fuzzy information processing. Fuzzy logic is an alternative logical foundation coming from artificial intelligence (AI) technology with several useful implications for spatial data handling. GIS has been found to have a crucial role in the performance of public sector organizations (PSO). However, the literature shows no universal model to support and shed light on GIS adoption, which lessens the chances for effective GIS adoption and usage. Therefore, a new model is needed for successful adoption and eventual enhanced organization's performance. Thus, there is a need to investigate the factors that can bring about GIS adoption. Models for GIS adoption in literature are few and far between, and the few that exist are not applicable as they do not cover all the significant factors that can contribute to adoption success. Hence, this paper brought a GIS adoption model for PSOs to promote their performance. The model was developed through the extension of the Technology Acceptance Model (TAM) in addition to the DeLone and McLean's Success Model. The study involved interviews with ten experts in ranking the extracted factors, and data was analyzed through thematic analysis. On the basis of the obtained analysis findings, the fundamental factors were found to be significant in their effects, and GIS adoption sufficiently related to the overall performance. Thus, the study contributes to the body of knowledge by filling the gap in the literature.

1. Introduction

Geographic information system (GIS) has become a top field of study owing to its several applications in different organizations in various fields [1]. Studies dedicated to it have been carried out in Saudi Arabia concerning GIS improvements, uses, and developments; remote sensing; determination of rainfall; management of floods; public safety; and urban planning [2–4].

Fuzzy logic provides an approach that allows expert semantic descriptions to be converted into a numerical, spatial model to predict the location of something of interest to GIS. Thus, both topics are closely related to each other. GIS has been shown to have several benefits to different types of organizations and industries (small, medium, and

large), with the five significant benefits being saving costs and increasing efficiency of the organization, optimum decision-making, enhanced communication, record-keeping efficiency, and geographical variables management [1]. Additionally, GIS can be combined with other IS frameworks present in the organization to store, control, and retrieve datasets. It can be used in applications in numerous areas.

Despite the richness of literature, a study has yet to be conducted on developing a GIS adoption model and its outcomes in the Saudi context, particularly in its public sector organizations. This highlights the impetus behind the current investigation to contribute to the extant literature on Technology Acceptance Models. Similar studies carried out in the Western countries context have had no significant

outcomes (Cakar [5]). Based on Mukred et al. [6] study, technology adoption and transplant to another culture would result in distinct outcomes. One such study in the Saudi context was conducted by Alharbi [7], where UTAUT was used and documented to have less variance than that it had in the US, which means the application of the same model may differ in terms of its use outcomes based on culture and outcomes. In the same line of argument, Oliveira et al. [8] mentioned that each technology needs to have a tailor-made model for the context area to shed accurate light on its adoption. Each technology needs its specific variables, and thus the current study proposes a model of GIS tailored to a specific case, which is the Saudi public sector. The research contributes to validating existing theories and literature concerning technology adoption and acceptance, focusing on GIS adoption.

The proposed GIS adoption research model is developed based on existing technology acceptance theories, modified for the Saudi public sector. The study combines the following acceptance theories: Diffusion of Innovation (DOI)/Innovation Diffusion Theory (IDT), Technology Acceptance Model (TAM, TAM2), Theory of Reasoned Action (TRA), Unified Theory of Acceptance and Use of Technology (UTAUT), and lastly Enhanced Model of Innovation Adoption [9–12]. The theories mentioned above were developed to examine individual intention and behaviors when adopting new technologies, with some of the factors being determinants of technology usage at the individual level. Thus, they highlight the importance of individuals to the process and their crucial role in the organizations, working together toward goal achievement.

Technology adoption in developing countries is still in urgent need of investigation [13–17]. In the Saudi case, technology acceptance theories have been the focus of several studies, such as Al-Gahtani [18], who recommended focusing on the influence of new factors on the acceptance of new technologies. For example, GIS has been used to manage the environment and assets by keeping track of harmful contaminants and identifying the locations of police, fire trucks, and snowplows. Similarly, there has been a corresponding rise in public participation and interaction. For example, citizens communicate with local governments using web applications on the phenomenon's location, like potholes and crimes, and provide feedback on the developing policy decisions [19].

This study focuses on determining the factors that affect technology adoption, specifically GIS adoption, using a combination of theories to develop a model for factors testing among Saudi public organization employees. First, the study identifies new factors for the model development based on a thorough review of literature, after which the model is tested in government organizations. The proposed model and its testing are expected to contribute to the theory concerning technology acceptance, with a specific focus on GIS adoption.

The rest of the paper is organized as follows: GIS subsections are discussed in Section 2. The past related works are presented in Section 3, followed by factor extractions in Section 4. Finally, Section 5 is allocated for conceptual

framework development, while Section 6 is devoted to the discussion and interpretations, followed by the conclusion in Section 7.

2. Geographic Information Systems (GIS)

GIS refers to an information system that works toward processing geographical data; it is a computer system that captures, stores, integrates, manipulates, analyzes, and displays spatially referenced data. It is also used to standardize, store, analyze, and model data for novel output and display maps and reports (output) [20].

In addition, GIS covers different spatially connected information types in software, hardware, and data capture, management, analysis, and reporting. It assists organizations in determining resolutions to issues through data understanding and sharing [21, 22].

Moreover, GIS has several definitions in the literature, among which is a system that stores, retrieves, analyzes, and displays data geographically/spatially. Thus, it is akin to a more extensive information system concept that handles geographical, spatial, or geospatial data for spatiotemporal application and geography investigation. “Geographic” is a general term described as relating to “geography”—a discipline of science that investigates lands, characteristics, inhabitants, and Earth phenomena (The American Heritage Dictionary, 2006). This paper adopts the GIS abbreviation for systems and technologies instead of geographical knowledge (GI Science), which is a more extensive umbrella covering other theoretical studies [23].

The composition of GIS includes applications like automated mapping and management of facilities and land information systems. GIS is commonly used to summarize various computer-based applications involving gathering, modifying, analyzing, and displaying geographic information with other linked services [24].

The following subsections are dedicated to presenting the GIS use, components, and system benefits.

2.1. The Areas of Geographic Information System Usage.

All sciences relating to Earth-related geographical locations consider GIS as relevant. These fields include climate changes, disparities in temperature and population distribution among regions, and distribution of diseases, in light of their measurement and management [25]. In addition, GIS contributes to identifying the distribution of crime in various areas/regions, using technology to understand the dispersion of plants and animals. More importantly, GIS assists in locating and managing services and resources through Global Positioning System (GPS) [20].

Moreover, GIS is valuable to building, digging, and burying pipes and cables; locating oil; and other activities that need their geographical coordinates documented and monitored. The process tracking method assists in drawing up relevant and valuable information. It is stored in GIS, considering the entire location of activities and functions in a given area that needs to be determined. In other words, GIS is used daily to solve problems related economic regions,

healthcare, and construction of infrastructure. Moreover, it shares data and information in a way that is impossible if the same information is gathered from individuals [26].

Innumerable organizations in almost all areas make use of GIS to create mapping to communicate, analyze, and share information and find resolutions to issues all over the globe, which has led to changes in the way things are done. Moreover, GIS is used for interpreting data visualization—as exemplified by Google Maps, a web-based GIS mapping solution for navigation in daily life. In the case of organizations, they can include geographical data into their tasks involving design, optimization, planning, and maintenance. For example, relevant data would work toward optimizing telecom processes through optimal customer-relationship management and location services. Data from GIS is also used to determine and enhance road networks through the use of data intelligence, and this contributes to keeping the roads safe and enhancing the management of traffic [27].

In addition to the above, GIS data is used to evaluate the growth of urban areas and the direction of such growth. Its effective use can set the basis for future development while considering important specifics for building success. Such data is also used in transport management, with companies enabled to plan new roads/rail routes by incorporating environmental and topical data into the GIS platform. GIS applications collect data, which is later utilized to conserve and protect natural resources and the whole environmental condition. For instance, impact reports are invaluable for evaluating the level to which issues in the environment are contributed by the GIS integration [28].

Furthermore, GIS data helps develop farming methods efficiency and soil data analysis, enhancing food production in different global areas. The use of an efficient GIS protects the environment and helps minimize risks and manage catastrophic events. Such data is also utilized in web-based navigation maps to provide the public with the required information. Web maps are kept in current condition daily based on GIS data and are extensively used [29].

On the whole, GIS is useful for mapping, telecommunication services, network services, analysis of accidents and hotspots, urban planning, and traffic management, which in turn makes it valuable for environment analysis, agricultural applications, management and minimization of catastrophes, navigation, estimates of floods, managing natural resources, taxation, banking, surveys, asset management and maintenance, geology, and even irrigation [20, 29].

Because of its varying applications, GIS data has had a profound impact on industries, businesses, and society. Without GIS data, a significant difference would be felt in our daily personal and professional lives. Moreover, GIS technologies have been increasingly transforming from a mere tool for resolving limited application problems to an element of territorial growth in public administration areas in emerging nations [30].

2.2. Geographic Information System Components. There are six components of GIS, with the network being the topmost significant component. This is because the transfer of data

and digital information can only be possible through the network. Hardware is the second GIS component, and it comprises equipment required for operating GIS [26]. The third GIS component is software, and service providers like Environmental Systems Research Institute sell business packages. These programs differ from one GIS supplier to the next in light of applications, complexity, and data size. The database is the fourth GIS component within which the complete information is stored and is useable for decision-making and problem-solving. This is followed by budget, as the fifth GIS component, and it encapsulates budget, capability, and accuracy techniques for GIS administration and meeting the needs of stakeholders. Finally, GIS users make up the sixth component and supply and update the GIS digital data for efficient and up-to-date functionalities [26, 31].

One of the leading GIS software suppliers is Esri, and it generates the ArcGIS line of products—it created a web-based GIS (ArcGIS Online) and a server (ArcGIS Server) to provide solutions that are state-of-the-art through advanced technology [32]. In the past two decades, Geographical Information Systems Science is a phrase that has been reported to have different qualities (technical and intellectual) and aspects [33]. In this regard, the GIS is referred to as an application, tool, and science in three major formats [33].

The topmost significant GIS software types are desktop software, web mapping, GIS server, virtual globe, developer, and handheld GIS software. Among them, desktop GIS software is the most extensively utilized, introduced initially on PC using Microsoft Windows operating system. However, other GIS software systems have also been functional and have efficiency [26, 32].

2.3. The Benefits of Geographic Information System (GIS). GIS function involves integrating software, hardware, and data to acquire and manage geographic data and its analysis. The finalized geographic information is displayed so that the users can access it for an easy and quick understanding of data and decision-making [34].

Moreover, when it comes to pandemics, GIS is invaluable in examining the spatial distribution of infectious diseases [35] to combat the pandemic and enhance care quality. In addition, it has become an essential tool used to analyze and visualize the COVID-19 spread [36].

Specifically, GIS and its different modes have been utilized in the scientific research and planning of strategic healthcare activities and decision-making to map out the spread of COVID-19 in light of spatial and temporal distribution and diffusion. Plenty of helpful information has come up from the system's geographical, geospatial, and geostatistical analysis and applications that have proven valuable for interpreting the dynamics, patterns, clusters, and trends to support the planning and activities of the healthcare sector. To this end, similar types of applications have multiplied owing to the web's synergistic nature of sharing and spreading data and information. In particular, GIS has led to enhanced and refined IT capabilities, public safety, and enhanced and efficient healthcare response. GIS

can be logically applied to monitor diseases and prepare for disasters in healthcare institutions [37].

Additionally, GIS is useful in mapping the geographical disease distribution and transmission trends and creating a spatial model of its environmental aspects and occurrences. Concerning this, Murad and Khashoggi [20] developed a GIS application to generate mapping and cluster modeling of three diseases, diabetes, asthma, and hypertension, in Jeddah, in Saudi Arabia. The authors proved the capability of GIS to assist in updating and mapping health events and to support surveillance and decision-making concerning the monitoring of health conditions and diseases.

The GIS adoption environment offers benefits like access to various spatial functions. It matches rapid development and prototyping. The approach can be used to trace and treat patients on time and adopt preemptive actions in the same area. The GIS approach can also be used to inform others about the details of the spread of the disease [38].

GIS's advantages can be categorized into six major categories: cost-saving, efficiency increase, informed decision-making, communication enhancement, efficient record-keeping, and geographical management [21, 39]. Furthermore, according to Margolis and Pauwels [21], GIS can be combined with an IS model.

Different areas also leverage GIS for storing, controlling, and retrieving datasets in what is referred to as a layer—a collection of information regarding roads, seas, buildings, and the like. Layers are stored in a GIS database based on coordinates. When they are in one location, with the exact coordinates, they are connected in what is referred to as spatial joins between datasets. These joins can assist in informed decision-making based on stored data in the database. This may be exemplified by the location of roads that may be flooded because of a specific river with similar geographical coordinates [1].

Other government sectors have also used comparable technologies, with the Saudi Post being one of the most notable examples. The National Address Project in 2010 encouraged enterprises to embrace such technology by pressing them to leverage project outcomes through agreements with various government and private sectors. This also made it easier to get spatial data via the Internet, notably in the circles of system and application developers, to ease the use of accurate and authentic maps and spatial data [40, 41].

3. Past Related Works on GIS Adoption and Research Gap

GIS diffusion is distinct from other technology diffusion in extant literature owing to one significant reason; technology in this study is adopted at the organization level rather than at the individual level. Consequently, the need to study the adoption of GIS by the organization and the way it is diffused in the organization needs examining to meet the organization's needs.

Regardless of the innumerable studies conducted concerning the diffusion of GIS technology, only a few were developed for the nation. For instance, Eria and McMaster

[42] focused on GIS adoption in Uganda through a mixed data collection and analysis approach underpinned by the Diffusion of Innovation (DOI) model. They found that GIS adoption among the Ugandan institutions took place in a classic diffusion pattern aligned with the propositions of the diffusion theory. The decision to adopt GIS was made based on its relative advantage over and alignment with current technologies. Because of bureaucratic intervention and patronage-based societal norms, the adoption rate followed an S-shaped diffusion curve.

Developing countries' governments have been undergoing continuous reforms that have urged their adoption and usage of GIS with other ICTs for the governance of urban areas. However, few studies have been carried out on GIS spatial knowledge construction in countries other than those in the West, as highlighted in Mukherjee's research [22], where he examined Surat Municipal Corporation (SMC). This corporation is one of India's leading urban local bodies, implementing e-governance strategies that entailed spatial information and GIS. Alkhofani [15] used an integrated method to study the spatial knowledge construct via GIS based on literature and adoption and diffusion studies. It was demonstrated that such a structure is formed not only by the corporation's role in finding a niche within the government's priorities and agendas but also by influential players who have played a leading role in bringing about innovation and dynamic changes.

Abousaeidi et al. [43] examined GIS modeling adoption in a similar study line to determine the fastest routes to deliver fresh vegetables. Fresh vegetables lose their freshness during regular routes because of the time consumed and the environment's temperature. Despite this, only a narrow focus has been placed on transportation issues in most Kuala Lumpur areas, and because perishable food generally has a short shelf life, timely delivery has a significant influence on delivery costs. Thus, the selection of the proper routes would minimize the costs of total transportation. The study employed a regression model to determine the route choice parameters concerning the fastest fresh vegetable delivery. The authors adopted ArcGIS software with network analyst extension to resolve complex network issues. Their conclusion mapped the timeliest routes with the shortest delivery time based on all variables examined.

Hebert and Root [44] examined and discussed the potential role of GIS in infection control in healthcare environments. The authors provided an overview of the GIS role in public health and reviewed work concerning the applications of the method in the setting. Potential opportunities and challenges for GIS adoption for infection prevention were enumerated, and literature concerning the topic was reviewed to demonstrate GIS use in the healthcare workplace. The complexity discussion was documented using the nonadoption, abandonment, scale-up, spread, and sustainability (NASSS) framework. The authors then obtained the challenges and opportunities from the process, and multiple challenges involving the domains of technology, organization, and adaptation were underlined. A transdisciplinary method was proposed to address the challenges, and more studies involving prospective, reproducible clinical trials for

the optimal assessment of the potential influence and effectiveness of GIS in hospital settings were called for. There are many GIS adoption studies in various countries listed in Table 1.

GIS technology has drawbacks and flaws such as a lack of data, a lack of GIS training skills, errors in geocoding, and structural limitations in its use due to personal data confidentiality and privacy. This demonstrates that GIS technology adoption varies by organization, even among those performing the same activities.

In the case of Saudi public sector organizations, efforts are being made to improve performance and effectiveness through the use of IT advances, including GIS. In this regard, most IT application initiatives are limited to individual departments rather than the entire institution's operations. As a result, applications and databases are fragmented. Consequently, for adoption success, public sector organizations require a guiding model to follow. Public sector organizations, on the other hand, have similar business functions and structures. In this regard, a general model is required to guide their GIS application adoption as soon as possible. This research aims to develop and propose a GIS adoption model for organizations in the Saudi public sector. First, the paper reviewed the literature for the top factors that determine GIS adoption. Then, they were used to develop a model that is tailor-made for Saudi public sector organizations. The GIS adoption model contains the requirements for organizations in their quest to adopt the system, and it highlights the role that other technologies play in supporting it. Nine major factors are contained within the model: information quality, system quality, service quality, change management, competitiveness pressure, pandemic pressure, security, perceived ease of use, and perceived usefulness. The factors were examined in terms of their effects on GIS adoption and the public organizations' overall performance. The details of factors extractions and model constructions are discussed in the coming sections.

4. Factors Extractions

Extant literature has delved into factors that influence the attitude of individuals toward innovation adoption [55], and based on the reviewed relevant studies, several factors have this potential [10, 11].

Studies in literature have adopted several methods to determine the GIS adoption factors, with the main ones enumerated in Table 2. In addition, every method with its strengths and weaknesses, among other details, was mentioned in Khandelwal and Ferguson [67] study, as depicted in the table.

Figure 1 depicts the study methodology, which is divided into three stages: conducting a thorough literature review and identifying the critical factors; consulting experts on GIS factors, emphasizing the most important; and then using them to develop the study framework.

The most commonly used method for determining the factors influencing GIS technology adoption was identified as the literature review method [69, 70]. Extraction of factors

entails several steps, beginning with reviewing the literature and progressing to interviews with experts.

4.1. Factors from the Intensive Literature Review. Technology adoption studies, particularly GIS, were reviewed to identify the general factors used by the authors, and from the review, 57 factors were highlighted; these factors are tabulated in Table 3.

The authors conducted a content-based literature review through Google Scholar, Scopus, and Web of Science databases. Several keywords were used in this regard such as "GIS adoption," "GIS factors," "GIS models," "GIS and organization's performance," "GIS in Saudi Arabia," and "GIS benefits."

Frequency depicts the number of times the factors are cited in literature; however, this does not imply their ordinary and common nature [71, 72]. Table 4 tabulates the 20 factors of GIS adoption based on the frequency table.

Aside from Table 4, the other factors were not as frequently cited and were thus not included in the table.

4.2. Factors Ranked by Experts. The above factors were exposed to experts to obtain their feedback on them in light of their influence on the integrated GIS. Consequently, nine factors were viewed to be the significant factors that influence behavioral intention to use GIS and, in fact, actual usage. This study followed Hawking and Sellitto [73] and Ahmad and Cuenca [74] studies to determine the significance of the factors (refer to Table 5).

The experts are selected from related fields such as technology adoption, GIS engineers, IS data analysts, and academicians. Table 6 shows the profile for the experts.

Experts were interviewed to learn about their perspectives on GIS adoption. Along with the interviews, the experts were also asked to answer various questions in a questionnaire about the items in each factor. A total of nine factors were identified as the most important factors of behavioral intention to use GIS and, ultimately, its role in improving organizational performance.

This study proposes a GIS model, which encapsulates determinants of intention to adopt GIS and the adoption's perceived organizational performance. The study uses variables from DeLone and McLean (informational quality, system quality, and service quality), variables from TAM (perceived ease of use, perceived usefulness), and additional factors from literature (change management, security, competitiveness pressure, and pandemic pressure) as recommended by experts.

5. Conceptual Framework Development

In this paper, ten hypotheses were formulated to propose a framework on the influence of factors on the behavioral intention to adopt GIS; nine propositions were formulated for each of the exogenous factors. The final proposition (P10) addresses the influence of GIS adoption on organizational performance. In contrast, the first nine hypotheses address the effect of GIS factors, namely, system quality,

TABLE 1: Past related work on GIS adoption.

No.	Author	Country	Gap
1	[42]	Uganda	Scarce research has been done in the developing countries
2	[22]	India	Scarce research has been done in the developing countries
3	[43]	Malaysia	There is a need for more investigation of GIS adoption
4	[45]	Malaysia	The need for more investigation of GIS adoption
5	[46]	Turkey	There is a need for more investigation of GIS adoption
6	[47]	Thailand	There is a need for more investigation of GIS adoption
7	[30]	Ukraine	GIS is useful and needs to be implemented in different fields
8	[48]	Sweden	GIS needs more research focus
9	[49]	Mozambique	GIT adoption is still in its early stage and needs more research to be done
10	[50]	India	GIS could be integrated with other tools to help many organizations. However, the research is still in the early stage
11	[51]	Saudi Arabia	GIS is useful and could be implemented in different fields
12	[52]	Saudi Arabia	There is a need for more investigation, especially at the organizational level
13	[53]	United States	The GIS adoption is still in its infancy and needs more attention. Therefore, there should be new research to be conducted and new factors to be included
14	[54]	Saudi Arabia	The GIS adoption is still in its infancy and needs more attention. Therefore, there should be new research to be conducted and new factors to be included
15	[44]	United States	More research should be conducted to examine the impacts of GIS adoption

TABLE 2: Techniques for factor extractions.

Research method	Authors
Action-research	[56]
Case studies	[57]
Structured interviewing	[58]
Scenario analysis	[59]
Multivariate analysis	[60]
Literature review	[61]
Group interviewing	[57]
Focus groups	[62]
Delphi technique	[63, 64]
Combination of methods	[65, 66]

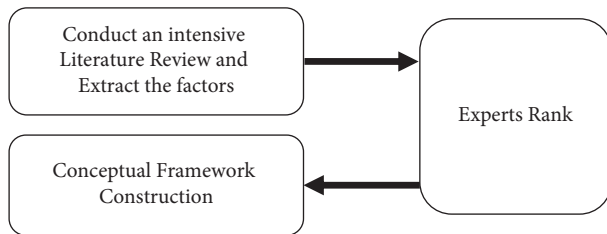


FIGURE 1: The methodology of the study as adopted from [68].

information quality, service quality, change management, competitiveness pandemic pressure, perceived ease of use, perceived usefulness, and security, on behavioral intention of adopting GIS. The factors must be aligned together to understand GIS adoption and, thus, such adoption's effect on the performance of the organization. GIS benefits cannot be fully reaped without matching the factors. The primary study constructs and their corresponding hypotheses are depicted in Figure 2 and Table 7.

Table 7 shows the propositions that the study comes up with.

TAM is the most extensively used model compared to the rest of the adoption models in the literature [75], and it has been examined in different contexts and versions. Lee

[76] claimed that it is the cited model for shedding light on technology acceptance in the past 20 years. Its intention-based model was exclusively developed for accepting IT.

When it comes to technology acceptance, TAM has been strongly established and well tested, and it has a robust ability to predict users' acceptance [77]. Users with different experience levels and various systems have been tested using TAM. It has been proved to have a high success rate in predicting technology use (40–70%) [78, 79].

In the context of public sector organizations, TAM was employed by many studies in the literature [80–82]. Specifically, in GIS adoption, TAM has been adopted by prior studies to examine acceptance/adoption with modifications to suit individual study objectives [79, 83–86].

Thus, in this study, TAM [9] is adopted as an underpinning model, with perceived ease of use (EAU) and perceived usefulness (PU) viewed as factors that affect intention to adopt GIS.

Furthermore, organizations generally adopt technologies for their employees' acceptance and usage. Therefore, in the present study, the attributes and characteristics of staff that affect their adoption of GIS are examined for enhanced performance of the whole organization. Based on the literature, models fall short of successfully explaining GIS adoption. This study addresses this literature gap by adopting theories and models to look into the adoption of GIS. The study proposes a GIS model capable of obtaining performance by analyzing a variety of factors that affect behavioral intention of adopting GIS. In sum, the study's underpinning models include TAM, and DeLone and McLean's IS Success Model, from which the study factors are extracted.

6. Discussion and Interpretation

The first exogenous latent variable in the brought forward conceptual model is system quality, which is described as the IS measurements [87], and it generates output in the

TABLE 3: Extracted factors from the intensive literature review.

Group of factors	Factors	Total of factors
Individual	Attitude, subjective norm, self-efficacy, satisfaction, motivation, personal beliefs, education, age, experience, gender	10
Technological	Trust, effort expectancy, perceived usefulness, features used, compatibility, privacy, information technology challenges, service quality, perceived ease of use, technological readiness, efficiency, reliability, IT infrastructure, interactivity, information quality, responsiveness, system quality	17
Organizational	Resources available, perceived financial cost, top management support, standardization, change management, outsourcing, social influence, facilitating conditions, training, effective communication	10
Environmental	Clear vision and planning, government role, policy, competitiveness pressure, security concerns, laws and legislations, pandemic pressure	7
Behavioral intention to use/adopt	Usage intentions, habit, intention to use, intention, user involvement, relationship with developers	6
Use/adopt GIS	Improved efficiency, user satisfaction, performance, output quality, organizational competence, perceived benefits, decision-making	7
Total		57

TABLE 4: Ranking of the extracted factors from literature review.

No.	Factor	Total
1	Top management support	33
2	User involvement	25
3	Perceived usefulness	30
4	Information quality	28
5	Effective communication	28
6	Clear vision and planning	27
7	Pandemic pressure	10
8	Perceived financial cost	25
9	Change management	25
10	System quality	25
11	Competitiveness pressure	23
12	Security	23
13	Policy	23
14	Service quality	20
15	Government role	19
16	Performance	17
17	Intention to use	17
18	Laws and legislations	17
19	Facilitating conditions	16
20	Perceived ease of use	28

TABLE 5: List of factors recommended by experts.

No.	Factor	Rank out of 5
1	System quality	4.2
2	Pandemic pressure	4.3
3	Information quality	4.2
4	Change management	4.3
5	Perceived ease of use	4.4
6	Security	4
7	Perceived usefulness	4.3
8	Service quality	4.1
9	Competitive pressure	4
10	Intention to use	4.1
11	Performance	4.3

information processing system. It also covers the processing quality of IS, with software and data components, and gauges the technical soundness of the system [88]. The system

quality is proposed to significantly affect IS effectiveness, which is the level to which IS can realize what it is meant to do. In GIS, system quality can be measured using appropriate access reports/services, and therefore the study proposes the following hypotheses for testing;

P1: System quality has a positive relationship with behavioral intention to adopt GIS.

With regard to information quality, it is described as a successful level to which IS meets the purpose of its creation [89]. Poor quality information can negatively affect the organization's tactics, strategies, and operations [88, 89]. The attributes of system quality are invaluable in assessing information processing systems, while those of informational quality assist in assessing the input and output of the system [90].

The generated quality of information from the system also has a hand in the perception of the system's usefulness; in other words, information quality has a significant relationship with system use according to the empirical findings reported by Mukred and Yusof [89].

P2: Information quality has a positive relationship with behavioral intention to adopt GIS.

The third factor proposed in the conceptual framework

is service quality. It refers to the difference between the user's normative expectations level of the service and his/her perception of the actual service performance level [90].

DeLone and McLean [91] updated the model viewed service quality as a new model dimension, and they categorized all the effect measures into one effect, known as a net benefit. The present study assesses the GIS service quality in light of the timely service provision, exemplary services, accurate services, and complete services. This study tests the following related proposition for this construct.

P3: Service quality has a positive relationship with behavioral intention to adopt GIS.

TABLE 6: Experts biography.

No.	Gender	Field and expertise	Affiliation
1	Female	Information science	King Saud University
2	Male	Technology adoption	University of King Abdulaziz
3	Male	Technology adoption	Ministry of Transport
4	Male	Technology adoption	Saudi Geological Survey
5	Male	Technology adoption and IS	State Properties General Authority
6	Female	Computer science	Ministry of Transport
7	Male	Information science	Department of Geography, Umm Al-Qura University
8	Male	Engineering	Saudi Geological Survey
9	Male	Engineering	Saudi Geographical Society
10	Male	Engineering	Ministry of Transport

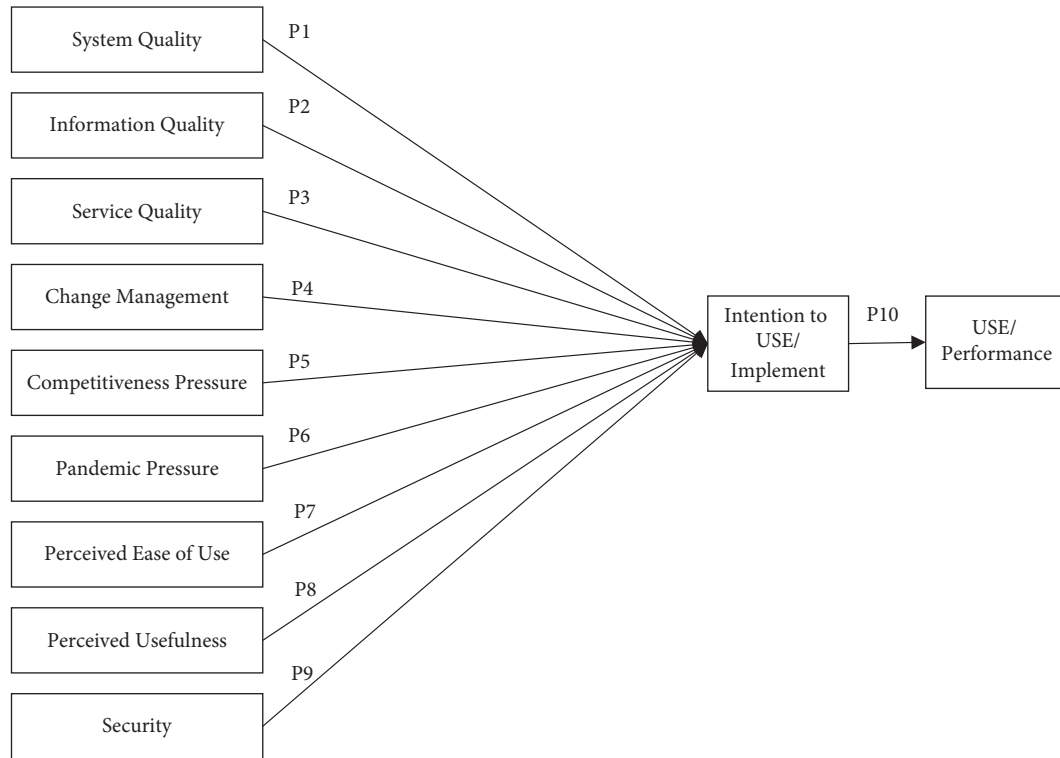


FIGURE 2: GIS proposed conceptual model and proposition.

TABLE 7: Research propositions.

No.	Proposition
P1	System quality has a positive relationship with the intention to use/implement GIS
P2	Information quality has a positive relationship with the intention to use/implement GIS
P3	Service quality has a positive relationship with the intention to use/implement GIS
P4	Change management has a positive relationship with the intention to use/implement GIS
P5	Competitiveness pressure has a positive relationship with the intention to use/implement GIS
P6	Pandemic pressure has a positive relationship with the intention to use/implement GIS
P7	Perceived ease of use has a positive relationship with the intention to use/implement GIS
P8	Perceived usefulness has a positive relationship with the intention to use/implement GIS
P9	Security has a positive relationship with the intention to use/implement GIS
P10	Intention to use/implement GIS has a positive relationship with use/performance

Moreover, change management is among the top evidence factors mentioned in the discussion, and the GIS application proves the changes wrought in the company. Essentially, change management is a method

used for effective management necessitating the shift from traditional models to current models. Change management is required for GIS as the employees need to be prepared to adapt to the changes. The

organization needs to have a formal change management initiative for issues resolution, such as employee resistance, redundancies, errors, and ambiguities, relating to implementing the new model [92]. Work-involved employees will be more privy to the benefits, enabling them to be more adept at using the new model, [93] and therefore the following hypotheses are proposed.

P4: Change management has a positive relationship with behavioral intention to adopt GIS.

Increased competitiveness at the local and global level is another factor that pressures organizations to seek ways to enhance their efficiency and effectiveness through strategies adoption [94]. In other words, dynamic competition and technological advancements throughout the world and digital technology developments have given governments a reason to take advantage of new methods to thrive. This has, in turn, led to the shift of government services from using traditional methods to using e-methods [95], and based on the above discussion of prior studies, this study proposes the following hypotheses.

P5: Competitiveness pressure has a positive relationship with behavioral intention to adopt GIS.

Still, another factor affecting the use of new technology is pandemic pressure, and using GIS calls for information and knowledge to be inculcated in users for solving various issues. User's training has a significant role in determining the overall GIS use success [96, 97]. Users armed with the knowledge of the new model concepts are more capable of being positively inclined to use it, are not adamant about resisting its use, and readily accept it. The involvement of users also helps in GIS configuration analysis, data conversion, and model testing [98].

Pandemic pressure requires an urgent training and is referred to as something that is a must for system adoption and for garnering benefits, and thus this study tests the following hypotheses.

P6: Pandemic pressure has a positive relationship with behavioral intention to adopt GIS.

In the TAM, perceived ease of use refers to an individual's conviction that utilizing a particular technology will be effortless [75]. Perceived ease of use refers to the manager's and employee's opinion of the minimal effort required to utilize GIS. Both intention to use and adoption are influenced by perceived ease of use [12].

Furthermore, several studies proved the significant effect of perceived ease of use on intention to use the system [99, 100]. Perceived ease of use significantly impacts intention to adopt the system [100]. Therefore, the following proposition is proposed to be tested.

P7: Perceived ease has a positive relationship with behavioral intention to adopt GIS.

Additionally, perceived usefulness refers to the degree to which an individual believes that utilizing a certain

system will improve his or her job performance [75]. In a parallel study, Vathanophas et al. [101] used TAM to assess naval officers' approval of Internet use. The data indicate that perceived utility has a major effect on users' intentions. Additionally, Venkatesh et al. [12] discovered that performance expectancy (perceived usefulness) and effort expectancy (perceived ease of use) are significant determinants of behavioral intention to adopt an information system.

Perceived usefulness is defined in this study as managers' and employees' perceptions of the GIS's usefulness. This characteristic was evaluated in terms of the system's capacity to improve performance, productivity, and effectiveness. Enormous empirical research has demonstrated that perceived usefulness influences an individual's inclination to utilize and adopt a system [102].

On the basis of the foregoing reasoning, the following proposition is put to the test.

P8: Perceived usefulness has a positive relationship with behavioral intention to adopt GIS.

Finally, even if the information system's security is started at the top, it affects every division of the firm. All of the company's records are accessible at any time and from any location. Because there is no way to know for sure if an organization is trustworthy when it comes to keeping confidential information and promoting information security, institutions such as banks, financial institutions, insurance companies, hospitals, and laboratories all keep personal and confidential records and credit card information that must be provided for Internet transactions [103].

The present study describes security as the features and processes that are followed in dealing with organizations' assets.

In connection with the above, several studies revealed that security significantly influences intention to adopt technologies [104, 105]. Based on this discussion, the following proposition is proposed by the researchers.

P9: Security has a positive relationship with behavioral intention to adopt GIS.

GIS adoption requires a plan and vision that could assist in directing the project and in developing a business model; in other words, the organizational strategy needs to be adopted after the implementation phase [72, 106]. The initiative needs to be viewed as the top priority, and decision-making relating to it needs to be conducted before its implementation through the management members' feedback [107].

Moving on to intention to use, we define it as the intention of the user to use new technology and the probability that the user will make use of a particular system [108]. TAM posits that intention to use influences actual new system use, and this was evidenced by Mukred et al. [82], who indicated that intention to behave determines actual usage. Therefore, it can be

stated that intention to use technology determines intention to adopt it.

Comparatively, business performance is the organization's responsibility to the shareholders and maximization of profits [96]. Therefore, as mentioned in prior studies, business performance conceptualization requires two dimensions: financial performance and market performance.

Evidently, behavioral intention directly affected GIS adoption [109] and thus, based on the discussion above, this study proposes the following proposition.

P10: Behavioral intention to adopt GIS has a direct significant and positive effect on Saudi's private sector GIS performance.

7. Conclusion

GIS plays a vital role in business management, which is why it is increasingly regarded as a key tool for improving organizational performance. However, determining the factors that contribute to new technology adoption is necessary, so the study examines the major issues related to the significant factors in adopting GIS among public organizations in Saudi Arabia. The focus of this study is on the level of importance of factors to the adoption of GIS; these factors have been obtained from theories and supported by reviewed literature and expert recommendations.

The underpinning theories for the study were TAM, and DeLone and McLean's Success Model. TAMs successfully explain 40% of the variance in behavioral intentions [98]. The model is appropriate for use in this study because TAM was used in IS adoption studies involving large-sized firms and institutions.

The paper went on to discuss the role of GIS in improving organizational performance as well as the significance of its model for adoption efficiency and effectiveness. Based on the reviewed literature, the paper detailed related system adoption issues in various contexts and the major role of GIS in organizations and as a firm asset promoting their performance as guided by the GIS adoption model.

Based on the reviewed literature and theories, the paper concluded a lack of studies concerning GIS adoption in public sector organizations. Thus, a model to guide such adoption is still lacking. The paper enumerated the limitations and gaps in the literature on GIS, emphasizing the importance of developing a GIS adoption model for successful adoption.

This study is expected to enrich the body of knowledge in the technology adoption field. The study comes up with new factors that might influence the GIS adoption in public sector organizations. Based on the literature review and experts, new factors and a new model are proposed. Nine major factors are contained within the model: information quality, system quality, service quality, change management, competitiveness pressure, pandemic pressure, security, perceived ease of use, and perceived usefulness.

Finally, the paper emphasizes the need for additional research into the role of GIS in performance in order to

develop an effective GIS adoption model. This study aims to create a model to guide GIS adoption and address issues that may arise during this process. According to the studies reviewed, there is a significant relationship between GIS adoption success and an effective model to guide such adoption.

Data Availability

Data are included in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Theta Omega Topological Operators and Some Product Theorems

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We introduce and investigate the concepts of θ_ω -limit points and θ_ω -interior points, and we use them to introduce two new topological operators. For a subset B of a topological space (Y, σ) , denote the set of all limit points of B (resp. θ -limit points of B , θ_ω -limit points of B , interior points of B , θ -interior points of B , and θ_ω -interior points of B) by $D(B)$ (resp. $D_\theta(B)$, $D_{\theta_\omega}(B)$, $\text{Int}(B)$, $\text{Int}_\theta(B)$, and $\text{Int}_{\theta_\omega}(B)$). Several results regarding the two new topological operators are given. In particular, we show that $D_{\theta_\omega}(B)$ lies strictly between $D(B)$ and $D_\theta(B)$ and $\text{Int}_{\theta_\omega}(B)$ lies strictly between $\text{Int}_\theta(B)$ and $\text{Int}(B)$. We show that $D(B) = D_{\theta_\omega}(B)$ (resp. $\text{Cl}_\theta(B) = \text{Cl}_{\theta_\omega}(B)$ and $D(B) = D_{\theta_\omega}(B) = D_\theta(B)$) for locally countable topological spaces (resp. antilocally countable topological spaces and regular topological spaces). In addition to these, we introduce several product theorems concerning metacompactness.

1. Introduction

In 1943, Fomin [1] introduced the notion of θ -continuity. For the purpose of studying the important class of H -closed spaces in terms of arbitrary filterbases, the notions of θ -open subsets, θ -closed subsets, and θ -closure were introduced by Velicko [2] in 1966, in which he showed that the family of θ -open sets in a topological space (Y, σ) forms a topology on Y denoted by σ_θ (see also [3]). The work of Velicko is continued by [3–26] and others. Hdeib [27] introduced the class of ω -closed sets by which he introduced and investigated the notion of ω -continuity. The family of all ω -open sets in (Y, σ) is denoted by σ_ω . It is known that σ_ω is a topology on Y which is finer than σ . Research related to ω -open sets is still a hot area of research [28–36]. In 2017, Al Ghour and Irshidat [37] introduced θ_ω -open subsets, θ_ω -closed subsets, and θ_ω -closure utilizing the topological spaces (Y, σ_θ) and (Y, σ_ω) . It is proved in [37] that σ_{θ_ω} forms a topology on Y which lies between σ_θ and σ , and that $\sigma_{\theta_\omega} = \sigma$ if and only if (Y, σ) is ω -regular. Also, in [37], ω - T_2 topological spaces were characterized via θ_ω -open sets. Authors in [35] introduced θ_ω -connectedness and some new separation axioms. Also, research in [37] was continued by various researchers in [28–31]. The notion of interior operators is important in the axiomatization of modal logics.

Judging from the importance of limit points in mathematical analysis, introducing a new limit point notion in any topological structure is still a hot area of research. The first goal of this paper is to introduce and investigate the concepts of θ_ω -limit points and θ_ω -interior points.

In general topology, several topological properties are not finitely productive, such as paracompactness, strong paracompactness, Lindelöfness, and metacompactness. The area of research regarding the problem “What conditions on (Y, σ) and (Z, δ) to insure that their product has property \mathcal{P} ” is still hot [38–45]. The second goal of this paper is to introduce several product theorems concerning metacompactness.

2. Preliminaries

From now on TS will denote topological space for simplicity. Let (Y, σ) and (Z, δ) be TS s and let $B \subseteq C \subseteq Y$ with C as nonempty. Then, B is called ω -open set in (Y, σ) [27] if for each $y \in B$, there is $M \in \sigma$ and a countable set $F \subseteq Y$ such that $y \in M - F \subseteq B$. The relative topology on C is denoted by σ_C , and the product topology on $Y \times Z$ is denoted by $\sigma \times \delta$. The closure of B in (Y, σ) (resp. (C, σ_C) , (Y, σ_ω)) is denoted by \overline{B} (resp. \overline{B}^C , \overline{B}_ω). A point $y \in Y$ is in θ -closure of B [2] ($y \in \text{Cl}_\theta(B)$) if for every $G \in \sigma$ with $y \in G$, $\overline{G} \cap B \neq \emptyset$. B is

called θ -closed [2] if $Cl_\theta(B) = B$. The complement of a θ -closed set is called a θ -open set. It is known that $\sigma_\theta = \sigma$ if and only if (Y, σ) is regular. A TS (Y, σ) is called ω -regular [37] if for each closed set C in (Y, σ) and $y \in Y - C$, there exist $G \in \sigma$ and $H \in \sigma_\omega$ such that $y \in G$, $C \subseteq H$, and $G \cap H = \emptyset$. In [37], the author defined θ_ω -closure operator as follows: a point $y \in Y$ is in θ_ω -closure of B ($y \in Cl_{\theta_\omega}(B)$) if for any $G \in \sigma$ with $y \in G$ we have $\overline{G}_\omega \cap B \neq \emptyset$. G is called θ_ω -closed if $Cl_{\theta_\omega}(G) = G$. The complement of a θ_ω -closed set is called a θ_ω -open set. A TS (Y, σ) is called metacompact [46] if every open cover of (Y, σ) has a point-finite open refinement.

The following sequence of definitions and theorems will be used in the sequel.

Definition 1 (see [47]). A TS (Y, σ) is called locally countable if for each $y \in Y$, there is $G \in \sigma$ such that G is countable and $y \in G$.

Definition 2 (see [48]). A TS (Y, σ) is called antilocally countable if each $G \in \sigma - \{\emptyset\}$ is uncountable.

Definition 3 (see [9]). Let (Y, σ) be a TS $B \subseteq Y$. A point $y \in Y$ is called θ -limit point of B if for each $G \in \sigma_\theta$ with $y \in G$, $G \cap (B - \{y\}) \neq \emptyset$. The set of all θ -limit points of B is called the θ -derived set of B and is denoted by $D_\theta(B)$.

Definition 4 (see [9]). Let (Y, σ) be a TS and $B \subseteq Y$. A point $y \in Y$ is called a θ -interior point of B if there exists $G \in \sigma$ such that $y \in G \subseteq \overline{G} \subseteq B$. The set of all θ -interior points of B is called the θ -interior of B and is denoted by $Int_\theta(B)$.

Theorem 1 (see [37]). If (Y, σ) is locally countable and $B \subseteq Y$, then $\overline{B} = Cl_{\theta_\omega}(B)$.

Theorem 2 (see [37]). If (Y, σ) is antilocally countable and $B \subseteq Y$, then $Cl_\theta(B) = Cl_{\theta_\omega}(B)$.

Theorem 3 (see [37]). For any TS (Y, σ) , $\sigma_\theta \subseteq \sigma_{\theta_\omega} \subseteq \sigma$.

Theorem 4 (see [2]). A TS (Y, σ) is regular if and only if $\sigma = \sigma_\theta$.

Theorem 5 (see [37]). Let (Y, σ) be a TS and $B \subseteq Y$. Then, B is θ_ω -open set if and only if for each $y \in B$, there exists $G \in \sigma$ such that

Definition 5. Let (Y, σ) and (Z, δ) be TSs and let $B \subseteq Y$. Then,

- (Y, σ) is called C -scattered if every $B \in \sigma^c - \{\emptyset\}$, there is $b \in B$ and a compact set K such that $b \in Int(K) \subseteq K \subseteq B$ [49]
- B is called strongly placed in $Y \times Z$ if for every $z \in Z$ and $H \in \sigma \times \delta$ with $B \times \{z\} \subseteq H$, there are $V \in \sigma$ and $W \in \delta$ such that $B \times \{z\} \subseteq V \times W \subseteq H$ [50]
- Y is called scattered relative to $Y \times Z$ if for each $B \in \sigma^c$, there exists $b \in B$ and $V \in \sigma_A$ such that \overline{V} is Lindelöf and strongly placed in $Y \times Z$ [50]

It is well known that if (Y, σ) and (Z, δ) are TSs and Y is C -scattered, then Y is scattered relative to $Y \times Z$ but not conversely.

Definition 6 (see [51]). A Hausdorff TS (Y, σ) is called ultraparacompact if every open cover of Y has a locally finite clopen refinement.

Ellis [51] showed that a Hausdorff space (Y, σ) is ultraparacompact if every open cover has a pairwise disjoint open refinement.

Theorem 6 (see [52]). Let $f: (Y, \sigma) \rightarrow (Z, \delta)$ be closed and continuous with (Y, σ) regular. If (Z, δ) is metacompact and $f^{-1}(z)$ is Lindelöf for each $z \in Z$, then (Y, σ) is metacompact.

Theorem 7 (see [50]). For any two TSs (Y, σ) and (Z, δ) , Y is strongly placed in $Y \times Z$ if and only if the projection $\pi: (Y \times Z, \sigma \times \delta) \rightarrow (Z, \delta)$ is closed.

Theorem 8 (see [35]). Let (Y, σ) and (Z, δ) be TSs and let $B \subseteq Y$. If B is strongly placed in $Y \times Z$ and $C \in \sigma \cap \sigma^c$, then $B \cap C$ is strongly placed in $Y \times Z$.

3. Theta Omega Limit Points

In this section, we explore the concept of θ_ω -limit points of a set and study its fundamental properties.

Definition 7. Let (Y, σ) be a TS and $B \subseteq Y$. A point $y \in Y$ is called θ_ω -limit point of B if for each $G \in \sigma_{\theta_\omega}$ with $y \in G$, $G \cap (B - \{y\}) \neq \emptyset$.

The set of all θ_ω -limit points of B is called the θ_ω -derived set of B and is denoted by $D_{\theta_\omega}(B)$.

The following result shows that θ_ω -derived set of a set B contains the derived set of B and contained in the θ_ω -derived set of B .

Theorem 9. Let (Y, σ) be a TS $B \subseteq Y$. The derived set of B is denoted by $D(B)$. Then, $D(B) \subseteq D_{\theta_\omega}(B) \subseteq D_\theta(B)$.

Proof. To see that $D(B) \subseteq D_{\theta_\omega}(B)$, let $y \notin D_{\theta_\omega}(B)$, then there exists $G \in \sigma_{\theta_\omega}$ such that $y \in G$ and $G \cap (B - \{y\}) = \emptyset$. By Theorem 3, $G \in \sigma$ and so $y \notin D(B)$. Therefore, we have $D(B) \subseteq D_{\theta_\omega}(B)$. To see that $D_{\theta_\omega}(B) \subseteq D_\theta(B)$, let $y \notin D_\theta(B)$, then there exists $G \in \sigma_\theta$ such that $y \in G$ and $G \cap (B - \{y\}) = \emptyset$. By Theorem 3, $G \in \sigma_{\theta_\omega}$ and so $y \notin D_{\theta_\omega}(B)$. Therefore, we have $D_{\theta_\omega}(B) \subseteq D_\theta(B)$.

The following example shows that the equality of each of the inclusions in Theorem 9 does not hold in general. \square

Example 1 (Example 2.26 of [37]). Let $X = \mathbb{R}$ and let $\sigma = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$. It is proved in [37] that $\sigma_{\theta_\omega} = \{\emptyset, \mathbb{R}, \mathbb{N}\}$ and $\sigma_\theta = \{\emptyset, \mathbb{R}\}$. Let $B = \{-n: n \in \mathbb{N}\} \cup \{0, 1\}$. Then, $D_\theta(B) = \mathbb{R}$, $D_{\theta_\omega}(B) = \mathbb{R} - \{1\}$, and $D(B) = \mathbb{Q} - \{1\}$.

Under the condition “regularity,” the θ_ω -derived set, the derived set, and the θ -derived set are all equal.

Theorem 10. Let (Y, σ) be a regular TS and $B \subseteq Y$. Then, $D(B) = D_{\theta_\omega}(B) = D_\theta(B)$.

Proof. It follows from Theorems 3, 4, and 9.

“Local countability” is a sufficient condition for the θ_ω -derived set and the derived set to be equal to each other: \square

Theorem 11. Let (Y, σ) be a locally countable TS and $B \subseteq Y$. Then, $D(B) = D_{\theta_\omega}(B)$.

Proof. By Theorem 9, we have $D(B) = D_{\theta_\omega}(B)$. To see that $D(B) = D_{\theta_\omega}(B)$, suppose to the contrary that there is $y \in D_{\theta_\omega}(B) - D(B)$. Since $x \notin D(B)$, there is $G \in \sigma$ such that $G \cap (B - \{y\}) = \emptyset$. By Theorem 1, $\text{Cl}_{\theta_\omega}(Y - G) = \overline{Y - G} = Y - G$ and so $G \in \sigma_{\theta_\omega}$. We conclude that $y \notin D_{\theta_\omega}(B)$, a contradiction.

“Antilocal countability” is a sufficient condition for the θ_ω -derived set and the θ -derived set to be equal to each other: \square

Theorem 12. Let (Y, σ) be an antilocally countable TS and $B \subseteq Y$. Then, $D_\theta(B) = D_{\theta_\omega}(B)$.

Proof. By Theorem 9, we have $D_{\theta_\omega}(B) \subseteq D_\theta(B)$. To see that $D_\theta(B) \subseteq D_{\theta_\omega}(B)$, suppose to the contrary that there is $y \in D_\theta(B) - D_{\theta_\omega}(B)$. Since $x \notin D_{\theta_\omega}(B)$, there is $G \in \sigma_{\theta_\omega}$ such that $G \cap (B - \{y\}) = \emptyset$. By Theorem 2, $\text{Cl}_\theta(Y - G) = \text{Cl}_{\theta_\omega}(Y - G) = Y - G$ and so $G \in \sigma_\theta$. We conclude that $y \notin D_\theta(B)$, a contradiction.

In Theorems 13–16, we give some natural properties for θ_ω -derived set. \square

Theorem 13. Let (Y, σ) be a TS. If $B \subseteq C \subseteq Y$, then $D_{\theta_\omega}(B) \subseteq D_{\theta_\omega}(C)$.

Proof. Let $y \notin D_{\theta_\omega}(C)$, there exists $G \in \sigma_{\theta_\omega}$ such that $y \in G$ and $G \cap (C - \{y\}) = \emptyset$. Since $B \subseteq C$, then $G \cap (B - \{y\}) = \emptyset$ and hence $y \notin D_{\theta_\omega}(B)$. It follows that $D_{\theta_\omega}(B) \subseteq D_{\theta_\omega}(C)$. \square

Theorem 14. Let (Y, σ) be a TS, and let A and B be subsets of Y . Then, $D_{\theta_\omega}(A) \cup D_{\theta_\omega}(B) = D_{\theta_\omega}(A \cup B)$.

Proof. By Theorem 13, $D_{\theta_\omega}(B) \subseteq D_{\theta_\omega}(A \cup B)$ and $D_{\theta_\omega}(A) \subseteq D_{\theta_\omega}(A \cup B)$. Therefore, $D_{\theta_\omega}(A) \cup D_{\theta_\omega}(B) \subseteq D_{\theta_\omega}(A \cup B)$. Now, let $y \notin (D_{\theta_\omega}(A) \cup D_{\theta_\omega}(B))$, then there exist θ_ω -open sets $G, H \in \sigma_{\theta_\omega}$ such that $y \in G \cap H$, $(G - \{y\}) \cap A = \emptyset$, and $(H - \{y\}) \cap B = \emptyset$. Let $W = G \cap H$. Then, $W \in \sigma_{\theta_\omega}$ and

$$\begin{aligned} (W - \{y\}) \cap (A \cup B) &= ((W - \{y\}) \cap A) \cup ((W - \{y\}) \cap B) \\ &= \emptyset \cup \emptyset \\ &= \emptyset. \end{aligned} \quad (1)$$

Thus, $y \notin D_{\theta_\omega}(A \cup B)$. \square

Theorem 15. Let (Y, σ) be a TS, and let A and B be subsets of Y . Then, $D_{\theta_\omega}(A \cap B) \subseteq D_{\theta_\omega}(A) \cap D_{\theta_\omega}(B)$.

Proof. By Theorem 13, $D_{\theta_\omega}(A \cap B) \subseteq D_{\theta_\omega}(B)$ and $D_{\theta_\omega}(A \cap B) \subseteq D_{\theta_\omega}(A)$. Then, $D_{\theta_\omega}(A \cap B) \subseteq D_{\theta_\omega}(A) \cap D_{\theta_\omega}(B)$.

The following example shows that the inclusion in Theorem 15 can not be replaced by equality in general. \square

Example 2 (Example 2.26 of [37]). Let $Y = \mathbb{R}$ and $\sigma = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$. It is proved in [37] that $\sigma_{\theta_\omega} = \{\mathbb{R}, \emptyset, \mathbb{N}\}$. Let $A = ((3/4), (3/2))$ and $B = ((5/4), (9/4))$. Then, $D_{\theta_\omega}(A) = \mathbb{R} - \{1\}$ and $D_{\theta_\omega}(B) = \mathbb{R} - \{2\}$, so $D_{\theta_\omega}(A) \cap D_{\theta_\omega}(B) = \mathbb{R} - \{1, 2\}$. On the other hand, $D_{\theta_\omega}(A \cap B) = D_{\theta_\omega}(((5/4), (3/2))) = \mathbb{R} - \mathbb{N}$.

Theorem 16. Let (Y, σ) be a TS and $B \subseteq Y$. Then, $D_{\theta_\omega}(D_{\theta_\omega}(B)) - B \subseteq D_{\theta_\omega}(B)$.

Proof. Let $y \in D_{\theta_\omega}(D_{\theta_\omega}(B)) - B$. Let $G \in \sigma_{\theta_\omega}$ with $y \in G$. Since $y \in D_{\theta_\omega}(D_{\theta_\omega}(B))$, $G \cap (D_{\theta_\omega}(B) - \{y\}) \neq \emptyset$. Choose $z \in G \cap (D_{\theta_\omega}(B) - \{y\})$. Since $z \in D_{\theta_\omega}(B)$ and $z \in G \in \sigma_{\theta_\omega}$, then $G \cap (B - \{z\}) \neq \emptyset$. Choose $w \in G \cap (B - \{z\})$. Since $w \in B$ and $y \notin B$, then $w \neq y$. Thus, $G \cap (B - \{y\}) \neq \emptyset$ and hence $y \in D_{\theta_\omega}(B)$.

The following example shows that the inclusion in Theorem 16 cannot be replaced by equality in general. \square

Example 3. Let $Y = \mathbb{R}$ and $\sigma = \{\mathbb{R}, \emptyset, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$. Let $B = ((3/4), (3/2))$. It is proved in [37] that $\sigma_{\theta_\omega} = \{\mathbb{R}, \emptyset, \mathbb{N}\}$. By Example 2, $D_{\theta_\omega}(B) = \mathbb{R} - \{1\}$. On the other hand,

$$\begin{aligned} D_{\theta_\omega}(D_{\theta_\omega}(B)) - B &= D_{\theta_\omega}(\mathbb{R} - \{1\}) - \left(\frac{3}{4}, \frac{3}{2}\right) \\ &= \mathbb{R} - \left(\frac{3}{4}, \frac{3}{2}\right). \end{aligned} \quad (2)$$

4. Theta Omega Interior Points

In this section, we explore the concept of θ_ω -interior points of a set and study its fundamental properties.

Definition 8. Let (Y, σ) be a TS and $B \subseteq Y$. A point $y \in Y$ is called a θ_ω -interior point of B if there exists $G \in \sigma$ such that $y \in G \subseteq \underline{G}_\omega \subseteq B$. The set of all θ_ω -interior points of B is called the θ_ω -interior of B and is denoted by $\text{Int}_{\theta_\omega}(B)$.

The following result shows that the θ_ω -interior of a set B contains the θ -interior B and contained in the θ_ω -interior of B .

Theorem 17. Let (Y, σ) be a TS and $B \subseteq Y$. Then, $\text{Int}_\theta(B) \subseteq \text{Int}_{\theta_\omega}(B) \subseteq \text{Int}(B)$.

The following example shows that each of the two inclusions in Theorem 17 cannot be replaced by equality in general.

Example 4 (Example 2.26 of [37]). Let $Y = \mathbb{R}$ and let $\sigma = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$. Let $A = \mathbb{N}$ and $B = \mathbb{Q}^c$. It is proved in [37] that $\sigma_{\theta_\omega} = \{\emptyset, \mathbb{R}, \mathbb{N}\}$ and $\sigma_\theta = \{\emptyset, \mathbb{R}\}$. We have

$\text{Int}_{\theta_\omega}(A) = \mathbb{N}$ but $\text{Int}_\theta(A) = \emptyset$. Also, we have $\text{Int}_{\theta_\omega}(B) = \emptyset$ but $\text{Int}(B) = \mathbb{Q}^c$.

Theorem 18. Let (Y, σ) be a TS and $B \subseteq Y$. If $G \in \sigma$ such that $G \subseteq \underline{G}_\omega \subseteq B$, then $G \subseteq \text{Int}_{\theta_\omega}(B)$.

Proof. It follows directly from the definition of $\text{Int}_{\theta_\omega}(B)$. θ_ω -interior is always open. \square

Theorem 19. Let (Y, σ) be a TS and $B \subseteq Y$. Then, $\text{Int}_{\theta_\omega}(B)$ is θ_ω -open.

Proof. By the definition of $\text{Int}_{\theta_\omega}(B)$ and Theorem 18, for every $y \in \text{Int}_{\theta_\omega}(B)$, there exists $G_y \in \sigma$ such that $y \in G_y \subseteq \underline{G}_y \subseteq \text{Int}_{\theta_\omega}(B)$. By Theorem 5, it follows that $\text{Int}_{\theta_\omega}(B) \text{ is } \theta_\omega\text{-open}$.

The following is a characterization of θ_ω -open via θ_ω -interior. \square

Theorem 20. Let (Y, σ) be a TS and $B \subseteq Y$. Then, B is θ_ω -open if and only if $B = \text{Int}_{\theta_\omega}(B)$.

Proof. Necessity: suppose that B is a θ_ω -open set. By the definition, we have $\text{Int}_{\theta_\omega}(B) \subseteq B$. To see that $B \subseteq \text{Int}_{\theta_\omega}(B)$, let $y \in B$. By Theorem 5, there exists $G \in \sigma$ such that $y \in G \subseteq \underline{G}_\omega \subseteq B$. Then, $y \in \text{Int}_{\theta_\omega}(B)$.

Sufficiency: suppose that $B = \text{Int}_{\theta_\omega}(B)$. Then, by Theorem 19, B is θ_ω -open.

The results in the rest of this section are some natural properties of θ_ω -interior. \square

Theorem 21. Let (Y, σ) be a TS and $B \subseteq Y$. Then, $\text{Int}_{\theta_\omega}[\text{Int}_{\theta_\omega}(B)] = \text{Int}_{\theta_\omega}(B)$.

Proof. Follows from Theorem 20. \square

Theorem 22. Let (Y, σ) be a TS and $B \subseteq Y$. Then, $Y - \text{Int}_{\theta_\omega}(B) = \text{Cl}_{\theta_\omega}(Y - B)$.

Proof. To see that $Y - \text{Int}_{\theta_\omega}(B) \subseteq \text{Cl}_{\theta_\omega}(Y - B)$, let $y \notin \text{Int}_{\theta_\omega}(B)$. Then, there is $G \in \sigma$ such that $y \in G$ and $\underline{G}_\omega \cap (Y - B) = \emptyset$, so we have $y \in G \subseteq \underline{G}_\omega \subseteq B$. This shows that $y \notin Y - \text{Int}_{\theta_\omega}(B)$. To see that $\text{Cl}_{\theta_\omega}(Y - B) \subseteq Y - \text{Int}_{\theta_\omega}(B)$, let $y \notin Y - \text{Int}_{\theta_\omega}(B)$. Then, $y \in \text{Int}_{\theta_\omega}(B)$, and so there is $G \in \sigma$ such that $y \in G \subseteq \underline{G}_\omega \subseteq B$. Therefore, we have $\underline{G}_\omega \cap (Y - B) = \emptyset$, and so $y \notin \text{Cl}_{\theta_\omega}(Y - B)$. \square

Theorem 23. Let (Y, σ) be a TS and $B \subseteq Y$. Then, $Y - \text{Cl}_{\theta_\omega}(B) = \text{Int}_{\theta_\omega}(Y - B)$.

Proof. By Theorem 22,

$$\begin{aligned} Y - \text{Cl}_{\theta_\omega}(B) &= Y - (Y - \text{Int}_{\theta_\omega}(Y - B)) \\ &= \text{Int}_{\theta_\omega}(Y - B). \end{aligned} \quad (3)$$

Theorem 24. Let (Y, σ) be a TS and let $A \subseteq B \subseteq Y$. Then, $\text{Int}_{\theta_\omega}(A) \subseteq \text{Int}_{\theta_\omega}(B)$.

Proof. Let $y \in \text{Int}_{\theta_\omega}(A)$. Then, there exists $G \in \sigma$ such that $y \in G \subseteq \underline{G}_\omega \subseteq A$. Since $A \subseteq B$, then $G \subseteq \underline{G}_\omega \subseteq B$. Thus, $y \in \text{Int}_{\theta_\omega}(B)$. \square

Theorem 25. Let (Y, σ) be a TS and let A and B be subsets of Y . Then, $\text{Int}_{\theta_\omega}(A) \cup \text{Int}_{\theta_\omega}(B) \subseteq \text{Int}_{\theta_\omega}(A \cup B)$.

Proof. By Theorem 24, we have $\text{Int}_{\theta_\omega}(A) \subseteq \text{Int}_{\theta_\omega}(A \cup B)$ and $\text{Int}_{\theta_\omega}(B) \subseteq \text{Int}_{\theta_\omega}(A \cup B)$. Thus, $\text{Int}_{\theta_\omega}(A) \cup \text{Int}_{\theta_\omega}(B) \subseteq \text{Int}_{\theta_\omega}(A \cup B)$. \square

Theorem 26. Let (Y, σ) be a TS, and let A and B be subsets of Y . Then, $\text{Int}_{\theta_\omega}(A \cap B) = \text{Int}_{\theta_\omega}(A) \cap \text{Int}_{\theta_\omega}(B)$.

Proof. By Theorem 24, we have $\text{Int}_{\theta_\omega}(A \cap B) \subseteq \text{Int}_{\theta_\omega}(A)$ and $\text{Int}_{\theta_\omega}(A \cap B) \subseteq \text{Int}_{\theta_\omega}(B)$. Thus, $\text{Int}_{\theta_\omega}(A \cap B) \subseteq \text{Int}_{\theta_\omega}(A) \cap \text{Int}_{\theta_\omega}(B)$. To see that $\text{Int}_{\theta_\omega}(A) \cap \text{Int}_{\theta_\omega}(B) \subseteq \text{Int}_{\theta_\omega}(A \cap B)$, let $y \in \text{Int}_{\theta_\omega}(A) \cap \text{Int}_{\theta_\omega}(B)$. Then, there exist $G, H \in \sigma$ such that $y \in G \subseteq \underline{G}_\omega \subseteq A$ and $y \in H \subseteq \underline{H}_\omega \subseteq B$. Let $W = G \cap H$. Then, $W \in \sigma$ and $y \in W \subseteq \underline{W}_\omega = \underline{G} \cap \underline{H}_\omega \subseteq \underline{G}_\omega \cap \underline{H}_\omega \subseteq A \cap B$. It follows that $y \in \text{Int}_{\theta_\omega}(A \cap B)$. \square

5. Metacompactness Product Theorems

In this section, we introduce several product theorems concerning metacompactness.

The following result will be used in the proof of Theorems 28 and 29.

Theorem 27. Let (Y, σ) and (Z, δ) be metacompact TSs. If for every $y \in Y$ there exists $W \in \sigma$ such that $y \in W$ and $\overline{W} \times Z$ is metacompact, then $(Y \times Z, \sigma \times \delta)$ is metacompact.

Proof. Let \mathcal{A} be an open cover of $(Y \times Z, \sigma \times \delta)$. For every $y \in Y$, choose $W_y \in \sigma$ such that $y \in W_y$ and $(\overline{W}_y \times Z, (\sigma \times \delta)_{\overline{W}_y \times Z})$ is metacompact. Since $\{W_y : y \in Y\}$ is an open cover of the metacompact TS (Y, σ) , then it has a point-finite open refinement $\{V_\beta : \beta \in \Gamma\}$. For each $\beta \in \Gamma$, $\overline{V}_\beta \times Z, (\sigma \times \delta)_{\overline{V}_\beta \times Z}$ is metacompact and has $\mathcal{M}_\beta = \{A \cap (\overline{V}_\beta \times Z) : A \in \mathcal{A}\}$ as an open cover, and hence \mathcal{M}_β has a point-finite open refinement \mathcal{H}_β . It is not difficult to see that $\{H \cap (V_\beta \times Z) : H \in \mathcal{H}_\beta, \beta \in \Gamma\}$ is a point-finite open refinement of \mathcal{A} . It follows that $(Y \times Z, \sigma \times \delta)$ is metacompact.

The following two product theorems concerning metacompactness will be used in the proof of Theorem 31 which is the main result of this section: \square

Theorem 28. Let (Y, σ) and (Z, δ) be regular metacompact TSs. If for every $y \in Y$ there exists $W \in \sigma$ such that $y \in W$, \overline{W} is strongly placed in $Y \times Z$, and $(\overline{W}, \sigma_{\overline{W}})$ is Lindelöf, then $(Y \times Z, \sigma \times \delta)$ is metacompact.

Proof. For each $y \in Y$, choose $W_y \in \sigma$ such that $y \in W_y$, \overline{W}_y is strongly placed in $Y \times Z$, and $(\overline{W}_y, \sigma_{\overline{W}_y})$ is Lindelöf. For every $y \in Y$, \overline{W}_y is strongly placed in $Y \times Z$ and so by Theorem 6, the projection function $\pi_y : (\overline{W}_y \times Z, (\sigma \times \delta)_{\overline{W}_y \times Z}) \rightarrow (Z, \delta)$ is a closed function. For every $y \in Y$, $(\overline{W}_y, \sigma_{\overline{W}_y})$ is Lindelöf and since $\pi_y^{-1}(z) = \overline{W}_y \times \{z\}$, then

$(\overline{W_y} \times \{z\}, (\sigma \times \delta)_{\overline{W_y} \times \{z\}})$ is Lindelöf. For every $y \in Y$, $(\overline{W_y}, \sigma_{\overline{W_y}})$ is metacompact and so by Theorem 6, $(\overline{W_y} \times Z, (\sigma \times \delta)_{\overline{W_y} \times Z})$ is metacompact. Thus, by Theorem 27, we have $(Y \times Z, \sigma \times \delta)$ is metacompact. \square

Theorem 29. Let (Y, σ) and (Z, δ) be metacompact TSs and let $B \subseteq Y$ such that B is closed in (Y, σ) , (B, σ_B) is Lindelöf, and B is strongly placed in $Y \times Z$. If for all $y \in Y - B$ there is $M \in \sigma_{Y-B}$ such that $(\overline{M}^{Y-B} \times Z, (\sigma \times \delta)_{\overline{M}^{Y-B} \times Z})$ is metacompact, then $(Y \times Z, \sigma \times \delta)$ is metacompact.

Proof. Let \mathcal{A} be an open cover of $(Y \times Z, \sigma \times \delta)$. For each $z \in Z$, $(M \times \{z\}, (\sigma \times \delta)_{M \times \{z\}})$ is Lindelöf with $M \times \{z\} \subset \bigcup \mathcal{A}$, and so there exists $\mathcal{A}_z \subseteq \mathcal{A}$ such that \mathcal{A} is countable and $M \times \{z\} \subseteq \bigcup \mathcal{A}_z$. Since M is strongly placed in $Y \times Z$, then for every $z \in Z$, there exist $U_z \in \sigma$ and $V_z \in \delta$ such that $M \times \{z\} \subseteq U_z \times V_z \subseteq \bigcup \mathcal{A}_z$. Since $\{V_z : z \in Z\}$ is an open cover of the metacompact TS (Z, δ) , then it has a point-finite open refinement $\{G_\beta : \beta \in \Gamma\}$. For each $\beta \in \Gamma$, choose $z(\beta)$ such that $G_\beta \subseteq V_{z(\beta)}$. Then, by Theorem 27 and the assumption, it is not difficult to see that $((Y - U_{z(\beta)}) \times Z, (\sigma \times \delta)_{(Y - U_{z(\beta)}) \times Z})$ is metacompact. Since $\{A \cap ((Y - U_{z(\beta)}) \times Z) : A \in \mathcal{A}\}$ is an open cover of $(Y - U_{z(\beta)}) \times Z$, then it has a point-finite open refinement \mathcal{A}_β . It is not difficult to check that

$$\begin{aligned} & \{A \cap (U_{z(\beta)} \times G_\beta) : A \in \mathcal{A}_{z(\beta)}, \beta \in \Gamma\} \\ & \cup \{G \cap (Y \times G_\beta) : G \in \mathcal{A}_\beta, \beta \in \Gamma\}, \end{aligned} \quad (4)$$

is a point-finite open refinement of \mathcal{A} . Therefore, $(Y \times Z, \sigma \times \delta)$ is metacompact. \square

Theorem 30. Let (Y, σ) be ultraparacompact and (Z, δ) be metacompact. Suppose there exists $D \subseteq Y$ such that D is closed in (Y, σ) and for every $y \in D$ there exists $W \in \sigma_D$ such that $y \in W$, \overline{W} is strongly placed in $Y \times Z$, and $(\overline{W}, \sigma_{\overline{W}})$ is Lindelöf, and for every $y \in Y - D$, there is $K \in \sigma_{Y-D}$ such that $y \in K$ and $\overline{K}^{Y-D} \times Y$ is metacompact. Then, $(Y \times Z, \sigma \times \delta)$ is metacompact.

Proof. By assumption there exists $\mathcal{A} \subseteq \sigma_D$ such that for all $A \in \mathcal{A}$, \overline{A} is strongly placed in $Y \times Z$ and $(\overline{A}, \sigma_{\overline{A}})$ is Lindelöf, and $\bigcup \mathcal{A} = D$. Since (D, σ_D) is ultraparacompact, then \mathcal{A} has a pairwise disjoint open refinement $\{C_\beta : \beta \in \Gamma\} \subseteq \sigma_D$. For every $\beta \in \Gamma$, choose $U_\beta \in \sigma$ such that $C_\beta = U_\beta \cap D$. Put $\mathcal{H} = \{U_\beta : \beta \in \Gamma\} \cup \{Y - D\}$. Since (Y, σ) is ultraparacompact and \mathcal{H} is an open cover of (Y, σ) , then \mathcal{H} has a pairwise disjoint open refinement $\{M_\gamma : \gamma \in \Delta\}$. For every $\gamma \in \Delta$, M_γ meets at most one member of $\{C_\beta : \beta \in \Gamma\}$. For every $\gamma \in \Delta$, let $(M_\gamma)^* = M_\gamma \cap (\bigcup \{C_\beta : \beta \in \Gamma \text{ and } C_\beta \cap M_\gamma \neq \emptyset\})$, then $(M_\gamma)^* = \emptyset$ or $(M_\gamma)^* = M_\gamma \cap \overline{A}$; for some $A \in \mathcal{A}$, it follows that $(M_\gamma)^*$ is closed in $(M_\gamma, \sigma_{M_\gamma})$ and $((M_\gamma)^*, \sigma_{(M_\gamma)^*})$ is Lindelöf and by Theorem 8; it is strongly placed in $M_\gamma \times Y$. By the assumption on $Y - D$ and Theorem 29, we conclude that $(M_\gamma \times Z, (\sigma \times \delta)_{M_\gamma \times Z})$ is metacompact. Since $Y \times Z = \bigcup_{\gamma \in \Delta} M_\gamma \times Z$ is metacompact, then $(Y \times Z, \sigma \times \delta)$ is metacompact.

Now, we are ready to state the main result of this section. \square

Theorem 31. Let (Y, σ) be ultraparacompact and (Z, δ) be regular and metacompact such that Y is scattered relative to the product $Y \times Z$, then $(Y \times Z, \sigma \times \delta)$ is metacompact.

Proof. Denote by $Y^{(0)} = Y$ and $Y^{(1)} = \{y \in Y : \text{there is no } U \in \sigma \text{ such that } y \in U \text{ and } \overline{U} \text{ is strongly placed in } Y \times Z \text{ and closure } (\overline{U}, \sigma_{\overline{U}}) \text{ is Lindelöf}\}$. If there is an ordinal $\alpha > 1$ such that $Y^{(\alpha)}$ has been defined and $\beta = \alpha + 1$, then $Y^{(\beta)} = (Y^{(\alpha)})^{(1)}$. If α is a limit ordinal, then $Y^{(\alpha)} = \bigcap_{\beta < \alpha} Y^{(\beta)}$. Since Y is scattered relative to $Y \times Z$, then there exists an ordinal α such that $Y^{(\alpha)} = \emptyset$.

The proof proceeds by transfinite induction on α . If $Y^{(1)} = \emptyset$, then for every $y \in Y$ there exists $U_y \in \sigma$ such that $y \in U_y$ and $\overline{U_y}$ is strongly placed in $Y \times Z$ and closure $(\overline{U_y}, \sigma_{\overline{U_y}})$ is Lindelöf. And by Theorem 28, $(Y \times Z, \sigma \times \delta)$ is metacompact. If $Y^{(\alpha+1)} = \emptyset$, then for every point $y \in Y^{(\alpha)}$ there exists $U_y \in \sigma_{Y^{(\alpha)}}$ such that $y \in U_y$, $\overline{U_y}$ is strongly placed in $Y \times Z$, and $(\overline{U_y}, \sigma_{\overline{U_y}})$ is Lindelöf and if $y \in Y - Y^{(\alpha)}$, choose a clopen set C_y such that $y \in C_y \subseteq Y - Y^{(\alpha)}$. Since $C_y^{(\alpha)} \subseteq (Y - Y^{(\alpha)})^{(\alpha)} = \emptyset$, then C_y is scattered relative to $C_y \times Z$ and (C_y, σ_{C_y}) is ultraparacompact, and by the inductive assumption, it follows that $(C_y \times Z, (\sigma \times \delta)_{C_y \times Z})$ is metacompact.

If $Y^{(\alpha)} = \emptyset$ for the limit ordinal α , then the open cover $\{Y - Y^{(\beta)} : \beta < \alpha\}$ has a pairwise disjoint open refinement $\{O_\gamma : \gamma \in \Gamma\}$. For each $\gamma \in \Gamma$, choose $\beta < \alpha$ such that $O_\gamma \subseteq Y - Y^{(\beta)}$. Therefore, $(O_\gamma)^{(\beta)} = \emptyset$, and hence $(O_\gamma \times Z, (\sigma \times \delta)_{O_\gamma \times Z})$ is metacompact. Since $Y \times Z = \bigcup_{\gamma \in \Gamma} O_\gamma \times Y$, it follows that $(Y \times Z, \sigma \times \delta)$ is metacompact. \square

Corollary 1. The product of an ultraparacompact C -scattered TS with a metacompact regular TS is again metacompact.

By the end of this paper, the authors found it is suitable to raise the following open question.

Question 1. Let (Y, σ) and (Z, δ) be regular and metacompact TSs such Y is scattered relative to the product $Y \times Z$. Is $(Y \times Z, \sigma \times \delta)$ metacompact?

6. Conclusion

In this work, the research via θ_ω -open sets is continued by introducing the notions of θ_ω -limit points and θ_ω -interior points. Several relationships regarding these two notions are introduced. Moreover, several product theorems concerning metacompactness are given. In future studies, the following topics could be considered: (1) define θ_ω -border, θ_ω -frontier, and θ_ω -exterior of a set using θ_ω -open sets and (2) try to solve Question 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Solving the Single-Valued Trapezoidal Neutrosophic Transportation Problems through the Novel Dhouib-Matrix-TP1 Heuristic

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The transportation problem has been widely studied in the field of supply chain management where circulation of products with a minimal transportation cost is an important issue. This paper presents the first adaptation of the Dhouib-Matrix-TP1 heuristic to solve the transportation problem in single-valued trapezoidal neutrosophic environment. Hence, the recently developed Dhouib-Matrix-TP1 heuristic is enriched with two functions to solve the neutrosophic transportation problem. On the one hand, a defuzzification function is exploited in order to convert the single-valued trapezoidal neutrosophic numbers to crisp numbers. On the other hand, an original metric function (*Average-Min*) is proposed with the intention of performing the nodes selection process. With an illustration from a literature example, we show to the decision maker the multiple advantages of the novel heuristic Dhouib-Matrix-TP1 which can be easily implemented in real-life industrial transportation under neutrosophic environment.

1. Introduction

The neutrosophic concept was introduced in 1995 by Smarandach [1] in order to make the chance of emulating the human thinking by using at the same time three membership functions: the Truth, the Falsity, and the Indeterminacy, noted, respectively, as (T) , (F) , and (I) . These three membership functions compose the neutrosophic set $\langle T, I, F \rangle$, where T , I and $F \in [0, 1]$.

The neutrosophic concept is becoming a major focus of several research papers. In fact, Ibrahim et al. [2] projected the idea of applying the neutrosophic analytical hierarchy process model to forecast the default of clients using five financial criteria (C_1 : the working capital, C_2 : the Liquidity, C_3 : the Profitability, C_4 : the Costs, and C_5 : the Customer obligation). The information was gathered in the form of neutrosophic datasets and evaluated in the credit department of one of private banks. In [3], the generalized assignment problem is considered in neutrosophic set theory, where elements of the cost matrix are presented as

trapezoidal fuzzy neutrosophic elements, and then, the problem is solved by the zero-suffix method. In [4], the assignment problem with costs as nonagonal number is optimized under three domains: fuzzy, intuitionistic, and neutrosophic. In [5], the neutrosophic assignment problem with the pentagonal neutrosophic number is solved using a new technique for ranking neutrosophic numbers into real numbers by the means of the magnitude function. In [6], the multicriteria assignment problem is solved using two methods where the different criteria have been considered as neutrosophic elements.

In [7], the order relation technique is applied to optimize the neutrosophic trapezoidal fuzzy assignment problem. In [8], the Branch and Bound method is used to optimize the neutrosophic assignment problem, where elements of the matrix are triangular fuzzy numbers. In [9, 10], the Branch and Bound technique and the Ones Assignment methods are, respectively, used to solve the neutrosophic Travelling Salesman Problem. Sikkannanl and Shanmugavel [11] introduce a new method to optimize the neutrosophic fuzzy

transportation problem using the Mean and Complete Contingency Cost Table.

The idea of trapezoidal neutrosophic is considered in [12], and the triangular fuzzy neutrosophic sets are proposed in [13]. Also, the Travelling Salesman Problem is solved under single-valued triangular neutrosophic parameters in [14]. A real-life TP under neutrosophic domain is solved in [15, 16], while the neutrosophic shortest path problem is optimized in [17]. Several operations on neutrosophic matrices are introduced in [18], and an application of the multicriteria group decision in a real-life problem is illustrated. The application of the neutrosophic N-structures to p-ideals of BCI-algebras is presented in [19]. The optimal hydrogen power plant site using a single-valued neutrosophic multiattribute decision-making technique is selected in [20].

In this paper, we focus to practically help the decision maker to handle a suitable solution for TP in neutrosophic domains and specially the case of single-valued trapezoidal numbers. Obviously, an easy and convivially neutrosophic decision support system is needed in order to depict graphically the crisp and neutrosophic solutions (graphical representation of the crisp and the single-valued trapezoidal neutrosophic solutions). Furthermore, this paper presents the first application of our novel heuristic Dhouib-Matrix-TP1 (DM-TP1) to solve the TP under the single-valued trapezoidal neutrosophic environment. The DM-TP1 was developed using Python programming language and enhanced with two techniques: on the one hand, the defuzzification score function in order to convert the neutrosophic trapezoidal fuzzy number to crisp number, and on the other hand, by the proposed original metric function (*Average-Min*) in order to drive the node (sources and destinations) selection process.

The remaining of this paper is organized as follows. Section 2 introduces the transportation problem. Section 3 presents some basic definitions on neutrosophic environment. Section 4 explains in detail the proposed heuristic DM-TP1. Section 5 depicts the stepwise application of the proposed method. Finally, conclusions with further research works are presented in Section 6.

2. The Transportation Problem

The transportation problem (TP) is a very well-studied topic in the field of supply chain management, and it deals with the minimization of the transportation cost of products from a certain number of sources to a certain number of destinations. The objective of the TP is to search for the optimal value of x_{ij} that will minimize the total transportation cost (see equation (1)) while satisfying the supply and demand restrictions (see equation (2)). The TP is mathematically formulated as follows.

Minimize

$$Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}, \quad (1)$$

which subjects to

$$\begin{aligned} \sum_{j=1}^n x_{ij} &= a_i; i = 1, 2, \dots, m, \\ \sum_{i=1}^m x_{ij} &= b_j; j = 1, 2, \dots, n, \\ x_{ij} &\geq 0 \text{ for all } i \text{ and } j. \end{aligned} \quad (2)$$

The notation of the TP is

m is the total number of supplies (sources)

n is the total number of demands (destinations)

a_i is the amount of supply at source i

b_j is the amount of demand at destination j

c_{ij} is the transportation cost from supply i to demand j

x_{ij} is the amount to be shipped from source i to destination j

The TP was firstly designed by Hitchcock in [21]. Charnes and Cooper in [22] proposed the Stepping Stone Method to find the optimal solution for TP. Dantzig, in [23], presented the simplex technique for the classical TP. Shell, in [24], introduced a solid transportation problem using three bounds: supply, demand, and conveyance that represent the different transport modes such as ships, goods train, trucks, and cargo flights. Haley, in [25], presented an enhanced version of the modified distribution method to optimize the solid transportation problem.

In real-life situations and in many cases, the decision maker has no precise information about the TP parameters: the transportation costs and the value of demands and supplies. In this situation, the corresponding elements defining the problem can be formulated by the means of fuzzy set, intuitionistic set, or neutrosophic set. Pandian and Natarajan, in [26], solved the TP under trapezoidal fuzzy numbers for all parameters. Kumar, in [27], designed the PSK method to solve the fuzzy TP type-1 and type-3. Mhaske and Bondar, in [28], solved the TP under triangular, pentagonal, and heptagonal fuzzy numbers and proposed the application of Lagrange's polynomial function for the nonagonal and hendecagonal fuzzy numbers. Dinagar and Thiripurasundari, in [29], solved the fuzzy TP under intuitionistic trapezoidal fuzzy numbers, and Sikkannan and Shanmugavel, in [30], used the magnitude ranking function with the Entire Contingency Cost Table method to solve the triangular fuzzy TP.

3. Basic Definitions

The neutrosophic concept allows to the decision maker, in real-world problems, to rely not only on true values but also on false ones as well as on indeterminacy membership.

Definition 1 (see [1]). Let us define by X the space of objects and define x as its generic element $x \in X$. The neutrosophic set N has the form $N = \{\langle x: T_N(x), I_N(x), F_N(x) \rangle, x \in X\}$, where the functions $T, I, F: X \rightarrow]^{-0}, 1^{+}[$ with the condition $^{-0} \leq T_N(x) + I_N(x) + F_N(x) \leq 3^{+}$.

Definition 2 (see [31]). The truth (T), indeterminacy (I), and falsity (F) membership functions for the neutrosophic trapezoidal fuzzy number are defined by

$$\begin{aligned} \mu(X) &= \begin{cases} \frac{(x-a)T_N}{b-a}, & a \leq x \leq b, \\ T_N & b \leq x \leq c, \\ \frac{(d-x)T_N}{d-c}, & c \leq x \leq d, \\ 0, & \text{otherwise,} \end{cases} \\ \theta(X) &= \begin{cases} \frac{b-x+(x-a)I_N}{b-a}, & a \leq x \leq b, \\ I_N & b \leq x \leq c, \\ \frac{x-c+(d-x)I_N}{d-c}, & c \leq x \leq d, \\ 0, & \text{otherwise,} \end{cases} \\ \lambda(X) &= \begin{cases} \frac{b-x+(x-a)F_N}{b-a}, & a \leq x \leq b, \\ F_N, & b \leq x \leq c, \\ \frac{x-c+(d-x)F_N}{d-c}, & c \leq x \leq d, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (3)$$

Here is an example (see Figure 1) of a graphical representation of a single-valued trapezoidal neutrosophic set $N = \langle (1, 3, 5, 7); 0.9, 0.2, 0.4 \rangle$.

Definition 3 (see [15]). Let us assume two neutrosophic trapezoidal fuzzy numbers $N = \langle (N_a, N_b, N_c, N_d); T_N, I_N, F_N \rangle$ and $M = \langle (M_a, M_b, M_c, M_d); T_M, I_M, F_M \rangle$. Then, their operations are defined as follows:

$$\begin{aligned}
M + N &= \langle (M_a + N_a, M_b + N_b, M_c + N_c, M_d + N_d); T_M \wedge T_N, I_M \vee I_N, F_M \vee F_N \rangle, \\
M - N &= \langle (M_a - N_d, M_b - N_c, M_c - N_b, M_d - N_a); T_M \wedge T_N, I_M \vee I_N, F_M \vee F_N \rangle, \\
M \times N &= \begin{cases} \langle (M_a \times N_a, M_b \times N_b, M_c \times N_c, M_d \times N_d); T_M \wedge T_N, I_M \vee I_N, F_M \vee F_N \rangle & \text{if } M_d > 0, N_d > 0, \\ \langle (M_a \times N_d, M_b \times N_c, M_c \times N_b, M_d \times N_a); T_M \wedge T_N, I_M \vee I_N, F_M \vee F_N \rangle & \text{if } M_d < 0, N_d > 0, \\ \langle (M_d \times N_d, M_c \times N_c, M_b \times N_b, M_a \times N_a); T_M \wedge T_N, I_M \vee I_N, F_M \vee F_N \rangle & \text{if } M_d < 0, N_d < 0, \end{cases} \\
\frac{M}{N} &= \begin{cases} \left\langle \left(\frac{M_a}{N_d}, \frac{M_b}{N_c}, \frac{M_c}{N_b}, \frac{M_d}{N_a} \right); T_M \wedge T_N, I_M \vee I_N, F_M \vee F_N \right\rangle & \text{if } M_d > 0, N_d > 0, \\ \left\langle \left(\frac{M_d}{N_d}, \frac{M_c}{N_c}, \frac{M_b}{N_b}, \frac{M_a}{N_a} \right); T_M \wedge T_N, I_M \vee I_N, F_M \vee F_N \right\rangle & \text{if } M_d < 0, N_d > 0, \\ \left\langle \left(\frac{M_d}{N_a}, \frac{M_c}{N_b}, \frac{M_b}{N_c}, \frac{M_a}{N_d} \right); T_M \wedge T_N, I_M \vee I_N, F_M \vee F_N \right\rangle & \text{if } M_d < 0, N_d < 0, \end{cases} \\
k * N &= \begin{cases} \langle (k * N_a, k * N_b, k * N_c, k * N_d); T_N, I_N, F_N \rangle & \text{if } k > 0, \\ \langle (k * N_d, k * N_c, k * N_b, k * N_a); T_N, I_N, F_N \rangle & \text{if } k < 0, \end{cases} \\
N^{-1} &= \left\langle \left(\frac{1}{N_d}, \frac{1}{N_c}, \frac{1}{N_b}, \frac{1}{N_a} \right); T_M, I_M, F_M \right\rangle, \text{ where } N \neq 0.
\end{aligned} \tag{4}$$

Definition 4 (see [16]). For a single-valued neutrosophic number $N = \langle (N_a, N_b, N_c, N_d); T_N, I_N, F_N \rangle$, its score function is defined by $M(\tilde{c}_{ij}^N)$ as follows:

$$\begin{aligned}
M(\tilde{c}_{ij}^N) &= \left(\min_{1 \leq i \leq n} (\mu_{c_{ij}}^N) + \min_{1 \leq i \leq n} (\vartheta_{c_{ij}}^N) + \max_{1 \leq i \leq n} (1 - \lambda_{c_{ij}}^N) \right) \\
&\times \sum_{i=1}^m \sum_{j=1}^n \left(\frac{S(\tilde{c}_{ij}^N x_{ij})}{\mu_{c_{ij}}^N + (1 - \vartheta_{c_{ij}}^N) + (1 - \lambda_{c_{ij}}^N)} \right), \tag{5}
\end{aligned}$$

where

$$\begin{aligned}
S(\tilde{c}_{ij}^N) &= \left(\frac{1}{16} \right) \times (N_a + N_b + N_c + N_d) \\
&\times (T_N + (1 - I_N) + (1 - F_N)). \tag{6}
\end{aligned}$$

4. The Proposed Method: Dhoub-Matrix-TP1 (DM-TP1)

Very recently, we invent in [32] a new constructive method entitled Dhoub-Matrix-TSP1 (DM-TSP1) to solve the classical Travelling Salesman Problem (TSP). The proposed method is based on several rules, and we handle it to solve the trapezoidal fuzzy TSP using the magnitude ranking function [33]. Then, in [34], we enhance the DM-TSP1 method with the α -Cut Technique to optimize the octagonal fuzzy TSP. Moreover, in [35], we generate a stochastic version, namely, the Dhoub-Matrix-TSP2 to solve the TSP.

In [36], we introduce a novel heuristic entitled Dhoub-Matrix-TP1 (DM-TP1) to solve the classical transportation problem (TP).

In this paper, we focus on the application of enhanced version of the DM-TP1 to solve the neutrosophic trapezoidal fuzzy TP. In fact, the DM-TP1 was presented in [18] using the standard deviation metric, whereas in this paper, we introduce a new original metric the (*Average-Min*) metric in order to drive the selection of sources and destinations. A stepwise application of this metric will be presented in the next section.

Furthermore, we enrich the DM-TP1 with the defuzzification score function [16] in order to convert the neutrosophic trapezoidal fuzzy parameters to crisp number. Thus, the score function for the neutrosophic trapezoidal fuzzy number $N = \langle (N_a, N_b, N_c, N_d); T_N, I_N, F_N \rangle$ is defined by equations (5) and (6).

The proposed method DM-TP1 is accomplished into nine steps (see Figure 2). Steps 1, 2, and 9 are executed only once, and Steps 3, 4, 5, 6, 7, and 8 are repeated until all columns are discarded.

Step 1 : transform the single-valued trapezoidal neutrosophic parameters to crisp parameters using the defuzzification functions described in equations (5) and (6).

Step 2 : balance the sum of supplies and demands for the transport matrix by adding fictive row and column with corresponding quantity. Next, in the transportation matrix, we add a row below,

entitled the Average Min Demand Column (AMDC), and insert a column at right, named the Average Min Supply Row (AMSR).

- Step 3 : compute the original function (*Average-Min*) for each row which affects the corresponding value for each element in the AMSR.
- Step 4 : apply the same operation on each column. Compute the original proposed function (*Average-Min*) for each column. Then, it affects the corresponding value for each element of the AMDC.
- Step 5 : identify the highest element among the AMSR and AMDC; if it is in AMSR, then select the minimal element (x_{ij}) of its corresponding row else check the minimal element (x_{ij}) of its corresponding column.
- Step 6 : if $a_i \leq b_j$ then allocate the a_i amount of units to x_{ij} , which affects $b'_j = b_j - a_i$, and discard row i .
- Step 7 : if $a_i > b_j$ then allocate the b_j amount of units to x_{ij} , which affects $a'_i = a_i - b_j$, and discard column j .
- Step 8 : repeat Steps 3, 4, 5, 6, and 7 until all columns are discarded.
- Step 9 : calculate the total minimal transportation cost.

5. Numerical Example

In this section, an example from [16] is used to prove the performance of the DM-TP1 heuristic in order to easily solve the TP under single-valued trapezoidal neutrosophic numbers. In fact, this problem was firstly introduced by [15] in order to present a peanut butter manufacturing company with three sources and four different destinations. In this problem, the cost parameters (\tilde{c}_{ij}^N) are presented as single-valued trapezoidal neutrosophic numbers, while the supply and demand quantities are presented as crisp numbers as given in Figure 3.

The mathematical formulation of this problem is given below:

$$\begin{aligned} \text{Minimize : } \tilde{Z}^N &= \sum_{i=1}^m \sum_{j=1}^n \tilde{c}_{ij}^N x_{ij}, \\ \text{Subject to :} \\ \sum_{j=1}^n x_{ij} &= a_i; \quad i = 1, 2, \dots, m, \\ \sum_{i=1}^m x_{ij} &= b_j; \quad j = 1, 2, \dots, n, \\ x_{ij} &\geq 0 \text{ for all } i \text{ and } j. \end{aligned} \quad (7)$$

The DM-TP1 starts by transforming the neutrosophic trapezoidal fuzzy number to crisp number using equation (5) and equation (6).

Here is an example for the first cost element: $N = \langle (3, 5, 6, 8); 0.6, 0.5, 0.4 \rangle$.

Then, $S(N) = (1/16) \times (3 + 5 + 6 + 8) \times (2 + 0.6 - 0.5 - 0.4) = 2.3$.

The second step is to calculate the AMSR and the AMDC for all rows and columns using the proposed formula (*Average-Min*). Let us give an example how to compute easily the first element of AMSR:

$$\text{AMSR}[1] = \text{Average}_{r1} - \text{Min}_{r1}, \quad (8)$$

where

$$\text{Average}_{r1} = \frac{(3 + 4 + 8 + 9)}{4} = 6, \quad (9)$$

$$\text{Min}_{r1} = (3, 4, 8, 9) = 3.$$

Thus, $\text{AMSR}[1] = 6 - 3 = 3$.

The same process is followed for the first element for AMDC:

$$\text{AMDC}[1] = \text{Average}_{c1} - \text{Min}_{c1}, \quad (10)$$

where

$$\text{Average}_{c1} = \frac{(3 + 1 + 4)}{3} = 2.66, \quad (11)$$

$$\text{Min}_{c1} = (3, 1, 4) = 1.$$

Thus, $\text{AMDC}[1] = 2.66 - 1 = 1.66$.

The next step is to select the highest element in AMSR and AMDC: 3.75 is the highest value and it is in the second row of AMSR (see Figure 4).

Thus, select the minimal element in the second row of AMSR which is equal to 1 at position d_{21} , affect 17 units (which represents the smallest element between demand 17 and supply 24), and discard the saturated element which is column 1. Hence, compute again the AMSR and AMDC indicators, select the highest one (3.33), and find its corresponding minimal element (3) at position d_{32} (see Figure 5).

Therefore, 28 units (the smallest element between demand 28 and supply 30) are affected and the saturated element which is column 3 is discarded. Next, calculate the AMSR and AMDC indicators, select the highest one (2.50), and find its corresponding minimal element (4) at position d_{12} (see Figure 6).

Consequently, 23 units (the smallest element between demand 23 and supply 26) are affected and the saturated element which is column 2 is discarded. Next, calculate the AMSR and AMDC indicators, select the highest one (1.67), and find its corresponding minimal element (5) at position d_{34} (see Figure 7).

Accordingly, 2 units (the smallest element between demand 12 and supply 2) are affected and the saturated element which is row 3 is discarded. Next, calculate the AMSR and AMDC indicators, select the highest one (1.50), and find its corresponding minimal element (6) at position d_{24} (see Figure 8).

Hence, 7 units (the smallest element between demand 10 and supply 7) are affected and the saturated element which is

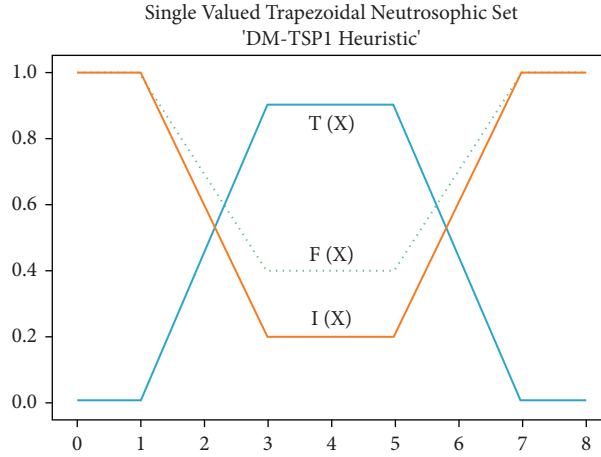


FIGURE 1: Graphical representation of a single-valued trapezoidal neutrosophic set.

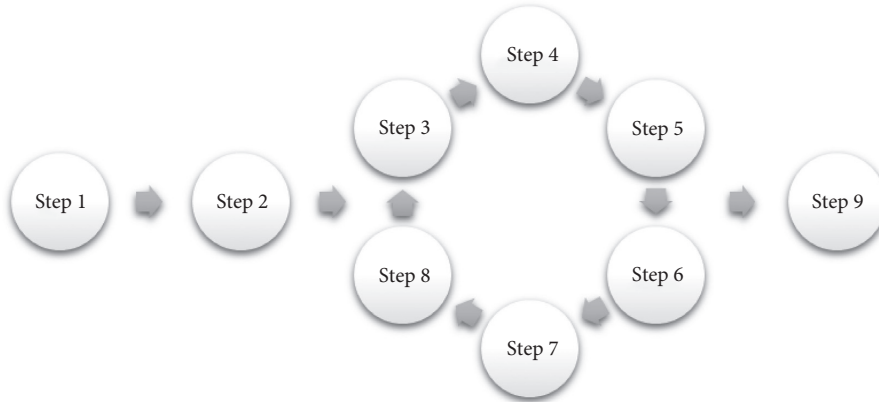


FIGURE 2: The flowchart of the proposed DM-TP1.

Source	Destination				Inventory
	$D1$	$D2$	$D3$	$D4$	
$O1$	(3,5,6,8); 0.6,0.5,0.4	(5,8,10,14); 0.3,0.6,0.6	(12,15,19,22); 0.6,0.4,0.5	(14,17,21,28); 0.8,0.2,0.6	26
$O2$	(0,1,3,6); 0.7,0.5,0.3	(5,7,9,11); 0.9,0.7,0.5	(15,17,19,22); 0.4,0.8,0.4	(9,11,14,16); 0.5,0.4,0.7	24
$O3$	(4,8,11,15); 0.6,0.3,0.2	(1,3,4,6); 0.6,0.3,0.5	(5,7,8,10); 0.5,0.4,0.7	(5,9,14,19); 0.3,0.7,0.6	30
Demand	17	23	28	12	

FIGURE 3: The single-valued trapezoidal neutrosophic TP.

row 2 is discarded. Finally, it remains only one element at position d_{14} , affects 3 units, and discards row 1 (see Figure 9).

Therefore, the obtained optimal solution using DM-TP1 heuristic is $x_{14} = 3$, $x_{24} = 7$, $x_{34} = 2$, $x_{12} = 23$, $x_{33} = 28$, and $x_{21} = 17$ with $Z = (9 * 3) + (6 * 7) + (5 * 2) + (4 * 23) + (3 * 28) + (1 * 17) = 272$.

Then, the DM-TP1 heuristic found the minimal total crisp cost equal to 272 and graphically presented in Figure 10. This result was also found by [16].

Consequently, the minimal total single-valued trapezoidal neutrosophic cost is

Source	Destination				Inventory	AMSR
	D1	D2	D3	D4		
D1	3	4	8	9	26	3.00
D2	1	4	8	6	24	3.75
D3	4	2	3	5	30	1.50
Demand	17	23	28	12		
AMDC	1.66	1.33	3.33	1.67		

FIGURE 4: After the defuzzification step.

Source	Destination				Inventory	AMSR
	D1	D2	D3	D4		
D1		4	8	9	26	3.00
D2		4	8	6	7	3.00
D3		2	3	5	30	1.33
Demand		23	28	12		
AMDC		1.33	3.33	1.67		

FIGURE 5: Discard column 1.

$$\begin{aligned}
 \text{Minimize: } \tilde{Z}^N &= \sum_{i=1}^3 \sum_{j=1}^4 \tilde{c}_{ij}^N x_{ij}, \\
 \tilde{Z}^N &= 3 \times \langle (14, 17, 21, 28); 0.3, 0.8, 0.7 \rangle, \\
 &+ 7 \times \langle (9, 11, 14, 16); 0.3, 0.8, 0.7 \rangle, \\
 &+ 2 \times \langle (5, 9, 14, 19); 0.3, 0.8, 0.7 \rangle, \quad (12) \\
 &+ 23 \times \langle (5, 8, 10, 14); 0.3, 0.8, 0.7 \rangle, \\
 &+ 28 \times \langle (5, 7, 8, 10); 0.3, 0.8, 0.7 \rangle, \\
 &+ 17 \times \langle (0, 1, 3, 6); 0.3, 0.8, 0.7 \rangle, \\
 &= \langle (370, 543, 694, 938); 0.3, 0.8, 0.7 \rangle.
 \end{aligned}$$

The graphical representation of the minimal total solution is depicted in Figure 11, where the total single-valued trapezoidal neutrosophic optimal solution will be greater than 370 and less than 938 with a level of acceptance 30% for the total neutrosophic cost lying between 543 and 694.

Therefore, a decision maker can conclude the minimal trapezoidal neutrosophic cost from the range 370 to 938, with its truth, indeterminacy, and falsity degrees. The truth membership function for the generated solution is denoted by

$$\mu(X) = \begin{cases} \frac{(x - 370) \times 0.3}{543 - 370}, & 370 \leq x \leq 543, \\ 0.3, & 543 \leq x \leq 694, \\ \frac{(938 - x) \times 0.3}{938 - 694}, & 694 \leq x \leq 938, \\ 0, & \text{otherwise.} \end{cases} \quad (13)$$

Correspondingly, the indeterminacy membership function for the minimal single-valued trapezoidal neutrosophic number cost is presented by

$$\vartheta(X) = \begin{cases} \frac{543 - x + (x - 370) \times 0.8}{543 - 370}, & 370 \leq x \leq 543, \\ 0.8, & 543 \leq x \leq 694, \\ \frac{x - 694 + (938 - x) \times 0.8}{938 - 694}, & 694 \leq x \leq 938, \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

Source	Destination				Inventory	AMSR
	<i>D1</i>	<i>D2</i>	<i>D3</i>	<i>D4</i>		
<i>D1</i>		4		9	26	2.50
<i>D2</i>		4		6	7	1.00
<i>D3</i>		2		5	2	1.50
Demand		23		12		
AMDC		1.33		1.67		

FIGURE 6: Discard column 3.

Source	Destination				Inventory	AMSR
	<i>D1</i>	<i>D2</i>	<i>D3</i>	<i>D4</i>		
<i>D1</i>				9	3	0.00
<i>D2</i>				6	7	0.00
<i>D3</i>				5	2	0.00
Demand				12		
AMDC				1.67		

FIGURE 7: Discard column 2.

Source	Destination				Inventory	AMSR
	<i>D1</i>	<i>D2</i>	<i>D3</i>	<i>D4</i>		
<i>D1</i>				9	3	0.00
<i>D2</i>				6	7	0.00
<i>D3</i>						
Demand				10		
AMDC				1.50		

FIGURE 8: Discard row 3.

Source	Destination				Inventory	AMSR
	<i>D1</i>	<i>D2</i>	<i>D3</i>	<i>D4</i>		
<i>D1</i>				9	3	0.00
<i>D2</i>						
<i>D3</i>						
Demand				3		
AMDC				0.00		

FIGURE 9: Discard row 2.

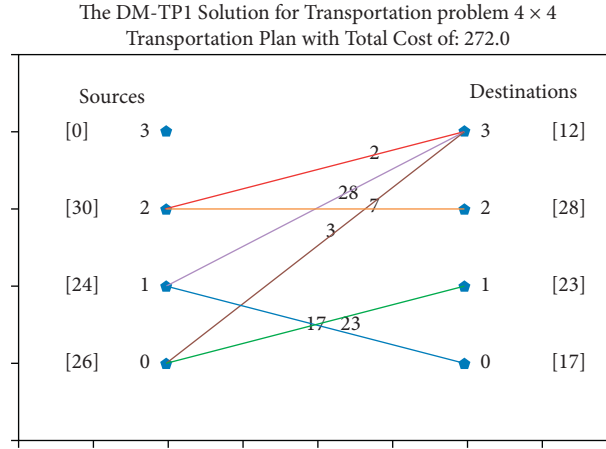


FIGURE 10: The generated transportation problem network diagram plan using DM-TP1 heuristic.

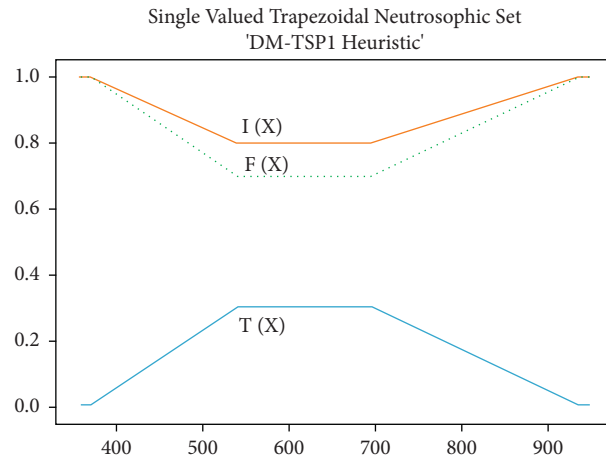


FIGURE 11: The minimal total single-valued trapezoidal neutrosophic cost using the DM-TP1 heuristic.

Consequently, the degree of falsity for the minimal single-valued trapezoidal neutrosophic number cost is depicted by

$$\lambda(X) = \begin{cases} \frac{543 - x + (x - 370) \times 0.7}{543 - 370}, & 370 \leq x \leq 543, \\ 0.7, & 543 \leq x \leq 694, \\ \frac{x - 694 + (938 - x) \times 0.7}{938 - 694}, & 694 \leq x \leq 938, \\ 0, & \text{otherwise.} \end{cases} \quad (15)$$

Based on the above result generated by the DM-TP1 heuristic, the decision maker can schedule the transportation network diagram plan.

6. Conclusion

The transportation problem is a special type in supply chain management where the aim is to minimize the total transportation cost of shipping products from sources to destinations. This paper addresses a transportation problem model under single-valued trapezoidal neutrosophic environment. To solve this problem, the novel Dhouib-Matrix-TP1 heuristic is enhanced at first with a score function in order to ensure the defuzzification of the neutrosophic trapezoidal fuzzy numbers into crisp numbers and at second with a new metric function (*Average-Min*) in order to drive the selection process.

Stepwise numerical applications are used to explain and to prove the performance of the proposed DM-TP1 heuristic. It also shows the effect of using the *Average* and the *Min* descriptive statistical metrics on the accuracy of the decision made. Further research will focus on the application

of the Dhouib-Matrix-TP1 to solve the multiobjective transportation problem in uncertain environment.

Data Availability

All data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

An Efficient Model for the Approximation of Intuitionistic Fuzzy Sets in terms of Soft Relations with Applications in Decision Making

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The basic notions in rough set theory are lower and upper approximation operators defined by a fixed binary relation. This paper proposes an intuitionistic fuzzy rough set (IFRS) model which is a combination of intuitionistic fuzzy set (IFS) and rough set. We approximate an IFS by using soft binary relations instead of fixed binary relations. By using this technique, we get two pairs of intuitionistic fuzzy (IF) soft sets, called the upper approximation and lower approximation with respect to foresets and aftersets. Properties of newly defined rough set model (IFRS) are studied. Similarity relations between IFS with respect to this rough set model (IFRS) are also studied. Finally, an algorithm is constructed depending on these approximations of IFSs and score function for decision-making problems, although a method of decision-making algorithm has been introduced for fuzzy sets already. But, this new IFRS model is more accurate to solve the problem because IFS has degree of nonmembership and degree of hesitant.

1. Introduction

To control uncertainty usually, probability theory is deliberated as an applicable tool, but for its practical work, a randomly stable system must be a very basic requirement. To establish such kind of system, a lot of time is needed. In today's speedy life, as everyone has shortage of time, in an unreliable environment, researchers have introduced many updated methods and techniques to solve uncertainties. Rough set (RS), fuzzy set (FS), and soft set (SS) are expressive methods to control uncertainty, vagueness, and incompleteness in the information systems. The above-mentioned sets have their own operations and properties. These sets are much applicable in real life, computer science, and artificial intelligence. We are encountered by different types of real-world problems everyday which have uncertainty and vagueness rather than preciseness. Precise and complete reasoning would not be possible if our

information data are inexact, vague, and incomplete. Recently, the gap between traditional mathematics with precise concepts and the world full of uncertainty become much smaller than earlier. In different fields, nature of vagueness can be different. Researchers are very active and interested to study many newly defined theories to solve this problem [1].

Fuzzy set (FS) theory was introduced by Zadeh in 1965 [2] which is a very revolutionary attempt to deal with uncertainty. The FS theory is a generalization of classical set theory. It has greater richness in application than the classical set theory. It has ability to translate human linguistic terms mathematically. Although FS has the membership degree, but often the nonmembership degree is required also to handle critical situations in real-world problems. Atanassov presented IFS in 1986 [3, 4] which is the generalization of FS. The IFS describes the fuzzy characteristic of things more comprehensively than FS and thus

is a powerful and successful tool to express fuzzy information of real-world problems. Elements of IFS are written in the form of ordered pairs and these ordered pairs are said to be intuitionistic fuzzy numbers or intuitionistic fuzzy values. Each intuitionistic fuzzy value is characterized by a membership degree, a nonmembership degree, and a hesitant degree. The sum of these three degrees is equal to 1. The IFSs are important in fuzzy mathematics due to its wide applications in real life, such as in pattern recognition, career determination, medical diagnosis, electoral system, and machine learning [5, 6].

In 1982, rough set (RS) theory introduced by Pawlak [7] is one of the untraditional methods to control uncertainty. A subset distinguished by lower approximation and upper approximation is known as RS. Pawlak used equivalence relations to prepare approximations in a set [7, 8]. However, the equivalence relations in RS seem to be very restrictive that may limit the scope of RS model. For instance, a frequent and significant problem in the medical field is the stomach pain in the children, which can be expected to some reasons and it is a demanding job to diagnose the reason correctly. The RS theory can help the doctors to diagnose the correct reason by discharge comments. Also, it has a broad scale of applications in image processing, knowledge finding, recognition of optical characters, and pattern recognition and to recognize various facial expressions in artificial intelligence, in data clustering, in decision-making problems with precised accuracy, and in business and finance because of their capacity to find the rule induction and knowledge. Figure 1 shows the graphical representation of RS with lower approximation and upper approximation. Upper approximation is a set which has elements having possible belonging with the target set and lower approximation set has objects having positive belonging with the target set [9–11].

Figure 1 shows that the target set is in red line circle, yellow box is the lower approximated set, blue box is the upper approximated set, and green box is the universal set.

In 1999, Molodtsov [12, 13] introduced the key notion of SS to deal with uncertainty. This new technique is free from the problems related with existing techniques of uncertainty. An appropriate number of parameters is available in this theory which makes it possible. The SS theory has a wide variety of applications in many fields, such as operational research, the smoothness of functions, Riemann integration, and game theory. Moreover, the SSs have a rich number of operations which are very helpful to deal with uncertainty in different types of situations. The concept of parametric reduction in SSs has been studied by many authors [14, 15]. Ali et al. [16] initiated some new operations in SS theory. Abbas et al. [17] initiated various generalized operations in SS theory by applying many relaxed conditions on parameters. Applications of SSs can be found in [14, 18–26].

Many extensions of SSs have been presented such as probabilistic SS theory, bijective SS theory, fuzzy bipolar SS theory, and intuitionistic fuzzy soft set (IFSS) theory. Akram et al. [27] introduced three hybrid models, namely, N-soft rough IF sets, IF N-soft sets, and IF N-soft RSs with real-life applications of decision-making algorithm. Alcantud et al. [28] presented covering-based fuzzy RS model by t-norm

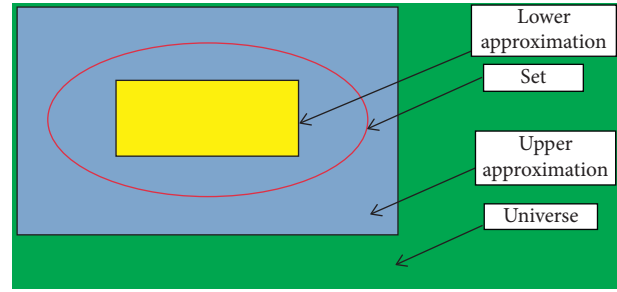


FIGURE 1: Graphical representation of rough set.

and fuzzy logical impicator. This fuzzy RS model is useful to characterize the covering-based optimistic and pessimistic multigranulation. They also presented two kinds of decision-making methods to analyze this model theoretically. Alcantud et al. [29] presented a tool which aggregates infinite chains of IF sets over time. They presented IF sets along an indefinitely long number of periods by using score and accuracy degrees of temporal IF elements.

A parameterized collection of ordinary binary relations is called a soft binary relation on a universe and this is generalization of binary relations. In RS theory, rough approximations just address single binary relations but rough approximations by using soft binary relations can deal with different binary relations. The idea of soft relation over U is given by Feng et al. [30] in 2013. Babitha and Sunil [31] presented some results on soft set relations. Many authors have generalized the notion of Pawlak RS model by using dominance relations, covering relations, similarity relations, tolerance relations, fuzzy relations, neighbourhood relations, and other indiscernibility relations, see [28, 32–42].

1.1. Related Works. Feng et al. [1] presented a hybrid model of SSs which is rough approximation of SS. They used an SS instead of an equivalence relation to granulate the universe. In the result, soft approximation space and soft RSs have been introduced as a deviation of rough approximation space. Furthermore, they also extended Dubois and Prade's RSs by approximating a FS in a soft approximation space and called soft rough FSs. Feng et al. have done a lot of work by combining SSs, RSs, and FSs and defined new models [1, 43, 44]. Ali and Shabir [45] studied fuzzy SSs. Roy and Maji [46] initiated the study of fuzzy SSs. In 2020, Bashir et al. introduced a model of rough fuzzy ternary semigroups based on three-dimensional congruence relation [47]. Many authors introduced RS approximations in IFSs [48, 49]. Kanwal and Shabir approximated the ideals and fuzzy sets in semigroups based on soft relations [50, 51]. Later in 2020, Shabir and Kanwal used soft relations to define lower and upper approximations of a set in [51]. We have generalized the concept in [51] in terms of IFS by introducing non-membership degree. Our model IFRS gives approximations corresponding to every attribute or parameter. In this way, we get more accurate results by reducing errors than all previous ones.

1.2. Connection of IFRS Model with Rough Sets. The IF set theory deals mainly with vagueness, while the RS theory deals with incompleteness. The study of the combination of these two theories is useful to deal with impreciseness. It means that the rough IF sets are useful to deal with both vagueness and incompleteness. Recently, RS approximations have been discussed in IF environment. In the result, IF rough sets, rough IF sets, and generalized IF rough sets have been presented. Zhou et al. studied different relation-based IF approximation operators in the axiomatic and constructive approaches [52].

1.3. Innovative Contribution. Some researchers have words that one theory is better than other theory to deal with inexact data. Majority of researchers admitted that RSs and FSs are very closely related with each other, but distinction is there that they model different types of vagueness. The RS is a coarsely described crisp set, whereas the FS is viewed as a class with blunt boundaries. Since the FS has only membership degree but the IF set has also degree of nonmembership which is more useful in medical science. To diagnose a disease, IF environment is better than fuzzy environment due to the presence of nonmembership degree. In 2020, Shabir et al. [51] presented a model of RS which is combination of SS and FS. They approximated FS in terms of soft binary relations. In our paper, we considered an IF set instead of a FS and an IF set has more accurate results than a FS in medical science. The IFS with other algebraic structures generalizes hybrid models which are very useful in medical science, computer science, and other fields. Samanta and Mondal [53] presented the IF rough set (A, B) which is generalized IFS in terms of fuzzy rough sets A and B . On the other hand, an IF rough set (A, B) presented by Chakrabarty et al. [48] is the generalization of fuzzy rough set in terms of IFSs A and B . Zhou [54] proposed IF rough sets induced by IF approximation spaces and discussed their properties. Recently, IF set has combined with rough set approximations and resulting sets are called IF rough sets and rough IF and generalized IF rough sets. In axiomatic and constructive approach, Zhou et al. presented a useful framework and studied different IF rough approximation operators by using a special type of IF triangular norm min. Zhou (2014) presented IF soft rough sets and soft rough IF sets. These newly presented models were very useful as new approaches for decision-making problems. By integrating IF sets with SSs, Maji et al. presented IF soft sets. Jiang et al. discussed an approach of IF soft sets-based decision making and they also presented interval-valued IF soft set model. IF soft set is an important combination of IF sets and SSs. It makes more accurate and realistic descriptions of materialistic world. Zhang presented a useful model combining IF soft sets with RSs [1, 52, 55, 56].

1.4. Motivation. In the present paper, we extend the concept given by Shabir et al. [51] in terms of FSs. We use IFSs instead of FSs which is more valuable to manage uncertainty in many scientific fields, such as medical diagnosis and pattern recognition. Our proposed model based on soft relations is very useful due to importance of IFS in real-life

situations [5, 6]. In our research, we propose a decision-making algorithm by using our model based on soft relations. Then, we present an example to illustrate the validity of our proposed decision-making algorithm. Our IFRS model is the combination of RS, IFS, and SS which is helpful to control impreciseness and uncertainty.

1.5. Organization of the Paper. The pattern of this paper is as follows. In Section 2, some foundational concepts are identified with FSs, IFS, RSs, and SSs and soft binary relations are described. In Section 3, we presented IFRS model based on soft relations and discussed some properties. Soft similarity relations have been examined in Section 4. In Section 5, we gave an approach to a decision-making problem based on an IFS. Moreover, an example is presented to illustrate this decision-making algorithm in Section 6.

2. Preliminaries and Basic Concepts

In this section, some basic notions about binary relations, IFS, soft sets, and intuitionistic fuzzy soft sets are given. Throughout this paper, U_1 and U_2 represent two nonempty finite sets unless stated otherwise.

A binary relation \mathcal{R} from U_1 to U_2 is a subset of $U_1 \times U_2$ and a subset of $U \times U$ is called a binary relation on U . If \mathcal{R} is a binary relation on U , then \mathcal{R} is said to be reflexive if $(u, u) \in \mathcal{R}$ for all $u \in U$, symmetric if $(u, v) \in \mathcal{R}$ implies $(v, u) \in \mathcal{R}$ for all $u, v \in U$, and transitive if $(u, v) \in \mathcal{R}$ and $(v, w) \in \mathcal{R}$ imply $(u, w) \in \mathcal{R}$ for all $u, v, w \in U$. If a binary relation \mathcal{R} is reflexive, symmetric, and transitive, then it is called an equivalence relation. An equivalence relation partitions the set into disjoint classes.

Let U be a nonempty universe. An IF set M in the universe U is an object having the form $M = \{\langle x, \mu_M(x), \gamma_M(x) \rangle : x \in U\}$, where $\mu_M: U \rightarrow [0, 1]$ and $\gamma_M: U \rightarrow [0, 1]$, satisfying $0 \leq \mu_M(x) + \gamma_M(x) \leq 1$ for all $x \in U$. The values $\mu_M(x)$ and $\gamma_M(x)$ are called degree of membership and degree of nonmembership of $x \in U$ to M , respectively. The number $\pi_M(x) = 1 - \mu_M(x) - \gamma_M(x)$ is called the degree of hesitancy of $x \in U$ to M . The collection of all IFSs in U is denoted by $IF(U)$. In the remaining paper, we shall write an IFS by $M = \langle \mu_M, \gamma_M \rangle$ instead of $M = \{\langle x, \mu_M(x), \gamma_M(x) \rangle : x \in U\}$. Let $M = \langle \mu_M, \gamma_M \rangle$ and $N = \langle \mu_N, \gamma_N \rangle$ be two IFSs in U . Then, $M \subseteq N$ if and only if $\mu_M(x) \leq \mu_N(x)$ and $\gamma_N(x) \leq \gamma_M(x)$ for all $x \in U$. Two IFSs M and N are said to be equal if and only if $M \subseteq N$ and $N \subseteq M$. The union and intersection of two IFSs M and N in U are denoted and defined by $M \cap N = \langle \mu_M \cap \mu_N, \gamma_M \cup \gamma_N \rangle$ and $M \cup N = \langle \mu_M \cup \mu_N, \gamma_M \cap \gamma_N \rangle$, where $(\mu_M \cap \mu_N)(x) = \inf\{\mu_M(x), \mu_N(x)\}$, $(\gamma_M \cup \gamma_N)(x) = \sup\{\gamma_M(x), \gamma_N(x)\}$, $(\mu_M \cup \mu_N)(x) = \sup\{\mu_M(x), \mu_N(x)\}$, $(\gamma_M \cap \gamma_N)(x) = \inf\{\gamma_M(x), \gamma_N(x)\}$.

Next, we define two special types of (IFSs) as follows:

The IF universe set $U = 1_U = \langle 1, 0 \rangle$ and IF empty set $\emptyset = 0_U = \langle 0, 1 \rangle$, where $1(x) = 1$ and $0(x) = 0$ for all $x \in U$. The complement of an IFS $M = \langle \mu, \gamma \rangle$ is denoted and defined as $M^c = \langle \gamma, \mu \rangle$ [3].

For a fixed $x \in U$, the pair $(\mu_M(x), \gamma_M(x))$ is called IF value or IF number. In order to define the order between two IFNs, Chen and Tan [57] presented the score function as $S(x) = \mu_M(x) - \gamma_M(x)$ and Hong and Choi [58] defined the accuracy function as $H(x) = \mu_M(x) + \gamma_M(x)$, where x is an IFV. Xu [59,60] combined the accuracy and score functions and formed the order relations between any pair of IFVs (x, y) as follows:

- (i) If $S(x) > S(y)$, then $x > y$;
- (ii) If $S(x) = S(y)$, then
 - (a) If $H(x) = H(y)$, then $x = y$;
 - (b) If $H(x) < H(y)$, then $x < y$.

A pair (F, A) is called a soft set over U if F is a mapping given by $F: A \rightarrow P(U)$, where A is a subset of E (the set of parameters) and $P(U)$ is the power set of U . Thus, $F(e)$ is a subset of U for all $e \in A$. Hence, a soft set over U is a parametrized collection of subsets of U . A pair (F, A) is called an intuitionistic fuzzy soft set over U if F is a mapping given by $F: A \rightarrow \text{IF}(U)$ and A is a subset of E (the set of parameters). Thus, $F(e)$ is an IF set in U for all $e \in A$. Hence, an IF soft set over U is a parametrized collection of IF sets in U . For two IF soft sets (F, A) and (G, B) over a common universe U , we say that (F, A) is an IF soft subset of (G, B) if (1) $A \subseteq B$ and (2) $F(e)$ is an IF subset of $G(e)$ for all $e \in A$. Two IF soft sets (F, A) and (G, B) over a common universe U are said to be IF soft equal if (F, A) is an IF soft subset of (G, B) and (G, B) is an IF soft subset of (F, A) . The union of two IF soft sets (F, A) and (G, A) over the common universe U is the IF soft set (H, A) , where $H(e) = F(e) \cup G(e)$ for all $e \in A$. The intersection of two IF soft sets (F, A) and (G, A) over the common universe U is the IF soft set (K, A) , where $K(e) = F(e) \cap G(e)$ for all $e \in A$ [12, 61, 62].

An IF soft set can be represented by a table, which is shown in the following example.

Example 1. Let $U = \{x, y, z, s, t\}$, $A = \{e_1, e_2\}$. Consider an IF soft set (F, A) over U defined by $F(e_1)(x) = (0.3, 0.4)$, $F(e_1)(y) = (0.4, 0.3)$, $F(e_1)(z) = (0.4, 0.2)$, $F(e_1)(s) = (0.8, 0.1)$, $F(e_1)(t) = (0.2, 0.6)$ and $(F(e_2))(x) = (0.9, 0)$, $F(e_2)(y) = (0.5, 0.5)$, $F(e_2)(z) = (0.4, 0.5)$, $F(e_2)(s) = (0.3, 0.7)$, $F(e_2)(t) = (0.6, 0.3)$.

The above intuitionistic fuzzy soft set can be represented as in Table 1.

3. Approximations of an IFS by Soft Binary Relation

In this section, we consider soft binary relation from U_1 to U_2 and approximate an IFS of U_2 by using aftersets and get two IF soft sets of U_1 . Similarly, we approximate an IFS of U_1 by using foresets and get two IF soft sets of U_2 . We also study some properties of these approximations.

Definition 1 (see [30]). A soft binary relation (σ, A) from U_1 to U_2 is a soft set over $U_1 \times U_2$, that is, $\sigma: A \rightarrow P(U_1 \times U_2)$, where A is a subset of the set of parameters E .

TABLE 1: Representation of IF soft set.

U	e_1	e_2
x	(0.3, 0.4)	(0.9, 0)
y	(0.4, 0.3)	(0.5, 0.5)
z	(0.4, 0.2)	(0.4, 0.5)
s	(0.8, 0.1)	(0.3, 0.7)
t	(0.2, 0.6)	(0.6, 0.3)

Of course, (σ, A) is a parameterized collection of binary relations from U_1 to U_2 . That is, for each $e \in A$, we have a binary relation $\sigma(e)$ from U_1 to U_2 .

Definition 2. Let (σ, A) be a soft binary relation from U_1 to U_2 and $M = \langle \mu_M, \gamma_M \rangle$ be an IFS in U_2 . Then, we define lower approximation $\underline{\sigma}^M = (\underline{\sigma}^{\mu_M}, \underline{\sigma}^{\gamma_M})$ and upper approximation $\overline{\sigma}^M = (\overline{\sigma}^{\mu_M}, \overline{\sigma}^{\gamma_M})$ of $M = \langle \mu_M, \gamma_M \rangle$ with respect to aftersets as follows:

$$\begin{aligned}
 \underline{\sigma}^{\mu_M}(e)(u_1) &= \begin{cases} \bigwedge_{a \in u_1 \sigma(e)} \mu_M(a), & \text{if } u_1 \sigma(e) \neq \emptyset, \\ 1, & \text{if } u_1 \sigma(e) = \emptyset, \end{cases} \\
 \underline{\sigma}^{\gamma_M}(e)(u_1) &= \begin{cases} \bigvee_{a \in u_1 \sigma(e)} \gamma_M(a), & \text{if } u_1 \sigma(e) \neq \emptyset, \\ 0, & \text{if } u_1 \sigma(e) = \emptyset, \end{cases} \\
 \overline{\sigma}^{\mu_M}(e)(u_1) &= \begin{cases} \bigvee_{a \in u_1 \sigma(e)} \mu_M(a), & \text{if } u_1 \sigma(e) \neq \emptyset, \\ 0, & \text{if } u_1 \sigma(e) = \emptyset, \end{cases} \\
 \overline{\sigma}^{\gamma_M}(e)(u_1) &= \begin{cases} \bigwedge_{a \in u_1 \sigma(e)} \gamma_M(a), & \text{if } u_1 \sigma(e) \neq \emptyset, \\ 1, & \text{if } u_1 \sigma(e) = \emptyset, \end{cases}
 \end{aligned} \tag{1}$$

where $u_1 \sigma(e) = \{a \in U_2 : (u_1, a) \in \sigma(e)\}$ and is called the afterset of u_1 for $u_1 \in U_1$ and $e \in A$.

- (i) $\underline{\sigma}^{\mu_M}(e)(u_1)$ indicates the degree to which u_1 definitely have the property e .
- (ii) $\underline{\sigma}^{\gamma_M}(e)(u_1)$ indicates the degree to which u_1 probably do not have the property e .
- (iii) $\overline{\sigma}^{\mu_M}(e)(u_1)$ indicates the degree to which u_1 probably have the property e .
- (iv) $\overline{\sigma}^{\gamma_M}(e)(u_1)$ indicates the degree to which u_1 definitely do not have the property e .

In Definition 2, soft binary relation from U_1 to U_2 is given and IFS in U_2 can be approximated as lower and upper approximations with respect to the aftersets. The resulting sets are two pairs of IF soft sets.

Definition 3. Let (σ, A) be a soft binary relation from U_1 to U_2 and $M = \langle \mu_M, \gamma_M \rangle$ be an IFS in U_1 . Then, we define lower approximation ${}^M \underline{\sigma} = (\mu_M \underline{\sigma}, \gamma_M \underline{\sigma})$ and upper approximation ${}^M \overline{\sigma} = (\mu_M \overline{\sigma}, \gamma_M \overline{\sigma})$ of $M = \langle \mu_M, \gamma_M \rangle$ with respect to foresets as follows:

$$\begin{aligned}
\mu_M \underline{\sigma}(e)(u_2) &= \begin{cases} \bigwedge_{a \in \sigma(e)u_2} \mu_M(a), & \text{if } \sigma(e)u_2 \neq \emptyset, \\ 1, & \text{if } \sigma(e)u_2 = \emptyset, \end{cases} \\
\gamma_M \underline{\sigma}(e)(u_2) &= \begin{cases} \bigvee_{a \in \sigma(e)u_2} \gamma_M(a), & \text{if } \sigma(e)u_2 \neq \emptyset, \\ 0, & \text{if } \sigma(e)u_2 = \emptyset, \end{cases} \\
\mu_M \overline{\sigma}(e)(u_2) &= \begin{cases} \bigvee_{a \in \sigma(e)u_2} \mu_M(a) & \text{if } \sigma(e)u_2 \neq \emptyset, \\ 0 & \text{if } \sigma(e)u_2 = \emptyset, \end{cases} \\
\gamma_M \overline{\sigma}(e)(u_2) &= \begin{cases} \bigwedge_{a \in \sigma(e)u_2} \gamma_M(a), & \text{if } \sigma(e)u_2 \neq \emptyset, \\ 1, & \text{if } \sigma(e)u_2 = \emptyset, \end{cases}
\end{aligned} \tag{2}$$

where $\sigma(e)u_2 = \{a \in U_1 : (a, u_2) \in \sigma(e)\}$ and is called the foreset of u_2 for $u_2 \in U_2$ and $e \in A$. Of course, $\underline{\sigma}^M: A \longrightarrow \text{IF}(U_1)$, $\overline{\sigma}^M: A \longrightarrow \text{IF}(U_1)$ and ${}^M \underline{\sigma}: A \longrightarrow \text{IF}(U_2)$, ${}^M \overline{\sigma}: A \longrightarrow \text{IF}(U_2)$. The following example explains these concepts.

In Definition 3, soft binary relation from U_1 to U_2 is given and IFS in U_1 can be approximated as lower and upper approximations with respect to the foresets. The resulting sets are two pairs of IF soft sets.

Example 2. Suppose that Mr. X wants to buy a shirt for his own use. Let $U_1 = \{\text{the set of all shirts designs}\} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ and $U_2 = \{\text{the colors of all designs}\} = \{c_1, c_2, c_3, c_4\}$ and the set of attributes be $A = \{e_1, e_2, e_3\} = \{\text{the set of stores near his house}\}$.

Define $\sigma: A \longrightarrow P(U_1 \times U_2)$ by

$$\begin{aligned}
\sigma(e_1) &= \left\{ (d_1, c_1), (d_1, c_2), (d_1, c_3), (d_2, c_2), (d_2, c_4), \right. \\
&\quad \left. (d_4, c_2), (d_4, c_3), (d_5, c_3), (d_5, c_4), (d_6, c_1) \right\}, \\
\sigma(e_2) &= \{(d_1, c_3), (d_2, c_3), (d_4, c_1), (d_5, c_1), (d_6, c_2), (d_6, c_3)\}, \\
\sigma(e_3) &= \{(d_2, c_4), (d_3, c_1), (d_3, c_3), (d_5, c_3), (d_5, c_4)\},
\end{aligned} \tag{3}$$

which represents the relation between designs and colors available on store e_i for $1 \leq i \leq 3$. Then,

$$\begin{aligned}
d_1 \sigma(e_1) &= \{c_1, c_2, c_3\}, \\
d_2 \sigma(e_1) &= \{c_2, c_4\}, \\
d_3 \sigma(e_1) &= \emptyset, \\
d_4 \sigma(e_1) &= \{c_2, c_3\}, \\
d_5 \sigma(e_1) &= \{c_3, c_4\}, \\
d_6 \sigma(e_1) &= \{c_1\} \\
d_1 \sigma(e_2) &= \{c_3\}, \\
d_2 \sigma(e_2) &= \{c_3\}, \\
d_3 \sigma(e_2) &= \emptyset, \\
d_4 \sigma(e_2) &= \{c_1\}, \\
d_5 \sigma(e_2) &= \{c_1\}, \\
d_6 \sigma(e_2) &= \{c_2, c_3\} \\
d_1 \sigma(e_3) &= \emptyset, \\
d_2 \sigma(e_3) &= \{c_4\}, \\
d_3 \sigma(e_3) &= \{c_1, c_3\}, \\
d_4 \sigma(e_3) &= \emptyset, \\
d_5 \sigma(e_3) &= \{c_3, c_4\}, \\
d_6 \sigma(e_3) &= \emptyset,
\end{aligned} \tag{4}$$

where $d_i \sigma(e_j)$ represents the color of the design d_i available on the store e_j .

Also,

$$\begin{aligned}
\sigma(e_1)c_1 &= \{d_1, d_6\}, \\
\sigma(e_1)c_2 &= \{d_1, d_2, d_4\}, \\
\sigma(e_1)c_3 &= \{d_1, d_4, d_5\}, \\
\sigma(e_1)c_4 &= \{d_2, d_5\}, \\
\sigma(e_2)c_1 &= \{d_4, d_5\}, \\
\sigma(e_2)c_2 &= \{d_6\}, \\
\sigma(e_2)c_3 &= \{d_1, d_2, d_6\}, \\
\sigma(e_2)c_4 &= \emptyset, \\
\sigma(e_3)c_1 &= \{d_3\}, \\
\sigma(e_3)c_2 &= \emptyset, \\
\sigma(e_3)c_3 &= \{d_3, d_5\}, \\
\sigma(e_3)c_4 &= \{d_2, d_5\},
\end{aligned} \tag{5}$$

where $\sigma(e_j)c_i$ represents the design of the color c_i available on the store e_j .

Define $M = \langle \mu_M, \gamma_M \rangle: U_2 \longrightarrow [0, 1]$ which represents the preference of the colors given by Mr. X such that

$$\begin{aligned}\mu_M(c_1) &= 0.9, \mu_M(c_2) = 0.8, \mu_M(c_3) = 0.4, \mu_M(c_4) = 0 \\ \gamma_M(c_1) &= 0.0, \gamma_M(c_2) = 0.2, \gamma_M(c_3) = 0.5, \gamma_M(c_4) = 0\end{aligned}$$

Define $N = \langle \mu_N, \gamma_N \rangle: U_1 \longrightarrow [0, 1]$ which represents the preference of the designs given by Mr. X such that

$$\begin{aligned}\mu_N(d_1) &= 1, \mu_N(d_2) = 0.7, \mu_N(d_3) = 0.5, \mu_N(d_4) = 0.1, \\ \mu_N(d_5) &= 0, \mu_N(d_6) = 0.4 \\ \gamma_N(d_1) &= 0, \gamma_N(d_2) = 0.2, \gamma_N(d_3) = 0.5, \gamma_N(d_4) = 0.7, \\ \gamma_N(d_5) &= 1, \gamma_N(d_6) = 0.5.\end{aligned}$$

Therefore, the lower and upper approximations (with respect to the aftersets as well as with respect to the foresets) are

$$\underline{\sigma}^M = \left(\begin{smallmatrix} \mu_M & \gamma_M \\ \underline{\sigma} & \underline{\sigma} \end{smallmatrix} \right) \text{ (given in Table 2),} \quad (7)$$

$$\overline{\sigma}^M = (\overline{\sigma}^{\mu_M}, \overline{\sigma}^{\gamma_M}) \text{ (given in Table 3),}$$

and

$$\begin{aligned}{}^N \underline{\sigma} &= \left(\begin{smallmatrix} \mu_N & \gamma_N \\ \underline{\sigma} & \underline{\sigma} \end{smallmatrix} \right) \text{ (given in Table 4)} {}^N \overline{\sigma} \\ &= (\mu_N \overline{\sigma}, \gamma_N \overline{\sigma}) \text{ (given in Table 5).}\end{aligned} \quad (8)$$

Table 2 shows the lower approximation of IFS M with respect to the aftersets by using Definition 2. Table 3 shows the upper approximation of IFS M with respect to the aftersets by using Definition 2. Table 4 shows the lower approximation of IFS N with respect to the foresets by using Definition 3. Table 5 shows the upper approximation of IFS N with respect to the foresets by using Definition 3.

Theorem 1. Let (σ, A) be a soft binary relation from U_1 to U_2 , that is, $\sigma: A \longrightarrow P(U_1 \times U_2)$. For any IFSs, $M = \langle \mu_M, \gamma_M \rangle$, $N = \langle \mu_N, \gamma_N \rangle$, and $P = \langle \mu_P, \gamma_P \rangle$ of U_2 , the following are true:

- (1) If $N \subseteq P$, then $\underline{\sigma}^N \subseteq \underline{\sigma}^P$;
- (2) If $N \subseteq P$, then $\overline{\sigma}^N \subseteq \overline{\sigma}^P$;
- (3) $\underline{\sigma}^N \cap \underline{\sigma}^P = \underline{\sigma}^{N \cap P}$;
- (4) $\overline{\sigma}^N \cap \overline{\sigma}^P \supseteq \overline{\sigma}^{N \cap P}$;
- (5) $\underline{\sigma}^N \cup \underline{\sigma}^P \subseteq \underline{\sigma}^{N \cup P}$;
- (6) $\overline{\sigma}^N \cup \overline{\sigma}^P = \overline{\sigma}^{N \cup P}$;
- (7) $\underline{\sigma}^{1_{U_2}} = 1_{U_1}$ if $u_1 \sigma(e) \neq \emptyset$;
- (8) $\overline{\sigma}^{1_{U_2}} = 1_{U_1}$ if $u_1 \sigma(e) \neq \emptyset$;
- (9) $\underline{\sigma}^M = (\overline{\sigma}^{M^c})^c$ if $u_1 \sigma(e) \neq \emptyset$;

$$(10) \overline{\sigma}^M = (\underline{\sigma}^{M^c})^c \text{ if } u_1 \sigma(e) \neq \emptyset;$$

$$(11) \underline{\sigma}^{0_{U_2}} = 0_{U_1} = \overline{\sigma}^{0_{U_2}} \text{ if } u_1 \sigma(e) \neq \emptyset.$$

Proof

- (1) Let $u_1 \in U_1$. If $u_1 \sigma(e) = \emptyset$, then $\underline{\sigma}^{\mu_N}(e)(u_1) = 1 = \underline{\sigma}^{\mu_P}(e)(u_1)$ and $\underline{\sigma}^{\gamma_N}(e)(u_1) = 0 = \underline{\sigma}^{\gamma_P}(e)(u_1)$. If $u_1 \sigma(e) \neq \emptyset$, then $\underline{\sigma}^{\mu_N}(e)(u_1) = \bigwedge_{a \in u_1 \sigma(e)} \mu_N(a) \leq \bigwedge_{a \in u_1 \sigma(e)} \mu_P(a)$ because $\mu_N(a) \leq \mu_P(a) = \underline{\sigma}^{\mu_P}(e)(u_1)$. Thus, $\underline{\sigma}^{\mu_N}(e)(u_1) \leq \underline{\sigma}^{\mu_P}(e)(u_1)$.

Also,

$$\begin{aligned}\underline{\sigma}^{\gamma_N}(e)(u_1) &= \bigvee_{a \in u_1 \sigma(e)} \gamma_N(a) \geq \bigvee_{a \in u_1 \sigma(e)} \gamma_P(a) \quad \text{because } \gamma_N(a) \geq \gamma_P(a) \\ &= \underline{\sigma}^{\gamma_P}(e)(u_1).\end{aligned}$$

Thus, $\underline{\sigma}^{\gamma_N}(e)(u_1) \geq \underline{\sigma}^{\gamma_P}(e)(u_1)$. Hence, $\underline{\sigma}^N \subseteq \underline{\sigma}^P$.

- (2) Let $u_1 \in U_1$. If $u_1 \sigma(e) = \emptyset$, then $\overline{\sigma}^{\mu_N}(e)(u_1) = 0 = \overline{\sigma}^{\mu_P}(e)(u_1)$ and $\overline{\sigma}^{\gamma_N}(e)(u_1) = 1 = \overline{\sigma}^{\gamma_P}(e)(u_1)$. If $u_1 \sigma(e) \neq \emptyset$, then

$$\overline{\sigma}^{\mu_N}(e)(u_1) = \bigvee_{a \in u_1 \sigma(e)} \mu_N(a) \leq \bigvee_{a \in u_1 \sigma(e)} \mu_P(a) \quad \text{because } \mu_N(a) \leq \mu_P(a) = \overline{\sigma}^{\mu_P}(e)(u_1).$$

Thus, $\overline{\sigma}^{\mu_N}(e)(u_1) \leq \overline{\sigma}^{\mu_P}(e)(u_1)$.

Also,

$$\begin{aligned}\overline{\sigma}^{\gamma_N}(e)(u_1) &= \bigwedge_{a \in u_1 \sigma(e)} \gamma_N(a) \geq \bigwedge_{a \in u_1 \sigma(e)} \gamma_P(a) \quad \text{because } \gamma_N(a) \geq \gamma_P(a) \\ &= \overline{\sigma}^{\gamma_P}(e)(u_1).\end{aligned}$$

Thus, $\overline{\sigma}^{\gamma_N}(e)(u_1) \geq \overline{\sigma}^{\gamma_P}(e)(u_1)$. Hence, $\overline{\sigma}^N \subseteq \overline{\sigma}^P$.

- (3) Let $u_1 \in U_1$. If $u_1 \sigma(e) = \emptyset$, then $\underline{\sigma}^{\mu_{N \cap P}}(e)(u_1) = 1 = \underline{\sigma}^{\mu_N}(e)(u_1) \cap \underline{\sigma}^{\mu_P}(e)(u_1)$ and $\underline{\sigma}^{\gamma_{N \cap P}}(e)(u_1) = 0 = \underline{\sigma}^{\gamma_N}(e)(u_1) \cup \underline{\sigma}^{\gamma_P}(e)(u_1)$. If $u_1 \sigma(e) \neq \emptyset$, then

$$\begin{aligned}(\underline{\sigma}^{\mu_N} \cap \underline{\sigma}^{\mu_P})(e)(u_1) &= \underline{\sigma}^{\mu_N}(e)(u_1) \cap \underline{\sigma}^{\mu_P}(e)(u_1) = (\bigwedge_{a \in u_1 \sigma(e)} \mu_N(a)) \cap (\bigwedge_{a \in u_1 \sigma(e)} \mu_P(a)) = \bigwedge_{a \in u_1 \sigma(e)} (\mu_N(a) \cap \mu_P(a)) \\ &= \bigwedge_{a \in u_1 \sigma(e)} (\mu_{N \cap P}(a)) = \underline{\sigma}^{\mu_{N \cap P}}(e)(u_1).\end{aligned}$$

TABLE 2: Lower approximation of intuitionistic fuzzy set M .

$\underline{\sigma}^M$	e_1	e_2	e_3
d_1	(0.4, 0.5)	(0.4, 0.5)	(1, 0)
d_2	(0, 0.8)	(0.4, 0.5)	(0, 0.8)
d_3	(1, 0)	(1, 0)	(0.4, 0.5)
d_4	(0.4, 0.5)	(0.9, 0)	(1, 0)
d_5	(0, 0.8)	(0.9, 0)	(0, 0.8)
d_6	(0.9, 0)	(0.4, 0.5)	(1, 0)

TABLE 3: Upper approximation of intuitionistic fuzzy set M .

$\overline{\sigma}^M$	e_1	e_2	e_3
d_1	(0.9, 0)	(0.4, 0.5)	(0, 1)
d_2	(0.8, 0.2)	(0.4, 0.5)	(0, 0.8)
d_3	(0, 1)	(0, 1)	(0.9, 0)
d_4	(0.8, 0.2)	(0.9, 0)	(0, 1)
d_5	(0.4, 0.5)	(0.9, 0)	(0.4, 0.5)
d_6	(0.9, 0)	(0.8, 0.2)	(0, 1)

TABLE 4: Lower approximation of intuitionistic fuzzy set N .

${}^N\underline{\sigma}$	e_1	e_2	e_3
c_1	(0.4, 0.5)	(0, 1)	(0.5, 0.5)
c_2	(0.1, 0.7)	(0.4, 0.5)	(1, 0)
c_3	(0, 1)	(0.4, 0.5)	(0, 1)
c_4	(0, 1)	(1, 0)	(0, 1)

TABLE 5: Upper approximation of intuitionistic fuzzy set N .

${}^N\overline{\sigma}$	e_1	e_2	e_3
c_1	(1, 0)	(0.1, 0.7)	(0.5, 0.5)
c_2	(1, 0)	(0.4, 0.5)	(0, 1)
c_3	(1, 0)	(1, 0)	(0.5, 0.5)
c_4	(0.7, 0.2)	(0, 1)	(0.7, 0.2)

Also,

$$\begin{aligned} (\underline{\sigma}^N \cup \underline{\sigma}^P)(e)(u_1) &= \underline{\sigma}^N(e)(u_1) \vee \underline{\sigma}^P(e)(u_1) = \\ &= (\bigvee_{a \in u_1 \sigma(e)} \gamma_N(a)) \vee (\bigvee_{a \in u_1 \sigma(e)} \gamma_P(a)) = \bigvee_{a \in u_1 \sigma(e)} (\gamma_N(a) \vee \gamma_P(a)) = \bigvee_{a \in u_1 \sigma(e)} (\gamma_N \vee \gamma_P)(a) = \bigvee_{a \in u_1 \sigma(e)} (\gamma_{N \cup P})(a) = \underline{\sigma}^{N \cup P}(e)(u_1). \end{aligned}$$

This shows that $\underline{\sigma}^N \cap \underline{\sigma}^P = \underline{\sigma}^{N \cap P}$.

(4) Since $N \cap P \subseteq N$ and $N \cap P \subseteq P$, we have from part (2) $\overline{\sigma}^{N \cap P} \subseteq \overline{\sigma}^N$ and $\overline{\sigma}^{N \cap P} \subseteq \overline{\sigma}^P$. Thus, $\overline{\sigma}^{N \cap P} \subseteq \overline{\sigma}^N \cap \overline{\sigma}^P$.

(5) Since $N \cup P \supseteq N$ and $N \cup P \supseteq P$, we have from part (1) $\underline{\sigma}^{N \cup P} \supseteq \underline{\sigma}^N$ and $\underline{\sigma}^{N \cup P} \supseteq \underline{\sigma}^P$. Thus, $\underline{\sigma}^{N \cup P} \supseteq \underline{\sigma}^N \cup \underline{\sigma}^P$.

(6) Let $u_1 \in U_1$. If $u_1 \sigma(e) = \emptyset$, then $\overline{\sigma}^{\mu_{N \cup P}}(e)(u_1) = 0 = \overline{\sigma}^{\mu_N}(e)(u_1) \cup \overline{\sigma}^{\mu_P}(e)(u_1)$ and $\overline{\sigma}^{\gamma_{N \cup P}}(e)(u_1) = 1 = \overline{\sigma}^{\gamma_N}(e)(u_1) \cap \overline{\sigma}^{\gamma_P}(e)(u_1)$. If $u_1 \sigma(e) \neq \emptyset$, then $\overline{\sigma}^{\mu_N \cup \mu_P}(e)(u_1) = \overline{\sigma}^{\mu_N}(e)(u_1) \vee \overline{\sigma}^{\mu_P}(e)(u_1) = (\bigvee_{a \in u_1 \sigma(e)} \mu_N(a)) \vee (\bigvee_{a \in u_1 \sigma(e)} \mu_P(a)) = \bigvee_{a \in u_1 \sigma(e)} (\mu_N(a) \vee \mu_P(a)) = \bigvee_{a \in u_1 \sigma(e)} (\mu_N \vee \mu_P)(a) = \bigvee_{a \in u_1 \sigma(e)} (\mu_{N \cup P})(a) = \overline{\sigma}^{\mu_{N \cup P}}(e)(u_1)$.

Also,

$$\begin{aligned} (\overline{\sigma}^{\gamma_N} \cap \overline{\sigma}^{\gamma_P})(e)(u_1) &= \overline{\sigma}^{\gamma_N}(e)(u_1) \wedge \overline{\sigma}^{\gamma_P}(e)(u_1) = \\ &= (\bigwedge_{a \in u_1 \sigma(e)} \gamma_N(a)) \wedge (\bigwedge_{a \in u_1 \sigma(e)} \gamma_P(a)) = \bigwedge_{a \in u_1 \sigma(e)} (\gamma_N(a) \wedge \gamma_P(a)) = \bigwedge_{a \in u_1 \sigma(e)} (\gamma_N \wedge \gamma_P)(a) = \bigwedge_{a \in u_1 \sigma(e)} (\gamma_{N \cap P})(a) = \overline{\sigma}^{\gamma_{N \cap P}}(e)(u_1). \end{aligned}$$

This shows that $\overline{\sigma}^N \cup \overline{\sigma}^P = \overline{\sigma}^{N \cup P}$.

(7) Consider $\underline{\sigma}^1(e)(u_1) = \bigwedge_{a \in u_1 \sigma(e)} 1(a) = \bigwedge_{a \in u_1 \sigma(e)} (1) = 1$, because $u_1 \sigma(e) \neq \emptyset$

and $\underline{\sigma}^0(e)(u_1) = \bigvee_{a \in u_1 \sigma(e)} 0(a) = \bigvee_{a \in u_1 \sigma(e)} (0) = 0$, because $u_1 \sigma(e) \neq \emptyset$.

Thus, $\underline{\sigma}^{1_{U_1}} = 1_{U_1}$.

(8) The proof is similar to the proof of part (7).

(9) Let $M = \langle \mu_M, \gamma_M \rangle$ be an IFS on U_2 . Then, $M^c = \langle \mu_{M^c}, \gamma_{M^c} \rangle = \langle \gamma_M, \mu_M \rangle$, that is, $\mu_{M^c} = \gamma_M$ and $\gamma_{M^c} = \mu_M$. Now, $\overline{\sigma}^{M^c} = (\overline{\sigma}^{\mu_{M^c}}, \overline{\sigma}^{\gamma_{M^c}}) = (\overline{\sigma}^{\gamma_M}, \overline{\sigma}^{\mu_M})$. Thus, $\overline{\sigma}^{\mu_{M^c}}(e)(u_1) = \bigvee_{a \in u_1 \sigma(e)} \mu_{M^c}(a) = \bigvee_{a \in u_1 \sigma(e)} \gamma_M(a) = \underline{\sigma}^{\gamma_M}(e)(u_1)$ and $\overline{\sigma}^{\gamma_{M^c}}(e)(u_1) = \bigwedge_{a \in u_1 \sigma(e)} \gamma_{M^c}(a) = \bigwedge_{a \in u_1 \sigma(e)} \mu_M(a) = \underline{\sigma}^{\mu_M}(e)(u_1)$. Hence, $\overline{\sigma}^{M^c} = (\underline{\sigma}^{\gamma_M}, \underline{\sigma}^{\mu_M}) = (\underline{\sigma}^{\gamma_M}, \underline{\sigma}^{\mu_M})^c$, that is, $(\overline{\sigma}^{M^c})^c = \overline{\sigma}^M$.

(10) follows from part (9).

(11) Straightforward.

Theorem 1 describes the properties of newly defined IFRS model based on soft relations. It shows that if an IFS N is the subset of IFS P , then the lower approximation of N is also a subset of the lower approximation of P , and if an IFS N is subset of IFS P , then the upper approximation of N is also a subset of the upper approximation of P . Similarly, the empirical relations among the operations union, intersection, and complement have been described. \square

Theorem 2. Let (σ, A) be a soft binary relation from U_1 to U_2 , that is, $\sigma: A \rightarrow P(U_1 \times U_2)$. For any IFSs, $M = \langle \mu_M, \gamma_M \rangle$, $N = \langle \mu_N, \gamma_N \rangle$, and $P = \langle \mu_P, \gamma_P \rangle$ of U_1 , the following are true:

- (1) If $N \subseteq P$, then ${}^N\underline{\sigma} \subseteq {}^P\underline{\sigma}$;
- (2) If $N \subseteq P$, then ${}^N\overline{\sigma} \subseteq {}^P\overline{\sigma}$;
- (3) ${}^N\underline{\sigma} \cap {}^P\underline{\sigma} = {}^{N \cap P}\underline{\sigma}$;
- (4) ${}^N\overline{\sigma} \cap {}^P\overline{\sigma} = {}^{N \cap P}\overline{\sigma}$;
- (5) ${}^N\underline{\sigma} \cup {}^P\underline{\sigma} \subseteq {}^{N \cup P}\underline{\sigma}$;
- (6) ${}^N\overline{\sigma} \cup {}^P\overline{\sigma} \subseteq {}^{N \cup P}\overline{\sigma}$;
- (7) $\underline{\sigma}^{1_{U_1}} = 1_{U_2}$ if $u_1 \sigma(e) \neq \emptyset$;
- (8) $\overline{\sigma}^{1_{U_1}} = 1_{U_2}$ if $u_1 \sigma(e) \neq \emptyset$;
- (9) ${}^M\underline{\sigma} = (M^c \overline{\sigma})^c$ if $u_1 \sigma(e) \neq \emptyset$;
- (10) ${}^M\overline{\sigma} = (M^c \underline{\sigma})^c$ if $u_1 \sigma(e) \neq \emptyset$;
- (11) ${}^{0_{U_1}}\underline{\sigma} = 0_{U_2} = {}^{0_{U_1}}\overline{\sigma}$.

Proof. The proof is similar to the proof of Theorem 1.

The following example shows that equality does not hold in (4) and (5) assertions of above theorems in general. \square

TABLE 6: Intuitionistic fuzzy set P .

	c_1	c_2	c_3	c_4
μ_P	0.1	0.6	0.5	1
γ_P	0.9	0.1	0.5	0

Example 3. Consider Example 2. Let $P = \langle \mu_P, \gamma_P \rangle$: $U_2 \rightarrow [0, 1]$ (given in Table 6).

Table 6 simply shows the degree of membership and degree of nonmembership of IFS P .

Then, $M \cup P = \langle \mu_{M \cup P}, \gamma_{M \cup P} \rangle = \langle \mu_M \cup \mu_P, \gamma_M \cap \gamma_P \rangle$ (given in Table 7) and $M \cap P = \langle \mu_{M \cap P}, \gamma_{M \cap P} \rangle = \langle \mu_M \cap \mu_P, \gamma_M \cup \gamma_P \rangle$ (given in Table 7).

Table 7 shows the calculations of union and intersection of two IFSs M and P , respectively.

Now, $\underline{\sigma}^P = (\underline{\sigma}^{\mu_P}, \underline{\sigma}^{\gamma_P})$ (given in Table 8) and $\underline{\sigma}^{M \cup P} = (\underline{\sigma}^{\mu_{M \cup P}}, \underline{\sigma}^{\gamma_{M \cup P}})$ (given in Table 9).

Now, $\underline{\sigma}^M \cup \underline{\sigma}^P$ (given in Table 10).

In Table 10, we calculated $\underline{\sigma}^M \cup \underline{\sigma}^P$. Tables 9 and 10 show that $\underline{\sigma}^M \cup \underline{\sigma}^P \neq \underline{\sigma}^{M \cup P}$.

Now, $\overline{\sigma}^P = (\overline{\sigma}^{\mu_P}, \overline{\sigma}^{\gamma_P})$ (given in Table 11).

In Table 11, we calculated upper approximation of P .

Now, $\overline{\sigma}^{M \cap P} = (\overline{\sigma}^{\mu_{M \cap P}}, \overline{\sigma}^{\gamma_{M \cap P}})$ (given in Table 12).

Now, $\overline{\sigma}^M \cap \overline{\sigma}^P$ (given in Table 13).

In Table 12, we calculated upper approximation of $M \cap P$. In Table 13, we calculated the intersection of upper approximations of M and P . Tables 12 and 13 show that $\overline{\sigma}^M \cap \overline{\sigma}^P \neq \overline{\sigma}^{M \cap P}$.

Theorem 3. Let (σ_1, A) and (σ_2, A) be two soft binary relations from U_1 to U_2 , such that $(\sigma_1, A) \subseteq (\sigma_2, A)$, that is, $\sigma_1(e) \subseteq \sigma_2(e)$ for all $e \in A$. Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U_2 , the following are true:

- (i) $\underline{\sigma}_2^M \subseteq \underline{\sigma}_1^M$;
- (ii) $\overline{\sigma}_1^M \subseteq \overline{\sigma}_2^M$.

Proof

- (1) Let $u_1 \in U_1$. If $u_1 \sigma_1(e) = \emptyset$, then $\underline{\sigma}_1^{\mu_M}(e)(u_1) = 1 \geq \underline{\sigma}_2^{\mu_M}(e)(u_1)$ and $\underline{\sigma}_1^{\gamma_M}(e)(u_1) = 0 \leq \underline{\sigma}_2^{\gamma_M}(e)(u_1)$. If $u_1 \sigma_1(e) \neq \emptyset$, then $u_1 \sigma_2(e) \neq \emptyset$, we have

$$\underline{\sigma}_1^{\mu_M}(e)(u_1) = \bigwedge_{a \in u_1 \sigma_1(e)} \mu_M(a) \geq \bigwedge_{a \in u_1 \sigma_2(e)} \mu_M(a) \text{ because } u_1 \sigma_1(e) \subseteq u_1 \sigma_2(e) = \underline{\sigma}_2^{\mu_M}(e)(u_1).$$

Also,

$$\underline{\sigma}_1^{\gamma_M}(e)(u_1) = \bigvee_{a \in u_1 \sigma_1(e)} \gamma_M(a) \leq \bigvee_{a \in u_1 \sigma_2(e)} \gamma_M(a) \text{ because } u_1 \sigma_1(e) \subseteq u_1 \sigma_2(e)$$

$$= \underline{\sigma}_2^{\gamma_M}(e)(u_1).$$

Hence, $\underline{\sigma}_2^M \subseteq \underline{\sigma}_1^M$.

- (2) Let $u_1 \in U_1$. If $u_1 \sigma_1(e) = \emptyset$, then $\overline{\sigma}_1^{\mu_M}(e)(u_1) = 0 \leq \overline{\sigma}_2^{\mu_M}(e)(u_1)$ and $\overline{\sigma}_1^{\gamma_M}(e)(u_1) = 1 \geq \overline{\sigma}_2^{\gamma_M}(e)(u_1)$. If $u_1 \sigma_1(e) \neq \emptyset$, then $u_1 \sigma_2(e) \neq \emptyset$, and we have

$$\overline{\sigma}_1^{\mu_M}(e)(u_1) = \bigvee_{a \in u_1 \sigma_1(e)} \mu_M(a) \leq \bigvee_{a \in u_1 \sigma_2(e)} \mu_M(a) \text{ because } u_1 \sigma_1(e) \subseteq u_1 \sigma_2(e) = \overline{\sigma}_2^{\mu_M}(e)(u_1).$$

TABLE 7: Intuitionistic fuzzy sets $M \cup P$, $M \cap P$.

	c_1	c_2	c_3	c_4
$M \cup P$	(0.9, 0)	(0.8, 0.1)	(0.5, 0.5)	(1, 0)
$M \cap P$	(0.1, 0.9)	(0.6, 0.2)	(0.4, 0.5)	(0, 0.8)

TABLE 8: Lower approximation of P .

$\underline{\sigma}^P$	e_1	e_2	e_3
d_1	(0.1, 0.9)	(0.5, 0.5)	(1, 0)
d_2	(0.6, 0.1)	(0.5, 0.5)	(1, 0)
d_3	(1, 0)	(1, 0)	(0.1, 0.9)
d_4	(0.5, 0.5)	(0.1, 0.9)	(1, 0)
d_5	(0.5, 0.5)	(0.1, 0.9)	(0.5, 0.5)
d_6	(0.1, 0.9)	(0.5, 0.5)	(1, 0)

TABLE 9: Lower approximation of $M \cup P$.

$\underline{\sigma}^{M \cup P}$	e_1	e_2	e_3
d_1	(0.5, 0.5)	(0.5, 0.5)	(1, 0)
d_2	(0.8, 0.1)	(0.5, 0.5)	(1, 0)
d_3	(1, 0)	(1, 0)	(0.5, 0.5)
d_4	(0.5, 0.5)	(0.9, 0)	(1, 0)
d_5	(0.5, 0.5)	(0.9, 0)	(0.5, 0.5)
d_6	(0.9, 0)	(0.5, 0.5)	(1, 0)

TABLE 10: Union of lower approximations of M and P .

$\underline{\sigma}^M \cup \underline{\sigma}^P$	e_1	e_2	e_3
d_1	(0.4, 0.5)	(0.5, 0.5)	(1, 0)
d_2	(0.6, 0.1)	(0.5, 0.5)	(1, 0)
d_3	(1, 0)	(1, 0)	(0.4, 0.5)
d_4	(0.5, 0.5)	(0.9, 0)	(1, 0)
d_5	(0.5, 0.5)	(0.9, 0)	(0.5, 0.5)
d_6	(0.9, 0)	(0.5, 0.5)	(1, 0)

TABLE 11: Upper approximation of P .

$\overline{\sigma}^P$	e_1	e_2	e_3
d_1	(0.6, 0.1)	(0.5, 0.5)	(0, 1)
d_2	(1, 0)	(0.5, 0.5)	(1, 0)
d_3	(0, 1)	(0, 1)	(0.5, 0.5)
d_4	(0.6, 0.1)	(0.1, 0.9)	(0, 1)
d_5	(1, 0)	(0.1, 0.9)	(1, 0)
d_6	(0.1, 0.9)	(0.6, 0.1)	(0, 1)

Also,

$$\overline{\sigma}_1^{\gamma_M}(e)(u_1) = \bigwedge_{a \in u_1 \sigma_1(e)} \gamma_M(a) \geq \bigwedge_{a \in u_1 \sigma_2(e)} \gamma_M(a) \text{ because } u_1 \sigma_1(e) \subseteq u_1 \sigma_2(e)$$

$$= \overline{\sigma}_2^{\gamma_M}(e)(u_1).$$

Hence, $\overline{\sigma}_1^M \subseteq \overline{\sigma}_2^M$.

Theorem 3 shows that if any soft relation $(\sigma_1, A) \subseteq (\sigma_2, A)$, then for any IFS M in U_2 , the lower approximation associated with (σ_2, A) is a subset of (σ_1, A) . Similarly, if any soft relation $(\sigma_1, A) \subseteq (\sigma_2, A)$, then for any

TABLE 12: Upper approximation of $M \cap P$.

$\overline{\sigma}^{M \cap P}$	e_1	e_2	e_3
d_1	(0.6, 0.2)	(0.4, 0.5)	(0, 1)
d_2	(0.6, 0.2)	(0.4, 0.5)	(0, 0.8)
d_3	(0, 1)	(0, 1)	(0.4, 0.5)
d_4	(0.6, 0.2)	(0.1, 0.9)	(0, 1)
d_5	(0.4, 0.5)	(0.1, 0.9)	(0.4, 0.5)
d_6	(0.1, 0.9)	(0.6, 0.2)	(0, 1)

TABLE 13: Intersection of upper approximations of M and P .

$\overline{\sigma}^M \cap \overline{\sigma}^P$	e_1	e_2	e_3
d_1	(0.6, 0.1)	(0.4, 0.5)	(0, 1)
d_2	(0.8, 0.2)	(0.4, 0.5)	(0, 0.8)
d_3	(0, 1)	(0, 1)	(0.5, 0.5)
d_4	(0.6, 0.2)	(0.1, 0.9)	(0, 1)
d_5	(0.4, 0.5)	(0.1, 0.9)	(0.4, 0.5)
d_6	(0.1, 0.9)	(0.6, 0.2)	(0, 1)

IFS M in U_2 , the upper approximation associated with (σ_1, A) is a subset of (σ_2, A) . \square

Theorem 4. Let (σ_1, A) and (σ_2, A) be two soft binary relations from U_1 to U_2 , such that $(\sigma_1, A) \subseteq (\sigma_2, A)$, that is, $\sigma_1(e) \subseteq \sigma_2(e)$ for all $e \in A$. Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U_1 , the following are true:

- (1) ${}^M \underline{\sigma}_2 \subseteq {}^M \underline{\sigma}_1$
- (2) ${}^M \overline{\sigma}_1 \subseteq {}^M \overline{\sigma}_2$

Proof

- (1) Let $u_2 \in U_2$. If $\sigma_1(e)u_2 = \emptyset$, then ${}^{\mu_M} \underline{\sigma}_1(e)(u_2) = 1 \geq {}^{\mu_M} \underline{\sigma}_2(e)(u_2)$ and ${}^{\gamma_M} \overline{\sigma}_1(e)(u_2) = 0 \leq {}^{\gamma_M} \overline{\sigma}_2(e)(u_2)$. If $\sigma_1(e)u_2 \neq \emptyset$, then $\sigma_2(e)u_2 \neq \emptyset$, we have

$${}^{\mu_M} \underline{\sigma}_1(e)(u_2) = \bigwedge_{a \in \sigma_1(e)u_2} \mu_M(a) \geq \bigwedge_{a \in \sigma_2(e)u_2} \mu_M(a) \text{ because } \sigma_1(e)u_2 \subseteq \sigma_2(e)u_2 = {}^{\mu_M} \underline{\sigma}_2(e)(u_2).$$

Also,

$${}^{\gamma_M} \overline{\sigma}_1(e)(u_2) = \bigvee_{a \in \sigma_1(e)u_2} \gamma_M(a) \leq \bigvee_{a \in \sigma_2(e)u_2} \gamma_M(a) \text{ because } \sigma_1(e)u_2 \subseteq \sigma_2(e)u_2$$

$$= {}^{\gamma_M} \overline{\sigma}_2(e)(u_2).$$

$$\text{Hence, } {}^M \underline{\sigma}_2 \subseteq {}^M \underline{\sigma}_1.$$

- (2) Let $u_2 \in U_2$. If $\sigma_1(e)u_2 = \emptyset$, then ${}^{\mu_M} \overline{\sigma}_1(e)(u_2) = 0 \leq {}^{\mu_M} \overline{\sigma}_2(e)(u_2)$ and ${}^{\gamma_M} \underline{\sigma}_1(e)(u_2) = 1 \geq {}^{\gamma_M} \underline{\sigma}_2(e)(u_2)$. If $\sigma_1(e)u_2 \neq \emptyset$, then $\sigma_2(e)u_2 \neq \emptyset$, and we have

$${}^{\mu_M} \overline{\sigma}_1(e)(u_2) = \bigvee_{a \in \sigma_1(e)u_2} \mu_M(a) \leq \bigvee_{a \in \sigma_2(e)u_2} \mu_M(a) \text{ because } \sigma_1(e)u_2 \subseteq \sigma_2(e)u_2 = {}^{\mu_M} \overline{\sigma}_2(e)(u_2).$$

Also,

$${}^{\gamma_M} \underline{\sigma}_1(e)(u_2) = \bigwedge_{a \in \sigma_1(e)u_2} \gamma_M(a) \geq \bigwedge_{a \in \sigma_2(e)u_2} \gamma_M(a) \text{ because } \sigma_1(e)u_2 \subseteq \sigma_2(e)u_2$$

$$= {}^{\gamma_M} \underline{\sigma}_2(e)(u_2).$$

$$\text{Hence, } {}^M \overline{\sigma}_1 \subseteq {}^M \overline{\sigma}_2. \quad \square$$

Theorem 5. Let (σ_1, A) and (σ_2, A) be two soft binary relations from U_1 to U_2 . Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U_2 , the following are true:

- (1) $\underline{\sigma}_1^M \subseteq (\sigma_1 \cap \sigma_2)^M$;
- (2) $\underline{\sigma}_2^M \subseteq (\sigma_1 \cap \sigma_2)^M$;
- (3) $(\sigma_1 \cap \sigma_2)^M \subseteq \overline{\sigma}_1^M$;
- (4) $(\sigma_1 \cap \sigma_2)^M \subseteq \overline{\sigma}_2^M$.

Proof

- (1) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$, therefore from Theorem 3 part (1), $\underline{\sigma}_1^M \subseteq (\sigma_1 \cap \sigma_2)^M$.
- (2) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, therefore from Theorem 3 part (1), $\underline{\sigma}_2^M \subseteq (\sigma_1 \cap \sigma_2)^M$.
- (3) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$, therefore from Theorem 3 part (2), $(\sigma_1 \cap \sigma_2)^M \subseteq \overline{\sigma}_1^M$.
- (4) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, therefore from Theorem 3 part (2), $(\sigma_1 \cap \sigma_2)^M \subseteq \overline{\sigma}_2^M$. \square

Theorem 6. Let (σ_1, A) and (σ_2, A) be two soft binary relations from U_1 to U_2 . Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U_1 , the following are true:

- (1) ${}^M \underline{\sigma}_1 \subseteq {}^M (\sigma_1 \cap \sigma_2)$;
- (2) ${}^M \underline{\sigma}_2 \subseteq {}^M (\sigma_1 \cap \sigma_2)$;
- (3) ${}^M (\sigma_1 \cap \sigma_2) \subseteq {}^M \overline{\sigma}_1$;
- (4) ${}^M (\sigma_1 \cap \sigma_2) \subseteq {}^M \overline{\sigma}_2$.

Proof

- (1) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$, therefore from Theorem 4 part (1), ${}^M \underline{\sigma}_1 \subseteq {}^M (\sigma_1 \cap \sigma_2)$.
- (2) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, therefore from Theorem 4 part (1), ${}^M \underline{\sigma}_2 \subseteq {}^M (\sigma_1 \cap \sigma_2)$.
- (3) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$, therefore from Theorem 4 part (2), ${}^M (\sigma_1 \cap \sigma_2) \subseteq {}^M \overline{\sigma}_1$.
- (4) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, therefore from Theorem 4 part (2), ${}^M (\sigma_1 \cap \sigma_2) \subseteq {}^M \overline{\sigma}_2$.

In Theorems 5 and 6, some empirical relations have been discussed about union and intersection of two soft relations (σ_1, A) and (σ_2, A) with respect to the aftersets and with respect to the foresets, respectively. \square

Definition 4. If (σ, A) is a soft set over $U \times U$, then (σ, A) is called a soft binary relation on U .

In fact, (σ, A) is a parameterized collection of binary relations on U . That is, for each parameter $e \in A$, we have a

binary relation $\sigma(e)$ on U . A soft binary relation (σ, A) on U is said to be soft reflexive relation on U if $\sigma(e)$ is a reflexive relation on U for all $e \in A$. If (σ, A) is a soft reflexive binary relation on U , then $u\sigma(e)$ (resp. $\sigma(e)u$) is nonempty and $u \in u\sigma(e)$ (resp. $u \in \sigma(e)u$). It is not necessary that $u\sigma(e) = \sigma(e)u$. A soft binary relation (σ, A) on U is said to be soft symmetric relation on U if $\sigma(e)$ is a symmetric relation on U for all $e \in A$. A soft binary relation (σ, A) on U is said to be soft transitive relation on U if $\sigma(e)$ is a transitive relation on U for all $e \in A$.

A soft binary relation (σ, A) over U is soft equivalence relation over U if it is soft reflexive, soft symmetric, and soft transitive relation over U . A soft binary relation (σ, A) over U is a soft equivalence relation over U if $\sigma(e)$ for all $e \in A$ is an equivalence relation over U . In this case, $u\sigma(e) = \sigma(e)u$ and $\{u\sigma(e) : u \in U\}$ is a partition of U . Also, in this case, ${}^M\bar{\sigma}(e) = \bar{\sigma}^M(e)$ and ${}^M\bar{\sigma}(e) = \bar{\sigma}^M(e)$, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U .

The approximation operators have additional properties with respect to soft reflexive binary relation as follows.

Theorem 7. Let (σ, A) be a soft reflexive binary relation on U . Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U , the following are true:

- (1) $\underline{\sigma}^{\mu_M}(e) \leq \mu_M$ for all $e \in A$;
- (2) $\mu_M \leq \bar{\sigma}^{\mu_M}(e)$ for all $e \in A$;
- (3) $\underline{\sigma}^{\gamma_M}(e) \leq \bar{\sigma}^{\gamma_M}(e)$ for all $e \in A$;
- (4) $\underline{\sigma}^{\gamma_M}(e) \geq \gamma_M$ for all $e \in A$;
- (5) $\gamma_M \geq \bar{\sigma}^{\gamma_M}(e)$ for all $e \in A$;
- (6) $\underline{\sigma}^{\gamma_M}(e) \geq \bar{\sigma}^{\gamma_M}(e)$ for all $e \in A$.

Proof

- (1) Let $u \in U$. Then,
 $\underline{\sigma}^{\mu_M}(e)(u) = \bigwedge_{a \in u\sigma_1(e)} \mu_M(a) \leq \mu_M(u)$ because $u \in u\sigma(e)$.
- (2) Hence, $\underline{\sigma}^{\mu_M}(e) \leq \mu_M$.
 Let $u \in U$. Then,
 $\bar{\sigma}^{\mu_M}(e)(u) = \bigvee_{a \in u\sigma_1(e)} \mu_M(a) \geq \mu_M(u)$ because $u \in u\sigma(e)$
 Hence, $\mu_M \leq \bar{\sigma}^{\mu_M}(e)$.
- (3) It follows from part (1) and part (2).
- (4) Let $u \in U$. Then,
 $\underline{\sigma}^{\gamma_M}(e)(u) = \bigvee_{a \in u\sigma_1(e)} \gamma_M(a) \geq \gamma_M(u)$ because $u \in u\sigma(e)$
 Hence, $\underline{\sigma}^{\gamma_M}(e) \geq \gamma_M$.
- (5) Let $u \in U$. Then,
 $\bar{\sigma}^{\gamma_M}(e)(u) = \bigwedge_{a \in u\sigma_1(e)} \gamma_M(a) \leq \gamma_M(u)$ because $u \in u\sigma(e)$
 Hence, $\gamma_M \geq \bar{\sigma}^{\gamma_M}(e)$.
 It follows from part (4) and part (5).

Theorem 7 shows the empirical relations between IFS M and a soft reflexive relation (σ, A) . \square

Theorem 8. Let (σ, A) be a soft reflexive binary relation on U . Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U , the following are true:

- (1) ${}^{\mu_M}\underline{\sigma}(e) \leq \mu_M$ for all $e \in A$;
- (2) $\mu_M \leq {}^{\mu_M}\bar{\sigma}(e)$ for all $e \in A$;
- (3) ${}^{\mu_M}\underline{\sigma}(e) \leq {}^{\mu_M}\bar{\sigma}(e)$ for all $e \in A$;
- (4) ${}^{\gamma_M}\underline{\sigma}(e) \geq \gamma_M$ for all $e \in A$;
- (5) $\gamma_M \geq {}^{\gamma_M}\bar{\sigma}(e)$ for all $e \in A$;
- (6) ${}^{\gamma_M}\underline{\sigma}(e) \geq {}^{\gamma_M}\bar{\sigma}(e)$ for all $e \in A$.

Proof

- (1) Let $u \in U$. Then,
 ${}^{\mu_M}\underline{\sigma}(e)(u) = \bigwedge_{a \in \sigma_1(e)u} \mu_M(a) \leq \mu_M(u)$ because $u \in \sigma(e)u$.
 Hence, ${}^{\mu_M}\underline{\sigma}(e) \leq \mu_M$.
- (2) Let $u \in U$. Then,
 ${}^{\mu_M}\bar{\sigma}(e)(u) = \bigvee_{a \in \sigma_1(e)u} \mu_M(a) \geq \mu_M(u)$ because $u \in \sigma(e)u$
 Hence, $\mu_M \leq {}^{\mu_M}\bar{\sigma}(e)$.
- (3) It follows from part (1) and part (2).
- (4) Let $u \in U$. Then,
 ${}^{\gamma_M}\underline{\sigma}(e)(u) = \bigvee_{a \in \sigma_1(e)u} \gamma_M(a) \geq \gamma_M(u)$ because $u \in \sigma(e)u$
 Hence, ${}^{\gamma_M}\underline{\sigma}(e) \geq \gamma_M$.
- (5) Let $u \in U$. Then,
 ${}^{\gamma_M}\bar{\sigma}(e)(u) = \bigwedge_{a \in \sigma_1(e)u} \gamma_M(a) \leq \gamma_M(u)$ because $u \in \sigma(e)u$
 (i) Hence, $\gamma_M \geq {}^{\gamma_M}\bar{\sigma}(e)$.
- (6) It follows from part (4) and part (5). \square

4. Similarity Relations

In this section, we define some relations between IFS of U_2 with the help of a soft relation from U_1 to U_2 . We say that two intuitionistic fuzzy sets in U_2 are related if the lower (upper) approximations in U_1 are equal. Similarly, we define relations between intuitionistic fuzzy sets of U_1 .

Definition 5. Let (σ, A) be a soft binary relation from U_1 to U_2 . Then, for any IFS $N = \langle \mu_N, \gamma_N \rangle$ and $P = \langle \mu_P, \gamma_P \rangle$ of U_2 , we define

$N \approx_A P$ if and only if $\underline{\sigma}^N = \underline{\sigma}^P$ $N \approx_A P$ if and only if $\bar{\sigma}^N = \bar{\sigma}^P$ $N \approx_A P$ if and only if $\underline{\sigma}^N = \underline{\sigma}^P$ and $\bar{\sigma}^N = \bar{\sigma}^P$.

Definition 6. Let (σ, A) be a soft binary relation from U_1 to U_2 . Then, for any IFS $N = \langle \mu_N, \gamma_N \rangle$ and $P = \langle \mu_P, \gamma_P \rangle$ of U_1 , we define

$N \approx_{\sigma} P$ if and only if ${}^N \underline{\sigma} = {}^P \underline{\sigma}$; $N \approx_{\sigma} P$ if and only if ${}^N \bar{\sigma} = {}^P \bar{\sigma}$.
 $N \approx_{\sigma} P$ if and only if ${}^N \underline{\sigma} = {}^P \underline{\sigma}$ and ${}^N \bar{\sigma} = {}^P \bar{\sigma}$.

These binary relations may be called the lower similarity relation, upper similarity relation, and similarity relation, respectively.

Definitions 5 and 6 show that if an IFS N has upper(lower) similarity relation with an IFS P , then its associated lower(upper) approximation has also upper(lower) similarity relation.

Proposition 1. *The relations \approx_A , \approx_{A^*} , and \approx_A are equivalence relations on $IF(U_2)$.*

Proof. \approx_A is reflexive: let N be an IFS of U_2 . Since $\underline{\sigma}^N = \underline{\sigma}^N$, so we have $N \approx_A N$. \approx_A is symmetric: let N and P be IFSs of U_2 such that $N \approx_P P$; this implies $\underline{\sigma}^N = \underline{\sigma}^P$, so $\underline{\sigma}^P = \underline{\sigma}^N$; this implies $P \approx_N N$.

\approx_A is transitive: let N, P and Q be IFSs of U_2 such that $N \approx_P P$ and $P \approx_Q Q$; this implies $\underline{\sigma}^N = \underline{\sigma}^P$ and $\underline{\sigma}^P = \underline{\sigma}^Q$, so $\underline{\sigma}^N = \underline{\sigma}^Q$; this implies $N \approx_Q Q$.

Thus, \approx_A is an equivalence relation on $IF(U_2)$.

Similarly, \approx_{A^*} and \approx_A are equivalence relations on $IF(U_2)$. \square

Proposition 2. *The relations \approx_{σ} , \approx_{σ^*} , and \approx_{σ} are equivalence relations on $IF(U_1)$.*

Proof. \approx_{σ} is reflexive: let N be an IFS of U_1 . Since ${}^N \underline{\sigma} = {}^N \underline{\sigma}$, so we have $N \approx_{\sigma} N$.

\approx_{σ} is symmetric: let N and P be IFSs of U_1 such that $N \approx_P P$; this implies ${}^N \underline{\sigma} = {}^P \underline{\sigma}$, so ${}^P \underline{\sigma} = {}^N \underline{\sigma}$; this implies $P \approx_N N$.

\approx_{σ} is transitive: let N, P , and Q be IFSs of U_1 such that $N \approx_P P$ and $P \approx_Q Q$; this implies ${}^N \underline{\sigma} = {}^P \underline{\sigma}$ and ${}^P \underline{\sigma} = {}^Q \underline{\sigma}$, so ${}^N \underline{\sigma} = {}^Q \underline{\sigma}$; this implies $N \approx_Q Q$.

Thus, \approx_{σ} is an equivalence relation on $IF(U_1)$.

Similarly, \approx_{σ^*} and \approx_{σ} are equivalence relations on $IF(U_1)$. \square

Theorem 9. *Let (σ, A) be a soft binary relation from U_1 to U_2 . Let N, P, Q , and T be IFSs of U_2 . Then, the following are true:*

- (1) $N \approx_A P$ if and only if $N \approx_A (N \cup P) \approx_A P$;
- (2) $N \approx_A P$ and $Q \approx_A T$ imply that $(N \cup Q) \approx_A (P \cup T)$;
- (3) $N \subseteq P$ and $P \approx_A 0_{U_2}$ imply that $N \approx_A 0$;
- (4) $(N \cup P) \approx_A 0_{U_2}$ if and only if $N \approx_A 0_{U_2}$ and $P \approx_A 0_{U_2}$;
- (5) $N \subseteq P$ and $N \approx_A 1_{U_2}$ imply that $P \approx_A 1$;
- (6) If $(N \cap P) \approx_A 1_{U_2}$, then $N \approx_A 1_{U_2}$ and $P \approx_A 1_{U_2}$.

Proof

- (1) Let $N \approx_A P$. Then, $\bar{\sigma}^N = \bar{\sigma}^P$. By Theorem 1, we get $\bar{\sigma}^{N \cup P} = \bar{\sigma}^N \cup \bar{\sigma}^P = \bar{\sigma}^N = \bar{\sigma}^P$. This implies that $N \approx_A (N \cup P) \approx_A P$. Conversely, it holds due to transitive property of relation \approx_A .
- (2) Let $N \approx_A P$ and $Q \approx_A T$. Then, $\bar{\sigma}^N = \bar{\sigma}^P$ and $\bar{\sigma}^Q = \bar{\sigma}^T$.

By Theorem 1, we get $\bar{\sigma}^{N \cup Q} = \bar{\sigma}^N \cup \bar{\sigma}^Q = \bar{\sigma}^P \cup \bar{\sigma}^T = \bar{\sigma}^{P \cup T}$. This implies that $(N \cup Q) \approx_A (P \cup T)$.

- (3) Let $N \subseteq P$ and $P \approx_A 0_{U_2}$. Then, $\bar{\sigma}^P = \bar{\sigma}^{0_{U_2}}$.

Also, by Theorem 1, $N \subseteq P$ implies that $\bar{\sigma}^N \subseteq \bar{\sigma}^P = \bar{\sigma}^{0_{U_2}}$. But $\bar{\sigma}^{0_{U_2}} \subseteq \bar{\sigma}^N$. Thus, $\bar{\sigma}^N = \bar{\sigma}^{0_{U_2}}$. This implies that $N \approx_A 0$.

- (4) If $N \approx_A 0_{U_2}$ and $P \approx_A 0_{U_2}$, then $\bar{\sigma}^N = \bar{\sigma}^{0_{U_2}}$ and $\bar{\sigma}^P = \bar{\sigma}^{0_{U_2}}$. Now, by Theorem 1, we have $\bar{\sigma}^{N \cup P} = \bar{\sigma}^N \cup \bar{\sigma}^P = \bar{\sigma}^{0_{U_2}} \cup \bar{\sigma}^{0_{U_2}} = \bar{\sigma}^{0_{U_2}}$. This implies that $(N \cup P) \approx_A 0_{U_2}$. Conversely, if $(N \cup P) \approx_A 0_{U_2}$, then by part (3), we have $N \approx_A 0_{U_2}$ and $P \approx_A 0_{U_2}$.

- (5) Suppose $N \approx_A 1_{U_2}$. Then, $\bar{\sigma}^N = \bar{\sigma}^{1_{U_2}}$. As

$N \subseteq P$, we have $\bar{\sigma}^P \supseteq \bar{\sigma}^N = \bar{\sigma}^{1_{U_2}}$. On the other hand, $P \subseteq 1_{U_2}$, so we have $\bar{\sigma}^P \subseteq \bar{\sigma}^{1_{U_2}}$. This implies that $\bar{\sigma}^P = \bar{\sigma}^{1_{U_2}}$, that is, $P \approx_A 1_{U_2}$.

- (6) It follows from (5).

Theorem 9 shows some lower similarity relations of union and intersection of IFSs N, P, Q , and T in U_2 with respect to the aftersets. \square

Theorem 10. *Let (σ, A) be a soft binary relation from U_1 to U_2 . Let N, P, Q , and T be IFSs of U_1 . Then, the following are true:*

- (1) $N \approx_{\sigma} P$ if and only if $N \approx_{\sigma} (N \cup P) \approx_{\sigma} P$;
- (2) $N \approx_{\sigma} P$ and $Q \approx_{\sigma} T$ imply that $(N \cup Q) \approx_{\sigma} (P \cup T)$;
- (3) $N \subseteq P$ and $P \approx_{\sigma} 0_{U_1}$ imply that $N \approx_{\sigma} 0_{U_1}$;
- (4) $(N \cup P) \approx_{\sigma} 0_{U_1}$ if and only if $N \approx_{\sigma} 0_{U_1}$ and $P \approx_{\sigma} 0_{U_1}$;
- (5) $N \subseteq P$ and $N \approx_{\sigma} 1_{U_1}$ imply that $P \approx_{\sigma} 1_{U_1}$;
- (6) If $(N \cap P) \approx_{\sigma} 1_{U_1}$, then $N \approx_{\sigma} 1_{U_1}$ and $P \approx_{\sigma} 1_{U_1}$.

Proof

- (1) Let $N \approx_{\sigma} P$. Then, ${}^N \bar{\sigma} = {}^P \bar{\sigma}$. By Theorem 2, we get ${}^{N \cup P} \bar{\sigma} = {}^N \bar{\sigma} \cup {}^P \bar{\sigma} = {}^N \bar{\sigma} = {}^P \bar{\sigma}$. This implies that $N \approx_{\sigma} (N \cup P) \approx_{\sigma} P$. Converse holds by the transitivity of the relation \approx_{σ} .

- (2) Let $N \approx_{\sigma} P$ and $Q \approx_{\sigma} T$. Then, ${}^N \bar{\sigma} = {}^P \bar{\sigma}$ and ${}^Q \bar{\sigma} = {}^T \bar{\sigma}$.

By Theorem 2, we get ${}^{N \cup Q} \bar{\sigma} = {}^N \bar{\sigma} \cup {}^Q \bar{\sigma} = {}^P \bar{\sigma} \cup {}^T \bar{\sigma} = {}^{P \cup T} \bar{\sigma}$. This implies that $(N \cup Q) \approx_{\sigma} (P \cup T)$.

- (3) Let $N \subseteq P$ and $P \approx_{\sigma} 0_{U_1}$. Then, ${}^P \bar{\sigma} = {}^{0_{U_1}} \bar{\sigma}$.

Also, by Theorem 2, $N \subseteq P$ implies that ${}^N \bar{\sigma} \subseteq {}^P \bar{\sigma} = {}^{0_{U_1}} \bar{\sigma}$. But, ${}^{0_{U_1}} \bar{\sigma} \subseteq {}^N \bar{\sigma}$. Thus, ${}^N \bar{\sigma} = {}^{0_{U_1}} \bar{\sigma}$. This implies that $N \approx_{\sigma} 0$.

- (4) If $N \approx_{\sigma} 0_{U_1}$ and $P \approx_{\sigma} 0_{U_1}$, then ${}^N \bar{\sigma} = {}^{0_{U_1}} \bar{\sigma}$ and ${}^P \bar{\sigma} = {}^{0_{U_1}} \bar{\sigma}$. Now, by Theorem 2, we have ${}^{N \cup P} \bar{\sigma} = {}^N \bar{\sigma} \cup {}^P \bar{\sigma} = {}^{0_{U_1}} \bar{\sigma} \cup {}^{0_{U_1}} \bar{\sigma} = {}^{0_{U_1}} \bar{\sigma}$. This implies that $(N \cup P) \approx_{\sigma} 0_{U_1}$. Conversely, if $(N \cup P) \approx_{\sigma} 0_{U_1}$, then by part (3), we have $N \approx_{\sigma} 0_{U_1}$ and $P \approx_{\sigma} 0_{U_1}$.

- (5) Suppose $N \approx_{\sigma} 1_{U_1}$. Then, ${}^N \bar{\sigma} = {}^{1_{U_1}} \bar{\sigma}$. As

$N \subseteq P$, we have ${}^P \bar{\sigma} \supseteq {}^N \bar{\sigma} = {}^{1_{U_1}} \bar{\sigma}$. On the other hand, $P \subseteq 1_{U_1}$, so we have ${}^P \bar{\sigma} \subseteq {}^{1_{U_1}} \bar{\sigma}$. This implies that ${}^P \bar{\sigma} = {}^{1_{U_1}} \bar{\sigma}$, that is, $P \approx_{\sigma} 1_{U_1}$.

(6) It follows from (5).

Theorem 10 shows some lower similarity relations of union and intersection of IFSs N, P, Q , and T in U_1 with respect to the foresets. \square

Theorem 11. Let (σ, A) be a soft binary relation from U_1 to U_2 . Let N, P, Q , and T be IFSs of U_2 . Then, the following are true:

- (1) $N \approx_A P$ if and only if $N \approx_A (N \cap P) \approx_A P$;
- (2) $N \approx_A P$ and $Q \approx_A T$ imply that $(N \cap Q) \approx_A (P \cap T)$;
- (3) $N \subseteq P$ and $P \approx_A 0_{U_2}$ imply that $N \approx_A 0_{U_2}$;
- (4) $(N \cap P) \approx_A 0_{U_2}$ if and only if $N \approx_A 0_{U_2}$ and $P \approx_A 0_{U_2}$;
- (5) $N \subseteq P$ and $N \approx_A 1_{U_2}$ imply that $P \approx_A 1_{U_2}$;
- (6) If $(N \cap P) \approx_A 1_{U_2}$, then $N \approx_A 1_{U_2}$ and $P \approx_A 1_{U_2}$.

Proof

- (1) Let $N \approx_A P$. Then, $\underline{\sigma}^N = \underline{\sigma}^P$. By Theorem 1, we get $\underline{\sigma}^{N \cap P} = \underline{\sigma}^N \cap \underline{\sigma}^P = \underline{\sigma}^N = \underline{\sigma}^P$. This implies that $N \approx_A (N \cap P) \approx_A P$. Converse holds by the transitivity of the relation \approx_A .
- (2) Let $N \approx_A P$ and $Q \approx_A T$. Then, $\underline{\sigma}^N = \underline{\sigma}^P$ and $\underline{\sigma}^Q = \underline{\sigma}^T$. By Theorem 1, we get $\underline{\sigma}^{N \cap Q} = \underline{\sigma}^N \cap \underline{\sigma}^Q = \underline{\sigma}^P \cap \underline{\sigma}^T = \underline{\sigma}^{P \cap T}$. This implies that $(N \cap Q) \approx_A (P \cap T)$.
- (3) Let $N \subseteq P$ and $P \approx_A 0_{U_2}$. Then, $\underline{\sigma}^P = \underline{\sigma}^{0_{U_2}}$. Also, by Theorem 1, $N \subseteq P$ implies that $\underline{\sigma}^N \subseteq \underline{\sigma}^P = \underline{\sigma}^{0_{U_2}}$. But, $\underline{\sigma}^{0_{U_2}} \subseteq \underline{\sigma}^N$. Thus, $\underline{\sigma}^N = \underline{\sigma}^{0_{U_2}}$. This implies that $N \approx_A 0_{U_2}$.
- (4) If $N \approx_A 0_{U_2}$ and $P \approx_A 0_{U_2}$, then $\underline{\sigma}^N = \underline{\sigma}^{0_{U_2}}$ and $\underline{\sigma}^P = \underline{\sigma}^{0_{U_2}}$. Now, by Theorem 1, we have $\underline{\sigma}^{N \cap P} = \underline{\sigma}^N \cap \underline{\sigma}^P = \underline{\sigma}^{0_{U_2}} \cap \underline{\sigma}^{0_{U_2}} = \underline{\sigma}^{0_{U_2}}$, so $\underline{\sigma}^{N \cap P} = \underline{\sigma}^{0_{U_2}}$. This implies that $(N \cap P) \approx_A 0_{U_2}$. Conversely, if $(N \cap P) \approx_A 0_{U_2}$, then by part (3), we have $N \approx_A 0_{U_2}$ and $P \approx_A 0_{U_2}$.
- (5) Suppose $N \approx_A 1_{U_2}$. Then, $\underline{\sigma}^N = \underline{\sigma}^{1_{U_2}}$. As $N \subseteq P$, we have $\underline{\sigma}^P \supseteq \underline{\sigma}^N = \underline{\sigma}^{1_{U_2}}$. On the other hand, $P \subseteq 1_{U_2}$, so we have $\underline{\sigma}^P \subseteq \underline{\sigma}^{1_{U_2}}$. This implies that $\underline{\sigma}^P = \underline{\sigma}^{1_{U_2}}$, that is, $P \approx_A 1_{U_2}$.
- (6) It follows from (5).

Theorem 11 shows some upper similarity relations of union and intersection of IFSs N, P, Q , and T in U_2 with respect to the aftersets. \square

Theorem 12. Let (σ, A) be a soft binary relation from U_1 to U_2 . Let N, P, Q , and T be IFSs of U_1 . Then, the following are true:

- (1) $N \approx_{\sigma} P$ if and only if $N \approx_{\sigma} (N \cap P) \approx_{\sigma} P$;
- (2) $N \approx_{\sigma} P$ and $Q \approx_{\sigma} T$ imply that $(N \cap Q) \approx_{\sigma} (P \cap T)$;
- (3) $N \subseteq P$ and $P \approx_{\sigma} 0_{U_1}$ imply that $N \approx_{\sigma} 0_{U_1}$;
- (4) $(N \cap P) \approx_{\sigma} 0_{U_1}$ if and only if $N \approx_{\sigma} 0_{U_1}$ and $P \approx_{\sigma} 0_{U_1}$;
- (5) $N \subseteq P$ and $N \approx_{\sigma} 1_{U_1}$ imply that $P \approx_{\sigma} 1_{U_1}$;
- (6) If $(N \cap P) \approx_{\sigma} 1_{U_1}$, then $N \approx_{\sigma} 1_{U_1}$ and $P \approx_{\sigma} 1_{U_1}$.

Proof

- (1) Let $N \approx_{\sigma} P$. Then, ${}^N \underline{\sigma} = {}^P \underline{\sigma}$. By Theorem 2, we get ${}^{N \cap P} \underline{\sigma} = {}^N \underline{\sigma} \cap {}^P \underline{\sigma} = {}^N \underline{\sigma} = {}^P \underline{\sigma}$. This implies that $N \approx_{\sigma} (N \cap P) \approx_{\sigma} P$. Converse holds by the transitivity of the relation \approx_{σ} .
- (2) Let $N \approx_{\sigma} P$ and $Q \approx_{\sigma} T$. Then, ${}^N \underline{\sigma} = {}^P \underline{\sigma}$ and ${}^Q \underline{\sigma} = {}^T \underline{\sigma}$. By Theorem 2, we get ${}^{N \cap Q} \underline{\sigma} = {}^N \underline{\sigma} \cap {}^Q \underline{\sigma} = {}^P \underline{\sigma} \cap {}^T \underline{\sigma} = {}^{P \cap T} \underline{\sigma}$. This implies that $(N \cap Q) \approx_{\sigma} (P \cap T)$.
- (3) Let $N \subseteq P$ and $P \approx_{\sigma} 0_{U_1}$. Then, ${}^P \underline{\sigma} = {}^{0_{U_1}} \underline{\sigma}$. Also, by Theorem 2, $N \subseteq P$ implies that ${}^N \underline{\sigma} \subseteq {}^P \underline{\sigma} = {}^{0_{U_1}} \underline{\sigma}$. But, ${}^{0_{U_1}} \underline{\sigma} \subseteq {}^N \underline{\sigma}$. Thus, ${}^N \underline{\sigma} = {}^{0_{U_1}} \underline{\sigma}$. This implies that $N \approx_{\sigma} 0_{U_1}$.
- (4) If $N \approx_{\sigma} 0_{U_1}$ and $P \approx_{\sigma} 0_{U_1}$, then ${}^N \underline{\sigma} = {}^{0_{U_1}} \underline{\sigma}$ and ${}^P \underline{\sigma} = {}^{0_{U_1}} \underline{\sigma}$. Now, by Theorem 2, we have ${}^{N \cap P} \underline{\sigma} = {}^N \underline{\sigma} \cap {}^P \underline{\sigma} = {}^{0_{U_1}} \underline{\sigma} \cap {}^{0_{U_1}} \underline{\sigma} = {}^{0_{U_1}} \underline{\sigma}$, so ${}^{N \cap P} \underline{\sigma} = {}^{0_{U_1}} \underline{\sigma}$. This implies that $(N \cap P) \approx_{\sigma} 0_{U_1}$. Conversely, if $(N \cap P) \approx_{\sigma} 0_{U_1}$, then by part (3), we have $N \approx_{\sigma} 0_{U_1}$ and $P \approx_{\sigma} 0_{U_1}$.
- (5) Suppose $N \approx_{\sigma} 1_{U_1}$. Then, ${}^N \underline{\sigma} = {}^{1_{U_1}} \underline{\sigma}$. As $N \subseteq P$, we have ${}^P \underline{\sigma} \supseteq {}^N \underline{\sigma} = {}^{1_{U_1}} \underline{\sigma}$. On the other hand, $P \subseteq 1_{U_1}$, so we have ${}^P \underline{\sigma} \subseteq {}^{1_{U_1}} \underline{\sigma}$. This implies that ${}^P \underline{\sigma} = {}^{1_{U_1}} \underline{\sigma}$, that is, $P \approx_{\sigma} 1_{U_1}$.
- (6) It follows from (5). \square

Theorem 13. Let (σ, A) be a soft binary relation from U_1 to U_2 . Let N, P, Q , and T be IFSs of U_2 . Then, the following are true:

- (1) $N \subseteq P$ and $P \approx_A 0_{U_2}$ imply that $N \approx_A 0_{U_2}$;
- (2) $N \subseteq P$ and $N \approx_A 1_{U_2}$ imply that $P \approx_A 1_{U_2}$;
- (3) $(N \cup P) \approx_A 0_{U_2}$, then $N \approx_A 0_{U_2}$ and $P \approx_A 0_{U_2}$;
- (4) $(N \cap P) \approx_A 1_{U_2}$, then $N \approx_A 1_{U_2}$ and $P \approx_A 1_{U_2}$;
- (5) $N \approx_A P$ if and only if $N \approx_A (N \cup P) \approx_A P$ and $N \approx_A (N \cap P) \approx_A P$.

Proof

- (1) Suppose $P \approx_A 0_{U_2}$, this implies that $\underline{\sigma}^N = \underline{\sigma}^{0_{U_2}}$ and $\bar{\sigma}^P = \bar{\sigma}^{0_{U_2}}$. As $N \subseteq P$, we have $\underline{\sigma}^N \subseteq \underline{\sigma}^P = \underline{\sigma}^{0_{U_2}}$ and $\bar{\sigma}^N \subseteq \bar{\sigma}^P = \bar{\sigma}^{0_{U_2}}$. On the other hand, $N \supseteq 0_{U_2}$; this implies that $\underline{\sigma}^N \supseteq \underline{\sigma}^{0_{U_2}}$ and $\bar{\sigma}^N \supseteq \bar{\sigma}^{0_{U_2}}$. So, $\underline{\sigma}^N = \underline{\sigma}^{0_{U_2}}$ and $\bar{\sigma}^N = \bar{\sigma}^{0_{U_2}}$, thus $N \approx_A 0_{U_2}$.
- (2) Suppose $N \approx_A 1_{U_2}$, this implies that $\underline{\sigma}^N = \underline{\sigma}^{1_{U_2}}$ and $\bar{\sigma}^P = \bar{\sigma}^{1_{U_2}}$. As $N \subseteq P$, we have $\underline{\sigma}^P \supseteq \underline{\sigma}^N = \underline{\sigma}^{1_{U_2}}$ and $\bar{\sigma}^P \supseteq \bar{\sigma}^N = \bar{\sigma}^{1_{U_2}}$. On the other hand, $P \subseteq 1_{U_2}$; this implies that $\underline{\sigma}^P \subseteq \underline{\sigma}^{1_{U_2}}$ and $\bar{\sigma}^P \subseteq \bar{\sigma}^{1_{U_2}}$. So, $\underline{\sigma}^P = \underline{\sigma}^{1_{U_2}}$ and $\bar{\sigma}^P = \bar{\sigma}^{1_{U_2}}$, thus $P \approx_A 1_{U_2}$.
- (3) It follows from part (1).
- (4) It follows from part (2).
- (5) Let $N \approx_A P$. Then, $\underline{\sigma}^N = \underline{\sigma}^P$ and $\bar{\sigma}^P = \bar{\sigma}^N$. By Theorem 1, we get $\underline{\sigma}^{N \cap P} = \underline{\sigma}^N \cap \underline{\sigma}^P = \underline{\sigma}^N = \underline{\sigma}^P$ and $\bar{\sigma}^{N \cup P} = \bar{\sigma}^N \cup \bar{\sigma}^P = \bar{\sigma}^N = \bar{\sigma}^P$. This implies that $N \approx_A (N \cap P) \approx_A P$ and $N \approx_A (N \cup P) \approx_A P$.

Converse holds by the transitivity of the relation \approx_A .

Theorem 13 shows some similarity relations of union and intersection of IFSs N, P, Q , and T in U_2 with respect to the aftersets. \square

Theorem 14. Let (σ, A) be a soft binary relation from U_1 to U_2 . Let N, P, Q , and T be IFSs of U_1 . Then, the following are true:

- (1) $N \subseteq P$ and $P \approx_{\sigma} 0_{U_1}$ imply that $N \approx_{\sigma} 0_{U_1}$;
- (2) $N \subseteq P$ and $N \approx_{\sigma} 1_{U_1}$ imply that $P \approx_{\sigma} 1_{U_1}$;
- (3) $(N \cup P) \approx_{\sigma} 0_{U_1}$, then $N \approx_{\sigma} 0_{U_1}$ and $P \approx_{\sigma} 0_{U_1}$;
- (4) $(N \cap P) \approx_{\sigma} 1_{U_1}$, then $N \approx_{\sigma} 1_{U_1}$ and $P \approx_{\sigma} 1_{U_1}$;
- (5) $N \approx_{\sigma} P$ if and only if $N \approx_{\sigma} (N \cup P) \approx_{\sigma} P$ and $N \approx_{\sigma} (N \cap P) \approx_{\sigma} P$.

Proof

- (1) Suppose $P \approx_{\sigma} 0_{U_1}$, this implies that ${}^N\sigma = {}^{0_{U_1}}\sigma$ and ${}^P\sigma = {}^{0_{U_1}}\sigma$. As $N \subseteq P$, we have ${}^N\sigma \subseteq {}^P\sigma = {}^{0_{U_1}}\sigma$ and ${}^N\sigma \subseteq {}^{0_{U_1}}\sigma$. On the other hand, $N \supseteq 0_{U_1}$; this implies that ${}^N\sigma \supseteq {}^{0_{U_1}}\sigma$ and ${}^N\sigma \supseteq {}^{0_{U_1}}\sigma$. So, ${}^N\sigma = {}^{0_{U_1}}\sigma$ and ${}^N\sigma = {}^{0_{U_1}}\sigma$, thus $N \approx_{\sigma} 0_{U_1}$.
- (2) Suppose $N \approx_{\sigma} 1_{U_1}$, this implies that ${}^N\sigma = {}^{1_{U_1}}\sigma$ and ${}^P\sigma = {}^{1_{U_1}}\sigma$. As $N \subseteq P$, we have ${}^P\sigma \supseteq {}^N\sigma = {}^{1_{U_1}}\sigma$ and ${}^P\sigma \supseteq {}^{1_{U_1}}\sigma$. On the other hand, $P \subseteq 1_{U_1}$; this implies that ${}^P\sigma \subseteq {}^{1_{U_1}}\sigma$ and ${}^P\sigma \subseteq {}^{1_{U_1}}\sigma$. So, ${}^P\sigma = {}^{1_{U_1}}\sigma$ and ${}^P\sigma = {}^{1_{U_1}}\sigma$, thus $P \approx_{\sigma} 1_{U_1}$.
- (3) It follows from part (1).
- (4) It follows from part (2).
- (5) Let $N \approx_{\sigma} P$. Then, ${}^N\sigma = {}^P\sigma$ and ${}^P\sigma = {}^N\sigma$. By Theorem 2, we get ${}^{N \cap P}\sigma = {}^N\sigma \cap {}^P\sigma = {}^N\sigma = {}^P\sigma$ and ${}^{N \cup P}\sigma = {}^N\sigma \cup {}^P\sigma = {}^N\sigma = {}^P\sigma$. This implies that $N \approx_{\sigma} (N \cap P) \approx_{\sigma} P$ and $N \approx_{\sigma} (N \cup P) \approx_{\sigma} P$.

Converse holds by the transitivity of the relation \approx_{σ} . \square

5. Application in Decision-Making Problem

A major area of study in all kinds of data analysis is decision making. Many experts and researchers introduced many methods to find a wise decision. RS theory [7], SS theory [12], and IFS theory [3] are the theories which are mostly used in the decision-making problems. In the above sections, we develop a rough set model using soft binary relations. We used soft binary relation to approximate an IFS. We used score function defined by Chen and Tan [57] and accuracy function defined by Hong and Choi [58] to define order between objects. Now, we present an algorithm for the approach to a decision-making problem and this problem is depended on IF soft rough set theory based on soft binary relations. This algorithm extends the already existing approach which is described by Kanwal and Shabir [51]. For our new approach, data information is only needed which is provided by the decision-making problem and no need of any additional information by any supplementary ways. So, the decision results can be avoided by the effect of subjective information. Therefore, the outcomes could avoid the inconsistent results for the same problem and could be better objective. The decision Algorithm 1 is as follows:

Now, we show this approach step by step to decision making which is proposed by using the following example. The following example discusses algorithm to make wise decision for the selection of a car.

Example 4. Suppose a person Mr. X wants to select a car out of available models. Let $U_1 = \{\text{the set of all models available in range}\} = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ and $U_2 = \{\text{the colors of all models}\} = \{c_1, c_2, c_3, c_4\}$ and the set of attributes be $A = \{e_1, e_2, e_3\} = \{\text{the set of brands}\} = \{e_1 = \text{Suzuki}, e_2 = \text{Toyota}, e_3 = \text{Honda}\}$.

Define $\sigma: A \longrightarrow P(U_1 \times U_2)$ by

$$\begin{aligned} \sigma(e_1) &= \left\{ (m_1, c_1), (m_1, c_2), (m_1, c_3), (m_2, c_2), (m_2, c_4), \right. \\ &\quad \left. (m_4, c_2), (m_4, c_3), (m_5, c_3), (m_5, c_4), (m_6, c_1) \right\}, \\ \sigma(e_2) &= \{(m_1, c_3), (m_2, c_3), (m_4, c_1), (m_5, c_1), (m_6, c_2), (m_6, c_3)\}, \\ \sigma(e_3) &= \{(m_3, c_3), (m_3, c_1), (m_2, c_4), (m_5, c_3), (m_5, c_4)\}, \end{aligned} \quad (9)$$

which represents the relation between models and colors available in brand e_i for $1 \leq i \leq 3$. Then,

$$\begin{aligned} m_1\sigma(e_1) &= \{c_1, c_2, c_3\}, m_2\sigma(e_1) = \{c_2, c_4\}, m_3\sigma(e_1) = \emptyset, \\ m_4\sigma(e_1) &= \{c_2, c_3\}, m_5\sigma(e_1) = \{c_3, c_4\}, m_6\sigma(e_1) = \{c_1\}, \\ m_1\sigma(e_2) &= \{c_3\}, m_2\sigma(e_2) = \{c_3\}, m_3\sigma(e_2) = \emptyset, \\ m_4\sigma(e_2) &= \{c_1\}, m_5\sigma(e_2) = \{c_1\}, m_6\sigma(e_2) = \{c_2, c_3\}, \\ m_1\sigma(e_3) &= \emptyset, m_2\sigma(e_3) = \{c_4\}, m_3\sigma(e_3) = \{c_1, c_3\}, \\ m_4\sigma(e_3) &= \emptyset, m_5\sigma(e_3) = \{c_3, c_4\}, m_6\sigma(e_3) = \emptyset, \end{aligned} \quad (10)$$

where $m_i\sigma(e_j)$ represents the color of the model m_i available in the brand e_j .

Also,

$$\begin{aligned} \sigma(e_1)c_1 &= \{m_1, m_6\}, \sigma(e_1)c_2 = \{m_1, m_2, m_4\}, \\ \sigma(e_1)c_3 &= \{m_1, m_4, m_5\}, \sigma(e_1)c_4 = \{m_2, m_5\}, \\ \sigma(e_2)c_1 &= \{m_4, m_5\}, \sigma(e_2)c_2 = \{m_6\}, \\ \sigma(e_2)c_3 &= \{m_1, m_2\}, \sigma(e_2)c_4 = \emptyset, \\ \sigma(e_3)c_1 &= \{m_3\}, \sigma(e_3)c_2 = \emptyset, \\ \sigma(e_3)c_3 &= \{m_3, m_5\}, \sigma(e_3)c_4 = \{m_2, m_5\}, \end{aligned} \quad (11)$$

- (1) Compute the upper IF soft set approximation $\bar{\sigma}^M$ and lower IF soft set approximation $\underline{\sigma}^M$ of an IF set $M = \langle \mu_M, \gamma_M \rangle$ with respect to the aftersets;
- (2) Compute the score values for each of the entries of the $\underline{\sigma}^M$ and $\bar{\sigma}^M$ and denote them by $\underline{S}_{ij}(x_i, e_j)$ and $\bar{S}_{ij}(x_i, e_j)$ for all i, j ;
- (3) Compute the aggregated score $\underline{S}(x_i) = \sum_{j=1}^n \underline{S}_{ij}(x_i, e_j)$ and $\bar{S}(x_i) = \sum_{j=1}^n \bar{S}_{ij}(x_i, e_j)$;
- (4) Compute $S(x_i) = \underline{S}(x_i) + \bar{S}(x_i)$;
- (5) The best decision is $x_k = \max_i S(x_i)$;
- (6) If k has more than one value, say k_1, k_2 , then we calculate the accuracy values $\underline{H}_{ij}(x_i, e_j)$ and $\bar{H}_{ij}(x_i, e_j)$ for only those x_k for which $S(x_k)$ are equal;
- (7) Compute $H(x_k) = \sum_{j=1}^n \underline{H}_{kj}(x_k, e_j) + \sum_{j=1}^n \bar{H}_{kj}(x_k, e_j)$ for $k = k_1, k_2$;
- (8) If $H(x_{k_1}) > H(x_{k_2})$, then we select x ;
- (9) If $H(x_{k_1}) = H(x_{k_2})$, then select any one of x_{k_1} and x_{k_2} .

ALGORITHM 1: Procedural steps for better decision with the help of score function.

TABLE 14: Lower approximation of M .

$(\underline{\sigma}^{\mu_M}, \underline{\sigma}^{\gamma_M})$	m_1	m_2	m_3	m_4	m_5	m_6
$\underline{\sigma}^{\mu_M}(e_1)$	0.4	0	1	0.4	0	0.9
$\underline{\sigma}^{\mu_M}(e_2)$	0.4	0.4	1	0.9	0.9	0.4
$\underline{\sigma}^{\mu_M}(e_3)$	1	0	0.4	1	0	1
$\underline{\sigma}^{\gamma_M}(e_1)$	0.5	0.8	0	0.5	0.8	0
$\underline{\sigma}^{\gamma_M}(e_2)$	0.5	0.5	0	0	0	0.5
$\underline{\sigma}^{\gamma_M}(e_3)$	0	0.8	0.5	0	0.8	0

where $\sigma(e_j)c_i$ represents the model of the color c_i available in the brand e_j .

Define $M = \langle \mu_M, \gamma_M \rangle: U_2 \longrightarrow [0, 1]$, which represents the preference of the colors given by Mr. X such that

$$\begin{aligned}\mu_M(c_1) &= 0.9, \mu_M(c_2) = 0.8, \mu_M(c_3) = 0.4, \mu_M(c_4) = 0 \\ \gamma_M(c_1) &= 0.0, \gamma_M(c_2) = 0.2, \gamma_M(c_3) = 0.5, \gamma_M(c_4) = 0.8.\end{aligned}$$

Define $N = \langle \mu_N, \gamma_N \rangle: U_1 \longrightarrow [0, 1]$, which represents the preference of the model given by Mr. X such that

$$\begin{aligned}\mu_N(m_1) &= 1, \mu_N(m_2) = 0.7, \mu_N(m_3) = 0.5, \mu_N(m_4) = 0.1, \\ \mu_N(m_5) &= 0, \mu_N(m_6) = 0.4 \\ \gamma_N(m_1) &= 0, \gamma_N(m_2) = 0.2, \gamma_N(m_3) = 0.5, \gamma_N(m_4) = 0.7, \\ \gamma_N(m_5) &= 1, \gamma_N(m_6) = 0.5.\end{aligned}$$

(12)

Therefore, the lower and upper approximations (with respect to the aftersets as well as with respect to the foresets) are as follows (Tables 14 and 15, respectively):

$$\begin{aligned}\underline{\sigma}^M &= \left(\underline{\sigma}^{\mu_M}, \underline{\sigma}^{\gamma_M} \right), \\ \bar{\sigma}^M &= \left(\bar{\sigma}^{\mu_M}, \bar{\sigma}^{\gamma_M} \right).\end{aligned}\tag{13}$$

The values of score function for car models are given in Table 16.

Table 16 shows that $S(m_4) = S(m_6)$, so we calculate accuracy values for m_4 and m_6 .

Hence, the values of accuracy function are given in Table 17.

Table 17 shows that $H(m_4) = H(m_6)$, so we can select any one, m_4 or m_6 .

Now, ${}^N\underline{\sigma} = (\mu_N \underline{\sigma}, \gamma_N \underline{\sigma})$ (given in Table 18) and ${}^N\bar{\sigma} = (\mu_N \bar{\sigma}, \gamma_N \bar{\sigma})$ (given in Table 19).

The values of score function for colors of cars are given in Table 20.

Table 20 shows that $S(c_3) = 0.5$ is maximum, so he will select color c_4 .

The flow chart of our decision-making algorithm is given in Figure 2.

TABLE 15: Upper approximation of M .

$(\bar{\sigma}^{\mu_M}, \bar{\sigma}^{\nu_M})$	m_1	m_2	m_3	m_4	m_5	m_6
$\bar{\sigma}^{\mu_M}(e_1)$	0.9	0.8	0	0.8	0.4	0.9
$\bar{\sigma}^{\mu_M}(e_2)$	0.4	0.4	0	0.9	0.9	0.8
$\bar{\sigma}^{\mu_M}(e_3)$	0	0	0.9	0	0.4	0
$\bar{\sigma}^{\nu_M}(e_1)$	0	0.2	1	0.2	0.5	0
$\bar{\sigma}^{\nu_M}(e_2)$	0.5	0.5	1	0	0	0.2
$\bar{\sigma}^{\nu_M}(e_3)$	1	0.8	0	1	0.5	1

TABLE 16: Values of score function for car models.

	$\underline{S}_{ij}(e_1)$	$\underline{S}_{ij}(e_2)$	$\underline{S}_{ij}(e_3)$	$\bar{S}_{ij}(e_1)$	$\bar{S}_{ij}(e_2)$	$\bar{S}_{ij}(e_3)$	$\underline{S}(x_i)$	$\bar{S}(x_i)$	$S(x_i)$
m_1	-0.1	-0.1	1	0.9	-0.1	-1	0.8	-0.2	0.6
m_2	-0.8	-0.1	-0.8	0.6	-0.1	-0.8	-1.7	-0.3	-2
m_3	1	1	-0.1	-1	-1	0.9	1.9	-1.1	0.8
m_4	-0.1	0.9	1	0.6	0.9	-1	1.8	0.5	2.3
m_5	-0.8	0.9	-0.8	-0.1	0.9	0.3	-0.7	1.1	0.4
m_6	0.9	-0.1	1	0.9	0.6	-1	1.8	0.5	2.3

TABLE 17: Values of accuracy function.

	$\underline{H}_{ij}(e_1)$	$\underline{H}_{ij}(e_2)$	$\underline{H}_{ij}(e_3)$	$\bar{H}_{ij}(e_1)$	$\bar{H}_{ij}(e_2)$	$\bar{H}_{ij}(e_3)$	H
m_4	0.9	0.9	1	1	0.9	1	5.7
m_6	0.9	0.9	1	0.9	1	1	5.7

TABLE 18: Lower approximation of N .

$(\mu_N \underline{\sigma}, \gamma_N \underline{\sigma})$	c_1	c_2	c_3	c_4
$\mu_N \underline{\sigma}(e_1)$	0.4	0.1	0	0
$\mu_N \underline{\sigma}(e_2)$	0	0.4	0.7	1
$\mu_N \underline{\sigma}(e_3)$	0.5	1	0	0
$\gamma_N \underline{\sigma}(e_1)$	0.5	0.7	1	1
$\gamma_N \underline{\sigma}(e_2)$	1	0.5	0.2	0
$\gamma_N \underline{\sigma}(e_3)$	0.5	0	1	1

TABLE 19: Upper approximation of N .

$(\mu_N \bar{\sigma}, \gamma_N \bar{\sigma})$	c_1	c_2	c_3	c_4
$\mu_N \bar{\sigma}(e_1)$	1	1	1	0.7
$\mu_N \bar{\sigma}(e_2)$	0.1	0.4	1	0
$\mu_N \bar{\sigma}(e_3)$	0.5	0	0.5	0.7
$\gamma_N \bar{\sigma}(e_1)$	0	0	0	0.2
$\gamma_N \bar{\sigma}(e_2)$	0.7	0.5	0	1
$\gamma_N \bar{\sigma}(e_3)$	0.5	1	0.5	0.2

TABLE 20: Values of score function for colors of cars.

	$\underline{S}_{ij}(e_1)$	$\underline{S}_{ij}(e_2)$	$\underline{S}_{ij}(e_3)$	$\bar{S}_{ij}(e_1)$	$\bar{S}_{ij}(e_2)$	$\bar{S}_{ij}(e_3)$	$\underline{S}(x_i)$	$\bar{S}(x_i)$	$S(x_i)$
c_1	-0.1	-1	0	1	-0.6	0	-1.1	0.4	-0.7
c_2	-0.6	-0.1	1	1	-0.1	-1	0.3	-0.1	0.2
c_3	-1	0.5	-1	1	1	0	-1.5	2	0.5
c_4	-1	1	-1	0.5	-1	0.5	-1	0	-1

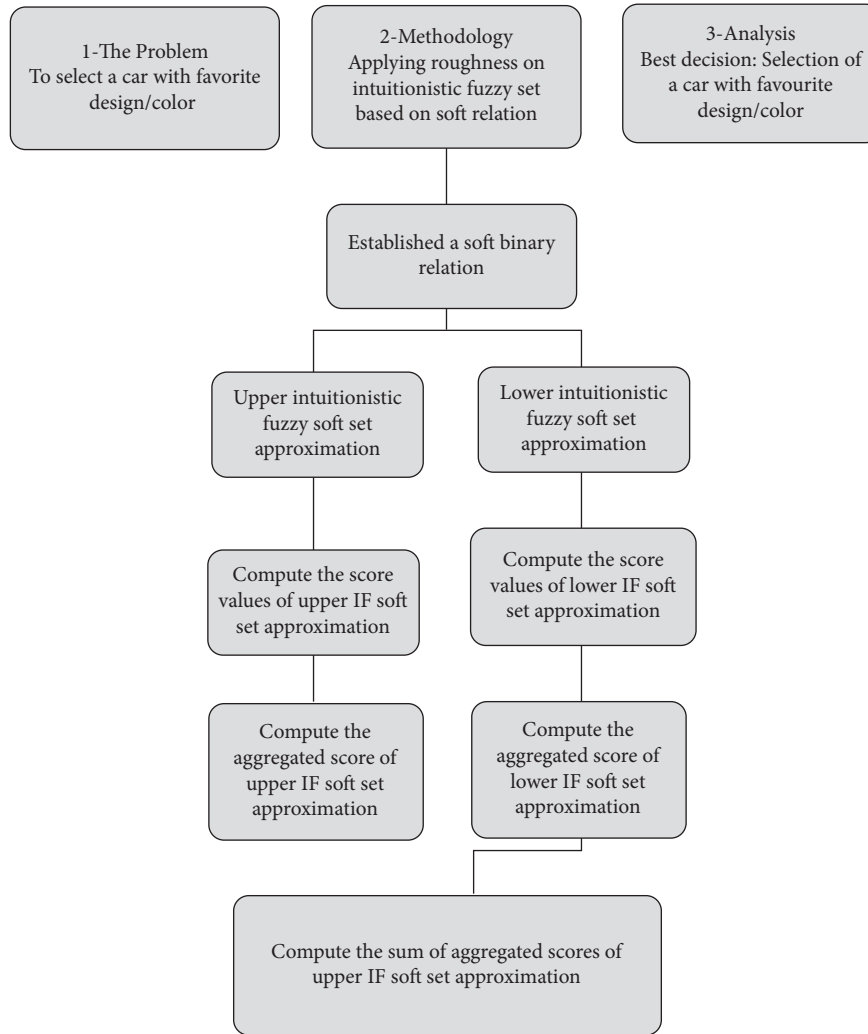


FIGURE 2: Flow chart of decision-making algorithm of IFRS proposed model.

In Figure 2, the flow chart shows that if a person wants to buy a car with his favourite design and color, then this algorithm helps him to make best and suitable decision according to his choice.

6. Comparison

First, we review existing approaches to intuitionistic fuzzy rough set (IFRS) model-based decision making and then finally we show that our newly proposed IFRS model is very useful than other existing theories. Since the combination of IFS with SS and RS is very helpful to deal with uncertainty and impreciseness, Maji et al. [24] presented a useful model of RS and SS. Chen et al. [20] used SS parameterization reduction and improved SSs-based decision making in [24]. Cagman et al. [63] presented a uni-int decision-making method by using redefined operations of soft sets. But all the above work in decision making is about only crisp soft set. Then, Roy et al. [46] solved recognition problems by using their newly proposed algorithm of fuzzy SSs [64]. Later, Kong et al. [65] modified Roy et al.' algorithm and proved that their algorithm was not able to obtain optimal choice

generally. Feng et al. [21] also worked on fuzzy soft set-based algorithm. Later, Jiang et al. [66] discussed intuitionistic fuzzy soft sets with an adjusted approach.

6.1. Maji and Roy's Method and Its Limitation. Maji et al. used concept of knowledge reduction in RSs with SSs to solve decision-making problems. This method consists of two steps. In first step, find one reduct soft set of the original SS based on the knowledge reduction of RSs, and then calculate the choice values of all elements and select the element with the maximum value as the optimum alternative. Chen et al. [20] claimed that soft set reduction in [24] has incorrect results in Step 1.

6.2. Cagman's Method and Its Limitation. Cagman et al. [63] proposed a soft max-min decision-making method. Optimum alternatives are selected from the alternatives set by this method. In this method, the noting point is that this method has its constitutive limitation. An algorithm of this method gets an empty optimum set.

Dubious and Prade presented rough FSs [35] and Feng et al. [1] extended their model in terms of SSs. Feng et al. [1] approximated a FS in a soft approximation space and extended a concept called soft rough FSs. The FSs and IF relations-based models are useful in many fields, but we used soft binary relations in our proposed model. Soft binary relation is the generalization of a binary relation and soft binary relation is the parameterized collection of ordinary binary relations. In soft binary relation, we can use different parameters according to the nature of problem. That is why our proposed model is more useful to manage uncertainty in different types of problems.

6.3. Advantages of Our Proposed IFRS Model

- (1) In our IFRS model based on soft binary relations, we also get information about what candidate is optimum alternative and what candidate should not be optimum alternative, whereas other existing theories only get optimum alternatives. That is why our newly proposed IFRS model is more precise and flexible for decision-making problems.
- (2) Our proposed model also gives a solution IFRS-based group decision making, whereas other existing approaches have no directions to discuss the intuitionistic fuzzy set group decision making.
- (3) This IFRS model based on soft relations can be applied to solve decision-making problems involving intuitionistic fuzzy sets in real life [56].

In 2012, Zhang [56] proposed a RS model based on ordinary binary relation induced by an IF relation over two universes and presented a decision-making algorithm based on RS model with IFSs. In 2020, Shabir et al. [51] proposed a RS model of FSs based on soft relations and presented decision-making algorithm. In comparison of these models, we proposed a RS model of IFSs based on soft relations and presented a decision-making algorithm based on IFRS which is a better technique to manage uncertainty and impreciseness. We used an IFS instead of a crisp set or a FS in our proposed model due to its importance in scientific fields and decision making, such as medical diagnosis, career determination, pattern recognition, and electoral system. An IFS has degree of membership and also degree of nonmembership which is helpful to make better decision in real-life problems.

7. Conclusion

Since the combination of IFS with SS and RS is very helpful to deal with uncertainty, Maji et al. presented a useful model of RS and SS. Chen et al. used SS parameterization reduction and improved SSs-based decision making. Cagman et al. presented a uni-int decision-making method by using redefined operations of SSs. But, all the above work in decision making is about only crisp SS. Then, Roy et al. solved recognition problems by using their newly proposed algorithm of fuzzy SSs. Later, Kong et al. modified Roy et al.'s

algorithm and proved that their algorithm was not able to obtain optimal choice generally. Feng et al. also worked on fuzzy SS-based algorithm. Later, Jiang et al. discussed IF soft sets with an adjusted approach. In our paper, we have given a generalization of [51] and we have approximated an IFS by soft binary relations. We used foresets and aftersets to approximate IFS. In this way, we get two pairs of intuitionistic fuzzy soft sets, called the lower approximation and upper approximation. Properties of these approximations are studied. Similarity relations between IFS with respect to this rough set model are also studied. Finally, we developed an algorithm for intuitionistic fuzzy rough sets (IFRS) based on decision making and an example is provided to illustrate the developed algorithm. Further study can be performed to investigate the roughness in interval-valued IFS and multigranulation roughness of IFS by using soft relations.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Interval Type-2 Fuzzy Standardized Cumulative Sum Control Charts in Production of Fertilizers

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Statistical process control is a method used for controlling processes in which causes of variations and correction actions can be observed. Control chart is one of the powerful tools of statistical process control that are used to control nonconforming products. Previous literature suggests that fuzzy charts are more sensitive than conventional control charts, and hence, they provide better quality and conformance of products. Nevertheless, some of the data used are more suitable to be presented in interval type-2 fuzzy numbers compared to type-1 fuzzy numbers as interval type-2 fuzzy numbers have more ability to capture uncertain and vague information. In this paper, we develop an interval type-2 fuzzy standardized cumulative sum (IT2F-SCUSUM) control chart and apply it to data of fertilizer production. This new approach combines the advantages of interval type-2 fuzzy numbers and standardized sample means which can control the variability. Twenty samples with a sample size of six were examined for testing the conformance. The proposed IT2F-SCUSUM control chart unveils that 15 samples are “out of control.” The results are also compared to the conventional CUSUM chart and type-1 fuzzy CUSUM chart. The conventional chart shows that 13 samples are “out of control.” In contrast, the type-1 fuzzy CUSUM chart shows that the process is “out of control” for 14 samples. In the analysis of average run length, the proposed IT2F-SCUSUM chart outperforms the other two CUSUM charts. Thus, we can conclude that the IT2F-SCUSUM chart is more sensitive and takes lesser number of observations to identify the shift in the process. The analyses suggest that the IT2F-SCUSUM chart is a promising tool in examining conformance of the quality of the fertilizer production.

1. Introduction

Over the past few decades, quality is one of the most imperative consumer decision factors in selecting the faultless products and services [1]. It had been evolutionary developed since 1900 through various improvements in the quality of the products [2]. Some of the definitions in the quality terms are discussed as a viewpoint as a need for the technical community in the various organization such as European Organization for Quality Control and the American Society for Quality Control [3]. Statistical process

control (SPC) is one of the techniques used for controlling processes to distinguish causes of variation and signal the need for corrective actions [2]. Walter Shewhart, Bell Telephone Laboratories, USA, in 1924 developed SPC methods for the improvement of manufacturing quality, and these methods were incorporated into a management philosophy [1]. In achieving process stability of the products and services, SPC is very useful to be applied in which variability of products can be reduced. The process is said to be in statistical control if disturbances or special causes of variation are eliminated.

In SPC, the most important tool that is useful in the process of monitoring technique is control chart. Control charts help to distinguish the products, that is, less or more than the control limits. The use of classical control chart that was proposed by Walter Andrew Shewhart in 1920s provides a graphical depiction and record of data series. It is suitable to analyse the variation in the process when the data are known precisely and exactly. The most crucial thing is the possibility of loss of information, and this situation has merely happened in the qualitative data [4]. However, it might not be possible to determine the data clearly especially when analysing vague and qualitative data. The classical control charts may not be applicable since they require certain information where human subjectivity plays an important role in defining the quality characteristics. Hence, fuzzy control charts were proposed, and it is believed that these types of control charts will provide a systematic base to deal with the scenario which is ambiguous or not well defined. Fuzzy charts are useful when the data are uncertain or vague, and the process is incomplete or includes human subjectivity [5]. The vagueness can be handled by transforming incomplete or nonprecise quantities to their representative values for control chart decisions as “in control” or “out of control” [6]. In fact, fuzzy set theory is a perfect means for modelling uncertainty (or imprecision) arising from mental phenomena which is neither random nor stochastic [7].

Nowadays, type-1 fuzzy set is widely used in the manufacturing and agricultural area. Mojtaba Zabihinpour et al. [8] constructed the fuzzy and s control charts with an unbiased estimation of standard deviation to monitor quality characteristics. The study on noodle production food industry proved that the proposed technique improves the detection of abnormal shift in process mean. Recently, Sabahno et al. [9] investigated the adaptive \bar{X} and R fuzzy control chart that allows all the chart parameters to adapt based on the process state in the sample. In a nutshell, they found that their adaptive scheme was able to detect the process shift faster than the classical one. Nevertheless, some of the data used in our daily lives can only be used for the type-2 fuzzy control chart and cannot be expressed by other fuzzy charts. This indicates that if the data are not suitable to be used in the type-1 fuzzy control chart or conventional chart, it might affect the number of defective products produced by the company. This means the company will have a high tendency to sell the defective products to customers. Any mistakes in intervention of the process, delay of alarming excessive product defects, and scraps or reworks of final products will result in an increase of production costs. In manufacturing industries, they need to reduce the percentage of nonconformities in order to reduce the costs and to fulfill consumer satisfaction. Therefore, they need to find the best method for getting the best result of the product's quality. The type-2 fuzzy control chart needs to be used if the data are suitable to its analysis. In fact, the efficiency of the type-2 fuzzy control chart is more than the analysing of crisp data, but it also gives essential alerts by means of linguistic terms.

Consequently, the type-2 fuzzy chart is much capable to detect the meaning of process shifts and hence it would help

managers to establish a predictable and consistent level of quality of the product of the company. Erginel et al. [10] examined the fraction nonconforming products by using the interval type-2 fuzzy control chart. Other than that, Şentürk and Antucheviciene [11] analysed the type-2 fuzzy nonconformities control charts. Type-2 fuzzy control charts came into consideration when the researcher wanted to investigate the imprecision of membership functions in three dimensions. The classical c control charts were not suitable to be used when the data were collected as the type-2 fuzzy numbers. Hence, they applied the interval type-2 fuzzy charts to reduce the vagueness and uncertainty of the observation data. Teksen and Anagün [4] explored type-2 fuzzy charts using likelihood and defuzzification methods. The different methods for analysing interval type-2 fuzzy \bar{X} and R charts are defuzzification, distance, ranking, and likelihood methods. Control charts are used to compare with the crisp number to choose the best method. Other than that, Kaya and Turgut [12] analysed the type-2 fuzzy variables control chart and applied it on a real case application from electronic industry. They concluded that type-2 fuzzy control charts can evaluate the process in more sensitive and precise way. Thus far, however, there has been little discussion about interval type-2 fuzzy standardized cumulative sum (IT2F-SCUSUM) control charts. Generally, the conventional control chart is often used as an alternative to cumulative sum (CUSUM) in diagnostics aspects of bringing an uncontrollable product to be “in control.” Nevertheless, the conventional chart, also known as Shewhart chart, is quite insensitive to small process shifts which means assignable causes do not result in large process disturbance. Therefore, CUSUM is the best chart to be used in detecting small process shifts in monitoring analysis. The CUSUM chart, also known as time-weighted control chart, is used for controlling cumulative sum of quality characteristics measurement. It helps in detecting small shifts in a process which is less than 1.5σ [1].

This study wishes to develop the IT2F-SCUSUM control charts as a new approach in quality control. This study also compares the IT2F-SCUSUM control chart with conventional standardized cumulative sum (SCUSUM) and type-1 fuzzy standardized cumulative sum (T1F-SCUSUM) control charts. The comparative analysis will be conducted to determine which control chart is the most sensitive as it would help the manufacturers to reduce the percentage of nonconformists, thereby reducing the manufacturing costs in their company. The proposed IT2F-SCUSUM control chart is a maiden study in CUSUM control charts. This paper is organised as follows. Section 2 provides the literature review of CUSUM charts. The theoretical structure of the proposed method is explained in Section 3, while Section 4 covers the application of the IT2F-SCUSUM control chart using fertilizer production data. The comparative analysis results are given in Section 5, whereas Section 6 concludes the study.

2. Literature Review

Over the past few decades, CUSUM charts have been widely used for monitoring process stability and capability in identifying small shifts in the process. It was first proposed

by Page [13] and is much better than conventional charts in perceiving small process shifts, and hence, it is a good chart to be used in analysing the data [1]. This section provides a review of past research that is related to CUSUM.

Very recently, Wang et al. [14] constructed a convolution model for oil leakage detection in electrohydraulic railway point systems in Beijing, China. The electrohydraulic railway point system is widely adopted due to its high efficiency and long service life. Nevertheless, its operation highly relies on a sufficient volume of oil which sometimes leads to leakage of oil and hence causes a failure to the electrohydraulic railway point system. In a nutshell, they conclude that the CUSUM chart is sensitive in detecting the small mean shift in the process.

Chen et al. [15] used multiple estimators to estimate the mean and variance of the population by using samples in phase I of their experiment. When the testing process in phase II is out of control, they analysed the influence of each estimator combination on the CUSUM chart based on running length distribution of control charts. They comprehensively analysed the average, standard deviation, and percentile of the control chart running length in four different environments in order to find the parameter combination that optimizes the control chart performance. The research results show that the CUSUM control chart based on $X\text{-}\sigma\text{IQR}$ and $WH\text{-}\sigma\text{IQR}$ estimators performs best in a polluted environment.

Yu and Cheng [16] studied the CUSUM charts in psychometric research to detect aberrant responses in a response sequence such as test speediness, inattentiveness, or cheating. They compared the CUSUM chart and change-point analysis (CPA) in detecting the test speediness. Simulation studies show that the performances of the statistics are affected by the underlying data generating model, the severity of the speediness, and the length of the test. In a nutshell, they conclude that CUSUM analysis shows better performance in a wide range of process means compared to the CPA method.

Xue and Qiu [17] developed the multivariate statistical process control (MSPC) based on some nonparametric distribution. They concluded that the CUSUM chart can accommodate stationary serial data correlation and perform well in different large process shifts. The CUSUM chart method was used by [18] for continuous monitoring of antifouling (AF) treatment. As a result, it showed that the CUSUM chart is the best tool to deal with reducing the operation and maintenance costs.

Boulloussa-Falces et al. [19] examined the validation of CUSUM chart for biofouling detection in heat exchangers. The CUSUM chart is very efficient in the early detection of slow and progressive changes within the process. They reported that CUSUM graphs demonstrated a greater capability to detect changes in the biological adherence process. Lawson [20] investigated monitoring a process mean by collecting one observation in every 12 minutes rather than a subgroup of five every 60 minutes. The results showed that the average time to signal (ATS) of both CUSUM and EWMA charts is substantially shorter than the ATS for the previous method, R chart. Volodarsky and Pototskiy [21]

compared the CUSUM charts with the overlay of the V-mask and the conventional charts of the mean values to the setting level offset. As a result, they found that CUSUM charts more sensitive to small displacements of the process setting level.

In Spain, Fortea-Sanchis and Escrig-Sos [22] applied CUSUM charts in monitoring clinical-care processes, a new aspect in clinical research. This study is really useful for studying learning curves which could not be observed with other methods. In medical research, Fortea-Sanchis et al. [23] studied the quality of nodal analysis in colon cancer and used a population registry cancer database to estimate the optimal number of lymph nodes for adequate prognostic analysis using CUSUM analysis.

A standardized CUSUM chart had been implemented by Ramasamy [24] to three different types of sample size which are variable sample size (VSS), fixed sample size (FSS), and Markov-dependent sample size (MDSS). He investigated the effect of the three types of sample size on the conventional control chart in monitoring small shifts in the process mean. He concluded that the standardized CUSUM chart with MDSS is the most vulnerable chart compared to other chart.

Over the past few years, fuzzy charts are being widely used in statistical process control research.

Al-Refaie et al. [25] analysed the CUSUM chart and EWMA chart in a manufacturing process using the triangular membership function. A set of three real case studies had been implemented to illustrate the proposed method which includes piston inside diameter, caps' angel, and tablet weight. The α -cut values show that the proposed CUSUM chart and EWMA chart efficiently help in monitoring the fuzzy observation in the process means. In a nutshell, the researcher revealed that the proposed charts have better detection ability in monitoring the quality characteristics of fuzzy observations and can be applied to other business applications.

Erginel and Şentürk [26] developed fuzzy EWMA and CUSUM control charts, and they reported that both conventional charts are not able to obtain the uncertainty in the case of incomplete data. Ghobadi et al. [27] constructed a fuzzy multivariate cumulative sum (CUSUM) control chart through a numerical comparison via a simulation study on the basis of the average run length (ARL). They concluded that the fuzzy multivariate cumulative sum (CUSUM) control chart performed better in detecting small- and medium-sized shifts in the process.

However, some of the data used in the manufacturing analysis not only can be expressed by type-1 fuzzy sets but are also more appropriate to be used in type-2 fuzzy sets. If there is no uncertainty, then a type-2 fuzzy set reduces to a type-1 fuzzy set, which is analogous to probability reducing to determinism when unpredictability vanishes. However, no study had been conducted on the type-2 fuzzy CUSUM control chart. Table 1 shows the summary of previous studies on the CUSUM control chart.

Based on the review in Table 1, we can conclude that majority of previous research focused on the conventional CUSUM chart and only two research studies had studied the T1F-CUSUM control charts. In fact, only one study towards the SCUSUM chart had been conducted. Therefore, this

TABLE 1: Summary of selected studies on the CUSUM control chart.

Existing literature	Application area	Findings	Type of CUSUM	Type of number used in analysis
Wang et al. [14]	Transportation sector (electrohydraulic railway point systems)	CUSUM control chart is constructed to monitor the residual signal since it is sensitive to the small mean shift	Conventional chart	Crisp numbers
Chen et al. [15]	Polluted environment	CUSUM control chart based on estimators performs best in a polluted environment	Conventional chart	Crisp numbers
Yu and Cheng [16]	Medical sector (psychometric research)	CUSUM analysis shows better performance in a wide range of conditions compared to change-point analysis (CPA) method	Conventional chart	Crisp numbers
Xue and Qiu [17]	Manufacturing sector	CUSUM chart accommodates stationary serial data correlation properly and it performs well in different cases	Conventional chart	Crisp numbers
Boullosa-Falces et al. [18]	Manufacturing sector (antifouling (AF) treatment of tubular heat exchangers)	CUSUM chart is the best tool for reducing the operation and maintenance costs	Conventional chart	Crisp numbers
Boullosa-Falces et al. [19]	Manufacturing sector (biofouling detection in heat exchangers)	CUSUM chart is simple and economical to be used	Conventional chart	Crisp numbers
Lawson [20]	Manufacturing sector	The average time to signal both CUSUM and EWMA charts are substantially better than the previous method, R chart	Conventional chart	Crisp numbers
Volodarsky and Pototskiy [21]	Manufacturing sector	CUSUM chart is more sensitive to small displacements	Conventional chart	Crisp numbers
Fortea-Sanchis and Escrig-Sos [22]	Medical sector (clinical-care processes)	Useful for assessing the quality-of-care outcomes by using learning curves	Conventional chart	Crisp numbers
Fortea-Sanchis et al. [23]	Medical sector	CUSUM has more appropriate cutoff point for diagnosing a high-quality prognosis in colon cancer patients	Conventional chart	Crisp numbers
Ramasamy [24]	Manufacturing sector (piston ring)	Standardized CUSUM chart is the most economical chart in detecting the small shift in the process	Conventional standardized chart	Crisp numbers
Al-Refaie et al. [25]	Manufacturing sector (piston ring, cap's angel, and tablet weight)	The proposed charts have better detection ability in monitoring the quality characteristics of fuzzy observations and can be applied to the other business applications	Type-1 fuzzy chart	Triangular fuzzy numbers
Erginel and Şentürk [26]	Manufacturing sector	Conventional EWMA and CUSUM control charts are not able to obtain the uncertainty of incomplete data	Type-1 fuzzy chart	Trapezoidal fuzzy numbers
Ghobadi et al. [27]	Manufacturing sector	The developed multivariate control chart shows better performance in detecting small- and medium-sized shifts in the process	Type-1 fuzzy chart	Trapezoidal fuzzy numbers

research differs from other studies by improving the fuzzy control chart for type-2 CUSUM control chart in agricultural sector. Different from most of the past studies where the CUSUM control chart was proposed, this study focuses on standardized CUSUM control charts. This study proposes a new standardized CUSUM control chart where interval type-2 fuzzy numbers are embedded in production data. This is the first identifiable study where the IT2F-SCUSUM control chart is proposed.

3. Proposed IT2F-SCUSUM Control Charts

In order to achieve the aforementioned objectives, this section revisits a brief conventional CUSUM control chart and type 1 fuzzy CUSUM control chart. More importantly,

this section proposes IT2F-SCUSUM control charts. CUSUM control charts are well recognized as a potentially advanced process monitoring tools because of their sensitivity against small and moderate shifts [28]. CUSUM is used to examine the process mean as it can easily detect even a small shift in the procedure.

In type-1 fuzzy control charts, the trapezoidal fuzzy numbers need to transform into crisp numbers. This transformation is called as defuzzification method. Four ways of representative (scalar) values for the fuzzy sets that transform fuzzy sets into crisp values are fuzzy mode, α -level fuzzy midrange, fuzzy median, and fuzzy average. Therefore, in this study, we will use the fuzzy midrange transformation method for the process of defuzzification of the data. Fuzzy midrange is the midpoint of the ends of the α -level cuts,

denoted as A^α , which is a nonfuzzy set that comprises all elements whose membership is greater than or equal to α -cuts [29]. There is no theoretical basis supporting any one specifically, and the selection between them should be mainly based on the ease of computation or preference of the user [11]. Therefore, in their study, they analysed the data using the α -level fuzzy midrange method because it is simpler to be used.

In this paper, we will analyse the data using conventional SCUSUM charts and T1F-SCUSUM control charts prior to proceeding with the proposed IT2F-SCUSUM control charts. In type-2 fuzzy sets, the membership functions are three dimensional which has a new third dimension that provides additional degrees of freedom that make it possible to directly model uncertainties [30]. An example of type-2 fuzzy sets is interval type-2 fuzzy sets. It is the most preferred type-2 fuzzy sets in scientific publications because computations with interval type-2 fuzzy sets are rather simple and manageable [31]. Therefore, the definitions of type-2 fuzzy sets, interval type-2 fuzzy sets, and some of interval type-2 arithmetic operations for two trapezoidal interval type-2 fuzzy sets are explained as follows.

Definition 1. A type-2 fuzzy set, denoted by \tilde{A} , is characterized by a type-2 membership function $\mu_{\tilde{A}}$ and presented as follows [31]:

$$\begin{aligned} \tilde{A} &= \{(x, u), \mu_{\tilde{A}}(x, u) | x \in X, u \in [0, 1]\}, \\ I_x &= \{u \in [0, 1] | (\mu_{\tilde{A}}(x, u) > 0)\}. \end{aligned} \quad (1)$$

An interval type-2 fuzzy set is a type-2 fuzzy set and can be expressed as follows:

$$I_x = \{u \in [0, 1] | (\mu_{\tilde{A}}(x, u) > 1)\}. \quad (2)$$

Therefore, interval type-2 trapezoidal fuzzy sets are called as closed interval type-2 fuzzy sets if I_x is the closed interval for every $x \in X$.

Definition 2. The upper membership function and the lower membership function of interval type-2 fuzzy sets are given in Figure 1 [32].

A trapezoidal interval type-2 fuzzy set is as follows:

$$\tilde{A}_i = (\tilde{A}_i^U, \tilde{A}_i^L) = \left((a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1(A_i^U), H_2(A_i^U)), (a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1(A_i^L), H_2(A_i^L)) \right), \quad (3)$$

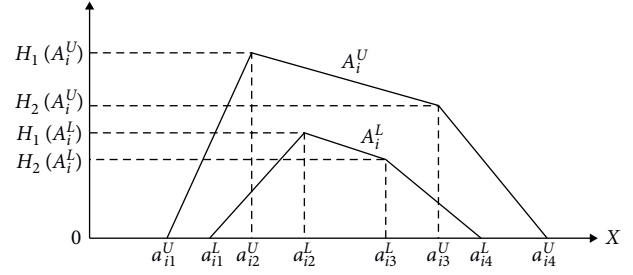


FIGURE 1: The membership functions of interval type-2 fuzzy set \tilde{A} .

where A_i^U and A_i^L denote type-1 fuzzy sets; $a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U$, $a_{i1}^L, a_{i2}^L, a_{i3}^L$, and a_{i4}^L are the reference points of the interval type-2 fuzzy \tilde{A}_i ; $H_j(A_i^U)$ signifies the membership value of the element $a_{i(j+1)}^U$ in the upper trapezoidal membership function A_i^U , $1 \leq j \leq 2$; and $H_j(A_i^L)$ indicates the membership value of the element $a_{i(j+1)}^L$ in the lower trapezoidal membership function A_i^L :

$$\begin{aligned} 1 &\leq j \leq 2, \\ H_1(A_i^U), H_2(A_i^U), H_1(A_i^L), H_2(A_i^L) &\in [0, 1], \\ 1 &\leq i \leq n. \end{aligned} \quad (4)$$

Let \tilde{A}_1 and \tilde{A}_2 be the two trapezoidal interval type-2 fuzzy sets:

$$\begin{aligned} \tilde{A}_1 &= (A_1^U, A_1^L) = \left((a_{11}^U, a_{12}^U, a_{13}^U, a_{14}^U; H_1(A_1^U), H_2(A_1^U)), (a_{11}^L, a_{12}^L, a_{13}^L, a_{14}^L; H_1(A_1^L), H_2(A_1^L)) \right), \\ \tilde{A}_2 &= (A_2^U, A_2^L) = \left((a_{21}^U, a_{22}^U, a_{23}^U, a_{24}^U; H_1(A_2^U), H_2(A_2^U)), (a_{21}^L, a_{22}^L, a_{23}^L, a_{24}^L; H_1(A_2^L), H_2(A_2^L)) \right). \end{aligned} \quad (5)$$

Then, the arithmetic operations for the two trapezoidal interval type-2 fuzzy sets are given as follows [32]:

Addition operation:

$$\begin{aligned} \tilde{A}_1 \oplus \tilde{A}_2 &= (A_1^U, A_1^L) \oplus (A_2^U, A_2^L) \\ &= \left((a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U; \min(H_1(A_1^U), H_1(A_2^U)), \min(H_2(A_1^U), H_2(A_2^U))), (a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L; \min(H_1(A_1^L), H_1(A_2^L)), \min(H_2(A_1^L), H_2(A_2^L))) \right). \end{aligned} \quad (6)$$

Subtraction operation:

$$\begin{aligned}\tilde{A}_1 \ominus \tilde{A}_2 &= (A_1^U, A_1^L) \ominus (A_2^U, A_2^L) \\ &= \left((a_{11}^U - a_{24}^U, a_{12}^U - a_{23}^U, a_{13}^U - a_{22}^U, a_{14}^U - a_{21}^U); \min(H_1(A_1^U); H_1(A_2^U)), \min(H_2(A_1^U); H_2(A_2^U)), \right. \\ &\quad \left. (a_{11}^L - a_{24}^L, a_{12}^L - a_{23}^L, a_{13}^L - a_{22}^L, a_{14}^L - a_{21}^L); \min(H_1(A_1^L); H_1(A_2^L)), \min(H_2(A_1^L); H_2(A_2^L)) \right). \quad (7)\end{aligned}$$

Multiplication operation:

$$\begin{aligned}\tilde{A}_1 \otimes \tilde{A}_2 &= (A_1^U, A_1^L) \otimes (A_2^U, A_2^L) \\ &= \left((a_{11}^U \times a_{21}^U, a_{12}^U \times a_{22}^U, a_{13}^U \times a_{23}^U, a_{14}^U \times a_{24}^U); \min(H_1(A_1^U); H_1(A_2^U)), \min(H_2(A_1^U); H_2(A_2^U)), \right. \\ &\quad \left. (a_{11}^L \times a_{21}^L, a_{12}^L \times a_{22}^L, a_{13}^L \times a_{23}^L, a_{14}^L \times a_{24}^L); \min(H_1(A_1^L); H_1(A_2^L)), \min(H_2(A_1^L); H_2(A_2^L)) \right). \quad (8)\end{aligned}$$

Arithmetic operations with crisp value k :

$$\begin{aligned}k \times \tilde{A}_1 &= \left((k \times a_{11}^U, k \times a_{12}^U, k \times a_{13}^U, k \times a_{14}^U); H_1(A_1^U), H_2(A_1^U), \right. \\ &\quad \left. (k \times a_{11}^L, k \times a_{12}^L, k \times a_{13}^L, k \times a_{14}^L); H_1(A_1^L), H_2(A_1^L), \right) \\ \frac{\tilde{A}_1}{k} &= \left(\left(\frac{1}{k} \times a_{11}^U, \frac{1}{k} \times a_{12}^U, \frac{1}{k} \times a_{13}^U, \frac{1}{k} \times a_{14}^U \right); H_1(A_1^U), H_2(A_1^U), \right. \\ &\quad \left. \left(\frac{1}{k} \times a_{11}^L, \frac{1}{k} \times a_{12}^L, \frac{1}{k} \times a_{13}^L, \frac{1}{k} \times a_{14}^L \right); H_1(A_1^L), H_2(A_1^L), \right) \quad (9)\end{aligned}$$

where $k > 0$.

There are some differences between type-1 fuzzy numbers and type-2 fuzzy numbers. Table 2 summarises the differences between them.

Based on Table 2, the type-1 fuzzy sets use a single membership function of data, whereas type-2 fuzzy sets use upper and lower membership functions. Next, in defuzzification techniques, there are four methods that can be used by using type-1 fuzzy sets, while in type-2 fuzzy sets, there are five methods that can be used. Besides that, the graph for type-1 fuzzy sets is in single trapezoid but the graph for type-2 fuzzy sets is in two trapezoids with three-dimensional sets.

3.1. IT2F-SCUSUM Control Chart. In the literature, the fuzzy approach to the CUSUM control chart was first introduced by Ghobadi et al. [27]. They developed the fuzzy CUSUM control chart by means of the fuzzy set theory

through a numerical example. Since then, there has been no study that combines type-2 fuzzy numbers with CUSUM control charts. In this section, we develop an IT2F-SCUSUM control chart where interval type-2 fuzzy numbers and standard sample means are combined.

The CUSUM chart is plot by the following quantity [1]:

$$C_i = \sum_{j=1}^i (\bar{x}_j - \mu_0), \quad (10)$$

where C_i is known as cumulative sum up to and including the i th sample, while μ_0 is the estimate of the in-control mean and \bar{x}_j is the mean of the j th sample. In this study, the number of defects was defined as trapezoidal number (a, b, c , and d). However, if $b = c$, the number of traps was converted into a triangular fuzzy number.

The interval type-2 fuzzy control chart uses fuzzy membership functions that have the grades themselves while constructing the limits of the control chart. The fuzzy sample mean is expressed as follows [34]:

TABLE 2: Differences between trapezoidal type-1 fuzzy numbers and type-2 fuzzy numbers.

	Type-1 fuzzy numbers [33]	Type-2 fuzzy numbers [32]
Fuzzy numbers	$A_i = (a_i, b_i, c_i, d_i)$	$(\tilde{A}_i^U, \tilde{A}_i^L) = \left((a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1(A_i^U), H_2(A_i^U)), (a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1(A_i^L), H_2(A_i^L)) \right)$
Defuzzification	(i) Fuzzy mode (ii) Fuzzy midrange (iii) Fuzzy median (iv) Fuzzy average	(i) Centroid method (ii) Indices method (iii) Ranking method (iv) Distance method (v) Likelihood method
Graph		

$$\begin{aligned} \tilde{A}_1 &= (A_1^U, A_1^L) = \left[\left(\overline{a_{i1}^U}, \overline{a_{i2}^U}, \overline{a_{i3}^U}, \overline{a_{i4}^U}; H_1(A_1^U), H_2(A_1^U) \right), \right. \\ &\quad \left. \left(\overline{a_{i1}^L}, \overline{a_{i2}^L}, \overline{a_{i3}^L}, \overline{a_{i4}^L}; H_1(A_1^L), H_2(A_1^L) \right) \right] \\ &= \left(\left(\frac{\sum_{i=1}^m \overline{a_{i1}^U}}{m}, \frac{\sum_{i=1}^m \overline{a_{i2}^U}}{m}, \frac{\sum_{i=1}^m \overline{a_{i3}^U}}{m}, \frac{\sum_{i=1}^m \overline{a_{i4}^U}}{m}; \min(H_1(A_1^U), H_2(A_1^U)) \right), \right. \\ &\quad \left. \left(\frac{\sum_{i=1}^m \overline{a_{i1}^L}}{m}, \frac{\sum_{i=1}^m \overline{a_{i2}^L}}{m}, \frac{\sum_{i=1}^m \overline{a_{i3}^L}}{m}, \frac{\sum_{i=1}^m \overline{a_{i4}^L}}{m}; \min(H_1(A_1^L), H_2(A_1^L)) \right) \right). \end{aligned} \quad (11)$$

Nevertheless, some researchers prefer to standardize the variable \bar{X}_i in equation (10) before performing the calculations because many CUSUM charts can have the same

value parameters. By standardizing the CUSUM chart, it can have the same values of κ and H ; hence, it leads naturally to a CUSUM for controlling variability by using

$$\begin{pmatrix} (Z_a^U, Z_b^U, Z_c^U, Z_d^U; H_1(A_i^U), H_2(A_i^U)) \\ (Z_a^L, Z_b^L, Z_c^L, Z_d^L; H_1(A_i^L), H_2(A_i^L)) \end{pmatrix} = \begin{pmatrix} \left(\frac{\overline{X}_a^U, \overline{X}_b^U, \overline{X}_c^U, \overline{X}_d^U}{\overline{X}_a^L, \overline{X}_b^L, \overline{X}_c^L, \overline{X}_d^L} \right) - \left(\frac{\mu_0^U}{\mu_0^L} \right) \\ \left(\frac{S_a^U, S_b^U, S_c^U, S_d^U}{S_a^L, S_b^L, S_c^L, S_d^L} \right) \end{pmatrix}, \quad \text{for } i = 1, 2, 3, \dots, n, \quad (12)$$

where

$$\begin{pmatrix} S_a^U, S_b^U, S_c^U, S_d^U \\ S_a^L, S_b^L, S_c^L, S_d^L \end{pmatrix} = \sqrt{\frac{\sum_{i=1}^n \left[\begin{pmatrix} X_a^U, X_b^U, X_c^U, X_d^U \\ X_a^L, X_b^L, X_c^L, X_d^L \end{pmatrix}_{ij} - \begin{pmatrix} \overline{X}_a^U, \overline{X}_b^U, \overline{X}_c^U, \overline{X}_d^U \\ \overline{X}_a^L, \overline{X}_b^L, \overline{X}_c^L, \overline{X}_d^L \end{pmatrix}_j \right]^2}{n-1}}, \quad (13)$$

an unbiased standard deviation for the i th sample. It can be seen that all sample means and standard deviations are presented in interval type-2 fuzzy numbers.

In representing the CUSUMs, there are two ways that can be used: the tabular or algorithmic form and the V-mask form of the CUSUM. The tabular CUSUM is more desirable

to be used in monitoring the process mean compared to V-mask because it is hard for practitioners to make an interpretation from the analysis [1]. Then, the tabular

standardized CUSUM works by accumulating deviations as follows [29]:

$$(C_{ai}^+, C_{bi}^+, C_{ci}^+, C_{di}^+) = (\max[0, Z_{ai} - \kappa + C_{ai-1}^+], \max[0, Z_{bi} - \kappa + C_{bi-1}^+], \max[0, Z_{ci} - \kappa + C_{ci-1}^+], \max[0, Z_{di} - \kappa + C_{di-1}^+]), \quad (14)$$

$$(C_{ai}^-, C_{bi}^-, C_{ci}^-, C_{di}^-) = (\max[0, Z_{ai} - \kappa + C_{ai-1}^-], \max[0, Z_{bi} - \kappa + C_{bi-1}^-], \max[0, Z_{ci} - \kappa + C_{ci-1}^-], \max[0, Z_{di} - \kappa + C_{di-1}^-]), \quad (15)$$

where κ is the reference value and C_i^+ and C_i^- are one-sided upper and lower SCUSUMs, respectively. Besides, $(C_{ai}^+, C_{bi}^+, C_{ci}^+, C_{di}^+)$ and $(C_{ai}^-, C_{bi}^-, C_{ci}^-, C_{di}^-)$ accumulate deviations from the target value that is greater than κ with both quantities reset to zero on becoming negative. This means the process is considered as “out of control” if either C_i^+ or C_i^- exceeds the decision interval H , based on N^+ and N^- . H is the recommended value of the decision interval as five times of the process standard deviation σ [1]. Based on this procedure, $H = h\sigma$ and $\kappa = k\sigma$, where σ is the standard deviation of the sample variable used in the CUSUM chart. In this procedure, $\kappa = 0.5$ and $H = 5$ are taken as decision parameters for optimum level [29]. In fact, Montgomery [1] and Ramasamy [35] also said that using the parameter of $H = 5$ and $\kappa = 0.5$ will provide a good run length value compared to a shift of about 1σ in the process mean.

However, some CUSUM charts can have the same values of κ and H , and these parameters are not scale dependent as they do not depend on S . Therefore, the standardized CUSUM chart is the best alternative as it leads naturally to a CUSUM for controlling variability. The head start or fast initial response (FIR) essentially sets the starting values C_i^+ and C_i^- to nonzero values, typically $H/2$ for effective detection of any shift in the mean.

3.2. Defuzzification Method for IT2F-SCUSUM Control Chart. Defuzzification provides the best representation value of interval type-2 fuzzy sets as it finds only one value for each fuzzy set. This means the output of the defuzzification is a crisp value. Interval type-2 fuzzy control charts generated by these methods are similar to control charts; hence, defuzzification is able to evaluate the process as “in control” and “out of control” in the same way as the classical method [36].

There are various methods that can be used in analysing the defuzzification process in type-2 fuzzy control charts such as centroid method [30], indices method [37], and best nonfuzzy performance (BNP) method [38]. Consequently, Ercan and Anagun [34] made a comparison between four methods used in analysing the interval type-2 fuzzy sets. The methods are Kahraman et al.’s defuzzification method [39], Qin and Liu’s ranking method [40], Chen’s distance method [41], and Chen and Lee’s likelihood method [42]. As a result, the researchers concluded that all the methods show a similar result in terms of “in control” or “out of control” situation. Therefore, in this research, we use Kahraman et al.’s defuzzification method [39] as it is much simpler and more flexible in evaluating the process. Nevertheless, Kahraman et al. [39] modified the BNP method for application with trapezoidal type-2 fuzzy sets as follows:

$$\text{DIT2}_{\text{Trap}(i)}^U = \frac{(\bar{u}_{a_4^U} - \bar{u}_{a_1^U}) + (H_2(A_1^U)\bar{u}_{a_2^U} - \bar{u}_{a_1^U}) + (H_1(A_1^U)\bar{u}_{a_3^U} - \bar{u}_{a_1^U})}{4} + \bar{u}_{a_1^U}, \quad (16)$$

$$\text{DIT2}_{\text{Trap}(i)}^L = \frac{(\bar{u}_{a_4^L} - \bar{u}_{a_1^L}) + (H_2(A_1^L)\bar{u}_{a_2^L} - \bar{u}_{a_1^L}) + (H_1(A_1^L)\bar{u}_{a_3^L} - \bar{u}_{a_1^L})}{4} + \bar{u}_{a_1^L}, \quad (17)$$

$$\text{DIT2}_{\text{Trap}(i)} = \frac{\text{DIT2}_{\text{Trap}(i)}^U + \text{DIT2}_{\text{Trap}(i)}^L}{2}, \quad (18)$$

where $H_1(A_1^U)$ and $H_2(A_1^U)$ are the maximum membership degree of the upper membership functions, while a_{i4}^U and a_{i1}^U are the largest and the least possible values of the upper membership function. a_{i2}^U and a_{i3}^U are the second and third parameters of the upper membership functions. a_{i4}^L and a_{i1}^L are the largest and the least possible values of the lower membership function, while a_{i2}^L and a_{i3}^L are the second and third parameters of the lower membership functions. On the other hand, $\text{DIT2}_{\text{Trap}(i)}$ is the defuzzification value of each

data of interval type-2 fuzzy number on standardized cumulative sum per unit.

3.3. Performance of Control Chart. The control chart’s performance will be analysed by using the average run length (ARL). ARL is the average number of points that must be plotted before a point indicates an “out-of control” condition [1]. It is the expectation of the time before the control

chart gives a false alarm that an “in control” process has gone “out of control” [43]. This signifies the control chart is defined as the most effective chart based on the least value of ARL in the process shifts. Previously, Roberts [44] developed monographs of ARLs for normally distributed observations, while Robinson and Ho [45] used a numeric procedure to determine the ARL. However, Crowder [46] calculates the ARL using a computer program and Lucas and Saccucci [47] presented table and graph of ARL values for different values of L and λ for the EWMA chart. In fact, they evaluated the run length properties as a continuous Markov chain. Markov chain is the best analysis that can be applied as it is fast and accurate to compute ARLs. Hence, in this study, simulation is carried out to calculate the ARL values using Sigma XL software based on Markov chain rule. The evaluation criteria of the ARL approximation are given as follows:

$$ARL = \frac{\exp(-2\Delta b) + 2\Delta b - 1}{2\Delta^2}. \quad (19)$$

For $\Delta \neq 0$, $\Delta = -\delta^* - k$ for the upper one-sided CUSUM C_i^+ , $\Delta = -\delta^* - k$ for the lower one-sided CUSUM C_i^- , $b = h + 1.166$, and $\delta^* = ((\mu_1 - \mu_0)/\sigma)$. If $\Delta = 0$, one can use $ARL = b^2$. On the other hand, the quantity δ^* denotes the shift in the mean, which is in the units of σ , for which the ARL is to be calculated. Hence, if $\delta^* = 0$, use equation (19) to calculate the ARL_0 , while if $\delta^* \neq 0$, the value of ARL_1 is calculated based on shift of size δ^* . The ARL^+ and ARL^- are calculated based on the following formula:

$$\frac{1}{ARL} = \frac{1}{ARL^+} + \frac{1}{ARL^-}. \quad (20)$$

The developed IT2F-SCUSUM is a complete version of quality control where the chart is being examined based on the small shift in the process and proved by the performance of the average run length. There are two types of ARLs: the “in-control” state, ARL_0 , and the “out-of-control” state, ARL_1 . The smaller value of ARL indicates that it is better in detecting the small shift of the analysis [48, 49]. Figure 2 shows the summary of flowchart on analysis of the IT2F-SCUSUM control chart.

The proposed work is subjected to a comparative analysis where the performance of IT2F-SCUSUM chart is being compared with the performance of T1F-SCUSUM chart and conventional SCUSUM chart to find out the best method for analysing the defects.

4. Case Study: Application to Fertilizer Production

The IT2F-SCUSUM control chart is applied to fertilizer production in an agricultural system. Fertilizers provide macro- and micronutrients to the plants. It is also a source of food for plants and soil to enable the plant growth. There must be a perfect balance of sun, water, and food for plants to grow successfully. In fact, the production of plant might be affected if the number of fertilizers used in a plant is not sufficient. For example, the essential nutrients needed by

plants are macronutrients which consist of the elements nitrogen, phosphorus, and potassium to keep the plants well nourished. There are two types of fertilizers which are organic and chemical. Organic fertilizers mean the production of fertilizers is based on natural products that are free from additives or chemical substances, for instance, leaf mould and cow manure. In contrast, chemical fertilizers typically contain some additives or nonorganic fillers. However, chemical fertilizers can give good improvement to the plants just in days, are easy to handle, and are not expensive.

Twenty samples of chemical fertilizers were collected every ten minutes for an hour from an agriculture and rural development company in Malaysia. The weights of the fertilizers are in grams with a sample size of six. Defective fertilizers might result in high toxic chemicals that may affect the soil pH. The company used two types of machines which are packaging machine to pack the soil and granulation machine to granulate the soil in mixing the materials and suitable speed in making the fertilizers. However, some uncertainty and vagueness can occur due to the operators' judgement or mechanical errors in handling the fertilizers as human cognitive decisions play an important role; hence, an IT2F-SCUSUM control chart modelled by membership functions is the inevitable tool for these uncertainties.

4.1. Implementation. The data are modelled as IT2F-SCUSUM fuzzy numbers using trapezoidal membership functions. The IT2F-SCUSUM control chart used the fuzzy membership functions that have the grades themselves while constructing the limits of the control chart.

The data of fertilizer production are shown in Table 3. Then, the data are fuzzified to interval type-2 fuzzy numbers as follows.

Values for upper interval type-2 fuzzy sets $(a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U)$ are defined as changes of $(a - \Delta, a, a + \Delta, a + 2\Delta)$. In this study, $\Delta = 0.1$ is chosen to show the flexibility of fuzzy numbers. An example of fuzzification for first sample is illustrated as follows:

$$\begin{aligned} a_{1a}^U &= 15.8 - 0.1 = 15.7, \\ a_{1b}^U &= 15.8, \\ a_{1c}^U &= 15.8 + 0.1 = 15.9, \\ a_{1d}^U &= 15.8 + 0.2 = 16. \end{aligned} \quad (21)$$

Next, values of lower interval type-2 fuzzy sets $(a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L)$ are the changes of $\Delta + 0.2$ to differentiate between A^U and A^L of interval type-2:

$$\begin{aligned} a_{1a}^L &= 15.7 + 0.1 = 15.8, \\ a_{1b}^L &= 15.8 + 0.1 = 15.9, \\ a_{1c}^L &= 15.9 + 0.1 = 16, \\ a_{1d}^L &= 16 + 0.1 = 16.1. \end{aligned} \quad (22)$$

All the fuzzified data results are shown in Tables 4 and 5. Table 4 shows the upper interval type-2 fuzzy number, and Table 5 shows the lower interval type-2 fuzzy number.

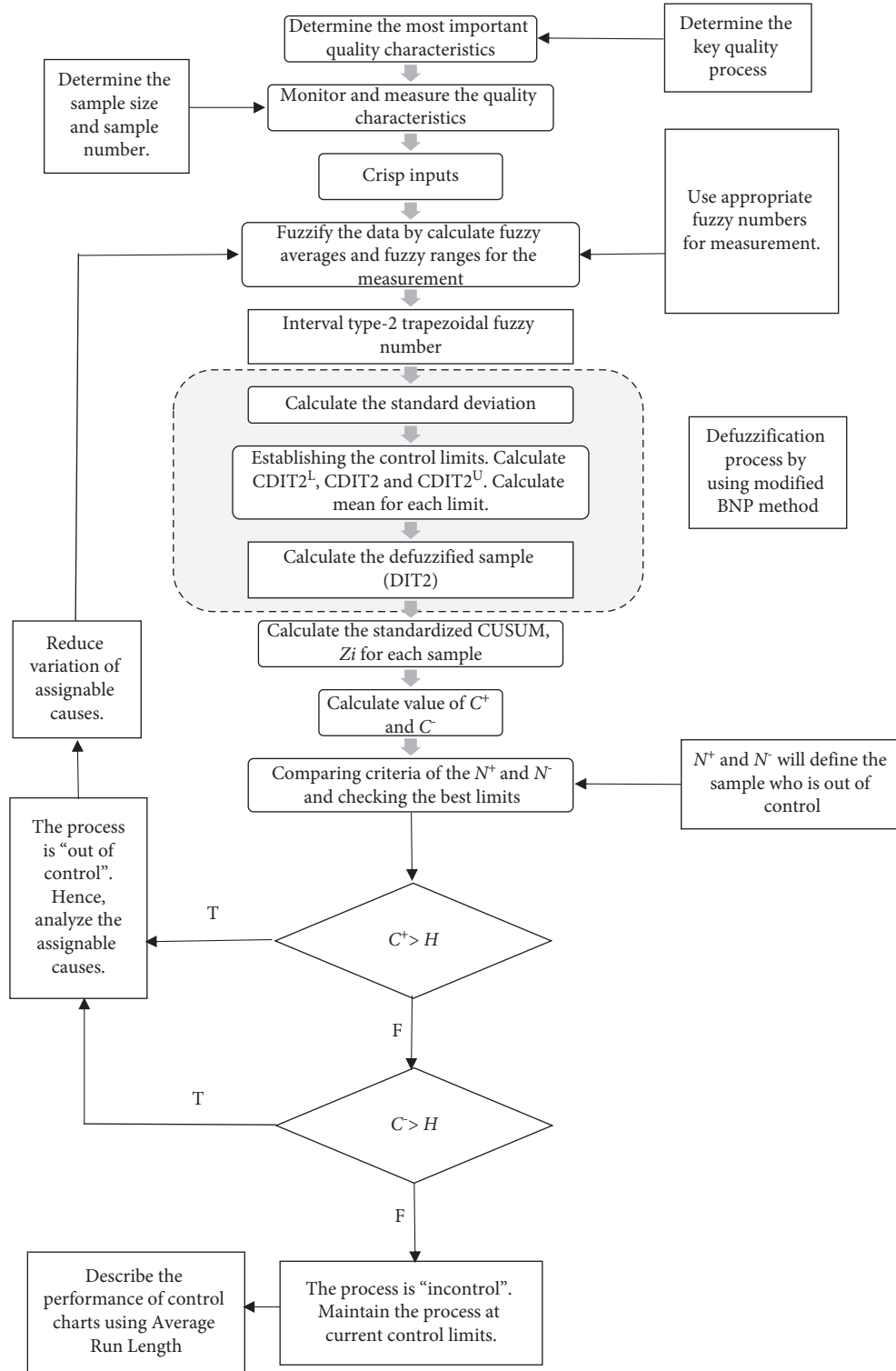


FIGURE 2: Flowchart of analysis of the IT2F-SCUSUM control chart.

TABLE 3: Data of 20 fertilizers' production in grams.

10 min	20 min	30 min	40 min	50 min	60 min
15.8	16.3	16.2	16.1	16.6	16.4
16.3	15.9	15.9	16.2	16.4	16.2
16.1	16.2	16.5	16.4	16.3	16.1
16.3	16.2	15.9	16.4	16.2	16
16.1	16.3	16.4	16.3	16	15.8
16.1	15.8	16.7	16.6	16.4	16.2
16.1	16.3	16.5	16.1	16.5	16.3
16.2	16.1	16.2	16.1	16.3	16.1
16.3	16.4	16.4	16.1	16.5	16.3
15.3	15.4	15.5	15.3	15.2	15.3
16.2	16.6	15.9	16.1	16.4	16.2
14.9	15.1	15.2	15.1	15.4	15.5
16.4	16.3	16.6	16.2	16.2	16
16.5	16.5	16.2	16.1	16.4	16.2
15.2	15.5	15.5	15.7	15.8	15.9
16	16.4	16.3	16.1	16.2	16
16.4	16	16.4	16.1	16.2	16
16	16.2	16.4	16.5	16.1	15.9
16.4	16.2	16.3	16.2	16.4	16.2
16.4	16.4	16.5	16	15.8	15.6

Then, the data from Tables 4 and 5 are calculated using equation (11), and the results are presented in Tables 6 and 7.

The calculation for the first sample of X_a from Table 6 is the mean of the first row of X_a in Table 4. Parts of the computations are shown as follows:

$$X_a = \frac{(15.7 + 16.2 + 16.1 + 16 + 16.5 + 16.3)}{6} = 16.1333. \quad (23)$$

For the second sample of X_b from Table 6, the calculation is as follows:

$$X_b = \frac{(15.8 + 16.3 + 16.2 + 16.1 + 16.6 + 16.4)}{6} = 16.2333. \quad (24)$$

For the third sample of X_c , the calculation is as follows:

$$X_c = \frac{(15.9 + 16.4 + 16.3 + 16.2 + 16.7 + 16.5)}{6} = 16.3333. \quad (25)$$

For the fourth sample of X_d , the calculation is as follows:

$$X_d = \frac{(16 + 16.5 + 16.4 + 16.3 + 16.8 + 16.6)}{6} = 16.4333. \quad (26)$$

Table 6 shows the upper interval type-2 fuzzy number for \bar{X} control chart, and Table 7 shows the lower interval type-2 fuzzy \bar{X} chart.

The calculation for S_1 standard deviation is as follows:

$$S_{1a} = \sqrt{\frac{\sum_{i=1}^n [(15.7, 16.2, 16.1, 16, 16.5, 16.3) - 16.1333]^2}{6-1}} = 0.2733. \quad (27)$$

Similar calculations are implemented for other values of standard deviations.

Then, standardized CUSUM and Z_i are calculated by using equation (12) for each of the sample, and the results are presented in Tables 8 and 9. In this study, we will use a tabular CUSUM with $\kappa = 0.5$ (because the shift size is 1σ and $\sigma = 1$), $\mu_0 = 16$, and $H = 5$ (as the recommended value of the decision interval is $H = 5\sigma = 5$). The calculation for the first sample of Z_a from Table 8 is as follows:

$$Z_{1a} = \frac{(16.1333 - 16)}{0.2733} = 0.4880. \quad (28)$$

Next, for the first sample of Z_b , the calculation is as follows:

$$Z_{1b} = \frac{(16.2333 - 16)}{0.2733} = 0.8539. \quad (29)$$

For the first sample of Z_c , the calculation is as follows:

$$Z_{1c} = \frac{(16.3333 - 16)}{0.2733} = 1.2199. \quad (30)$$

Afterwards, the calculation for Z_{1d} is as follows:

$$Z_{1d} = \frac{(16.4333 - 16)}{0.2733} = 1.5858. \quad (31)$$

Table 8 shows the upper IT2F-SCUSUM control chart, and Table 9 shows the lower IT2F-SCUSUM control chart.

Next, each sample of the standardized CUSUM control chart has been defuzzified based on the methods suggested by [39] for the evaluation of the process control using equations (16)–(18) as follows. The results for all samples are given in Table 10.

For first sample of $DIT2_{Upper}$, $DIT2_{Lower}$, and $DIT2$, the calculations are as follows:

TABLE 4: Upper IT2F-SCUSUM number for 20 subgroups.

	X_a						X_b						X_c						X_d					
	10	20	30	40	50	60	10	20	30	40	50	60	10	20	30	40	50	60	10	20	30	40	50	60
15.7	16.2	16.1	16	16.5	16.3	15.8	16.3	16.2	16.1	16.6	16.4	16.4	15.9	16.4	16.3	16.2	16.7	16.5	16	16.5	16.4	16.3	16.8	16.6
16.2	15.8	15.8	16.1	16.3	16.1	16.3	15.9	15.9	16.2	16.4	16.2	16.4	16	16	16	16.3	16.5	16.3	16.5	16.1	16.1	16.4	16.6	16.4
16	16.1	16.4	16.3	16.2	16	16.1	16.2	16.5	16.4	16.3	16.1	16.2	16.3	16.3	16.6	16.5	16.4	16.2	16.3	16.4	16.7	16.6	16.5	16.3
16.2	16.1	15.8	16.3	16.1	15.9	16.3	16.2	15.9	16.4	16.2	16	16	16.4	16.3	16	16.5	16.3	16.1	16.5	16.4	16.1	16.6	16.4	16.2
16	16.2	16.3	16.2	15.9	15.7	16.1	16.3	16.4	16.3	16	15.8	16.2	16.4	16.3	16.5	16.4	16.1	15.9	16.3	16.5	16.6	16.5	16.2	16
16	15.7	16.6	16.5	16.3	16.1	16.1	15.8	16.7	16.6	16.4	16.2	16.3	16.2	15.9	16.8	16.7	16.5	16.3	16.3	16	16.9	16.8	16.6	16.4
16	16.2	16.4	16	16.4	16.2	16.1	16.3	16.5	16.1	16.5	16.3	16.3	16.2	16.4	16.6	16.2	16.6	16.4	16.3	16.5	16.7	16.3	16.7	16.5
16.1	16	16.1	16	16.2	16	16.2	16.1	16.2	16.1	16.3	16.1	16.3	16.3	16.2	16.3	16.2	16.4	16.2	16.4	16.3	16.4	16.3	16.5	16.3
16.2	16.3	16.3	16	16.4	16.2	16.3	16.4	16.4	16.1	16.5	16.3	16.3	16.4	16.5	16.5	16.2	16.6	16.4	16.5	16.6	16.6	16.3	16.7	16.5
15.2	15.3	15.4	15.2	15.1	15.2	15.3	15.4	15.5	15.3	15.2	15.3	15.3	15.4	15.5	15.6	15.4	15.3	15.4	15.5	15.6	15.7	15.5	15.4	15.5
16.1	16.5	15.8	16	16.3	16.1	16.2	16.6	15.9	16.1	16.4	16.2	16.2	16.3	16.7	16	16.2	16.5	16.3	16.4	16.8	16.1	16.3	16.6	16.4
14.8	15	15.1	15	15.3	15.4	14.9	15.1	15.2	15.1	15.4	15.5	15	15	15.2	15.3	15.2	15.5	15.6	15.1	15.3	15.4	15.3	15.6	15.7
16.3	16.2	16.5	16.1	16.1	15.9	16.4	16.3	16.6	16.2	16.2	16	16	16.5	16.4	16.7	16.3	16.3	16.1	16.6	16.5	16.8	16.4	16.4	16.2
16.4	16.4	16.1	16	16.3	16.1	16.5	16.5	16.2	16.1	16.4	16.2	16.2	16.6	16.6	16.3	16.2	16.5	16.3	16.7	16.7	16.4	16.3	16.6	16.4
15.1	15.4	15.4	15.6	15.7	15.8	15.2	15.5	15.5	15.7	15.8	15.9	15.9	15.3	15.6	15.6	15.8	15.9	16	15.4	15.7	15.7	15.9	16	16.1
15.9	16.3	16.2	16	16.1	15.9	16	16.4	16.3	16.1	16.2	16	16	16.1	16.5	16.4	16.2	16.3	16.1	16.2	16.6	16.5	16.3	16.4	16.2
16.3	15.9	16.3	16	16.1	15.9	16.4	16	16.4	16.1	16.2	16	16	16.5	16.1	16.5	16.2	16.3	16.1	16.6	16.2	16.6	16.3	16.4	16.2
15.9	16.1	16.3	16.4	16	15.8	16	16.2	16.4	16.5	16.1	15.9	15.9	16.1	16.3	16.5	16.6	16.2	16	16.2	16.4	16.6	16.7	16.3	16.1
16.3	16.1	16.2	16.1	16.3	16.1	16.4	16.2	16.3	16.2	16.4	16.2	16.2	16.5	16.3	16.4	16.3	16.5	16.3	16.6	16.4	16.5	16.4	16.6	16.4
16.3	16.3	16.4	15.9	15.7	15.5	16.4	16.4	16.5	16	15.8	15.6	15.6	16.5	16.5	16.6	16.1	15.9	15.7	16.6	16.6	16.7	16.2	16	15.8

TABLE 5: Lower IT2F-SCUSUM number for 20 subgroups.

X_a						X_b						X_c						X_d					
10	20	30	40	50	60	10	20	30	40	50	60	10	20	30	40	50	60	10	20	30	40	50	60
15.8	16.3	16.2	16.1	16.6	16.4	15.9	16.4	16.3	16.2	16.7	16.5	16	16.5	16.4	16.3	16.8	16.6	16.1	16.6	16.5	16.4	16.9	16.7
16.3	15.9	15.9	16.2	16.4	16.2	16.4	16	16	16.3	16.5	16.3	16.5	16.1	16.1	16.4	16.6	16.4	16.6	16.2	16.2	16.5	16.7	16.5
16.1	16.2	16.5	16.4	16.3	16.1	16.2	16.3	16.6	16.5	16.4	16.2	16.3	16.4	16.7	16.6	16.5	16.3	16.4	16.5	16.8	16.7	16.6	16.4
16.3	16.2	15.9	16.4	16.2	16	16.4	16.3	16	16.5	16.3	16.1	16.5	16.4	16.1	16.6	16.4	16.2	16.6	16.5	16.2	16.7	16.5	16.3
16.1	16.3	16.4	16.3	16	15.8	16.2	16.4	16.5	16.4	16.1	15.9	16.3	16.5	16.6	16.5	16.2	16	16.4	16.6	16.7	16.6	16.3	16.1
16.1	15.8	16.7	16.6	16.4	16.2	16.2	15.9	16.8	16.7	16.5	16.3	16.3	16	16.9	16.8	16.6	16.4	16.4	16.1	17	16.9	16.7	16.5
16.1	16.3	16.5	16.1	16.5	16.3	16.2	16.4	16.6	16.2	16.6	16.4	16.3	16.5	16.7	16.3	16.7	16.5	16.4	16.6	16.8	16.4	16.8	16.6
16.2	16.1	16.2	16.1	16.3	16.1	16.3	16.2	16.3	16.2	16.4	16.2	16.4	16.3	16.4	16.3	16.5	16.3	16.5	16.4	16.5	16.4	16.6	16.4
16.3	16.4	16.4	16.1	16.5	16.3	16.4	16.5	16.5	16.2	16.6	16.4	16.5	16.6	16.6	16.3	16.7	16.5	16.6	16.7	16.7	16.4	16.8	16.6
15.3	15.4	15.5	15.3	15.2	15.3	15.4	15.5	15.6	15.4	15.3	15.4	15.5	15.6	15.7	15.5	15.4	15.5	15.6	15.7	15.8	15.6	15.5	15.6
16.2	16.6	15.9	16.1	16.4	16.2	16.3	16.7	16	16.2	16.5	16.3	16.4	16.8	16.1	16.3	16.6	16.4	16.5	16.9	16.2	16.4	16.7	16.5
14.9	15.1	15.2	15.1	15.4	15.5	15	15.2	15.3	15.2	15.5	15.6	15.1	15.3	15.4	15.3	15.6	15.7	15.2	15.4	15.5	15.4	15.7	15.8
16.4	16.3	16.6	16.2	16.2	16	16.5	16.4	16.7	16.3	16.3	16.1	16.6	16.5	16.8	16.4	16.4	16.2	16.7	16.6	16.9	16.5	16.5	16.3
16.5	16.5	16.2	16.1	16.4	16.2	16.6	16.6	16.3	16.2	16.5	16.3	16.7	16.7	16.4	16.3	16.6	16.4	16.8	16.8	16.5	16.4	16.7	16.5
15.2	15.5	15.5	15.7	15.8	15.9	15.3	15.6	15.6	15.8	15.9	16	15.4	15.7	15.7	15.9	16	16.1	15.5	15.8	15.8	16	16.1	16.2
16	16.4	16.3	16.1	16.2	16	16.1	16.5	16.4	16.2	16.3	16.1	16.2	16.6	16.5	16.3	16.4	16.2	16.3	16.7	16.6	16.4	16.5	16.3
16.4	16	16.4	16.1	16.2	16	16.5	16.1	16.5	16.2	16.3	16.1	16.6	16.2	16.6	16.3	16.4	16.2	16.7	16.3	16.7	16.4	16.5	16.3
16	16.2	16.4	16.5	16.1	15.9	16.1	16.3	16.5	16.6	16.2	16	16.2	16.4	16.6	16.7	16.3	16.1	16.3	16.5	16.7	16.8	16.4	16.2
16.4	16.2	16.3	16.2	16.4	16.2	16.5	16.3	16.4	16.3	16.5	16.3	16.6	16.4	16.5	16.4	16.6	16.4	16.7	16.5	16.6	16.5	16.7	16.5
16.4	16.4	16.5	16	15.8	15.6	16.5	16.5	16.6	16.1	15.9	15.7	16.6	16.6	16.7	16.2	16	15.8	16.7	16.7	16.8	16.3	16.1	15.9

TABLE 6: Upper interval type-2 fuzzy \bar{X} of 20 subgroups.

a	b	c	d	H_1	H_2
16.1333	16.2333	16.3333	16.4333	1	1
16.0500	16.1500	16.2500	16.3500	1	1
16.1667	16.2667	16.3667	16.4667	1	1
16.0667	16.1667	16.2667	16.3667	1	1
16.0500	16.1500	16.2500	16.3500	1	1
16.2000	16.3000	16.4000	16.5000	1	1
16.2000	16.3000	16.4000	16.5000	1	1
16.0667	16.1667	16.2667	16.3667	1	1
16.2333	16.3333	16.4333	16.5333	1	1
15.2333	15.3333	15.4333	15.5333	1	1
16.1333	16.2333	16.3333	16.4333	1	1
15.1000	15.2000	15.3000	15.4000	1	1
16.1833	16.2833	16.3833	16.4833	1	1
16.2167	16.3167	16.4167	16.5167	1	1
15.5000	15.6000	15.7000	15.8000	1	1
16.0667	16.1667	16.2667	16.3667	1	1
16.0833	16.1833	16.2833	16.3833	1	1
16.0833	16.1833	16.2833	16.3833	1	1
16.1833	16.2833	16.3833	16.4833	1	1
16.0167	16.1167	16.2167	16.3167	1	1

TABLE 7: Lower interval type-2 fuzzy \bar{X} of 20 subgroups

a	b	c	d	H_1	H_2
16.2333	16.3333	16.4333	16.5333	0.6	0.5
16.1500	16.2500	16.3500	16.4500	0.7	0.6
16.2667	16.3667	16.4667	16.5667	0.7	0.6
16.1667	16.2667	16.3667	16.4667	0.6	0.5
16.1500	16.2500	16.3500	16.4500	0.6	0.6
16.3000	16.4000	16.5000	16.6000	0.6	0.5
16.3000	16.4000	16.5000	16.6000	0.5	0.7
16.1667	16.2667	16.3667	16.4667	0.7	0.5
16.3333	16.4333	16.5333	16.6333	0.8	0.6
15.3333	15.4333	15.5333	15.6333	0.7	0.4
16.2333	16.3333	16.4333	16.5333	0.6	0.6

TABLE 7: Continued.

a	b	c	d	H_1	H_2
15.2000	15.3000	15.4000	15.5000	0.7	0.5
16.2833	16.3833	16.4833	16.5833	0.6	0.6
16.3167	16.4167	16.5167	16.6167	0.8	0.6
15.6000	15.7000	15.8000	15.9000	0.6	0.8
16.1667	16.2667	16.3667	16.4667	0.7	0.6
16.1833	16.2833	16.3833	16.4833	0.6	0.4
16.1833	16.2833	16.3833	16.4833	0.6	0.6
16.2833	16.3833	16.4833	16.5833	0.8	0.4
16.1167	16.2167	16.3167	16.4167	0.7	0.6

TABLE 8: Upper IT2F-SCUSUM of 20 subgroups.

Z_a	Z_b	Z_c	Z_d	H_1	H_2
0.4880	0.8539	1.2199	1.5858	1	1
0.2411	0.7234	1.2056	1.6878	1	1
1.0206	1.6330	2.2454	2.8577	1	1
0.3581	0.8951	1.4322	1.9693	1	1
0.2214	0.6642	1.1070	1.5498	1	1
0.5976	0.8964	1.1952	1.4940	1	1
1.1180	1.6771	2.2361	2.7951	1	1
0.8165	2.0412	3.2660	4.4907	1	1
1.7078	2.4398	3.1717	3.9036	1	1
-7.4232	-6.4550	-5.4867	-4.5185	1	1
0.5505	0.9633	1.3762	1.7891	1	1
-4.1079	-3.6515	-3.1950	-2.7386	1	1
0.8981	1.3880	1.8779	2.3678	1	1
1.2579	1.8385	2.4191	2.9997	1	1
-1.9764	-1.5811	-1.1859	-0.7906	1	1
0.4082	1.0206	1.6330	2.2454	1	1
0.4542	0.9992	1.5442	2.0892	1	1
0.3597	0.7914	1.2231	1.6547	1	1
1.8647	2.8818	3.8989	4.9160	1	1
0.0449	0.3144	0.5840	0.8535	1	1

TABLE 9: Lower IT2F-SCUSUM of 20 subgroups.

Z_a	Z_b	Z_c	Z_d	H_1	H_2
1.2199	1.5858	1.9518	2.3178	0.6	0.7
1.2056	1.6878	2.1701	2.6523	0.7	0.6
2.2454	2.8577	3.4701	4.0825	0.7	0.6
1.4322	1.9693	2.5064	3.0435	0.6	0.7
1.1070	1.5498	1.9926	2.4354	0.6	0.6
1.1952	1.4940	1.7928	2.0917	0.6	0.7
2.2361	2.7951	3.3541	3.9131	0.8	0.7
3.2660	4.4907	5.7155	6.9402	0.7	0.8
3.1717	3.9036	4.6355	5.3675	0.8	0.6
-5.4867	-4.5185	-3.5502	-2.5820	0.7	0.9
1.3762	1.7891	2.2019	2.6148	0.6	0.6
-3.1950	-2.7386	-2.2822	-1.8257	0.7	0.8
1.8779	2.3678	2.8577	3.3476	0.6	0.6
2.4191	2.9997	3.5803	4.1609	0.8	0.6
-1.1859	-0.7906	-0.3953	0.0000	0.6	0.8
1.6330	2.2454	2.8577	3.4701	0.7	0.6
1.5442	2.0892	2.6342	3.1792	0.6	0.9
1.2231	1.6547	2.0864	2.5181	0.6	0.6
3.8989	4.9160	5.9331	6.9502	0.8	0.9
0.5840	0.8535	1.1230	1.3925	0.7	0.6

TABLE 10: IT2F-SCUSUM control charts for 20 subgroups.

No.	DIT2 _{Upper}	DIT2 _{Lower}	DIT2	C ⁺	N ⁺	C ⁻	N ⁻
1	1.0369	1.4547	1.2458	0.746	1	0.000	0
2	0.9645	1.5974	1.2810	1.527	2	0.000	0
3	1.9392	2.6179	2.2785	3.305	3	0.000	0
4	1.1637	1.8395	1.5016	4.307	4	0.000	0
5	0.8856	1.4170	1.1513	4.958	5	0.000	0
6	1.0458	1.3521	1.1990	5.657	6	0.000	0
7	1.9566	2.6973	2.3269	7.484	7	0.000	0
8	2.6536	4.4499	3.5518	10.536	8	0.000	0
9	2.8057	3.6474	3.2266	13.262	9	0.000	0
10	-5.9708	-3.6551	-4.8130	7.949	10	4.313	1
11	1.1698	1.5964	1.3831	8.832	11	2.430	0
12	-3.4233	-2.2023	-2.8128	5.520	12	4.743	1
13	1.6330	2.0902	1.8616	6.881	13	2.381	0
14	2.1288	2.8110	2.4699	8.851	14	0.000	0
15	-1.3835	-0.5139	-0.9487	7.403	15	0.449	1
16	1.3268	2.1127	1.7197	8.622	16	0.000	2
17	1.2717	2.0460	1.6589	9.781	17	0.000	0
18	1.0072	1.4964	1.2518	10.533	18	0.000	0
19	3.3903	5.0050	4.1976	14.231	19	0.000	0
20	0.4492	0.8187	0.6339	14.365	20	0.000	0

TABLE 11: Result for the conventional SCUSUM control chart and T1F-SCUSUM control chart.

No.	Conventional SCUSUM chart					T1F-SCUSUM chart				
	Z _i	C ⁺	N ⁺	C ⁻	N ⁻	Z _i	C ⁺	N ⁺	C ⁻	N ⁻
1	0.8539	0.3539	1	0.0000	0	1.0369	0.5369	1	0.0000	0
2	0.7234	0.5773	2	0.0000	0	0.9645	1.0014	2	0.0000	0
3	1.6330	1.7103	3	0.0000	0	1.9392	2.4406	3	0.0000	0
4	0.8951	2.1054	4	0.0000	0	1.1637	3.1042	4	0.0000	0
5	0.6642	2.2696	5	0.0000	0	0.8856	3.4899	5	0.0000	0
6	0.8964	2.6660	6	0.0000	0	1.0458	4.0357	6	0.0000	0
7	1.6771	3.8431	7	0.0000	0	1.9566	5.4922	7	0.0000	0
8	2.0412	5.3843	8	0.0000	0	2.6536	7.6459	8	0.0000	0
9	2.4398	7.3241	9	0.0000	0	2.8057	9.9516	9	0.0000	0
10	-6.4550	0.3691	10	6.9550	1	-5.9708	3.4807	10	6.4708	1
11	0.9633	0.8325	11	6.4916	2	1.1698	4.1505	11	5.8011	2
12	-3.6515	0.0000	0	10.6431	3	-3.4233	0.2272	12	9.7243	3
13	1.3880	0.8880	1	9.7551	4	1.6330	1.3602	13	8.5913	4
14	1.8385	2.2266	2	8.4166	5	2.1288	2.9890	14	6.9625	5
15	-1.5811	0.1454	3	10.4977	6	-1.3835	1.1055	15	8.8460	6
16	1.0206	0.6660	4	9.9771	7	1.3268	1.9323	16	8.0192	7
17	0.9992	1.1652	5	9.4779	8	1.2717	2.7040	17	7.2476	8
18	0.7914	1.4566	6	9.1865	9	1.0072	3.2112	18	6.7403	9
19	2.8818	3.8384	7	6.8047	10	3.3903	6.1016	19	3.8500	10
20	0.3144	3.6528	8	6.9903	11	0.4492	6.0508	20	3.9008	11

$$\text{DIT2}_{\text{Trap}(1)}^U = \frac{(1.5858 - 0.4880) + (1 \times 0.8539 - 0.4880) + (1 \times 1.2199 - 0.4880)}{4} + 0.4880 = 1.0369,$$

$$\text{DIT2}_{\text{Trap}(1)}^L = \frac{(2.3178 - 1.2199) + (0.7 \times 1.5858 - 1.2199) + (0.6 \times 1.9518 - 1.2199)}{4} + 1.2199 = 1.4547,$$

$$\text{DIT2}_{\text{Trap}(1)} = \frac{1.0369 + 1.4547}{2} = 1.2458.$$

(32)

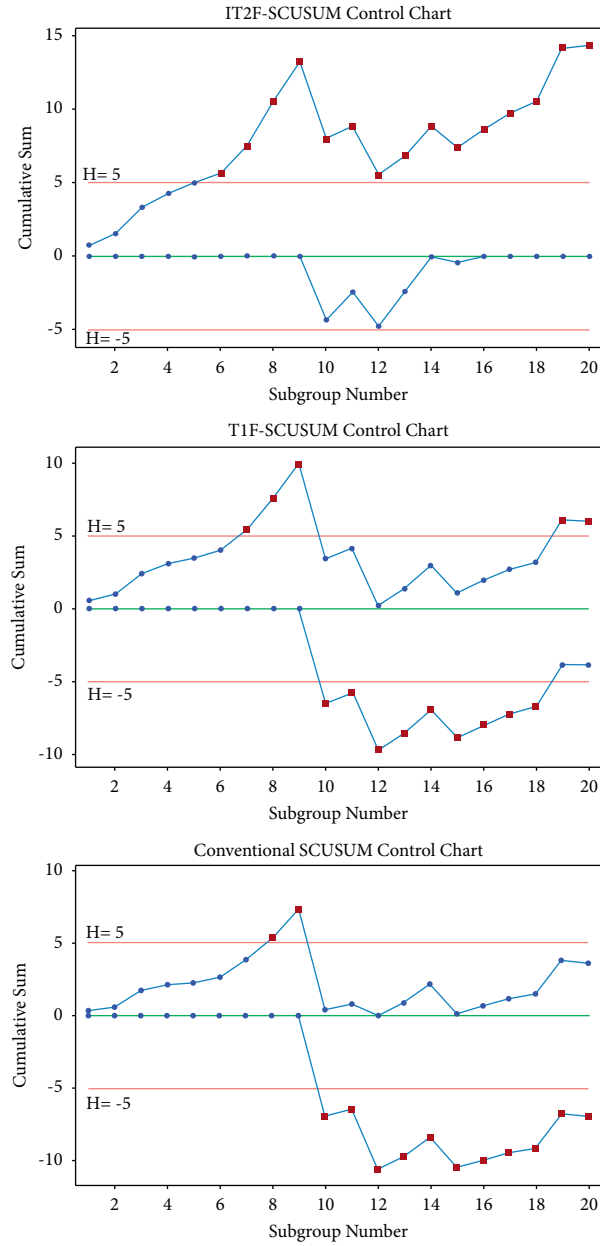


FIGURE 3: Control charts of conventional SCUSUM, T1F-SCUSUM, and IT2F-SCUSUM charts.

TABLE 12: Average run length (ARL) for conventional SCUSUM, T1F-SCUSUM, and IT2F-SCUSUM charts.

Shift in mean (multiple of sigma)	IT2F-SCUSUM 2	T1F-SCUSUM	Conventional SCUSUM
0	301.69	302.69	310.44
0.25	107.40	107.62	109.30
0.5	32.91	32.94	33.23
0.75	15.43	15.44	15.53
1	9.52	9.53	9.58
1.25	6.82	6.83	6.86
1.5	5.32	5.32	5.35
1.75	4.37	4.37	4.39
2	3.72	3.73	3.74
2.25	3.25	3.26	3.27
2.5	2.90	2.90	2.91
2.75	2.63	2.63	2.64
3	2.41	2.41	2.42
3.5	2.10	2.10	2.11
4	1.90	1.90	1.91
4.5	1.72	1.73	1.73
5	1.53	1.53	1.54

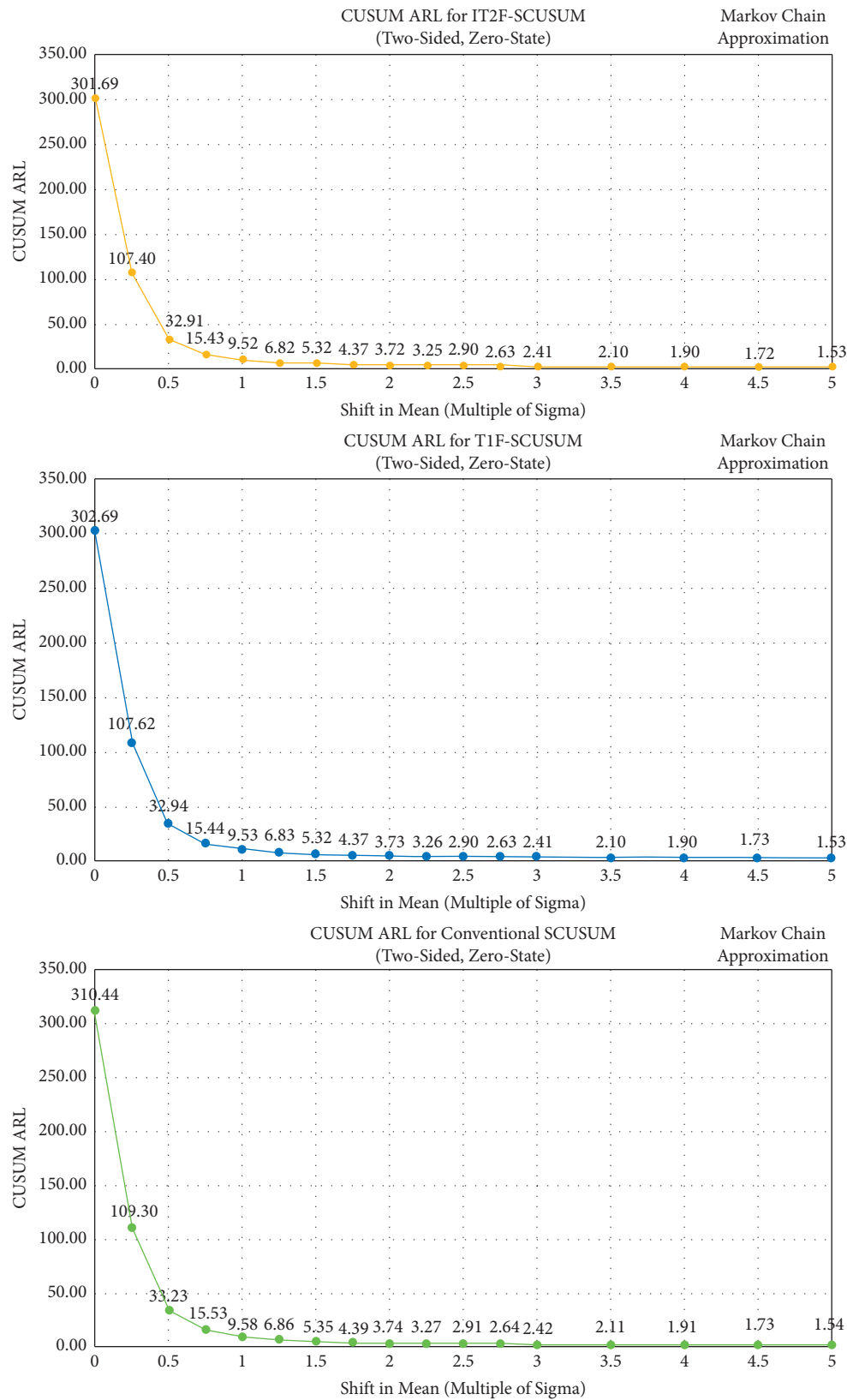


FIGURE 4: Graph of ARL on SCUSUM, T1F-SCUSUM, and IT2F-SCUSUM charts.

Then, we calculate C_i^+ and C_i^- by using equations (14) and (15). The initial value is $C^+(0) = 0 = C^-(0)$. For the first sample, $C^+ = C_1^+ = \max[0, 1.2458 - 0.5 + 0] = 0.746$ and $C^- = C_1^- = \max[0, -0.5 - 1.2458 + 0] = 0$.

For the second sample, $C^+ = C_2^+ = \max[0, 1.2810 - 0.5 + 0.746] = 1.527$ and $C^- = C_1^- = \max[0, -0.5 - 1.2810 + 0] = 0$.

Based on the result of N^+ , we start to count from 1 since the result of C_1^+ is 0.746. Then, C_6^+ shows the result is 5.657 which is greater than the limits of $H = 5$. Therefore, we can conclude that the process is out of control from sample 6 until sample 20 since the value of N^+ is greater than $H = 5$. In this study, $H = 5$ is taken as decision parameter for optimal level as recommended by [1, 35].

5. Comparative Analysis

To see the stability of the three types of control charts, this section provides a comparative result of the analysis. Table 11 shows the result for the conventional SCUSUM control chart and T1F-SCUSUM control chart.

Figure 3 presents the control charts using conventional SCUSUM, T1F-SCUSUM, and the proposed IT2F-SCUSUM.

Based on the conventional SCUSUM chart in Table 11 and Figure 3, we can see that, samples 8–20 are “out of control” since they exceed the control limits which means 13 points are uncontrolled. However, for the T1F-SCUSUM control chart, 14 samples are “out of control,” and 15 samples are “out of control” for the IT2F-SCUSUM control chart. Hence, this shows that the IT2F-SCUSUM control chart is more sensitive than the conventional SCUSUM chart and T1F-SCUSUM control chart since it captures the least number of samples compared to other charts.

Next, ARL with different values of shift is used to evaluate the control chart’s performance. ARL is the average number of samples taken before any signal of “out of control” condition is detected in the control chart. It was determined using Sigma XL, and the outcomes are provided in Table 12.

From Table 12, we can see that the “in-control” ARL of the IT2F-SCUSUM chart is 301.69, which is lower than the “in-control” ARL of the conventional SCUSUM chart ($ARL_0 = 310.44$) and T1F-SCUSUM chart ($ARL_0 = 302.69$). This concludes that if the process is in control, we expect to get a signal every 301 samples on average which is faster than the other two types of charts. In fact, the “out-of-control” ARL for the IT2F-SCUSUM chart is 9.52, also lower than the out-of-control ARL of the conventional SCUSUM chart and T1F-SCUSUM chart. This indicates that the IT2F-SCUSUM chart control chart is quicker in indicating small shifts in the process compared to other charts. It also proves that when the magnitude of the shifts increases, the power of control charts is then augmented. In fact, it shows that the control chart’s performance is better when fuzzy numbers are being implemented.

Figure 4 is presented to visualise the three ARLs that are obtained using Markov chain approximation where the shifts in the control process can be observed.

From Table 12 and Figure 4, we can see that the IT2F-SCUSUM chart has lower value of ARL compared to the conventional SCUSUM chart and T1F-SCUSUM chart. This indicates that the IT2F-SCUSUM chart control chart is quicker in indicating small shifts in the process compared to other charts. It shows that the control chart’s performance is better when fuzzy numbers are being implemented.

6. Conclusions

Fuzzy control charts have been widely used in sociological, medical, engineering, economics, service, and management research. The fuzzy set theory has the capability of systematic dealing with fuzzy data. Most previous studies designed fuzzy control chart for linguistic data and fuzzy numbers since fuzzy logic helps in explaining vague and imprecise data. Traditionally, the conventional control chart is used to identify the process shift with real-value data. The efficiency of the IT2F-SCUSUM charts is more than analysing of crisp data, but it also gives essential alerts by means of flexibility of interval type-2 fuzzy numbers. The CUSUM control chart through a numerical comparison via a simulation study shows better performance in detecting small- and medium-sized shifts in the process. In fact, it is the best chart for detecting small process shifts which are less than 1.5σ . Besides, CUSUM charts are used for controlling cumulative sum of quality characteristics measurement in monitoring analysis.

The contributions of this paper are threefold: (1) we proposed to use IT2F-SCUSUM sets since this method can model higher levels of uncertainty compared to T1F-SCUSUM sets. Pertaining to the SCUSUM charts, there are some studies on ordinary fuzzy control charts, yet there is no study on IT2F-SCUSUM charts so far. Indeed, modelling the fuzzy control charts using the IT2F-SCUSUM sets may contribute to more accuracy in monitoring the process as the membership functions of the data are already imprecise. (2) Then, we compare the new method with the conventional SCUSUM chart and T1F-SCUSUM control chart. This comparative analysis is not only to gain insights about the performance of the three charts but also to help researchers make decisions either in reducing or eliminating the need for inspection in the products. (3) Last but not the least, we computed the ARL for each chart to evaluate the exact probabilities of false alarm in designing the best charts among them. The simulation study shows better performance in detecting small- and medium-sized shifts in the process which are less than 1.5σ .

Based on the case study results, we can conclude that the IT2F-SCUSUM control chart is more sensitive to examine the variations of the fertilizer production characteristics compared to the T1F-SCUSUM control chart and conventional SCUSUM since the IT2F-SCUSUM control chart found 15 defects, but the T1F-SCUSUM control chart found 14 defects and conventional SCUSUM found 13 defects in fertilizers. Hence, this means the defect should be removed, and if the company includes the defects, the fertilizers will be low in quality and it might affect the plantations of the consumers. The company should consider the quality of the product to increase the level of customer satisfaction.

In a nutshell, we can conclude that the IT2F-SCUSUM control chart is more sensitive to monitor the variations of the fertilizer production compared to the T1F-SCUSUM control chart and conventional SCUSUM chart. Further research can investigate the IT2F-SCUSUM control chart for monitoring defects using a variable sample size and fuzzy theory control charts using hesitant fuzzy theory, intuitionistic fuzzy theory, or neutrosophic fuzzy theory. Additionally, future research can use high number of samples to see more variabilities in the analysis. Other than that, the stated proposal may be extended in the future by using the CUSUM structure proposed by Faisal et al. [50], where a link relative variable transformation could be introduced to IT2F-CUSUM.

Data Availability

The data used to support the findings of this study are included within the article.

Disclosure

Publication of this research is part of the requirement for graduation from University Malaysia Terengganu.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this manuscript and read and approved the final manuscript.

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Retraction

Retracted: Global Dynamics of Sixth-Order Fuzzy Difference Equation

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] A. Khaliq, M. Adnan, and A. Q. Khan, "Global Dynamics of Sixth-Order Fuzzy Difference Equation," *Mathematical Problems in Engineering*, vol. 2021, Article ID 9769093, 16 pages, 2021.

Research Article

Global Dynamics of Sixth-Order Fuzzy Difference Equation

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Across many fields, such as engineering, ecology, and social science, fuzzy differences are becoming more widely used; there is a wide variety of applications for difference equations in real-life problems. Our study shows that the fuzzy difference equation of sixth order has a nonnegative solution, an equilibrium point and asymptotic behavior. $y_{i+1} = (Dy_{i-1}y_{i-2}/(E + Fy_{i-3} + Gy_{i-4} + Hy_{i-5})), i = 0, 1, 2, \dots$, where y_i is the sequence of fuzzy numbers and the parameter D, E, F, G, H and the initial condition $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0$ are nonnegative fuzzy number. The characterization theorem is used to convert each single fuzzy difference equation into a set of two crisp difference equations in a fuzzy environment. So, a pair of crisp difference equations is formed by converting the difference equation. The stability of the equilibrium point of a fuzzy system has been evaluated. By using variational iteration techniques and inequality skills as well as a theory of comparison for fuzzy difference equations, we investigated the governing equation dynamics, such as its boundedness, existence, and local and global stability analysis. In addition, we provide some numerical solutions for the equation describing the system to verify our results.

1. Introduction

The fuzzy differential equation was first proposed by Zadeh [1]. The fuzzy difference equation was solved in their analysis by Chang and Zadeh [2]; the original value question has been analyzed thoroughly. Their analysis shows that the nonnegative solution is bounded and proceeds.

Fuzzy difference equations are difference equations whose initial values, constants, and solutions are all fuzzy numbers (see preliminaries). By using the fuzzy analog of concepts understood from the theory of ordinary difference equations, we extend these solutions to parametric fuzzy difference equations as a means of verifying the behavior of the fuzzy difference equation. According to our findings, the behavior of the parametric fuzzy difference equation does not mirror that of the coinciding parametric ordinary difference equation [3].

Since the data on the differential equation model describing many practical issues are incomplete and fluffy setting theory is an effective tool for modifying uncertainty

and processing vague subjective information in mathematical models, we ought to look into the behavior of the solution to the flouted equation where the parameter is relevant.

We cited that Deeba et al. [4], in 1996, investigated the 1st degree difference equation for the historical background for the equation we are studying in this research:

$$C_{n+1} = aC_n + b, \quad n = 0, 1, 2, 3, \dots, \quad (1)$$

where a nonnegative fuzzy number is C_n and C_0, b, a is a fuzzy number that occurs in the genetic population.

In addition, to calculate the concentration of CO_2 in the blood, the following linearized form of 2nd degree linear fuzzy equation is considered by Deeba and Korvin [3]:

$$B_{n+1} = B_n - ghB_{n-1} + p, \quad n = 0, 1, 2, \dots, \quad (2)$$

where g, h, p, B_0 , and B_1 are fuzzy number and B_n is a sequence of a fuzzy nonnegative number.

Moreover, Papaschinopoulos and Stefanidou [5] handle the existence, the persistence, the uniqueness, and the

boundedness of nonnegative results of the succeeding fuzzy difference equation:

$$C_{n+1} = \sum_{k=0}^i \frac{B_k}{C_{n-k}^{q_k}}, \quad (3)$$

where $i \in 1, 2, 3, \dots$, the parameters $B_i, k \in 0, 1, 2, \dots, i$, are fuzzy nonnegative numbers, the parameters $q_k, k \in 0, 1, 2, \dots, i$, are real nonnegative constant, and the initial values $C_k, k \in -i, -i+1, \dots, 0$, are fuzzy nonnegative numbers.

Moreover, in 2006, Papaschinopoulos and Stefanidou [6] consider the periodicity of the solution of the following fuzzy difference equation of max-type:

$$Y_{i+1} = \max \left[\frac{D_0}{Y_{i-k}}, \frac{D_1}{Y_{i-m}} \right], \quad (4)$$

where the primary values $Y_i, i \in -d, -d+1, \dots, -1, d = \max\{k, m\}$ are fuzzy nonnegative numbers, k and m are nonnegative integers, and D_0 and D_1 are the fuzzy nonnegative numbers.

More recent, Zhang et al. [7] study the asymptomatic behavior and the existence of nonnegative results of the following nonlinear fuzzy difference equation:

$$y_{i+1} = \frac{A y_i + Y_{i-1}}{B + y_{i-1}}, \quad i = 0, 1, 2, \dots, \quad (5)$$

where y_i is the sequence of fuzzy nonnegative number, A and B are nonnegative fuzzy numbers, and the initial conditions y_{-1} and y_0 are nonnegative fuzzy numbers.

In 2004, Zhang et al. [8] deal with asymptomatic behavior, the boundedness, and the existence of the nonnegative solution for the 1st degree Ricatti difference equation:

$$z_{n+1} = \frac{C + z_n}{D + z_n}, \quad n = 0, 1, 2, 3, \dots, \quad (6)$$

where z_n is a sequence of a fuzzy nonnegative number, the initial value z_0 , and C and D are fuzzy nonnegative number.

In recent, in 2006, Zhang et al. [9] inspected the global behavior, persistence, and boundedness of nonnegative result of the 3rd degree rational fuzzy difference equation:

$$y_{i+1} = \frac{B + y_{i-1}}{y_{i-1} y_{i-2}}, \quad i = 0, 1, 2, 3, \dots, \quad (7)$$

where initial values y_0, y_{-1}, y_{-2} , and B are fuzzy nonnegative numbers.

Moreover, in 2007, Khastan et al. [10] investigated global behavior, the uniqueness, and the existence of solution for next two nonequivalent fuzzy difference equation:

$$z_{i+1} - p = a z_i, \quad i = 0, 1, 2, 3, \dots, \quad (8)$$

where z_i is a sequence of fuzzy nonnegative numbers and z_0, p , and a are fuzzy nonnegative numbers. It is easy to see that equations (1) and (8) are in-equivalent.

In more recent, Changyou Wang, Ping Liu, Xiaolin Su, Rui Li, and Xiaohom Hu [11] investigate the uniqueness and

existence of trivial solution and the asymptotical behavior of the equilibrium point of fifth-order nonlinear fuzzy difference equation:

$$\beta_{i+1} = \frac{A \beta_{i-1} \beta_{i-2}}{B + C \beta_{i-3} + D \beta_{i-4}}, \quad i = 0, 1, 2, \dots, \quad (9)$$

where β_i is the sequence of fuzzy numbers, the initial values $\beta_{-4}, \beta_{-3}, \beta_{-2}, \beta_{-1}$, and β_0 , and the parameter A, B, C , and D are fuzzy nonnegative number. In theoretical study, we have some numerical simulation.

Motivated by the recent conversation, we want to analyze the optimality of constructive results and the asymptotic behavior for the equilibrium points of the nonlinear fuzzy differences equations:

$$y_{i+1} = \frac{D y_{i-1} y_{i-2}}{E + F y_{i-3} + G y_{i-4} + H y_{i-5}}, \quad i = 0, 1, 2, \dots, \quad (10)$$

where y_i is the sequence of fuzzy numbers, the parameter D, E, F, G, H , and the initial condition $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 are nonnegative fuzzy number. When the parameters and the initial values are positive real numbers, Wang et al. [12] considered the global attractivity of the equilibrium point and the asymptotic behavior of the solutions of the difference equation. To demonstrate our theoretical study, we have some numerical simulations.

2. Preliminaries and Definitions

We provide the following definitions and preliminaries result for the reader's convenience. In this section, we will discuss about the fundamental ideas, notations, and definitions of fuzzy difference equation. Some examples will also be given to explain the concepts of result.

Definition 1 (membership function, see [13]). For a set A , we define a membership function μ_A such as

$$\mu_A(x) = \begin{cases} 1, & \text{iff } x \in A, \\ 0, & \text{iff } x \notin A. \end{cases} \quad (11)$$

We can say that the function μ_A maps the elements in the universal set X to the set $[0, 1]$. Membership function μ_A in crisp set maps whole members in universal set X to set $[0, 1]$:

$$\mu_A: X \longrightarrow [0, 1], \quad (12)$$

as shown in Figure 1.

Definition 2 (fuzzy set, see [1]). Fuzzy set was for the first proposed by Zadeh in 1965 as an extension of classical notion of a set. The word “fuzzy” means “uncertainty or imprecise.” If the information is not clearly defined, then we introduce fuzziness.

A fuzzy set is a collection of elements which correspond to the definition of ‘A’ in the reliability degree equal to 1 or equal to the value belonging to interval $[0, 1]$. In fuzzy sets, each element is mapped to $[0, 1]$ by the membership function:

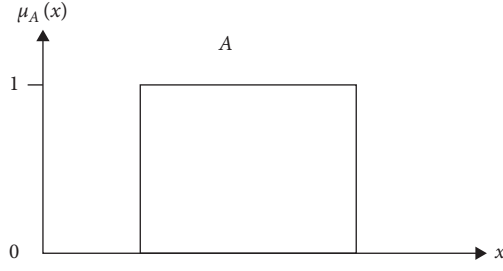


FIGURE 1: Membership function.

$$\mu_A: Y \longrightarrow [0, 1], \quad (13)$$

where $[0, 1]$ is valid for 0 to 1 (including 0, 1). The fuzzy set is thus “the ambiguous boundary set” compared to the crisp set.

Definition 3 (fuzzy number, see [14]). Consider a set Y ; we denoted the closure of Y as \bar{Y} . We call a function $D: R \longrightarrow [0, 1]$ is a fuzzy number if it fulfills the characteristics:

- (i) D is normal which means that there exist $y \in R$ s.t $D(y) = 1$.
- (ii) D is fuzzy convex which means

$$[p + q]_\alpha = [p]_\alpha + [q]_\alpha,$$

$$[\omega p]_\alpha = \omega [p]_\alpha, \quad \forall \alpha \in [0, 1],$$

$$[pq]_\alpha = \left[\min(p_{l,\alpha} q_{l,\alpha}, p_{l,\alpha} q_{r,\alpha}, p_{r,\alpha} q_{l,\alpha}, p_{r,\alpha} q_{r,\alpha}), \max(p_{l,\alpha} q_{l,\alpha}, p_{l,\alpha} q_{r,\alpha}, p_{r,\alpha} q_{l,\alpha}, p_{r,\alpha} q_{r,\alpha}) \right], \quad (16)$$

$$\left[\frac{p}{q} \right]_\alpha = \left[\min\left(\frac{p_{l,\alpha}}{q_{l,\alpha}}, \frac{p_{l,\alpha}}{q_{r,\alpha}}, \frac{p_{r,\alpha}}{q_{l,\alpha}}, \frac{p_{r,\alpha}}{q_{r,\alpha}} \right), \max\left(\frac{p_{l,\alpha}}{q_{l,\alpha}}, \frac{p_{l,\alpha}}{q_{r,\alpha}}, \frac{p_{r,\alpha}}{q_{l,\alpha}}, \frac{p_{r,\alpha}}{q_{r,\alpha}} \right) \right], \quad 0 \in [q]_\alpha.$$

Definition 4 (LR-fuzzy number, see [15]). A fuzzy number \bar{B} on R is said to be LR-fuzzy number. If there exist a real numbers $p, q \geq 0$ such that

$$\mu_B(y) = \begin{cases} L\left(\frac{m-y}{p}\right), & y \leq m, \\ R\left(\frac{y-m}{q}\right), & y \geq m, \end{cases} \quad (17)$$

in which $L(y)$ and $R(y)$ are continuous and nondecreasing function on the real line $L(1) = R(1) = 0$. L and R are left and right reference functions, respectively, m is the mean value, and p and q are called left and right spreads on the membership function.

“A LR-fuzzy number \bar{B} is represented by 3 real number p, q , and m as whose meaning are defined in Figure 2.

$$D(ty_1 + (1-t)y_2) \geq \min[D(y_1), D(y_2)], \quad (14)$$

$$\forall D \in [0, 1], y_1, y_2 \in R.$$

(iii) D is upper semicontinuous on R .

(iv) D is compactly supported which means $\sup\{y \in R: D(y) > 0\}$ is compact.

Now, consider the set of all fuzzy numbers is represented by R_f , with $\alpha \in (0, 1]$ and $D \in R_f$. We expressed fuzzy number D with α -cuts as

$$[D]_\alpha = [y \in R: D(y) \geq \alpha], \quad (15)$$

$$[D]_0 = [y \in R: D(y) \geq 0].$$

We consider $[D]_0$ the fuzzy number D with support and represent this by $\sup(u)$. Clearly, with $[D]_\alpha$ limited to R for closed interval, we assumed that D is a nonnegative fuzzy number if D set $(0, \infty)$. It is clear that if D is trivial fuzzy number (real nonnegative number), then D is a fuzzy trivial number with $[D]_\alpha = [D, D]$. For $p, q \in R_f$, $[p]_\alpha = [p_{l,\alpha}, p_{r,\alpha}]$, $[q]_\alpha = [q_{l,\alpha}, q_{r,\alpha}]$, and $\omega \in R$, the addition $p + q$, the scalar product ωp , the product pq , and division (p/q) in the SIA (Standard Interval Arithmetic) setting are defined as

Definition 5 (triangular fuzzy number, see [16]). Consider fuzzy number denoted by 3 points as follows:

$$B = (u_1, u_2, u_3). \quad (18)$$

It is denoted as a membership function as seen in Figure 3:

$$\mu_B(y) = \begin{cases} (0), & y < u_1, \\ \left(\frac{y-u_1}{u_2-u_1} \right), & u_1 \leq y \leq u_2, \\ \left(\frac{y-u_2}{u_3-u_2} \right), & u_2 \leq y \leq u_3, \\ (0), & y > u_3. \end{cases} \quad (19)$$

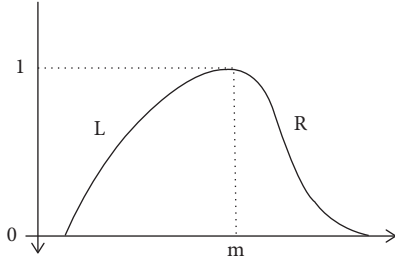
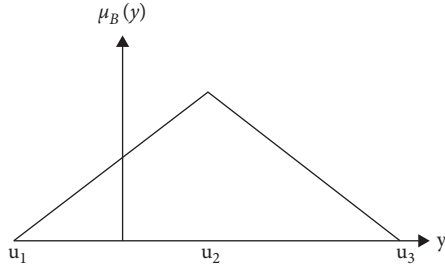


FIGURE 2: LR-fuzzy number.

FIGURE 3: Triangular fuzzy number $B = (u_1, u_2, u_3)$.

Now, if we get crisp interval by α -cut operation, interval B_α shall be obtained. $\forall \alpha \in [0, 1]$ from

$$\begin{aligned} \left(\frac{u_1^{(\alpha)} - u_1}{u_2 - u_1} \right) &= \alpha, \\ \left(\frac{u_3^{(\alpha)} - u_3}{u_2 - u_3} \right) &= \alpha. \end{aligned} \quad (20)$$

We obtain

$$\begin{aligned} u_1^\alpha &= (u_2 - u_1)\alpha + u_1, \\ u_3^\alpha &= -(u_3 - u_2)\alpha + u_3. \end{aligned} \quad (21)$$

Thus,

$$\begin{aligned} B_\alpha &= [u_1^\alpha, u_3^\alpha] \\ &= [(u_2 - u_1)\alpha + u_1, -(u_3 - u_2)\alpha + u_3]. \end{aligned} \quad (22)$$

Definition 6 (value of fuzzy number, see [15]). \bar{B} is the fuzzy number having α -cut denoted $[L^{-1}(\alpha), R^{-1}(\alpha)]$. S is a decreasing function; then, the value of \bar{B} is determined by

$$\text{Val}(\bar{B}) = \int_0^1 f(\alpha) [L^{-1}(\alpha) + R^{-1}(\alpha)] d\alpha. \quad (23)$$

Definition 7 (boundedness, see [17]). A series of $\Delta(Y) = (\Delta Y_n)$ of the fuzzy number seems to be bounded if the set $(\Delta Y_n; n \in \mathbb{N})$ of the fuzzy number is bounded, where $\Delta Y = (|Y_n - Y_{n+1}|)$. Let $m(\Delta)$ represented the set of all bounded difference series of fuzzy numbers.

Definition 8 (metric on fuzzy number, see [11]). Let p and q be the fuzzy number with

$$\begin{aligned} [p]_\alpha &= [p_{l,\alpha}, p_{r,\alpha}], \\ [q]_\alpha &= [q_{l,\alpha}, q_{r,\alpha}], \quad \forall \alpha \in [0, 1]. \end{aligned} \quad (24)$$

Then, we define the metric on fuzzy numbers as follows:

$$B(p, q) = \sup \max[|p_{l,\alpha} - q_{l,\alpha}|, |p_{r,\alpha} - q_{r,\alpha}|], \quad (25)$$

where \sup is applied, for all $\alpha \in [0, 1]$. Moreover, (R_f, B) is the complete metric space. For future analysis, we express $\hat{0} \in R_f$ as

$$\hat{0}(y) = \begin{cases} 1, & \text{if } y = 0, \\ 0, & \text{if } y \neq 0. \end{cases} \quad (26)$$

Thus, $[\hat{0}]_\alpha = [0, 0]$, $0 < \alpha \leq 1$.

Lemma 1 (see [11]). Let I_u and I_v be some intervals of real numbers and $f: I_u^{k+1} \times I_v^{k+1} \rightarrow I_u$ and $g: I_u^{k+1} \times I_v^{k+1} \rightarrow I_v$ be continuously differentiable function. Thus, for every set of initial conditions $(u_n, v_j) \in I_u \times I_v$, $(n = -k, -k+1, \dots, 0, j = -l, -l+1, \dots, 0)$, the following system of difference equations,

$$\begin{cases} u_{i+1} = f(i_n, i_{n-1}, \dots, i_{n-k}, v_i, v_{i-1}, \dots, v_{i-l}), \\ v_{i+1} = g(u_i, u_{i-1}, \dots, u_{i-k}, v_i, v_{i-1}, \dots, v_{i-l}), \end{cases} \quad i = 0, 1, 2, \dots, \quad (27)$$

has a unique solution $(u_n, v_j)_{n=-k, j=-l}^{+\infty, +\infty}$.

Definition 9 (equilibrium points, see [11]). A point $(u, v) \in I_u \times I_v$ is called an equilibrium point of system (27) if $u = f(u, u, \dots, u, v, v, \dots, v)$, $v = g(u, u, \dots, u, v, v, \dots, v)$. That is, $(u_i, v_i) = (u, v)$, for $i \geq 0$, is the result of system (27), or equivalent, (u, v) is allotted point (f, g) of the vector map.

For system (27), we consider the equilibrium point (u, v) . Then, we have

- (i) The equilibrium point (\bar{u}, \bar{v}) said to be locally stable if each $\epsilon > 0$, there exist $\delta > 0$, such that, for any initial conditions $(u_i, v_j) \in I_u \times I_v$, $(i = -n, -n+1, \dots, 0, j = -k, -k+1, \dots, 0)$ with $\sum_{i=-n}^0 |u_i - \bar{u}| < \epsilon$ and $\sum_{j=-k}^0 |v_j - \bar{v}| < \epsilon$; we have $|u_n - \bar{u}| < \epsilon$ and $|v_n - \bar{v}| < \epsilon$, for any $n < 0$.
- (ii) (\bar{u}, \bar{v}) , the equilibrium point, is said to be attractor if $\lim_{n \rightarrow \infty} u_n = u$ and $\lim_{n \rightarrow \infty} v_n = v$, for any initial conditions $(u_i, v_j) \in I_u \times I_v$, $(i = -n, -n+1, \dots, 0, j = -k, -k+1, \dots, 0)$.
- (iii) If (\bar{u}, \bar{v}) is attractor and stable, then the equilibrium point is said to be asymptotically stable.
- (iv) If (\bar{u}, \bar{v}) is locally unstable, then equilibrium point is said to be unstable.

Note 1. To calculate the stability criteria of the system for a fuzzy difference equation, the equilibrium points are very important.

To calculate the equilibrium point of a fuzzy system, the methods are follows:

- (i) Convert the fuzzy system into according crisp system
- (ii) From the crisp system calculate the equilibrium point

Definition 10 (equilibrium points of a vector map, see [11]). Let (\bar{u}, \bar{v}) be equilibrium point of a vector map $F = (f, u_i, u_{i-1}, \dots, u_{i-k}, g, v_i, v_{i-1}, \dots, v_{i-l})$, where f and g are continuously differential function at (\bar{u}, \bar{v}) . The linearized system of (27) about equilibrium point (\bar{u}, \bar{v}) is $U_{i+1} = F(U_i) = (F_j \cdot U_i)$, where F_j is the Jacobian matrix of system (27) about (\bar{u}, \bar{v}) and $U_i = (u_i, u_{i-1}, \dots, u_{i-k}, v_i, v_{i-1}, \dots, v_{i-l})^T$.

Definition 11 (trivial solution, see [11]). The trivial solution $j = \hat{0}$ of equation (10)

- (i) The result of $j_n \in D_\varepsilon, n > 0$ is stable; if given $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ with $D(j_n, \hat{0}) < \delta, n = -5, -4, \dots, 0$, which implies $D(j_n, \hat{0}) < \varepsilon$, for $n > 0$, such that $j_n \in D_\delta, n = -5, -4, \dots, 0$
- (ii) The result of $j_n \in D_\varepsilon, n > 0$ is attractive if there is a $\delta > 0$ such that $D(j_n, \hat{0}) < \delta, n = -5, -4, \dots, 0$, one has $\lim_{i \rightarrow \infty} D(j_n, \hat{0}) = 0$.
- (iii) If (i) and (ii) hold concurrently, then it is asymptotically stable

Definition 12 (monotone, see [11]). Let a, b, c , and d be the 4 nonnegative whole number such that $a + b = n$ and $c + d = m$. Splitting $u = (u_1, u_2, \dots, u_n)$ into $u = (u_a, u_b)$ and $v = (v_1, v_2, \dots, v_m)$ into $v = (v_c, v_d)$, where $[u]_\sigma$ denotes the σ -components of u . We say that the function $f(u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m)$ hold a mixed monotone property in subsets $I_u^n \times I_v^m$ of $R^n \times R^m$ if $f(u_a, u_b, v_c, v_d)$ is a nondecreasing monotone in every element of (u_a, v_c) and is nonincreasing monotone in every element of (u_b, v_d) , for $(u, v) \in I_u^n \times I_v^m$. In specific, if $b = 0$ and $d = 0$, then it is called nondecreasing monotone in $I_u^n \times I_v^m$.

Lemma 2 (see [11]). Assume that $U(n+1) = G(U_n), n = 0, 1, 2, \dots$, is a system of differential equations and U is the equilibrium point of this system, i.e., $G(U) = U$. Then, we have

- (i) If all eigenvalues of Jacobian matrix J_G about U lies inside the open unit disk $|\omega| < 1$, then U is locally asymptotically stable
- (ii) If all eigenvalues of Jacobian matrix J_G about U lies outside the open unit disk $|\omega| > 1$, then U is unstable

Theorem 1 (characterization theorem, see [16]). Let us consider the fuzzy difference equation problem:

$$\bar{y}_{i+1} = \bar{f}(x_n, i), \quad (28)$$

with initial condition $\bar{y}_{i=0} = \bar{y}_0$, where $f: E^* \times Z_{\geq 0} \rightarrow E^*$ such that

- (1) The parametric form of the function is

$$[f((y_i, i))_\alpha] = [\underline{f}(\underline{y}_n(\alpha), \bar{y}_n(\alpha), n, \alpha), \bar{f}(\underline{y}_n(\alpha), \bar{y}_n(\alpha), n, \alpha)], \quad (29)$$

- (2) The functions $\underline{f}(\underline{y}_n(\alpha), \bar{y}_n(\alpha), n, \alpha)$ and $\bar{f}(\underline{y}_n(\alpha), \bar{y}_n(\alpha), n, \alpha)$ are taken as continuous functions if, for any $\varepsilon_1 > 0$, there exist a $\delta_1 > 0$ such that

$$|\underline{f}(\underline{y}_n(\alpha), \bar{y}_n(\alpha), n) - \bar{f}(\underline{y}_{n1}(\alpha), \bar{y}_{n1}(\alpha), n_1)| < \varepsilon_1, \quad \forall \alpha \in [0, 1], \quad (30)$$

with

$$|(\underline{y}_n(\alpha), \bar{y}_n(\alpha), n) - (\underline{y}_{n1}(\alpha), \bar{y}_{n1}(\alpha), n_1)| < \delta_1, \quad \forall \alpha \in [0, 1], \quad (31)$$

and $\varepsilon_2 > 0$; there exists a $\delta_2 > 0$ such that

$$|\underline{f}(\underline{y}_n(\alpha), \bar{y}_n(\alpha), n, \alpha) - \bar{f}(\underline{y}_{n2}(\alpha), \bar{y}_{n2}(\alpha), n_2)| < \varepsilon_2, \quad \forall \alpha \in [0, 1], \quad (32)$$

with

$$|(\underline{y}_n(\alpha), \bar{y}_n(\alpha), n) - (\underline{y}_{n2}(\alpha), \bar{y}_{n2}(\alpha), n_2)| < \delta_2, \quad \forall \alpha \in [0, 1]. \quad (33)$$

Then, the difference equation (28) reduces the system of 2 difference equations as

$$\begin{aligned} \underline{y}_{n+1}(\alpha) &= \underline{f}(\underline{y}_n(\alpha), \bar{y}_n(\alpha), n, \alpha), \\ \bar{y}_{n+1}(\alpha) &= \bar{f}(\underline{y}_n(\alpha), \bar{y}_n(\alpha), n, \alpha), \end{aligned} \quad (34)$$

under initial conditions

$$\begin{aligned} \underline{y}_{n=0}(\alpha) &= \underline{y}_0(\alpha), \\ \bar{y}_{n=0}(\alpha) &= \bar{y}_0(\alpha). \end{aligned} \quad (35)$$

Note 2. By using characterization theorem, the single fuzzy difference equation is changed into the system of 2 crisp difference equations. In this paper, in the environment of fuzzy, we take a single fuzzy difference equation. Hence, the difference equation changed into 2 crisp difference equation.

Lemma 3 (see [11]). Assume that $U(i+1) = G(Y_i), i = z^+$, is a differential equation's system and the equilibrium point of the proceeding system is U . Then, about equilibrium point U , characteristics equation of the proceeding system is $Q(\omega) = b_0\omega^i + b_1\omega^{i-1} + \dots + b_{i-1}\omega + b_i = 0$, with the real coefficient $b_0 > 0$. Therefore, total answers of the equation $Q(\omega)$ lies inside the open disk $|\omega| > 1$ iff $\Delta_h > 0, h = 1, 2, \dots, n$,

where Δ_h is the principal minor of order h of the $m \times m$ matrix:

$$\Delta_m = \begin{bmatrix} b_1 & b_3 & b_5 & \dots & 0 \\ b_0 & b_2 & b_4 & \dots & 0 \\ 0 & b_1 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_m \end{bmatrix}. \quad (36)$$

3. Main Results

We needed the following lemmas for investigating the uniqueness and existence of a nonnegative solution of (10).

Lemma 4 (see [18]). Consider that f be the continuous function from $R^+ \times R^+ \rightarrow R^+$ and D, E, F, G , and H be the fuzzy numbers. Then,

$$[f(D, E, F, G, H)]_\alpha = f([D]_\alpha, [E]_\alpha, [F]_\alpha, [G]_\alpha, [H]_\alpha), \quad \alpha \in (0, 1]. \quad (37)$$

Lemma 5 (see [18]). Consider that $p \in R_f$ express $[p]_\alpha = [p_{l,\alpha}, p_{r,\alpha}]$, $\alpha \in (0, 1]$. Therefore, $p_{l,\alpha}$ and $p_{r,\alpha}$ can be regarded as function on $(0, 1]$ which holds

- (i) $p_{l,\alpha}$ is nondecreasing and continuous on left
- (ii) $p_{r,\alpha}$ is nonincreasing and continuous on right

(iii) $p_{l,\alpha} \leq p_{r,\alpha}$

Alternately, for any function $u(\alpha)$ and $v(\alpha)$ belong to $(0, 1]$ which hold (i)-(iii) in for the proceeding, there exist a unique $p \in R_f$ such that $p(\alpha) = [u(\alpha), v(\alpha)]$, for any $\alpha \in (0, 1]$.

Theorem 2. Consider equation (10), where y_i is the sequence of fuzzy numbers, the parameter D, E, F, G , and H and the initial condition $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 are nonnegative fuzzy numbers. There exist a unique nonnegative solution y_i of equation (10) under initial conditions $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 .

Proof. Suppose that there exist a sequence of fuzzy numbers y_i satisfying the equation (10) under initial conditions $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 .

Consider the α -cuts, $\alpha \in (0, 1]$:

$$\begin{aligned} [D]_\alpha &= [D_{l,\alpha}, D_{r,\alpha}], \\ [E]_\alpha &= [E_{l,\alpha}, E_{r,\alpha}], \\ [F]_\alpha &= [F_{l,\alpha}, F_{r,\alpha}], \\ [G]_\alpha &= [G_{l,\alpha}, G_{r,\alpha}], \\ [H]_\alpha &= [H_{l,\alpha}, H_{r,\alpha}], \\ [J]_\alpha &= [J_{l,\alpha}, J_{r,\alpha}], \quad i = -5, -4, \dots \end{aligned} \quad (38)$$

Then, from (10) and Lemma 4, it follows that

$$\begin{aligned} [y_{i+1}]_\alpha &= [L_{i+1,\alpha}, R_{i+1,\alpha}] = \left[\frac{Dy_{i-1}y_{i-2}}{E + Fy_{i-3} + Gy_{i-4} + Hy_{i-5}} \right]_\alpha \\ &= \left[\frac{[Dy_{i-1}y_{i-2}]_\alpha}{[E]_\alpha + [Fy_{i-3}]_\alpha + [Gy_{i-4}]_\alpha + [Hy_{i-5}]_\alpha} \right] \\ &= \frac{[D_{l,\alpha}, D_{r,\alpha}][L_{i-1,\alpha}, R_{i-1,\alpha}][L_{i-2,\alpha}, R_{i-2,\alpha}]}{[E_{l,\alpha}, E_{r,\alpha}] + [F_{l,\alpha}, F_{r,\alpha}][L_{i-3,\alpha}, R_{i-3,\alpha}] + [G_{l,\alpha}, G_{r,\alpha}][L_{i-4,\alpha}, R_{i-4,\alpha}] + [H_{l,\alpha}, H_{r,\alpha}][L_{i-5,\alpha}, R_{i-5,\alpha}]} \\ &= \frac{[D_{l,\alpha}L_{i-1,\alpha}L_{i-2,\alpha}, D_{r,\alpha}R_{i-1,\alpha}R_{i-2,\alpha}]}{[E_{l,\alpha} + F_{l,\alpha}L_{i-3,\alpha} + G_{l,\alpha}L_{i-4,\alpha} + H_{l,\alpha}L_{i-5,\alpha}, E_{r,\alpha} + F_{r,\alpha}R_{i-3,\alpha} + G_{r,\alpha}R_{i-4,\alpha} + H_{r,\alpha}R_{i-5,\alpha}]} \\ &= \left[\frac{D_{l,\alpha}L_{i-1,\alpha}L_{i-2,\alpha}}{E_{r,\alpha} + F_{r,\alpha}R_{i-3,\alpha} + G_{r,\alpha}R_{i-4,\alpha} + H_{r,\alpha}R_{i-5,\alpha}}, \frac{D_{r,\alpha}R_{i-1,\alpha}R_{i-2,\alpha}}{E_{l,\alpha} + F_{l,\alpha}L_{i-3,\alpha} + G_{l,\alpha}L_{i-4,\alpha} + H_{l,\alpha}L_{i-5,\alpha}} \right], \end{aligned} \quad (39)$$

from the above equation, for $i = -5, -4, \dots$, and we have

$$\begin{aligned} L_{i+1,\alpha} &= \frac{D_{l,\alpha}L_{i-1,\alpha}L_{i-2,\alpha}}{E_{r,\alpha} + F_{r,\alpha}R_{i-3,\alpha} + G_{r,\alpha}R_{i-4,\alpha} + H_{r,\alpha}R_{i-5,\alpha}}, \\ R_{i+1,\alpha} &= \frac{D_{r,\alpha}R_{i-1,\alpha}R_{i-2,\alpha}}{E_{l,\alpha} + F_{l,\alpha}L_{i-3,\alpha} + G_{l,\alpha}L_{i-4,\alpha} + H_{l,\alpha}L_{i-5,\alpha}}. \end{aligned} \quad (40)$$

Then, from Lemma 1, it is evident that, for any $(L_{n,\alpha}, R_{n,\alpha})$, $n = -5, -4, -3, -2, -1, 0$ of the proceeding system (40) under primary conditions, $(L_{i,\alpha}, R_{i,\alpha})$, $i = -5, -4, -3, -2, -1, 0$, $\alpha \in (0, 1]$, has a unique solution $(L_{i,\alpha}, R_{i,\alpha})$.

Alternately, we want to show that $[L_{i+1,\alpha}, R_{i+1,\alpha}]$, $\alpha \in (0, 1]$, where $(L_{i,\alpha}, R_{i,\alpha})$ is the solution of

system (40) with initial conditions $(L_{n,\alpha}, R_{n,\alpha})$, $n = -5, -4, -3, -2, -1, 0$, determines the solution y_i of equation (10) with initial conditions $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 such that

$$[y_i]_\alpha = [L_{i+1,\alpha}, R_{i+1,\alpha}], \quad \alpha \in (0, 1], i = -5, -4, \dots \quad (41)$$

By Lemma 5 and as D, E, F, G , and H and y_i , $i = -5, -4, -3, -2, -1, 0$, are fuzzy nonnegative numbers, for any $\alpha_1, \alpha_2 \in (0, 1]$ and $\alpha_1 > \alpha_2$, we obtain

$$\begin{aligned} 0 < D_{l,\alpha_1} &\leq D_{l,\alpha_2} \leq D_{r,\alpha_2} \leq D_{r,\alpha_1}, \\ 0 < E_{l,\alpha_1} &\leq E_{l,\alpha_2} \leq E_{r,\alpha_2} \leq E_{r,\alpha_1}, \\ 0 < F_{l,\alpha_1} &\leq F_{l,\alpha_2} \leq F_{r,\alpha_2} \leq F_{r,\alpha_1}, \\ 0 < G_{l,\alpha_1} &\leq G_{l,\alpha_2} \leq G_{r,\alpha_2} \leq G_{r,\alpha_1}, \\ 0 < H_{l,\alpha_1} &\leq H_{l,\alpha_2} \leq H_{r,\alpha_2} \leq H_{r,\alpha_1}, \\ 0 < L_{i,\alpha_1} &\leq L_{i,\alpha_2} \leq R_{i,\alpha_2} \leq R_{i,\alpha_1}, \quad i = -5, -4, -3, -2, -1, 0. \end{aligned} \quad (42)$$

We solved by mathematical induction that

$$0 < L_{i,\alpha_1} \leq L_{i,\alpha_2} \leq R_{i,\alpha_2} \leq R_{i,\alpha_1}, \quad i = -5, -4, -3, -2, -1, 0. \quad (43)$$

From (42), we find that (43) holds, for $i = -5, -4, -3, -2, -1, 0$.

Consider equation (43) is verifiable, for $i \leq j$, $j \in \{1, 2, 3, \dots\}$, then, by using (41)–(43), it pursues that, for $i = j + 1$,

$$\begin{aligned} L_{j+1,\alpha_1} &= \frac{D_{l,\alpha_1} L_{j-1,\alpha_1} L_{j-2,\alpha_1}}{E_{r,\alpha_1} + F_{r,\alpha_1} R_{j-3,\alpha_1} + G_{r,\alpha_1} R_{j-4,\alpha_1} + H_{r,\alpha_1} R_{j-5,\alpha_1}} \\ &\leq \frac{D_{l,\alpha_2} L_{j-1,\alpha_2} L_{j-2,\alpha_2}}{E_{r,\alpha_2} + F_{r,\alpha_2} R_{j-3,\alpha_2} + G_{r,\alpha_2} R_{j-4,\alpha_2} + H_{r,\alpha_2} R_{j-5,\alpha_2}} = L_{j+1,\alpha_2} \\ &\leq \frac{D_{r,\alpha_2} R_{j-1,\alpha_2} R_{j-2,\alpha_2}}{E_{l,\alpha_2} + F_{l,\alpha_2} L_{j-3,\alpha_2} + G_{l,\alpha_2} L_{j-4,\alpha_2} + H_{l,\alpha_2} L_{j-5,\alpha_2}} = R_{j+1,\alpha_2} \\ &\leq \frac{D_{r,\alpha_1} R_{j-1,\alpha_1} R_{j-2,\alpha_1}}{E_{l,\alpha_1} + F_{l,\alpha_1} L_{j-3,\alpha_1} + G_{l,\alpha_1} L_{j-4,\alpha_1} + H_{l,\alpha_1} L_{j-5,\alpha_1}} = R_{j+1,\alpha_1}. \end{aligned} \quad (44)$$

Therefore, (42) holds.

Moreover, from (40), we obtain

$$\begin{aligned} L_{i+1,\alpha} &= \frac{D_{l,\alpha} L_{i-1,\alpha} L_{i-2,\alpha}}{E_{r,\alpha} + F_{r,\alpha} R_{i-3,\alpha} + G_{r,\alpha} R_{i-4,\alpha} + H_{r,\alpha} R_{i-5,\alpha}}, \\ R_{i+1,\alpha} &= \frac{D_{r,\alpha} R_{i-1,\alpha} R_{i-2,\alpha}}{E_{l,\alpha} + F_{l,\alpha} L_{i-3,\alpha} + G_{l,\alpha} L_{i-4,\alpha} + H_{l,\alpha} L_{i-5,\alpha}}, \quad \alpha \in (0, 1]. \end{aligned} \quad (45)$$

From Lemma 5 and since D, E, F, G , and H and y_i , $i = -5, -4, -3, -2, -1, 0$, are the fuzzy nonnegative numbers, we obtain

$$\begin{aligned} D_{l,\alpha}, D_{r,\alpha}, E_{l,\alpha}, E_{r,\alpha}, F_{l,\alpha}, F_{r,\alpha}, G_{l,\alpha}, G_{r,\alpha}, H_{l,\alpha}, H_{r,\alpha}, L_{-1,\alpha}, \\ R_{-1,\alpha}, L_{-2,\alpha}, R_{-2,\alpha}, L_{-3,\alpha}, R_{-3,\alpha}, L_{-4,\alpha}, R_{-4,\alpha}, L_{-5,\alpha}, R_{-5,\alpha} \end{aligned} \quad (46)$$

are left continuous. As a result of (45), we get $L_{1,\alpha}$ and $R_{1,\alpha}$ both are left continuous. Then, we want to show that $L_{i,\alpha}$ and $R_{i,\alpha}$, $i = 1, 2, 3, \dots$, also left continuous by mathematical induction.

Now, we can show that the support of y_i , $\text{supp } y_i = \bigcup_{\alpha \in (0,1]} [L_{i,\alpha}, R_{i,\alpha}]$, is compact. It is abundant to show that $\bigcup_{\alpha \in (0,1]} [L_{i,\alpha}, R_{i,\alpha}]$ is bounded.

Consider $i = 1$; since D, E, F, G , and H and y_i , $i = -5, -4, -3, -2, -1, 0$, are the fuzzy nonnegative numbers, there exist constants $P_i, Q_i > 0$, $i = 1, 2, 3, 4, 5$ such that, for all $\alpha \in (0, 1]$,

$$\begin{aligned} [D_{l,\alpha}, D_{r,\alpha}] &\subset [P_1, Q_1], \\ [E_{l,\alpha}, E_{r,\alpha}] &\subset [P_2, Q_2], \\ [F_{l,\alpha}, F_{r,\alpha}] &\subset [P_3, Q_3], \\ [G_{l,\alpha}, G_{r,\alpha}] &\subset [P_4, Q_4], \\ [H_{l,\alpha}, H_{r,\alpha}] &\subset [P_5, Q_5], \\ [L_{i,\alpha}, R_{i,\alpha}] &\subset [P_i, Q_i], \quad i = -5, -4, -3, -2, -1, 0. \end{aligned} \quad (47)$$

Therefore, from (45) and (47), we can prove that

$$\begin{aligned} [L_{1,\alpha}, R_{1,\alpha}] &\subset \left[\frac{P_1 P_{-1} P_{-2}}{Q_5 + Q_4 Q_{-3} + Q_3 Q_{-4} + Q_2 Q_{-5}}, \right. \\ &\quad \left. \frac{Q_1 Q_{-1} Q_{-2}}{P_5 + P_4 P_{-3} + P_3 P_{-4} + P_2 P_{-5}} \right], \quad \alpha \in (0, 1], \end{aligned} \quad (48)$$

from which it is obvious that

$$\begin{aligned} \bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}] &\subset \left[\frac{P_1 P_{-1} P_{-2}}{Q_5 + Q_4 Q_{-3} + Q_3 Q_{-4} + Q_2 Q_{-5}}, \right. \\ &\quad \left. \frac{Q_1 Q_{-1} Q_{-2}}{P_5 + P_4 P_{-3} + P_3 P_{-4} + P_2 P_{-5}} \right], \alpha \in (0, 1]. \end{aligned} \quad (49)$$

Relation (49) shows that $\overline{\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}]}$ is compact and $\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subset (0, \infty)$. Then, from mathematical induction, we want to show that $\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}]$ is compact and

$$\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subset (0, \infty), \quad i = 1, 2, 3, \dots \quad (50)$$

So, by Lemma 5, relations (43) and (50) and $L_{i,\alpha}$ and $R_{i,\alpha}$ are left continuous, we get $[L_{i,\alpha}, R_{i,\alpha}]$; calculate the sequence of fuzzy nonnegative numbers y_i as equation (10) holds.

Now, we show that y_i is the solution of equation (10) under initial conditions $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 . Then, for all $\alpha \in (0, 1]$,

$$\begin{aligned}
[y_{i+1}]_\alpha &= [L_{i+1,\alpha}, R_{i+1,\alpha}] \\
&= \left[\frac{D_{l,\alpha} L_{i-1,\alpha} L_{i-2,\alpha}}{E_{r,\alpha} + F_{r,\alpha} R_{i-3,\alpha} + G_{r,\alpha} R_{i-4,\alpha} + H_{r,\alpha} R_{i-5,\alpha}}, \frac{D_{r,\alpha} R_{i-1,\alpha} R_{i-2,\alpha}}{E_{l,\alpha} + F_{l,\alpha} L_{i-3,\alpha} + G_{l,\alpha} L_{i-4,\alpha} + H_{l,\alpha} L_{i-5,\alpha}} \right], \\
[y_{i+1}]_\alpha &= [L_{i+1,\alpha}, R_{i+1,\alpha}] = \left[\frac{D j_{i-1} j_{i-2}}{E + F j_{i-3} + G j_{i-4} + H j_{i-5}} \right]_\alpha,
\end{aligned} \tag{51}$$

we get that y_i is the solution of equation (10) under initial conditions $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 .

Consider that there exists one more solution y_i^* of equation (10) initial values $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 ; then, we can easily show by explaining as above that

$$[y_i^*]_\alpha = [L_{i+1,\alpha}, R_{i+1,\alpha}], \quad \alpha \in (0, 1], i = 1, 2, 3, \dots \tag{52}$$

Then, from (41) and (52), we get that $[y_i]_\alpha = [y_i^*]_\alpha, \alpha \in (0, 1], i = -5, -4, -3, \dots$, from which it satisfies $y_i = y_i^*, \alpha \in (0, 1], i = -5, -4, -3, \dots$. Hence, proved.

With the use of the following theorem, we are investigating the behavior of asymptotic of equation (10) at equilibrium points.

If $\{y_i\}$ is the unique nonnegative solution of equation (10) under the initial conditions $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 such that

$$[y_i]_\alpha = [L_{i,\alpha}, R_{i,\alpha}], \quad \alpha \in (0, 1], i = 1, 2, 3, \dots, \tag{53}$$

then we see that $(L_{i,\alpha}, R_{i,\alpha})$ is the member of the system of ordinary difference equation family:

$$\begin{aligned}
L_{i+1,\alpha} &= \frac{D_{l,\alpha} L_{i-1,\alpha} L_{i-2,\alpha}}{E_{r,\alpha} + F_{r,\alpha} R_{i-3,\alpha} + G_{r,\alpha} R_{i-4,\alpha} + H_{r,\alpha} R_{i-5,\alpha}}, \\
R_{i+1,\alpha} &= \frac{A_{r,\alpha} R_{i-1,\alpha} R_{i-2,\alpha}}{B_{l,\alpha} + C_{l,\alpha} L_{i-3,\alpha} + D_{l,\alpha} L_{i-4,\alpha} + E_{l,\alpha} L_{i-5,\alpha}}, \quad \alpha \in (0, 1].
\end{aligned} \tag{54}$$

We assume the succeeding system of the ordinary parametric difference equations, in order to investigate the asymptotically behavior of equation (10). Then, from (54),

$$\begin{aligned}
u_{i+1} &= \frac{du_{i-1}u_{i-2}}{e + f v_{i-3} + g v_{i-4} + h v_{i-5}}, \\
v_{i+1} &= \frac{p v_{i-1} v_{i-2}}{q + r u_{i-3} + s u_{i-4} + t u_{i-5}}, \quad i = 0, 1, 2, \dots,
\end{aligned} \tag{55}$$

where the parameter $d, e, f, g, h, p, q, r, s$, and t are the real constant of the nonnegative number and $u_{-5}, u_{-4}, u_{-3}, u_{-2}, u_{-1}, u_0, v_{-5}, v_{-4}, v_{-3}, v_{-2}, v_{-1}$, and v_0 are initial conditions of nonnegative real constant.

By Lemma 1, we came to know that (55) is the system of the ordinary parameter difference equation and has a unique solution (u_i, v_i) under any initial conditions.

Moreover, we can easily calculate the equilibrium points (u_n, v_n) of any initial conditions of system (55). There are three equilibrium points of system (55):

$$\begin{aligned}
\overline{Y}_1 &= (\overline{u}_1, \overline{v}_1) = (0, 0), \\
\overline{Y}_2 &= (\overline{u}_2, \overline{v}_2) = \left(\frac{e}{d}, 0 \right), \\
\overline{Y}_3 &= (\overline{u}_3, \overline{v}_3) = \left(0, \frac{q}{p} \right).
\end{aligned} \tag{56}$$

If $dp > (f + g + h)(r + s + t)$, then the fourth nonnegative equilibrium points of system (55) are

$$\overline{Y}_4 = (\overline{u}_4, \overline{v}_4) = \left(\frac{ep + q(f + g + h)}{dp - (f + g + h)(r + s + t)}, \frac{dq + b(r + s + t)}{dp - (f + g + h)(r + s + t)} \right). \tag{57}$$

Theorem 3. The system of equation (55) is locally asymptotically stable at equilibrium point \overline{Y}_1 .

Proof. Let $T: (R^5)^+ \rightarrow R^+$ and $U: (R^5)^+ \rightarrow R^+$ be the multivariable function defined as

$$\begin{aligned}
T(u_{i-1}, u_{i-2}, v_{i-3}, v_{i-4}, v_{i-5}) &= \frac{du_{i-1}u_{i-2}}{e + f v_{i-3} + g v_{i-4} + h v_{i-5}}, \\
S(v_{i-1}, v_{i-2}, u_{i-3}, u_{i-4}, u_{i-5}) &= \frac{p v_{i-1} v_{i-2}}{q + r u_{i-3} + s u_{i-4} + t u_{i-5}}.
\end{aligned} \tag{58}$$

Moreover, about the equilibrium point \overline{Y}_1 , we can easily determine the linear equation of system (55) such as

$$\varphi_1 = D_1 \varphi_i, \quad (59)$$

where

$$\varphi_i = \begin{bmatrix} u_i \\ u_{i-1} \\ u_{i-2} \\ u_{i-3} \\ u_{i-4} \\ u_{i-5} \\ v_i \\ v_{i-1} \\ v_{i-2} \\ v_{i-3} \\ v_{i-4} \\ v_{i-5} \end{bmatrix}, \quad (60)$$

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristics polynomial with (59) is

$$\omega^{12} = 0. \quad (61)$$

Since we get $|\omega| < 1$, from Lemma 2, we see that the equilibrium point \overline{Y}_1 of system (55) is locally asymptotically stable, and hence, proved. \square

Theorem 4. The system of equation (55) is unstable at the equilibrium point \overline{Y}_2 .

Proof. From (59), about the equilibrium point \overline{Y}_2 , we can easily calculate the linear equation of system (55) as

$$\varphi_{i+1} = D_2 \varphi_i, \quad (62)$$

where

$$\varphi_i = \begin{bmatrix} u_i \\ u_{i-1} \\ u_{i-2} \\ u_{i-3} \\ u_{i-4} \\ u_{i-5} \\ v_i \\ v_{i-1} \\ v_{i-2} \\ v_{i-3} \\ v_{i-4} \\ v_{i-5} \end{bmatrix}, \quad (63)$$

$$D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristics polynomial with (62) is

$$\omega^9 (\omega^3 - \omega - 1) = 0. \quad (64)$$

Since, we get $|\omega| > 1$ so that $\omega^9 (\omega^3 - \omega - 1) = 0$; it clears that one of the root of characteristic polynomial (59) lies

outside the unit disk; therefore, by Lemma 2, we computed the equilibrium point \bar{Y}_2 of system of equation (55) is unstable, and hence, proved. \square

Theorem 5. *The system of equation (55) is unstable at the equilibrium point \bar{Y}_3 .*

Proof. From (59), about the equilibrium point \bar{Y}_3 , we can easily calculate the linear equation of system (55) as

$$\varphi_{i+1} = D_3 \varphi_i, \quad (65)$$

where

$$\varphi_i = \begin{bmatrix} u_i \\ u_{i-1} \\ u_{i-2} \\ u_{i-3} \\ u_{i-4} \\ u_{i-5} \\ v_i \\ v_{i-1} \\ v_{i-2} \\ v_{i-3} \\ v_{i-4} \\ v_{i-5} \end{bmatrix}, \quad (66)$$

$$D_3 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{f}{d} & -\frac{g}{d} & -\frac{h}{d} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristics polynomial with (65) is

$$\omega^9(\omega^3 - \omega - 1) = 0. \quad (67)$$

Since we have $|\omega| > 1$ so that $\omega^9(\omega^3 - \omega - 1) = 0$, it is the same as equation (64), such that roots of the characteristics polynomial (67) lie outside the unit disk; therefore, by Lemma 2, we get that the equilibrium point \bar{Y}_3 of equation (55) is not stable, and hence, proved. \square

Theorem 6. *The system of equation (55) is not stable about the equilibrium point \bar{Y}_4 .*

Proof. From (59), about the equilibrium point \bar{Y}_4 , we can easily calculate the linear equation of system (55) as

$$\varphi_{i+1} = D_4 \varphi_i, \quad (68)$$

where

$$\varphi_i = \begin{bmatrix} u_i \\ u_{i-1} \\ u_{i-2} \\ u_{i-3} \\ u_{i-4} \\ u_{i-5} \\ v_i \\ v_{i-1} \\ v_{i-2} \\ v_{i-3} \\ v_{i-4} \\ v_{i-5} \end{bmatrix}, \quad (69)$$

$$D_4 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{f}{d} & -\frac{g}{d} & -\frac{h}{d} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial with (68) is

$$\omega^{12} - 2\omega^{10} - 2\omega^9 + \omega^8 + 2\omega^7 + \omega^6 - \frac{fr}{dp}\omega^4 - \frac{(gr+fs)}{dp}\omega^3 - \frac{(hr+gs+ft)}{dp}\omega^2 - \frac{(hs+gt)}{dp}\omega - \frac{ht}{dp} = 0. \quad (70)$$

From (70), we obtained

$$\Delta_{12} = \begin{bmatrix} 0 & -2 & -2 & 0 & -\frac{(gr+fs)}{dp} & -\frac{(hs+gt)}{dp} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 1 & \frac{fr}{dp} & \frac{(hr+gs+ht)}{dp} & \frac{ht}{dp} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & -\frac{(gr+fs)}{dp} & -\frac{(hs+gt)}{dp} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 & \frac{fr}{dp} & \frac{(hr+gs+ht)}{dp} & \frac{ht}{dp} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & -\frac{(gr+fs)}{dp} & -\frac{(hs+gt)}{dp} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \frac{fr}{dp} & \frac{(hr+gs+ht)}{dp} & \frac{ht}{dp} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{(gr+fs)}{dp} & -\frac{(hs+gt)}{dp} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{fr}{dp} & \frac{(hr+gs+ht)}{dp} & \frac{ht}{dp} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{(gr+fs)}{dp} & -\frac{(hs+gt)}{dp} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{fr}{dp} & \frac{(hr+gs+ht)}{dp} & \frac{ht}{dp} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{(gr+fs)}{dp} & -\frac{(hs+gt)}{dp} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(hr+gs+ht)}{dp} & \frac{ht}{dp} \end{bmatrix}. \quad (71)$$

We notice that all $\Delta_i \neq 0, i = 1, 2, 3, \dots, 12$; by Lemmas 2 and 3, we computed that \bar{Y}_4 is not stable, and hence, proved. \square

Theorem 7. Let I_u and I_v be some period of real number, and consider that $f = I_u^{k+1} \times I_v^{k+1} \longrightarrow I_u$ and $g = I_u^{k+1} \times I_v^{k+1} \longrightarrow I_v$ are satisfying mixed monotone property and hence continuously differentiable. If there exist

$$\begin{cases} m_0 \leq \min\{u_{-k}, \dots, u_0, v_{-l}, \dots, v_0\} \leq \max\{u_{-k}, \dots, u_0, v_{-l}, \dots, v_0\} \leq M_0, \\ n_0 \leq \min\{u_{-k}, \dots, u_0, v_{-l}, \dots, v_0\} \leq \max\{u_{-k}, \dots, u_0, v_{-l}, \dots, v_0\} \leq N_0, \end{cases} \quad (72)$$

such that

$$\begin{cases} m_0 \leq f([m_0]_a, [M_0]_b, [n_0]_s, [N_0]_t) \leq f([M_0]_a, [m_0]_b, [N_0]_s, [n_0]_t) \leq M_0, \\ n_0 \leq g([m_0]_{a_1}, [M_0]_{b_1}, [n_0]_{s_1}, [N_0]_{t_1}) \leq g([M_0]_{a_1}, [m_0]_{b_1}, [N_0]_{s_1}, [n_0]_{t_1}) \leq N_0. \end{cases} \quad (73)$$

Then, there exist $(m, M) \in [m_0, M_0]^2$ and $(n, N) \in [n_0, N_0]^2$ holding

$$\begin{aligned} m &= f([m_0]_a, [M_0]_b, [n_0]_s, [N_0]_t), \\ M &= f([M_0]_a, [m_0]_b, [N_0]_s, [n_0]_t), \\ n &= g([m_0]_{a_1}, [M_0]_{b_1}, [n_0]_{s_1}, [N_0]_{t_1}), \\ N &= g([M_0]_{a_1}, [m_0]_{b_1}, [N_0]_{s_1}, [n_0]_{t_1}). \end{aligned} \quad (74)$$

Moreover, if $m = M, n = N$, then equation (27) has a unique equilibrium point $(\bar{u}, \bar{v}) \in [m_0, M_0] \times [n_0, N_0]$ and every solution of (27) converges to (\bar{u}, \bar{v}) .

Proof. Using m_0, M_0, n_0 , and N_0 as two couples of initial iteration, we construct four sequences $\{m_i\}, \{M_i\}, \{n_i\}$, and $\{N_i\}$ ($i = 1, 2, 3, \dots$) from the following equations:

$$\begin{aligned} m_i &= f([m_{i-1}]_a, [M_{i-1}]_b, [n_{i-1}]_s, [N_{i-1}]_t), \\ M_i &= f([M_{i-1}]_a, [m_{i-1}]_b, [N_{i-1}]_s, [n_{i-1}]_t), \\ n_i &= g([m_{i-1}]_{a_1}, [M_{i-1}]_{b_1}, [n_{i-1}]_{s_1}, [N_{i-1}]_{t_1}), \\ N_i &= g([M_{i-1}]_{a_1}, [m_{i-1}]_{b_1}, [N_{i-1}]_{s_1}, [n_{i-1}]_{t_1}). \end{aligned} \quad (75)$$

It is obvious from the mixed monotone property of f and g that the sequences $\{m_k\}, \{M_k\}, \{n_k\}$, and $\{N_k\}$ possess the following monotone property:

$$\begin{cases} m_0 \leq m_1 \leq \dots \leq m_i \leq M_i \leq \dots \leq M_1 \leq M_0, \\ n_0 \leq n_1 \leq \dots \leq n_i \leq N_i \leq \dots \leq N_1 \leq N_0, \end{cases} \quad (76)$$

where $i = 0, 1, 2, \dots$, and

$$\begin{aligned} m_i &\leq u_q \leq M_i, \\ n_i &\leq v_p \leq N_i, \quad \text{for } q \geq (i+1)(k+1), p \geq (i+1)(l+1), i = 0, 1, 2, \dots \end{aligned} \quad (77)$$

Set

$$\begin{aligned} m &= \lim_{i \rightarrow \infty} m_i, \\ M &= \lim_{i \rightarrow \infty} M_i, \\ n &= \lim_{i \rightarrow \infty} n_i, \\ N &= \lim_{i \rightarrow \infty} N_i. \end{aligned} \quad (78)$$

Then,

$$\begin{aligned} m &\leq \lim_{i \rightarrow \infty} \inf u_i \leq \lim_{i \rightarrow \infty} \sup u_i \leq M, \\ n &\leq \lim_{i \rightarrow \infty} \inf v_i \leq \lim_{i \rightarrow \infty} \sup v_i \leq N, \end{aligned} \quad (79)$$

by continuity of f and g , one has

$$\begin{cases} m = f([m_0]_a, [M_0]_b, [n_0]_s, [N_0]_t), \\ M = f([M_0]_a, [m_0]_b, [N_0]_s, [n_0]_t), \\ n = g([m_0]_{a_1}, [M_0]_{b_1}, [n_0]_{s_1}, [N_0]_{t_1}), \\ N = g([M_0]_{a_1}, [m_0]_{b_1}, [N_0]_{s_1}, [n_0]_{t_1}). \end{cases} \quad (80)$$

Moreover, if $m = M$ and $n = N$, then $m = M = \lim_{i \rightarrow \infty} u_i = \bar{u}$ and $n = N = \lim_{i \rightarrow \infty} v_i = \bar{v}$, and hence, proved. \square

Theorem 8. If $d = p, e = q, f = r, g = s$, and $h = t$, then $(0, 0)$ is the equilibrium point of system (55) is the global attractor for all conditions:

$$(u_{-1}, v_{-1}) \in \left(0, \frac{b}{2a}\right) \times \left(0, \frac{b}{2a}\right), \quad j = -5, -4, -3, -2, -1, 0. \quad (81)$$

Proof. Since $d = p, e = q, f = r, g = s$, and $h = t$, hence, system (55) converts as

$$u_{j+1} = \frac{du_{j-1}u_{j-2}}{e + fv_{j-3} + gv_{j-4} + hv_{j-5}}, \quad (82)$$

$$v_{j+1} = \frac{dv_{j-1}v_{j-2}}{e + fu_{j-3} + gu_{j-4} + hu_{j-5}}, \quad j = 0, 1, 2, \dots$$

Let $(f, g) \in (0, (e/2d))^{12} \times (0, (e/2d))^{12} \rightarrow (0, \infty) \times (0, \infty)$ be a function expressed as

$$\begin{aligned} &f(u_j, u_{j-i}, u_{j-2}, u_{j-3}, u_{j-4}, u_{j-5}, v_j, v_{j-i}, v_{j-2}, v_{j-3}, v_{j-4}, v_{j-5}) \\ &= \frac{du_{j-1}u_{j-2}}{e + fv_{j-3} + gv_{j-4} + hv_{j-5}}, \\ &g(u_j, u_{j-i}, u_{j-2}, u_{j-3}, u_{j-4}, u_{j-5}, v_j, v_{j-i}, v_{j-2}, v_{j-3}, v_{j-4}, v_{j-5}) \\ &= \frac{dv_{j-1}v_{j-2}}{e + fu_{j-3} + gu_{j-4} + hu_{j-5}}. \end{aligned} \quad (83)$$

Set

$$\begin{aligned} f &= \frac{d_{uv}}{e + f\gamma + g\alpha + h\beta}, \\ g &= \frac{d_{u^*v^*}}{e + f\gamma^* + g\alpha^* + h\beta^*}, \end{aligned} \quad (84)$$

and we can obtain that

$$\begin{aligned} f_u &= \frac{dv}{e + f\alpha + g\beta + h\gamma} > 0, \\ f_v &= \frac{du}{e + f\alpha + g\beta + h\gamma} > 0, \\ f_\alpha &= -\frac{df_{uv}}{(e + f\alpha + g\beta + h\gamma)^2} < 0, \\ f_\beta &= -\frac{ag_{uv}}{(e + f\alpha + g\beta + h\gamma)^2} < 0, \\ f_\gamma &= \frac{dh_{uv}}{(e + f\alpha + g\beta + h\gamma)^2} < 0, \\ g_{u^*} &= \frac{dv^*}{e + f\alpha^* + g\beta^* + h\gamma^*} > 0, \\ g_{v^*} &= \frac{du^*}{e + f\alpha^* + g\beta^* + h\gamma^*} > 0, \\ g_{\alpha^*} &= -\frac{df_{u^*v^*}}{(e + f\alpha^* + g\beta^* + h\gamma^*)^2} < 0, \\ g_{\beta^*} &= -\frac{dg_{u^*v^*}}{(e + f\alpha^* + g\beta^* + h\gamma^*)^2} < 0, \\ g_{\gamma^*} &= \frac{dh_{u^*v^*}}{(e + f\alpha^* + g\beta^* + h\gamma^*)^2} < 0, \end{aligned} \quad (85)$$

which indicate that f and g hold a mixed monotone characteristic:

$$\begin{aligned} M_0 = N_0 = \max\{u_{-5}, u_{-4}, u_{-3}, u_{-2}, u_{-1}, \\ u_0, v_{-5}, v_{-4}, v_{-3}, v_{-2}, v_{-1}, v_0\}, \end{aligned} \quad (86)$$

$$\frac{dM_0 - e}{f + g + h} < m_0 = n_0 < 0,$$

We have

$$\begin{aligned} m_0 &\leq \frac{dm_0^2}{e + fN_0 + gN_0 + hN_0} \leq \frac{dM_0^2}{e + fm_0 + gm_0 + hm_0} \leq M_0, \\ n_0 &\leq \frac{dn_0^2}{e + fM_0 + gM_0 + hM_0} \leq \frac{dN_0^2}{e + fm_0 + gm_0 + hm_0} \leq N_0. \end{aligned} \quad (87)$$

It is obvious that $m_j = n_j$ and $M_j = N_j$, $j = 0, 1, 2, \dots$; then, by the proceeding system (55) and Theorem 2.6, $\exists m, M \in [m_0, M_0]$, $n = m$, and $N = M$ satisfy

$$\begin{aligned} m &= \frac{dm^2}{e + fN + gN + hN}, \\ n &= \frac{dn^2}{e + fM + gM + hM}, \\ M &= \frac{dM^2}{e + fn + gn + hn}, \\ N &= \frac{dN^2}{e + fm + gm + hm}. \end{aligned} \quad (88)$$

Thus, $[e - d(m + M)](m - M) = 0$. In vision of $2dM_0 < e$, we get $e - d(m + M) > 0$. So, $M = m, N = n$. From Theorem 7, we get that $(0, 0)$ is the equilibrium point of system (55) which is a global attractor, and hence, proved.

Now, we establish stability of the fuzzy difference equation (10) in terms of positive results of standard difference equation (55). For this justification, we initiate the succeeding view of equation (10) for stability. It manifests that equation (10) has a trivial solution $\hat{0}$. \square

Theorem 9. If the parameter D, E, F, G , and H are positive fuzzy numbers, i.e., nonnegative real numbers and the primary conditions are nonnegative fuzzy numbers with $[y_j]_\alpha \subset (0, E/2D)$, $j = -5, -4, -3, -2, -1, 0$, $\alpha \in [0, 1]$, then the trivial solution $y = \hat{0}$ of equation (10) is asymptotically stable with regard to E as $j \rightarrow \infty$.

Proof. This is proved by the result of Theorems 3 and 8. \square

4. Numerical Problem

In this section, the numerical example is performed for confirmation of the result discussed in the previous section and for support of the theoretical discussion. These examples show the asymptotically behavior of the results of equation (10).

Example 1. Consider the following fuzzy difference equation:

$$y_{i+1} = \frac{Dy_{i-1}y_{i-2}}{E + Fy_{i-3} + Gy_{i-4} + Hy_{i-5}}, \quad i = 0, 1, 2, \dots, \quad (89)$$

where D, E, F, G , and H are positive trivial fuzzy numbers. By Theorem 9, we take $[D]_\alpha = [D, D] = 0.2$, $[E]_\alpha = [E, E] = 17$, $[F]_\alpha = [F, F] = 2$, $[G]_\alpha = [G, G] = 2$, and $[H]_\alpha = [H, H] = 3$, $\alpha \in (0, 1]$; in addition, from Theorem 9, the initial conditions $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 with $[y_i]_\alpha \subset (0, E/2D)$, $i = -5, -4, -3, -2, -1, 0$ and $\alpha \in (0, 1]$, are represented such that

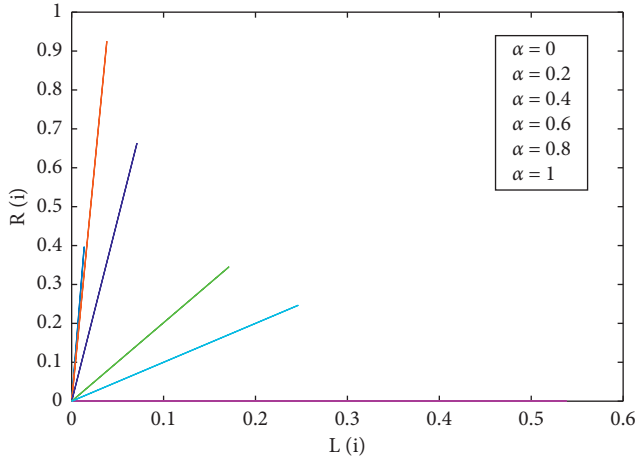
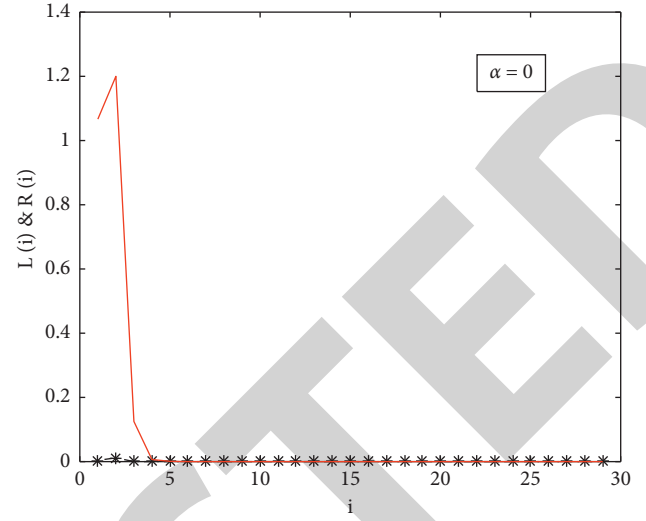
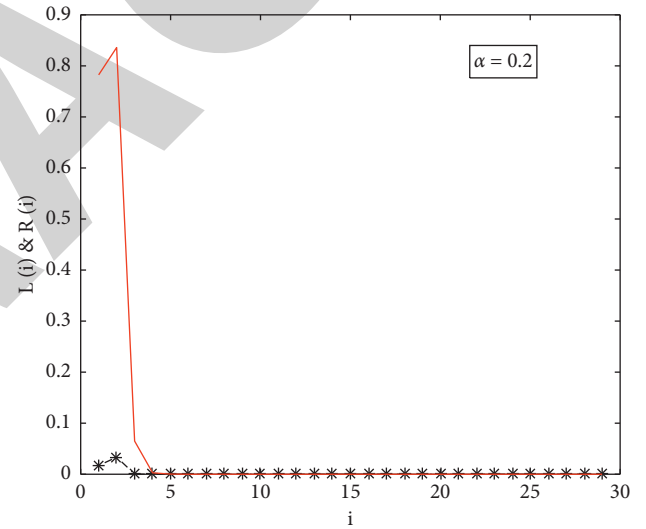


FIGURE 4: The dynamics of equations (91).

FIGURE 5: The dynamics of equations (91), when $\alpha = 0$.

$$\begin{aligned}
 y_0(y) &= \begin{cases} \frac{y-4}{2}, & 4 \leq y \leq 6, \\ \frac{-y+11}{5}, & 6 \leq y \leq 11, \end{cases} \\
 y_{-1}(y) &= \begin{cases} \frac{y-3}{5}, & 3 \leq y \leq 8, \\ \frac{-y+15}{7}, & 8 \leq y \leq 15, \end{cases} \\
 y_{-2}(y) &= \begin{cases} \frac{y-2}{5}, & 2 \leq y \leq 7, \\ \frac{-y+12}{5}, & 7 \leq y \leq 12, \end{cases} \\
 y_{-3}(y) &= \begin{cases} \frac{y-2}{7}, & 2 \leq y \leq 9, \\ \frac{-y+16}{7}, & 9 \leq y \leq 16, \end{cases} \\
 y_{-4}(y) &= \begin{cases} \frac{y-1}{7}, & 1 \leq y \leq 8, \\ \frac{-y+12}{4}, & 8 \leq y \leq 12, \end{cases} \\
 y_{-5}(y) &= \begin{cases} \frac{y-7}{3}, & 7 \leq y \leq 10, \\ \frac{-y+17}{7}, & 10 \leq y \leq 17. \end{cases}
 \end{aligned} \tag{90}$$

From (40), the triangular fuzzy number is obtained:

FIGURE 6: The dynamics of equations (91), when $\alpha = 0.2$.

$$\begin{aligned}
 [y_0]_\alpha &= [4 + 2\alpha, 11 - 5\alpha], \\
 [y_{-1}]_\alpha &= [3 + 5\alpha, 15 - 7\alpha], \\
 [y_{-2}]_\alpha &= [2 + 5\alpha, 12 - 5\alpha], \\
 [y_{-3}]_\alpha &= [2 + 7\alpha, 16 - 7\alpha], \\
 [y_{-4}]_\alpha &= [1 + 7\alpha, 12 - 4\alpha], \\
 [y_{-5}]_\alpha &= [7 + 3\alpha, 17 - 7\alpha].
 \end{aligned} \tag{91}$$

From (41), the parameter D, E, F, G , and H and initial conditions $y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 satisfy the following system of nonlinear difference equation with parameter α :

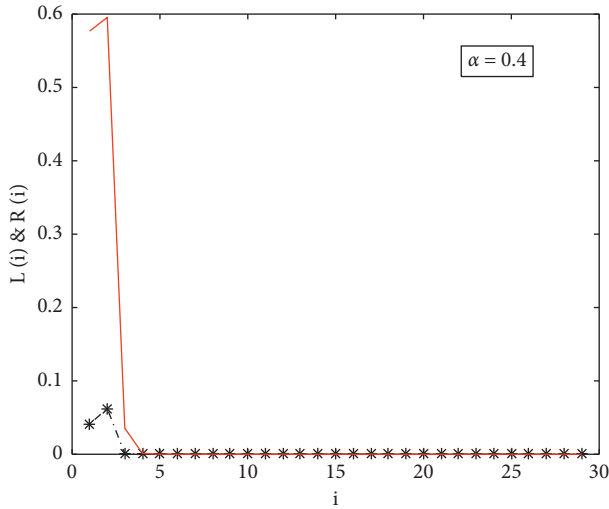


FIGURE 7: The dynamics of equations (91), when $\alpha = 0.4$.

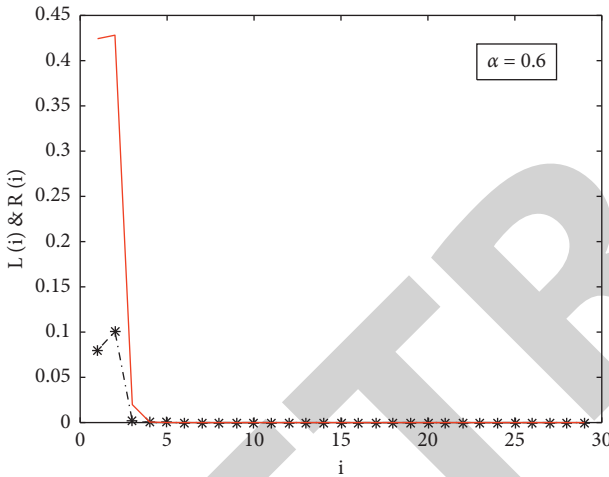


FIGURE 8: The dynamics of equations (91), when $\alpha = 0.6$.

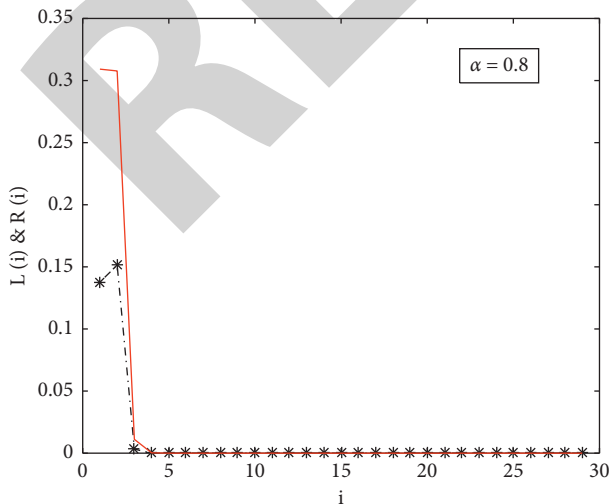


FIGURE 9: The dynamics of equations (91), when $\alpha = 0.8$.

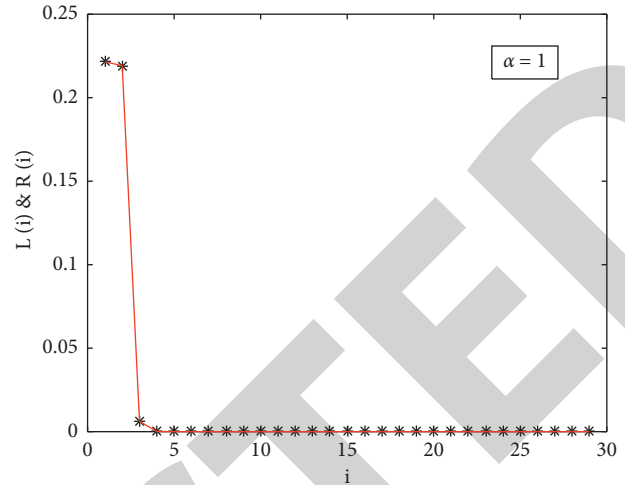


FIGURE 10: The dynamics of equations (91), when $\alpha = 1$.

$$L_{i+1,\alpha} = \frac{0.2L_{i-1,\alpha}L_{i-2,\alpha}}{17 + 2R_{i-3,\alpha} + 2R_{i-4,\alpha} + 3R_{i-5,\alpha}}, \quad (92)$$

$$R_{i+1,\alpha} = \frac{0.2R_{i-1,\alpha}R_{i-2,\alpha}}{17 + 2L_{i-3,\alpha} + 2L_{i-4,\alpha} + 3L_{i-5,\alpha}}, \quad \alpha \in (0, 1].$$

It is easy to prove that $[y_i]_\alpha \subset (0, E/2D)$, $i = -5, -4, -3, -2, -1, 0$, for $\alpha \in (0, 1]$, namely, the condition of Theorem 9 holds. So, from Theorem 9, we have that the trivial solution $y = \hat{0}$ of equation (10) is asymptotically stable with respect to D as $i \rightarrow \infty$; Figures 4–10 shows the dynamics of system (91), where L and R is left and right reference functions, respectively.

5. Conclusion

In this work, we demonstrate how to use the variational iteration technique to solve a system of fuzzy nonlinear difference equations. In physics, this is a powerful technique to solve nonlinear differential equations with fuzzy outcomes. According to the mathematical analysis, the solution is very satisfying. Variational iteration technique offers a powerful tool to drive nonlinear formations. Calculations are achieved by utilizing the package of MATLAB 2014(a). The dynamics action of high-order fuzzy nonlinear difference equation is examined in this work. Initially, we prove the existence and the uniqueness of fuzzy solutions through nonnegative fuzzy calculations. Then, using the linearization technique, we compute the nonzero equilibrium points of corresponding equation (55) that is not stable. At last, for equation (10), we compute the nonnegative solution $[\hat{0}]$ is stable when D, E, F, G , and H are nonnegative fuzzy numbers. The computational conclusions are illustrated in a few exemplifying examples. Specifically, the conditions which we derive in the association are so easy, which enabled flexibility in investigating, experimenting, and implementing the fuzzy nonlinear difference equation.

Research Article

Picture Fuzzy Maclaurin Symmetric Mean Operators and Their Applications in Solving Multiattribute Decision-Making Problems

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To evaluate objects under uncertainty, many fuzzy frameworks have been designed and investigated so far. Among them, the frame of picture fuzzy set (PFS) is of considerable significance which can describe the four possible aspects of expert's opinion using a degree of membership (DM), degree of nonmembership (DNM), degree of abstinence (DA), and degree of refusal (DR) in a certain range. Aggregation of information is always challenging especially when the input arguments are interrelated. To deal with such cases, the goal of this study is to develop the notion of the Maclaurin symmetric mean (MSM) operator as it aggregates information under uncertain environments and considers the relationship of the input arguments, which make it unique. In this paper, we studied the theory of MSM operators in the layout of PFSs and discussed their applications in the selection of the most suitable enterprise resource management (ERP) scheme for engineering purposes. We developed picture fuzzy MSM (PFMSM) operators and investigated their validity. We developed the multiattribute decision-making (MADM) algorithm based on the PFMSM operators to examine the performance of the ERP systems using picture fuzzy information. A numerical example to evaluate the performance of ERP systems is studied, and the effects of the associated parameters are discussed. The proposed aggregated results using PFMSM operators are found to be reliable as it takes into account the interrelationship of the input information, unlike traditional aggregation operators. A comparative study of the proposed PFMSM operators is also studied.

1. Introduction

Expert opinion under uncertain situation is a challenging task, and fuzzy set (FS) [1] developed by Zadeh is a useful tool to express the DM of elements. Later, the notion of FS was equipped by the DNM, and the frame of intuitionistic FS (IFS) was developed by Atanassov [2]. The frame of IFS was a flexible approach and has been utilized on a large account in the various real-life phenomena. The frame of IFS was strengthened by Yager [3, 4] as he introduces the notion of Pythagorean FS (PyFS) and q-rung orthopair FS (qROFS) by giving some flexibility in assigning the DM and the DNM. The frame of IFS, PyFS, and qROFS is based on the DM and the DNM only that accounts for describing two aspects of human opinion. It was suggested by Cuong and Kreinovich [5] who claim that the notion of IFS cannot describe the human perception properly as a human opinion have two

more aspects known as DA and DR alongside the DM and the DNM and introduce the idea of PFS. Hence, expressing the human opinion using IFS, PyFS, and qROFS leads to loss of information in the absence of the DA and DR while the frame of PFS covered that possibly lost information shows the supremacy of PFS over the IFS, PyFS, and qROFS. From the application point of view, FS and its extension have some useful roles in parametric analysis and some differential equations as well [6, 7].

Aggregation of information is one of the challenging tasks in uncertain environments, and several aggregation operators (AOs) have been introduced in various fuzzy settings. Averaging and geometric AOs [8, 9] based on algebraic t-norms (t-conorms) provide the aggregation of information and consider their weights into consideration. Einstein AOs [10, 11] are also used in the aggregation of information based on Einstein t-norm (t-conorm) and

consider the weights of input information in the aggregation process. Dombi t-norms-based Dombi AOs [12–14] also handle uncertain information by taking into account their weights. By considering the prioritization among the input information, many prioritized AOs [15–17] have been studied for practical use in many fuzzy frameworks. Among the discussion of AOs, the theory of Hamacher AOs [18, 19] is also a prominent one, based on the Hamacher t-norm (t-conorm), that takes the weight of the input arguments into account in aggregation steps. All the AOs discussed so far only aggregate the information and took only the weights into account. None of them consider the interrelationship of the input information. Due to this, some other types of aggregation operators are being developed including Heronian mean (HM) operators [20, 21], Bonferroni mean (BM) operators [22], hammy mean operators [23], power AOs [24], and MSM operators [25] which somehow relate the input information during the aggregation process. For some other significant work on MADM and aggregation theory, we refer to [26, 27].

MSM operators are among the widely studied topic in the theory of aggregation. Maclaurin [28] gave the idea of MSM operators for the first time and was popularized by DeTemple and Robertson [29]. The main feature of the MSM operator is that it takes into account the interrelationship of more than two input arguments at a time unlike BM operators and other traditional AOs, and hence the information fusion due to MSM operators is more robust and flexible. Intuitionistic fuzzy MSM (IFMSM) operators were proposed by Qin and Liu [30] while partitioned MSM operators for MADM purposes in the frame of IFSs have been studied by Liu et al. [31]. A study of Pythagorean fuzzy MSM (PyFMSM) operators and their application in the commercialization of the technology was investigated by Wei and Lu [32] while Yang and Pang [33] revisited PyFMSM operators by developing interactive PyFMSM operators for MADM purposes. Wei et al. [34] developed some q-rung orthopair fuzzy MSM (QROFMSM) operators for the evaluation of ERP systems using the MADM approach. Liu et al. [35] proposed power QROFMSM operators by merging power AO with QROFMSM operators for decision-making approaches. Wang et al. [36] extended the notion of QROFMSM operators to the frame of interval-valued qROFMs by expressing the DM and the DNM using a closed subinterval of $[0, 1]$. The MSM operators in hesitant fuzzy settings were proposed by Qin et al. [37] while the MSM operators for linguistic variables in the intuitionistic fuzzy layout were discussed by Liu and Qin [38]. For some other useful work on the theory and applications of MSM operators, one can refer to [39–41].

The ability of PFS of describing uncertain information using DM, DNM, DA, and DR signifies its importance in dealing with the problems of MADM, pattern recognition, clustering, and medical diagnosis. Further, it has been noticed that the traditional aggregation operators only give us aggregation of information but do not correlate the input information and hence lost credibility to some extent. MSM operators overcome this disadvantage of the traditional aggregation operators by correlating the input information.

By keeping in mind, the diverse structure of PFSs where four aspects of uncertain information can be described that reduces the chances of information loss and the significance of MSM operators, our aim in this paper is to develop the notion of MSM operators for PFSs. The newly developed PFMSM operators can aggregate uncertain information by considering the relationship of more than two input arguments at a time. This type of AOs provides us robustness and flexibility and reduces information loss. The main contributions of this paper are summarized as follows:

- (1) The notion of the MSM operator is introduced in the frame of PFSs to describe the four possible aspects of human opinion under uncertainty
- (2) The superiority of the proposed MSM operators is shown using some remarks and using a comparative study of proposed and previous approaches
- (3) The significance of the PFMSM operators is shown using a MADM problem numerically by discussing the performance of ERP systems.

The manuscript is designed such that Section 1 summarizes a history of different AOs and their drawbacks and the significance of the newly developed PFMSM operators. In Section 2, some basic definitions are recalled. In Section 3, we developed the theory of the PFMSM operator and picture fuzzy weighted MSM (PFWMSM) operator. In Section 4, we proposed a picture fuzzy dual MSM (PFDMSM) operator and a picture fuzzy weighted dual MSM (PFWDMSM) operator. In Section 5, it is proved that the proposed MSM operators are advanced than the MSM operators of IFSs. In Section 6, the MADM procedure is elaborated and a comprehensive numerical example is discussed where the selection of the best ERP scheme is carried out. In Section 7, we analyzed the comparative study of the existing and newly developed operators numerically. Section 8 consists of some conclusive remarks.

2. Preliminaries

In this part of the paper, some preliminary concepts are studied as a recall. We recall the notion of PFS and its supremacy over IFS. We also discuss the basic algebraic operations of PFSs that are used in our new study. The idea of ranking of picture fuzzy numbers (PFNs) is also recalled. It is to be noted that in this paper, by \check{X} , we mean a non-empty set and trios (m', \check{a}, d) denotes the DM, DA, and DNM of PFNs. Further, $J = 1, 2, 3, \dots, n$ and $k = 1, 2, 3, \dots, n$ shall be used as indexing terms.

Definition 1 (see [5]). A PFS is of the form $p' = \{(\check{X}, (m'(\check{X}), \check{a}(\check{X}), d(\check{X}))) : \check{X} \in \check{X}\}$ with $0 \leq m'^q(\check{X}) + \check{a}^q(\check{X}) + d^q(\check{X}) \leq 1$, $q \in \mathbb{Z}^+$. Also, $r(\check{X}) = \sqrt[q]{1 - (m'^q(\check{X}) + \check{a}^q(\check{X}) + d^q(\check{X}))}$ is termed as the DR.

Definition 2 (see [9]). Let $p' = (m', \check{a}, d)$, $p'_1 = (m'_1, \check{a}_1, d_1)$, and $p'_2 = (m'_2, \check{a}_2, d_2)$ be three PFNs and $y > 0$. Then,

$$\begin{aligned}
p_1' \oplus p_2' &= (m_1' + m_2' - m_1' m_2', \mathfrak{a}_1 \mathfrak{a}_2, d_1 d_2), \\
p_1' \oplus p_2' &= (m_1' m_2', \mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_1 \cdot \mathfrak{a}_2, d_1 + d_2 - d_1 \cdot d_2), \\
y p' &= (1 - (1 - m')^y, \mathfrak{a}^y, d^y), \\
(p')^y &= (m'^y, 1 - (1 - \mathfrak{a})^y, 1 - (1 - d)^y), \\
(p'^c) &= (d, \mathfrak{a}, m').
\end{aligned} \quad (1)$$

For the comparison of two PFNs $p_i' = (m_i', \mathfrak{a}_i, d_i)$ and $p_2' = (m_2', \mathfrak{a}_2, d_2)$, we have the following score function:

$$S'(p') = p' = m_i' - d_i \cdot r_i, S'(p') \in [-1, 1]. \quad (2)$$

Definition 3 (see [28]). For a family of positive numbers P'_j , the MSM operator is given by

$$\text{MSM}^K(P'_1, P'_2, \dots, P'_n) = \left(\frac{\prod_{j=1}^k P'_{i_j}}{\zeta_n'^k} \right)^{1/k}. \quad (3)$$

The term $\zeta_n'^k$ denotes a binominal coefficient throughout this paper. The above-defined MSM operator is likely to satisfy the following:

$$\begin{aligned}
\text{MSM}^K(0, 0, \dots, 0) &= 0, \\
\text{MSM}^K(P', P', \dots, P') &= P', \\
\text{MSM}^K(P'_1, P'_2, \dots, P'_n) &\leq \text{MSM}^K(Q_1, Q_2, \dots, Q_n) \text{ if } P'_i \leq Q_i \forall i, \\
m' \text{ in } \{P'_i\} &\leq \text{MSM}^K(P'_1, P'_2, \dots, P'_n) \leq m' \mathfrak{a} x \{P'_i\}.
\end{aligned} \quad (4)$$

Definition 4 (see [32]). For a family of positive numbers $P'_{j'}$, the dual MSM (DMSM) operator is given by

$$\text{DMSM}^K(P'_1, P'_2, \dots, P'_n) = \frac{1}{k} \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(\sum_{j=1}^k P'_{i_j} \right)^{1/\zeta_n'^k} \right). \quad (5)$$

Now, we present some MSM operators that are discussed already and analyze their weaknesses. The notion of MSM operator has been discussed by several authors including Qin and Liu [30], Liu et al. [31], Wei and Lu [32], Yang and Pang [33], Wei et al. [34], Liu et al. [35], Wang et al. [36], Qin et al. [37], Liu and Qin [38], Wang et al. [39], Yu et al. [40], and Ju et al. [41]. We already discussed the significance of the MSM operators earlier. Our aim here is to point out the shortcomings of the previously defined MSM operators and to clear the objectives of proposing the new MSM operators in a picture fuzzy frame.

First, we present the MSM operators of IFs [30] as follows:

$$\begin{aligned}
&\text{PFMSM}(P'_1, P'_2, P'_3, \dots, P'_n) \\
&= \left(\left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k m_{i_j}' \right) \right) \right)^{1/\zeta_n'^k} \right)^{1/k}, 1 - \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (1 - d_{i_j}') \right) \right) \right)^{1/\zeta_n'^k} \right)^{1/k} \right).
\end{aligned} \quad (6)$$

There are two issues with the MSM operators of IFs; first, it has a strict condition in assigning the DM and DNM hence provide very little flexibility to decision-makers.

Second, it discusses only two aspects of human opinion using the DM and the DNM; hence, information loss occurs.

Next, we present the MSM operators of PyFSs [32] as follows:

$$\begin{aligned}
&\text{PyFMSM}(P'_1, P'_2, P'_3, \dots, P'_n) \\
&= \left(\sqrt[1/k]{\left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k m_{i_j}'^2 \right) \right) \right)^{1/\zeta_n'^k}} \right)^{1/k}, \sqrt[1/k]{1 - \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (1 - d_{i_j}'^2) \right) \right) \right)^{1/\zeta_n'^k}} \right)^{1/k}} \right).
\end{aligned} \quad (7)$$

The MSM operators in the Pythagorean fuzzy frame only enlarge the range for assigning the DM, and the DNM is comparative to the MSM operators of IFSs but it also has the case of information loss due to the absence of the DA and

DR. To relax the restriction on the DM and DNM, Wei et al. [34] proposed the MSM operators in q-rung orthopair fuzzy settings given as follows:

$$\begin{aligned} & \text{qROFMSM}(P'_1, P'_2, P'_3, \dots, P'_n) \\ &= \left(\sqrt[q]{1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k m'_{i_j} \right)^q \right) \right)^{1/C_n^k}} \right)^{1/k}, \sqrt[q]{1 - \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (1 - d'^q_{i_j}) \right) \right)^{1/C_n^k} \right)^{1/k}} \right)^{1/k}}. \end{aligned} \quad (8)$$

The MSM operators of qROFSs also have the problem of information loss due to the absence of DA and DR. Due to this fact, this paper aims to introduce the notion of MSM operators in a picture fuzzy setting. The main objective is to reduce information loss by incorporating the four aspects of uncertain information.

3. Picture Fuzzy MSM Operators

The goal of this section is to introduce the MSM operators using a DM, DNM, DA, and DR in the layout of PFNs. We develop PFMSM and PFWMSM operators using the notion of MSM and WMSM in the layout of PFSS. In our onward study, we shall mean by $\omega_j = (\omega_1, \omega_2, \dots, \omega_n)^T$ the weight vector of p'_j where $\omega_j > 0$, and $\sum_{j=1}^n \omega_j = 1$.

Definition 5. Let $P'_j = (m'_j, a_j, d_j)$ denote a collection of PFNs. Then, the PFMSM operator is given by

$$\text{PFMSM}^k(P'_1, P'_2, P'_3, \dots, P'_n) = \left(\frac{\bigoplus_{1 \leq i_1 \leq \dots \leq i_k \leq n} \left(\bigotimes_{j=1}^k P'_{i_j} \right)}{\dot{C}_n^k} \right)^{1/k}. \quad (9)$$

Theorem 1. Let $P'_j = (m'_j, a_j, d_j)$ denote a collection of PFNs. Then, using PFMSM operators, we have

$$\begin{aligned} \text{PFMSM}(P'_1, P'_2, P'_3, \dots, P'_n) &= \left(\left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k m'_{i_j} \right) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k}, \left(1 - \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (1 - a_{i_j}) \right) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k} \right. \\ &\quad \left. 1 - \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (1 - d_{i_j}) \right) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k} \right). \end{aligned} \quad (10)$$

Proof. By using Definition 5, we have

$$\begin{aligned}
& \bigotimes_{j=1}^k P_{i_j}' = \left(\prod_{j=1}^k m_{i_j}', \prod_{j=1}^k a_{i_j}, 1 - \prod_{j=1}^k (1 - d_{i_j}) \right), \\
& \bigoplus_{1 \leq i_1 \leq \dots < i_k \leq n} \left(\bigotimes_{j=1}^k P_{i_j}' \right) = \left(1 - \prod_{1 \leq i_1 \leq \dots < i_k \leq n} \left(1 - \prod_{j=1}^k m_{i_j}' \right), \prod_{1 \leq i_1 \leq \dots < i_k \leq n} 1 - \prod_{j=1}^k (1 - a_{i_j}), \prod_{1 \leq i_1 \leq \dots < i_k \leq n} 1 - \prod_{j=1}^k (1 - d_{i_j}) \right), \\
& \frac{1}{\hat{C}_n^k} \bigoplus_{1 \leq i_1 \leq \dots < i_k \leq n} \left(\bigotimes_{j=1}^k P_{i_j}' \right) = \left(1 - \left(\prod_{1 \leq i_1 \leq \dots < i_k \leq n} \left(1 - \prod_{j=1}^k m_{i_j}' \right) \right)^{1/\hat{C}_n^k}, \left(\prod_{1 \leq i_1 \leq \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - a_{i_j}) \right) \right)^{1/\hat{C}_n^k}, \right. \\
& \quad \left. \left(\prod_{1 \leq i_1 \leq \dots < i_k \leq n} \left(1 - \prod_{j=1}^k d_{i_j} \right) \right)^{1/\hat{C}_n^k} \right). \tag{11}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \text{PFMSM}(P_1', P_2', \dots, P_n') = \\
& \left(\left(1 - \left(\prod_{1 \leq i_1 \leq \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k m_{i_j}' \right) \right) \right)^{1/\hat{C}_n^k} \right)^{1/k}, 1 - \left(1 - \left(\prod_{1 \leq i_1 \leq \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (1 - a_{i_j}) \right) \right) \right)^{1/\hat{C}_n^k} \right)^{1/k}, \right. \\
& \quad \left. 1 - \left(1 - \left(\prod_{1 \leq i_1 \leq \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (1 - d_{i_j}) \right) \right) \right)^{1/\hat{C}_n^k} \right)^{1/k} \right). \tag{12}
\end{aligned}$$

The above-defined PFMSM operator satisfies the basic characteristics of aggregation given as follows. \square

$$\text{PFMSM}(P_1', P_2', P_3', \dots, P_n') \geq \text{PFMSM}(P_1', P_2', \dots, P_3'). \tag{14}$$

Property 1. Let $P_j' = (m_j', a_j, d_j)$ denote a collection of PFNs. If $P_j' = P'$, then

$$\text{PFMSM}(P_1', P_2', P_3', \dots, P_n') = P'. \tag{13}$$

Property 2. Let P_j' and P_3' denote the collections of PFNs if $m_j' \geq m_3'$, $a_j \leq a_3$, and $d_j \leq d_3$. Then,

Property 3. Let $P_j' = (m_j', a_j, d_j)$ denote a collection of PFNs such that

$$\begin{aligned}
P_j'^- &= \min P_j' = (\min m_j', \max a_j, \max d_j), \\
P_j'^+ &= \max P_j' = (\max m_j', \min a_j, \min d_j). \tag{15}
\end{aligned}$$

Then,

$$P'^- \leq \text{PFMSM}(P'_1, P'_2, P'_3, \dots, P'_n) \leq P'^+ \quad (16)$$

Weighted aggregation operators have always their significance and incorporate the weight vector of an aggregation phenomenon. In the following, we define weighted MSM operators for PFNs.

Definition 6. Let $P'_j = (m'_j, a_j, d_j)$ denote a collection of PFNs. Then, the PFWMSM operator is given by

$$\text{PFWMSM}(P'_1, P'_2, \dots, P'_n) = \left(\frac{\bigoplus_{1 \leq i_1 \leq \dots \leq i_k \leq n} \left(\bigotimes_{j=1}^k P'_{i_j} \right)^{w_{i_j}}}{\dot{C}_n^k} \right)^{1/k} \quad (17)$$

Theorem 2. Let $P'_j = (m'_j, q_j, d_j)$ denote a collection of PFNs. Then, using the PFWMSM operator, we have

$$\text{PFWMSM}(P'_1, P'_2, P'_3, \dots, P'_n) = \left(\left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (m'_{i_j})^{w_{i_j}} \right) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k}, \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (1 - a_{i_j})^{w_{i_j}} \right) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k}, \right. \\ \left. 1 - \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (1 - d_{i_j})^{w_{i_j}} \right) \right) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k} \right) \quad (18)$$

Proof. Similar to Theorem 1. \square

4. Picture Fuzzy Dual MSM Operators

This section aims to introduce the notion of PFSMSM operators using the DM, DNM, DA, and DR. We propose PFDMSM operator and PFDWMSM operator.

Definition 7. Let $P'_j = (m'_j, a_j, d_j)$ denote a collection of PFNs. Then, the PFDMSM operator is given by

$$\text{PFDMSM}^K(P'_1, P'_2, \dots, P'_n) = \frac{1}{k} \left(\bigoplus_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(\bigotimes_{j=1}^k P'_{i_j} \right)^{1/\dot{C}_n^k} \right) \quad (19)$$

Theorem 3. Let $P'_j = (m'_j, q_j, d_j)$ denote a collection of PFNs. Then, by using PFDMSM, we have

$$\text{PFDMSM}(P'_1, P'_2, \dots, P'_n) = \left(\left(1 - \left(1 - \prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \prod_{j=1}^k (1 - m'_{i_j}) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k}, \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k a_{i_j} \right) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k}, \right. \\ \left. \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k d_{i_j} \right) \right) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k} \right) \quad (20)$$

Proof. Using Definition 7, we have

$$\begin{aligned}
& \bigotimes_{j=1}^k P_{i_j}' = \left(1 - \prod_{j=1}^k (1 - m_{i_j}'), \prod_{j=1}^k a_{i_j}, \prod_{j=1}^k d_{i_j} \right), \\
& \left(\bigotimes_{j=1}^k P_{i_j}' \right)^{1/\dot{C}_n^k} = \left(\left(1 - \prod_{j=1}^k (1 - m_{i_j}') \right)^{1/\dot{C}_n^k}, 1 - \left(1 - \left(\prod_{j=1}^k a_{i_j} \right) \right)^{1/\dot{C}_n^k}, 1 - \left(1 - \left(\prod_{j=1}^k d_{i_j} \right) \right)^{1/\dot{C}_n^k} \right), \\
& \bigotimes_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(\bigotimes_{j=1}^k P_{i_j}' \right)^{1/\dot{C}_n^k} = \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \prod_{j=1}^k (1 - m_{i_j}') \right)^{1/\dot{C}_n^k}, 1 - \prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k a_{i_j} \right) \right)^{1/\dot{C}_n^k}, \right. \\
& \quad \left. 1 - \prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k d_{i_j} \right) \right)^{1/\dot{C}_n^k} \right), \\
& \text{PFDMMSM}(P_1', P_2', \dots, P_n') = \left(1 - \left(1 - \prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \prod_{j=1}^k (1 - m_{i_j}') \right)^{1/\dot{C}_n^k} \right)^{1/k}, \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k a_{i_j} \right) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k}, \right. \\
& \quad \left. \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k d_{i_j} \right) \right) \right)^{1/\dot{C}_n^k} \right)^{1/k} \right). \tag{21}
\end{aligned}$$

The above-defined PFDMMSM operator is likely to satisfy the following characteristics of aggregation. \square

Property 4. Let $P_j' = (m_j', a_j, d_j)$ denote a collection of PFNs. If $P_j' = P'$, then

$$\text{PFDMMSM}(P_1', P_2', P_3', \dots, P_n') = P'. \tag{22}$$

Property 5. Let P_j' and P_3' denote the collections of PFNs if $m_j' \geq m_3'$, $a_j \leq a_3$, and $d_j \leq d_3$. Then,

$$\text{PFDMMSM}(P_1', P_2', \dots, P_j') \geq \text{PFDMMSM}(P_1', P_2', \dots, P_3'). \tag{23}$$

Property 6. Let $P_j' = (m_j', a_j, d_j)$ denote a collection of PFNs such that

$$\begin{aligned}
P_j'^- &= \min P_j' = (\min m_j', \max a_j, \max d_j), \\
P_j'^+ &= \max P_j' = (\max m_j', \min a_j, \min d_j). \tag{24}
\end{aligned}$$

Then,

$$P_j'^- \leq \text{PFDMMSM}(P_1', P_2', P_3', \dots, P_n') \leq P_j'^+. \tag{25}$$

Definition 8. Let $P_j' = (m_j', a_j, d_j)$ denote a collection of PFNs. Then, the PFWDMMSM operator is given by

$$\text{PFWDMMSM}(P_1', P_2', \dots, P_n') = \frac{1}{k} \left(\bigoplus_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(\bigotimes_{j=1}^k (\omega_{i_j} \otimes P_{i_j}') \right)^{1/\dot{C}_n^k} \right). \tag{26}$$

Theorem 4. Let $P_j' = (m_j', a_j, d_j)$ denote a collection of PFNs. Then, using the PFWDMMSM operator, we have

$$\text{TSFWDMSM}^k(P'_1, P'_2, P'_3, \dots, P'_n) = \left(\begin{array}{c} 1 - \left(1 - \prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \prod_{j=1}^k (1 - m_{i_j}')^{w_{i_j}} \right)^{1/C_n^k} \right)^{1/k}, \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (a_{i_j})^{w_{i_j}} \right) \right) \right)^{1/C_n^k} \right)^{1/k} \\ \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (d_{i_j})^{w_{i_j}} \right) \right) \right)^{1/C_n^k} \right)^{1/k} \end{array} \right). \quad (27)$$

Proof. Similar as Theorem 3.

□

Consider the PFMSM and PFDMSM operators as follows:

5. Consequences of the PFMSM Operator

This section aims to show the generalization of the PFMSM and PFDMSM operators over the MSM operators of IFSSs.

$$\begin{aligned} \text{PFMSM}(P'_1, P'_2, \dots, P'_n) &= \left(\begin{array}{c} \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k m_{i_j}' \right) \right) \right)^{1/C_n^k} \right)^{1/k}, 1 - \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (1 - a_{i_j}) \right) \right) \right)^{1/C_n^k} \right)^{1/k} \\ 1 - \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (1 - d_{i_j}) \right) \right) \right)^{1/C_n^k} \right)^{1/k} \end{array} \right), \\ \text{PFDMSM}(P'_1, P'_2, \dots, P'_n) &= \left(\begin{array}{c} 1 - \left(1 - \prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \prod_{j=1}^k (1 - m_{i_j}')^{w_{i_j}} \right)^{1/C_n^k} \right)^{1/k}, \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (a_{i_j})^{w_{i_j}} \right) \right) \right)^{1/C_n^k} \right)^{1/k} \\ \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k (d_{i_j})^{w_{i_j}} \right) \right) \right)^{1/C_n^k} \right)^{1/k} \end{array} \right). \quad (28) \end{aligned}$$

For $a_{i_j} = 0$, the PFMSM and PFDMSM operators are reduced into the MSM operators of IFSSs proposed by [30] and given as follows:

$$\begin{aligned}
\text{IFMSM}(P'_1, P'_2, P'_3, \dots, P'_n) &= \left(\left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k m_{i_j}' \right)^1 \right) \right)^{1/\tilde{C}_n^k} \right)^{1/k} \right) \\
&\quad \left(1 - \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k \left(1 - (d_{i_j})^1 \right) \right) \right) \right)^{1/\tilde{C}_n^k} \right)^{1/k} \right), \\
\text{IFDMSM}(P'_1, P'_2, \dots, P'_n) &= \left(1 - \left(1 - \prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \prod_{j=1}^k (1 - m_{i_j}') \right)^{1/\tilde{C}_n^k} \right)^{1/k} \right), \left(1 - \left(\prod_{\substack{1 \leq i_1 \leq \dots \\ < i_k \leq n}} \left(1 - \left(\prod_{j=1}^k d_{i_j} \right) \right)^{1/\tilde{C}_n^k} \right)^{1/k} \right).
\end{aligned} \tag{29}$$

6. Picture Fuzzy MADM Procedure

The goal of this paper is to study the procedure for MADM using picture fuzzy information based on MSM operators proposed in the earlier sections.

MADM is a procedure for the selection of the finest option, under uncertain environments, among a list of finite alternatives based on some attributes. The MADM procedure that we aim to adapt is based on the PFMSM operator and PFDMSM operators. Assume the set of alternatives + among which the best is to be chosen and the attribute set under observation be $G = \{G_1, G_2, \dots, G_{m'}\}$. The uncertain information is based on PFNs where the four facets of an expert's opinion are considered to evaluate all the alternatives. In the proposed approach, the experts are asked to evaluate the alternatives using PFNs in the form of a

decision matrix and then the PFMSM and PFDMSM operators are utilized to aggregate the initial evaluation of the experts for the selection of the finest alternatives. The attributes used in the approach have their weights based on the expert's preferences. The detailed steps of the algorithm are given as follows:

Step 1. Initially, the experts evaluate the possible alternatives under the decided set of attributes using PFNs where they gave their opinion using a DM, DNM, DA, and DR under the restraints of PFSs.

Step 2. To deal with risk/cost factors, the risk is minimized by using the below-given equation where each risk/cost type of attribute is converted into a benefit type of attribute. The phenomenon is known as normalization.

$$P_{i_j}' = \left(\hat{m}_{i_j}', \hat{a}_{i_j}', \hat{d}_{i_j}' \right) = \begin{cases} \left(m_{i_j}', a_{i_j}', d_{i_j}' \right), & \text{benefit type of attributes,} \\ \left(d_{i_j}', a_{i_j}', m_{i_j}' \right), & \text{for risk/cost type of attributes.} \end{cases} \tag{30}$$

Step 3. Once done with normalizations, we utilize the following two MSM operators to aggregate the initial information normalized in Step 2.

Step 4. After the aggregation of picture fuzzy information, we use the score value of the PFNs (using Definition 2) to arrange the alternatives in ascending/descending order based on the collected information.

Step 5. The last step involves the ranking of the given alternatives based on the score values obtained in the preceding step.

Technology commercialization is a challenging problem that has been discussed widely by various authors. In the current paper, we consider the ERP system problem discussed by [32]. ERP systems are of great capability in many

engineering manufacturing companies. It eases human efforts by handling many tasks using its dynamic design.

The following numerical example is to demonstrate the MADM steps in detail.

Example 1. Consider a set of ERP schemes $\{\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5\}$ that needs evaluation by experts. These ERP schemes are evaluated under a finite set of attributes denoted by $\{G_1, G_2, \dots, G_{m'}\}$. The target is to choose the optimum ERP schemes that suit best for the company. The attribute set is defined as follows: G_1 represents technical advancement; G_2 represents reducing human resources, G_3 represents economic advantages, and G_4 represents development of the company. Each attribute $G_1, G_2, \dots, G_{m'}$ is

TABLE 1: Evaluation remarks about ERP Schemes based on PFNs.

	$\bar{\bar{A}}_1$			$\bar{\bar{A}}_2$			$\bar{\bar{A}}_3$			$\bar{\bar{A}}_4$			$\bar{\bar{A}}_5$		
	DM	DA	DNM	DM	DA	DNM	DM	DA	DNM	DM	DA	DNM	DM	DA	DNM
G ₁	0.3	0.2	0.5	0.6	0.1	0.2	0.3	0.3	0.3	0.2	0.7	0.1	0.4	0.4	0.1
G ₂	0.4	0.3	0.1	0.8	0.1	0.1	0.2	0.5	0.3	0.6	0.3	0.1	0.3	0.4	0.3
G ₃	0.7	0.1	0.2	0.4	0.2	0.4	0.3	0.1	0.2	0.3	0.4	0.3	0.5	0.1	0.3
G ₄	0.2	0.4	0.3	0.2	0.2	0.5	0.4	0.3	0.3	0.7	0.1	0.1	0.5	0.3	0.2

TABLE 2: Aggregated results of Table 1 using picture fuzzy MSM operators.

	$\bar{\bar{A}}_1$			$\bar{\bar{A}}_2$			$\bar{\bar{A}}_3$			$\bar{\bar{A}}_4$			$\bar{\bar{A}}_5$		
	DM	DA	DNM	DM	DA	DNM	DM	DA	DNM	DM	DA	DNM	DM	DA	DNM
G ₁	0.36	0.32	0.29	0.44	0.15	0.32	0.29	0.31	0.28	0.40	0.42	0.15	0.42	0.31	0.23
G ₂	0.10	0.10	0.08	0.15	0.04	0.08	0.08	0.10	0.08	0.12	0.13	0.04	0.12	0.10	0.06
G ₃	0.44	0.32	0.23	0.56	0.15	0.25	0.30	0.31	0.27	0.49	0.42	0.13	0.43	0.31	0.21
G ₄	0.12	0.07	0.06	0.20	0.03	0.06	0.08	0.08	0.08	0.16	0.13	0.59	0.13	0.08	0.05

TABLE 3: The score of the aggregated PFNs, obtained in Table 2.

	Score($\bar{\bar{A}}_1$)	Score($\bar{\bar{A}}_2$)	Score($\bar{\bar{A}}_3$)	Score($\bar{\bar{A}}_4$)	Score($\bar{\bar{A}}_5$)
PFMSM	0.35121	0.414797	0.258776	0.394238	0.405949
PFWMSM	0.03978	0.088813	0.020347	0.09474	0.075623
PFDMSM	0.433926	0.54734	0.273624	0.496403	0.419766
PFWDMSM	0.073764	0.158918	0.024447	0.134875	0.085704

associated by a weight given the weight vector $\omega_j = (0.29, 0.31, 0.17, 0.23)^T$. The selection process of the most suitable ERP scheme is elaborated as follows.

Step 1. The decision panel of the company, after analyzing initially securitized 5 ERP schemes, evaluated their performances and gave their opinion using PFNs. The evaluation information is given in Table 1.

Step 2. This step involves normalization but, in this case, it does not occur as all the attributes are of benefit type.

Step 3. This step involves the aggregation of uncertain information based on PFNs using PFMSM, PFWMSM, PFDMSM, and PFWDMSM operators. We apply the four newly developed MSM operators, and the results are portrayed in Table 2.

Step 4. This step involves the definition of the ranking function. We computed the scores of the data obtained in Table 2, and the results are given in Table 3.

Step 5. This step involves the ranking of the ERP schemes in order to get the optimum result. All the ERP schemes are ranked in Table 4.

For better understanding, the ranking results of Table 4 are shown in Figure 1.

After analyzing Table 4 and Figure 1, it is evident that PFMSM, PFDMSM, and PFWDMSM operators give us the ERP scheme $\bar{\bar{A}}_2$ as the most effective one while the PFWMSM operators give us $\bar{\bar{A}}_4$ as best one. The selection of any MSM operators is based on the preference of the experts.

TABLE 4: Ranking of ERP schemes.

Methods	Ranking values
PFMSM	$\bar{\bar{A}}_2 \geq \bar{\bar{A}}_5 \geq \bar{\bar{A}}_4 \geq \bar{\bar{A}}_1 \geq \bar{\bar{A}}_3$
TSFWMSM	$\bar{\bar{A}}_4 \geq \bar{\bar{A}}_2 \geq \bar{\bar{A}}_4 \geq \bar{\bar{A}}_1 \geq \bar{\bar{A}}_3$
TSFDMSM	$\bar{\bar{A}}_2 \geq \bar{\bar{A}}_4 \geq \bar{\bar{A}}_1 \geq \bar{\bar{A}}_5 \geq \bar{\bar{A}}_3$
TSFWDMSM	$\bar{\bar{A}}_2 \geq \bar{\bar{A}}_4 \geq \bar{\bar{A}}_3 \geq \bar{\bar{A}}_1 \geq \bar{\bar{A}}_5$

It is to be noted that all of them can give us different results, and getting the same results is not a necessary option. Further, we took $k = 4$ in our discussed problem to take into account the relationship of all four ERP schemes. Varying k may also vary the results.

7. Comparative Analysis

The goal of this section is to set up a comparative study of the newly developed MSM operators with other existing AOs of PFNs.

PFS is an advanced frame where uncertain information can be expressed with less information loss in a certain range. A great number of AOs are being developed in the few years. Here, we aim to investigate the comparison of the MSM operators with other AOs of PFNs numerically to see the validity of the proposed MSM operators. We apply the Hamacher AOs of PFN proposed by Wei [19], picture fuzzy arithmetic and geometric AOs proposed by Garg [9], picture fuzzy Dombi AOs proposed by Jana et al. [12] to the problem discussed in Example 1, and portray the results in Table 5.

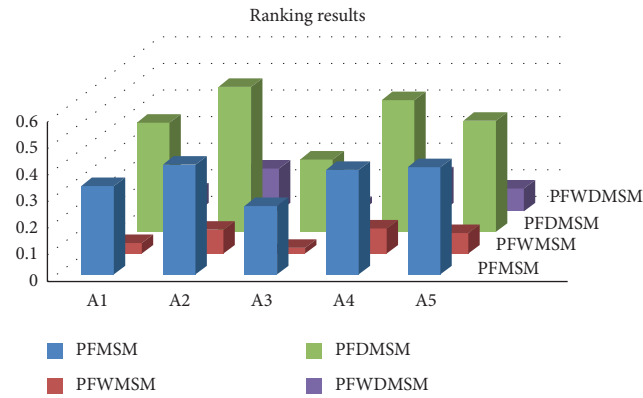


FIGURE 1: Ranking results of Table 4.

TABLE 5: Comparative analysis of the proposed operators with some existing operators.

Operators	$\dot{S}(\bar{A}_1)$	$\dot{S}(\bar{A}_2)$	$\dot{S}(\bar{A}_3)$	$\dot{S}(\bar{A}_4)$	$\dot{S}(\bar{A}_5)$	Ranking
PFHWA [19]	0.385	0.571	0.275	0.480	0.405	$\bar{A}_2 \geq \bar{A}_4 \geq \bar{A}_5 \geq \bar{A}_1 \geq \bar{A}_3$
PFHWG [19]	0.438	0.609	0.298	0.630	0.523	$\bar{A}_4 \geq \bar{A}_2 \geq \bar{A}_5 \geq \bar{A}_3 \geq \bar{A}_1$
PFWA [9]	0.384	0.584	0.259	0.488	0.397	$\bar{A}_2 \geq \bar{A}_4 \geq \bar{A}_5 \geq \bar{A}_1 \geq \bar{A}_3$
PFWG [9]	0.420	0.439	0.248	0.397	0.383	$\bar{A}_2 \geq \bar{A}_1 \geq \bar{A}_4 \geq \bar{A}_5 \geq \bar{A}_3$
PFDWA [12]	0.413	0.641	0.251	0.521	0.394	$\bar{A}_2 \geq \bar{A}_4 \geq \bar{A}_1 \geq \bar{A}_5 \geq \bar{A}_3$
PFDWG [12]	0.317	0.366	0.249	0.346	0.382	$\bar{A}_5 \geq \bar{A}_4 \geq \bar{A}_3 \geq \bar{A}_1 \geq \bar{A}_2$
PFMSM	0.334	0.415	0.259	0.394	0.406	$\bar{A}_2 \geq \bar{A}_5 \geq \bar{A}_4 \geq \bar{A}_1 \geq \bar{A}_3$
PFWMSM	0.038	0.089	0.020	0.095	0.076	$\bar{A}_4 \geq \bar{A}_2 \geq \bar{A}_4 \geq \bar{A}_1 \geq \bar{A}_3$
PFDMSM	0.411	0.547	0.274	0.496	0.420	$\bar{A}_2 \geq \bar{A}_4 \geq \bar{A}_1 \geq \bar{A}_5 \geq \bar{A}_3$
PFWDMSM	0.074	0.159	0.024	0.135	0.086	$\bar{A}_2 \geq \bar{A}_4 \geq \bar{A}_3 \geq \bar{A}_1 \geq \bar{A}_5$
IFMSM [30]			NA			Unable to specify
PyFMSM [32]			NA			Unable to specify
QROFMSM [34]			NA			Unable to specify

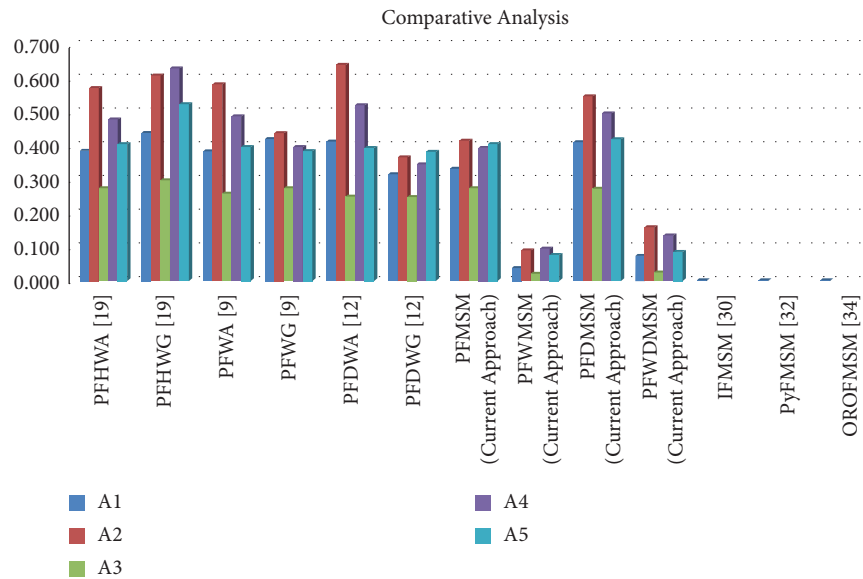


FIGURE 2: Comparative results obtained in Table 5 at a glance.

In the results of Table 5, we see that Hamacher AOs, Dombi AOs, and arithmetic and geometric AOs of PFSs provide us different results from the results obtained using

the proposed approach. The reason behind this is that those operators do not take into account the interrelationship of the input arguments; however, the current MSM operators

of PFNs do. The analysis also shows that the MSM operators or any other AOs of IFs, PyFSs, and qROFSs are unable to deal with such kinds of advanced data. The reason behind the failure of previous work is the absence of the DA and the DR which leads to information loss. The reason behind the different results using the proposed approach is the incorporation of the relationship of the input argument and the absence of such a relationship in the traditional aggregation operators. For better understanding, a summary of the above-obtained results is depicted in Figure 2.

In Figure 2, the results of the comparative study section can be seen at once. It is clearly shown that the proposed MSM operators of PFs can be used to investigate the MADM problems in a better way than the MSM operators of IFs, PyFSs, and qROFSs.

8. Conclusion

In the manuscript, we proposed some MSM operators using picture fuzzy information. PFMSM operators have two main advantages. It uses four kinds of degrees, i.e., DM, DNM, DA, and DR to express the uncertain information as an expert's opinion for solving MADM problems. These operators take into account the relationship of more than two input values, unlike other traditional aggregation operators. The two characteristics make the notion of MSM of PFs stronger than the previously discussed MSM operators. The MSM operators of PFs can be used in MADM and multiattribute group decision making (MAGDM) efficiently as in the presented problem in Section 6. These types of MSM operators can further be improved if equipped with a parameter for enlarging the space of the DM, DA, DNM, and DR. Such work is aimed to be achieved in near future in the frame of interval-valued PFs [5], SFs, and TSFs [42], interval-valued TSFs [14], and complex TSFs [43]. The MSM operators upon combining with other Dombi t-norms, Einstein t-norms, and Frank t-norms can also be some future work. The notion of TSFs can also be extended to introduce TSF matrices [44, 45] and to study their several aspects. Furthermore, such a concept can also be very beneficial when equipped with the frame of picture hesitant FS due to the ambiguous nature of uncertain information.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

A Hybrid Z-Based MADM Model for the Evaluation of Urban Resilience

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Natural disasters and man-made incidents have many negative impacts on major cities, including casualties, economic losses, disruption of social order, and environmental contamination. Cities need to be resilient in order to protect people's lives and property. Although research on urban resilience has been rapidly emerging in recent years, there are still some research gaps. The interplay of attributes for assessing urban resilience has not been explored, and the Multiple Attribute Decision Making- (MADM-) based framework for evaluating urban resilience is rarely studied. Therefore, this study proposes a novel model to evaluate urban resilience, the Z number-based Decision-Making Trial, and Evaluation Laboratory (Z-DEMATEL), to identify the mutual influential relationships and the weights of the attributes. In addition, the Z number-based Reference Ideal Method (Z-RIM) is used to determine the resilience capacity of cities and to suggest improvements for decision makers to develop appropriate strategies. In this study, we not only use trapezoidal fuzzy numbers to reflect the uncertainty of information but also measure the reliability/confidence of experts in the assessment. The integrated methodology is presented for the first time in this study, and we use the firmness data of six major metropolitan cities in Taiwan as an example of model demonstration. The results of the study show that population density, value of business activity, healthcare facilities, electricity supply, and number of business registrations are the most important attributes influencing the resilience of cities. Taipei City and Taichung City are the two major cities with better resilience in Taiwan based on the analysis of this study.

1. Introduction

Many disasters caused by extreme weather and human factors negatively affect human life, and this situation is becoming more and more frequent and serious. Cities need to not only build a defensive system that can withstand disasters, but also protect the lives and property of their residents. Cities need to be sufficiently defensible, resilient, and able to adapt to numerous disturbances [1]. In recent years, research on urban resilience has been rapidly emerging, and the number of topics related to the terms “urban resilience,” “resilient city,” and “resilient cities” is growing every year [2]. The definition of “resilience” has always been a controversial issue, leading to conceptual and definitional differences in many studies. How to measure

“resilience” is of interest to researchers [2–4], and Meerow et al. [5] provide a clear definition of urban resilience through an extensive literature review. Urban resilience is the ability of an urban system and all its constituent organizations (including cross-functional departments, societies, and technologies) to maintain or rapidly recover required functions in the face of turbulence, to adapt to change, and to rapidly transform the system's current limits or future adaptive capacity [5, 6]. Ribeiro and Pena Jardim Gonçalves [7] argue that enhancing the adaptive capacity of cities is one of the most important needs of urban communities, especially those areas that are often at risk. Although natural or man-made disasters are always tragic, postdisaster recovery can provide a rare opportunity for societies to rebuild, address serious structural problems, and

prevent affected people from suffering the same hardships again, thereby increasing the resilience of local communities. The strength of urban resilience determines the social and economic resilience of cities and the speed of recovery [3].

In recent years, there has been a growing body of research on urban resilience assessment. Researchers' research topics cover communities and cities and even extend to different levels such as cross-regional level. The research methods involve both mathematical and spatial analysis methods [8]. For example, Cariolet et al. [9] use maps and geographic information systems (GIS) to assist in mapping urban resilience. McClymont et al. [10] developed a multiobjective optimization system for urban resilience and green infrastructure systems, using policy, performance, connectivity, and socializing as the four main dimensions of the system. Meerow and Newell [2] further define and discuss urban resilience through an extensive literature review and compilation. However, most scholars have focused on urban performance in response to individual disasters (e.g., earthquakes, floods, epidemics, and transportation disasters) [11], but participation in comprehensive urban resilience assessment is still limited. Furthermore, it is necessary to consider the mutual influences of various factors/attributes in the overall assessment of urban resilience [8].

In this study, we discuss the following four key urban resilience assessment questions: (i) Which attributes are suitable for assessing urban resilience? (ii) What are the mutually influential relationships of attributes? And what is their importance? (iii) How can an assessment model be constructed to measure the resilience of a city? (iv) Finally, how can poorly resilient cities be improved? These questions constitute our research motivation. The four points make up a typical MADM problem, and the MADM approach has excellent analytical performance in complex evaluation environments. It does not require the assumptions of normal distribution and independence that traditional statistics require for its use [12, 13]. The process performed by MADM includes identifying assessment attributes, calculating attribute weights, and integrating the performance of the evaluated items [14]. The urban resilience evaluation framework relies on the judgment of multiple experts. Only a few studies have used the MADM concept for urban resilience assessment [15]. In addition, most of the studies using qualitative surveys lack the exploration of the interaction of factors affecting urban resilience. Urban communities can be viewed as complex and dynamic interactions of physical, facility, social, economic, and environmental systems. People living in cities move, work, and engage in activities among communities regularly. Therefore, the resilience of a neighborhood cannot be considered completely independent of the resilience of its surrounding neighborhoods [3].

In the past decades, MADM techniques have been playing an important role in decision-making issues in many research areas, such as supplier selection [16], renewable power sources evaluation [17], sustainable sports tourism planning [18], and risk assessment [19]. When evaluating alternatives based on criteria/attributes/indicators, experts/

decision makers usually refer to extreme values of attributes (the-larger-the-better and the-smaller-the-better characteristics) in many MADM methods, such as COMplex PROportional ASsessment (COPRAS), Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS), VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR), ELimination Et Choice Translating REALity (ELECTRE), and Preference Ranking Organization METHod for Enrichment Evaluation (PROMETHEE). In these methods, TOPSIS and VIKOR use the same principles to determine positive and negative ideal solutions based on the extreme values of the attributes [20]. However, in the real world, many attributes do not belong to the-larger-the-better or the-smaller-the-better characteristics, or there are ranges of preferences for these attributes rather than simply extreme values, e.g., temperature and humidity for human habitation, optimal age for athletes, human blood pressure and heart rate, and height of ideal partners. Obviously, the optimal values of these attributes are not extreme, and they should be of the nominal-the-best characteristic; in addition, these values should have a specific interval range. In these cases, the expert cannot make the most appropriate decision based on extreme values. Moreover, when the number of alternatives increases or decreases, it may affect the ranking results of these traditional MADM methods, most often by reversing the ranking [21]. Cables et al. [22] proposed the Reference Ideal Method (RIM) to overcome these problems. The RIM provides a reasonable way of calculating the attributes of the nominal-the-best characteristic, which can set the ideal solution considered by the experts as the reference ideal solution and set the reference ideal solution as an interval value to avoid the problem of ranking reversal of evaluated items.

However, in the current complex and uncertain decision-making environment, expert judgments and opinions are difficult to interpret using crisp values [23]. Furthermore, when experts are faced with problems that they cannot fully grasp, the confidence level of their assessments may not be 100%. Many researchers have proposed various fuzzy theories to describe the ambiguities and uncertainty of the information expressed by experts. Common fuzzy theories include general fuzzy set theory, intuitionistic fuzzy set theory, hesitation fuzzy set theory, Fermatean fuzzy set theory, and Pythagorean fuzzy set theory. However, little attention has been paid to the reliability or confidence of the messages of these theories. A theory called Z-number theory, proposed by Zadeh [24], takes into account these limitations by using two sets of fuzzy numbers to evaluate the events and to measure the reliability of the evaluated values. Z-number has been widely used in various decision-making problems based on uncertain environments. For example, Peng et al. [25] used Z-TOPSIS to select the most suitable location for an inland nuclear power station in Hunan Province, China. Hsu et al. [26] proposed a modified Z-number-based Decision-Making Trial and Evaluation Laboratory (Z-DEMATEL) to explore the interrelationship and priority of Taiwan's medical industry development trends. Garg et al. [27] developed granulated Z-VIKOR to provide a novel failure mode and effect analysis model in the field of risk

management. Although MADM methods combined with Z-number overcome the problem of reliability measurement in expert assessment, they only consider the maximum or minimum value of the attribute when evaluating the attribute and do not reasonably address the attribute of the nominal-the-best characteristic. In addition, the evaluated items do not necessarily have to meet the extreme values for some attributes of the-larger-the-better or the-smaller-the-better characteristic, they only have to reach a certain range to be accepted. In fact, it is costly to achieve the attribute extremes.

Therefore, in order to fill this research gap, this paper proposes a novel MADM model that combines Z-DEMATEL and Z-RIM to evaluate urban resilience. The proposed modified Z-DEMATEL extends the study of Hsu et al. [26] by using trapezoidal fuzzy numbers instead of the general triangular fuzzy numbers in assessing the mutually influential relationships between attributes. Compared to triangular fuzzy numbers, trapezoidal fuzzy numbers may cover a wider range of ambiguities. The improved Z-DEMATEL determines which attributes are the main factors affecting the resilience system and generates an influential network relationship map (INRM). Besides, the improved Z-RIM extends the concept of TOPSIS by using ideal and nonideal solutions to map the relative position of each evaluation item when determining the variation in each alternative relative to the normalized reference ideal. The improved Z-RIM not only overcomes the limitations of Z-VIKOR and Z-TOPSIS in practical applications, but also facilitates the examination of which attributes of the evaluated items are underperforming [22]. The proposed model is valid and reliable for assessing urban resilience. This study provides information on which attributes are the main factors affecting the resilience of cities, and the government and relevant ministries can focus on these attributes to develop relevant regulations. The results of the RIM can be also used to observe which attributes are underperforming in their current state, and decision makers can develop improvement strategies to enhance city resilience based on these underperforming attributes.

The MADM model proposed in this study provides a novel soft calculation method for urban resilience assessment. As far as we know, the proposed model is novel, and no other studies have proposed this integration method. In addition, both the conventional DEMATEL and RIM have been improved in this study. The characteristics, innovations, and contributions of this work are summarized as follows.

- (i) This paper introduces an effective integrated model, which provides a reference for the government and relevant ministries to evaluate urban resilience.
- (ii) This study uses Z-number to measure the reliability of experts in the evaluation. Both Z-DEMATEL and

Z-RIM methods reflect the decision-making process in an uncertain environment.

- (iii) Experts could understand the causal relationship between the attributes more clearly through the INRM constructed by Z-DEMATEL. In addition, underperforming attributes can be observed through Z-RIM. Therefore, decision makers can formulate strategies for improving urban resilience.
- (iv) In this study, six major metropolitan cities in Taiwan are taken as examples to demonstrate the feasibility of the model.

The other sections are arranged as follows: Section 2 introduces the description of the evaluation framework and attributes. Section 3 explains the concept and calculation of trapezoidal fuzzy numbers and Z-number theory. Section 4 introduces the proposed MADM model, including calculation procedures describing Z-DEMATEL and Z-RIM. Section 5 uses survey data from six major cities in Taiwan as a practical application case to prove the feasibility and practicability of the proposed model. Section 6 discusses and describes the implications for management and finally gives conclusions and future research directions.

2. Description of the Evaluation Framework and Attributes

This section reviews the literature related to urban resilience. Many researchers have developed some indicators, frameworks, and conceptual models to quantify resilience analysis [3, 28–35]. Our first task is to establish a suitable evaluation framework to measure the resilience of cities and then, by collecting and sorting out relevant data sets, to create an urban resilience knowledge system. We identify and select widely available data sources.

Very few scholars have explored the research on urban resilience assessment related to Taiwan. In particular, the evaluation framework constructed with the MADM concept has not yet been fully studied. This study establishes the initial evaluation attributes suitable for assessing urban resilience in Taiwan based on relevant academic literature and opinions from disaster prevention-related ministries (including police and firefighting units, relevant departments of universities, and research institutions) and then selects relatively important attributes to be included in the evaluation system to reflect the characteristics and connotations of urban resilience. The main framework consists of four dimensions, namely, social infrastructure and community connectivity (SI), contingency capacity (CC), economic strength (ES), and environmental conditions (GC). Each of these dimensions can be divided into several attributes, and a total of 24 attributes are used to build the evaluation framework, as shown in Table 1. The proposed city resilience attributes can be used to examine the resilience of the evaluated cities.

TABLE 1: The evaluation framework.

Dimensions	Attributes	Description	References
Social infrastructure and community connectivity (SI)	Population density in urban area (SI1)	Planned population per unit of land area in the urban planning area (unit: people/square kilometer)	Monteiro et al. [28]; Borsekova et al. [29]; Kontokosta and Malik [3]; Zheng et al. [30]
	Population over 65 years old (SI2)	Number of people aged 65 or older registered in the household register (unit: number of persons)	Monteiro et al. [28]; Kontokosta and Malik [3]; Zheng et al. [30]
	Population under 5 years old (SI3)	Number of people under the age of 5 registered in the household register (unit: number of persons)	Kontokosta and Malik [3]; Zheng et al. [30]
	Population of disabled people (SI4)	The number of people with disability cards or certificates (unit: number of persons)	Kontokosta and Malik [3]; Zheng et al. [30]
	Population with Bachelor's degree (SI5)	Percentage of the population aged 15 or above with college education to the population aged 15 or above (unit: %)	Kontokosta and Malik [3]
	Number of families with low income (SI6)	The number of people whose average household income is below the minimum cost of living standard (unit: number of persons)	Monteiro et al. [28]
	Number of single-person households (SI7)	The number of people living alone who need care (the aged living alone booked for caring persons) (unit: number of persons)	Kontokosta and Malik [3]
Contingency capacity (CC)	Number of healthcare services (CC1)	Medical institutions including public and nonpublic hospitals and clinics (unit: number of institutions)	Zheng et al. [30]; Ghouchani et al. [34]
	Number of emergency shelters (CC2)	The number of emergency shelters in the city that can accommodate people (unit: number of shelters)	Chen et al. [32]
	Integrity of evacuation roads and routes (CC3)	The length of roads including national highways, provincial highways, county roads, rural roads, special highways, and urban roads; the length of urban roads refers to the length of roads with a road width of 6 meters or more in the urban planning area of each county and city (unit: km)	Monteiro et al. [28]; Zheng et al. [30]
	Stability of electricity supply (CC4)	It refers to the number of households other than lighting customers of Taipower, including the number of households supplied by packaged electricity and electricity consumption (low voltage, high voltage, and extra-high voltage) (unit: number of households)	Almeida et al. [33]
	Stability of water supply (CC5)	The ratio of the actual number of water users to the population of the administrative area (unit: %)	Almeida et al. [33]
	Ratio of police officers to population (CC6)	Ratio of local police officers to the population (unit: %)	Ghouchani et al. [34]
	Ratio of firefighters to population (CC7)	Ratio of local firefighters * 100,000 to the population (unit: %)	Ghouchani et al. [34]

TABLE 1: Continued.

Dimensions	Attributes	Description	References
Economic strength (ES)	Unemployed population (ES1)	The percentage of the unemployed population in the labor force (unit: %)	Kontokosta and Malik [3]
	Gini index for income inequality (ES2)	Gini index or Gini concentration coefficient is a measure of the ratio of the area contained between the Lorenz distribution curve and the perfect parity line to the area of the entire triangle below the perfect parity line and is sometimes called the concentration ratio or the inequality coefficient (unit: from 0 to 1; the larger the coefficient, the higher the degree of inequality in distribution; the smaller the coefficient, the lower the degree of inequality)	Kontokosta and Malik [3]
	Outcome of business activities (ES3)	The sales amount declared or approved by companies registered as for-profit businesses under regulations (unit: thousand New Taiwan dollars)	Okada [35]
	Number of existing businesses in urban area (ES4)	The number of companies registered (unit: number of companies)	Okada [35]
	Disposable income on average households (ES5)	Average disposable income per household (unit: New Taiwan dollars)	Zheng et al. [30]; Okada [35]; Ghouchani et al. [34]
Environmental conditions (GC)	Coverage of flood zone (GC1)	The maximum possible flooding area in the jurisdiction is calculated by 650 mm of rain per hour for 24 consecutive hours (unit: meter square)	Borsekova et al. [29]; Caldarice et al. [31]
	Coverage of earthquake fault zone (GC2)	The length of the fault zone in the jurisdiction (unit: meters)	Borsekova et al. [29]
	Number of fires per 10,000 households (GC3)	The average number of fires per 10,000 households in a certain period (unit: number of times)	Caldarice et al. [31]
	Density of buildings (GC4)	Number of buildings per square kilometer (unit: total number of buildings per square kilometer)	Monteiro et al. [28]; Kontokosta and Malik [3]; Ghouchani et al. [34]
	Number of buildings over 30 years old (GC5)	Number of buildings more than 30 years old (unit: number of buildings)	Monteiro et al. [28]

3. Preliminaries

In this section, the basic concepts and calculation logic of trapezoidal fuzzy numbers and Z-number are introduced.

3.1. The Definition and Operation Laws of Trapezoidal Fuzzy Number. In the real world, most qualitative evaluations involve ambiguity and uncertainty due to the presence of many unknown and unidentified information in the decision-making process. Moreover, in an uncertain environment, ambiguity and subjective judgment can greatly affect the decision-making process. To alleviate this problem, researchers have proposed the fuzzy theory to express the information uncertainty encountered in decision making [36]. Often, linguistic variables are used to describe information about an expert's evaluation, which is a convenient way for humans to express their evaluation ideas. Linguistic variables are effective in converting qualitative content into fuzzy forms of quantitative data [37]. Many studies use trapezoidal fuzzy numbers for modeling fuzzy information. A trapezoidal fuzzy number can be symmetric or

asymmetric, and it covers a wider range of uncertainty than the conventional triangular fuzzy number [38].

A fuzzy set E on a universe discourse X can be written as a pair of (x, μ_E) , where $\mu_E: X \in [0, 1]$ is the membership function. The fuzzy number \tilde{E} on the real set S can be defined as a trapezoidal fuzzy number; the membership function $\mu_{\tilde{E}}(x)$ is shown in the following equation:

$$\mu_{\tilde{E}}(x) = \begin{cases} 0 & x < e_1, \\ \frac{(x - e_1)}{(e_2 - e_1)} & e_1 \leq x < e_2, \\ 1 & e_2 \leq x < e_3, \\ \frac{(e_4 - x)}{(e_4 - e_3)} & e_3 \leq x < e_4, \\ 0 & x > e_4. \end{cases} \quad (1)$$

A trapezoidal fuzzy number can be denoted as $\tilde{E} = (e_1, e_2, e_3, e_4)$, where $e_1 < e_2 < e_3 < e_4$. Suppose there are two trapezoidal fuzzy numbers $\tilde{P} = (p_1, p_2, p_3, p_4)$ and $\tilde{Q} = (q_1, q_2, q_3, q_4)$, and θ is a positive constant number. Then, the operation of the trapezoidal fuzzy number can be defined as follows [20]:

(i) Addition:

$$\tilde{P} + \tilde{Q} = (p_1 + q_1, p_2 + q_2, p_3 + q_3, p_4 + q_4). \quad (2)$$

(ii) Subtraction:

$$\tilde{P} - \tilde{Q} = (p_1 - q_4, p_2 - q_3, p_3 - q_2, p_4 - q_1). \quad (3)$$

(iii) Division:

$$\tilde{P} \div \tilde{Q} = \left(\frac{p_1}{q_4}, \frac{p_2}{q_3}, \frac{p_3}{q_2}, \frac{p_4}{q_1} \right). \quad (4)$$

(iv) Multiplication:

$$\tilde{P} \times \tilde{Q} = (p_1 q_1, p_2 q_2, p_3 q_3, p_4 q_4). \quad (5)$$

(v) Multiplication by a positive constant number:

$$\theta \times \tilde{P} = (\theta p_1, \theta p_2, \theta p_3, \theta p_4). \quad (6)$$

3.2. The Transformation Rules of Z-Number. Zadeh [24] proposed a variation of fuzzy numbers called Z-number, which adds “reliability/confidence” as a parameter in fuzzy

operations. A Z-number covers two kinds of fuzzy information: one is the judgment of the experts/decision makers (\tilde{E}), and the other is the reliability of the judgment (\tilde{R}). The Z-number can be recorded as $Z = \langle \tilde{E}, \tilde{R} \rangle$, where \tilde{E} is the trapezoidal fuzzy number of the judgment value and \tilde{R} is a measure of the reliability of the fuzzy number \tilde{E} , and they can be expressed as $\tilde{E} = (e, \mu_{\tilde{E}}^-) | x \in [0, 1]$ and $\tilde{R} = (x, \mu_{\tilde{R}}^-) | x \in [0, 1]$. In this paper, \tilde{E} and \tilde{R} are the trapezoidal fuzzy number and triangular fuzzy number, respectively. The reliability of the Z-number \tilde{R} can be converted into a reliability weight α by the following equation:

$$\alpha = \frac{\int_x \mu_{\tilde{R}}^- dx}{\int_{\mu_{\tilde{R}}^-} dx}. \quad (7)$$

Next, the reliability weights α obtained according to (7) are integrated into the judgment value \tilde{E} , and the weighted Z-number is as in the following equation:

$$Z^\alpha = \left\{ (x, \mu_{\tilde{E}}^-) | \mu_{\tilde{E}}^-(x) = \alpha \mu_{\tilde{E}}^-(x), x \in \sqrt{\alpha} x \right\}. \quad (8)$$

A simple example is used to illustrate the procedure of Z-number calculation. Assuming a Z-number with the judgment value $\tilde{E} = (0.3, 0.45, 0.55, 0.7)$ and reliability $\tilde{R} = (0.1, 0.3, 0.5)$, it forms $Z = \langle (0.3, 0.45, 0.55, 0.7), (0.1, 0.3, 0.5) \rangle$. According to (7), the reliability weight α is calculated as follows:

$$\alpha = \frac{\int_x \mu_{\tilde{R}}^- dx}{\int_{\mu_{\tilde{R}}^-} dx} = \frac{\int_{0.1}^{0.3} x(x - 0.1/0.3 - 0.1) dx + \int_{0.3}^{0.5} x(0.5 - x/0.5 - 0.3) dx}{\int_{0.1}^{0.3} (x - 0.1/0.3 - 0.1) dx + \int_{0.3}^{0.5} (0.5 - x/0.5 - 0.3) dx} = 0.3003. \quad (9)$$

Then, integrate α with the judgment value \tilde{E} , and Z^α is as follows:

$$Z^\alpha = \{(0.3, 0.45, 0.55, 0.7) | \alpha = 0.3003\}. \quad (10)$$

The Z-number is converted to a regular fuzzy number Z' . $Z' = (\sqrt{0.3003} \cdot 0.3, \sqrt{0.3003} \cdot 0.45, \sqrt{0.3003} \cdot 0.55, \sqrt{0.3003} \cdot 0.7) = (0.1644, 0.2466, 0.3014, 0.3836)$. In this paper, the linguistic variables to measure the reliability of experts' judgment are based on the research of Hsu et al. [26], as shown in Table 2. More Z-number calculation examples can be found in Zadeh [24] and Dong et al. [20].

4. The Proposed MADM Model

This section introduces the proposed MADM model. First, the influence weights for evaluating the attributes are obtained through Z-DEMATEL. Then, the results of the Z-DEMATEL analysis are incorporated into the Z-RIM algorithm to calculate the performance of the alternative solutions.

4.1. Z-DEMATEL. DEMATEL is a technology that defines the mutually influential relationships between attributes. It can construct a structured INRM to help decision makers

understand the complex relationships of the system, to identify which factors are the main factors that affect others and which are the factors that are affected [26]. In an environment full of uncertainty, it is difficult to use crisp values to reflect the true judgments and ideas of experts. Although many fuzzy theory methods incorporate DEMATEL to overcome the problem of uncertainty evaluation, there is a lack of reliability measurement for these evaluation values. In this study, Z-number is introduced into DEMATEL; not only can we know the reliability of experts in the evaluation process, but also we can use trapezoidal fuzzy numbers to evaluate the uncertainty of the influential relationships of attributes. The detailed steps of the modified Z-DEMATEL technique are as follows.

Step 1. Identify a set of attributes suitable for the evaluation issue.

The potential evaluation attributes are compiled through literature review, and then a decision-making team is formed by experts to screen the attributes to create an appropriate set of evaluation attributes a_j , $j = 1, 2, \dots, n$.

Step 2. Construct the average direct relation matrix \tilde{A} .

TABLE 2: The evaluation scale of reliability in expert judgment and the corresponding membership function [26].

Linguistic variable (abbreviation)	Membership function
Very low (VL)	(0, 0, 0.3)
Low (L)	(0.1, 0.3, 0.5)
Medium (M)	(0.3, 0.5, 0.7)
High (H)	(0.5, 0.7, 0.9)
Very high (VH)	(0.7, 1, 1)

There are n attributes that need to be evaluated for influence. Each expert evaluates the direct influence degree of the j th attribute on the j' th attribute according to the evaluation level of Table 3; $j, j' = 1, 2, \dots, n$. Next, the confidence level in the evaluation is measured according to the reliability level in Table 2.

TABLE 3: Influence evaluation scale and corresponding membership function [39].

Linguistic variable (abbreviation)	Membership function
No influence (N)	(0, 0, 0, 0)
Negligible influence (E)	(0, 0, 0.05, 0.2)
Low influence (L)	(0.05, 0.2, 0.3, 0.45)
Medium influence (M)	(0.3, 0.45, 0.55, 0.7)
High influence (H)	(0.55, 0.7, 0.8, 0.95)
Very high influence (VH)	(0.8, 0.95, 1, 1)

According to the judgment of expert k ($k = 1, 2, \dots, K$), the direct relation matrix $\tilde{D}^{(k)}$ can be constructed. The arithmetic mean is used to integrate an average direct relation matrix $\tilde{D}^{(k)}$, as shown in the following equation:

$$\tilde{D}^{(k)} = \left[\langle \tilde{d}_{jj'}^E, \tilde{d}_{jj'}^R \rangle \right]_{n \times n}^{(k)} = \begin{bmatrix} \langle \tilde{d}_{11}^E, \tilde{d}_{11}^R \rangle & \langle \tilde{d}_{12}^E, \tilde{d}_{12}^R \rangle & \cdots & \langle \tilde{d}_{1n}^E, \tilde{d}_{1n}^R \rangle \\ \langle \tilde{d}_{21}^E, \tilde{d}_{21}^R \rangle & \langle \tilde{d}_{22}^E, \tilde{d}_{22}^R \rangle & \cdots & \langle \tilde{d}_{2n}^E, \tilde{d}_{2n}^R \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{d}_{n1}^E, \tilde{d}_{n1}^R \rangle & \langle \tilde{d}_{n2}^E, \tilde{d}_{n2}^R \rangle & \cdots & \langle \tilde{d}_{nn}^E, \tilde{d}_{nn}^R \rangle \end{bmatrix}_{n \times n}^{(k)}, \quad j = j' = 1, 2, \dots, n; k = 1, 2, \dots, K, \quad (11)$$

where $\tilde{d}_{jj'}^E = (d_{jj'}^{E,L}, d_{jj'}^{E,M1}, d_{jj'}^{E,M2}, d_{jj'}^{E,U})$ and $\tilde{d}_{jj'}^R = (d_{jj'}^{R,L}, d_{jj'}^{R,M}, d_{jj'}^{R,U})$. Here, DEMATEL requires the diagonal elements in the matrix $\tilde{D}^{(k)}$ to be 0; that is, $\tilde{d}_{jj}^E = 0$ (when $j = j'$).

According to the Z-number operation described in Section 3.2, the direct relation matrix $\tilde{D}^{(k)}$ is

transformed into the matrix $\tilde{Q}^{(k)}$ as shown in (12). $\alpha_{jj'}$ is the reliability weight of the evaluation attribute j to the attribute j' . The matrix $\tilde{Q}^{(k)}$ of the k experts is integrated into the average direct relation matrix \tilde{A} by the arithmetic mean, as shown in equation (13).

$$\tilde{Q}^{(k)} = [\tilde{q}_{jj'}]_{n \times n}^{(k)} = \begin{bmatrix} \tilde{d}_{11}^E \cdot \alpha_{11} & \tilde{d}_{12}^E \cdot \alpha_{12} & \cdots & \tilde{d}_{1n}^E \cdot \alpha_{1n} \\ \tilde{d}_{21}^E \cdot \alpha_{21} & \tilde{d}_{22}^E \cdot \alpha_{22} & \cdots & \tilde{d}_{2n}^E \cdot \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{d}_{n1}^E \cdot \alpha_{n1} & \tilde{d}_{n2}^E \cdot \alpha_{n2} & \cdots & \tilde{d}_{nn}^E \cdot \alpha_{nn} \end{bmatrix}_{n \times n}^{(k)}, \quad j = j' = 1, 2, \dots, n; k = 1, 2, \dots, K, \quad (12)$$

$$\tilde{A} = \frac{1}{K} \left[[\tilde{q}_{jj'}]^{(1)} + [\tilde{q}_{jj'}]^{(2)} + \cdots + [\tilde{q}_{jj'}]^{(K)} \right] = [\tilde{a}_{jj'}]_{n \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}, \quad j = j' = 1, 2, \dots, n; k = 1, 2, \dots, K, \quad (13)$$

where $\tilde{a}_{jj'} = (a_{jj'}^L, a_{jj'}^{M1}, a_{jj'}^{M2}, a_{jj'}^L)$.

Step 3. Obtain the normalized direct relation matrix \tilde{X} through normalization.

Equation (14) is used for normalization calculation to obtain the normalized direct relation matrix \tilde{X} .

$$\tilde{X} = \varepsilon \cdot \tilde{Q}, \quad (14)$$

where

$$\varepsilon = \min[(1/\max_j \sum_{j'=1}^n a_{jj'}), (1/\max_{j'} \sum_{j=1}^n a_{jj'})], \\ j=j'=1, 2, \dots, n.$$

Step 4. Construct the total influence matrix \tilde{T} .

The total influence matrix \tilde{T} , (15), uses (16) to aggregate all the direct and indirect influence relationships of the normalized direct relation matrix \tilde{X} . The accumulation from the first power to the infinite power of matrix \tilde{X} can reflect all potential influence relationships. Because the operating procedure of (16) is cumbersome, a faster solution can be derived from equation (17).

$$\tilde{T} = [\tilde{t}_{jj'}]_{n \times n} = \begin{bmatrix} \tilde{t}_{11} & \tilde{t}_{12} & \cdots & \tilde{t}_{1n} \\ \tilde{t}_{21} & \tilde{t}_{22} & \cdots & \tilde{t}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{t}_{n1} & \tilde{t}_{n2} & \cdots & \tilde{t}_{nn} \end{bmatrix}, \quad j = j' = 1, 2, \dots, n, \quad (15)$$

where $\tilde{t}_{jj'} = (t_{jj'}^L, t_{jj'}^{M1}, t_{jj'}^{M2}, t_{jj'}^U)$.

$$\tilde{T} = \tilde{X} + \tilde{X}^2 + \cdots + \tilde{X}^\infty, \quad (16)$$

$$\begin{aligned} \tilde{T} &= \tilde{X} + \tilde{X}^2 + \cdots + \tilde{X}^\infty = \tilde{X}(I + \tilde{X} + \tilde{X}^2 + \cdots + \tilde{X}^{\infty-1}) \\ &= \tilde{X}(I - \tilde{X}^\infty)(I - \tilde{X})^{-1} = \tilde{X}(I - \tilde{X})^{-1}, \end{aligned} \quad (17)$$

where $\tilde{X}^\infty = [0]_{n \times n}$ and I are unit matrices. The superscript “-1” indicates the inverse matrix.

Step 5. Plot INRM to identify the mutually influential relationships between attributes.

Each column and each row of the total influence matrix \tilde{T} is summed up to obtain (18) and (19). \tilde{r}_j is the influence of attribute j on other attributes, and \tilde{s}_j is the influence on attribute j by other attributes.

$$\tilde{r} = [\tilde{r}_j]_{n \times 1} = (\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_n), \quad (18)$$

$$\tilde{s} = [\tilde{s}_{j'}]_{1 \times n} = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n)^T, \quad (19)$$

where $[\tilde{r}_j]_{n \times 1} = [\sum_{j'=1}^n \tilde{t}_{jj'}]_{n \times 1}$, $[\tilde{s}_{j'}]_{1 \times n} = [\sum_{j=1}^n \tilde{t}_{jj'}]_{1 \times n}$ and $\tilde{r}_j = (r_j^L, r_j^{M1}, r_j^{M2}, r_j^U)$, and $\tilde{s}_{j'} = (s_{j'}^L, s_{j'}^{M1}, s_{j'}^{M2}, s_{j'}^U)$. The superscript “T” represents the matrix transposition.

$\tilde{r}_j + \tilde{s}_j$ is the index of the strength of influences given and received, which is called the total influence. On the other hand, $\tilde{r}_j - \tilde{s}_j$ represents the net influence. The larger $\tilde{r}_j + \tilde{s}_j$, the greater the influence of the attribute j on the evaluation system. If $\tilde{r}_j - \tilde{s}_j > 0$, this means that attribute j has a more significant influence on other attributes, which is called a causal factor; conversely, if

$\tilde{r}_j - \tilde{s}_j < 0$, this means that attribute j is more influenced by other attributes, which is called an affected factor. In this paper, the centroid method is used to defuzzify the fuzzy values (e.g., $\tilde{\varphi} = (\varphi^L, \varphi^{M1}, \varphi^{M2}, \varphi^U)$) to obtain the crisp value (φ) as shown in the following equation [20].

$$\varphi = \frac{(\varphi^L + 2\varphi^{M1} + 2\varphi^{M2} + \varphi^U)}{6}. \quad (20)$$

Next, \tilde{r}_j and \tilde{s}_j are obtained as r_j and s_j , respectively, by the defuzzification procedure of (20). The relative coordinate positions of each attribute are clearly plotted using $r_j + s_j$ as the horizontal axis and $r_j - s_j$ as the vertical axis. The matrix \tilde{T} is used to identify the influence between each attribute, and the arrows (indicating the direction of influence) are drawn to generate a systematic INRM.

Step 6. Obtain the influence weights of the attributes.

Here, $r_j + s_j$ reflects the total influence of the attribute on the evaluation system, so the influence weight of the attribute, $w_j = \{w_1, w_2, \dots, w_n\}$, can be constructed by (21). Here, the total weight is required to be 1.

$$w_j = \frac{(r_j + s_j)}{\sum_{j=1}^n (r_j + s_j)}. \quad (21)$$

4.2. Z-RIM. Z-number is introduced into the traditional RIM method, and a practical MADM method is proposed to rank the alternatives, called Z-RIM. The proposed Z-RIM method can handle the uncertainty of the performance rank given by the decision maker. At the same time, the ranking reversal problem can be effectively avoided. The main steps of the proposed Z-RIM method are as follows.

Step 1. Set the Z-RIM parameter.

First, the required execution parameters of Z-RIM are established, and the following parameters are set for each attribute a_j [22]:

- (i) The value range of the attribute a_j : $[\omega^L, \omega^U]$
- (ii) The reference ideal (RI) of the attribute a_j : $[\tau^L, \tau^U]$
- (iii) The weight of the attribute a_j : w_j

Step 2. Obtain the evaluation matrix \tilde{Y} .

Assuming that there are m evaluated items $V_i = \{V_1, V_2, \dots, V_m\}$ and n attributes $a_j = \{a_1, a_2, \dots, a_n\}$, each performance value of evaluated item V_i under attribute a_j is investigated. In this paper, the data are collected based on the public information provided by the police and fire agencies of the Ministry of Internal Affairs. Some of the attribute data are ambiguous in nature, so Z-numbers are added to reflect their uncertainty. After converting the Z-numbers in Section 3.2, the evaluation matrix \tilde{Y} is shown in the following equation:

$$\tilde{Y} = [\tilde{y}_{ij}]_{m \times n} = \begin{bmatrix} \tilde{y}_{11} & \tilde{y}_{12} & \cdots & \tilde{y}_{1n} \\ \tilde{y}_{21} & \tilde{y}_{22} & \cdots & \tilde{y}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{y}_{m1} & \tilde{y}_{m2} & \vdots & \tilde{y}_{mn} \end{bmatrix}_{m \times n}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n, \quad (22)$$

where $\tilde{y}_{ij} = (y_{ij}^L, y_{ij}^{M1}, y_{ij}^{M2}, y_{ij}^U)$.

Step 3. Calculate the normalized evaluation matrix **F**.

It is calculated by referring to the fuzzy RIM normalization program proposed by Cables et al. [40] to

obtain the normalized evaluation matrix **F**, as shown in the following equation:

$$F = [f_{ij}]_{m \times n} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \vdots & f_{mn} \end{bmatrix}_{m \times n}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n, \quad (23)$$

where f_{ij} is the crisp value obtained by normalization. Fuzzy numbers are converted to crisp numbers in the normalization process.

Step 4. Construct the weighted normalized evaluation matrix **G**.

Taking into account the different importance of each attribute, the attribute weights w_j obtained from Z-DEMATEL are multiplied with matrix **F** to obtain the weighted normalized evaluation matrix **G**, as shown in the following equation:

$$G = [g_{ij}]_{m \times n} = \begin{bmatrix} f_{11} \cdot w_1 & f_{12} \cdot w_2 & \cdots & f_{1n} \cdot w_n \\ f_{21} \cdot w_1 & f_{22} \cdot w_2 & \cdots & f_{2n} \cdot w_n \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} \cdot w_1 & f_{m2} \cdot w_2 & \vdots & f_{mn} \cdot w_n \end{bmatrix}_{m \times n}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n, \quad (24)$$

Step 5. Calculate the distance to the ideal and nonideal evaluated items.

The definition of Euclidean distance is used to calculate the distance to the ideal and nonideal evaluated items, as shown in the following equations:

$$e_i^+ = \sqrt{\sum_{j=1}^n (g_{ij} - w_j)^2}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n, \quad (25)$$

$$e_i^- = \sqrt{\sum_{j=1}^n (g_{ij})^2} \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n, \quad (26)$$

Step 6. Calculate the relative index to the reference ideal of each evaluated item.

The ranking index of the evaluated item can be calculated as (27). The closer I_i is to 1, the closer it is to the reference ideal. On the contrary, the closer I_i is to 0, the

closer the performance is to be extremely poor.

$$I_i = \frac{e_i^-}{e_i^+ + e_i^-}, \quad i = 1, 2, \dots, m. \quad (27)$$

5. Illustration of a Real Case

5.1. Problem Description. Taiwan is located in the eastern part of Asia, at the intersection of Northeast Asia and Southeast Asia. Due to the unique geographical location of Taiwan, natural disasters occur frequently. Typhoons generated from tropical cyclones bring instantaneous rainfall, resulting in heavy rains, floods, and landslides. Taiwan is located at the junction of plates (in the Pacific Rim seismic zone), so earthquakes occur frequently, with an average of over 100 felt earthquakes per year. Besides, Taiwan's high population density makes some human-induced disasters also prone to high casualties [41]. Therefore, it is an urgent task to evaluate the resilience of Taiwan's major urban areas. This study focuses on four questions: What are the interdependencies of urban resilience attributes? What are their

TABLE 4: The background of 13 experts.

Expert no.	Institution	Years of working experience	Degree
Expert 1	University	More than 10 years	Ph.D.
Expert 2	Firefighting department	More than 10 years	Master
Expert 3	University	More than 10 years	Ph.D.
Expert 4	University	More than 10 years	Ph.D.
Expert 5	University	More than 10 years	Ph.D.
Expert 6	University	More than 10 years	Ph.D.
Expert 7	Firefighting department	More than 10 years	Bachelor
Expert 8	Police department	More than 10 years	Ph.D.
Expert 9	City government	More than 10 years	Master
Expert 10	Firefighting department	More than 10 years	Bachelor
Expert 11	Firefighting department	More than 10 years	Master
Expert 12	Police department	More than 10 years	Master
Expert 13	University	More than 10 years	Ph.D.

weights? How resilient are these cities? And how can the resilience of cities be improved and strengthened? The answers to these questions will help the government to develop strategies to improve the disaster preparedness of Taiwan. We extensively invite experts on urban resilience and disaster prevention in Taiwan to join the research group. A total of 13 experts were surveyed for this study, including those from police and firefighting units, relevant departments of universities, and research institutions. These experts have sufficient expertise and years of experience in urban resilience. They have worked on the topic for more than 10 years. In addition, these experts have contributed to the promotion of critical infrastructure protection policies in Taiwan. Table 4 shows the organization, working experience, and academic qualifications of the experts.

This study analyzed the evaluated projects according to Section 2 evaluation framework. According to the definition of municipality promulgated by the Executive Yuan of Taiwan, cities belonging to municipalities directly under the Central Government are included in the evaluation items, including Taipei City (V1), New Taipei City (V2), Taoyuan City (V3), Taichung City (V4), Tainan City (V5), and Kaohsiung City (V6), commonly known as the “six direct-controlled municipalities”. Direct-controlled municipalities are the first-level administrative divisions of Taiwan and are directly under the jurisdiction of the Executive Yuan. According to the Local Government Act, municipalities are established in areas with a population of over 1.25 million and special needs in political, economic, cultural, and metropolitan development.

5.2. Using Z-DEMATEL to Generate Attribute Weights and Identify Their Causal Relationships. In Section 4.1, the calculation process of Z-DEMATEL is described, and the data from the 13 experts' surveys are executed according to the process to identify which attributes are influential. The level of consensus among experts can be examined by the average sample gap index $((n(n-1))^{-1} \times \sum_{j=1}^n \sum_{j'=1}^n (|d_{jj'}^{E,(k)} - d_{jj'}^{E,(k-1)}|/d_{jj'}^{E,(k)}) \times 100\%)$, where n is the sample size, k is the number of experts, and $d_{jj'}^{E,(k)}$ is the evaluation value in the matrix $\tilde{D}^{(k)}$. The average gap of 13 experts calculated by the

index is 4.7%, which means that there is a 95.3% confidence level that these experts have a consensus of about 95%. Tables 5 and 6 present the average direct relation matrix \tilde{A} and the normalized direct relation matrix \tilde{X} , respectively. The diagonal elements of the matrix are all 0. In addition, all elements are Z-numbers, and the total influence relationship of the attributes can be seen in the total influence matrix \tilde{T} (Table 7), where the elements of the matrix encompass the direct and indirect influences of the attributes.

Table 8 shows the analysis results of Z-DEMATEL, including the attributes' total influence ($r_j + s_j$) and net influence ($r_j - s_j$). The greater the net influence, the greater the influence of the attribute on other attributes. $r_j + s_j$ can indicate the total influence in the overall evaluation system to express the importance ratio (the weights of the attributes). The top five attributes' weights generated by Z-DEMATEL are ranked as SI1 > ES3 > CC1 > GC4 > ES4.

Moreover, with $r_j + s_j$ as the horizontal axis and $r_j - s_j$ as the vertical axis, the INRM of the attributes is plotted, as shown in Figure 1. In Figure 1, the attributes on the upper right indicate high total and net influences, being the main causes. On the other hand, the attributes on the lower left indicate lower total and net influences, being the effects [18]. Obviously, the cause attributes are SI1, ES3, GC4, ES4, ES5, and GC5. In addition, CC3, CC2, SI2, CC6, GC3, SI4, SI7, and SI3 are factors that are more likely to be influenced by other attributes. The management implications derived from the Z-DEMATEL analysis are discussed in Section 6.

5.3. Applying Z-RIM to Integrate the Resilience Performance of the Six Major Cities. The performance values of the six evaluated objects (six major cities in Taiwan) in the proposed evaluation framework were investigated as shown in Tables 9–12. These data were collected from publicly available data provided by the relevant government agencies in Taiwan, and the parameters required by Z-RIM were set by 13 experts. Most of the attributes are quantitative data from the actual survey and are therefore of the crisp type. For example, the survey value of V1 under the SI1 attribute is 9731.6 (person/km²). Nevertheless, GC1 and GC2 are uncertain data and thus are Z-numbers. For example, the Z-numbers of V1 under the GC1 attribute are (16937697,

TABLE 5: The average direct relation matrix \tilde{A} .

	SI1	SI2	SI3	...	GC5
SI1	(0.00, 0.00, 0.00, 0.00)	(0.19, 0.32, 0.40, 0.53)	(0.27, 0.40, 0.48, 0.58)	...	(0.35, 0.47, 0.54, 0.64)
SI2	(0.22, 0.34, 0.43, 0.55)	(0.00, 0.00, 0.00, 0.00)	(0.14, 0.23, 0.30, 0.43)	...	(0.22, 0.32, 0.39, 0.49)
SI3	(0.31, 0.43, 0.50, 0.60)	(0.15, 0.22, 0.29, 0.41)	(0.00, 0.00, 0.00, 0.00)	...	(0.10, 0.17, 0.23, 0.35)
⋮	⋮	⋮	⋮	⋮	⋮
GC5	(0.31, 0.43, 0.50, 0.60)	(0.27, 0.38, 0.45, 0.56)	(0.17, 0.25, 0.32, 0.44)	...	(0.00, 0.00, 0.00, 0.00)

TABLE 6: The normalized direct relation matrix \tilde{X} .

	SI1	SI2	SI3	...	GC5
SI1	(0.00, 0.00, 0.00, 0.00)	(0.01, 0.02, 0.03, 0.04)	(0.02, 0.03, 0.03, 0.04)	...	(0.02, 0.03, 0.04, 0.04)
SI2	(0.01, 0.02, 0.03, 0.04)	(0.00, 0.00, 0.00, 0.00)	(0.01, 0.02, 0.02, 0.03)	...	(0.02, 0.02, 0.03, 0.03)
SI3	(0.02, 0.03, 0.03, 0.04)	(0.01, 0.02, 0.02, 0.03)	(0.00, 0.00, 0.00, 0.00)	...	(0.01, 0.01, 0.02, 0.02)
⋮	⋮	⋮	⋮	⋮	⋮
GC5	(0.02, 0.03, 0.03, 0.04)	(0.02, 0.03, 0.03, 0.04)	(0.01, 0.02, 0.02, 0.03)	...	(0.00, 0.00, 0.00, 0.00)

TABLE 7: The total influence matrix \tilde{T} .

	SI1	SI2	SI3	...	GC5
SI1	(0.02, 0.05, 0.09, 0.24)	(0.03, 0.06, 0.10, 0.24)	(0.03, 0.06, 0.10, 0.23)	...	(0.04, 0.07, 0.10, 0.23)
SI2	(0.03, 0.06, 0.09, 0.23)	(0.01, 0.03, 0.05, 0.16)	(0.02, 0.04, 0.07, 0.18)	...	(0.02, 0.04, 0.07, 0.18)
SI3	(0.03, 0.06, 0.08, 0.20)	(0.02, 0.04, 0.06, 0.16)	(0.01, 0.02, 0.04, 0.13)	...	(0.01, 0.03, 0.05, 0.15)
⋮	⋮	⋮	⋮	⋮	⋮
GC5	(0.04, 0.07, 0.10, 0.24)	(0.03, 0.06, 0.09, 0.21)	(0.02, 0.04, 0.07, 0.19)	...	(0.01, 0.03, 0.05, 0.16)

TABLE 8: Z-DEMATEL results.

	\tilde{r}_j	\tilde{s}_j	r_j	s_j	$r_j + s_j$	$r_j - s_j$	Weight	Rank
SI1	(0.93, 1.65, 2.48, 5.75)	(0.91, 1.62, 2.44, 5.68)	2.490	2.451	4.941	0.039	0.053	1
SI2	(0.56, 1.10, 1.77, 4.57)	(0.65, 1.23, 1.94, 4.85)	1.812	1.973	3.785	-0.160	0.041	12
SI3	(0.42, 0.84, 1.41, 3.86)	(0.56, 1.08, 1.74, 4.46)	1.463	1.778	3.240	-0.315	0.035	24
SI4	(0.50, 0.99, 1.62, 4.31)	(0.54, 1.05, 1.70, 4.41)	1.669	1.739	3.408	-0.069	0.037	19
SI5	(0.62, 1.16, 1.84, 4.67)	(0.45, 0.93, 1.55, 4.16)	1.880	1.596	3.477	0.284	0.037	18
SI6	(0.59, 1.13, 1.81, 4.64)	(0.58, 1.13, 1.82, 4.67)	1.853	1.859	3.712	-0.005	0.040	15
SI7	(0.45, 0.94, 1.57, 4.20)	(0.54, 1.07, 1.74, 4.51)	1.613	1.780	3.393	-0.167	0.037	20
CC1	(0.77, 1.39, 2.14, 5.16)	(0.84, 1.50, 2.28, 5.43)	2.168	2.305	4.474	-0.137	0.048	3
CC2	(0.58, 1.06, 1.69, 4.37)	(0.78, 1.41, 2.17, 5.24)	1.743	2.194	3.937	-0.451	0.042	11
CC3	(0.63, 1.15, 1.81, 4.58)	(0.80, 1.41, 2.16, 5.17)	1.857	2.186	4.043	-0.329	0.043	10
CC4	(0.72, 1.31, 2.04, 4.97)	(0.82, 1.44, 2.01, 5.15)	2.066	2.199	4.264	-0.133	0.046	6
CC5	(0.70, 1.28, 1.98, 4.87)	(0.83, 1.44, 2.19, 5.18)	2.013	2.212	4.225	-0.199	0.045	7
CC6	(0.59, 1.09, 1.73, 4.40)	(0.66, 1.23, 1.95, 4.88)	1.774	1.983	3.757	-0.208	0.040	14
CC7	(0.72, 1.27, 1.96, 4.76)	(0.73, 1.33, 2.07, 5.08)	1.991	2.101	4.092	-0.110	0.044	9
ES1	(0.51, 1.00, 1.63, 4.30)	(0.51, 1.00, 1.64, 4.32)	1.675	1.686	3.361	-0.010	0.036	21
ES2	(0.52, 1.04, 1.70, 4.48)	(0.52, 1.04, 1.70, 4.45)	1.747	1.744	3.490	0.003	0.038	17
ES3	(0.84, 1.52, 2.34, 5.58)	(0.76, 1.40, 2.16, 5.24)	2.355	2.186	4.541	0.169	0.049	2
ES4	(0.80, 1.46, 2.25, 5.40)	(0.68, 1.28, 2.01, 4.97)	2.271	2.039	4.310	0.232	0.046	5
ES5	(0.74, 1.37, 2.13, 5.21)	(0.64, 1.23, 1.96, 4.90)	2.157	1.987	4.144	0.170	0.045	8
GC1	(0.68, 1.23, 1.92, 4.78)	(0.40, 0.77, 1.29, 3.63)	1.961	1.360	3.321	0.600	0.036	22
GC2	(0.66, 1.20, 1.87, 4.65)	(0.42, 0.80, 1.32, 3.66)	1.911	1.388	3.299	0.524	0.035	23
GC3	(0.57, 1.07, 1.72, 4.43)	(0.60, 1.12, 1.78, 4.57)	1.763	1.829	3.592	-0.067	0.039	16
GC4	(0.79, 1.44, 2.22, 5.33)	(0.77, 1.37, 2.12, 5.15)	2.243	2.150	4.393	0.092	0.047	4
GC5	(0.67, 1.25, 1.97, 4.93)	(0.55, 1.07, 1.72, 4.43)	2.007	1.760	3.767	0.247	0.041	13

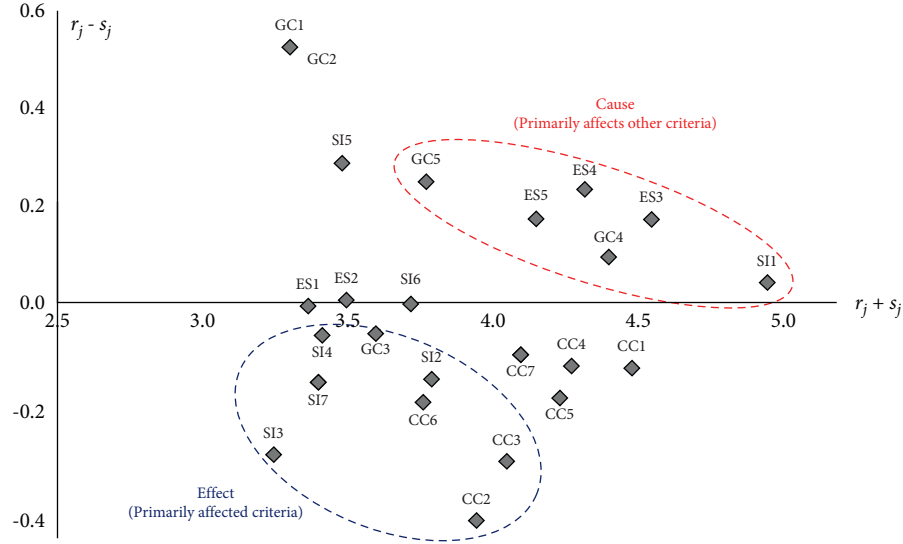


FIGURE 1: INRM.

TABLE 9: Data and parameters of Z-RIM implementation for six major cities (dimension 1: SI1 to SI7).

	SI1	SI2	SI3	SI4	SI5 (%)	SI6	SI7
V1	9731.6	578511	153472	121171	74.9	44984	5507
V2	3067.6	477944	187532	169935	46.4	40305	3922
V3	5050.2	272348	139225	85291	43.7	23822	2297
V4	4239.3	362249	160968	125999	47.8	44049	3074
V5	2957.8	295947	88143	98640	41.9	18964	2278
V6	5923.1	438452	128678	143508	46.2	36844	4900
ω^L	2957.8	272348	88143	85291	41.9	18964	2278
ω^U	9731.6	578511	187532	169935	74.9	44984	5507
τ^L	2957.8	272348	88143	85291	68.7	18964	2278
τ^U	4214.2	330597	104942	100529	74.9	24419	2954

TABLE 10: Data and parameters of Z-RIM implementation for six major cities (dimension 2: CC1 to CC7).

	CC1	CC2	CC3	CC4	CC5 (%)	CC6	CC7
V1	3662	209987	1232	23391	99.8	280.1	67.1
V2	3371	449659	3667	36315	97.9	186.9	55.9
V3	1612	63538	3301	17146	95.7	172.1	67.7
V4	3507	387532	4334	29198	96.0	227.3	55.7
V5	1955	278234	4584	30274	99.1	213.2	60.4
V6	3047	263715	4889	33978	96.2	249.4	56.3
ω^L	1612	63538	1232	17146	95.7	172.1	55.7
ω^U	3662	449659	4889	36315	99.8	280.1	67.7
τ^L	3221	381744	4224	17146	98.9	260.2	64.9
τ^U	3662	449659	4889	20680	99.8	280.1	67.7

TABLE 11: Data and parameters of Z-RIM implementation for six major cities (dimension 3: ES1 to ES5).

	ES1 (%)	ES2	ES3	ES4	ES5
V1	3.70	0.29	13529701956	235828	1422400
V2	3.80	0.28	5027810552	228575	1102332
V3	3.80	0.32	3918229936	118827	1147356
V4	3.70	0.31	4288428451	197491	1082584
V5	3.70	0.34	2355635718	114058	904114
V6	3.70	0.35	4686304754	167495	1014869
ω^L	3.70	0.28	2355635718	114058	904114
ω^U	3.80	0.35	13529701956	235828	1422400
τ^L	3.70	0.28	11540860551	209393	1335450
τ^U	3.73	0.30	13529701956	235828	1422400

TABLE 12: Data and parameters of Z-RIM implementation for six major cities (dimension 4: GC1 to GC5).

	GC1	GC2	GC3	GC4	GC5
V1	(16937697, 16985131, 17032565, 17079999)	(34068, 34542, 35016, 35491)	20.3	12185.4	543661
V2	(31645608, 31693042, 31740477, 31787911)	(112237, 112711, 113186, 113660)	13.5	2523.7	620749
V3	(71301063, 71348498, 71395932, 71443366)	(68841, 69316, 69790, 70264)	19.6	2218.0	245462
V4	(72795467, 72842901, 72890336, 72937770)	(116981, 117455, 117929, 118404)	31.9	1158.1	325935
V5	(282820089, 282867523, 282914958, 282962392)	(79582, 80056, 80531, 81005)	32.2	623.8	291395
V6	(145625637, 145673072, 145720506, 145767940)	(125370, 125844, 126318, 126793)	25.0	883.9	453675
ω^L	(16937697, 16985131, 17032565, 17079999)	(34068, 34542, 35016, 35491)	13.5	623.8	245462
ω^U	(282820089, 282867523, 282914958, 282962392)	(125370, 125844, 126318, 126793)	32.2	12185.4	620749
τ^L	(16937697, 16985131, 17032565, 17079999)	(34068, 34542, 35016, 35491)	13.5	623.8	245462
τ^U	(66215423, 66262858, 66310292, 66357726)	(51613, 52087, 52562, 53036)	17.2	2840.8	320403

TABLE 13: Results of Z-RIM calculations.

	e_i^+	e_i^-	I_i	Rank
V1	0.1167	0.1577	0.5746	1
V2	0.1273	0.1390	0.5220	5
V3	0.1231	0.1461	0.5428	4
V4	0.1111	0.1490	0.5727	2
V5	0.1235	0.1474	0.5441	3
V6	0.1223	0.1212	0.4978	6

TABLE 14: The calculation results of multiple models.

	Z-SAW	Z-SWASPAS	Z-COPRAS	Z-TOPSIS	Z-VIKOR	Z-RIM
V1	1	1	1	1	1	1
V2	5	5	5	5	5	5
V3	3	3	3	4	4	4
V4	2	2	2	2	2	2
V5	4	4	4	3	3	3
V6	6	6	6	6	6	6

TABLE 15: The results of the nine runs’ sensitivity analysis.

[illegible]

16985131, 17032565, 17079999). The overall performance (ranking index) of the evaluated objects was obtained by following the Z-RIM execution steps described in Section 4.2.

The degree of separation of the evaluated objects from the ideal and nonideal solutions can be determined. What is certain is that the evaluated objects with better performance will be closer to the ideal solution, while the objects with poor performance will be far away from the ideal solution. On the contrary, the evaluated items with better performance will be far away from the nonideal solution. Table 13 shows the calculation results of Z-RIM. The performance of the evaluated objects is $V1 > V4 > V5 > V3 > V2 > V6$ in order.

6. Discussion and Conclusions

The analysis of urban resilience is an important and urgent task. Advanced countries have invested huge amounts of money in this issue to strengthen the resilience of cities. In order to improve the level of urban resilience, many researchers have proposed advanced methodologies to effectively assess urban resilience and provide feasible improvement strategies as a reference [3, 5, 6]. Based on the literature review, previous studies rarely used MADM models to explore the mutual influences and weights of urban resilience attributes. In addition, an urban resilience evaluation framework suitable for Taiwan has not yet been fully established. This paper proposes a novel MADM model to bring a more complete evaluation framework and analysis method to the problem of urban resilience measurement. Firstly, through a large number of literature reviews and expert interviews, 4 dimensions and 24 attributes were determined to establish an evaluation framework. Secondly, this paper uses Z-DEMATEL to obtain the mutually influential relationships and weights of attributes and then generates an INRM so that decision makers can easily identify the key influencing factors in the evaluation system. The Z-DEMATEL proposed in this paper is an extension of the research of Hsu et al. [26]. It uses trapezoidal fuzzy numbers to envelop a wider range of uncertainty and effectively measures the confidence of experts in the evaluation. Finally, we also improved the conventional RIM [22], combined with Z-numbers to form the Z-RIM, and further optimized the practicability of the conventional RIM in a fuzzy environment.

Based on the Z-DEMATEL results in Table 8, in terms of total influence ($r_j + s_j$), the top five are “population density in the urban area (SI1) (4.941),” “outcome of business activities (ES3) (4.541),” “number of healthcare services (CC1) (4.474),” “density of buildings (GC4) (4.393),” and “number of existing businesses in the area (ES4) (4.310)”. Besides, the total weight of these five attributes is approximately 0.25 ($0.053 + 0.049 + 0.048 + 0.047 + 0.046$), which accounts for about a quarter of the overall evaluation system. The INRM shows that “population density in the urban area (SI1),” “outcome of business activities (ES3),” “density of buildings

(GC4),” “number of existing businesses in the area (ES4),” “disposable income on average households (ES5),” and “number of buildings over 30 years old (GC5)” are the main factors affecting the relationship system. The results of Z-DEMATEL analysis echo the research of Monteiro et al. [28], Borsekova et al. [29], Kontokosta and Malik [3], and Zheng et al. [30]. This study focuses on the results of Z-DEMATEL analysis. Several feasible improvement measures for resilience are proposed and described as follows.

- (i) The population density in urban areas: Many resilience-related documents mention the impact of population density and population composition on urban resilience, and most of them focus on the vulnerable population, such as the elderly, the young, and the disadvantaged. These populations have higher vulnerability and lower resilience in the face of natural disasters [42–45]. In terms of improvement measures, the possibility of urban migration can be considered. Taipei is the capital of Taiwan, and capitals are often the economic, political, and cultural centers of countries. However, as the scale of capital cities continues to expand, many city capacity problems will arise, for example, huge problems such as traffic congestion, environmental pollution, overconcentration of population, and disaster multiplication. Relocating the capital city, or moving part of the government to a different city, can solve the problems of uneven economic development and urban disaster multiplication in a country and strengthen the resilience of the city, which is a better policy choice. There are many successful cases in the world, such as the Seoul metropolitan area, which has nearly 50% of the country’s population, and the problems of metropolitan areas are getting more and more serious, such as high housing prices, traffic congestion, and environmental pollution. In order to solve the urban development problem, Korea decentralized Seoul’s urban capacity and relocated the capital to become the first choice to decentralize Seoul’s politics and economy. Therefore, the National Assembly passed the New Administrative Capital Special Act in December 2003 to provide legal protection for the relocation of Korea’s capital.
- (ii) The outcome of business activities and the number of existing businesses in the urban area: The output value of urban industrial activities and the number of industrial companies in the urban area are important indicators to measure urban resilience. Different industries have different influences on disaster resilience. For example, agriculture may be less resilient than the commercial and industrial sectors because of its lower value and less attention received [43]. For example, the key focus of Taipei

City is the development of high-tech industries such as communications, electronics, information, computers, and machinery, located mainly in the Neihu and Nangang districts of Taipei City, which are in line with the current key construction of Taipei's urban development and the creation of high-tech industrial clusters. The impact of large-scale disasters on the industry is relatively large for sure. In particular, earthquakes and floods in Taiwan have a significant impact on urban resilience. The stability of electricity is also an important determinant of industrial development and urban resilience. Therefore, it is recommended that the government should pay attention to the industrial development characteristics of a city, such as Taipei City as a high-tech development city, and also pay attention to the industrial clustering effect, the more industrial clusters, what is more favorable to the development of industries, what is more favorable to the city income, and what is more favorable to the city resilience.

- (iii) Density of buildings/number of buildings over 30 years old: The distribution of buildings affects the impact level of earthquakes and relevant disasters (e.g., floods), as well as the response and recovery ability during and after disasters. In general, the higher the number of buildings and age of buildings, the lower the urban resilience; on the contrary, stronger buildings are expected to have a positive relationship with urban resilience [46]. Strategies for improvement include accelerating urban renewal and introducing disaster-resistant urban building plans, such as the design of sponge cities and seismic retrofitting of 30-year-old houses, to strengthen urban resilience and disaster resistance.
- (iv) Number of healthcare services: In the event of a disaster, many casualties may occur, and the distribution of healthcare facilities is important in measuring the disaster response capability of a region. In terms of disaster prevention and relief facilities, in addition to the number and distribution of medical institutions, it is also necessary to consider the distribution of fire and police agencies and the number of disaster response sites. The more these facilities are adequate, the more the public resources and space are available to flexibly deal with or recover from disaster-related impacts and therefore have a higher degree of urban resilience.
- (v) Disposable income on average households: When a disaster occurs, having more disposable income allows people to purchase different disaster prevention and relief equipment or to have more flexibility in disaster avoidance and relief, to reduce

the occurrence of damage and to have more resources to rebuild after the disaster. The higher the disposable income, the higher the resilience.

It is feasible and effective to use MADM methodologies to discuss the comprehensive assessment of alternatives [47–49]. The results of the Z-DEMATEL analysis were shared and fed back to all experts, who indicated that the information assisted them in their decision making in the strategic alliance. In addition, the results of the Z-RIM analysis show that Taipei City (V1) and Taichung City (V4) are relatively more resilient in terms of urban capacity. We use index I_i as attributes to determine the resilience of the cities. I_1 to I_5 are all greater than 0.5, which means that the resilience of the five cities is above the average of the assessment system; only I_6 of V6 is 0.4978, which is less than 0.5. Therefore, appropriate improvement strategies should be given to Kaohsiung City (V6). Kaohsiung City has a diverse landscape and is a large industrial city in southern Taiwan and therefore faces complex and diverse types of disasters. Every year, Kaohsiung is threatened by typhoons, earthquakes, and factory explosions. Highly developed urban areas need to be protected from flooding, hillside areas have the potential for landslides and collapses, and ocean-front areas need to be protected from tsunamis. In addition to natural disasters, there are also threats from man-made fires, gas, firecrackers, and dangerous waters. In terms of improving policies, the Kaohsiung government should conduct disaster potential surveys, use and update disaster risk maps, deepen local disaster prevention and relief capabilities, establish various promotion mechanisms for disaster prevention, conduct education and training, improve the capabilities of relevant disaster prevention and relief personnel, and enhance the local government's ability to promote disaster prevention and relief work. Furthermore, Kaohsiung should also strengthen the chain of industrial activities, including introducing foreign investment, strengthening the formation of industrial clusters (such as science parks), and strengthening the stability of water and electricity infrastructure. Moreover, the care and immediate evacuation of vulnerable and low resilience populations (such as the elderly, the young, and the disabled) will help to improve the resilience of the city.

This study performed a model comparison, including Z-SAW (Simple Additive Weighting), Z-WASPAS (Weighted Aggregated Sum Product ASsessment), Z-COPRAS, Z-TOPSIS, Z-VIKOR, and Z-RIM methods. The calculation results of multiple models are shown in Table 14. Obviously, the three methods Z-TOPSIS, Z-VIKOR, and Z-RIM have the same sorting result because they are all based on the concept of compromise sorting. However, Z-RIM considers more potential management information than Z-TOPSIS and Z-VIKOR. In addition, we also performed a sensitivity analysis to check the robustness of the proposed model. The weight of the SI1 (the most important attribute) was changed from 0.1 to 0.9, and the other

attributes were weighted proportionally. Table 15 shows the ranking results of the nine runs. It can be confirmed that the sensitivity of the proposed model will not be affected by the single attribute weight.

Our proposed MADM model provides a systematic analysis process to provide a complete resilience evaluation for the six major cities in Taiwan. In summary, this study provides an effective tool for cities on the issue of resilience evaluation. This effective soft calculation method can be combined with expert/decision maker's judgment to propose more realistic management implications. The methodology of this study has not yet been developed in the academic community. We prove the validity and reliability of the proposed model. Several advantages of the model integrating Z-DEMATEL and Z-RIM are as follows: (i) Z-number takes into account the uncertainty and reliability in expert judgment; (ii) Z-DEMATEL explores the mutually influential relationships of urban resilience attributes and generates influence weights for the attributes; (iii) Z-RIM overcomes the shortcomings of conventional TOPSIS, which can be applied to the-larger-the-better, the-smaller-the-better, and the-nominal-the-best attributes; (iv) it helps decision makers to be more systematic in the decision-making process. Moreover, the proposed model is suitable for various evaluation and selection problems based on multiple criteria, especially under uncertain information environment, for example, site selection, corporate performance evaluation, and talent recruitment.

Although the study makes an important contribution to solving the problem of urban resilience, it still has some limitations. The proposed model uses the arithmetic mean to integrate the survey data provided by multiple experts, which makes it impossible to detect any anomalies in the data. In the future, it can be combined with the rough set theory to overcome the aforementioned problem. Furthermore, the concept of aspiration level is not introduced in Z-RIM, so it is impossible to determine the gap between the evaluated items and the aspiration level. At present, the experts suggested evaluating the six major cities in Taiwan. In the future, the model can be applied to evaluate more cities.

Data Availability

All data generated or analysed during the study are included in this published article.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

H.-W. L. conceptualized the study and developed the methodology. C.-N. H. investigated the study and wrote, reviewed, and edited the manuscript. Both authors prepared the original draft and read and agreed to the published version of the manuscript.

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Retraction

Retracted: Analysis of the Shortest Path in Spherical Fuzzy Networks Using the Novel Dijkstra Algorithm

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] Z. Ullah, H. Bashir, R. Anjum, S. A. AlQahtani, S. Al-Hadhrani, and A. Ghaffar, "Analysis of the Shortest Path in Spherical Fuzzy Networks Using the Novel Dijkstra Algorithm," *Mathematical Problems in Engineering*, vol. 2021, Article ID 7946936, 15 pages, 2021.

Research Article

Analysis of the Shortest Path in Spherical Fuzzy Networks Using the Novel Dijkstra Algorithm

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The concept of fuzzy graph (FG) and its generalized forms has been developed to cope with several real-life problems having some sort of imprecision like networking problems, decision making, shortest path problems, and so on. This paper is based on some developments in generalization of FG theory to deal with situation where imprecision is characterized by four types of membership grades. A novel concept of T-spherical fuzzy graph (TSFG) is proposed as a common generalization of FG, intuitionistic fuzzy graph (IFG), and picture fuzzy graph (PFG) based on the recently introduced concept of T-spherical fuzzy set (TSFS). The significance and novelty of proposed concept is elaborated with the help of some examples, graphical analysis, and results. Some graph theoretic terms are defined and their properties are studied. Specially, the famous Dijkstra algorithm is proposed in the environment of TSFGs and is applied to solve a shortest path problem. The comparative analysis of the proposed concept and existing theory is made. In addition, the advantages of the proposed work are discussed over the existing tools.

1. Introduction

In the past decades, the development of graph theory, specifically the fuzzy graph (FG) theory, and its applications in numerous scientific subjects indicates its significance. The addition of FGs in graph theory is of worth as it increases the viability of graph theory. From application point of view, FGs have been widely utilized in practical problems, for example, reference [1] provided a list of possible regions handled by FGs and fuzzy hypergraphs, reference [2] modelled some traffic problems using FGs, reference [3] utilized FGs in optimization of networks, reference [4] is based on application of telecommunication system in FGs, and reference [5] applied FGs in fuzzy neural networks. The

theory of FGs has been initiated in [6] but briefly elaborated in [7] by Rosenfield after the remarkable work of Zadeh [8] on fuzzy sets (FSs). For some works on FGs, one may refer to [9–16].

An FS only described the membership grade of an event/object while the non-membership grade is obtained by subtracting the membership grade from 1, i.e., the non-membership grade could not be chosen independently. Therefore, Atanassov [17] developed the theory of intuitionistic fuzzy set (IFS) as an advanced form of FSs and provided an opening for the theory of IFGs which was proposed in [18]. Atanassov's tool of IFSs gave strength to Zadeh's FSs, and in the same way, theory of IFG generalizes FGs and makes it more valuable. For some quality work on

IFGs and its applications, one may refer to [19–23]. IFs could not model human opinion properly as described in [24, 25], and hence a new tool of picture fuzzy set (PFS) was introduced describing not only yes or no type situations but also situations having some abstinence or refusal grade involved like in voting situation. PFS strongly generalizes FSs and IFs, and some useful work in this direction could be found in [26–30]. The idea of PFGs was developed in [31] generalizing the FGs and IFGs.

If we observe the structure of PFSs, it is clear that they generalize the FSs and IFs. They know how to handle the situations or data that FSs or IFs might not. But the structure of PFS has some limitations. Its constraint on the membership, abstinence, and non-membership grades states that their sum must be less than or equal to one. Due to this formation of PFSs, one is unable to assign the values to these membership, abstinence, and non-membership functions by their own choice. Keeping this issue in mind, Mahmood et al. [32] proposed the concept of spherical fuzzy sets (SFSs) and consequently T-spherical fuzzy sets (TSFSs), which improves the construction of PFS and does not have limitations at all. Such type of framework of TSFSs not only models human opinion other than yes or no but also can deal with any form of data without any limitations. For example, if we look at the constraint of PFSs and TSFSs, then it becomes very much clear that the framework of TSFSs has no limitations. The constraints of IFs, PFSs, and TSFSs are as follows:

- (i) For IFs $A = \{x_i, (s(x_i), d(x_i))\}$, we have $0 \leq s(x_i) + d(x_i) \leq 1$.
- (ii) For PFSs $A = \{x_i, (s(x_i), i(x_i), d(x_i))\}$, we have $0 \leq s(x_i) + i(x_i) + d(x_i) \leq 1$.
- (iii) For TSFSs $A = \{x_i, (s(x_i), i(x_i), d(x_i))\}$, we have $0 \leq s^n(x_i) + i^n(x_i) + d^n(x_i) \leq 1$ for some $n \in \mathbb{Z}^+$.

The diverse structure and novelty of TSFSs is clear from its constraints and comparison with existing structures. Further diversity of proposed structure is discussed in Section 2 with the aid of some pictorial representations in Figures 1–5.

The problem of the shortest path is one of the well-known problems that has been discussed prominently in various extended structures of FSs. Okada and Soper [33] worked out the shortest path problems utilizing fuzzy arcs, and Deng et al. [34] presented the Dijkstra algorithm that is the technique for finding out the shortest path. References [35–37] provide some good work on fuzzy shortest path problems. Gani and Jabarulla [38] also studied the shortest paths in the environment of IFs, and for details on finding out the shortest path in an IFG using Dijkstra algorithm, see [39]. Plenty of works have been carried out on the topic of shortest path problems (one may refer to [40–43]).

As discussed, the framework of TSFS is more generalized than FS, IFs, and PFS. Therefore, the graph of TSFS could be more useful in dealing with uncertain situations. Keeping in view the developments in FG, IFG, and PFG and their several real-life applications, the aim of this study is to propose the graphs of TSFSs named as TSFGs. The TSFG

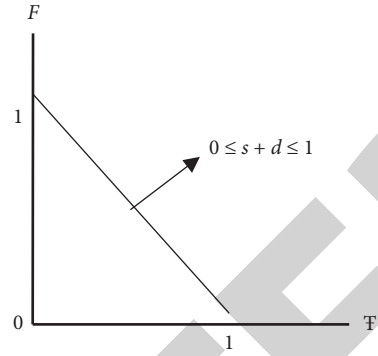


FIGURE 1: Intuitionistic fuzzy space.

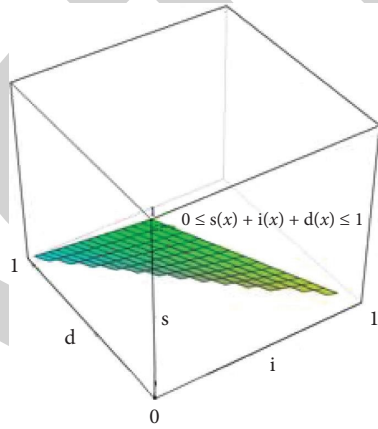


FIGURE 2: Pictorial fuzzy space.

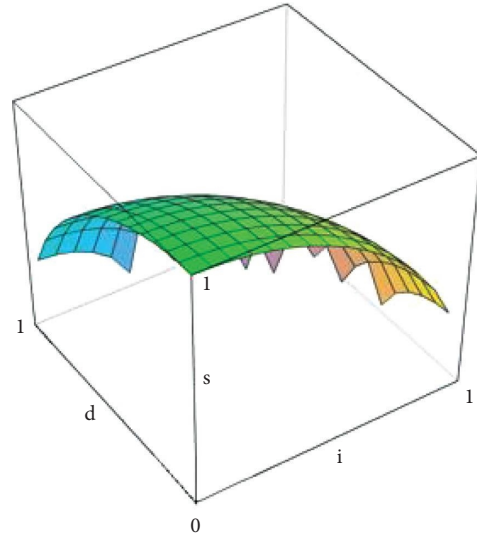
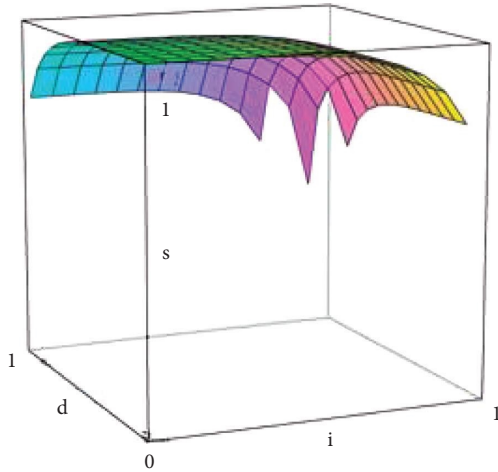
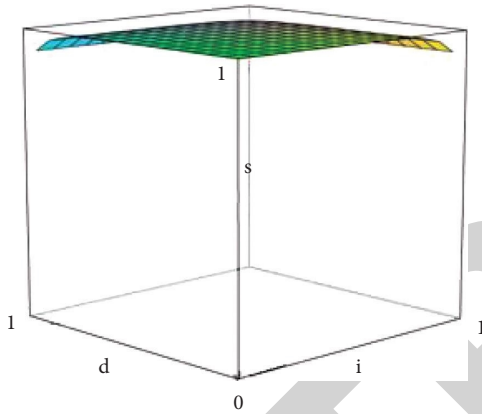


FIGURE 3: Space of spherical fuzzy sets [32].

generalizes the FG, IFG, and PFG. It discusses the membership, abstinence, and non-membership grades of an entity. Moreover, there is complete freedom for a decision maker; they can assign any fuzzy number as the membership, abstinence, or non-membership grades. Unlike PFGs, there are no limitations in the structure of TSFGs. The TSFG

FIGURE 4: TSFS for $n = 5$.FIGURE 5: TSFS for $n = 10$.

can handle any problem that its predecessors could handle. It is the most powerful modelling tool among all the existing tools. The new structure of TSFG is investigated and some related terms are defined. The terms subgraph, complement, degree, and strength are defined for TSFGs and supported with examples. Several operations are also defined for TSFG, and some examples are discussed to support the defined concepts. To discuss the diversity and significance of TSFGs, a shortest path problem in the environment of TSFSs and TSFGs is also studied.

This paper is organized as follows. Section 1 is based on some history and motivation for proposing TSFGs. In Section 2, the basic definitions of IFS, PFS, SFS, TSFS, IFG, PFG, and the novelty of the proposed idea are discussed with the help of geometrical shapes. Also, in this section, TSFGs are proposed along with some basic graph theoretic terms like complement of TSFGs, order, degree and size of TSFGs and subgraphs of TSFGs, and the results of proposed notions are studied. In Section 3, operations of join and union are defined for TSFGs along with Cartesian product and composition of TSFGs. In Section 4, we propose a modified Dijkstra algorithm for the TSF shortest path that is then applied to find out the shortest path in a network.

Furthermore, a comparative study is provided. In Section 5, we discussed the summary of our work along with its advantages and some future directions.

2. Preliminaries

In this section, the basic definitions of IFSs, PFSs, and TSFSs are reviewed and their spaces are geometrically described. Some elementary definitions of graphs of IFS and PFS are also discussed and explained with the help of some examples.

Definition 1 (see [17]). Let X be a universal set. An IFS on X is characterized by two mappings \bar{F} and F on $[0, 1]$ given that $0 \leq s(x) + d(x) \leq 1$. The values of s and d in the unit interval described the grade of membership and grade of non-membership of an element x in X . Also, $1 - (s(x) + d(x))$ denotes the hesitancy of $x \in X$. Moreover, the duplet (s, d) is said to be an intuitionistic fuzzy number (IFN). The range of the IFNs is portrayed in Figure 1.

In the voting situations, we might end up with four types of statuses, i.e., vote against, vote in favour, refusal, and abstain (nor in favour nor against). IFSs cannot cope with issues like this. Realizing this, a novel concept of PFS was developed by B. C. Cuong in 2013.

Definition 2 (see [25]). Let X be a universal set. A PFS on X is characterized by three mappings s , i , and d on $[0, 1]$ provided that $0 \leq s(x) + i(x) + d(x) \leq 1$. The value of s , i , and d in the interval $[0, 1]$ describes the membership, abstinence, and non-membership grades of x in X . Also, $1 - (s(x) + i(x) + d(x))$ denoted the refusal grade of $x \in X$. The triplet (s, i, d) is called the picture fuzzy number (PFN). The space of PFNs is depicted in Figure 2.

The problem with the framework of PFS is its check on the grade mappings, as depicted in Figure 2. Realizing this concern, Tahir et al. [32] proposed SFSs and consequently TSFSs. The definition of TSFSs is described below. Moreover, in order to make the point clear that TSFSs generalize IFSs and PFSs, a pictorial representation is given.

Definition 3 (see [32]). An SFS on X (a universal set) consists of three mappings s , i , and d on $[0, 1]$ provided that $0 \leq s^2(x) + i^2(x) + d^2(x) \leq 1$. The values of s , i , and d in the interval $[0, 1]$ describe the membership, abstinence, and non-membership grades of x in X . Also, the refusal grade of $x \in X$ is denoted by $r(x) = \sqrt{1 - (s^2(x) + i^2(x) + d^2(x))}$. The triplet (s, i, d) is called a spherical fuzzy number (SFN).

Definition 4 (see [32]). A TSFS on X (a universal set) consists of three mappings s , i , and d on $[0, 1]$ provided that $0 \leq s^n(x) + i^n(x) + d^n(x) \leq 1$ for some $n \in \mathbb{Z}$. The values of s , i , and d in the interval $[0, 1]$ describe the grade of membership, grade of abstinence, and grade of non-membership of x in X . Also, the refusal grade of $x \in X$ is denoted by $r(x) = \sqrt[n]{1 - (s^n(x) + i^n(x) + d^n(x))}$. The triplet (s, i, d) is called a T-spherical fuzzy number (TSFN).

The following figures described SFSs and TSFSs geometrically presenting their innovation and diverse structure. Figures 3–5 also show that TSFSs have no limitation.

From all the observations in this section, we conclude that the concept of TSFSs is the generalization of FSs, IFs, PFSs, and SFs and their structure does not have any limitations. Now, some elementary definitions associated with the graphs of IFs and PFS are discussed providing a base for the proposed work.

Definition 5 (see [25]). A pair $\mathcal{G} = (N, \mathbb{E})$ is known as IFG if

- (i) $N = \{x_1, x_2, x_3, x_4, \dots, x_n\}$ is the collection of vertices such that $s_1: N \rightarrow [0, 1]$ and $d_1: N \rightarrow [0, 1]$ denote the grade of membership and grade of non-membership of the element $x_i \in N$, respectively, with the condition that $0 \leq s_1 + d_1 \leq 1$ for all $x_i \in N$, ($i \in I$).
- (ii) $\check{\mathbb{E}} \subseteq N \times N$ where $s_2: N \times N \rightarrow [0, 1]$ and $d_2: N \times N \rightarrow [0, 1]$ denote the grade of membership and grade of non-membership of the element $(x_i, x_j) \in \check{\mathbb{E}}$ such that $s_2(x_i, x_j) \leq \min\{s_1(x_i), s_1(x_j)\}$ and $d_2(x_i, x_j) \leq \max\{d_1(x_i), d_1(x_j)\}$ with the condition $0 \leq s_2(x_i, x_j) + d_2(x_i, x_j) \leq 1$ for all $(x_i, x_j) \in \check{\mathbb{E}}$, ($i \in I$).

Example 1. Figure 6 is an example of IFG.

Definition 6 (see [31]). A pair $\mathcal{G} = (N, \mathbb{E})$ is said to be a PFG if

- (i) $N = \{x_1, x_2, x_3, x_4, \dots, x_n\}$ is the set of vertices such that $s_1: N \rightarrow [0, 1]$ describes the grade of membership, $i_1: N \rightarrow [0, 1]$ describes the grade of abstinence, and $d_1: N \rightarrow [0, 1]$ describes the grade of non-membership of the element $x_i \in N$ on the condition that $0 \leq s_1(x_i) + i_1(x_i) + d_1(x_i) \leq 1$ for all $x_i \in N$, ($i \in I$), and $1 - (s_{1i} + i_{1i} + d_{1i})$ is known as refusal grade of x in N .
- (ii) $\check{\mathbb{E}} \subseteq N \times N$ where $s_2: N \times N \rightarrow [0, 1]$ describes the grade of membership, $i_2: N \times N \rightarrow [0, 1]$ describes the grade of abstinence, and $d_2: N \times N \rightarrow [0, 1]$ describes the grade of non-membership of the element $(x_i, x_j) \in \check{\mathbb{E}}$ such that $s_2(x_i, x_j) \leq \min\{s_1(x_i), s_1(x_j)\}$, $i_2(x_i, x_j) \leq \min\{i_1(x_i), i_1(x_j)\}$ and $d_2(x_i, x_j) \leq \max\{d_1(x_i), d_1(x_j)\}$ with the condition $0 \leq s_2(x_i, x_j) + i_2(x_i, x_j) + d_2(x_i, x_j) \leq 1$ for all $(x_i, x_j) \in \check{\mathbb{E}}$, ($i \in I$), and $1 - s_2(x_i, x_j) - i_2(x_i, x_j) - d_2(x_i, x_j)$ is known as refusal grade of (x_i, x_j) in $\check{\mathbb{E}}$.

Example 2. Let $\mathcal{G} = (N, \mathbb{E})$ represent a graph with the collection of vertices N and the collection of edges \mathbb{E} . Figure 7 is an example of PFG.

2.1. T-Spherical Fuzzy Graphs

Definition 7 (see [43]). A pair $\mathcal{G} = (N, \mathbb{E})$ is said to be TSFG if

- (i) $N = \{x_1, x_2, x_3, \dots, x_n\}$ is the set of vertices such that $s_1: N \rightarrow [0, 1]$ describes the grade of membership, $i_1: N \rightarrow [0, 1]$ describes the grade of abstinence, and $d_1: N \rightarrow [0, 1]$ describes the grade of non-membership of the element $x_i \in N$ on the condition that for some positive integers n $0 \leq s_1^n(x_i) + i_1^n(x_i) + d_1^n(x_i) \leq 1$ for all $x_i \in N$ ($i \in I$), and $\sqrt[n]{1 - (s_1^n(x_i) + i_1^n(x_i) + d_1^n(x_i))}$ is known as refusal grade of x in N .
- (ii) $\mathbb{E} \subseteq N \times N$ where $s_2: N \times N \rightarrow [0, 1]$, $i_2: N \times N \rightarrow [0, 1]$ and $d_2: N \times N \rightarrow [0, 1]$ describes the grades of membership, abstinence, and non-membership of the element $(x_i, x_j) \in \mathbb{E}$ such that $s_2(x_i, x_j) \leq \min\{s_1(x_i), s_1(x_j)\}$, $i_2(x_i, x_j) \leq \min\{i_1(x_i), i_1(x_j)\}$ and $d_2(x_i, x_j) \leq \max\{d_1(x_i), d_1(x_j)\}$ with the condition $0 \leq s_2^n(x_i, x_j) + i_2^n(x_i, x_j) + d_2^n(x_i, x_j) \leq 1$ for all $(x_i, x_j) \in \mathbb{E}$, and $\sqrt[n]{1 - (s_2^n(x_i, x_j) + i_2^n(x_i, x_j) + d_2^n(x_i, x_j))}$ is known as refusal grade of (x_i, x_j) in \mathbb{E} .

Example 3. Let $\check{\mathcal{G}} = (N, \check{\mathbb{E}})$ represent a graph with the collection of vertices N and the collection of edges $\check{\mathbb{E}}$.

The vertices shown in Figures 8 and 9 are purely T-spherical fuzzy numbers (TSFNs) for $n = 5$.

Remark 1. PFG and SFG are TSFGs, but generally, the converse is not true.

Example 4. The graph in Figure 8 is clearly TSFG, but it is neither PFG nor SFG. Consider $(0.8, 0.9, 0.8)$; then, $0.8 + 0.9 + 0.8 = 2.5 \not\leq 1$ and $0.8^2 + 0.9^2 + 0.8^2 = 2.09 \not\leq 1$.

Definition 8 (see [43]). For TSFNs $A = \{s_A, i_A, d_A\}$ and $B = \{s_B, i_B, d_B\}$, we define

$$A \oplus B = \left\{ \left\{ t, \left(\sqrt[n]{s_A^n(x) + s_B^n(x) - s_A^n(x) \cdot s_B^n(x)}, \sqrt[n]{i_A^n(x) + i_B^n(x) - i_A^n(x) \cdot i_B^n(x)} \right) \right\} \right\}_{i \in A \cdot d_B}, \quad (1)$$

$$A \otimes B = \left\{ \left\{ x, \left((s_A(x) \cdot s_B(x)), i(i_A(x) \cdot i_B(x)), i \sqrt[n]{d_A^n(x) + d_B^n(x) - d_A^n(x) \cdot d_B^n(x)} \right) \right\} \right\}.$$

In FS theory, the rules of comparison have always been a challenge. For IFs, several score functions have been

established regularly. These score functions fall under the title of comparison rules. A better score function (SF) for

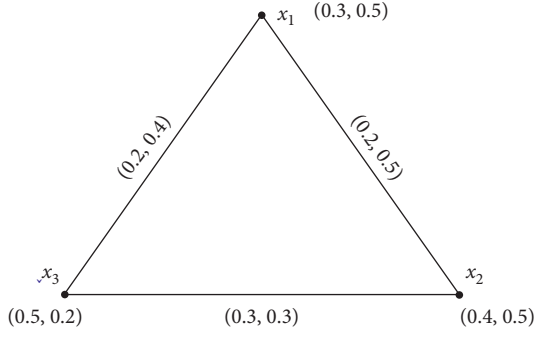


FIGURE 6: Intuitionistic fuzzy graph.

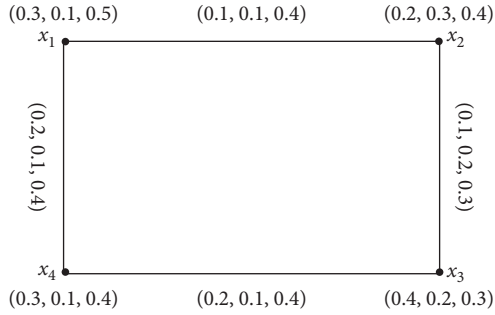


FIGURE 7: Picture fuzzy graph.

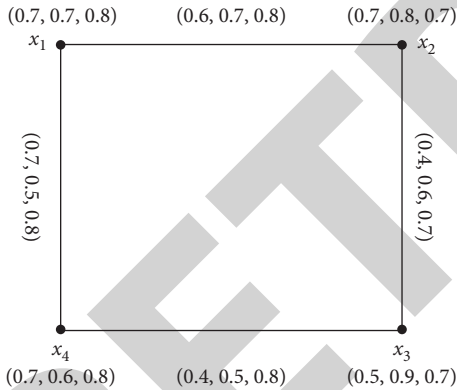


FIGURE 8: T-spherical fuzzy graph.

IFSs is established in [39] and it discusses the limitations of existing score functions which is demonstrated using examples. Further, work done on PFSs is significantly less; hence, in literature, there does not exist any SFs. Therefore, this article establishes a novel SF as a generalized SF proposed in [39]. In Section 4, this SF shall be utilized in the problems of the shortest path.

Definition 9 (see [43]). The SF for a TSFN $A = (s, i, d)$ is defined as

$$SC(A) = \frac{(s)^n (1 - (i)^n - (d)^n)}{3}, \quad SC(A) \in [0, 1]. \quad (2)$$

Remark 2. Replacing $i = 0$ and $n = 1$ reduces the defined score function in the environment of IFSs.

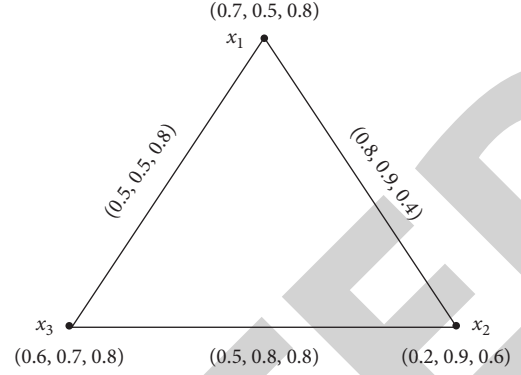


FIGURE 9: An example of not T-spherical fuzzy graph.

Definition 10 (see [43]). A pair $H = (N^*, \mathbb{E}^*)$ is said to be T-spherical fuzzy subgraph (TSFSG) of TSFG $G = (N, \mathbb{E})$ if $N^* \subseteq N$ and $\mathbb{E}^* \subseteq \mathbb{E}$, that is, $s_{1i}^* \leq s_{1i}$, $i_{1i}^* \leq i_{1i}$, $d_{1i}^* \geq d_{1i}$ and $s_{2ij}^* \leq s_{2ij}$, $i_{2ij}^* \leq i_{2ij}$, $d_{2ij}^* \geq d_{2ij}$ for all $i, j = 1, 2, \dots, n$.

Definition 11 (see [43]). The complement of a TSFG $G = (N, \mathbb{E})$ is defined as

- (i) $\overline{N} = N$.
- (ii) $\overline{s}_i = s_i$, $\overline{i}_i = i_i$ and $\overline{d}_i = d_i$ for $i = 1, 2, \dots, n$.
- (iii) $\overline{s}_{2ij} = \min(s_i, s_j) - s_{2ij}$, $\overline{i}_{2ij} = \min(i_i, i_j) - s_{2ij}$ and $\overline{d}_{2ij} = \max(d_i, d_j) - d_{2ij}$ for any $i, j = 1, 2, \dots, n$.

Example 5. Figures 10 and 11 are examples of complement of TSFG.

The vertices are purely TSFNs for $n = 3$ in Figures 10 and 11.

Definition 12 (see [43]). The degree of a TSFG $G = (N, \mathbb{E})$ is denoted and is defined by $\tilde{d}(x) = (\tilde{d}_s(x), \tilde{d}_i(x), \tilde{d}_d(x))$, where $\tilde{d}_s(x) = \sum_{y \neq x} s_2(x, y)i$, $\tilde{d}_i(x) = \sum_{y \neq x} i_2(x, y)i$ and $\tilde{d}_d(x) = \sum_{y \neq x} \tilde{d}_2(x, y)i$ for $i, x, y \in N$.

Example 6. Let $G = (N, \mathbb{E})$ represent a graph with the collection of vertices N and the collection of edges \mathbb{E} .

The vertices are purely TSFNs for $n = 4$ in Figure 12.

The degree of vertices shown in Figure 12 is

$$\begin{aligned} \tilde{d}(x_1) &= (0.9, 1.1, 1.3), \\ \tilde{d}(x_2) &= (1, 1, 1.1), \\ \tilde{d}(x_3) &= (0.8, 0.8, 1.4), \\ \tilde{d}(x_4) &= (0.7, 0.9, 1.6). \end{aligned} \quad (3)$$

Definition 13 (see [43]). A pair $G = (N, \mathbb{E})$ is said to be strong TSFG if

- (i) $N = \{x_1, x_2, x_3, \dots, x_n\}$ is the set of vertices such that $s: N \rightarrow [0, 1]$ denotes the grade of membership, $i: N \rightarrow [0, 1]$ denotes the grade of abstinance, and $d: N \rightarrow [0, 1]$ represents the grade of non-membership of the element $x_i \in N$ with the

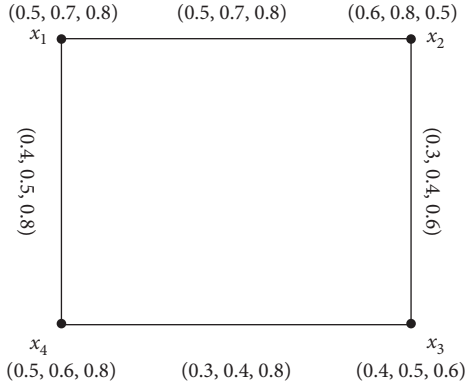
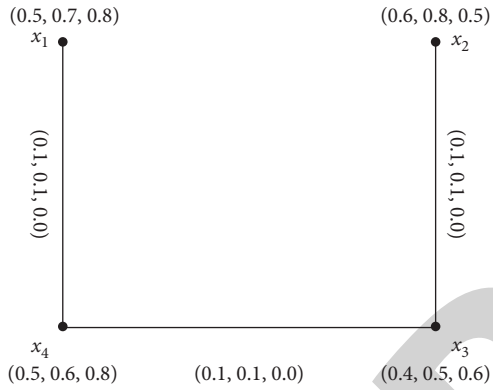
FIGURE 10: T-spherical fuzzy graph for $n = 3$.

FIGURE 11: Complement of Figure 10.

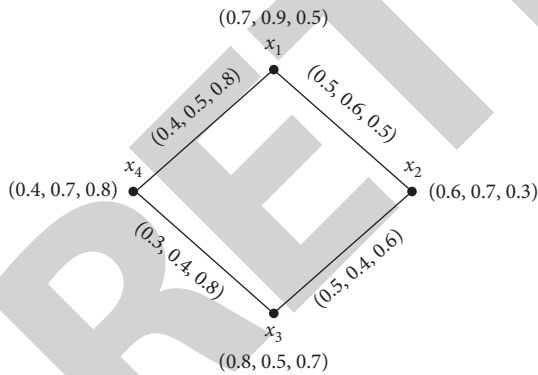


FIGURE 12: For the degree of TSFG.

condition that for some positive integers n $0 \leq s_1^n(x_i) + i_1^n(x_i) + d_1^n(x_i) \leq 1$ for all $x_i \in N$ ($i \in I$), and $\sqrt[n]{1 - (s_1^n(x_i) + i_1^n(x_i) + d_1^n(x_i))}$ is known as refusal grade of x in N .

- (ii) $\mathbb{E} \subseteq N \times N$ where $s_2: N \times N \rightarrow [0, 1]$ denotes the grade of membership, $i_2: N \times N \rightarrow [0, 1]$ describes the grade of abstention, and $d_2: N \times N \rightarrow [0, 1]$ represents the grade of non-membership of the element $(x_i, x_j) \in \mathbb{E}$ such that $s_2(x_i, x_j) = \min\{s_1(x_i), s_1(x_j)\}$, $i_2(x_i, x_j) = \min\{i_1(x_i), i_1(x_j)\}$ and $d_2(x_i,$

$x_j) = \max\{d_1(x_i), d_1(x_j)\}$ with the condition $0 \leq s_2^n(x_i, x_j) + i_2^n(x_i, x_j) + d_2^n(x_i, x_j) \leq 1$ for all $(x_i, x_j) \in \mathbb{E}$, and $\sqrt[n]{1 - (s_2^n(x_i, x_j) + i_2^n(x_i, x_j) + d_2^n(x_i, x_j))}$ is known as refusal grade of (x_i, x_j) in \mathbb{E} .

Example 7. Figure 13 is an example of strong TSFG.

The vertices are purely TSFNs for $n = 4$ in Figure 13.

Definition 14 (see [43]). An edge (x_i, x_j) in a TSFG $\mathcal{G} = (N, \mathbb{E})$ is known to be a bridge, if by removal of that edge decreases the strength of the connectedness among any pair of vertices in \mathcal{G} .

Example 8. Let $\mathcal{G} = (N, \mathbb{E})$ represent a graph with the collection of vertices N and the collection of edges \mathbb{E} .

Here, (x_1, x_4) is a bridge.

The vertices shown in Figure 14 are purely TSFNs for $n = 5$.

Definition 15 (see [43]). A vertex x_i in a TSFG $\mathcal{G} = (N, \mathbb{E})$ is known to be cut vertex, if the removal of that vertex decreases the strength of the connectedness among any pair of vertices.

Example 9. Let $\mathcal{G} = (N, \mathbb{E})$ represent a graph with the collection of vertices N and the collection of edges \mathbb{E} .

Here, x_1 is a cut vertex.

The vertices are purely TSFNs for $n = 3$ in Figure 15.

3. Operations on T-Spherical Fuzzy Graphs

In this section, the operations on T-spherical fuzzy graph are defined and their results are studied.

Definition 18. The union of a TSFG $\mathcal{G}_1 = (N_1, \mathbb{E}_1)$ and $\mathcal{G}_2 = (N_2, \mathbb{E}_2)$ with $N_1 \cap N_2 = \emptyset$ and $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 = (N_1 \cup N_2, \mathbb{E}_1 \cup \mathbb{E}_2)$ is defined by

$$(s_1 \cup s_1')(x) = \begin{cases} s_1(x) & \text{if } x \in N_1 - N_2 \\ s_1'(x) & \text{if } x \in N_2 - N_1 \end{cases},$$

$$(i_1 \cup i_1')(x) = \begin{cases} i_1(x) & \text{if } x \in N_1 - N_2 \\ i_1'(x) & \text{if } x \in N_2 - N_1 \end{cases},$$

$$(d_1 \cup d_1')(x) = \begin{cases} d_1(x) & \text{if } x \in N_1 - N_2 \\ d_1'(x) & \text{if } x \in N_2 - N_1 \end{cases},$$

$$(s_2 \cup s_2')(x_i, x_j) = \begin{cases} s_{2ij} & \text{if } \hat{e}_{ij} \in \mathbb{E}_1 - \mathbb{E}_2 \\ s_{2ij}' & \text{if } \hat{e}_{ij} \in \mathbb{E}_2 - \mathbb{E}_1 \end{cases},$$

$$(i_2 \cup i_2')(x_i, x_j) = \begin{cases} i_{2ij} & \text{if } \hat{e}_{ij} \in \mathbb{E}_1 - \mathbb{E}_2 \\ i_{2ij}' & \text{if } \hat{e}_{ij} \in \mathbb{E}_2 - \mathbb{E}_1 \end{cases},$$

$$(d_2 \cup d_2')(x_i, x_j) = \begin{cases} d_{2ij} & \text{if } \hat{e}_{ij} \in \mathbb{E}_1 - \mathbb{E}_2 \\ d_{2ij}' & \text{if } \hat{e}_{ij} \in \mathbb{E}_2 - \mathbb{E}_1 \end{cases},$$

(4)

where (s_1, i_1, d_1) and (s_1', i_1', d_1') represent the vertices of truth membership, abstention membership, and false membership of \mathcal{G}_1 and \mathcal{G}_2 , respectively, and (s_2, i_2, d_2) and (s_2', i_2', d_2') represent the edges of truth, abstention, and false memberships \mathcal{G}_1 and \mathcal{G}_2 , respectively.

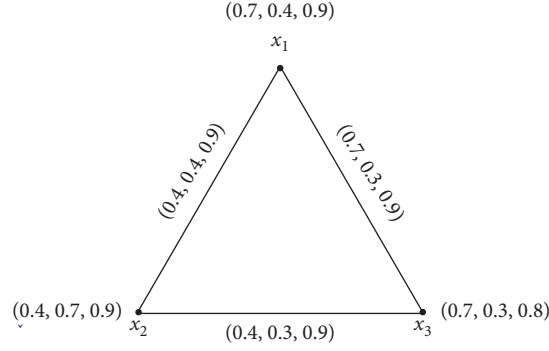


FIGURE 13: Strong T-spherical fuzzy graph.

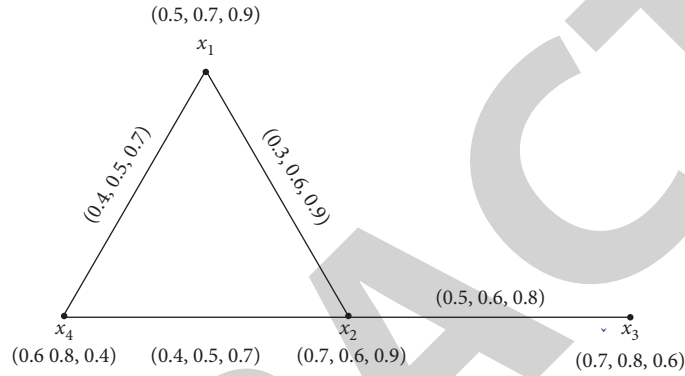


FIGURE 14: TSFG for a bridge.

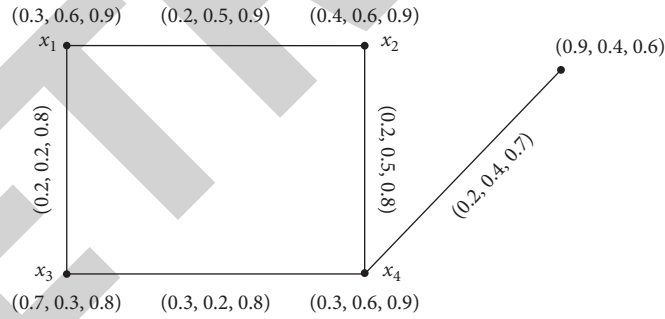


FIGURE 15: T-spherical fuzzy graph for cut vertex.

Example 10. Let $\mathcal{G} = (N, \mathbb{E})$ represent a graph with the collection of vertices N and the collection of edges \mathbb{E} . Figures 16–18 are examples of the union of two TSFGs.

The vertices are purely TSFNs for $n = 5$ in Figures 16–18.

Definition 19. Let \mathcal{G}_1 and \mathcal{G}_2 be two TSFGs. Then, the join of \mathcal{G}_1 and \mathcal{G}_2 is a TSFG, $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 = (N_1 \cup N_2, \mathbb{E}_1 \cup \mathbb{E}_2 \cup \mathbb{E}')$ defined by: $(s_1 + s'_1)(x) = (s_1 \cup s'_1)(x)$ if $x \in N_1 \cup N_2$, $(i_1 + i'_1)(x) = (i_1 \cup i'_1)(x)$ if $x \in N_1 \cup N_2$, $(d_1 + d'_1)(x) = (d_1 \cup d'_1)(x)$ if $x \in N_1 \cup N_2$ and $(s_2 + s'_2)(x_i x_j) = (s_1 \cup s'_1)(x_i x_j)$ if $x \in \mathbb{E}_1 \cup \mathbb{E}_2 = \min(s_1(x_i), s'_1(x_j))$ if $x_i x_j \in \mathbb{E}'$, and $(i_2 + i'_2)(x_i x_j) = (i_1 \cup i'_1)(x_i x_j)$ if $x \in \mathbb{E}_1 \cup \mathbb{E}_2 = \min(i_1(x_i), i'_1(x_j))$ if $x_i x_j \in \mathbb{E}'$ and $(d_2 + d'_2)(x_i x_j) = (d_1 \cup d'_1)(x_i x_j)$ if $x \in \mathbb{E}_1 \cup \mathbb{E}_2 = \max(d_1(x_i), d'_1(x_j))$ if $x_i x_j \in \mathbb{E}'$.

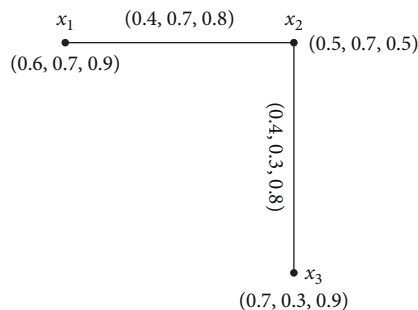
Theorem 4. If $\mathcal{G}_1 = (N_1, \mathbb{E}_1)$ and $\mathcal{G}_2 = (N_2, \mathbb{E}_2)$ are two TSFGs, then

- (i) $\overline{\mathcal{G}_1 + \mathcal{G}_2} \cong \overline{\mathcal{G}_1} \cup \overline{\mathcal{G}_2}$
- (ii) $\overline{\mathcal{G}_1} \cup \overline{\mathcal{G}_2} \cong \overline{\mathcal{G}_1} + \overline{\mathcal{G}_2}$

Proof. Let $I: N_1 \cup N_2 \rightarrow N_1 \cup N_2$ be the identity map. The following steps are calculated to prove (i)

- (a) $\overline{(s_1 + s'_1)}(x_i) = \overline{s_1} \cup \overline{s'_1}(x_i)$, $\overline{(i_1 + i'_1)}(x_i) = \overline{i_1} \cup \overline{i'_1}(x_i)$, $\overline{(d_1 + d'_1)}(x_i) = \overline{d_1} \cup \overline{d'_1}(x_i)$.
- (b) $\overline{(s_2 + s'_2)}(x_i, x_j) = \overline{s_2} \cup \overline{s'_2}(x_i, x_j)$, $\overline{(i_2 + i'_2)}(x_i, x_j) = \overline{i_2} \cup \overline{i'_2}(x_i, x_j)$, $\overline{(d_2 + d'_2)}(x_i, x_j) = \overline{d_2} \cup \overline{d'_2}(x_i, x_j)$.

Now to prove (a).



A triangle representing a game with three players. The vertices are labeled x_1 , x_2 , and x_3 . The edges are labeled with probability vectors: $(0.7, 0.5, 0.7)$ for the edge x_1x_2 , $(0.4, 0.7, 0.8)$ for the edge x_2x_3 , and $(0.3, 0.5, 0.8)$ for the edge x_3x_1 .

$$\begin{aligned}
\text{(i)} \quad & \overline{(s_1 + s'_1)}(x_i) = (s_1 + s'_1)(x_i), \text{ by definition } = \{s_1 \\
& (x_i) \text{ if } (x_i) \in N_1 s'_1(x_i) \text{ if } (x_i) \in N_2\} i = \{ \overline{s_1}(x_i) \\
& \text{if } (x_i) \in N_1 \overline{s'_1}(x_i) \text{ if } i(x_i) \in N_2 \} = (\overline{s_1} \cup \overline{s'_1})(x_i) \\
& = \overline{i(i_1 + i'_1)}, (x_i) = (i_1 + i'_1) \quad (x_i) = \{i_1(x_i) \text{ if } (x_i) \in \\
& N_1 i'_1(x_i) \text{ if } (x_i) \in N_2\} = \{ \overline{i_1}(\mathbf{y}_i) \text{ if } (x_i) \in N_1 \} \\
& = \{ \overline{i_1} \cup \overline{i'_1} \} i(x_i), \overline{(d_2 + d'_2)}(x_i) = (d_2 + d'_2)(x_i) = \{d_2(\mathbf{y}_i) \text{ if } (x_i) \\
& \in N_1 d'_1(\mathbf{y}_i) \text{ if } (x_i) \in N_2\} i = \{ \overline{d_1} \} (x_i) \text{ if } (x_i) \in \\
& N_1 \overline{d'_1} i(x_i) \text{ if } (x_i) \in N_2 = (\overline{d_1} \cup \overline{d'_1})(x_i) \\
\text{(ii)} \quad & \overline{i(s_2 + s'_2)} i(x_i, x_j) = \min i((s_1 + s'_1)(x_i), i(s_1 + s'_1) \\
& (x_j) i) - i(s_1 + s'_1)(x_i, x_j) = \{ \min((s_1 + s'_1)(x_i), (s_1 \\
& + s'_1)(x_j) - ((s_2 \cup s'_2)(x_i, x_j) \text{ if } (x_i, x_j) \in E_1 \cup \\
& E_2 \min((s_1 \cup s'_1)(x_i), i(s_1 \cup s'_1)(x_j)) - \min(s_1 \\
& (x_i), s'_1(x_j)) \text{ if } (x_i, x_j) \in E \} = \{ \min(s_1(x_i), s'_1(x_j) - \\
& s_2(x_i, x_j) \text{ if } (x_i, x_j) \in E_1 \min(s_1(x_i), s'_1(x_j) - s_2 \\
& (x_i, x_j) \text{ if } (x_i, x_j) \in E_2 \min(s_1(x_i), s'_1(x_j)) - \min \\
& (s_1(x_i), s'_1(x_j) \text{ if } i(x_i, x_j) \in E \} = \{ \overline{s_2}(x_i, x_j)
\end{aligned}$$
$$\begin{aligned}
\text{(a)} \quad & \overline{(s_1 \cup s'_1)}(x_i) = (s_1 \cup s'_1)(x_i), \quad \text{by definition} = \\
& \left\{ \begin{array}{l} s_1(x_i) \text{ if } i(x_i) \in N_1 \\ s'_1(x_i) \text{ if } i(x_i) \in N_2 \end{array} \right\} \quad i = \left\{ \begin{array}{l} \overline{s_1}(x_i) \text{ if } i(x_i) \in N_1 \\ \overline{s'_1}(x_i) \text{ if } i(x_i) \in N_2 \end{array} \right\} \\
i = & \overline{(s_1 + s'_1)}(x_i), \quad i(i_1 \cup i'_1)(x_i) = (i_1 \cup i'_1)(x_i) = \\
& \left\{ \begin{array}{l} i_1(x_i) \text{ if } (x_i) \in N_1 \\ i'_1(x_i) \text{ if } (x_i) \in N_2 \end{array} \right\} i = \left\{ \begin{array}{l} \overline{i_1}(\mathcal{V}_i) \text{ if } (x_i) \in N_1 \\ \overline{i'_1}(\mathcal{V}_i) \text{ if } (x_i) \in N_2 \end{array} \right\} = \\
& (\overline{i_1} + \overline{i'_1})(x_i), \quad \overline{(d_2 \cup d'_2)}(x_i) = (d_2 \cup d'_2)(x_i) = i\{d_1 \\
& (\mathcal{V}_i) \text{ if } i(x_i) \in N_1, d'_1(\mathcal{V}_i) \text{ if } i(x_i) \in N_2\} i = \{\overline{d_1} \quad (x_i) \\
\text{if } (x_i) \in N_1, \overline{d'_1}(x_i) \text{ if } (x_i) \in N_2\} \quad (\overline{d_1} + \overline{d'_1})(x_i).
\end{aligned}$$

(a) $(s_1 \circ s'_1)(u_1, u_2) = \min(s_1(u_1), s'_1(u_2))$ for every $u_1, u_2 \in N_1 \times N_2$, $(i_1 \circ i'_1)(u_1, u_2) = \min(i_1(u_1), i'_1(u_2))$ for every $u_1, u_2 \in N_1 \times N_2$, and $(d_1 \circ d'_1)(u_1, u_2) = \min(d_1(u_1), d'_1(u_2))$ for every $u_1, u_2 \in N_1 \times N_2$.

(b) $(s_2 \circ s'_2)(u, u_2)(u, x_2) = \min(s_1(u), s_2(u, x_2))$ for every $u \in N_1$, and $u_2, x_2 \in \mathbb{E}_2$, $(i_2 \circ i'_2)(u, u_2)(u, x_2) = \min(i_1(u), i_2(u, x_2))$ for every $u \in N_1$, and $u_2, x_2 \in \mathbb{E}_2$, $(d_2 \circ d'_2)(u, u_2)(u, x_2) = \max(d_1(u), d_2(u, x_2))$ for every $u \in N_1$, and $u_2, x_2 \in \mathbb{E}_2$. And $(s_2 \circ s'_2)(u_1, w)(x_1, w) = \min(s_1(w), s_2(u_1, w))$ for every $w \in N_2$, $u_1, x_1 \in \mathbb{E}_1$, $(i_2 \circ i'_2)(u_1, w)(x_1, w) = \min(i_1(w), i_2(u_1, x_1))$ for every $w \in N_2$, $u_1, x_1 \in \mathbb{E}_1$, $(d_2 \circ d'_2)(u_1, w)(x_1, w) = \max(d_1(w), d_2(u_1, x_1))$ for every $w \in \tilde{V}_2$, $u_1, x_1 \in \mathbb{E}_1$. $(s_2 \circ s'_2)(u_1, u_2)(x_1, x_2) = \min(s'_1(u_2), s'_1(x_2), s_2(u_1, x_1))$ for every $(u_1, u_2)(x_1, x_2) \in \tilde{E} - \tilde{E}''$, $(i_2 \circ i'_2)(u_1, u_2)(x_1, x_2) = \min(i'_1(u_2), i'_1(x_2), i_2(u_1, x_1))$ for every $(u_1, u_2)(x_1, x_2) \in \mathbb{E} - \mathbb{E}''$, $(d_2 \circ d'_2)(u_1, u_2)(x_1, x_2) = \max(d'_1(u_2), d'_1(x_2), d_2(u_1, x_1))$ for every $(u_1, u_2)(x_1, x_2) \in \mathbb{E} - \mathbb{E}''$, where $\mathbb{E}'' = \{(u, u_2)(u, x_2): u \in N_1 \text{ for every } u_2, x_2 \in \mathbb{E}_2\} \cup \{(u_1, w)(x_1, w): w \in N_2 \text{ for every } u_1, x_1 \in \mathbb{E}_1\}$.

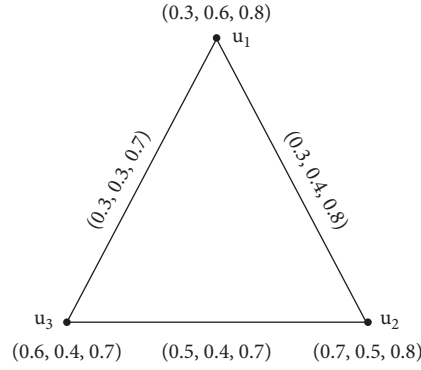


FIGURE 19: TSFG-A for product.

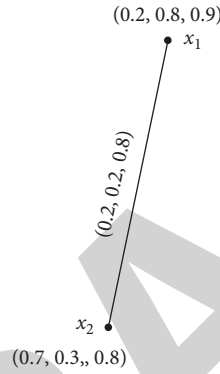


FIGURE 20: TSFG-B for product.

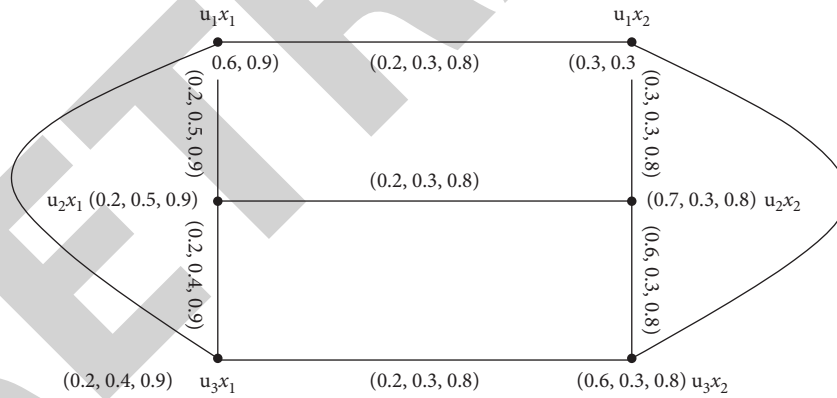


FIGURE 21: Product of two TSFGs.

problems has been given great importance in various fuzzy algebraic structures, which results in the development of several new approaches. Our aim is to follow the Dijkstra algorithm and apply it to a network of nodes where the path information has been provided in the form of TSFNs.

In this section, we assume a network where the shortest path must be computed from source node (SN) to destination node (DN) and the information about path between every two nodes is provided in the form of TSFNs. Usually, the shortest path is the one which is less costly or requires less time or the one on which one must travel less distance between SN and DN. The Dijkstra algorithm in T-spherical fuzzy environment is demonstrated briefly in the following.

4.1. T-Spherical Fuzzy Dijkstra Algorithm. The most reasonable approach to find shortest path in a network is to follow Dijkstra algorithm which is the successful algorithm used by many researchers such as [39–42]. The detailed steps of Dijkstra algorithm is for T-spherical fuzzy network are stated as follows.

- (i) The source node is marked as permanent node (P). Moreover, it is labelled as $((0, 0, 1), -)$. Therefore, this node is involved in shortest path by default and distance travelled is zero at this stage.
- (ii) Compute the label $[v_i \oplus d_{ij}, i]$ if j is not a permanent node where j is a node whose path is

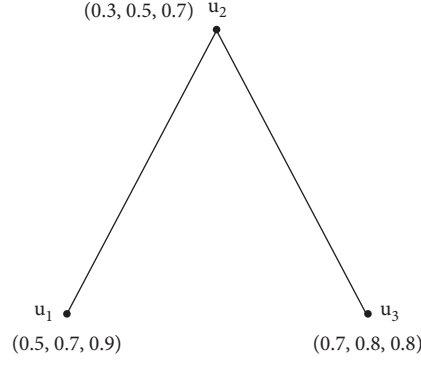


FIGURE 22: TSFG for composition-A.

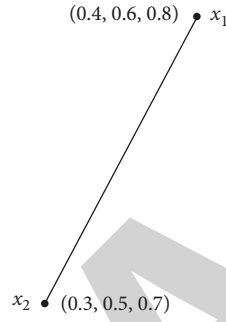


FIGURE 23: TSFG for composition-B.

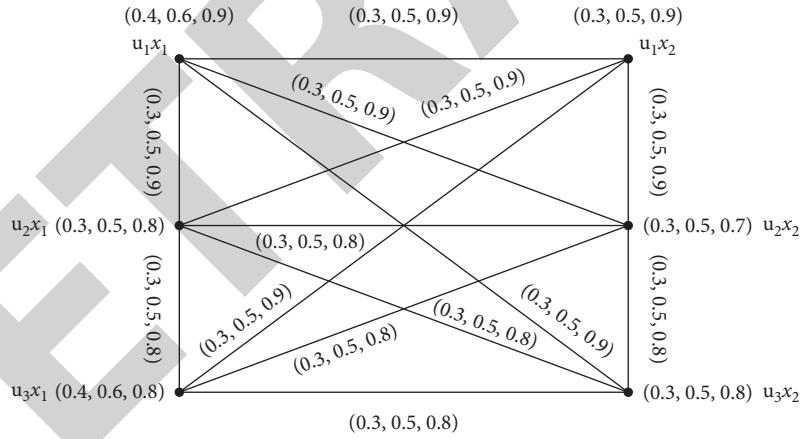


FIGURE 24: Composition of TSFG.

from node i . Furthermore, if j is labelled as $[v_j, ik]$ through some other node, then replace $[v_j, ik]$ by $[v_i \oplus d_{ij}, i]$ only if $SC(v_i \oplus d_{ij})$ is less than $SC(v_j)$.

- (iii) If all of the nodes are labelled permanently, then the algorithm terminates. Otherwise, choose $[v_r, is]$ having shortest distance v_r and repeat Step 2 by setting $i = r$.
- (iv) Using the information of the label, find the shortest path from SN to DN.

The flowchart the algorithm is shown in Figure 25.

Remark 4. $[v_i \oplus d_{ij}, i]$ is a label which states that the current location is node i and we travelled a distance $v_i \oplus d_{ij}$. Further, it is to be noted that the process cannot be continued to a permanent node but can be reversed. For two directly connected adjacent nodes i and j , node i is considered as the predecessor of node j if the path connecting them is directed from i to j .

Example 13. In Figure 26, a network is portrayed which is composed of 6 nodes and 8 edges. The aim is to find out the shortest path from SN (N_1) to DN (N_2) using modified Dijkstra algorithm.

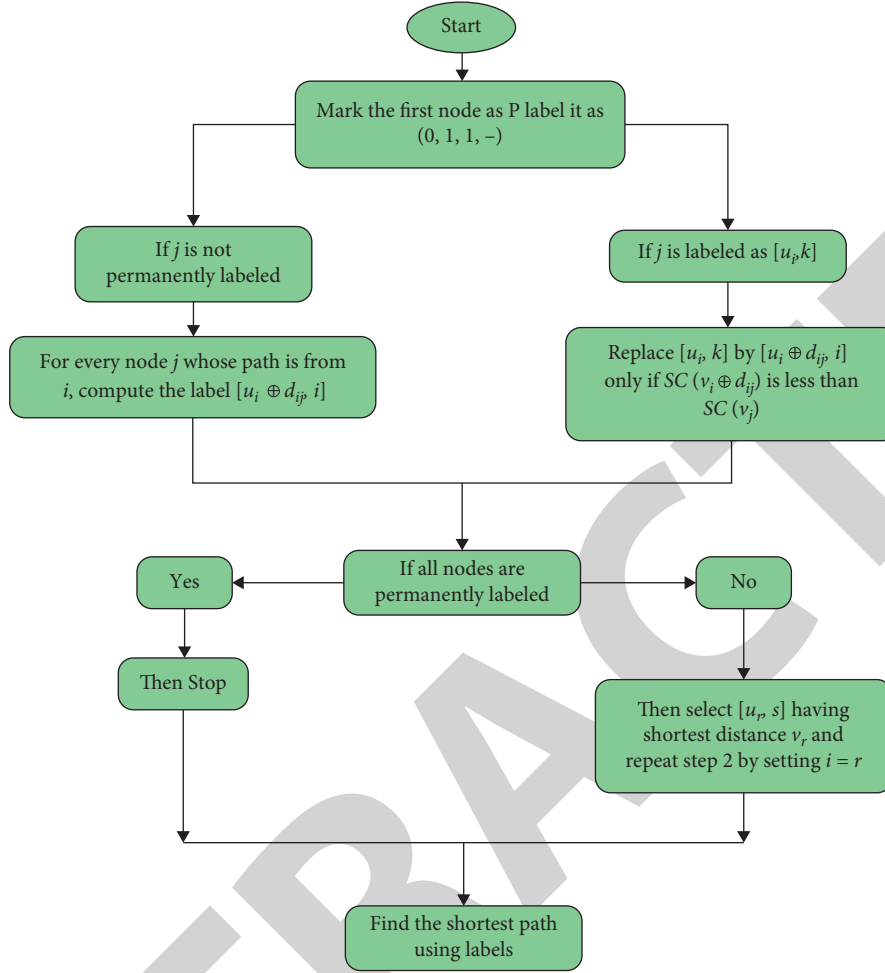


FIGURE 25: Flowchart of modified Dijkstra algorithm for computing the shortest path.

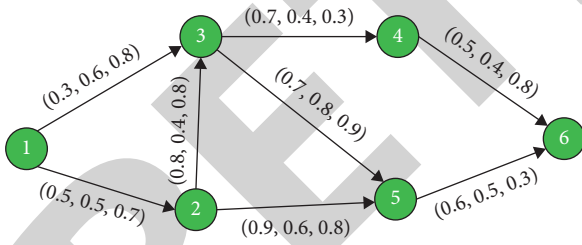


FIGURE 26: T-spherical fuzzy network.

The list of edges involved in this network is given in Table 1.

Now, we apply the modified Dijkstra algorithm and carry out the step-wise computations.

Step 1. Node 1 is in the shortest path by default, and thus we mark it as a permanent node.

Step 2. Node 1 is connected to two other nodes, and thus there are two ways, i.e., we might move from node 1 to node 3 or from node 1 to node 2. Hence, the list of nodes is given in Table 2.

Now we compute the scores by Definition 9 of $(0.3, 0.6, 0.8)$ and $(0.5, 0.5, 0.7)$.

$$\begin{aligned} SC(0.3, 0.6, 0.8) &= 0.00245, \\ SC(0.5, 0.5, 0.7) &= 0.022. \end{aligned} \quad (6)$$

As the score of $(0.3, 0.6, 0.8)$ is less than $(0.5, 0.5, 0.7)$, we mark node 3 as $((0.3, 0.6, 0.8), N_1)$ and label it as permanent.

Step 3. Again there are two ways to initiate from node 3, i.e., we might move from node 3 to node 5 or from node 3 to node 4. Hence, the list of nodes is given in Table 3.

Now we compute the scores of $(0.54, 0.24, 0.24)$ and $(0.54, 0.48, 0.72)$ as follows:

$$\begin{aligned} SC(0.54, 0.24, 0.24) &= 0.086, \\ SC(0.54, 0.48, 0.72) &= 0.028. \end{aligned} \quad (7)$$

As the score of $(0.54, 0.48, 0.72)$ is less than $(0.54, 0.24, 0.24)$, we mark node 5 as $((0.54, 0.48, 0.72), N_3)$ and label it as permanent.

Step 4. The only way out of node 5 leads to node 6. Hence, the list of nodes is given in Table 4.

Since there is a single way from node 5 to 6, node 6 is marked as $((0.37, 0.24, 0.23), N_5)$ and labelled as permanent.

TABLE 1: Weights of edges.

Edges	T-spherical distances
(N_1, N_2)	$(0.5, 0.5, 0.7)$
(N_1, N_3)	$(0.3, 0.6, 0.8)$
(N_2, N_3)	$(0.8, 0.4, 0.8)$
(N_2, N_5)	$(0.9, 0.6, 0.8)$
(N_3, N_4)	$(0.7, 0.4, 0.3)$
(N_3, N_5)	$(0.7, 0.8, 0.9)$
(N_4, N_6)	$(0.5, 0.4, 0.8)$
(N_5, N_6)	$(0.6, 0.5, 0.3)$

TABLE 2: List of nodes.

Nodes	Label	Status
N_1	$((0, 0, 1), -)$	Permanent
N_2	$((0.5, 0.5, 0.7), N_1)$	Temporary
N_3	$((0.3, 0.6, 0.8), N_1)$	Temporary

TABLE 3: List of nodes.

Nodes	Label	Status
N_1	$((0, 0, 1), -)$	Permanent
N_2	$((0.5, 0.5, 0.7), N_1)$	Temporary
N_3	$((0.3, 0.6, 0.8), N_1)$	Permanent
N_4	$((0.54, 0.24, 0.24), N_3)$	Temporary
N_5	$((0.54, 0.48, 0.72), N_3)$	Temporary

Step 5. Nodes 4 and 2 are the temporary nodes left over; therefore, their status is altered to permanent and the following list of nodes is obtained (Table 5).

Step 6. Table 6 implies the following sequence of shortest path from SN to DN, i.e., from node 1 to node 6.

Hence, according to modified Dijkstra algorithm, the shortest path is

$$N_1 \longrightarrow N_3 \longrightarrow N_5 \longrightarrow N_6. \quad (8)$$

4.1.1. Comparative Study. In this section, our aim is to analyse and compare the networks of TSFGs with existing concepts and prove the superiority of T-spherical fuzzy Dijkstra algorithm over existing approaches.

We take a network in the environment of IFNs where the information of paths is provided in IFNs. The network presented in Figure 27 is based on IFNs as all the values of paths are in IFNs and such information could be very easily converted into TSFGs if we assume the value of $i = 0$. Hence, we can determine the shortest path using the proposed approach.

Similarly, a network where information is in the form of FNs can also be transformed to a network of TSFGs by assuming the values of $i = d = 0$. For example, the network in Figure 28 is based on fuzzy information, and hence using the proposed approach of T-spherical fuzzy Dijkstra

TABLE 4: List of nodes.

Nodes	Label	Status
N_1	$((0, 0, 1), -)$	Permanent
N_2	$((0.5, 0.5, 0.7), N_1)$	Temporary
N_3	$((0.3, 0.6, 0.8), N_1)$	Permanent
N_4	$((0.54, 0.24, 0.24), N_3)$	Temporary
N_5	$((0.54, 0.48, 0.72), N_3)$	Permanent
N_6	$((0.37, 0.24, 0.23), N_5)$	Permanent

TABLE 5: List of nodes.

Nodes	Label	Status
N_1	$((0, 0, 1), -)$	Permanent
N_2	$((0.5, 0.5, 0.7), N_1)$	Permanent
N_3	$((0.3, 0.6, 0.8), N_1)$	Permanent
N_4	$((0.54, 0.24, 0.24), N_3)$	Permanent
N_5	$((0.54, 0.48, 0.72), N_3)$	Permanent
N_6	$((0.37, 0.24, 0.23), N_5)$	Permanent

TABLE 6: List of nodes.

N_6	$((0.37, 0.24, 0.23), N_5)$
N_5	$((0.54, 0.48, 0.72), N_3)$
N_3	$((0.3, 0.6, 0.8), N_1)$

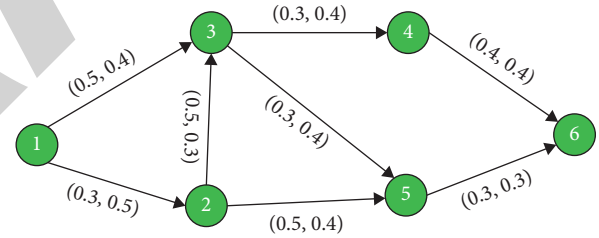


FIGURE 27: Intuitionistic fuzzy network.

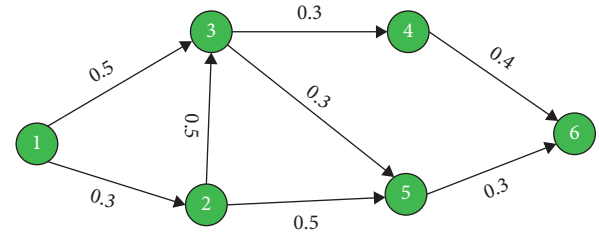


FIGURE 28: Fuzzy network.

algorithm, we can easily compute the shortest path from source node to destination node.

5. Conclusion

In this paper, the concept of TSFG is introduced based on the novel theory of TSFGs. In view of the novelty of TSFGs, the importance of TSFGs is elaborated and it is discussed that TSFGs are generalizations of IFGs and PFGs and can be

applicable in those situations where the frameworks of IFG and PFG failed to be applied. Some very basic graph theoretic terms like complement of TSFGs, T-spherical fuzzy subgraph, degree of vertices in TSFGs, strength of TSFGs, and bridges in TSFGs are defined. A study of operations of TSFGs is also established and related results are studied. The famous Dijkstra algorithm for TSFGs has been developed and the shortest path in a network of TSFGs has been solved. The main benefit of the proposed work is that it could be applied in the conditions that are handled by using the concepts of IFG or PFG, but these structures are incapable of handling the information given in the T-spherical fuzzy environment. In the near future, the framework of TSFGs could prove to be very useful tool that can be applied in the traffic signal problems, optimization in networks, and other problems of computer sciences and engineering. Additionally, the proposed work can be extended to interval-valued and cubic-valued frameworks that will give rise to much stronger and interesting structures with extended range of applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A Multiattribute Decision-Making Framework: VIKOR Method with Complex Spherical Fuzzy N -Soft Sets

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In this paper, we set forth a framework for solving a multiattribute group decision-making (MAGDM) problem, namely, the selection of a firm for participation in a Saudi oil refinery project in Pakistan. This project will prove a key success factor for the economic growth of Pakistan due to its enormous economic impact on the energy sector, industrial development, commerce, transportation, and so on. This multiplicity justifies that several intricate components comprising both intrinsic and external attributes should be adequately evaluated for the selection of such a firm, that is, the formulation of this question as a MAGDM problem. Nonbinary evaluation with two-dimensional ambiguity and uncertainty in the parameters are general concerns in modern literature, and they fit into this problem. Within this context, one of the most superior and amenable theories (complex spherical fuzzy N -soft sets, henceforth $CSFNS_f$ s) shall be used to formulate a new comprehensive method, known as complex spherical fuzzy N -soft-VIKOR ($CSFNS_f$ -VIKOR) method. According to the general spirit of the benchmark technique, the normalized Euclidean distances and the weights of the attributes are jointly handled, and as consequence, two main features ("maximum group utility" and "minimum individual regret") are acquired. The coefficient strategy with reference to group utility measure and individual regret measure of opponents are employed for the compromise measure. Armed with this novel tool, we single out the most feasible firm according to the preference order of the alternatives examined by the decision-makers on the subject of linear normalized weights of experts and attributes. Furthermore, a comparative analysis justifies the CSF -VIKOR method, and some results prove its capabilities and validity. Moreover, a sensitivity test certifies the stability of the proposed method.

1. Introduction

In 1998, the VIKOR approach was drafted by Opricovic [1] as a multiattribute decision-making (MADM) method. In the setting of civil engineering, this system attempted to find a compromise solution based on two dominant principles (group utility measure and individual regret of opponent), where the compromise solution means a decision done by generic involvement. Due to the paradigm of maximum group utility and minimum individual regret, the feasible solution closest to the best values and most distant from the worst value is resolved by the recourse to the L_p -metric as an aggregation function, according to Opricovic. Opricovic and

Tzeng [2] extended the VIKOR theory for MADM for the postearthquake reconstruction problem in Central Taiwan and the selection of destination for the mountain climber, respectively. Yazdani-Chamzini et al. [3] employed the modified version of the VIKOR method along with the combination of TOPSIS, MOORA, additive weighting, and ratio assessment techniques in order to address a multi-criteria decision-making (MCDM) problem relating to renewable energy resources.

People habitually live with real-life properties that are not sufficiently precise and not fully objective, such as "beautiful," "tall," or "experienced." Many decision-related problems must take into account such properties. To

contend with such vague and uncertain data, Zadeh [4] presented the notion of fuzzy set. Suh et al. [5] assessed the mobile service quality through the proposed ground-breaking fuzzy-VIKOR method using an integrated weighted approach. Lee et al. [6] extended the VIKOR technique for the MAGDM problem of detection of flood risks under the fuzzy frame. Li and Liu [7] presented the VIKOR with the combination of the QUALIFLEX method on the basis of trapezoidal fuzzy numbers corresponded to each two-dimensional linguistic data. Chang [8] disclosed the situation about the private and governmental hospital agencies in Taiwan with the help of the new fuzzy-VIKOR approach. Wang and Chang [9] resolved the MAGDM problem through fuzzy-VIKOR methodology. Another MAGDM problem, the selection of machine tools evaluated by the fuzzy-VIKOR approach is introduced by Wu et al. [10].

In accordance with fuzzy descriptions, the degree of truthiness (or satisfaction), collected from the closed unit interval $[0, 1]$, denotes the correspondence of an object related to a parameter. Under these interpretations, its inability to represent the nonassociation of the object related to that particular parameter is quite obvious. For this reason, in 1986, Atanassov [11] eradicated this obstacle through the extended concept of intuitionistic fuzzy set (IFS) and confronted the vague opinion through both a degree of truthiness ϕ and a degree of falsity χ , which are jointly subject to the constraint: $\phi + \chi \leq 1$. Roostaei et al. [12] developed the theoretical background for an IF-VIKOR method to rank the suppliers following the opinions of decision-makers, and Gupta et al. [13] extended the IF-VIKOR ideology for allocating the best environment for plantation. Krishankumar et al. [14] resolved a personnel selection problem through the extended version of the IF-VIKOR method.

Pythagorean fuzzy sets (P_yFS s) evolved as an extension of IFSs. Introduced by Yager [15], P_yFS also deal with both degree of truth ϕ and degree of falsity χ , but they are under the modified condition: $\phi^2 + \chi^2 \leq 1$. Gul et al. [16] extended the VIKOR based approach within the field of P_yFS s and evaluated the safety risks in the mine industry. Rani et al. [17] proposed the P_yF -VIKOR method within the tool of entropy and divergence measures for the selection of renewable energy technology in India. Ma et al. [18] introduced the group decision-making framework using complex Pythagorean fuzzy information.

People habitual nature sometimes has a neutral judgment. This is generally exhibited in certain problems or events like voting situations. Therefore, along with the judgments of yes or no, there is often a need for an abstain part of the opinion (possibly related to the satisfaction of a particular parameter). Since the Pythagorean fuzzy set is not able to handle such part of the decisional attitude, Cuong [19] introduced picture fuzzy set (PFS) along with the degree of truth ϕ , remain neutral ψ , and degree of falsity χ but with their range limited by an inequality: $\phi + \psi + \chi \leq 1$. Meksavang et al. [20] extended the decision-making approach based on the VIKOR methodology for the picture fuzzy environment, which was applied for sustainable suppliers along with an application in the beef industry. Liu and You

[21] presented the PF-VIKOR technique for green supplier evaluation and selection.

However, PFSs are very reliable to deal with imprecision and fuzziness and still useless when the sum of the degrees of truth, neutrality, and falsity exceeds 1. To get over this drawback, Gundogdu and Kahraman [22–25] worked out a model with a relaxed condition, introduced the idea of spherical fuzzy sets (\mathcal{SFS} s), and employed them on MADM problems. Later on, Mahmood et al. [26] proposed T -spherical fuzzy sets, as an extension of \mathcal{SFS} s. Gundogdu et al. [27, 28] developed the theory for the \mathcal{SF} -VIKOR method and applied it to the MADM problem for waste management problems and selection of site, respectively.

The aforementioned models of fuzzy knowledge have a one-dimensional structure. This pattern is incompatible with certain types of problems. Ramot et al. [29] developed a fuzzy set model in two-directional frames, known as complex fuzzy set (CFS). In this model, the degree of truth is $\phi = te^{i2\pi\zeta}$, which is divided into amplitude term t and periodic term ζ that belongs to the unit closed interval. For the sake of decision-making, complex fuzzy aggregation operators are defined by Akram and Bashir [30]. Alkouri and Salleh [31] generalized the ideology to a complex intuitionistic fuzzy set (CIFS), which accounts for both degree of truthiness $\phi = te^{i2\pi\zeta}$ and falsity $\chi = fe^{i2\pi\rho}$ along with the constraints $t + f \leq 1$ and $\zeta + \rho \leq 1$. Ullah et al. [32] introduced a complex Pythagorean fuzzy set (CP_yFS) with degrees of truth $\phi = te^{i2\pi\zeta}$ and falsity $\chi = fe^{i2\pi\rho}$ within the complex unit circle and restricted by the constraints $t^2 + f^2 \leq 1$ and $\zeta^2 + \rho^2 \leq 1$. Currently, Akram et al. [33] presented the model of a complex spherical fuzzy set ($C\mathcal{SFS}$) representing three degrees of truth $\phi = te^{i2\pi\zeta}$, neutrality $\psi = qe^{i2\pi\varrho}$, and falsity $\chi = fe^{i2\pi\rho}$ subjected to conditions $t^2 + q^2 + f^2 \leq 1$ and $\zeta^2 + \varrho^2 + \rho^2 \leq 1$. They also extended the VIKOR method under the $C\mathcal{SF}$ environment. The yielding and tractable conditions of the $C\mathcal{SF}$ representation make it a privileged framework for the modelization of two-dimensional ambiguous knowledge. In the advantageous $C\mathcal{SF}$ -VIKOR technique, decision-makers scrutinize the competencies of the feasible choices with reference to the preferred criteria and indicate initial observation through the use of linguistic information, which further enhanced the virtues of complex spherical fuzzy numbers.

Though $C\mathcal{SFS}$ s are highly competent and skillful, they neglect the possibility of nonbinary parameterized information. In this panorama, Akram et al. [34] presented the concept of complex spherical fuzzy N -soft sets ($C\mathcal{SFNS}_fS$ s) within the TOPSIS methodology, which has the ability to look over the modern real-life ranking problems in every field of science. In relation to this, the idea of N -soft sets (NS_fS s) was presented by [35] as an extension of soft set theory [36], which accommodates all kind of attributes. Soon afterwards, Akram et al. [37] explored the scope of N -soft sets and put forward hybrid models, namely, fuzzy N -soft sets (FNS_fS s), intuitionistic fuzzy N -soft sets ($IFNS_fS$ s), and hesitant N -soft sets. Recently, Zhang et al. [38] combined N -soft theory with P_yFS s and proposed the novel model of Pythagorean fuzzy N -soft set (P_yFNS_fS). The $C\mathcal{SFNS}_f$ model was presented (by Akram et al. [34]) within

research conducive to an extension of the TOPSIS method that can deal with MAGDM problems based on $CSFNS_f$ information. For other notations and applications, the readers are referred to [39–44].

In modern times, ranking evaluation systems (instead of linguistic information) can be in use as a primary source in furtherance of decision-making problems and surveys. The CSF -VIKOR technique (proposed by Akram et al. [33]) is unqualified for situations comprising parameterized ranking information. Therefore, we develop a novel technique, namely, the $CSFNS_f$ -VIKOR method. The delineation and explanation related to this technique do not pertain to any of the existing techniques; hence, it requires a concrete analysis. Of course, the proposed model successfully evaluates decision-making problems related to the modern era. The purpose of our concept is to extend the VIKOR method under the circumstances of $CSFNS_f$ Ss. Its motivations are given as follows:

- (i) The two-dimensional influential technique of CSF -VIKOR is inadequate to operate with data comprising ordered grades along with two or several parameters.
- (ii) The fuzzy N -soft models, along with intuitionistic fuzzy N -soft, Pythagorean fuzzy N -soft, and complex Pythagorean fuzzy N -soft environments provide an array of models that operate under parameterized ranking systems. But they are all useless when it comes to incorporating uncertain data containing neutral opinions.
- (iii) The $CSFNS_f$ -TOPSIS method uses a different methodology but ultimately computes a feasible solution, which merely does not take into account the relative importance of distances from the ideal solution.
- (iv) These limitations provoke the development of $CSFNS_f$ -VIKOR method that gives us the ability to incorporate information regarding ordered grades among complex valued degrees of truth, neutrality, and falsity.

The main aim of our proposed idea is to develop the methodology of VIKOR within the hybrid model of $CSFNS_f$ Ss specifically for the solution of MAGDM problems precisely. The contributions of this paper are as follows:

- (i) We introduce a hybrid MAGDM VIKOR approach whose structure is based on $CSFNS_f$ Ss, and it is known as $CSFNS_f$ -VIKOR technique. This methodology qualifies for situations that comprise parameterized ranking information; hence, it successfully evaluates a large proportion of existing MAGDM problems, as well as decision-making problems with nonbinary parameterized information as an initial assessment.
- (ii) Linear normalized weights of experts and attributes are defined along with normalized Euclidean distance for sake of maximum group utility measure,

individual regret, and compromise ranking whose values are arranged in ascending order for final decisions.

- (iii) The presented technique is supported through a real-life application from the grounds of economy and business assisted by comparative analysis and sensitivity tests.

The rest of the article is structured as follows. Section 2 contains preliminaries from $CSFNS_f$ model [34] with some operations, comparison rule, and averaging operator. In Section 3, a descriptive theory for $CSFNS_f$ -VIKOR method is developed along with a flowchart. Section 4 interpreted a real-life MAGDM problem of firm's selection for the Saudi oil refinery project. Section 5 narrated the proposed method by a sensitivity test. In Section 6, we compare our proposed method with the existing techniques. In Section 7, the merits of the $CSFNS_f$ -VIKOR method are clarified. In the end, we present concluding remarks and mention some future research work in Section 8.

2. Complex Spherical Fuzzy N -Soft Sets

Definition 1 (see [34]). Let J be a nonempty set and $\mathcal{X} \subseteq T$, whereas T be a set of parameters (or attributes), and $C = \{0, 1, 2, \dots, N-1\}$ be a set of grades level with $N \in \{2, 3, \dots\}$. Then a complex spherical fuzzy N -soft set ($CSFNS_f$ S) on \mathcal{X} is denoted by a triplet $(\ddot{E}_H, \mathcal{X}, N)$, defined as follows:

$$\begin{aligned} \ddot{E}_H(x_a) &= \{ \langle (\ddot{E}(x_a), H(x_a)) \rangle : x_a \in \mathcal{X} \} \\ &= \{ \langle (j_b, c_a^b), (\phi_{ba}, \psi_{ba}, \chi_{ba}) \rangle \} \\ &= \{ \langle (j_b, c_a^b), (t_{ws} e^{i2\pi\zeta_{ws}}, q_{ba} e^{i2\pi\varrho_{ba}}, f_{ba} e^{i2\pi\rho_{ba}}) \rangle \}, \end{aligned} \quad (1)$$

along with the assumption of $\ddot{E}: \mathcal{X} \longrightarrow 2^{J \times H}$, where NS_f S be defined on J , and H is a function from \mathcal{X} to $CSFN$. The notation $CSFN$ denotes the collection of all complex spherical fuzzy numbers of J , and $t_{ba}, \zeta_{ba}, q_{ba}, \varrho_{ba}, f_{ba}$, and ρ_{ba} will be taken from unit closed interval, with constraint

$$\begin{aligned} 0 \leq t_{ba}^2 + q_{ba}^2 + f_{ba}^2 &\leq 1, \\ 0 \leq \zeta_{ba}^2 + \varrho_{ba}^2 + \rho_{ba}^2 &\leq 1. \end{aligned} \quad (2)$$

The term c_a^b denotes the ranking of attributes for the alternative j_b , for all $j_b \in J$.

Definition 2 (see [34]). Let $\ddot{E}_H(x_a) = ((j_b, c_a^b), t_{ba} e^{i2\pi\zeta_{ba}}, q_{ba} e^{i2\pi\varrho_{ba}}, f_{ba} e^{i2\pi\rho_{ba}})$ be a $CSFNS_f$ S. Then the complex spherical fuzzy N -soft number ($CSFNS_fN$) is defined as follows:

$$F_{ba} = (c_a^b, t_{ba} e^{i2\pi\zeta_{ba}}, q_{ba} e^{i2\pi\varrho_{ba}}, f_{ba} e^{i2\pi\rho_{ba}}), \quad (3)$$

and the hesitancy degree is defined as follows:

$$\Omega_{F_{ba}} = \sqrt{1 - (t_{ba}^2 + q_{ba}^2 + f_{ba}^2)} e^{i2\pi \sqrt{1 - (\zeta_{ba}^2 + \varrho_{ba}^2 + \rho_{ba}^2)}}. \quad (4)$$

Definition 3. (see [34]). Let $F_{ba} = (c_a^b, t_{ba} e^{i2\pi \zeta_{ba}}, q_{ba} e^{i2\pi \varrho_{ba}}, f_{ba} e^{i2\pi \rho_{ba}})$ be $CSFNS_f N$. The score function $Sc(F_{ba})$ is defined as follows:

$$Sc_{F_{ba}} = \left(\frac{c_a^b}{N-1} \right)^2 + (t_{ba}^2 - q_{ba}^2 - f_{ba}^2) + [\zeta_{ba}^2 - \varrho_{ba}^2 - \rho_{ba}^2], \quad (5)$$

where $Sc_{F_{ba}} \in [-2, 3]$. The accuracy function $Ac(F_{ba})$ is defined as follows:

$$Ac_{F_{ba}} = \left(\frac{c_a^b}{N-1} \right)^2 + (t_{ba}^2 + q_{ba}^2 + f_{ba}^2) + [\zeta_{ba}^2 + \varrho_{ba}^2 + \rho_{ba}^2], \quad (6)$$

where $Ac_{F_{ba}} \in [0, 3]$.

Definition 4 (see [34]). Let $F_{bl} = (c_l^b, t_{lj} e^{i2\pi \zeta_{bl}}, q_{bl} e^{i2\pi \varrho_{bl}}, f_{bl} e^{i2\pi \rho_{bl}})$ and $F_{ba} = (c_a^b, t_{ba} e^{i2\pi \zeta_{ba}}, q_{ba} e^{i2\pi \varrho_{ba}}, f_{ba} e^{i2\pi \rho_{ba}})$ be two $CSFNS_f N$ s.

- (1) If $Sc_{F_{bl}} < Sc_{F_{ba}}$, then $F_{bl} < F_{ba}$ (F_{bl} is inferior to F_{ba})
- (2) If $Sc_{F_{bl}} > Sc_{F_{ba}}$, then $F_{bl} > F_{ba}$ (F_{bl} is superior to F_{ba})
- (3) If $Sc_{F_{bl}} = Sc_{F_{ba}}$, then
 - (i) $Ac_{F_{bl}} < Ac_{F_{ba}}$, then $F_{bl} < F_{ba}$ (F_{bl} is inferior to F_{ba})
 - (ii) $Ac_{F_{bl}} > Ac_{F_{ba}}$, then $F_{bl} > F_{ba}$ (F_{bl} is superior to F_{ba})
 - (iii) $Ac_{F_{bl}} = Ac_{F_{ba}}$, then $F_{bl} \sim F_{ba}$ (F_{bl} is equivalent to F_{ba})

Definition 5 (see [34]). Let $F_{bl} = (c_l^b, t_{lj} e^{i2\pi \zeta_{bl}}, q_{bl} e^{i2\pi \varrho_{bl}}, f_{bl} e^{i2\pi \rho_{bl}})$ and $F_{ba} = (c_a^b, t_{ba} e^{i2\pi \zeta_{ba}}, q_{ba} e^{i2\pi \varrho_{ba}}, f_{ba} e^{i2\pi \rho_{ba}})$ be two $CSFNS_f N$ s and $\lambda > 0$. Some operations for $CSFNS_f N$ s are as follows:

$$\begin{aligned} \lambda F_{bl} &= \left(c_l^b, \sqrt{1 - (1 - t_{lj}^2)^\lambda} e^{i2\pi \sqrt{1 - (1 - \zeta_{bl}^2)^\lambda}}, q_{bl}^\lambda e^{i2\pi \varrho_{bl}^\lambda}, f_{bl}^\lambda e^{i2\pi \rho_{bl}^\lambda} \right), \\ F_{bl}^\lambda &= \left(c_l^b, t_{lj}^\lambda e^{i2\pi \zeta_{bl}^\lambda}, \sqrt{1 - (1 - v_{lj}^2)^\lambda} e^{i2\pi \sqrt{1 - (1 - \varrho_{bl}^2)^\lambda}}, \sqrt{1 - (1 - f_{lj}^2)^\lambda} e^{i2\pi \sqrt{1 - (1 - \rho_{bl}^2)^\lambda}} \right), \\ F_{bl} \oplus F_{ba} &= \left(\max(c_l^b, c_a^b), \sqrt{t_{lj}^2 + t_{ba}^2 - t_{lj}^2 t_{ba}^2} e^{i2\pi \sqrt{\zeta_{lj}^2 + \zeta_{ba}^2 - \zeta_{lj}^2 \zeta_{ba}^2}}, q_{lj} q_{ba} e^{i2\pi \varrho_{lj} \varrho_{ba}}, f_{lj} f_{ba} e^{i2\pi \rho_{lj} \rho_{ba}} \right), \\ F_{bl} \otimes F_{ba} &= \left(\min(c_l^b, c_a^b), t_{lj} t_{ba} e^{i2\pi \zeta_{lj} \zeta_{ba}}, \sqrt{v_{lj}^2 + v_{ba}^2 - v_{lj}^2 v_{ba}^2} e^{i2\pi \sqrt{\varrho_{lj}^2 + \varrho_{ba}^2 - \varrho_{lj}^2 \varrho_{ba}^2}}, \sqrt{f_{lj}^2 + f_{ba}^2 - f_{lj}^2 f_{ba}^2} e^{i2\pi \sqrt{\rho_{lj}^2 + \rho_{ba}^2 - \rho_{lj}^2 \rho_{ba}^2}} \right). \end{aligned} \quad (7)$$

Proposition 1 (see [34]). Let $F_{ba} = (c_a^b, t_{ba} e^{i2\pi \zeta_{ba}}, q_{ba} e^{i2\pi \varrho_{ba}}, f_{ba} e^{i2\pi \rho_{ba}})$ and $F_{bl} = (c_l^b, t_{lj} e^{i2\pi \zeta_{bl}}, q_{bl} e^{i2\pi \varrho_{bl}}, f_{bl} e^{i2\pi \rho_{bl}})$ be two $CSFNS_f N$ s and $\lambda > 0$, then the following properties hold:

- (1) $F_{ba} \oplus F_{bl} = F_{bl} \oplus F_{ba}$
- (2) $F_{ba} \otimes F_{bl} = F_{bl} \otimes F_{ba}$
- (3) $\lambda F_{ba} \oplus \lambda F_{bl} = \lambda (F_{bl} \oplus F_{ba}), \lambda > 0$

- (4) $\lambda_1 F_{ba} \oplus \lambda_2 F_{ba} = (\lambda_1 + \lambda_2) F_{ba}, \lambda_1, \lambda_2 > 0$
- (5) $F_{ba}^\lambda \otimes F_{bl}^\lambda = (F_{bl} \otimes F_{ba})^\lambda, \lambda > 0$
- (6) $F_{ba}^{\lambda_1} \otimes F_{ba}^{\lambda_2} = F_{ba}^{(\lambda_1 + \lambda_2)}, \lambda_1, \lambda_2 > 0$

Proof

$$\begin{aligned} (1) F_{ba} \oplus F_{bl} &= \left(\max(c_a^b, c_l^b), \left[\sqrt{t_{ba}^2 + t_{bl}^2 - t_{ba}^2 t_{bl}^2} \right] e^{i2\pi \left[\sqrt{\zeta_{ba}^2 + \zeta_{bl}^2 - \zeta_{ba}^2 \zeta_{bl}^2} \right]}, [q_{ba} q_{bl}] e^{i2\pi [\varrho_{ba} \varrho_{bl}]}, [f_{ba} f_{bl}] e^{i2\pi [\rho_{ba} \rho_{bl}]} \right), \\ &= \left(\max(c_l^b, c_a^b), \left[\sqrt{t_{bl}^2 + t_{ba}^2 - t_{bl}^2 t_{ba}^2} \right] e^{i2\pi \left[\sqrt{\zeta_{bl}^2 + \zeta_{ba}^2 - \zeta_{bl}^2 \zeta_{ba}^2} \right]}, [q_{bl} q_{ba}] e^{i2\pi [\varrho_{bl} \varrho_{ba}]}, [f_{bl} f_{ba}] e^{i2\pi [\rho_{bl} \rho_{ba}]} \right), \\ &= F_{bl} \oplus F_{ba}. \end{aligned} \quad (8)$$

$$\begin{aligned}
(2) \quad & F_{ba} \otimes F_{bl} \\
&= \left(\min(c_a^b, c_l^b), [t_{ba}t_{bl}]e^{i2\pi[\zeta_{ba}\zeta_{bl}]}, \left[\sqrt{q_{ba}^2 + q_{bl}^2 - q_{ba}^2 q_{bl}^2} \right] e^{i2\pi \left[\sqrt{\varrho_{ba}^2 + \varrho_{bl}^2 - \varrho_{ba}^2 \varrho_{bl}^2} \right]}, \left[\sqrt{f_{ba}^2 + f_{bl}^2 - f_{ba}^2 f_{bl}^2} \right] e^{i2\pi \left[\sqrt{\rho_{ba}^2 + \rho_{bl}^2 - \rho_{ba}^2 \rho_{bl}^2} \right]} \right) \\
&= \left(\min(c_l^b, c_a^b), [t_{bl}t_{ba}]e^{i2\pi[\zeta_{bl}\zeta_{ba}]}, \left[\sqrt{q_{bl}^2 + q_{ba}^2 - q_{bl}^2 q_{ba}^2} \right] e^{i2\pi \left[\sqrt{\varrho_{bl}^2 + \varrho_{ba}^2 - \varrho_{bl}^2 \varrho_{ba}^2} \right]}, \left[\sqrt{f_{bl}^2 + f_{ba}^2 - f_{bl}^2 f_{ba}^2} \right] e^{i2\pi \left[\sqrt{\rho_{bl}^2 + \rho_{ba}^2 - \rho_{bl}^2 \rho_{ba}^2} \right]} \right) \\
&= F_{bl} \otimes F_{ba}.
\end{aligned} \tag{9}$$

$$\begin{aligned}
(3) \quad & \lambda F_{ba} \oplus \lambda F_{bl} \\
&= \left(c_a^b, \left[\sqrt{1 - (1 - t_{ba}^2)^\lambda} \right] e^{i2\pi \left[\sqrt{1 - (1 - \zeta_{ba}^2)^\lambda} \right]}, [q_{ba}^\lambda] e^{i2\pi[\varrho_{ba}^\lambda]}, [f_{ba}^\lambda] e^{i2\pi[\rho_{ba}^\lambda]} \right) \\
&\quad \oplus \left(c_l^b, \left[\sqrt{1 - (1 - t_{bl}^2)^\lambda} \right] e^{i2\pi \left[\sqrt{1 - (1 - \zeta_{bl}^2)^\lambda} \right]}, [q_{bl}^\lambda] e^{i2\pi[\varrho_{bl}^\lambda]}, [f_{bl}^\lambda] e^{i2\pi[\rho_{bl}^\lambda]} \right) \\
&= \left(\max(c_a^b, c_l^b), \left[\sqrt{1 - (1 - t_{ba}^2)^\lambda} + 1 - (1 - t_{bl}^2)^\lambda - \left[1 - (1 - t_{ba}^2)^\lambda \right] \left[1 - (1 - t_{bl}^2)^\lambda \right]} \right] e^{i2\pi \left[\sqrt{1 - (1 - \zeta_{ba}^2)^\lambda} + 1 - (1 - \zeta_{bl}^2)^\lambda - \left[1 - (1 - \zeta_{ba}^2)^\lambda \right] \left[1 - (1 - \zeta_{bl}^2)^\lambda \right]} \right], \\
&\quad [q_{ba}^\lambda q_{bl}^\lambda] e^{i2\pi[\varrho_{ba}^\lambda \varrho_{bl}^\lambda]}, [f_{ba}^\lambda f_{bl}^\lambda] e^{i2\pi[\rho_{ba}^\lambda \rho_{bl}^\lambda]} \right) \\
&= \left(\max(c_a^b, c_l^b), \left[\sqrt{1 - (1 - t_{ba}^2 + t_{bl}^2 - t_{ba}^2 t_{bl}^2)^\lambda} \right] e^{i2\pi \left[\sqrt{1 - (1 - \zeta_{ba}^2 + \zeta_{bl}^2 - \zeta_{ba}^2 \zeta_{bl}^2)^\lambda} \right]}, (q_{ba} q_{bl})^\lambda e^{i2\pi(\varrho_{ba} \varrho_{bl})^\lambda}, (f_{ba} f_{bl})^\lambda e^{i2\pi(\rho_{ba} \rho_{bl})^\lambda} \right) \\
&= \lambda \left(\max(c_a^b, c_l^b), \left[\sqrt{t_{ba}^2 + t_{bl}^2 - t_{ba}^2 t_{bl}^2} \right] e^{i2\pi \left[\sqrt{\zeta_{ba}^2 + \zeta_{bl}^2 - \zeta_{ba}^2 \zeta_{bl}^2} \right]}, (q_{ba} q_{bl}) e^{2\pi(\varrho_{ba} \varrho_{bl})}, (f_{ba} f_{bl}) e^{2\pi(\rho_{ba} \rho_{bl})} \right) \\
&= \lambda(F_{ba} \oplus F_{bl}).
\end{aligned} \tag{10}$$

$$\begin{aligned}
(4) \quad & \lambda_1 F_{ba} \oplus \lambda_2 F_{ba} \\
&= \left(c_a^b, \left[\sqrt{1 - (1 - t_{ba}^2)^{\lambda_1}} \right] e^{i2\pi \left[\sqrt{1 - (1 - \zeta_{ba}^2)^{\lambda_1}} \right]}, [q_{ba}^{\lambda_1}] e^{i2\pi[\varrho_{ba}^{\lambda_1}]}, [f_{ba}^{\lambda_1}] e^{i2\pi[\rho_{ba}^{\lambda_1}]} \right) \\
&\quad \oplus \left(c_l^b, \left[\sqrt{1 - (1 - t_{ba}^2)^{\lambda_2}} \right] e^{i2\pi \left[\sqrt{1 - (1 - \zeta_{ba}^2)^{\lambda_2}} \right]}, [q_{ba}^{\lambda_2}] e^{i2\pi[\varrho_{ba}^{\lambda_2}]}, [f_{ba}^{\lambda_2}] e^{i2\pi[\rho_{ba}^{\lambda_2}]} \right) \\
&= \left(\max(c_a^b, c_l^b), \left[\sqrt{1 - (1 - t_{ba}^2)^{\lambda_1}} + 1 - (1 - t_{ba}^2)^{\lambda_2} - \left[1 - (1 - t_{ba}^2)^{\lambda_1} \right] \left[1 - (1 - t_{ba}^2)^{\lambda_2} \right]} \right] e^{i2\pi \left[\sqrt{1 - (1 - \zeta_{ba}^2)^{\lambda_1}} + 1 - (1 - \zeta_{ba}^2)^{\lambda_2} - \left[1 - (1 - \zeta_{ba}^2)^{\lambda_1} \right] \left[1 - (1 - \zeta_{ba}^2)^{\lambda_2} \right]} \right], \\
&\quad [q_{ba}^{\lambda_1} q_{ba}^{\lambda_2}] e^{i2\pi[\varrho_{ba}^{\lambda_1} \varrho_{ba}^{\lambda_2}]}, [f_{ba}^{\lambda_1} f_{ba}^{\lambda_2}] e^{i2\pi[\rho_{ba}^{\lambda_1} \rho_{ba}^{\lambda_2}]} \right) \\
&= \left(\max(c_a^b, c_l^b), \left[\sqrt{1 - (1 - t_{ba}^2)^{(\lambda_1 + \lambda_2)}} \right] e^{i2\pi \left[\sqrt{1 - (1 - \zeta_{ba}^2)^{(\lambda_1 + \lambda_2)}} \right]}, (q_{ba})^{(\lambda_1 + \lambda_2)} e^{i2\pi(\varrho_{ba})^{(\lambda_1 + \lambda_2)}}, (f_{ba})^{(\lambda_1 + \lambda_2)} e^{i2\pi(\rho_{ba})^{(\lambda_1 + \lambda_2)}} \right) \\
&= (\lambda_1 + \lambda_2) \left(c_a^b, t_{ba} e^{i2\pi \zeta_{ba}}, (q_{ba}) e^{2\pi(\varrho_{ba})}, (f_{ba}) e^{2\pi(\rho_{ba})} \right) \\
&= (\lambda_1 + \lambda_2) F_{ba}.
\end{aligned} \tag{11}$$

Similarly, we can clarify (5) and (6). \square

Definition 6 (see [34]). Let $F_{ba} = (c_a^b, t_{ba}e^{i2\pi\zeta_{ba}}, q_{ba}e^{i2\pi\varrho_{ba}}, f_{ba}e^{i2\pi\rho_{ba}})$ ($a = 1, 2, \dots, v$) be a collection of $C\mathcal{SFNS}_fNs$,

then complex spherical fuzzy N -soft weighted average ($C\mathcal{SFNS}_fWA$) operator is defined as follows:

$$\begin{aligned} C\mathcal{SFNS}_fWA(F_{b1}, F_{b2}, \dots, F_{bv}) &= \left(\bigoplus_{a=1}^v \lambda_a F_{ba} \right) \\ &= \left(\max_{a=1}^v (c_a^b), \sqrt{\left[1 - \prod_{a=1}^v (1 - t_{ba}^2)^{\lambda_a}\right]} e^{i2\pi \sqrt{\left[1 - \prod_{a=1}^v (1 - \zeta_{ba}^2)^{\lambda_a}\right]}}, \right. \\ &\quad \left. \left[\prod_{a=1}^v (q_{ba})^{\lambda_a} \right] e^{i2\pi \left[\prod_{a=1}^v (\varrho_{ba})^{\lambda_a} \right]}, \left[\prod_{a=1}^v (f_{ba})^{\lambda_a} \right] e^{i2\pi \left[\prod_{a=1}^v (\rho_{ba})^{\lambda_a} \right]} \right), \end{aligned} \quad (12)$$

where $\lambda_a = (\lambda_1, \lambda_1, \dots, \lambda_v)^T$ is a weighted vector of F_{ba} with the property that $\lambda_a > 0$ and $\sum_{a=1}^v \lambda_a = 1$, for all ($b = 1, 2, \dots, m$).

Theorem 1. Let $F_{ba} = (c_a^b, t_{ba}e^{i2\pi\zeta_{ba}}, q_{ba}e^{i2\pi\varrho_{ba}}, f_{ba}e^{i2\pi\rho_{ba}})$ ($a = 1, 2, \dots, v$) be a collection of $C\mathcal{SFNS}_fNs$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_v)^T$ be the weight vector then the aggregated value by applying the $C\mathcal{SFNS}_fWA$ operator is also a $C\mathcal{SFNS}_fN$ formulated as follows:

$$\begin{aligned} C\mathcal{SFNS}_fWA(F_{b1}, F_{b2}, \dots, F_{bv}) &= \left(\bigoplus_{a=1}^v \lambda_a F_{ba} \right) \\ &= \left(\max_{a=1}^v (c_a^b), \sqrt{\left[1 - \prod_{a=1}^v (1 - t_{ba}^2)^{\lambda_a}\right]} e^{i2\pi \sqrt{\left[1 - \prod_{a=1}^v (1 - \zeta_{ba}^2)^{\lambda_a}\right]}}, \right. \\ &\quad \left. \left[\prod_{a=1}^v (q_{ba})^{\lambda_a} \right] e^{i2\pi \left[\prod_{a=1}^v (\varrho_{ba})^{\lambda_a} \right]}, \left[\prod_{a=1}^v (f_{ba})^{\lambda_a} \right] e^{i2\pi \left[\prod_{a=1}^v (\rho_{ba})^{\lambda_a} \right]} \right). \end{aligned} \quad (13)$$

Proof. The proof is followed by mathematical induction. \square

Case 1. When $a = 1$, the $C\mathcal{SFNS}_fWA$ operator gives the following equation:

$$\begin{aligned} C\mathcal{SFNS}_fWA(F_{b1}, F_{b2}, \dots, F_{bv}) &= \lambda_1 F_{b1} \\ &= F_{b1}, \end{aligned} \quad (14)$$

where ($\lambda_1 = 1$), and equation (13) becomes

$$\begin{aligned} &= \left((c_1^b), \sqrt{\left[1 - (1 - t_{b1}^2)\right]} e^{i2\pi \sqrt{\left[1 - (1 - \zeta_{b1}^2)\right]}}, [(q_{b1})] e^{i2\pi [(\varrho_{b1})]}, [(f_{ba})] e^{i2\pi [(\rho_{ba})]} \right) \\ &= (c_a^b, t_{ba}e^{i2\pi\zeta_{ba}}, q_{ba}e^{i2\pi\varrho_{ba}}, f_{ba}e^{i2\pi\rho_{ba}}). \end{aligned} \quad (15)$$

The result is true for $a = 1$, as equation (15) is a $C\mathcal{SFNS}_fN$.

Case 2. Suppose that the result hold for $a = n$, n is a natural number, and therefore, equation (13) becomes

$$\begin{aligned} C\mathcal{SFNS}_fWA(F_{b1}, F_{b2}, \dots, F_{bn}) &= \left(\bigoplus_{a=1}^n \lambda_a F_{ba} \right) \\ &= \left(\max_{a=1}^n (c_a^b), \sqrt{\left[1 - \prod_{a=1}^n (1 - t_{ba}^2)^{\lambda_a}\right]} e^{i2\pi \sqrt{\left[1 - \prod_{a=1}^n (1 - \zeta_{ba}^2)^{\lambda_a}\right]}}, \right. \\ &\quad \left. \left[\prod_{a=1}^n (q_{ba})^{\lambda_a} \right] e^{i2\pi \left[\prod_{a=1}^n (\varrho_{ba})^{\lambda_a} \right]}, \left[\prod_{a=1}^n (f_{ba})^{\lambda_a} \right] e^{i2\pi \left[\prod_{a=1}^n (\rho_{ba})^{\lambda_a} \right]} \right). \end{aligned} \quad (16)$$

Now we prove for $a = n + 1$. Consider

$$\begin{aligned}
 CSFNS_f WA(F_{b1}, F_{b2}, \dots, F_{b(n+1)}) &= \left(\bigoplus_{a=1}^{n+1} \lambda_a F_{ba} \right) \\
 &= \left(\max_{a=1}^n (c_a^b), \sqrt{\left[1 - \prod_{a=1}^n (1 - t_{ba}^2)^{\lambda_a} \right]} e^{i2\pi \sqrt{\left[1 - \prod_{a=1}^n (1 - \zeta_{ba}^2)^{\lambda_a} \right]}}, \right. \\
 &\quad \left. \left[\prod_{a=1}^n (q_{ba})^{\lambda_a} \right] e^{i2\pi \left[\prod_{a=1}^n (\varrho_{ba})^{\lambda_a} \right]}, \left[\prod_{a=1}^n (f_{ba})^{\lambda_a} \right] e^{i2\pi \left[\prod_{a=1}^n (\rho_{ba})^{\lambda_a} \right]} \right) \\
 &\quad \oplus \left(c_n^b + 1, \sqrt{\left[1 - (1 - t_{b(n+1)}^2) \right]} e^{i2\pi \sqrt{\left[1 - (1 - \zeta_{b(n+1)}^2) \right]}}, \left[(q_{b(n+1)}) \right] e^{i2\pi \left[(\varrho_{b(n+1)}) \right]}, \right. \\
 &\quad \left. \left[(f_{b(n+1)}) \right] e^{i2\pi \left[(\rho_{b(n+1)}) \right]} \right) \\
 &= \left(\max_{a=1}^{n+1} (c_a^b), \sqrt{\left[1 - \prod_{a=1}^{n+1} (1 - t_{ba}^2)^{\lambda_a} \right]} e^{i2\pi \sqrt{\left[1 - \prod_{a=1}^{n+1} (1 - \zeta_{ba}^2)^{\lambda_a} \right]}}, \right. \\
 &\quad \left. \left[\prod_{a=1}^{n+1} (q_{ba})^{\lambda_a} \right] e^{i2\pi \left[\prod_{a=1}^{n+1} (\varrho_{ba})^{\lambda_a} \right]}, \left[\prod_{a=1}^{n+1} (f_{ba})^{\lambda_a} \right] e^{i2\pi \left[\prod_{a=1}^{n+1} (\rho_{ba})^{\lambda_a} \right]} \right).
 \end{aligned} \tag{17}$$

Result holds for $n + 1$; hence, it is held for all natural numbers.

Theorem 2 (idempotency property). Let $F_{ba} = (c_a^b, t_{ba} e^{i2\pi \zeta_{ba}}, q_{ba} e^{i2\pi \varrho_{ba}}, f_{ba} e^{i2\pi \rho_{ba}})$ ($a = 1, 2, \dots, v$) be a collection of $CSFNS_f$ Ns and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_v)^T$ be the weight vector of F_{ba} , if $F_{ba} = F_{11}$ for all b, a ; then

$$\begin{aligned}
 CSFNS_f WA(F_{b1}, F_{b2}, \dots, F_{bv}) &= \bigoplus_{a=1}^v (\lambda_a F_{ba}) \\
 &= (F_{11}).
 \end{aligned} \tag{18}$$

Theorem 3 (boundedness property). Let $F_{ba} = (c_a^b, t_{ba} e^{i2\pi \zeta_{ba}}, q_{ba} e^{i2\pi \varrho_{ba}}, f_{ba} e^{i2\pi \rho_{ba}})$ be a collection of $CSFNS_f$ Ns and for all b, a ; then

$$F^- \leq CSFNS_f WA \leq F^+, \tag{19}$$

where, $F^+ = (\max_a c_a^b, \max_a [t_{ba}] e^{i2\pi \max_a [\zeta_{ba}]}, \min_a [q_{ba}] e^{i2\pi \min_a [\varrho_{ba}]}, \min_a [f_{ba}] e^{i2\pi \min_a [\rho_{ba}]})$ and $F^- = (\min_a c_a^b, \min_a [t_{ba}] e^{i2\pi \min_a [\zeta_{ba}]}, \max_a [q_{ba}] e^{i2\pi \max_a [\varrho_{ba}]}, \max_a [f_{ba}] e^{i2\pi \max_a [\rho_{ba}]})$.

Theorem 4 (monotonicity property). Consider $F_{ba} = (c_a^b, t_{ba} e^{i2\pi \zeta_{ba}}, q_{ba} e^{i2\pi \varrho_{ba}}, f_{ba} e^{i2\pi \rho_{ba}})$ and $\bar{F}_{ba} = (\bar{c}_a^b, \bar{t}_{ba} e^{i2\pi \bar{\zeta}_{ba}}, \bar{q}_{ba} e^{i2\pi \bar{\varrho}_{ba}}, \bar{f}_{ba} e^{i2\pi \bar{\rho}_{ba}})$ are two collections of $CSFNS_f$ Ns along with $c_a^b \leq \bar{c}_a^b$, $t_{ba} \leq \bar{t}_{ba}$, $\zeta_{ba} \leq \bar{\zeta}_{ba}$, $q_{ba} \geq \bar{q}_{ba}$, $\varrho_{ba} \geq \bar{\varrho}_{ba}$, $f_{ba} \geq \bar{f}_{ba}$, and $\rho_{ba} \geq \bar{\rho}_{ba}$; then

$$CSFNS_f WA(F_{b1}, F_{b2}, \dots, F_{bv}) \leq CSFNS_f WA(\bar{F}_{b1}, \bar{F}_{b2}, \dots, \bar{F}_{bv}). \tag{20}$$

3. Complex Spherical Fuzzy N-Soft- VIKOR Method

This section delineates the VIKOR method to resolve the MAGDM problems employing the domain of $CSFNS_f$ Ss, which is appropriate to deal with two-dimensional data. The proposed method works out to compromise a solution in which an agreement is accomplished by mutual conductance that possesses maximum group utility and minimum individual regret.

Let $J = \{J_1, J_2, J_3, \dots, J_m\}$ be the set of m possible choices whose expertise, possibility, and utility are figured out by k decision-experts $\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3, \dots, \mathbb{Y}_k$, with the help of

v crucial elements $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \dots, \mathcal{X}_v$, treated as attributes, and let $l_1, l_2, l_3, \dots, l_v$ be considered as the normalized weight vectors of the attributes in accordance with related MAGDM problem. Let $\lambda_s \in [0, 1]$ be the weight of the s -th decision-expert \mathbb{Y}_s ; therefore, the normalized weight vector for the experts is $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k)^T$. The strategy for complex spherical fuzzy N-soft-VIKOR ($CSFNS_f$ -VIKOR) method is described as follows.

3.1. Complex Spherical Fuzzy N-Soft Performance Matrix. The alternatives are analyzed by decision-experts on the basis of the selected attributes and initially admeasured by numerical labels representing linguistic information relative

to the MAGDM, such as 5 for “best,” 4 for “good,” 3 for “bad,” and so. These numerical labels are further epitomized by $C\mathcal{SFNS}_fNs$ with constraints of grading criteria, which moves us towards the $kC\mathcal{SFNS}_fSs$ classified by k decision-

experts. Furthermore, the $kC\mathcal{SFNS}_fSs$ regarding k decision-experts are systemized in the form of complex spherical fuzzy N -soft performance matrices $C\mathcal{SFNS}_fPMs\mathcal{Z}^s = (\mathcal{Z}_{ba}^{(s)})_{m \times v}$ as follows:

$$\mathcal{Z}^s = \begin{pmatrix} (c_1^{(s)}, \phi_{11}^{(s)}, \psi_{11}^{(s)}, \chi_{11}^{(s)}) & (c_2^{(s)}, \phi_{12}^{(s)}, \psi_{12}^{(s)}, \chi_{12}^{(s)}) & \dots & (c_v^{(s)}, \phi_{1v}^{(s)}, \psi_{1v}^{(s)}, \chi_{1v}^{(s)}) \\ (c_1^{(s)}, \phi_{21}^{(s)}, \psi_{21}^{(s)}, \chi_{21}^{(s)}) & (c_2^{(s)}, \phi_{22}^{(s)}, \psi_{22}^{(s)}, \chi_{22}^{(s)}) & \dots & (c_v^{(s)}, \phi_{2v}^{(s)}, \psi_{2v}^{(s)}, \chi_{2v}^{(s)}) \\ \vdots & \vdots & \ddots & \vdots \\ (c_1^{(s)}, \phi_{m1}^{(s)}, \psi_{m1}^{(s)}, \chi_{m1}^{(s)}) & (c_2^{(s)}, \phi_{m2}^{(s)}, \psi_{m2}^{(s)}, \chi_{m2}^{(s)}) & \dots & (c_v^{(s)}, \phi_{mv}^{(s)}, \psi_{mv}^{(s)}, \chi_{mv}^{(s)}) \end{pmatrix}, \quad (21)$$

where $\mathcal{Z}_{ba}^{(s)} = ((c_a^b)^{(s)}, \phi_{ba}^{(s)}, \psi_{ba}^{(s)}, \chi_{ba}^{(s)}) = ((c_a^b)^{(s)}, t_{ba}^{(s)} e^{i2\pi\zeta_{ba}^{(s)}}, q_{ba}^{(s)} e^{i2\pi\varrho_{ba}^{(s)}}, f_{ba}^{(s)} e^{i2\pi\rho_{ba}^{(s)}})$, and $b = \{1, 2, 3, \dots, m\}$, $a = \{1, 2, 3, \dots, v\}$, and $s = \{1, 2, 3, \dots, k\}$ stand for the alternatives J_b , attributes \mathcal{X}_a , and decision-experts \mathcal{Y}_s , respectively.

3.2. Aggregated Complex Spherical Fuzzy N -Soft Performance Matrix. All decision-experts are adequate to manipulate the individual opinion using $C\mathcal{SFNS}_fWA$ operator and resulting matrix known as aggregated complex spherical fuzzy N -soft performance matrix is $(AC\mathcal{SFNS}_fPM)$ evaluated as follows:

$$\begin{aligned} \mathcal{Z}_{ba} &= C\mathcal{SFNS}_fA_\lambda(\mathcal{Z}_{ba}^{(1)}, \mathcal{Z}_{ba}^{(2)}, \dots, \mathcal{Z}_{ba}^{(k)}) \\ &= \lambda_1 \mathcal{Z}_{ba}^{(1)} \oplus \lambda_2 \mathcal{Z}_{ba}^{(2)} \oplus \dots \oplus \lambda_s \mathcal{Z}_{ba}^{(k)} \\ &= \left(\max_{s=1}^k (c_a^b)^{(s)}, \sqrt[1 - \prod_{s=1}^k (1 - (t_{ba}^{(s)})^2)^{\lambda_s}} e^{i2\pi \sqrt[1 - \prod_{s=1}^k (1 - (\zeta_{ba}^{(s)})^2)^{\lambda_s}]}, \left(\prod_{s=1}^k q_{ba}^{(s)} \right) e^{i2\pi \prod_{s=1}^k \varrho_{ba}^{(s)}}, \prod_{s=1}^k f_{ba}^{(s)} e^{i2\pi \prod_{s=1}^k \rho_{ba}^{(s)}} \right) \\ &= (c_a^b, t_{ba} e^{i2\pi\zeta_{ba}}, q_{ba} e^{i2\pi\varrho_{ba}}, f_{ba} e^{i2\pi\rho_{ba}}). \end{aligned} \quad (22)$$

The $AC\mathcal{SFNS}_fPM$ is as follows:

$$\mathcal{P} = \begin{pmatrix} (c_1^1, \phi_{11}, \psi_{11}, \chi_{11}) & (c_2^1, \phi_{12}, \psi_{12}, \chi_{12}) & \dots & (c_v^1, \phi_{1v}, \psi_{1v}, \chi_{1v}) \\ (c_1^2, \phi_{21}, \psi_{21}, \chi_{21}) & (c_2^2, \phi_{22}, \psi_{22}, \chi_{22}) & \dots & (c_v^2, \phi_{2v}, \psi_{2v}, \chi_{2v}) \\ \vdots & \vdots & \ddots & \vdots \\ (c_1^m, \phi_{m1}, \psi_{m1}, \chi_{m1}) & (c_2^m, \phi_{m2}, \psi_{m2}, \chi_{m2}) & \dots & (c_v^m, \phi_{mv}, \psi_{mv}, \chi_{mv}) \end{pmatrix}. \quad (23)$$

3.3. Selection of $C\mathcal{SFNS}_f$ Best Value and $C\mathcal{SFNS}_f$ Worst Value. The $C\mathcal{SFNS}_f$ best value is assessed by the following formula denoted by $\hat{\mathcal{Z}}_a$:

$$\hat{\mathcal{Z}}_a = \begin{cases} \max_b \mathcal{Z}_{ba}, & \text{if } \mathcal{X}_a \in \Omega_B, \\ \min_b \mathcal{Z}_{ba}, & \text{if } \mathcal{X}_a \in \Omega_C, \end{cases} \quad (24)$$

where $\hat{\mathcal{Z}}_a = (\hat{c}_a^b, \hat{t}_{ba} e^{i2\pi\hat{\zeta}_{ba}}, \hat{q}_{ba} e^{i2\pi\hat{\varrho}_{ba}}, \hat{f}_{ba} e^{i2\pi\hat{\rho}_{ba}})$.

The $C\mathcal{SFNS}_f$ worst value is assessed by the following formula denoted by $\check{\mathcal{Z}}_a$:

$$\check{\mathcal{Z}}_a = \begin{cases} \max_b \mathcal{Z}_{ba}, & \text{if } \mathcal{X}_a \in \Omega_C, \\ \min_b \mathcal{Z}_{ba}, & \text{if } \mathcal{X}_a \in \Omega_B, \end{cases} \quad (25)$$

where $\check{\mathcal{Z}}_a = (\check{c}_a^b, \check{t}_{ba} e^{i2\pi\check{\zeta}_{ba}}, \check{q}_{ba} e^{i2\pi\check{\varrho}_{ba}}, \check{f}_{ba} e^{i2\pi\check{\rho}_{ba}})$.

The score value $\mathbb{S}(\mathcal{Z}_{ba})$ and accuracy value $\mathbb{A}(\mathcal{Z}_{ba})$ are utilized to compare the $C\mathcal{SFNS}_fNs$ in $AC\mathcal{SFNS}_fPM$.

3.4. Evaluating \mathbb{S}_b and \mathbb{R}_b . The normalized Euclidean distance is utilized to evaluate the group utility measure \mathbb{S}_b and

individual regret measure \mathbb{R}_b with the provision of normalized weights of attributes as follows:

$$\mathbb{S}_b = \sum_{a=1}^v l_a \frac{d(\hat{\mathcal{X}}_a, \mathcal{X}_{ba})}{d(\check{\mathcal{X}}_a, \mathcal{X}_{ba})}, \quad (26)$$

$$\mathbb{R}_b = \max_{1 < a < v} l_a \frac{d(\hat{\mathcal{X}}_a, \mathcal{X}_{ba})}{d(\check{\mathcal{X}}_a, \mathcal{X}_{ba})}. \quad (27)$$

The normalized Euclidean distance $d(\hat{\mathcal{X}}_a, \mathcal{X}_{ba})$ and $d(\check{\mathcal{X}}_a, \mathcal{X}_{ba})$ are calculated as follows:

$$d(\hat{\mathcal{X}}_a, \mathcal{X}_{ba}) = \left(\frac{1}{4} \left[\left(\left(\frac{\hat{c}_a^b}{N-1} \right)^2 - \left(\frac{c_a^b}{N-1} \right)^2 \right)^2 + (\hat{t}_{ba}^2 - t_{ba}^2)^2 + (\hat{q}_{ba}^2 - q_{ba}^2)^2 + (\hat{f}_{ba}^2 - f_{ba}^2)^2 \right. \right. \\ \left. \left. + (\hat{\zeta}_{ba}^2 - \zeta_{ba}^2)^2 + (\hat{\varrho}_{ba}^2 - \varrho_{ba}^2)^2 + (\hat{\rho}_{ba}^2 - \rho_{ba}^2)^2 \right] \right)^{1/2}, \quad (28)$$

$$d(\check{\mathcal{X}}_a, \mathcal{X}_{ba}) = \left(\frac{1}{4} \left[\left(\left(\frac{\check{c}_a^b}{N-1} \right)^2 - \left(\frac{c_a^b}{N-1} \right)^2 \right)^2 + (\check{t}_{ba}^2 - t_{ba}^2)^2 + (\check{q}_{ba}^2 - q_{ba}^2)^2 + (\check{f}_{ba}^2 - f_{ba}^2)^2 \right. \right. \\ \left. \left. + (\check{\zeta}_{ba}^2 - \zeta_{ba}^2)^2 + (\check{\varrho}_{ba}^2 - \varrho_{ba}^2)^2 + (\check{\rho}_{ba}^2 - \rho_{ba}^2)^2 \right] \right)^{1/2}. \quad (29)$$

The optimal values of \mathbb{S}_b and \mathbb{R}_b are as follows:

$$\begin{aligned} \hat{\mathbb{S}} &= \min_b \mathbb{S}_b, \\ \check{\mathbb{S}} &= \max_b \mathbb{S}_b, \\ \hat{\mathbb{R}} &= \min_b \mathbb{R}_b, \\ \check{\mathbb{R}} &= \max_b \mathbb{R}_b. \end{aligned} \quad (30)$$

The $\hat{\mathbb{S}}$ and $\hat{\mathbb{R}}$ correspond to a maximum majority rule index and a minimum regret of opponent strategy, respectively.

3.5. Compromise Measure \mathbb{Q}_b . The configurations of utility measure and regret measure of feasible choice J_b link up for the compromise measure \mathbb{Q}_b as follows:

$$\mathbb{Q} = \xi \left(\frac{\mathbb{S}_b - \hat{\mathbb{S}}_b}{\check{\mathbb{S}}_b - \hat{\mathbb{S}}_b} \right) + (1 - \xi) \left(\frac{\mathbb{R}_b - \hat{\mathbb{R}}_b}{\check{\mathbb{R}}_b - \hat{\mathbb{R}}_b} \right), \quad (31)$$

where $\xi \in [0, 1]$ is the coefficient strategy of the majority of the attributes and in most cases taken as 0.5, usually for the sake of equal weightage of both the configurations, $\xi = 0.5$. Moreover, $\xi = 1$, representing that the compromise solution is biased towards the maximum group utility. On the other hand, $\xi = 0$ shows the biasness towards minimum individual regret.

3.6. Ranking of Alternatives. The values of ranking measures \mathbb{S}_b , \mathbb{R}_b , and \mathbb{Q}_b , corresponding to each alternative, are arranged in ascending order so that we get three ranking lists that further play an important role in finding a compromise

solution. Moreover, the alternative with minimum value regarding three ranking lists is considered as the best feasible option.

3.7. Compromise Solution. For compromise solution consisting of alternative $J^{(1)}$ with a minimum value of ranking measure \mathbb{Q} , the following conditions should hold, which are described as follows:

\mathbb{C}_1 : acceptable advantage:

$$\mathbb{Q}(J^{(2)}) - \mathbb{Q}(J^{(1)}) \geq \frac{1}{m-1}, \quad (32)$$

where $J^{(1)}$ and $J^{(2)}$ are the alternatives with the initial and subsequent position in the ranking list and m representing the number of alternatives.

\mathbb{C}_2 : acceptable stability: $J^{(1)}$ should be ranked first with respect to \mathbb{S} or \mathbb{R} . The compromise solution within the proposed method can be assumed stable in all possible situations of “voting by majority rule ($\xi > 0.5$),” “by consensus ($\xi = 0.5$),” or “by veto ($\xi < 0.5$).”

Moreover, if the condition \mathbb{C}_1 is not satisfied, the compromise solution set contains the alternatives satisfying the following inequality:

$$\mathbb{Q}(J^{(b)}) - \mathbb{Q}(J^{(1)}) < \frac{1}{m-1}, \quad \forall 1 \leq b \leq m. \quad (33)$$

But, if condition \mathbb{C}_2 is not fulfilled, both alternatives $J^{(1)}$ and $J^{(2)}$ are known as compromise solutions of the MAGDM problem.

The flowchart of the proposed $C\mathcal{SFNS}_f$ -VIKOR method is given in Figure 1.

4. Application to Group Decision-Making

In this section, the proposed $C\mathcal{SFNS}_f$ -VIKOR method is carried out for the MAGDM problem in the domain of business administration. Precisely, the proposed $C\mathcal{SFNS}_f$ -VIKOR method is implemented for the selection of firms for the Saudi oil refinery project in Pakistan.

4.1. Selection of Firm for the Saudi Oil Refinery Project in Pakistan. China-Pakistan Economic Corridor (CPEC) is a prudent economic project that upgrades the financial activities in Pakistan and China, serving as a gateway for the Middle East, Europe, and Africa. Specifically, there are advantageous aspects for the people of Gwadar as this project has industrious and everlasting benefits. In a sign of its enormous appliances, Saudi Arabia made a decision to establish a mega oil city in Pakistan at the spot of Gwadar and get involved in business and investment under the flagship of CPEC. A firm or company will be hired for the formation of the mega oil city, as the oil will be imported from Gulf and will be stored at the proposed Gwadar oil city, within the time period of one year. For this purpose, a team of three decision-makers \mathbb{Y}_1 , \mathbb{Y}_2 , and \mathbb{Y}_3 are chosen, and the selected concise list of five firms is as follows:

- J_1 : Rabigh Refining and Petrochemical firm
- J_2 : Petromin Company
- J_3 : Saudi Aramco Total Refining and Petrochemical firm
- J_4 : Alkhorayef Lubricants Company
- J_5 : Yanbu Aramco Sinopec Refining Company (YAS-REF) Ltd.

According to the proficiency and expertise of the MAGDM problem, weights for the decision-makers \mathbb{Y}_1 , \mathbb{Y}_2 , and \mathbb{Y}_3 are 0.34, 0.31, and 0.35, respectively.

Decision-makers thoroughly study the five firms according to the requirements of the project and captured data from the websites <http://www.dnb.com> (Dun and Bradstreet Data Cloud) and <http://www.investopedia.com>, on the basis of precise characteristics or parameters discussed in Table 1.

Decision-makers assigned weight vector $(0.26, 0.24, 0.05, 0.15, 0.30)^T$ to the attributes taking into account the excellence and type of the attributes. For the selection of the best firm, we adopt the $C\mathcal{SFNS}_f$ -VIKOR method as follows:

Step 1: decision-makers modeled six-soft set, arranged in Table 2, related to attributes of the MAGDM problem, where five asterisks mean "excellent performance," four asterisks mean "great performance," three asterisks mean "good performance," two asterisks mean "average performance," one asterisk means "bad performance," and big dot means "very bad performance."

The criteria for the association of the $C\mathcal{SFNS}$ s to 6-soft sets of decision-makers is given in Table 3, so that the opinions of the decision-makers in the form of $C\mathcal{SFNS}_fPMs\mathcal{X}^1$, \mathcal{X}^2 , and \mathcal{X}^3 are arranged in Tables 4, 5, and 6, simultaneously.

Step 2: the judgments of all performance matrices of decision-makers amalgamated into $AC\mathcal{SFNS}_fPM$ using $C\mathcal{SFNS}_fA_\lambda$ operator are summarized in Table 7. Step 3: for the evaluation of the worst and the best values, the score values of the entries of $AC\mathcal{SFNS}_fPM$ are calculated, keeping in view the benefit and cost type criteria where all the attributes are benefit type except \mathcal{X}_3 . The worst and best values are calculated from equations (24) and (25) that are shown in Table 8.

Step 4: the normalized Euclidean distance is formulated by equations (28) and (29), which is utilized to measure the separation of best value from each entry of $C\mathcal{SFNS}_fPM$ and also from worst value simultaneously, organized in Table 9. These distances are further employed in equations (26) and (27) to investigate the group utility measure \mathbb{S} and individual regret measure \mathbb{R} , calculated in Table 10.

Step 5: the compromise measure \mathbb{Q}_b related to each alternative or firm, with $\xi = 0.50$, is estimated through equation (31) and pinned up in Table 10.

Step 6: furthermore, the alternatives are ranked on the basis of the group utility, regret measure, and compromise measure that is illuminated in Table 11.

The firm J_3 has rank order 1 on the basis of the compromise measure \mathbb{Q} , and also, the remaining two conditions of $C\mathcal{SFNS}_f$ -VIKOR method are satisfied and calculated as follows:

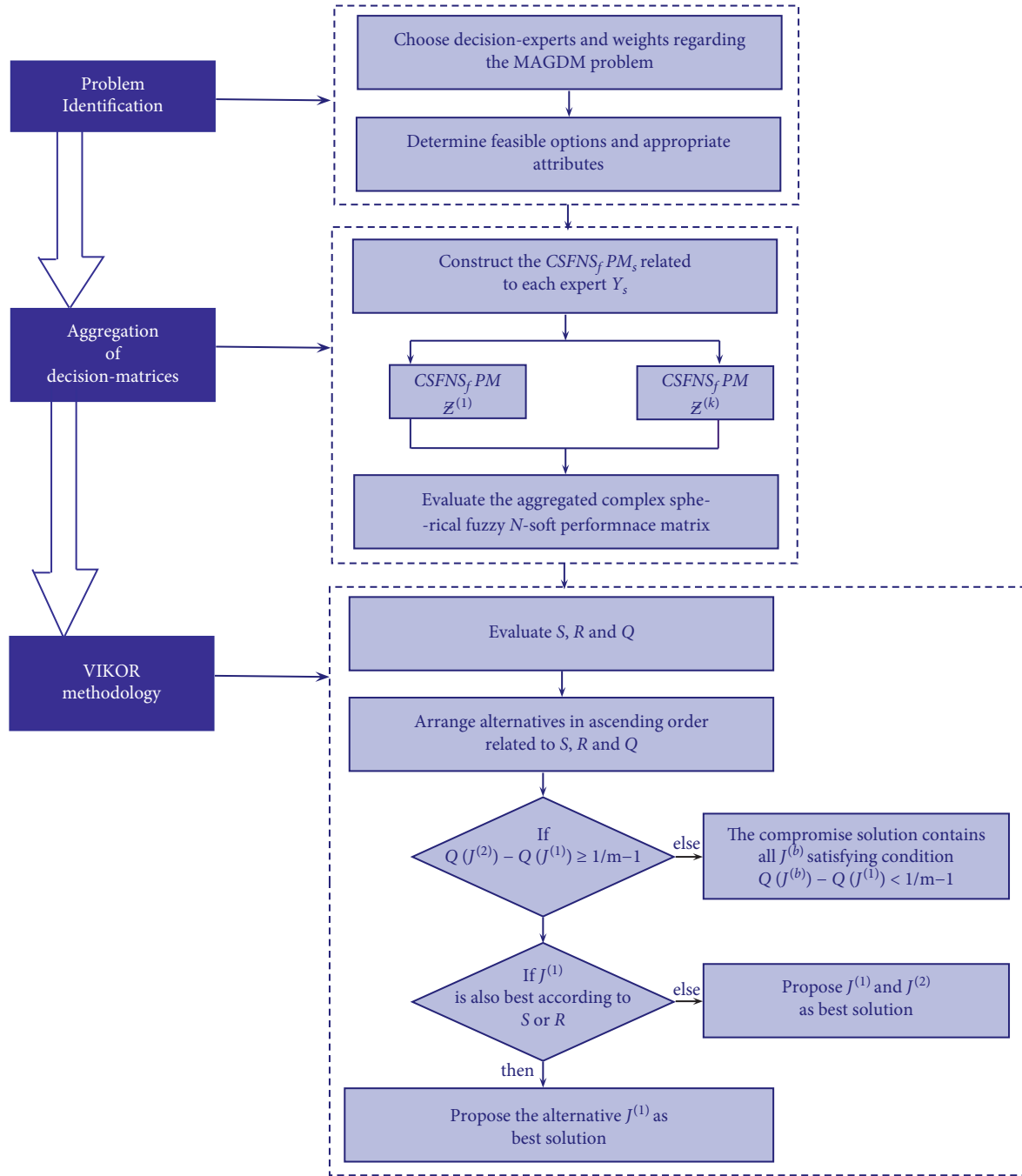
- (1) $\mathbb{Q}(J_4) - \mathbb{Q}(J_3) = 0.5559 - 0 = 0.5559 \geq (1/6 - 1) = 0.2$
- (2) J_3 is the best firm regarding \mathbb{S} and \mathbb{R} , and ranking orders are $J_3 > J_4 > J_5 > J_1 > J_2$

Thus, the firm J_3 is best for the establishment of the Saudi oil refinery project in Pakistan.

5. Sensitivity Test

In the sensitivity test, the role of coefficient strategy ξ playing as the weight of group utility measure and $1 - \xi$ as the weight of individual regret measure is investigated to demonstrate the ability of the proposed $C\mathcal{SFNS}_f$ -VIKOR model and the stability of the solution, which is evaluated by applying the presented methodology.

ξ is assigned different values from the unit interval $[0, 1]$, and then we analyze the strength and potency of the computed decisions from the proposed methodology. In this case, it is assumed that the variation in the coefficient strategy weight occurs because the MAGDM panel can prioritize both (group utility or individual regret) measures, according to the future necessities of the proposed case study. However, $\xi \in [0.6, 1]$ illustrates that the compromise solution is subjective to maximum group utility, and

FIGURE 1: Flowchart of the $CSFNS_f$ -VIKOR method.

$\xi \in [0, 0.4]$ represents the biasness towards minimum individual regret.

Figure 2 illustrate that the variation in ξ from 0 to 1 affects the value of the compromise measure Q of alternatives, but the ranking order of the alternatives and the best solution (i.e., J_3) are static in all cases.

6. Comparative Analysis

This section elaborates the superiority and caliber of the proposed $CSFNS_f$ -VIKOR method by solving the proposed MAGDM problem “Selection of firm for Saudi oil

refinery project in Pakistan” by complex spherical fuzzy- CSF -VIKOR (CSF -VIKOR) method, proposed by Akram et al. [33]. For the selection of the best firm, we adopt the CSF -VIKOR method as follows:

Step 1: the initial judgments of the decision-makers are in the form of grade level as shown in Table 2 that further transformed into $CSFNs$, following the grading criteria defined in Table 3. The personal assessment of the decision-makers related to the parameters are denoted by $CSFNS_f PMs$ (\mathcal{Z}^1 , \mathcal{Z}^2 , and \mathcal{Z}^3) and arranged in Tables 12, 13, and 14 respectively.

TABLE 1: Study of attributes for the MAGDM problem.

\mathcal{X}_1	Strong risk Management	It is imperative to select a business firm that is financially strong and fully endorsed.	Benefit type
\mathcal{X}_2	Modern technology:	It includes the latest apparatus and machinery, up-to-date software, and other high-tech innovations.	Benefit type
\mathcal{X}_3	Cost price	This attribute includes initial cost, maintenance cost, and production cost. The firm with the lowest cost is preferable.	Cost type
\mathcal{X}_4	Plan and vision	It is necessary to choose a firm whose strategy related to the project is impactful and profitable.	Benefit type
\mathcal{X}_5	Experience	It includes business-related experience, knowledge, and expertise with fluctuating track record and success.	Benefit type

TABLE 2: Decision-makers opinions regarding attributes.

Attributes	Alternatives	\mathbb{Y}_1	\mathbb{Y}_2	\mathbb{Y}_3
\mathcal{X}_1	J_1	$\ast = 1$	$\ast \ast = 2$	$\cdot = 0$
	J_2	$\ast = 1$	$\cdot = 0$	$\ast = 1$
	J_3	$\ast \ast \ast \ast = 5$	$\ast \ast \ast \ast = 4$	$\ast \ast \ast \ast = 5$
	J_4	$\ast \ast \ast \ast = 4$	$\ast \ast \ast = 3$	$\ast \ast \ast \ast = 4$
	J_5	$\ast \ast \ast = 3$	$\ast \ast = 2$	$\ast \ast \ast = 3$
\mathcal{X}_2	J_1	$\cdot = 0$	$\ast = 1$	$\cdot = 0$
	J_2	$\ast = 1$	$\ast = 1$	$\ast = 1$
	J_3	$\ast \ast \ast \ast = 4$	$\ast \ast \ast \ast \ast = 5$	$\ast \ast \ast \ast \ast = 5$
	J_4	$\ast \ast \ast = 3$	$\ast \ast \ast \ast = 4$	$\ast \ast \ast \ast = 4$
	J_5	$\ast \ast = 2$	$\ast \ast \ast = 3$	$\ast \ast \ast = 3$
\mathcal{X}_3	J_1	$\cdot = 0$	$\cdot = 0$	$\cdot = 0$
	J_2	$\cdot = 0$	$\ast = 1$	$\ast = 1$
	J_3	$\ast \ast \ast \ast = 4$	$\ast \ast \ast \ast \ast = 5$	$\ast \ast \ast \ast \ast = 5$
	J_4	$\ast \ast \ast \ast = 4$	$\ast \ast \ast \ast \ast = 5$	$\ast \ast \ast \ast = 4$
	J_5	$\ast \ast \ast = 3$	$\ast \ast \ast = 3$	$\ast \ast \ast = 3$
\mathcal{X}_4	J_1	$\cdot = 0$	$\cdot = 0$	$\ast = 1$
	J_2	$\ast \ast = 2$	$\ast = 1$	$\ast \ast = 2$
	J_3	$\ast \ast \ast \ast \ast = 5$	$\ast \ast \ast \ast \ast = 5$	$\ast \ast \ast \ast = 4$
	J_4	$\ast \ast \ast \ast = 4$	$\ast \ast \ast = 3$	$\ast \ast \ast \ast \ast = 5$
	J_5	$\ast \ast \ast = 3$	$\ast \ast \ast = 3$	$\ast \ast = 2$
\mathcal{X}_5	J_1	$\ast = 1$	$\cdot = 0$	$\cdot = 0$
	J_2	$\ast = 1$	$\ast = 1$	$\ast = 1$
	J_3	$\ast \ast \ast \ast \ast = 5$	$\ast \ast \ast \ast \ast = 5$	$\ast \ast \ast \ast \ast = 5$
	J_4	$\ast \ast \ast \ast = 4$	$\ast \ast \ast \ast = 4$	$\ast \ast \ast \ast = 4$
	J_5	$\ast \ast \ast = 3$	$\ast \ast \ast = 3$	$\ast \ast \ast = 3$

TABLE 3: Grading criteria for $CSF6SS$.

h_z^w/J	Degree of yes		Neutral membership		Degree of no	
Grades	t_{ba}	$2\pi\zeta_{ba}$	q_{ba}	$2\pi\varrho_{ba}$	f_{ba}	$2\pi\rho_{ba}$
$c_a^b = 0$		$[0, 0.30\pi)$	$[0, 0.0170)$	$[0, 0.0340\pi)$	$(0.90, 1.00]$	$[1.80\pi, 2.00\pi]$
$c_a^b = 1$	$[0.15, 0.30)$	$[0.30\pi, 0.60\pi)$	$[0.0170, 0.0819)$	$[0.0340\pi, 0.1638\pi)$	$(0.75, 0.90]$	$[1.50\pi, 1.80\pi]$
$c_a^b = 2$	$[0.30, 0.50)$	$[0.60\pi, 1.00\pi)$	$[0, 0.0170)$	$[0, 0.0340\pi)$	$(0.50, 0.75]$	$[1.00\pi, 1.50\pi]$
$c_a^b = 3$	$[0.50, 0.75)$	$[1.00\pi, 1.50\pi)$	$[0.0170, 0.0819)$	$[0.0340\pi, 0.1638\pi)$	$(0.30, 0.50]$	$[0.60\pi, 1.00\pi]$
$c_a^b = 4$	$[0.75, 0.90)$	$[1.50\pi, 1.80\pi)$	$[0, 0.0170)$	$[0, 0.0340\pi)$	$(0.15, 0.30]$	$[0.30\pi, 0.60\pi]$
$c_a^b = 5$	$[0.90, 1.00]$	$[1.80\pi, 2.00\pi]$	$[0.0170, 0.0819)$	$[0.0340\pi, 0.1638\pi)$	$[0, 0.15]$	$[0, 0.30\pi)$

TABLE 4: $C\mathcal{EFNS}_f$ PM of decision-maker \mathbb{V}_1 .

	\mathcal{X}_1	\mathcal{X}_2	\mathcal{X}_3	\mathcal{X}_4	\mathcal{X}_5
J_1	$(1, (0.160e^{j0.340\pi}, 0.021e^{j0.044\pi}, 0.092e^{j1.860\pi}))$	$(0, (0.070e^{j0.120\pi}, 0.015e^{j0.028\pi}, 0.985e^{j1.972\pi}))$	$(0, (0.08e^{j0.140\pi}, 0.016e^{j0.030\pi}, 0.986e^{j1.974\pi}))$	$(0, (0.090e^{j0.160\pi}, 0.017e^{j0.032\pi}, 0.987e^{j1.976\pi}))$	$(1, (0.170e^{j0.360\pi}, 0.022e^{j0.026\pi}, 0.930e^{j1.880\pi}))$
J_2	$(1, (0.180e^{j0.380\pi}, 0.230e^{j0.480\pi}, 0.940e^{j1.900\pi}))$	$(1, (0.190e^{j0.400\pi}, 0.025e^{j0.052\pi}, 0.950e^{j1.920\pi}))$	$(0, (0.130e^{j0.240\pi}, 0.012e^{j0.026\pi}, 0.994e^{j1.990\pi}))$	$(2, (0.310e^{j0.640\pi}, 0.011e^{j0.024\pi}, 0.076e^{j1.54\pi}))$	$(1, (0.190e^{j0.400\pi}, 0.024e^{j0.050\pi}, 0.950e^{j1.920\pi}))$
J_3	$(5, (0.900e^{j1.820\pi}, 0.020e^{j0.360\pi}, 0.010e^{j0.040\pi}))$	$(4, (0.760e^{j1.540\pi}, 0.010e^{j0.022\pi}, 0.200e^{j0.420\pi}))$	$(4, (0.770e^{j1.560\pi}, 0.011e^{j0.024\pi}, 0.220e^{j0.420\pi}))$	$(5, (0.910e^{j1.840\pi}, 0.018e^{j0.038\pi}, 0.010e^{j0.040\pi}))$	$(5, (0.920e^{j1.860\pi}, 0.019e^{j0.040\pi}, 0.020e^{j0.060\pi}))$
J_4	$(3, (0.510e^{j1.04\pi}, 0.028e^{j0.058\pi}, 0.450e^{j0.920\pi}))$	$(2, (0.320e^{j0.660\pi}, 0.013e^{j0.028\pi}, 0.770e^{j1.560\pi}))$	$(3, (0.520e^{j1.060\pi}, 0.020e^{j0.060\pi}, 0.46e^{j0.940\pi}))$	$(3, (0.530e^{j1.080\pi}, 0.031e^{j0.064\pi}, 0.470e^{j0.960\pi}))$	$(3, (0.540e^{j1.100\pi}, 0.032e^{j0.066\pi}, 0.480e^{j0.980\pi}))$
J_5	$(1, (0.200e^{j0.42\pi}, 0.026e^{j0.054\pi}, 0.960e^{j1.940\pi}))$	$(2, (0.350e^{j0.680\pi}, 0.014e^{j0.030\pi}, 0.780e^{j1.580\pi}))$	$(1, (0.210e^{j0.420\pi}, 0.027e^{j0.056\pi}, 0.970e^{j1.920\pi}))$	$(2, (0.340e^{j0.700\pi}, 0.015e^{j0.032\pi}, 0.790e^{j1.600\pi}))$	$(2, (0.350e^{j0.720\pi}, 0.016e^{j0.034\pi}, 0.810e^{j1.640\pi}))$

TABLE 5: $C\mathcal{EFNS}_f$ PM of decision-maker \mathbb{V}_2 .

	\mathcal{X}_1	\mathcal{X}_2	\mathcal{X}_3	\mathcal{X}_4	\mathcal{X}_5
J_1	$(2, (0.360e^{j0.740\pi}, 0.012e^{j0.022\pi}, 0.820e^{j1.660\pi}))$	$(1, (0.220e^{j0.460\pi}, 0.028e^{j0.058\pi}, 0.960e^{j1.900\pi}))$	$(0, (0.100e^{j0.180\pi}, 0.013e^{j0.028\pi}, 0.989e^{j1.980\pi}))$	$(0, (0.110e^{j0.200\pi}, 0.014e^{j0.030\pi}, 0.991e^{j1.984\pi}))$	$(0, (0.120e^{j0.220\pi}, 0.015e^{j0.032\pi}, 0.993e^{j1.988\pi}))$
J_2	$(0, (0.110e^{j0.180\pi}, 0.013e^{j0.028\pi}, 0.990e^{j1.978\pi}))$	$(1, (0.230e^{j0.480\pi}, 0.030e^{j0.062\pi}, 0.950e^{j1.880\pi}))$	$(1, (0.240e^{j0.500\pi}, 0.031e^{j0.064\pi}, 0.940e^{j1.86\pi}))$	$(1, (0.250e^{j0.520\pi}, 0.033e^{j0.068\pi}, 0.930e^{j1.84\pi}))$	$(1, (0.260e^{j0.540\pi}, 0.034e^{j0.070\pi}, 0.920e^{j1.860\pi}))$
J_3	$(4, (0.820e^{j1.660\pi}, 0.017e^{j0.032\pi}, 0.250e^{j0.320\pi}))$	$(5, (0.930e^{j1.88\pi}, 0.020e^{j0.042\pi}, 0.030e^{j0.080\pi}))$	$(5, (0.940e^{j1.900\pi}, 0.021e^{j0.040\pi}, 0.040e^{j0.100\pi}))$	$(5, (0.950e^{j1.920\pi}, 0.023e^{j0.048\pi}, 0.050e^{j0.120\pi}))$	$(5, (0.960e^{j1.940\pi}, 0.024e^{j0.050\pi}, 0.060e^{j0.140\pi}))$
J_4	$(2, (0.390e^{j0.800\pi}, 0.014e^{j0.026\pi}, 0.840e^{j1.700\pi}))$	$(3, (0.570e^{j1.160\pi}, 0.036e^{j0.074\pi}, 0.530e^{j1.040\pi}))$	$(3, (0.580e^{j1.180\pi}, 0.037e^{j0.076\pi}, 0.530e^{j1.040\pi}))$	$(3, (0.590e^{j1.200\pi}, 0.038e^{j0.078\pi}, 0.530e^{j1.080\pi}))$	$(2, (0.380e^{j0.780\pi}, 0.013e^{j0.024\pi}, 0.830e^{j1.680\pi}))$
J_5	$(1, (0.270e^{j0.560\pi}, 0.036e^{j0.074\pi}, 0.940e^{j1.860\pi}))$	$(1, (0.280e^{j0.580\pi}, 0.037e^{j0.076\pi}, 0.950e^{j1.88\pi}))$	$(2, (0.410e^{j0.840\pi}, 0.015e^{j0.028\pi}, 0.850e^{j1.720\pi}))$	$(2, (0.420e^{j0.860\pi}, 0.016e^{j0.034\pi}, 0.860e^{j1.94\pi}))$	$(2, (0.430e^{j0.880\pi}, 0.017e^{j0.032\pi}, 0.870e^{j1.760\pi}))$

TABLE 6: $C\mathcal{EFNS}_f$ PM of decision-maker \mathbb{V}_3 .

	\mathcal{X}_1	\mathcal{X}_2	\mathcal{X}_3	\mathcal{X}_4	\mathcal{X}_5
J_1	$(0, (0.030e^{j0.040\pi}, 0.011e^{j0.060\pi}, 0.981e^{j1.564\pi}))$	$(0, (0.040e^{j0.060\pi}, 0.012e^{j0.026\pi}, 0.982e^{j1.966\pi}))$	$(0, (0.05e^{j0.080\pi}, 0.013e^{j0.028\pi}, 0.983e^{j0.968\pi}))$	$(1, (0.270e^{j0.520\pi}, 0.040e^{j0.082\pi}, 0.960e^{j1.900\pi}))$	$(0, (0.060e^{j0.100\pi}, 0.014e^{j0.030\pi}, 0.984e^{j1.970\pi}))$
J_2	$(1, (0.290e^{j0.480\pi}, 0.038e^{j0.078\pi}, 0.940e^{j1.86\pi}))$	$(1, (0.280e^{j0.540\pi}, 0.039e^{j0.080\pi}, 0.950e^{j1.88\pi}))$	$(1, (0.270e^{j0.520\pi}, 0.041e^{j0.084\pi}, 0.960e^{j1.900\pi}))$	$(2, (0.440e^{j0.900\pi}, 0.016e^{j0.030\pi}, 0.880e^{j1.780\pi}))$	$(1, (0.260e^{j0.500\pi}, 0.042e^{j0.086\pi}, 0.970e^{j1.920\pi}))$
J_3	$(5, (0.990e^{j1.960\pi}, 0.071e^{j0.144\pi}, 0.100e^{j0.220\pi}))$	$(5, (0.980e^{j1.940\pi}, 0.072e^{j0.146\pi}, 0.110e^{j0.240\pi}))$	$(5, (0.970e^{j1.920\pi}, 0.073e^{j0.148\pi}, 0.120e^{j0.260\pi}))$	$(4, (0.840e^{j1.700\pi}, 0.014e^{j0.026\pi}, 0.270e^{j0.460\pi}))$	$(5, (0.960e^{j1.900\pi}, 0.074e^{j0.150\pi}, 0.130e^{j0.280\pi}))$
J_4	$(3, (0.630e^{j1.280\pi}, 0.043e^{j0.084\pi}, 0.560e^{j1.140\pi}))$	$(3, (0.640e^{j1.300\pi}, 0.044e^{j0.086\pi}, 0.570e^{j1.160\pi}))$	$(3, (0.650e^{j1.320\pi}, 0.043e^{j0.088\pi}, 0.580e^{j1.180\pi}))$	$(2, (0.480e^{j0.980\pi}, 0.012e^{j0.022\pi}, 0.860e^{j1.700\pi}))$	$(3, (0.620e^{j1.260\pi}, 0.042e^{j0.082\pi}, 0.310e^{j0.640\pi}))$
J_5	$(2, (0.450e^{j0.920\pi}, 0.015e^{j0.028\pi}, 0.890e^{j1.760\pi}))$	$(2, (0.460e^{j0.940\pi}, 0.014e^{j0.026\pi}, 0.880e^{j1.740\pi}))$	$(2, (0.470e^{j0.960\pi}, 0.013e^{j0.024\pi}, 0.870e^{j1.720\pi}))$	$(3, (0.600e^{j1.220\pi}, 0.039e^{j0.082\pi}, 0.540e^{j1.100\pi}))$	$(2, (0.490e^{j0.940\pi}, 0.011e^{j0.020\pi}, 0.850e^{j1.720\pi}))$

TABLE 7: Aggregated complex spherical fuzzy N -soft performance matrix.

	\mathcal{X}_1	\mathcal{X}_2	\mathcal{X}_3	\mathcal{X}_4	\mathcal{X}_5
J_1	(2, (0.2254e ^{j0.4656π} , 0.0133e ^{j0.0280π} , 0.0168e ^{j0.2698π} , 0.0168e ^{j0.0340π} , 0.9761e ^{j1.9472π}))	(1, (0.1321e ^{j0.2698π} , 0.0168e ^{j0.0340π} , 0.9761e ^{j1.9472π}))	(0, (0.0784e ^{j0.1376π} , 0.0139e ^{j0.0286π} , 0.9858e ^{j1.9736π}))	(1, (0.1802e ^{j0.3426π} , 0.0216e ^{j0.0436π} , 0.9786e ^{j1.9516π}))	(1, (0.1249e ^{j0.2506π} , 0.0166e ^{j0.0352π} , 0.9680e ^{j1.9444π}))
J_2	(1, (0.2114e ^{j0.4130π} , 0.0229e ^{j0.0480π} , 0.9552e ^{j1.9094π}))	(1, (0.2372e ^{j0.4776π} , 0.0309e ^{j0.0638π} , 0.9500e ^{j1.8934π}))	(1, (0.2222e ^{j0.4392π} , 0.0247e ^{j0.0518π} , 0.9651e ^{j1.9174π}))	(2, (0.3487e ^{j0.7170π} , 0.0176e ^{j0.0388π} , 0.8516e ^{j1.7118π}))	(1, (0.2387e ^{j0.4822π} , 0.0325e ^{j0.0670π} , 0.9475e ^{j1.9012π}))
J_3	(5, (0.9475e ^{j1.8722π} , 0.0296e ^{j0.0678π} , 0.0607e ^{j0.1600π}))	(5, (0.9329e ^{j1.8540π} , 0.0247e ^{j0.0530π} , 0.0901e ^{j0.2064π}))	(5, (0.9270e ^{j1.8492π} , 0.0260e ^{j0.0546π} , 0.1049e ^{j0.2274π}))	(5, (0.9088e ^{j1.8176π} , 0.0178e ^{j0.0388π} , 0.0522e ^{j1.4160π}))	(5, (0.9494e ^{j1.9044π} , 0.0328e ^{j0.0680π} , 0.0541e ^{j1.3368π}))
J_4	(3, (0.5305e ^{j1.0812π} , 0.0262e ^{j0.0514π} , 0.5895e ^{j1.1996π}))	(3, (0.5288e ^{j1.0976π} , 0.0273e ^{j0.0560π} , 0.6172e ^{j1.2402π}))	(3, (0.5892e ^{j1.1984π} , 0.0358e ^{j0.0738π} , 0.5182e ^{j1.0564π}))	(3, (0.5344e ^{j1.0890π} , 0.0236e ^{j0.0488π} , 0.6027e ^{j1.2054π}))	(3, (0.5337e ^{j1.0876π} , 0.0266e ^{j0.0520π} , 0.5966e ^{j1.2136π}))
J_5	(2, (0.3316e ^{j0.6826π} , 0.0237e ^{j0.0472π} , 0.9288e ^{j1.8505π}))	(2, (0.3766e ^{j0.7608π} , 0.0189e ^{j0.0380π} , 0.8649e ^{j1.7248π}))	(2, (0.3842e ^{j0.7878π} , 0.0174e ^{j0.0336π} , 0.8963e ^{j1.7854π}))	(3, (0.4769e ^{j0.9738π} , 0.0214e ^{j0.0426π} , 0.7099e ^{j1.4402π}))	(2, (0.4294e ^{j0.8583π} , 0.0142e ^{j0.0276π} , 0.8422e ^{j1.7044π}))

TABLE 8: $CSFNS_f$ best and worst values related to the attributes.

Attribute	Best value	Worst value
\mathcal{X}_1	$(5, (0.9475e^{i1.8722\pi}, 0.0296e^{i0.0678\pi}, 0.0607e^{i0.1608\pi}))$	$(1, (0.2114e^{i0.4130\pi}, 0.0229e^{i0.0480\pi}, 0.9552e^{i1.9094\pi}))$
\mathcal{X}_2	$(5, (0.9329e^{i1.8540\pi}, 0.0247e^{i0.0520\pi}, 0.0901e^{i0.2064\pi}))$	$(1, (0.1321e^{i0.2698\pi}, 0.0168e^{i0.0340\pi}, 0.9761e^{i1.9472\pi}))$
\mathcal{X}_3	$(0, (0.0784e^{i0.1376\pi}, 0.0139e^{i0.0286\pi}, 0.9858e^{i1.9736\pi}))$	$(5, (0.9270e^{i1.8492\pi}, 0.0260e^{i0.0546\pi}, 0.1049e^{i0.2274\pi}))$
\mathcal{X}_4	$(5, (0.9088e^{i1.8176\pi}, 0.0178e^{i0.0356\pi}, 0.0522e^{i1.4160i\pi}))$	$(1, (0.1802e^{i0.3426\pi}, 0.0216e^{i0.0436\pi}, 0.9786e^{i1.9516i\pi}))$
\mathcal{X}_5	$(5, (0.9494e^{i1.9044\pi}, 0.0328e^{i0.0680\pi}, 0.0541e^{i1.3360i\pi}))$	$(1, (0.1249e^{i0.2506\pi}, 0.0166e^{i0.0352\pi}, 0.9680e^{i1.9444i\pi}))$

TABLE 9: Normalized Euclidean distance.

Alternative	$d(\hat{\mathcal{X}}_1, \mathcal{Z}_{b1})$	$d(\hat{\mathcal{X}}_2, \mathcal{Z}_{b2})$	$d(\hat{\mathcal{X}}_3, \mathcal{Z}_{b3})$	$d(\hat{\mathcal{X}}_4, \mathcal{Z}_{b4})$	$d(\hat{\mathcal{X}}_5, \mathcal{Z}_{b5})$
J_1	0.9304	0.9660	0.00049	1.0035	1.0316
J_2	0.9986	0.9124	0.0497	0.8319	0.9954
J_3	0	0	1.0362	0	0
J_4	0.5860	0.5596	0.5806	0.5628	0.5957
J_5	0.9169	0.7890	0.1785	0.6430	0.8308

TABLE 10: The values of \mathbb{S}_b , \mathbb{R}_b , and \mathbb{Q}_b .

Alternative	\mathbb{S}_b	\mathbb{R}_b	\mathbb{Q}_b
J_1	0.9322	0.2422	0.9014
J_2	0.9029	0.2894	0.9833
J_3	0.05	0.05	0
J_4	0.5769	0.1732	0.5559
J_5	0.7810	0.2416	0.7418

TABLE 11: Ranking of firms.

Alternatives	J_1	J_2	J_3	J_4	J_5
Ranking order of \mathbb{S}	5	4	1	2	3
Ranking order of \mathbb{R}	5	4	1	2	3
Ranking order of \mathbb{Q}	5	4	1	2	3

Step 2: the aggregated complex spherical fuzzy performance matrix $ACSFPM$ securing the collective opinion of the decision-makers through a complex

spherical fuzzy weighted average ($CSFWA$) operator is employing as follows:

$$\mathcal{Z}_{ba} = \left(\sqrt{1 - \prod_{s=1}^k \left(1 - (t_{ba}^{(s)})^2 \right)^{\lambda_s}} e^{i2\pi \sqrt{1 - \prod_{s=1}^k \left(1 - (\zeta_{ba}^{(s)})^2 \right)^{\lambda_s}}}, \left(\prod_{s=1}^k q_{ba}^{(s)} \right) e^{i2\pi \prod_{s=1}^k \varrho_{ba}^{(s)}}, \prod_{s=1}^k f_{ba}^{(s)} e^{i2\pi \prod_{s=1}^k \rho_{ba}^{(s)}} \right). \quad (34)$$

The $ACSFPM$ shown in Table 15.

Step 3: furthermore, for the evaluation of worst and best attributes in $ACSFPM$, equations (24) and (25) are brought into play, despite that the comparison of complex spherical fuzzy numbers in $ACSFPM$ is computed by exploiting the score function of each entry as follows:

$$Sc(\mathcal{Z}_{ba}) = [t_{ba}^2 - q_{ba}^2 - f_{ba}^2] + [\zeta_{ba}^2 - \varrho_{ba}^2 - \rho_{ba}^2]. \quad (35)$$

CSF 's best and worst values related to the attributes are given in Table 16.

Step 4: the group utility measure \mathbb{S} and individual regret measure \mathbb{R} of each alternative J_b are valued through equations (26) and (27) in the light of

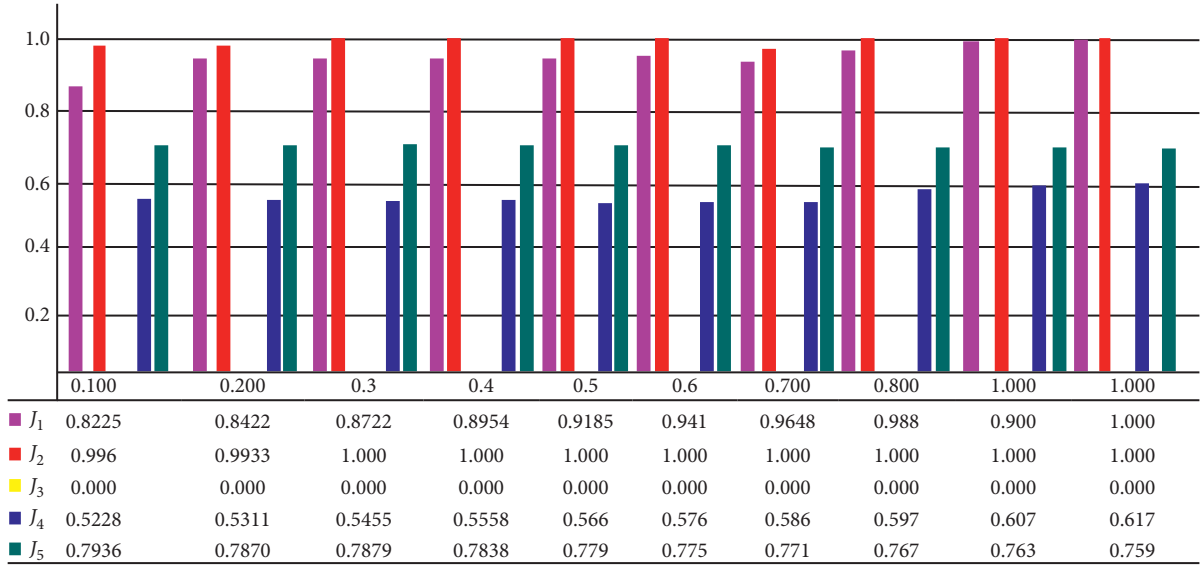


FIGURE 2: Sensitivity test.

normalized distance measures, indicated in the following equations and worthy attributes.

$$d(\hat{\mathcal{X}}_a, \mathcal{X}_{ba}) = \sqrt{\left[\frac{1}{3} \left[(\hat{t}_{ba}^2 - t_{ba}^2)^2 + (\hat{q}_{ba}^2 - q_{ba}^2)^2 + (\hat{f}_{ba}^2 - f_{ba}^2)^2 + (\hat{\zeta}_{ba}^2 - \zeta_{ba}^2)^2 + (\hat{\varrho}_{ba}^2 - \varrho_{ba}^2)^2 + (\hat{\rho}_{ba}^2 - \rho_{ba}^2)^2 \right] \right]}. \quad (36)$$

$$d(\tilde{\mathcal{X}}_a, \mathcal{X}_{ba}) = \sqrt{\left[\frac{1}{3} \left[(\tilde{t}_{ba}^2 - t_{ba}^2)^2 + (\tilde{q}_{ba}^2 - q_{ba}^2)^2 + (\tilde{f}_{ba}^2 - f_{ba}^2)^2 + (\tilde{\zeta}_{ba}^2 - \zeta_{ba}^2)^2 + (\tilde{\varrho}_{ba}^2 - \varrho_{ba}^2)^2 + (\tilde{\rho}_{ba}^2 - \rho_{ba}^2)^2 \right] \right]}. \quad (37)$$

The normalized distance in equations (36) and (37) is symmetric as well as it gives positive values corresponding to each entry in $AC\mathcal{SFPM}$, listed in Table 17. In addition to that, the group utility measure \mathbb{S} , individual regret measure \mathbb{R} and compromise ranking \mathbb{Q} (around with equation (31) and $\xi = 0.50$) are calculated in Table 18.

Step 5: following the values of \mathbb{S} , \mathbb{R} , and \mathbb{Q} from Table 1 in ascending order, the priority order of alternatives is presented in Table 19.

Step 6: the firm J_3 has rank order one on the basis of the compromise measure \mathbb{Q} , and also, the remaining two conditions of the $C\mathcal{SF}$ -VIKOR method are satisfied and calculated as follows:

- (1) $\mathbb{Q}(J_4) - \mathbb{Q}(J_3) = 0.5262 - 0 = 0.5262 \geq (1/6 - 1) = 0.2$
- (2) J_3 is best firm regarding \mathbb{S} and \mathbb{R}

Thus, the firm J_3 is also the best feasible choice for the establishment of the Saudi oil refinery project in Pakistan with respect to $C\mathcal{SF}$ -VIKOR method.

6.1. Discussion. The comparison provides the following results:

- (1) The compromise solution assessed by the $C\mathcal{SFNS}_f$ -VIKOR and $C\mathcal{SF}$ -VIKOR methods is the same exact, as indicated in Table 19, which implies that J_3 is the best firm for the development of the Saudi oil refinery at the place of Gwadar. Together with the same results, the rankings of the alternatives in both methods are unvarying. So, we can claim that the outcomes of the proposed $C\mathcal{SFNS}_f$ -VIKOR method are equivalent to the $C\mathcal{SF}$ -VIKOR method. The correlation between $C\mathcal{SFNS}_f$ -VIKOR and $C\mathcal{SF}$ -VIKOR methods is illustrated in Table 20, and their graphical representations are put forward in Figure 3.
- (2) We also implement the techniques of \mathcal{SF} -VIKOR [27] and fuzzy-VIKOR [8] on the proposed MAGDM problem. The transparency and accuracy in the outcomes improve and glorify the level of trust and confidence about the proposed method. The comparison results are pinned up in Table 21.
- (3) The comparative study of the $C\mathcal{SFNS}_f$ -VIKOR method with the existing techniques demonstrates and certifies the superiority of the proposed method as fuzzy-VIKOR, \mathcal{SF} -VIKOR, and $C\mathcal{SF}$ -VIKOR methods have some imperfections and limitations

TABLE 12: Complex spherical fuzzy performance matrix of decision-maker \mathbb{V}_1 .

	\mathcal{X}_1	\mathcal{X}_2	\mathcal{X}_3	\mathcal{X}_4	\mathcal{X}_5
J_1	$(0.160e^{j0.340\pi}, 0.021e^{j0.044\pi}, 0.092e^{j1.860\pi})$	$(0.070e^{j0.120\pi}, 0.015e^{j0.028\pi}, 0.985e^{j1.972\pi})$	$(0.08e^{j0.140\pi}, 0.016e^{j0.030\pi}, 0.986e^{j1.974\pi})$	$(0.09e^{j0.160\pi}, 0.017e^{j0.032\pi}, 0.987e^{j1.976\pi})$	$(0.170e^{j0.360\pi}, 0.022e^{j0.026\pi}, 0.930e^{j1.88\pi})$
J_2	$(0.180e^{j0.380\pi}, 0.230e^{j0.480\pi}, 0.940e^{j1.900\pi})$	$(0.190e^{j0.400\pi}, 0.025e^{j0.052\pi}, 0.950e^{j1.920\pi})$	$(0.130e^{j0.240\pi}, 0.012e^{j0.026\pi}, 0.994e^{j1.990\pi})$	$(0.310e^{j0.640\pi}, 0.011e^{j0.024\pi}, 0.076e^{j1.54\pi})$	$(0.190e^{j0.400\pi}, 0.024e^{j0.050\pi}, 0.950e^{j1.920\pi})$
J_3	$(0.900e^{j1.820\pi}, 0.02e^{j0.060\pi}, 0.010e^{j0.040\pi})$	$(0.760e^{j1.540\pi}, 0.010e^{j0.022\pi}, 0.200e^{j0.420\pi})$	$(0.770e^{j1.560\pi}, 0.011e^{j0.024\pi}, 0.220e^{j0.420\pi})$	$(0.910e^{j1.840\pi}, 0.018e^{j0.038\pi}, 0.010e^{j0.040\pi})$	$(0.920e^{j1.860\pi}, 0.019e^{j0.040\pi}, 0.020e^{j0.060\pi})$
J_4	$(0.510e^{j1.04\pi}, 0.028e^{j0.058\pi}, 0.450e^{j0.920\pi})$	$(0.320e^{j0.660\pi}, 0.013e^{j0.028\pi}, 0.770e^{j1.560\pi})$	$(0.520e^{j1.060\pi}, 0.020e^{j0.060\pi}, 0.46e^{j0.940\pi})$	$(0.530e^{j1.080\pi}, 0.031e^{j0.064\pi}, 0.470e^{j0.960\pi})$	$(0.540e^{j1.100\pi}, 0.032e^{j0.066\pi}, 0.480e^{j0.980\pi})$
J_5	$(0.200e^{j0.42\pi}, 0.026e^{j0.054\pi}, 0.960e^{j1.94\pi})$	$(0.350e^{j0.680\pi}, 0.014e^{j0.030\pi}, 0.780e^{j1.58\pi})$	$(0.210e^{j0.420\pi}, 0.027e^{j0.056\pi}, 0.970e^{j1.92\pi})$	$(0.340e^{j0.700\pi}, 0.015e^{j0.032\pi}, 0.790e^{j1.600\pi})$	$(0.350e^{j0.720\pi}, 0.016e^{j0.034\pi}, 0.810e^{j1.64\pi})$

TABLE 13: Complex spherical fuzzy performance matrix of decision-maker \mathbb{V}_2 .

	\mathcal{X}_1	\mathcal{X}_2	\mathcal{X}_3	\mathcal{X}_4	\mathcal{X}_5
J_1	$(0.360e^{j0.740\pi}, 0.012e^{j0.022\pi}, 0.820e^{j1.660\pi})$	$(0.220e^{j0.460\pi}, 0.028e^{j0.058\pi}, 0.960e^{j1.900\pi})$	$(0.100e^{j0.180\pi}, 0.013e^{j0.028\pi}, 0.989e^{j1.980\pi})$	$(0.110e^{j0.200\pi}, 0.014e^{j0.030\pi}, 0.991e^{j1.984\pi})$	$(0.120e^{j0.220\pi}, 0.015e^{j0.032\pi}, 0.993e^{j1.988\pi})$
J_2	$(0.110e^{j0.180\pi}, 0.013e^{j0.028\pi}, 0.990e^{j1.978\pi})$	$(0.230e^{j0.480\pi}, 0.030e^{j0.062\pi}, 0.950e^{j1.88\pi})$	$(0.240e^{j0.500\pi}, 0.031e^{j0.064\pi}, 0.940e^{j1.86\pi})$	$(0.250e^{j0.520\pi}, 0.033e^{j0.068\pi}, 0.930e^{j1.84\pi})$	$(0.260e^{j0.540\pi}, 0.034e^{j0.070\pi}, 0.920e^{j1.860\pi})$
J_3	$(0.820e^{j1.660\pi}, 0.017e^{j0.032\pi}, 0.250e^{j0.320\pi})$	$(0.930e^{j1.88\pi}, 0.020e^{j0.040\pi}, 0.030e^{j0.080\pi})$	$(0.940e^{j1.900\pi}, 0.021e^{j0.040\pi}, 0.040e^{j0.100\pi})$	$(0.950e^{j1.920\pi}, 0.023e^{j0.048\pi}, 0.050e^{j0.120\pi})$	$(0.960e^{j1.940\pi}, 0.024e^{j0.050\pi}, 0.060e^{j0.140\pi})$
J_4	$(0.390e^{j0.800\pi}, 0.014e^{j0.026\pi}, 0.840e^{j1.700\pi})$	$(0.570e^{j1.16\pi}, 0.036e^{j0.074\pi}, 0.530e^{j1.04\pi})$	$(0.580e^{j1.18\pi}, 0.037e^{j0.076\pi}, 0.530e^{j1.04\pi})$	$(0.590e^{j1.200\pi}, 0.038e^{j0.078\pi}, 0.530e^{j1.08\pi})$	$(0.380e^{j0.78\pi}, 0.013e^{j0.024\pi}, 0.830e^{j1.680\pi})$
J_5	$(0.27e^{j0.560\pi}, 0.036e^{j0.074\pi}, 0.940e^{j1.860\pi})$	$(0.280e^{j0.580\pi}, 0.037e^{j0.076\pi}, 0.950e^{j1.88\pi})$	$(0.410e^{j0.840\pi}, 0.015e^{j0.028\pi}, 0.850e^{j1.72\pi})$	$(0.420e^{j0.860\pi}, 0.016e^{j0.034\pi}, 0.860e^{j1.94\pi})$	$(0.430e^{j0.880\pi}, 0.017e^{j0.032\pi}, 0.870e^{j1.76\pi})$

TABLE 14: Complex spherical fuzzy performance matrix of decision-maker \mathbb{V}_3 .

	\mathcal{X}_1	\mathcal{X}_2	\mathcal{X}_3	\mathcal{X}_4	\mathcal{X}_5
J_1	$(0.030e^{j0.040\pi}, 0.011e^{j0.060\pi}, 0.981e^{j1.564\pi})$	$(0.040e^{j0.060\pi}, 0.012e^{j0.026\pi}, 0.982e^{j1.566\pi})$	$(0.05e^{j0.080\pi}, 0.013e^{j0.028\pi}, 0.983e^{j0.968\pi})$	$(0.270e^{j0.520\pi}, 0.040e^{j0.082\pi}, 0.960e^{j1.900\pi})$	$(0.060e^{j0.100\pi}, 0.014e^{j0.030\pi}, 0.984e^{j1.570\pi})$
J_2	$(0.290e^{j0.480\pi}, 0.038e^{j0.078\pi}, 0.940e^{j1.86\pi})$	$(0.280e^{j0.540\pi}, 0.039e^{j0.080\pi}, 0.950e^{j1.88\pi})$	$(0.270e^{j0.52\pi}, 0.041e^{j0.084\pi}, 0.960e^{j1.900\pi})$	$(0.440e^{j0.900\pi}, 0.016e^{j0.030\pi}, 0.880e^{j1.78\pi})$	$(0.260e^{j0.500\pi}, 0.042e^{j0.086\pi}, 0.970e^{j1.92\pi})$
J_3	$(0.990e^{j1.960\pi}, 0.071e^{j0.144\pi}, 0.100e^{j0.220\pi})$	$(0.980e^{j1.940\pi}, 0.072e^{j0.146\pi}, 0.110e^{j0.240\pi})$	$(0.970e^{j1.920\pi}, 0.073e^{j0.148\pi}, 0.120e^{j0.260\pi})$	$(0.840e^{j1.700\pi}, 0.014e^{j0.026\pi}, 0.270e^{j0.460\pi})$	$(0.960e^{j1.900\pi}, 0.074e^{j0.150\pi}, 0.130e^{j0.280\pi})$
J_4	$(0.630e^{j1.280\pi}, 0.043e^{j0.084\pi}, 0.560e^{j1.140\pi})$	$(0.640e^{j1.300\pi}, 0.044e^{j0.086\pi}, 0.570e^{j1.160\pi})$	$(0.650e^{j1.320\pi}, 0.043e^{j0.088\pi}, 0.580e^{j1.180\pi})$	$(0.480e^{j0.980\pi}, 0.012e^{j0.022\pi}, 0.860e^{j1.700\pi})$	$(0.620e^{j1.260\pi}, 0.042e^{j0.082\pi}, 0.310e^{j0.640\pi})$
J_5	$(0.450e^{j0.920\pi}, 0.015e^{j0.028\pi}, 0.890e^{j1.760\pi})$	$(0.460e^{j0.940\pi}, 0.014e^{j0.026\pi}, 0.880e^{j1.740\pi})$	$(0.470e^{j0.960\pi}, 0.013e^{j0.024\pi}, 0.870e^{j1.720\pi})$	$(0.600e^{j1.220\pi}, 0.039e^{j0.082\pi}, 0.540e^{j1.100\pi})$	$(0.490e^{j0.940\pi}, 0.011e^{j0.020\pi}, 0.850e^{j1.720\pi})$

TABLE 15: Aggregated complex spherical fuzzy performance matrix.

	\mathcal{A}_1	\mathcal{A}_2	\mathcal{A}_3	\mathcal{A}_4	\mathcal{A}_5
J_1	$(0.2254e^{j0.4656\pi}, 0.0133e^{j0.0280\pi}, 0.09079e^{j1.8300\pi})$	$(0.1321e^{j0.2698\pi}, 0.0168e^{j0.0340\pi}, 0.9761e^{j1.9472\pi})$	$(0.0784e^{j0.1376\pi}, 0.0139e^{j0.0286\pi}, 0.9858e^{j1.9736\pi})$	$(0.1802e^{j0.3426\pi}, 0.0216e^{j0.0436\pi}, 0.9786e^{j1.9516\pi})$	$(0.1249e^{j0.2506\pi}, 0.0166e^{j0.0352\pi}, 0.9680e^{j1.9444\pi})$
J_2	$(0.2114e^{j0.4130\pi}, 0.0229e^{j0.0480\pi}, 0.9552e^{j1.9094\pi})$	$(0.2372e^{j0.4776\pi}, 0.0309e^{j0.0638\pi}, 0.9500e^{j1.8934\pi})$	$(0.2222e^{j0.4392\pi}, 0.0247e^{j0.0518\pi}, 0.9651e^{j1.9174\pi})$	$(0.3487e^{j0.7170\pi}, 0.0176e^{j0.0358\pi}, 0.8516e^{j1.718\pi})$	$(0.2387e^{j0.4822\pi}, 0.0325e^{j0.0670\pi}, 0.9475e^{j1.9012\pi})$
J_3	$(0.9475e^{j1.8722\pi}, 0.0296e^{j0.0678\pi}, 0.0607e^{j0.1608\pi})$	$(0.9329e^{j1.8540\pi}, 0.0247e^{j0.0520\pi}, 0.0901e^{j0.2064\pi})$	$(0.9270e^{j1.8492\pi}, 0.0260e^{j0.0546\pi}, 0.1049e^{j0.2274\pi})$	$(0.9088e^{j1.8176\pi}, 0.0178e^{j0.0356\pi}, 0.0522e^{j1.4160\pi})$	$(0.9494e^{j1.9044\pi}, 0.0328e^{j0.0680\pi}, 0.0541e^{j1.336\pi})$
J_4	$(0.5305e^{j1.0812\pi}, 0.0262e^{j0.0514\pi}, 0.5895e^{j1.1996\pi})$	$(0.5288e^{j1.0976\pi}, 0.0273e^{j0.0560\pi}, 0.6172e^{j1.2402\pi})$	$(0.5892e^{j1.1984\pi}, 0.0358e^{j0.0738\pi}, 0.5182e^{j1.0564\pi})$	$(0.5344e^{j1.0890\pi}, 0.0236e^{j0.0468\pi}, 0.6027e^{j1.2054\pi})$	$(0.5337e^{j1.0876\pi}, 0.0266e^{j0.0520\pi}, 0.5966e^{j1.2136\pi})$
J_5	$(0.3316e^{j0.6826\pi}, 0.0237e^{j0.0472\pi}, 0.9288e^{j1.8505\pi})$	$(0.3766e^{j0.7608\pi}, 0.0189e^{j0.0380\pi}, 0.8649e^{j1.7248\pi})$	$(0.3842e^{j0.7878\pi}, 0.0174e^{j0.0336\pi}, 0.8963e^{j1.7854\pi})$	$(0.4769e^{j0.9738\pi}, 0.0214e^{j0.0426\pi}, 0.7099e^{j1.4402\pi})$	$(0.4294e^{j0.8583\pi}, 0.0142e^{j0.0276\pi}, 0.8422e^{j1.7044\pi})$

TABLE 16: $C\mathcal{SF}$ best and worst values related to the attributes.

Attribute	Best value	Worst value
\mathcal{X}_1	$(0.9475e^{i1.8722\pi}, 0.0296e^{i0.0678\pi}, 0.0607e^{i0.1608\pi})$	$(0.2114e^{i0.4130\pi}, 0.0229e^{i0.0480\pi}, 0.9552e^{i1.9094\pi})$
\mathcal{X}_2	$(0.9329e^{i1.8540\pi}, 0.0247e^{i0.0520\pi}, 0.0901e^{i0.2064\pi})$	$(0.1321e^{i0.2698\pi}, 0.0168e^{i0.0340\pi}, 0.9761e^{i1.9472\pi})$
\mathcal{X}_3	$(0.0784e^{i0.1376\pi}, 0.0139e^{i0.0286\pi}, 0.9858e^{i1.9736\pi})$	$(0.9270e^{i1.8492\pi}, 0.0260e^{i0.0546\pi}, 0.1049e^{i0.2274\pi})$
\mathcal{X}_4	$(0.9088e^{i1.8176\pi}, 0.0178e^{i0.0356\pi}, 0.0522e^{i1.4160\pi})$	$(0.1802e^{i0.3426\pi}, 0.0216e^{i0.0436\pi}, 0.9786e^{i1.9516\pi})$
\mathcal{X}_5	$(0.9494e^{i1.9044\pi}, 0.0328e^{i0.0680\pi}, 0.0541e^{i1.3360\pi})$	$(0.1249e^{i0.2506\pi}, 0.0166e^{i0.0352\pi}, 0.9680e^{i1.9444\pi})$

TABLE 17: Differences of alternatives from best values of $AC\mathcal{SFPM}$.

Alternative	$d(\widehat{\mathcal{X}}_1, \mathcal{X}_{b1})$	$d(\widehat{\mathcal{X}}_2, \mathcal{X}_{b2})$	$d(\widehat{\mathcal{X}}_3, \mathcal{X}_{b3})$	$d(\widehat{\mathcal{X}}_4, \mathcal{X}_{b4})$	$d(\widehat{\mathcal{X}}_5, \mathcal{X}_{b5})$
J_1	0.9585	0.9680	0.00005	1.0175	1.0544
J_2	1.0111	0.8960	0.0526	0.8292	1.0070
J_3	0	0	1.0480	0	0
J_4	0.5668	0.5302	0.6371	0.5346	0.5803
J_5	0.9412	0.7714	0.1844	0.6441	0.8277

TABLE 18: The values of \mathbb{S}_b , \mathbb{R}_b , and \mathbb{Q}_b .

J_1	0.9364	0.2464	0.9152
J_2	0.98934	0.2865	0.9757
J_3	0.05	0.05	0
J_4	0.5515	0.1651	0.5262
J_5	0.7725	0.2420	0.81346

TABLE 19: Ranking of firms with respect to $C\mathcal{SF}$ -VIKOR.

Alternatives	J_1	J_2	J_3	J_4	J_5
Ranking order of \mathbb{S}	5	4	1	2	3
Ranking order of \mathbb{R}	5	4	1	2	3
Ranking order of \mathbb{Q}	5	4	1	2	3

TABLE 20: Comparison with $C\mathcal{SF}$ -VIKOR.

Method	Ranking of firms	Best firm
$C\mathcal{SFNS}_f$ -VIKOR (proposed)	$J_3 > J_4 > J_5 > J_1 > J_2$	J_3
$C\mathcal{SF}$ -VIKOR [33]	$J_3 > J_4 > J_5 > J_1 > J_2$	J_3

TABLE 21: Comparison with \mathcal{SF} -VIKOR and fuzzy-VIKOR.

Method	Ranking of firms	Best firm
$C\mathcal{SFNS}_f$ -VIKOR (proposed)	$J_3 > J_4 > J_5 > J_1 > J_2$	J_3
\mathcal{SF} -VIKOR [27]	$J_3 > J_4 > J_5 > J_1 > J_2$	J_3
Fuzzy-VIKOR [8]	$J_3 > J_4 > J_5 > J_1 > J_2$	J_3

7. Advantages of Proposed Method

- The principle of the VIKOR method relies on group utility measure and individual regret measure of each feasible choice, obtained using the normalized weighted vector of attributes within the normalized Euclidean distances, compromise the effects of grades level along with the two-dimensional frame.
- Decision-making using the VIKOR method provides compromise solution in close proximity of the ideal solution in conjunction with a two-dimensional fusion of $C\mathcal{SFNS}_f$ model that enjoys four opinions of true, neutral, false, and level of the alternatives in reference to the soft information. Moreover, by keeping the level of the alternatives zero, two-dimensional information of four perspectives diversified into the $C\mathcal{SF}$ data that can be readily managed by the presented model.
- The proposed strategic model even accommodates one-dimensional MAGDM problems within the ambit of spherical fuzzy information and picture fuzzy information by exerting periodic terms and ordered grades zero and raises the assurance of the proposed method via the accuracy of the results.

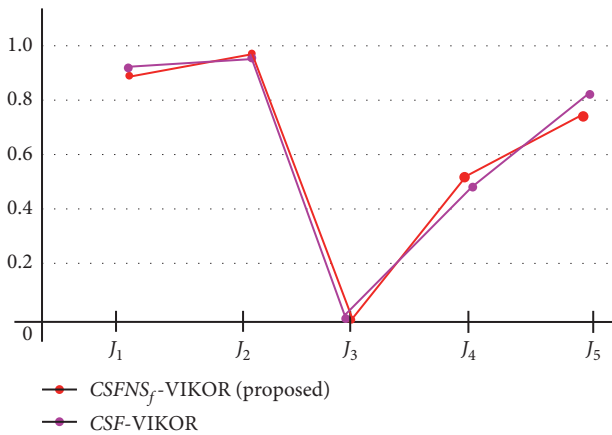


FIGURE 3: Comparative analysis.

that they all are impotent to understand the grades level with parameterized figures of such contemporary MAGDM problems.

- (iv) Additionally, the proficiency about the neutral part elongates the presented model as the abstraction of the fuzzy N -soft models including intuitionistic fuzzy N -soft, Pythagorean fuzzy N -soft, and complex Pythagorean fuzzy N -soft environments.

8. Conclusion

Many real MAGDM problems have a complex pattern. A hybrid decision-making model has been introduced in this paper, which is based on the VIKOR method but allows us to solve problems posed in the elaborate form of complex spherical fuzzy N -soft sets. This model was developed by Akram et al. [34]. An advantage of decision-making within the two-dimensional frame of $CSFNS_f$ Ss is that the experts are free to use four conjectures of true, neutral, false, and level of the alternatives in reference to the soft information. The proposed $CSFNS_f$ -VIKOR method has been in discussion corresponding to the conflicting criteria under different methodologies. The linear normalized weights of the attributes and normalized Euclidean distances have been interpolated together, for the sake of two main features of the acclaimed VIKOR methodology, known as maximum group utility and minimum individual regret. Moreover, the coefficient of weight strategy pertaining to majority opinions and minimum regret of opponents have been exhibited for the compromise measure (ranking function). Furthermore, by keeping the level of the alternatives at zero, two-dimensional information comprising four conjectures becomes CSF knowledge that can therefore be embedded in the proposed model.

The applicability and adequacy of the proposed method have been exemplified through a MAGDM problem whereupon a feasible firm is required for the development of the Saudi oil refinery project in Pakistan. For the implementation of the proposed $CSFNS_f$ -VIKOR method, experts' inputs have been supported in the form of a nonbinary evaluation system. The $CSFNS_f$ WA operator has been deployed for the construction of $ACSFNS_f$ PM, and score degree has been established for the comparison of two complex entities in $ACSFNS_f$ PM. In the end, a comparison has been executed, which strengthens the presented method through the results' transparency. The one-dimensional MAGDM problems under the auspices of spherical fuzzy information and picture fuzzy information can be conciliated with this structure by setting phase terms and ordered grades at zero. This speaks for the strength and validity of the presented approach. The proposed model, in accordance with neutral information, constitutes a generalization of fuzzy N -soft knowledge including intuitionistic fuzzy N -soft, Pythagorean fuzzy N -soft, and complex Pythagorean fuzzy N -soft data.

For future direction, the $CSFNS_f$ -VIKOR method can be employed for selections in other types of MAGDM problems, including solar panel selection, sites for hotel business selection, in the field of medical and sustainable suppliers. Moreover, the limitations of the proposed model invite us to formulate new paradigms. For instance, the $CSFNS_f$ -VIKOR method is deficient to settle conditions in

which the sum of squares of truth membership, neutral membership, and falsity membership of amplitude terms (or phase terms) exceeds 1. Therefore, we can work for establishing a complex T -spherical fuzzy N -soft-VIKOR methodology to extend the current boundary constraints. Moreover, a computer program can be created to handle the difficulties that appear in the presence of large numbers of alternatives and attributes.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Estimating Lane Change Duration for Overtaking in Nonlane-Based Driving Behavior by Local Linear Model Trees (LOLIMOT)

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
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The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] E. Ramezani-Khansari, M. Tabibi, and F. Moghadas Nejad, "Estimating Lane Change Duration for Overtaking in Nonlane-Based Driving Behavior by Local Linear Model Trees (LOLIMOT)," *Mathematical Problems in Engineering*, vol. 2021, Article ID 4388776, 7 pages, 2021.

Research Article

Estimating Lane Change Duration for Overtaking in Nonlane-Based Driving Behavior by Local Linear Model Trees (LOLIMOT)

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Lane change (LC) is one of the main maneuvers in traffic flow. Many studies have estimated LC duration directly by using lane-based data. The current research presents an estimate of LC duration for overtaking maneuver in nonlane-based traffic flow. In this paper, the LC duration is estimated implicitly by modeling lateral speed and applying the length of required lateral movement to complete the LC maneuver. In lateral speed modeling, the local linear model tree is applied which consists of three variables: the initial lateral distance, longitudinal speed, and time to collision (TTC), which itself is a function of the relative speed of follower and the distance between the two vehicles. The initial lateral distance is the relative transverse distance from which the following vehicle initializes the LC. The range of lateral speed was estimated between 0.5 and 5 km/h, which resulted in the LC duration between 2.5 and 24 sec. The results indicate that the lateral and longitudinal speed would be inversely related, while the lateral speed and the initial transverse distance as well as TTC would be directly related. The findings also indicate that TTC can be assumed as the most important factor affecting lateral speed. TTC at 8 sec can be considered as the threshold for its effect on the LC duration since at longer TTCs, and the lateral speed has remained almost constant. When TTC is longer than 8 sec, it would not affect the LC duration.

1. Introduction

LC manoeuvre is referred to as a driving manoeuvre in which a vehicle moves from current lane to another where both of them are in same direction. Lane-changing, car-following, and gap acceptance models are the core of all micro-simulation softwares [1]. Driving behaviors can vary based on the country, region, and so on. Investigating LC behavior helps the traffic engineers to better estimate, predict, and plan for traffic flow. Due to the importance of the LC maneuver in traffic flow, many researchers have studied its modeling and decision-making process [2–5].

Finnegan and Green reported that lane-changing takes between 4.9–7.6 sec depending on the traffic condition and direction of lane-changing [6]. Hetrick used an instrumented vehicle to investigate LC behavior. The range of LC duration was measured about 3.4–13.6 sec by navigating 16 drivers to a predefined route. His research also concluded that the LC duration for young drivers was significantly shorter than elderly drivers. The average LC time was reported about 6 sec [7]. Hanowski in Virginia transportation institute used an instrumented vehicle to assess of the impact of fatigue on occurring accidents. Results driven from 42 drivers in the research indicated that LC duration was about

1.1–16.5 sec. LC initiation rule in his research considered the moment when the wheel of the subject vehicle crossed the lane marking [8].

In another research performed by Salvucci and Liu, 11 participants were examined by using a driving simulator. The simulated driving environment included a two-way, two-lane highway without any off-ramp or on-ramp in a medium-fidelity driving simulator. Recorded data included steer signals, throttle position, eye movement, and speed. Data were driven from 401 participants using the simulator resulted in a mean LC duration of about 5.14 sec. They also concluded that drivers slow down slightly before initiating the LC and increase the speed after initiating the LC [9].

Toledo et al. investigated LC durations by NGSIM dataset. These data were gathered from I-80 Emeryville in California by a camera that was mounted in a high-rise building. They extracted vehicle trajectory data from the observed segment. Variables including traffic characteristics and position of subject vehicle (related to the surrounding vehicle) were determined for each lane-changing. They found that LC behavior of heavy vehicles was different from passenger cars. Therefore, they made completely two separate models for each type of vehicles with specific parameters. They showed that traffic density was the most important variable that affected LC duration. The higher the traffic density, the greater the LC duration. The range of LC duration was measured between 1–13.3 sec, with a mean value of 4.6 sec and a standard deviation of 2.3 sec [10].

Moridpoor et al. conducted research to investigate the increase/decrease in speed in passenger cars compared with heavy vehicles at the time of lane-changing. They selected two segments of highways in California. They applied 42 LCs for each type of vehicle. The LC duration of passenger cars was reported between 1.1–8.9 sec with a mean of 4.8 sec and a standard deviation of 3.7 sec. However, the LC value for heavy vehicles was reported 1.6–16.2 sec with an average value of 8 sec and a standard deviation of 3.7 sec. Their results showed that the required time for heavy vehicles for LC maneuver was 70% greater than passenger cars [11].

Cao et al. modeled LC duration in an urban arterial road. They used a camera that mounted on a high-rise building for recording video of a road segment with a length of 140 m. The rule for starting the LC was the time instant by which the subject vehicle moves out of the current lane and enters to the target lane. They proved that time of lane-changing for heavy vehicles would be much longer than passenger cars. They reported the LC duration for passenger cars about 1–6.8 sec with a mean value of 2.54 sec and a standard deviation of 1.29 sec. For heavy vehicles, this was between 2–9.8 sec with the mean value of 4 sec and the standard deviation of 1.25 sec [12].

The definitions of initiation and completion of the LC maneuver have been different among various studies. In some cases, the lane-changing started when the first wheel passed lines (the lane markings) whereas in other cases it has been measured when the driver decides to change the lane [12, 13]. In Toledo's study lane-changing starts when the subject vehicle begins to move laterally relative to the current lane. The completion of maneuver is when the vehicle reaches the center of the target lane [10].

Li et al. studied LC duration by using data of 11000 vehicles, which had been extracted from naturalistic vehicle trajectory HighD dataset. By applying comparative univariate and regression analysis, they found that generalized Gamma distribution has high degree of coincidence with the nonparametric method in estimating the survival function [14]. Ataelmanan et al. employed the instrumented vehicle to examine LC duration in a highway in Kuala Lumpur. A total of 174 LC manoeuvre incidences were observed in this study. LC data ranged from 0.90 to 10.52 seconds with mean and standard deviation values of 3.02 and 1.32 seconds, respectively. They found Lognormal distribution appropriate for recorded data [15].

Here, two aspects of LC duration have examined which were not considered in aforementioned investigations. First, many of them have assumed lane-based traffic flow, while in many developing countries, such as Iran, nonlane-based behavior is very common. In nonlane-based behavior, drivers do not pay attention to road markings, and it is more difficult to distinguish lane-changing maneuver. Second, LC duration of personal cars (PCs) was estimated by lateral speed indirectly and using microscopic characteristics. In other words, the studies have measured LC duration by considering macroscopic characteristics of the traffic flow.

The next section describes data collection. The third section describes the methodology for modeling LC duration. The fourth and fifth sections have been devoted to the local linear model trees (LOLIMOT) model and lateral speed modeling, respectively. The sixth section is about the results of this study. Finally, conclusions of the research have been addressed.

2. Data Collection

Data were collected from a length of about 100 m of Tehran-Karaj rural freeway, along Iranian road network. A 20-meter-high camera was used to record the video (Figure 1). Films were analyzed by the semiautomated image processing method at intervals of 0.1 sec. In this method, an operator selects each vehicle, and the image processing program only tracks selected vehicles by using kernelized correlation filter. Although the method is very time-consuming and requires high processing sources, it was much more accurate than fully automated methods. Trajectory data of 2580 vehicles were extracted in 300 minutes. The maximum and the minimum speeds driven from data analysis were 127 and 61 km/h, respectively.

The longitudinal distance (or headway) and lateral distances have been measured, as depicted in Figure 2. Because of the selected road had limits and obligations for heavy freight vehicles, it has been only focused on the PC-PC situations. Therefore, other situations in which a heavy vehicle was involved have not been taken into account in this study.

3. Methodology

In this research, the lateral speed has been modeled to implicitly estimate the LC duration for overtaking. This method would be more convenient for traffic flow with



FIGURE 1: Snapshot of recorded films.

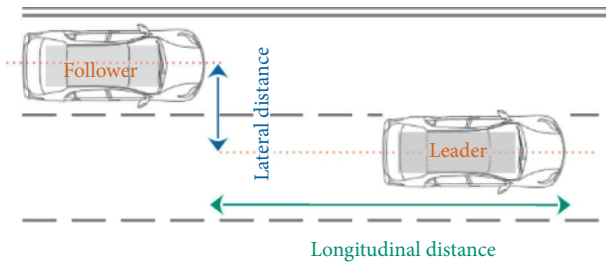


FIGURE 2: Lateral and longitudinal distance.

nonlane-based behavior. In nonlane-based behavior, the current and target lanes of LC maneuver cannot be defined accurately because the drivers pay attention to the leading vehicle instead of following lanes and obeying road markings.

To ensure that LC behavior was being carried out for overtaking, the average speed of the follower must be at least 10 km/h faster than the leader. The selected segment was far from any ramp or loop. By doing so, it can be concluded that almost all LCs were discretionary, and due to the speed advantages, because in the case of mandatory LCs, drivers may behave differently and take more risks [16, 17]. In this research, lane-changing starts when the subject vehicle begins to move laterally relative to the preceding vehicle.

Ramezani-Khansari et al.'s research demonstrated that Iranian drivers terminate overtaking maneuver at the lateral distance of 3.3 m related to the leading vehicle [18]. This means that due to the nonlane-based driving behavior, the driver could overtake without entering adjacent lane, instead she/he only requires adequately lateral distance, e.g., 3.3 m. Equation (1) reflects how the LC duration can be calculated having the lateral speed:

$$\text{lane change duration (sec)} = \frac{3.3 \text{ m}}{\text{lateral speed (m/sec)}}. \quad (1)$$

Longitudinal speed, initial lateral distance (lateral distance at the beginning of the LC), and time to collision (TTC) have been used for modeling the lateral distance. These variables have been explained as follows:

Initial Lateral Distance. The lateral speed of the follower may differ if she/he starts LC at different lateral distances. In other words, when the follower has some lateral distance, she/he has already done a part of the overtaking and lane-changing. Therefore, the initial lateral distance can be important to be recognized well and has been considered in the model.

Longitudinal Speed. Vehicle dynamics and driver behavior can vary at different longitudinal speeds. The reason why it has been considered is the sensitivity of drivers to changes in lateral position with respect to longitudinal speed.

TTC. It may be considered an important factor in two ways. On the one hand, TTC includes the relative speed of the following vehicle and headway (or distance) (equation (2)). Both of these factors may be considered in the driver's choice of lateral speed:

$$\text{time to collision (TTC)} = \frac{\text{headway} - \text{vehicle length}}{\text{follower speed} - \text{leader speed}} \quad (2)$$

On the other hand, two factors have been expressed in one factor by using TTC, which simplifies the final model of lateral speed. On the other hand, lateral speed can be affected by the lead-lag spacing in the target lane. The effect of lead-lag spacing can be considered by TTC implicitly. It has been postulated that the driver has to accept higher risk (shorter TTC) for the LC if there is not enough lead-lag spacing. Due to lack of enough lead-lag spacing, the driver may approach the preceding vehicle, so she/he must change the lane at faster lateral speed. It can be concluded that there would be an inverse relation between the lead-lag spacing and TTC because it is not logical to tolerate shorter TTC when there is enough lead-lag spacing for LC. It is worth noting that TTC is in terms of time, which makes it independent of the speed and easier to build the model.

Figure 3 shows the relationship between the dependent and independent variables for estimating the lateral speed.

Given that the lateral speed model can be nonlinear and multiregime, linear and conventional mathematical models may not be appropriate. Therefore, the local linear model tree (LOLIMOT) has been used, which is a subset of neurofuzzy models whose flexibility in solving complex engineering problems has been evaluated [19–22].

LOLIMOT is an algorithm based on a problem-solving strategy in which a complex problem is solved by dividing the main problem into some smaller and simpler problems. Then, they are solved relatively independently by linear models (other regression models). Locally linear neurofuzzy structure with M neurons and P input is shown in Figure 4.

The most important issue in the success of such method is the initial problem segmentation algorithm. The LOLIMOT algorithm divides the input space using vertical axes. Each neuron contains a local linear model (LLM) and a validation function that specify the validity range of the LLM. For a model with P inputs, the output of each LLM is as follows:

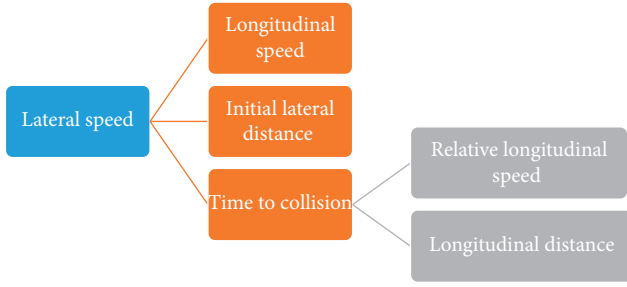


FIGURE 3: Lateral speed modeling.

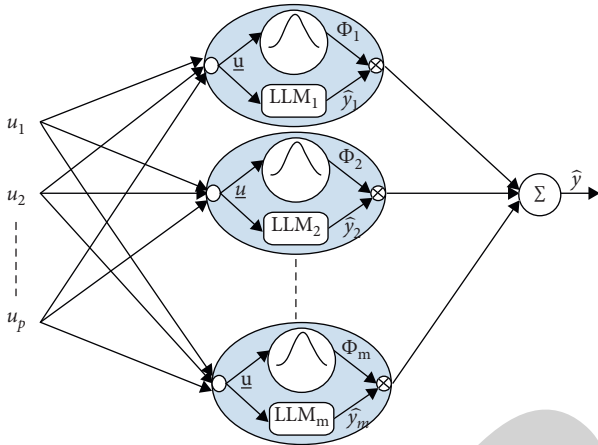


FIGURE 4: Locally linear neurofuzzy structure.

$$\hat{y}_i = \omega_{i0} + \omega_{i1} u_1 + \omega_{i2} u_2 + \dots + \omega_{ip} u_p, \quad (3)$$

where ω and u are the LLM parameters for neuron i and network inputs, respectively. The parameters of the LLMs are estimated independently by the least-squares error method, which makes the network less sensitive to noise because the noise affects locally. The validation functions (Φ_i) are usually selected as normalized Gaussian functions as follows:

$$\Phi_i(\underline{u}) = \frac{\mu_i(\underline{u})}{\sum_{j=1}^M \mu_j(\underline{u})}. \quad (4)$$

μ_i in equation (5) refers to Gaussian functions whose centers and standard deviations are c_{ij} and σ_{ij} , respectively. Equation (6) shows the final output of the network.

$$\mu_i(\underline{u}) = \exp \left[-\frac{1}{2} \left[\frac{(u_1 - c_{i1})^2}{\sigma_{i1}^2} + \dots + \frac{(u_p - c_{ip})^2}{\sigma_{ip}^2} \right] \right], \quad (5)$$

$$\hat{y} = \sum_{i=1}^M (\omega_{i0} + \omega_{i1} u_1 + \omega_{i2} u_2 + \dots + \omega_{ip} u_p) \Phi_i(\underline{u}). \quad (6)$$

The output of the network is the weighted sum of the LLMs, and therefore the network interpolates between the different LLMs with the validation functions. At the beginning of LOLIMOT, the entire input space has one neuron, but in each iteration, the input space is divided and

another neuron is added. In each iteration, the worst LLM with more local error is selected and divided into two parts based on each input. The direction that creates the least modeling error is selected [23]. This procedure is repeated until achieving the desired answer. The LOLIMOT model is an iterative method and requires a stopping criterion. Here, the improvement in the coefficient of determination (R-squared) has been used as stopping criteria. The number of LLMs increases until it reaches the threshold of 1% progress of R-squared. In other words, the number of LLMs has been optimized based on the progress of R-squared.

4. Results and Discussion

In the LOLIMOT model, the lateral speed has been the dependent variable, where the independent variables have been assumed to be the longitudinal speed, TTC, and initial lateral distance. LOLIMOT tool box in MATLAB was used [23]. The trajectory data had some noises which have been seen in many data extracted from films [24]. To overcome this variation, the data are smoothed in each 0.5 and 1 sec, by applying a moving average method. The data set was splitted into training, validation, and testing by the percentage of 70, 15, and 15, respectively.

Figure 5 shows that using 70 LLM can result in an acceptable solution (R-square = 0.69) by using training and validation data. In Figure 5, blue and red lines (respectively, marked with triangular and squares) are R-squared and its progress, respectively.

Figure 6 also depicts the histogram error of estimating the lateral speed using LOLIMOT by using test data. The RMSE (root mean square error) index of the model has been addressed equal to 0.59.

Since the model includes four variables (three independent variables and one dependent variable), it is not possible to display these four variables in one figure simultaneously. Hence, one independent variable has been considered constant to picture the output of the model. Two variables have been placed on the horizontal and vertical axes. The fourth variable has been plotted inside the chart (referring to the legend). By doing so, four variables could be shown simultaneously.

Figures 7 and 8 show the relationship between the lateral speed and TTC when the initial lateral distance of the following vehicle has been categorized to 0–0.5 and 1.8–2.2 m, respectively.

It can be seen that the lateral speed has decreased by increasing the TTC. As the initial lateral distance has increased, the lateral speed slope has decreased. Figure 9, for instance, shows the relationship between lateral speed and initial lateral distance when TTC equals to 5 sec.

According to the results obtained from the LOLIMOT model, it can be seen that there has been inverse relationship between TTC and the lateral speed. It also can be concluded that by increasing TTC, the lateral speed has decreased, too. The figures also reflect that as the TTC increased and also the initial lateral distance increased, the slope of the lateral speed graph has decreased. It also indicates that an inverse relationship would exist between lateral and longitudinal speed,

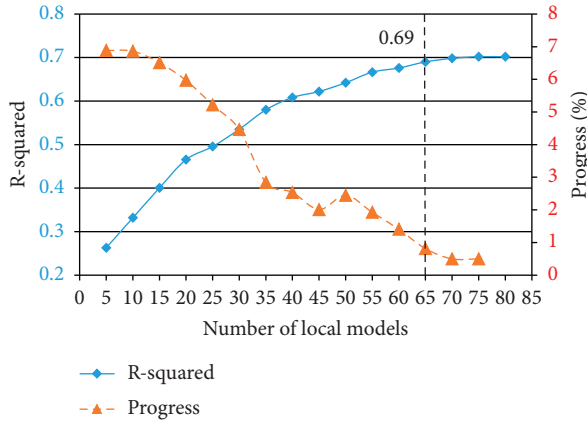


FIGURE 5: Coefficient of determination of the estimated LOLIMOT model.

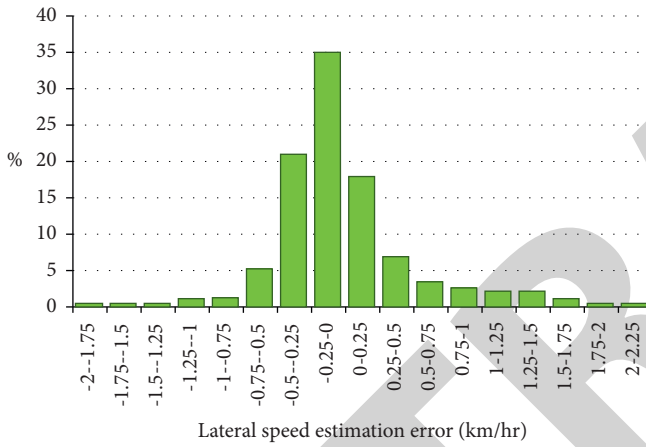


FIGURE 6: RMSE histogram.

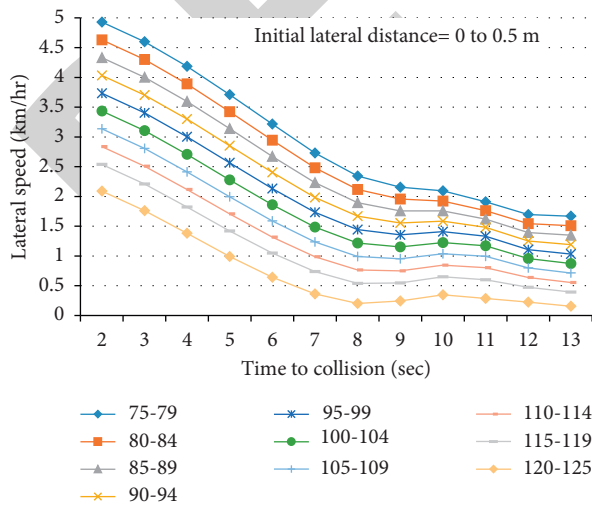


FIGURE 7: Relationship between the lateral speed and TTC in different speeds (assuming lateral distance varies between 0 and 0.5 m).

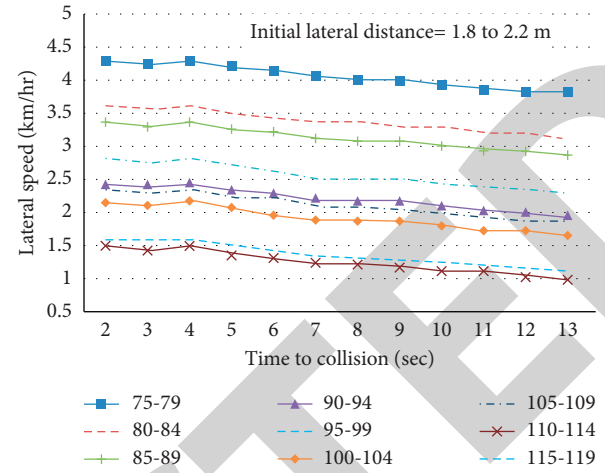


FIGURE 8: Relationship between the lateral speed and TTC in different speeds (assuming lateral distance varies between 1.8 and 2.2 m).

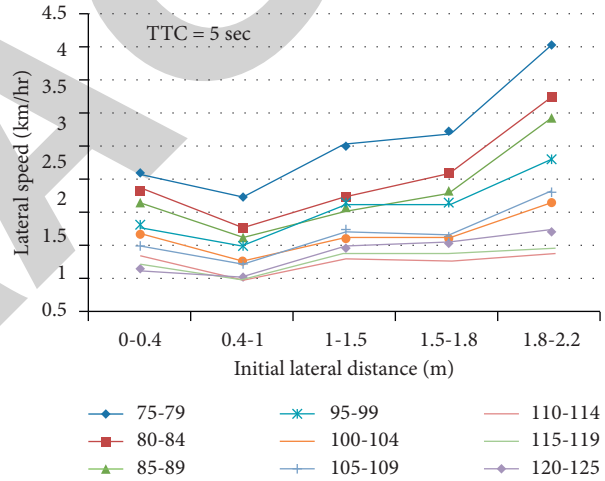


FIGURE 9: Relationship between the lateral speed and initial lateral distance in different speeds (assuming TTC equals to 5 sec).

in which the initial lateral distance could affect it. Findings also imply that at higher longitudinal speeds, drivers' sensitivity to lateral speeds would increase. In other words, when the longitudinal speed is high, the driver would apply less lateral speed to ensure safety and comfort. By increasing the longitudinal speed, the driver's inclination to higher lateral speed would decrease. When the vehicle had a large initial lateral distance (1.8 to 2.2 m), it has been seen that the slope of the lateral speed and TTC relationship chart was very low. It can be concluded that when the driver does not have to travel a longer distance to complete the LC and overtaking maneuver, she/he would be less affected by the TTC. In other words, increasing the initial lateral distance has reduced the effect of the TTC on the lateral speed.

It was observed that when the TTC has become more than 8 sec (the initial lateral distance between 0 and 0.5 m), the slope of the lateral speed diagrams has decreased and has become almost a straight line.

This may indicate that the effect of TTC would diminish when it has passed 8 sec. It would correspond to free driving conditions in car-following behavior in which by increasing time headway to a specific value, the follower does not react to the leader.

According to the relationship between lateral speed and initial lateral distance (Figure 9), it can be seen that by increasing initial lateral distance, the lateral speed would increase. It can be seen that when the initial lateral distance has been larger and the required lateral distance for completing the LC maneuver has been shorter simultaneously, drivers have applied higher lateral speed because they have spent less time in LC maneuver. It can be said that the tendency of the drivers to higher lateral speed has increased with decreasing lateral distance. Accordingly, when the initial lateral distance has been smaller and the drivers have required longer lateral movement, simultaneously, the tendency to higher lateral movement has decreased.

In the proposed model, the lateral speed has been considered constant and equal to the average recorded lateral speed. It is expected that the lateral speed is lower at the beginning and end of the LC maneuver because it is a function of the angle of the wheels. In the LC, the angle of the wheels gradually increases and reaches a constant value and decreases again [25].

The LC duration has been estimated by using lateral speed and lateral distance, so it was expected that using the mean lateral speed would not have a significant effect on results. Due to the length of the recorded road segment, some LCs have been recorded incompletely. Using the average lateral speed could help to estimate them as well.

In this study, the aim of obtaining the lateral speed has been to estimate the LC duration; taking into account the estimated lateral speed has been between 5 and 0.5 km/h and the length of lateral movement for LC has been assumed equal to 3.3 m, the LC duration has been calculated between 2.5 and 24 sec.

The main limitation in this article was the length of recording data along highway, which was not so long to cover entire of all LC maneuvers. In other words, there were LCs that recorded incompletely. It should be mentioned as the average of the lateral speed was used, the effect of this limitation should be reduced significantly. Another limitation was the volume of traffic flow. The recorded data set only included uncongested condition (level of service A to C). So, congested condition should be examined.

5. Conclusions

LC is one of the main manoeuvre in the traffic flow and is essential for all simulation programs. Studying LC duration in nonlane-based traffic flow is much difficult than lane-based because drivers do not consider road markings and lanes. So, it is not appropriate to estimate directly. By estimating LC duration indirectly, the estimate will be dependent on the microscopic characteristics of traffic flow rather than macroscopic characteristics, which make the estimation more specific for each vehicle and is more accurate. Here, LC duration has been achieved indirectly by

using the lateral speed and the length of required lateral movement to complete LC maneuver. The LOLIMOT method has been used, by which the problem is divided into smaller local spaces and be solved through applying linear models to solve equations of these spaces.

It was found that there would be an inverse relationship between lateral and longitudinal speed, whereas a direct relationship was found between the lateral speed and both the initial transverse distance and TTC. TTC would be addressed as the most important factor affecting the lateral speed, but it affects the lateral speed up to a threshold. TTC was considered as it is a function of the relative speed of the follower and the distance between the two consecutive vehicles.

In this article, LC was modeled by using microscopic characteristics of subject and surrounding vehicles. However, it is interesting to consider driver factors such as age and gender. Furthermore, the vehicle and road type can affect results which can be studied.

Data Availability

The data used to support this study are available upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Some Topological Approaches for Generalized Rough Sets via Ideals

Mathematical Problems in Engineering

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Research Article

Some Topological Approaches for Generalized Rough Sets via Ideals

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The idea of neighborhood systems is induced from the geometric idea of “near,” and it is primitive in the topological structures. Now, the idea of neighborhood systems has been extensively applied in rough set theory. The master contribution of this manuscript is to generate various topologies by means of the concepts of j -adhesion neighborhoods and ideals. Then, we define a new rough set model derived from these topologies and discussed main features. We show that these topologies are finer than those given in the previous ones under arbitrary binary relations. In addition, we elucidate that these topologies are finer than those topologies initiated based on different neighborhoods and ideals under reflexive relations. Several examples are provided to validate that our model is better than the previous ones.

1. Introduction

The idea of rough set theory (RST) was first put forth by Pawlak [1], where imperfect information gives rise to indiscernibility of objects. RST proves its adequacy to treat and model a lot of real-life issues that were constructed as a method to overcome imperfectness and ambiguity of information. The classical RST was characterized by a pair of approximation operators called lower and upper approximations which are established using equivalence classes. But at times, equivalence relations are tricky to be acquired in real-life issues due to the incompleteness of human information. So, a lot of ideas and articles have been introduced to generalizing the classical theory of RSs, for further specifics, see reference [2]. For interpretation of the granules, both Lin [3] and Yao [4] studied the RSs utilizing neighborhood systems. In fact, they freed RST from an equivalence relation which is a very inflexible obligation that restricts the real-life implementation scope of rough sets philosophy.

Quite recently, Abd El-Monsef et al. [5] applied j -neighborhoods to generalize the classical RST. Allam et al. presented the concepts of minimal right neighborhoods and minimal left neighborhoods in [6, 7], respectively. Dai et al. [8] presented new rough set models using maximal neighborhoods induced from similarity relations. Al-shami [9] defined containment neighborhoods and applied to protect medical staff from infected diseases. Also, Al-shami et al. [10] initiated several types of lower and upper approximations using N_j -neighborhoods. In [11], El-Bably and Fleifel investigated new topological structures by relations. El-Bably et al. [12] introduced some closure operators using arbitrary binary relation and generated some topologies from any binary relation without using sub-base or base.

Study of the rough set theory via topology is an enjoyable topic that received the attention of many researchers, see, for example [2, 13–19]. An ideal \mathcal{I} [20] on a nonempty finite set (universe) \mathcal{U} is a nonempty family of subsets of \mathcal{U} with

heredity property, as well as it is closed under finite unions. Some interesting papers studied ideals via rough set theory such as [15, 21]. Hosny [22] replaced Pawlak's approximations (lower and upper) by topological operators (interior and closure). Then, she [23] presented different methods to establish new rough set models using ideals. In 2020, Kandil et al. [24] offered the collection $\langle \mathcal{I}, \mathcal{F} \rangle$ generated by two ideals \mathcal{I} and \mathcal{F} and proved that it is also an ideal on a universe.

From topology's view, our work discusses the notions of rough sets based on the topological spaces generated by j -adhesion neighborhoods and ideals. That is, it studies the concepts of lower and upper approximations in terms of j -adhesion neighborhoods via ideals. This approach minimizes the vagueness of uncertainty regions at their borders by increasing the lower approximation and decreasing the upper approximation which automatically implies increasing the accuracy measure of the uncertainty regions.

The layout of this paper is as follows. In Section 2, we recall three classes of neighborhood systems called N_j -neighborhoods, \mathcal{E}_j -neighborhoods, and \mathcal{P}_j -neighborhoods as well as the results that show the way of generating topologies using these classes of neighborhoods. In Section 3, we apply the notions of ideal and \mathcal{P}_j -neighborhood systems to establish new types of topologies. Based on these topologies, we propose new kind of rough set models and compare them with the previous ones under reflexive and arbitrary relations. We devote Section 4 to discuss the approximations and accuracy measures generated by \mathcal{E}_j -neighborhoods and \mathcal{P}_j -neighborhoods with ideals. Eventually, Section 5 gives conclusions and some directions for future works.

2. Preliminaries

In this part, we present the j -neighborhood space, which depends on a finite number of various kinds of arbitrary binary relations. So, we produce eight diverse topologies and investigate the relationships among these topologies. Then, we get eight techniques to find the lower and upper approximations of rough sets. Comparisons between the accuracy of these types of new approximations are attained.

Henceforward, we consider $j \in \{r, l, \langle r \rangle, \langle l \rangle, i, u, \langle i \rangle, \langle u \rangle\}$, unless otherwise specified.

2.1. Approximations and Topologies Generated by Different j -Neighborhoods

Definition 1 (see [5]). Let R be a binary relation on \mathcal{U} . The eight sorts of j -neighborhoods of any point $h \in \mathcal{U}$, say $N_j(h)$, are given by

- (1) $N_r(h) = \{z \in \mathcal{U} : hRz\}$.
- (2) $N_l(h) = \{z \in \mathcal{U} : zRh\}$.
- (3) $N_i(h) = N_r(h) \cap N_l(h)$.
- (4) $N_u(h) = N_r(h) \cup N_l(h)$.

- (5) $N_{\langle r \rangle}(h) = \cap \{N_r(z) : h \in N_r(z)\}$; if $N_r(z)$ does not exist there such that $h \in N_r(z)$, then $N_{\langle r \rangle}(h) = \emptyset$.
- (6) $N_{\langle l \rangle}(h) = \cap \{N_l(z) : h \in N_l(z)\}$; if $N_l(z)$ does not exist there such that $h \in N_l(z)$, then $N_{\langle l \rangle}(h) = \emptyset$.
- (7) $N_{\langle i \rangle}(h) = N_{\langle r \rangle}(h) \cap N_{\langle l \rangle}(h)$.
- (8) $N_{\langle u \rangle}(h) = N_{\langle r \rangle}(h) \cup N_{\langle l \rangle}(h)$.

Definition 2 (see [5]). The triple (\mathcal{U}, R, ψ_j) is called a j -neighborhood space (in short, N_jS), where ψ_j is a mapping from \mathcal{U} to $P(\mathcal{U})$ which assigns for each point in \mathcal{U} its j -neighborhood.

Proposition 1 (see [25]). Let (\mathcal{U}, R, ψ_j) be a N_jS and $h \in \mathcal{U}$. Then,

- (1) The reflexivity of R implies that $N_j(h)$ is a nonempty set.
- (2) The reflexivity of R implies that $N_{\langle j \rangle}(h)$ is a subset of $N_j(h)$, for every $j \in \{r, l, i, u\}$.
- (3) If R is a transitive relation, then $N_j(h)$ is a subset of $N_{\langle j \rangle}(h)$, for every $j \in \{r, l, i, u\}$.
- (4) The symmetry of R implies that $N_r(h) = N_l(h) = N_i(h) = N_u(h)$ and $N_{\langle r \rangle}(h) = N_{\langle l \rangle}(h) = N_{\langle i \rangle}(h) = N_{\langle u \rangle}(h)$.

Theorem 1 (see [5]). Let (\mathcal{U}, R, ψ_j) be a N_jS . Then, the family $\tau_j = \{M \subseteq \mathcal{U} : \forall h \in M, N_j(h) \subseteq M\}$ forms a topology on \mathcal{U} .

Definition 3 (see [5]). Let (\mathcal{U}, R, ψ_j) be a N_jS . Then, we called a subset $M \subseteq \mathcal{U}$ a j -open set if $M \in \tau_j$, and we called its complement a j -closed set. The class Γ_j of all j -closed sets of a j -neighborhood space is given by $\Gamma_j = \{F \subseteq \mathcal{U} : F^c \in \tau_j\}$.

Definition 4 (see [5]). Let τ_j be a topology induced from j -neighborhoods. The j -lower, j -upper approximations, and j -accuracy of $M \subseteq \mathcal{U}$ are respectively given by

- (1) $\underline{R}_j(M) = \cup \{O \in \tau_j : O \subseteq M\} = \text{int}_j(M)$.
- (2) $\overline{R}_j(M) = \cap \{F \in \Gamma_j : M \subseteq F\} = \text{cl}_j(M)$.
- (3) $\alpha_j(M) = (|\underline{R}_j(M)|/|\overline{R}_j(M)|)$, where $|\overline{R}_j(M)| \neq 0$.

2.2. Approximations and Topologies Generated by Different \mathcal{E}_j -Neighborhoods

Definition 5 (see [26]). Let R be a binary relation on \mathcal{U} . The \mathcal{E}_j -neighborhood of a point $h \in \mathcal{U}$ (briefly, $\mathcal{E}_j(h)$) is formulated as

- (1) $\mathcal{E}_r(h) = \{z \in \mathcal{U} : \text{the intersection of } N_r(z) \text{ and } N_r(h) \text{ is nonempty}\}$.
- (2) $\mathcal{E}_l(h) = \{z \in \mathcal{U} : \text{the intersection of } N_l(z) \text{ and } N_l(h) \text{ is nonempty}\}$.
- (3) $\mathcal{E}_i(h) = \mathcal{E}_r(h) \cap \mathcal{E}_l(h)$.
- (4) $\mathcal{E}_u(h) = \mathcal{E}_r(h) \cup \mathcal{E}_l(h)$.

- (5) $\mathcal{E}_{\langle r \rangle}(h) = \{z \in \mathcal{U} : \text{the intersection of } N_{\langle r \rangle}(z) \text{ and } N_{\langle r \rangle}(h) \text{ is nonempty}\}.$
- (6) $\mathcal{E}_{\langle l \rangle}(h) = \{z \in \mathcal{U} : \text{the intersection of } N_{\langle l \rangle}(z) \text{ and } N_{\langle l \rangle}(h) \text{ is nonempty}\}.$
- (7) $\mathcal{E}_{\langle i \rangle}(h) = \mathcal{E}_{\langle r \rangle}(h) \cap \mathcal{E}_{\langle l \rangle}(h).$
- (8) $\mathcal{E}_{\langle u \rangle}(h) = \mathcal{E}_{\langle r \rangle}(h) \cup \mathcal{E}_{\langle l \rangle}(h).$

Theorem 2 (see [26]). Let R be a binary relation on \mathcal{U} and $h \in \mathcal{U}$. Then,

- (1) $h \in \mathcal{E}_j(z)$ iff $z \in \mathcal{E}_j(h).$
- (2) The reflexivity of \mathcal{R} implies that $\mathcal{E}_{\langle j \rangle}(h)$ is a subset of $\mathcal{E}_j(h)$ and $\mathcal{N}_j(h)$ is a subset of $\mathcal{E}_j(h)$ for every j .
- (3) The symmetry of \mathcal{R} implies that $\mathcal{E}_r(h), \mathcal{E}_l(h), \mathcal{E}_i(h)$, and $\mathcal{E}_u(h)$ are equal, and $\mathcal{E}_{\langle r \rangle}(h), \mathcal{E}_{\langle l \rangle}(h), \mathcal{E}_{\langle i \rangle}(h)$, and $\mathcal{E}_{\langle u \rangle}(h)$ are equal.
- (4) The transitivity of \mathcal{R} implies that $\mathcal{E}_j(h)$ is a subset of $\mathcal{E}_{\langle j \rangle}(h)$ for each $j \in \{u, i, r, l\}.$
- (5) If \mathcal{R} is transitive and symmetric, then $\mathcal{E}_j(h) = \mathcal{N}_j(h)$ and $\mathcal{E}_j(h) \subseteq \mathcal{E}_j(z)$ (if $h \in \mathcal{E}_j(z)$), for each j .
- (6) If \mathcal{R} is preorder, then $\mathcal{E}_j(h) = \mathcal{E}_{\langle j \rangle}(h)$, for all $j \in \{r, l, i, u\}.$
- (7) If \mathcal{R} is an equivalence, then $\mathcal{E}_j(h)$ are equal for each j , and $\mathcal{E}_j(h) = \mathcal{N}_j(h).$

Theorem 3 (see [26]). For each j , the family $\tau_{\mathcal{E}_j} = \{M \subseteq \mathcal{U} : \forall h \in M, \mathcal{E}_j(h) \subseteq M\}$ forms a topology on \mathcal{U} .

Definition 6 (see [26]). Let (\mathcal{U}, R, ψ_j) be an \mathcal{E}_j S. We called a set $M \subseteq \mathcal{U}$ an \mathcal{E}_j -open set if $M \in \tau_{\mathcal{E}_j}$, and we called its complement an \mathcal{E}_j -closed set. The class $\Gamma_{\mathcal{E}_j}$ of all \mathcal{E}_j -closed sets is given by $\Gamma_{\mathcal{E}_j} = \{K \subseteq \mathcal{U} : K^c \in \tau_{\mathcal{E}_j}\}.$

Definition 7 (see [26]). Consider $\tau_{\mathcal{E}_j}$ as a topology induced by \mathcal{E}_j neighborhoods. For each j , the \mathcal{E}_j lower, \mathcal{E}_j upper approximations, and \mathcal{E}_j accuracy of $M \subseteq \mathcal{U}$ are respectively given by

- (1) $L_j^\circ(M) = \bigcup \{O \in \tau_{\mathcal{E}_j} : O \subseteq M\} = \text{int}_{\mathcal{E}_j}(M).$
- (2) $U_j^\circ(M) = \bigcap \{F \in \Gamma_{\mathcal{E}_j} : M \subseteq F\} = \text{cl}_{\mathcal{E}_j}(M).$
- (3) $\mu_j(M) = (|L_j^\circ(M)|/|U_j^\circ(M)|), \text{ where } |U_j^\circ(M)| \neq 0.$

2.3. Approximations and Topologies Generated by Different j -Adhesion Neighborhoods

Definition 8 (see [27]). Let R be a binary relation on \mathcal{U} . The j -adhesion (\mathcal{P}_j) neighborhood of any point $h \in \mathcal{U}$ (denoted by $\mathcal{P}_j(h)$) is defined as

- (1) r -adhesion neighborhood: $\mathcal{P}_r(h) = \{z \in \mathcal{U} : N_r(z) \text{ is equal to } N_r(h)\}.$
- (2) l -adhesion neighborhood: $\mathcal{P}_l(h) = \{z \in \mathcal{U} : N_l(z) \text{ is equal to } N_l(h)\}.$
- (3) i -adhesion neighborhood: $\mathcal{P}_i(h) = \mathcal{P}_r(h) \cap \mathcal{P}_l(h).$

- (4) u -adhesion neighborhood: $\mathcal{P}_u(h) = \mathcal{P}_r(h) \cup \mathcal{P}_l(h).$
- (5) $\langle r \rangle$ -adhesion neighborhood: $\mathcal{P}_{\langle r \rangle}(h) = \{z \in \mathcal{U} : N_{\langle r \rangle}(z) \text{ is equal to } N_{\langle r \rangle}(h)\}.$
- (6) $\langle l \rangle$ -adhesion neighborhood: $\mathcal{P}_{\langle l \rangle}(h) = \{z \in \mathcal{U} : N_{\langle l \rangle}(z) \text{ is equal to } N_{\langle l \rangle}(h)\}.$
- (7) $\langle i \rangle$ -adhesion neighborhood: $\mathcal{P}_{\langle i \rangle}(h) = \mathcal{P}_{\langle r \rangle}(h) \cap \mathcal{P}_{\langle l \rangle}(h).$
- (8) $\langle u \rangle$ -adhesion neighborhood: $\mathcal{P}_{\langle u \rangle}(h) = \mathcal{P}_{\langle r \rangle}(h) \cup \mathcal{P}_{\langle l \rangle}(h).$

Remark 1. It should be noted that the concept of \mathcal{P}_j -adhesion neighborhood of any point in \mathcal{U} in [27] is the same as the notion of core of neighborhood systems induced by R in [28].

Proposition 2 (see [25]). Let R be a binary relation on \mathcal{U} and $h \in \mathcal{U}$. Then, \mathcal{P}_j -neighborhoods have the next properties:

- (1) $\mathcal{P}_{\langle r \rangle}(h) = \mathcal{P}_l(h), \mathcal{P}_{\langle l \rangle}(h) = \mathcal{P}_r(h).$
- (2) $\mathcal{P}_{\langle i \rangle}(h) = \mathcal{P}_i(h), \mathcal{P}_{\langle u \rangle}(h) = \mathcal{P}_u(h).$

Lemma 1 (see [25]). Let R be a binary relation on \mathcal{U} and $h, z \in \mathcal{U}$. Then,

- (1) $h \in \mathcal{P}_j(h)$ for all j .
- (2) $z \in \mathcal{P}_j(h)$ iff $\mathcal{P}_j(z) = \mathcal{P}_j(h)$ for every $j \in \{i, \langle i \rangle, r, \langle r \rangle, l, \langle l \rangle\}.$

Corollary 1 (see [25]). Let R be a binary relation on \mathcal{U} . The class $\wp(\mathcal{U}) = \{\mathcal{P}_j(y) | y \in \mathcal{U}\}$ forms a partition for \mathcal{U} in the cases of $j \in \{i, \langle i \rangle, r, \langle r \rangle, l, \langle l \rangle\}.$

Proposition 3 (see [25]). Consider R as a reflexive relation on \mathcal{U} and $h \in \mathcal{U}$. Then, $\mathcal{P}_j(h) \subseteq N_j(h) \subseteq E_j(h)$ for each j .

Proposition 4 (see [25]). Consider R as an equivalence relation on \mathcal{U} and $h \in \mathcal{U}$. Then, $\mathcal{P}_j(h) = N_j(h) = E_j(h)$ for each j .

Theorem 4 (see [25]). Let (\mathcal{U}, R, ψ_j) be a j -adhesion neighborhood space $(\mathcal{P}_j$ S). Then, for each $j \in \{r, l, i, u\}$, the class $\tau_{\mathcal{P}_j} = \{M \subseteq \mathcal{U} : \forall h \in M, \mathcal{P}_j(h) \subseteq M\}$ forms a topology on \mathcal{U} .

Definition 9 (see [25]). Let (\mathcal{U}, R, ψ_j) be a j -adhesion neighborhood space. We called a set $M \subseteq \mathcal{U}$ a j -adhesion open set if $M \in \tau_{\mathcal{P}_j}$, and we called its complement a j -adhesion closed set. The family $\Gamma_{\mathcal{P}_j}$ of all j -adhesion closed sets of a j -neighborhood space is defined by $\Gamma_{\mathcal{P}_j} = \{K \subseteq \mathcal{U} : K^c \in \tau_{\mathcal{P}_j}\}.$

Remark 2 (see [25]). Let (\mathcal{U}, R, ψ_j) be a j -adhesion neighborhood space. Then,

- (1) The collection $\tau_{\mathcal{P}_j}$ is a quasidiscrete (clopen) topology on \mathcal{U} , $\forall j \in \{i, \langle i \rangle, r, \langle r \rangle, l, \langle l \rangle\}.$
- (2) For each $j, \tau_j \subseteq \tau_{\mathcal{P}_j}$ provided that R is reflexive.

Definition 10 (see [25]). Let (\mathcal{U}, R, ψ_j) be a \mathcal{P}_j S and $\tau_{\mathcal{P}_j}$ be a topology generated by j -adhesion neighborhoods. The j -adhesion lower, j -adhesion upper approximations, and j -adhesion accuracy of $M \subseteq \mathcal{U}$ are respectively given by

- (1) $\mathfrak{L}_j(M) = \bigcup \left\{ O \in \tau_{\mathcal{P}_j} : O \subseteq M \right\} = \text{int}_{\mathcal{P}_j}(M)$.
- (2) $\mathfrak{U}_j(M) = \bigcap \left\{ F \in \Gamma_{\mathcal{P}_j} : M \subseteq F \right\} = \text{cl}_{\mathcal{P}_j}(M)$.
- (3) $\theta_j(M) = (|\mathfrak{L}_j(M)|/|\mathfrak{U}_j(M)|)$, where $|\mathfrak{U}_j(M)| \neq 0$.

2.4. Approximations and Topologies Generated by Different j -Neighborhoods and Ideals. In [23], Hosny presented an idea depending on generating different topologies by using j -neighborhoods and ideals and studied some of their properties.

Theorem 5 (see [23]). Consider (\mathcal{U}, R, ψ_j) as a N_j S and \mathcal{I} as an ideal on \mathcal{U} . Then, the family $\tau_j^\mathcal{I} = \{M \subseteq \mathcal{U} : \forall h \in M, N_j(h) - M \in \mathcal{I}\}$ forms a topology on \mathcal{U} for each j .

Theorem 6 (see [23]). Let (\mathcal{U}, R, ψ_j) be a N_j S and \mathcal{I} be an ideal on \mathcal{U} . Then, $\tau_j^\mathcal{I}$ for each j .

Definition 11 (see [23]). Let (\mathcal{U}, R, ψ_j) be a N_j S and \mathcal{I} be an ideal on \mathcal{U} . We called a set $M \subseteq \mathcal{U}$ an \mathcal{I}_j -open set if $M \in \tau_j^\mathcal{I}$, and we called its complement an \mathcal{I}_j -closed set. The family $\Gamma_j^\mathcal{I}$ of all \mathcal{I}_j -closed sets of a j -neighborhood space is given by $\Gamma_j^\mathcal{I} = \{K \subseteq \mathcal{U} : K^c \in \tau_j^\mathcal{I}\}$.

Definition 12 (see [23]). Let (\mathcal{U}, R, ψ_j) be a N_j S and \mathcal{I} be an ideal on \mathcal{U} . The \mathcal{I}_j -lower, \mathcal{I}_j -upper approximations, and \mathcal{I}_j -accuracy of the approximation of $M \subseteq \mathcal{U}$ are respectively given by

- (1) $\underline{R}_j^\mathcal{I}(M) = \bigcup \left\{ O \in \tau_j^\mathcal{I} : O \subseteq M \right\} = \text{int}_j^\mathcal{I}(M)$, where $\text{int}_j^\mathcal{I}(M)$ is an \mathcal{I}_j interior of M .
- (2) $\overline{R}_j^\mathcal{I}(M) = \bigcap \left\{ F \in \Gamma_j^\mathcal{I} : M \subseteq F \right\} = \text{cl}_j^\mathcal{I}(M)$, where $\text{cl}_j^\mathcal{I}(M)$ is an \mathcal{I}_j closure of M .
- (3) $\alpha_j^\mathcal{I}(M) = (|\underline{R}_j^\mathcal{I}(M)|/|\overline{R}_j^\mathcal{I}(M)|)$, where $|\overline{R}_j^\mathcal{I}(M)| \neq 0$.

2.5. Approximations and Topologies Generated by Different \mathcal{E}_j -Neighborhoods and Ideals. For any binary relation, Hosny et al. [29] utilized the concepts of \mathcal{E}_j -neighborhoods and ideal \mathcal{I} to output various topologies $\zeta_j^\mathcal{I}$ which are finer than the previous one generated by \mathcal{E}_j -neighborhoods due to [26].

Theorem 7 (see [29]). The class $\zeta_j^\mathcal{I} = \{M \subseteq \mathcal{U} : \forall h \in M, \mathcal{E}_j(h) - M \in \mathcal{I}\}$ forms a topology on \mathcal{U} for each j .

Definition 13 (see [29]). Let (\mathcal{U}, R, ψ_j) be a E_j S and \mathcal{I} be an ideal on \mathcal{U} . We called a set $M \subseteq \mathcal{U}$ a $\zeta_j^\mathcal{I}$ -open set if $M \in \zeta_j^\mathcal{I}$, and we called its complement a $\zeta_j^\mathcal{I}$ -closed set. The class $\Pi_j^\mathcal{I}$ of all $\zeta_j^\mathcal{I}$ -closed sets is given by $\Pi_j^\mathcal{I} = \{K \subseteq \mathcal{U} : K^c \in \zeta_j^\mathcal{I}\}$.

Theorem 8 (see [29]). Consider (\mathcal{U}, R, ψ_j) as an E_j S and \mathcal{I} as an ideal on \mathcal{U} . Then,

- (1) $\tau_{\mathcal{E}_j} \subseteq \zeta_j^\mathcal{I}$.

- (2) The reflexivity of \mathcal{R} implies that $\zeta_j^\mathcal{I} \subseteq \zeta_{\langle j \rangle}^\mathcal{I}$ for each $j \in \{i, u, r, l\}$.
- (3) If \mathcal{R} is a symmetric, then $\zeta_r^\mathcal{I} = \zeta_l^\mathcal{I} = \zeta_i^\mathcal{I} = \zeta_u^\mathcal{I}$ and $\zeta_{\langle r \rangle}^\mathcal{I} = \zeta_{\langle l \rangle}^\mathcal{I} = \zeta_{\langle i \rangle}^\mathcal{I} = \zeta_{\langle u \rangle}^\mathcal{I}$.
- (4) The transitivity of \mathcal{R} implies that $\zeta_{\langle j \rangle}^\mathcal{I} \subseteq \zeta_j^\mathcal{I}$ for each $j \in \{i, u, r, l\}$.
- (5) If \mathcal{R} is a preorder, then $\zeta_{\langle j \rangle}^\mathcal{I} = \zeta_j^\mathcal{I}$ for each $j \in \{i, u, r, l\}$.
- (6) If \mathcal{R} is an equivalence, then all $\zeta_j^\mathcal{I}$ are equal, and $\tau_j^\mathcal{I} = \zeta_j^\mathcal{I}$ for each j .

Lemma 2 (see [29]). For any binary relation \mathcal{R} on \mathcal{U} , we have $\mathcal{N}_{\langle j \rangle}(h) \subseteq E_{\langle j \rangle}(h)$ for each $h \in \mathcal{U}$ and $j \in \{i, u, r, l\}$.

Definition 14 (see [29]). Consider $\zeta_j^\mathcal{I}$ as a topology generated by E_j -neighborhoods and ideal \mathcal{I} . Then, for each j , \mathcal{I}_{E_j} -lower, \mathcal{I}_{E_j} -upper approximations, and \mathcal{I}_{E_j} -accuracy of a subset $M \subseteq \mathcal{U}$ are defined respectively as follows:

- (1) $L_j^{\mathcal{I}_{E_j}}(M) = \text{int}_{E_j}^{\mathcal{I}}(M)$, where $\text{int}_{E_j}^{\mathcal{I}}(M)$ forms the interior points of M in $\zeta_j^\mathcal{I}$.
- (2) $U_j^{\mathcal{I}_{E_j}}(M) = \text{cl}_{E_j}^{\mathcal{I}}(M)$, where $\text{cl}_{E_j}^{\mathcal{I}}(M)$ forms the closure points of M in $\zeta_j^\mathcal{I}$.
- (3) $\sigma_j^{\mathcal{I}}(M) = (|L_j^{\mathcal{I}_{E_j}}(M)|/|U_j^{\mathcal{I}_{E_j}}(M)|)$, where $|U_j^{\mathcal{I}_{E_j}}(M)| \neq 0$.

3. Novel Topologies Generated from j -Adhesion Neighborhoods via Ideals

Now, we deal with ideals and four different j -adhesion neighborhoods generated from eight different j -neighborhoods via the same binary relation. By using them, new topologies are generated that generalize these topologies generated by j -adhesion neighborhoods. Several properties and relationships between these topologies are obtained.

Throughout this paper, a \mathcal{P}_j S with ideal $\mathcal{I}(\mathcal{U}, R, \mathcal{I}, \psi_j)$ is denoted by $\mathcal{P}_j^\mathcal{I}$ S.

Theorem 9. Let $(\mathcal{U}, R, \mathcal{I}, \psi_j)$ be a $\mathcal{P}_j^\mathcal{I}$ S. Then, the collection $\rho_j^\mathcal{I} = \{M \subseteq \mathcal{U} : \forall h \in M, \mathcal{P}_j(h) - M \in \mathcal{I}\}$ forms a topology on \mathcal{U} for each j .

Proof. First, let $M_\alpha \in \rho_j^\mathcal{I}$, $\alpha \in \Delta$, and $z \in \bigcup_{\alpha \in \Delta} M_\alpha$. Then, there is an $\alpha_0 \in \Delta$ s.t. $z \in M_{\alpha_0}$. Therefore, $[\mathcal{P}_j(z) - M_{\alpha_0}] \in \mathcal{I}$. Since $-(\bigcup_{\alpha \in \Delta} M_\alpha) \subseteq -M_{\alpha_0}$, $[\mathcal{P}_j(z) - (\bigcup_{\alpha \in \Delta} M_\alpha)] \in \mathcal{I}$, i.e., $\bigcup_{\alpha \in \Delta} M_\alpha \in \rho_j^\mathcal{I}$. Second, let M_1, M_2 be members of $\rho_j^\mathcal{I}$ and z belong to the intersection of M_1 and M_2 . Then, $[\mathcal{P}_j(z) - M_1] \in \mathcal{I}$ and $[\mathcal{P}_j(z) - M_2] \in \mathcal{I}$. According to the definition of \mathcal{I} , we obtain $[\mathcal{P}_j(z) - M_1] \cup [\mathcal{P}_j(z) - M_2] \in \mathcal{I}$. Hence, $[\mathcal{P}_j(z) - (M_1 \cap M_2)] \in \mathcal{I}$. This means that $M_1 \cap M_2 \in \rho_j^\mathcal{I}$. Finally, it is easy to see that $\emptyset, \mathcal{U} \in \rho_j^\mathcal{I}$, $\forall j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$. Consequently, $\rho_j^\mathcal{I}$ is a topology on \mathcal{U} . \square

Lemma 3. If \mathcal{I}, \mathcal{J} are ideals on a \mathcal{P}_j S (\mathcal{U}, R, ψ_j) such that \mathcal{I} is a subset of \mathcal{J} , then $\rho_j^\mathcal{I} \subseteq \rho_j^\mathcal{J}$.

Proof. Direct to prove.

The fundamental goal of the next results is to deduce the relations between the topologies generated by j -adhesion neighborhoods, topologies generated by j -neighborhoods and ideals, and topologies generated by j -adhesion neighborhoods and ideals, as it is shown in the following theorems.

Our new types of topologies, which were generated by j -adhesion neighborhoods and ideals, are finer than the previous one generated by j -adhesion neighborhoods due to [28] for any relation. \square

Theorem 10. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$. If $j \in \{r, l, i, u\}$, then $\tau_{\mathcal{P}_j} \subseteq \rho_j^{\mathcal{F}}$.

Proof. Straightforward. \square

Example 1. Let $\mathcal{U} = \{a, b, c, d\}$ and $R = \{(c, a), (c, b), (c, d), (d, a), (d, c), (d, d)\}$. Then, we obtain the next topologies

$$\begin{aligned}
 \tau_r &= \{\{a\}, \{b\}, \{a, b\}, \emptyset, \mathcal{U}\}, \\
 \tau_l &= \{\{c, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_i &= \{\{a\}, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_u &= \{\emptyset, \mathcal{U}\}, \\
 \tau_{\langle r \rangle} &= \tau_{\langle u \rangle} = \{\{a, d\}, \{a, b, d\}, \{a, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_{\langle l \rangle} &= \tau_{\langle i \rangle} = P(U), \\
 \tau_{\mathcal{P}_r} &= \tau_{\mathcal{P}_{\langle l \rangle}} = \{\{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}, \emptyset, \mathcal{U}\}, \\
 \tau_{\mathcal{P}_l} &= \tau_{\mathcal{P}_{\langle r \rangle}} = \{\{b\}, \{c\}, \{a, d\}, \{b, c\}, \{a, b, d\}, \{a, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_{\mathcal{P}_i} &= \tau_{\mathcal{P}_{\langle i \rangle}} = P(U), \\
 \tau_{\mathcal{P}_u} &= \tau_{\mathcal{P}_{\langle u \rangle}} = \{\{c\}, \{a, b, d\}, \emptyset, \mathcal{U}\}.
 \end{aligned} \tag{1}$$

If $\mathcal{F} = \{\emptyset, \{a\}\}$, then

$$\begin{aligned}
 \tau_r^{\mathcal{F}} &= \{\{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_l^{\mathcal{F}} &= \{\{c, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_i^{\mathcal{F}} &= \{\{a\}, \{b\}, \{c, d\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_u^{\mathcal{F}} &= \{\{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_{\langle r \rangle}^{\mathcal{F}} &= \tau_{\langle u \rangle}^{\mathcal{F}} = \{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_{\langle l \rangle}^{\mathcal{F}} &= \tau_{\langle i \rangle}^{\mathcal{F}} = P(U), \\
 \rho_r^{\mathcal{F}} &= \rho_{\langle l \rangle}^{\mathcal{F}} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \rho_l^{\mathcal{F}} &= \rho_{\langle r \rangle}^{\mathcal{F}} = \{\{b\}, \{c\}, \{d\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \rho_i^{\mathcal{F}} &= \rho_{\langle i \rangle}^{\mathcal{F}} = P(U), \\
 \rho_u^{\mathcal{F}} &= \rho_{\langle u \rangle}^{\mathcal{F}} = \{\{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}.
 \end{aligned} \tag{2}$$

If $\widehat{\mathcal{F}} = \{\emptyset, \{d\}\}$, then

$$\begin{aligned}
 \widehat{\tau}_r^{\mathcal{F}} &= \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \emptyset, \mathcal{U}\}, \\
 \widehat{\tau}_l^{\mathcal{F}} &= \{\{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \widehat{\tau}_i^{\mathcal{F}} &= \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \widehat{\tau}_u^{\mathcal{F}} &= \{\{a, b, c\}, \emptyset, \mathcal{U}\}, \\
 \widehat{\tau}_{\langle r \rangle}^{\mathcal{F}} &= \widehat{\tau}_{\langle u \rangle}^{\mathcal{F}} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \emptyset, \mathcal{U}\}, \\
 \widehat{\tau}_{\langle l \rangle}^{\mathcal{F}} &= \widehat{\tau}_{\langle i \rangle}^{\mathcal{F}} \\
 \widehat{\rho}_r^{\mathcal{F}} &= \widehat{\rho}_{\langle l \rangle}^{\mathcal{F}} = \{\{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \emptyset, \mathcal{U}\},
 \end{aligned}$$

$$\begin{aligned}
\hat{\rho}_l^{\mathcal{F}} &= \hat{\rho}_{\langle r \rangle}^{\mathcal{F}} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \emptyset, \mathcal{U}\}, \\
\hat{\rho}_i^{\mathcal{F}} &= \hat{\rho}_{\langle i \rangle}^{\mathcal{F}} = P(U), \\
\hat{\rho}_u^{\mathcal{F}} &= \hat{\rho}_{\langle u \rangle}^{\mathcal{F}} = \{\{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \emptyset, \mathcal{U}\}.
\end{aligned} \tag{3}$$

If $\mathcal{F} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$, then

$$\begin{aligned}
\tau_r^{\mathcal{F}} &= \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
\tau_l^{\mathcal{F}} &= \{\{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
\tau_i^{\mathcal{F}} &= \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
\tau_u^{\mathcal{F}} &= \{\{b, c\}, \{a, b, c\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
\tau_{\langle r \rangle}^{\mathcal{F}} &= \tau_{\langle u \rangle}^{\mathcal{F}} = \tau_{\langle l \rangle}^{\mathcal{F}} = \tau_{\langle i \rangle}^{\mathcal{F}} = P(U), \\
\rho_r^{\mathcal{F}} &= \rho_{\langle l \rangle}^{\mathcal{F}} = \rho_u^{\mathcal{F}} = \rho_{\langle u \rangle}^{\mathcal{F}} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
\rho_l^{\mathcal{F}} &= \rho_{\langle r \rangle}^{\mathcal{F}} = \rho_i^{\mathcal{F}} = \rho_{\langle i \rangle}^{\mathcal{F}} = P(U).
\end{aligned} \tag{4}$$

Remark 3. If $\mathcal{F} = \{\emptyset, \{a\}\}$ in Example 1, then

- (1) $\rho_r^{\mathcal{F}} \neq \rho_i^{\mathcal{F}}$ and $\rho_l^{\mathcal{F}} \neq \rho_i^{\mathcal{F}}$.
- (2) $\rho_r^{\mathcal{F}} \neq \rho_u^{\mathcal{F}}$ and $\rho_l^{\mathcal{F}} \neq \rho_u^{\mathcal{F}}$.
- (3) $\rho_i^{\mathcal{F}} \neq \rho_u^{\mathcal{F}}$.

Remark 4. In view of Example 1,

- (1) In general, the collection $\rho_j^{\mathcal{F}}$ need not be a quasi-discrete (clopen) topology on \mathcal{U} , although $\tau_{\mathcal{F}_j}$ is quasidiscrete topology on \mathcal{U} , $\forall j \in \{i, \langle i \rangle, r, \langle r \rangle, l, \langle l \rangle\}$.
- (2) If $\forall j \in \{u, \langle u \rangle\}$, then $\rho_j^{\mathcal{F}}$ is not quasidiscrete topology on \mathcal{U} .
- (3) $\rho_r^{\mathcal{F}}$ need not be dual topology to $\rho_l^{\mathcal{F}}$.
- (4) If R is any relation, then $\tau_j^{\mathcal{F}}, \rho_j^{\mathcal{F}}$ are not comparable, for each j .
- (5) If $\mathcal{F} = \{\emptyset\}$ in Theorem 9, then the present generated topologies coincide with the previous one in

Proposition 4.1 in [28]. So, the current work is considered as a generalization of work of [25, 27, 28].

Theorem 11 illustrates that the reflexivity condition is necessary to create a relationship between the topologies $\tau_j^{\mathcal{F}}$ and $\rho_j^{\mathcal{F}}$.

Theorem 11. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$, where R is reflexive. Then, $\tau_j^{\mathcal{F}} \subseteq \rho_j^{\mathcal{F}}$, for any ideal \mathcal{F} .

Proof. Let \mathcal{F} be any ideal on \mathcal{U} . According to Proposition 3, $\tau_j^{\mathcal{F}} \subseteq \rho_j^{\mathcal{F}}$. \square

Remark 5. The new forms of topologies, which were generated by j -adhesion neighborhoods and ideals, are finer than the topologies generated by j -neighborhoods and ideals due to [23] for reflexive relation.

Example 2. Let $\mathcal{U} = \{a, b, c, d\}$ and $R = \blacktriangle \cup \{(b, c), (b, d), (c, a), (d, b), (c, d), (d, c)\}$.

If $\mathcal{F} = \{\emptyset, \{a\}\}$, then

$$\begin{aligned}
\tau_r^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{a\}, \{b, c, d\}\}, \\
\tau_l^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{b, c, d\}\}, \\
\tau_i^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{a\}, \{b, c, d\}\}, \\
\tau_u^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{b, c, d\}\}, \\
\tau_{\langle r \rangle}^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}, \\
\tau_{\langle l \rangle}^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{c\}, \{a, c\}, \{b, d\}, \{b, c, d\}\},
\end{aligned}$$

$$\begin{aligned}
\tau_{\langle i \rangle}^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}, \\
\tau_{\langle u \rangle}^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{b, c, d\}\}, \\
\rho_r^{\mathcal{F}} &= \rho_{\langle l \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}, \\
\rho_l^{\mathcal{F}} &= \rho_{\langle r \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}, \\
\rho_i^{\mathcal{F}} &= \rho_{\langle i \rangle}^{\mathcal{F}} = P(U), \\
\rho_u^{\mathcal{F}} &= \rho_{\langle u \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{a\}, \{b, c, d\}\}.
\end{aligned} \tag{5}$$

Theorem 12. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$ and R be a symmetric relation on \mathcal{U} . Then, $\rho_r^{\mathcal{F}} = \rho_l^{\mathcal{F}} = \rho_i^{\mathcal{F}} = \rho_u^{\mathcal{F}} = \rho_{\langle r \rangle}^{\mathcal{F}} = \rho_{\langle l \rangle}^{\mathcal{F}} = \rho_{\langle i \rangle}^{\mathcal{F}} = \rho_{\langle u \rangle}^{\mathcal{F}}$.

Proof. Since R is a symmetric relation on \mathcal{U} , then all j -adhesion neighborhoods coincide. So, $\rho_r^{\mathcal{F}} = \rho_l^{\mathcal{F}} = \rho_i^{\mathcal{F}} = \rho_u^{\mathcal{F}} = \rho_{\langle r \rangle}^{\mathcal{F}} = \rho_{\langle l \rangle}^{\mathcal{F}} = \rho_{\langle i \rangle}^{\mathcal{F}} = \rho_{\langle u \rangle}^{\mathcal{F}}$. \square

Proposition 5. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$. Then, the following statements hold:

- (1) $\rho_r^{\mathcal{F}} = \rho_{\langle l \rangle}^{\mathcal{F}}, \rho_l^{\mathcal{F}} = \rho_{\langle r \rangle}^{\mathcal{F}}$.
- (2) $\rho_i^{\mathcal{F}} = \rho_{\langle i \rangle}^{\mathcal{F}}, \rho_u^{\mathcal{F}} = \rho_{\langle u \rangle}^{\mathcal{F}}$.
- (3) $\rho_u^{\mathcal{F}} \subseteq \rho_r^{\mathcal{F}} \cap \rho_l^{\mathcal{F}}$.
- (4) $\rho_r^{\mathcal{F}} \cup \rho_l^{\mathcal{F}} \subseteq \rho_i^{\mathcal{F}}$.
- (5) $\rho_{\langle u \rangle}^{\mathcal{F}} \subseteq \rho_{\langle r \rangle}^{\mathcal{F}} \cap \rho_{\langle l \rangle}^{\mathcal{F}}$.
- (6) $\rho_{\langle r \rangle}^{\mathcal{F}} \cup \rho_{\langle l \rangle}^{\mathcal{F}} \subseteq \rho_{\langle i \rangle}^{\mathcal{F}}$.

Proof. We prove (3) and one can prove the other cases in a similar way. Let $M \in \rho_u^{\mathcal{F}}$, then $[\mathcal{P}_u(z) - M] \in \mathcal{F}, \forall z \in M$. Hence, $[(\mathcal{P}_r(z) \cup \mathcal{P}_l(z)) - M] \in \mathcal{F}, \forall z \in M$. So, $[\mathcal{P}_r(z) - M] \in \mathcal{F}$ and $[\mathcal{P}_l(z) - M] \in \mathcal{F}, \forall z \in M$. Consequently, $M \in \rho_r^{\mathcal{F}} \cap \rho_l^{\mathcal{F}}$. \square

Definition 15. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$. A set $M \subseteq \mathcal{U}$ is called $\rho_j^{\mathcal{F}}$ -open set if $M \in \rho_j^{\mathcal{F}}$, and its complement is called $\rho_j^{\mathcal{F}}$ -closed set. The family $\Upsilon_j^{\mathcal{F}}$ of all $\rho_j^{\mathcal{F}}$ -closed sets is defined as $\Upsilon_j^{\mathcal{F}} = \{K \subseteq \mathcal{U} : K^c \in \rho_j^{\mathcal{F}}\}$.

Definition 16. Consider $\rho_j^{\mathcal{F}}$ as a topology generated by j -adhesion neighborhoods and ideals. The \mathcal{F}_{P_j} -lower, \mathcal{F}_{P_j} -upper approximations, \mathcal{F}_{P_j} -boundary regions, and \mathcal{F}_{P_j} -accuracy of $M \subseteq \mathcal{U}$ are respectively given as follows:

- (1) $\mathfrak{L}_j^{\mathcal{F}}(M) = \cup \{O \in \rho_j^{\mathcal{F}} : O \subseteq M\} = \text{int}_{\mathcal{F}_j}^{\mathcal{F}}(M)$, where $\text{int}_{\mathcal{F}_j}^{\mathcal{F}}(M)$ represents interior of M wrt $\rho_j^{\mathcal{F}}$.
- (2) $\mathfrak{U}_j^{\mathcal{F}}(M) = \cap \{F \in \Upsilon_j^{\mathcal{F}} : M \subseteq F\} = \text{cl}_{\mathcal{F}_j}^{\mathcal{F}}(M)$, where $\text{cl}_{\mathcal{F}_j}^{\mathcal{F}}(M)$ represents closure of M wrt $\rho_j^{\mathcal{F}}$.
- (3) $\mathfrak{B}_j^{\mathcal{F}}(M) = \mathfrak{U}_j^{\mathcal{F}}(M) - \mathfrak{L}_j^{\mathcal{F}}(M)$.
- (4) $\theta_j^{\mathcal{F}}(M) = (|\mathfrak{L}_j^{\mathcal{F}}(M)|/|\mathfrak{U}_j^{\mathcal{F}}(M)|)$, where $|\mathfrak{U}_j^{\mathcal{F}}(M)| \neq 0$.

Remark 6. Table 1 displays the comparison between j -approximations and j -accuracy for $j = r$ depending on Definitions 10, 12, and 16 by using any relation R and ideal $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 1.

Remark 7. Table 2 offers the comparison between j -approximations and j -accuracy for $j = l$ depending on Definitions 12 and 16 by using a reflexive relation R and ideal $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 2.

The next proposition that demonstrates the fundamental properties of $\mathfrak{L}_j^{\mathcal{F}}, \mathfrak{U}_j^{\mathcal{F}}$ operators are understandable by observing that $\text{int}_{\mathcal{F}_j}^{\mathcal{F}}, \text{cl}_{\mathcal{F}_j}^{\mathcal{F}}$ fulfill all properties of the topological interior and closure operators, respectively.

Proposition 6. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$ and $j \in \{i, u, r, l\}$. If $M, M \subseteq \mathcal{U}$, then the following conditions hold:

- (L₁) $\mathfrak{L}_j^{\mathcal{F}}(M) = (\mathfrak{U}_j^{\mathcal{F}}(M^c))^c$.
- (U₁) $\mathfrak{U}_j^{\mathcal{F}}(M) = (\mathfrak{L}_j^{\mathcal{F}}(M^c))^c$.
- (L₂) $\mathfrak{L}_j^{\mathcal{F}}(\mathcal{U}) = \mathcal{U}$.
- (U₂) $\mathfrak{U}_j^{\mathcal{F}}(\emptyset) = \emptyset$.
- (L₃) if $M \subseteq M$, then $\mathfrak{L}_j^{\mathcal{F}}(M) \subseteq \mathfrak{L}_j^{\mathcal{F}}(M)$.
- (U₃) if $M \subseteq M$, then $\mathfrak{U}_j^{\mathcal{F}}(M) \subseteq \mathfrak{U}_j^{\mathcal{F}}(M)$.
- (L₄) $\mathfrak{L}_j^{\mathcal{F}}(M \cap M) = \mathfrak{L}_j^{\mathcal{F}}(M) \cap \mathfrak{L}_j^{\mathcal{F}}(M)$.
- (U₄) $\mathfrak{U}_j^{\mathcal{F}}(M \cup M) = \mathfrak{U}_j^{\mathcal{F}}(M) \cup \mathfrak{U}_j^{\mathcal{F}}(M)$.
- (L₅) $\mathfrak{L}_j^{\mathcal{F}}(M \cup M) \supseteq \mathfrak{L}_j^{\mathcal{F}}(M) \cup \mathfrak{L}_j^{\mathcal{F}}(M)$.
- (U₅) $\mathfrak{U}_j^{\mathcal{F}}(M \cap M) \subseteq \mathfrak{U}_j^{\mathcal{F}}(M) \cap \mathfrak{U}_j^{\mathcal{F}}(M)$.
- (L₆) $\mathfrak{L}_j^{\mathcal{F}}(\emptyset) = \emptyset$.
- (U₆) $\mathfrak{U}_j^{\mathcal{F}}(\mathcal{U}) = \mathcal{U}$.
- (L₇) $\mathfrak{L}_j^{\mathcal{F}}(M) \subseteq M$.
- (U₇) $M \subseteq \mathfrak{U}_j^{\mathcal{F}}(M)$.
- (L₈) $\mathfrak{L}_j^{\mathcal{F}}(M) = \mathfrak{L}_j^{\mathcal{F}}(\mathfrak{L}_j^{\mathcal{F}}(M))$.
- (U₈) $\mathfrak{U}_j^{\mathcal{F}}(M) = \mathfrak{U}_j^{\mathcal{F}}(\mathfrak{U}_j^{\mathcal{F}}(M))$.
- (L₉) $\mathfrak{L}_j^{\mathcal{F}}(M) \subseteq \mathfrak{U}_j^{\mathcal{F}}(\mathfrak{L}_j^{\mathcal{F}}(M))$.
- (U₉) $\mathfrak{U}_j^{\mathcal{F}}(M) \supseteq \mathfrak{L}_j^{\mathcal{F}}(\mathfrak{U}_j^{\mathcal{F}}(M))$.

Remark 8. If $\mathcal{F} = \{\emptyset, \{a\}\}$ and $j = r$, then Table 1 shows that the converse of (L₃), (L₅), (L₇), (L₉) does not hold.

TABLE 1: Comparison between j -approximations and j -accuracy for each $j = r$ depending on Definitions 10, 12, and 16 by using any relation R and $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 1.

$M \subseteq \mathcal{U}$	Our method, Definition 16			Definition 10			Definition 12		
	$\mathfrak{R}_r^{\mathcal{F}}(M)$	$\mathfrak{U}_r^{\mathcal{F}}(M)$	$\theta_r^{\mathcal{F}}(M)$	$\mathfrak{R}_r(M)$	$\mathfrak{U}_r(M)$	$\theta_r(M)$	$\underline{R}_r^{\mathcal{F}}(M)$	$\overline{R}_r^{\mathcal{F}}(M)$	$\alpha_r^{\mathcal{F}}(M)$
$\{a\}$	\emptyset	$\{a\}$	0	\emptyset	$\{a, b\}$	0	$\{a\}$	$\{a\}$	1
$\{b\}$	$\{b\}$	$\{a, b\}$	1/2	\emptyset	$\{a, b\}$	0	$\{b\}$	$\{b, c, d\}$	1/3
$\{c\}$	$\{c\}$	$\{c\}$	1	$\{c\}$	$\{c\}$	1	\emptyset	$\{c, d\}$	0
$\{d\}$	$\{d\}$	$\{d\}$	1	$\{d\}$	$\{d\}$	1	\emptyset	$\{c, d\}$	0
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	1	$\{a, b\}$	$\{a, b\}$	1	$\{a, b\}$	\mathcal{U}	1/2
$\{a, c\}$	$\{c\}$	$\{a, c\}$	1/2	$\{c\}$	$\{a, b, c\}$	1/3	$\{a\}$	$\{a, c, d\}$	1/3
$\{a, d\}$	$\{d\}$	$\{a, d\}$	1/2	$\{d\}$	$\{a, b, d\}$	1/3	$\{a\}$	$\{a, c, d\}$	1/3
$\{b, c\}$	$\{b, c\}$	$\{a, b, c\}$	2/3	$\{c\}$	$\{a, b, c\}$	1/3	$\{b\}$	$\{b, c, d\}$	1/3
$\{b, d\}$	$\{b, d\}$	$\{a, b, d\}$	2/3	$\{d\}$	$\{a, b, d\}$	1/3	$\{b\}$	$\{b, c, d\}$	1/3
$\{c, d\}$	$\{c, d\}$	$\{c, d\}$	1	$\{c, d\}$	$\{c, d\}$	1	\emptyset	$\{c, d\}$	0
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	1	$\{a, b, c\}$	$\{a, b, c\}$	1	$\{a, b\}$	\mathcal{U}	1/2
$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b\}$	\mathcal{U}	1/2
$\{a, c, d\}$	$\{c, d\}$	$\{a, c, d\}$	2/3	$\{c, d\}$	\mathcal{U}	1/2	$\{a\}$	$\{a, c, d\}$	1/3
$\{b, c, d\}$	$\{b, c, d\}$	\mathcal{U}	3/4	$\{c, d\}$	\mathcal{U}	1/2	$\{b, c, d\}$	$\{b, c, d\}$	1
\mathcal{U}	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1

TABLE 2: Comparison between j -approximations and j -accuracy for each $j = l$ depending on Definitions 12 and 16 by using a reflexive relation R and $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 2.

$M \subseteq \mathcal{U}$	Our method, Definition 16			Definition 12		
	$\mathfrak{R}_l^{\mathcal{F}}(M)$	$\mathfrak{U}_l^{\mathcal{F}}(M)$	$\theta_l^{\mathcal{F}}(M)$	$\underline{R}_l^{\mathcal{F}}(M)$	$\overline{R}_l^{\mathcal{F}}(M)$	$\alpha_l^{\mathcal{F}}(M)$
$\{a\}$	$\{a\}$	$\{a\}$	1	\emptyset	$\{a\}$	0
$\{b\}$	$\{b\}$	$\{b\}$	1	\emptyset	\mathcal{U}	0
$\{c\}$	\emptyset	$\{c, d\}$	0	\emptyset	\mathcal{U}	0
$\{d\}$	\emptyset	$\{c, d\}$	0	\emptyset	\mathcal{U}	0
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	1	\emptyset	\mathcal{U}	0
$\{a, c\}$	$\{a\}$	$\{a, c, d\}$	1/3	\emptyset	\mathcal{U}	0
$\{a, d\}$	$\{a\}$	$\{a, c, d\}$	1/3	\emptyset	\mathcal{U}	0
$\{b, c\}$	$\{b\}$	$\{b, c, d\}$	1/3	\emptyset	\mathcal{U}	0
$\{b, d\}$	$\{b\}$	$\{b, c, d\}$	1/3	\emptyset	\mathcal{U}	0
$\{c, d\}$	$\{c, d\}$	$\{c, d\}$	1	\emptyset	\mathcal{U}	0
$\{a, b, c\}$	$\{a, b\}$	\mathcal{U}	1/2	\emptyset	\mathcal{U}	0
$\{a, b, d\}$	$\{a, b\}$	\mathcal{U}	1/2	\emptyset	\mathcal{U}	0
$\{a, c, d\}$	$\{a, c, d\}$	$\{a, c, d\}$	1	\emptyset	\mathcal{U}	0
$\{b, c, d\}$	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	\mathcal{U}	3/4
\mathcal{U}	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1

- (1) If $M = \{a\}$, $M = \{b\}$, then $\mathfrak{R}_r^{\mathcal{F}}(M) \subseteq \mathfrak{R}_r^{\mathcal{F}}(M)$, but $M \not\subseteq M$. Consequently, the converse of (L_3) is not true.
- (2) If $M = \{a\}$, $M = \{b\}$, then $\mathfrak{R}_r^{\mathcal{F}}(M \cup M) = \{a, b\} \neq \{b\} = \mathfrak{R}_r^{\mathcal{F}}(M) \cup \mathfrak{R}_r^{\mathcal{F}}(M)$. Consequently, the converse of (L_5) is not true.
- (3) If $M = \{a, c\}$, then the converse of (L_7) is not true.
- (4) The equality of properties (L_9) , (U_9) does not hold wrt the topology $\rho_j^{\mathcal{F}}$, although they are true according to the topology $\tau_{\mathcal{F}}$. Here, if $M = \{b\}$, then $\mathfrak{R}_j^{\mathcal{F}}(M) = \{b\}$ and $\mathfrak{U}_j^{\mathcal{F}}(\mathfrak{R}_j^{\mathcal{F}}(M)) = \{a, b\}$. Consequently, the converse of (L_9) does not hold.

By using any binary relation and without totting any supplement conditions as Proposition 6 claims, one might observe that our approximation approach satisfies all features of the classical theory of rough set [1]. Thus, we can say

that our presented method exemplifies a noteworthy generalization to RST.

Proposition 7. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$ and $M \subseteq \mathcal{U}$. If $j \in \{r, l, i, \langle r \rangle, \langle l \rangle, \langle i \rangle\}$, then the following conditions hold: $\mathfrak{R}_j^{\mathcal{F}}(M) = M = \mathfrak{U}_j^{\mathcal{F}}(M)$, $\forall M \in \wp(\mathcal{U})$.

Proposition 8. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$. If R is a reflexive relation on \mathcal{U} , then $\mathfrak{R}_j^{\mathcal{F}}(M) = \mathfrak{U}_j^{\mathcal{F}}(\mathfrak{R}_j^{\mathcal{F}}(M))$ and $\mathfrak{U}_j^{\mathcal{F}}(M) = \mathfrak{R}_j^{\mathcal{F}}(\mathfrak{U}_j^{\mathcal{F}}(M))$.

The following propositions are obvious and the proof is omitted.

Proposition 9. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$ and $M \subseteq \mathcal{U}$. If $j \in \{r, l, i, \langle r \rangle, \langle l \rangle, \langle i \rangle\}$, then the following statements hold:

- (1) $\mathfrak{R}_u^{\mathcal{F}}(M) \subseteq \mathfrak{R}_r^{\mathcal{F}}(M) \subseteq \mathfrak{R}_i^{\mathcal{F}}(M)$ and $\mathfrak{R}_u^{\mathcal{F}}(M) \subseteq \mathfrak{R}_l^{\mathcal{F}}(M) \subseteq \mathfrak{R}_i^{\mathcal{F}}(M)$.
- (2) $\mathfrak{U}_i^{\mathcal{F}}(M) \subseteq \mathfrak{U}_r^{\mathcal{F}}(M) \subseteq \mathfrak{U}_u^{\mathcal{F}}(M)$ and $\mathfrak{U}_i^{\mathcal{F}}(M) \subseteq \mathfrak{U}_l^{\mathcal{F}}(M) \subseteq \mathfrak{U}_u^{\mathcal{F}}(M)$.
- (3) $\theta_u^{\mathcal{F}}(M) \leq \theta_r^{\mathcal{F}}(M) \leq \theta_i^{\mathcal{F}}(M)$ and $\theta_u^{\mathcal{F}}(M) \leq \theta_l^{\mathcal{F}}(M) \leq \theta_i^{\mathcal{F}}(M)$.

Proposition 10. Let \mathcal{F}, \mathcal{F} be two ideals on a $\mathcal{P}_j^{\mathcal{F}}S(\mathcal{U}, R, \psi_j)$ and $M \subseteq \mathcal{U}$. If $\mathcal{F} \subseteq \mathcal{F}$, then the following statements hold:

- (1) $\mathfrak{R}_j^{\mathcal{F}}(M) \subseteq \mathfrak{R}_j^{\mathcal{F}}(M)$.
- (2) $\mathfrak{U}_j^{\mathcal{F}}(M) \supseteq \mathfrak{U}_j^{\mathcal{F}}(M)$.
- (3) $\theta_j^{\mathcal{F}}(M) \leq \theta_j^{\mathcal{F}}(M)$.

In Tables 3 and 4, we make comparisons between j -approximations and j -accuracy for each $j \in \{i, u, r, l\}$.

Definition 17. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$ and $j \in \{i, \langle i \rangle, r, \langle r \rangle, l, \langle l \rangle\}$. A subset M of \mathcal{U} is called

- (1) Totally \mathcal{F}_j definable, if $\mathfrak{R}_j^{\mathcal{F}}(M) = M = \mathfrak{U}_j^{\mathcal{F}}(M)$.

TABLE 3: Comparison between j -approximations and j -accuracy for each $j \in \{i, u, r, l\}$ depending on Definition 16 by using any relation R and $\mathcal{F} = \{\emptyset, \{a\}\}$.

$M \subseteq \mathcal{U}$	$\mathfrak{R}_r^{\mathcal{F}}(M)$	$\mathfrak{U}_r^{\mathcal{F}}(M)$	$\theta_r^{\mathcal{F}}(M)$	$\mathfrak{R}_l^{\mathcal{F}}(M)$	$\mathfrak{U}_l^{\mathcal{F}}(M)$	$\theta_l^{\mathcal{F}}(M)$	$\mathfrak{R}_i^{\mathcal{F}}(M)$	$\mathfrak{U}_i^{\mathcal{F}}(M)$	$\theta_i^{\mathcal{F}}(M)$	$\mathfrak{R}_u^{\mathcal{F}}(M)$	$\mathfrak{U}_u^{\mathcal{F}}(M)$	$\theta_u^{\mathcal{F}}(M)$
$\{a\}$	\emptyset	$\{a\}$	0	\emptyset	$\{a\}$	0	$\{a\}$	$\{a\}$	1	\emptyset	$\{a\}$	0
$\{b\}$	$\{b\}$	$\{a, b\}$	1/2	$\{b\}$	$\{b\}$	1	$\{b\}$	$\{b\}$	1	$\{b\}$	$\{a, b\}$	1/2
$\{c\}$	$\{c\}$	$\{c\}$	1	$\{c\}$	$\{c\}$	1	$\{c\}$	$\{c\}$	1	$\{c\}$	$\{c\}$	1
$\{d\}$	$\{d\}$	$\{d\}$	1	$\{d\}$	$\{a, d\}$	1/2	$\{d\}$	$\{d\}$	1	$\{d\}$	$\{a, d\}$	1/2
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	1	$\{b\}$	$\{a, b\}$	1/2	$\{a, b\}$	$\{a, b\}$	1	$\{b\}$	$\{a, b\}$	1/2
$\{a, c\}$	$\{c\}$	$\{a, c\}$	1/2	$\{c\}$	$\{a, c\}$	1/2	$\{a, c\}$	$\{a, c\}$	1	$\{c\}$	$\{a, c\}$	1/2
$\{a, d\}$	$\{d\}$	$\{a, d\}$	1/2	$\{a, d\}$	$\{a, d\}$	1	$\{a, d\}$	$\{a, d\}$	1	$\{d\}$	$\{a, d\}$	1/2
$\{b, c\}$	$\{b, c\}$	$\{a, b, c\}$	2/3	$\{b, c\}$	$\{b, c\}$	1	$\{b, c\}$	$\{b, c\}$	1	$\{b, c\}$	$\{a, b, c\}$	2/3
$\{b, d\}$	$\{b, d\}$	$\{a, b, d\}$	2/3	$\{b, d\}$	$\{a, b, d\}$	2/3	$\{b, d\}$	$\{b, d\}$	1	$\{b, d\}$	$\{a, b, d\}$	2/3
$\{c, d\}$	$\{c, d\}$	$\{c, d\}$	1	$\{c, d\}$	$\{a, c, d\}$	2/3	$\{c, d\}$	$\{c, d\}$	1	$\{c, d\}$	$\{a, c, d\}$	2/3
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	1	$\{b, c\}$	$\{a, b, c\}$	2/3	$\{a, b, c\}$	$\{a, b, c\}$	1	$\{b, c\}$	$\{a, b, c\}$	2/3
$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b, d\}$	$\{a, b, d\}$	1
$\{a, c, d\}$	$\{c, d\}$	$\{a, c, d\}$	2/3	$\{a, c, d\}$	$\{a, c, d\}$	1	$\{a, c, d\}$	$\{a, c, d\}$	1	$\{c, d\}$	$\{a, c, d\}$	2/3
$\{b, c, d\}$	$\{b, c, d\}$	\mathcal{U}	3/4	$\{b, c, d\}$	\mathcal{U}	3/4	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	\mathcal{U}	3/4
\mathcal{U}	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1

TABLE 4: Comparison between j -approximations and j -accuracy for each $j \in \{r, l, i, u\}$ depending on Definition 16 by using reflexive relation R and $\mathcal{F} = \{\emptyset, \{a\}\}$.

$M \subseteq \mathcal{U}$	$\mathfrak{R}_r^{\mathcal{F}}(M)$	$\mathfrak{U}_r^{\mathcal{F}}(M)$	$\theta_r^{\mathcal{F}}(M)$	$\mathfrak{R}_l^{\mathcal{F}}(M)$	$\mathfrak{U}_l^{\mathcal{F}}(M)$	$\theta_l^{\mathcal{F}}(M)$	$\mathfrak{R}_i^{\mathcal{F}}(M)$	$\mathfrak{U}_i^{\mathcal{F}}(M)$	$\theta_i^{\mathcal{F}}(M)$	$\mathfrak{R}_u^{\mathcal{F}}(M)$	$\mathfrak{U}_u^{\mathcal{F}}(M)$	$\theta_u^{\mathcal{F}}(M)$
$\{a\}$	$\{a\}$	$\{a\}$	1	$\{a\}$	$\{a\}$	1	$\{a\}$	$\{a\}$	1	$\{a\}$	$\{a\}$	1
$\{b\}$	\emptyset	$\{b, d\}$	0	$\{b\}$	$\{b\}$	1	$\{b\}$	$\{b\}$	1	\emptyset	$\{b, c, d\}$	0
$\{c\}$	$\{c\}$	$\{c\}$	1	\emptyset	$\{c, d\}$	0	$\{c\}$	$\{c\}$	1	\emptyset	$\{b, c, d\}$	0
$\{d\}$	\emptyset	$\{b, d\}$	0	\emptyset	$\{c, d\}$	0	$\{d\}$	$\{d\}$	1	\emptyset	$\{b, c, d\}$	0
$\{a, b\}$	$\{a\}$	$\{a, b, d\}$	1/3	$\{a, b\}$	$\{a, b\}$	1	$\{a, b\}$	$\{a, b\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	1	$\{a\}$	$\{a, c, d\}$	1/3	$\{a, c\}$	$\{a, c\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{a, d\}$	$\{a\}$	$\{a, b, d\}$	1/3	$\{a\}$	$\{a, c, d\}$	1/3	$\{a, d\}$	$\{a, d\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{b, c\}$	$\{c\}$	$\{b, c, d\}$	1/3	$\{b\}$	$\{b, c, d\}$	1/3	$\{b, c\}$	$\{b, c\}$	1	\emptyset	$\{b, c, d\}$	0
$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	1	$\{b\}$	$\{b, c, d\}$	1/3	$\{b, d\}$	$\{b, d\}$	1	\emptyset	$\{b, c, d\}$	0
$\{c, d\}$	$\{c\}$	$\{b, c, d\}$	1/3	$\{c, d\}$	$\{c, d\}$	1	$\{c, d\}$	$\{c, d\}$	1	\emptyset	$\{b, c, d\}$	0
$\{a, b, c\}$	$\{a, c\}$	\mathcal{U}	1/2	$\{a, b\}$	\mathcal{U}	1/2	$\{a, b, c\}$	$\{a, b, c\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b\}$	\mathcal{U}	1/2	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{a, c, d\}$	$\{a, c\}$	\mathcal{U}	1/2	$\{a, c, d\}$	$\{a, c, d\}$	1	$\{a, c, d\}$	$\{a, c, d\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{b, c, d\}$	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	$\{b, c, d\}$	1
\mathcal{U}	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1

- (2) Internally \mathcal{F}_j definable, if $\mathfrak{R}_j^{\mathcal{F}}(M) = M$ and $\mathfrak{U}_j^{\mathcal{F}}(M) \neq M$.
- (3) Externally \mathcal{F}_j definable, if $\mathfrak{R}_j^{\mathcal{F}}(M) \neq M$ and $\mathfrak{U}_j^{\mathcal{F}}(M) = M$.
- (4) \mathcal{F}_j rough set, if $\mathfrak{R}_j^{\mathcal{F}}(M) \neq M$ and $\mathfrak{U}_j^{\mathcal{F}}(M) \neq M$.

Remark 9. According to Table 3, $\{c\}$ is totally \mathcal{F}_r -definable, $\{d\}$ is internally \mathcal{F}_l -definable, and $\{a\}$ is externally \mathcal{F}_u -definable.

4. Comparison of Approximations and Accuracy Measures Generated by \mathcal{E}_j -Neighborhoods and \mathcal{P}_j -Neighborhoods with Ideals

In this part, the comparison of approximations and accuracy measures generated by \mathcal{E}_j -neighborhoods and \mathcal{P}_j -neighborhoods with ideals is discussed.

In view of Example 1, we conclude that $\zeta_j^{\mathcal{F}}$ and $\rho_j^{\mathcal{F}}$ are not comparable, for each j , as we see in the next example.

Example 3. Continued from Example 1, if $\mathcal{F} = \{\emptyset, \{a\}\}$, then

$$\zeta_r^{\mathcal{F}} = \zeta_i^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\},$$

$$\zeta_l^{\mathcal{F}} = \zeta_u^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{c, d\}, \{b, c, d\}\},$$

$$\zeta_{\langle r \rangle}^{\mathcal{F}} = \zeta_{\langle u \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{b, c, d\}\},$$

$$\zeta_{\langle l \rangle}^{\mathcal{F}} = \zeta_{\langle i \rangle}^{\mathcal{F}} = P(\mathcal{U}).$$

(6)

According to Proposition 3, our new sorts of topologies, which were generated by j -adhesion neighborhoods and ideal, are finer than the topologies generated by \mathcal{E}_j -neighborhoods and ideals due to [29] for reflexive relation, as we see in the next theorem.

Theorem 13. Let R be any arbitrary binary relation on \mathcal{U} . Then, for any ideal \mathcal{F} , the following statements are true:

- (1) If R is a reflexive relation on \mathcal{U} , then $\zeta_j^{\mathcal{F}} \subseteq \tau_{\mathcal{F}} \subseteq \rho_j^{\mathcal{F}}$, for each j .

TABLE 5: Comparison between j -approximations and j -accuracy for each $j = \langle i \rangle$ depending on Definitions 14 and 16 by using a reflexive relation R and $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 2.

$M \subseteq \mathcal{U}$	Our method, Definition 16			Definition 14		
	$\mathfrak{L}_{\langle i \rangle}^{\mathcal{F}}(M)$	$\mathfrak{U}_{\langle i \rangle}^{\mathcal{F}}(M)$	$\theta_{\langle i \rangle}^{\mathcal{F}}(M)$	$L_{\langle i \rangle}^{\mathcal{F}\ominus}(M)$	$U_{\langle i \rangle}^{\mathcal{F}\oplus}(M)$	$\sigma_{\langle i \rangle}^{\mathcal{F}}(M)$
$\{a\}$	$\{a\}$	$\{a\}$	1	$\{a\}$	$\{a\}$	1
$\{b\}$	$\{b\}$	$\{b\}$	1	\emptyset	$\{b, d\}$	0
$\{c\}$	$\{c\}$	$\{c\}$	1	$\{c\}$	$\{c\}$	1
$\{d\}$	$\{d\}$	$\{d\}$	1	\emptyset	$\{b, d\}$	0
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	1	$\{a\}$	$\{a, b, d\}$	1/3
$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	1	$\{a, c\}$	$\{a, c\}$	1
$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	1	$\{a\}$	$\{a, b, d\}$	1/3
$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	1	$\{c\}$	$\{b, c, d\}$	1/3
$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	1	$\{b, d\}$	$\{b, d\}$	1
$\{c, d\}$	$\{c, d\}$	$\{c, d\}$	1	$\{c\}$	$\{b, c, d\}$	1/3
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	1	$\{a, c\}$	\mathcal{U}	1/2
$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b, d\}$	$\{a, b, d\}$	1
$\{a, c, d\}$	$\{a, c, d\}$	$\{a, c, d\}$	1	$\{a, c\}$	\mathcal{U}	1/2
$\{b, c, d\}$	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	$\{b, c, d\}$	1
\mathcal{U}	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1

(2) If R is an equivalence relation on \mathcal{U} , then $\rho_j^{\mathcal{F}} = \tau_j^{\mathcal{F}} = \zeta_j^{\mathcal{F}}$, for each j .

(1) $L_j^{\mathcal{F}\ominus}(M) \subseteq \mathfrak{L}_j(M) \subseteq \mathfrak{L}_j^{\mathcal{F}}(M)$.

(2) $\mathfrak{U}_j^{\mathcal{F}}(M) \subseteq \mathfrak{U}_j(M) \subseteq U_j^{\mathcal{F}\oplus}(M)$.

(3) $\sigma_j^{\mathcal{F}}(M) \leq \theta_j(M) \leq \theta_j^{\mathcal{F}}(M)$.

Proof. Direct to prove. \square

Proposition 11. Let R (respectively \mathcal{F}) be a reflexive binary relation (respectively an ideal) on \mathcal{U} and $M \subseteq \mathcal{U}$. If $j \in \{r, l, i, \langle r \rangle, \langle l \rangle, \langle i \rangle\}$, then the following statements hold:

To uphold the acquired outcomes, we consider a reflexive relation R on \mathcal{U} given in Example 2. If $\mathcal{F} = \{\emptyset, \{a\}\}$, then

$$\zeta_r^{\mathcal{F}} = \zeta_u^{\mathcal{F}} = \zeta_{\langle u \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{b, c, d\}\},$$

$$\zeta_l^{\mathcal{F}} = \zeta_i^{\mathcal{F}} = \zeta_{\langle r \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{a\}, \{b, c, d\}\},$$

$$\zeta_{\langle l \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{c\}, \{a, c\}, \{b, d\}, \{b, c, d\}\},$$

$$\zeta_{\langle i \rangle}^{\mathcal{F}} = \{\{\emptyset, \mathcal{U}, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}\}.$$

(7)

Remark 10. Table 5 offers the comparison between j -approximations and j -accuracy for $j = \langle i \rangle$ depending on Definitions 14 and 16 by using a reflexive relation R and ideal $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 2.

5. Conclusions and Future Works

To deal with uncertainty issues, Pawlak [1] proposed a non-statistical approach called rough set. Its idea is based on collection of the elements that have the same values according the required attribute. Accuracy measures and approximations represent the essential ideas in rough set theory which give details of the boundary region in terms of size and structure.

In this paper, we have suggested new techniques to generate new types of accuracy measures and approximations for a rough set under any arbitrary relation. These techniques are induced from hybridization of topological structures and ideals. In addition to studying their basic

properties, we have compared between them. Then, we have compared them with last methods to show their importance to improve the approximations and maximize the accuracy measures.

In the forthcoming articles, we are going to introduce further methods that help to obtain better approximation and higher accuracy measures than those given in the literature. Also, we search how these methods can be applied to model real-life issues.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Retraction

Retracted: Multicriteria Decision-Making Approach for Pythagorean Fuzzy Hypersoft Sets' Interaction Aggregation Operators

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

- [1] R. M. Zulqarnain, I. Siddique, R. Ali, F. Jarad, and A. Iampan, "Multicriteria Decision-Making Approach for Pythagorean Fuzzy Hypersoft Sets' Interaction Aggregation Operators," *Mathematical Problems in Engineering*, vol. 2021, Article ID 9964492, 17 pages, 2021.

Research Article

Multicriteria Decision-Making Approach for Pythagorean Fuzzy Hypersoft Sets' Interaction Aggregation Operators

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In this paper, we examine the multicriteria decision-making (MCDM) difficulties for Pythagorean fuzzy hypersoft sets (PFHSSs). The PFHSSs are a suitable extension of the Pythagorean fuzzy soft sets (PFSSs) which deliberates the parametrization of multi-subattributes of considered parameters. It is a most substantial notion for describing fuzzy information in the decision-making (DM) procedure to accommodate more vagueness comparative to existing PFSSs and intuitionistic fuzzy hypersoft sets (IFHSSs). The core objective of this study is to plan some innovative operational laws considering the interaction for Pythagorean fuzzy hypersoft numbers (PFHSNs). Also, based on settled interaction operational laws, two aggregation operators (AOs) i.e., Pythagorean fuzzy hypersoft interaction weighted average (PFHSIWA) and Pythagorean fuzzy hypersoft interaction weighted geometric (PFHSIWG) operators for PFHSSs operators have been presented with their fundamental properties. Furthermore, an MCDM technique has been established using planned interaction AOs. To ensure the strength and practicality of the developed MCDM method, a mathematical illustration has been presented. The usefulness, influence, and versatility of the developed method have been demonstrated via comparative analysis with the help of some conventional studies.

1. Introduction

Multicriteria decision-making (MCDM) is a prerequisite for decision science. The goal is to distinguish between the most essential of the possible choices. The decision maker must assess the selection specified by different types of diagnostic circumstances such as intervals and numbers. However, in numerous circumstances, it is difficult for one person to do it because of various uncertainties within the data. One is because of the shortcoming of professional knowledge or contraventions. Hence, to measure given hazards and think about the method, a series of theories have been proposed. Zadeh presented the theory of fuzzy sets (FSs) [1] to resolve the complex problem of anxiety along with ambiguity. Usually, we need to observe membership as a

nonmembership degree to indicate objects for which FSs cannot handle. To conquer the current concern, Atanassov anticipated the concept of intuitionistic fuzzy sets (IFSs) [2]. Atanassov's IFSs competently deal with insufficient data because of membership and nonmembership values, but IFSs are not able to influence incompatible and imprecise information. The theories declared over had been fairly advised by specialists, along with the sum up of two membership and nonmembership values cannot overreach one because the above work is regarded as to visualize the environment of linear inequality between the degree of membership (MD) and the degree of nonmembership (NMD). If the experts considered the MD and NMD such as $MD = 0.4$ and $NMD = 0.7$, then $0.4 + 0.7 \neq 1$ and IFSs cannot handle the situation. Yager [3, 4] prolonged the idea of IFSs

to Pythagorean fuzzy sets (PFSs) to overcome the above-discussed difficulties by amending $MD + NMD \leq 1$ to $MD^2 + NMD^2 \leq 1$. Succeeding the construction of PFSs, Zhang and Xu [5] planned operational rules for PFSs and set up the DM strategy to address the MCDM problem. Sanam et al. [6] presented the induced intuitionistic fuzzy Einstein hybrid AOs and discussed their desired properties. Wang and Li [7] offered some novel operational laws and AOs for PFSs considering the interaction with their desirable properties. Gao et al. [8] prolonged the notion of PFSs and developed numerous AOs considering the interaction. They also established a multiattribute decision-making (MADM) approach based on their established operators.

Wei [9] developed some novel operational laws for Pythagorean fuzzy numbers (PFNs) considering the interaction and proposed AOs for PFSs based on their developed operational laws. Talukdar et al. [10] utilized the linguistic PFSs for medical diagnoses and introduced some distance measures and accuracy function. They also proposed a DM technique to solve multiple criteria group decision-making (MCGDM) complications utilizing PFNs. Wang et al. [11] extended the concept of PFSs, proposed the interactive Hamacher AOs, and established a MADM method to resolve DM complications. Ejegwa et al. [12] established a correlation measure for IFs and presented an MCDM approach. Peng and Yang [13] offered various essential operations for PFSs along with their basic characteristics. Garg [14] proposed some AOs for PFSs based on his developed logarithmic operational laws. Arora and Garg [15] introduced prioritized AOs for linguistic IFs based on their developed operational laws. Ma and Xu [16] established novel AOs for PFSs and offered the comparison laws for PFNs. Current theories and their progressed DM strategies have been utilized in various aspects of life. However, these theories fail to cope with the parameters of alternatives.

The above-presented theories with their DM techniques are used in many fields of life such as medical diagnoses, artificial intelligence, and economics. But these theories have some limitations because of their inability with the parameterization tool. Molodtsov [17] introduced the notion of soft sets (SSs) to accommodate the abovementioned drawbacks considering the parameterization of the alternatives. Maji et al. [18] prolonged the idea of SSs with several necessary operations along with their appropriate possessions and established a DM method to resolve DM issues utilizing their developed operations [19]. Maji et al. [20] merged the two existing theories such as FSs and SSs and offered the concept of fuzzy soft sets (FSSs) with some elementary operations and their desired properties. Maji et al. [21] extended the notion of FSSs and proposed the idea of intuitionistic fuzzy soft sets with some operations and properties. Xu [22] introduced a method for IFs to compare intuitionistic fuzzy numbers utilizing score and accuracy functions. Xu and Yager [23] proposed the weighted average and ordered weighted average operators for IFs with their examples and properties. They also presented a DM approach to solve MADM complications utilizing their developed operators. Garg and Arora [24] proposed the generalized form of IFSSs with AOs and established a DM methodology based on their developed AOs

to resolve DM issues. Garg and Arora [25] developed the correlation coefficient (CC) and weighted correlation coefficient (WCC) for IFSSs. They also presented the TOPSIS methodology to resolve MADM issues utilizing their developed correlation measures. Zulqarnain et al. [26] extended the notion of interval-valued IFSSs and proposed AOs for interval-valued IFSSs. They also presented the CC and WCC for interval-valued IFSSs and constructed the TOPSIS approach to resolve the MADM complications based on their presented correlation measures.

Peng et al. [27] introduced the theory of PFSSs by merging two existing theories such as PFSs and SSs. They also presented some fundamental operations of PFSSs and discussed their desirable properties. Athira et al. [28] extended the notion of PFSSs, introduced some novel distance measures for PFSSs, and established a DM method based on presented distance measures to solve complicated problems. Zulqarnain et al. [29] developed the operational laws for Pythagorean fuzzy soft numbers (PFSNs) and proposed the AOs for PFSNs. They also presented a MADM method to resolve DM concerns using their developed AOs. Riaz et al. [30] defined the concept of m polar PFSSs and developed the TOPSIS method to solve MCGDM problems. Riaz et al. [31] presented the similarity measures for PFSSs and discussed their essential properties. They also proposed the weighted AOs for m -polar PFSs [32] and established a decision-making approach to solve DM concerns. Zulqarnain et al. [33] extended the idea of PFSSs and developed the TOPSIS method based on the CC. They also presented an MCGDM approach and utilized their developed approach for the selection of suppliers in green supply chain management. Mehmood et al. [34] proposed the AOs for T -spherical fuzzy sets and developed a DM approach to solving MADM issues. Wang and Garg [35] introduced some novel operational laws considering the interaction and established the AOs based on their developed rules. Batool et al. [36] introduced the TOPSIS method for Pythagorean probabilistic hesitant fuzzy sets and entropy measures under considered environment. Ullah et al. [37] developed the complex PFSs with some novel distance measures and their desirable properties. Hussain et al. [38] introduced the soft rough PFSs and Pythagorean fuzzy soft rough set with some necessary operators and properties.

The existing studies are unable to accommodate the situation when any parameters of a set of attributes have corresponding subattributes. Smarandache [39] developed the concept of hypersoft sets (HSSs) which replace the function f of a parameter with a multi-subattribute, that is, characterized on the Cartesian product of n attributes. The developed HSS competently deals with the uncertainty and vagueness comparative to SS. He also presented many other extensions of HSS such as crisp HSS, fuzzy HSS, intuitionistic fuzzy HSS, neutrosophic HSS, and plithogenic HSS. Zulqarnain et al. [40] developed the theory of neutrosophic hypersoft matrices with some logical operators. They also proposed the MADM approach to solve DM concerns. The authors presented the generalized AOs for NHSSs [41]. Zulqarnain et al. [42] developed the CC and WCC for IFHSSs and proposed the TOPSIS method using developed CC. Zulqarnain et al. [43] proposed some AOs and CC for

PFHSSs and discussed their properties. They also developed the TOPSIS approach for PFHSSs based on their presented CC. However, the above-discussed theories only deal with the uncertainty utilizing MD and NMD of subattributes. If experts consider MD = 0.6 and NDM = 0.7, then $0.6 + 0.7 \geq 1$ of any subattribute of the alternatives. We will check that it cannot be addressed by the above strategies. To overwhelm the above restrictions, we introduced some AOs for PFHSSs by modifying the condition $\mathcal{T}_{\mathcal{F}(\check{a})}(\delta) + \mathcal{J}_{\mathcal{F}(\check{a})}(\delta) \leq 1$ to $(\mathcal{T}_{\mathcal{F}(\check{a})}(\delta))^2 + (\mathcal{J}_{\mathcal{F}(\check{a})}(\delta))^2 \leq 1$. The main purpose of the succeeding study is to originate new AOs for the PFHSSs considering interactions, which may also observe the assertions of PFHSSs. Moreover, an MCDM method with a numerical example has been presented which shows the effectiveness of the planned methodology.

Supplier selection and valuation are a crucial prospect of business routine. Due to variations in management strategies, the selection of suppliers is considered from multiple perspectives, which included environmental and social necessities. Therefore, in the literature, this query is stated as a reference question for MCGDM as a sustainable supplier selection. Continuing, there are several papers [44–47] that carried the MCDM approach for the selection of sustainable suppliers according to relevant data and considerations that appropriately reflect the preferences of decision makers. However, all the above methods are not appropriate for summarizing the abovementioned methodologies and cannot deliberate the interaction among Mem and NMem functions. Particularly, we can say that the influence of other levels of Mem or NMem on the conforming geometric or average AOs does not have any influence on the aggregation process. In addition, it has been stated from the above-discussed models that the overall Mem (NMem) function level is independent of its corresponding NMem (Mem) function level. So, the consequences corresponding to those models are not favorable, so no reasonable order of preference is given for alternatives. Therefore, how to add these PFHSSNs through interaction relations is an interesting topic. To solve this problem, in this article, we are going to develop some interaction AOs such as PFHSIWA and PFHSIWG operators for PFHSSs. An algorithm is planned to resolve the DM problem based on our established operators. A numerical example has been presented to ensure the practicality of the developed DM approach.

The rest of the research can be summarized as follows: In Section 2, we presented the necessary concepts such as SSs, FSSs, HSSs, IFHSSs, and PFHSSs which can support us to construct the subsequent research organization. In Section 3, we defined some novel operational laws for PFHSSs considering interaction and developed some AOs based on interaction operational laws such as PFHSIWA and PFHSIWG operators using presented operational laws with their desirable properties. In Section 4, an MCDM method is developed utilizing the proposed operators. A numerical example is provided to ensure the implementation of the setup MCDM method. Moreover, we used some of the existing methods to present comparative analysis with our planned approach. Also, we present the benefits, simplicity, flexibility, as well as effectiveness of the planned method in Section 5, and we organized a comprehensive debate and

comparison among some available techniques and our established methodology.

2. Preliminaries

In this section, we recollect some fundamental notions such as SSs, FSSs, HSSs, IFHSSs, and PFHSSs.

Definition 1 (see [17]). Let \mathcal{U} and \mathcal{E} be the universe of discourse and set of attributes, respectively. Let $\mathcal{P}(\mathcal{U})$ be the power set of \mathcal{U} and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called SSs over \mathcal{U} , and its mapping is expressed as follows:

$$\mathcal{F}: \mathcal{A} \longrightarrow \mathcal{P}(\mathcal{U}). \quad (1)$$

Also, it can be defined as follows:

$$(\mathcal{F}, \mathcal{A}) = \{\mathcal{F}(\mathbf{e}) \in \mathcal{P}(\mathcal{U}): \mathbf{e} \in \mathcal{E}, \mathcal{F}(\mathbf{e}) = \emptyset, \text{ if } \mathbf{e} \notin \mathcal{A}\}. \quad (2)$$

Definition 2 (see [20]). Let \mathcal{U} and \mathcal{E} be a universe of discourse and set of attributes, respectively, and $\mathcal{F}(\mathcal{U})$ be a power set of \mathcal{U} . Let $\mathcal{A} \subseteq \mathcal{E}$; then, $(\mathcal{F}, \mathcal{A})$ is FSSs over \mathcal{U} , and its mapping can be stated as follows:

$$\mathcal{F}: \mathcal{A} \longrightarrow \mathcal{F}(\mathcal{U}). \quad (3)$$

Definition 3 (see [39]). Let \mathcal{U} be a universe of discourse, $\mathcal{P}(\mathcal{U})$ be a power set of \mathcal{U} , $k = \{k_1, k_2, k_3, \dots, k_n\}$, $n \geq 1$, and K_i represents the set of attributes and their corresponding subattributes such as $K_i \cap K_j = \emptyset$, where $i \neq j$ for each $n \geq 1$ and i and $j \in \{1, 2, 3, \dots, n\}$. Assume $K_1 \times K_2 \times K_3 \times \dots \times K_n = \mathcal{A} = \{d_{1h} \times d_{2k} \times \dots \times d_{nl}\}$ is a collection of subattributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, $1 \leq l \leq \gamma$, and α, β , and $\gamma \in \mathbb{N}$. Then, the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \dots \times K_n = (\mathcal{F}, \mathcal{A})$ is known as HSSs, defined as follows:

$$\mathcal{F}: K_1 \times K_2 \times K_3 \times \dots \times K_n = \mathcal{A} \longrightarrow \mathcal{P}(\mathcal{U}). \quad (4)$$

It is also defined as

$$(\mathcal{F}, \mathcal{A}) = \left\{ \check{d}, \mathcal{F}_{\check{A}}(\check{d}): \check{d} \in \mathcal{A}, \mathcal{F}_{\check{A}}(\check{d}) \in \mathcal{P}(\mathcal{U}) \right\}. \quad (5)$$

Definition 4 (see [39]). Let \mathcal{U} be a universe of discourse, $\mathcal{P}(\mathcal{U})$ be a power set of \mathcal{U} , $k = \{k_1, k_2, k_3, \dots, k_n\}$, $n \geq 1$, and K_i represents the set of attributes and their corresponding subattributes such as $K_i \cap K_j = \emptyset$ where $i \neq j$ for each $n \geq 1$ and i and $j \in \{1, 2, 3, \dots, n\}$. Assume $K_1 \times K_2 \times K_3 \times \dots \times K_n = \mathcal{A} = \{d_{1h} \times d_{2k} \times \dots \times d_{nl}\}$ is a collection of subattributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, $1 \leq l \leq \gamma$, and α, β , and $\gamma \in \mathbb{N}$, and let $\text{IFS}^{\mathcal{U}}$ be a collection of all fuzzy subsets over \mathcal{U} . Then, the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \dots \times K_n = (\mathcal{F}, \mathcal{A})$ is known as IFHSSs, defined as follows:

$$\mathcal{F}: K_1 \times K_2 \times K_3 \times \dots \times K_n = \mathcal{A} \longrightarrow \text{IFS}^{\mathcal{U}}. \quad (6)$$

It is also defined as $(\mathcal{F}, \mathcal{A}) = \left\{ (\check{d}, \mathcal{F}_{\check{A}}(\check{d})): \check{d} \in \mathcal{A}, \mathcal{F}_{\check{A}}(\check{d}) \in \text{IFS}^{\mathcal{U}} \in [0, 1] \right\}$, where $\mathcal{F}_{\check{A}}(\check{d}) = \{\delta, \mathcal{T}_{\mathcal{F}(\check{a})}(\delta), \mathcal{J}_{\mathcal{F}(\check{a})}$

$(\delta): \delta \in \mathcal{U}$, where $\mathcal{T}_{\mathcal{F}(\tilde{a})}(\delta)$ and $\mathcal{J}_{\mathcal{F}(\tilde{a})}(\delta)$ signify the Mem and NMem values of the attributes:

$$\mathcal{T}_{\mathcal{F}(\tilde{a})}(\delta), \mathcal{J}_{\mathcal{F}(\tilde{a})}(\delta) \in [0, 1], 0 \leq \mathcal{T}_{\mathcal{F}(\tilde{a})}(\delta) + \mathcal{J}_{\mathcal{F}(\tilde{a})}(\delta) \leq 1. \quad (7)$$

Remark 1. If $(\mathcal{T}_{\mathcal{F}(\tilde{a})}(\delta))^2 + (\mathcal{J}_{\mathcal{F}(\tilde{a})}(\delta))^2 \leq 1$ and $\mathcal{T}_{\mathcal{F}(\tilde{a})}(\delta) + \mathcal{J}_{\mathcal{F}(\tilde{a})}(\delta) \leq 1$ are satisfied, then PFHSSs are reduced to IFHSSs [42].

The PFHSSNs $\mathcal{F}_{\delta_i}(\tilde{a}_j) = \{(\mathcal{T}_{\mathcal{F}(\tilde{a}_j)}(\delta_i), \mathcal{J}_{\mathcal{F}(\tilde{a}_j)}(\delta_i)) | \delta_i \in \mathcal{U}\}$ can be express as $\mathfrak{F}_{\tilde{a}_{ij}} = \mathcal{T}_{\mathcal{F}(\tilde{a}_{ij})}, \mathcal{J}_{\mathcal{F}(\tilde{a}_{ij})}$. To compute the alternatives, ranking score function of $\mathfrak{F}_{\tilde{a}_{ij}}$ can be defined as follows:

$$\mathbb{S}(\mathfrak{F}_{\tilde{a}_{ij}}) = \mathcal{T}_{\mathcal{F}(\tilde{a}_{ij})}^2 - \mathcal{J}_{\mathcal{F}(\tilde{a}_{ij})}^2, \mathbb{S}(\mathfrak{F}_{\tilde{a}_{ij}}) \in [-1, 1]. \quad (8)$$

But, sometimes the scoring function such as $\mathfrak{F}_{\tilde{a}_{11}} = \langle 0.4, 0.7 \rangle$ and $\mathfrak{F}_{\tilde{a}_{12}} = \langle 0.5, 0.8 \rangle$ cannot compare two PFHSSNs. It is impossible to claim that which alternative is most suitable $\mathbb{S}(\mathfrak{F}_{\tilde{a}_{11}}) = 0.3 = \mathbb{S}(\mathfrak{F}_{\tilde{a}_{12}})$. To overcome such difficulties, we need to introduce the accuracy function as follows:

$$H(\mathfrak{F}_{\tilde{a}_{ij}}) = \mathcal{T}_{\mathcal{F}(\tilde{a}_{ij})}^2 + \mathcal{J}_{\mathcal{F}(\tilde{a}_{ij})}^2, \quad H(\mathfrak{F}_{\tilde{a}_{ij}}) \in [0, 1]. \quad (9)$$

Hence, some rules have been introduced in the following for the comparison among two PFHSSNs $\mathfrak{F}_{\tilde{a}_{ij}}$ and $\mathfrak{F}_{\tilde{a}_{ij}}$.

- (1) If $\mathbb{S}(\mathfrak{F}_{\tilde{a}_{ij}}) > \mathbb{S}(\mathfrak{F}_{\tilde{a}_{ij}})$, then $\mathfrak{F}_{\tilde{a}_{ij}} > \mathfrak{F}_{\tilde{a}_{ij}}$
- (2) If $\mathbb{S}(\mathfrak{F}_{\tilde{a}_{ij}}) = \mathbb{S}(\mathfrak{F}_{\tilde{a}_{ij}})$, then

$$\text{If } H(\mathfrak{F}_{\tilde{a}_{ij}}) > H(\mathfrak{F}_{\tilde{a}_{ij}}), \text{ then } \mathfrak{F}_{\tilde{a}_{ij}} > \mathfrak{F}_{\tilde{a}_{ij}}$$

$$\text{If } H(\mathfrak{F}_{\tilde{a}_{ij}}) = H(\mathfrak{F}_{\tilde{a}_{ij}}), \text{ then } \mathfrak{F}_{\tilde{a}_{ij}} = \mathfrak{F}_{\tilde{a}_{ij}}$$

Observe that the overall difference between PFHSSNs and IFHSSNs lies in their distinguishing limits. The Pythagorean membership degree area is larger than either the intuitionistic membership degree area. PFHSSNs cannot only model IFHSSNs' ability to capture DM scenarios anywhere the sum of Mem as well as NMem of subattributes of the considered parameters is equal to or less than 1 but it is also unable to handle the circumstances where IFHSSNs are not able to characterize the sum of Mem as well as NMem of multi-subattributes of the considered attributes exceeding 1. On the contrary, PFHSSNs accommodate more uncertainty considering Mem as well as NMem of multi-subattributes of the considered attributes, and the sum of their squares is equal to or less than 1.

Definition 5 (see [43]). Let $\mathfrak{F}_{\tilde{a}_k} = (\mathcal{T}_{\tilde{a}_k}, \mathcal{J}_{\tilde{a}_k})$, $\mathfrak{F}_{\tilde{a}_{11}} = (\mathcal{T}_{\tilde{a}_{11}}, \mathcal{J}_{\tilde{a}_{11}})$, and $\mathfrak{F}_{\tilde{a}_{12}} = (\mathcal{T}_{\tilde{a}_{12}}, \mathcal{J}_{\tilde{a}_{12}})$ be three PFHSSNs and α be a positive real number; by algebraic norms, we have

- (1) $\mathfrak{F}_{\tilde{a}_{11}} \oplus \mathfrak{F}_{\tilde{a}_{12}} = \left\langle \sqrt{\mathcal{T}_{\tilde{a}_{11}}^2 + \mathcal{T}_{\tilde{a}_{12}}^2 - \mathcal{T}_{\tilde{a}_{11}}^2 \mathcal{T}_{\tilde{a}_{12}}^2}, \sqrt{\mathcal{J}_{\tilde{a}_{11}}^2 + \mathcal{J}_{\tilde{a}_{12}}^2 - \mathcal{J}_{\tilde{a}_{11}}^2 \mathcal{J}_{\tilde{a}_{12}}^2} \right\rangle$
- (2) $\mathfrak{F}_{\tilde{a}_{11}} \otimes \mathfrak{F}_{\tilde{a}_{12}} = \left\langle \mathcal{T}_{\tilde{a}_{11}} \mathcal{T}_{\tilde{a}_{12}}, \sqrt{\mathcal{J}_{\tilde{a}_{11}}^2 + \mathcal{J}_{\tilde{a}_{12}}^2 - \mathcal{J}_{\tilde{a}_{11}}^2 \mathcal{J}_{\tilde{a}_{12}}^2} \right\rangle$
- (3) $\alpha \mathfrak{F}_{\tilde{a}_k} = \left\langle \sqrt{1 - (1 - \mathcal{T}_{\tilde{a}_k}^2)^\alpha}, \mathcal{J}_{\tilde{a}_k}^\alpha \right\rangle$
- (4) $\mathfrak{F}_{\tilde{a}_k}^\alpha = \left\langle \mathcal{T}_{\tilde{a}_k}^\alpha, \sqrt{1 - (1 - \mathcal{J}_{\tilde{a}_k}^2)^\alpha} \right\rangle$

For the collection of PFHSSNs $\mathfrak{F}_{\tilde{a}_{ij}}$, where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, $\Omega = (\Omega_1, \Omega_2, \dots, \Omega_n)^T$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_m)^T$ be weight vectors for experts and attributes $\Omega_i > 0$, $\sum_{i=1}^n \Omega_i = 1$, $\gamma_j > 0$, and $\sum_{j=1}^m \gamma_j = 1$, respectively. Zulqarnain et al. [43] presented the averaging and geometric AOs as follows:

$$\text{PFHSSWA}(\mathfrak{F}_{\tilde{a}_{11}}, \mathfrak{F}_{\tilde{a}_{12}}, \dots, \mathfrak{F}_{\tilde{a}_{nm}}) = \left\langle \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n (1 - \mathcal{T}_{\tilde{a}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}}, \prod_{j=1}^m \left(\prod_{i=1}^n (\mathcal{J}_{\tilde{a}_{ij}}^{\Omega_i})^{\gamma_j} \right) \right\rangle, \quad (10)$$

$$\text{PFHSSWG}(\mathfrak{F}_{\tilde{a}_{11}}, \mathfrak{F}_{\tilde{a}_{12}}, \dots, \mathfrak{F}_{\tilde{a}_{nm}}) = \left\langle \prod_{j=1}^m \left(\prod_{i=1}^n (\mathcal{T}_{\tilde{a}_{ij}}^{\Omega_i})^{\gamma_j} \right), \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n (1 - \mathcal{J}_{\tilde{a}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}} \right\rangle. \quad (11)$$

3. Interaction Aggregation Operators for Pythagorean Fuzzy Hypersoft Numbers

In this section, we introduce interaction AOs for PFHSSNs. In it, some fundamental properties have been discussed based on defined interaction PFHSSWA and PFHSSWG operators for PFHSSNs.

3.1. Interaction Operational Laws for PFHSSNs

Definition 6. Let $\mathfrak{F}_{\tilde{a}_k} = (\mathcal{T}_{\tilde{a}_k}, \mathcal{J}_{\tilde{a}_k})$, $\mathfrak{F}_{\tilde{a}_{11}} = (\mathcal{T}_{\tilde{a}_{11}}, \mathcal{J}_{\tilde{a}_{11}})$, and $\mathfrak{F}_{\tilde{a}_{12}} = (\mathcal{T}_{\tilde{a}_{12}}, \mathcal{J}_{\tilde{a}_{12}})$ be three PFHSSNs and α be a positive real

number; by algebraic norms, considering the interaction, we have

- (1) $\mathfrak{F}_{\tilde{a}_{11}} \oplus \mathfrak{F}_{\tilde{a}_{12}} = \left\langle \sqrt{\mathcal{T}_{\tilde{a}_{11}}^2 + \mathcal{T}_{\tilde{a}_{12}}^2 - \mathcal{T}_{\tilde{a}_{11}}^2 \mathcal{T}_{\tilde{a}_{12}}^2}, \sqrt{\mathcal{J}_{\tilde{a}_{11}}^2 + \mathcal{J}_{\tilde{a}_{12}}^2 - \mathcal{J}_{\tilde{a}_{11}}^2 \mathcal{J}_{\tilde{a}_{12}}^2} \right\rangle$
- (2) $\mathfrak{F}_{\tilde{a}_{11}} \otimes \mathfrak{F}_{\tilde{a}_{12}} = \left\langle \sqrt{\mathcal{T}_{\tilde{a}_{11}}^2 + \mathcal{T}_{\tilde{a}_{12}}^2 - \mathcal{T}_{\tilde{a}_{11}}^2 \mathcal{T}_{\tilde{a}_{12}}^2}, \sqrt{\mathcal{J}_{\tilde{a}_{11}}^2 + \mathcal{J}_{\tilde{a}_{12}}^2 - \mathcal{J}_{\tilde{a}_{11}}^2 \mathcal{J}_{\tilde{a}_{12}}^2} \right\rangle$
- (3) $\alpha \mathfrak{F}_{\tilde{a}_k} = \left\langle \sqrt{1 - (1 - \mathcal{T}_{\tilde{a}_k}^2)^\alpha}, \sqrt{(1 - \mathcal{T}_{\tilde{a}_k}^2)^\alpha - [1 - (\mathcal{T}_{\tilde{a}_k}^2 + \mathcal{J}_{\tilde{a}_k}^2)]^\alpha} \right\rangle$

$$(4) \mathfrak{F}_{\check{d}_k}^\alpha = \left\langle \sqrt{(1 - \mathcal{F}_{\check{d}_k}^2)^\alpha - [1 - (\mathcal{T}_{\check{d}_k}^2 + \mathcal{J}_{\check{d}_k}^2)]^\alpha}, \sqrt{1 - (1 - \mathcal{F}_{\check{d}_k}^2)^\alpha} \right\rangle$$

Based on the above-defined operational laws, now we introduce some interaction AOs for PFHSNs' Δ .

Definition 7. Let $\mathfrak{F}_{\check{d}_{ij}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}})$ be PFHSNs and Ω_i and γ_j represent the weights of expert's and multi-subattributes along with stated conditions $\Omega_i > 0$, $\sum_{i=1}^n \Omega_i = 1$, $\gamma_j > 0$, and

$\sum_{j=1}^m \gamma_j = 1$. Then, PFHSIWA: $\Delta^n \longrightarrow \Delta$ is defined as follows:

$$\text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{nm}}) = \oplus_{j=1}^m \gamma_j \left(\oplus_{i=1}^n \Omega_i \mathfrak{F}_{\check{d}_{ij}} \right). \quad (12)$$

Theorem 1. Let $\mathfrak{F}_{\check{d}_{ij}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}})$ be PFHSNs, where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Then, the attained aggregated values using equation (12) is also a PFHSN and

$$\text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{nm}}) = \left\langle \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n (1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}}, \sqrt{\prod_{j=1}^m \left(\prod_{i=1}^n (1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^m \left(\prod_{i=1}^n [1 - (\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{J}_{\check{d}_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j}} \right\rangle, \quad (13)$$

where Ω_i and γ_j represent the expert's and subattributes' weights with certain circumstances $\Omega_i > 0$, $\sum_{i=1}^n \Omega_i = 1$, $\gamma_j > 0$, and $\sum_{j=1}^m \gamma_j = 1$.

Proof. The PFHSIWA operator can be proved using the principle of mathematical induction as follows:

For $n = 1$, we get $\Omega_1 = 1$. Then, we have

$$\begin{aligned} \text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{nm}}) &= \oplus_{j=1}^m \gamma_j \mathfrak{F}_{\check{d}_{1j}}, \\ \text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{nm}}) &= \left\langle \sqrt{1 - \prod_{j=1}^m \left((1 - \mathcal{T}_{\check{d}_{1j}}^2) \right)^{\gamma_j}}, \sqrt{\prod_{j=1}^m (1 - \mathcal{T}_{\check{d}_{1j}}^2)^{\gamma_j} - \prod_{j=1}^m (1 - (\mathcal{T}_{\check{d}_{1j}}^2 + \mathcal{J}_{\check{d}_{1j}}^2))^{\gamma_j}} \right\rangle, \\ &= \left\langle \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^1 (1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}}, \sqrt{\prod_{j=1}^m \left(\prod_{i=1}^1 (1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^m \left(\prod_{i=1}^1 [1 - (\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{J}_{\check{d}_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j}} \right\rangle. \end{aligned} \quad (14)$$

For $m = 1$, we get $\gamma_1 = 1$. Then, we have

$$\begin{aligned} \text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{nm}}) &= \oplus_{i=1}^n \Omega_i \mathfrak{F}_{\check{d}_{i1}}, \\ &= \left\langle \sqrt{1 - \prod_{i=1}^n (1 - \mathcal{T}_{\check{d}_{i1}}^2)^{\Omega_i}}, \sqrt{\prod_{i=1}^n (1 - \mathcal{T}_{\check{d}_{i1}}^2)^{\Omega_i} - \prod_{i=1}^n [1 - (\mathcal{T}_{\check{d}_{i1}}^2 + \mathcal{J}_{\check{d}_{i1}}^2)]^{\Omega_i}} \right\rangle \\ &= \left\langle \sqrt{1 - \prod_{j=1}^1 \left(\prod_{i=1}^n (1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}}, \sqrt{\prod_{j=1}^1 \left(\prod_{i=1}^n (1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^1 \left(\prod_{i=1}^n [1 - (\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{J}_{\check{d}_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j}} \right\rangle. \end{aligned} \quad (15)$$

The above justification shows that equation (13) holds for $n = 1$ and $m = 1$. Now, assume that equation (13) also holds for $m = \beta_1 + 1$, $n = \beta_2$, $m = \beta_1$, and $n = \beta_2 + 1$:

$$\begin{aligned}\oplus_{j=1}^{\beta_1+1} \gamma_j \left(\oplus_{i=1}^{\beta_2} \Omega_i \mathfrak{F}_{d_{ij}} \right) &= \left\langle \sqrt{1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} (1 - \mathcal{T}_{d_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}}, \sqrt{\prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} (1 - \mathcal{T}_{d_{ij}}^2)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} [1 - (\mathcal{T}_{d_{ij}}^2 + \mathcal{J}_{d_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j}} \right\rangle, \\ \oplus_{j=1}^{\beta_1} \gamma_j \left(\oplus_{i=1}^{\beta_2+1} \Omega_i \mathfrak{F}_{d_{ij}} \right) &= \left\langle \sqrt{1 - \prod_{j=1}^{\beta_1} \left(\prod_{i=1}^{\beta_2+1} (1 - \mathcal{T}_{d_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}}, \sqrt{\prod_{j=1}^{\beta_1} \left(\prod_{i=1}^{\beta_2+1} (1 - \mathcal{T}_{d_{ij}}^2)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^{\beta_1} \left(\prod_{i=1}^{\beta_2+1} [1 - (\mathcal{T}_{d_{ij}}^2 + \mathcal{J}_{d_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j}} \right\rangle.\end{aligned}\quad (16)$$

For $m = \beta_1 + 1$ and $n = \beta_2 + 1$, we have

$$\begin{aligned}\oplus_{j=1}^{\beta_1+1} \gamma_j \left(\oplus_{i=1}^{\beta_2+1} \Omega_i \mathfrak{F}_{d_{ij}} \right) &= \oplus_{j=1}^{\beta_1+1} \gamma_j \left(\oplus_{i=1}^{\beta_2} \Omega_i \mathfrak{F}_{d_{ij}} \oplus \Omega_{\beta_2+1} \mathfrak{F}_{d_{(\beta_2+1)j}} \mathfrak{F}_{d_{(\beta_2+1)j}} \right), \\ &= \oplus_{j=1}^{\beta_1+1} \gamma_j \left(\oplus_{i=1}^{\beta_2} \Omega_i \mathfrak{F}_{d_{ij}} \oplus \gamma_j \Omega_{\beta_2+1} \mathfrak{F}_{d_{(\beta_2+1)j}} \right), \\ &= \left\langle \sqrt{1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} (1 - \mathcal{T}_{d_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}} \oplus \sqrt{1 - \prod_{j=1}^{\beta_1+1} \left((1 - \mathcal{T}_{d_{(\beta_2+1)j}}^2)^{\Omega_{\beta_2+1}} \right)^{\gamma_j}} \right. \\ &\quad \left. \sqrt{\prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} (1 - \mathcal{T}_{d_{ij}}^2)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} [1 - (\mathcal{T}_{d_{ij}}^2 + \mathcal{J}_{d_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j}} \right. \\ &\quad \left. \oplus \sqrt{\prod_{j=1}^{\beta_1+1} \left((1 - \mathcal{T}_{d_{(\beta_2+1)j}}^2)^{\Omega_{\beta_2+1}} \right)^{\gamma_j} - \prod_{j=1}^{\beta_1+1} \left([1 - (\mathcal{T}_{d_{(\beta_2+1)j}}^2 + \mathcal{J}_{d_{(\beta_2+1)j}}^2)]^{\Omega_{\beta_2+1}} \right)^{\gamma_j}} \right\rangle, \\ &= \left\langle \sqrt{1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2+1} (1 - \mathcal{T}_{d_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}}, \sqrt{\prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2+1} (1 - \mathcal{T}_{d_{ij}}^2)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2+1} [1 - (\mathcal{T}_{d_{ij}}^2 + \mathcal{J}_{d_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j}} \right\rangle.\end{aligned}\quad (17)$$

Hence, it is true for $m = \beta_1 + 1$ and $n = \beta_2 + 1$. \square

Example 1. Let $\mathcal{U}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a set of experts whose weights are given as $\Omega_i = (0.243, 0.514, 0.343)^T$. Experts evaluate the beauty of a house under a considered set of attributes $\mathfrak{F}' = \{d_1 = \text{lawn}, d_2 = \text{security system}\}$ with their corresponding subattributes $\text{lawn} = d_1 = \{d_{11} = \text{with grass}, d_{12} = \text{without grass}\}$ and $\text{security system} = d_2 = \{d_{21} = \text{guards}, d_{22} = \text{cameras}\}$. Let $\mathfrak{F}' = d_1 \times d_2$ be a set of subattributes $\mathfrak{F}' = d_1 \times d_2 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} = \{(d_{11}, d_{21}), (d_{11}, d_{22}), (d_{12}, d_{21}), (d_{12}, d_{22})\}$ and $\mathfrak{F}' = \{d_1, d_2, d_3, d_4\}$

represent the set subattributes with weights $\gamma_j = (0.25, 0.15, 0.2, 0.4)^T$. Experts' opinion for each multi-subattribute in the form of PFHSNs $(\mathfrak{F}, \mathfrak{F}') = \langle \mathcal{T}_{d_{ij}}, \mathcal{J}_{d_{ij}} \rangle_{3 \times 4}$ is given as follows:

$$(\mathfrak{F}, \mathfrak{F}') = \begin{bmatrix} (0.3, 0.8) & (0.4, 0.6) & (0.3, 0.6) & (0.5, 0.6) \\ (0.8, 0.3) & (0.7, 0.4) & (0.7, 0.3) & (0.4, 0.8) \\ (0.3, 0.6) & (0.5, 0.7) & (0.6, 0.5) & (0.5, 0.4) \end{bmatrix}.\quad (18)$$

By using equation (13),

$$\begin{aligned}
\text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mm}}) &= \left\langle \sqrt{1 - \prod_{j=1}^4 \left(\prod_{i=1}^3 (1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}}, \sqrt{\prod_{j=1}^4 \left(\prod_{i=1}^3 (1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^4 \left(\prod_{i=1}^3 [1 - (\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{J}_{\check{d}_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j}} \right\rangle, \\
&= \left\langle \sqrt{1 - \left\{ (0.91)^{0.243} (0.36)^{0.514} (0.91)^{0.343} \right\}^{0.25} \left\{ (0.84)^{0.243} (0.51)^{0.514} (0.75)^{0.343} \right\}^{0.15} \left\{ (0.91)^{0.243} (0.51)^{0.514} (0.64)^{0.343} \right\}^2 \left\{ (0.75)^{0.243} (0.84)^{0.514} (0.75)^{0.343} \right\}^4} \right\}, \\
&\sqrt{\left\{ (0.91)^{0.243} (0.36)^{0.514} (0.91)^{0.343} \right\}^{0.25} \left\{ (0.84)^{0.243} (0.51)^{0.514} (0.75)^{0.343} \right\}^{0.15} \left\{ (0.91)^{0.243} (0.51)^{0.514} (0.64)^{0.343} \right\}^{0.2} \left\{ (0.75)^{0.243} (0.84)^{0.514} (0.75)^{0.343} \right\}^{0.4}} - \\
&\sqrt{\left\{ (0.27)^{0.243} (0.27)^{0.514} (0.55)^{0.343} \right\}^{0.25} \left\{ (0.48)^{0.243} (0.35)^{0.514} (0.26)^{0.343} \right\}^{0.15} \left\{ (0.55)^{0.243} (0.42)^{0.514} (0.39)^{0.343} \right\}^{0.2} \left\{ (0.39)^{0.243} (0.20)^{0.514} (0.59)^{0.343} \right\}^{0.4}} \right\}, \\
&= 0.58759, 0.58241.
\end{aligned} \tag{19}$$

Hence, some fundamental properties utilizing the planned PFHSIWA operator for the collection of PFHSNs are established based on Theorem 1.

3.2. Properties of PFHSIWA Operator

3.2.1. Idempotency. If $\mathfrak{F}_{\check{d}_{ij}} = \mathfrak{F}_{\check{d}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}})$, $\forall i, j$, then

$$\text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mm}}) = \mathfrak{F}_{\check{d}}. \tag{20}$$

Proof. As we know that all $\mathfrak{F}_{\check{d}_{ij}} = \mathfrak{F}_{\check{d}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}})$, using equation (13), we have

$$\begin{aligned}
\text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mm}}) &= \left\langle \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n (1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}}, \right. \\
&\left. \sqrt{\prod_{j=1}^m \left(\prod_{i=1}^n (1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^m \left(\prod_{i=1}^n [1 - (\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{J}_{\check{d}_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j}} \right\rangle \\
&= \left\langle \sqrt{1 - \left((1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\sum_{i=1}^n \Omega_i} \right)^{\sum_{j=1}^m \gamma_j}}, \right. \\
&\left. \sqrt{\left((1 - \mathcal{T}_{\check{d}_{ij}}^2)^{\sum_{i=1}^n \Omega_i} \right)^{\sum_{j=1}^m \gamma_j} - \left([1 - (\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{J}_{\check{d}_{ij}}^2)]^{\sum_{i=1}^n \Omega_i} \right)^{\sum_{j=1}^m \gamma_j}} \right\rangle \\
&= \left\langle \sqrt{1 - (1 - \mathcal{T}_{\check{d}_{ij}}^2)}, \sqrt{(1 - \mathcal{T}_{\check{d}_{ij}}^2) - [1 - (\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{J}_{\check{d}_{ij}}^2)]} \right\rangle = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}) = \mathfrak{F}_{\check{d}}. \quad \square
\end{aligned} \tag{21}$$

3.2.2. Boundedness. Let $\mathfrak{F}_{\check{d}_{ij}}$ be a collection of PFHSNs, $\mathfrak{F}_{\check{d}_{ij}}^- = \min_j \min_i \{\mathcal{T}_{\check{d}_{ij}}\}$, $\max_j \max_i \{\mathcal{J}_{\check{d}_{ij}}\}$, and $\mathfrak{F}_{\check{d}_{ij}}^+ = \max_j$

$\max_i \{\mathcal{T}_{\check{d}_{ij}}\}$, $\min_j \min_i \{\mathcal{J}_{\check{d}_{ij}}\}$; then, $\mathfrak{F}_{\check{d}_{ij}}^- \leq \text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mm}}) \leq \mathfrak{F}_{\check{d}_{ij}}^+$.

Proof. As we know that $\mathfrak{F}_{\check{d}_{ij}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{F}_{\check{d}_{ij}})$ is a collection of PFHSNs, then

$$\begin{aligned}
 & \min_j \min_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \leq \mathcal{T}_{\check{d}_{ij}}^2 \leq \max_j \max_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \\
 & \Rightarrow 1 - \max_j \max_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \leq 1 - \mathcal{T}_{\check{d}_{ij}}^2 \leq 1 - \min_j \min_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \\
 & \Leftrightarrow \left(1 - \max_j \max_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \right)^{\Omega_i} \leq \left(1 - \mathcal{T}_{\check{d}_{ij}}^2 \right)^{\Omega_i} \leq \left(1 - \min_j \min_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \right)^{\Omega_i} \\
 & \Leftrightarrow \left(1 - \max_j \max_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \right)^{\sum_{i=1}^n \Omega_i} \leq \prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2 \right)^{\Omega_i} \leq \left(1 - \min_j \min_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \right)^{\sum_{i=1}^n \Omega_i} \\
 & \Leftrightarrow \left(1 - \max_j \max_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \right)^{\sum_{j=1}^m \gamma_j} \leq \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2 \right)^{\Omega_i} \right)^{\gamma_j} \leq \left(1 - \min_j \min_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \right)^{\sum_{j=1}^m \gamma_j} \quad (22) \\
 & \Leftrightarrow 1 - \max_j \max_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \leq \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2 \right)^{\Omega_i} \right)^{\gamma_j} \leq 1 - \min_j \min_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \\
 & \Leftrightarrow \min_j \min_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \leq 1 - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2 \right)^{\Omega_i} \right)^{\gamma_j} \leq \max_j \max_i \left\{ \mathcal{T}_{\check{d}_{ij}}^2 \right\} \\
 & \Leftrightarrow \min_j \min_i \left\{ \mathcal{T}_{\check{d}_{ij}} \right\} \leq \sqrt[1 - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2 \right)^{\Omega_i} \right)^{\gamma_j}]{} \leq \max_j \max_i \left\{ \mathcal{T}_{\check{d}_{ij}} \right\}.
 \end{aligned}$$

Similarly,

$$\min_j \min_i \left\{ \mathcal{F}_{\check{d}_{ij}} \right\} \leq \sqrt[1 - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2 \right)^{\Omega_i} \right)^{\gamma_j}]{} - \prod_{j=1}^m \left(\prod_{i=1}^n \left[1 - \left(\mathcal{F}_{\check{d}_{ij}} \right) \right]^{\Omega_i} \right)^{\gamma_j} \leq \max_j \max_i \left\{ \mathcal{F}_{\check{d}_{ij}} \right\}. \quad (23)$$

Let PFHSIWA $(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mm}}) = \mathcal{T}_{\delta}$ and $\mathcal{F}_{\delta} = \mathfrak{F}_{\delta}$; then, inequalities (22) and (23) can be changed into the subsequent arrangement $\min_j \min_i \left\{ \mathcal{T}_{\check{d}_{ij}} \right\} \leq \mathcal{T}_{\delta} \leq$

$\max_j \max_i \left\{ \mathcal{T}_{\check{d}_{ij}} \right\}$ and $\min_j \min_i \left\{ \mathcal{F}_{\check{d}_{ij}} \right\} \leq \mathcal{F}_{\delta} \leq \max_j \max_i \left\{ \mathcal{F}_{\check{d}_{ij}} \right\}$, respectively. Operating equation (8), we get

$$\begin{aligned}
 \mathcal{S}(\mathfrak{F}_{\delta}) &= \mathcal{T}_{\delta}^2 - \mathcal{F}_{\delta}^2 \leq \max_j \max_i \left\{ \mathcal{T}_{\check{d}_{ij}} \right\} - \min_j \min_i \left\{ \mathcal{F}_{\check{d}_{ij}} \right\} = \mathcal{S}(\mathfrak{F}_{\check{d}_{ij}}^+), \\
 \mathcal{S}(\mathfrak{F}_{\delta}) &= \mathcal{T}_{\delta}^2 - \mathcal{F}_{\delta}^2 \geq \min_j \min_i \left\{ \mathcal{T}_{\check{d}_{ij}} \right\} - \max_j \max_i \left\{ \mathcal{F}_{\check{d}_{ij}} \right\} = \mathcal{S}(\mathfrak{F}_{\check{d}_{ij}}^-). \quad (24)
 \end{aligned}$$

Then, by order relation among two PFSNs, we have

$$\mathfrak{F}_{\check{d}_{ij}}^- \leq \text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mm}}) \leq \mathfrak{F}_{\check{d}_{ij}}^+. \quad (25)$$

3.2.3. Homogeneity. Prove that $\text{PFHSIWA}(\alpha \mathfrak{F}_{\check{d}_{11}}, \alpha \mathfrak{F}_{\check{d}_{12}}, \dots, \alpha \mathfrak{F}_{\check{d}_{mm}}) = \alpha \cdot \text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mm}})$ for any positive real number α .

Proof. Let $\mathfrak{F}_{\check{d}_{ij}}$ be a collection of PFHSNs and $\alpha > 0$; then, by using Definition 6 (10), we have

$$\alpha \mathfrak{F}_{\check{d}_{ij}} = \left\langle \sqrt{1 - \left(1 - \mathcal{T}_{\check{d}_{ij}}^2\right)^\alpha}, \sqrt{\left(1 - \mathcal{T}_{\check{d}_{ij}}^2\right)^\alpha - \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2\right)\right]^\alpha} \right\rangle. \quad (26)$$

So,

$$\begin{aligned} \text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{nm}}) &= \left\langle \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2\right)^{\alpha \Omega_i} \right)^{\gamma_j}}, \right. \\ &\quad \left. \sqrt{\prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2\right)^{\alpha \Omega_i} \right)^{\gamma_j} - \prod_{j=1}^m \left(\prod_{i=1}^n \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2\right)\right]^{\alpha \Omega_i} \right)^{\gamma_j}} \right\rangle \\ &= \left\langle \sqrt{1 - \left(\prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i} \right)^{\gamma_j} \right)^\alpha}, \right. \\ &\quad \left. \sqrt{\left(\prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i} \right)^{\gamma_j} \right)^\alpha - \left(\prod_{j=1}^m \left(\prod_{i=1}^n \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i} \right)^{\gamma_j} \right)^\alpha} \right\rangle \\ &= \alpha \text{PFHSIWA}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{nm}}). \end{aligned} \quad (27)$$

The proof is completed. \square

Definition 8. Let $\mathfrak{F}_{\check{d}_{ij}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{F}_{\check{d}_{ij}})$ be PFHSNs and Ω_i and γ_j represent the weights of experts and multi-subattributes along with stated conditions $\Omega_i > 0$, $\sum_{i=1}^n \Omega_i = 1$, $\gamma_j > 0$, and $\sum_{j=1}^m \gamma_j = 1$. Then, PFHSIWG: $\Delta^n \rightarrow \Delta$ is defined as follows:

$$\text{PFHSIWG}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{nm}}) = \otimes_{j=1}^m \left(\otimes_{i=1}^n \mathfrak{F}_{\check{d}_{ij}}^{\Omega_i} \right)^{\gamma_j}. \quad (28)$$

Theorem 2. Let $\mathfrak{F}_{\check{d}_{ij}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{F}_{\check{d}_{ij}})$ be a collection of PFHSNs, where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Then, utilizing equation (28), we get PFHSN and

$$\text{PFHSIWG}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{nm}}) = \left\langle \sqrt{\prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{F}_{\check{d}_{ij}}^2\right)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^m \left(\prod_{i=1}^n \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i} \right)^{\gamma_j}}, \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i} \right)^{\gamma_j}} \right\rangle, \quad (29)$$

where Ω_i and γ_j represent the expert's and subattributes' weights with certain circumstances $\Omega_i > 0$, $\sum_{i=1}^n \Omega_i = 1$, $\gamma_j > 0$, and $\sum_{j=1}^m \gamma_j = 1$.

Proof. The PFHSIWG operator can be proved using the principle of mathematical induction as follows:

For $n = 1$, we get $\Omega_1 = 1$. Then, we have

$$\text{PFHSIWG}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mm}}) = \otimes_{j=1}^m \mathfrak{F}_{\check{d}_{1j}}^{\gamma_j},$$

$$\begin{aligned} \text{PFHSIWG}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mm}}) &= \left\langle \sqrt{\prod_{j=1}^m \left(1 - \mathcal{F}_{\check{d}_{1j}}^2\right)^{\gamma_j}} - \prod_{j=1}^m \left(1 - \left(\mathcal{T}_{\check{d}_{1j}}^2 + \mathcal{F}_{\check{d}_{1j}}^2\right)\right)^{\gamma_j}, \sqrt{1 - \prod_{j=1}^m \left(\left(1 - \mathcal{F}_{\check{d}_{1j}}^2\right)\right)^{\gamma_j}} \right\rangle, \\ &= \left\langle \sqrt{\prod_{j=1}^m \left(\prod_{i=1}^1 \left(1 - \mathcal{F}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} - \prod_{j=1}^m \left(\prod_{i=1}^1 \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}, \right. \\ &\quad \left. \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^1 \left(1 - \mathcal{F}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} \right\rangle. \end{aligned} \quad (30)$$

For $m = 1$, we get $\gamma_1 = 1$. Then, we have

$$\begin{aligned} \text{PFHSIWG}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mm}}) &= \otimes_{i=1}^n \Omega_i \mathfrak{F}_{\check{d}_{11}} \\ &= \left\langle \sqrt{\prod_{i=1}^n \left(1 - \mathcal{F}_{\check{d}_{1i}}^2\right)^{\Omega_i}} - \prod_{i=1}^n \left[1 - \left(\mathcal{T}_{\check{d}_{1i}}^2 + \mathcal{F}_{\check{d}_{1i}}^2\right)\right]^{\Omega_i}, \sqrt{1 - \prod_{i=1}^n \left(1 - \mathcal{F}_{\check{d}_{1i}}^2\right)^{\Omega_i}} \right\rangle \\ &= \left\langle \sqrt{\prod_{j=1}^1 \left(\prod_{i=1}^n \left(1 - \mathcal{F}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} - \prod_{j=1}^1 \left(\prod_{i=1}^n \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}, \sqrt{1 - \prod_{j=1}^1 \left(\prod_{i=1}^n \left(1 - \mathcal{F}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} \right\rangle. \end{aligned} \quad (31)$$

The above justification shows that equation (10) holds for $n = 1$ and $m = 1$. Now, assume that equation (10) also holds for $m = \beta_1 + 1$, $n = \beta_2$, $m = \beta_1$, and $n = \beta_2 + 1$:

$$\begin{aligned} \otimes_{j=1}^{\beta_1+1} \gamma_j \left(\otimes_{i=1}^{\beta_2} \Omega_i \mathfrak{F}_{\check{d}_{ij}} \right) &= \left\langle \sqrt{\prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} \left(1 - \mathcal{F}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}, \sqrt{1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} \left(1 - \mathcal{F}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} \right\rangle, \\ \otimes_{j=1}^{\beta_1} \gamma_j \left(\otimes_{i=1}^{\beta_2+1} \Omega_i \mathfrak{F}_{\check{d}_{ij}} \right) &= \left\langle \sqrt{\prod_{j=1}^{\beta_1} \left(\prod_{i=1}^{\beta_2+1} \left(1 - \mathcal{F}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} - \prod_{j=1}^{\beta_1} \left(\prod_{i=1}^{\beta_2+1} \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}, \sqrt{1 - \prod_{j=1}^{\beta_1} \left(\prod_{i=1}^{\beta_2+1} \left(1 - \mathcal{F}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} \right\rangle. \end{aligned} \quad (32)$$

For $m = \beta_1 + 1$ and $n = \beta_2 + 1$, we have

$$\begin{aligned}
 \otimes_{j=1}^{\beta_1+1} \gamma_j \left(\otimes_{i=1}^{\beta_2+1} \Omega_i \mathfrak{F}_{\check{d}_{ij}} \right) &= \otimes_{j=1}^{\beta_1+1} \gamma_j \left(\otimes_{i=1}^{\beta_2} \Omega_i \mathfrak{F}_{\check{d}_{ij}} \otimes \Omega_{\beta_2+1} \mathfrak{F}_{\check{d}_{(\beta_2+1)j}} \right) \\
 &= \otimes_{j=1}^{\beta_1+1} \otimes_{i=1}^{\beta_2} \gamma_j \Omega_i \mathfrak{F}_{\check{d}_{ij}} \otimes_{j=1}^{\beta_1+1} \gamma_j \Omega_{\beta_2+1} \mathfrak{F}_{\check{d}_{(\beta_2+1)j}} \\
 &= \left\langle \sqrt{\prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} (1 - \mathcal{F}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}} - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} [1 - (\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j} \right. \\
 &\quad \oplus \sqrt{\prod_{j=1}^{\beta_1+1} \left((1 - \mathcal{F}_{\check{d}_{(\beta_2+1)j}}^2)^{\Omega_{\beta_2+1}} \right)^{\gamma_j}} - \prod_{j=1}^{\beta_1+1} \left([1 - (\mathcal{T}_{\check{d}_{(\beta_2+1)j}}^2 + \mathcal{F}_{\check{d}_{(\beta_2+1)j}}^2)]^{\Omega_{\beta_2+1}} \right)^{\gamma_j}} \\
 &\quad \left. \sqrt{1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} (1 - \mathcal{F}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}} \oplus \sqrt{1 - \prod_{j=1}^{\beta_1+1} \left((1 - \mathcal{F}_{\check{d}_{(\beta_2+1)j}}^2)^{\Omega_{\beta_2+1}} \right)^{\gamma_j}} \right\rangle \\
 &= \left\langle \sqrt{\prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2+1} (1 - \mathcal{F}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}} - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2+1} [1 - (\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j} \right. \\
 &\quad \left. \sqrt{1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2+1} (1 - \mathcal{F}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}} \right\rangle.
 \end{aligned} \tag{33}$$

Hence, it is true for $m = \beta_1 + 1$ and $n = \beta_2 + 1$. \square

Example 2. Let $\mathcal{U}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a set of experts whose weights are given as $\Omega_i = (0.243, 0.514, 0.343)^T$. Experts evaluate the beauty of a house under a considered set of attributes $\mathfrak{F}' = \{d_1 = \text{lawn}, d_2 = \text{security system}\}$ with their corresponding subattributes $\text{lawn} = d_1 = \{d_{11} = \text{with grass}, d_{12} = \text{without grass}\}$ and security system = $d_2 = \{d_{21} = \text{guards}, d_{22} = \text{cameras}\}$. Let $\mathfrak{F}' = d_1 \times d_2$ be a set of subattributes $\mathfrak{F}' = d_1 \times d_2 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} = \{(d_{11}, d_{21}), (d_{11}, d_{22}), (d_{12}, d_{21}), (d_{12}, d_{22})\}$ and $\mathfrak{F}' = \{d_1, d_2\}$.

$\check{d}_3, \check{d}_4\}$ represents the set subattributes with weights $\gamma_j = (0.25, 0.15, 0.2, .4)^T$. Experts' opinion for each multi-subattribute in the form of PFHSNs $(\mathfrak{F}, \mathfrak{F}') = \langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{F}_{\check{d}_{ij}} \rangle_{3 \times 4}$ is given as follows:

$$(\mathfrak{F}, \mathfrak{F}') = \begin{bmatrix} (0.3, 0.8) & (0.4, 0.6) & (0.3, 0.6) & (0.5, 0.6) \\ (0.8, 0.3) & (0.7, 0.4) & (0.7, 0.3) & (0.4, 0.8) \\ (0.3, 0.6) & (0.5, 0.7) & (0.6, 0.5) & (0.5, 0.4) \end{bmatrix}. \tag{34}$$

By using equation (13),

$$\begin{aligned}
 \text{PFHSIWG}(\mathfrak{F}_{\check{d}_{11}}, \mathfrak{F}_{\check{d}_{12}}, \dots, \mathfrak{F}_{\check{d}_{mn}}) &= \left\langle \sqrt{\prod_{j=1}^4 \left(\prod_{i=1}^3 (1 - \mathcal{F}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}} - \prod_{j=1}^4 \left(\prod_{i=1}^3 [1 - (\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{F}_{\check{d}_{ij}}^2)]^{\Omega_i} \right)^{\gamma_j} \right. \\
 &\quad \left. \sqrt{1 - \prod_{j=1}^4 \left(\prod_{i=1}^3 (1 - \mathcal{F}_{\check{d}_{ij}}^2)^{\Omega_i} \right)^{\gamma_j}} \right\rangle, \\
 &= \left\langle \sqrt{\left(\{(0.36)^{0.243} (0.91)^{0.514} (0.64)^{0.343}\}^{0.25} \{(0.64)^{0.243} (0.84)^{0.514} (0.51)^{0.343}\}^{0.15} \{(0.64)^{0.243} (0.91)^{0.514} (0.75)^{0.343}\}^{0.2} \{(0.64)^{0.243} (0.36)^{0.514} (0.84)^{0.343}\}^{0.4} \right) - \right. \\
 &\quad \left. \sqrt{\left(\{(0.27)^{0.243} (0.27)^{0.514} (0.55)^{0.343}\}^{0.25} \{(0.48)^{0.243} (0.35)^{0.514} (0.26)^{0.343}\}^{0.15} \{(0.55)^{0.243} (0.42)^{0.514} (0.39)^{0.343}\}^{0.2} \{(0.39)^{0.243} (0.20)^{0.514} (0.59)^{0.343}\}^{0.4} \right)} \right. \\
 &\quad \left. \sqrt{1 - \left(\{(0.36)^{0.243} (0.91)^{0.514} (0.64)^{0.343}\}^{0.25} \{(0.64)^{0.243} (0.84)^{0.514} (0.51)^{0.343}\}^{0.15} \{(0.64)^{0.243} (0.91)^{0.514} (0.75)^{0.343}\}^{0.2} \{(0.64)^{0.243} (0.36)^{0.514} (0.84)^{0.343}\}^{0.4} \right)} \right\rangle, \\
 &\langle 0.53653, 0.62976.
 \end{aligned} \tag{35}$$

Hence, some basic properties for PFHSNs using the PFHSWG operator are established using Theorem 2.

3.3. Properties of PFHSIWG Operator

3.3.1. *Idempotency.* $\mathfrak{F}_{\tilde{d}_{ij}} = \mathfrak{F}_{\tilde{d}} = (\mathcal{T}_{\tilde{d}_{ij}}, \mathcal{J}_{\tilde{d}_{ij}}), \forall i, j$, then

$$\text{PFHSIWG}(\mathfrak{F}_{\tilde{d}_{11}}, \mathfrak{F}_{\tilde{d}_{12}}, \dots, \mathfrak{F}_{\tilde{d}_{nm}}) = \mathfrak{F}_{\tilde{d}}. \quad (36)$$

3.3.2. *Boundedness.* Let $\mathfrak{F}_{\tilde{d}_{ij}}$ be a collection of PFHSNs, $\mathfrak{F}_{\tilde{d}_{ij}}^- = \min_j \min_i \{\mathcal{T}_{\tilde{d}_{ij}}\}, \max_j \max_i \{\mathcal{J}_{\tilde{d}_{ij}}\}$, and $\mathfrak{F}_{\tilde{d}_{ij}}^+ = \max_j \max_i \{\mathcal{T}_{\tilde{d}_{ij}}\}, \min_j \min_i \{\mathcal{J}_{\tilde{d}_{ij}}\}$; then,

$$\mathfrak{F}_{\tilde{d}_{ij}}^- \leq \text{PFHSIWG}(\mathfrak{F}_{\tilde{d}_{11}}, \mathfrak{F}_{\tilde{d}_{12}}, \dots, \mathfrak{F}_{\tilde{d}_{nm}}) \leq \mathfrak{F}_{\tilde{d}_{ij}}^+. \quad (37)$$

3.3.3. *Homogeneity.* Prove that $\text{PFHSIWG}(\alpha \mathfrak{F}_{\tilde{d}_{11}}, \alpha \mathfrak{F}_{\tilde{d}_{12}}, \dots, \alpha \mathfrak{F}_{\tilde{d}_{nm}}) = \alpha \cdot \text{PFHSIWG}(\mathfrak{F}_{\tilde{d}_{11}}, \mathfrak{F}_{\tilde{d}_{12}}, \dots, \mathfrak{F}_{\tilde{d}_{nm}})$ for any positive real number α .

4. An MCDM Approach Based on Interaction Aggregation Operators for PFHSSs

An MCDM approach is established here under the developed operators and presented a comprehensive comparative

analysis to prove the usefulness and practicality of our established method.

4.1. Proposed MCDM Approach. Consider $\chi = \{\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \dots, \chi^{(s)}\}$ to be a set of s alternatives and $\mathcal{U} = \{\delta_1, \delta_2, \delta_3, \dots, \delta_n\}$ to be a set n experts. The weights of experts are given as $\Omega = (\Omega_1, \Omega_2, \dots, \Omega_n)^T$ and $\Omega_i > 0, \sum_{i=1}^n \Omega_i = 1$. Let $\mathfrak{Q} = \{d_1, d_2, \dots, d_m\}$ represent the set attributes with their corresponding multi-subattributes such as $\mathfrak{F}' = \{(d_{1\rho} \times d_{2\rho} \times \dots \times d_{m\rho}) \text{ for all } \rho \in \{1, 2, \dots, t\}\}$ with weights $\gamma = (\gamma_{1\rho}, \gamma_{2\rho}, \gamma_{3\rho}, \dots, \gamma_{m\rho})^T$ such as $\gamma_\rho > 0, \sum_{\rho=1}^t \gamma_\rho = 1$, and can be stated as $\mathfrak{F} = \{\tilde{d}_\partial: \partial \in \{1, 2, \dots, m\}\}$. The group of experts $\{\kappa^i: i = 1, 2, \dots, n\}$ assess the alternatives $\{\aleph^{(z)}: z = 1, 2, \dots, s\}$ under the chosen sub-attributes $\{\tilde{d}_\partial: \partial = 1, 2, \dots, k\}$ in the form of PFHSNs such as $(\chi_{\tilde{d}_{ik}}^{(z)})_{n \times m} = (\mathcal{T}_{\tilde{d}_{ij}}, \mathcal{J}_{\tilde{d}_{ij}})_{n \times m}$ where $0 \leq \mathcal{T}_{\tilde{d}_{ij}}, \mathcal{J}_{\tilde{d}_{ij}} \leq 1$, and $0 \leq (\mathcal{T}_{\tilde{d}_{ij}})^2 + (\mathcal{J}_{\tilde{d}_{ij}})^2 \leq 1$ for all i and k . Utilizing the proposed PFHSIWA, PFHSIWG operators develop aggregated PFHSNs \mathcal{L}_ϕ for each alternative according to the expert's preferences. Finally, utilizing equation (8), compute the score function. The above-presented approach can be concise as follows:

Step 1. Develop decision matrices for each alternative $\{D^{(z)}: z = 1, 2, \dots, s\}$ as follows:

$$(\chi^{(z)}, \mathfrak{F}')_{n \times \partial} = \begin{pmatrix} \delta_1 \left(\begin{pmatrix} \mathcal{T}_{\tilde{d}_{11}}^{(z)}, \mathcal{J}_{\tilde{d}_{11}}^{(z)} \\ \mathcal{T}_{\tilde{d}_{12}}^{(z)}, \mathcal{J}_{\tilde{d}_{12}}^{(z)} \\ \dots \\ \mathcal{T}_{\tilde{d}_{1\partial}}^{(z)}, \mathcal{J}_{\tilde{d}_{1\partial}}^{(z)} \end{pmatrix} \right) \\ \delta_2 \left(\begin{pmatrix} \mathcal{T}_{\tilde{d}_{21}}^{(z)}, \mathcal{J}_{\tilde{d}_{21}}^{(z)} \\ \mathcal{T}_{\tilde{d}_{22}}^{(z)}, \mathcal{J}_{\tilde{d}_{22}}^{(z)} \\ \dots \\ \mathcal{T}_{\tilde{d}_{2\partial}}^{(z)}, \mathcal{J}_{\tilde{d}_{2\partial}}^{(z)} \end{pmatrix} \right) \\ \vdots \\ \delta_n \left(\begin{pmatrix} \mathcal{T}_{\tilde{d}_{n1}}^{(z)}, \mathcal{J}_{\tilde{d}_{n1}}^{(z)} \\ \mathcal{T}_{\tilde{d}_{n2}}^{(z)}, \mathcal{J}_{\tilde{d}_{n2}}^{(z)} \\ \dots \\ \mathcal{T}_{\tilde{d}_{n\partial}}^{(z)}, \mathcal{J}_{\tilde{d}_{n\partial}}^{(z)} \end{pmatrix} \right) \end{pmatrix}. \quad (38)$$

Step 2. Obtain normalized decision matrices for alternatives utilizing the normalization rule:

$$H_{ij} = \begin{cases} \mathfrak{F}_{\tilde{d}_{ij}}^c; & \text{cost type parameter,} \\ \mathfrak{F}_{\tilde{d}_{ij}}; & \text{benefit type parameter.} \end{cases} \quad (39)$$

Step 3. Establish a collective decision matrix \mathcal{L}_k for each alternative using developed AOs

Step 4. Using equation (8), compute the score values for each alternative

Step 5. Select the most suitable alternative with the maximum score value

Step 6. Rank the alternatives

The graphical representation of the presented approach can be expressed in following Figure 1.

4.2. Case Study. The problem of supplier selection is an essential part at both a logical and practical level. This is an ongoing problem for the organization because the most

suitable choice of suppliers is the basis for effective supply chain management and also the basis of reasonable benefit, which includes environmental management standards and includes more features of sustainable improvement in environmental management standards and supplier selection procedures. Depending on the visible horizon of substantial or social activities, supplier selection is typically known as “sustainable supplier selection” in the literature. This is a multidimensional consequence along with conflicting specifications. The self-assessment process needs to deliberate several features. From these perspectives, the issue of supplier selection is often considered a “reference” issue in the literature, with a wide range of methods used to support incorporative decisions. The problem of choosing and assessing a sustainable supplier is solved in lots of the best ways. This example of sustainable supplier selection results in a set of five parameters, using the analysis of [44–53]. These are d_1 , supremacy of service, d_2 , delivery, d_3 , environmental efficiently, d_4 , troposphere, and d_5 , corporate social concern.

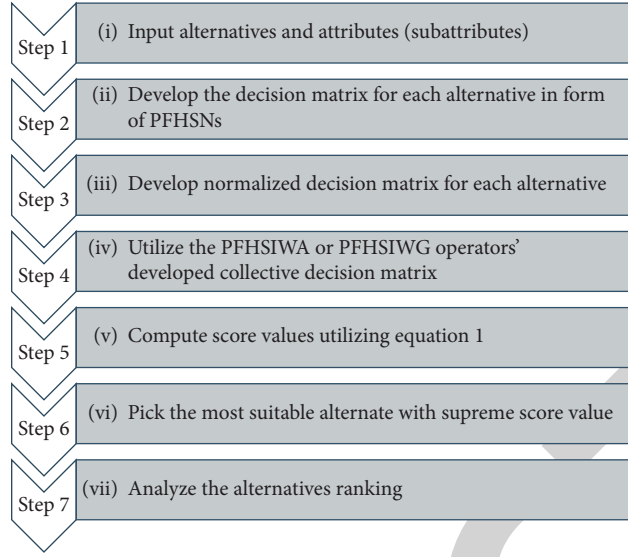


FIGURE 1: Flowchart of presented PFHSIWA or PFHSIWG operators.

Consider $\{\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \chi^{(4)}, \chi^{(5)}\}$ to be a set of alternatives, and $\mathfrak{L} = \{d_1 = \text{supremacy}, d_2 = \text{delivery}, d_3 = \text{environmental efficiently}, d_4 = \text{troposphere}, d_5 = \text{corporate societal concern}\}$ represents the collection of considered parameters prearranged as supremacy = $d_1 = \{d_{11} = \text{national level}, d_{12} = \text{international level}\}$, delivery =

$d_2 = \{d_{21} = \text{by carrier}, d_{22} = \text{by hand}\}$, environmental efficiently = $d_3 = \{d_{31} = \text{environmental efficiently}\}$, troposphere = $d_4 = \{d_{41} = \text{friendly}, d_{42} = \text{non serious}\}$, and corporate social concern = $d_5 = \{d_{51} = \text{corporate social concern}\}$. Let $\mathfrak{F}' = d_1 \times d_2 \times d_3 \times d_4 \times d_5$ be a set of subattributes:

$$\begin{aligned} \mathfrak{F}' &= d_1 \times d_2 \times d_3 \times d_4 \times d_5 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} \times \{d_{31}, d_{32}\} \times \{d_{41}\} \times \{d_{51}\}, \\ &= \{(d_{11}, d_{21}, d_{31}, d_{41}, d_{51}), (d_{11}, d_{21}, d_{32}, d_{41}, d_{51}), (d_{11}, d_{22}, d_{31}, d_{41}, d_{51}), (d_{11}, d_{22}, d_{32}, d_{41}, d_{51}), \\ &\quad (d_{12}, d_{21}, d_{31}, d_{41}, d_{51}), (d_{12}, d_{21}, d_{32}, d_{41}, d_{51}), (d_{12}, d_{22}, d_{31}, d_{41}, d_{51}), (d_{12}, d_{22}, d_{32}, d_{41}, d_{51})\}, \end{aligned} \quad (40)$$

where $\mathfrak{F}' = \{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4, \check{d}_5, \check{d}_6, \check{d}_7, \check{d}_8\}$ represents the set of all subattributes with weights $(0.12, 0.18, 0.1, 0.15, 0.05, 0.22, 0.08, 0.1)^T$. Let $\{u_1, u_2, u_3\}$ be a set of experts with weights $(0.243, 0.514, 0.343)^T$ to evaluate the optimum alternative. Specialists contribute their predilections in the form of PFHSNs under multi-subattributes of considered attributes.

4.2.1. By Using PFHSIWA Operator

Step 1. Experts access the matters to illustrate the PFHSN. A summary of the many subattributes of the perceived attributes as well as their score values is given in Tables 1–3.

Step 2. All attributes are of the same type, so no need to normalize them.

Step 3. Experts' opinion can be summarized utilizing equation (13) as follows:

$$\mathcal{L}_1 = 0.6009, 0.4342, \quad \mathcal{L}_2 = 0.6499, 0.4078, \quad \mathcal{L}_3 = 0.6179, 0.3506, \quad \mathcal{L}_4 = 0.6076, 0.3527, \quad \text{and} \quad \mathcal{L}_5 = 0.5493, 0.4345.$$

Step 4. Compute the score values using equation (8):

$$\mathbb{S}(\mathcal{L}_1) = 0.1667, \quad \mathbb{S}(\mathcal{L}_2) = 0.2421, \quad \mathbb{S}(\mathcal{L}_3) = 0.2673, \quad \mathbb{S}(\mathcal{L}_4) = 0.2549, \quad \text{and} \quad \mathbb{S}(\mathcal{L}_5) = 0.1148.$$

Step 5. $\chi^{(3)}$ has greatest score value, so $\chi^{(3)}$ is the finest option.

Step 6. Alternatives' ranking using the PFHSIWA operator is given as follows:

$$\mathbb{S}(\mathcal{L}_3) > \mathbb{S}(\mathcal{L}_4) > \mathbb{S}(\mathcal{L}_2) > \mathbb{S}(\mathcal{L}_1) > \mathbb{S}(\mathcal{L}_5). \text{ So, } \chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}.$$

4.2.2. By Using PFHSIWG Operator

Step 1 and Step 2. They are the same as Section 4.2.1.

Step 3. Experts' opinion can be summarized utilizing equation (29) as follows:

$$\mathcal{L}_1 = 0.4679, 0.5590, \quad \mathcal{L}_2 = 0.5157, 0.5289, \quad \mathcal{L}_3 = 0.4892, 0.4387, \quad \mathcal{L}_4 = 0.4910, 0.4751, \quad \text{and} \quad \mathcal{L}_5 = 0.4440, 0.6407$$

Step 4. Compute the score values using equation (8):

$$\mathbb{S}(\mathcal{L}_1) = -0.0911, \quad \mathbb{S}(\mathcal{L}_2) = -0.0132, \quad \mathbb{S}(\mathcal{L}_3) = 0.0505, \quad \mathbb{S}(\mathcal{L}_4) = 0.0159, \quad \text{and} \quad \mathbb{S}(\mathcal{L}_5) = -0.1967$$

TABLE 1: PFHS decision matrix for u_1 .

	\tilde{d}_1	\tilde{d}_2	\tilde{d}_3	\tilde{d}_4	\tilde{d}_5	\tilde{d}_6	\tilde{d}_7	\tilde{d}_8
$\chi^{(1)}$	(0.3, 0.8)	(0.7, 0.3)	(0.6, 0.7)	(0.5, 0.4)	(0.2, 0.4)	(0.4, 0.6)	(0.5, 0.8)	(0.9, 0.3)
$\chi^{(2)}$	(0.6, 0.7)	(0.4, 0.6)	(0.3, 0.4)	(0.9, 0.2)	(0.3, 0.8)	(0.2, 0.4)	(0.7, 0.5)	(0.4, 0.5)
$\chi^{(3)}$	(0.7, 0.3)	(0.2, 0.5)	(0.1, 0.6)	(0.3, 0.4)	(0.4, 0.6)	(0.8, 0.4)	(0.6, 0.7)	(0.2, 0.5)
$\chi^{(4)}$	(0.8, 0.4)	(0.2, 0.9)	(0.2, 0.4)	(0.4, 0.6)	(0.6, 0.5)	(0.5, 0.6)	(0.4, 0.5)	(0.8, 0.3)
$\chi^{(5)}$	(0.5, 0.7)	(0.8, 0.5)	(0.7, 0.4)	(0.4, 0.3)	(0.4, 0.9)	(0.2, 0.4)	(0.8, 0.4)	(0.7, 0.5)

TABLE 2: PFHS decision matrix for u_2 .

	\tilde{d}_1	\tilde{d}_2	\tilde{d}_3	\tilde{d}_4	\tilde{d}_5	\tilde{d}_6	\tilde{d}_7	\tilde{d}_8
$\chi^{(1)}$	(0.7, 0.6)	(0.3, 0.4)	(0.6, 0.5)	(0.3, 0.9)	(0.5, 0.4)	(0.4, 0.6)	(0.7, 0.5)	(0.4, 0.8)
$\chi^{(2)}$	(0.8, 0.5)	(0.7, 0.4)	(0.9, 0.2)	(0.7, 0.4)	(0.4, 0.5)	(0.9, 0.3)	(0.2, 0.7)	(0.3, 0.8)
$\chi^{(3)}$	(0.3, 0.7)	(0.4, 0.5)	(0.4, 0.8)	(0.3, 0.4)	(0.6, 0.7)	(0.3, 0.4)	(0.9, 0.2)	(0.7, 0.2)
$\chi^{(4)}$	(0.5, 0.4)	(0.7, 0.6)	(0.9, 0.3)	(0.8, 0.5)	(0.9, 0.2)	(0.2, 0.4)	(0.4, 0.6)	(0.6, 0.5)
$\chi^{(5)}$	(0.8, 0.5)	(0.7, 0.4)	(0.8, 0.5)	(0.5, 0.2)	(0.5, 0.7)	(0.7, 0.5)	(0.7, 0.6)	(0.6, 0.4)

TABLE 3: PFHS decision matrix for u_3 .

	\tilde{d}_1	\tilde{d}_2	\tilde{d}_3	\tilde{d}_4	\tilde{d}_5	\tilde{d}_6	\tilde{d}_7	\tilde{d}_8
$\chi^{(1)}$	(0.5, 0.7)	(0.8, 0.5)	(0.7, 0.4)	(0.4, 0.3)	(0.4, 0.9)	(0.2, 0.4)	(0.8, 0.4)	(0.7, 0.5)
$\chi^{(2)}$	(0.8, 0.5)	(0.7, 0.4)	(0.8, 0.5)	(0.5, 0.2)	(0.5, 0.7)	(0.7, 0.5)	(0.7, 0.6)	(0.6, 0.4)
$\chi^{(3)}$	(0.6, 0.8)	(0.4, 0.5)	(0.6, 0.5)	(0.6, 0.4)	(0.7, 0.5)	(0.8, 0.4)	(0.5, 0.8)	(0.4, 0.5)
$\chi^{(4)}$	(0.5, 0.7)	(0.9, 0.3)	(0.3, 0.5)	(0.5, 0.7)	(0.3, 0.5)	(0.8, 0.5)	(0.7, 0.5)	(0.2, 0.5)
$\chi^{(5)}$	(0.5, 0.4)	(0.4, 0.8)	(0.5, 0.6)	(0.3, 0.4)	(0.7, 0.6)	(0.7, 0.5)	(0.4, 0.9)	(0.5, 0.2)

TABLE 4: Comparison of PFHSSs with some prevailing models.

	Set	Truthiness	Falsity	Parametrization	Attributes	Subattributes
Zadeh [1]	FS	✓	×	×	✓	×
Atanassov [2]	IFS	✓	✓	×	✓	×
Maji et al. [21]	IFSS	✓	✓	✓	✓	×
Peng et al. [27]	PFSS	✓	✓	✓	✓	×
Zulqarnain et al. [42]	IFHSS	✓	✓	✓	✓	✓
Proposed approach	PFHSS	✓	✓	✓	✓	✓

Step 5. $\chi^{(3)}$ has the greatest score value, so $\chi^{(3)}$ is the finest option

Step 6. Alternatives' ranking using the PFHSIWG operator is given as follows:

$$\mathbb{S}(\mathcal{L}_3) > \mathbb{S}(\mathcal{L}_4) > \mathbb{S}(\mathcal{L}_2) > \mathbb{S}(\mathcal{L}_1) > \mathbb{S}(\mathcal{L}_5). \quad \text{So,} \\ \chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}.$$

5. Comparative Analysis and Discussion

In the following section, we will discuss quality, naivety, and tractability by means of the planned method. We also gave a brief overview of the following: the proposed approach with some existing methods.

5.1. Superiority of the Proposed Method. Through this study, along with association, it is resolute that the concerns attained with the proposed method are rather extragenereal than either technique. However, the developed MCDM

approach has been provided more information to cope with the hesitancy in the DM procedure related to the existing MCDM strategies. Besides, the numerous mixed structures of FSs have grown into a unique feature of PFHSSs; after adding some proper conditions, the general facts about the component can be stated precisely and logically, as shown in Table 4. It is observed that the obtained results deliver extrainformation comparative to existing studies. The developed PFHSSs accurately accommodate more information considering the multi-subattributes of the parameters. It is quite an easy tool to mix inexact and unsure information within the DM process. Hence, the proposed methodology is pragmatic, diffident, and distinctive from available hybrid structures of fuzzy sets.

5.2. Discussion. Zadeh's [1] FSs only addressed the rough and vague facts using MD considering the subattributes for each alternative. But, the FSs are unable to deal with the

TABLE 5: Comparative analysis with existing operators.

Method	Score values for alternatives					Ranking order
	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	$\chi^{(4)}$	$\chi^{(5)}$	
PFIWA [8]	0.55374	0.33901	0.60019	0.52007	0.36813	$\chi^{(3)} > \chi^{(1)} > \chi^{(4)} > \chi^{(5)} > \chi^{(2)}$
PFIWG [9]	0.49325	0.41837	0.73000	0.48906	0.46524	$\chi^{(3)} > \chi^{(1)} > \chi^{(4)} > \chi^{(5)} > \chi^{(2)}$
PFSWA [10]	0.21173	0.22017	0.33215	0.27008	0.21893	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(5)} > \chi^{(1)}$
PFSWG [10]	0.20587	0.23066	0.32902	0.25462	0.21727	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(5)} > \chi^{(1)}$
PFEWA [54]	0.51686	0.54833	0.60467	0.59021	0.51235	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}$
PFEWG [54]	0.54219	0.56597	0.62190	0.59381	0.52209	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}$
SPFWA [16]	0.08158	0.07674	0.14762	0.09959	0.07985	$\chi^{(3)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)} > \chi^{(4)}$
IFHSA [55]	0.49830	0.41735	0.40935	0.46175	0.43247	$\chi^{(3)} > \chi^{(2)} > \chi^{(4)} > \chi^{(1)} > \chi^{(5)}$
IFHSWG [55]	0.42615	0.36175	0.35635	0.40790	0.40635	$\chi^{(3)} > \chi^{(2)} > \chi^{(4)} > \chi^{(1)} > \chi^{(5)}$
Proposed PFHSIWA operator	0.1667	0.2421	0.2673	0.2549	0.1148	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}$
Proposed PFHSIWG operator	-0.0911	-0.0132	0.0505	0.0159	-0.1967	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}$

NMD of parameters. Atanassov's [2] IFSSs accommodate the unclear and undefined objects using MD and NMD. However, IFSSs are unable to handle the circumstances when $MD + NMD > 1$; on the contrary, our presented idea expertly compacts with such complications. Maji et al. [21] proposed the theory of IFSSs; the presented idea conducts the anxiety of the object in which the characteristics of MD and NMD can be used appropriately along with their parameterization with the following condition $MD + NMD \leq 1$. To handle these consequences, Peng et al. [27] suggested the idea of PFSSs by amending the condition $MD + NMD \leq 1$ to $MD^2 + NMD^2 \leq 1$ with their parameterization. But there is no information about the subattributes of the attributes under consideration in all the above studies. Therefore, the above theories are unable to address the situation when their subattributes are associated with the attributes. All prevailing hybrid structures of FSs cannot handle the NMem values of subattributes of considered n -tuple attributes. Zulqarnain et al. [42] extended the IFSSs to IFHSSs and proposed the CC and WCC for IFHSSs in which $MD + NMD \leq 1$ for each subattribute. But IFHSSs cannot provide any information on the Mem and NMem values of the subattribute of the considered attribute when $MD + NMD \geq 1$. It can be seen the finest choice of the projected approach simulates itself and ensures the success of the developed method as well as the responsibility.

5.3. Comparative Analysis. We endorse a new algorithmic rule for PFHSSs using developed PFHSIWA and PFHSIWG operators within the succeeding section. Consequently, we used the proposed algorithmic rule for any veridical problem, that is to say, supplier selection in SSCM. Results demonstrate that algorithmic governance is effective as well as sensible. From the above calculation, it can be observed that $\chi^{(3)}$ supplier is the premium alternative for SSCM. From the exploration, it is terminated that the results attained from the proposed viewpoint are more than the consequences of the planned theories. Thus, compared to available techniques, established AOs addressed unsure and unclear information efficiently. However, under available MCDM methods, the most important benefit of the proposed approach is that it can serve more information in the data than

the available methodology. The comparison between existing AOs and our developed operators is given in following Table 5. The presented approach contemplates the interaction among the Mem and NMem function of PFHSSNs, which can attain the more realistic decision effects considering the parametric values of the multi-subattributes of the parameters.

The existing PFIWA [8], PFIWG [9], PFEWA, PFEWG [54], and SPFWA [16] operators are not capable of dealing with the parametrization of the alternatives. The PFSWA and PFSWG [10] operators handle the parametric values of the alternatives but these operators cannot accommodate the multi-subattributes of the considered parameters. The prevailing IFHSA and IFHSWG [55] operators competently deals the multi-sub attributes of the parameters comparative to above discuss operators. But IFHSSs cannot handle the situation when the sum of Mem and NMem values of the subattribute of the considered attribute exceeds 1. On the contrary, our proposed PFHSIWA and PFHSIWG operators competently accommodate the abovementioned shortcomings. So, we claim that our established operators are extraordinary than existing operators to solve imprecise as well as vague facts in DM procedure. The assistance of the deliberated approach along with related measures over present approaches is evading conclusions grounded on adverse reasons. Therefore, it is a useful tool for combining inaccurate and uncertain information in the DM process.

6. Conclusion

In this article, PFHSSs consider solving the complexities of information related to unsatisfactory, instability, and deviation by considering MD and NMD on the n -tuple subattributes of the suggested attributes. The core objective of this research is to propose novel operational laws considering the interaction. We also presented interaction aggregation operators, i.e., PFHSIWA and PFHSIWG, utilizing developed operational laws and discussed their desirable properties. Furthermore, based on developed interaction AOs, an MCDM approach has been established to solve real-life complications. To certify the applicability and practicality of our anticipated method, we

planned an ephemeral comparative analysis of our developed methodology with some existing studies. From the obtained results, it can be decided absolutely that the predetermined methodology indicates that the experts have high stability and accessibility in the process of DM. The subsequent study will clarify the presentation of DM techniques using a number of other initiatives under PFHSSs, such as entropy and similarity measures. Furthermore, many other structures can be established and proposed, such as topological structure, algebraic structure, and configurable structure. In the future, PFHSSs can be extended to q -rung orthopair fuzzy hypersoft sets and spherical and T -spherical fuzzy hypersoft sets with their several AOs and decision-making approaches.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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Retraction

Retracted: A Public-Participation-Based Mixed Multiattribute Decision-Making Approach for Major Public Affairs

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] C. Cai, Y. Luo, G. Zhu, and H. Zou, "A Public-Participation-Based Mixed Multiattribute Decision-Making Approach for Major Public Affairs," *Mathematical Problems in Engineering*, vol. 2021, Article ID 7550055, 14 pages, 2021.

Research Article

A Public-Participation-Based Mixed Multiattribute Decision-Making Approach for Major Public Affairs

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The decision-making activities of major public affairs are closely related to the public, so the decision results of such affairs must be supported by the public. The public must participate in decision-making activities to ensure their effectiveness, which further increases the complexity. In addition, attribute information and public opinion usually present different forms of expression in this type of problem, making decision-making more difficult. Therefore, a suitable decision approach must be chosen to deal with this type of decision problem. This paper addresses the decision-making characteristics of major public affairs and proposes a public-participation-based decision-making approach for mixed multiattribute decision-making problems in major public affairs. The proposed approach can work with entirely unknown attribute weights and decision-making values represented in multiple formats. First, the statistical distribution of public opinions is determined based on the expectation of the attributes, resulting in decision-making reference points for various attributes. The different forms of attributes and reference points are unified. Then, the values of the attributes and reference points are standardized. Afterward, the attribute prospect value for each alternative is calculated using the attribute value and corresponding reference points. The attribute weight intervals are determined based on the importance information of the attributes provided by the public. An optimization model is established to determine the attribute weights to maximize the alternative attribute deviation. Next, the comprehensive prospect value of each alternative is obtained to determine the ranking of the alternatives. Finally, a case analysis is conducted with a method comparison and sensitivity analysis, and the feasibility and effectiveness of the proposed approach are verified. In the proposed method, the reference points for each attribute are set according to the distribution characteristics and ambiguity of public expectations, guaranteeing that public expectations can be effectively reflected in the attribute reference points. In the process of attribute weighting, based on the information for the attribute importance given by the public, the range of attribute weights is determined. Then, we obtain the exact value of the attribute weights using an optimization model to maximize the alternative attribute deviation. The final result of the attribute weights ensures the full expression of public opinion and can improve the differentiation of decision results, which is convenient for ranking alternatives. During evaluation of the alternatives based on prospect theory, the expression forms of attributes and reference points are unified. Subsequently, the values of them are normalized, which satisfy the decision-making requirement of major public affairs.

1. Introduction

Major public affairs have a profound influence and are closely related to the vital interests of the public. Consequently, they attract a considerable amount of societal attention. Once the public does not support the final decision

results, it is easy to generate social risks, leading to a series of social problems [1, 2]. Numerous public opinions must be gathered across different strata of society to ensure that the decision-making results regarding major public affairs fully reflect the majority public opinion, determining the decision outcomes [3, 4]. In the process of decision-making, given the

variations in individual opinions on major public affairs, the distribution of public opinions presents a distinct characteristic of discreteness. In addition, the scale of public opinion is usually immense, which makes it challenging to collect, process, and analyze public opinions. Thus, effective decision-making on major public affairs based on collecting and analyzing public opinions is an important concern for governmental authorities.

The department of public affairs management uses various means to encourage the public to participate in decision-making to ensure that information collected from the public is effective and extensive. Popular public-participation methods include questionnaire surveys, online platform messages, and on-site hearings [5–8]. Some researchers have introduced statistical theory for sorting and analyzing public opinions, identifying representative public opinions based on their statistical distribution characteristics, and providing tools for identifying and processing public opinions [9]. Most citizens lack professional knowledge, making it difficult for them to provide objective guidance and suggestions when contributing to decision-making. The opinions they provide focus more on expressing their personal expectations or concerns. In general, the stronger the public expectations or appeals for certain aspects of a topic, the more the attention this aspect needs to receive [10–12]. Therefore, the public expectations should be included in the decision-making of public affairs as an important reference standard for alternative evaluation. It is crucial to set the decision-making reference points effectively based on public expectations. Previous research on decision-making reference points has focused on the effectiveness evaluation of setting reference points [13, 14], the distance measurement between the reference points and alternative attributes [15], and the dynamic evolution characteristics of the reference points [16]. Most existing methods for setting reference points are based on known individual data or a small amount of known data. How to set the reference points based on large amount of public opinions requires further study.

For the decision problems, the results of attribute weight have a great influence on decision-making, so choosing a proper weighting method is necessary for decision-making activities. Common weighting methods include analytic hierarchy process (AHP) [17], entropy weight [18], and maximum deviation methods [19]. In an actual weighting operation process, the attribute weighting methods of decision-making problems are primarily chosen based on several factors: the decision-making problem type, attribute data expression type, distribution characteristics of the attribute data, and others [20, 21]. For decision-making in public affairs, in order to ensure the effectiveness of attribute weighting, the public is introduced to decision-making activities. Public opinions on the importance of the attributes must be considered when determining attribute weights to ensure the effectiveness of attribute weighting. However, the existing attribute weighting approaches are seldom involved in public opinion. Therefore, further research is required on how to effectively weight the attributes considering public opinions.

Different attributes reflect the various contents of each alternative; hence, according to the actual needs of the attribute expression, each attribute is usually expressed in a different form, such as a crisp value, interval value, and linguistic value. For a decision problem, if the attributes are expressed in different forms, we call this decision problem as mixed multiattribute decision problem [22–24]. Public decision-making activities are usually complicated; in order to ensure the effectiveness of the decision, different types of attributes are introduced to explain the decision alternatives. Thus, public decision-making activities usually have the characteristics of mixed multiattribute decision-making problems; the mixed multiattribute characteristics of decision-making problems must be considered when dealing with public decision problems.

In the research regarding decision-making methods, some scholars have introduced fuzzy number operators to decision-making activities, which can provide methodical support for the operation of decision-making information [25–29]. Moreover, some decision methods, such as the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [30], Vlekriterijumsko KOMPromisno Rangiranje (VIKOR) method [31], Elimination and Choice Expressing Reality (ELECTRE) method [32, 33], and the prospect theory [34], are widely used in various types of decision problems. Some scholars have used social network analysis to rank schemes based on the relationship and trust between decision-makers [35, 36]. These research achievements provide necessary technical and methodological support for solving various kinds of decision-making problems.

Major public affairs' decision-making has a high degree of complexity and uncertainty. To ensure decision-making effectiveness, we must consider the actual characteristics of the decision-making problems before the decision method is determined.

According to the above, the existing research can provide necessary references for the study of major public affairs. However, the decision-making problem of major public affairs requires public participation; therefore, the characteristics of public opinions must be considered in the aspects of attribute reference point setting, attribute weighting, and alternative ranking. Relatively, few research achievements exist on public-participation decision-making problems, so further research is needed regarding this aspect. Based on the above analysis, a mixed multiattribute decision-making approach with public participation is proposed in this paper. Compared to the existing research, the contribution of this paper is reflected in the three main aspects.

First, we set attribute reference points based on public expectations. Public expectations have a large volume and high dispersion characteristics, which must be collated and analyzed. In addition, public expectations are subjective and uncertain, making it more challenging to deal with this information. Existing research results rarely consider public expectations when setting reference points, so the existing reference point setting methods are unsuitable for this decision-making background. Based on the above analysis, we propose a reference point setting method considering public

expectations. We determine the distribution and ambiguity of public expectation according to the expectation information given by the public, and, on this basis, the reference point for each attribute is set.

Second, we propose an attribute weighting method considering public opinions. As we know, the value of attribute weights has a direct effect on the outcomes of decision-making. Existing attribute weighting methods rarely consider public opinion during the decision-making process in major public affairs. To ensure that attribute weighting result is acceptable to the public, we propose an attribute weighting method considering public opinions. In this attribute weighting method, we determine the value range of attribute weights according to public opinion and establish the attribute weighting optimization model to determine the final attribute weights. The result of the attribute weights obtained using this method reflects the public opinion and guarantees the validity of weighting.

Third, we introduce the decision-making method of the prospect theory to rank the alternatives in public-participation scenarios. The prospect theory considers the uncertainty of events and the decision risk and explains the structural effect, preference nonlinearity, resource dependence, risk pursuit, and loss avoidance reflected in people's choices [37]. Therefore, the prospect theory is suitable for the decision-making needs of major public affairs. However, the existing research on the prospect theory is not involved in public-participation scenarios. Thus, the existing decision-making method of prospect theory must be improved according to the characteristics of public participation in major public affairs to make it suitable for this type of decision-making problem. Based on the above analysis, we introduce an improved decision-making method of the prospect theory, which requires taking the expectation of public groups as the reference point to guarantee that the decision-making results better reflect public opinion. In the decision-making process, the expression forms of attribute information and public information vary to ensure the implementation of decision-making. Different expression forms of attribute information and public opinions must be unified into an interval before the normalized operation.

The rest of the paper is organized as follows. In Section 2, the preliminaries summarize the knowledge that forms the basis of the paper, and in Section 3, the method and principle are elucidated, explaining the proposed method. Section 4 presents the case analysis to verify the rationality and validity of the approach. In Section 5, a comparison of methods and sensitivity analysis are discussed. Finally, Section 6 provides the conclusions.

2. Preliminaries

Definition 1 (see [38]). Given a linguistic set $S = \{s_0, s_1, \dots, s_T\}$, s_0 and s_T are the lower and upper limits of the linguistic variables, respectively. The conversion relationship between the linguistic variables s_t ($t = 0, 1, 2, \dots, T$) and interval numbers $a_t = [a_t^L, a_t^U]$ is as follows, where $0 \leq a_t^L \leq a_t^U \leq 1$,

$$\begin{cases} a_0^L = 0; \\ a_t^U = a_t^L + \frac{1}{T}, & 0 \leq t \leq T; \\ a_t^L = a_{t-1}^U, & 1 \leq t \leq T; \\ a_T^U = 1. \end{cases} \quad (1)$$

Definition 2. Given the uncertain linguistic variable $s = [s_{t_1}, s_{t_2}]$, $s_{t_1}, s_{t_2} \in S$, where $S = \{s_0, s_1, \dots, s_T\}$ ($t_1, t_2 = 0, 1, 2, \dots, T; 0 \leq t_1 \leq t_2 \leq T$). The conversion relationship between the uncertain linguistic variable $s = [s_{t_1}, s_{t_2}]$ and interval number $a_t = [a_t^L, a_t^U]$ ($0 \leq a_t^L \leq a_t^U \leq 1$) is as follows:

$$\begin{cases} a_t^L = \frac{t_1}{T}, & 0 \leq t_1 \leq T, \\ a_t^U = \frac{t_2}{T}, & 0 \leq t_2 \leq T. \end{cases} \quad (2)$$

Definition 3 (see [39]). Suppose that the interval number $a = [a^L, a^U]$ is a nonnegative interval number (i.e., $0 \leq a^L \leq a^U$); then, the midpoint of a is defined by $\bar{a} = (a^L + a^U)/2$, and the interval number radius of a is defined by $c_a = (a^U - a^L)/2$.

3. Method and Principle

3.1. Problem Description. The decision-making problems in this study satisfy the following basic assumptions:

- (1) The set of alternatives is certain
- (2) The set of attributes to describe the alternatives is certain
- (3) The expression form of each attribute is known
- (4) The public individuals provide expectations based on their psychological perceptions, and the form of expression of public expectations is consistent with the form of expression of the attributes
- (5) The public provides the attribute evaluation value of importance in the form of linguistic or uncertain linguistic variables

Suppose that the set of decision-making alternatives for a major public affair is $Z = \{z_1, z_2, \dots, z_M\}$ with attribute set $G = \{g_1, g_2, \dots, g_N\}$ and attribute weight $\omega = (\omega_1, \omega_2, \dots, \omega_N)^T$, $\sum_{j=1}^N \omega_j = 1$. The value of attribute j in alternative i is y_j^i , $i = 1, 2, \dots, M; j = 1, 2, \dots, N$. Moreover, y_j^i can be expressed in the form of crisp numbers, interval numbers, linguistic variables, or uncertain linguistic variables. The public provides expectation values and evaluates the importance values of the attributes as per their preferences. Suppose that the number of individuals participating in the evaluation of the expected value of attribute

j is H_j . The expected value given by individual k on attribute j is r_j^k , where $k = 1, 2, \dots, H_j$, which can be in crisp numbers, interval numbers, linguistic variables, or uncertain linguistic variables. Suppose that the number of public individuals participating in evaluating the importance of attribute j is ζ_j . The evaluation value of importance provided by public individual k on attribute j is b_j^k , where $k = 1, 2, \dots, \zeta_j$, which can be in linguistic or uncertain linguistic variables depending on the public's actual need for attribute expression.

The study problem is determining the reference point for each attribute and the attribute weights according to the attribute values for each alternative, the public expected values for each attribute, and the public-evaluated importance value for each attribute. In addition, this study assesses how to choose a practical decision-making approach to rank all alternatives to determine the optimal alternative.

3.2. Determination of Attribute Reference Points Based on Public Opinions. The expression form of public expected values is primarily related to two factors: the attribute expression form of the alternative and the accuracy of individual expected values. If the attribute value of the alternative is expressed in real numbers (crisp numbers, interval numbers, etc.), the public expected value of the attribute is also expressed in real numbers (crisp numbers, interval numbers, etc.). In contrast, if the attribute value of the alternative is expressed in linguistic or uncertain linguistic variables, the public expected value of the attribute is also expressed in linguistic or uncertain linguistic variables.

The other factor is the accuracy of the individual expected values. When the public provides their expected values of an attribute, some of the public can express their expected values accurately. Consequently, this portion of the public chooses to provide their expected value of the attribute in crisp numbers or linguistic variables. In contrast, the other portion of the public is affected by internal or external factors and cannot accurately express their expected values for the attributes. This portion of the public usually expresses their expected values in interval numbers or uncertain linguistic variables.

According to the above analysis, the expected values of attributes given by the public also have various forms of expression and thus are expressed in different forms. The expression forms must be normalized to determine the specific distribution and ambiguity of the public expected values and obtain the reference point for each attribute. The reference points for attributes can be determined as follows:

- (1) The initial public expected values of the attribute are processed, which is achieved as follows.

Step 1. The public expected values for the attribute from linguistic or uncertain linguistic variables are converted into an interval value using Definitions 1 and 2.

Step 2. The converted public expected values are expressed in crisp or interval numbers. According to Definition 3, if the public

individual's expected value, r_j^k , is an interval number, described as $r_j^k = [r_j^{kL}, r_j^{kU}]$, its ambiguity is $c_j^k = (r_j^{kU} - r_j^{kL})/2$. If r_j^k is a crisp number, its ambiguity is $c_j^k = 0$. The comprehensive expected ambiguity is defined as the average expected ambiguity of all public individuals for attribute j (i.e., $c_j = (1/H_j) \sum_{k=1}^{H_j} c_j^k$).

Step 3. Definition 3 converts the interval form of the public individual expected values, r_j^k , into crisp numbers, denoted by \bar{r}_j^k , $\bar{r}_j^k = (r_j^{kL} + r_j^{kU})/2$. The crisp form of public expected values, r_j^k , retains the original form, which does not need to be converted. After the conversion operation, the expectation given by individual k on attribute j is a crisp value, defined as \bar{r}_j^k .

- (2) Based on the distribution of \bar{r}_j^k , the probability distribution of the expected values on different attributes can be determined. According to a previous study [40], large-scale public opinions usually follow a normal distribution.
- (3) The attribute reference points are determined:

Step 1. Under the normal distribution situation for \bar{r}_j^k , the mean of public expectation distribution over \bar{r}_j^k is defined as $\mu(\bar{r}_j^k)$, which can be determined based on the distribution of \bar{r}_j^k . The distribution variance of the public expected value on attribute j is defined as $\sigma(\bar{r}_j^k)$.

Step 2. The attribute reference point \bar{r}_j^* is confirmed, which is expressed in the form of interval numbers:

$\bar{r}_j^* = [\bar{r}_j^{*L}, \bar{r}_j^{*U}] = [\mu(\bar{r}_j^k) - c_j, \mu(\bar{r}_j^k) + c_j]$. The attribute reference point \bar{r}_j^* can be determined using the mean of public expectation distribution $\mu(\bar{r}_j^k)$ and the comprehensive ambiguity of the public expected values c_j .

3.3. Calculation of the Alternative Prospect Value

3.3.1. Normalization of the Attribute Values and Reference Points. According to Section 3.2, the finalized form of the attribute reference point is an interval number, whereas the attribute value of the alternative can be expressed as a crisp number, interval number, linguistic variable, or uncertain linguistic variable. The dimensions of various attributes are inconsistent, so the attribute values and reference points of all alternatives must be normalized.

The attribute values are unified in the form of interval numbers. If the attribute value of the alternative is a crisp number, it is rewritten as an interval number with equal upper and lower limits. If the attribute value of the alternative is a linguistic or uncertain linguistic variable, then Definitions 1 and 2 can convert it into an interval number. The interval form of y_j^i is defined as $y_j^i = [y_j^{iL}, y_j^{iU}]$.

Next, the attribute values and attribute reference points are normalized. Equations (3)–(6) are used to normalize the attribute values and reference points in interval numbers to eliminate the dimension influence of the original data. The

normalized attribute value of y_j^i is $p_{ij} = [p_{ij}^L, p_{ij}^U]$, $0 \leq p_{ij}^L \leq p_{ij}^U \leq 1$, and the normalized attribute reference point of \tilde{r}_j^* is $q_j = [q_j^L, q_j^U]$, $0 \leq q_j^L \leq q_j^U \leq 1$.

Attribute g_j is a profit index:

$$\begin{cases} p_{ij}^L = \frac{y_j^{iL} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}, \\ p_{ij}^U = \frac{y_j^{iU} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}, \end{cases} \quad (3)$$

$$\begin{cases} q_j^L = \frac{\tilde{r}_j^{*L} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}, \\ q_j^U = \frac{\tilde{r}_j^{*U} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}. \end{cases} \quad (4)$$

In addition, attribute g_j is a cost index:

$$\begin{cases} p_{ij}^L = \frac{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - y_j^{iU}}{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}, \\ p_{ij}^U = \frac{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - y_j^{iL}}{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}, \end{cases} \quad (5)$$

$$\begin{cases} q_j^L = \frac{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - \tilde{r}_j^{*U}}{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}, \\ q_j^U = \frac{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - \tilde{r}_j^{*L}}{\max\{\max_{1 \leq i \leq M}\{y_j^{iU}\}, \tilde{r}_j^{*U}\} - \min\{\min_{1 \leq i \leq M}\{y_j^{iL}\}, \tilde{r}_j^{*L}\}}. \end{cases} \quad (6)$$

3.3.2. Calculation of Prospect Profit and Loss Value over Different Attributes. The profit, G_{ij} , and loss, F_{ij} , of attribute value p_{ij} are calculated according to the relationship between the normalized attribute value, p_{ij} , and the normalized reference point, q_j . The equations to calculate G_{ij} and F_{ij} are presented in Table 1. The values of $v_{ij}^{(+)}$ and $v_{ij}^{(-)}$ are determined based on the prospect theory, as given in equation (7). According to a previous study [41], the coefficients of α, β, λ are $\alpha = \beta = 0.88$ and $\lambda = 2.25$:

$$\begin{cases} v_{ij}^{(+)} = (G_{ij})^\alpha, & G_{ij} \geq 0, \\ v_{ij}^{(-)} = -\lambda(-F_{ij})^\beta, & F_{ij} < 0. \end{cases} \quad (7)$$

The prospect profit-loss matrix of the attribute can be constructed as follows: $VM = [v_{ij}]_{M \times N}$, where the value of v_{ij} can be obtained using the following equation:

$$v_{ij} = v_{ij}^{(+)} + v_{ij}^{(-)}. \quad (8)$$

3.4. Determination of Attribute Weights. First, the public evaluation information of the attribute importance is processed. Public individuals give their evaluation values of the importance of different attributes in the form of linguistic or uncertain linguistic variables. Some individuals can express their opinions more accurately, so they choose to evaluate them in linguistic variables. Others feel a certain degree of ambiguity or uncertainty over the evaluation results of their

TABLE 1: Equations to calculate G_{ij} and F_{ij} .

No.	The relationship between p_{ij} and q_j	The loss F_{ij}	The profit G_{ij}
1	$p_{ij}^U < q_j^L$	$0.5(p_{ij}^L + p_{ij}^U) - q_j^L$	0
2	$q_j^U < p_{ij}^L$	0	$0.5(p_{ij}^L + p_{ij}^U) - q_j^U$
3	$p_{ij}^L < q_j^L \leq p_{ij}^U < q_j^U$	$0.5(p_{ij}^L - q_j^L)$	0
4	$q_j^L < p_{ij}^L \leq q_j^U < p_{ij}^U$	0	$0.5(p_{ij}^U - q_j^U)$
5	$p_{ij}^L < q_j^L < q_j^U < p_{ij}^U$	$0.5(p_{ij}^L - q_j^L)$	$0.5(p_{ij}^U - q_j^U)$
6	$q_j^L \leq p_{ij}^L < p_{ij}^U \leq q_j^U$	0	0

given attribute importance, so they choose to express their opinions using uncertain linguistic variables. According to the expression characteristics of the public, Definitions 1 and 2 can be used to convert the public evaluation values from linguistic or uncertain linguistic variables into interval numbers. If the number of individuals participating in the importance evaluation of attribute j is ζ_j , the importance evaluation value given by individual k over attribute j is ε_j^k , $k = 1, 2, \dots, \zeta_j$. Based on Definitions 1 and 2, ε_j^k can be converted into the interval type $b_j^k = [b_j^{kL}, b_j^{kU}]$, $k = 1, 2, \dots, \zeta_j$. We take the average value of the importance evaluation of the public on attribute j as the comprehensive importance evaluation value of attribute j (i.e., $b_j = [b_j^L, b_j^U]$, where $b_j^L = (1/\zeta_j) \sum_{k=1}^{\zeta_j} b_j^{kL}$, $b_j^U = (1/\zeta_j) \sum_{k=1}^{\zeta_j} b_j^{kU}$, $0 \leq b_j^L \leq b_j^U \leq 1$).

Second, the value range of the attribute weights ω_j is determined. The value range of the attribute weights ω_j is assumed to be $\omega_j \in [\omega_j^L, \omega_j^U]$. Based on the public comprehensive evaluation value of the importance over attribute j , the upper and lower limits of ω_j are determined:

$$\begin{cases} \omega_j^L = \frac{b_j^L}{b_j^L + \sum_{e=1, e \neq j}^{N-1} b_e^U}, \\ \omega_j^U = \frac{b_j^U}{b_j^U + \sum_{e=1, e \neq j}^{N-1} b_e^L}. \end{cases} \quad (9)$$

Theorem 1. For $\omega_j \in [\omega_j^L, \omega_j^U]$, ω_j must exist that meets the constraints of $0 \leq \omega_j \leq 1$ and $\sum_{j=1}^N \omega_j = 1$.

Proof. Because $0 \leq b_j^L \leq (b_j^L + \sum_{e=1, e \neq j}^{N-1} b_e^U)$ and $0 \leq b_j^U \leq (b_j^U + \sum_{e=1, e \neq j}^{N-1} b_e^L)$, $0 \leq (b_j^L / (b_j^L + \sum_{e=1, e \neq j}^{N-1} b_e^U)) \leq 1$ and $0 \leq (b_j^U / (b_j^U + \sum_{e=1, e \neq j}^{N-1} b_e^L)) \leq 1$. Thus, it can be deduced that $0 \leq \omega_j \leq 1$. As $0 \leq b_j^L \leq b_j^U \leq 1$ and $\omega_j^L = (b_j^L / (b_j^L + \sum_{e=1, e \neq j}^{N-1} b_e^U)) \leq (b_j^L / \sum_{j=1}^N b_j^L)$, and it can be deduced that $(\sum_{j=1}^N \omega_j^L = \sum_{j=1}^N [b_j^L / (b_j^L + \sum_{e=1, e \neq j}^{N-1} b_e^U)]) \leq \sum_{j=1}^N (b_j^L / \sum_{j=1}^N b_j^L) = 1$. Similarly, it is deduced that $\sum_{j=1}^N \omega_j^U \geq 1$. As the value of ω_j is continuous within the range $[\omega_j^L, \omega_j^U]$ and $0 \leq \omega_j^L \leq \omega_j^U \leq 1$, so $\sum_{j=1}^N \omega_j = 1$ must exist.

Third, we determine the attribute weights. An optimization model is constructed to solve the attribute weights to maximize the dispersion of attributes on all alternatives:

$$\begin{aligned} \max \psi(\omega_j) &= \frac{1}{M} \sum_{i=1}^M \sum_{j=1}^N \left(\frac{|p_{ij}^L - \tilde{p}_j^L| + |p_{ij}^U - \tilde{p}_j^U|}{2} \right) \omega_j \\ \text{s.t.} \quad &\begin{cases} \omega_j^L \leq \omega_j \leq \omega_j^U \\ \sum_{j=1}^N \omega_j = 1 \\ \tilde{p}_j = [\tilde{p}_j^L, \tilde{p}_j^U] = \left[\frac{1}{M} \sum_{i=1}^M p_{ij}^L, \frac{1}{M} \sum_{i=1}^M p_{ij}^U \right]. \end{cases} \end{aligned} \quad (10)$$

□

Theorem 2. *Model (10) must have an optimal solution.*

Proof. Under the constraint $\omega_j \in [\omega_j^L, \omega_j^U]$, there must be a reasonable value of ω_j that satisfies $\sum_{j=1}^N \omega_j = 1$, so the feasible domain of the attribute weight is a nonempty set. In addition, $0 \leq p_{ij} \leq 1$ and $0 \leq \tilde{p}_j \leq 1$, so it is easy to deduce that $0 \leq ((|p_{ij}^L - \tilde{p}_j^L| + |p_{ij}^U - \tilde{p}_j^U|)/2) \leq 1$. As known, $0 \leq \omega_j \leq 1$, $\sum_{j=1}^N \omega_j = 1$; thus, it can be deduced that $0 \leq \sum_{j=1}^N ((|p_{ij}^L - \tilde{p}_j^L| + |p_{ij}^U - \tilde{p}_j^U|)/2) \omega_j \leq 1$ and $0 \leq \psi(\omega_j) = (1/M) \sum_{i=1}^M \sum_{j=1}^N ((|p_{ij}^L - \tilde{p}_j^L| + |p_{ij}^U - \tilde{p}_j^U|)/2) \omega_j \leq 1$. As $\psi(\omega_j)$ is a bounded continuous function, the constraint condition of the attribute weight is a bounded closed set, so Model (10) must have an optimal solution.

To sum up, the specific steps of the proposed decision-making approach are as follows:

Step 1. The public expected values of attributes with various expressions are converted into crisp numbers.

Step 2. The distribution of the public expected values is determined on all attributes according to the public expected opinion in the crisp number type. Next, the attribute reference points are obtained based on the distribution mean of the public expectation and comprehensive ambiguity of public expected values.

Step 3. The attribute value and reference points are normalized. The attribute prospect value of each alternative is calculated based on the prospect theory.

Step 4. The value range of the attribute weights is determined using equation (9) and Model (10) to determine the attribute weight.

Step 5. The comprehensive prospect values of different alternatives are obtained using equation (11) to realize the ranking of alternative alternatives:

$$V_i = \sum_{j=1}^N \omega_j v_{ij}. \quad (11)$$

4. Case Analysis

We take a subway construction project as an example to verify the rationality and effectiveness of the method proposed in this paper. A provincial capital city plans to extend the No. 2 subway line to the west, for which three alternatives can be selected. The extension of the subway line will make public transportation more convenient for residents along the line. However, the subway construction will take a long time to be completed, occupy a large amount of public space, and generate substantial dust, which interferes with the daily life of the surrounding residents. According to the construction requirements and characteristics of the subway line, the organizers of the decision-making activity selected four attributes to evaluate the alternatives: the average distance between the stations and densely populated areas along the line (g_1 , units: m, cost-based index), estimated

construction time (g_2 , units: month, cost-based index), enclosed public area for construction (g_3 , units: m^2 , cost-based index), and dust and sand treatment effect (g_4 , qualitative index, profit-based index). Among these alternatives, g_1 is expressed in crisp numbers, g_2 and g_3 are expressed in interval numbers, and g_4 is expressed in linguistic or uncertain linguistic variables. The conversion standard between the dust and sand treatment effect and linguistic variables is presented in Table 2, and the attribute values of different alternatives are listed in Table 3.

The primary public group affected by the construction and operation of the subway is urban residents. Therefore, during the decision-making process, the opinions of the public group directly affected by the subway must be fully considered. Various media-driven methods were used to publicize the project to enable the public to understand the actual subway project situation better. The public could express their opinions on the subway project using different methods, such as online platforms, telephone, and mail questionnaires. The public provides two pieces of evaluation information based on their opinions of the attributes: their expectations and the importance evaluation value. The conversion standard between the evaluation values of the importance and linguistic variables is given in Table 2.

When the public opinion survey was finished, the organizers of the public opinion survey identified and counted the public individuals who effectively participated. The statistical results of the public expectations are listed in Table 4, and the statistical results of the public importance evaluation values of different attributes are presented in Table 5. Numerous individuals participated in the survey effectively. Due to the space limitations of the article, we only present the partial statistical results of the public opinions in Tables 4 and 5.

The original public expected values of the attributes in Table 4 were processed. First, the public expected values of the attributes were converted into interval numbers. Next, the comprehensive ambiguity of the expected values of the attributes was calculated which was shown in Table 4. Then, the public expected values of the attributes in interval numbers were converted into crisp numbers in Definition 3; the details of the public expected values for different attributes in the form of crisp numbers are shown in Table 6.

Finally, based on the relevant content in Table 6, the distribution of public expected values for various attributes was examined. The distribution of public expected values on various attributes was examined. The fittings of the distributions are illustrated in Figures 1–4. According to the statistical distribution results and comprehensive ambiguity of the public expected values, the reference point of each attribute was determined, as presented in Table 7.

Equations (3)–(6) were used to normalize the attribute reference points and attribute values of different alternatives. The normalized attribute reference points and alternative attribute values are listed in Table 8.

Next, equations (7) and (8) were used to calculate the prospect values of the attributes, and the prospect profit-loss matrix of the attributes is expressed:

TABLE 2: Conversion standards between dust and sand treatment (attribute importance) and linguistic variables.

Dust and sand treatment effect (attribute importance)	Extremely poor (can be ignored)	Terribly poor (extremely unimportant)	Very poor (very unimportant)	Poor (unimportant)	Fair (good)
Linguistic variable	s_0	s_1	s_2	s_3	s_4
Dust and sand treatment effect (attribute importance)	Good (important)	Very good (very important)	Extremely good (extremely important)	Perfect (maximum importance)	—
Linguistic variable	s_5	s_6	s_7	s_8	—

TABLE 3: Attribute values of various alternatives.

Alternatives	Average distance between the station and the population gathering area along the line (m)	Estimated construction time of the project (months)	Construction enclosed public areas (m ²)	Dust and sand treatment effect (qualitative index)
1	330	[64, 71]	[28732, 29849]	[S_6 , S_7]
2	379	[61, 65]	[25373, 27711]	S_7
3	336	[64, 68]	[28064, 29849]	[S_6 , S_7]

TABLE 4: Statistics of public expected values for different attributes.

Average distance between the station and the population gathering area along the line (m)			Estimated construction time of the project (months)			Construction-enclosed public areas (m ²)			Dust and sand treatment effect (qualitative index)		
Number of effective participants			Number of effective participants			Number of effective participants			Number of effective participants		
5472			6070			5518			5491		
No.	Expected values	Ambiguity	No.	Expected value	Ambiguity	No.	Expected value	Ambiguity	No.	Expected value	Ambiguity
1	[453, 499]	23	1	[55, 58]	1.5	1	[30528, 30789]	130.5	1	[S_4 , S_5]	0.0625
2	[377, 377]	0	2	[64, 66]	1	2	[26212, 29284]	1536	2	[S_5 , S_5]	0
...
5471	[346, 352]	3	6069	[67, 74]	3.5	5517	[30049, 33846]	1898.5	5490	[S_5 , S_6]	0.0625
5472	[338, 374]	18	6070	[70, 78]	4	5518	[28175, 31285]	1555	5491	[S_6 , S_6]	0

TABLE 5: Public evaluation results of the importance for different attributes.

Average distance between the station and the population gathering area along the line (m)		Estimated construction time of the project (months)		Construction-enclosed public areas (m ²)		Dust and sand treatment effect (qualitative index)	
Number of effective participants		Number of effective participants		Number of effective participants		Number of effective participants	
4186		4182		4186		4185	
No.	Importance	No.	Importance	No.	Importance	No.	Importance
1	[S_3 , S_4]	1	[S_1 , S_2]	1	[S_3 , S_3]	1	[S_3 , S_5]
2	[S_3 , S_4]	2	[S_1 , S_2]	2	[S_3 , S_3]	2	[S_3 , S_5]
...
4185	[S_3 , S_6]	4181	[S_2 , S_2]	4185	[S_1 , S_4]	4184	[S_4 , S_5]
4186	[S_3 , S_6]	4182	[S_0 , S_1]	4186	[S_1 , S_4]	4185	[S_4 , S_5]

TABLE 6: The public expected values for different attributes in the form of crisp numbers.

Average distance between the station and the population gathering area along the line (m)		Estimated construction time of the project (months)		Construction-enclosed public areas (m ²)		Dust and sand treatment effect (qualitative index)	
No.	Expected value	No.	Expected value	No.	Expected value	No.	Expected value
1	476	1	56	1	30658.5	1	0.5625
2	377	2	65	2	27748	2	0.5000
...
5471	349	6069	70.5	5517	31947.5	5490	0.6875
5472	356	6070	74	5518	29730	5491	0.7500

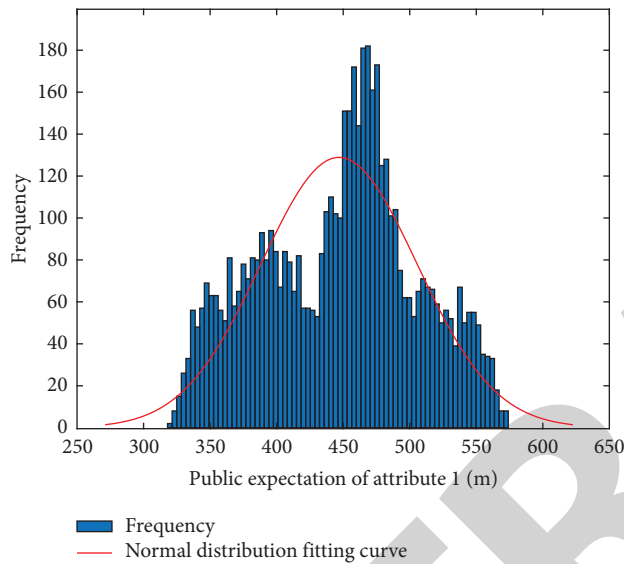


FIGURE 1: Fitting curve for public expected values of attribute 1.

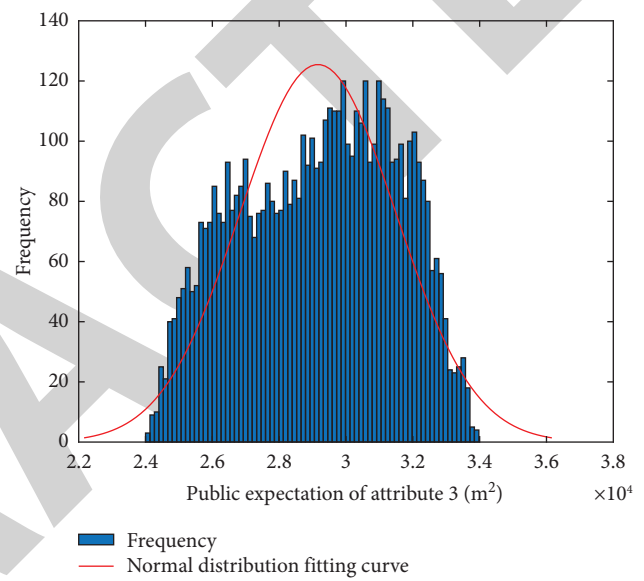


FIGURE 3: Fitting curve for public expected values of attribute 3.

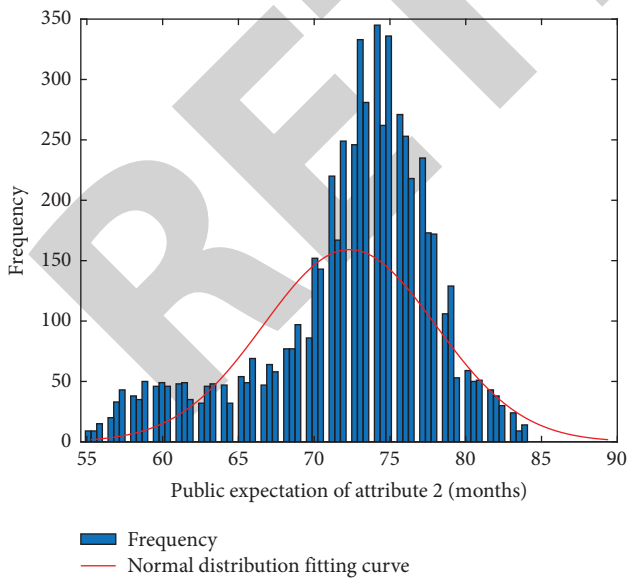


FIGURE 2: Fitting curve for public expected values of attribute 2.

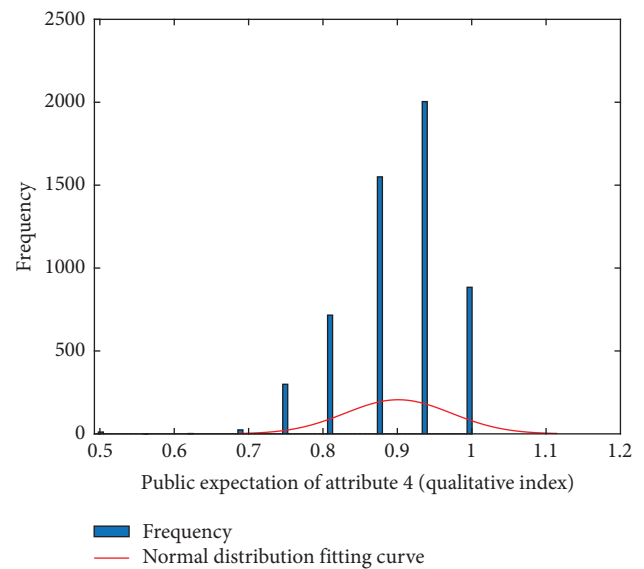


FIGURE 4: Fitting curve for public expected values of attribute 4.

TABLE 7: Results of public expectation reference points for different attributes.

Attributes	Average distance between the station and the population gathering area along the line (m)	Estimated construction time of the project (months)	Construction-enclosed public areas (m ²)	Dust and sand treatment effect (qualitative index)
Statistical distribution	Normal distribution $N(446.0, 74.41^2)$	Normal distribution $N(69.49, 8.48^2)$	Normal distribution $N(28990, 2899^2)$	Normal distribution $N(0.75, 0.15^2)$
Comprehensive ambiguity of the expectations	12.34	1.99	1006.21	0.06
Attribute reference point	[433.67, 458.34]	[67.50, 71.48]	[27983.79, 29996.21]	[0.69, 0.81]

TABLE 8: Normalized attribute reference points and attribute values for different alternatives.

Alternative	Average distance between the station and the population gathering area along the line (m)	Estimated construction time of the project (months)	Construction-enclosed public areas (m ²)	Dust and sand treatment effect (qualitative index)
1	[1.0000, 1.0000]	[0.0458, 0.7137]	[0.0318, 0.2734]	[0.3243, 1.0000]
2	[0.6182, 0.6182]	[0.6183, 1.0000]	[0.4943, 1.0000]	[1.0000, 1.0000]
3	[0.9532, 0.9532]	[0.3321, 0.7137]	[0.0318, 0.4179]	[0.3243, 1.0000]
Attribute reference points	[0.0000, 0.1922]	[0.0000, 0.3798]	[0.0000, 0.4353]	[0.0000, 0.6486]

$$VM = \begin{bmatrix} 0.8287 & 0.2070 & 0.0000 & 0.2164 \\ 0.4719 & 0.4752 & 0.3587 & 0.3983 \\ 0.7864 & 0.2070 & 0.0000 & 0.2164 \end{bmatrix}. \quad (12)$$

The value range of the attribute weights can be determined using equation (9) based on the conversion between linguistic variables and interval numbers: $\omega_1 \in [0.24, 0.55]$; $\omega_2 \in [0.07, 0.26]$; $\omega_3 \in [0.10, 0.34]$; $\omega_4 \in [0.17, 0.44]$. Attribute weights can be determined using Model (10), $\omega_1 = 0.42$; $\omega_2 = 0.07$; $\omega_3 = 0.34$; $\omega_4 = 0.17$.

Based on the attribute prospect value and attribute weights, the comprehensive prospect value of each alternative can be obtained using equation (11), where $V_1 = 0.3994$, $V_2 = 0.4211$, and $V_3 = 0.3816$. The ranking of three alternatives is $V_2 > V_1 > V_3$. Therefore, V_2 is the optimal alternative.

5. Comparison of Methods and Sensitivity Analysis

5.1. Comparison of Methods. To verify the effectiveness of the proposed approach, we introduce two existing decision-making methods. The first method is the double-reference point decision-making method based on the prospect theory [42]. The second method is the TOPSIS method [43]. In the two mentioned methods, positive and negative ideal points are set as reference points used as the basis of the alternative evaluation. The positive and negative reference points of attribute j are defined as \tilde{r}_j^{*+} and \tilde{r}_j^{*-} , respectively. Based on the prospect theory and TOPSIS method, the comparison between the attribute value of the alternative and the corresponding reference points is analyzed, and the evaluation value of each alternative is obtained. In the double-reference point decision-making method based on the prospect

theory, the maximum and minimum values of public expectation are taken as positive and negative reference points.

Public expectations obey the normal distribution; therefore, the maximum and minimum values of public expectations can be obtained using the three-sigma (3σ) theorem of normal distribution [44]. For attribute j , the maximum and minimum values of public expectations are obtained using the 3σ theorem of normal distribution, denoted by $\mu(\tilde{r}_j^k) + 3\sigma(\tilde{r}_j^k)$ and $\mu(\tilde{r}_j^k) - 3\sigma(\tilde{r}_j^k)$, respectively. According to Definitions 1 and 2, the conversion values of linguistic variables or uncertain linguistic variables are in the range $[0, 1]$. In the conversion crisp values' distribution of linguistic variables or uncertain linguistic variables, in order to ensure the maximum and minimum values, we meet the range requirements in this type of distribution; we define the maximum and minimum values as follows: if $\mu(\tilde{r}_j^k) + 3\sigma(\tilde{r}_j^k) > 1$, we set maximum value as 1; if $\mu(\tilde{r}_j^k) - 3\sigma(\tilde{r}_j^k) < 0$, we set minimum value as 0. For the other scenarios, the maximum and minimum values are set as $\mu(\tilde{r}_j^k) + 3\sigma(\tilde{r}_j^k)$ and $\mu(\tilde{r}_j^k) - 3\sigma(\tilde{r}_j^k)$. To satisfy the calculation need of prospect value, we set the positive and negative reference points of the attribute as interval numbers with equal upper and lower limits, expressed as $\tilde{r}_j^{*+} = [(\mu(\tilde{r}_j^k) + 3\sigma(\tilde{r}_j^k)), (\mu(\tilde{r}_j^k) + 3\sigma(\tilde{r}_j^k))]$ and $\tilde{r}_j^{*-} = [(\mu(\tilde{r}_j^k) - 3\sigma(\tilde{r}_j^k)), (\mu(\tilde{r}_j^k) - 3\sigma(\tilde{r}_j^k))]$. In the TOPSIS method, we set the positive and negative ideal points according to the maximum and minimum values of each attribute in all alternatives. For example, the positive and negative reference points of attribute j are, respectively, expressed as $\tilde{r}_j^{*+} = [\max_{i=1,2,\dots,M} y_j^{iU}, \max_{i=1,2,\dots,M} y_j^{iU}]$ and $\tilde{r}_j^{*-} = [\min_{i=1,2,\dots,M} y_j^{iL}, \min_{i=1,2,\dots,M} y_j^{iL}]$. The alternative ranking results of different decision methods are listed in Table 9.

According to Table 9, the alternative ranking results corresponding to the existing methods proposed are

TABLE 9: Alternative ranking results of different decision methods.

Methods	Attribute reference points	Decision results	Ranking of the alternatives
Double-reference point decision-making method based on prospect theory	$\bar{r}_1^{**} = [222.77, 222.77], \bar{r}_2^{**} = [44.05, 44.05],$ $\bar{r}_3^{**} = [20293.00, 20293.00], \bar{r}_4^{**} = [1.00, 1.00];$ $\bar{r}_1^{*-} = [669.23, 669.23], \bar{r}_2^{*-} = [94.93, 94.93],$ $\bar{r}_3^{*-} = [37687.00, 37687.00], \bar{r}_4^{*-} = [0.00, 0.00]$	$V_1 = -0.1738,$ $V_2 = -0.0983,$ $V_3 = -0.1654$	$V_2 > V_3 > V_1$
TOPSIS	$\bar{r}_1^{**} = [330.00, 330.00], \bar{r}_2^{**} = [61.00, 61.00],$ $\bar{r}_3^{**} = [25373.00, 25373.00], \bar{r}_4^{**} = [0.875, 0.875];$ $\bar{r}_1^{*-} = [379.00, 379.00], \bar{r}_2^{*-} = [71.00, 71.00],$ $\bar{r}_3^{*-} = [29849.00, 29849.00], \bar{r}_4^{*-} = [0.75, 0.75]$	$V_1 = 0.5719,$ $V_2 = 0.4772,$ $V_3 = 0.5564$	$V_1 > V_3 > V_2$
The proposed method	$\bar{r}_1^* = [433.67, 458.34], \bar{r}_2^* = [67.50, 71.48],$ $\bar{r}_3^* = [27983.79, 29996.21], \bar{r}_4^* = [0.69, 0.81]$	$V_1 = 0.3994,$ $V_2 = 0.4211,$ $V_3 = 0.3816$	$V_2 > V_1 > V_3$

inconsistent with the results obtained in this study. The main reason for the inconsistency of decision results is that the reference point setting in each decision method is different.

The setting principle of reference points for the two existing methods is that reference points are obtained according to the public expectation or the attribute value of the alternative. The setting principle for the reference points used in the two existing methods is relatively simple, but it does not consider the characteristics of public expectations. For decision-making activities in major public affairs, public expectations must be fully considered to ensure the effectiveness of the decision-making results. Therefore, public expectations must be considered when setting reference points.

Based on the above analysis, we develop the setting principle for reference points by considering public expectations. First, the information on public expectations is discrete and has the characteristics of a normal distribution; thus, the mean value of public expectations is taken as the standard setting for the reference point so that the reference point can effectively reflect the expectations of the public group. Second, due to the uncertainty of the expectation evaluation by public individuals, the expected value is usually expressed in an interval number, which reflects the ambiguity of public expectations. The comprehensive expected ambiguity of the public is used to represent the uncertain characteristics of public expectations. The reference point is set by the mean value of public expectations and the comprehensive expected ambiguity of the public to better reflect the actual situation of public expectation expression. The results of the reference points are presented as interval numbers. The value of the reference points reflects the expectations of the public group and considers the uncertainty of the public evaluation.

If an attribute value in interval form falls within the range of the corresponding reference points in whole or in part, we consider that part of the attribute value that falls within the range of the reference points exactly meets public expectations, which means that the prospect value for this part of the attribute is zero. Therefore, when the prospect value is calculated based on the interval reference

point, the inclusion or cross relationship between the attribute interval and corresponding reference point interval should be considered. For example, Table 8 indicates that the normalized values of attribute 3 for alternatives 1 and 3 are $p_{13} = [0.0318, 0.2734]$ and $p_{33} = [0.0318, 0.4179]$, respectively. Moreover, p_{13} and p_{33} are not equal, and the normalized reference point for attribute 3 is $q_3 = [0.0000, 0.4353]$. It is apparent that p_{13} and p_{33} are entirely within the interval range of q_3 . We can affirm that the prospect values of attribute 3 for alternatives 1 and 3 are both zero ($v_{13} = v_{33} = 0$). In addition, we use equations (7) and (8) and Table 8 to calculate the prospect values of p_{13} and p_{33} ; we can also obtain the same result. If we choose the double-reference point decision-making method to calculate the prospect value of p_{13} and p_{33} , the results of the prospect value for p_{13} and p_{33} are $v_{13} = -0.7329$ and $v_{33} = -0.6732$, which is different from the result of the proposed method. Therefore, a different setting principle for the reference point leads to different decision-making results, explaining why the sorting results are different in Table 9. Therefore, if we need to set reference points for decision-making, we must set them according to the actual situation for the decision problem and decision requirements to guarantee the effectiveness of the decision results.

5.2. Sensitivity Analysis of the Reference Point Interval Range.

According to the content above, the comprehensive ambiguity of public expectations is related to the interval range of attribute reference points, affecting the decision-making results. The interval adjustment coefficient of reference point θ ($0 \leq \theta \leq 1$) is introduced in the expression of reference points to study the relationship between the value range of the reference points and decision results further. The reference point of attribute j is defined as $\bar{r}_j^* = [\bar{r}_j^{*L}, \bar{r}_j^{*U}] = [\mu(\bar{r}_j^k) - \theta c_j, \mu(\bar{r}_j^k) + \theta c_j]$. The comprehensive ambiguity of public expectations for each attribute c_j is known. When the values of θ are 0, 0.25, 0.5, 0.75, and 1, the corresponding value range of reference points changes

TABLE 10: Decision results for different conditions of reference points.

θ	Reference points	Comprehensive prospect value	Alternatives ranking
0	$\bar{r}_1^* = [446.00, 446.00], \bar{r}_2^* = [69.49, 69.49],$ $\bar{r}_3^* = [28990.00, 28990.00], \bar{r}_4^* = [0.75, 0.75]$	$V_1 = 0.4365,$ $V_2 = 0.6769,$ $V_3 = 0.4699$	$V_2 > V_3 > V_1$
0.25	$\bar{r}_1^* = [442.92, 449.09], \bar{r}_2^* = [69.00, 70.00],$ $\bar{r}_3^* = [28738.45, 29241.55], \bar{r}_4^* = [0.74, 0.77]$	$V_1 = 0.4138,$ $V_2 = 0.6068,$ $V_3 = 0.4442$	$V_2 > V_3 > V_1$
0.50	$\bar{r}_1^* = [439.83, 452.17], \bar{r}_2^* = [68.51, 70.50],$ $\bar{r}_3^* = [28486.90, 29493.11], \bar{r}_4^* = [0.72, 0.78]$	$V_1 = 0.4104,$ $V_2 = 0.5425,$ $V_3 = 0.4231$	$V_2 > V_3 > V_1$
0.75	$\bar{r}_1^* = [436.75, 455.26], \bar{r}_2^* = [68.01, 70.99],$ $\bar{r}_3^* = [28235.34, 29744.66], \bar{r}_4^* = [0.71, 0.80]$	$V_1 = 0.4141,$ $V_2 = 0.4827,$ $V_3 = 0.4066$	$V_2 > V_1 > V_3$
1	$\bar{r}_1^* = [433.67, 458.34], \bar{r}_2^* = [67.50, 71.48],$ $\bar{r}_3^* = [27983.79, 29996.21], \bar{r}_4^* = [0.69, 0.81]$	$V_1 = 0.3994,$ $V_2 = 0.4211,$ $V_3 = 0.3816$	$V_2 > V_1 > V_3$

accordingly. The decision results for different conditions of reference points are presented in Table 10.

As listed in Table 10, when the interval adjustment coefficient of the reference points increases gradually, the interval range of the reference points for different attributes is also constantly enlarged, so the alternative ranking results are not the same. As observed in Table 10, if $\theta \in [0, 0.5]$, the sorting result is $V_2 > V_3 > V_1$. Moreover, if $\theta \in [0.75, 1]$, the sorting result is $V_2 > V_1 > V_3$. In addition, the alternative ranking results in Table 10 with the continuous expansion of the interval for reference points indicate that the comprehensive prospect value of each alternative also constantly decreases. In addition, the differentiation degree of the comprehensive prospect value between alternatives also gradually decreases. Therefore, to ensure the effectiveness of setting attribute reference points, we must consider the influence of the interval range of the reference points for the decision results to reduce the difficulty of decision-making and improve the accuracy of decision results.

6. Conclusion

This paper proposed a new decision-making approach for a mixed multiattribute decision-making problem with unknown attribute weights. The advantages of this approach are summarized as follows. First, the reference point for each attribute is set based on the distribution and comprehensive ambiguity of public expectations, making the attribute reference points better reflect the public's expected group opinions and expectation uncertainty. The effectiveness of the decision result is guaranteed. Second, in solving the attribute weights, the attribute importance given by the public is used to determine the value range of the attribute weights so that the weighting results are in accordance with public opinions, making the results of the attribute weight more acceptable. Then, the exact values of the attribute weights are determined to maximize the attribute information deviation of all the alternatives, improving the discrimination of alternatives. Third, each alternative is

evaluated based on the prospect theory to satisfy the calculation needs of the prospect value. The different expression forms of the attribute information and reference point are unified, and normalization is performed. The operation above can eliminate the influence of the expression form and dimension on decision-making, making the decision operations smoother.

The proposed approach also has certain limitations. First, in the actual decision-making process, the decision-making scenarios in some decision-making problems are not static. Changes in the decision-making scenarios easily cause decision risk, which negatively affects the decision activity. Our proposed approach does not consider dynamic decision scenarios, so the proposed approach must be further expanded and improved, making it suitable for public-participation decision-making problems under changing scenarios. Second, according to the setting principle of reference points, the interval range of the attribute reference points affects the decision-making results. The interval range of the attribute reference points is large, and the discrimination of the evaluation results for different alternatives is less obvious. Because the interval range of attribute reference points is set by the comprehensive ambiguity of public expectations, if the dispersion degree of public expectations is overly high (i.e., the comprehensive ambiguity of public expectations is overly high), the interval range of attribute reference points is also set over wide corresponding values. This outcome may lead to a lack of differentiation in the decision results. Therefore, to ensure the effectiveness of the decision-making results, reasonable interval ranges must be set for attribute reference points according to the characteristics of public expectation information, which is also worth an in-depth study.

Data Availability

The data used to support the findings of this study are included within the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

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Review Article

On Different Types of Single-Valued Neutrosophic Covering Rough Set with Application in Decision-Making

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This paper aims to propose the notion of Type-1 single-valued neutrosophic complementary β -neighborhood (briefly, Type-1 SVN complementary β -neighborhood) and use it to introduce a novel class of 1-single-valued neutrosophic β -covering rough set (briefly, 1-SVN β -CRS). Then, we will merge the 1-SVN β -neighborhood and 1-SVN complementary β -neighborhood to create new two models of 1-SVN β -CRS. Furthermore, we will discuss the relationships between the present work and Wang and Zhang's work. For further study on Type-2 Wang and Zhang's models, we will define the 2-SVN complementary β -neighborhood and use it to present a novel class of 2-SVN β -CRS. Also, we combine the 2-SVN β -neighborhood and 2-SVN complementary β -neighborhood to investigate the new two models of 2-SVN β -CRS. Lately, we will demonstrate two illustrative examples as real problems to show the differences between two of our approaches and Wang and Zhang's approach.

1. Introduction

In 1982, the world-known new notion called rough sets (briefly, RSs) dealt with uncertain data on the hand of Pawlak [1, 2]. This notion helped researchers in several areas of research to develop these areas through RS, for instance, there are many published papers (see [3–19]). The famed generalization of RS is covering rough sets (briefly, CRSs). The CRSs were studied by many specialists from different views which made an evolution in many fields such as computer science, mathematics, and chemistry. Some of the relevant studies helped scholars to solve many life problems [20–33]. Consequently, in 1990, the notions of fuzzy rough sets (briefly, FRs) and rough fuzzy sets (briefly, RFSs) are defined by Dubois and Prade [34] from the merging between the CRS and the fuzzy sets (briefly, FSs) which appeared by Zadeh [35]. From this point of view, the new kinds of covering fuzzy rough sets (briefly, CFRs) through the fuzzy β -neighborhoods were called fuzzy β covering rough sets (briefly, β CRSs) (see [36]). To complete this study, Yang

et al. [37, 38] defined several basic notions of fuzzy complementary β -neighborhoods, fuzzy β -minimal description, and fuzzy β -maximal description to establish new classes of β CRSs.

The notion of single-valued neutrosophic sets (briefly, SVNS) was developed by Wang et al. [39]. SVNS is a natural extension of the intuitionistic fuzzy set (briefly, IFS) [40]. Smarandache [41] investigated a new set called neutrosophic set as a generalization of mathematical tools (i.e., fuzzy set [35], interval-valued fuzzy set [42], IFS [40], and interval-valued intuitionistic fuzzy set [43]). In 2015, Mondal and Pramanik [44] demonstrated a new terminology called rough neutrosophic set. By using SVN relation, Yang et al. [45] introduced the SVN rough set model, and based on the notion of Type-1 SVN β -neighborhoods, Wang and Zhang [46] proposed two models of Type-1 SVN β -covering rough sets (briefly, SVN β -CRS). Furthermore, they presented a new kind of SVN β -CRS called Type-2 SVN β -CRS utilizing Type-2 SVN β -neighborhoods in [47]. The notions of

neutrosophic soft rough sets and its generalizations are presented in [48–53].

By the above discussion and extend the other work (see [46, 47]) in SVN β -CRS. We will generalize these methods in [46, 47] by boosting the lower approximation and minimizing the upper approximation, which is a big challenge to every author. Consequently, the motivation of this paper is to improve this area is obtained by introducing the notion of 1-SVN complementary β -neighborhood (resp., 2-SVN complementary β -neighborhood) to build novel classes of 1-SVN β -CRS (resp., 2-SVN β -CRS). And, by joining 1-SVN β -neighborhoods (resp., 2-SVN β -neighborhoods) and 1-SVN complementary β -neighborhood (resp., 2-SVN complementary β -neighborhood), we obtain two new SVN β -neighborhoods which establish two new models of 1-SVN β -CRS (resp., 2-SVN β -CRS). Also, we discuss the properties of the two proposed covering methods. Finally, we apply our work (i.e., two proposed methods) to solve decision-making problems.

The organization of this article is as follows. In Section 2, we give a basic notion about the presented study. Section 3 establishes the definition of 1-SVN complementary β -neighborhood, and hence, a new model of 1-SVN β -CRS is proposed. Also, by merging between the 1-SVN β -neighborhoods and its complementary, we set up two other models of 1-SVN β -CRS. Thus, the relevant characteristics are also studied. Section 4 constructs the notion of 2-SVN complementary β -neighborhood, and thus, a new model of 2-SVN β -CRS is proposed. By merging between the 2-SVN β -neighborhoods and its complementary, we also set up two other models of 2-SVN β -CRS. Then, the relevant properties are also discussed. The decision-making approaches to the two methods are mentioned in Sections 3 and 4 are investigated in Section 5. Also, in this section, we compare between our approach and Wang and Zhang's approach. Section 6 shows the overall benefits of our study.

2. Preliminaries

In this section, we review some basic terminologies about the subject of this study.

Definition 1 (Cf. [26]). Assume that Ω is a universe and $\tilde{\Gamma}$ is a family of subsets of Ω . If no element in $\tilde{\Gamma}$ is empty and $\Omega = \cup_{\tilde{C} \in \tilde{\Gamma}} \tilde{C}$, then $\tilde{\Gamma}$ is called a covering of Ω , and $(\Omega, \tilde{\Gamma})$ is called a covering approximation space (briefly, CAS).

Definition 2 (Cf. [54, 55]). Assume that Ω is a universe. We say $\tilde{\Gamma} = \{\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_m\}$, with $\tilde{C}_i \in \mathcal{F}(\Omega)$ ($i = 1, 2, \dots, m$), a fuzzy covering (briefly, FC) of Ω if $(\cup_{i=1}^m \tilde{C}_i)(x) = 1$, for each $x \in \Omega$.

The notion of fuzzy β -covering was discovered by Ma [36] ($0 < \beta \leq 1$). This notion is considered as a generalization

of FC. If $\tilde{\Gamma} = \{\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_m\}$, with $\tilde{C}_i \in \mathcal{F}(\Omega)$ ($i = 1, 2, \dots, m$), then $(\Omega, \tilde{\Gamma})$ is called a fuzzy β -covering approximation space (briefly, F β CAS).

Definition 3 (Cf. [54, 55]). Assume that Ω is not an empty set. For each $x \in \Omega$, define the SVN set $\mathcal{A} \subseteq \Omega$ as the following formula:

$$\mathcal{A} = \{\langle x, \mathcal{T}_{\mathcal{A}}(x), \mathcal{I}_{\mathcal{A}}(x), \mathcal{F}_{\mathcal{A}}(x) \rangle\}. \quad (1)$$

where $\mathcal{T}_{\mathcal{A}}: \Omega \rightarrow [0, 1]$ is the degree of truth membership of the element x to \mathcal{A} , $\mathcal{I}_{\mathcal{A}}: \Omega \rightarrow [0, 1]$ is the degree of indeterminacy membership of the element x to \mathcal{A} , and $\mathcal{F}_{\mathcal{A}}: \Omega \rightarrow [0, 1]$ is the degree of falsity membership. These variables satisfy $0 \leq \mathcal{T}_{\mathcal{A}} + \mathcal{I}_{\mathcal{A}} + \mathcal{F}_{\mathcal{A}} \leq 3$.

In 2018, Wang et al. [39] established the notion of SVN β -covering approximation space, and then, Wang and Zhang [46, 47] used this notion to create two types of the covering method as in the following definition.

Definition 4 (Cf. [46, 47]). Let Ω be a universe and SVN (Ω) be the SVN power set of Ω . For a SVN number $\beta = (a, b, c)$, we call $\tilde{\Gamma} = \{\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_m\}$, with $\tilde{C}_i \in \text{SVN}(\Omega)$ ($i = 1, 2, \dots, m$), a Type-1 SVN β -covering (a Type-2 SVN β -covering) of Ω if $\tilde{C}_i(x) \geq \beta$ ($\tilde{C}_i(x) \geq \beta$), for each $x \in \Omega$. Moreover, $(\Omega, \tilde{\Gamma})$ is called a Type-1 SVN β -covering approximation space (a Type-2 SVN β -covering approximation space) (briefly, 1-SVN β CAS (2-SVN β CAS)).

If $\mathcal{A} = \langle a_1, b_1, c_1 \rangle$ and $\mathcal{B} = \langle a_2, b_2, c_2 \rangle$ are two SVN numbers, then

- (i) $\mathcal{A} \leq \mathcal{B} \Leftrightarrow a_1 \leq a_2, b_1 \geq b_2, c_1 \geq c_2$
- (ii) $\mathcal{A} \geq \mathcal{B} \Leftrightarrow a_1 \geq a_2, b_1 \leq b_2, c_1 \leq c_2$
- (iii) $\mathcal{A} < \mathcal{B} \Leftrightarrow a_1 \leq a_2, b_1 \leq b_2, c_1 \geq c_2$
- (iv) $\mathcal{A} \geq \mathcal{B} \Leftrightarrow a_1 \geq a_2, b_1 \geq b_2, c_1 \leq c_2$

Here, $\forall \mathcal{A}, \mathcal{B} \in \text{SVN}(\Omega)$, and we have the following relation, union, and intersection operations.

For Type-1,

- (1) $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \mathcal{T}_{\mathcal{A}} \leq \mathcal{T}_{\mathcal{B}}, \mathcal{I}_{\mathcal{A}} \geq \mathcal{I}_{\mathcal{B}}, \mathcal{F}_{\mathcal{A}} \leq \mathcal{F}_{\mathcal{B}} \forall x \in \Omega$
- (2) $\mathcal{A} \cap \mathcal{B} = \{\langle x, \mathcal{T}_{\mathcal{A}} \wedge \mathcal{T}_{\mathcal{B}}, \mathcal{I}_{\mathcal{A}} \vee \mathcal{I}_{\mathcal{B}}, \mathcal{F}_{\mathcal{A}} \vee \mathcal{F}_{\mathcal{B}} \rangle\}$
- (3) $\mathcal{A} \cup \mathcal{B} = \{\langle x, \mathcal{T}_{\mathcal{A}} \vee \mathcal{T}_{\mathcal{B}}, \mathcal{I}_{\mathcal{A}} \wedge \mathcal{I}_{\mathcal{B}}, \mathcal{F}_{\mathcal{A}} \wedge \mathcal{F}_{\mathcal{B}} \rangle\}$

For Type-2,

- (1) $\mathcal{A} \sqsubseteq \mathcal{B} \Leftrightarrow \mathcal{T}_{\mathcal{A}} \leq \mathcal{T}_{\mathcal{B}}, \mathcal{I}_{\mathcal{A}} \geq \mathcal{I}_{\mathcal{B}}, \mathcal{F}_{\mathcal{A}} \leq \mathcal{F}_{\mathcal{B}} \forall x \in \Omega$
- (2) $\mathcal{A} \sqcap \mathcal{B} = \{\langle x, \mathcal{T}_{\mathcal{A}} \wedge \mathcal{T}_{\mathcal{B}}, \mathcal{I}_{\mathcal{A}} \wedge \mathcal{I}_{\mathcal{B}}, \mathcal{F}_{\mathcal{A}} \vee \mathcal{F}_{\mathcal{B}} \rangle\}$
- (3) $\mathcal{A} \sqcup \mathcal{B} = \{\langle x, \mathcal{T}_{\mathcal{A}} \vee \mathcal{T}_{\mathcal{B}}, \mathcal{I}_{\mathcal{A}} \vee \mathcal{I}_{\mathcal{B}}, \mathcal{F}_{\mathcal{A}} \wedge \mathcal{F}_{\mathcal{B}} \rangle\}$

Definition 5 (Cf. [46, 47]). Let $(\Omega, \tilde{\Gamma})$ be a 1-SVN β CAS with $\tilde{\Gamma} = \{\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_m\}$ for some $\beta = \langle a, b, c \rangle$. Then, for each $x \in \Omega$, define the Type-1 SVN β -neighborhood (the Type-2 SVN β -neighborhood) of x as follows:

$$\begin{aligned}
{}_1\widetilde{\mathcal{N}}_x^\beta &= \cap \{ \widetilde{C}_i \in \widetilde{\Gamma}: \widetilde{C}_i(x) \geq \beta \} \\
&= \cap \{ \widetilde{C}_i \in \widetilde{\Gamma}: \mathcal{T}_{\widetilde{C}_i} \geq a, \mathcal{F}_{\widetilde{C}_i} \leq b, \mathcal{F}_{\widetilde{C}_i} \leq c \}, \\
{}_2\widetilde{\mathcal{N}}_x^\beta &= \cap \{ \widetilde{C}_i \in \widetilde{\Gamma}: \widetilde{C}_i(x) \geq \beta \} \\
&= \cap \{ \widetilde{C}_i \in \widetilde{\Gamma}: \mathcal{T}_{\widetilde{C}_i} \geq a, \mathcal{F}_{\widetilde{C}_i} \geq b, \mathcal{F}_{\widetilde{C}_i} \leq c \}.
\end{aligned} \tag{2}$$

Definition 6 (Cf. [46, 47]). Let $(\Omega, \widetilde{\Gamma})$ be a 1-SVN β CAS (resp., 2-SVN β CAS) for some $\beta = \langle a, b, c \rangle$. For each $x \in \Omega$ and $\mathcal{A} \in \text{SVN}(\Omega)$, define the first type of Type-1 SVN lower approximation (1-1-SVNLA) $\mathcal{L}_1^1(\mathcal{A})$ (resp., the first type of Type-2 SVN lower approximation (1-2-SVNLA) $\mathcal{L}_1^2(\mathcal{A})$) and the first type of Type-1 SVN upper approximation (1-1-SVNUA) $\mathcal{U}_1^1(\mathcal{A})$ (resp., the first type of Type-2 SVN upper approximation (1-2-SVNUA) $\mathcal{U}_1^2(\mathcal{A})$) as follows:

$$\begin{aligned}
\mathcal{L}_1^1(\mathcal{A}) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{{}_1\widetilde{\mathcal{N}}_x^\beta}(y) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{{}_1\widetilde{\mathcal{N}}_x^\beta}(y) \right) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{T}_{{}_1\widetilde{\mathcal{N}}_x^\beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}, \\
\mathcal{U}_1^1(\mathcal{A}) &= \left\{ \langle x, \vee_{y \in \Omega} \left(\mathcal{T}_{{}_1\widetilde{\mathcal{N}}_x^\beta}(y) \wedge \mathcal{T}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{{}_1\widetilde{\mathcal{N}}_x^\beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\mathcal{F}_{{}_1\widetilde{\mathcal{N}}_x^\beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}, \\
\mathcal{L}_1^2(\mathcal{A}) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{{}_2\widetilde{\mathcal{N}}_x^\beta}(y) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{{}_2\widetilde{\mathcal{N}}_x^\beta}(y) \right) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{T}_{{}_2\widetilde{\mathcal{N}}_x^\beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}, \\
\mathcal{U}_1^2(\mathcal{A}) &= \left\{ \langle x, \vee_{y \in \Omega} \left(\mathcal{T}_{{}_2\widetilde{\mathcal{N}}_x^\beta}(y) \wedge \mathcal{T}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{{}_2\widetilde{\mathcal{N}}_x^\beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\mathcal{F}_{{}_2\widetilde{\mathcal{N}}_x^\beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}.
\end{aligned} \tag{3}$$

If $\mathcal{L}_1^1(\mathcal{A})$ (resp., $\mathcal{L}_1^2(\mathcal{A})$) $\neq \mathcal{U}_1^1(\mathcal{A})$ (resp., $\mathcal{U}_1^2(\mathcal{A})$), then \mathcal{A} is called the first type of Type-1 SVN β -covering rough sets (resp., the first type of Type-2 SVN β -covering rough sets) (briefly, 1-1-SVN β CRSs (resp., 1-2-SVN β CRSs)).

3. Type-1 SVN Complementary β -Neighborhood and Three New Kinds of Type-1 SVN β -CRS

We will propose the concept of a type-1 SVN complementary β -neighborhood and three new kinds of Type-1 SVN β -CRS and introduce several definitions, propositions, and examples as indicated below.

Definition 7. Let $(\Omega, \widetilde{\Gamma})$ be a 1-SVN β CAS with $\widetilde{\Gamma} = \{\widetilde{C}_1, \widetilde{C}_2, \dots, \widetilde{C}_m\}$, for some $\beta = \langle a, b, c \rangle$. Then, for each $x \in \Omega$, define the type-1 SVN complementary β -neighborhood of x as follows:

$${}_1\widetilde{\mathcal{M}}_x^\beta(y) = {}_1\widetilde{\mathcal{N}}_y^\beta(x), \quad \forall y \in \Omega. \tag{4}$$

Example 1. Let $(\Omega, \widetilde{\Gamma})$ be a 1-SVN β CAS, $\Omega = \{x_1, x_2, x_3, x_4, x_5\}$ and $\widetilde{\Gamma} = \{\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4\}$, where $\beta = \langle 0.5, 0.3, 0.8 \rangle$ are summarized in Table 1.

In Table 1, ${}_1\widetilde{\mathcal{N}}_{x_1}^\beta = \widetilde{C}_1 \cap \widetilde{C}_2$, ${}_1\widetilde{\mathcal{N}}_{x_2}^\beta = \widetilde{C}_1 \cap \widetilde{C}_2 \cap \widetilde{C}_4$, ${}_1\widetilde{\mathcal{N}}_{x_3}^\beta = \widetilde{C}_3 \cap \widetilde{C}_4$, ${}_1\widetilde{\mathcal{N}}_{x_4}^\beta = \widetilde{C}_1 \cap \widetilde{C}_4$, and ${}_1\widetilde{\mathcal{N}}_{x_5}^\beta = \widetilde{C}_2 \cap \widetilde{C}_3 \cap \widetilde{C}_4$.

Table 2 contains the results of type-1 SVN β -neighborhood.

Thus, we can obtain the values of type-1 SVN complementary β -neighborhood as in Table 3.

Hence, we can merge ${}_1\widetilde{\mathcal{N}}_x^\beta$ and ${}_1\widetilde{\mathcal{M}}_x^\beta$ and compute ${}_1\widetilde{\mathcal{N}}_x^\beta \cap {}_1\widetilde{\mathcal{M}}_x^\beta$ as Table 4.

Also, we can compute ${}_1\widetilde{\mathcal{N}}_x^\beta \cup {}_1\widetilde{\mathcal{M}}_x^\beta$ as set in Table 5.

Proposition 1. Let $(\Omega, \widetilde{\Gamma})$ be a 1-SVN β CAS, for some $\beta = \langle a, b, c \rangle$ and for each $x, y, z \in \Omega$. Then, the following statements hold:

- (1) ${}_1\widetilde{\mathcal{M}}_x^\beta(x) \geq \beta$
- (2) If ${}_1\widetilde{\mathcal{M}}_x^\beta(y) \geq \beta$ and ${}_1\widetilde{\mathcal{M}}_y^\beta(z) \geq \beta$, then ${}_1\widetilde{\mathcal{M}}_x^\beta(z) \geq \beta$
- (3) $0 < \beta_1 \leq \beta_2 \leq \beta$, then ${}_1\widetilde{\mathcal{M}}_x^{\beta_1} \subseteq {}_1\widetilde{\mathcal{M}}_x^{\beta_2}$

Proof

- (1) It follows directly from Definitions 5 and 7.
- (2) Since ${}_1\widetilde{\mathcal{M}}_x^\beta(y) \geq \beta$, then ${}_1\widetilde{\mathcal{N}}_y^\beta(x) \geq \beta$. If $\widetilde{C}_i(x) \geq \beta$, then $\widetilde{C}_i(y) \geq \beta$, and since ${}_1\widetilde{\mathcal{M}}_y^\beta(z) \geq \beta$, then ${}_1\widetilde{\mathcal{N}}_z^\beta(y) \geq \beta$. If $\widetilde{C}_i(y) \geq \beta$, then $\widetilde{C}_i(z) \geq \beta$. Therefore, ${}_1\widetilde{\mathcal{M}}_x^\beta(z) \geq \beta$.
- (3) For each $x \in \Omega, 0 < \beta_1 \leq \beta_2 \leq \beta$, then $\cap \{ \widetilde{C}_i \in \widetilde{\Gamma}: \widetilde{C}_i(x) \geq \beta_1 \} \supseteq \cap \{ \widetilde{C}_i \in \widetilde{\Gamma}: \widetilde{C}_i(x) \geq \beta_2 \}$. Thus, by Definition 7, we have ${}_1\widetilde{\mathcal{M}}_x^{\beta_1} \subseteq {}_1\widetilde{\mathcal{M}}_x^{\beta_2}$. \square

Proposition 2. Let $(\Omega, \widetilde{\Gamma})$ be a 1-SVN β CAS, for some $\beta = \langle a, b, c \rangle$. For each $x, y \in \Omega$,

$${}_1\widetilde{\mathcal{M}}_x^\beta(y) \geq \beta \Leftrightarrow {}_1\widetilde{\mathcal{M}}_y^\beta(x) \geq \beta. \tag{5}$$

Proof. Let ${}_1\widetilde{\mathcal{M}}_x^\beta(y) \geq \beta$, $\mathcal{T}_{{}_1\widetilde{\mathcal{M}}_x^\beta}(y) = \mathcal{T} \cap \widetilde{C}_i(y) = \mathcal{F}_{\widetilde{C}_i}(y) \geq a$, $\mathcal{F}_{{}_1\widetilde{\mathcal{M}}_x^\beta}(y) = \mathcal{F} \cap \widetilde{C}_i(y) = \mathcal{V} \vee \mathcal{F}_{\widetilde{C}_i}(y) \leq b$, and $\mathcal{F}_{{}_1\widetilde{\mathcal{M}}_x^\beta}(y) = \mathcal{F} \cap \widetilde{C}_i(y) = \mathcal{V} \vee \mathcal{F}_{\widetilde{C}_i}(y) \leq c$. Then, $\{ \widetilde{C}_i \in \widetilde{\Gamma}: \mathcal{T}_{\widetilde{C}_i}(x) \geq a, \mathcal{F}_{\widetilde{C}_i}(x) \leq b, \mathcal{F}_{\widetilde{C}_i}(x) \leq c \} \subseteq \{ \widetilde{C}_i \in \widetilde{\Gamma}: \mathcal{T}_{\widetilde{C}_i}(y) \geq a, \mathcal{F}_{\widetilde{C}_i}(y) \leq b, \mathcal{F}_{\widetilde{C}_i}(y) \leq c \}$.

TABLE 1: $(\Omega, \tilde{\Gamma})$.

	\tilde{C}_1	\tilde{C}_2	\tilde{C}_3	\tilde{C}_4
x_1	$\langle 0.7, 0.2, 0.5 \rangle$	$\langle 0.6, 0.2, 0.4 \rangle$	$\langle 0.4, 0.1, 0.5 \rangle$	$\langle 0.1, 0.5, 0.6 \rangle$
x_2	$\langle 0.5, 0.3, 0.2 \rangle$	$\langle 0.5, 0.2, 0.8 \rangle$	$\langle 0.4, 0.5, 0.4 \rangle$	$\langle 0.6, 0.1, 0.7 \rangle$
x_3	$\langle 0.4, 0.5, 0.2 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.5, 0.2, 0.4 \rangle$	$\langle 0.6, 0.3, 0.4 \rangle$
x_4	$\langle 0.6, 0.1, 0.7 \rangle$	$\langle 0.4, 0.5, 0.7 \rangle$	$\langle 0.3, 0.6, 0.5 \rangle$	$\langle 0.5, 0.3, 0.2 \rangle$
x_5	$\langle 0.3, 0.2, 0.6 \rangle$	$\langle 0.7, 0.3, 0.5 \rangle$	$\langle 0.6, 0.3, 0.5 \rangle$	$\langle 0.8, 0.1, 0.2 \rangle$

TABLE 2: ${}_1\tilde{\mathcal{N}}_{x_i}^\beta, \forall i = 1, 2, \dots, 5$.

	x_1	x_2	x_3	x_4	x_5
${}_1\tilde{\mathcal{N}}_{x_1}^\beta$	$\langle 0.6, 0.2, 0.5 \rangle$	$\langle 0.5, 0.3, 0.8 \rangle$	$\langle 0.2, 0.5, 0.6 \rangle$	$\langle 0.4, 0.5, 0.7 \rangle$	$\langle 0.3, 0.3, 0.6 \rangle$
${}_1\tilde{\mathcal{N}}_{x_2}^\beta$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.5, 0.3, 0.8 \rangle$	$\langle 0.2, 0.5, 0.6 \rangle$	$\langle 0.4, 0.5, 0.7 \rangle$	$\langle 0.3, 0.3, 0.6 \rangle$
${}_1\tilde{\mathcal{N}}_{x_3}^\beta$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.4, 0.5, 0.7 \rangle$	$\langle 0.5, 0.3, 0.4 \rangle$	$\langle 0.3, 0.6, 0.5 \rangle$	$\langle 0.6, 0.3, 0.5 \rangle$
${}_1\tilde{\mathcal{N}}_{x_4}^\beta$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.5, 0.3, 0.7 \rangle$	$\langle 0.4, 0.5, 0.4 \rangle$	$\langle 0.5, 0.3, 0.7 \rangle$	$\langle 0.3, 0.2, 0.6 \rangle$
${}_1\tilde{\mathcal{N}}_{x_5}^\beta$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.4, 0.5, 0.8 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.3, 0.6, 0.7 \rangle$	$\langle 0.6, 0.3, 0.5 \rangle$

TABLE 3: ${}_1\tilde{\mathcal{M}}_{x_i}^\beta, \forall i = 1, 2, \dots, 5$.

	x_1	x_2	x_3	x_4	x_5
${}_1\tilde{\mathcal{M}}_{x_1}^\beta$	$\langle 0.5, 0.8, 0.6 \rangle$	$\langle 0.8, 0.7, 0.5 \rangle$	$\langle 0.6, 0.5, 0.2 \rangle$	$\langle 0.7, 0.5, 0.4 \rangle$	$\langle 0.6, 0.7, 0.3 \rangle$
${}_1\tilde{\mathcal{M}}_{x_2}^\beta$	$\langle 0.6, 0.5, 0.1 \rangle$	$\langle 0.8, 0.7, 0.5 \rangle$	$\langle 0.6, 0.5, 0.2 \rangle$	$\langle 0.7, 0.5, 0.4 \rangle$	$\langle 0.6, 0.7, 0.3 \rangle$
${}_1\tilde{\mathcal{M}}_{x_3}^\beta$	$\langle 0.6, 0.5, 0.1 \rangle$	$\langle 0.7, 0.5, 0.4 \rangle$	$\langle 0.4, 0.7, 0.5 \rangle$	$\langle 0.5, 0.4, 0.3 \rangle$	$\langle 0.5, 0.7, 0.6 \rangle$
${}_1\tilde{\mathcal{M}}_{x_4}^\beta$	$\langle 0.6, 0.5, 0.1 \rangle$	$\langle 0.7, 0.7, 0.5 \rangle$	$\langle 0.4, 0.5, 0.4 \rangle$	$\langle 0.7, 0.7, 0.5 \rangle$	$\langle 0.6, 0.8, 0.3 \rangle$
${}_1\tilde{\mathcal{M}}_{x_5}^\beta$	$\langle 0.6, 0.5, 0.1 \rangle$	$\langle 0.8, 0.5, 0.4 \rangle$	$\langle 0.6, 0.7, 0.2 \rangle$	$\langle 0.7, 0.4, 0.3 \rangle$	$\langle 0.5, 0.7, 0.6 \rangle$

TABLE 4: ${}_1\tilde{\mathcal{N}}_{x_i}^\beta \cap {}_1\tilde{\mathcal{M}}_{x_i}^\beta, \forall i = 1, 2, \dots, 5$.

	x_1	x_2	x_3	x_4	x_5
${}_1\tilde{\mathcal{N}}_{x_1}^\beta \cap {}_1\tilde{\mathcal{M}}_{x_1}^\beta$	$\langle 0.5, 0.8, 0.6 \rangle$	$\langle 0.5, 0.7, 0.8 \rangle$	$\langle 0.2, 0.5, 0.6 \rangle$	$\langle 0.4, 0.5, 0.7 \rangle$	$\langle 0.3, 0.7, 0.6 \rangle$
${}_1\tilde{\mathcal{N}}_{x_2}^\beta \cap {}_1\tilde{\mathcal{M}}_{x_2}^\beta$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.5, 0.7, 0.8 \rangle$	$\langle 0.2, 0.5, 0.6 \rangle$	$\langle 0.4, 0.5, 0.7 \rangle$	$\langle 0.3, 0.7, 0.6 \rangle$
${}_1\tilde{\mathcal{N}}_{x_3}^\beta \cap {}_1\tilde{\mathcal{M}}_{x_3}^\beta$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.4, 0.5, 0.7 \rangle$	$\langle 0.4, 0.7, 0.5 \rangle$	$\langle 0.3, 0.6, 0.5 \rangle$	$\langle 0.5, 0.7, 0.6 \rangle$
${}_1\tilde{\mathcal{N}}_{x_4}^\beta \cap {}_1\tilde{\mathcal{M}}_{x_4}^\beta$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.5, 0.7, 0.7 \rangle$	$\langle 0.4, 0.5, 0.4 \rangle$	$\langle 0.5, 0.7, 0.7 \rangle$	$\langle 0.3, 0.8, 0.6 \rangle$
${}_1\tilde{\mathcal{N}}_{x_5}^\beta \cap {}_1\tilde{\mathcal{M}}_{x_5}^\beta$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.4, 0.5, 0.8 \rangle$	$\langle 0.2, 0.7, 0.6 \rangle$	$\langle 0.3, 0.6, 0.7 \rangle$	$\langle 0.5, 0.7, 0.6 \rangle$

TABLE 5: ${}_1\tilde{\mathcal{N}}_{x_i}^\beta \cup {}_1\tilde{\mathcal{M}}_{x_i}^\beta, \forall i = 1, 2, \dots, 5$.

	x_1	x_2	x_3	x_4	x_5
${}_1\tilde{\mathcal{N}}_{x_1}^\beta \cup {}_1\tilde{\mathcal{M}}_{x_1}^\beta$	$\langle 0.6, 0.2, 0.5 \rangle$	$\langle 0.8, 0.3, 0.5 \rangle$	$\langle 0.6, 0.5, 0.2 \rangle$	$\langle 0.7, 0.5, 0.4 \rangle$	$\langle 0.6, 0.3, 0.3 \rangle$
${}_1\tilde{\mathcal{N}}_{x_2}^\beta \cup {}_1\tilde{\mathcal{M}}_{x_2}^\beta$	$\langle 0.6, 0.5, 0.1 \rangle$	$\langle 0.8, 0.3, 0.5 \rangle$	$\langle 0.6, 0.5, 0.2 \rangle$	$\langle 0.7, 0.5, 0.4 \rangle$	$\langle 0.6, 0.3, 0.3 \rangle$
${}_1\tilde{\mathcal{N}}_{x_3}^\beta \cup {}_1\tilde{\mathcal{M}}_{x_3}^\beta$	$\langle 0.6, 0.5, 0.1 \rangle$	$\langle 0.7, 0.5, 0.4 \rangle$	$\langle 0.5, 0.3, 0.4 \rangle$	$\langle 0.5, 0.4, 0.3 \rangle$	$\langle 0.6, 0.3, 0.5 \rangle$
${}_1\tilde{\mathcal{N}}_{x_4}^\beta \cup {}_1\tilde{\mathcal{M}}_{x_4}^\beta$	$\langle 0.6, 0.5, 0.1 \rangle$	$\langle 0.7, 0.3, 0.5 \rangle$	$\langle 0.4, 0.5, 0.4 \rangle$	$\langle 0.7, 0.3, 0.5 \rangle$	$\langle 0.6, 0.2, 0.3 \rangle$
${}_1\tilde{\mathcal{N}}_{x_5}^\beta \cup {}_1\tilde{\mathcal{M}}_{x_5}^\beta$	$\langle 0.6, 0.5, 0.1 \rangle$	$\langle 0.8, 0.5, 0.4 \rangle$	$\langle 0.6, 0.3, 0.2 \rangle$	$\langle 0.7, 0.4, 0.3 \rangle$	$\langle 0.6, 0.3, 0.5 \rangle$

$b, \mathcal{F}_{\tilde{C}_i}(y) \leq c\}$. Thus, ${}_1\tilde{\mathcal{M}}_y^\beta \subseteq {}_1\tilde{\mathcal{M}}_x^\beta$. On the contrary, let ${}_1\tilde{\mathcal{M}}_y^\beta \subseteq {}_1\tilde{\mathcal{M}}_x^\beta$. Then, $\mathcal{T}_{{}_1\tilde{\mathcal{M}}_x^\beta}(y) \geq \mathcal{T}_{{}_1\tilde{\mathcal{M}}_y^\beta}(y) \geq a, \mathcal{F}_{{}_1\tilde{\mathcal{M}}_x^\beta}(y) \leq \mathcal{F}_{{}_1\tilde{\mathcal{M}}_y^\beta}(y) \leq b$, and $\mathcal{F}_{{}_1\tilde{\mathcal{M}}_x^\beta}(y) \leq \mathcal{F}_{{}_1\tilde{\mathcal{M}}_y^\beta}(y) \leq c$. Hence, ${}_1\tilde{\mathcal{M}}_x^\beta(y) \geq \beta_y$. \square

Now, we present the three new types of 1-1-SVN β CRSs based on Definitions 5 and 7 as indicated below.

Definition 8. Consider $(\Omega, \tilde{\Gamma})$ is a 1-SVN β CAS with $\tilde{\Gamma} = \{\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_m\}$, for some $\beta = \langle a, b, c \rangle$. For each $x \in \Omega$ and $\mathcal{A} \in \text{SVN}(\Omega)$, then we have the following paradigms:

Paradigm 1: the second type of Type-1 SVN lower approximation (2-1-SVNLA) $\mathcal{L}_2^1(\mathcal{A})$ and the second type of Type-1 SVN upper approximation (2-1-SVNUA) $\mathcal{U}_2^1(\mathcal{A})$ are as follows:

$$\begin{aligned}\mathcal{L}_2^1(\mathcal{A}) &= \left\{ \langle x, \bigwedge_{y \in \Omega} \left(\mathcal{F}_{{}_1\tilde{\mathcal{M}}_x^\beta}(y) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \bigwedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{{}_1\tilde{\mathcal{M}}_x^\beta}(y) \right) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \bigvee_{y \in \Omega} \left(\mathcal{T}_{{}_1\tilde{\mathcal{M}}_x^\beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}, \\ \mathcal{U}_2^1(\mathcal{A}) &= \left\{ \langle x, \bigvee_{y \in \Omega} \left(\mathcal{T}_{{}_1\tilde{\mathcal{M}}_x^\beta}(y) \wedge \mathcal{T}_{\mathcal{A}}(y) \right), \bigvee_{y \in \Omega} \left(\mathcal{T}_{{}_1\tilde{\mathcal{M}}_x^\beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \bigwedge_{y \in \Omega} \left(\mathcal{F}_{{}_1\tilde{\mathcal{M}}_x^\beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}.\end{aligned}\quad (6)$$

If $\mathcal{L}_2^1(\mathcal{A}) \neq \mathcal{U}_2^1(\mathcal{A})$, then \mathcal{A} is called the second type of Type-1 SVN β -covering rough sets (briefly, 2-1-SVN β CRSs).

Paradigm 2: the third type of Type-1 SVN lower approximation (3-1-SVNLA) $\mathcal{L}_3^1(\mathcal{A})$ and the third type of Type-1 SVN upper approximation (3-1-SVNUA) $\mathcal{U}_3^1(\mathcal{A})$ are introduced as follows:

$$\begin{aligned}\mathcal{L}_3^1(\mathcal{A}) &= \left\{ \langle x, \bigwedge_{y \in \Omega} \left(\mathcal{F}_{{}_1\tilde{\mathcal{N}}_x^\beta \cap {}_1\tilde{\mathcal{M}}_x^\beta}(y) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \bigwedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{{}_1\tilde{\mathcal{N}}_x^\beta \cap {}_1\tilde{\mathcal{M}}_x^\beta}(y) \right) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \bigvee_{y \in \Omega} \left(\mathcal{T}_{{}_1\tilde{\mathcal{N}}_x^\beta \cap {}_1\tilde{\mathcal{M}}_x^\beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}, \\ \mathcal{U}_3^1(\mathcal{A}) &= \left\{ \langle x, \bigvee_{y \in \Omega} \left(\mathcal{T}_{{}_1\tilde{\mathcal{N}}_x^\beta \cap {}_1\tilde{\mathcal{M}}_x^\beta}(y) \wedge \mathcal{T}_{\mathcal{A}}(y) \right), \bigvee_{y \in \Omega} \left(\mathcal{T}_{{}_1\tilde{\mathcal{N}}_x^\beta \cap {}_1\tilde{\mathcal{M}}_x^\beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \bigwedge_{y \in \Omega} \left(\mathcal{F}_{{}_1\tilde{\mathcal{N}}_x^\beta \cap {}_1\tilde{\mathcal{M}}_x^\beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}.\end{aligned}\quad (7)$$

If $\mathcal{L}_3^1(\mathcal{A}) \neq \mathcal{U}_3^1(\mathcal{A})$, then \mathcal{A} is called the third type of Type-1 SVN β -covering rough sets (briefly, 3-1-SVN β CRSs).

Paradigm 3: the fourth type of Type-1 SVN lower approximation (4-1-SVNLA) $\mathcal{L}_4^1(\mathcal{A})$ and the fourth type of Type-1 SVN upper approximation (4-1-SVNUA) $\mathcal{U}_4^1(\mathcal{A})$ are proposed as follows:

$$\begin{aligned}\mathcal{L}_4^1(\mathcal{A}) &= \left\{ \langle x, \bigwedge_{y \in \Omega} \left(\mathcal{F}_{{}_1\tilde{\mathcal{N}}_x^\beta \cup {}_1\tilde{\mathcal{M}}_x^\beta}(y) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \bigwedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{{}_1\tilde{\mathcal{N}}_x^\beta \cup {}_1\tilde{\mathcal{M}}_x^\beta}(y) \right) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \bigvee_{y \in \Omega} \left(\mathcal{T}_{{}_1\tilde{\mathcal{N}}_x^\beta \cup {}_1\tilde{\mathcal{M}}_x^\beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}, \\ \mathcal{U}_4^1(\mathcal{A}) &= \left\{ \langle x, \bigvee_{y \in \Omega} \left(\mathcal{T}_{{}_1\tilde{\mathcal{N}}_x^\beta \cup {}_1\tilde{\mathcal{M}}_x^\beta}(y) \wedge \mathcal{T}_{\mathcal{A}}(y) \right), \bigvee_{y \in \Omega} \left(\mathcal{T}_{{}_1\tilde{\mathcal{N}}_x^\beta \cup {}_1\tilde{\mathcal{M}}_x^\beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \bigwedge_{y \in \Omega} \left(\mathcal{F}_{{}_1\tilde{\mathcal{N}}_x^\beta \cup {}_1\tilde{\mathcal{M}}_x^\beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}.\end{aligned}\quad (8)$$

If $\mathcal{L}_4^1(\mathcal{A}) \neq \mathcal{U}_4^1(\mathcal{A})$, then \mathcal{A} is called the fourth type of Type-1 SVN β -covering rough sets (briefly, 4-1-SVN β CRSs).

Example 2. Consider Example 1 if $\beta = \langle 0.5, 0.3, 0.8 \rangle$ and $\mathcal{A} = ((0.5, 0.3, 0.6)/x_1) + ((0.4, 0.5, 0.1)/x_2) + ((0.3, 0.2,$

$0.6)/x_3) + ((0.5, 0.3, 0.4)/x_4) + ((0.7, 0.2, 0.3)/x_5)$; then, we have the following results:

$$\begin{aligned}
 \mathcal{L}_1^1(\mathcal{A}) &= \{\langle x_1, 0.6, 0.5, 0.5 \rangle, \langle x_2, 0.6, 0.5, 0.4 \rangle, \langle x_3, 0.4, 0.4, 0.5 \rangle, \langle x_4, 0.4, 0.5, 0.4 \rangle, \langle x_5, 0.6, 0.4, 0.3 \rangle\}, \\
 \mathcal{U}_1^1(\mathcal{A}) &= \{\langle x_1, 0.6, 0.3, 0.5 \rangle, \langle x_2, 0.4, 0.3, 0.6 \rangle, \langle x_3, 0.6, 0.5, 0.5 \rangle, \langle x_4, 0.5, 0.3, 0.6 \rangle, \langle x_5, 0.6, 0.5, 0.5 \rangle\}, \\
 \mathcal{L}_2^1(\mathcal{A}) &= \{\langle x_1, 0.3, 0.3, 0.6 \rangle, \langle x_2, 0.3, 0.3, 0.6 \rangle, \langle x_3, 0.4, 0.3, 0.5 \rangle, \langle x_4, 0.4, 0.2, 0.4 \rangle, \langle x_5, 0.3, 0.3, 0.6 \rangle\}, \\
 \mathcal{U}_2^1(\mathcal{A}) &= \{\langle x_1, 0.6, 0.5, 0.3 \rangle, \langle x_2, 0.6, 0.5, 0.3 \rangle, \langle x_3, 0.6, 0.5, 0.4 \rangle, \langle x_4, 0.6, 0.5, 0.3 \rangle, \langle x_5, 0.6, 0.5, 0.4 \rangle\}, \\
 \mathcal{L}_3^1(\mathcal{A}) &= \{\langle x_1, 0.6, 0.3, 0.5 \rangle, \langle x_2, 0.6, 0.3, 0.4 \rangle, \langle x_3, 0.5, 0.3, 0.4 \rangle, \langle x_4, 0.4, 0.2, 0.3 \rangle, \langle x_5, 0.6, 0.3, 0.3 \rangle\}, \\
 \mathcal{U}_3^1(\mathcal{A}) &= \{\langle x_1, 0.5, 0.5, 0.6 \rangle, \langle x_2, 0.4, 0.5, 0.6 \rangle, \langle x_3, 0.5, 0.5, 0.5 \rangle, \langle x_4, 0.5, 0.5, 0.6 \rangle, \langle x_5, 0.5, 0.5, 0.6 \rangle\}, \\
 \mathcal{L}_4^1(\mathcal{A}) &= \{\langle x_1, 0.3, 0.5, 0.6 \rangle, \langle x_2, 0.3, 0.5, 0.6 \rangle, \langle x_3, 0.4, 0.5, 0.5 \rangle, \langle x_4, 0.4, 0.5, 0.5 \rangle, \langle x_5, 0.3, 0.5, 0.6 \rangle\}, \\
 \mathcal{U}_4^1(\mathcal{A}) &= \{\langle x_1, 0.6, 0.3, 0.3 \rangle, \langle x_2, 0.6, 0.3, 0.3 \rangle, \langle x_3, 0.6, 0.5, 0.4 \rangle, \langle x_4, 0.6, 0.3, 0.3 \rangle, \langle x_5, 0.6, 0.5, 0.4 \rangle\}.
 \end{aligned} \tag{9}$$

Next, we will present Proposition 3 for the 2-1-SVN β CRS model; also, it satisfies in case of the 3-1-SVN β CRS and the 4-1-SVN β CRS models.

Proposition 3. Let (Ω, \tilde{T}) be a 1-SVN β CAS, for some $\beta = \langle a, b, c \rangle$. For each $x, y, z \in \Omega$ and $\mathcal{A}, \mathcal{B} \in \text{SVN}(\Omega)$. Then, the following statements hold:

- (1) (SVNL1) $\mathcal{L}_2^1(\mathcal{A}^c) = (\mathcal{U}_2^1(\mathcal{A}))^c$.
- (SVNU1) $\mathcal{U}_2^1(\mathcal{A}^c) = (\mathcal{L}_2^1(\mathcal{A}))^c$.
- (2) If $\mathcal{A} \subseteq \mathcal{B}$, then
- (SVNL2) $\mathcal{L}_2^1(\mathcal{A}) \subseteq \mathcal{L}_2^1(\mathcal{B})$.

$$(SVNU2) \mathcal{U}_2^1(\mathcal{A}) \subseteq \mathcal{U}_2^1(\mathcal{B}).$$

$$(3) (SVNL3) \mathcal{L}_2^1(\mathcal{A} \cap \mathcal{B}) = \mathcal{L}_2^1(\mathcal{A}) \cap \mathcal{L}_2^1(\mathcal{B}).$$

$$(SVNU3) \mathcal{U}_2^1(\mathcal{A} \cap \mathcal{B}) \subseteq \mathcal{U}_2^1(\mathcal{A}) \cap \mathcal{U}_2^1(\mathcal{B}).$$

$$(4) (SVNL4) \mathcal{L}_2^1(\mathcal{A} \cup \mathcal{B}) \supseteq \mathcal{L}_2^1(\mathcal{A}) \cup \mathcal{L}_2^1(\mathcal{B}).$$

$$(SVNU4) \mathcal{U}_2^1(\mathcal{A} \cup \mathcal{B}) = \mathcal{U}_2^1(\mathcal{A}) \cup \mathcal{U}_2^1(\mathcal{B}).$$

Proof. We shall only prove (SVNL1), (SVNL2), (SVNL3), and (SVNL4).

(SVNL1):

$$\begin{aligned}
 \mathcal{L}_2^1(\mathcal{A}^c) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{A}^c}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{F}_{\mathcal{A}^c}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}^c}(y) \right) \rangle \right\} \\
 &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee (1 - \mathcal{F}_{\mathcal{A}}(y)) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\} \\
 &= (\mathcal{U}_2^1(\mathcal{A}))^c.
 \end{aligned} \tag{10}$$

(SVNL2): let $\mathcal{A}, \mathcal{B} \in \text{SVN}(\Omega)$ such that $\mathcal{A} \subseteq \mathcal{B}$ (i.e., $\mathcal{F}_{\mathcal{A}} \leq \mathcal{F}_{\mathcal{B}}, \mathcal{F}_{\mathcal{B}} \leq \mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{B}} \leq \mathcal{F}_{\mathcal{A}}$) and $x \in \Omega$. Then, we get the following result:

$$\begin{aligned}
 \mathcal{L}_2^1(\mathcal{A})(x) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}, \\
 \mathcal{F}_{\mathcal{L}_2^1(\mathcal{A})} &= \wedge_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \leq \wedge_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{B}}(y) \right) = \mathcal{F}_{\mathcal{L}_2^1(\mathcal{B})}, \\
 \mathcal{F}_{\mathcal{L}_2^1(\mathcal{A})} &= \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \geq \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{F}_{\mathcal{B}}(y) \right) = \mathcal{F}_{\mathcal{L}_2^1(\mathcal{B})}, \\
 \mathcal{F}_{\mathcal{L}_2^1(\mathcal{A})} &= \vee_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \geq \vee_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{B}}(y) \right) = \mathcal{F}_{\mathcal{L}_2^1(\mathcal{B})}.
 \end{aligned} \tag{11}$$

Therefore, $\mathcal{L}_2^1(\mathcal{A}) \subseteq \mathcal{L}_2^1(\mathcal{B})$.

(SVNL3): if $x \in \Omega$, then we have

$$\begin{aligned}
\mathcal{L}_2^1(\mathcal{A})(x) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \vee \mathcal{F}_{\mathcal{A} \cap \mathcal{B}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \right) \vee \mathcal{F}_{\mathcal{A} \cap \mathcal{B}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \wedge \mathcal{F}_{\mathcal{A} \cap \mathcal{B}}(y) \right) \rangle \right\} \\
&= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \vee (\mathcal{F}_{\mathcal{A}}(y) \wedge \mathcal{F}_{\mathcal{B}}(y)) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \right) \vee (\mathcal{F}_{\mathcal{A}}(y) \vee \mathcal{F}_{\mathcal{B}}(y)) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \wedge (\mathcal{F}_{\mathcal{A}}(y) \vee \mathcal{F}_{\mathcal{B}}(y)) \right) \rangle \right\} \\
&= \left\{ \langle x, \wedge_{y \in \Omega} \left(\left(\mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \wedge \left(\mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \vee \mathcal{F}_{\mathcal{B}}(y) \right) \right), \wedge_{y \in \Omega} \left(\left(\left(1 - \mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \right) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \vee \left(\left(1 - \mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \right) \vee \mathcal{F}_{\mathcal{B}}(y) \right) \right), \right. \\
&\quad \left. \vee_{y \in \Omega} \left(\left(\mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \vee \left(\mathcal{F}_{1, \mathcal{M}_x}^\beta(y) \wedge \mathcal{F}_{\mathcal{B}}(y) \right) \right) \rangle \right\} \\
&= \mathcal{L}_2^1(\mathcal{A}) \cap \mathcal{L}_2^1(\mathcal{B}).
\end{aligned} \tag{12}$$

(SVNL4): since $\mathcal{A} \cup \mathcal{B} \supseteq \mathcal{A}$, then, by (SVNL2), we have $\mathcal{L}_2^1(\mathcal{A} \cup \mathcal{B}) \supseteq \mathcal{L}_2^1(\mathcal{A})$. Similarly, $\mathcal{A} \cup \mathcal{B} \supseteq \mathcal{B}$; then, by (SVNL2), we have $\mathcal{L}_2^1(\mathcal{A} \cup \mathcal{B}) \supseteq \mathcal{L}_2^1(\mathcal{B})$. Thus, $\mathcal{L}_2^1(\mathcal{A} \cup \mathcal{B}) \supseteq \mathcal{L}_2^1(\mathcal{A}) \cup \mathcal{L}_2^1(\mathcal{B})$. \square

Now, we proceed to explain some relationships among these models.

Proposition 4. Let $(\Omega, \tilde{\Gamma})$ be a 1-SVN β CAS and $\mathcal{A} \in \text{SVN}(\Omega)$. Then, we have the following properties:

- (1) $\mathcal{L}_4^1(\mathcal{A}) \leq \mathcal{L}_2^1(\mathcal{A}) \leq \mathcal{L}_3^1(\mathcal{A})$
- (2) $\mathcal{L}_4^1(\mathcal{A}) \leq \mathcal{L}_1^1(\mathcal{A}) \leq \mathcal{L}_3^1(\mathcal{A})$
- (3) $\mathcal{U}_3^1(\mathcal{A}) \leq \mathcal{U}_2^1(\mathcal{A}) \leq \mathcal{L}_4^1(\mathcal{A})$
- (4) $\mathcal{U}_3^1(\mathcal{A}) \leq \mathcal{U}_1^1(\mathcal{A}) \leq \mathcal{U}_4^1(\mathcal{A})$

Proof. The proof is clear from Definition 8. \square

Proposition 5. Let $(\Omega, \tilde{\Gamma})$ be a 1-SVN β CAS and $\mathcal{A} \in \text{SVN}(\Omega)$. Then, we have the following properties:

- (1) $\mathcal{L}_3^1(\mathcal{A}) \geq \mathcal{L}_1^1(\mathcal{A}) \cup \mathcal{L}_2^1(\mathcal{A})$
- (2) $\mathcal{U}_3^1(\mathcal{A}) \leq \mathcal{U}_1^1(\mathcal{A}) \cap \mathcal{U}_2^1(\mathcal{A})$
- (3) $\mathcal{L}_4^1(\mathcal{A}) \leq \mathcal{L}_1^1(\mathcal{A}) \cap \mathcal{L}_2^1(\mathcal{A})$
- (4) $\mathcal{U}_4^1(\mathcal{A}) \geq \mathcal{U}_1^1(\mathcal{A}) \cup \mathcal{U}_2^1(\mathcal{A})$

Proof (straightforward) \square

4. Type-2 SVN Complementary β -Neighborhood and Three New Kinds of Type-2 SVN β -CRS

Definition 9. Let $(\Omega, \tilde{\Gamma})$ be a 2-SVN β CAS with $\tilde{\Gamma} = \{\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_m\}$, for some $\beta = \langle a, b, c \rangle$. Then, for each $x \in \Omega$, define the type-2 SVN complementary β -neighborhood of x as follows:

$${}_2\tilde{\mathcal{M}}_x^\beta(y) = {}_2\tilde{\mathcal{N}}_y^\beta(x), \quad \forall y \in \Omega. \tag{13}$$

Example 3. Consider Example 1, $\beta = \langle 0.5, 0.1, 0.8 \rangle$ and $\mathcal{A} = ((0.6, 0.3, 0.5)/x_1) + ((0.4, 0.5, 0.1)/x_2) + ((0.3, 0.2, 0.6)/x_3) + ((0.5, 0.3, 0.4)/x_4) + ((0.7, 0.2, 0.3)/x_5)$. Then, the values of type-2 SVN β -neighborhood are seen in Table 6:

$$\begin{aligned}
{}_2\tilde{\mathcal{N}}_{x_1}^\beta &= \tilde{C}_1 \cap \tilde{C}_2, \\
{}_2\tilde{\mathcal{N}}_{x_2}^\beta &= \tilde{C}_1 \cap \tilde{C}_2 \cap \tilde{C}_4, \\
{}_2\tilde{\mathcal{N}}_{x_3}^\beta &= \tilde{C}_3 \cap \tilde{C}_4, \\
{}_2\tilde{\mathcal{N}}_{x_4}^\beta &= \tilde{C}_1 \cap \tilde{C}_4, \\
{}_2\tilde{\mathcal{N}}_{x_5}^\beta &= \tilde{C}_2 \cap \tilde{C}_3 \cap \tilde{C}_4.
\end{aligned} \tag{14}$$

Also, we compute type-2 SVN complementary β -neighborhood as in Table 7.

Thus, we can merge ${}_2\tilde{\mathcal{M}}_x^\beta$ and ${}_2\tilde{\mathcal{M}}_x^\beta$ and calculate ${}_2\tilde{\mathcal{N}}_x^\beta \sqcap {}_2\tilde{\mathcal{M}}_x^\beta$ as Table 8.

Furthermore, we can calculate ${}_2\tilde{\mathcal{N}}_x^\beta \sqcup {}_2\tilde{\mathcal{M}}_x^\beta$, as set in Table 9.

Proposition 6. Let $(\Omega, \tilde{\Gamma})$ be a 2-SVN β CAS, for some $\beta = \langle a, b, c \rangle$ and for each $x, y, z \in \Omega$. Then, the following statements hold:

- (1) ${}_2\tilde{\mathcal{M}}_x^\beta(x) \geq \beta$
- (2) If ${}_2\tilde{\mathcal{M}}_x^\beta(y) \geq \beta$ and ${}_2\tilde{\mathcal{M}}_y^\beta(z) \geq \beta$, then ${}_2\tilde{\mathcal{M}}_x^\beta(z) \geq \beta$
- (3) $0 < \beta_1 \leq \beta_2 \leq \beta$, then ${}_2\tilde{\mathcal{M}}_x^{\beta_1} \sqsubseteq {}_2\tilde{\mathcal{M}}_x^{\beta_2}$

Proof

(1) follows directly from Definitions 5 and 9.

(2) Since ${}_2\tilde{\mathcal{M}}_x^\beta(y) \geq \beta$, then ${}_2\tilde{\mathcal{N}}_y^\beta(x) \geq \beta$. If $\tilde{C}_i(x) \geq \beta$, then $\tilde{C}_i(y) \geq \beta$, and since ${}_2\tilde{\mathcal{M}}_y^\beta(z) \geq \beta$, then ${}_2\tilde{\mathcal{N}}_z^\beta(y) \geq \beta$. If $\tilde{C}_i(y) \geq \beta$, then $\tilde{C}_i(z) \geq \beta$. Therefore, ${}_2\tilde{\mathcal{M}}_x^\beta(z) \geq \beta$.

(3) For each $x \in \Omega, 0 < \beta_1 < \beta_2 < \beta$, then $\cap\{\tilde{C}_i \in \tilde{\Gamma} : \tilde{C}_i(x) \geq \beta_1\} \sqsupseteq \cap\{\tilde{C}_i \in \tilde{\Gamma} : \tilde{C}_i(x) \geq \beta_2\}$. Thus, by Definition 9, we have ${}_2\tilde{\mathcal{M}}_x^{\beta_1} \sqsubseteq {}_2\tilde{\mathcal{M}}_x^{\beta_2}$. \square

TABLE 6: $\widetilde{\mathcal{N}}_{x_i}^\beta, \forall i = 1, 2, \dots, 5.$

	x_1	x_2	x_3	x_4	x_5
$\widetilde{\mathcal{N}}_{x_1}^\beta$	$\langle 0.6, 0.2, 0.5 \rangle$	$\langle 0.5, 0.2, 0.8 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.3, 0.2, 0.6 \rangle$
$\widetilde{\mathcal{N}}_{x_2}^\beta$	$\langle 0.1, 0.2, 0.6 \rangle$	$\langle 0.5, 0.1, 0.8 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.3, 0.1, 0.6 \rangle$
$\widetilde{\mathcal{N}}_{x_3}^\beta$	$\langle 0.1, 0.1, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.5, 0.2, 0.4 \rangle$	$\langle 0.3, 0.3, 0.5 \rangle$	$\langle 0.6, 0.1, 0.5 \rangle$
$\widetilde{\mathcal{N}}_{x_4}^\beta$	$\langle 0.1, 0.2, 0.6 \rangle$	$\langle 0.5, 0.1, 0.7 \rangle$	$\langle 0.4, 0.3, 0.4 \rangle$	$\langle 0.5, 0.1, 0.7 \rangle$	$\langle 0.3, 0.1, 0.6 \rangle$
$\widetilde{\mathcal{N}}_{x_5}^\beta$	$\langle 0.1, 0.1, 0.6 \rangle$	$\langle 0.4, 0.1, 0.8 \rangle$	$\langle 0.2, 0.2, 0.6 \rangle$	$\langle 0.3, 0.3, 0.7 \rangle$	$\langle 0.6, 0.1, 0.5 \rangle$

TABLE 7: $\widetilde{\mathcal{M}}_{x_i}^\beta, \forall i = 1, 2, \dots, 5.$

	x_1	x_2	x_3	x_4	x_5
$\widetilde{\mathcal{M}}_{x_1}^\beta$	$\langle 0.5, 0.8, 0.6 \rangle$	$\langle 0.8, 0.8, 0.5 \rangle$	$\langle 0.6, 0.7, 0.2 \rangle$	$\langle 0.7, 0.9, 0.4 \rangle$	$\langle 0.6, 0.8, 0.3 \rangle$
$\widetilde{\mathcal{M}}_{x_2}^\beta$	$\langle 0.6, 0.8, 0.1 \rangle$	$\langle 0.8, 0.9, 0.5 \rangle$	$\langle 0.6, 0.7, 0.2 \rangle$	$\langle 0.7, 0.9, 0.4 \rangle$	$\langle 0.6, 0.9, 0.3 \rangle$
$\widetilde{\mathcal{M}}_{x_3}^\beta$	$\langle 0.6, 0.9, 0.1 \rangle$	$\langle 0.7, 0.9, 0.4 \rangle$	$\langle 0.4, 0.8, 0.5 \rangle$	$\langle 0.5, 0.7, 0.3 \rangle$	$\langle 0.5, 0.9, 0.6 \rangle$
$\widetilde{\mathcal{M}}_{x_4}^\beta$	$\langle 0.6, 0.8, 0.1 \rangle$	$\langle 0.7, 0.9, 0.5 \rangle$	$\langle 0.4, 0.7, 0.4 \rangle$	$\langle 0.7, 0.9, 0.5 \rangle$	$\langle 0.6, 0.9, 0.3 \rangle$
$\widetilde{\mathcal{M}}_{x_5}^\beta$	$\langle 0.6, 0.9, 0.1 \rangle$	$\langle 0.8, 0.9, 0.4 \rangle$	$\langle 0.6, 0.8, 0.2 \rangle$	$\langle 0.7, 0.7, 0.3 \rangle$	$\langle 0.5, 0.9, 0.6 \rangle$

TABLE 8: $\widetilde{\mathcal{N}}_{x_i}^\beta \sqcap_2 \widetilde{\mathcal{M}}_{x_i}^\beta, \forall i = 1, 2, \dots, 5.$

	x_1	x_2	x_3	x_4	x_5
$\widetilde{\mathcal{N}}_{x_1}^\beta \sqcap_2 \widetilde{\mathcal{M}}_{x_1}^\beta$	$\langle 0.5, 0.2, 0.6 \rangle$	$\langle 0.5, 0.2, 0.8 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.3, 0.2, 0.6 \rangle$
$\widetilde{\mathcal{N}}_{x_2}^\beta \sqcap_2 \widetilde{\mathcal{M}}_{x_2}^\beta$	$\langle 0.1, 0.2, 0.6 \rangle$	$\langle 0.5, 0.1, 0.8 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.3, 0.1, 0.6 \rangle$
$\widetilde{\mathcal{N}}_{x_3}^\beta \sqcap_2 \widetilde{\mathcal{M}}_{x_3}^\beta$	$\langle 0.1, 0.1, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.4, 0.2, 0.5 \rangle$	$\langle 0.3, 0.3, 0.5 \rangle$	$\langle 0.5, 0.1, 0.6 \rangle$
$\widetilde{\mathcal{N}}_{x_4}^\beta \sqcap_2 \widetilde{\mathcal{M}}_{x_4}^\beta$	$\langle 0.1, 0.2, 0.6 \rangle$	$\langle 0.5, 0.1, 0.7 \rangle$	$\langle 0.4, 0.3, 0.4 \rangle$	$\langle 0.5, 0.1, 0.7 \rangle$	$\langle 0.3, 0.1, 0.6 \rangle$
$\widetilde{\mathcal{N}}_{x_5}^\beta \sqcap_2 \widetilde{\mathcal{M}}_{x_5}^\beta$	$\langle 0.1, 0.1, 0.6 \rangle$	$\langle 0.4, 0.1, 0.8 \rangle$	$\langle 0.2, 0.2, 0.6 \rangle$	$\langle 0.3, 0.3, 0.7 \rangle$	$\langle 0.5, 0.1, 0.6 \rangle$

TABLE 9: $\widetilde{\mathcal{N}}_{x_i}^\beta \sqcup_2 \widetilde{\mathcal{M}}_{x_i}^\beta, \forall i = 1, 2, \dots, 5.$

	x_1	x_2	x_3	x_4	x_5
$\widetilde{\mathcal{N}}_{x_1}^\beta \sqcup_2 \widetilde{\mathcal{M}}_{x_1}^\beta$	$\langle 0.6, 0.8, 0.5 \rangle$	$\langle 0.8, 0.8, 0.5 \rangle$	$\langle 0.6, 0.7, 0.2 \rangle$	$\langle 0.7, 0.9, 0.4 \rangle$	$\langle 0.6, 0.8, 0.3 \rangle$
$\widetilde{\mathcal{N}}_{x_2}^\beta \sqcup_2 \widetilde{\mathcal{M}}_{x_2}^\beta$	$\langle 0.6, 0.8, 0.1 \rangle$	$\langle 0.8, 0.9, 0.5 \rangle$	$\langle 0.6, 0.7, 0.2 \rangle$	$\langle 0.7, 0.9, 0.4 \rangle$	$\langle 0.6, 0.9, 0.3 \rangle$
$\widetilde{\mathcal{N}}_{x_3}^\beta \sqcup_2 \widetilde{\mathcal{M}}_{x_3}^\beta$	$\langle 0.6, 0.9, 0.1 \rangle$	$\langle 0.7, 0.9, 0.4 \rangle$	$\langle 0.5, 0.8, 0.4 \rangle$	$\langle 0.5, 0.7, 0.3 \rangle$	$\langle 0.6, 0.9, 0.5 \rangle$
$\widetilde{\mathcal{N}}_{x_4}^\beta \sqcup_2 \widetilde{\mathcal{M}}_{x_4}^\beta$	$\langle 0.6, 0.8, 0.1 \rangle$	$\langle 0.7, 0.9, 0.5 \rangle$	$\langle 0.4, 0.7, 0.4 \rangle$	$\langle 0.7, 0.9, 0.5 \rangle$	$\langle 0.6, 0.9, 0.3 \rangle$
$\widetilde{\mathcal{N}}_{x_5}^\beta \sqcup_2 \widetilde{\mathcal{M}}_{x_5}^\beta$	$\langle 0.6, 0.9, 0.1 \rangle$	$\langle 0.8, 0.9, 0.4 \rangle$	$\langle 0.6, 0.8, 0.2 \rangle$	$\langle 0.7, 0.7, 0.3 \rangle$	$\langle 0.6, 0.9, 0.5 \rangle$

Proposition 7. Let $(\Omega, \widetilde{\Gamma})$ be a 1-SVN β CAS, for some $\beta = \langle a, b, c \rangle$. For each $x, y \in \Omega$, $\widetilde{\mathcal{M}}_x^\beta(y) \geq \beta \Leftrightarrow \widetilde{\mathcal{M}}_y^\beta \sqsubseteq_2 \widetilde{\mathcal{M}}_x^\beta$.

Proof. Let $\widetilde{\mathcal{M}}_x^\beta(y) \geq \beta$, $\mathcal{F}_{\widetilde{\mathcal{M}}_x^\beta}(y) = \mathcal{F} \sqcap \widetilde{\mathcal{C}}_i(y) = \bigwedge \mathcal{F}_{\widetilde{\mathcal{C}}_i}(y) \geq a$, $\mathcal{F}_{\widetilde{\mathcal{M}}_x^\beta}(y) = \mathcal{F} \sqcap \widetilde{\mathcal{C}}_i(y) = \bigvee \mathcal{F}_{\widetilde{\mathcal{C}}_i}(y) \geq b$, and $\mathcal{F}_{\widetilde{\mathcal{M}}_x^\beta}(y) = \mathcal{F} \sqcap \widetilde{\mathcal{C}}_i(y) = \bigvee \mathcal{F}_{\widetilde{\mathcal{C}}_i}(y) \leq c$. Then, $\{\widetilde{\mathcal{C}}_i \in \widetilde{\Gamma} : \mathcal{F}_{\widetilde{\mathcal{C}}_i}(x) \geq a, \mathcal{F}_{\widetilde{\mathcal{C}}_i}(x) \geq b, \mathcal{F}_{\widetilde{\mathcal{C}}_i}(x) \leq c\} \subseteq \{\widetilde{\mathcal{C}}_i \in \widetilde{\Gamma} : \mathcal{F}_{\widetilde{\mathcal{C}}_i}(y) \geq a, \mathcal{F}_{\widetilde{\mathcal{C}}_i}(y) \geq b, \mathcal{F}_{\widetilde{\mathcal{C}}_i}(y) \leq c\}$. Thus, $\widetilde{\mathcal{M}}_y^\beta \sqsubseteq_2 \widetilde{\mathcal{M}}_x^\beta$. On the contrary, let $\widetilde{\mathcal{M}}_y^\beta \sqsubseteq_2 \widetilde{\mathcal{M}}_x^\beta$. Then, $\mathcal{F}_{\widetilde{\mathcal{M}}_x^\beta}(y) \geq \mathcal{F}_{\widetilde{\mathcal{M}}_y^\beta}(y) \geq a$,

$\mathcal{F}_{\widetilde{\mathcal{M}}_x^\beta}(y) \geq \mathcal{F}_{\widetilde{\mathcal{M}}_y^\beta}(y) \geq b$, and $\mathcal{F}_{\widetilde{\mathcal{M}}_x^\beta}(y) \leq \mathcal{F}_{\widetilde{\mathcal{M}}_y^\beta}(y) \leq c$. Hence, $\widetilde{\mathcal{M}}_x^\beta(y) \geq \beta$. \square

Here, we construct three new types of 2-1-SVN β CRSs based on Definitions 5 and 9 as seen below.

Definition 10. Consider $(\Omega, \widetilde{\Gamma})$ be a 2-SVN β CAS with $\widetilde{\Gamma} = \{\widetilde{\mathcal{C}}_1, \widetilde{\mathcal{C}}_2, \dots, \widetilde{\mathcal{C}}_m\}$ for some $\beta = \langle a, b, c \rangle$. For each $x \in \Omega$ and $\mathcal{A} \in \text{SVN}(\Omega)$, then we have the following paradigms.

Paradigm 1: The second type of Type-2 SVN lower approximation (2-2-SVNLA) $\mathcal{L}_2^2(\mathcal{A})$ and the second

type of Type-2 SVN upper approximation (2-2-SVNUA) $\mathcal{U}_2^2(\mathcal{A})$ are found as follows:

$$\begin{aligned}\mathcal{L}_2^2(\mathcal{A}) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{\mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}, \\ \mathcal{U}_2^2(\mathcal{A}) &= \left\{ \langle x, \vee_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}.\end{aligned}\quad (15)$$

If $\mathcal{L}_2^2(\mathcal{A}) \neq \mathcal{U}_2^2(\mathcal{A})$, then \mathcal{A} is called the second type of Type-2 SVN β -covering rough sets (briefly, 2-2-SVN β CRSs)

Paradigm 2: the third type of Type-2 SVN lower approximation (3-2-SVNLA) $\mathcal{L}_3^2(\mathcal{A})$ and the third type of Type-2 SVN upper approximation (3-2-SVNUA) $\mathcal{U}_3^2(\mathcal{A})$ are introduced as follows:

$$\begin{aligned}\mathcal{L}_3^2(\mathcal{A}) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x \cap \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{\mathcal{M}_x \cap \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x \cap \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}, \\ \mathcal{U}_3^2(\mathcal{A}) &= \left\{ \langle x, \vee_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x \cap \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x \cap \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x \cap \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}.\end{aligned}\quad (16)$$

If $\mathcal{L}_3^2(\mathcal{A}) \neq \mathcal{U}_3^2(\mathcal{A})$, then \mathcal{A} is called the third type of Type-2 SVN β -covering rough sets (briefly, 3-2-SVN β CRSs).

Paradigm 3: the fourth type of Type-2 SVN lower approximation (4-2-SVNLA) $\mathcal{L}_4^2(\mathcal{A})$ and the fourth type of Type-2 SVN upper approximation (4-2-SVNUA) $\mathcal{U}_4^2(\mathcal{A})$ are proposed as follows:

$$\begin{aligned}\mathcal{L}_4^2(\mathcal{A}) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x \sqcup \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{\mathcal{M}_x \sqcup \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{F}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x \sqcup \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}, \\ \mathcal{U}_4^2(\mathcal{A}) &= \left\{ \langle x, \vee_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x \sqcup \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x \sqcup \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x \sqcup \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{F}_{\mathcal{A}}(y) \right) \rangle \right\}.\end{aligned}\quad (17)$$

If $\mathcal{L}_4^2(\mathcal{A}) \neq \mathcal{U}_4^2(\mathcal{A})$, then \mathcal{A} is called the fourth type of Type-2 SVN β -covering rough sets (briefly, 4-2-SVN β CRSs).

Example 4. Consider Example 1 if $\beta = \langle 0.5, 0.1, 0.8 \rangle$ and $\mathcal{A} = ((0.6, 0.3, 0.5)/x_1) + ((0.4, 0.5, 0.1)/x_2) + ((0.3, 0.2, 0.6)/x_3) + ((0.5, 0.3, 0.4)/x_4) + ((0.7, 0.2, 0.3)/x_5)$; then, we have the following results:

$$\begin{aligned}\mathcal{L}_1^2(\mathcal{A}) &= \{ \langle x_1, 0.6, 0.7, 0.5 \rangle, \langle x_2, 0.6, 0.7, 0.4 \rangle, \langle x_3, 0.4, 0.7, 0.5 \rangle, \langle x_4, 0.4, 0.7, 0.4 \rangle, \langle x_5, 0.6, 0.7, 0.3 \rangle \}, \\ \mathcal{U}_1^2(\mathcal{A}) &= \{ \langle x_1, 0.6, 0.2, 0.5 \rangle, \langle x_2, 0.4, 0.2, 0.6 \rangle, \langle x_3, 0.6, 0.3, 0.5 \rangle, \langle x_4, 0.5, 0.2, 0.6 \rangle, \langle x_5, 0.6, 0.3, 0.5 \rangle \}, \\ \mathcal{L}_2^2(\mathcal{A}) &= \{ \langle x_1, 0.3, 0.2, 0.6 \rangle, \langle x_2, 0.3, 0.2, 0.6 \rangle, \langle x_3, 0.4, 0.2, 0.5 \rangle, \langle x_4, 0.4, 0.2, 0.4 \rangle, \langle x_5, 0.3, 0.2, 0.6 \rangle \}, \\ \mathcal{U}_2^2(\mathcal{A}) &= \{ \langle x_1, 0.6, 0.5, 0.3 \rangle, \langle x_2, 0.6, 0.5, 0.3 \rangle, \langle x_3, 0.6, 0.5, 0.4 \rangle, \langle x_4, 0.6, 0.5, 0.3 \rangle, \langle x_5, 0.6, 0.5, 0.4 \rangle \}, \\ \mathcal{L}_3^2(\mathcal{A}) &= \{ \langle x_1, 0.6, 0.7, 0.5 \rangle, \langle x_2, 0.6, 0.7, 0.4 \rangle, \langle x_3, 0.5, 0.7, 0.4 \rangle, \langle x_4, 0.4, 0.7, 0.3 \rangle, \langle x_5, 0.6, 0.7, 0.3 \rangle \}, \\ \mathcal{U}_3^2(\mathcal{A}) &= \{ \langle x_1, 0.5, 0.2, 0.6 \rangle, \langle x_2, 0.4, 0.2, 0.6 \rangle, \langle x_3, 0.5, 0.3, 0.5 \rangle, \langle x_4, 0.5, 0.2, 0.6 \rangle, \langle x_5, 0.5, 0.3, 0.6 \rangle \}, \\ \mathcal{L}_4^2(\mathcal{A}) &= \{ \langle x_1, 0.3, 0.2, 0.6 \rangle, \langle x_2, 0.3, 0.2, 0.6 \rangle, \langle x_3, 0.4, 0.2, 0.5 \rangle, \langle x_4, 0.4, 0.2, 0.5 \rangle, \langle x_5, 0.3, 0.2, 0.6 \rangle \}, \\ \mathcal{U}_4^2(\mathcal{A}) &= \{ \langle x_1, 0.6, 0.5, 0.3 \rangle, \langle x_2, 0.6, 0.5, 0.3 \rangle, \langle x_3, 0.6, 0.5, 0.4 \rangle, \langle x_4, 0.6, 0.5, 0.3 \rangle, \langle x_5, 0.6, 0.5, 0.4 \rangle \}.\end{aligned}\quad (18)$$

In the following, we will propose Proposition 8 for the 2-2-SVN β CRS model; also, it fulfills in case of the 3-2-SVN β CRS and the 4-2-SVN β CRS models.

Proposition 8. Let (Ω, \tilde{T}) be a 1-SVN β CAS, for some $\beta = \langle a, b, c \rangle$. For each $x, y, z \in \Omega$ and $\mathcal{A}, \mathcal{B} \in \text{SVN}(\Omega)$, then the following statements hold:

$$(1) \text{ (SVNL1): } \mathcal{L}_2^2(\mathcal{A}^c) = (\mathcal{U}_2^2(\mathcal{A}))^c.$$

$$\text{(SVNU1): } \mathcal{U}_2^2(\mathcal{A}^c) = (\mathcal{L}_2^2(\mathcal{A}))^c.$$

(2) If $\mathcal{A} \subseteq \mathcal{B}$, then

$$\text{(SVNL2): } \mathcal{L}_2^2(\mathcal{A}) \subseteq \mathcal{L}_2^2(\mathcal{B}).$$

$$\text{(SVNU2): } \mathcal{U}_2^2(\mathcal{A}) \subseteq \mathcal{U}_2^2(\mathcal{B}).$$

$$(3) \text{ (SVNL3): } \mathcal{L}_2^2(\mathcal{A} \cap \mathcal{B}) = \mathcal{L}_2^2(\mathcal{A}) \cap \mathcal{L}_2^2(\mathcal{B}).$$

$$\text{(SVNU3): } \mathcal{U}_2^2(\mathcal{A} \cap \mathcal{B}) \subseteq \mathcal{U}_2^2(\mathcal{A}) \cap \mathcal{U}_2^2(\mathcal{B}).$$

$$(4) \text{ (SVNL4): } \mathcal{L}_2^2(\mathcal{A} \cup \mathcal{B}) \supseteq \mathcal{L}_2^2(\mathcal{A}) \cup \mathcal{L}_2^2(\mathcal{B}).$$

$$\text{(SVNU4): } \mathcal{U}_2^2(\mathcal{A} \cap \mathcal{B}) = \mathcal{U}_2^2(\mathcal{A}) \cap \mathcal{U}_2^1(\mathcal{B}).$$

$$(5) \text{ (SVNL5): } \mathcal{L}_2^2(\Omega) = \Omega.$$

$$\text{(SVNU5): } \mathcal{U}_2^2(\emptyset) = \emptyset.$$

Proof. We shall only prove (SVNL1), (SVNL2), (SVNL3), and (SVNL4).

(SVNL1):

$$\begin{aligned} \mathcal{L}_2^2(\mathcal{A}^c) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{T}_{\mathcal{A}^c}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{T}_{\mathcal{A}^c}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{T}_{\mathcal{A}^c}(y) \right) \rangle \right\} \\ &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee (1 - \mathcal{T}_{\mathcal{A}}(y)) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{T}_{\mathcal{A}}(y) \right) \rangle \right\} \\ &= (\mathcal{U}_2^2(\mathcal{A}))^c. \end{aligned} \quad (19)$$

(SVNL2): let $\mathcal{A}, \mathcal{B} \in \text{SVN}(\Omega)$ such that $\mathcal{A} \subseteq \mathcal{B}$ (i.e., $\mathcal{T}_{\mathcal{A}} \leq \mathcal{T}_{\mathcal{B}}, \mathcal{F}_{\mathcal{B}} \leq \mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{B}} \leq \mathcal{F}_{\mathcal{A}}$) and $x \in \Omega$. Then, we get the following result:

$$\begin{aligned} \mathcal{L}_2^2(\mathcal{A})(x) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{T}_{\mathcal{A}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{T}_{\mathcal{A}}(y) \right) \rangle \right\}, \\ \mathcal{T}_{\mathcal{L}_2^2(\mathcal{A})} &= \wedge_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{T}_{\mathcal{A}}(y) \right) \leq \wedge_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{T}_{\mathcal{B}}(y) \right) = \mathcal{T}_{\mathcal{L}_2^2(\mathcal{B})}, \\ \mathcal{F}_{\mathcal{L}_2^2(\mathcal{A})} &= \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{T}_{\mathcal{A}}(y) \right) \leq \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{T}_{\mathcal{B}}(y) \right) = \mathcal{F}_{\mathcal{L}_2^2(\mathcal{B})}, \\ \mathcal{F}_{\mathcal{L}_2^2(\mathcal{A})} &= \vee_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{T}_{\mathcal{A}}(y) \right) \geq \vee_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{T}_{\mathcal{B}}(y) \right) = \mathcal{F}_{\mathcal{L}_2^2(\mathcal{B})}. \end{aligned} \quad (20)$$

Therefore, $\mathcal{L}_2^2(\mathcal{A}) \subseteq \mathcal{L}_2^2(\mathcal{B})$.

(SVNL3): If $x \in \Omega$, then we have

$$\begin{aligned} \mathcal{L}_2^2(\mathcal{A})(x) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{T}_{\mathcal{A} \cap \mathcal{B}}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{T}_{\mathcal{A} \cap \mathcal{B}}(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{T}_{\mathcal{A} \cap \mathcal{B}}(y) \right) \rangle \right\} \\ &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \vee (\mathcal{T}_{\mathcal{A}} \wedge \mathcal{T}_{\mathcal{B}})(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee (\mathcal{T}_{\mathcal{A}} \wedge \mathcal{T}_{\mathcal{B}})(y) \right), \vee_{y \in \Omega} \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \wedge (\mathcal{T}_{\mathcal{A}} \vee \mathcal{T}_{\mathcal{B}})(y) \right) \rangle \right\} \\ &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{T}_{\mathcal{A}}(y) \right) \wedge \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{T}_{\mathcal{B}}(y) \right) \right), \wedge_{y \in \Omega} \left(\left(\left(1 - \mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{T}_{\mathcal{A}}(y) \right) \wedge \left(\left(1 - \mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{T}_{\mathcal{B}}(y) \right) \right), \right. \\ &\quad \left. \vee_{y \in \Omega} \left(\left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{T}_{\mathcal{A}}(y) \right) \vee \left(\mathcal{F}_{2, \mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{T}_{\mathcal{B}}(y) \right) \right) \rangle \right\} \\ &= \mathcal{L}_2^2(\mathcal{A}) \cap \mathcal{L}_2^2(\mathcal{B}). \end{aligned} \quad (21)$$

(SVNL4): since $\mathcal{A} \sqcup \mathcal{B} \supseteq \mathcal{A}$, then by SVNL2 we have $\mathcal{L}_2^2(\mathcal{A} \sqcup \mathcal{B}) \supseteq \mathcal{L}_2^2(\mathcal{A})$. Similarly, $\mathcal{A} \sqcup \mathcal{B} \supseteq \mathcal{B}$; then, by SVNL2,

we have $\mathcal{L}_2^2(\mathcal{A} \sqcup \mathcal{B}) \supseteq \mathcal{L}_2^2(\mathcal{B})$. Thus, $\mathcal{L}_2^2(\mathcal{A} \sqcup \mathcal{B}) \supseteq \mathcal{L}_2^2(\mathcal{A}) \sqcup \mathcal{L}_2^2(\mathcal{B})$.

(SVNL5): since SVN universe is $\Omega = \langle x, 1, 1, 0 \rangle$ and SVN empty set is $\emptyset = \langle x, 0, 0, 1 \rangle$, then we have

$$\begin{aligned}\mathcal{L}_2^2(\Omega) &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x}^{\sim \beta}(y) \vee \mathcal{T}_{\Omega}(y) \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{\mathcal{M}_x}^{\sim \beta}(y) \right) \vee \mathcal{T}_{\Omega}(y) \right), \vee_{y \in \Omega} \left(\mathcal{T}_{\mathcal{M}_x}^{\sim \beta}(y) \wedge \mathcal{F}_{\Omega}(y) \right) \rangle \right\} \\ &= \left\{ \langle x, \wedge_{y \in \Omega} \left(\mathcal{F}_{\mathcal{M}_x}^{\sim \beta}(y) \vee 1 \right), \wedge_{y \in \Omega} \left(\left(1 - \mathcal{F}_{\mathcal{M}_x}^{\sim \beta}(y) \right) \vee 1 \right), \vee_{y \in \Omega} \left(\mathcal{T}_{\mathcal{M}_x}^{\sim \beta}(y) \wedge 0 \right) \rangle \right\} \\ &= \{\langle x, 1, 1, 0 \rangle\} \\ &= \Omega.\end{aligned}\quad (22)$$

□

In the following, we give some relationships among these models.

Proposition 9. Let $(\Omega, \tilde{\Gamma})$ be a 2-SVN β CAS and $\mathcal{A} \in \text{SVN}(\Omega)$. Then, we have the following properties:

- (1) $\mathcal{L}_4^2(\mathcal{A}) \leq \mathcal{L}_2^2(\mathcal{A}) \leq \mathcal{L}_3^2(\mathcal{A})$
- (2) $\mathcal{L}_4^2(\mathcal{A}) \leq \mathcal{L}_1^2(\mathcal{A}) \leq \mathcal{L}_3^2(\mathcal{A})$
- (3) $\mathcal{U}_3^2(\mathcal{A}) \leq \mathcal{U}_2^2(\mathcal{A}) \leq \mathcal{U}_4^2(\mathcal{A})$
- (4) $\mathcal{U}_3^2(\mathcal{A}) \leq \mathcal{U}_1^2(\mathcal{A}) \leq \mathcal{U}_4^2(\mathcal{A})$

Proof. The proof is clear from Definition 10. □

Proposition 10. Let $(\Omega, \tilde{\Gamma})$ be a 2-SVN β CAS and $\mathcal{A} \in \text{SVN}(\Omega)$. Then, we have the following properties:

- (1) $\mathcal{L}_3^2(\mathcal{A}) \geq \mathcal{L}_1^2(\mathcal{A}) \sqcup \mathcal{L}_2^2(\mathcal{A})$
- (2) $\mathcal{U}_3^2(\mathcal{A}) \leq \mathcal{U}_1^2(\mathcal{A}) \sqcap \mathcal{U}_2^2(\mathcal{A})$
- (3) $\mathcal{L}_4^2(\mathcal{A}) \geq \mathcal{L}_1^2(\mathcal{A}) \sqcap \mathcal{L}_2^2(\mathcal{A})$
- (4) $\mathcal{U}_4^2(\mathcal{A}) \leq \mathcal{U}_1^2(\mathcal{A}) \sqcup \mathcal{U}_2^2(\mathcal{A})$

Proof. (clear). □

5. Decision-Making Approach to DM Based on SVN β CRSs

5.1. Description and Process

5.1.1. Method I. Assume that $\Omega = \{x_r: r = 1, \dots, k\}$ is the set of alternatives (patients), m is main attributes (symptoms) (e.g., cough and fever) $V = \{y_i: i = 1, 2, \dots, m\}$ of A disease, $\tilde{\mathcal{E}}_i(x_r) = \langle \mathcal{T}_{\tilde{\mathcal{E}}_i}(x_r), \mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r), \mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r) \rangle$ indicates the symptom value for each patient which is known by a doctor D , for some $\beta = \langle a, b, c \rangle$, and $(\Omega, \tilde{\Gamma})$ is a Type-1 SVN β -CRS, where $\mathcal{T}_{\tilde{\mathcal{E}}_i}(x_r) \in [0, 1]$ (i.e., the degree that doctor D confirms the patient x_r has symptom y_i), $\mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r) \in [0, 1]$ (i.e., the degree that doctor D is not sure if the patient x_r has symptom y_i), $\mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r) \in [0, 1]$ (i.e., the degree that doctor D confirms the patient x_r does not have any symptom y_i), and $0 \leq \mathcal{T}_{\tilde{\mathcal{E}}_i}(x_r) + \mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r) + \mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r) \leq 3$. According to the presented covering methods, we propose a decision-making algorithm to obtain the result by the following steps:

Step 1: consider, for each $x_r \in \Omega$, there is at least one $y_i \in V$ such that the symptom value \mathcal{E}_i for patient x_r is not less than β , where β is a critical value.

Step 2: consider $\mathcal{A}(x_r) = \langle d, e, f \rangle$ is the evaluation by a decision maker D , where d is a possible degree, e is an indeterminacy degree, and f is an impossible degree of A disease.

Step 3: based on this information, use Definition 8 and 3-1-SVN β CRSs model to calculate the lower and upper approximation of \mathcal{A} .

Step 4: calculate $\mathfrak{R}_{\mathcal{A}}$ by the following equation:

$$\mathfrak{R}_{\mathcal{A}} = \mathcal{U}_3^1(\mathcal{A}) \oplus \mathcal{L}_3^1(\mathcal{A}), \quad (23)$$

where $\mathcal{A} \oplus \mathcal{B} = \{ \langle x, \mathcal{T}_{\mathcal{A}}(x) + \mathcal{T}_{\mathcal{B}}(x) - \mathcal{F}_{\mathcal{A}}(x)^* \mathcal{F}_{\mathcal{B}}(x), \mathcal{F}_{\mathcal{A}}(x)^* \mathcal{F}_{\mathcal{B}}(x), \mathcal{F}_{\mathcal{A}}(x)^* \mathcal{F}_{\mathcal{B}}(x) \rangle : x \in \Omega \}$.

Step 5: calculate the decision method by the following formula:

$$\mathcal{S}(x) = \frac{\mathcal{T}_{\mathfrak{R}_{\mathcal{A}}}(x)}{\sqrt{(\mathcal{T}_{\mathfrak{R}_{\mathcal{A}}}(x))^2 + (\mathcal{F}_{\mathfrak{R}_{\mathcal{A}}}(x))^2 + (\mathcal{F}_{\mathfrak{R}_{\mathcal{A}}}(x))^2}}, \quad (24)$$

hence, ranking the alternatives.

Based on these steps, we give an algorithm to solve the decision-making problems based on Definition 8. The steps corresponding to it are summarized in Algorithm 1.

5.1.2. Method II. Suppose that $\Omega = \{x_r: r = 1, \dots, k\}$ is the set of alternatives (papers), m is main attributes (symptoms) (e.g., spot and steak) $V = \{y_i: i = 1, 2, \dots, m\}$ of A paper trouble, $\tilde{\mathcal{E}}_i(x_r) = \langle \mathcal{T}_{\tilde{\mathcal{E}}_i}(x_r), \mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r), \mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r) \rangle$ indicates the symptom value for each paper which known by an investigator I , for some $\beta = \langle a, b, c \rangle$, and $(\Omega, \tilde{\Gamma})$ is a Type-2 SVN β -CRS, where $\mathcal{T}_{\tilde{\mathcal{E}}_i}(x_r) \in [0, 1]$ (i.e., the degree that the investigator I asserts the paper x_r has symptom y_i), $\mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r) \in [0, 1]$ (i.e., the degree that the investigator I is not sure whether the paper x_r has symptom y_i), $\mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r) \in [0, 1]$ (i.e., the degree that investigator I affirms paper x_r does not have any symptom y_i), and $0 \leq \mathcal{T}_{\tilde{\mathcal{E}}_i}(x_r) + \mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r) + \mathcal{F}_{\tilde{\mathcal{E}}_i}(x_r) +$

Input: SVN decision information system $(\Omega, \tilde{\Gamma}, \beta, \mathcal{A})$.

Output: Decision-making.

- (1) Enter \mathcal{A}, β and Ω .
- (2) From Definition 5, compute the 1-SVN β -neighborhood ${}_1\tilde{\mathcal{N}}_x^\beta$.
- (3) From **Step 2** and by Definition 7, compute 1-SVN complementary β -neighborhood ${}_1\tilde{\mathcal{M}}_x^\beta$.
- (4) From **Steps 2 and 3** and by Definition 8, compute 3-1-SVN β CRSs $\mathcal{L}_3^1(\mathcal{A})$ and $\mathcal{U}_3^1(\mathcal{A})$.
- (5) Compute $\mathfrak{R}_{\mathcal{A}}$.
- (6) Compute the cosine similarity measure $\mathcal{S}(x)$.
- (7) Obtain the decision.

ALGORITHM 1: Algorithm for a 1-SVN β CRSs to make a decision.

Input: SVN decision information system $(\Omega, \tilde{\Gamma}, \beta, \mathcal{A})$.

Output: Decision-making.

- (1) Enter \mathcal{A}, β , and Ω .
- (2) From Definition 5, compute the 2-SVN β -neighborhood ${}_2\tilde{\mathcal{N}}_x^\beta$.
- (3) From **Step 2** and by Definition 9, compute the 2-SVN complementary β -neighborhood ${}_2\tilde{\mathcal{M}}_x^\beta$.
- (4) From **Steps 2 and 3** and by Definition 10, compute 3-2-SVN β CRSs $\mathcal{L}_3^2(\mathcal{A})$ and $\mathcal{U}_3^2(\mathcal{A})$.
- (5) Compute $\mathfrak{R}_{\mathcal{A}}$.
- (6) Compute the cosine similarity measure $\mathcal{S}(x)$.
- (7) Obtain the decision.

ALGORITHM 2: Algorithm for 2-SVN β CRSs to make a decision.

$\mathcal{F}_{\mathcal{C}_i}(x_r) \leq 3$. According to the presented covering methods, we propose a decision-making algorithm to obtain the result by the following steps:

Step 1: consider, for each $x_r \in \Omega$, there is at least one $y_i \in V$ such that the symptom value \mathcal{C}_i for paper x_r is not less than β (i.e., $\tilde{C}_i(x) \geq \beta$), where β is a critical value.

Step 2: consider $\mathcal{A}(x_r) = \langle d, e, f \rangle$ is the evaluation by a decision maker I , where d is a possible degree, e is an indeterminacy degree, and f is an impossible degree of A disease.

Step 3: based on this information, use Definition 10 and 3-2-SVN β CRSs model to calculate the lower and upper approximation of \mathcal{A} .

Step 4: calculate $\mathfrak{R}_{\mathcal{A}}$ by the following equation:

$$\mathfrak{R}_{\mathcal{A}} = \mathcal{U}_3^2(\mathcal{A}) \oplus \mathcal{L}_3^2(\mathcal{A}), \quad (25)$$

where $\mathcal{A} \oplus \mathcal{B} = \{ \langle x, \mathcal{T}_{\mathcal{A}}(x) + \mathcal{T}_{\mathcal{B}}(x) - \mathcal{T}_{\mathcal{A}}(x)^* \mathcal{T}_{\mathcal{B}}(x), \mathcal{F}_{\mathcal{A}}(x)^* \mathcal{F}_{\mathcal{B}}(x), \mathcal{F}_{\mathcal{A}}(x)^* \mathcal{F}_{\mathcal{B}}(x) \rangle : x \in \Omega \}$.

Step 5: calculate the decision method by the following formula.

$$\mathcal{S}(x) = \frac{\mathcal{T}_{\mathfrak{R}_{\mathcal{A}}}(x)}{\sqrt{(\mathcal{T}_{\mathfrak{R}_{\mathcal{A}}}(x))^2 + (\mathcal{F}_{\mathfrak{R}_{\mathcal{A}}}(x))^2 + (\mathcal{F}_{\mathfrak{R}_{\mathcal{A}}}(x))^2}}, \quad (26)$$

hence, ranking the alternatives.

Based on these steps, we give an algorithm to solve the decision-making problems based on Definition 10. The steps corresponding to it are summarized in Algorithm 2.

5.2. Numerical Example

Example 5. Diseased people form a set $\Omega = \{x_1, x_2, x_3, x_4, x_5\}$ and their relevant symptoms are collected by the attribute set $V = \{\text{cough}(y_1), \text{fever}(y_2), \text{sore}(y_3), \text{headache}(y_4)\}$ for A disease. Here, the following steps of the algorithm described are implemented.

Step 1: under the attribute set, doctor D estimates each patient and presents its decisions with suitable values which are summarized in Table 1.

Step 2: consider $\beta = \langle 0.5, 0.3, 0.8 \rangle$ is a critical and $\tilde{\Gamma} = \{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4\}$ is a Type-1 SVN β CRS. Then, we compute the Type-1 SVN β -neighborhood ${}_1\tilde{\mathcal{N}}_x^\beta$ and the Type-1 SVN complementary β -neighborhood ${}_1\tilde{\mathcal{M}}_x^\beta$, as shown in Tables 2 and 3.

Consider $\mathcal{A} = ((0.6, 0.3, 0.5)/x_1) + ((0.4, 0.5, 0.1)/x_2) + ((0.3, 0.2, 0.6)/x_3) + ((0.5, 0.3, 0.4)/x_4) + ((0.7, 0.2, 0.3)/x_5)$.

Step 3: by Definition 8 and 3-1-SVN β RSs model, we have the following results:

$$\begin{aligned}\mathcal{L}_3^1(\mathcal{A}) &= \{\langle x_1, 0.6, 0.3, 0.5 \rangle, \langle x_2, 0.6, 0.3, 0.4 \rangle, \langle x_3, 0.5, 0.3, 0.4 \rangle, \langle x_4, 0.4, 0.2, 0.3 \rangle, \langle x_5, 0.6, 0.3, 0.3 \rangle\}, \\ \mathcal{U}_3^1(\mathcal{A}) &= \{\langle x_1, 0.5, 0.5, 0.6 \rangle, \langle x_2, 0.4, 0.5, 0.6 \rangle, \langle x_3, 0.5, 0.5, 0.5 \rangle, \langle x_4, 0.5, 0.5, 0.6 \rangle, \langle x_5, 0.5, 0.5, 0.6 \rangle\}.\end{aligned}\quad (27)$$

Step 4: compute $\mathfrak{R}_{\mathcal{A}}$ as follows:

$$\mathfrak{R}_{\mathcal{A}} = \mathcal{L}_3^1(\mathcal{A}) \oplus \mathcal{U}_3^1(\mathcal{A}) = \{\langle x_1, 0.8, 0.15, 0.3 \rangle, \langle x_2, 0.76, 0.15, 0.24 \rangle, \langle x_3, 0.75, 0.15, 0.2 \rangle, \langle x_4, 0.7, 0.1, 0.18 \rangle, \langle x_5, 0.8, 0.15, 0.18 \rangle\}.\quad (28)$$

Step 5: according to the above information, we get $\mathcal{S}(x)$ as follows:

$$\begin{aligned}\mathcal{S}(x_1) &= 0.923, \\ \mathcal{S}(x_2) &= 0.938, \\ \mathcal{S}(x_3) &= 0.949, \\ \mathcal{S}(x_4) &= 0.959, \\ \mathcal{S}(x_5) &= 0.964,\end{aligned}\quad (29)$$

and hence, we get the ranking order as

$$\mathcal{S}(x_5) > \mathcal{S}(x_4) > \mathcal{S}(x_3) > \mathcal{S}(x_2) > \mathcal{S}(x_1).\quad (30)$$

So, by the above computation, the verdict of the decision maker D is x_5 .

Example 6. Let $\Omega = \{x_1, x_2, x_3, x_4, x_5\}$ be the set of papers, and their relevant symptoms are collected by the attribute set

$$\begin{aligned}\mathcal{L}_3^2(\mathcal{A}) &= \{\langle x_1, 0.6, 0.7, 0.5 \rangle, \langle x_2, 0.6, 0.7, 0.4 \rangle, \langle x_3, 0.5, 0.7, 0.4 \rangle, \langle x_4, 0.4, 0.7, 0.3 \rangle, \langle x_5, 0.6, 0.7, 0.3 \rangle\}, \\ \mathcal{U}_3^2(\mathcal{A}) &= \{\langle x_1, 0.5, 0.2, 0.6 \rangle, \langle x_2, 0.4, 0.2, 0.6 \rangle, \langle x_3, 0.5, 0.3, 0.5 \rangle, \langle x_4, 0.5, 0.2, 0.6 \rangle, \langle x_5, 0.5, 0.3, 0.6 \rangle\}.\end{aligned}\quad (31)$$

Step 4: compute $\mathfrak{R}_{\mathcal{A}}$ as follows:

$$\mathfrak{R}_{\mathcal{A}} = \mathcal{L}_3^2(\mathcal{A}) \oplus \mathcal{U}_3^2(\mathcal{A}) = \{\langle x_1, 0.8, 0.14, 0.3 \rangle, \langle x_2, 0.76, 0.14, 0.24 \rangle, \langle x_3, 0.75, 0.21, 0.2 \rangle, \langle x_4, 0.7, 0.14, 0.18 \rangle, \langle x_5, 0.8, 0.21, 0.18 \rangle\}.\quad (32)$$

Step 5: according to above information, we get $\mathcal{S}(x)$ as follows:

$$\begin{aligned}\mathcal{S}(x_1) &= 0.924, \\ \mathcal{S}(x_2) &= 0.939, \\ \mathcal{S}(x_3) &= 0.933, \\ \mathcal{S}(x_4) &= 0.951, \\ \mathcal{S}(x_5) &= 0.945,\end{aligned}\quad (33)$$

and hence, we get the ranking order as

$V = \{\text{spot}(y_1), \text{steak}(y_2), \text{crater}(y_3), \text{fracture}(y_4)\}$ for A paper error. Here, the following steps of the algorithm described are implemented.

Step 1: under the attribute set, investigator I estimates each paper and presents its decisions with suitable values which are summarized in Table 1.

Step 2: consider $\beta = \langle 0.5, 0.1, 0.8 \rangle$ is a critical and $\tilde{\Gamma} = \{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4\}$ is a 2-SVN β CRS. Then, we compute the Type-2 SVN β -neighborhood ${}_2\tilde{\mathcal{N}}_x^\beta$ and the Type-2 SVN complementary β -neighborhood ${}_2\tilde{\mathcal{M}}_x^\beta$, as shown in Tables 6 and 7. Consider $\mathcal{A} = ((0.6, 0.3, 0.5)/x_1) + ((0.4, 0.5, 0.1)/x_2) + ((0.3, 0.2, 0.6)/x_3) + ((0.5, 0.3, 0.4)/x_4) + ((0.7, 0.2, 0.3)/x_5)$.

Step 3: by Definition 10 and 3-2-SVN β RSs model, we have the following results:

$$\mathcal{S}(x_4) > \mathcal{S}(x_5) > \mathcal{S}(x_2) > \mathcal{S}(x_3) > \mathcal{S}(x_1).\quad (34)$$

So, by the above calculations, the verdict of the decision maker I is x_4 .

5.3. Comparative Analysis. The major purpose of our presented work is eligible to raise the lower approximation and reduce the upper approximation of the previous study by Wang and Zhang's methods [46, 47], as visible in Examples 2 and 4. To clarify the comparisons between Wang and Zhang's methods [46, 47] and our methods, the sorting outcomes of these decision-making models are listed in Table 10 for 1-SVN β CAS and Table 11 for 2-SVN β CAS.

TABLE 10: Sorting outcomes for 1-SVN β CAS.

Different methods	Obtain a decision
Wang and Zhang's model [46]	$x_5 > x_1 > x_2 > x_3 > x_4$
Our model	$x_5 > x_4 > x_3 > x_2 > x_1$

TABLE 11: Sorting outcomes for 2-SVN β CAS.

Different methods	Obtain a decision
Wang and Zhang's model [47]	$x_5 > x_1 > x_2 > x_4 > x_3$
Our model	$x_4 > x_5 > x_2 > x_3 > x_1$

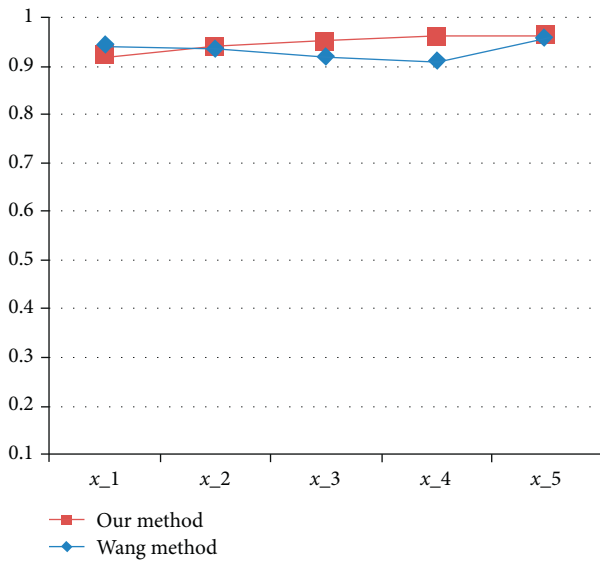


FIGURE 1: The representations of the results by using our model and Wang and Zhang's model [46].

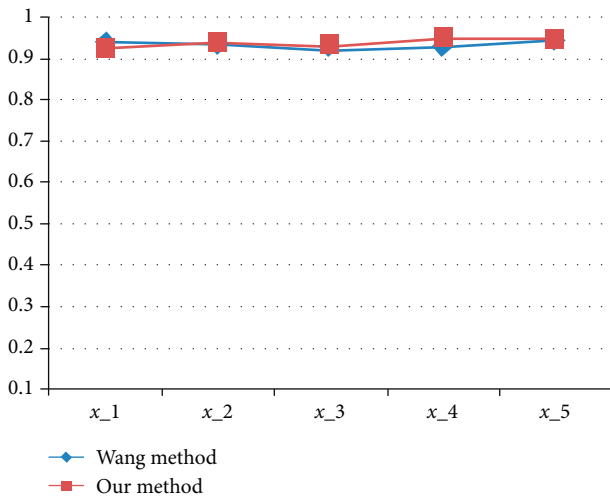


FIGURE 2: The representations of the results by using our model and Wang and Zhang's model [47].

An easy way to explain these outcomes, see Figures 1 and 2 which simplify the comparisons between our presented method and the previous one.

Figure 1 explained the differences between the outcomes using our model (3-1-SVN β CAS) and the last one (1-1-SVN β CRSs). Furthermore, Figure 2 illustrated the comparisons between the values through our model (3-2-SVN β CAS) and the previous one (1-2-SVN β CRSs). Thus, there are slight differences among these distinct methods, and these variations made our model better than others.

6. Conclusion

This work is extended to Wang and Zhang's studies in [46, 47]. We presented the definitions of 1-SVN complementary β -neighborhoods and 2-SVN complementary β -neighborhoods. We use them to set up new models of 1-SVN β -CRS and 2-SVN β -CRS, respectively. Moreover, by merging the Type-1 neighborhoods (resp., Type-2 neighborhoods) and Type-1 complementary neighborhoods (resp., Type-2 complementary neighborhoods), we obtain two new types of Type-1 neighborhoods and Type-2 neighborhoods, respectively. Thus, two new classes of 1-SVN β -CRS and 2-SVN β -CRS are investigated. To explain the differences between these new and older types of covering methods, see Examples 2 and 4. For more clarification about them, see Figures 1 and 2. There are some issues in these two covering methods:

- (1) If $\beta = (0.5, 0.1, 0.8)$ in Example 2, then $\tilde{\Gamma}$ is not 1-SVN β CRSs, but it is applicable in 2-SVN β CRSs
- (2) If $\beta = (0.5, 0.3, 0.8)$ in Example 4, then $\tilde{\Gamma}$ is not 2-SVN β CRSs, but it is applicable in 1-SVN β CRSs

In short, the two methods are considered complementary to each other, which means if there are some failures in 1-SVN β CRSs, the 2-SVN β CRSs is working instead and vice versa.

In the future, we can extend the results of this study as a combination between 1-SVN (or 2-SVN) complementary β -neighborhoods and published papers (see [50–55]). In addition, one may investigate further based on 1-SVN (or 2-SVN) complementary β -neighborhoods with some links to topology as in [26, 48]. Finally, there are many areas (for example, several comparative of this proposed method) which can be presented by researchers in the next paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Intuitionistic Fuzzy Dombi Hybrid Decision-Making Method and Their Applications to Enterprise Financial Performance Evaluation

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] C. Jana, G. Muhiuddin, M. Pal, and D. Al-Kadi, "Intuitionistic Fuzzy Dombi Hybrid Decision-Making Method and Their Applications to Enterprise Financial Performance Evaluation," *Mathematical Problems in Engineering*, vol. 2021, Article ID 3218133, 14 pages, 2021.

Research Article

Intuitionistic Fuzzy Dombi Hybrid Decision-Making Method and Their Applications to Enterprise Financial Performance Evaluation

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In the era of the knowledge-based economy, the active branch of information technology plays a crucial role. The enterprise administration covers efficient changes, and it has been entered in the age of reasonable management argument. The standard enterprise financial review evaluation centers on the importance of bondholders. The investor takes operational data as an issue and pays surveillance to the study of material attraction and the result. Otherwise, it is not intelligent to adjust in a modern marketplace period. Therefore, enterprise financial directing the interests of shareholders and business policies that are taking into account stakeholders' needs is continually investigated in the future lively competition. In that view, accumulating data is an essential research tool to draw the researchers' recent attention during the knowledge investigation. In this research, multiple attribute decision-making (MADM) approach has been proposed for the enterprise financial performance evaluation. To this view, financial performance evaluation has been done with intuitionistic fuzzy arguments. We apply new Dombi hybrid operators such as the intuitionistic fuzzy Dombi hybrid average (IFDHPWA) operator and intuitionistic fuzzy Dombi hybrid geometric (IFDHPWG) operator. These operators have a good advantage of adaptability in the working parameter. Finally, a realistic instance for enterprise financial performance is reported following comment on the benefit or utility of the recommended output.

1. Introduction

Financial control is an integral part of administration examination which shows how companies exploit proper approaches to generate and keep up reasonable dominance. Nowadays, the study on contest has enlarged exponentially. Mintzberg et al. [1] condemned excessively explanatory orientation, upper administration slant, and an absence of thoughtfulness regarding taking movement and learning and disregarding the components that prompt those formations of methodologies. Shrivastava [2] focused on his examination of hierarchical learning forms; this can offer bits of knowledge into these apparent disadvantages. The hierarchical information is justified as the groundwork for gaining defended aggressive capabilities and a resolution

alternative of the exhibition of firm performance [3]. Moreover, a few studies set up evidence of the beneficial connection between policy-making learning and obstinate execution. For example, the learning intention has an immediate impact on firm production [4]. Ussahawanitchakit [5] executed advanced model research on the economy and obtained related issues.

The enterprise financial performance appraisal is not only a growth of the market economy at a particular time. But, there also a precise approach to supervising the enterprises for a nation with the contemporary market economy. It is the direction to manage successfully with modern enterprise management. The enterprise management team must learn about the foreign market economy, which can supervise its performance. The methods of

contemporary enterprise management systems change rapidly and face the challenges of a rapidly developing economy. Also, a policy of offering judgment in which our business community plays an especial operation improving the management level, sharpening the business competitiveness, and further increasing the industrial growth quality is shown. Numerous MCDM or MADM approaches (review from business management, financial, and strategic management) have been considered using aggregation operators (AOs) in probabilistic notion [6–8]. The ordered weighted averaging induces geometric operators and then utilizes these to construct strategic decision theory [9]. The management study [10] based on game theory has been developed in the IFS environment. The survey of aggregation under the IFS environment is a vital research machine in decision theory. In the next part, we shortly overview some related decision-making-based problems. It is hard to take absolute values of attributes because of complexity level to be higher in the field of decision science. Zadeh [11] started the hypothesis of fuzzy sets (FSs) new mathematical knowledge of handling MADM issues efficiently and multiattribute group decision-making (MAGDM) problems.

However, FS cannot handle complex fuzzy information because it expresses only membership degree. In that situation, Atanassov [12] defined IFSSs, which easily handle complex FS information. IFS addressed an object characterized by membership degree (MD) and non-membership degree (NMD). In the preceding several decades, researchers have more thought about IFSSs and interval-valued intuitionistic fuzzy sets (IVIFSSs) because these thoughts have been connected to many practical results, such as medical science, decision-making, pattern recognition, and clustering. Recently, many tremendous works have been developed in the IFS environment, such as De et al. [13] defined operation on IFSSs, Szmidt and Kacprzyk [14] measured the similarity between IFSSs, and Guo and Song [15] proposed entropy between IFSSs. It is generally seen that the theory of IFSSs is used to manage the MAGDM issues and clustering algorithms for economic risk evaluation utilizing MCDM procedures [16], Li et al. [17] introduced an MADM method using Hassdraf's distance measure generalized fuzzy numbers, Garg [18] proposed a generalized improved score function of IVIFSSs and applied it in expert systems, and Chen and Chiou [19] solved MADM problems based on IVIFSSs using PSO procedures and evidential logic methodology. Kumar and Garg [20] utilized the technique for order of preference by similarity to ideal solution (TOPSIS) process based on set pair evaluation relationship under the IVIFSSs environment. Li and Pen [21] pointed out the MADM method using the amount and reliability of IFS information. Lourenzutti and Krohling [22] studied TODIM (an acronym in Portuguese for interactive multicriteria decision making) problems based on IFS random methodology.

Nowadays, information AOs are a major research topic in the multiattribute group decision-making (MAGDM) environment and become a concentration of the researchers to developed critical works [23–30]. Some traditional results

[31–38] have been developed based on aggregation operators. At present, some research studies have been made using extended AOs; for instance, Zhang et al. [39] delivered power AOs on IFSSs and Liu and Yu [40] used density operators for IFSSs or IVIFSSs, respectively. Wu and Su [41] introduced prioritized AOs for IFSSs that bear in mind priorities of characteristics using precedence weights. In [42, 43], a few AOs based on algebraic operational laws for IFSSs are suggested, which is an exceptional case of triangular norms. The Archimedean triangular norms are the Hamacher norms, Algebraic norms, Einstein norms, Frank norms, and Dombi norms. Liu [44] utilized Hamacher AOs on IVIFSSs and developed a MAGDM model. Zhang [45] proposed Frank AOs for IVIFSSs and their applications to MAGDM procedures. Einstein hybrid AOs [46] for IFSSs are applied to the MADM approach. Yu [47] introduced Choquet AOs based on Einstein norms for IFS. Dombi triangular norms are general triangular norms, which can deal with the data collection process more adaptable by a parameter. Dombi [48] provided Dombi norms which have good operational flexibility to find the results. For this advantage, Bonferroni operations to IFSSs are applied it to develop a MAGDM [49] problem. Chen and Ye [50] referred MADM problem using Dombi AOs in the SVN environment. Some research studies [51, 52] are developed based on Dombi norms in the different uncertain fuzzy environment, and hybrid aggregation operators [53, 54] in intuitionistic ambiguous environment have been motivated to study the proposed work. They are gripping in mind that the IFS has a powerful technique to model the vague and imprecise knowledge that appears in real-world problems. The decision-making problems in a complex fuzzy environment under Dombi operations present sufficient motivation to improve our present paper. The central object of this paper is to develop an MADM approach based on intuitionistic Dombi hybrid operators. The objectives of this paper are as follows:

- A new approach is developed in connecting with intuitionistic fuzzy Dombi hybrid operators

- The proposed operators are utilized for IFMADM approach

- An illustrative example is demonstrated by a numerical example

- Superiority of the proposed method is verified numerically.

The organization of the paper is sorted as follows: in the next section, some basic ideas of IFNs are briefly reviewed, and their operational laws and some operators are presented. Intuitionistic fuzzy Dombi weighted averaging, order weighted averaging, and hybrid weighted averaging operator are defined in Section 3. In Section 4, intuitionistic fuzzy Dombi hybrid weighted geometric operator, order weighted geometric operator, and hybrid weighted geometric operators are mentioned. In Section 5, we have applied IFDHW and IFDHWG operators to construct MADM problems. An illustrative example is given for the preference of the best choice of the alternatives in Section 6. Section 7 incorporates

assured comparative discussions with different broadly used MADM procedures. In Section 8, some comments are given to the paper.

2. Preliminaries

In this section, we recall a few fundamental ideas associated with IFSs over the universe of discourse X .

Definition 1 (see [55]). An IFS over the fixed set X is interpreted as

$$\tilde{I} = \{ \langle x, \mu(x), \nu(x) \rangle \mid x \in X \}, \quad (1)$$

where $\mu(x): X \rightarrow [0, 1]$ follows MD and $\nu(x): X \rightarrow [0, 1]$ indicates NMD for an element $x \in X$ to an IFS. $\pi(x) = 1 - \mu(x) - \nu(x)$ is the depicted degree of indeterminacy for an element x to the set \tilde{I} . $\langle (\mu, \nu) \rangle$ is denoted as intuitionistic fuzzy elements (IFE) or intuitionistic fuzzy values (IFVs).

Reference [32] introduced the IFWA operator, IFOWA operator, and IFHWA operator as follows. for which $(\sigma(1), \sigma(2), \dots, \sigma(\eta))$ is a permutation of $(1, 2, \dots, \eta)$, where $\tilde{I}_{\sigma(\phi-1)} \geq \tilde{I}_{\sigma(\phi)}$ for every $\phi = 1, 2, \dots, \eta$. where $\tilde{\Omega}_{\sigma(\phi)}$ is the ϕ -th largest weighted IFV ($\tilde{\Omega}_{\phi} = \eta \psi_{\phi} \tilde{\Omega}_{\phi}$, $(\phi = 1, 2, \dots, \eta)$ and $\psi = (\psi_1, \psi_2, \dots, \psi_{\eta})^T$ be the weight vector of $\tilde{\Omega}_{\phi}$ with $\psi_{\phi} > 0$ and $\sum_{\phi=1}^{\eta} \psi_{\phi} = 1$.

Definition 2 (see [32]). Let $\tilde{\Omega}_{\phi} = (\mu_{\phi}, \nu_{\phi})$ ($\phi = 1, 2, \dots, \eta$) be a group of IFEs. The IFWA operator of dimension η is a function $\tilde{IFE}^{\eta} \rightarrow \tilde{IFE}$ with weighting vector $\psi = (\psi_1, \psi_2, \dots, \psi_{\eta})^T$ and react as $\psi > 0$ and $\sum_{t=1}^{\eta} \psi_{\phi} = 1$; then,

$$\begin{aligned} \text{IFWA}_{\psi}(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_{\eta}) &= \oplus_{\phi=1}^{\eta} (\psi_{\phi} \tilde{\Omega}_{\phi}) \\ &= \left(1 - \prod_{\phi=1}^{\eta} (1 - \mu_{\phi})^{\psi_{\phi}}, \prod_{\phi=1}^{\eta} \nu_{\phi}^{\psi_{\phi}} \right). \end{aligned} \quad (2)$$

Definition 3 (see [32]). Let $\tilde{\Omega}_{\phi} = (\mu_{\phi}, \nu_{\phi})$ ($\phi = 1, 2, \dots, \eta$) be a group of IFEs. An IFOWA operator is a function with dimension η defined as $\tilde{IFE}^{\eta} \rightarrow \tilde{IFE}$ with acting weight vector $\psi = (\psi_1, \psi_2, \dots, \psi_{\eta})^T$ such that $\psi > 0$ and $\sum_{\phi=1}^{\eta} \psi_{\phi} = 1$. Furthermore,

$$\begin{aligned} \text{IFOWA}_{\psi}(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_{\eta}) &= \oplus_{\phi=1}^{\eta} (\psi_{\phi} \tilde{\Omega}_{\sigma(\phi)}) \\ &= \left(1 - \prod_{\phi=1}^{\eta} (1 - \mu_{\sigma(\phi)})^{\psi_{\phi}}, \prod_{\phi=1}^{\eta} \nu_{\sigma(\phi)}^{\psi_{\phi}} \right), \end{aligned} \quad (3)$$

Definition 4 (see [32]). Let $\tilde{\Omega}_{\phi} = (\mu_{\phi}, \nu_{\phi})$ ($\phi = 1, 2, \dots, \eta$) be a group of IFEs. A function IFHWA of dimension η defined as $\tilde{IFE}^{\eta} \rightarrow \tilde{IFE}$ is weight vector acting $\psi = (\psi_1, \psi_2, \dots, \psi_{\eta})^T$ with $\psi > 0$ and $\sum_{\phi=1}^{\eta} \psi_{\phi} = 1$. Furthermore,

$$\begin{aligned} \text{IFHWA}_{\psi}(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_{\eta}) &= \oplus_{\phi=1}^{\eta} (\psi_{\phi} \tilde{\Omega}_{\sigma(\phi)}) \\ &= \left(1 - \prod_{\phi=1}^{\eta} (1 - \mu_{\sigma(\phi)})^{\psi_{\phi}}, \prod_{\phi=1}^{\eta} \nu_{\sigma(\phi)}^{\psi_{\phi}} \right), \end{aligned} \quad (4)$$

Xu [33] also used weighted geometric operators such as IFWG, IFOWG, and IFHWG operators.

We defined score and accuracy functions [54] as follows.

Definition 5 (see [36]). Let $U = (\mu_U, \nu_U)$ be an IFEs. Then, score \tilde{U} for IFEs is computed as follows:

$$\Lambda(U) = \frac{1 + \mu_U - \nu_U}{2}, \quad \Lambda(U) \in [0, 1], \quad (5)$$

and accuracy function Φ for IFEs is evaluated as follows:

$$\Phi(U) = \frac{\mu_U + \nu_U}{2}, \quad \Phi(U) \in [0, 1]. \quad (6)$$

On the basis of $\Lambda(U)$ and $\Phi(U)$, we used order relation between two IFEs $U = (\langle \mu_U(x), \nu_U(x) \rangle)$ and $V = (\langle \mu_V(x), \nu_V(x) \rangle)$ as follows.

Definition 6 (see [36]). Let U and V be any two IFEs. Then,

- (i) If $\Lambda(U) < \Lambda(V)$, then $U < V$
- (ii) If $\Lambda(U) > \Lambda(V)$, then $U > V$
- (iii) If $\Lambda(U) = \Lambda(V)$, then
 - (1) If $\Phi(U) < \Phi(V)$, then $U < V$
 - (2) If $\Phi(U) > \Phi(V)$, then $U > V$
 - (3) If $\Phi(U) = \Phi(V)$, then $U \sim V$

In the following section, we defined Dombi operator [48].

2.1. Dombi Operations on IFEs. Dombi triangular norms and conorms are defined as follows.

Definition 7 (see [48]). Let us take p and q as any two real numbers. Then,

$$\begin{aligned} \text{Dom}(p, q) &= \frac{1}{1 + \{((1-p)/p)^e + ((1-q)/q)^e\}^{1/e}}, \\ \text{Dom}^*(p, q) &= 1 - \frac{1}{1 + \{(p/(1-p))^e + (q/(1-q))^e\}^{1/e}}, \end{aligned} \quad (7)$$

where $e \geq 1$ and $(p, q) \in [0, 1] \times [0, 1]$.

In view of Dom-norms and Dom-conorms, we explained Dombi operations with respect to IFEs.

Let $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ be two IFEs and $\lambda > 0$, then Dom-product and Dom-sum of $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are, respectively, denoted as $(\tilde{\Omega}_1 \otimes \tilde{\Omega}_2)$ and $(\tilde{\Omega}_1 \oplus \tilde{\Omega}_2)$ given as follows:

$$(i) \tilde{\Omega}_1 \oplus \tilde{\Omega}_2 = \left\langle 1 - \frac{1}{1 + \{(\mu_1/(1 - \mu_1))^e + (\mu_2/(1 - \mu_2))^e\}^{1/e}}, \frac{1}{1 + \{((1 - \nu_1)/\nu_1)^e + ((1 - \nu_2)/\nu_2)^e\}^{1/e}} \right\rangle, \quad (8)$$

$$(ii) \tilde{\tilde{\Omega}}_1 \otimes \tilde{\Omega}_2 = \left\langle \frac{1}{1 + \{((1 - \mu_1)/\mu_1)^e + ((1 - \mu_2)/\mu_2)^e\}^{1/e}}, 1 - \frac{1}{1 + \{(\nu_1/(1 - \nu_1))^e + (\nu_2/(1 - \nu_2))^e\}^{1/e}} \right\rangle, \quad (9)$$

$$(iii) \lambda \cdot \tilde{\Omega}_1 = \left\langle 1 - \frac{1}{1 + \{\lambda(\mu_1/(1 - \mu_1))^e\}^{1/e}}, \frac{1}{1 + \{\lambda((1 - \nu_1)/\nu_1)^e\}^{1/e}} \right\rangle, \quad (10)$$

$$(iv) (\tilde{\Omega}_1)^\lambda = \left\langle \frac{1}{1 + \{\lambda((1 - \mu_1)/\mu_1)^e\}^{1/e}}, 1 - \frac{1}{1 + \{\lambda(\nu_1/(1 - \nu_1))^e\}^{1/e}} \right\rangle. \quad (11)$$

3. Intuitionistic Fuzzy Dombi Aggregation Operator

To this section, we propose Dombi arithmetic AOs with IFEs such as intuitionistic fuzzy Dombi weighted averaging (IFDWA) operator, intuitionistic fuzzy Dombi ordered weighted averaging (IFDOWA) operator, and intuitionistic fuzzy Dombi hybrid weighted averaging (IFDHWA) operator.

3.1. Intuitionistic Fuzzy Dombi Hybrid Averaging Operator. In this section, we introduce IFDHWA operator.

Definition 8 (see [55]). Let $\tilde{\Omega}_\phi = (\mu_\phi, \nu_\phi)$ ($\phi = 1, 2, \dots, \eta$) be a group of IFEs. Then, IFDWA operator is a function $\text{IFE}^\eta \rightarrow \text{IFE}$ such that

$$\begin{aligned} \text{IFDWA}_\psi(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_\eta) &= \oplus_{\phi=1}^\eta (\psi_\phi \tilde{\Omega}_\phi) \\ &= \left\langle 1 - \frac{1}{1 + \{\sum_{\phi=1}^\eta \psi_\phi (\mu_\phi/(1 - \mu_\phi))^e\}^{1/e}}, \frac{1}{1 + \{\sum_{\phi=1}^\eta \psi_\phi ((1 - \nu_\phi)/\nu_\phi)^e\}^{1/e}} \right\rangle, \end{aligned} \quad (12)$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_\eta)^T$ be the weight vector of $\tilde{\Omega}_\phi$ ($\phi = 1, 2, \dots, \eta$) with $\psi_\phi > 0$ and $\sum_{\phi=1}^\eta \psi_\phi = 1$.

Now, we introduce IFDOWA operator.

Definition 9 (see [36]). Let $\tilde{\Omega}_\phi = (\mu_\phi, \nu_\phi)$ ($t = 1, 2, \dots, n$) be a number of IFEs. The IFDOWA operator of dimension η is a function IFDOWA: $\text{IFE}^\eta \rightarrow \text{IFE}$ with related vector $\psi = (\psi_1, \psi_2, \dots, \psi_\eta)^T$ such that $\psi_t > 0$ and $\sum_{\phi=1}^\eta \psi_\phi = 1$. Therefore,

$$\begin{aligned} \text{IFDOWA}_\psi(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_\eta) &= \oplus_{\phi=1}^\eta (\psi_\phi \tilde{\Omega}_{\phi(\sigma)}) \\ &= \left\langle 1 - \frac{1}{1 + \{\sum_{\phi=1}^\eta \psi_\phi (\mu_{\phi(\sigma)}/(1 - \mu_{\phi(\sigma)}))^e\}^{1/e}}, \frac{1}{1 + \{\sum_{\phi=1}^\eta \psi_\phi ((1 - \nu_{\phi(\sigma)})/\nu_{\phi(\sigma)})^e\}^{1/e}} \right\rangle, \end{aligned} \quad (13)$$

where $(\sigma(1), \sigma(2), \dots, \sigma(\eta))$ are the permutation of $(\phi = 1, 2, \dots, \eta)$, for which $\tilde{\Omega}_{\sigma(\phi-1)} \geq \tilde{\Omega}_{\sigma(\phi)}$ for all $\phi = 1, 2, \dots, \eta$.

In Definitions 8 and 9, the IFDWA considered the weights of the IFVs whereas the IFDOWA operator used the weights in their ordered locations of IFVs instead of IFVs weighting themselves. Thus, weighting for IFDWA and IFDOWA is used in different aspects. However, they are

produced one time only. To overcome this happening, we focus on IFDHA operator.

Definition 10. The IFDHWA operator of dimension η is a function IFDHWA: $\text{IFE}^\eta \rightarrow \text{IFE}$, with related weight vector $W = (w_1, w_2, \dots, w_\eta)$ such that $w_\phi > 0$ and $\sum_{\phi=1}^\eta w_\phi = 1$. Therefore, IFDHWA operator can be evaluated as

$$\text{IFDHW A}_{\psi}(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_{\eta}) = \bigoplus_{t=1}^{\eta} \left(\psi_t \dot{\tilde{\Omega}}_{\sigma(\phi)} \right) \quad (14)$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} w_{\phi} \left(\dot{\mu}_{\sigma(\phi)} / (1 - \dot{\mu}_{\sigma(\phi)}) \right)^{\eta} \right\}^{1/\eta}}, \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} w_{\phi} \left((1 - \dot{\nu}_{\sigma(\phi)}) / \dot{\nu}_{\sigma(\phi)} \right)^{\eta} \right\}^{1/\eta}} \right\rangle,$$

where, $\dot{\tilde{\Omega}}_{\sigma(\phi)}$ is the ϕ^{th} biggest weighted IFVs $\tilde{\Omega}_{\phi}$ ($\tilde{\Omega}_{\phi} = \eta \psi_{\phi} \Omega_{\phi}$), $\phi = 1, 2, \dots, \eta$, and weighting vector be $\psi = (\psi_1, \psi_2, \dots, \psi_{\eta})^T$ of $\tilde{\Omega}_{\phi}$ with $\psi_{\phi} > 0$ and $\sum_{\phi=1}^{\eta} \psi_{\phi} = 1$, where η is the equipoise coefficient.

Example 1. There are four IFEs $\tilde{\Omega}_1 = (0.5, 0.3)$, $\tilde{\Omega}_2 = (0.6, 0.3)$, $\tilde{\Omega}_3 = (0.7, 0.3)$, and $\tilde{\Omega}_4 = (0.2, 0.4)$ and $\psi = (0.20, 0.30, 0.30, 0.20)^T$ is the weight vector of these four IFEs and $W = (0.2, 0.1, 0.3, 0.4)^T$ is the associated weight vector. Then, by Definition 10, for aggregated of IFEs for $\kappa = 3$, by the way,

$$\begin{aligned} \dot{\tilde{\Omega}}_1 &= \left\langle \left(1 - \frac{1}{1 + \{4 \times 0.20 \times (0.5 / (1 - 0.5))^3\}^{1/3}}, \frac{1}{1 + \{4 \times 0.20 \times ((1 - 0.3) / 0.3)^3\}^{1/3}} \right) \right\rangle \\ &= \langle 0.4814, 0.3158 \rangle, \\ \dot{\tilde{\Omega}}_2 &= \left\langle \left(1 - \frac{1}{1 + \{4 \times 0.30 \times (0.6 / (1 - 0.6))^3\}^{1/3}}, \frac{1}{1 + \{4 \times 0.30 \times ((1 - 0.3) / 0.3)^3\}^{1/3}} \right) \right\rangle \\ &= \langle 0.6145, 0.2874 \rangle, \\ \dot{\tilde{\Omega}}_3 &= \left\langle \left(1 - \frac{1}{1 + \{4 \times 0.30 \times (0.7 / (1 - 0.7))^3\}^{1/3}}, \frac{1}{1 + \{4 \times 0.30 \times ((1 - 0.3) / 0.3)^3\}^{1/3}} \right) \right\rangle \\ &= \langle 0.7126, 0.2874 \rangle, \\ \dot{\tilde{\Omega}}_4 &= \left\langle \left(1 - \frac{1}{1 + \{4 \times 0.20 \times (0.2 / (1 - 0.2))^3\}^{1/3}}, \frac{1}{1 + \{4 \times 0.20 \times ((1 - 0.4) / 0.4)^3\}^{1/3}} \right) \right\rangle \\ &= \langle 0.1884, 0.4180 \rangle. \end{aligned} \quad (15)$$

Scores of $\tilde{\Omega}_t$ ($t = 1, 2, 3, 4$) are calculated as follows:

$$\begin{aligned} \Lambda(\dot{\tilde{\Omega}}_1) &= \frac{1 + 0.4814 - 0.3158}{2} = 0.5828, \\ \Lambda(\dot{\tilde{\Omega}}_2) &= \frac{1 + 0.6145 - 0.2874}{2} = 0.6636, \\ \Lambda(\dot{\tilde{\Omega}}_3) &= \frac{1 + 0.7126 - 0.2874}{2} = 0.7126, \\ \Lambda(\dot{\tilde{\Omega}}_4) &= \frac{1 + 0.1884 - 0.4180}{2} = 0.3852. \end{aligned} \quad (16)$$

Since

$$\Lambda(\dot{\tilde{\Omega}}_3) > \Lambda(\dot{\tilde{\Omega}}_2) > \Lambda(\dot{\tilde{\Omega}}_1) > \Lambda(\dot{\tilde{\Omega}}_4), \quad (17)$$

then

$$\begin{aligned} \dot{\tilde{\Omega}}_{\sigma(1)} &= \dot{\tilde{\Omega}}_3 = (0.7126, 0.2874), \\ \dot{\tilde{\Omega}}_{\sigma(2)} &= \dot{\tilde{\Omega}}_2 = (0.6145, 0.2874), \\ \dot{\tilde{\Omega}}_{\sigma(3)} &= \dot{\tilde{\Omega}}_1 = (0.4814, 0.3158), \\ \dot{\tilde{\Omega}}_{\sigma(4)} &= \dot{\tilde{\Omega}}_4 = (0.1884, 0.4180). \end{aligned} \quad (18)$$

Therefore, aggregated values of IFEs for $\varrho = 3$ by IFDHW operator are as follows:

$$\begin{aligned}
\text{IFDHWG}_w(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_4) &= \oplus_{q=1}^4 \left(\omega_q \dot{\tilde{\Omega}}_{\sigma(q)} \right) \\
&= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{\phi=1}^4 w_{\phi} \left(\dot{\mu}_{\sigma(\phi)} / (1 - \dot{\mu}_{\sigma(\phi)}) \right)^3 \right\}^{1/3}}, \frac{1}{1 + \left\{ \sum_{\phi=1}^4 w_{\phi} \left((1 - \dot{\nu}_{\sigma(\phi)}) / \dot{\nu}_{\sigma(\phi)} \right)^3 \right\}^{1/3}} \right\rangle \\
&= \left\langle 1 - \frac{1}{1 + \left\{ 0.2 \times (0.7126 / (1 - 0.7126))^3 + 0.1 \times (0.6145 / (1 - 0.6145))^3 + 0.3 \times (0.4814 / (1 - 0.4814))^3 + 0.4 \times (0.1884 / (1 - 0.1884))^3 \right\}^{1/3}}, \right. \\
&\quad \left. \frac{1}{1 + \left\{ 0.2 \times ((1 - 0.2874) / 0.2874)^3 + 0.1 \times ((1 - 0.2874) / 0.2874)^3 + 0.3 \times ((1 - 0.3158) / 0.3158)^3 + 0.4 \times ((1 - 0.4180) / 0.4180)^3 \right\}^{1/3}} \right\rangle \\
&= \langle (0.6073, 0.3271) \rangle.
\end{aligned} \tag{19}$$

4. Intuitionistic Fuzzy Dombi Geometric Aggregation Operators

To this section, we propose Dombi geometric AOs with IFEs such as intuitionistic fuzzy Dombi weighted geometric (IFDWG) operator, intuitionistic fuzzy Dombi ordered weighted geometric (IFDOWG) operator, and intuitionistic fuzzy Dombi hybrid weighted geometric (IFDHWG) operator.

4.1. Intuitionistic Fuzzy Dombi Hybrid Geometric Operator. In this section, we introduce IFDHWG operator.

Definition 11 (see [36]). Assume that $\tilde{\Omega}_{\phi} = (\mu_{\phi}, \nu_{\phi})$ ($\phi = 1, 2, \dots, \eta$) is a group of IFEs. Then, IFDWG operator is a function $\text{IFE}^{\eta} \rightarrow \text{IFE}$ such that

$$\begin{aligned}
\text{IFDWG}_{\psi}(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_{\eta}) &= \otimes_{\phi=1}^{\eta} (\tilde{\Omega}_{\phi})^{\psi_{\phi}} \\
&= \left\langle \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} \psi_{\phi} \left((1 - \mu_{\phi}) / \mu_{\phi} \right)^{\eta} \right\}^{1/\eta}}, 1 - \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} \psi_{\phi} \left(\nu_{\phi} / (1 - \nu_{\phi}) \right)^{\eta} \right\}^{1/\eta}} \right\rangle,
\end{aligned} \tag{20}$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_{\eta})^T$ be the weight vector of $\tilde{\Omega}_{\phi}$ ($\phi = 1, 2, \dots, \eta$) such that $\psi_{\phi} > 0$ and $\sum_{\phi=1}^{\eta} \psi_{\phi} = 1$.

Now, we introduce IFDOWG operator.

Definition 12 (see [36]). Let $\tilde{\Omega}_{\phi} = (\mu_{\phi}, \nu_{\phi})$ ($\phi = 1, 2, \dots, \eta$) be a group of IFEs. An IFDOWG operator of dimension η is a function IFDOWG: $\text{IFE}^{\eta} \rightarrow \text{IFE}$ with related weight vector $w = (w_1, w_2, \dots, w_{\eta})^T$ such that $w_{\phi} > 0$ and $\sum_{\phi=1}^{\eta} w_{\phi} = 1$. Therefore,

$$\begin{aligned}
\text{IFDOWG}_w(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_{\eta}) &= \otimes_{\phi=1}^{\eta} (\tilde{\Omega}_{\phi(\sigma)})^{w_{\phi}} \\
&= \left\langle \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} w_{\phi} \left((1 - \mu_{\sigma(\phi)}) / \mu_{\sigma(\phi)} \right)^{\eta} \right\}^{1/\eta}}, 1 - \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} w_{\phi} \left(\nu_{\sigma(\phi)} / (1 - \nu_{\sigma(\phi)}) \right)^{\eta} \right\}^{1/\eta}} \right\rangle,
\end{aligned} \tag{21}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(\eta))$ are the permutation of $(\phi = 1, 2, \dots, \eta)$, for which $\tilde{\Omega}_{\sigma(\phi-1)} \geq \tilde{\Omega}_{\sigma(\phi)}$ for all $\phi = 1, 2, \dots, \eta$.

In Definitions 11 and 12, the IFDWG considered the weights of the IFVs, however, in IFDOWG weights considered the ordered ground of the IFVs in view of weights of the IFVs among. Thus, the weighting for IFDWG and IFDOWG is

utilized in various ways. However, they are produced only one time. To overcome this case, we introduce IFDHWG operator.

Definition 13. An IFDHWG operator of dimension η is a function IFDHWG: $\text{IFE}^{\eta} \rightarrow \text{IFE}$, with related weight vector $w = (w_1, w_2, \dots, w_{\eta})$ such that $w_{\phi} > 0$ and $\sum_{\phi=1}^{\eta} w_{\phi} = 1$. Therefore, IFDHWG operator can be evaluated as

$$\begin{aligned} \text{IFDHWG}_{\psi, w}(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_\eta) &= \otimes_{\phi=1}^{\eta} \left(\dot{\tilde{\Omega}}_{\sigma(\phi)} \right)^{w_\phi} \\ &= \left\langle \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} w_\phi \left((1 - \dot{\mu}_{\sigma(\phi)}) / \dot{\mu}_{\sigma(\phi)} \right)^{\varrho} \right\}^{1/\varrho}}, 1 - \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} w_\phi \left(\dot{\nu}_{\sigma(\phi)} / (1 - \dot{\nu}_{\sigma(\phi)}) \right)^{\varrho} \right\}^{1/\varrho}} \right\rangle, \end{aligned} \quad (22)$$

where $\dot{\tilde{\Omega}}_{\sigma(\varrho)}$ is the ϕ^{th} largest weighted IFVs ($\dot{\tilde{\Omega}}_\phi = \eta \psi_\phi I_\phi$, $\phi = 1, 2, \dots, \eta$) and weight vector be $\psi = (\psi_1, \psi_2, \dots, \psi_\eta)^T$ of \tilde{I}_ϕ with $\psi_\phi > 0$ and $\sum_{\phi=1}^{\eta} \psi_\phi = 1$, where η is the equipoise coefficient.

Example 2. There are four IFEs $\tilde{\Omega}_1 = (0.5, 0.3)$, $\tilde{\Omega}_2 = (0.6, 0.3)$, $\tilde{\Omega}_3 = (0.7, 0.3)$, and $\tilde{\Omega}_4 = (0.2, 0.4)$, and $\psi = (0.20, 0.30, 0.30, 0.20)^T$ is the weight vector of these four IFEs and $W = (0.2, 0.1, 0.3, 0.4)^T$ is the associated weight vector. Then, by Definition 13, for aggregated of IFEs for ($\kappa = 3$), by the way,

$$\begin{aligned} \dot{\tilde{\Omega}}_1 &= \left\langle \left(\frac{1}{1 + \{4 \times 0.20 \times ((1 - 0.5)/0.5)^3\}^{1/3}}, 1 - \frac{1}{1 + \{4 \times 0.20 \times (0.3/(1 - 0.3))^3\}^{1/3}} \right) \right\rangle \\ &= \langle (0.5186, 0.2846) \rangle, \\ \dot{\tilde{\Omega}}_2 &= \left\langle \left(\frac{1}{1 + \{4 \times 0.30 \times ((1 - 0.6)/0.6)^3\}^{1/3}}, 1 - \frac{1}{1 + \{4 \times 0.30 \times (0.3/(1 - 0.3))^3\}^{1/3}} \right) \right\rangle \\ &= \langle (0.5853, 0.3129) \rangle, \\ \dot{\tilde{\Omega}}_3 &= \left\langle \left(\frac{1}{1 + \{4 \times 0.30 \times ((1 - 0.7)/0.7)^3\}^{1/3}}, 1 - \frac{1}{1 + \{4 \times 0.30 \times (0.3/(1 - 0.3))^3\}^{1/3}} \right) \right\rangle \\ &= \langle (0.6871, 0.3129) \rangle, \\ \dot{\tilde{\Omega}}_4 &= \left\langle \left(\frac{1}{1 + \{4 \times 0.20 \times ((1 - 0.2)/0.2)^3\}^{1/3}}, 1 - \frac{1}{1 + \{4 \times 0.20 \times (0.4/(1 - 0.4))^3\}^{1/3}} \right) \right\rangle \\ &= \langle (0.2122, 0.3823) \rangle. \end{aligned} \quad (23)$$

Scores of $\tilde{\Omega}_\phi$ ($\phi = 1, 2, 3, 4$) are calculated as follows:

$$\begin{aligned} \Lambda(\dot{\tilde{\Omega}}_1) &= \frac{1 + 0.5186 - 0.2846}{2} = 0.6170, \\ \Lambda(\dot{\tilde{\Omega}}_2) &= \frac{1 + 0.5853 - 0.3129}{2} = 0.6362, \\ \Lambda(\dot{\tilde{\Omega}}_3) &= \frac{1 + 0.6871 - 0.3129}{2} = 0.6871, \\ \Lambda(\dot{\tilde{\Omega}}_4) &= \frac{1 + 0.2122 - 0.3823}{2} = 0.4150. \end{aligned} \quad (24)$$

Since

$$\Lambda(\dot{\tilde{\Omega}}_3) > \Lambda(\dot{\tilde{\Omega}}_2) > \Lambda(\dot{\tilde{\Omega}}_1) > \Lambda(\dot{\tilde{\Omega}}_4), \quad (25)$$

then

$$\begin{aligned} \dot{\tilde{\Omega}}_{\sigma(1)} &= \dot{\tilde{\Omega}}_3 = (0.6871, 0.3129), \\ \dot{\tilde{\Omega}}_{\sigma(2)} &= \dot{\tilde{\Omega}}_2 = (0.5853, 0.3129), \\ \dot{\tilde{\Omega}}_{\sigma(3)} &= \dot{\tilde{\Omega}}_1 = (0.5186, 0.2846), \\ \dot{\tilde{\Omega}}_{\sigma(4)} &= \dot{\tilde{\Omega}}_4 = (0.2122, 0.3823). \end{aligned} \quad (26)$$

Therefore, aggregated values of IFEs ($\varrho = 3$) by IFDHWG operator are as follows:

$$\begin{aligned}
\text{IFDHWG}_w(\tilde{\Omega}_1, \tilde{\Omega}_2, \dots, \tilde{\Omega}_4) &= \oplus_{\phi=1}^4 \left(\omega_{\phi} \dot{\tilde{\Omega}}_{\sigma(\phi)} \right) \\
&= \left\langle \frac{1}{1 + \left\{ \sum_{\phi=1}^4 \omega_{\phi} \left((1 - \dot{\mu}_{\sigma(\phi)}) / \dot{\mu}_{\sigma(\phi)} \right)^3 \right\}^{1/3}}, 1 - \frac{1}{1 + \left\{ \sum_{\phi=1}^4 \omega_{\phi} \left(\dot{\nu}_{\sigma(\phi)} / (1 - \dot{\nu}_{\sigma(\phi)}) \right)^3 \right\}^{1/3}} \right\rangle \\
&= \left\langle 1 - \frac{1}{1 + \left\{ 0.2 \times ((1 - 0.6871) / 0.6871)^3 + 0.1 \times ((1 - 0.5853) / 0.5853)^3 + 0.3 \times ((1 - 0.5186) / 0.5186)^3 + 0.4 \times ((1 - 0.3823) / 0.3823)^3 \right\}^{1/3}}, \right. \\
&\quad \left. \frac{1}{1 + \left\{ 0.2 \times (0.3129 / (1 - 0.3129))^3 + 0.1 \times (0.3129 / (1 - 0.3129))^3 + 0.3 \times (0.2846 / (1 - 0.2846))^3 + 0.4 \times (0.3823 / (1 - 0.3823))^3 \right\}^{1/3}} \right\rangle \\
&= \langle (0.3582, 0.3429) \rangle.
\end{aligned} \tag{27}$$

5. MADM Model Based on Hybrid Operators

For the section, we form an MADM approach applying IFS AOs in which the weight of the attributes is in real numbers, and one can apply different existing methods for the evaluation of the attribute's weight vector. Here, MADM problem is used to develop for enterprise financial performance evaluation under intuitionistic fuzzy information. Let a group of alternative be $Q = \{Q_1, Q_2, \dots, Q_{\xi}\}$ and attributes be $G = \{G_1, G_2, \dots, G_{\eta}\}$. Let a set of weight vector be $\psi = (\psi_1, \psi_2, \dots, \psi_{\eta})$ of Q_{ϕ} which are real-values where $\psi_{\phi} > 0$ and $\sum_{\phi=1}^{\eta} \psi_{\phi} = 1$. Suppose the decision

matrix $\tilde{D} = (\mu_{\xi\phi}, \nu_{\xi\phi})_{\delta \times \eta}$ where $\mu_{\xi\phi}$ is the MD considered for the alternative Q_{ϕ} satisfied under the attribute G_{ϕ} proposed by the DMs, and $\nu_{\xi\phi}$ implied NMD for the alternative Q_{ξ} does not comfort G_{ϕ} stated by the DMs, where $\mu_{\xi\phi} \in [0, 1]$ and $\nu_{\xi\phi} \in [0, 1]$ such that $0 \leq \mu_{\xi\phi} + \nu_{\xi\phi} \leq 1$, ($\xi = 1, 2, \dots, \delta$) and ($\phi = 1, 2, \dots, \eta$).

In the following algorithm, IFDHWG and IFDHWG operators are applied to solve an MADM problem:

Step 1. We operate the decision said matrix D and apply the IFDHWG operator as follows:

$$\begin{aligned}
\beta_{\xi} &= \text{IFDHWG}(\beta_{\xi 1}, \beta_{\xi 2}, \dots, \beta_{\xi \eta}) = \oplus_{\phi=1}^{\eta} (\omega_{\phi} \beta_{\xi \phi}) \\
&= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} \omega_{\phi} \left(\dot{\mu}_{\sigma(\phi)} / (1 - \dot{\mu}_{\sigma(\phi)}) \right)^q \right\}^{1/q}}, \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} \omega_{\phi} \left((1 - \dot{\nu}_{\sigma(\phi)}) / \dot{\nu}_{\sigma(\phi)} \right)^q \right\}^{1/q}} \right\rangle,
\end{aligned} \tag{28}$$

$$\begin{aligned}
\text{or } \beta_{\xi} &= \text{IFDHWG}(\beta_{\xi 1}, \beta_{\xi 2}, \dots, \beta_{\xi \eta}) = \oplus_{\phi=1}^{\eta} (\beta_{\xi \phi})^{\omega_{\phi}} \\
&= \left\langle \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} \omega_{\phi} \left((1 - \dot{\mu}_{\sigma(\phi)}) / \dot{\mu}_{\sigma(\phi)} \right)^q \right\}^{1/q}}, 1 - \frac{1}{1 + \left\{ \sum_{\phi=1}^{\eta} \omega_{\phi} \left(\dot{\nu}_{\sigma(\phi)} / (1 - \dot{\nu}_{\sigma(\phi)}) \right)^q \right\}^{1/q}} \right\rangle,
\end{aligned} \tag{29}$$

to achieve the overall decision values β_{ξ} ($\xi = 1, 2, \dots, \delta$) of the alternative Q_{ϕ} .

Step 2. Measure the score $\Lambda(\beta_{\xi})$ ($\xi = 1, 2, \dots, \delta$) approaching overall IFVs β_{ξ} ($\xi = 1, 2, \dots, \delta$) in established to list all the alternatives Q_{ξ} ($\xi = 1, 2, \dots, \delta$) to get favorable Q_{ξ} . If there is no divergence among score functions $\Lambda(\beta_{\xi})$ and $\Lambda(\beta_{\phi})$, then we proceed to compute accuracy $\Phi(\beta_{\xi})$ and $\Phi(\beta_{\phi})$ based on overall IFVs of β_{ξ} and β_{ϕ} and rank the alternatives Q_{ξ} depending on accuracy degrees of $\Phi(\beta_{\xi})$ and $\Phi(\beta_{\phi})$.

Step 3. List all Q_{ξ} ($\xi = 1, 2, \dots, \delta$) to get the leading one(s) conforming with $\Lambda(\beta_{\xi})$ ($\xi = 1, 2, \dots, \delta$).

Step 4. Stop.

6. Case Study

The long-time steady increase of enterprise hindered due to specific problems: improvement of production, sounding

environmental impairment, reduced class production, loss of sources, and reduced security of the benefits. As a result, its employees lose interest in spending their wealth, and they move to special-purpose funding in the organization and support the funding opportunity. Thus, the enterprise's growth and survival rely upon its capacity to manage the relationship among different investors. The necessary management expert's steadily understood that it is small-minded conduct for enterprises if they want to achieve the aim of stockholder value in manufacturing procedure, regardless of the importance of different stakeholder situations. Finally, they reach the strategic control purpose of an enterprise. In this article, we will present a project to select the best enterprise alternative(s). Here, we investigate the financial performance of five possible enterprise for their ranking Q_{ϕ} ($\phi = 1, 2, 3, 4, 5$). An organization invests its money in an enterprise with the enterprise financial shows and to maximize the demanded profit. In that direction, it is required to measure the enterprise financial

performance of five possible enterprises to select attractive ones. The funding enterprise makes a decision depending on the consequent four attributes as follows: G_1 : debt paying ability; G_2 : management capability; G_3 : earning capacity; and G_4 : development and proper application. Enterprise financial performance of five possible enterprises Q_ϕ ($\phi = 1, 2, \dots, 5$) is evaluated using IFS data by the DMs under the IFDHW and IFDHWG operators in which triangular Dombi norms are used which have a good advantage of flexibility for the operation of working parameter ϱ based on G_ϕ and their weight vector $\psi = (0.2, 0.1, 0.3, 0.4)^T$, for resulting matrix $\tilde{D} = (\beta_{\xi\phi})_{5 \times 4}$ given in Table 1, where $\beta_{\xi\phi}$ is IFEs.

In order to find most desirable enterprise Q_ϕ ($\phi = 1, 2, \dots, \eta$), we utilize the IFDHA operator to develop an MADM theory with intuitionistic fuzzy data, which can be evaluated as follows:

Step 1. Utilizing the decision data stated in matrix \tilde{D} , $\tilde{I}_{\xi\phi} = \eta\psi\tilde{I}_{\xi\phi}$, we get

$$\begin{aligned}\tilde{\Omega}_{11} &= (0.6512, 0.3488), \\ \tilde{\Omega}_{12} &= (0.2857, 0.3846), \\ \tilde{\Omega}_{13} &= (0.6429, 0.1724), \\ \tilde{\Omega}_{14} &= (0.8649, 0.1351), \\ \tilde{\Omega}_{21} &= (0.5455, 0.2381), \\ \tilde{\Omega}_{22} &= (0.4828, 0.3846), \\ \tilde{\Omega}_{23} &= (0.5455, 0.0847), \\ \tilde{\Omega}_{24} &= (0.5161, 0.4839), \\ \tilde{\Omega}_{31} &= (0.3478, 0.3488), \\ \tilde{\Omega}_{32} &= (0.3750, 0.3846), \\ \tilde{\Omega}_{33} &= (0.6429, 0.3571), \\ \tilde{\Omega}_{34} &= (0.7059, 0.2113), \\ \tilde{\Omega}_{41} &= (0.5455, 0.1220), \\ \tilde{\Omega}_{42} &= (0.2857, 0.3846), \\ \tilde{\Omega}_{43} &= (0.8276, 0.0847), \\ \tilde{\Omega}_{44} &= (0.6154, 0.2113), \\ \tilde{\Omega}_{51} &= (0.5455, 0.4545), \\ \tilde{\Omega}_{52} &= (0.2857, 0.2174), \\ \tilde{\Omega}_{53} &= (0.6429, 0.1724), \\ \tilde{\Omega}_{54} &= (0.5161, 0.0649).\end{aligned}\tag{30}$$

TABLE 1: Intuitionistic fuzzy elements.

	Q_1	Q_2	Q_3	Q_4	Q_5
G_1	(0.7, 0.3)	(0.6, 0.2)	(0.4, 0.3)	(0.6, 0.1)	(0.6, 0.4)
G_2	(0.5, 0.2)	(0.7, 0.2)	(0.6, 0.2)	(0.5, 0.2)	(0.5, 0.1)
G_3	(0.6, 0.2)	(0.5, 0.1)	(0.6, 0.4)	(0.8, 0.1)	(0.6, 0.2)
G_4	(0.8, 0.2)	(0.4, 0.6)	(0.6, 0.3)	(0.5, 0.3)	(0.4, 0.1)

Then, by score function, we obtained

$$\begin{aligned}\tilde{\Omega}_{\sigma(11)} &= (0.8649, 0.1351), \\ \tilde{\Omega}_{\sigma(12)} &= (0.6429, 0.1724), \\ \tilde{\Omega}_{\sigma(13)} &= (0.6512, 0.3488), \\ \tilde{\Omega}_{\sigma(14)} &= (0.2857, 0.3846), \\ \tilde{\Omega}_{\sigma(21)} &= (0.5455, 0.0847), \\ \tilde{\Omega}_{\sigma(22)} &= (0.5455, 0.2381), \\ \tilde{\Omega}_{\sigma(23)} &= (0.4828, 0.3846), \\ \tilde{\Omega}_{\sigma(24)} &= (0.5161, 0.4839), \\ \tilde{\Omega}_{\sigma(31)} &= (0.7059, 0.2113), \\ \tilde{\Omega}_{\sigma(32)} &= (0.6429, 0.3571), \\ \tilde{\Omega}_{\sigma(33)} &= (0.3478, 0.3488), \\ \tilde{\Omega}_{\sigma(34)} &= (0.3750, 0.3846), \\ \tilde{\Omega}_{\sigma(41)} &= (0.8276, 0.0847), \\ \tilde{\Omega}_{\sigma(42)} &= (0.5455, 0.1220), \\ \tilde{\Omega}_{\sigma(43)} &= (0.6154, 0.2113), \\ \tilde{\Omega}_{\sigma(44)} &= (0.2857, 0.3846), \\ \tilde{\Omega}_{\sigma(51)} &= (0.6429, 0.1724), \\ \tilde{\Omega}_{\sigma(52)} &= (0.5161, 0.0649), \\ \tilde{\Omega}_{\sigma(53)} &= (0.5455, 0.4545), \\ \tilde{\Omega}_{\sigma(54)} &= (0.2857, 0.2174).\end{aligned}\tag{31}$$

Step 2. Utilizing the \tilde{D} using the IFDHW operator and applying weighted vector $w = (0.20, 0.30, 0.30, 0.20)^T$, then obtain overall decision assessments β_ξ for Q_ξ ($\xi = 1, 2, \dots, 5$) as follows: $\beta_1 = (0.7110, 0.2174)$, $\beta_2 = (0.5223, 0.2077)$, $\beta_3 = (0.5652, 0.3158)$, $\beta_4 = (0.6528, 0.1479)$, and $\beta_5 = (0.5283, 0.1358)$.

Step 3. Compute the score of $\Lambda(\beta_\xi)$ ($\xi = 1, 2, \dots, 5$) of each $Q_\xi \beta_\xi$ ($\xi = 1, 2, \dots, 5$) as follows:

$$\begin{aligned}\Lambda(\beta_1) &= 0.7468, \\ \Lambda(\beta_2) &= 0.6573, \\ \Lambda(\beta_3) &= 0.6247, \\ \Lambda(\beta_4) &= 0.7525, \\ \Lambda(\beta_5) &= 0.6963.\end{aligned}\quad (32)$$

Step 4. Rank Q_ξ ($\xi = 1, 2, \dots, 5$) in accordance with the value of $\Lambda(\beta_\xi)$ as follows: $Q_4 > Q_1 > Q_5 > Q_2 > Q_3$.

Step 5. Thus, Q_4 is finalized as the favorable enterprise.

If IFDWG operator is applied, proceed similarly as above.

Step 1. Utilizing the decision matrix \tilde{D} , $\tilde{I}_{\xi\phi} = \eta\psi\tilde{I}_{\xi\phi}$, we get

$$\begin{aligned}\tilde{\Omega}_{11} &= (0.7447, 0.2553), \\ \tilde{\Omega}_{12} &= (0.7143, 0.0909), \\ \tilde{\Omega}_{13} &= (0.5556, 0.2308), \\ \tilde{\Omega}_{14} &= (0.7143, 0.2857), \\ \tilde{\Omega}_{21} &= (0.6522, 0.1667), \\ \tilde{\Omega}_{22} &= (0.8537, 0.0909), \\ \tilde{\Omega}_{23} &= (0.4545, 0.1176), \\ \tilde{\Omega}_{24} &= (0.2941, 0.7059), \\ \tilde{\Omega}_{31} &= (0.4545, 0.2553), \\ \tilde{\Omega}_{32} &= (0.7895, 0.0909), \\ \tilde{\Omega}_{33} &= (0.5556, 0.4444), \\ \tilde{\Omega}_{34} &= (0.4839, 0.4068), \\ \tilde{\Omega}_{41} &= (0.6523, 0.0816), \\ \tilde{\Omega}_{42} &= (0.7143, 0.0909), \\ \tilde{\Omega}_{43} &= (0.7692, 0.1176), \\ \tilde{\Omega}_{44} &= (0.3846, 0.4068), \\ \tilde{\Omega}_{51} &= (0.6522, 0.3478), \\ \tilde{\Omega}_{52} &= (0.7143, 0.0426), \\ \tilde{\Omega}_{53} &= (0.5556, 0.2308), \\ \tilde{\Omega}_{54} &= (0.3846, 0.1509).\end{aligned}\quad (33)$$

Then, by score function, we obtained

$$\begin{aligned}\dot{\tilde{\Omega}}_{\sigma(11)} &= (0.7143, 0.0909), \\ \dot{\tilde{\Omega}}_{\sigma(12)} &= (0.7447, 0.2553), \\ \dot{\tilde{\Omega}}_{\sigma(13)} &= (0.7143, 0.2857), \\ \dot{\tilde{\Omega}}_{\sigma(14)} &= (0.5556, 0.2308), \\ \dot{\tilde{\Omega}}_{\sigma(21)} &= (0.8537, 0.0909), \\ \dot{\tilde{\Omega}}_{\sigma(22)} &= (0.6522, 0.1667), \\ \dot{\tilde{\Omega}}_{\sigma(23)} &= (0.4545, 0.1176), \\ \dot{\tilde{\Omega}}_{\sigma(24)} &= (0.2941, 0.7059), \\ \dot{\tilde{\Omega}}_{\sigma(31)} &= (0.7895, 0.0909), \\ \dot{\tilde{\Omega}}_{\sigma(32)} &= (0.4545, 0.2553), \\ \dot{\tilde{\Omega}}_{\sigma(33)} &= (0.5556, 0.4444), \\ \dot{\tilde{\Omega}}_{\sigma(34)} &= (0.4839, 0.4068), \\ \dot{\tilde{\Omega}}_{\sigma(41)} &= (0.7692, 0.1176), \\ \dot{\tilde{\Omega}}_{\sigma(42)} &= (0.7143, 0.0909), \\ \dot{\tilde{\Omega}}_{\sigma(43)} &= (0.6523, 0.0816), \\ \dot{\tilde{\Omega}}_{\sigma(44)} &= (0.3846, 0.4068), \\ \dot{\tilde{\Omega}}_{\sigma(51)} &= (0.7143, 0.0426), \\ \dot{\tilde{\Omega}}_{\sigma(52)} &= (0.5556, 0.2308), \\ \dot{\tilde{\Omega}}_{\sigma(53)} &= (0.6522, 0.3478), \\ \dot{\tilde{\Omega}}_{\sigma(54)} &= (0.3846, 0.1509).\end{aligned}\quad (34)$$

Step 2. Utilizing matrix \tilde{D} using the IFDHWAG operator and applying weighted vector $w = (0.20, 0.30, 0.30, 0.20)^T$, we get overall decision costs β_ϕ of Q_ϕ ($\phi = 1, 2, \dots, 5$) as follows:

$$\begin{aligned}\beta_1 &= (0.6836, 0.2324), \\ \beta_2 &= (0.4915, 0.3750), \\ \beta_3 &= (0.5357, 0.3333), \\ \beta_4 &= (0.6024, 0.1807), \\ \beta_5 &= (0.5556, 0.2274).\end{aligned}\quad (35)$$

TABLE 2: Comparison analysis with some of the existing method.

Methods	$\Lambda(\beta_1)$	$\Lambda(\beta_2)$	$\Lambda(\beta_3)$	$\Lambda(\beta_4)$	$\Lambda(\beta_5)$	Ordering
Xu [32] method by IFWA	0.4704	0.1983	0.2280	0.4819	0.3135	$Q_4 > Q_1 > Q_5 > Q_3 > Q_2$
Liao and Xu [53] by IFHWA	0.7192	0.6175	0.6151	0.6937	0.6363	$Q_1 > Q_4 > Q_5 > Q_3 > Q_2$
Zhao and Wei [46] by IFEHWA	0.4718	0.2362	0.2356	0.4647	0.3518	$Q_1 > Q_4 > Q_5 > Q_2 > Q_3$
Huang [54] by IFHHWA	0.6819	0.5029	0.5406	0.6206	0.5331	$Q_1 > Q_4 > Q_3 > Q_5 > Q_2$
Proposed method by IFDHWG	0.7468	0.6573	0.6247	0.7525	0.6963	$Q_4 > Q_1 > Q_5 > Q_2 > Q_3$
Proposed method by IFDHWG	0.7256	0.5583	0.6012	0.7109	0.6641	$Q_1 > Q_4 > Q_5 > Q_3 > Q_2$

TABLE 3: Characteristic comparisons of different methods.

Methods	Fuzzy information easier	Aggregation flexibility
Xu [32] method by IFWA	Yes	No
Liao and Xu [53] by IFHWA	Yes	No
Zhao and Wei [46] by IFEHWA	Yes	No
Huang [54] by IFHHWA	Yes	No
Proposed method	Yes	Yes

TABLE 4: Aggregate value of overall evaluation by IFDHWG, IFDHWG, and other methods.

Methods	β_1	β_2	β_3	β_4	β_5
Reference [32] method by IFWA	$\langle 0.6956, 0.2252 \rangle$	$\langle 0.5177, 0.3194 \rangle$	$\langle 0.5587, 0.3307 \rangle$	$\langle 0.6592, 0.1773 \rangle$	$\langle 0.5231, 0.2096 \rangle$
Liao and Xu [53] by IFHWA	$\langle 0.6956, 0.2204 \rangle$	$\langle 0.5177, 0.2215 \rangle$	$\langle 0.5749, 0.3208 \rangle$	$\langle 0.6291, 0.1600 \rangle$	$\langle 0.5263, 0.1713 \rangle$
Zhao and Wei [46] by IFEHWA	$\langle 0.6927, 0.2209 \rangle$	$\langle 0.5160, 0.2798 \rangle$	$\langle 0.5570, 0.3214 \rangle$	$\langle 0.6278, 0.1631 \rangle$	$\langle 0.5337, 0.1819 \rangle$
Huang [54] by IFHHWA	$\langle 0.6915, 0.0096 \rangle$	$\langle 0.5163, 0.0134 \rangle$	$\langle 0.5559, 0.0153 \rangle$	$\langle 0.6276, 0.0070 \rangle$	$\langle 0.5409, 0.0078 \rangle$
Proposed method by IFDHWG	$\langle 0.7110, 0.2174 \rangle$	$\langle 0.5223, 0.2077 \rangle$	$\langle 0.5652, 0.3158 \rangle$	$\langle 0.6528, 0.1479 \rangle$	$\langle 0.5283, 0.1358 \rangle$
Proposed method by IFDHWG	$\langle 0.6836, 0.2324 \rangle$	$\langle 0.4915, 0.3750 \rangle$	$\langle 0.5357, 0.3333 \rangle$	$\langle 0.6024, 0.1807 \rangle$	$\langle 0.5556, 0.2274 \rangle$

Step 3. Calculate the value of $\Lambda(\beta_\phi)$ ($\phi = 1, 2, \dots, 5$) of each β_ϕ ($\phi = 1, 2, \dots, 5$) as follows:

$$\begin{aligned}
 \Lambda(\beta_1) &= 0.7256, \\
 \Lambda(\beta_2) &= 0.5583, \\
 \Lambda(\beta_3) &= 0.6012, \\
 \Lambda(\beta_4) &= 0.7109, \\
 \Lambda(\beta_5) &= 0.6641.
 \end{aligned} \tag{36}$$

Step 4. Now, enterprise ranking Q_ϕ ($\xi = 1, 2, \dots, 5$) as per score values $\Lambda(\beta_\phi)$ is $Q_1 > Q_4 > Q_5 > Q_3 > Q_2$.

Step 5. Thus, Q_1 is selected as attractive enterprise.

7. Comparative Analysis

A comparative study was directed to approve the after-effects of the proposed technique with those of the other approach. The relative outcomes appear in Tables 2 and 3. Therefore, with regard to the same MADM problem for assessing enterprise financial performance, this paper contrasted the outcomes acquired with those of the Xu method, the Liao and Xu method, the Zhao and Wei method, and the Huang methods inside the intuitionistic fuzzy surroundings. Table 4 gives an aggregate value of

overall evaluation by IFDHWG, IFDHWG, and other methods. Table 2 gives a synopsis of solution consequences for the choice issue of enterprise financial performances.

As indicated in Table 2, the Xu method and the proposed method by IFDHWG yielded the same complete-preference ranking result of the financial performance of five possible enterprises: $Q_4 > Q_1 > Q_5 > Q_2 > Q_3$. The distinction exists within the preference relation between Q_2 and Q_3 . The comparative outcomes exhibit that the recommended method by IFDHWG is powerful and legitimate for addressing the interpretative issue within the intuitionistic fuzzy surroundings. By using the Liao and Xu method, we get the complete-preference ranking result of the alternatives as $Q_1 > Q_4 > Q_5 > Q_3 > Q_2$, which is the same as our obtained result by IFDHWG operator. Again, by applying the Zhao and Wei method and the Huang method, we get the complete-preference ranking outcome of the alternatives as $Q_1 > Q_4 > Q_5 > Q_2 > Q_3$ and $Q_1 > Q_4 > Q_3 > Q_5 > Q_2$, respectively. The difference exists in the preference relation between Q_2 , Q_3 , and Q_5 . For this proposed approach, we apply intuitionistic fuzzy Dombi hybrid averaging and geometric operators which have an advantage of flexibility for the operation of \otimes ; on the other hand, the existing operators IFWA [32], IFHWA [53], IFEHWA [46], and IFHHWA [54] have no

such advantages provided in Table 3. Thus, our proposed methods are more general and more flexible than some existing methods to control intuitionistic fuzzy MADM problems.

8. Conclusions

Enterprises are an important key factor of employees, stockholders, customer, creditor, government, and other stakeholder. In the financial performance of enterprises, double characters should be considered: economic and society, so we should consider all stakeholder's benefit in performance of enterprise evaluating time. We set up a performance evaluating system on basis of stakeholder benefits. In this article, we have studied MADM problems for enterprise financial performance with intuitionistic fuzzy data. We employ IFDHW and IFDHWG operators to develop MADM approach for evaluating enterprises financial performance. In these operators, we use triangular Dombi norms which have a good advantage of flexibility for the operation of working parameter ρ . For this proposed approach, we apply intuitionistic fuzzy Dombi hybrid averaging and geometric operators which have an advantage of flexibility for the operation of ρ ; on the other hand, the existing operators IFWA [32], IFHWA [53], IFEHWA [46], and IFHHWA [54] have no such advantages. In future, proposed approach can be applied in (1) green supplier selection [56], (2) economic risk evaluation [8], (3) hybrid rough approach study [57], and other domains under ambiguous environments.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Retraction

Retracted: New Soft Structure: Infra Soft Topological Spaces

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] T. M. Al-shami, "New Soft Structure: Infra Soft Topological Spaces," *Mathematical Problems in Engineering*, vol. 2021, Article ID 3361604, 12 pages, 2021.

Research Article

New Soft Structure: Infra Soft Topological Spaces

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It is always convenient to find the weakest conditions that preserve some topologically inspired properties. To this end, we introduce the concept of an infra soft topology which is a collection of subsets that extend the concept of soft topology by dispensing with the postulate that the collection is closed under arbitrary unions. We study the basic concepts of infra soft topological spaces such as infra soft open and infra soft closed sets, infra soft interior and infra soft closure operators, and infra soft limit and infra soft boundary points of a soft set. We reveal the main properties of these concepts with the help of some elucidative examples. Then, we present some methods to generate infra soft topologies such as infra soft neighbourhood systems, basis of infra soft topology, and infra soft relative topology. We also investigate how we initiate an infra soft topology from crisp infra topologies. In the end, we explore the concept of continuity between infra soft topological spaces and determine the conditions under which the continuity is preserved between infra soft topological space and its parametric infra topological spaces.

1. Introduction

This paper is at the junction of two disciplines, namely, infra topology and soft set theory. Their hybridization has produced an interesting structure called infra soft topology which is the framework for our contribution. Let us summarize the antecedents and state-of-the-art of the topic.

In 1999, Molodtsov [1] proposed the concept of soft sets as a new mathematical approach to cope with problems containing uncertainties, and he explained the potentiality of soft sets to handle many problems in different areas. This theory has gained much attention from researchers and scientists because of its diverse applications. It is possible to see a rapid growth in soft sets' research in the last few years, see, for example, [2, 3].

In 2003, Maji et al. [4] put forward some soft operations such as union and intersection and subset and equality relations between two soft sets. They also defined the null and absolute soft sets as a soft version of the empty and universal crisp sets. Ali et al. [5] showed some shortcomings given in [4], defined certain new operations on soft sets, and explored their main properties. Abbas et al. [6] and Qin and Hong [7] described new types of soft equality, which they applied to introduce new types of algebraic structures.

Recently, Al-shami and El-Shafei [8] have defined and discussed new types of operations between soft sets.

In 2011, Shabir and Naz [9] introduced a topological structure on soft setting. They defined the fundamental notions of soft topologies such as soft open and closed sets, soft subspaces, and belonging and nonbelonging relations which are used to initiate soft separation axioms. Zorlutuna et al. [10] came up with the idea of soft point which helps to study some properties of soft interior points and soft neighbourhood systems. This concept was independently reformulated by Samanta et al. [11, 12], while Das and Samanta [11] applied the new version of the soft point to study the concept of soft metric spaces and Nazmul and Samanta [12] used it to discuss soft neighbourhood systems and reveal some relations of soft limit points of a soft set. Many scholars analyzed the properties of soft topologies and compared their performance with the case of classical topologies, see, for example, [13–23]. Generalizations of open sets were investigated in soft topologies, see [24, 25]. In [26], we corrected some alleged results concerning soft separation axioms, especially those defined using soft points.

Some generalizations of a soft topology were given by weakening a soft topology's conditions. For example, in 2014, El-Sheikh and Abd El-Latif [27] established the

concept of supra soft topological spaces by neglecting a finite intersection condition of a soft topology. This path, therefore, attracted a lot of researchers who studied essential notions related to supra soft topologies, see, for example, [28, 29]. Thomas and John [30] formulated the concept of soft generalized topological spaces which are defined as a family of soft sets that satisfy an arbitrary union condition of a soft topology, and Zakari et al. [31] originated the concepts of soft weak structures which are defined as a family of soft sets that contain the null soft set Φ . Ittanagi [32] studied the concept of soft bitopological space which can be regarded as a soft topological space when the two soft topologies are identical. Lately, Al-shami et al. [33] have constructed soft topology on ordered setting as an extension of soft topology. Similarly, Al-shami and El-Shafei [34] studied supra soft topology on ordered setting.

We note that many properties of soft topological spaces are still valid on infra soft topological spaces, and initiating examples that show some relationships between certain topological concepts are easier on infra soft topological spaces. Therefore, we aim in this paper to perform an exhaustive analysis of infra soft topological spaces.

This paper is structured as follows: after this introduction, Section 2 addresses some definitions and properties that help the reader to well understand this manuscript. In Section 3, we introduce the concept of infra soft topological spaces and disclose the main properties of infra soft interior, infra soft closure, infra soft limit, and infra soft boundary points of a soft set. In Section 4, we tackle some techniques of generating infra soft topology such as infra soft neighbourhoods and infra soft subspaces. In Section 5, we formulate the concept of infra continuous maps between infra soft topological spaces and determine the conditions under which the continuity is preserved between infra soft topological space and its parametric infra topological spaces. We give some conclusions and make a plan for future works in Section 6.

2. Preliminaries

In this section, we recall the technical concepts that we need in this paper.

The notation 2^X refers to the set of subsets of X .

Definition 1 (see [1]). A pair (G, E) is a soft set over a nonempty set X provided that G is a map from the set of parameters E to 2^X .

For the sake of brevity and ease, henceforth, a soft set is symbolized by G_E instead of (G, E) . It is identified as $G_E = \{(e, G(e)): e \in E \text{ and } G(e) \in 2^X\}$.

The set of all soft sets over X under a set of parameters E is symbolized by $S(X_E)$.

Definition 2 (see [11]). A soft set G_E is called finite (resp., countable) if $G(e)$ is finite (resp., countable) for each $e \in E$.

Definition 3 (see [35]). The relative complement of a soft set G_E is a soft set G_E^c , where $G^c: E \rightarrow 2^X$ is a map defined by $G^c(e) = X \setminus G(e)$ for each $e \in E$.

Definition 4 (see [4]). A soft set G_E over X is said to be the null soft set, symbolized by Φ , if $G(e) = \emptyset$ for each $e \in E$. Its relative complement is said to be the absolute soft set, symbolized by \tilde{X} .

Definition 5 (see [11, 12]). A soft point P_E over X is a soft set such that $P(e)$ is a singleton set and $P(e')$ is the empty set for each $e' \neq e$. This soft point will be briefly symbolized by P_e^x .

Since a soft topological space and its generalizations, which are the theme of this manuscript, are defined under a fixed set of parameters, we will mention the definitions and findings given in the previous studies under a fixed set of parameters.

Definition 6 (see [4]). A soft set G_E is a soft subset of a soft set F_E , symbolized by $G_E \subseteq F_E$, if $G(e) \subseteq F(e)$ for all $e \in E$.

The soft sets G_E and F_E are called soft equal if each is a soft subset of the other.

Definition 7 (see [5]). The intersection of two soft sets G_E and F_E over X , symbolized by $G_E \cap F_E$, is a soft set H_E , where a map $H: E \rightarrow 2^X$ is given by $H(e) = G(e) \cap F(e)$ for each $e \in E$.

Definition 8 (see [4]). The union of two soft sets G_E and F_E over X , symbolized by $G_E \sqcup F_E$, is a soft set H_E , where a map $H: E \rightarrow 2^X$ is given by $H(e) = G(e) \cup F(e)$ for each $e \in E$.

Definition 9 (see [9, 19]). For a soft set G_E over X and $x \in X$, we say that

- (i) $x \in G_E$ (it reads as x totally belongs to G_E) if $x \in G(e)$ for each $e \in E$
- (ii) $x \notin G_E$ (it reads as x does not partially belong to G_E) if $x \notin G(e)$ for some $e \in E$
- (iii) $x \in G_E$ (it reads as x partially belongs to G_E) if $x \in G(e)$ for some $e \in E$
- (iv) $x \notin G_E$ (it reads as x does not totally belong to G_E) if $x \notin G(e)$ for each $e \in E$

Soft maps are recalled in the next two definitions with some modifications to be convenient for defining the concepts of soft continuous maps.

Definition 10 (see [36]). A soft mapping between $S(X_E)$ and $S(Y_E)$ is a pair (f, φ) , denoted also by f_φ , of mappings such that $f: X \rightarrow Y$ and $\varphi: E \rightarrow E$. Let G_E and H_E be subsets of $S(X_E)$ and $S(Y_E)$, respectively. Then, the image of G_E and preimage of H_E are defined as follows:

- (i) $f_\varphi(G_E) = (f(G), E)$ is a soft set in $S(Y_E)$ such that $f(G)(e) = \bigcup_{a \in \varphi^{-1}(e)} f(G(a)), \varphi^{-1}(e) \neq \emptyset, \emptyset, \varphi^{-1}(e) = \emptyset$.

(1)

for each $e \in E$.

- (ii) $f_\varphi^{-1}(H_E) = (f^{-1}(H), E)$ is a soft set in $S(X_E)$ such that

$$f^{-1}(H)(e) = f^{-1}(H(\varphi(e))) \quad \text{for each } e \in E. \quad (2)$$

Remark 1. Henceforth, a soft map $f_\varphi: S(X_E) \longrightarrow S(Y_E)$ implies a map f from the universal set X to the universal set Y and a map φ from the set of parameters E to itself.

Definition 11 (see [10]). A soft map $f_\varphi: S(X_E) \longrightarrow S(Y_E)$ is said to be injective (resp. surjective and bijective) if f and φ are injective (resp. surjective and bijective).

Definition 12 (see [37]). An infratopology on X is a collection τ of subsets of X , that is, closed under finite intersections and satisfies $\emptyset \in \tau$.

Definition 13 (see [9]). The collection ϑ of soft sets over X under a fixed set of parameters E is called a soft topology on X if it satisfies the following axioms:

- (i) \tilde{X} and Φ belong to ϑ
- (ii) The union of an arbitrary family of soft sets in ϑ belongs to ϑ
- (iii) The intersection of a finite family of soft sets in ϑ belongs to ϑ

The triple (X, ϑ, E) is called a soft topological space. The term given to each member of ϑ is a soft open set, and the relative complement of each member of ϑ is a soft closed set.

3. Infra Soft Topological Spaces

In this section, we introduce the concept of infra soft topology as a class of soft sets, that is, closed under finite soft intersection and contains the null soft set. It lies between soft topology and soft weak structure, and it is independent of a supra soft topology. We define the main concepts of infra soft topology and reveal their main properties. One of the merits of infra soft topology is that many results of soft topology are still valid on infra soft topology, especially those are related to the soft interior and closure operators. A number of examples are provided to validate the obtained results.

Definition 14. The collection ϑ of soft sets over X under a fixed set of parameters E is said to be an infra soft topology on X if it is closed under finite soft intersection and the null soft set is a member of ϑ .

The triple (X, ϑ, E) is called an infra soft topological space. Every member of ϑ is called an infra soft open set, and its relative complement is called an infra soft closed set.

The following examples elucidate the uniqueness of infra soft topology than the other celebrated soft structures.

Example 1. Let $E = \{e_1, e_2\}$ be a set of parameters and $X = \{a, b\}$ be the universal set. Then, $\vartheta_a = \{\tilde{X}, F_E \sqsubseteq \tilde{X}: a \notin F_E\}$ is an infra soft topology on X ; we called ϑ_a an excluding point infra soft topology. On the contrary, $U_E = \{(e_1, \{a\}), (e_2, \emptyset)\}$ and $V_E = \{(e_1, \emptyset), (e_2, \{a\})\}$ are infra soft open sets, but their

union is not an infra soft open set. Therefore, ϑ_a is neither supra soft topology nor generalized soft topology. Hence, it is not a soft topology.

Example 2. Let $E = \{e_1, e_2\}$ be a set of parameters and $X = \{a, b\}$ be the universal set. Then, $\vartheta_a = \{\Phi, F_E \sqsubseteq \tilde{X}: a \in F_E\}$ is a supra soft topology on X ; we called ϑ_a a particular point supra soft topology. On the contrary, $U_E = \{(e_1, \{b\}), (e_2, X)\}$ and $V_E = \{(e_1, X), (e_2, \{b\})\}$ are supra soft open sets, but their intersection is not a supra soft open set. Therefore, ϑ_a is not an infra soft topology. Moreover, it is not a soft topology.

Remark 2. It can be examined that the intersection of any family of infra soft topologies is always infra soft topology. But the union of two infra soft topologies need not be an infra soft topology. We show this fact in the following example.

Example 3. Consider the soft sets below defined on $X = \{a, b, c\}$ with a set of parameters $E = \{e_1, e_2\}$:

$$\begin{aligned} F_{1E} &= \{(e_1, \{a, b\}), (e_2, \{c\})\}, \\ F_{2E} &= \{(e_1, \{c\}), (e_2, \{b\})\}, \\ F_{3E} &= \{(e_1, \{a\}), (e_2, \emptyset)\}, \\ F_{4E} &= \{(e_1, \{b, c\}), (e_2, \{b\})\}. \end{aligned} \quad (3)$$

Then, $\vartheta_1 = \{\Phi, \tilde{X}, F_{1E}, F_{2E}\}$ and $\vartheta_2 = \{\Phi, \tilde{X}, F_{3E}, F_{4E}\}$ are two infra soft topologies on X . But $\vartheta_1 \cup \vartheta_2 = \{\Phi, \tilde{X}, F_{1E}, F_{2E}, F_{3E}, F_{4E}\}$ is not an infra soft topology on X because $F_{1E} \cap F_{4E} \notin \vartheta_1 \cup \vartheta_2$.

The following results present two techniques to originate infra soft topology using soft maps.

Proposition 1. Let $f_\varphi: S(X_E) \longrightarrow S(Y_E)$ be a soft map. If ϑ is an infra soft topology on Y , then a class $\theta = \{f_\varphi^{-1}(G_E) \sqsubseteq \tilde{X}: G_E \in \vartheta\}$ is an infra soft topology on X .

Proof. Since Φ and $\tilde{Y} \in \vartheta$, then $f_\varphi^{-1}(\Phi) = \Phi$ and $f_\varphi^{-1}(\tilde{Y}) = \tilde{X} \in \theta$. Let F_{1E} and $F_{2E} \in \theta$. Then, there exist H_{1E} and $H_{2E} \in \vartheta$ such that $f_\varphi^{-1}(H_{1E}) = F_{1E}$ and $f_\varphi^{-1}(H_{2E}) = F_{2E}$. Therefore, $F_{1E} \cap F_{2E} = f_\varphi^{-1}(H_{1E}) \cap f_\varphi^{-1}(H_{2E}) = f_\varphi^{-1}(H_{1E} \cap H_{2E})$. Since $H_{1E} \cap H_{2E} \in \vartheta$, then $F_{1E} \cap F_{2E} \in \theta$. Hence, the proof is complete. \square

In a similar manner, one can prove the following result.

Proposition 2. Let $f_\varphi: S(X_E) \longrightarrow S(Y_E)$ be a bijective soft map. If ϑ is an infra soft topology on X , then a class $\theta = \{f_\varphi^{-1}(G_E) \sqsubseteq \tilde{Y}: G_E \in \vartheta\}$ is an infra soft topology on Y .

Definition 15. We define the infra soft interior points and infra soft closure points of a soft subset H_E of (X, ϑ, E) which are, respectively, denoted by $\text{Int}(H_E)$ and $\text{Cl}(H_E)$ as follows:

- (i) $\text{Int}(H_E)$ is the union of all infra soft open sets that are contained in H_E
- (ii) $\text{Cl}(H_E)$ is the intersection of all infra soft closed sets containing H_E

The following example illustrates that the infra soft interior points and infra soft closure points of a soft set need not be infra soft open and infra soft closed sets, respectively.

Example 4. Let $E = \{e_1, e_2\}$ be a set of parameters. Then, $\{\mathcal{G} = \mathbb{R}, F_E \subseteq \mathbb{R} : F_E \text{ is finite}\}$ is an infra soft topology on the set of real numbers \mathbb{R} . It is clear that $(\mathbb{R}, \mathcal{G}, E)$ is not a soft topological space. Let $G_E = \{(e_1, \mathbb{N}), (e_2, \mathbb{N})\}$, where \mathbb{N} is the set of natural numbers. We find that $\text{Int}(G_E) = \text{Cl}(G_E)$ equals to G_E which is neither an infra soft open set nor an infra soft closed set.

Proposition 3. Let H_E be a soft subset of (X, \mathcal{G}, E) . Then, the following properties hold.

- (i) If H_E is an infra soft open set, then $\text{Int}(H_E) = H_E$
- (ii) If H_E is an infra soft closed set, then $\text{Cl}(H_E) = H_E$

Proof. It immediately follows from Definition 15. \square

Example 4 shows that the converse of the two properties given in the above proposition need not be true in general. These two properties are an example of some soft topological properties that are losing on infra soft topological spaces.

Proposition 4. Let H_E be a soft subset of (X, \mathcal{G}, E) . Then, the following properties hold.

- (i) $P_e^x \in \text{Int}(H_E)$ if and only if there exists an infra soft open set G_E such that $P_e^x \in G_E \subseteq H_E$
- (ii) $P_e^x \in \text{Cl}(H_E)$ if and only if $H_E \cap G_E \neq \Phi$ for every infra soft open set G_E containing P_e^x

Proof.

- (i) Straightforward.
- (ii) Necessity: let $P_e^x \in \text{Cl}(H_E)$. Then, P_e^x belongs to every infra soft closed set containing H_E . Suppose that there exists an infra soft open set G_E containing P_e^x such that $H_E \cap G_E = \Phi$, so $H_E \subseteq G_E^c$. This is a contradiction. Thus, the necessary part holds.

Sufficiency: let $P_e^x \notin \text{Cl}(H_E)$. Then, $(\text{Cl}(H_E))^c$ is an infra soft open set containing P_e^x such that $H_E \cap (\text{Cl}(H_E))^c = \Phi$. Thus, the proof is complete. \square

Proposition 5. Let H_E be a soft subset of (X, \mathcal{G}, E) . Then,

- (i) $(\text{Int}(H_E))^c = \text{Cl}(H_E^c)$
- (ii) $(\text{Cl}(H_E))^c = \text{Int}(H_E^c)$

Proof.

- (i) $(\text{Int}(H_E))^c = \{\cup_{i \in I} (G_{iE}) : G_{iE} \text{ is an infra soft open set included in } H_E\}^c = \cap_{i \in I} \{G_E^c : G_E^c \text{ is an infra soft closed set including } H_E^c\} = \text{Cl}(H_E^c)$

In a similar manner, we prove (ii). \square

Proposition 6. Let H_E be a soft subset of (X, \mathcal{G}, E) . Then,

- (i) If U_E is an infra soft open set, then $U_E \cap \text{Cl}(H_E) \subseteq \text{Cl}(U_E \cap H_E)$
- (ii) If F_E is an infra soft closed set, then $\text{Int}(F_E \sqcup H_E) \subseteq F_E \sqcup \text{Int}(H_E)$

Proof

- (i) Let $P_e^x \in U_E \cap \text{Cl}(H_E)$. Then, $P_e^x \in U_E$ and $P_e^x \in \text{Cl}(H_E)$. This means that, for each infra soft open set V_E containing P_e^x , we have $V_E \cap H_E \neq \Phi$. Since \mathcal{G} is infra soft topology, then $U_E \cap V_E$ is an infra soft open set containing P_e^x ; consequently, $(V_E \cap U_E) \cap H_E \neq \Phi \iff V_E \cap (U_E \cap H_E) \neq \Phi$. Thus, $P_e^x \in \text{Cl}(U_E \cap H_E)$. Hence, $U_E \cap \text{Cl}(H_E) \subseteq \text{Cl}(U_E \cap H_E)$.
- (ii) Let $P_e^x \notin F_E \sqcup \text{Int}(H_E)$. Then, $P_e^x \notin F_E$ and $P_e^x \notin \text{Int}(H_E)$. Therefore, there is an infra soft open set U_E such that $P_e^x \in U_E \subseteq F_E^c$. Suppose that $P_e^x \in \text{Int}(F_E \sqcup H_E)$. Then, there is an infra soft open set V_E such that $P_e^x \in V_E \subseteq F_E \sqcup H_E$. Now, we have $U_E \cap V_E$ which is an infra soft open set containing P_e^x such that $U_E \cap V_E \subseteq F_E^c$ and $U_E \cap V_E \subseteq F_E \sqcup H_E$. This means that $P_e^x \in \text{Int}(H_E)$. But this contradicts $P_e^x \notin \text{Int}(H_E)$. Thus, $P_e^x \notin \text{Int}(F_E \sqcup H_E)$. Hence, $\text{Int}(U_E \sqcup G_E) \subseteq F_E \sqcup \text{Int}(G_E)$. \square

Theorem 1. Let F_E and G_E be soft subsets of (X, \mathcal{G}, E) . Then, the following properties hold:

- (i) $\text{Int}(\tilde{X}) = \tilde{X}$
- (ii) $\text{Int}(F_E) \subseteq F_E$
- (iii) If $G_E \subseteq F_E$, then $\text{Int}(G_E) \subseteq \text{Int}(F_E)$
- (iv) $\text{Int}(\text{Int}(F_E)) = \text{Int}(F_E)$
- (v) $\text{Int}(F_E \cap G_E) = \text{Int}(F_E) \cap \text{Int}(G_E)$

Proof. The proofs of (i), (ii), and (iii) are obvious.

- (iv) It follows from (ii) that $\text{Int}(\text{Int}(F_E)) \subseteq \text{Int}(F_E)$. Conversely, let $P_e^x \in \text{Int}(F_E)$. Then, there exists an infra soft open set G_E such that $P_e^x \in G_E \subseteq F_E$. By (iii), we have $P_e^x \in \text{Int}(G_E) = G_E \subseteq \text{Int}(F_E)$. Therefore, $P_e^x \in \text{Int}(\text{Int}(F_E))$. This ends the proof of $\text{Int}(\text{Int}(F_E)) = \text{Int}(F_E)$.
- (v) It is clear that $\text{Int}(F_E \cap G_E) \subseteq \text{Int}(F_E) \cap \text{Int}(G_E)$. Conversely, let $P_e^x \in \text{Int}(F_E) \cap \text{Int}(G_E)$. Then, there exist two infra soft open sets U_E and V_E such that $P_e^x \in U_E \subseteq F_E$ and $P_e^x \in V_E \subseteq G_E$. Now, $U_E \cap V_E$ is an infra soft open set containing P_e^x such that $P_e^x \in U_E \cap V_E \subseteq F_E \cap G_E$. Therefore, $P_e^x \in \text{Int}(F_E \cap G_E)$. Thus, $\text{Int}(F_E) \cap \text{Int}(G_E) \subseteq \text{Int}(F_E \cap G_E)$. Hence, the proof is complete. \square

One can prove the following result similarly.

Theorem 2. Let F_E and G_E be soft subsets of (X, \mathcal{G}, E) . Then, the following properties hold:

- (i) $Cl(\Phi) = \Phi$
- (ii) $F_E \subseteq Cl(F_E)$
- (iii) If $G_E \subseteq F_E$, then $Cl(G_E) \subseteq Cl(F_E)$
- (iv) $Cl(Cl(F_E)) = Cl(F_E)$
- (v) $Cl(F_E \sqcup G_E) = Cl(F_E) \sqcup Cl(G_E)$

The following example clarifies that the following two equalities are not always true.

- (i) $\text{Int}(\cap_{n=1}^{\infty} G_{nE}) = \cap_{n=1}^{\infty} \text{Int}(G_{nE})$
- (ii) $Cl(\sqcup_{n=1}^{\infty} G_{nE}) = \sqcup_{n=1}^{\infty} Cl(G_{nE})$

Example 5. Let $E = \{e_1, e_2\}$ be a set of parameters and $\vartheta = \{\Phi, F_E \subseteq \mathbb{R}: F_E^c \text{ is finite}\}$ be a family of soft sets on the set of real numbers \mathbb{R} . Then, $(\mathbb{R}, \vartheta, E)$ is an infra soft topological space. Let $H_{nE} = \{(e_1, \mathbb{R} \setminus \{n\}), (e_2, \mathbb{R} \setminus \{n\})\}$. Then, $\text{Int}(H_{nE}) = H_{nE}$. Therefore, $\cap_{n=1}^{\infty} \text{Int}(G_{nE}) = \{(e_1, \mathbb{R} \setminus \mathbb{N}), (e_2, \mathbb{R} \setminus \mathbb{N})\}$. But $\text{Int}(\cap_{n=1}^{\infty} G_{nE}) = \Phi$. Also, let $G_{nE} = (e_1, \{n\}), (e_2, \{n\})$. Then, $Cl(G_{nE}) = G_{nE}$ for each n . Therefore, $\sqcup_{n=1}^{\infty} Cl(G_{nE}) = \{(e_1, \mathbb{N}), (e_2, \mathbb{N})\}$. But $Cl(\sqcup_{n=1}^{\infty} G_{nE}) = \{(e_1, \mathbb{R}), (e_2, \mathbb{R})\}$.

Proposition 7. If F_E and G_E are soft subsets of (X, ϑ, E) such that $Cl(F_E) \cap Cl(G_E) = \Phi$, then $\text{Int}(F_E \sqcup G_E) = \text{Int}(F_E) \sqcup \text{Int}(G_E)$.

Proof. It follows from (iii) of Theorem 1 that $\text{Int}(F_E) \sqcup \text{Int}(G_E) \subseteq \text{Int}(F_E \sqcup G_E)$. To prove that $\text{Int}(F_E \sqcup G_E) \subseteq \text{Int}(F_E) \sqcup \text{Int}(G_E)$, let $P_e^x \in \text{Int}(F_E \sqcup G_E)$. Then, there exists an infra soft open set U_E containing P_e^x such that $P_e^x \in U_E \subseteq F_E \sqcup G_E$. Now, we have three cases: \square

Case 1. $U_E \subseteq F_E$. Then, $P_e^x \in \text{Int}(F_E)$.

Case 2. $U_E \subseteq G_E$. Then, $P_e^x \in \text{Int}(G_E)$.

Case 3. $U_E \subseteq F_E$ and $U_E \subseteq G_E$. Then, $U_E \cap F_E \neq \Phi$ and $U_E \cap G_E \neq \Phi$. This implies that, for each infra soft open set V_E containing P_e^x , we obtain $P_e^x \in Cl(F_E)$ and $P_e^x \in Cl(G_E)$. But this contradicts $Cl(F_E) \cap Cl(G_E) = \Phi$. Therefore, this case is impossible.

Thus, Cases 1 and 2 are only valid, and they imply that $\text{Int}(F_E \sqcup G_E) \subseteq \text{Int}(F_E) \sqcup \text{Int}(G_E)$. Hence, the proof is complete.

The following result explains two methods of generating crisp infra topologies from an infra soft topology.

Proposition 8. Let (X, ϑ, E) be an infra soft topological space. Then,

- (i) (X, ϑ_e) is an infra topological space for each $e \in E$, where $\vartheta_e = \{F(e): F_E \in \vartheta\}$
- (ii) (E, ϑ_x) is an infra topological space for each $x \in X$, where $\vartheta_x = \{\{e: x \in F(e)\}: F_E \in \vartheta\}$

Proof

- (i) Since Φ and $\tilde{X} \in \vartheta$, then \emptyset and $X \in \vartheta_e$. To prove that ϑ_e is closed under finite intersection, let U and $V \in \vartheta_e$. Then, there exists two infra soft open subsets F_E and G_E of (X, ϑ, E) such that $F(e) = U$ and $G(e) = V$. Owing to the fact that $F_E \cap G_E \in \vartheta$, we obtain $U \cap V \in \vartheta_e$, as required.

Following similar arguments, one can prove (ii). \square

We called (X, ϑ_e) given in the above proposition “a parametric infra topological space.”

The converse of the above proposition need not be true, as shown in the following example.

Example 6. Consider the soft sets below defined on $X = \{a, b, c\}$ with a set of parameters $E = \{e_1, e_2\}$:

$$\begin{aligned} F_{1E} &= \{(e_1, \{a\}), (e_2, \{b, c\})\}, \\ F_{2E} &= \{(e_1, \{a\}), (e_2, \{a\})\}. \end{aligned} \quad (4)$$

Then, $\vartheta = \{\Phi, \tilde{X}, F_{1E}, F_{2E}\}$ is not an infra soft topology on X . On the contrary, $\vartheta_{e_1} = \{\emptyset, X, \{a\}\}$ and $\vartheta_{e_2} = \{\emptyset, X, \{a\}, \{b, c\}\}$ are two infra soft topologies on X . Also, $\vartheta_a = \{\emptyset, E, \{e_1\}\}$, $\vartheta_b = \{\emptyset, E, \{e_2\}\}$, and $\vartheta_c = \{\emptyset, E\}$ are three infra soft topologies on E .

Definition 16. Let H_E be a soft subset of (X, ϑ, E) . Then,

- (i) $(Cl(H))_E$ is a soft set given by $(Cl(H))(e) = Cl(H(e))$, where $Cl(H(e))$ is the closure of $H(e)$ in (X, ϑ_e) for each $e \in E$
- (ii) $(\text{Int}(H))_E$ is a soft set given by $(\text{Int}(H))(e) = \text{Int}(H(e))$, where $\text{Int}(H(e))$ is the interior of $H(e)$ in (X, ϑ_e) for each $e \in E$

Proposition 9. Let H_E be a soft subset of (X, ϑ, E) . Then,

- (i) $\text{Int}(H_E) \subseteq (\text{Int}(H))_E$
- (ii) $(Cl(H))_E \subseteq Cl(H_E)$

Proof

- (i) Let $P_e^x \in \text{Int}(H_E)$. Then, there exists an infra soft open set U_E containing P_e^x such that $P_e^x \in U_E \subseteq H_E$. Now, $U(e)$ is an infra open subset of (X, ϑ_e) for every $e \in E$ such that $U(e) \subseteq H(e)$. Therefore, $P_e^x \in \text{Int}(H(e))$; thus, $P_e^x \in (\text{Int}(H))_E$, as required.

Following similar arguments to those given in the proof of (i), one can prove (ii). \square

The converse of properties (i) and (ii) in the above theorem is not always true as explained in the following example.

Example 7. Assume that (X, ϑ_1, E) is the infra soft topological space given in Example 3. Let $U_E = \{(e_1, \{a, c\}), (e_2, \{b, c\})\}$ and $V_E = \{(e_1, \{a, c\}), (e_2, \{a, b\})\}$. By calculating, we find $\text{Int}(U_E) = e_1, \{c\}, (e_2, \{b\})$ and $Cl(V_E) = \tilde{X}$. On the contrary, $(\text{Int}(H))_E$

$= \{(e_1, \{c\}), (e_2, \{b, c\})\}$ and $(Cl(H))_E = \{(e_1, X), (e_2, \{a, b\})\}$. Hence, $(Int(H))_E \subseteq Int(H_E)$ and $Cl(H_E) \subseteq (Cl(H))_E$.

Definition 17. Let H_E be a soft subset of an infra soft topological space (X, \mathcal{I}, E) . A soft point P_e^x is said to be an infra soft limit point of H_E if $[U_E \setminus P_e^x] \cap H_E \neq \Phi$ for each infra soft open set U_E containing P_e^x .

All infra soft limit points of H_E are said to be an infra-derived soft set of H_E and denoted by H_E^i .

Proposition 10. If G_E and H_E are soft subsets of (X, \mathcal{I}, E) , then the following properties hold:

- (i) $\Phi^{i'} = \Phi$ and $\tilde{X}^{i'} \subseteq \tilde{X}$
- (ii) If $G_E \subseteq H_E$, then $G_E^i \subseteq H_E^i$
- (iii) If $P_e^x \in G_E^i$, then $x \in (G_E \setminus P_e^x)^{i'}$
- (iv) $G_E^i \sqcup H_E^i = (G_E \sqcup H_E)^{i'}$

Proof. The proofs of (i), (ii), and (iii) are obvious.

- (iv) It follows from (ii) that $G_E^i \sqcup H_E^i \subseteq (G_E \sqcup H_E)^{i'}$. Conversely, let $P_e^x \notin G_E^i \sqcup H_E^i$. Then, there exist infra soft open sets U_E and V_E containing P_e^x such that $[U_E \setminus P_e^x] \cap G_E = \Phi$ and $[V_E \setminus P_e^x] \cap H_E = \Phi$. It is clear that $U_E \cap V_E$ is an infra soft open set containing P_e^x such that $[(U_E \cap V_E) \setminus P_e^x] \cap (G_E \sqcup H_E) = \Phi$. This means that $P_e^x \notin (G_E \sqcup H_E)^{i'}$. Thus, $(G_E \sqcup H_E)^{i'} \subseteq G_E^i \sqcup H_E^i$. Hence, the proof is complete. \square

The next result investigates the role of infra soft limit points in studying the infra soft closure points of a soft set.

Theorem 3. Let F_E be a soft subset of (X, \mathcal{I}, E) . Then,

- (i) If F_E is infra soft closed, then $F_E^i \subseteq F_E$.
- (ii) $(F_E \sqcup F_E^i)^{i'} \subseteq F_E \sqcup F_E^i$.
- (iii) $Cl(F_E) = F_E \sqcup F_E^i$.

Proof.

- (i) Let F_E be an infra soft closed set and $P_e^x \notin F_E$. Then, $P_e^x \in F_E^c$. Now, F_E^c is an infra soft open set such that $F_E^c \cap F_E = \Phi$. Therefore, $P_e^x \notin F_E^i$. Thus, $F_E^i \subseteq F_E$, as required.
- (ii) Let $P_e^x \notin (F_E \sqcup F_E^i)^{i'}$. Then, $P_e^x \notin F_E$ and $P_e^x \notin F_E^i$. Therefore, there exists an infra soft open set G_E such that

$$G_E \cap F_E = \Phi. \quad (5)$$

On the contrary, for each $P_e^x \in G_E$, we have $P_e^x \notin F_E$. Then, $[G_E \setminus P_e^x] \cap F_E = \Phi$. Therefore, $P_e^x \notin F_E^i$, and this implies that

$$G_E \cap F_E^i = \Phi. \quad (6)$$

From (1) and (2), we get $G_E \cap (F_E \sqcup F_E^i)^{i'} = \Phi$. Thus, $P_e^x \notin (F_E \sqcup F_E^i)^{i'}$. Hence, $(F_E \sqcup F_E^i)^{i'} \subseteq (F_E \sqcup F_E^i)$, as required.

- (iii) It follows from (ii) of Proposition 4 that $F_E^i \subseteq Cl(F_E)$; also, it is clear that $F_E \subseteq Cl(F_E)$. Then, $F_E \sqcup F_E^i \subseteq Cl(F_E)$. On the contrary, let $P_e^x \in Cl(F_E)$. Then, $F_E \cap G_E \neq \Phi$ for every infra soft open set containing P_e^x . Without loss of generality, suppose that $P_e^x \notin F_E$. Then, $[F_E \setminus P_e^x] \cap G_E \neq \Phi$. Therefore, $P_e^x \in F_E^i$. Hence, we obtain the desired result. \square

The converse of property (i) in the above theorem is not always true as explained in the following example.

Example 8. From Example 4, we demonstrate that $Cl(G_E) = G_E$. This directly means that $G_E^i \subseteq G_E$. But G_E is not an infra soft closed set.

Definition 18. Let H_E be a soft subset of (X, \mathcal{I}, E) . The infra soft boundary points of H_E , denoted by $B(H_E)$, are all soft points which belong to the relative complement of $Int(H_E) \sqcup Int(H_E^c)$.

Proposition 11. Let H_E be a soft subset of (X, \mathcal{I}, E) . Then,

- (i) $B(H_E) = Cl(H_E) \cap Cl(H_E^c)$
- (ii) $B(H_E) = Cl(H_E) \setminus Int(H_E)$

Proof.

- (i) $B(H_E) = \{P_e^x \in \tilde{X} : P_e^x \notin Int(H_E) \text{ and } P_e^x \notin Int(H_E^c)\}$
 $= \{P_e^x \in \tilde{X} : P_e^x \notin (Cl(H_E^c))^c \text{ and } P_e^x \notin (Cl(H_E))^c\}$
 $= \{P_e^x \in \tilde{X} : P_e^x \in Cl(H_E^c) \text{ and } P_e^x \in Cl(H_E)\} = Cl(H_E) \cap Cl(H_E^c)$
- (ii) $B(H_E) = Cl(H_E) \cap Cl(H_E^c) = Cl(H_E) \cap (Int(H_E))^c = Cl(H_E) \setminus Int(H_E)$ \square

Corollary 1. Let H_E be a soft subset of (X, \mathcal{I}, E) . Then,

- (i) $B(H_E) = B(H_E^c)$
- (ii) $Cl(H_E) = Int(H_E) \sqcup B(H_E)$
- (iii) If H_E is infra soft open, then $B(H_E) \cap H_E = \Phi$
- (iv) If H_E is infra soft closed, then $B(H_E) \subseteq H_E$
- (v) If H_E is both infra soft open and infra soft closed, then $B(H_E) = \Phi$

4. Methods of Generating Infra Soft Topologies

In this section, we present some methods of generating infra soft topologies different from those given in Propositions 1 and 2. These methods are infra soft neighbourhood systems, infra soft basis, infra soft subspace, and crisp infra topologies. We research these methods with the help of elucidative examples.

4.1. Infra Soft Neighbourhood and Infra Soft Neighbourhood Systems

Definition 19. A soft subset W_E of (X, \mathcal{I}, E) is said to be an infra soft neighbourhood of a soft point $P_e^x \in \tilde{X}$ if there exists an infra soft open set G_E such that $P_e^x \in G_E \subseteq W_E$.

Definition 20. The infra soft neighbourhood system of a soft point $P_e^x \in \tilde{X}$, denoted by $\mathcal{FN}_{P_e^x}$, is the class of all infra soft neighbourhoods of P_e^x . In other words, $\mathcal{FN}_{P_e^x} = \{W_E \sqsubseteq \tilde{X} : W_E \text{ is an infra soft neighbourhood of a soft point } P_e^x\}$.

Proposition 12. If G_E is an infra soft open subset of (X, ϑ, E) , then it is an infra soft neighbourhood of its all soft points.

Proof. Straightforward. \square

To demonstrate that the converse of the above proposition fails, consider Example 5. It is clear that every infinite infra soft set is an infra soft neighbourhood of its all soft points, but it is not an infra soft open set.

Theorem 4. The infra soft neighbourhood system of a soft point $P_e^x \in (X, \vartheta, E)$ satisfies the following properties.

- (IN1): $\mathcal{FN}_{P_e^x} \neq \emptyset$
- (IN2): if $W_E \in \mathcal{FN}_{P_e^x}$ and $W_E \sqsubseteq N_E$, then $N_E \in \mathcal{FN}_{P_e^x}$
- (IN3): if $V_E, W_E \in \mathcal{FN}_{P_e^x}$, then $V_E \sqcap W_E \in \mathcal{FN}_{P_e^x}$
- (IN4): for each $W_E \in \mathcal{FN}_{P_e^x}$, there is an infra soft neighbourhood V_E of a soft point P_a^y contained in W_E such that $V_E \in \mathcal{FN}_{P_a^y}$ for each $P_a^y \in V_E$

Proof.

- (IN1): since \tilde{X} is an infra soft open set containing every soft point P_e^x , then it is an infra soft neighbourhood of its all soft points. Therefore, $\tilde{X} \in \mathcal{FN}_{P_e^x}$ for every P_e^x . Thus, $\mathcal{FN}_{P_e^x} \neq \emptyset$.
- (IN2): let $W_E \in \mathcal{FN}_{P_e^x}$. Then, there exists an infra soft open set G_E such that $P_e^x \in G_E \sqsubseteq W_E$. If $W_E \sqsubseteq N_E$, then $P_e^x \in G_E \sqsubseteq N_E$. Therefore, $N_E \in \mathcal{FN}_{P_e^x}$.
- (IN3): let V_E and $W_E \in \mathcal{FN}_{P_e^x}$. Then, there exist two infra soft open sets F_E and G_E such that $P_e^x \in F_E \sqsubseteq V_E$ and $P_e^x \in G_E \sqsubseteq W_E$. Since ϑ is an infra soft topology, then $F_E \sqcap G_E$ is an infra soft open set. Obviously, $P_e^x \in F_E \sqcap G_E \sqsubseteq V_E \sqcap W_E$. Thus, $\mathcal{FN}_{P_e^x} \neq \emptyset$.
- (IN4): let $W_E \in \mathcal{FN}_{P_e^x}$. Then, there exists an infra soft open set G_E such that $P_e^x \in G_E \sqsubseteq W_E$. Putting $V_E = G_E$, we obtain the desired result. \square

Theorem 5. Let $\mathcal{FN}_{P_e^x}$ is the class of all families satisfying the four properties given in Theorem 4. Then, $\mathcal{M} = \{U_E \sqsubseteq \tilde{X} : \text{for each } P_e^x \in U_E \Rightarrow U_E \in \mathcal{FN}_{P_e^x}\}$ forms an infra soft topology on the universal set X . Moreover, it forms a soft topology.

Proof. It is well known that \tilde{X} is an infra soft neighbourhood of all its soft points; then, $\tilde{X} \in \mathcal{M}$; also, $\Phi \in \mathcal{M}$. Let U_E and $V_E \in \mathcal{M}$. For each $P_e^x \in U_E \sqcap V_E$, we have $P_e^x \in U_E$ and $P_e^x \in V_E$. Then, $U_E \in \mathcal{FN}_{P_e^x}$ and $V_E \in \mathcal{FN}_{P_e^x}$. It follows

from (IN3) that $U_E \sqcap V_E \in \mathcal{FN}_{P_e^x}$. Thus, $U_E \sqcap V_E \in \mathcal{M}$. Hence, \mathcal{M} is an infra soft topology.

To prove that \mathcal{M} is a soft topology, let $U_{iE} \in \mathcal{M}$ for each $i \in I$. Suppose that $P_e^x \in \sqcup_{i \in I} U_{iE}$. Then, there exists $j \in I$ such that $P_e^x \in U_{jE}$. Therefore, $U_{jE} \in \mathcal{FN}_{P_e^x}$. Since $U_{jE} \sqsubseteq \sqcup_{i \in I} U_{iE}$, then, by (IN2), we obtain $\sqcup_{i \in I} U_{iE} \in \mathcal{FN}_{P_e^x}$; hence, $\sqcup_{i \in I} U_{iE} \in \mathcal{M}$, as required. \square

4.2. Basis of Infra Soft Topology

Definition 21. Let (X, ϑ, E) be an infra soft topological space. A class $\mathcal{B} \subseteq \vartheta$ is said to be a basis for ϑ if the finite soft intersection of members of \mathcal{B} forms ϑ .

It is clear that ϑ is a basis for itself.

Proposition 13. Every subclass \mathcal{B} of $S(X_E)$ containing Φ (or two disjoint infra soft open sets) is a basis for a unique infra soft topology on X .

Proof. Let ϑ be a family of soft sets generated by \mathcal{B} . Since the empty soft intersection is the absolute soft set, then $\tilde{X} \in \vartheta$. By hypothesis, $\Phi \in \mathcal{B}$, so $\Phi \in \vartheta$. It follows from the definition of ϑ , which is generated by \mathcal{B} , that ϑ is closed under finite soft intersection. Hence, we obtain the desired result.

To prove the uniqueness, let ϑ_1 be another infra soft topology generated by \mathcal{B} . Let $F_E \in \vartheta_1$. Then, F_E can be expressed as a finite soft intersection of members of \mathcal{B} . Therefore, $F_E \in \vartheta$. Thus, $\vartheta_1 \subseteq \vartheta$. Similarly, one can prove that $\vartheta \subseteq \vartheta_1$. Hence, $\vartheta = \vartheta_1$, as required. \square

Example 9. Consider the class $\mathcal{B} = \{F_{iE} : i = 1, 2, 3\}$ defined on $X = \{a, b, c\}$ with a set of parameters $E = \{e_1, e_2\}$, where

$$\begin{aligned} F_{1E} &= \{(e_1, X), (e_2, \{c\})\}, \\ F_{2E} &= \{(e_1, \{a, b\}), (e_2, \{a, c\})\}, \\ F_{3E} &= \{(e_1, \{c\}), (e_2, \{b\})\}. \end{aligned} \quad (7)$$

By taking the finite soft intersection of the members of \mathcal{B} , we obtain $\theta = \{\Phi, \tilde{X}, F_{1E}, F_{2E}, F_{3E}, H_{1E}, H_{2E}\}$, where

$$\begin{aligned} H_{1E} &= \{(e_1, \{a, b\}), (e_2, \{c\})\}, \\ H_{2E} &= \{(e_1, \{c\}), (e_2, \emptyset)\}. \end{aligned} \quad (8)$$

Obviously, θ is an infra soft topology on X .

Remark 3. The basis for an infra soft topology need not be unique. In other words, there are more than one basis for an infra soft topology in general.

4.3. Subspace

Definition 22. Let (X, ϑ, E) be an infra soft topological space and Y be a nonempty subset of X . A class $\vartheta_Y = \{\tilde{Y} \sqcap G_E : G_E \in \vartheta\}$ is called an infra soft relative topology on Y , and (Y, ϑ_Y, E) is called an infra soft subspace of (X, ϑ, E) .

It can be easily checked that ϑ_Y given in the above definition is an infra soft topology on Y .

Theorem 6. Let (Y, ϑ_Y, E) be an infra soft subspace of (X, ϑ, E) . Then, H_E is an infra soft closed subset of (Y, ϑ_Y, E) if and only if there exists an infra soft closed subset F_E of (X, ϑ, E) such that $H_E = \tilde{Y} \cap F_E$.

Proof. Necessity: let H_E be an infra soft closed subset of (Y, ϑ_Y, E) . Then, there exists an infra soft open set U_E in (Y, ϑ_Y, E) such that $U_E = \tilde{Y} \setminus H_E$. This means that there exists an infra soft open set V_E in (X, ϑ, E) such that $U_E = \tilde{Y} \cap V_E$. Therefore, $H_E = \tilde{Y} \setminus (\tilde{Y} \cap V_E) = \tilde{Y} \cap V_E^c$. Putting $F_E = V_E^c$ ends the proof of the necessary part.

Sufficiency: let $H_E = \tilde{Y} \cap F_E$ such that F_E is an infra soft closed set in (X, ϑ, E) . Then, $\tilde{Y} \setminus H_E = \tilde{Y} \setminus (\tilde{Y} \cap F_E) = (\tilde{Y} \cap \tilde{X}) \setminus (\tilde{Y} \cap F_E) = \tilde{Y} \cap (\tilde{X} \setminus F_E)$. Since $\tilde{X} \setminus F_E$ is an infra soft open set in (X, ϑ, E) , then $\tilde{Y} \setminus H_E$ is an infra soft open set in (Y, ϑ_Y, E) . Therefore, H_E is an infra soft closed set in (Y, ϑ_Y, E) . Hence, the proof is complete. \square

The proofs of the following two propositions are straightforward, and thus, they are omitted.

Proposition 14. Let \tilde{Y} be an infra soft open subset of (X, ϑ, E) . Then, U_E is an infra soft open subset of (Y, ϑ_Y, E) if and only if it is an infra soft open subset of (X, ϑ, E) .

Proposition 15. Let (Y, ϑ_Y, E) be an infra soft subspace of (X, ϑ, E) . Then, V_E is an infra soft neighbourhood of P_e^y in (Y, ϑ_Y, E) if and only if there exists an infra soft neighbourhood W_E of P_e^y in (X, ϑ, E) such that $V_E = \tilde{Y} \cap W_E$.

Theorem 7. Let (Y, ϑ_Y, E) be an infra soft subspace of (X, ϑ, E) such that $H_E \subseteq \tilde{Y}$. Let Int_Y , Cl_Y , and y_i' be, respectively, the infra soft interior, infra soft closure, and infra soft limit points of a soft set in (Y, ϑ_Y, E) , and let Int , Cl , and i' be, respectively, the infra soft interior, infra soft closure, and infra soft limit points of a soft set in (X, ϑ, E) . Then,

- (i) $\text{Int}(H_E) = \text{Int}_Y(H_E) \cap \text{Int}(\tilde{Y})$
- (ii) $\text{Cl}_Y(H_E) = \text{Cl}(H_E) \cap \tilde{Y}$
- (iii) $(H_E)^{y_i'} = (H_E)^{i'} \cap \tilde{Y}$

Proof. We only prove (i). One can prove the other cases using similar techniques.

Let $P_e^x \in \text{Int}(H_E)$. Then, there exists an infra soft open subset U_E of (X, ϑ, E) such that $P_e^x \in U_E \subseteq H_E \subseteq \tilde{Y}$. Therefore, $P_e^x \in U_E \cap \tilde{Y} \subseteq H_E$. Thus, $P_e^x \in \text{Int}_Y(H_E)$ and $P_e^x \in \text{Int}(\tilde{Y})$. Hence, $\text{Int}(H_E) \subseteq \text{Int}_Y(H_E) \cap \text{Int}(\tilde{Y})$. Conversely, let $P_e^x \in \text{Int}_Y(H_E) \cap \text{Int}(\tilde{Y})$. Then, there exist two infra soft open subsets U_E and V_E of (X, ϑ, E) such that $P_e^x \in U_E \subseteq \tilde{Y}$ and $P_e^x \in V_E \cap \tilde{Y} \subseteq H_E$. Since ϑ is an infra soft topology, then $U_E \cap V_E$ is an infra soft open set containing

P_e^x such that $U_E \cap V_E \subseteq F_E$. Consequently, $P_e^x \in \text{Int}(H_E)$. Thus, $\text{Int}_Y(H_E) \cap \text{Int}(\tilde{Y}) \subseteq \text{Int}(H_E)$. Hence, the proof is complete. \square

4.4. Producing Infra Soft Topology from Crisp Infra Topologies

Proposition 16. Suppose that $\Psi = \{\Omega_e\}_{e \in E}$ is a family of crisp infra topologies on X . Then,

$$\vartheta(\Psi) = \{e, F(e) : e \in E\} \in S(X_E) \text{ such that } F(e) \in \Omega_e \text{ for each } e \in E, \quad (9)$$

which defines an infra soft topology on X .

Proof. Let the given assumptions be satisfied. Since \emptyset and $X \in \Omega_e$ for each $e \in E$, then Φ and $\tilde{X} \in \vartheta(\Psi)$. To prove that $\vartheta(\Psi)$ is closed under finite soft intersection, let U_E and $V_E \in \vartheta(\Psi)$. According to the structure of $\vartheta(\Psi)$, we have $U(e)$ and $V(e) \in \Omega_e$ for each $e \in E$. Since Ω_e is infra topology, then $U(e) \cap V(e) \in \Omega_e$. This implies that $U_E \cap V_E \in \vartheta(\Psi)$, as required. Hence, $X, \vartheta(\Psi), E$ is an infra soft topological space. \square

Definition 23. The infra soft topological space given in the above proposition is called the infra soft topology on X generated by Ψ .

We write $\vartheta(\Psi) = \vartheta(\Omega)$ if $\Omega_e = \Omega_{e'} = \Omega$ for each e and $e' \in E$.

Remark 4. The largest infra soft topology whose parametric infra topologies are $\Psi = \{\Omega_e\}_{e \in E}$ is $\vartheta(\Psi)$.

We explain in the following example how we can apply Proposition 16 to construct an infra soft topology from crisp infra topologies.

Example 10. Let $\Omega_{e_1} = \emptyset, X, \{a\}, \{b\}, \{a, c\}$ and $\Omega_{e_2} = \{\emptyset, X, \{a\}, \{c\}\}$ be two crisp infra topologies on $X = \{a, b, c\}$. To produce an infra soft topology from Ω_{e_1} and Ω_{e_2} , we construct infra soft open sets H_{iE} by choosing any set in Ω_{e_1} as an image of e_1 , say, \emptyset . Then, we can choose the image of e_2 by four different ways because the number of infra open sets in Ω_{e_2} is four. Therefore, we obtain the following four infra soft sets:

$$\begin{aligned} H_{1E} &= \{(e_1, \emptyset), (e_2, \emptyset)\}, \\ H_{2E} &= \{(e_1, \emptyset), (e_2, X)\}, \\ H_{3E} &= \{(e_1, \emptyset), (e_2, \{a\})\}, \\ H_{4E} &= \{(e_1, \emptyset), (e_2, \{c\})\}. \end{aligned} \quad (10)$$

We repeat this manner with each set in Ω_{e_1} . Then, we obtain the following infra soft open sets:

$$\begin{aligned}
H_{5E} &= \{(e_1, X), (e_2, \emptyset)\}, \\
H_{6E} &= \{(e_1, X), (e_2, X)\}, \\
H_{7E} &= \{(e_1, X), (e_2, \{a\})\}, \\
H_{8E} &= \{(e_1, X), (e_2, \{c\})\}, \\
H_{9E} &= \{(e_1, \{a\}), (e_2, \emptyset)\}, \\
H_{10E} &= \{(e_1, \{a\}), (e_2, X)\}, \\
H_{11E} &= \{(e_1, \{a\}), (e_2, \{a\})\}, \\
H_{12E} &= \{(e_1, \{a\}), (e_2, \{c\})\}, \\
H_{13E} &= \{(e_1, \{b\}), (e_2, \emptyset)\}, \\
H_{14E} &= \{(e_1, \{b\}), (e_2, X)\}, \\
H_{15E} &= \{(e_1, \{b\}), (e_2, \{a\})\}, \\
H_{17E} &= \{(e_1, \{a, c\}), (e_2, \emptyset)\}, \\
H_{17E} &= \{(e_1, \{a, c\}), (e_2, \emptyset)\}, \\
H_{18E} &= \{(e_1, \{a, c\}), (e_2, X)\}, \\
H_{19E} &= \{(e_1, \{a, c\}), (e_2, \{a\})\}, \\
H_{20E} &= \{(e_1, \{a, c\}), (e_2, \{c\})\}.
\end{aligned} \tag{11}$$

Hence, $\vartheta = \{\Phi, \tilde{X}, H_{iE}; i = 1, 2, \dots, 20\}$ is an infra soft topology on $X = \{a, b, c\}$.

Proposition 17. Every infra soft topology generated by infra topologies $\{\Omega_e\}_{e \in E}$ contains all soft sets, in which their e -components are X or \emptyset .

Proof. Since \emptyset and $X \in \Omega_e$ for each $e \in E$, then any soft set G_E defined as $G(e)$ is X or \emptyset , which is a member of $\vartheta(\Psi)$. \square

The converse of the above proposition fails as illustrated in the following example.

Example 11. Let $E = \{e_1, e_2\}$ and $\vartheta = \{\Phi, \tilde{X}, F_{iE}; i = 1, 2, 3\}$ be an infra soft topology on $X = \{a, b, c\}$, where

$$\begin{aligned}
F_{1E} &= \{(e_1, X), (e_2, \emptyset)\}, \\
F_{2E} &= \{(e_1, \emptyset), (e_2, X)\}, \\
F_{3E} &= \{(e_1, \{a\}), (e_2, \emptyset)\}.
\end{aligned} \tag{12}$$

It is clear that ϑ contains all soft sets, in which their e -components are X or \emptyset . But ϑ does not generate from crisp infra topologies because $\{a\} \in \vartheta_{e_1}$ and $X \in \vartheta_{e_2}$; however, $e_1, \{a\}, (e_2, X) \notin \vartheta$.

In the following two examples, we show how we can examine whether the infra soft topology is generated by crisp infra topologies or not?

Example 12. Let $E = \{e_1, e_2\}$ and $\vartheta = \{\Phi, \tilde{X}, F_E, H_E\}$ be an infra soft topology on $X = \{a, b, c\}$, where

$$\begin{aligned}
F_E &= \{(e_1, \{a\}), (e_2, \{c\})\}, \\
H_E &= \{(e_1, \{b\}), (e_2, \{b\})\}.
\end{aligned} \tag{13}$$

Then, $\vartheta_{e_1} = \{\emptyset, X, \{a\}, \{b\}\}$ and $\vartheta_{e_2} = \{\emptyset, X, \{b\}, \{c\}\}$ are the parametric (crisp) infra topologies of an infra soft topology ϑ . Consider $\Omega_{e_1} = \vartheta_{e_1}$ and $\Omega_{e_2} = \vartheta_{e_2}$. Now, $\{a\} \in \Omega_{e_1}$ and $\{b\} \in \Omega_{e_2}$; however, $e_1, \{a\}, (e_2, \{b\}) \notin \vartheta$. Thus, ϑ is not generated from crisp infra topologies.

Example 13. Consider the soft sets below defined on $X = \{a, b, c\}$ with a set of parameters $E = \{e_1, e_2\}$:

$$\begin{aligned}
H_{1E} &= \{(e_1, \{a\}), (e_2, X)\}, \\
H_{2E} &= \{(e_1, \{b\}), (e_2, X)\}, \\
H_{3E} &= \{(e_1, \{a\}), (e_2, \emptyset)\}, \\
H_{4E} &= \{(e_1, \{b\}), (e_2, \emptyset)\}, \\
H_{5E} &= \{(e_1, X), (e_2, \emptyset)\}, \\
H_{6E} &= \{(e_1, \emptyset), (e_2, X)\}.
\end{aligned} \tag{14}$$

Then, $\vartheta = \{\Phi, \tilde{X}, H_{iE}; i = 1, 2, \dots, 6\}$ is an infra soft topology on $X = \{a, b, c\}$. It is clear that $\vartheta_{e_1} = \{\emptyset, X, \{a\}, \{b\}\}$ and $\vartheta_{e_2} = \{\emptyset, X\}$ are the parametric (crisp) infra topologies of an infra soft topology ϑ . Consider $\Omega_{e_1} = \vartheta_{e_1}$ and $\Omega_{e_2} = \vartheta_{e_2}$. It can be seen that ϑ is generated from the crisp infra topologies Ω_{e_1} and Ω_{e_2} .

5. Continuity between Infra Soft Topological Spaces

In this section, we define the concept of continuity between infra soft topological spaces and then give its equivalent conditions using infra soft open and infra soft closed sets. Also, we discuss losing some equivalent conditions of soft continuity on infra soft topology with the help of an illustrative example. We close this section by studying “transmission” of continuity between an infra soft topological space and its parametric infra topological spaces.

Definition 24. Definition 24A soft map f_φ from (X, ϑ, E) to (Y, μ, E) is said to be infra soft continuous at a soft point $P_e^x \in \tilde{X}$ if for each infra soft open set U_E containing $f_\varphi(P_e^x)$, there is an infra soft open set V_E containing P_e^x such that $f_\varphi(V_E) \subseteq U_E$.

Definition 25. A soft map f_φ from (X, ϑ, E) to (Y, μ, E) is said to be infra soft continuous if it is infra soft continuous for all $P_e^x \in \tilde{X}$.

Theorem 8. A soft map $f_\varphi: (X, \vartheta, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous if and only if the inverse image of each infra soft open set is an infra soft open set.

Proof. Necessity: let U_E be an infra soft open subset of (Y, μ, E) . Without loss of generality, consider $f_\varphi^{-1}(U_E) \neq \Phi$. Then, for each $P_e^x \in f_\varphi^{-1}(U_E)$, we have an infra soft open subset V_E of (X, ϑ, E) containing P_e^x such that $f_\varphi(V_E) \subseteq U_E$.

Thus, $P_e^x \in V_E \subseteq f_\varphi^{-1}(U_E)$ and $\sqcup \{V_E\} = f_\varphi^{-1}(U_E)$. Hence, $f_\varphi^{-1}(U_E)$ is infra soft open.

Sufficiency: suppose that $P_e^x \in \tilde{X}$ and U_E is an infra soft open set containing $f_\varphi(P_e^x)$. Then, $f_\varphi^{-1}(U_E)$ is an infra soft open set containing P_e^x such that $f_\varphi(f_\varphi^{-1}(U_E)) \subseteq U_E$. Therefore, f_φ is infra soft continuous at P_e^x which we choose arbitrarily; hence, f_φ is infra soft continuous. \square

Corollary 2. A soft map $f_\varphi: (X, \vartheta, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous if and only if the inverse image of each infra soft closed set is an infra soft closed set.

Proof. Necessity: let G_E be an infra soft closed subset of \tilde{Y} . Then, G_E^c is infra soft open. Therefore, $f_\varphi^{-1}(G_E^c) = \tilde{X} \setminus f_\varphi^{-1}(G_E)$ is an infra soft open set; hence, $f_\varphi^{-1}(G_E)$ is infra soft closed.

Sufficiency: following similar arguments, we prove the sufficient part. \square

The following properties are equivalent to soft continuity on soft topological and supra soft topological spaces.

- (i) $\text{Cl}(f_\varphi^{-1}(H_E)) \subseteq f_\varphi^{-1}(\text{Cl}(H_E))$ for each $H_E \subseteq \tilde{Y}$
- (ii) $f_\varphi(\text{Cl}(F_E)) \subseteq \text{Cl}(f_\varphi(F_E))$ for each $F_E \subseteq \tilde{X}$
- (iii) $f_\varphi^{-1}(\text{Int}(H_E)) \subseteq \text{Int}(f_\varphi^{-1}(H_E))$ for each $H_E \subseteq \tilde{Y}$

But they are not equivalent to soft continuity on the infra soft topological spaces.

Definition 26. Let $f_\varphi: (X, \vartheta, E) \longrightarrow (Y, \mu, E)$ be a soft map and Z be a nonempty subset of X . A soft map $f_{\varphi|_Z}$ from (Z, ϑ_Z, E) to (Y, μ, E) , which is given by $f_{\varphi|_Z}(P_e^z) = f_\varphi(P_e^z)$ for each $P_e^z \in \tilde{Z}$, is called the restriction soft map of f_φ on Z .

Theorem 9. If $f_\varphi: (X, \vartheta, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous, then the restriction soft map $f_{\varphi|_Z}: (Z, \vartheta_Z, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous.

Proof. Let F_E be an infra soft open subset of (Y, μ, E) . Then, $f_{\varphi|_Z}^{-1}(F_E) = f_\varphi^{-1}(F_E) \cap \tilde{Z}$. By hypothesis, $f_\varphi^{-1}(F_E)$ is an infra soft open subset of (X, ϑ, E) ; therefore, $f_\varphi^{-1}(F_E) \cap \tilde{Z}$ is an infra soft open subset of (Z, ϑ_Z, E) . Hence, $f_{\varphi|_Z}$ is an infra soft continuous map, as required. \square

It is easy to prove the following two results; thus, their proofs will be omitted.

Proposition 18. If $f_\varphi: (X, \vartheta, E) \longrightarrow (Y, \mu, E)$ and $g_\varphi: (Y, \mu, E) \longrightarrow (Z, \nu, E)$ are infra soft continuous maps, then $g_\varphi \circ f_\varphi$ is an infra soft continuous map.

Proposition 19. Let $\{\vartheta_i: i \in I\}$ be a family of infra soft topologies on X with a set of parameters E . If $f_\varphi: (X, \vartheta_i, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous for each i , then $f_\varphi: (X, \sqcap_{i \in I} \vartheta_i, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous.

Proposition 20. Let f_φ be a soft map of an infra soft topological space (X, ϑ, E) to $S_E(Y)$. Then, $\mu = \{G_E \subseteq \tilde{Y}: f_\varphi^{-1}(G_E) \in \vartheta\}$ is an infra soft topology on Y .

Moreover, μ is the strongest infra soft topology on Y which makes f_φ as infra soft continuous.

Proof. Since $f_\varphi^{-1}(\Phi), f_\varphi^{-1}(\tilde{Y}) \in \vartheta$, then Φ and $\tilde{Y} \in \mu$. Let G_{1E} and $G_{2E} \in \mu$. Then, there exist $f_\varphi^{-1}(G_{1E})$ and $f_\varphi^{-1}(G_{2E}) \in \vartheta$. Since ϑ is infra soft topology, then $f_\varphi^{-1}(G_{1E}) \sqcap f_\varphi^{-1}(G_{2E}) = f_\varphi^{-1}(G_{1E} \sqcap G_{2E}) \in \vartheta$. Therefore, $G_{1E} \sqcap G_{2E} \in \mu$; thus, μ is infra soft topology on Y . To prove that μ is the strongest infra soft topology on Y , it makes f_φ as infra soft continuous. Suppose that ν is an infra soft topology on Y such that $f_\varphi: (X, \vartheta, E) \longrightarrow (Y, \nu, E)$ is infra soft continuous. Let $H_E \in \nu$; then, $f_\varphi^{-1}(H_E) \in \vartheta$. This implies that $H_E \in \mu$; thus, $\mu \sqsubset \nu$. Hence, the proof is complete. \square

We complete this section by investigating the concept of infra soft continuity between the soft map and crisp map.

Theorem 10. If a soft map $f_\varphi: (X, \vartheta, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous, then a map $f: (X, \vartheta_e) \longrightarrow (Y, \mu_{\varphi(e)})$ is infra continuous.

Proof. Let U be an infra open subset of $(Y, \mu_{\varphi(e)})$. Then, there exists an infra soft open subset G_E of (Y, μ, E) such that $G(\varphi(e)) = U$. Since f_φ is an infra soft continuous map, then $f_\varphi^{-1}(G_E)$ is an infra soft open subset of (X, ϑ, E) . It follows from Definition 10 that $f_\varphi^{-1}(G_E) = (f^{-1}(G))_E$, where $f^{-1}(G)(e) = f^{-1}(G(\varphi(e)))$; this implies that $f^{-1}(U) = f^{-1}(G(\varphi(e)))$ in the infra open subset of (X, ϑ_e) . Hence, we obtain the desired result. \square

We explain that the converse of the above theorem fails in the example below.

Example 14. Let $E = \{e_1, e_2\}$ and $\vartheta = \{\Phi, \tilde{X}, F_E\}$ and $\mu = \{\Phi, \tilde{X}, H_E\}$ be two infra soft topologies on $X = \{a, b\}$, where

$$\begin{aligned} F_E &= \{(e_1, \{a\}), (e_2, \emptyset)\}, \\ H_E &= \{(e_1, \emptyset), (e_2, \{a\})\}. \end{aligned} \quad (15)$$

Consider $\varphi: E \longrightarrow E$ and $f: X \longrightarrow X$ are identity maps. Then, $f: (X, \vartheta_e) \longrightarrow (X, \mu_{\varphi(e)=e})$ is infra continuous for each $e \in E$. But $f_\varphi: (X, \vartheta, E) \longrightarrow (X, \mu, E)$ is not an infra soft continuous map because $f_\varphi^{-1}(F_E) = F_E \notin \vartheta$.

We show under which condition the converse of Theorem 10 holds.

Theorem 11. Let ϑ be an infra soft topology generated from the crisp infra topologies. Then, a soft map $f_\varphi: (X, \vartheta, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous if and only if a map $f: (X, \vartheta_e) \longrightarrow (Y, \mu_{\varphi(e)})$ is infra continuous.

Proof. The necessary part is proved in Theorem 10.

To prove the sufficient part, let U_E be an infra soft open subset of (Y, μ, E) . Then, $f_\varphi^{-1}(U_E) = (f^{-1}(U))_E$ is a soft subset of (X, ϑ, E) such that $f^{-1}(U)(e) = f^{-1}(U(\varphi(e)))$ for each $e \in E$. Since $U(\varphi(e))$ is an infra open subset of $(Y, \mu_{\varphi(e)})$ and a map f is infra continuous, then

$f^{-1}(U(\varphi(e)))$ is the infra open subset of (X, ϑ_e) . By hypothesis, ϑ is generated from the crisp infra topologies, so $f_{\varphi}^{-1}(U_E)$ is an infra soft open subset of (X, ϑ, E) , as required. \square

6. Conclusion

This study has introduced the concept of an infra soft topology as a new structure is weaker than a soft topology. The most important goal of investigating this concept is to keep some soft topological properties under fewer conditions than topology.

We have contributed to improve the knowledge about this area in three aspects. First, we have established the basic concepts of infra soft topological spaces and scrutinized properties. We have noted that most properties of interior and closure operators are valid on infra soft topological spaces, while most of them are losing on other generalizations of soft topology such as supra soft topology. Second, we have proposed some techniques of producing infra soft topologies such as soft maps, soft neighbourhood systems, infra soft basis, infra soft subspace, and crisp infra topologies. In fact, the techniques of soft neighbourhood systems and soft operators initiate soft topology which is due to the identical between their properties on soft topology and infra soft topology. Third, we have introduced and investigated the concept of continuity between infra soft topological spaces. We have described this concept using infra soft open and infra soft closed sets. Moreover, we have showed that some characterizations of continuity on soft topology are losing on the frame of infra soft topology, especially those that are based on the interior and closure operators.

In future works, we plan to formulate the soft topological concepts such as separation axioms, compactness, and connectedness on the frame of infra soft topology. In particular, we shed light on discovering which ones of their properties are still valid on the infra soft topologies.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Retraction

Retracted: Knowledge Management System Adoption to Improve Decision-Making Process in Higher Learning Institutions in the Developing Countries: A Conceptual Framework

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

In addition, our investigation has also shown that one or more of the following human-subject reporting requirements has not been met in this article: ethical approval by an Institutional Review Board (IRB) committee or equivalent, patient/participant consent to participate, and/or agreement to publish patient/participant details (where relevant).

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

- [1] H. M. Oumran, R. B. Atan, R. N. H. Binti Nor, S. B. Abdullah, and M. Mukred, "Knowledge Management System Adoption to Improve Decision-Making Process in Higher Learning Institutions in the Developing Countries: A Conceptual Framework," *Mathematical Problems in Engineering*, vol. 2021, Article ID 9698773, 15 pages, 2021.

Research Article

Knowledge Management System Adoption to Improve Decision-Making Process in Higher Learning Institutions in the Developing Countries: A Conceptual Framework

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Currently, higher learning institutions (HLIs) are facing their most challenging problem in inefficient information management. The knowledge management system (KMS) application calls for providing several benefits to lecturers and students, producing daily information, documenting records for evidence of a transaction, and eventually improving the decision-making process. Knowledge management can be coupled with fuzzy logic to deal with imprecision and uncertainty of data in a KMS. The ICT dynamic development has shifted the HLI operations from manual to electronic-based handling of related information. KMS is one of the systems that are of significant consideration in this regard. Nevertheless, such a system has not been extensively adopted as expected due to users' rejection of its use. In the present paper, the factors affecting the decision to adopt/reject KMS are highlighted. The study is qualitative and entails a critical review of the related literature concerning the topic, backed by interviews. KMS experts working with highly reputable HLI were interviewed. A total of 11 factors were focused on in light of their effect on the decision to adopt/reject KMS, as argued by the technological adoption theories and literature review. All the factors were validated and placed in ranks by the experts. From the results, a novel conceptual framework of KMS adoption was developed for Libyan HLIs to bring about technology adoption and improved decision-making.

1. Introduction

Many countries have realized the need to adopt an outcome-based approach to provide ongoing educational improvement for increasing unemployed graduates. Higher educational institutions have responded by concentrating on students' sufficient professional and career preparation through the stress on market demands of specific outcomes or abilities. Such outcome-based approaches are directed towards assessing the students' performance and knowledge, mitigating the gap between university learning and practice in their actual careers [1–3].

HLIs in various countries generally acknowledge the value of information in the management and decision-

making processes. This has paved the way for further systems development, computer hardware, software, and Internet usage. Furthermore, an information system refers to an organized integration of people, hardware, software, channels of communication, and data resources, functioning in tandem to collect, transform, and spread information in the organization [4]. In this combination, HLIs will find the OBE application and implementation, aided by its evaluation and particular system, invaluable [3]. In the KMS, information is furnished for education institutions to make decisions and assessments and oversee and evaluate educational activities [5].

In the context of educational institutions, adopting KMS can minimize the education demand-supply gap [6, 7], and

such notion has resulted in heightened awareness and investment in KMS innovation in the majority of nations to enhance their system of education [8, 9]. Additionally, KMS adoption that constitutes the education provision has been considered a set of processes to be implemented to enhance the effectiveness of HLI in terms of its performance and objectives achievement. In literature, several barriers to KMS adoption have been evidenced by studies in the context of developing countries, including Garrett [10], Shroff et al. [11], and Alharthi [12] in the Libyan context.

Such studies revealed that the adoption of technology and systems, specifically KMS, is still at the initial stages [9, 12–14]. Most studies of this caliber have stressed three main barrier categories, namely, human-related, organization-related, and technology-related barriers (e.g., [15, 16]). Heeks [17] reported that information systems, combined with technical, social, organizational, and environmental factors, have been successful, although evidence backed by theory regarding adopting KMS at the individual and environmental level is still scarce.

In Arab nations, Gholam and Kobeissi [18] reported the absence of technology implementation for evaluation to support professional development. In this regard, Alfahadi et al. [19] and Alharthi [12] presented a critical look at the implemented evaluation process in Libya that lacks tools and techniques, leading to an ambiguous view of the students' performance. Evaluation procedures in Libyan institutions need reformation for validation, realism, and authentic implementation and use [20].

The stress of the above discussion is the requirement of examining innovation and technology adoption to allow higher education institutions competitiveness and ability to develop into global leaders in the educational platform. Thus, a clearer picture of such adoption is called to extend and promote learning innovations adoption and usage [21].

Moreover, KMS use and adoption in educational institutions for their improvement are part of the advancement of technology. Research studies of this caliber have highlighted KMSs as a crucial tool in assessing the process of evaluation (e.g., Bartlett [22]).

This manuscript is structured to include KMS and the decision-making process in Section 2 after the introduction. Section 3 presents related works on KMS adoption, and Section 4 provides the methodology. Discussion and interpretations of the finding are presented in Section 5, and Section 6 is dedicated for the conclusion.

2. Knowledge Management Systems and Decision-Making Process

Information system (IS) refers to integrating a group of components used to gather, store, and process data to distribute the information and knowledge obtained [23]. It refers to a combination of hardware, software, and telecommunication networks built to collect, create, and distribute required data, generally in organizations. On the other hand, KMS consists of a class of information systems employed to manage the organization's knowledge [24]. KMS is a category of IS used in organizations to manage

knowledge with the help of IT-based systems, created to provide support and improvement to the processes involving the creation, storage, retrieval, transfer, conversion, protection, and application of knowledge [25, 26].

KMS refers to an IT-based system developed to provide support and enhancement to the processes of organizations relating to the creation, storage, transfer, and application of knowledge [24]. It was similarly described by Alatawi et al. [27] as a system created and designed to provide the knowledge needed for decision-making and tasks undertaking among decision-makers and users [27]. As Alavi and Leidner [24] definition corresponds to the present study's objective of examining the adoption of KMS in Libyan universities, it best describes the university practices, settings, and processes when it comes to KMS adoption. Therefore, their definition is adopted. Initiatives of KMS depend mainly on IT, which enables and supports KM in several ways including knowledge sharing and collaboration in a virtual environment by team members, accessing prior project information, and documenting knowledge sources through online directories and search databases [28, 29].

Related studies in the literature (e.g., [29]) examine the critical success factors (CSFs) following KMS adoption and implementation and their significance to the system. The study found organizational readiness to affect KMS adoption or continued intention towards such adoption significantly. In this regard, potential adopters with high behavioral uncertainty need to ensure consistency between themselves and the process of KMS. The two subgroups in Shrafat's [29] study indicated that expected advantages had significant impacts on the intention towards adoption or continued use of KMS. This is empirical evidence confirming that perceived benefits have a substantial role in adopting and diffusing innovation-related activities. This also ensures that KMS adoption and continued use success boost experimentation and risk-taking. In contrast, organization-environmental interaction requirements can be established via dialogue, interaction, and participative decision-making process. The study findings supported the relationship between organizational readiness and KMS adoption or continued use and intention among potential adopters compared to current ones [29].

Decision-making (DM) is considered one of the top executive roles, and available authentic knowledge sources play a crucial role in DMP. Knowledge sources may take the form of oral, written, or computer-based sources. KMS is created to enable users to access knowledge that is essential in achieving their activities on the job. The premise of using computer-based systems to support DM has existed throughout the years, and the issue of how computer-based systems can be utilized to provide support to DM under the DSS nomenclature can be traced back to the later years of the 1970s [30].

On the whole, organizations have increased their complexity and stressed decentralized DM, which tends to lead to using KMS with DSS to support decision-making success. According to Turban et al. [31], DSS covers a knowledge component that is useful for supporting DMP. Suitable DSS integration with KMS will thus help the

interaction and develop new opportunities to enhance the quality of support provided by the system [32]. Meanwhile, other authors like Martinsons and Davison [33] are convinced that KMS and IS success in providing DMP support is dependent on the way IT applications are enhanced and adapted to match the users' decision styles. Therefore, KMS and IS global implementation should be flexible enough to satisfy various decision styles and fit DMP [31, 34, 35].

In Bolloju et al.'s [32] related study, the authors mentioned the advantages of DSS-KMS integration. They included improving support quality in real-time, adaptive active decision support, supporting acquisition, exploitation, creation, gathering knowledge in organizations, facilitation of patterns/trends discovery in the gathered knowledge, and supporting the development means and tools in the organizational memory.

Along a similar line of study, Turban et al. [31] demonstrated that DSS employment could facilitate several advantages: provision of support in all DMP phases and managerial levels (individuals, groups, and organization), improvement of DM effectiveness, mitigation of the requirement for training, enhancement of management control, facilitation of communication, saving effort of the user, ease of costs, and enabling DM objectives.

In addition to the above advantages, DSS can also be utilized by management, analysts, and even intermediaries. Bals et al. [36] emphasize technology as a tool that decision-makers and users can use to leverage their knowledge to achieve the work at hand. Nevertheless, most organizations administering KMS initiatives display various success levels. Thus, the perception of decision-makers and users of technology and their interaction play a key role in KMS and DM initiatives' success.

However, decision-making can be defined as evaluating, assessing, and developing human performance in an organization (HLI as an example). Performance evaluation at educational institutions or organizations has been the subject of several studies in the literature. As a result, performance evaluation is critical in both research and instruction. At educational institutions, performance evaluations are commonly undertaken regularly. Universities and research organizations frequently use the outcomes of evaluations in making decisions such as promoting lecturers or funding research. Without reliable performance evaluation tools, good performers may not receive enough positive feedback, feel upset, and depart, resulting in high recruitment expenses for the firm [37].

The input data for a performance can be from multiple periods. Hence, a dynamic decision-making procedure that uses fuzzy logic is required [38]. In such a method, alternative and importance weights of criteria are represented as triangular fuzzy integers in time sequence [39]. Fuzziness is typical in challenges on decision-making as well as fuzzy logic's advantages. Fuzzy decision-making occurs when single or several criteria are used to discover the ideal option [40].

In this context, Tong et al. [41] presented a method for comparison that is notably ambiguous. Using the fuzzy

extent analysis method (F-EAM), the relative weight of each parameter was measured.

The steps of the fuzzy extended analysis included in their study are as follows:

Let $Z = \{Z_1, Z_2, \dots, Z_n\}$ be an object set

let $V = \{V_1, V_2, \dots, V_n\}$ be an object set

Following this, the value calculation results for each i th object in each stage are obtained and shown in the following form:

$$M_i^j, \quad (1)$$

where $i = 1, 2, \dots, n; j = 1, 2, \dots, m$.

Step 1. Utilize values of fuzzy extended analysis synthesis to acquire priority weights:

$$s_i = \sum_{j=1}^m a_j \sum_{k=1}^m b_k \sum_{l=1}^m c_l \left\{ \frac{1}{\sum_{j=1}^m a_j}, \frac{1}{\sum_{k=1}^m b_k}, \frac{1}{\sum_{l=1}^m c_l} \right\}. \quad (2)$$

Step 2. The following is the expression for comparing degrees of possibility by the degree of probability of $N_2 \geq N_1$:

$V(N_2 \geq N_1)$

$$a \begin{cases} 0, & \text{if } b_2 \leq b_1, \\ 1, & \text{if } b_1 \geq c_1, \\ \frac{a_1 - c_1}{(b_2 - c_2) - (b_1 - c_1)}, & \text{otherwise.} \end{cases} \quad (3)$$

Step 3. Assume that you want to obtain the weight vector $d'(B_i) = \min V(T_i \geq T_k)$ for $k = 1, 2, \dots, n, k \neq i$; then, the weight vector is defined as follows: $W' (d'(B_1), d'(B_2), \dots, d'(B_n))^T$, where $B_i (i = 1, 2, \dots, n)$ are n elements.

Step 4. Calculate the vector of normalized weights: $W (d(B_1), d(B_2), \dots, d(B_n))^T$.

Step 5. After determining the component weights, the components are ranked.

3. Related Works in Knowledge Management System Adoption

Following the adoption of new technology and becoming a trend in organizations, even individuals new to the technology will adopt it for their survival and competitiveness. In other words, adoption is a crucial premise when it comes to technology. According to Shih et al. [42], adoption is a technology diffusion step involving the inclination of the organization and the individual to select and use the technology.

In Arpacı's study [43], the primary objective was to examine the antecedents and outcomes of cloud computing adoption in education to achieve knowledge management.

The study focused on implementing cloud computing in an actual learning environment to support KM practices and provide training and education to participants. Their research focused on the causal relationship between the KM practices expectations and the perceived usefulness of cloud computing services. Based on the obtained results, there is a significant relationship between the perceived usefulness and the creation/discovery, storage, and sharing knowledge expectations. Among them, knowledge storage and sharing expectations have stronger relationships with perceived usefulness. In addition, innovativeness and training and education were significantly related to the promotion of cloud computing adoption in education by enhancing KM practices awareness.

Furthermore, Al-Rahmi et al. [44] proposed a model for measuring sustainability in the education sector and included big data adoption and knowledge management sharing as variables. Based on their findings, behavioral intention to use big data supported big data adoption sustainability in education, and knowledge management influenced the intention to use big data and educational sustainability. Their study used UTAUT and knowledge management sharing factors to examine behavioral intention towards big data use and adoption for sustainable education. The study contributed to the literature on big data adoption and knowledge management sharing for sustainable education, proposing combining knowledge management sharing and UTAUT model to obtain the overall results.

Along with the same study caliber, Tsai and Hung [45] employed empirical methods to examine KMS adoption determinants based on a national survey. They found KMS adoption to be influenced by the organization's characteristics, enablers of KM, and attributes of KMS. According to them, KMS adoption is rife with complexity as it is highly dependent on KM enablers and characteristics of the organization instead of just system characteristics. Their findings had several implications for theory and practice, with the conclusions supporting a majority of the proposed hypotheses. Overall, the KMS adoption determinants can be considered through the characteristics of the organization, the KM enablers, and the characteristics of KMS.

In a related empirical study, Shrafat [29] examined the differential impact of three contextual variables, organizational readiness, expected benefits, and organizational learning capability, on KMS adoption or decisions for continued use. The author gathered data from 220 senior executives working in major Taiwanese firms and tested various relationships in the research model through PLS analysis. Based on the results, organizational readiness expected benefits and organizational learning capability significantly influence KMS adoption and intention towards continued use. Their study also supported the organizational readiness-KMS adoption or intention towards continued use relationship, which was more significant for potential adopters than for current ones. In theory, the study contributed a model that successfully explained the KMS adoption or inclination towards continued use determinants in light of potential and current adopters. Based on the

managerial viewpoint, the findings obtained establish guidelines for companies willing to adopt KMS by overcoming possible barriers and leveraging the most benefits in the preadoption and postadoption phases. The potential for KMS adoption has been focused on by SMEs, but limited studies dedicated to the little topic information are known [5, 29]. Hence, the present study contributes to explaining and clarifying the factors driving the adoption of KMS among SMEs.

In the same line of study, Rohendi [26] revealed that KMS enables the organization and documentation of the institution's knowledge. The study developed a prototype of KMS to organize and document knowledge in universities and carry out document aggregation based on a total number of subjects and writers. The prototype was developed using SharePoint to collect, store, and publish digital data at the university to make them accessible online. Aggregation is a process that uses the percentage of the number of documents, subjects, and writers. The result of such aggregation among the number of digital files was compared to the number of courses and lecturers, which equated to below 10% each. The university was recommended to boost lecturers to increase the gathering of digital files that could indirectly enhance the quality of educational services. The author highlighted the need to examine and determine various factors to contribute to the enlightenment of the field.

In Nigeria, Salami and Suhaimi [5] focused on the factors relating to KMS adoption among academicians, using an explanatory quantitative survey approach. According to the obtained findings, individual and management support factors have a crucial role in KMS adoption in Nigeria compared to organizational and technological factors. The study results can assist future studies in verifying and exploring these factors, particularly management support and individual factors. The study focused on structure, government support, culture, and organizational infrastructure from the organizational factors. The individual factors, knowledge, personal innovativeness, experience, and attitude were included, and management support, training, management initiatives, and management were included. Lastly, their study focused on trialability, compatibility, visibility, and complexity for the technological factors.

In Libya, Alhaj [46] investigated the effects of organizational factors on innovation among oil firms (public and private) while determining the role of social capital and knowledge sharing using the integrative and comprehensive conceptual model. The focus was on the direct and indirect effects of organizational factors on innovation, using a sample of 418 employees from the public and private oil sectors. Data were analyzed using PLS-SEM, and the author recommended that future authors add factors that could have a mediating role in the effect of organizational factors on innovation. A longitudinal study could improve the information on indirect effects accuracy and evaluate its effectiveness due to the long-term outcomes related to such factors.

Concerning the above studies, Haque et al. [47] looked into the factors influencing knowledge management and

knowledge sharing and their potential benefits to the decision-making process and the overall performance. Their study primarily aimed to examine antecedents of academics' knowledge management and knowledge sharing intention among universities. Further studies were recommended to validate and generalize the findings using a greater sample size in the cross-national university contexts.

Meanwhile, Alshahrani's study [48] aimed to determine the critical success factors (CSFs) for effective knowledge management in universities, using Nonaka's model and comparing the Western Sydney University (WSU) in Australia and King Fahd Security College (KFSC). The authors extended Nonaka et al. [49] study to include CSFs in proper KM implementation. Such extension provided a significant practical and theoretical foundation for the examination of KM among universities. In addition, the authors conducted a comparison of the two universities' implementation, excluding other factors that may contribute to KM implementation success. The study found knowledge production and distribution in universities of both countries not to be an explicit activity and one that is not limited within one static framework in that it has a contextual and dynamic nature. Added to the prior highlighted CSFs of KM, other major factors were also proved to affect four knowledge conversion modes. Those elements, involving several rational, cognitive, and intuitive processes and practices, have several characteristics and dynamics mutually facilitating knowledge generation and distribution. According to the findings obtained, effective KM practices and initiatives implementation in both countries originate from the complexity of factors and behaviors linked to the knowledge environment. There were 14 internal and six external factors that substantially contributed to Nonaka's knowledge conversion model (i.e., socialization, externalization, combination, and internalization) to manage KM properly. His study's internal factors included leadership, organizational structure, organizational rules, responsibilities of the employees, information technology infrastructure, training, teamwork, and measurement.

4. Methodology

The study methodology comprises four stages (see Figure 1) which are conducting a thorough literature review and identifying the critical factors, consulting the experts' information on the KMS factors, and stressing the most significant of them, which are used to develop the study framework.

In the method, the researchers conducted a literature review. It determined the critical variables to assess behavioral intention towards KMS adoption among the HLIs in Libya, following which the field experts reviewed the factors. The following are the details of all the steps in the methodology.

4.1. Factor Extraction through Literature Review. In this paper, literature was analyzed using KMS adoption factors, factors for technology adoption in education, and KMS and

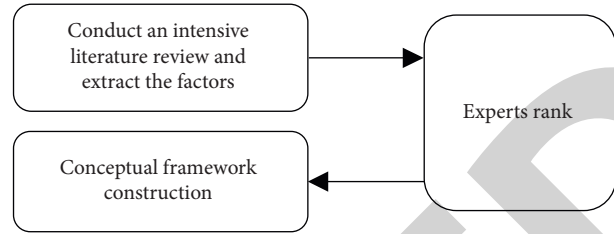


FIGURE 1: The methodology of the study as adopted from Mukred et al. [50].

education decision-making. A review of relevant studies regarding KMS was conducted to determine the relevant factors highlighted by the authors. The factors were then classified into dimensions and provided to the experts for perusal and review.

The authors selected the libraries and the main keywords related to KMS adoption, so that the searched words and terms remained in the research range. The keywords included KMS adoption, KMS factors, KMS frameworks, KMS adoption, decision-making, and KMS education. The sources provided information based on the keywords typed in, and thus the information was utilized to develop a pathway for developing and validating the keywords themselves. Different publishers were included in this stage.

Moreover, KMS-dedicated studies that were reviewed to determine the general factors used by the authors unearthed 65 factors. Table 1 lists the highlighted factors from which the determination of the top mentioned factors in literature can be discerned.

Frequency refers to the number of citations for each of the extracted factors mentioned in the previous works of literature, and it does not reflect the typical and common characteristics of factors [51].

Though a total of 65 factors were identified, the study was limited to the top-cited factors (24 factors) concerning KMS and technology adoption, specifically in the educational field. Table 2 shows the most cited factors that were extracted from the literature review.

Only 24 factors out of the 65 extracted factors were the most cited ones. The rest of the factors were only cited a few times in literature and, therefore, were not included in the final list of frequencies. The present study defined KMS from technical and nontechnical perspectives. It adopted a categorization type that has its basis on TOE theory, which covers technological, organizational, and environmental factors.

4.2. Experts Consultation and Factors Ranking. As the list of 24 most cited factors that affect KMS adoption was forwarded to the experts (lecturers who use KMS and are familiar with it), interviews were conducted with them to gain their perception of KMS of education. Along with the interviews, the experts also answered different questions in a questionnaire regarding the items of each factor. A total of 10 factors were identified to be the top essential factors regarding behavioral intention towards KMS use and eventually its actual use. Recommendations provided by

TABLE 1: Extracted factors from the literature review.

Dimension	Factors	No. of factors
Individual	Attitude, gender, education, age, experience, training, subjective norm, self-efficacy, satisfaction, motivation, personal normative belief	12
Technological	Reliability, perceived performance expectancy, service quality, perceived effort expectancy, features used, system quality, perceived ease of use, IT infrastructure, perceived usefulness, self-identity, trust, compatibility, privacy, efficiency, interactivity, information quality, usability, efficiency	18
Organizational	Training, motivation, policy, social influence, perceived financial support, change management, information need, competition, top management support, facilitating conditions, effective communication, organization readiness, standardization, outsourcing	14
Environmental	Clear vision and planning, big data analytics, laws and legislations, cloud computing, policy, competitiveness pressure, security concerns, safety	8
Behavioral intention	Intention to use, intention to adopt, habit, user expectations, extrinsic motivation	5
Use	User satisfaction, decision-making, organizational competency, user involvement, perceived benefits, overall satisfaction, performance, output quality	8
Total		65

TABLE 2: Frequency of the extracted factors.

No.	Factor	Total
1	Top management support	33
2	Big data	25
3	Perceived usefulness	30
4	Competitive pressure	28
5	Effective communication	28
6	Clear vision and planning	27
7	Training	27
8	Gender	25
9	Change management	25
10	User involvement	25
11	Government role	24
12	Cloud computing	24
13	Social influence	23
14	Perceived effort expectancy	23
15	Usability	23
16	System quality	20
17	Policy	19
18	Service quality	17
19	Perceived performance expectancy	17
20	Financial support	17
21	Information quality	16
22	Intention to adopt	15
23	Teamwork and composition	14
24	Decision-making	13

TABLE 3: Experts' profiles.

	Gender	Specialist areas	Years of experience
E01	Male	Information science	8
E02	Male	Technology adoption and education	12
E03	Female	Technology adoption and education	8
E04	Male	Technology adoption and education	7
E05	Male	Technology adoption and education	10
E06	Male	Computer science	10
E07	Male	Information science	9
E08	Male	Technology adoption and engineering	11
E09	Male	Technology adoption and engineering	14
E10	Male	Technology adoption and engineering	9

Hawking and Sellitto [52] and Ahmad and Pinedo Cuenca [53] were followed when determining the significant factors. Ten experts, who work in higher educational institutions and are familiar with KMS technology adoption, were consulted for their knowledge. The experts are Ph.D. holders working in different affiliations in Libya, Yemen, Malaysia, and Saudi Arabia. The experts' profile is listed in Table 3.

The experts highlighted 12 of the top factors that might influence the behavioral intention towards adopting and

using KMS. The experts dissected the factors based on selection criteria and interviews. The aim was to assess the factors that influence KMS adoption.

Based on the literature, 24 factors were confirmed, but 12 factors were dropped as the experts had mixed feedback about them following further validation. The list of factors ranked by the experts is provided in Table 4.

Table 5 shows the final list of factors that were extracted based on the interviewees. The table also shows the source of each factor with the overall percentage after the analysis.

The calculation of the percentage and validity belonging to all questions is done by the following the equation suggested by Mukred et al. [70]:

$$V_{\text{Total}} = \sum_{i=1}^{10} v_i * i, \quad (4)$$

where i is the rank given from 1 to 10 and v_i is the number of experts for each rank value.

TABLE 4: The experts ranking for the factors and items.

Factor	No.	Questions	1	2	3	4	5	Rank
Perceived effort expectancy		How easy is KMS to use?						
	1	KMS is easy to use			5	2	3	76
	2	KMS can be used without referring to a user manual	1	3	2	4		78
	3	KMS is flexible to interact with		4	2	4		80
	4	It is easy to get information using KMS to do what I want to do	1	3	2	4		78
	5	It is easy to detect and correct errors in student records using KMS	1	1	4	4		82
Perceived performance expectancy		How useful is KMS?						
	1	KMS enhances my work effectiveness		3	2	5		84
	2	KMS increases my productivity in my work		3	1	6		86
	3	KMS enables me to accomplish tasks more quickly		3	1	6		86
	4	KMS makes my work easier		3	2	5		84
	5	KMS gives me greater control over my work		3	1	6		86
IT infrastructure		IT infrastructure for adopting KMS:						
	1	It provides remote users with seamless access to centralized data		2	3	5		86
	2	It captures data that is made available to everyone in our organization in real-time		2	3	5		86
	3	It can easily incorporate software applications and can be used across multiple platforms		3	3	4		82
	4	It provides interfaces that give transparent access to all platforms and applications		3	3	4		82
	5	It offers multiple interfaces or entry points to external users		3	3	4		82
Training		Training on KMS:						
	1	It should be developed to meet the requirements of users		2	2	6		88
	2	It should have customized materials for each specific job		2	3	5		86
	3	It should have materials for the entire business task of the system		2	2	6		88
	4	It should be tracked to ensure that employees have received the appropriate training		2	1	7		90
	5	It should be adequate for all involved staff		3	1	6		86
Financial support		Financial support for adopting KMS is important for:						
	1	Purchasing a system		1	4	5		88
	2	Incentive payments		1	4	5		88
	3	Securing infrastructure and equipment		1	3	6		90
	4	Technical assistance cost		1	4	5		88
	5	Maintenance cost		1	4	5		88
Organization's readiness		How ready is your organization to adopt KMS?						
	1	If we have the system to engage in the knowledge management, we will not hesitate	1	1	1	7		88
	2	We feel comfortable (regarding security, privacy, etc.); thus, we will adopt it	1	1	3	5		84
	3	We are willing to adopt the KMS completely	1	1	1	7		88
	4	We consider it essential to engage in the system	1	1	1	7		88
	5	We consider it essential to improve coordination and collaboration regarding the use of knowledge	1	1	3	5		84
Change management		Change management in KMS adoption:						
	1	It ensures that employees understand how their work fits into the system	1	2	2	5		82
	2	It receives input from employees about how their jobs will change	1	1	3	5		84
	3	It actively works to alleviate employee concerns	1	2	1	6		84
	4	It makes available a support group to answer concerns about job changes	1	1	2	6		86
	5	The roles of all employees are communicated	1	1	2	6		86
Competitiveness pressure		With KMS adoption:						
	1	My job frequently requires me to rely on the KMS	1		1	3	5	82
	2	My everyday work tasks require me to need the support of the KMS frequently	1		2	2	5	80
	3	I frequently have to use the KMS to meet my work obligations	1		2	3	4	78
	4	I am expected to use the KMS all the time to meet my work obligations	1		2	2	5	80
	5	KMS is vital to ensure competitiveness	1		1	1	6	76

TABLE 4: Continued.

Factor	No.	Questions	1	2	3	4	5	Rank
Big data analytics		The use of big data should have						
	1	Ability to save huge volumes of information			2	3	5	86
	2	Ability to handle real-time data processing			2	3	5	86
	3	Data integration			2	2	6	88
	4	Rapid and interactive analysis			2	3	5	86
	5	Flexibility to consolidate data from various sources into one single place			2	2	6	88
Cloud integration		The cloud feature of KMS:						
	1	It provides a high degree of interconnectivity			1	4	5	88
	2	It is sufficiently flexible to incorporate electronic connections to external parties			2	3	5	86
	3	It is a factor that determines whether or not to choose KMS	1		3	2	4	78
	4	It captures data that is made available to everyone in our organization in real-time			4	1	5	82
	5	It provides remote users with seamless access to centralized data			3	3	4	82
Intention to adopt kms		My intention regarding KMS adoption is:						
	1	Assuming I have the KMS, I intend to adopt it			2	3	5	86
	2	Given that I have the KMS, I predict that I would adopt it			2	2	6	88
	3	In my work, if I have KMS, I want to use it as much as possible			2	3	5	86
	4	I prefer to use electronic records even though I can do my work with other tools			2	3	5	86
	5	KMS is essential to my work, and I need to adopt it			2	2	6	88
Decision-making		KMS gives decisions that provide the following:						
	1	Quality			2	1	7	90
	2	Effectiveness			2	1	7	90
	3	Accuracy			2	1	7	90
	4	Performance			2	1	7	90
	5	Transparency			2	1	7	90
	6	Integrity			2	1	7	90
	7	Accountability			2	1	7	90

4.3. Framework Development. An essential aspect in conducting any study is examining and determining the theories/models underpinning the study topic so that they can be used for guidance in developing a premise of the constructs' relationships during framework development [3]. In a study of KMS adoption, the level of adoption can be enhanced if the determinants of such adoption are determined and examined. Prior literature on the topic has thus proposed several theories and models [71–73] to examine the technology adoption in institutions. The major theories used and reviewed included the Technology Acceptance Model (TAM), Unified Theory of Acceptance and Use of Technology (UTAUT), Theory of Planned Behavior (TPB), Diffusion of Innovation (DOI) theory, and Technology-Organization-Environment (TOE) framework.

In this study, the KMS framework is developed and proposed by identifying five interrelated variables (technological dimensions, organizational dimensions, environmental dimensions, KMS adoption intention, and educational institutions' decision-making; see Figure 2). The variables are examined and categorized under technology adoption factors in the study framework.

This study reviews the unified theories and models to choose the most appropriate to achieve the study's objectives. Top extensively used models in literature in education included TAM, TOE, UTAUT, and DOI, as Alharbi [71] and Al-Jabri [72] mentioned.

Accordingly, UTAUT was validated in the reviewed literature as a robust model. UTAUT was selected because of

its use, suitability, validity, and reliability in examining technology adoption in different contexts [74–77]. The present study used UTAUT to examine the factors that influence KMS adoption consistent with the suggestion by Abdullah et al. in the case of Libyan HLIs. Thus, the main UTAUT features include technological differences, characteristics of the organization, and environmental settings—these are all viewed as determinants of KMS adoption behavior in HLIs in Libya. UTAUT is suitable for the underpinning theory of the present study in light of its objectives and context.

5. Discussion and Interpretation

The interviewed experts agreed that perceived effort expectancy and perceived performance expectancy are significant factors that influence KMS adoption. Regarding the users, the majority of them are inclined to use the system if they are convinced that it can enhance their work quality and is easy to use. Other factors such as financial support and training were also included in the top-listed factors. Furthermore, three experts (E2, E7, and E8) perceived that big data facility and cloud computing ability could potentially influence KMS adoption. In contrast, others proposed financial support for such adoption in the HLI sector of Libya.

In addition, experts E1 and E6 suggested that the environmental dimension factors may also be considered new factors to be included in the conceptual framework based on which successful and timely adoption can occur. Experts E3,

TABLE 5: List of factors recommended by experts.

No.	Factor		Percentage %
1	Perceived effort expectancy	[54–56]	79.00
2	Perceived performance expectancy	[54–56]	82.2
3	IT infrastructure	[57–59]	83.6
4	Training	[60]	87.6
5	Financial support	[61, 62]	88.4
6	Organization readiness	[63]	86.4
7	Change management	[2, 64]	84.4
8	Competitive pressure	[65–67]	79.2
9	Big data analytics	[59]	86.8
10	Cloud integration	[68]	83.2
11	Behavioral intention (intention to adopt)	[54–56]	86.8
12	Decision-making	[69]	90.00

E4, and E5 also agreed that competitiveness pressure is one of the top influencing factors of KMS adoption in the HLIs of Libya to get expected exceptional outcomes. Thus, this factor was included in the present study. The experts agreed on the importance of perceived effort expectancy, perceived performance expectancy, and IT infrastructure as essential determinants of KMS adoption. Thus, they were included in the study framework. Moreover, E9 and E10 stressed the importance of testing the influence of the identified factors on behavioral intention towards adopting KMS as the role of system adoption in improving decision-making has yet to be confirmed.

The proposed study's conceptual framework is displayed in Figure 2. The framework was developed using ten identified factors validated and ranked by experts in the field, and the factors are arranged based on underlying theories.

The proposed framework was examined in light of the influence of the factors on KMS adoption, and the factors include those adopted from the UTAUT framework (perceived effort expectancy and perceived performance expectancy), which directly determine behavioral intention to adopt KMS. Other factors include IT infrastructure, training, financial support, organization readiness, change management, cloud computing, and big data analytics.

The propositions and description of each factor included in the proposed conceptual framework are detailed in the following sections.

5.1. Technological Variables. In any sector, technology use provides the potential for enhancing service quality provided and the workforce efficiency and effectiveness and minimizing the organization's costs. Thus, technology adoption is essential in institutions as it has been evidenced and highlighted as a critical issue [78]. Although several studies in the literature revealed that technology adoption positively influences organizations, empirical works presented barriers and challenges to technology adoption in educational institutions. Therefore, it is pertinent to examine factors that influence technology adoption for successful technology implementation and use [69, 79].

In this study, the technology dimension factor refers to the level to which the user believes that using a specific system would enhance his/her job performance [54].

In the line of this study, Ahmed and Ward [80] adopted UTAUT in their measurement of KMS acceptance among academic and professional development department's employees. Based on their findings, perceived performance expectancy has a significant effect on the intention of the users. Meanwhile, performance expectancy and effort expectancy are the two main predictors of behavioral intention towards IS adoption, as evidenced by Venkatesh et al. [81].

In this study, perceived performance expectancy is referred to as the perception of managers and employees of the usefulness of KMS. This variable has been examined in light of the system's ability to enhance productivity, effectiveness, and performance at work. Empirical findings also showed that perceived effort expectancy is a determinant of intention towards system use and adoption [82, 83].

In a similar line of study, Tarcan et al. [84] concentrated on the factors that affect intention to use IT among academicians. They found effort expectancy to be one of the top factors. Elkaseh et al. [85] also found that intention towards IT use and adoption among users is affected by the users' perceptions and beliefs, including effort expectancy and performance expectancy. These are the two significant IS adoption antecedents [74].

IT infrastructure, which is another factor in the technological dimension, is significant [2, 69]. It includes the IT plans, business aims, IT architecture, and IT workforce skills consistency. In this regard, Broadbent and Weill [86] revealed that the capabilities of IT infrastructure enable various applications to reinforce the present and potential organization's objectives and its competitive status in the business market.

Based on the above definition and discussion of IT infrastructure, it is clear that there are two components of the variable: technical IT infrastructure and human IT infrastructure. The first one is made up of data, technology, and application. The second one is made up of knowledge and capabilities for IT resources management [86].

KMS has been studied in several empirical works [5, 26, 27, 29], each with its objectives and conclusions, but the general trend among the studies is that technological factors of perceived performance expectancy, perceived effort expectancy, and IT infrastructure have the potential to influence KMS adoption. Based on the above discussion and the importance of the factors in boosting KMS adoption, this study proposes the following proposition for testing:

(P1) Technological factors have a positive influence on the intention to adopt KMS in HLIs in Libya.

5.2. Organizational Variables. Generally speaking, the successful adoption of KMS depends on the engagement of the whole organization. Therefore, senior management needs to promote new records management system as part of the change management initiative. In addition, organizational implementation methods of further KMS vary, but the focus should not be on IT alone. According to Binyamin et al. [87],

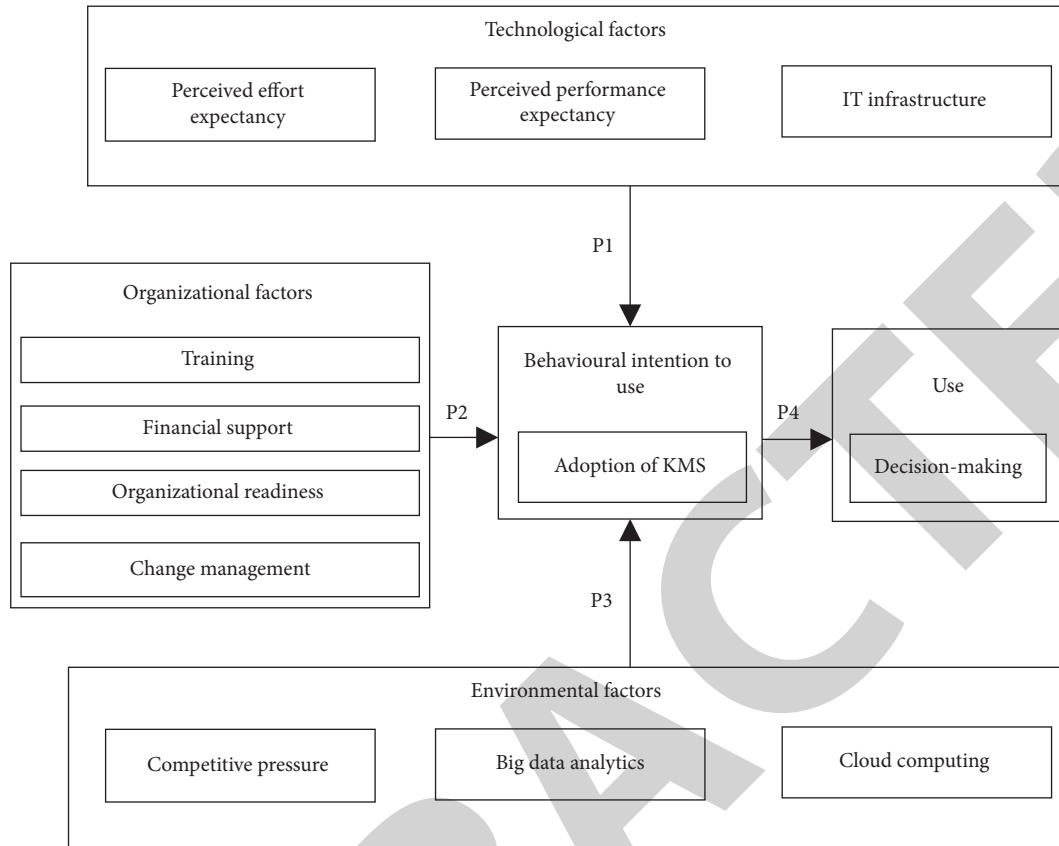


FIGURE 2: The proposed framework for KMS adoption in HLIs.

organizational factors are as significant as their technological counterparts when it comes to adopting technology in the institutions of higher learning. The authors found that organizational support plays a crucial role in successful IS adoption and use.

In this regard, [3] adopted a mixed explanatory approach to continuously explore the experienced education staff who managed to transition beyond adopting the technology stage in their practices. Based on this study, some factors prevent the adoption of technology in the form of challenges, including learning to use a computer. Technology optimal usage could be enabled by assessing and enhancing the user's computer skills, working towards data entry and system use consistently via training [88].

Staff training ensures that risks that crop up are overcome. Otherwise, such risks could prevent the successful KMS implementation and adoption [89]. Moreover, the lack or absence of training and support could cause a barrier to system adoption among users [90]. Insufficient training could also lead to discomfort at dealing with system and computer, and eventually it may lead to implementation failure [89, 90].

Another organizational factor that has a crucial role in technology adoption is financial support [91]. Technology adoption has become increasingly dependent on financial support, and therefore financial aid has a positive effect on the successful adoption of technology towards enhancing future efforts in information [92]. Thus, in the present study,

financial support is examined in terms of its influence on KMS adoption in educational institutions.

Readiness is another crucial factor for KMS adoption. It refers to the level of inclination of a country to be a part of the networked global village by evaluating its development in different aspects of ICT adoption [50]. Readiness is described as the capacity to meet the organization's required institutional, legal framework and ICT infrastructure. Additionally, according to Griffiths et al. [93], readiness is one of the factors with which progress is measured in contrast to the overall ability of organizations to adopt or use the systems. It is therefore a vital driver for assessing the behavioral intention to adopt the KMS among HLIs.

On top of that, researchers commonly acknowledged change management as a necessary factor, and in the case of KMS application, the organization is faced with several changes. In this situation, change management is a method/strategy adopted for the proper management of the transition from traditional frameworks to newer ones. Thus, in using the KMS aspect, the organization and the employees need to be ready for any eventual change that needs to happen. This is particularly true when it comes to the need of the organization to develop such management as early as possible to tackle issues (e.g., employee resistance, redundancies and confusion, and the errors that crop up during the implementation of the framework [79, 94]).

In the same line of argument, administrators may be the basis of change management initiatives but not IT initiatives

[95]. In addition, in cases where individuals are work involved, they acquire a more extensive view of the advantages, leading to more acceptance of the novel technology framework [2].

As a whole, factors related to the organization were the most often cited reasons for the limited KMS usage. These factors include organization readiness, financial support, training, and change management. Thus, this study proposes the following proposition:

(P2) Organizational factors have a positive influence on the adoption of KMS in HLIs in Libya.

5.3. Environmental Factors. Prior literature dedicated to KMS adoption mainly studied organizational and technological factors and human and individual factors [5]. In the field of education, KMS adoption should also focus on the environmental dimension. Thus, the present study considers such dimensions and factors: competitive pressure, big data analytics, and cloud computing.

To this end, competitive pressure is a significant factor under the environmental dimension at the local and global levels. This pressure forces the organization to search for ways to enhance its efficiency and effectiveness through technology adoption [96]. Both dynamic competition and digital technology advancement have left governments worldwide wide open to leveraging new methods for developmental progress. Awareness of such technology adoption has resulted in the transition of government services from outdated approaches to e-methods in the current decade [97].

Other environmental factors that influence KMS adoption are cloud computing and big data analytics, as mentioned and illustrated by Mohamad et al. [98], Medvedeva et al. [99], and Dening et al. [100]. Prior studies also indicated that the used software and hardware when implementing KMS are among the factors. This issue has more to do with developing KMS software with a user interface that could be customized for cloud computing capability. Cloud computing is an alternative solution to help keep data and help HLIs use KMS at any time. In other words, an effective KMS system should be compatible with any platform and database and maintenance-friendly. This is because an inefficient hanging system could minimize users as they refuse to waste their time and effort to achieve their goals. As a result, it is critical to select an appropriate and efficient technology compatible with the application and hardware to facilitate the institution's implementation [101, 102].

In sum, environmental factors are crucial for successful KMS adoption [103]. Based on the above discussion, this study proposes the following proposition for testing:

P3. Environmental factors have a positive influence on the adoption of KMS in HLIs in Libya.

5.4. Intention to Adopt KMS Factors. Behavioral intention indicates the readiness of the individual to perform a specific behavior, and it is proposed to be an antecedent of behavior [104]. In the present study, intention is defined as the

willingness of the individual to try or the effort they are willing to exert to perform a future behavior.

According to Venkatesh et al. [81], behavioral intention towards technology is the primary determinant of actual behavior. The three factors predicting intention to use are attitude, subjective norms, and perceived behavioral control.

In the same study, Ahmed and Ward [105] compared competing technology acceptance frameworks to examine personal, academic, and professional portfolio acceptance behavior. The authors revealed a positive direct effect of perceived ease of use on perceived usefulness. Furthermore, perceived ease of use was found to have an immediate positive impact on intention.

Overall, the identification of content and context dimensions offers a suitable method to shed light on the current adoption state of KMS in educational institutions and the barriers that prevent such adoption [43, 106].

Therefore, there is a need to examine intention to adopt based on technological, organizational, and environmental factors in KMS adoption. This study thus proposes the following proposition for testing:

P4. Behavioral intention to adopt KMS has a significant relationship with the decision-making process in HLIs in Libya.

6. Conclusion

In the present work, the lack of studies dedicated to examining KMS adoption and its key role in supporting and enhancing the performance of educational institutions by improving the decision-making process is highlighted. The study also highlighted the limitations of the existing studies when it comes to such examination, and thus it developed and proposed a conceptual framework. KMS adoption in HLIs requires a robust framework. Accordingly, the present work reviewed the literature concerning KMS use, adoption, and implementation and the factors included in the study framework. The study conducted a thorough review of KMS factors' literature, extracted them, and forwarded them to experts for validation. The factors were categorized into three dimensions: technological, organizational, and environmental dimensions. The panel of experts perused the factors and recognized the significance of KMS initiatives in educational institutions. Based on the highlighted factors, the study developed a conceptual framework that is appropriate to examine the factors influencing the adoption of KMS in educational institutions. However, the framework was based on the reviewed literature, which had its limitations. According to the confirmation of experts, ten factors were found to influence the adoption of KMS in the Libyan HLIs, with two adopted from UTAUT and eight adopted from a literature review. The examined factors included perceived effort expectancy, perceived performance expectancy, IT infrastructure, training, financial support, organization readiness, change management, competitiveness pressure, cloud computing, and big data analytics—all these factors were tested for their significant influence on KMS adoption in HLIs in Libya. The study revealed the role of KMS in enhancing the decision-making process. The present

study contributes to the literature by identifying the factors influencing behavioral intention to adopt and use KMS. Moreover, it contributes to practice by directing limited management resources to the significant areas that would make successful and smooth system adoption.

Data Availability

The qualitative data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Optimization of LR-Type Fully Bipolar Fuzzy Linear Programming Problems

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Optimization of LR-Type Fully Bipolar Fuzzy Linear Programming Problems

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In this study, we present a technique to solve LR-type fully bipolar fuzzy linear programming problems (FBFLPPs) with equality constraints. We define LR-type bipolar fuzzy numbers and their arithmetic operations. We discuss multiplication of LR-type bipolar fuzzy numbers. Furthermore, we develop a method to solve LR-type FBFLPPs with equality constraints involving LR-type bipolar fuzzy numbers as parameters and variables. Moreover, we define ranking for LR-type bipolar fuzzy numbers which transform the LR-type FBFLPP into a crisp linear programming problem. Finally, we consider numerical examples to illustrate the proposed method.

1. Introduction

Zadeh [1–3]’s fuzzy set (FS) theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. FSs handle such situations by attributing a degree to which a certain object belongs to a set. In 1994, Zhang [4] initiated the concept of bipolar fuzzy sets (BFSs) as an extension of FSs. The BFS representation is useful when irrelevant elements and contrary elements are needed to be discriminated. Furthermore, Zhang [5] introduced NPN fuzzy sets and NPN qualitative algebra. On the other hand, Akram and Arshad [6] defined bipolar fuzzy numbers and proposed a novel trapezoidal bipolar fuzzy TOPSIS method for group decision making, and Singh [7] discussed two schemes based on the properties of next neighbors and Euclidean distance in a bipolar fuzzy environment. Chakraborty et al. [8] introduced pentagonal neutrosophic numbers and further analyzed their properties. They studied the definitions of score function and accuracy function which transform pentagonal neutrosophic numbers into crisp numbers and also presented transportation models in a neutrosophic environment.

In mathematical programming models, the simplest model is a linear programming problem. Linear programming is a mathematical modeling technique in which a linear function is maximized or minimized when subjected to various constraints. Uncertainty is also introduced in the linear programming problem, which is widely studied by many scholars. Bellman and Zadeh [9] studied about objectives and human decision making. Zimmerman [10] presented a scheme to solve the fuzzy linear programming (FLP) problem by using multiobjective function. Tanaka et al. [11] suggested a method and obtained solution of FLP problems. Later, several methods have been developed in [12–17] to solve FLP problems and fuzzy systems of linear equations. Recently, certain methods have been developed in [18–20] to solve bipolar fuzzy linear system (BFLS) of equations. The notion of LR-FN was introduced by Dubois and Prade [21]. Dehghan et al. [22] presented a technique and obtained solution of fully fuzzy linear system $\tilde{D}\tilde{X} = \tilde{E}$ in which the coefficient matrix \tilde{D} and right hand column vector \tilde{E} contain LR-fuzzy numbers. Kaur and Kumar [15] defined arithmetic operations of LR-FNs and suggested Mehar’s method to solve fully FLP problems by using these numbers as

variables and parameters. Buckley [23] gave the idea of fuzzy complex set and fuzzy complex number. Akram et al. [18] proposed a technique to solve LR-BFLS, LR-complex BFLS with real coefficients, and LR-complex BFLS with complex coefficients of equations. Recently, Akram et al. [24, 25] presented methods to solve Pythagorean FLP problems. Mehmood et al. [26] proposed a method for solving fully BFL programming problems. In this research article, we present a technique to solve LR-type fully bipolar fuzzy linear programming problems (FBFLPPs) with equality constraints. We define LR-type bipolar fuzzy numbers and their arithmetic operations. We discuss multiplication of LR-type bipolar fuzzy numbers. Furthermore, we develop a method to solve LR-type FBFLPPs with equality constraints involving LR-type bipolar fuzzy numbers as parameters and variables. Moreover, we define ranking for LR-type bipolar fuzzy numbers which transform the LR-type FBFLPP into a crisp linear programming problem. Finally, we consider numerical examples to illustrate the proposed method.

We have organized the research article as follows: In Section 2, some preliminary concepts are presented. In Section 3, arithmetic operations are introduced. Section 4 presents a method to solve LR-type FBFLPPs with equality constraints in which variables and parameters are LR-type BFNs. In Section 5, the numerical example and model are illustrated. Conclusion is given in Section 6. The list of acronyms used in the research article is given in Table 1.

2. Preliminaries

Definition 1 [4]). Let $Z \neq \emptyset$. A BFS \tilde{K} in Z is an object having the form

$$\tilde{K} = (\eta_{\tilde{K}}^P, \eta_{\tilde{K}}^N) = \{(z, \eta_{\tilde{K}}^P(z), \eta_{\tilde{K}}^N(z)) | z \in Z\}, \quad (1)$$

which is characterized by the functions $\eta_{\tilde{K}}^P: Z \rightarrow [0, 1]$ and $\eta_{\tilde{K}}^N: Z \rightarrow [-1, 0]$, known as the truth membership function and falsity membership function, respectively. Here, truth or positive membership degree $\eta_{\tilde{K}}^P(z) \in [0, 1]$ indicates the satisfaction degree of those elements $z \in Z$ which fulfill a certain property relating to BFS \tilde{K} , and negative membership degree $\eta_{\tilde{K}}^N(z) \in [-1, 0]$ indicates the satisfaction degree of those elements $z \in Z$ which fulfill some counter property relating to BFS \tilde{K} .

Definition 2 [18]). A BFN, $\tilde{K} = \langle P, N \rangle = \langle [p_1, p_2, p_3, p_4], [n_1, n_2, n_3, n_4] \rangle$ is a BFS of the mapping $\eta: \mathbb{R} \rightarrow [0, 1] \times [-1, 0]$, with satisfaction degree η_p and dissatisfaction degree η_n such that

TABLE 1: List of acronyms.

Acronyms	Representation
BFNs	Bipolar fuzzy numbers
LPPs	Linear programming problems
FBFLPPs	Fully bipolar fuzzy linear programming problems
BFOS	Bipolar fuzzy optimal solution

$$\eta_p(x) = \begin{cases} \frac{x - p_1}{p_2 - p_1}, & \text{if } x \in [p_1, p_2], \\ 1, & \text{if } x \in [p_2, p_3], \\ \frac{p_4 - x}{p_4 - p_3}, & \text{if } x \in [p_3, p_4], \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

$$\eta_n(x) = \begin{cases} \frac{n_1 - x_1}{n_2 - n_1}, & \text{if } x \in [n_1, n_2], \\ -1, & \text{if } x \in [n_2, n_3], \\ \frac{x - n_4}{n_4 - n_3}, & \text{if } x \in [n_3, n_4], \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3 [6]). Let $\tilde{K} = \langle (p_1, p_2, p_3, p_4), (n_1, n_2, n_3, n_4) \rangle$ be a BFN. Then, its (μ, ν) -cut is defined as

$$\tilde{K}^{(\mu, \nu)} = \{[(p_2 - p_1)\mu + p_1, -(p_4 - p_3)\mu + p_4]; [-\nu(n_2 - n_1) + n_1, \nu(n_4 - n_3) + n_4]\}, \quad (3)$$

where $\mu \in [0, 1]$, $\nu \in [-1, 0]$.

Definition 4. [18]). A BFN is said to be an LR-bipolar fuzzy number of the form $\tilde{K} = \langle [m^*, \alpha^*, \beta^*]_{LR}, [m', \alpha', \beta']_{LR} \rangle$, where m^*, m' are real numbers and $\alpha^*, \beta^* > 0$ and $\alpha', \beta' > 0$, if its membership function $(\tau_{\tilde{K}})$ and nonmembership function $(\omega_{\tilde{K}})$, are represented by

$$\tau_{\tilde{K}}(x) = \begin{cases} L\left(\frac{m^* - x}{\alpha^*}\right), & \text{if } x \leq m^*, \alpha^* > 0, \\ R\left(\frac{x - m^*}{\beta^*}\right), & \text{if } x \geq m^*, \beta^* > 0, \end{cases} \quad (4)$$

$$\omega_{\tilde{K}}(x) = \begin{cases} L'\left(\frac{m' - x}{\alpha'}\right), & \text{if } x \leq m', \alpha' > 0, \\ R'\left(\frac{x - m'}{\beta'}\right), & \text{if } x \geq m', \beta' > 0, \end{cases}$$

where m^*, α^* , and β^* are called the mean value and left and right spreads of the positive side of K , respectively, while m', α' , and β' are called the mean value and left and right spreads of negative side of K , respectively. Also, L and R are continuous and decreasing functions from \mathfrak{R}^+ to $[0, 1]$, while L' and R' are continuous and increasing functions from \mathfrak{R}^+ to $[-1, 0]$ such that

- (1) $L(0) = R(0) = 1$
- (2) $\lim_{x \rightarrow \infty} L(x) = \lim_{x \rightarrow \infty} R(x) = 0$
- (3) $L'(0) = R'(0) = -1$
- (4) $\lim_{x \rightarrow \infty} L'(x) = \lim_{x \rightarrow \infty} R'(x) = 0$

Remark 1. If we put

$$L(x) = R(x) = \begin{cases} 1 - x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

$$L'(x) = R'(x) = \begin{cases} x - 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

in Definition 4, then $\tilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ converts to an LR-type bipolar fuzzy number.

$$\text{Rank}(\tilde{K}) = \frac{1}{4} \left(\int_0^1 (m^* - \alpha^* L^{-1}(\gamma)) d\gamma + \int_0^1 (m^* + \beta^* R^{-1}(\gamma)) d\gamma \right) + \frac{1}{4} \int_0^{-1} (m' - \alpha' L'^{-1}(\delta)) d\delta + \int_0^{-1} (m' + \beta' R'^{-1}(\delta)) d\delta, \quad (7)$$

where $\gamma \in [0, 1]$ and $\delta \in [-1, 0]$.

Let \tilde{K}_1 and \tilde{K}_2 be two LR-type BFNs; then,

- (i) $\tilde{K}_1 < \tilde{K}_2$, if $\mathfrak{R}(\tilde{K}_1) < \mathfrak{R}(\tilde{K}_2)$
- (ii) $\tilde{K}_1 > \tilde{K}_2$, if $\mathfrak{R}(\tilde{K}_1) > \mathfrak{R}(\tilde{K}_2)$
- (iii) $\tilde{K}_1 \approx \tilde{K}_2$, if $\mathfrak{R}(\tilde{K}_1) = \mathfrak{R}(\tilde{K}_2)$

For other concepts and applications, refer to [27–32].

3. Arithmetic Operations

In this section, we study about arithmetic operations for LR-type BFNs.

Definition 5. An LR-type BFN $\tilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ is said to be nonnegative if $m^* - \alpha^* \geq 0$ and $m' - \alpha' \geq 0$ and is said to be nonpositive if $m^* + \beta^* \leq 0$ and $m' + \beta' \leq 0$.

Definition 6. An LR-type BFN $\tilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ is positive if $m^* - \alpha^* > 0$ and $m' - \alpha' > 0$ and is negative if $m^* + \beta^* < 0$ and $m' + \beta' < 0$.

Definition 7. An LR-type BFN $\tilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ is said to be unrestricted if $m^* - \alpha^*$ and $m' - \alpha'$ are real numbers.

Definition 8. An LR-type BFNs $\tilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ is said to be zero if and only if $m^* = 0, \alpha^* = 0, \beta^* = 0, m' = 0, \alpha' = 0$, and $\beta' = 0$.

Definition 9. Two LR-type BFNs $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle$ are equal if $m_1^* = m_2^*, \alpha_1^* = \alpha_2^*, \beta_1^* = \beta_2^*, m_1' = m_2', \alpha_1' = \alpha_2', \beta_1' = \beta_2'$.

Definition 10. Let $\tilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ be an LR-type BFN; then, their γ -cut and δ -cut are given as follows:

$$\begin{aligned} \tilde{K}^\gamma &= [m^* - \alpha^* L^{-1}(\gamma), m^* + \beta^* R^{-1}(\gamma)], \\ \tilde{K}^\delta &= [m' - \alpha' L'^{-1}(\delta), m' + \beta' R'^{-1}(\delta)], \end{aligned} \quad (6)$$

where $\gamma \in [0, 1]$ and $\delta \in [-1, 0]$.

Definition 11. Let $\tilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ be an LR-type BFN; then, ranking of \tilde{K} , represented by $\mathfrak{R}(\tilde{K})$, is defined as

Theorem 1. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle$ be two LR-type BFNs; then,

$$\begin{aligned} \tilde{K}_1 \oplus \tilde{K}_2 &= \langle [m_1^* + m_2^*, \alpha_1^* + \alpha_2^*, \beta_1^* + \beta_2^*]_{\text{LR}}, \\ &[m_1' + m_2', \alpha_1' + \alpha_2', \beta_1' + \beta_2']_{\text{LR}} \rangle. \end{aligned} \quad (8)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle$ be two LR-type BFNs; then,

their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned}\tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)].\end{aligned}\quad (9)$$

So,

$$\tilde{K}_1^\gamma + \tilde{K}_2^\gamma = [m_1^* - \alpha_1^* L^{-1}(\gamma) + m_2^* - \alpha_2^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) + m_2^* + \beta_2^* R^{-1}(\gamma)]. \quad (10)$$

By setting $\gamma = 1$ in equation (170), we get

$$(\tilde{K}_1 + \tilde{K}_2)^{\gamma=1} = m_1^* + m_2^*. \quad (11)$$

By setting $\gamma = 0$ in equation (170), we get

$$(\tilde{K}_1 + \tilde{K}_2)^{\gamma=0} = [m_1^* + m_2^* - \alpha_1^* - \alpha_2^*, m_1^* + m_2^* + \beta_1^* + \beta_2^*]. \quad (12)$$

Now,

$$\tilde{K}_1^\delta + \tilde{K}_2^\delta = [m_1' - \alpha_1' L'^{-1}(\delta) + m_2' - \alpha_2' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta) + m_2' + \beta_2' R'^{-1}(\delta)]. \quad (13)$$

By setting $\delta = -1$ in equation (181), we get

$$(\tilde{K}_1 + \tilde{K}_2)^{\delta=-1} = m_1' + m_2'. \quad (14)$$

By setting $\delta = 0$ in equation (181), we get

$$(\tilde{K}_1 + \tilde{K}_2)^{\delta=0} = [m_1' + m_2' - \alpha_1' - \alpha_2', m_1' + m_2' + \beta_1' + \beta_2'], \quad (15)$$

On combining the equations (173), (178), (185), and (15), the result follows. \square

Theorem 2. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be two LR-type BFNs; then,

$$\tilde{K}_1 \ominus \tilde{K}_2 = \langle [m_1^* - m_2^*, \alpha_1^* + \beta_2^*, \alpha_2^* + \beta_1^*]_{LR}, [m_1' - m_2', \alpha_1' + \beta_2', \alpha_2' + \beta_1']_{LR} \rangle. \quad (16)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be two LR-type BFNs; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned}\tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)].\end{aligned}\quad (17)$$

So,

$$\tilde{K}_1^\gamma - \tilde{K}_2^\gamma = [m_1^* - \alpha_1^* L^{-1}(\gamma) - m_2^* - \beta_2^* R^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) - m_2^* + \alpha_2^* L^{-1}(\gamma)]. \quad (18)$$

By setting $\gamma = 1$ in equation (18), we get

$$(\tilde{K}_1 - \tilde{K}_2)^{\gamma=1} = m_1^* - m_2^*. \quad (19)$$

By setting $\gamma = 0$ in equation (18), we get

$$(\tilde{K}_1 - \tilde{K}_2)^{\gamma=0} = [m_1^* - m_2^* - \alpha_1^* - \beta_2^*, m_1^* - m_2^* + \alpha_2^* + \beta_1^*]. \quad (20)$$

Now,

$$\tilde{K}_1^\delta - \tilde{K}_2^\delta = [m_1' - \alpha_1' L'^{-1}(\delta) - m_2' - \beta_2' R'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta) - m_2' + \alpha_2' L'^{-1}(\delta)]. \quad (21)$$

By setting $\delta = -1$ in equation (21), we get

$$(\tilde{K}_1 - \tilde{K}_2)^{\delta=-1} = m_1' - m_2'. \quad (22)$$

By setting $\delta = 0$ in equation (21), we get

$$(\tilde{K}_1 - \tilde{K}_2)^{\delta=0} = [m_1' - m_2' - \alpha_1' - \beta_2', m_1' - m_2' + \alpha_2' + \beta_1']. \quad (23)$$

On combining equations (19), (20), (22), and (23), the result follows. \square

Theorem 3. Let $\tilde{K} = \langle [m^*, \alpha^*, \beta^*]_{LR}, [m', \alpha', \beta']_{LR} \rangle$ be an LR-type BFN and ϵ be an arbitrary real number; then,

$$\epsilon \tilde{K} = \begin{cases} \langle [\epsilon m^*, \epsilon \alpha^*, \epsilon \beta^*]_{LR}, [\epsilon m', \epsilon \alpha', \epsilon \beta']_{LR} \rangle, & \text{if } \epsilon \geq 0, \\ \langle [\epsilon m^*, -\epsilon \beta^*, -\epsilon \alpha^*]_{LR}, [\epsilon m', -\epsilon \beta', -\epsilon \alpha']_{LR} \rangle, & \text{if } \epsilon < 0. \end{cases} \quad (24)$$

Proof. Let $\tilde{K} = \langle [m^*, \alpha^*, \beta^*]_{LR}, [m', \alpha', \beta']_{LR} \rangle$ be an LR-type BFN and ϵ be an arbitrary real number; then, their γ -cut and δ -cut, $\forall \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \tilde{K}^\gamma &= [m^* - \alpha^* L^{-1}(\gamma), m^* + \beta^* R^{-1}(\gamma)], \\ \tilde{K}^\delta &= [m' - \alpha' L'^{-1}(\delta), m' + \beta' R'^{-1}(\delta)]. \end{aligned} \quad (25)$$

Now, if $\epsilon \geq 0$, then

$$\epsilon \tilde{K}^\gamma = [\epsilon m^* - \epsilon \alpha^* L^{-1}(\gamma), \epsilon m^* + \epsilon \beta^* R^{-1}(\gamma)]. \quad (26)$$

By setting $\gamma = 1$ in equation (26), we get

$$\epsilon \tilde{K}^{\gamma=1} = \epsilon m^*. \quad (27)$$

By setting $\gamma = 0$ in equation (26), we get

$$\epsilon \tilde{K}^{\gamma=0} = [\epsilon m^* - \epsilon \alpha^*, \epsilon m^* + \epsilon \beta^*]. \quad (28)$$

Also,

$$\epsilon \tilde{K}^\delta = [\epsilon m' - \epsilon \alpha' L'^{-1}(\delta), \epsilon m' + \epsilon \beta' R'^{-1}(\delta)]. \quad (29)$$

By setting $\delta = -1$ in equation (29), we get

$$\epsilon \tilde{K}^{\delta=-1} = \epsilon m'. \quad (30)$$

By setting $\delta = 0$ in equation (29), we get

$$\epsilon \tilde{K}^{\delta=0} = [\epsilon m' - \epsilon \alpha', \epsilon m' + \epsilon \beta']. \quad (31)$$

On combining the equations (27), (28), (30), and (31), the case $\epsilon \geq 0$ follows.

If $\epsilon < 0$, then

$$\epsilon \tilde{K}^\gamma = [\epsilon m^* + \epsilon \beta^* R^{-1}(\gamma), \epsilon m^* - \epsilon \alpha^* L^{-1}(\gamma)]. \quad (32)$$

By setting $\gamma = 1$ in equation (32), we get

$$\epsilon \tilde{K}^{\gamma=1} = \epsilon m^*. \quad (33)$$

By setting $\gamma = 0$ in equation (32), we get

$$\epsilon \tilde{K}^{\gamma=0} = [\epsilon m^* + \epsilon \beta^*, \epsilon m^* - \epsilon \alpha^*]. \quad (34)$$

Also,

$$\epsilon \tilde{K}^\delta = [\epsilon m' + \epsilon \beta' R'^{-1}(\delta), \epsilon m' - \epsilon \alpha' L'^{-1}(\delta)]. \quad (35)$$

By setting $\delta = -1$ in equation (35), we get

$$\epsilon \tilde{K}^{\delta=-1} = \epsilon m'. \quad (36)$$

By setting $\delta = 0$ in equation (35), we get

$$\epsilon \tilde{K}^{\delta=0} = [\epsilon m' + \epsilon \beta', \epsilon m' - \epsilon \alpha']. \quad (37)$$

On combining equations (33), (34), (36), and (37), the case $\epsilon < 0$ follows. Thus result is as required. \square

Theorem 4. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be two nonnegative LR-type BFNs; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \begin{bmatrix} m_1^* m_2^*, m_1^* \alpha_2^* + \alpha_1^* m_2^* - \alpha_1^* \alpha_2^*, m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^* \\ m_1' m_2', m_1' \alpha_2' + \alpha_1' m_2' - \alpha_1' \alpha_2', m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2' \end{bmatrix}_{LR} \right\rangle. \quad (38)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be two nonnegative LR-type BFNs; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)]. \end{aligned} \quad (39)$$

So,

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = [(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))]. \quad (40)$$

By setting $\gamma = 1$ in equation (40), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (41)$$

By setting $\gamma = 0$ in equation (40), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = [(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)], \quad (42)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = [m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*]. \quad (43)$$

Also,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = [(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta))]. \quad (44)$$

By setting $\delta = -1$ in equation (44), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'. \quad (45)$$

By setting $\delta = 0$ in equation (44), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = [(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' + \beta_2')], \quad (46)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = [m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2']. \quad (47)$$

On combining the equations (41), (43), (45), and (47), the result follows. \square

Theorem 5. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be a nonnegative LR-type BFN

and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be a non-positive LR-type BFN, then

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left(\begin{bmatrix} m_1^* m_2^*, m_1^* \alpha_2^* - \beta_1^* m_2^* + \beta_1^* \alpha_2^*, m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^* \\ m_1' m_2', m_1' \alpha_2' - \beta_1' m_2' + \beta_1' \alpha_2', m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2' \end{bmatrix}_{LR} \right) \right\rangle. \quad (48)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be a non-negative LR-type BFN and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be a nonpositive LR-type BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m_1' - \alpha_1' L^{-1}(\delta), m_1' + \beta_1' R^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m_2' - \alpha_2' L^{-1}(\delta), m_2' + \beta_2' R^{-1}(\delta)]. \end{aligned} \quad (49)$$

So,

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = [(m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))]. \quad (50)$$

By setting $\gamma = 1$ in equation (50), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (51)$$

By setting $\gamma = 0$ in equation (50), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = [(m_1^* + \beta_1^*)(m_2^* - \alpha_2^*), (m_1^* - \alpha_1^*)(m_2^* + \beta_2^*)], \quad (52)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = [m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*]. \quad (53)$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = [(m_1' + \beta_1' R^{-1}(\delta))(m_2' - \alpha_2' L^{-1}(\delta)), (m_1' - \alpha_1' L^{-1}(\delta))(m_2' + \beta_2' R^{-1}(\delta))]. \quad (54)$$

By setting $\delta = -1$ in equation (54), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'. \quad (55)$$

By setting $\delta = 0$ in equation (54), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = [(m_1' + \beta_1')(m_2' - \alpha_2'), (m_1' - \alpha_1')(m_2' + \beta_2')], \quad (56)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = [m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2', m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2']. \quad (57)$$

On combining equations (51), (53), (55), and (57), the result follows. \square

Theorem 6. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be a nonpositive LR-type BFN and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be a nonnegative LR-type BFN; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left(\begin{bmatrix} m_1^* m_2^*, -m_1^* \beta_2^* + \alpha_1^* m_2^* + \alpha_1^* \beta_2^*, -m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^* \\ m_1' m_2', -m_1' \beta_2' + \alpha_1' m_2' + \alpha_1' \beta_2', -m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2' \end{bmatrix}_{LR} \right) \right\rangle. \quad (58)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be a non-positive LR-type BFN and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be a nonnegative LR-type BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned}\tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m_1' - \alpha_1' L^{-1}(\delta), m_1' + \beta_1' R^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m_2' - \alpha_2' L^{-1}(\delta), m_2' + \beta_2' R^{-1}(\delta)].\end{aligned}\quad (59)$$

So,

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = [(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))]. \quad (60)$$

By setting $\gamma = 1$ in equation (60), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (61)$$

By setting $\gamma = 0$ in equation (60), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = [(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)], \quad (62)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = [m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*]. \quad (63)$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = [(m_1' - \alpha_1' L^{-1}(\delta))(m_2' + \beta_2' R^{-1}(\delta)), (m_1' + \beta_1' R^{-1}(\delta))(m_2' - \alpha_2' L^{-1}(\delta))]. \quad (64)$$

By setting $\delta = -1$ in equation (64), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'. \quad (65)$$

By setting $\delta = 0$ in equation (64), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = [(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' - \alpha_2')], \quad (66)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = [m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2']. \quad (67)$$

On combining the equations (61), (63), (65), and (67), the result follows. \square

Theorem 7. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be two nonpositive LR-type BFNs; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \begin{bmatrix} m_1^* m_2^*, -m_1^* \beta_2^* - \beta_1^* m_2^* - \beta_1^* \beta_2^*, -m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^* \\ m_1' m_2', -m_1' \beta_2' - \beta_1' m_2' - \beta_1' \beta_2', -m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2' \end{bmatrix}_{LR} \right\rangle. \quad (68)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be two nonpositive LR-type

BFNs; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned}
\tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\
\tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\
\tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\
\tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)].
\end{aligned} \quad (69)$$

So,

$$\begin{aligned}
\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma &= [(m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), \\
&\quad (m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))].
\end{aligned} \quad (70)$$

By setting $\gamma = 1$ in equation (70), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (71)$$

By setting $\gamma = 0$ in equation (70), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = [(m_1^* + \beta_1^*)(m_2^* + \beta_2^*), (m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*)], \quad (72)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = [m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*]. \quad (73)$$

Now,

$$\tilde{K}_1^\gamma \otimes \tilde{K}_2^\delta = [(m_1' + \beta_1' R'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta)), (m_1' - \alpha_1' L'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta))]. \quad (74)$$

By setting $\delta = -1$ in equation (74), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'. \quad (75)$$

By setting $\delta = 0$ in equation (74), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = [(m_1' + \beta_1')(m_2' + \beta_2'), (m_1' - \alpha_1')(m_2' - \alpha_2')], \quad (76)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = [m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2', m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2']. \quad (77)$$

On combining the equations (191), (73), (75), and (77), the result follows. \square

Theorem 8. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' - \alpha_1' \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left(\left[\begin{aligned} & \left(\begin{aligned} & m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ & \max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{aligned} \right) \end{aligned} \right]_{LR} \right) \right\rangle. \quad (78)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' - \alpha_1' \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned}
\tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\
\tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\
\tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\
\tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)].
\end{aligned} \quad (79)$$

So,

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = \left[\left(\begin{array}{l} \min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\} \\ \max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \end{array} \right) \right]. \quad (80)$$

By setting $\gamma = 1$ in equation (80), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (81)$$

By setting $\gamma = 0$ in equation (80), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{l} \min\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\} \\ \max\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \end{array} \right) \right]. \quad (82)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{l} \min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \end{array} \right) \right]. \quad (83)$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = \left[\left(\begin{array}{l} \min\{(m_1' - \alpha_1' L^{-1}(\delta))(m_2' - \alpha_2' L^{-1}(\delta)), (m_1' + \beta_1' R^{-1}(\delta))(m_2' - \alpha_2' L^{-1}(\delta))\} \\ \max\{(m_1' - \alpha_1' L^{-1}(\delta))(m_2' + \beta_2' R^{-1}(\delta)), (m_1' + \beta_1' R^{-1}(\delta))(m_2' + \beta_2' R^{-1}(\delta))\} \end{array} \right) \right]. \quad (84)$$

By setting $\delta = -1$ in equation (84), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'. \quad (85)$$

By setting $\delta = 0$ in equation (84), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{l} \min\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\} \\ \max\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \end{array} \right) \right]. \quad (86)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{l} \min\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \end{array} \right) \right]. \quad (87)$$

On combining the equations (81), (83), (85), and (87), the result follows. \square

Theorem 9. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' - \alpha_1' \leq 0$, $m_1' + \beta_1' \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left[\begin{array}{l} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \\ m_1' m_2', m_1' m_2' - \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} - m_1' m_2' \end{array} \right]_{LR} \right\rangle. \quad (88)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' - \alpha_1' \leq 0$, $m_1' + \beta_1' \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

So,

$$\begin{aligned}\tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)].\end{aligned}\quad (89)$$

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = \left[\left(\min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\} \right), \right. \\ \left. \left(\max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \right) \right]. \quad (90)$$

By setting $\gamma = 1$ in equation (90), we get

By setting $\gamma = 0$ in equation (90), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (91)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\left(\min\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\} \right), \right. \\ \left. \left(\max\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \right) \right], \quad (92)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\left(\min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \right), \right. \\ \left. \left(\max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \right) \right]. \quad (93)$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = \left[\left(\min\left\{ (m_2' + \beta_2' R'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta)) \right\} \right), \right. \\ \left. \left(\max\left\{ (m_1' - \alpha_1' L'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta)) \right\} \right) \right]. \quad (94)$$

By setting $\delta = -1$ in equation (94), we get

By setting $\delta = 0$ in equation (94), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'. \quad (95)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\min\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\} \right), \right. \\ \left. \left(\max\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \right) \right]. \quad (96)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \right), \right. \\ \left. \left(\max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \right) \right]. \quad (97)$$

On combining the equations (91), (93), (95), and (97), the result follows. \square

Theorem 10. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' + \beta_1' \leq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left[\begin{array}{c} \left(m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \right) \\ \max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{array} \right]_{LR}, \right. \\ \left. \left[\begin{array}{c} \left(m_1' m_2', m_1' m_2' - \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \right) \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1^* \alpha_2^*, m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1^* \alpha_2^*\} - m_1' m_2' \end{array} \right]_{LR} \right] \right\rangle. \quad (98)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' + \beta_1' \leq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

So,

$$\begin{aligned} \tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)]. \end{aligned} \quad (99)$$

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = \left[\begin{array}{c} \min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\}, \\ \max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \end{array} \right]. \quad (100)$$

By setting $\gamma = 1$ in equation (100), we get

By setting $\gamma = 0$ in equation (100), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (101)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\begin{array}{c} \min\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\}, \\ \max\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \end{array} \right], \quad (102)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\begin{array}{c} \min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \end{array} \right]. \quad (103)$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = \left[\begin{array}{c} \min\{(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta))\}, \\ \max\{(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta))\} \end{array} \right]. \quad (104)$$

By setting $\delta = -1$ in equation (104), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m'_1 m'_2. \quad (105)$$

By setting $\delta = 0$ in equation (104), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{l} \min\{(m'_1 - \alpha'_1)(m'_2 + \beta'_2), (m'_1 + \beta'_1)(m'_2 + \beta'_2)\}, \\ \max\{(m'_1 - \alpha'_1)(m'_2 - \alpha'_2), (m'_1 + \beta'_1)(m'_2 - \alpha'_2)\} \end{array} \right) \right], \quad (106)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{l} \min\{m'_1 m'_2 + m'_1 \beta'_2 - \alpha'_1 m'_2 - \alpha'_1 \beta'_2, m'_1 m'_2 + m'_1 \beta'_2 + \beta'_1 m'_2 + \beta'_1 \beta'_2\}, \\ \max\{m'_1 m'_2 - m'_1 \alpha'_2 - \alpha'_1 m'_2 + \alpha'_1 \alpha'_2, m'_1 m'_2 - m'_1 \alpha'_2 + \beta'_1 m'_2 - \beta'_1 \alpha'_2\} \end{array} \right) \right]. \quad (107)$$

On combining equations (101), (103), (105), and (107), the result follows. \square

Theorem 11. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m'_1, \alpha'_1, \beta'_1]_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \leq 0$, $m_1^* + \beta_1^* \geq 0$, $m'_1 - \alpha'_1 \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m'_2, \alpha'_2, \beta'_2]_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left[\left(\begin{array}{l} m_1^* m_2^*, m'_1 m'_2 - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{array} \right) \right]_{LR} \right\rangle. \quad (108)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m'_1, \alpha'_1, \beta'_1]_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \leq 0$, $m_1^* + \beta_1^* \geq 0$, $m'_1 - \alpha'_1 \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m'_2, \alpha'_2, \beta'_2]_{LR} \rangle$ be an unrestricted

LR-type BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m'_1 - \alpha'_1 L'^{-1}(\delta), m'_1 + \beta'_1 R'^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m'_2 - \alpha'_2 L'^{-1}(\delta), m'_2 + \beta'_2 R'^{-1}(\delta)]. \end{aligned} \quad (109)$$

So,

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = \left[\left(\begin{array}{l} \min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\} \\ \max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \end{array} \right) \right]. \quad (110)$$

By setting $\gamma = 1$ in equation (110), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (111)$$

By setting $\gamma = 0$ in equation (110), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{l} \min\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\}, \\ \max\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \end{array} \right) \right], \quad (112)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{l} \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\}, \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \end{array} \right) \right]. \quad (113)$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = \left[\left(\begin{array}{l} \min\{(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta))\}, \\ \max\{(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta))\} \end{array} \right) \right]. \quad (114)$$

By setting $\delta = -1$ in equation (114), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'. \quad (115)$$

By setting $\delta = 0$ in equation (114), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{l} \min\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\}, \\ \max\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \end{array} \right) \right], \quad (116)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{l} \min\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\}, \\ \max\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \end{array} \right) \right]. \quad (117)$$

On combining the equations (111), (113), (115), and (117), the result follows. \square

Theorem 12. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \leq 0$, $m_1^* + \beta_1^* \geq 0$, $m_1' - \alpha_1' \leq 0$, $m_1' + \beta_1' \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left[\left(\begin{array}{l} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{array} \right) \right]_{LR}, \right. \\ \left. \left[\left(\begin{array}{l} m_1' m_2', m_1' m_2' - \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} - m_1' m_2' \end{array} \right) \right]_{LR} \right] \rangle. \quad (118)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \leq 0$, $m_1^* + \beta_1^* \geq 0$, $m_1' - \alpha_1' \leq 0$, $m_1' + \beta_1' \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an

unrestricted LR-type BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)]. \end{aligned} \quad (119)$$

So,

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = \left[\left(\min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\}, \right. \right. \\ \left. \left. \max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \right) \right]. \quad (120)$$

By setting $\gamma = 1$ in equation (120), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (121)$$

By setting $\gamma = 0$ in equation (120), we get

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = \left[\left(\min\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\}, \right. \right. \\ \left. \left. \max\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \right) \right], \quad (122)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\left(\min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\}, \right. \right. \\ \left. \left. \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \right) \right]. \quad (123)$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = \left[\left(\min\{(m_1' - \alpha_1' L^{-1}(\gamma))(m_2' + \beta_2' R^{-1}(\gamma)), (m_1' + \beta_1' R^{-1}(\gamma))(m_2' - \alpha_2' L^{-1}(\gamma))\}, \right. \right. \\ \left. \left. \max\{(m_1' - \alpha_1' L^{-1}(\gamma))(m_2' - \alpha_2' L^{-1}(\gamma)), (m_1' + \beta_1' R^{-1}(\gamma))(m_2' + \beta_2' R^{-1}(\gamma))\} \right) \right]. \quad (124)$$

By setting $\delta = -1$ in equation (124), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'. \quad (125)$$

By setting $\delta = 0$ in equation (124), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\min\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\}, \right. \right. \\ \left. \left. \max\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \right) \right]. \quad (126)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\}, \right. \right. \\ \left. \left. \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \right) \right]. \quad (127)$$

On combining the equations (121), (123), (125), and (127), the result follows. \square

Theorem 13. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \leq 0$, $m_1^* + \beta_1^* \geq 0$, $m_1' + \beta_1' \leq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left[\left(\begin{aligned} & \left(m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \right) \\ & \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{aligned} \right) \right]_{LR}, \right. \\ \left. \left[\left(\begin{aligned} & m_1' m_2', m_1' m_2' - \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ & \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} - m_1' m_2' \end{aligned} \right) \right]_{LR} \right] \right\rangle. \quad (128)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \leq 0, m_1^* + \beta_1^* \geq 0, m_1' + \beta_1' \leq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted

LR-type BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned}\tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)].\end{aligned}\tag{129}$$

So,

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = \left[\left(\min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \right), \right. \\ \left. \left(\max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\} \right) \right].\tag{130}$$

By setting $\gamma = 1$ in equation (130), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*.\tag{131}$$

By setting $\gamma = 0$ in equation (130), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\left(\min\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \right), \right. \\ \left. \left(\max\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\} \right) \right],\tag{132}$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\left(\min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \right), \right. \\ \left. \left(\max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \right) \right].\tag{133}$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = \left[\left(\min\{(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta))\} \right), \right. \\ \left. \left(\max\{(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta))\} \right) \right].\tag{134}$$

By setting $\delta = -1$ in equation (134), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'.\tag{135}$$

By setting $\delta = 0$ in equation (135), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\min\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \right), \right. \\ \left. \left(\max\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\} \right) \right],\tag{136}$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \right), \right. \\ \left. \left(\max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \right) \right].\tag{137}$$

On combining the equations (131), (133), (135), and (137), the result follows. \square

Theorem 14. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m_1' - \alpha_1' \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left[\begin{array}{c} \left(\begin{array}{c} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} - m_1^* m_2^* \end{array} \right) \end{array} \right]_{LR} \right\rangle. \quad (138)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m_1' - \alpha_1' \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type

BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)]. \end{aligned} \quad (139)$$

So,

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = \left[\begin{array}{c} \min\{(m_1^* - \alpha_1^* L^{*-1}(\gamma))(m_2^* + \beta_2^* R^{*-1}(\gamma)), (m_1^* + \beta_1^* R^{*-1}(\gamma))(m_2^* + \beta_2^* R^{*-1}(\gamma))\} \\ \max\{(m_1^* - \alpha_1^* L^{*-1}(\gamma))(m_2^* - \alpha_2^* L^{*-1}(\gamma)), (m_1^* + \beta_1^* R^{*-1}(\gamma))(m_2^* - \alpha_2^* L^{*-1}(\gamma))\} \end{array} \right]. \quad (140)$$

By setting $\gamma = 1$ in equation (140), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (141)$$

By setting $\gamma = 0$ in equation (140), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\begin{array}{c} \min\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \\ \max\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\} \end{array} \right], \quad (142)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\begin{array}{c} \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \end{array} \right]. \quad (143)$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = \left[\begin{array}{c} \min\{(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta))\} \\ \max\{(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta))\} \end{array} \right]. \quad (144)$$

By setting $\delta = -1$ in equation (144), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m'_1 m'_2. \quad (145)$$

By setting $\delta = 0$ in equation (144), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{l} \min\{(m'_1 - \alpha'_1)(m'_2 - \alpha'_2), (m'_1 + \beta'_1)(m'_2 - \alpha'_2)\} \\ \max\{(m'_1 - \alpha'_1)(m'_2 + \beta'_2), (m'_1 + \beta'_1)(m'_2 + \beta'_2)\} \end{array} \right) \right], \quad (146)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{l} \min\{m'_1 m'_2 - m'_1 \alpha'_2 - \alpha'_1 m'_2 + \alpha'_1 \alpha'_2, m'_1 m'_2 - m'_1 \alpha'_2 + \beta'_1 m'_2 - \beta'_1 \alpha'_2\} \\ \max\{m'_1 m'_2 + m'_1 \beta'_2 - \alpha'_1 m'_2 - \alpha'_1 \beta'_2, m'_1 m'_2 + m'_1 \beta'_2 + \beta'_1 m'_2 + \beta'_1 \beta'_2\} \end{array} \right) \right]. \quad (147)$$

On combining equations (141), (143), (145), and (147), the result follows. \square

Theorem 15. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m'_1, \alpha'_1, \beta'_1]_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m'_1 - \alpha'_1 \leq 0$, $m'_1 + \beta'_1 \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m'_2, \alpha'_2, \beta'_2]_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left[\left(\begin{array}{l} m_1^* m_2^*, m'_1 m'_2 - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{array} \right) \right]_{LR}, \right. \\ \left. \left[\left(\begin{array}{l} m'_1 m'_2, m'_1 m'_2 - \min\{m'_1 m'_2 + m'_1 \beta'_2 - \alpha'_1 m'_2 - \alpha'_1 \beta'_2, m'_1 m'_2 - m'_1 \alpha'_2 + \beta'_1 m'_2 - \beta'_1 \alpha'_2\} \\ \max\{m'_1 m'_2 - m'_1 \alpha'_2 - \alpha'_1 m'_2 + \alpha'_1 \alpha'_2, m'_1 m'_2 + m'_1 \beta'_2 + \beta'_1 m'_2 + \beta'_1 \beta'_2\} - m'_1 m'_2 \end{array} \right) \right]_{LR} \right] \right\rangle. \quad (148)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m'_1, \alpha'_1, \beta'_1]_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m'_1 - \alpha'_1 \leq 0$, $m'_1 + \beta'_1 \geq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m'_2, \alpha'_2, \beta'_2]_{LR} \rangle$ be an unrestricted

LR-type BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\ \tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\ \tilde{K}_1^\delta &= [m'_1 - \alpha'_1 L'^{-1}(\delta), m'_1 + \beta'_1 R'^{-1}(\delta)], \\ \tilde{K}_2^\delta &= [m'_2 - \alpha'_2 L'^{-1}(\delta), m'_2 + \beta'_2 R'^{-1}(\delta)]. \end{aligned} \quad (149)$$

So,

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = \left[\left(\begin{array}{l} \min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \\ \max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\} \end{array} \right) \right]. \quad (150)$$

By setting $\gamma = 1$ in equation (150), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \quad (151)$$

By setting $\gamma = 0$ in equation (150), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\begin{array}{c} \min\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \\ \max\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\} \end{array} \right], \quad (152)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\begin{array}{c} \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \end{array} \right]. \quad (153)$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = \left[\begin{array}{c} \min\{(m_1' - \alpha_1'^{-1}(\gamma))(m_2' + \beta_2'^{-1}(\gamma)), (m_1' + \beta_1'^{-1}(\gamma))(m_2' - \alpha_2'^{-1}(\gamma))\} \\ \max\{(m_1' - \alpha_1'^{-1}(\gamma))(m_2' - \alpha_2'^{-1}(\gamma)), (m_1' + \beta_1'^{-1}(\gamma))(m_2' + \beta_2'^{-1}(\gamma))\} \end{array} \right]. \quad (154)$$

By setting $\delta = -1$ in equation (154), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'. \quad (155)$$

By setting $\delta = 0$ in equation (154), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\begin{array}{c} \min\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\} \\ \max\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \end{array} \right], \quad (156)$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\begin{array}{c} \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \end{array} \right]. \quad (157)$$

On combining equations (151), (153), (155), and (157), the result follows. \square

Theorem 16. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m_1' + \beta_1' \leq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\tilde{K}_1 \otimes \tilde{K}_2 = \left\langle \left[\begin{array}{c} \left(\begin{array}{c} m_1^* m_2^* + m_1^* \beta_2^* - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \end{array} \right)_{LR} \\ \left(\begin{array}{c} m_1' m_2' + m_1' \beta_2' - \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \end{array} \right)_{LR} \end{array} \right] \right\rangle. \quad (158)$$

Proof. Let $\tilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m_1' + \beta_1' \leq 0$ and $\tilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type

BFN; then, their γ -cut and δ -cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned}
\tilde{K}_1^\gamma &= [m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)], \\
\tilde{K}_2^\gamma &= [m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)], \\
\tilde{K}_1^\delta &= [m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)], \\
\tilde{K}_2^\delta &= [m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)].
\end{aligned} \tag{159}$$

So,

$$\tilde{K}_1^\gamma \times \tilde{K}_2^\gamma = \left[\begin{array}{c} \min\{(m_1^* - \alpha_1^* L^{*-1}(\gamma))(m_2^* + \beta_2^* R^{*-1}(\gamma)), (m_1^* + \beta_1^* R^{*-1}(\gamma))(m_2^* + \beta_2^* R^{*-1}(\gamma))\} \\ \max\{(m_1^* - \alpha_1^* L^{*-1}(\gamma))(m_2^* - \alpha_2^* L^{*-1}(\gamma)), (m_1^* + \beta_1^* R^{*-1}(\gamma))(m_2^* - \alpha_2^* L^{*-1}(\gamma))\} \end{array} \right]. \tag{160}$$

By setting $\gamma = 1$ in equation (160), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \tag{161}$$

By setting $\gamma = 0$ in equation (160), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\begin{array}{c} \min\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \\ \max\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\} \end{array} \right], \tag{162}$$

$$(\tilde{K}_1 \tilde{K}_2)^{\gamma=0} = \left[\begin{array}{c} \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \end{array} \right]. \tag{163}$$

Now,

$$\tilde{K}_1^\delta \times \tilde{K}_2^\delta = \left[\begin{array}{c} \min\{(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' + \beta_2' R'^{-1}(\delta))\} \\ \max\{(m_1' - \alpha_1' L'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta)), (m_1' + \beta_1' R'^{-1}(\delta))(m_2' - \alpha_2' L'^{-1}(\delta))\} \end{array} \right]. \tag{164}$$

By setting $\delta = -1$ in equation (164), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=-1} = m_1' m_2'. \tag{165}$$

By setting $\delta = 0$ in equation (164), we get

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\begin{array}{c} \min\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \\ \max\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\} \end{array} \right], \tag{166}$$

$$(\tilde{K}_1 \tilde{K}_2)^{\delta=0} = \left[\begin{array}{c} \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \end{array} \right]. \tag{167}$$

On combining equations (161), (163), (165), and (167), the result follows. \square

4. LR-Type FBFLPP

In this section, we study about the LR-type fully bipolar fuzzy linear programming problem with equality constraints in which all the variables are represented by LR-type BFNs. Consider an LR-type FBFLPP with LR-type BFNs

$$\text{Maximize/Minimize } \sum_{j=1}^n \tilde{B}_j \otimes \tilde{Y}_j, \quad (168)$$

subject to

$$\sum_{j=1}^n \tilde{F}_{ij} \otimes \tilde{Y}_j = \tilde{E}_i, \quad \forall i = 1, 2, 3, \dots, m, \quad (169)$$

where \tilde{Y}_j , \tilde{B}_j , \tilde{E}_i , and \tilde{F}_{ij} are LR-type BFNs.

Definition 12. A bipolar fuzzy optimal solution of LR-type FBFLPP (168) will be LR-type BFN \tilde{Y}_j if

- (1) \tilde{Y}_j are LR-type BFNs
- (2) $\sum_{j=1}^n \tilde{F}_{ij} \otimes \tilde{Y}_j = \tilde{E}_i, \forall i = 1, 2, 3, \dots, m$
- (3) If there exists any LR-type BFN \tilde{Y}_j satisfying the constraints, then

In case of a maximization problem,
 $\Re \sum_{j=1}^n (\tilde{B}_j \otimes \tilde{Y}_j) > \Re (\sum_{j=1}^n (\tilde{B}_j \otimes \tilde{Y}_j))$

In case of a minimization problem,
 $\Re \sum_{j=1}^n (\tilde{B}_j \otimes \tilde{Y}_j) < \Re (\sum_{j=1}^n (\tilde{B}_j \otimes \tilde{Y}_j))$

4.1. Methodology.

$$\text{Maximize/Minimize } \sum_{j=1}^n \tilde{B}_j \otimes \tilde{Y}_j, \quad (170)$$

subject to

$$\sum_{j=1}^n \tilde{F}_{ij} \otimes \tilde{Y}_j = \tilde{E}_i, \quad \forall i = 1, 2, 3, \dots, m, \quad (171)$$

where \tilde{Y}_j , \tilde{E}_i , \tilde{F}_{ij} , and \tilde{B}_j LR-type BFNs.

Step 1. assume that

$$\begin{aligned} \tilde{F}_{ij} &= \langle [f_{ij}^*, \epsilon_{ij}^*, \eta_{ij}^*]_{\text{LR}}, [f'_{ij}, \epsilon'_{ij}, \eta'_{ij}]_{\text{LR}} \rangle, \\ \tilde{E}_i &= \langle [e_i^*, \phi_i^*, \vartheta_i^*]_{\text{LR}}, [e'_i, \phi'_i, \vartheta'_i]_{\text{LR}} \rangle, \\ \tilde{B}_j &= \langle [b_j^*, \zeta_j^*, \psi_j^*]_{\text{LR}}, [b'_j, \zeta'_j, \psi'_j]_{\text{LR}} \rangle, \\ \tilde{Y}_j &= \langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m'_j, \alpha'_j, \beta'_j]_{\text{LR}} \rangle. \end{aligned} \quad (172)$$

The LR-type FBFLPP (170) can be transformed into the following problem:

$$\text{Maximize/Minimize } \sum_{j=1}^n \left(\langle [b_j^*, \zeta_j^*, \psi_j^*]_{\text{LR}}, [b'_j, \zeta'_j, \psi'_j]_{\text{LR}} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m'_j, \alpha'_j, \beta'_j]_{\text{LR}} \rangle \right), \quad (173)$$

subject to

$$\begin{aligned} \sum_{j=1}^n \left(\langle [f_{ij}^*, \epsilon_{ij}^*, \eta_{ij}^*]_{\text{LR}}, [f'_{ij}, \epsilon'_{ij}, \eta'_{ij}]_{\text{LR}} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m'_j, \alpha'_j, \beta'_j]_{\text{LR}} \rangle \right), \\ = \langle [e_i^*, \phi_i^*, \vartheta_i^*]_{\text{LR}}, [e'_i, \phi'_i, \vartheta'_i]_{\text{LR}} \rangle, \end{aligned} \quad (174)$$

where $\langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m'_j, \alpha'_j, \beta'_j]_{\text{LR}} \rangle$ is an LR-type BFN, $\forall j = 1, 2, 3, \dots, n$.

Step 2. By using the product of LR-type BFNs given in Section 3, we suppose that

$$\langle [f_{ij}^*, \epsilon_{ij}^*, \eta_{ij}^*]_{\text{LR}}, [f'_{ij}, \epsilon'_{ij}, \eta'_{ij}]_{\text{LR}} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m'_j, \alpha'_j, \beta'_j]_{\text{LR}} \rangle = \langle [t_{ij}^{**}, \chi_{ij}^{**}, \sigma_{ij}^{**}]_{\text{LR}}, [m''_{ij}, \chi''_{ij}, \sigma''_{ij}]_{\text{LR}} \rangle. \quad (175)$$

The LR-type FBFLPP (173) transforms to a problem as follows:

$$\text{Maximize/Minimize } \sum_{j=1}^n \left(\langle [b_j^*, \zeta_j^*, \psi_j^*]_{\text{LR}}, [b'_j, \zeta'_j, \psi'_j]_{\text{LR}} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m'_j, \alpha'_j, \beta'_j]_{\text{LR}} \rangle \right), \quad (176)$$

subject to

$$\langle [t_{ij}^{**}, \chi_{ij}^{**}, \sigma_{ij}^{**}]_{LR}, [m_{ij}'', \chi_{ij}'', \sigma_{ij}'']_{LR} \rangle = \langle [e_i^*, \phi_i^*, \vartheta_i^*]_{LR}, [e_i', \phi_i', \vartheta_i']_{LR} \rangle, \quad (177)$$

where $\langle [m_j^*, \alpha_j^*, \beta_j^*]_{LR}, [m_j', \alpha_j', \beta_j']_{LR} \rangle$ is an LR-type BFN, $\forall j = 1, 2, 3, \dots, n$.

Step 3. By using arithmetic operations, given in Section 3, the LR-type FBFLPP (176) transforms into a problem as follows:

$$\text{Maximize/Minimize } \sum_{j=1}^n (\langle [b_j^*, \zeta_j^*, \psi_j^*]_{LR}, [b_j', \zeta_j', \psi_j']_{LR} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{LR}, [m_j', \alpha_j', \beta_j']_{LR} \rangle), \quad (178)$$

subject to

$$\begin{aligned} \sum_{j=1}^n t_{ij}^{**} &= e_i^*, \quad \sum_{j=1}^n t_{ij}'' = e_i', \\ \sum_{j=1}^n \chi_{ij}^{**} &= \phi_i^*, \quad \sum_{j=1}^n \chi_{ij}'' = \phi_i', \\ \sum_{j=1}^n \sigma_{ij}^{**} &= \vartheta_i^*, \quad \sum_{j=1}^n \sigma_{ij}'' = \vartheta_i', \end{aligned} \quad (179)$$

$\forall i = 1, 2, 3, \dots, m$

$$\begin{aligned} \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \\ \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \end{aligned} \quad (180)$$

$\forall j = 1, 2, 3, \dots, n$.

Step 4. By using ranking function, the LR-type FBFLPP converts into a crisp mathematical problem as follows:

Maximize/Minimize \mathfrak{R}

$$\left(\sum_{j=1}^n (\langle [b_j^*, \zeta_j^*, \psi_j^*]_{LR}, [b_j', \zeta_j', \psi_j']_{LR} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{LR}, [m_j', \alpha_j', \beta_j']_{LR} \rangle) \right), \quad (181)$$

subject to

$$\begin{aligned} \sum_{j=1}^n t_{ij}^{**} &= e_i^*, \quad \sum_{j=1}^n t_{ij}'' = e_i', \\ \sum_{j=1}^n \chi_{ij}^{**} &= \phi_i^*, \quad \sum_{j=1}^n \chi_{ij}'' = \phi_i', \\ \sum_{j=1}^n \sigma_{ij}^{**} &= \vartheta_i^*, \quad \sum_{j=1}^n \sigma_{ij}'' = \vartheta_i', \end{aligned} \quad (182)$$

$\forall i = 1, 2, 3, \dots, m$

$$\begin{aligned} \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \\ \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \end{aligned} \quad (183)$$

$\forall j = 1, 2, 3, \dots, n$.

Step 5. Considering

$$\langle [b_j^*, \zeta_j^*, \psi_j^*]_{LR}, [b_j', \zeta_j', \psi_j']_{LR} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{LR}, [m_j', \alpha_j', \beta_j']_{LR} \rangle = \langle [u_j^{**}, \theta_j^{**}, \varsigma_j^{**}]_{LR}, [u_j'', \theta_j'', \varsigma_j'']_{LR} \rangle, \quad (184)$$

the crisp mathematical problem (181) can be transformed into the following problem:

$$\text{Maximize/Minimize } \Re \sum_{j=1}^n \langle [u_j^{* *}, \theta_j^{* *}, \varsigma_j^{* *}]_{\text{LR}}, [u_j'', \theta_j'', \varsigma_j'']_{\text{LR}} \rangle, \quad (185)$$

subject to

$$\begin{aligned} \sum_{j=1}^n t_{ij}^{* *} &= e_i^*, \sum_{j=1}^n t_{ij}'' = e_i', \\ \sum_{j=1}^n \chi_{ij}^{* *} &= \phi_i^*, \sum_{j=1}^n \chi_{ij}'' = \phi_i', \\ \sum_{j=1}^n \sigma_{ij}^{* *} &= \vartheta_i^*, \sum_{j=1}^n \sigma_{ij}'' = \vartheta_i', \end{aligned} \quad (186)$$

$\forall i = 1, 2, 3, \dots, m$

$$\begin{aligned} \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \\ \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \end{aligned} \quad (187)$$

$\forall j = 1, 2, 3, \dots, n.$

Step 6. By applying the linearity property $\Re(\sum_{j=1}^n \tilde{H}_i) = \sum_{j=1}^n \Re(\tilde{H}_i)$, here let \tilde{H}_i be a BFN, the crisp mathematical problem (185) can be transformed into the following problem:

$$\text{Maximize/Minimize } \sum_{j=1}^n \Re \langle [u_j^{* *}, \theta_j^{* *}, \varsigma_j^{* *}]_{\text{LR}}, [u_j'', \theta_j'', \varsigma_j'']_{\text{LR}} \rangle, \quad (188)$$

subject to

$$\begin{aligned} \sum_{j=1}^n t_{ij}^{* *} &= e_i^*, \sum_{j=1}^n t_{ij}'' = e_i', \\ \sum_{j=1}^n \chi_{ij}^{* *} &= \phi_i^*, \sum_{j=1}^n \chi_{ij}'' = \phi_i', \\ \sum_{j=1}^n \sigma_{ij}^{* *} &= \vartheta_i^*, \sum_{j=1}^n \sigma_{ij}'' = \vartheta_i', \end{aligned} \quad (189)$$

$\forall i = 1, 2, 3, \dots, m$

$$\begin{aligned} \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \\ \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \end{aligned} \quad (190)$$

$\forall j = 1, 2, 3, \dots, n.$

Step 7. By applying ranking for LR-type BFNs (2.11), the crisp mathematical problem (188) becomes

Maximize/Minimize

$$\begin{aligned} &\left(\frac{1}{4} \left(\int_0^1 (u_j^{* *} - \theta_j^{* *} L^{-1}(\gamma)) d\gamma + \int_0^1 (u_j^{* *} + \varsigma_j^{* *} R^{-1}(\gamma)) d\gamma \right) \right. \\ &\quad \left. + \frac{1}{4} \left(\int_{-1}^0 (u_j'' - \theta_j'' L'^{-1}(\delta)) d\delta + \int_{-1}^0 (u_j'' + \varsigma_j'' R'^{-1}(\delta)) d\delta \right) \right) \end{aligned} \quad (191)$$

subject to

$$\begin{aligned} \sum_{j=1}^n t_{ij}^{* *} &= e_i^*, \sum_{j=1}^n t_{ij}'' = e_i', \\ \sum_{j=1}^n \chi_{ij}^{* *} &= \phi_i^*, \sum_{j=1}^n \chi_{ij}'' = \phi_i', \\ \sum_{j=1}^n \sigma_{ij}^{* *} &= \vartheta_i^*, \sum_{j=1}^n \sigma_{ij}'' = \vartheta_i', \end{aligned} \quad (192)$$

$\forall i = 1, 2, 3, \dots, m$

$$\begin{aligned}
\alpha_j^* &\geq 0, \\
\beta_j^* &\geq 0, \\
\alpha_j' &\geq 0, \\
\beta_j' &\geq 0, \\
\alpha_j^* &\geq 0, \\
\beta_j^* &\geq 0, \\
\alpha_j' &\geq 0, \\
\beta_j' &\geq 0,
\end{aligned} \tag{193}$$

$\forall j = 1, 2, 3, \dots, n.$

Step 8. By solving the crisp mathematical problem (190), we get the optimal solution $m_j^*, \alpha_j^*, \beta_j^*, m_j', \alpha_j', \beta_j', \forall j = 1, 2, 3, \dots, n.$

Step 9. find the exact LR-type bipolar fuzzy optimal solution \tilde{Y}_j^* of LR-type FBFLPP by assigning the values of

$m_j^*, \alpha_j^*, \beta_j^*, m_j', \alpha_j', \beta_j'$ in $\tilde{Y}_j^* = \langle [m_j^*, \alpha_j^*, \beta_j^*]_{LR}, [m_j', \alpha_j', \beta_j']_{LR} \rangle, \forall j = 1, 2, 3, \dots, n.$

Step 10. find the LR-type bipolar fuzzy optimal value by putting the values of \tilde{Y}_j^* in $\sum_{j=1}^n \tilde{B}_j \otimes \tilde{Y}_j, \forall j = 1, 2, 3, \dots, n.$

Thus, we state the existence condition for the optimal solution of bipolar fuzzy LPP in the following theorem.

Theorem 17. *The solution of LR-type FBFLPP.*
Maximize/Minimize

$$Z = \sum_{j=1}^n B_j \otimes Y_j, \tag{194}$$

subject to,

$$\sum_{j=1}^n F_{ij} \otimes Y_j = E_i, \quad \forall i = 1, \dots, m.$$

$B_j, Y_j, F_{ij},$ and E_i are LR-type BFNs, which exist with the solution of the associated crisp mathematical problem.

Maximize/Minimize

$$\Re(Z) = \Re \left(\sum_{j=1}^n \left(\langle [b_j^*, \zeta_j^*, \psi_j^*]_{LR}, [b_j', \zeta_j', \psi_j']_{LR} \rangle \otimes \left(\langle [m_j^*, \alpha_j^*, \beta_j^*]_{LR}, [m_j', \alpha_j', \beta_j']_{LR} \rangle \right) \right) \right), \tag{195}$$

subject to

$$\begin{aligned}
\sum_{j=1}^n p_{ij} &= e_i^*, \\
\sum_{j=1}^n q_{ij} &= \phi_i^*, \\
\sum_{j=1}^n r_{ij} &= \vartheta_i^*, \\
\sum_{j=1}^n s_{ij} &= e_i', \\
\sum_{j=1}^n t_{ij} &= \phi_i', \\
\sum_{j=1}^n u_{ij} &= \vartheta_i' \\
\alpha_j^* &\geq 0, \\
\beta_j^* &\geq 0, \\
\alpha_j' &\geq 0, \\
\beta_j' &\geq 0,
\end{aligned} \tag{196}$$

$\forall i = 1, \dots, m$ exists. Otherwise, there is no guarantee that the LR-type bipolar fuzzy optimal solution exists.

Proof. The proof is straightforward. \square

5. Numerical Examples

In this section, the methodology presented in Section 4 is illustrated by solving a numerical example and model.

Example 1.

$$\text{Maximize } (\langle [9, 6, 3]_{\text{LR}}, [8, 5, 1]_{\text{LR}} \rangle \otimes \tilde{Y}_1 \oplus \langle [6, 3, 4]_{\text{LR}}, [5, 4, 2]_{\text{LR}} \rangle \otimes \tilde{Y}_2), \quad (197)$$

subject to

$$\begin{aligned} \langle [9, 5, 3]_{\text{LR}}, [4, 1, 5]_{\text{LR}} \rangle \otimes \tilde{Y}_1 \oplus \langle [8, 5, 2]_{\text{LR}}, [4, 2, 3]_{\text{LR}} \rangle \otimes \tilde{Y}_2 &= \langle [94, 68, 86]_{\text{LR}}, [44, 30, 100]_{\text{LR}} \rangle, \\ \langle [6, 2, 5]_{\text{LR}}, [6, 3, 5]_{\text{LR}} \rangle \otimes \tilde{Y}_1 \oplus \langle [8, 6, 4]_{\text{LR}}, [9, 2, 1]_{\text{LR}} \rangle \otimes \tilde{Y}_2 &= \langle [76, 52, 106]_{\text{LR}}, [78, 59, 111]_{\text{LR}} \rangle, \end{aligned} \quad (198)$$

where Y_1 and Y_2 are LR-type BFNs and $L(x) = R(x) = \max\{0, 1 - x\}$, $L'(x) = R'(x) = \min\{0, x - 1\}$.

Step 1:

Let $\tilde{Y}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle$ and $\tilde{Y}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle$ be LR-type BFNs; then,

$$\text{Maximize } \left(\begin{aligned} &\langle [9, 6, 3]_{\text{LR}}, [8, 5, 1]_{\text{LR}} \rangle \otimes \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle \oplus \\ &\langle [6, 3, 4]_{\text{LR}}, [5, 4, 2]_{\text{LR}} \rangle \otimes \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle \end{aligned} \right), \quad (199)$$

subject to

$$\begin{aligned} &\langle [9, 5, 3]_{\text{LR}}, [4, 1, 5]_{\text{LR}} \rangle \otimes \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle \oplus \langle [8, 5, 2]_{\text{LR}}, [4, 2, 3]_{\text{LR}} \rangle \\ &\otimes \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle = \langle [94, 68, 86]_{\text{LR}}, [44, 30, 100]_{\text{LR}} \rangle, \\ &\langle [6, 2, 5]_{\text{LR}}, [6, 3, 5]_{\text{LR}} \rangle \otimes \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle \oplus \langle [8, 6, 4]_{\text{LR}}, [9, 2, 1]_{\text{LR}} \rangle \\ &\otimes \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle = \langle [76, 52, 106]_{\text{LR}}, [78, 59, 111]_{\text{LR}} \rangle, \end{aligned} \quad (200)$$

where $\tilde{Y}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle$ and $\tilde{Y}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle$ are LR-type BFNs

Step 2: by using the product of LR-type BFNs given in Section 3, the LR-type FBFLPP converts as

Maximize

$$\left(\begin{aligned} &\langle [9m_1^*, 9\alpha_1^* + 6m_1^* - 6\alpha_1^*, 9\beta_1^* + 3m_1^* + 3\beta_1^*]_{\text{LR}}, [8m_1', 8\alpha_1' + 5m_1' - 5\alpha_1', 8\beta_1' + 1m_1' + 1\beta_1']_{\text{LR}} \rangle \oplus \\ &\langle [6m_2^*, 6\alpha_2^* + 3m_2^* - 3\alpha_2^*, 6\beta_2^* + 4m_2^* + 4\beta_2^*]_{\text{LR}}, [5m_2', 5\alpha_2' + 4m_2' - 4\alpha_2', 5\beta_2' + 2m_2' + 2\beta_2']_{\text{LR}} \rangle \end{aligned} \right), \quad (201)$$

subject to

$$\begin{aligned}
& \langle [9m_1^*, 9\alpha_1^* + 5m_1^* - 5\alpha_1^*, 9\beta_1^* + 3m_1^* + 3\beta_1^*]_{LR}, [4m_1', 4\alpha_1' + 1m_1' - 1\alpha_1', 4\beta_1' + 5m_1' + 5\beta_1']_{LR} \rangle \oplus \\
& \langle [8m_2^*, 8\alpha_2^* + 5m_2^* - 5\alpha_2^*, 8\beta_2^* + 2m_2^* + 2\beta_2^*]_{LR}, [4m_2', 4\alpha_2' + 2m_2' - 2\alpha_2', 4\beta_2' + 3m_2' + 3\beta_2']_{LR} \rangle \\
& = \langle [94, 68, 86]_{LR}, [44, 30, 100]_{LR} \rangle, \\
& \langle [6m_1^*, 6\alpha_1^* + 2m_1^* - 2\alpha_1^*, 6\beta_1^* + 5m_1^* + 5\beta_1^*]_{LR}, [6m_1', 6\alpha_1' + 3m_1' - 3\alpha_1', 6\beta_1' + 5m_1' + 5\beta_1']_{LR} \rangle \oplus \\
& \langle [8m_2^*, 8\alpha_2^* + 6m_2^* - 6\alpha_2^*, 8\beta_2^* + 4m_2^* + 4\beta_2^*]_{LR}, [9m_2', 9\alpha_2' + 2m_2' - 2\alpha_2', 9\beta_2' + 1m_2' + 1\beta_2']_{LR} \rangle \\
& = \langle [76, 52, 106]_{LR}, [78, 59, 111]_{LR} \rangle,
\end{aligned}$$

$$\begin{aligned}
& \alpha_1^* \geq 0, \\
& \beta_1^* \geq 0, \\
& \alpha_1' \geq 0, \\
& \beta_1' \geq 0, \\
& \alpha_2^* \geq 0, \\
& \beta_2^* \geq 0, \\
& \alpha_2' \geq 0, \\
& \beta_2' \geq 0
\end{aligned} \tag{202}$$

Step 3: using arithmetic operations (12), the LR-type FBFLPP converts as

Maximize

$$\left(\left\langle \begin{aligned} & [9m_1^* + 6m_2^*, 3\alpha_1^* + 6m_1^* + 3\alpha_2^* + 3m_2^*, 12\beta_1^* + 3m_1^* + 10\beta_2^* + 4m_2^*]_{LR} \\ & [8m_1' + 5m_2', 3\alpha_1' + 5m_1' + 1\alpha_2' + 4m_2', 9\beta_1' + 1m_1' + 7\beta_2' + 2m_2']_{LR} \end{aligned} \right\rangle \right), \tag{203}$$

subject to

$$\begin{aligned}
& \langle [9m_1^* + 8m_2^*, 4\alpha_1^* + 5m_1^* + 3\alpha_2^* + 5m_2^*, 12\beta_1^* + 3m_1^* + 10\beta_2^* + 2m_2^*]_{LR}, \\
& \langle [4m_1' + 4m_2', 3\alpha_1' + 1m_1' + 2\alpha_2' + 2m_2', 9\beta_1' + 5m_1' + 7\beta_2' + 3m_2']_{LR} \rangle, \\
& = \langle [94, 68, 86]_{LR}, [44, 30, 100]_{LR} \rangle, \\
& \langle [6m_1^* + 8m_2^*, 4\alpha_1^* + 2m_1^* + 2\alpha_2^* + 6m_2^*, 11\beta_1^* + 5m_1^* + 12\beta_2^* + 4m_2^*]_{LR}, \\
& \langle [6m_1' + 9m_2', 3\alpha_1' + 3m_1' + 7\alpha_2' + 2m_2', 11\beta_1' + 5m_1' + 10\beta_2' + 1m_2']_{LR} \rangle \\
& = \langle [76, 52, 106]_{LR}, [78, 59, 111]_{LR} \rangle,
\end{aligned}$$

$$\begin{aligned}
& \alpha_1^* \geq 0, \\
& \beta_1^* \geq 0, \\
& \alpha_1' \geq 0, \\
& \beta_1' \geq 0, \\
& \alpha_2^* \geq 0, \\
& \beta_2^* \geq 0, \\
& \alpha_2' \geq 0, \\
& \beta_2' \geq 0
\end{aligned} \tag{204}$$

Step 4: by applying ranking function and using (2.8),
the LR-type FBFLPP converts as

Maximize

$$\begin{aligned} & \Re \left(\left\langle \begin{pmatrix} [9m_1^* + 6m_2^*, 3\alpha_1^* + 6m_1^* + 3\alpha_2^* + 3m_2^*, 12\beta_1^* + 3m_1^* + 10\beta_2^* + 4m_2^*]_{LR} \\ [8m_1' + 5m_2', 3\alpha_1' + 5m_1' + 1\alpha_2' + 4m_2', 9\beta_1' + 1m_1' + 7\beta_2' + 2m_2']_{LR} \end{pmatrix} \right\rangle \right) \\ & 9m_1^* + 8m_2^* = 94, \\ & 4\alpha_1^* + 5m_1^* + 3\alpha_2^* + 5m_2^* = 68, \\ & 12\beta_1^* + 3m_1^* + 10\beta_2^* + 2m_2^* = 86, \\ & 4m_1' + 4m_2' = 44, \\ & 3\alpha_1' + 1m_1' + 2\alpha_2' + 2m_2' = 30, \\ & 9\beta_1' + 5m_1' + 7\beta_2' + 3m_2' = 100, \\ & 6m_1^* + 8m_2^* = 76, \\ & 4\alpha_1^* + 2m_1^* + 2\alpha_2^* + 6m_2^* = 52, \\ & 11\beta_1^* + 5m_1^* + 12\beta_2^* + 4m_2^* = 106, \\ & 6m_1' + 9m_2' = 78, \\ & 3\alpha_1' + 3m_1' + 7\alpha_2' + 2m_2' = 59, \\ & 11\beta_1' + 5m_1' + 10\beta_2' + 1m_2' = 111, \\ & \alpha_1^* \geq 0, \\ & \beta_1^* \geq 0, \\ & \alpha_1' \geq 0, \\ & \beta_1' \geq 0, \\ & \alpha_2^* \geq 0, \\ & \beta_2^* \geq 0, \\ & \alpha_2' \geq 0, \\ & \beta_2' \geq 0 \end{aligned} \tag{205}$$

Step 5: using ranking (2.11) for LR-type BFNs, the
problem converts to crisp LPP as

Maximize

$$\left(\frac{33}{8}m_1^* + \frac{25}{8}m_2^* - \frac{3}{8}\alpha_1^* - \frac{3}{8}\alpha_2^* + \frac{6}{4}\beta_1^* + \frac{5}{4}\beta_2^* + \frac{14}{4}m_1' + \frac{9}{4}m_2' - \frac{3}{8}\alpha_1' - \frac{1}{8}\alpha_2' + \frac{9}{8}\beta_1' + \frac{7}{8}\beta_2' \right), \tag{206}$$

subject to

TABLE 2: Fitness problem.

Ingredients	Mutton	Beef	Maximum availability (grams)
Protein (grams)	$\langle [7, 6, 5]_{LR}, [8, 4, 1]_{LR} \rangle$	$\langle [6, 2, 3]_{LR}, [8, 7, 2]_{LR} \rangle$	$\langle [101, 86, 151]_{LR}, [128, 112, 55]_{LR} \rangle$
Minerals (grams)	$\langle [12, 6, 2]_{LR}, [10, 8, 2]_{LR} \rangle$	$\langle [-9, 3, 1]_{LR}, [-7, 4, 2]_{LR} \rangle$	$\langle [-39, 135, 141]_{LR}, [-10, 116, 74]_{LR} \rangle$

$$\begin{aligned}
9m_1^* + 8m_2^* &= 94, \\
4\alpha_1^* + 5m_1^* + 3\alpha_2^* + 5m_2^* &= 68, \\
12\beta_1^* + 3m_1^* + 10\beta_2^* + 2m_2^* &= 86, \\
4m_1' + 4m_2' &= 44, \\
3\alpha_1' + 1m_1' + 2\alpha_2' + 2m_2' &= 30, \\
9\beta_1' + 5m_1' + 7\beta_2' + 3m_2' &= 100, \\
6m_1^* + 8m_2^* &= 76, \\
4\alpha_1^* + 2m_1^* + 2\alpha_2^* + 6m_2^* &= 52, \\
11\beta_1^* + 5m_1^* + 12\beta_2^* + 4m_2^* &= 106, \\
6m_1' + 9m_2' &= 78, \\
3\alpha_1' + 3m_1' + 7\alpha_2' + 2m_2' &= 59, \\
11\beta_1' + 5m_1' + 10\beta_2' + 1m_2' &= 111, \\
\alpha_1^* &\geq 0, \\
\beta_1^* &\geq 0, \\
\alpha_1' &\geq 0, \\
\beta_1' &\geq 0, \\
\alpha_2^* &\geq 0, \\
\beta_2^* &\geq 0, \\
\alpha_2' &\geq 0, \\
\beta_2' &\geq 0
\end{aligned} \tag{207}$$

Step 6: by using software Maple, an optimal solution of the crisp LPP is $m_1^* = 6, \alpha_1^* = 1, \beta_1^* = 4, m_2^* = 5, \alpha_2^* = 3, \beta_2^* = 1, m_1' = 7, \alpha_1' = 3, \beta_1' = 2, m_2' = 4, \alpha_2' = 3, \beta_2' = 5$

Step 7: the exact LR-type BFOS is $\tilde{Y}_1 = \langle (6, 1, 4), (7, 3, 2) \rangle, \tilde{Y}_2 = \langle (5, 3, 1), (4, 3, 5) \rangle$

Step 8: the bipolar fuzzy optimal value of LR-type FBFLPP is $\langle (84, 63, 96), (76, 63, 68) \rangle$

Example 2. (Fitness Problem). A chef purchases beef and mutton for the players of a football team. Both types of meat

possess plenty of proteins and minerals. The cost of beef and mutton per kilograms is Rs. $\langle [12, 7, 5]_{LR}, [11, 9, 3]_{LR} \rangle$ and Rs. $\langle [10, 5, 2]_{LR}, [14, 8, 2]_{LR} \rangle$, respectively. Each player must have to take $\langle [101, 86, 151]_{LR}, [128, 112, 55]_{LR} \rangle$ grams of proteins and $\langle [-39, 135, 141]_{LR}, [-10, 116, 74]_{LR} \rangle$ grams of minerals daily to maintain the physical fitness standards, and further details are presented in Table 2. How many units of beef and mutton should be used to fulfill the demand of each player at the minimum cost?

We apply the proposed method to solve this problem.

Let Y_1 and Y_2 units be taken of beef and mutton; then, the given problem converts to an LR-type FBFLPP as

$$\text{Minimize } (\langle [12, 7, 5]_{LR}, [11, 9, 3]_{LR} \rangle \otimes \tilde{Y}_1 \oplus \langle [10, 5, 2]_{LR}, [14, 8, 2]_{LR} \rangle \otimes \tilde{Y}_2), \tag{208}$$

subject to

$$\begin{aligned} &\langle [7, 6, 5]_{LR}, [8, 4, 1]_{LR} \rangle \otimes \tilde{Y}_1 \oplus \langle [6, 2, 3]_{LR}, [8, 7, 2]_{LR} \rangle \otimes \tilde{Y}_2 = \langle [101, 86, 151]_{LR}, [128, 112, 55]_{LR} \rangle, \\ &\langle [12, 6, 2]_{LR}, [10, 8, 2]_{LR} \rangle \otimes \tilde{Y}_1 \oplus \langle [-9, 3, 1]_{LR}, [-7, 4, 2]_{LR} \rangle \otimes \tilde{Y}_2 = \langle [-39, 135, 141]_{LR}, [-10, 116, 74]_{LR} \rangle, \end{aligned} \quad (209)$$

where Y_1 and Y_2 are LR-type BFNs and $L(x) = R(x) = \max\{0, 1 - x\}$, $L'(x) = R'(x) = \min\{0, x - 1\}$.

Step 1: let $\tilde{Y}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\tilde{Y}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be LR-type BFNs; then,

$$\text{Minimize} \left(\begin{aligned} &\langle [12, 7, 5]_{LR}, [11, 9, 3]_{LR} \rangle \otimes \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle \oplus \\ &\langle [10, 5, 2]_{LR}, [14, 8, 2]_{LR} \rangle \otimes \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle \end{aligned} \right), \quad (210)$$

subject to

$$\begin{aligned} &\langle [7, 6, 5]_{LR}, [8, 4, 1]_{LR} \rangle \otimes \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle \oplus \langle [6, 2, 3]_{LR}, [8, 7, 2]_{LR} \rangle \\ &\otimes \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle = \langle [101, 86, 151]_{LR}, [128, 112, 55]_{LR} \rangle, \\ &\langle [12, 6, 2]_{LR}, [10, 8, 2]_{LR} \rangle \otimes \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle \oplus \langle [-9, 3, 1]_{LR}, [-7, 4, 2]_{LR} \rangle \\ &\otimes \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle = \langle [-39, 135, 141]_{LR}, [-10, 116, 74]_{LR} \rangle, \end{aligned} \quad (211)$$

where $\tilde{Y}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\tilde{Y}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ are LR-type BFNs

Step 2: by using the product of LR-type BFNs given in Section 3, LR-type FBFLPP converts as
Minimize

$$\left(\left\langle \begin{aligned} &\left[\begin{aligned} &12m_1^*, 12m_1^* - \min\{12m_1^* - 12\alpha_1^* - 7m_1^* + 7\alpha_1^*, 12m_1^* - 12\alpha_1^* + 5m_1^* - 7\alpha_1^*\}, \\ &\max\{12m_1^* + 12\beta_1^* + 5m_1^* + 5\beta_1^*, 12m_1^* + 12\beta_1^* - 7m_1^* - 7\beta_1^*\} - 12m_1^* \end{aligned} \right]_{LR} \\ &\left[\begin{aligned} &11m_1', 11m_1' - \min\{11m_1' - 11\alpha_1' - 9m_1' + 9\alpha_1', 11m_1' - 11\alpha_1' + 3m_1' - 3\alpha_1'\} \\ &\max\{11m_1' + 11\beta_1' + 3m_1' + 3\beta_1', 11m_1' + 11\beta_1' - 9m_1' - 9\beta_1'\} - 11m_1' \end{aligned} \right]_{LR} \end{aligned} \right\rangle \oplus \right. \\ &\left. \left\langle \begin{aligned} &\left[\begin{aligned} &10m_2^*, 10m_2^* - \min\{10m_2^* - 10\alpha_2^* - 5m_2^* + 5\alpha_2^*, 10m_2^* - 10\alpha_2^* + 2m_2^* - 2\alpha_2^*\} \\ &\max\{10m_2^* + 10\beta_2^* + 2m_2^* + 2\beta_2^*, 10m_2^* + 10\beta_2^* - 5m_2^* - 5\beta_2^*\} - 10m_2^* \end{aligned} \right]_{LR} \\ &\left[\begin{aligned} &14m_2', 14m_2' - \min\{14m_2' - 14\alpha_2' - 8m_2' + 8\alpha_2', 14m_2' - 14\alpha_2' + 2m_2' - 2\alpha_2'\} \\ &\max\{14m_2' + 14\beta_2' + 2m_2' + 2\beta_2', 14m_2' + 14\beta_2' - 8m_2' - 8\beta_2'\} - 14m_2' \end{aligned} \right]_{LR} \end{aligned} \right\rangle \right) \quad (212)$$

subject to

$$\left\langle \begin{aligned} &\left[\begin{aligned} &7m_1^*, 7m_1^* - \min\{7m_1^* - 7\alpha_1^* - 6m_1^* + 6\alpha_1^*, 7m_1^* - 7\alpha_1^* + 5m_1^* - 5\alpha_1^*\}, \\ &\max\{7m_1^* + 7\beta_1^* + 5m_1^* + 5\beta_1^*, 7m_1^* + 7\beta_1^* - 6m_1^* - 6\beta_1^*\} - 7m_1^* \end{aligned} \right]_{LR} \\ &\left[\begin{aligned} &8m_1', 8m_1' - \min\{8m_1' - 8\alpha_1' - 4m_1' + 4\alpha_1', 8m_1' - 8\alpha_1' + 1m_1' - 1\alpha_1'\}, \\ &\max\{8m_1' + 8\beta_1' + 1m_1' + 1\beta_1', 8m_1' + 8\beta_1' - 4m_1' - 4\beta_1'\} - 8m_1' \end{aligned} \right]_{LR} \end{aligned} \right\rangle \oplus \\ &\left\langle \begin{aligned} &\left[\begin{aligned} &6m_2^*, 6m_2^* - \min\{6m_2^* - 6\alpha_2^* - 2m_2^* + 2\alpha_2^*, 6m_2^* - 6\alpha_2^* + 3m_2^* - 3\alpha_2^*\}, \\ &\max\{6m_2^* + 6\beta_2^* + 3m_2^* + 3\beta_2^*, 6m_2^* + 6\beta_2^* - 2m_2^* - 2\beta_2^*\} - 6m_2^* \end{aligned} \right]_{LR} \\ &\left[\begin{aligned} &8m_2', 8m_2' - \min\{8m_2' - 8\alpha_2' - 7m_2' + 7\alpha_2', 8m_2' - 8\alpha_2' + 2m_2' - 2\alpha_2'\}, \\ &\max\{8m_2' + 8\beta_2' + 2m_2' + 2\beta_2', 8m_2' + 8\beta_2' - 7m_2' - 7\beta_2'\} - 8m_2' \end{aligned} \right]_{LR} \end{aligned} \right\rangle \end{aligned}$$

$$\begin{aligned}
& \left\langle \left(\begin{bmatrix} 12m_1^*, 12m_1^* - \min\{12m_1^* - 12\alpha_1^* - 6m_1^* + 6\alpha_1^*, 12m_1^* - 12\alpha_1^* + 2m_1^* - 2\alpha_1^*\} \\ \max\{12m_1^* + 12\beta_1^* + 2m_1^* + 2\beta_1^*, 12m_1^* + 12\beta_1^* - 6m_1^* - 6\beta_1^*\} - 12m_1^* \end{bmatrix}_{LR} \right) \oplus \right. \\
& \left. \left(\begin{bmatrix} 10m_1', 10m_1' - \min\{10m_1' - 10\alpha_1' - 8m_1' + 8\alpha_1', 10m_1' - 10\alpha_1' + 2m_1' - 2\alpha_1'\} \\ \max\{10m_1' + 10\beta_1' + 2m_1' + 2\beta_1', 10m_1' + 10\beta_1' - 8m_1' - 8\beta_1'\} - 10m_1' \end{bmatrix}_{LR} \right) \right\rangle \oplus \\
& \left\langle \begin{bmatrix} -9m_2^*, 9\beta_2^* + 3m_2^* + 3\beta_2^*, 9\alpha_2^* + 1m_2^* - 1\alpha_2^* \\ -7m_2', 7\beta_2' + 4m_2' + 4\beta_2', 7\alpha_2' + 2m_2' - 2\alpha_2' \end{bmatrix}_{LR} \right\rangle = \langle [-39, 135, 141]_{LR}, [-10, 116, 74]_{LR} \rangle, \\
& \alpha_1^* \geq 0, \\
& \beta_1^* \geq 0, \\
& \alpha_1' \geq 0, \\
& \beta_1' \geq 0, \\
& \alpha_2^* \geq 0, \\
& \beta_2^* \geq 0, \\
& \alpha_2' \geq 0, \\
& \beta_2' \geq 0
\end{aligned} \tag{213}$$

Step 3: using arithmetic operations (12), the LR-type FBFLPP converts as

Minimize

$$\left\langle \begin{bmatrix} 12m_1^* + 10m_2^*, 12m_1^* - \min\{5m_1^* - 5\alpha_1^*, 17m_1^* - 17\alpha_1^*\} \\ +10m_2^* - \min\{5m_2^* - 5\alpha_2^*, 12m_2^* - 12\alpha_2^*\} \\ \max\{17m_1^* + 17\beta_1^*, 5m_1^* + 5\beta_1^*\} \\ -12m_1^* + \max\{12m_2^* + 12\beta_2^*, 5m_2^* + 5\beta_2^*\} - 10m_2^* \end{bmatrix}_{LR} \right\rangle, \tag{214}$$

$$\left\langle \begin{bmatrix} 11m_1' + 14m_2', 11m_1' - \min\{2m_1' - 2\alpha_1', 14m_1' - 14\alpha_1'\} \\ +14m_2' - \min\{6m_2' - 6\alpha_2', 16m_2' - 16\alpha_2'\} \\ \max\{14m_1' + 14\beta_1', 2m_1' + 2\beta_1'\} \\ -11m_1' + \max\{16m_2' + 16\beta_2', 6m_2' + 6\beta_2'\} - 14m_2' \end{bmatrix}_{LR} \right\rangle$$

subject to

$$\left\langle \begin{bmatrix} 7m_1^* + 6m_2^*, 7m_1^* - \min\{1m_1^* - 1\alpha_1^*, 12m_1^* - 12\alpha_1^*\} \\ +6m_2^* - \min\{4m_2^* - 4\alpha_2^*, 9m_2^* - 9\alpha_2^*\} \\ \max\{12m_1^* + 12\beta_1^*, 1m_1^* + 1\beta_1^*\} \\ -7m_1^* + \max\{9m_2^* + 9\beta_2^*, 4m_2^* + 4\beta_2^*\} - 6m_2^* \end{bmatrix}_{LR} \right\rangle$$

$$\left\langle \begin{bmatrix} 8m_1' + 8m_2', 8m_1' - \min\{4m_1' - 4\alpha_1', 9m_1' - 9\alpha_1'\} \\ +8m_2' - \min\{1m_2' - 1\alpha_2', 10m_2' - 10\alpha_2'\}, \\ \max\{9m_1' + 9\beta_1', 4m_1' + 4\beta_1'\} \\ -8m_1' + \max\{10m_2' + 10\beta_2', 1m_2' + 1\beta_2'\} - 8m_2' \end{bmatrix}_{LR} \right\rangle$$

$$\begin{aligned}
&= \langle [101, 86, 151]_{LR}, [128, 112, 55]_{LR} \rangle, \\
&\left\langle \left(\begin{bmatrix} 12m_1^* - 9m_2^*, 12m_1^* - \min\{6m_1^* - 6\alpha_1^*, 14m_1^* - 14\alpha_1^*\} + 12\beta_2^* + 3m_2^*, \\ \max\{14m_1^* + 14\beta_1^*, 6m_1^* + 6\beta_1^*\} - 12m_1^* + 8\alpha_2^* + 1m_2^* \end{bmatrix}_{LR} \right) \right\rangle \\
&\left(\begin{bmatrix} 10m_1' - 7m_2', 10m_1' - \min\{2m_1' - 2\alpha_1', 12m_1' - 12\alpha_1'\} + 11\beta_2' + 4m_2' \\ \max\{12m_1' + 12\beta_1', 2m_1' + 2\beta_1'\} - 10m_1' + 5\alpha_2' + 2m_2' \end{bmatrix}_{LR} \right) \rangle \\
&= \langle [-39, 135, 141]_{LR}, [-10, 116, 74]_{LR} \rangle, \\
&\alpha_1^* \geq 0, \\
&\beta_1^* \geq 0, \\
&\alpha_1' \geq 0, \\
&\beta_1' \geq 0, \\
&\alpha_2^* \geq 0, \\
&\beta_2^* \geq 0, \\
&\alpha_2' \geq 0, \\
&\beta_2' \geq 0
\end{aligned} \tag{215}$$

Step 4: by applying ranking function and using
 $\min(p, q) = ((p+q)/2) - |(p-q)/2|$ and
 $\max(p, q) = ((p+q)/2) + |(p-q)/2|$, the LR-type
 FBLPP converts as

Minimize

$$\mathfrak{R} \left\langle \left(\begin{bmatrix} 12m_1^* + 10m_2^*, 12m_1^* - \frac{22m_1^* - 22\alpha_1^*}{2} + \left| \frac{-12m_1^* + 12\alpha_1^*}{2} \right| \\ + 10m_2^* - \frac{17m_2^* - 17\alpha_2^*}{2} + \left| \frac{-7m_2^* + 7\alpha_2^*}{2} \right|, \\ \left[\frac{22m_1^* + 22\beta_1^*}{2} + \left| \frac{12m_1^* + 12\beta_1^*}{2} \right| - 12m_1^* + \frac{17m_2^* + 17\beta_2^*}{2} + \left| \frac{7m_2^* + 7\beta_2^*}{2} \right| - 10m_2^* \right]_{LR} \end{bmatrix}, \right. \tag{216}$$

subject to

$$\begin{aligned}
& \left(\left[\begin{array}{l} 7m_1^* + 6m_2^*, 7m_1^* - \left\{ \frac{13m_1^* - 13\alpha_1^*}{2} - \left| \frac{-11m_1^* + 11\alpha_1^*}{2} \right| \right\} \\ + 6m_2^* - \left\{ \frac{13m_2^* - 13\alpha_2^*}{2} - \left| \frac{-5m_2^* + 5\alpha_2^*}{2} \right| \right\}, \\ \left\{ \frac{13m_1^* + 13\beta_1^*}{2} + \left| \frac{11m_1^* + 11\beta_1^*}{2} \right| \right\} - 7m_1^* + \left\{ \frac{13m_2^* + 13\beta_2^*}{2} + \left| \frac{5m_2^* + 5\beta_2^*}{2} \right| \right\} - 6m_2^* \end{array} \right]_{\text{LR}} \right) \\
& \left(\left[\begin{array}{l} 8m_1' + 8m_2', 8m_1' - \left\{ \frac{13m_1' - 13\alpha_1'}{2} - \left| \frac{-5m_1' + 5\alpha_1'}{2} \right| \right\} \\ + 8m_2' - \left\{ \frac{11m_2' - 11\alpha_2'}{2} - \left| \frac{-9m_2' + 9\alpha_2'}{2} \right| \right\}, \\ \left\{ \frac{13m_1' + 13\beta_1'}{2} + \left| \frac{5m_1' + 5\beta_1'}{2} \right| \right\} - 8m_1' + \left\{ \frac{11m_2' + 11\beta_2'}{2} + \left| \frac{9m_2' + 9\beta_2'}{2} \right| \right\} - 8m_2' \end{array} \right]_{\text{LR}} \right) \\
& = \langle [101, 86, 151]_{\text{LR}}, [128, 112, 55]_{\text{LR}} \rangle, \\
& \left(\left[\begin{array}{l} 12m_1^* - 9m_2^*, 12m_1^* - \left\{ \frac{20m_1^* - 20\alpha_1^*}{2} - \left| \frac{-8m_1^* + 8\alpha_1^*}{2} \right| \right\} + 12\beta_2^* + 3m_2^* \\ \left\{ \frac{20m_1^* + 20\beta_1^*}{2} + \left| \frac{8m_1^* + 8\beta_1^*}{2} \right| \right\} - 12m_1^* + 8\alpha_2^* + 1m_2^* \end{array} \right]_{\text{LR}} \right) \\
& \left(\left[\begin{array}{l} 10m_1' - 7m_2', 10m_1' - \left\{ \frac{14m_1' - 14\alpha_1'}{2} - \left| \frac{-10m_1' + 10\alpha_1'}{2} \right| \right\} + 11\beta_2' + 4m_2', \\ \left\{ \frac{14m_1' + 14\beta_1'}{2} + \left| \frac{10m_1' + 10\beta_1'}{2} \right| \right\} - 10m_1' + 5\alpha_2' + 2m_2' \end{array} \right]_{\text{LR}} \right) \\
& = \langle [-39, 135, 141]_{\text{LR}}, [-10, 116, 74]_{\text{LR}} \rangle, \\
& \alpha_1^* \geq 0, \\
& \beta_1^* \geq 0, \\
& \alpha_1' \geq 0, \\
& \beta_1' \geq 0, \\
& \alpha_2^* \geq 0, \\
& \beta_2^* \geq 0, \\
& \alpha_2' \geq 0, \\
& \beta_2' \geq 0
\end{aligned} \tag{217}$$

Step 5: using ranking (2.11) for LR-type BFNs, the LR-type FBFLPP converts to a crisp non-LPP as

Minimize

$$\left(\frac{23}{4}m_1^* + \frac{37}{8}m_2^* - \frac{5}{8}\alpha_1^* - \frac{5}{8}\alpha_2^* + \frac{17}{8}\beta_1^* + \frac{3}{2}\beta_2^* + \frac{19}{4}m_1' + \frac{25}{4}m_2' - \frac{1}{4}\alpha_1' - \frac{3}{4}\alpha_2' + \frac{7}{4}\beta_1' + 2\beta_2' \right), \quad (218)$$

subject to

$$7m_1^* + 6m_2^* = 101,$$

$$7m_1^* - \left\{ \frac{13m_1^* - 13\alpha_1^*}{2} - \left| \frac{-11m_1^* + 11\alpha_1^*}{2} \right| \right\} + 6m_2^* - \left\{ \frac{13m_2^* - 13\alpha_2^*}{2} - \left| \frac{-5m_2^* + 5\alpha_2^*}{2} \right| \right\} = 86,$$

$$\left\{ \frac{13m_1^* + 13\beta_1^*}{2} + \left| \frac{11m_1^* + 11\beta_1^*}{2} \right| \right\} - 7m_1^* + \left\{ \frac{13m_2^* + 13\beta_2^*}{2} + \left| \frac{5m_2^* + 5\beta_2^*}{2} \right| \right\} - 6m_2^* = 151,$$

$$8m_1' + 8m_2' = 128,$$

$$8m_1' - \left\{ \frac{13m_1' - 13\alpha_1'}{2} - \left| \frac{-5m_1' + 5\alpha_1'}{2} \right| \right\} + 8m_2' - \left\{ \frac{11m_2' - 11\alpha_2'}{2} - \left| \frac{-9m_2' + 9\alpha_2'}{2} \right| \right\} = 112,$$

$$\left\{ \frac{13m_1' + 13\beta_1'}{2} + \left| \frac{5m_1' + 5\beta_1'}{2} \right| \right\} - 8m_1' + \left\{ \frac{11m_2' + 11\beta_2'}{2} + \left| \frac{9m_2' + 9\beta_2'}{2} \right| \right\} - 8m_2' = 55,$$

$$12m_1^* - 9m_2^* = -39,$$

$$12m_1^* - \left\{ \frac{20m_1^* - 20\alpha_1^*}{2} - \left| \frac{-8m_1^* + 8\alpha_1^*}{2} \right| \right\} + 12\beta_2^* + 3m_2^* = 135,$$

$$\left\{ \frac{20m_1^* + 20\beta_1^*}{2} + \left| \frac{8m_1^* + 8\beta_1^*}{2} \right| \right\} - 12m_1^* + 8\alpha_2^* + 1m_2^* = 141,$$

$$10m_1' - 7m_2' = -10,$$

$$10m_1' - \left\{ \frac{14m_1' - 14\alpha_1'}{2} - \left| \frac{-10m_1' + 10\alpha_1'}{2} \right| \right\} + 11\beta_2' + 4m_2' = 116,$$

$$\left\{ \frac{14m_1' + 14\beta_1'}{2} + \left| \frac{10m_1' + 10\beta_1'}{2} \right| \right\} - 10m_1' + 5\alpha_2' + 2m_2' = 74,$$

$$\alpha_1^* \geq 0,$$

$$\beta_1^* \geq 0,$$

$$\alpha_1' \geq 0,$$

$$\beta_1' \geq 0,$$

$$\alpha_2^* \geq 0,$$

$$\beta_2^* \geq 0,$$

$$\alpha_2' \geq 0,$$

$$\beta_2' \geq 0$$

(219)

Step 6: minimize

$$\left(\frac{23}{4}m_1^* + \frac{37}{8}m_2^* - \frac{5}{8}\alpha_1^* - \frac{5}{8}\alpha_2^* + \frac{17}{8}\beta_1^* + \frac{3}{2}\beta_2^* + \frac{19}{4}m_1' + \frac{25}{4}m_2' - \frac{1}{4}\alpha_1' - \frac{3}{4}\alpha_2' + \frac{7}{4}\beta_1' + 2\beta_2' \right), \quad (220)$$

subject to

$$7m_1^* + 6m_2^* = 101,$$

$$\frac{1}{2}m_1^* - \frac{1}{2}m_2^* + \frac{13}{2}\alpha_1^* + \frac{13}{2}\alpha_2^* + \left| \frac{-11m_1^* + 11\alpha_1^*}{2} \right| + \left| \frac{-5m_2^* + 5\alpha_2^*}{2} \right| = 86,$$

$$-\frac{1}{2}m_1^* + \frac{1}{2}m_2^* + \frac{13}{2}\beta_1^* + \frac{13}{2}\beta_2^* + \left| \frac{11m_1^* + 11\beta_1^*}{2} \right| + \left| \frac{5m_2^* + 5\beta_2^*}{2} \right| = 151,$$

$$8m_1' + 8m_2' = 128,$$

$$\frac{3}{2}m_1' + \frac{5}{2}m_2' + \frac{13}{2}\alpha_1' + \frac{11}{2}\alpha_2' + \left| \frac{-5m_1' + 5\alpha_1'}{2} \right| + \left| \frac{-9m_2' + 9\alpha_2'}{2} \right| = 112,$$

$$-\frac{3}{2}m_1' - \frac{5}{2}m_2' + \frac{13}{2}\beta_1' + \frac{11}{2}\beta_2' + \left| \frac{5m_1' + 5\beta_1'}{2} \right| + \left| \frac{9m_2' + 9\beta_2'}{2} \right| = 55,$$

$$12m_1^* - 9m_2^* = -39,$$

$$2m_1^* + 3m_2^* + 10\alpha_1^* + 12\beta_2^* + \left| \frac{-8m_1^* + 8\alpha_1^*}{2} \right| = 135,$$

$$-2m_1^* + 1m_2^* + 8\alpha_2^* + 10\beta_1^* + \left| \frac{8m_1^* + 8\beta_1^*}{2} \right| = 141,$$

(221)

$$10m_1' - 7m_2' = -10,$$

$$3m_1' + 4m_2' + 7\alpha_1' + 11\beta_2' + \left| \frac{-10m_1' + 10\alpha_1'}{2} \right| = 116,$$

$$-3m_1' + 2m_2' + 5\alpha_2' + 7\beta_1' + \left| \frac{10m_1' + 10\beta_1'}{2} \right| = 74,$$

$$\alpha_1^* \geq 0,$$

$$\beta_1^* \geq 0,$$

$$\alpha_1' \geq 0,$$

$$\beta_1' \geq 0,$$

$$\alpha_2^* \geq 0,$$

$$\beta_2^* \geq 0,$$

$$\alpha_2' \geq 0,$$

$$\beta_2' \geq 0$$

Step 7: by using software Maple, Optimal solution of the crisp non-LPP is $m_1^* = 5, \alpha_1^* = 2, \beta_1^* = 4, m_2^* = 11, \alpha_2^* = 8, \beta_2^* = 5, m_1' = 6, \alpha_1' = 3, \beta_1' = 1, m_2' = 10, \alpha_2' = 6, \beta_2' = 2$

Step 8: the exact LR-type BFOS is $\tilde{Y}_1 = \langle (5, 2, 4), (6, 3, 1) \rangle, \tilde{Y}_2 = \langle (11, 8, 5), (10, 6, 2) \rangle$

Step 9: the bipolar fuzzy optimal value of the LR-type fully bipolar fuzzy linear programming problem is $\langle (170, 140, 175), (206, 176, 84) \rangle$

Thus, $\langle (5, 2, 4), (6, 3, 1) \rangle$ units of mutton and $\langle (11, 8, 5), (10, 6, 2) \rangle$ units of beef should be given to each player at a minimum cost of Rs. $\langle (170, 140, 175), (206, 176, 84) \rangle$.

6. Conclusions

Linear programming is applied to optimize an objective function subject to constraints. It has vast application in many fields such as science, marketing, industry, business, agriculture, and telecommunication. In this research article, we have defined LR-type BFNs and their arithmetic operations, and particularly by considering different cases, multiplication of LR-type BFNs is discussed. We have introduced the ranking for LR-type BFNs that transform the LR-type FBFLPP into a crisp linear programming problem. The proposed technique is applied to solve LR-type FBFLPPs with equality constraints involving variables and parameters as LR-type BFNs. The given method has been interpreted with a numerical example and a model. The obtained optimal solution satisfies all the constraints of the LR-type FBFLPP and justifies that the proposed scheme is accurately designed. In future, our scheme can be extended to a complex bipolar fuzzy LPP.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Interval-Valued m -Polar Fuzzy Positive Implicative Ideals in BCK -Algebras

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Interval-Valued m -Polar Fuzzy Positive Implicative Ideals in BCK -Algebras

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In this paper, the notion of interval-valued m -polar fuzzy positive implicative ideals in BCK -algebras is presented. Then, the relationships between interval-valued m -polar fuzzy positive implicative ideals and interval-valued m -polar fuzzy ideals are investigated. After that, the concepts of interval-valued m -polar $(\epsilon, \in \vee q_k)$ -fuzzy positive implicative ideals and interval-valued m -polar $(\epsilon, \in \vee q_k)$ -fuzzy ideals are defined and some equivalent conditions are provided. Furthermore, we show that interval-valued m -polar $(\epsilon, \in \vee q_k)$ -fuzzy positive implicative ideals are interval-valued m -polar $(\epsilon, \in \vee q_k)$ -fuzzy ideals, but the converse need not be true in general and an example is given in this aim.

1. Introduction

As an extension of fuzzy sets, Zadeh [1] defined fuzzy sets with an interval-valued membership function proposing the concept interval-valued fuzzy sets. This concept has been studied from various points of view in different algebraic structures as BCK -algebras and some of its generalization (see, for example, [2–7]), groups (see, for example, [8–10]), and rings (see, for example, [11–13]). Jun [14] studied interval-valued fuzzy ideals in BCI -algebras. Zhan et al. [15, 16] studied $(\epsilon, \in \vee q)$ -fuzzy ideals of BCI -algebras. The concept of “quasi-coincidence” of an interval-valued fuzzy point together with “belongingness” within an interval-valued fuzzy set were used in the studies made by Ma et al. in [17, 18], where they discussed properties of some types of $(\epsilon, \in \vee q)$ -interval-valued fuzzy ideals of BCI -algebras. Also, in [19–24], some more general ideas on bipolar fuzzy sets’ related ideals were considered.

The m -polar fuzzy set, an extension of the bipolar fuzzy set, was introduced by Chen et al. [25] in 2014. When more than one agreement has to work with the m -polar fuzzy model, it offers the system more accuracy, flexibility, and compatibility. The investigation of m -polar fuzzy algebraic

structures started with the idea of textit m -polar fuzzy subalgebras proposed by Akram et al. [26]. Following that, Akram and Farooq [27] in lie subalgebras introduced the theory of m -polar fuzzy lie ideal. A concept proposed by [28] for the m -polar fuzzy subgroups. The notions of m -polar fuzzy ideals and m -polar fuzzy commutative ideals on BCK/BCI -algebras were introduced by Al-Masarwah and Ahmad [29]. The concepts of $(\epsilon, \in \vee q)$ -fuzzy ideals and $(\epsilon, \in \vee q)$ -fuzzy commutative ideals have been considered by Al-Masarwah and Ahmad in [30]. In [31], Muhiuddin et al. introduced and characterized the notion of m -polar (ϵ, \in) -fuzzy q -ideal in BCI -algebras. Takallo et al. [32] proposed the notion of (ϵ, \in) -fuzzy p -ideal in BCI -algebras and studied related properties of m -polar (ϵ, \in) -fuzzy ideals and m -polar (ϵ, \in) -fuzzy p -ideals in BCI -algebras. Recently, by generalizing the concept of m -polar fuzzy positive implicative ideals of BCK -algebras, Al-Masarwah et al. [33] introduced the notions of $(\epsilon, \in \vee q)$ -fuzzy positive implicative ideals and $(\bar{\epsilon}, \bar{\in} \vee \bar{q})$ -fuzzy positive implicative ideals in BCK -algebras. Also, different kinds of concepts, related to this study, were investigated in various ways (see, for example, [34–40]).

In this paper, the notion of interval-valued m -polar fuzzy positive implicative ideals in BCK -algebras is presented. We

prove that every interval-valued m -polar fuzzy positive implicative ideal of BCK -algebras is an interval-valued m -polar fuzzy ideal but the converse statement is not true in general and an example is given in this aim. Moreover, the concepts of interval-valued m -polar $(\in, \in \vee q_{\kappa}^-)$ -fuzzy positive implicative ideals and interval-valued m -polar $(\in, \in \vee q_{\kappa}^-)$ -fuzzy ideals are defined and some equivalent conditions are provided. Furthermore, we show that interval-valued m -polar $(\in, \in \vee q_{\kappa}^-)$ -fuzzy positive implicative ideals are interval-valued m -polar $(\in, \in \vee q_{\kappa}^-)$ -fuzzy ideals, but converse need not be true in general and an example is given in this aim.

2. Preliminaries

An algebra $(\tilde{\mathcal{X}}; *, 0)$ of type $(2, 0)$ is called a BCK -algebra if, for all $\vartheta, \omega, h \in \tilde{\mathcal{X}}$,

- (i) $((\vartheta * \omega) * (\vartheta * h)) \leq (h * \omega)$.
- (ii) $(\vartheta * (\vartheta * \omega)) \leq \omega$.
- (iii) $\vartheta * \vartheta = 0$.
- (iv) $0 * \vartheta = 0$.
- (v) $\vartheta \leq \omega$ and $\omega \leq \vartheta$ imply $\vartheta = \omega$, where \leq can be presented by $\vartheta \leq \omega \Leftrightarrow \vartheta * \omega = 0$. Every BCK -algebra $\tilde{\mathcal{X}}$ satisfies the following axioms, for all $\vartheta, \omega, h \in \tilde{\mathcal{X}}$:
 - (1) $\vartheta * 0 = \vartheta$.
 - (2) $(\vartheta * \omega) * h = (\vartheta * h) * \omega$.

A subset $(\varnothing \neq) A$ of $\tilde{\mathcal{X}}$ is called a subalgebra if, for all $\vartheta, \omega \in \tilde{\mathcal{X}}$, $\vartheta * \omega \in A$ and is called an ideal of $\tilde{\mathcal{X}}$ if $0 \in A$ and, for all $\vartheta, \omega \in \tilde{\mathcal{X}}$, $\vartheta * \omega \in A$, $\omega \in A$ implies $\vartheta \in A$.

$$\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) = ([\mathcal{U}_1^{\mathcal{P}-}(\vartheta), \mathcal{U}_1^{\mathcal{P}+}(\vartheta)], [\mathcal{U}_2^{\mathcal{P}-}(\vartheta), \mathcal{U}_2^{\mathcal{P}+}(\vartheta)], \dots, [\mathcal{U}_m^{\mathcal{P}-}(\vartheta), \mathcal{U}_m^{\mathcal{P}+}(\vartheta)]), \quad (2)$$

for all $\vartheta \in \tilde{\mathcal{X}}$, where $\mathcal{U}_i^{\mathcal{P}-}$ and $\mathcal{U}_i^{\mathcal{P}+}$ are fuzzy sets of $\tilde{\mathcal{X}}$ with $\mathcal{U}_i^{\mathcal{P}-}(\vartheta) \leq \mathcal{U}_i^{\mathcal{P}+}(\vartheta)$, for all $\vartheta \in \tilde{\mathcal{X}}$ and $i \in \{1, 2, \dots, m\}$.

The i^{th} projection map \tilde{q}_i is order preserving and vice versa, i.e.,

$$\vartheta \leq \omega \Leftrightarrow \tilde{q}_i(\vartheta) \leq \tilde{q}_i(\omega), \quad \forall i \in \{1, 2, \dots, m\}. \quad (3)$$

Definition 3 (see [40]). An $IVmPF$ set $\widetilde{\mathcal{U}}^{\mathcal{P}}$ of $\tilde{\mathcal{X}}$ is called an $IVmPF$ ideal of $\tilde{\mathcal{X}}$ if, for any $\vartheta, \omega \in \tilde{\mathcal{X}}$,

- (1) $\widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta)$
- (2) $\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq r \min\{\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega), \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega)\}$

That is,

- (1) $\tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq \tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta)$
- (2) $\tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq r \min\{\tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega), \tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega)\},$
 $\forall i = 1, 2, \dots, m$

Definition 4 (see [40]). The set $\widetilde{\mathcal{U}}^{\mathcal{P}}|_{[\alpha, \beta]}$ is called an $IVmPF$ set of $\tilde{\mathcal{X}}$ if $\{\vartheta \in \tilde{\mathcal{X}} | \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq [\alpha, \beta]\}$, where $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an $IVmPF$ set of $\tilde{\mathcal{X}}$

Definition 1 (see [33]). A subset $(\varnothing \neq) \mathcal{P}$ of $\tilde{\mathcal{X}}$ is called a positive implicative ideal of $\tilde{\mathcal{X}}$ if $\forall \vartheta, \omega, h \in \tilde{\mathcal{X}}$:

- (i) $0 \in \mathcal{P}$
- (ii) $(\vartheta * \omega) * h \in \mathcal{P}$ and $\omega * h \in \mathcal{P}$ imply $\vartheta * h \in \mathcal{P}$

The interval number \tilde{t} is the interval $[t^-, t^+]$, where $0 \leq t^- \leq t^+ \leq 1$, and $D[0, 1]$ is the set of all interval numbers. For the interval numbers $\tilde{t}_i = [t_i^-, t_i^+]$, $\tilde{d}_i = [d_i^-, d_i^+] \in D[0, 1]$, $i \in I$, we describe

- (a) $r \min\{\tilde{t}_i, \tilde{d}_i\} = [\min\{t_i^-, d_i^-\}, \min\{t_i^+, d_i^+\}]$
- (b) $r \max\{\tilde{t}_i, \tilde{d}_i\} = [\min\{t_i^-, d_i^-\}, \min\{t_i^+, d_i^+\}]$
- (c) $\tilde{t}_1 \leq \tilde{t}_2 \Leftrightarrow t_1^- \leq t_2^-$ and $t_1^+ \leq t_2^+$
- (d) $\tilde{t}_1 = \tilde{t}_2 \Leftrightarrow t_1^- = t_2^-$ and $t_1^+ = t_2^+$

A mapping $\widetilde{\mathcal{U}}^{\mathcal{P}}: \tilde{\mathcal{X}} \rightarrow D[0, 1]$ is called an interval-valued fuzzy set of $\tilde{\mathcal{X}}$, where $\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) = [\mathcal{U}^{\mathcal{P}-}(\vartheta), \mathcal{U}^{\mathcal{P}+}(\vartheta)]$, for all $\vartheta \in \tilde{\mathcal{X}}$, where $\mathcal{U}^{\mathcal{P}-}$ and $\mathcal{U}^{\mathcal{P}+}$ are fuzzy sets of $\tilde{\mathcal{X}}$ with $\mathcal{U}^{\mathcal{P}-}(\vartheta) \leq \mathcal{U}^{\mathcal{P}+}(\vartheta)$, for all $\vartheta \in \tilde{\mathcal{X}}$.

Definition 2. A mapping $\widetilde{\mathcal{U}}^{\mathcal{P}}: \tilde{\mathcal{X}} \rightarrow D[0, 1]^m$ is called an interval-valued m -polar fuzzy set (briefly, $IVmPF$ set) of $\tilde{\mathcal{X}}$ and is defined as

$$\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) = (\tilde{q}_1 \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), \tilde{q}_2 \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), \dots, \tilde{q}_m \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta)), \quad (1)$$

where $\tilde{q}_i: D[0, 1]^m \rightarrow D[0, 1]$ is the i^{th} projection mapping for $i \in \{1, 2, \dots, m\}$. That is,

is called the level cut subset of $\widetilde{\mathcal{U}}^{\mathcal{P}}$, $\forall [\alpha, \beta] = [\alpha_1, \beta_1, [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m]] \in D(0, 1)^m$.

Lemma 1 (see [40]). Every $IVmPF$ ideal $\widetilde{\mathcal{U}}^{\mathcal{P}}$ of $\tilde{\mathcal{X}}$ satisfies the following assertion, $\forall \vartheta, \omega \in \tilde{\mathcal{X}}$:

$$\vartheta \leq \omega \Rightarrow \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega). \quad (4)$$

3. Interval-Valued m -Polar Fuzzy Positive Implicative Ideals

Definition 5. An $IVmPF$ set $\widetilde{\mathcal{U}}^{\mathcal{P}}$ of $\tilde{\mathcal{X}}$ is called an $IVmPFPI$ ideal of $\tilde{\mathcal{X}}$ if, for any $\vartheta, \omega, h \in \tilde{\mathcal{X}}$,

- (1) $\widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta)$
- (2) $\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq r \min\{\widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h)\}$

That is,

$$(1) \tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq \tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta)$$

$$(2) \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq r \min \left\{ \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega * h)), \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\omega * h) \right\}, \forall i = 1, 2, \dots, m$$

Example 1. Consider a BCK-algebra $\tilde{\mathcal{X}} = \{0, 1, 2, 3, 4\}$ with the Cayley table (Table 1).

Let $\tilde{\mathcal{U}}^{\mathcal{P}}$ be an IV4PF set defined as

$$\tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) = \begin{cases} ([0.7, 0.8], [0.4, 0.5], [0.9, 0.1], [0.7, 0.8]), & \text{if } \vartheta = 0, \\ ([0.6, 0.7], [0.3, 0.4], [0.8, 0.9], [0.6, 0.7]), & \text{if } \vartheta = 1, \\ ([0.5, 0.6], [0.2, 0.3], [0.6, 0.7], [0.5, 0.6]), & \text{if } \vartheta = 2, \\ ([0.4, 0.5], [0.1, 0.2], [0.3, 0.4], [0.4, 0.5]), & \text{if } \vartheta = 3, \\ ([0.3, 0.4], [0.2, 0.3], [0.5, 0.6], [0.3, 0.4]), & \text{if } \vartheta = 4. \end{cases} \quad (5)$$

It is straightforward to check that $\tilde{\mathcal{U}}^{\mathcal{P}}$ is an IV4PFPI ideal of $\tilde{\mathcal{X}}$.

Theorem 1. Every IV4PFPI ideal of $\tilde{\mathcal{X}}$ is an IVmPF ideal of $\tilde{\mathcal{X}}$.

Proof. Let $\tilde{\mathcal{U}}^{\mathcal{P}}$ be an IV4PFPI ideal of $\tilde{\mathcal{X}}$. Then, condition (1) of Definition 5 holds. By assumption, we have

$$\tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq r \min \left\{ \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega * h)), \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\omega * h) \right\}. \quad (6)$$

Put $h = 0$, so

$$\tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq r \min \left\{ \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega), \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\omega) \right\}. \quad (7)$$

Hence, $\tilde{\mathcal{U}}^{\mathcal{P}}$ is an IVmPF ideal of $\tilde{\mathcal{X}}$.

As shown by the following example, the converse of the preceding Theorem 1 is not valid in general. \square

Example 2. Consider a BCK-algebra $\tilde{\mathcal{X}} = \{0, 1, 2, 3\}$ with the Cayley table (Table 2).

Now, define an IV3PF set $\tilde{\mathcal{U}}^{\mathcal{P}}$ as follows:

$$\tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) = \begin{cases} ([0.6, 0.7], [0.6, 0.7], [0.9, 0.9]), & \text{if } \vartheta = 0, \\ ([0.5, 0.6], [0.5, 0.6], [0.8, 0.8]), & \text{if } \vartheta = 1, 2, \\ ([0.3, 0.3], [0.3, 0.3], [0.3, 0.3]), & \text{if } \vartheta = 3. \end{cases} \quad (8)$$

It is straightforward to check that $\tilde{\mathcal{U}}^{\mathcal{P}}$ is an IV3PF ideal of $\tilde{\mathcal{X}}$, but it is not an IV3PFPI ideal of $\tilde{\mathcal{X}}$ since $\tilde{q}_1 \circ \tilde{\mathcal{U}}^{\mathcal{P}}(2 * 1) = \tilde{q}_1 \circ \tilde{\mathcal{U}}^{\mathcal{P}}(1) = [0.5, 0.6] < r \min \left\{ \tilde{q}_1 \circ \tilde{\mathcal{U}}^{\mathcal{P}}((2 * 1) * 1), \tilde{q}_1 \circ \tilde{\mathcal{U}}^{\mathcal{P}}(1 * 1) \right\} = r \min \left\{ \tilde{q}_1 \circ \tilde{\mathcal{U}}^{\mathcal{P}}(0), \tilde{q}_1 \circ \tilde{\mathcal{U}}^{\mathcal{P}}(0) \right\} = \tilde{q}_1 \circ \tilde{\mathcal{U}}^{\mathcal{P}}(0) = [0.6, 0.7]$.

Theorem 2. An IVmPF set of $\tilde{\mathcal{X}}$ is an IVmPFPI ideal of $\tilde{\mathcal{X}}$ if and only if it is an IVmPF ideal of $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega) \geq \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \omega) \forall \vartheta, \omega \in \tilde{\mathcal{X}}$.

Proof. (\Rightarrow) Suppose $\tilde{\mathcal{U}}^{\mathcal{P}}$ is an IVmPFPI ideal of $\tilde{\mathcal{X}}$. By Theorem 1, $\tilde{\mathcal{U}}^{\mathcal{P}}$ is an IVmPF ideal of $\tilde{\mathcal{X}}$. By assumption, we have

$$\tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq r \min \left\{ \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\omega * h) \right\}. \quad (9)$$

Now, replace h by ω ; then,

$$\begin{aligned} \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega) &\geq r \min \left\{ \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \omega), \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\omega * \omega) \right\} \\ &= r \min \left\{ \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \omega), \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(0) \right\} \\ &= \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \omega), \end{aligned} \quad (10)$$

$\forall \vartheta, \omega \in \tilde{\mathcal{X}}$.

(\Leftarrow) Suppose that $\tilde{\mathcal{U}}^{\mathcal{P}}$ is an IVmPF ideal of $\tilde{\mathcal{X}}$. Then, condition (1) of Definition 5 holds. As $((\vartheta * h) * h) * (\omega * h) \leq (\vartheta * h) * \omega = (\vartheta * \omega) * h \forall \vartheta, \omega \in \tilde{\mathcal{X}}$, so by Lemma 1, we have

$$\tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(((\vartheta * h) * h) * (\omega * h)) \geq \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h). \quad (11)$$

Now, by assumption,

$$\begin{aligned} \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) &\geq \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * h) * h) \\ &\geq r \min \left\{ \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(((\vartheta * h) * h) * (\omega * h)), \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\omega * h) \right\} \\ &\geq r \min \left\{ \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\omega * h) \right\}. \end{aligned} \quad (12)$$

Hence, $\tilde{\mathcal{U}}^{\mathcal{P}}$ is an IVmPFPI ideal of $\tilde{\mathcal{X}}$. \square

Theorem 3. An IVmPF set $\tilde{\mathcal{U}}^{\mathcal{P}}$ of $\tilde{\mathcal{X}}$ is an IVmPFPI ideal of $\tilde{\mathcal{X}}$ if and only if $\tilde{\mathcal{U}}^{\mathcal{P}} \sim_{[\alpha, \beta]} \neq \emptyset$ is a positive implicative ideal of $\tilde{\mathcal{X}}$, $\forall [\alpha, \beta] = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m]) \in D(0, 1)^m$.

Proof. (\Rightarrow) Suppose that $\tilde{\mathcal{U}}^{\mathcal{P}}$ is an IVmPFPI ideal of $\tilde{\mathcal{X}}$. Let $[\alpha, \beta] = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m]) \in D(0, 1)^m$ be such that $\vartheta \in \tilde{\mathcal{U}}^{\mathcal{P}} \sim_{[\alpha, \beta]}$. Then, $\tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(0) \geq \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq [\alpha_i, \beta_i]$, and we have $0 \in \tilde{\mathcal{U}}^{\mathcal{P}} \sim_{[\alpha, \beta]}$. Let $\vartheta, \omega, h \in \tilde{\mathcal{X}}$ be such that $(\vartheta * \omega) * h \in \tilde{\mathcal{U}}^{\mathcal{P}} \sim_{[\alpha, \beta]}$ and $\omega * h \in \tilde{\mathcal{U}}^{\mathcal{P}} \sim_{[\alpha, \beta]}$. Then, $\tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h) \geq [\alpha_i, \beta_i]$ and $\tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\omega * h) \geq [\alpha_i, \beta_i]$. It follows from Definition 5 (2) that

$$\begin{aligned} \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) &\geq r \min \left\{ \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \tilde{q}_i \circ \tilde{\mathcal{U}}^{\mathcal{P}}(\omega * h) \right\} \\ &\geq [\alpha_i, \beta_i]. \end{aligned} \quad (13)$$

Thus, $\vartheta * h \in \tilde{\mathcal{U}}^{\mathcal{P}} \sim_{[\alpha, \beta]}$. Hence, $\tilde{\mathcal{U}}^{\mathcal{P}} \sim_{[\alpha, \beta]}$ is a positive implicative ideal of $\tilde{\mathcal{X}}$.

TABLE 1: Cayley table of the binary operation*.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	4	4	4	0

(\Leftarrow) Assume that $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is a positive implicative ideal of $\widetilde{\mathcal{X}}$, $\forall [\alpha, \beta] = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m]) \in D(0, 1)^m$. If $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) < \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(t)$ for some $t \in \widetilde{\mathcal{X}}$, then $\exists [\theta, \lambda] = ([\theta_1, \lambda_1], [\theta_2, \lambda_2], \dots, [\theta_m, \lambda_m]) \in D(0, 1)^m$ such that $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) < [\theta_i, \lambda_i] \leq \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(t)$. It implies that $0 \in \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\theta, \lambda]}$, a contradiction. Thus, $\widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta)$, $\forall \vartheta \in \widetilde{\mathcal{X}}$.

Again, if $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) < r \min\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h)\}$, for some $\vartheta, \omega, h \in \widetilde{\mathcal{X}}$; then, $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) < [\rho_i, \sigma_i] \leq r \min\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h)\}$, (14)

for some $[\rho, \sigma] = ([\rho_1, \sigma_1], [\rho_2, \sigma_2], \dots, [\rho_m, \sigma_m]) \in D(0, 1)^m$. It follows that $(\vartheta * \omega) * h \in \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]}$ and $\omega * h \in \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]}$, but $\vartheta * h \notin \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]}$. This is a contradiction. Thus, $\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq r \min\{\widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h)\} \forall \vartheta, \omega, h \in \widetilde{\mathcal{X}}$. Hence, $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an IVmPFPI ideal of $\widetilde{\mathcal{X}}$. \square

4. m-Polar $(\in, \in \vee q_k)$ -Fuzzy Positive Implicative Ideals

An IVmPF set $\widetilde{\mathcal{U}}^{\mathcal{P}}$ of $\widetilde{\mathcal{X}}$ of the form

$$\mathcal{U}(h) = \begin{cases} ([0.9, 0.8], [0.8, 0.7], [0.7, 0.6], [0.6, 0.5], [0.5, 0.4]), & \text{if } h = 0, \\ ([0.6, 0.5], [0.5, 0.4], [0.4, 0.3], [0.3, 0.2], [0.2, 0.1]), & \text{if } h \in \{1, 2, 3\}. \end{cases} \quad (16)$$

Choose $k = [0.9, 0.9]$. Then, with direct computation, we find that $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an IV5P $(\in, \in \vee q_k)$ -F ideal of $\widetilde{\mathcal{X}}$.

Theorem 4. An IVmPF set $\widetilde{\mathcal{U}}^{\mathcal{P}}$ of $\widetilde{\mathcal{X}}$ is an IVmP $(\in, \in \vee q_k)$ -F ideal of $\widetilde{\mathcal{X}} \Leftrightarrow$

- (1) $\widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq r \min\{\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), ((\widetilde{1} - \widetilde{k})/2)\}$
- (2) $\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq r \min\{\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h), \widetilde{\mathcal{U}}^{\mathcal{P}}(h), ((\widetilde{1} - \widetilde{k})/2)\}$, $\forall \vartheta, \omega, h \in \widetilde{\mathcal{X}}$

Proof. (\Rightarrow) Suppose, on the contrary, that $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) < r \min\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), ((1 - k_i)/2)\}$; then,

$$\widetilde{\mathcal{U}}^{\mathcal{P}}(h) = \begin{cases} [\alpha, \beta] = [\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m] \in D(0, 1)^m, & \text{if } h = \vartheta, \\ \widetilde{0} = ([0, 0], [0, 0], \dots, [0, 0]), & \text{if } h \neq \vartheta, \end{cases} \quad (15)$$

is called an IVmPF point, denoted as $\vartheta \sim_{[\alpha, \beta]}$, with support ϑ and value $[\alpha, \beta] = [\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m]$. An IVmPF point $\vartheta \sim_{[\alpha, \beta]}$

- (1) Belongs to $\widetilde{\mathcal{U}}^{\mathcal{P}}$, written as $\vartheta \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$, if $\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq [\alpha, \beta]$, i.e., $\forall i = 1, 2, \dots, m$, $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq [\alpha_i, \beta_i]$
- (2) Is quasi-coincidence with $\widetilde{\mathcal{U}}^{\mathcal{P}}$, written as $\vartheta \sim_{[\alpha, \beta]} q_k \widetilde{\mathcal{U}}^{\mathcal{P}}$, if $\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) + [\alpha, \beta] + \widetilde{k} > \widetilde{1}$, i.e., $\forall i = 1, 2, \dots, m$, $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) + [\alpha_i, \beta_i] + [\kappa_i^+, \kappa_i^-] > 1$, where $\widetilde{k} = (\kappa_1, \kappa_2, \dots, \kappa_m)$ and $\widetilde{1} = ([1, 1], [1, 1], \dots, [1, 1])$ in which $\kappa_i = [\kappa_i^+, \kappa_i^-]$ and $1 = [1, 1]$

Assume $\widetilde{0} \leq \widetilde{k} < \widetilde{1}$. We write

- (1) $\vartheta \sim_{[\alpha, \beta]} \not\sim \widetilde{\mathcal{U}}^{\mathcal{P}}$ if $\vartheta \sim_{[\alpha, \beta]} \not\sim \widetilde{\mathcal{U}}^{\mathcal{P}}$ does not hold
- (2) $\vartheta \sim_{[\alpha, \beta]} \in \vee q_k \widetilde{\mathcal{U}}^{\mathcal{P}}$ (resp. $\vartheta \sim_{[\alpha, \beta]} \in \wedge q_k \widetilde{\mathcal{U}}^{\mathcal{P}}$) if $\vartheta \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$ or $\vartheta \sim_{[\alpha, \beta]} q_k \widetilde{\mathcal{U}}^{\mathcal{P}}$ (resp. $\vartheta \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$ and $\vartheta \sim_{[\alpha, \beta]} q_k \widetilde{\mathcal{U}}^{\mathcal{P}}$)

Definition 6. An IVmPF set $\widetilde{\mathcal{U}}^{\mathcal{P}}$ of a BCK-algebra $\widetilde{\mathcal{X}}$ is called an IVmP $(\in, \in \vee q_k)$ -F ideal of $\widetilde{\mathcal{X}}$ if

- (1) $\vartheta \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$ implies $0 \sim_{[\alpha, \beta]} \in \vee q_k \widetilde{\mathcal{U}}^{\mathcal{P}}$
- (2) $(\vartheta * h) \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$ and $h \sim_{[\rho, \sigma]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$ imply $\vartheta \sim_{r \min\{[\alpha, \beta], [\rho, \sigma]\}} \in \vee q_k \widetilde{\mathcal{U}}^{\mathcal{P}}$, $\forall \vartheta, \omega, h \in \widetilde{\mathcal{X}}$, and $[\alpha, \beta], [\rho, \sigma] \in D(0, 1)^m$

Example 3. Consider a BCK-algebra $\widetilde{\mathcal{X}} = \{0, 1, 2, 3\}$ with the Cayley table (Table 3).

Define an IV5PF set $\widetilde{\mathcal{U}}^{\mathcal{P}}$: $\widetilde{\mathcal{X}} \rightarrow D[0, 1]^5$ as

$\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) < [\alpha_i, \beta_i] \leq r \min\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), (1 - k_i/2)\}$ for some $[\alpha, \beta] = [\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m] \in D(0, 1)^m$ and $1 \leq i \leq m$. This implies that $\vartheta \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$, but $0 \sim_{[\alpha, \beta]} \notin \widetilde{\mathcal{U}}^{\mathcal{P}}$, a contradiction. Thus, $\widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq r \min\{\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), ((\widetilde{1} - \widetilde{k})/2)\}$.

Again, suppose the contrary that $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) < r \min\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(h), ((1 - k_i)/2)\}$. Then, $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) < [\alpha_i, \beta_i] \leq r \min\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(h), ((1 - k_i)/2)\}$ for some $[\alpha, \beta] = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m]) \in D(0, 1)^m$. This implies that

TABLE 2: Cayley table of the binary operation*.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

$(\vartheta * h) \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$ and $h \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$, but $\vartheta \sim_{[\alpha, \beta]} \in \overline{\vee q_k} \widetilde{\mathcal{U}}^{\mathcal{P}}$,
a contradiction. Hence,

$\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq r \min\{\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h), \widetilde{\mathcal{U}}^{\mathcal{P}}(h), ((\tilde{1} - \tilde{k})/2)\}.$
(\Leftarrow) Suppose that $h \in \widetilde{\mathcal{X}}$ such that $h \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$. Then,
 $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(h) \geq [\alpha_i, \beta_i]$. So,

$$\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq \min\left\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(h), \frac{1 - \kappa_i}{2}\right\} \geq \min\left\{[\alpha_i, \beta_i], \frac{1 - \kappa_i}{2}\right\}. \quad (17)$$

Now, if $[\alpha_i, \beta_i] \leq ((1 - \kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq [\alpha_i, \beta_i]$.
Therefore, $0 \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$. On the contrary, if
 $[\alpha_i, \beta_i] > ((1 - \kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq ((1 - \kappa_i)/2)$. So,
 $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) + [\alpha_i, \beta_i] > ((1 - \kappa_i)/2) + ((1 - \kappa_i)/2) = 1 - \kappa_i$.
This implies that $0 \sim_{[\alpha, \beta]} \in \overline{\vee q_k} \widetilde{\mathcal{U}}^{\mathcal{P}}$.

Hence, $0 \sim_{[\alpha, \beta]} \in \overline{\vee q_k} \widetilde{\mathcal{U}}^{\mathcal{P}}$. Let $(\vartheta * h) \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$ and
 $h \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$, $\forall [\theta, \lambda] = ([\theta_1, \lambda_1], [\theta_2, \lambda_2], \dots, [\theta_m, \lambda_m]),$
 $[\rho, \sigma] = ([\rho_1, \sigma_1], [\rho_2, \sigma_2], \dots, [\rho_m, \sigma_m]) \in D(0, 1)^m$. Then,
 $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq [\theta_i, \lambda_i]$ and $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(h) \geq [\rho_i, \sigma_i]$. Thus,

$$\begin{aligned} \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) &\geq r \min\left\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(h), \frac{1 - \kappa_i}{2}\right\}, \\ &\geq r \min\left\{[\theta_i, \lambda_i], [\rho_i, \sigma_i], \frac{1 - \kappa_i}{2}\right\}. \end{aligned} \quad (18)$$

Now, if $r \min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\} \leq ((1 - \kappa_i)/2)$, then
 $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq r \min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\}$ implies $\vartheta \sim_{r \min\{[\theta, \lambda], [\rho, \sigma]\}} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$;
otherwise, when $r \min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\} > ((1 - \kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq ((1 - \kappa_i)/2)$. So, we have

$$\begin{aligned} \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) + r \min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\} &> \frac{1 - \kappa_i}{2} + \frac{1 - \kappa_i}{2} \\ &= 1 - \kappa_i. \end{aligned} \quad (19)$$

This implies that $\vartheta \sim_{r \min\{[\theta, \lambda], [\rho, \sigma]\}} \in \overline{\vee q_k} \widetilde{\mathcal{U}}^{\mathcal{P}}$. Hence,
 $\vartheta \sim_{r \min\{[\theta, \lambda], [\rho, \sigma]\}} \in \overline{\vee q_k} \widetilde{\mathcal{U}}^{\mathcal{P}}$, as required. \square

Lemma 2. Let $\widetilde{\mathcal{U}}^{\mathcal{P}}$ be an $IVmP(\epsilon, \in \vee q_k^-)$ -F ideal of $\widetilde{\mathcal{X}}$
and $\vartheta, \omega \in \widetilde{\mathcal{X}}$ such that $\vartheta \leq \omega$. Then,

$$\vartheta \leq \omega \implies \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq r \min\left\{\widetilde{\mathcal{U}}^{\mathcal{P}}(\omega), \frac{\tilde{1} - \tilde{k}}{2}\right\}. \quad (20)$$

Proof. Let $\vartheta, \omega \in \widetilde{\mathcal{X}}$ such that $\vartheta \leq \omega$. Then, we have

$$\begin{aligned} \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) &\geq r \min\left\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega), \frac{1 - k_i}{2}\right\}, \\ &= r \min\left\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega), \frac{1 - k_i}{2}\right\} \\ &= r \min\left\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega), \frac{1 - k_i}{2}\right\}. \end{aligned} \quad (21)$$

Hence, $\vartheta \leq \omega \implies \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq r \min\left\{\widetilde{\mathcal{U}}^{\mathcal{P}}(\omega), ((\tilde{1} - \tilde{k})/2)\right\}$. \square

Definition 7. An $IVmPF$ set $\widetilde{\mathcal{U}}^{\mathcal{P}}$ of a BCK -algebra $\widetilde{\mathcal{X}}$ is
called an $IVmP_{(\epsilon, \in \vee q_k^-)}$ -FPI ideal of $\widetilde{\mathcal{X}}$ if

- (1) $\vartheta \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$ implies $0 \sim_{[\alpha, \beta]} \in \overline{\vee q_k} \widetilde{\mathcal{U}}^{\mathcal{P}}$
- (2) $((\vartheta * \omega) * h) \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$ and $(\omega * h) \sim_{[\rho, \sigma]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$
imply $(\vartheta * h) \sim_{r \min\{[\alpha, \beta], [\rho, \sigma]\}} \in \overline{\vee q_k} \widetilde{\mathcal{U}}^{\mathcal{P}}$,
 $\forall \vartheta, \omega, h \in \widetilde{\mathcal{X}}$ and $[\alpha, \beta], [\rho, \sigma] \in D(0, 1)^m$

Example 4. Consider a BCK -algebra $\widetilde{\mathcal{X}} = \{0, 1, 2, 3\}$ which
is given in Example 2. Let $\widetilde{\mathcal{U}}^{\mathcal{P}}$ be an $IVmPF$ set defined as

$$\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) = \begin{cases} ([0.5, 0.5], [0.5, 0.5], \dots, [0.5, 0.5]), & \text{if } \vartheta = 0, \\ ([0.4, 0.4], [0.4, 0.4], \dots, [0.4, 0.4]), & \text{if } \vartheta = 1, 2, 3. \end{cases} \quad (22)$$

Choose $\kappa = [0.4, 0.3]$. Then, $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an $IVmP_{(\epsilon, \in \vee q_k^-)}$ -FPI
ideal of $\widetilde{\mathcal{X}}$.

Theorem 5. An $IVmPF$ set $\widetilde{\mathcal{U}}^{\mathcal{P}}$ of $\widetilde{\mathcal{X}}$ is an $IVmP_{(\epsilon, \in \vee q_k^-)}$ -
FPI ideal of $\widetilde{\mathcal{X}} \iff \forall \vartheta, \omega, h \in \widetilde{\mathcal{X}}$:

- (1) $\widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq r \min\left\{\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), ((\tilde{1} - \tilde{k})/2)\right\}$
- (2) $\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq r \min\left\{\widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h), ((\tilde{1} - \tilde{k})/2)\right\}$

Proof. (\Rightarrow) Suppose that $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an $IVmP_{(\epsilon, \in \vee q_k^-)}$ -FPI ideal
of $\widetilde{\mathcal{X}}$. If $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) < r \min\left\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), ((1 - k_i)/2)\right\}$, then
 $\exists [\alpha, \beta] = [\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m] \in D(0, 1)^m$ such
that $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) < [\alpha_i, \beta_i] \leq r \min\left\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), (1 - k_i/2)\right\}$.

This implies that $\vartheta \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$, but $0 \sim_{[\alpha, \beta]} \notin \widetilde{\mathcal{U}}^{\mathcal{P}}$, a contradic-
tion. Hence, $\widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq r \min\left\{\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), ((\tilde{1} - \tilde{k})/2)\right\}$.

If we assume that $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) < r \min\left\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h), (1 - k_i/2)\right\}$, then \exists
 $[\alpha, \beta] = [\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m] \in D(0, 1)^m$ such that
 $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) < [\alpha_i, \beta_i] \leq r \min\left\{\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h), ((1 - k_i)/2)\right\}$. This implies that
 $((\vartheta * \omega) * h) \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$ and $(\omega * h) \sim_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$, but

TABLE 3: Cayley table of the binary operation*.

*	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

$(\vartheta * h)_{[\alpha, \beta]} \in \overline{\mathcal{U}}_k^{\mathcal{P}}$, a contradiction. Hence, $\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq r \min \left\{ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h), ((1 - \kappa_i)/2) \right\}$.

(\Leftarrow) Let $h \in \widetilde{\mathcal{X}}$ such that $h_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$. Then, $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(h) \geq [\alpha_i, \beta_i]$. So,

$$\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq r \min \left\{ \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(h), \frac{1 - \kappa_i}{2} \right\} \geq r \min \left\{ [\alpha_i, \beta_i], \frac{1 - \kappa_i}{2} \right\}. \quad (23)$$

Now, if $[\alpha_i, \beta_i] \leq ((1 - \kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq [\alpha_i, \beta_i]$. Therefore, $0_{[\alpha, \beta]} \in \widetilde{\mathcal{U}}^{\mathcal{P}}$. On the contrary, if $[\alpha_i, \beta_i] > ((1 - \kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq ((1 - \kappa_i)/2)$. So, $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) + [\alpha_i, \beta_i] > ((1 - \kappa_i)/2) + ((1 - \kappa_i)/2) = 1 - \kappa_i$. This implies that $0_{[\alpha, \beta]} \notin \widetilde{\mathcal{U}}^{\mathcal{P}}$. Hence, $0_{[\alpha, \beta]} \in \mathcal{V}q_k \widetilde{\mathcal{U}}^{\mathcal{P}}$.

Let $((\vartheta * \omega) * h)_{[\theta, \lambda]} \in \mathcal{U}$ and $(\omega * h)_{[\rho, \sigma]} \in \mathcal{U}$, $\forall [\theta, \lambda]$ and $[\rho, \sigma]$. Then, $[(\theta_1, \lambda_1), (\theta_2, \lambda_2), \dots, (\theta_m, \lambda_m)]$ and $[(\rho_1, \sigma_1), (\rho_2, \sigma_2), \dots, (\rho_m, \sigma_m)] \in D(0, 1]^m$.

$$((\vartheta * \omega) * h)_{[\theta, \lambda]} \in \widetilde{\mathcal{U}}^{\mathcal{P}} \text{ and } (\omega * h)_{[\rho, \sigma]} \in \widetilde{\mathcal{U}}^{\mathcal{P}} \text{ imply } (\vartheta * h)_{r \min \{ \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\theta, \lambda]}, \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]} \}} \in \mathcal{V}q_k \widetilde{\mathcal{U}}^{\mathcal{P}}. \quad (26)$$

Put $h = 0$, so,

$$((\vartheta * \omega) * h)_{[\theta, \lambda]} \in \widetilde{\mathcal{U}}^{\mathcal{P}} \text{ and } (\omega * 0)_{[\rho, \sigma]} \in \widetilde{\mathcal{U}}^{\mathcal{P}} \text{ imply } (\vartheta * 0)_{r \min \{ \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\theta, \lambda]}, \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]} \}} \in \mathcal{V}q_k \widetilde{\mathcal{U}}^{\mathcal{P}}. \quad (27)$$

Thus,

$$(\vartheta * \omega)_{[\theta, \lambda]} \in \widetilde{\mathcal{U}}^{\mathcal{P}} \text{ and } \omega_{[\rho, \sigma]} \in \widetilde{\mathcal{U}}^{\mathcal{P}} \text{ imply } (\vartheta * \omega)_{r \min \{ \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\theta, \lambda]}, \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]} \}} \in \mathcal{V}q_k \widetilde{\mathcal{U}}^{\mathcal{P}}. \quad (28)$$

Hence, $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an $\text{IV}mP_{(\epsilon, \in \mathcal{V}q_k)} - F$ ideal of $\widetilde{\mathcal{X}}$.

As shown by the following example, the converse of the preceding Theorem 6 is not valid in general. \square

Example 5. Reconsider the BCK-algebras $\widetilde{\mathcal{X}}$ given in Example 2. Define an IV3PF set $\widetilde{\mathcal{U}}^{\mathcal{P}}$ as

$$\widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) = \begin{cases} ([0.8, 0.7], [0.7, 0.6], [0.6, 0.5]), & \text{if } \vartheta = 0, \\ ([0.5, 0.4], [0.4, 0.3], [0.3, 0.2]), & \text{if } \vartheta = 1, 2, \\ ([0.4, 0.3], [0.3, 0.2], [0.2, 0.1]), & \text{if } \vartheta = 3. \end{cases} \quad (29)$$

$\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h) \geq [\theta_i, \lambda_i]$ and $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h) \geq [\rho_i, \sigma_i]$. Thus,

$$\begin{aligned} \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) &\geq r \min \left\{ \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h), \frac{1 - \kappa_i}{2} \right\}, \\ &\geq r \min \left\{ [\theta_i, \lambda_i], [\rho_i, \sigma_i], \frac{1 - \kappa_i}{2} \right\}. \end{aligned} \quad (24)$$

Now, if $r \min \{ [\theta_i, \lambda_i], [\rho_i, \sigma_i] \} \leq ((1 - \kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq r \min \{ [\theta_i, \lambda_i], [\rho_i, \sigma_i] \}$ and $(\vartheta * h)_{r \min \{ \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\theta, \lambda]}, \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]} \}} \in \mathcal{U}$; otherwise, when $r \min \{ [\theta_i, \lambda_i], [\rho_i, \sigma_i] \} > ((1 - \kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq ((1 - \kappa_i)/2)$. So, we have

$$\begin{aligned} \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) + r \min \{ [\theta_i, \lambda_i], [\rho_i, \sigma_i] \} &> \frac{1 - \kappa_i}{2} + \frac{1 - \kappa_i}{2} \\ &= 1 - \kappa_i. \end{aligned} \quad (25)$$

This implies that $(\vartheta * h)_{r \min \{ \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\theta, \lambda]}, \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]} \}} \in \mathcal{V}q_k \widetilde{\mathcal{U}}^{\mathcal{P}}$, as required. \square

Theorem 6. Every $\text{IV}mP_{(\epsilon, \in \mathcal{V}q_k)}\text{FPI}$ ideal of $\widetilde{\mathcal{X}}$ is an $\text{IV}mP_{(\epsilon, \in \mathcal{V}q_k)} - F$ ideal of $\widetilde{\mathcal{X}}$.

Proof. Let $\text{IV}mP_{(\epsilon, \in \mathcal{V}q_k)}\text{FPI}$ be an $\text{IV}mP_{(\epsilon, \in \mathcal{V}q_k)}\text{FPI}$ ideal of $\widetilde{\mathcal{X}}$. Then, condition (1) of Definition 6 holds. By assumption, we have

Choose $\kappa = [0.1, 0.1]$. Clearly, $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an $\text{IV}3P_{(\epsilon, \in \mathcal{V}q_k)} - \text{FI}$ of $\widetilde{\mathcal{X}}$, but is not an $\text{IV}3P_{(\epsilon, \in \mathcal{V}q_k)} - \text{FPI}$ ideal of $\widetilde{\mathcal{X}}$ because $\widetilde{q}_1 \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(2 * 1) = \widetilde{q}_1 \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(1) = [0.4, 0.3] < r \min \left\{ \widetilde{q}_1 \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((2 * 1) * 1), \widetilde{q}_1 \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(1 * 1), (1 - \kappa/2) \right\} = r \min \left\{ \widetilde{q}_1 \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0), \widetilde{q}_1 \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0), (1 - \kappa/2) \right\} = [0.45, 0.45]$.

Theorem 7. Let $\widetilde{\mathcal{U}}^{\mathcal{P}}$ be an $\text{IV}mP_{(\epsilon, \in \mathcal{V}q_k)} - F$ ideal of $\widetilde{\mathcal{X}}$. Then, $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an $\text{IV}mP_{(\epsilon, \in \mathcal{V}q_k)} - \text{FPI}$ ideal of $\widetilde{\mathcal{X}} \Leftrightarrow \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega) \geq r \min \left\{ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \omega), ((1 - \kappa)/2) \right\}, \forall \vartheta, \omega \in \widetilde{\mathcal{X}}$.

Proof. (\Rightarrow) Assume $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an $IVmP_{(\epsilon, \in \vee q_k)} - FPI$ ideal of $\widetilde{\mathcal{L}}$. Now, replace \hbar by ω in Theorem 5 (2); then,

$$\begin{aligned} \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega) &\geq r \min \left\{ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \omega), \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * \omega), \frac{\bar{1} - \bar{k}}{2} \right\}, \\ &= r \min \left\{ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \omega), \widetilde{\mathcal{U}}^{\mathcal{P}}(0), \frac{\bar{1} - \bar{k}}{2} \right\}, \\ &= r \min \left\{ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \omega), \frac{\bar{1} - \bar{k}}{2} \right\}, \end{aligned} \quad (30)$$

$\forall \vartheta, \omega \in \widetilde{\mathcal{L}}$. (\Leftarrow) Let $\widetilde{\mathcal{U}}^{\mathcal{P}}$ be an $IVmP_{(\epsilon, \in \vee q_k)} - F$ ideal of $\widetilde{\mathcal{L}}$. Then, condition (1) holds. As $((\vartheta * \hbar) * \hbar) * (\omega * \hbar) \leq (\vartheta * \hbar) * \omega = (\vartheta * \omega) * \hbar, \forall \vartheta, \omega \in \widetilde{\mathcal{L}}$. By Lemma 2, we have

$$\widetilde{\mathcal{U}}^{\mathcal{P}}(((\vartheta * \hbar) * \hbar) * (\omega * \hbar)) \geq r \min \left\{ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \hbar), \frac{\bar{1} - \bar{k}}{2} \right\}. \quad (31)$$

Since $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an $IVmP_{(\epsilon, \in \vee q_k)} - F$ ideal, so

$$\begin{aligned} \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \hbar) &\geq r \min \left\{ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \hbar) * \hbar), \frac{\bar{1} - \bar{k}}{2} \right\}, \\ &\geq r \min \left\{ \widetilde{\mathcal{U}}^{\mathcal{P}}(((\vartheta * \hbar) * \hbar) * (\omega * \hbar)), \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * \hbar), \frac{\bar{1} - \bar{k}}{2}, \frac{\bar{1} - \bar{k}}{2} \right\}, \\ &\geq r \min \left\{ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \hbar) * \hbar), \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * \hbar), \frac{\bar{1} - \bar{k}}{2} \right\}. \end{aligned} \quad (32)$$

$$\begin{aligned} \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \hbar) &\geq r \min \left\{ \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \hbar), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * \hbar), \frac{\bar{1} - \bar{k}}{2} \right\} \\ &\geq r \min \left\{ [\alpha_i, \beta_i], [\alpha_i, \beta_i], \frac{1 - k_i}{2} \right\} \\ &= [\alpha_i, \beta_i]. \end{aligned} \quad (34)$$

Therefore, $\vartheta * \hbar \in \widetilde{\mathcal{U}}^{\mathcal{P}} \widetilde{\sim}_{[\alpha, \beta]}$. Hence, $\widetilde{\mathcal{U}}^{\mathcal{P}} \widetilde{\sim}_{[\alpha, \beta]}$ is a positive implicative ideal of $\widetilde{\mathcal{L}}$.

(\Leftarrow) Suppose, on the contrary, that $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) < r \min \left\{ \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), (1 - k/2) \right\}$ for some $\vartheta \in \widetilde{\mathcal{L}}$. Choose $[\bar{\theta}, \bar{\lambda}] = [\theta_1, \lambda_1], [\theta_2, \lambda_2], \dots, [\theta_m, \lambda_m] \in D(0, (1 - k)/2)^m$ such that

$$\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) < [\theta_i, \lambda_i] \leq r \min \left\{ \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), \frac{1 - k_i}{2} \right\}. \quad (35)$$

Hence, $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an $IVmP_{(\epsilon, \in \vee q_k)} - FPI$ ideal of $\widetilde{\mathcal{L}}$. \square

Theorem 8. An $IVmPF$ set $\widetilde{\mathcal{U}}^{\mathcal{P}}$ of $\widetilde{\mathcal{L}}$ is an $IVmP_{(\epsilon, \in \vee q_k)} - F$ ideal of a BCK-algebra $\widetilde{\mathcal{L}} \Leftrightarrow \widetilde{\mathcal{U}}^{\mathcal{P}} \widetilde{\sim}_{[\alpha, \beta]} \neq \phi$ is a positive implicative ideal of $\widetilde{\mathcal{L}}, \forall [\alpha, \beta] = [\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m] \in D(0, ((1 - k)/2))^m$.

Proof. (\Rightarrow) Let $\vartheta \in \widetilde{\mathcal{U}}^{\mathcal{P}} \widetilde{\sim}_{[\alpha, \beta]}$ for $[\alpha, \beta] = [\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m] \in D(0, ((1 - k)/2))^m$. Then, $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta) \geq [\alpha_i, \beta_i]$. It follows from Theorem 5 (i) that

$$\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq r \min \left\{ \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), \frac{1 - k_i}{2} \right\} = [\alpha_i, \beta_i]. \quad (33)$$

Thus, $0 \in \widetilde{\mathcal{U}}^{\mathcal{P}} \widetilde{\sim}_{[\alpha, \beta]}$.

Next, suppose that $(\vartheta * \omega) * \hbar \in \widetilde{\mathcal{U}}^{\mathcal{P}} \widetilde{\sim}_{[\alpha, \beta]}$ and $\omega * \hbar \in \widetilde{\mathcal{U}}^{\mathcal{P}} \widetilde{\sim}_{[\alpha, \beta]}$. Then, $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \hbar) \geq [\alpha_i, \beta_i]$ and $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * \hbar) \geq [\alpha_i, \beta_i]$. Again, it follows from Theorem 5 (ii) that

It follows that $\vartheta \in \widetilde{\mathcal{U}}^{\mathcal{P}} \widetilde{\sim}_{[\theta, \lambda]}$, but $0 \notin \widetilde{\mathcal{U}}^{\mathcal{P}} \widetilde{\sim}_{[\theta, \lambda]}$, a contradiction. Therefore, $\widetilde{\mathcal{U}}^{\mathcal{P}}(0) \geq r \min \left\{ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta), (\bar{1} - \bar{k}/2) \right\}, \forall \vartheta \in \widetilde{\mathcal{L}}$. Suppose that

$$\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega) < r \min \left\{ \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * \hbar), \widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega), \frac{\bar{1} - \bar{k}}{2} \right\}, \quad (36)$$

for some $\vartheta, \omega, \hbar \in \widetilde{\mathcal{L}}$. Then, $\exists [\bar{\rho}, \bar{\sigma}] = [\rho_1, \sigma_1], [\rho_2, \sigma_2], \dots, [\rho_m, \sigma_m] \in D(0, 1 - k/2)^m$ such that $\widetilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega) < [\rho_i, \sigma_i]$.

$\sigma_i] \leq r \min\{\{\tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * \omega), ((1 - k_i)/2)\}$ implies that $(\vartheta * \omega) * h \in \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]}$

and $\vartheta * \omega \in \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]}$, but $\vartheta * \omega \notin \widetilde{\mathcal{U}}^{\mathcal{P}}_{[\rho, \sigma]}$, which is not possible. Thus,

$$\tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\vartheta * h) \geq r \min\left\{\tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}((\vartheta * \omega) * h), \tilde{q}_i \circ \widetilde{\mathcal{U}}^{\mathcal{P}}(\omega * h), \frac{1 - k_i}{2}\right\}, \quad \forall \vartheta, \omega, h \in \tilde{\mathcal{X}}. \quad (37)$$

Hence, by Theorem 5, $\widetilde{\mathcal{U}}^{\mathcal{P}}$ is an $\text{IVmP}_{(\epsilon, \in \vee q_k)} - \text{F ideal}$ of $\tilde{\mathcal{X}}$. \square

5. Conclusion

We applied the theory of interval-valued fuzzy sets on positive implication ideals of BCK-algebras. In this aim, the concept of interval-valued m -polar fuzzy positive implicative ideals in BCK-algebras is introduced. The related properties of interval-valued m -polar fuzzy positive implicative ideals and interval-valued m -polar fuzzy ideals are investigated. In addition, the concepts of interval-valued m -polar $(\epsilon, \in \vee q_k)$ -fuzzy positive implicative ideals and interval-valued m -polar $(\epsilon, \in \vee q_k)$ -fuzzy ideals are defined and characterized. Furthermore, we have shown that interval-valued m -polar $(\epsilon, \in \vee q_k)$ -fuzzy positive implicative ideals are interval-valued m -polar $(\epsilon, \in \vee q_k)$ -fuzzy ideals, but converse is not valid and an illustration is provided in this support.

In future work, one may extend these concepts to various algebraic structures such as rings, hemirings, LA-semigroups, semihypergroups, semihyperrings, BL-algebras, MTL-algebras, R0-algebras, MV-algebras, EQ-algebras, and lattice implication algebras.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Spherical Cubic Fuzzy Extended TOPSIS Method and Its Application in Multicriteria Decision-Making

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Spherical Cubic Fuzzy Extended TOPSIS Method and Its Application in Multicriteria Decision-Making

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This paper aims to propose a new methodology for spherical cubic fuzzy (SCF) multicriteria decision-making (MCDM) utilizing the TOPSIS method that uses incomplete weight information. At first, the maximum deviation model is suggested to determine the criteria of weight values. An MCDM methodology is introduced using SCF information, based on the proposed method. Also, to validate the effectiveness of the proposed information, a numerical example is given. Finally, a comprehensive and structured analysis of existing work in comparison with previous work is given.

1. Introduction

MCGDM (multicriteria group decision-making) is a useful tool for selecting the most important option from a collection of alternatives in the evaluation and selection process. It has been used extensively in several real-life circumstances. Zhang and Guo [1] developed uncertain preference ordinals and incomplete weight information to solve group decision-making (GDM) problems in the VIKOR-based process. Zhang et al. established in [2] a new computational model based on comprehensive linguistic hierarchies that not only can be used to execute multi-granular linguistic distribution evaluations but also can provide decision-makers with interpretable linguistic results. Yu et al. [3] provided an extensive analysis of the various approaches to consensus-building processes (CRP) and set out a variety of CRPs. When production methods and specifications become more complex, it can be difficult for decision-makers (DMs) or experts to consider all relevant factors during the evaluation and selection process. To solve a problem of real decision-making that characterizes membership, Zadeh [4] proposed fuzzy sets. After the

introduction of fuzzy sets, many authors have extended the concept and applied it to MCGDM problems. Zhang et al. proposed a method for obtaining a priority weight vector from an incomplete HFPR (hesitant fuzzy preference relations) and the sum of the membership and nonmembership degrees using the least-squares logarithmic method in [5].

We assumed that participants in failure mode and effect analysis linguistically provide their preferences, using possibly hesitant fuzzy linguistic information, to classify failure modes into several ordinary risk groups. Zhang et al. [6] developed a consensus-based group decision-making process for failure mode and effect analysis. On the condition that the number of membership and nonmembership degrees is less than or equal to one, fuzzy sets were generalized into intuitionistic fuzzy sets (IFS) [7] defined by membership and nonmembership degrees. Zeshui Xu [8] developed intuitionistic fuzzy aggregation operators. Zhang and Guo [9] proposed group decisions based on intuitionistic choice multiplication relationships. Yager et al. [10–13] extended the Pythagorean fuzzy set (PFS) by allowing it to be determined by membership degrees and nonmembership degrees, and the sum of the square of the membership and

nonmembership degrees is less than or equal to one. PFS, as shown by Yager and Abbasov [13], can model forms of inaccuracy in decision-making problems that IFS cannot.

Garg [14] presented exponential operational law and its aggregation operators for interval-valued Pythagorean fuzzy set and has further generalized PFS by modifying terms of membership and degrees of nonmembership to the point that their number of squares is less than or equal to one. Researchers have created many extensions to those techniques and models to deal with the difficulty in MCDM problems.

To generalize the concepts of PFS and IVPFS, Mahmood et al. [15] introduced the idea of a spherical fuzzy set, in which the sum of squares of membership, neutral, and nonmembership degrees is less than or equal to one.

Gündoğdu and Kahraman [16] presented the application of decision-making by using the idea of spherical fuzzy sets. Kutlu Gündoğdu and Kahraman [17] introduced the new idea by combining the spherical fuzzy sets with the TOPSIS method and discussed their applications. In [18–21], many researchers presented the applications of spherical fuzzy sets in decision-making.

TOPSIS is also an essential aspect of the decision-making process. iTOPSIS collects and compares data from the groups by assigning a weight to each criterion and applying a distance calculation formula to find the best solution to a decision-making problem. The TOPSIS method assumes that the criterion function is monotonic. When the TOPSIS method parameters are not as important as normalization in MCDM problems, the advantage of TOPSIS is that it allows the substitution of unnecessary parameters in situations where other models are insufficient to solve a variety of decision-making problems. Gündoğdu and Kahraman [22] have given a novel fuzzy TOPSIS method using interval-valued spherical fuzzy sets. Garg et al. [23] presented an algorithm for T-spherical fuzzy multiattribute decision-making by utilizing improved interactive aggregation operators. Sajjad Ali Khan et al. [24, 25] used an integral Choquet TOPSIS technology with IVPFNs to solve MCGDM problems and the IVPF GRA MCDM method. In addition, many authors discussed the TOPSIS method for dealing with MADM in spherical fuzzy environments.

The definition of IFS was generalized in June 2012 and the notion of the cubic set was initiated. Many researchers presented the idea of an intuitionistic cubic fuzzy set (ICFS) [26] and its applications in decision-making. Naeem et al. [27] developed the new idea of Pythagorean m-polar fuzzy sets with the TOPSIS method and their applications in the selection of advertisement mode. After that, Ayaz et al. [28] introduced the generalized idea of SCFS, which is the generalization of ICFS and PCFS, and discussed its application in the selection of enterprise performance. The maximum deviation methodology of weighted aggregation operator for multiple criteria group decision analysis is presented in [29]. The flowchart represents the generalization of SCFS in Figure 1.

Section 2 reviews essential SFS and SCFS properties. Section 3 presents a new SCFS approach to extending the TOPSIS handling process. In Section 4, we will discuss an example of practical application of MCGDM. Section 5

compares the suggested technique to other well-known decision-making approaches to demonstrate the approach's reliability.

2. Preliminaries

In this section, we present some basic definitions and important properties.

Definition 1 (see [7]). Let $\hat{X} \neq \emptyset$ be a universal set, and an intuitionistic fuzzy set (IFS) \tilde{I} on \hat{X} is given as follows:

$$\tilde{I} = \{ \hat{x}, \langle \hat{\alpha}_{\tilde{I}}(\hat{x}), \hat{\beta}_{\tilde{I}}(\hat{x}) \rangle | \hat{x} \in \hat{X} \}, \quad (1)$$

where $\hat{\alpha}_{\tilde{I}}(\hat{x}): \hat{X} \rightarrow [0, 1]$ represents the membership degree and $\hat{\beta}_{\tilde{I}}(\hat{x}): \hat{X} \rightarrow [0, 1]$ represents the nonmembership degree with the specified condition $0 \leq \hat{\alpha}_{\tilde{I}}(\hat{x}) + \hat{\beta}_{\tilde{I}}(\hat{x}) \leq 1$ for all $\hat{x} \in \hat{X}$.

Definition 2 (see [28]). Let $\hat{X} \neq \emptyset$ be a universal set, and a cubic set \tilde{C} on \hat{X} is given as follows:

$$\tilde{C} = \{ \hat{x}, \langle \neg \hat{\alpha}_{\tilde{I}}(\hat{x}), \hat{\beta}_{\tilde{I}}(\hat{x}) \rangle | \hat{x} \in \hat{X} \}, \quad (2)$$

where $\neg \hat{\alpha}_{\tilde{I}}(\hat{x})$ represents an interval fuzzy set and $\hat{\beta}_{\tilde{I}}(\hat{x})$ represents the simple fuzzy set in \hat{X} .

Definition 3 (see [10]). Let $\hat{X} \neq \emptyset$ be a universal set, and a Pythagorean fuzzy set (PFS) \tilde{P} on \hat{X} is given as follows:

$$\tilde{P} = \{ \hat{x}, \langle \hat{\alpha}_{\tilde{P}}(\hat{x}), \hat{\beta}_{\tilde{P}}(\hat{x}) \rangle | \hat{x} \in \hat{X} \}, \quad (3)$$

where $\hat{\alpha}_{\tilde{P}}(\hat{x}): \hat{X} \rightarrow [0, 1]$ represents the membership degree and $\hat{\beta}_{\tilde{P}}(\hat{x}): \hat{X} \rightarrow [0, 1]$ represents the nonmembership degree with the specified condition $0 \leq (\hat{\alpha}_{\tilde{P}}(\hat{x}))^2 + (\hat{\beta}_{\tilde{P}}(\hat{x}))^2 \leq 1$ for all $\hat{x} \in \hat{X}$.

Definition 4 (see [26]). Let $\hat{X} \neq \emptyset$ be a universal set, and an intuitionistic cubic fuzzy set (ICFS) \tilde{I}_c on \hat{X} is given as follows:

$$\tilde{I}_c = \{ \hat{x}, \langle \hat{\alpha}_{\tilde{I}_c}(\hat{x}), \hat{\beta}_{\tilde{I}_c}(\hat{x}) \rangle | \hat{x} \in \hat{X} \}, \quad (4)$$

where $\hat{\alpha}_{\tilde{I}_c}(\hat{x}) = [\tilde{a}^-, \tilde{a}^+]$, $\tilde{\lambda}$ and $\hat{\beta}_{\tilde{I}_c}(\hat{x}) = [\tilde{b}^-, \tilde{b}^+]$, $\tilde{\mu}$ represent the membership and nonmembership degrees, respectively, with the specified conditions $0 \leq \tilde{\lambda} + \tilde{\mu} \leq 1$ and $0 \leq \sup([\tilde{a}^-, \tilde{a}^+]) + \sup([\tilde{b}^-, \tilde{b}^+]) \leq 1$. The ICFS's degree of indeterminacy can be described as

$$\tilde{\pi}_{\tilde{I}_c} = \left\langle \sqrt{1 - (\sup([\tilde{a}^-, \tilde{a}^+]) + \sup([\tilde{b}^-, \tilde{b}^+]))}, \sqrt{1 - (\tilde{\lambda} + \tilde{\mu})} \right\rangle. \quad (5)$$

Definition 5 (see [24]). Let $\hat{X} \neq \emptyset$ be a universal set, and a Pythagorean cubic fuzzy set (PCFS) \tilde{P}_c on \hat{X} is given as follows:

$$\tilde{P}_c = \{ \hat{x}, \langle \hat{\alpha}_{\tilde{P}_c}(\hat{x}), \hat{\beta}_{\tilde{P}_c}(\hat{x}) \rangle | \hat{x} \in \hat{X} \}, \quad (6)$$

where $\hat{\alpha}_{\tilde{P}_c}(\hat{x}) = [\tilde{a}^-, \tilde{a}^+]$, $\tilde{\lambda}$ and $\hat{\beta}_{\tilde{P}_c}(\hat{x}) = [\tilde{b}^-, \tilde{b}^+]$, $\tilde{\mu}$ represent the membership and nonmembership degrees, respectively,

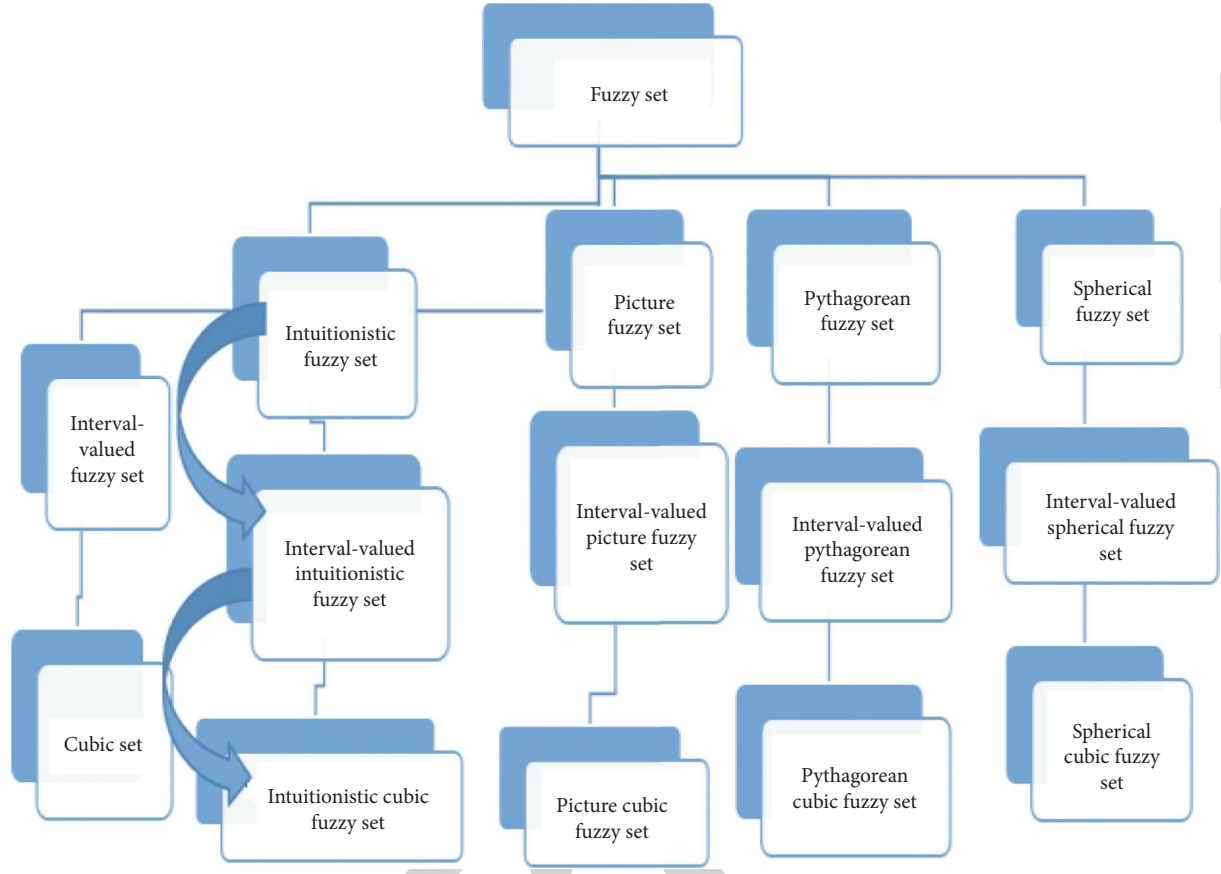


FIGURE 1: Flowchart.

with the specified conditions $0 \leq \tilde{\lambda}^2 + \tilde{\mu}^2 \leq 1$ and $0 \leq (\sup([\tilde{a}^-, \tilde{a}^+]))^2 + (\sup([\tilde{b}^-, \tilde{b}^+]))^2 \leq 1$. The PCFS's degree of indeterminacy can be described as

$$\tilde{\pi}_{P_c} = \left\langle \frac{\sqrt{1 - ((\sup([\tilde{a}^-, \tilde{a}^+]))^2 + (\sup([\tilde{b}^-, \tilde{b}^+]))^2)}}{\sqrt{1 - (\tilde{\lambda}^2 + \tilde{\mu}^2)}} \right\rangle. \quad (7)$$

Definition 6 (see [28]). Let $\tilde{X} \neq \emptyset$ be a universal set, and a spherical cubic fuzzy set (SCFS) \tilde{S}_c on \tilde{X} is given as follows:

$$\tilde{S}_c = \left\{ \hat{x}, \left\langle \hat{\alpha}_{\tilde{S}_c}(\hat{x}), \hat{\eta}_{\tilde{S}_c}, \hat{\beta}_{\tilde{S}_c}(\hat{x}) \right\rangle | \hat{x} \in \tilde{X} \right\}, \quad (8)$$

where $\hat{\alpha}_{\tilde{S}_c}(\hat{x}) = [\hat{a}^-, \hat{a}^+]$, $\tilde{\lambda}, \hat{\eta}_{\tilde{S}_c} = [\hat{n}^-, \hat{n}^+]$, $\tilde{\mu}$, and $\hat{\beta}_{\tilde{S}_c}(\hat{x}) = [\hat{b}^-, \hat{b}^+]$, $\tilde{\vartheta}_{\tilde{S}_c}$ represent the membership neutral and non-membership degrees, respectively, with the specified conditions $0 \leq \tilde{\lambda}^2 + \tilde{\mu}^2 + \tilde{\vartheta}^2 \leq 1$ and $0 \leq (\sup([\hat{a}^-, \hat{a}^+]))^2 + (\sup([\hat{n}^-, \hat{n}^+]))^2 + (\sup([\hat{b}^-, \hat{b}^+]))^2 \leq 1$.

The SCFSs degree of indeterminacy can be described as

$$\tilde{\pi}_{\tilde{S}_c} = \left\langle \frac{\sqrt{1 - ((\sup([\hat{a}^-, \hat{a}^+]))^2 + (\sup([\hat{n}^-, \hat{n}^+]))^2 + (\sup([\hat{b}^-, \hat{b}^+]))^2)}}{\sqrt{1 - (\tilde{\lambda}^2 + \tilde{\vartheta} + \tilde{\mu}^2)}} \right\rangle. \quad (9)$$

For our convenience, we write SCFS as $\tilde{S}_c = ([\hat{a}^-, \hat{a}^+], \tilde{\lambda}, [\hat{n}^-, \hat{n}^+], \tilde{\mu}, [\hat{b}^-, \hat{b}^+], \tilde{\vartheta})$.

Definition 7 (see [28]). Let $\tilde{S}_{c_1} = ([\hat{a}_1^-, \hat{a}_1^+], \tilde{\lambda}_1, [\hat{n}_1^-, \hat{n}_1^+], \tilde{\mu}_1, [\hat{b}_1^-, \hat{b}_1^+], \tilde{\vartheta}_1)$ and $\tilde{S}_{c_2} = ([\hat{a}_2^-, \hat{a}_2^+], \tilde{\lambda}_2, [\hat{n}_2^-, \hat{n}_2^+], \tilde{\mu}_2, [\hat{b}_2^-, \hat{b}_2^+], \tilde{\vartheta}_2)$

be two spherical cubic fuzzy sets (SCFSs) and $\tau > 0$, then the following operation holds: (1)

$$\tilde{S}_{c_1} \oplus \tilde{S}_{c_2} = \left\{ \left\langle \left[\sqrt{(\tilde{a}_1^-)^2 + (\tilde{a}_2^-)^2 - (\tilde{a}_1^-)^2 (\tilde{a}_2^-)^2}, \sqrt{(\tilde{a}_1^+)^2 + (\tilde{a}_2^+)^2 - (\tilde{a}_1^+)^2 (\tilde{a}_2^+)^2} \right], \right. \right. \\ \left. \left. \frac{\sqrt{(\tilde{\lambda}_1)^2 + (\tilde{\lambda}_2)^2 - (\tilde{\lambda}_1)^2 (\tilde{\lambda}_2)^2}}{\sqrt{(\tilde{\lambda}_1)^2 + (\tilde{\lambda}_2)^2 - (\tilde{\lambda}_1)^2 (\tilde{\lambda}_2)^2}}, \langle [\tilde{n}_1^- \tilde{n}_2^-, \tilde{n}_2^+ \tilde{n}_1^+], \tilde{\mu}_1 \tilde{\mu}_2 \rangle, \langle [\tilde{b}_1^- \tilde{b}_2^-, \tilde{b}_1^+ \tilde{b}_2^+], \tilde{\vartheta}_1 \tilde{\vartheta}_2 \rangle \right\rangle \right\}. \quad (10)$$

(2)

$$\tilde{S}_{c_1} \otimes \tilde{S}_{c_2} = \left\{ \left\langle \left[\sqrt{(\tilde{b}_1^-)^2 + (\tilde{b}_2^-)^2 - (\tilde{b}_1^-)^2 (\tilde{b}_2^-)^2}, \sqrt{(\tilde{b}_1^+)^2 + (\tilde{b}_2^+)^2 - (\tilde{b}_1^+)^2 (\tilde{b}_2^+)^2} \right], \right. \right. \\ \left. \left. \frac{\sqrt{(\tilde{\vartheta}_1)^2 + (\tilde{\vartheta}_2)^2 - (\tilde{\vartheta}_1)^2 (\tilde{\vartheta}_2)^2}}{\sqrt{(\tilde{\vartheta}_1)^2 + (\tilde{\vartheta}_2)^2 - (\tilde{\vartheta}_1)^2 (\tilde{\vartheta}_2)^2}}, \langle [\tilde{a}_1^- \tilde{a}_2^-, \tilde{a}_1^+ \tilde{a}_2^+], \tilde{\lambda}_1 \tilde{\lambda}_2, [\tilde{n}_1^- \tilde{n}_2^-, \tilde{n}_2^+ \tilde{n}_1^+], \tilde{\mu}_1 \tilde{\mu}_2, \right\rangle \right\}. \quad (11)$$

(3)

$$\tau \cdot \tilde{S}_{c_1} = \left\{ \left\langle \left[\sqrt{1 - (1 - (\tilde{a}_1^-)^2)^\tau}, \sqrt{1 - (1 - (\tilde{a}_1^+)^2)^\tau} \right], \sqrt{1 - (1 - (\tilde{\lambda}_1)^2)^\tau} \right\rangle, \right. \\ \left. \langle [(\tilde{n}_1^-)^\tau, (\tilde{n}_1^+)^\tau, (\tilde{\mu}_1)^\tau], \langle [(\tilde{b}_1^-)^\tau, (\tilde{b}_1^+)^\tau, (\tilde{\vartheta}_1)^\tau] \rangle \right\}. \quad (12)$$

(4)

$$\tilde{S}_{c_1}^\tau = \left\{ \left\langle [(\tilde{a}_1^-)^\tau, (\tilde{a}_1^+)^\tau, (\tilde{\lambda}_1)^\tau], \langle [(\tilde{b}_1^-)^\tau, (\tilde{b}_1^+)^\tau, (\tilde{\vartheta}_1)^\tau] \right\rangle, \right. \\ \left. \left\langle \left[\sqrt{1 - (1 - (\tilde{b}_1^-)^2)^\tau}, \sqrt{1 - (1 - (\tilde{b}_1^+)^2)^\tau} \right], \sqrt{1 - (1 - (\tilde{\vartheta}_1)^2)^\tau} \right\rangle \right\}. \quad (13)$$

Definition 8 (see [28]). Let $\tilde{S}_{c_1} = ([\tilde{a}_1^-, \tilde{a}_1^+], \tilde{\lambda}_1, [\tilde{n}_1^-, \tilde{n}_1^+], \tilde{\mu}_1, [\tilde{b}_1^-, \tilde{b}_1^+], \tilde{\vartheta}_1)$ and $\tilde{S}_{c_2} = ([\tilde{a}_2^-, \tilde{a}_2^+], \tilde{\lambda}_2, [\tilde{n}_2^-, \tilde{n}_2^+], \tilde{\mu}_2, [\tilde{b}_2^-, \tilde{b}_2^+], \tilde{\vartheta}_2)$ be two spherical cubic fuzzy sets (SCFSs) and $\tau_1, \tau_2 > 0$, then the following will hold:

- (1) $\tilde{S}_{c_1} \oplus \tilde{S}_{c_2} = \tilde{S}_{c_2} \oplus \tilde{S}_{c_1}$
- (2) $\tilde{S}_{c_1} \otimes \tilde{S}_{c_2} = \tilde{S}_{c_2} \otimes \tilde{S}_{c_1}$
- (3) $\tau(\tilde{S}_{c_1} \oplus \tilde{S}_{c_2}) = \tau(\tilde{S}_{c_1}) \oplus \tau(\tilde{S}_{c_2})$
- (4) $(\tau_1 + \tau_2)\tilde{S}_{c_1} = \tau_1\tilde{S}_{c_1} \oplus \tau_2\tilde{S}_{c_1}$
- (5) $(\tilde{S}_{c_1} \otimes \tilde{S}_{c_2})^\tau = (\tilde{S}_{c_1})^\tau \otimes (\tilde{S}_{c_2})^\tau$
- (6) $\tilde{S}_{c_1}^{(\tau_1 + \tau_2)} = (\tilde{S}_{c_1})^{\tau_1} \otimes (\tilde{S}_{c_2})^{\tau_2}$

We defined score and accuracy function and its basic properties in order to compare two SCFNs.

Definition 9 (see [28]). Let $\tilde{S}_c = ([\tilde{a}^-, \tilde{a}^+], \tilde{\lambda}, [\tilde{n}^-, \tilde{n}^+], \tilde{\mu}, [\tilde{b}^-, \tilde{b}^+], \tilde{\vartheta})$ be an SCFN. The score function of \tilde{S}_c is defined as

$$\text{score}(\tilde{S}_c) = \frac{1}{9} \left[(\tilde{a}^- + \tilde{a}^+ + \tilde{\lambda})^2 + (\tilde{n}^- + \tilde{n}^+ + \tilde{\mu})^2 - (\tilde{b}^- + \tilde{b}^+ + \tilde{\vartheta})^2 \right], \quad (14)$$

where $\text{score}(\tilde{S}_c) \in [-1, 1]$.

Definition 10 (see [28]). Let $\tilde{S}_c = ([\tilde{a}^-, \tilde{a}^+], \tilde{\lambda}, [\tilde{n}^-, \tilde{n}^+], \tilde{\mu}, [\tilde{b}^-, \tilde{b}^+], \tilde{\vartheta})$ be an SCFN. The accuracy function of \tilde{S}_c is defined as

$$\text{accuracy}(\tilde{S}_c) = \frac{1}{9} \left[(\tilde{a}^- + \tilde{a}^+ + \tilde{\lambda})^2 + (\tilde{n}^- + \tilde{n}^+ + \tilde{\mu})^2 + (\tilde{b}^- + \tilde{b}^+ + \tilde{\vartheta})^2 \right], \quad (15)$$

where $\text{accuracy}(\tilde{S}_c) \in [0, 1]$.

Definition 11 (see [28]). Let $\tilde{S}_{c_1} = ([\tilde{a}_1^-, \tilde{a}_1^+], \tilde{\lambda}_1, [\tilde{n}_1^-, \tilde{n}_1^+], \tilde{\mu}_1, [\tilde{b}_1^-, \tilde{b}_1^+], \tilde{\vartheta}_1)$ and $\tilde{S}_{c_2} = ([\tilde{a}_2^-, \tilde{a}_2^+], \tilde{\lambda}_2, [\tilde{n}_2^-, \tilde{n}_2^+], \tilde{\mu}_2, [\tilde{b}_2^-, \tilde{b}_2^+], \tilde{\vartheta}_2)$ be two spherical cubic fuzzy sets (SCFSSs), then the comparison of \tilde{S}_{c_1} and \tilde{S}_{c_2} is defined as follows:

- (1) If $\text{score}(\tilde{S}_{c_1}) < \text{score}(\tilde{S}_{c_2})$, then $\tilde{S}_{c_1} < \tilde{S}_{c_2}$
- (2) If $\text{score}(\tilde{S}_{c_1}) > \text{score}(\tilde{S}_{c_2})$, then $\tilde{S}_{c_1} > \tilde{S}_{c_2}$
- (3) If $\text{score}(\tilde{S}_{c_1}) = \text{score}(\tilde{S}_{c_2})$, then
 - (a) If $H(\tilde{S}_{c_1}) < H(\tilde{S}_{c_2})$, then $\tilde{S}_{c_1} < \tilde{S}_{c_2}$
 - (b) If $H(\tilde{S}_{c_1}) > H(\tilde{S}_{c_2})$, then $\tilde{S}_{c_1} > \tilde{S}_{c_2}$
 - (c) If $H(\tilde{S}_{c_1}) = H(\tilde{S}_{c_2})$, then $\tilde{S}_{c_1} = \tilde{S}_{c_2}$

Definition 12 (see [28]). Let \tilde{S}_{c_1} and \tilde{S}_{c_2} be two SCFNs, then the distance function of \tilde{S}_{c_1} and \tilde{S}_{c_2} is denoted by $\tilde{d}(\tilde{S}_{c_1}, \tilde{S}_{c_2})$ and defined as follows:

$$\tilde{d}(\tilde{S}_{c_1}, \tilde{S}_{c_2}) = \frac{1}{9} \left[\begin{aligned} &|(\tilde{a}_1^-)^2 - (\tilde{a}_2^-)^2| + |(\tilde{a}_1^+)^2 - (\tilde{a}_2^+)^2| + |(\tilde{\lambda}_1)^2 - (\tilde{\lambda}_2)^2| + \\ &|(\tilde{n}_1^-)^2 - (\tilde{n}_2^-)^2| + |(\tilde{n}_1^+)^2 - (\tilde{n}_2^+)^2| + |(\tilde{\mu}_1)^2 - (\tilde{\mu}_2)^2| + \\ &|(\tilde{b}_1^-)^2 - (\tilde{b}_2^-)^2| + |(\tilde{b}_1^+)^2 - (\tilde{b}_2^+)^2| + |(\tilde{\vartheta}_1)^2 - (\tilde{\vartheta}_2)^2| \end{aligned} \right] \quad (16)$$

Definition 13 (see [28]). Let $\tilde{S}_{c_{ij}} = ([\tilde{a}_{ij}^-, \tilde{a}_{ij}^+], \tilde{\lambda}_{ij}, [\tilde{n}_{ij}^-, \tilde{n}_{ij}^+], \tilde{\mu}_{ij}, [\tilde{b}_{ij}^-, \tilde{b}_{ij}^+], \tilde{\vartheta}_{ij})$ be a collection of SCFNs and let $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of \tilde{S}_{c_i} with the specified condition that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. Then (SCFWA) operator is a function SCFWA: $\text{SCFN}^n \rightarrow \text{SCFN}$, defined as

$$\text{SCFWA}(\tilde{S}_{c_1}, \tilde{S}_{c_2}, \dots, \tilde{S}_{c_n}) = \omega_1 \tilde{S}_{c_1} \oplus \omega_2 \tilde{S}_{c_2} \oplus \dots \oplus \omega_n \tilde{S}_{c_n}.$$

$$\text{SCFWA}(\tilde{S}_{c_1}, \tilde{S}_{c_2}, \dots, \tilde{S}_{c_n}) = \left(\begin{aligned} &\left\langle \left[\sqrt{1 - \prod_{i=1}^n (1 - (\tilde{a}_i^-)^2)^{\omega_i}}, \sqrt{1 - \prod_{i=1}^n (1 - (\tilde{a}_i^+)^2)^{\omega_i}} \right] \sqrt{1 - \prod_{i=1}^n (1 - (\tilde{\lambda}_i)^2)^{\omega_i}} \right\rangle, \\ &\left\langle \left[\prod_{i=1}^n (\tilde{n}_i^-)^{\omega_i}, \prod_{i=1}^n (\tilde{n}_i^+)^{\omega_i} \right], \prod_{i=1}^n (\tilde{\mu}_i)^{\omega_i} \right\rangle, \\ &\left\langle \left[\prod_{i=1}^n (\tilde{b}_i^-)^{\omega_i}, \prod_{i=1}^n (\tilde{b}_i^+)^{\omega_i} \right], \prod_{i=1}^n (\tilde{\vartheta}_i)^{\omega_i} \right\rangle, \end{aligned} \right) \quad (17)$$

and the SCFWA operator is known as a spherical cubic fuzzy weighted averaging operator, which is also an SCFN.

Definition 14 (see [28]). Let $\tilde{S}_{c_{ij}} = ([\tilde{a}_{ij}^-, \tilde{a}_{ij}^+], \tilde{\lambda}_{ij}, [\tilde{n}_{ij}^-, \tilde{n}_{ij}^+], \tilde{\mu}_{ij}, [\tilde{b}_{ij}^-, \tilde{b}_{ij}^+], \tilde{\vartheta}_{ij})$ be a collection of SCFNs and let

$\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of \tilde{S}_{c_i} with the specified condition that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. Then (SCFWG) operator is a function SCFWA: $\text{SCFN}^n \rightarrow \text{SCFN}$, defined as

$$\text{SCFWG}(\tilde{S}_{c_1}, \tilde{S}_{c_2}, \dots, \tilde{S}_{c_n}) = \tilde{S}_{c_1}^{\omega_1} \otimes \tilde{S}_{c_2}^{\omega_2} \otimes \dots \otimes \tilde{S}_{c_n}^{\omega_n},$$

$$\text{SCFWG}(\tilde{S}_{c_1}, \tilde{S}_{c_2}, \dots, \tilde{S}_{c_n}) = \left(\begin{aligned} &\left\langle \left[\prod_{i=1}^n (\tilde{a}_i^-)^{\omega_i}, \prod_{i=1}^n (\tilde{a}_i^+)^{\omega_i} \right], \prod_{i=1}^n (\tilde{\lambda}_i)^{\omega_i} \right\rangle, \\ &\left\langle \left[\prod_{i=1}^n (\tilde{n}_i^-)^{\omega_i}, \prod_{i=1}^n (\tilde{n}_i^+)^{\omega_i} \right], \prod_{i=1}^n (\tilde{\mu}_i)^{\omega_i} \right\rangle, \\ &\left\langle \left[\sqrt{1 - \prod_{i=1}^n (1 - (\tilde{b}_i^-)^2)^{\omega_i}}, \sqrt{1 - \prod_{i=1}^n (1 - (\tilde{b}_i^+)^2)^{\omega_i}} \right] \sqrt{1 - \prod_{i=1}^n (1 - (\tilde{\vartheta}_i)^2)^{\omega_i}} \right\rangle, \end{aligned} \right) \quad (18)$$

and the SCFWG operator is known as a spherical cubic fuzzy weighted geometric operator, which is also an SCFN.

3. Multicriteria Decision-Making Based on Spherical Cubic Fuzzy Numbers

This section discusses a multicriteria decision-making strategy based on the spherical cubic fuzzy TOPSIS system with unknown weight.

3.1. Formulation of the Problem. The MCDM problems are described as a decision-making mechanism that provides the attributes with ranking information in relation to the criteria. We suggest a spherical cubic fuzzy decision-making mechanism that not only describes the data on the Z_i alternatives that fulfill the A_j criterion, the data on the Z_i alternatives that keep the A_j criterion unchanged, and the extent to which Z_i fails to meet A_j criterion. Suppose that we have an MCDM function with an $Z = \{Z_1, Z_2, \dots, Z_m\}$ set

of m alternatives. Let Z be a set of alternatives and let $A = \{A_1, A_2, \dots, A_n\}$ be the set of criteria. In order to measure the efficiency of the i th alternative Z_i in the j th criterion A_j , the decision-maker must use knowledge of the fulfillment of criteria A_j by alternative Z_i 's but of its nonfulfillment of A_j and keep A_j unchanged. $\tilde{\alpha}_{\tilde{S}_{c_{ij}}}$, $\tilde{\eta}_{\tilde{S}_{c_{ij}}}$ and $\tilde{\beta}_{\tilde{S}_{c_{ij}}}$ represent the membership, neutral, and nonmembership degrees, respectively. The alternative efficiency based on criteria A_j is represented by $\tilde{S}_{c_{ij}} = \tilde{\alpha}_{\tilde{S}_{c_{ij}}}, \tilde{\eta}_{\tilde{S}_{c_{ij}}}, \tilde{\beta}_{\tilde{S}_{c_{ij}}} = (\langle [\tilde{a}_{ij}^-, \tilde{a}_{ij}^+], \tilde{\lambda}_{ij} \rangle, \langle [\tilde{n}_{ij}^-, \tilde{n}_{ij}^+], \tilde{\mu}_{ij} \rangle, \langle [\tilde{b}_{ij}^-, \tilde{b}_{ij}^+], \tilde{\vartheta}_{ij} \rangle)$ with the specified conditions $0 \leq \tilde{\lambda}_{ij}^2 + \tilde{\mu}_{ij}^2 + \tilde{\vartheta}_{ij}^2 \leq 1$ and $0 \leq (\sup([\tilde{a}_{ij}^-, \tilde{a}_{ij}^+])^2 + (\sup([\tilde{n}_{ij}^-, \tilde{n}_{ij}^+])^2 + (\sup([\tilde{b}_{ij}^-, \tilde{b}_{ij}^+])^2 \leq 1$. The decision matrix \tilde{D} of SCF is shown as follows:

$$\tilde{D} = \begin{bmatrix} \tilde{S}_{c_{11}} & \tilde{S}_{c_{12}} & \dots & \tilde{S}_{c_{1n}} \\ \tilde{S}_{c_{21}} & \tilde{S}_{c_{22}} & \dots & \tilde{S}_{c_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{S}_{c_{m1}} & \tilde{S}_{c_{m2}} & \dots & \tilde{S}_{c_{mn}} \end{bmatrix}. \quad (19)$$

Taking the different degrees attributes, the weight vector given in decision matrix satisfied the condition $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. The weight attribute is unknown due to uncertainty in practical decision-making problems and inherent human thinking nature. For simplicity, let $\tilde{\Delta}$ represent the weight information [38], where the construction of $\tilde{\Delta}$ for $i \neq j$ is shown below:

- (1) Weak ranking criteria $\{\omega_i \geq \omega_j\}$
- (2) Strict ranking criteria $\{\omega_i - \omega_j \geq (\tilde{\lambda}_i > 0)\}$
- (3) Ranking criteria with scaling $\{\omega_i \geq \tilde{\lambda}_i \omega_j\}, \tilde{\lambda}_i \in [0, 1]$
- (4) Interval formation $\{\tilde{\lambda}_i \leq \omega_i \leq \tilde{\lambda}_i + \tilde{\sigma}_i\}, 0 \leq \tilde{\lambda}_i \leq \tilde{\lambda}_i + \tilde{\sigma}_i \leq 1$

3.2. Maximum Deviation Methodology for Optimal Weight.

The optimal weight is critical in a multicriteria decision-making process. To illustrate the MCDM problem with numerical information, we present a technique of maximizing deviation to define the criteria weights. A higher weight must be allocated to the criteria with a higher deviation value compared to the alternatives. Hence, by using the method of maximizing deviation, we construct an optimization model for the determination of optimal attribute weight in cubic spherical fuzzy environment. The distance between Z_i 's options can be defined as follows for the criteria $A_j \in A$:

$$\tilde{D}_{ij}(\omega) = \sum_{z=1}^m \omega_z \tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_{zj}}), \quad i = 1, 2, 3, \dots, m, \quad j = 1, 2, 3, \dots, n, \quad (20)$$

where

$$\tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_{zj}}) = \frac{1}{9} \left[\begin{aligned} &|(\tilde{a}_{ij}^-)^2 - (\tilde{a}_{zj}^-)^2| + |(\tilde{a}_{ij}^+)^2 - (\tilde{a}_{zj}^+)^2| + |(\tilde{\lambda}_{ij})^2 - (\tilde{\lambda}_{zj})^2| + \\ &|(\tilde{n}_{ij}^-)^2 - (\tilde{n}_{zj}^-)^2| + |(\tilde{n}_{ij}^+)^2 - (\tilde{n}_{zj}^+)^2| + |(\tilde{\mu}_{ij})^2 - (\tilde{\mu}_{zj})^2| + \\ &|(\tilde{b}_{ij}^-)^2 - (\tilde{b}_{zj}^-)^2| + |(\tilde{b}_{ij}^+)^2 - (\tilde{b}_{zj}^+)^2| + |(\tilde{\vartheta}_{ij})^2 - (\tilde{\vartheta}_{zj})^2| \end{aligned} \right] \quad (21)$$

represent the spherical cubic fuzzy distance measure between $\tilde{S}_{c_{ij}}$ and $\tilde{S}_{c_{zj}}$.

Definition 15. Let $\tilde{D}_j(\omega) = \sum_{i=1}^m \tilde{D}_{ij}(\omega)$.

$$\sum_{i=1}^m \sum_{z=1}^m \omega_j \left(\frac{1}{9} \left[\begin{aligned} &|(\tilde{a}_{ij}^-)^2 - (\tilde{a}_{zj}^-)^2| + |(\tilde{a}_{ij}^+)^2 - (\tilde{a}_{zj}^+)^2| + |(\tilde{\lambda}_{ij})^2 - (\tilde{\lambda}_{zj})^2| + \\ &|(\tilde{n}_{ij}^-)^2 - (\tilde{n}_{zj}^-)^2| + |(\tilde{n}_{ij}^+)^2 - (\tilde{n}_{zj}^+)^2| + |(\tilde{\mu}_{ij})^2 - (\tilde{\mu}_{zj})^2| + \\ &|(\tilde{b}_{ij}^-)^2 - (\tilde{b}_{zj}^-)^2| + |(\tilde{b}_{ij}^+)^2 - (\tilde{b}_{zj}^+)^2| + |(\tilde{\vartheta}_{ij})^2 - (\tilde{\vartheta}_{zj})^2| \end{aligned} \right] \right), \quad (22)$$

where $j = 1, 2, \dots, n$. $\tilde{D}_j(\omega)$ then denotes the distance for the parameters $A_j \in A$, from the alternatives. The choice of the

weight vector ω , which maximizes the deviation, is based on the proposed model to describe a nonlinear model.

3.2.1. First Model

$$\begin{aligned} \max \quad & \tilde{D}_j(\omega) = \sum_{j=1}^n \sum_{i=1}^m \sum_{z=1}^m \omega_j \tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_{zj}}), \\ \text{s.that} \quad & \sum_{j=1}^n \omega_j = 1. \end{aligned} \quad (23)$$

We have this model to clarify

$$\tilde{L}(\omega, \varrho) = \sum_{j=1}^n \sum_{i=1}^m \sum_{z=1}^m \omega_j \tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_{zj}}) + \frac{\varrho}{2} \left(\sum_{j=1}^n \omega_j^2 - 1 \right) = 0, \quad (24)$$

which shows the Lagrange function of the problem of restricted optimization of first model, where ϱ is a real number, denoting the variable of Lagrange multiplier. Now \tilde{L} 's partial derivatives are determined as follows:

$$\begin{aligned} \frac{\partial \tilde{L}(\omega, \varrho)}{\partial \omega_j} &= \sum_{i=1}^m \sum_{z=1}^m \omega_j \tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_{zj}}) + \varrho \sum_{j=1}^n \omega_j = 1 = 0, \\ \frac{\partial \tilde{L}(\omega, \varrho)}{\partial \varrho} &= \frac{1}{2} \left(\sum_{j=1}^n \omega_j^2 - 1 \right) = 0. \end{aligned} \quad (25)$$

We get

$$\omega_j = \frac{\sum_{i=1}^m \sum_{z=1}^m \omega_j \tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_{zj}})}{\varrho}. \quad (26)$$

Using the above equations, we get

$$\varrho = \sqrt{\sum_{j=1}^n \sum_{i=1}^m \sum_{z=1}^m \left(\tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_{zj}}) \right)^2}, \quad (27)$$

where $\sqrt{\sum_{j=1}^n \sum_{i=1}^m \sum_{z=1}^m \left(\tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_{zj}}) \right)^2}$ means the sum of deviations of all the alternatives with respect to all the criteria.

From equations (26) and (27), we get

$$\omega_j = \frac{\sum_{j=1}^n \sum_{i=1}^m \sum_{z=1}^m \tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_{zj}})}{\sqrt{\sum_{j=1}^n \left(\sum_{i=1}^m \sum_{z=1}^m \left(\tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_{zj}}) \right)^2 \right)}} \quad (28)$$

By normalization of ω_j , we make sum into unity, and we get

$$\omega_j = \frac{\sum_{i=1}^m \sum_{z=1}^m \left((1/9) \left[\left| (\tilde{a}_{ij}^-)^2 - (\tilde{a}_{zj}^-)^2 \right| + \left| (\tilde{a}_{ij}^+)^2 - (\tilde{a}_{zj}^+)^2 \right| + \left| (\tilde{\lambda}_{ij})^2 - (\tilde{\lambda}_{zj})^2 \right| + \left| (\tilde{n}_{ij}^-)^2 - (\tilde{n}_{zj}^-)^2 \right| + \left| (\tilde{n}_{ij}^+)^2 - (\tilde{n}_{zj}^+)^2 \right| + \left| (\tilde{\mu}_{ij})^2 - (\tilde{\mu}_{zj})^2 \right| + \left| (\tilde{b}_{ij}^-)^2 - (\tilde{b}_{zj}^-)^2 \right| + \left| (\tilde{b}_{ij}^+)^2 - (\tilde{b}_{zj}^+)^2 \right| + \left| (\tilde{\vartheta}_{ij})^2 - (\tilde{\vartheta}_{zj})^2 \right| \right] \right)}{\sum_{j=1}^n \left(\sum_{i=1}^m \sum_{z=1}^m \left((1/9) \left[\left| (\tilde{a}_{ij}^-)^2 - (\tilde{a}_{zj}^-)^2 \right| + \left| (\tilde{a}_{ij}^+)^2 - (\tilde{a}_{zj}^+)^2 \right| + \left| (\tilde{\lambda}_{ij})^2 - (\tilde{\lambda}_{zj})^2 \right| + \left| (\tilde{n}_{ij}^-)^2 - (\tilde{n}_{zj}^-)^2 \right| + \left| (\tilde{n}_{ij}^+)^2 - (\tilde{n}_{zj}^+)^2 \right| + \left| (\tilde{\mu}_{ij})^2 - (\tilde{\mu}_{zj})^2 \right| + \left| (\tilde{b}_{ij}^-)^2 - (\tilde{b}_{zj}^-)^2 \right| + \left| (\tilde{b}_{ij}^+)^2 - (\tilde{b}_{zj}^+)^2 \right| + \left| (\tilde{\vartheta}_{ij})^2 - (\tilde{\vartheta}_{zj})^2 \right| \right] \right) \right)} \quad (29)$$

There are, however, real cases where the weight vector information is not totally unknown but is slightly modified. For partially known weight information, we construct the following constrained optimization model.

3.2.2. Second Model

$$\begin{aligned} \max \quad & \tilde{D}_j(\omega) = \sum_{j=1}^n \left(\sum_{i=1}^m \sum_{z=1}^m \left((1/9) \left[\left| (\tilde{a}_{ij}^-)^2 - (\tilde{a}_{zj}^-)^2 \right| + \left| (\tilde{a}_{ij}^+)^2 - (\tilde{a}_{zj}^+)^2 \right| + \left| (\tilde{\lambda}_{ij})^2 - (\tilde{\lambda}_{zj})^2 \right| + \left| (\tilde{n}_{ij}^-)^2 - (\tilde{n}_{zj}^-)^2 \right| + \left| (\tilde{n}_{ij}^+)^2 - (\tilde{n}_{zj}^+)^2 \right| + \left| (\tilde{\mu}_{ij})^2 - (\tilde{\mu}_{zj})^2 \right| + \left| (\tilde{b}_{ij}^-)^2 - (\tilde{b}_{zj}^-)^2 \right| + \left| (\tilde{b}_{ij}^+)^2 - (\tilde{b}_{zj}^+)^2 \right| + \left| (\tilde{\vartheta}_{ij})^2 - (\tilde{\vartheta}_{zj})^2 \right| \right] \right) \right), \\ \text{s.that} \quad & \sum_{j=1}^n \omega_j = 1. \end{aligned} \quad (30)$$

The weight value ω_j is also a set of restricted conditions where $\tilde{\Delta}$ is the criteria that should be satisfied. The second model given in equation (29) is a linear programming model that can be implemented using the software LINGO 11.0. We get the optimal solution $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, which can be used as the weight vector of criteria by solving this model.

3.3. Proposed Technique. In the spherical cubic fuzzy aggregation operator process [28], too much information is lost due to the difficulty of the spherical cubic fuzzy aggregating process, which means a lack of consistency in the final results. We have therefore expanded the TOPSIS approach to take into account spherical cubic information in

order to address this limitation and have used the distance measurements of SCFNs to obtain the final ranking of the alternatives. TOPSIS is a kind of method for solving MCDM problems with the shortest distance from the positive ideal solution ($\check{P}IS$) to select the alternative. The farthest distance from the negative ideal solution ($\check{N}IS$) is commonly used in actual situations to deal with the ranking problems. Under the notation of SCF, the spherical cubic fuzzy positive ideal solution (SCF $\check{P}IS$) is expressed by \check{p}^+ , and it is possible to write the spherical negative ideal solution with (SCF $\check{N}IS$) expressed by \check{p}^- .

Let $\tilde{S}_{c_1} = (\langle [\tilde{a}_1^-, \tilde{a}_1^+], \tilde{\lambda}_1 \rangle, \langle [\tilde{n}_1^-, \tilde{n}_1^+], \tilde{\mu}_1 \rangle, \langle [\tilde{b}_1^-, \tilde{b}_1^+], \tilde{\vartheta}_1 \rangle)$ and $\tilde{S}_{c_2} = (\langle [\tilde{a}_2^-, \tilde{a}_2^+], \tilde{\lambda}_2 \rangle, \langle [\tilde{n}_2^-, \tilde{n}_2^+], \tilde{\mu}_2 \rangle, \langle [\tilde{b}_2^-, \tilde{b}_2^+], \tilde{\vartheta}_2 \rangle)$ be two spherical cubic fuzzy sets (SCFSs), then

$$\check{p}^+ = \left\langle \max_i \{[\tilde{a}_1^-, \tilde{a}_1^+], [\tilde{a}_2^-, \tilde{a}_2^+]\}, \max_i \{[\tilde{n}_1^-, \tilde{n}_1^+], [\tilde{n}_2^-, \tilde{n}_2^+]\}, \max_i \{[\tilde{b}_1^-, \tilde{b}_1^+], [\tilde{b}_2^-, \tilde{b}_2^+]\}, \right. \\ \left. \min_i \{\tilde{\lambda}_1, \tilde{\lambda}_2\}, \min_i \{\tilde{\mu}_1, \tilde{\mu}_2\}, \min_i \{\tilde{\vartheta}_1, \tilde{\vartheta}_2\} \right\rangle, \quad (31)$$

$$\check{p}^- = \left\langle \min_i \{[\tilde{a}_1^-, \tilde{a}_1^+], [\tilde{a}_2^-, \tilde{a}_2^+]\}, \min_i \{[\tilde{n}_1^-, \tilde{n}_1^+], [\tilde{n}_2^-, \tilde{n}_2^+]\}, \min_i \{[\tilde{b}_1^-, \tilde{b}_1^+], [\tilde{b}_2^-, \tilde{b}_2^+]\}, \right. \\ \left. \max_i \{\tilde{\lambda}_1, \tilde{\lambda}_2\}, \max_i \{\tilde{\mu}_1, \tilde{\mu}_2\}, \max_i \{\tilde{\vartheta}_1, \tilde{\vartheta}_2\} \right\rangle. \quad (32)$$

The separated distance measures \check{d}^+ and \check{d}^- for alternatives of (SCF $\check{P}IS$) \check{p}^+ and (SCF $\check{N}IS$) \check{p}^- are formulated as

$$\check{d}^+ = \sum_{j=1}^n \omega_j \tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_j}^+), \\ = \frac{1}{9} \sum_{j=1}^n \omega_j \left[\begin{aligned} &|(\tilde{a}_{ij}^-)^2 - \left(\tilde{\tilde{a}}_j^+\right)^2| + |(\tilde{a}_{ij}^+)^2 - \left(\tilde{\tilde{a}}_j^+\right)^2| + |(\tilde{\lambda}_{ij})^2 - \left(\tilde{\tilde{\lambda}}_j^+\right)^2| + \\ &|(\tilde{n}_{ij}^-)^2 - \left(\tilde{\tilde{n}}_j^+\right)^2| + |(\tilde{n}_{ij}^+)^2 - \left(\tilde{\tilde{n}}_j^+\right)^2| + |(\tilde{\mu}_{ij})^2 - \left(\tilde{\tilde{\mu}}_j^+\right)^2| + \\ &|(\tilde{b}_{ij}^-)^2 - \left(\tilde{\tilde{b}}_j^+\right)^2| + |(\tilde{b}_{ij}^+)^2 - \left(\tilde{\tilde{b}}_j^+\right)^2| + |(\tilde{\vartheta}_{ij})^2 - \left(\tilde{\tilde{\vartheta}}_j^+\right)^2| \end{aligned} \right], \quad (33)$$

$$\check{d}^- = \sum_{j=1}^n \omega_j \tilde{d}(\tilde{S}_{c_{ij}}, \tilde{S}_{c_j}^-), \\ = \frac{1}{9} \sum_{j=1}^n \omega_j \left[\begin{aligned} &|(\tilde{a}_{ij}^-)^2 - \left(\tilde{\tilde{a}}_j^-\right)^2| + |(\tilde{a}_{ij}^+)^2 - \left(\tilde{\tilde{a}}_j^-\right)^2| + |(\tilde{\lambda}_{ij})^2 - \left(\tilde{\tilde{\lambda}}_j^-\right)^2| + \\ &|(\tilde{n}_{ij}^-)^2 - \left(\tilde{\tilde{n}}_j^-\right)^2| + |(\tilde{n}_{ij}^+)^2 - \left(\tilde{\tilde{n}}_j^-\right)^2| + |(\tilde{\mu}_{ij})^2 - \left(\tilde{\tilde{\mu}}_j^-\right)^2| + \\ &|(\tilde{b}_{ij}^-)^2 - \left(\tilde{\tilde{b}}_j^-\right)^2| + |(\tilde{b}_{ij}^+)^2 - \left(\tilde{\tilde{b}}_j^-\right)^2| + |(\tilde{\vartheta}_{ij})^2 - \left(\tilde{\tilde{\vartheta}}_j^-\right)^2| \end{aligned} \right]. \quad (34)$$

Relative coefficient of closeness of Z_i to $(SCF\check{P}IS)\check{p}^+$ is

$$\bar{C}_i = \frac{\check{d}_i^-}{\check{d}_i^+ + \check{d}_i^-}, \quad (35)$$

where $\bar{C}_i \in [0, 1]$. Alternative Z_i is clearly similar to $(SCF\check{P}IS)\check{p}^+$ and further from $(SCF\check{N}IS)\check{p}^-$ as \bar{C}_i approaches 1. Therefore, we decide the ranking orders of all alternatives according to the closeness coefficient \bar{C}_i and choose the best one from a set of feasible alternatives. We will develop an effective approach to solving MCDM problems based on the above models in which the attribute weight information is incomplete or entirely unknown, and the attribute values take the form of SCFNs.

The following are the steps of our proposed technique:

Step 1: first of all, we will construct the decision matrices $\check{D} = (\check{S}_{c_{ij}})_{m \times n} = (\langle [\check{a}_{ij}^-, \check{a}_{ij}^+], \check{\lambda}_1 \rangle, \langle [\check{n}_{ij}^-, \check{n}_{ij}^+], \check{\mu}_{ij} \rangle, \langle [\check{b}_{ij}^-, \check{b}_{ij}^+], \check{\vartheta}_{ij} \rangle)_{m \times n}$, where $(i = 1, 2, 3, \dots, m), j = (1, 2, 3, \dots, n)$ are SCFNs, for the alternative Z_i and the criteria A_j

Step 2: we use the SCFWG operator to aggregate all the spherical cubic fuzzy decision matrices

Step 3: if the knowledge of the criteria weights is absolutely unknown, the first model can be used to obtain them; and if the knowledge of the criteria weights is not completely known but is partially known, then the criteria weights can be determined using the second model

Step 4: using equations (30) and (31), we will find $(SCF\check{P}IS)\check{p}^+$ and $(SCF\check{N}IS)\check{p}^-$

Step 5: using equations (33) and (34), we will find \check{d}_i^+ and \check{d}_i^-

Step 6: we rank all the alternatives Z_i and select the best one

4. Illustrative Description

In this section, we will present a numerical example to demonstrate the potential assessment of the commercialization of emerging technology with spherical cubic fuzzy information to illustrate the approach proposed in this paper. There is a panel with four potential $Z_i (i = 1, 2, 3, 4)$ new technology companies to select. To assess the three potential emerging technology enterprises, the experts select four attributes: (1) A_1 is the technological advance; (2) A_2 is the potential demand and market risk; (3) A_3 is the infrastructure for industrialization; and (4) A_4 represents the human economic and financial conditions. Three decision-makers whose weighted vector is $(0.25, 0.30, 0.45)^T$ and the spherical cubic fuzzy decision matrices are provided in Tables 1–3. The four potential emerging technology companies should be evaluated using the spherical cubic fuzzy information.

Assume that the information about the attribute weights is completely unknown; according to the following steps, we get the most suitable alternatives.

Step 1: In Tables 1–3, the decision-makers have decision.

Step 2: The SCFWG operator is used to aggregate all the spherical cubic fuzzy decision matrices.

Step 3: We obtain the weight vector by using equation (29) with the matrix in Table 4:

$$\omega = (0.2110, 0.2884, 0.2670, 0.2336)^T. \quad (36)$$

Step 4: The $SCF\check{P}IS \check{p}^+$ and $SCF\check{N}IS \check{p}^-$ are given by equation (31) and equation (32):

$$\begin{aligned} \check{p}^+ &= \left\{ \begin{pmatrix} \langle [0.6000, 0.7000], 0.6000 \rangle, \\ \langle [0.2000, 0.4026], 0.3249 \rangle, \\ \langle [0.3000, 0.4000], 0.5000 \rangle \end{pmatrix}, \begin{pmatrix} \langle [0.4676, 0.6000], 0.3669 \rangle, \\ \langle [0.3085, 0.4315], 0.4925 \rangle, \\ \langle [0.2351, 0.3341], 0.4734 \rangle \end{pmatrix} \right\}, \\ \check{p}^- &= \left\{ \begin{pmatrix} \langle [0.5681, 0.7000], 0.6284 \rangle, \\ \langle [0.1625, 0.4517], 0.1231 \rangle, \\ \langle [0.3341, 0.4752], 0.3000 \rangle \end{pmatrix}, \begin{pmatrix} \langle [0.4277, 0.5281], 0.5681 \rangle, \\ \langle [0.2633, 0.4290], 0.2158 \rangle, \\ \langle [0.2744, 0.4530], 0.2000 \rangle \end{pmatrix} \right\}, \\ \check{p}^- &= \left\{ \begin{pmatrix} \langle [0.3669, 0.5918], 0.1793 \rangle, \\ \langle [0.5000, 0.2912], 0.2000 \rangle, \\ \langle [0.4752, 0.5761], 0.7320 \rangle \end{pmatrix}, \begin{pmatrix} \langle [0.3085, 0.4315], 0.2781 \rangle, \\ \langle [0.2633, 0.4676], 0.3249 \rangle, \\ \langle [0.3294, 0.4275], 0.5373 \rangle \end{pmatrix} \right\}, \\ \check{p}^- &= \left\{ \begin{pmatrix} \langle [0.4676, 0.6284], 0.6684 \rangle, \\ \langle [0.5757, 0.3497], 0.2000 \rangle, \\ \langle [0.4337, 0.5337], 0.4257 \rangle \end{pmatrix}, \begin{pmatrix} \langle [0.3249, 0.4290], 0.2912 \rangle, \\ \langle [0.2624, 0.1621], 0.4000 \rangle, \\ \langle [0.2351, 0.4000], 0.5265 \rangle \end{pmatrix} \right\} \end{aligned} \quad (37)$$

Step 5: for calculating \check{d}_i^+ and \check{d}_i^- , equation (33) and equation (34) are used:

TABLE 1: 1st spherical cubic fuzzy decision-making.

	A_1	A_2	A_3	A_4
Z_1	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$
Z_2	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$
Z_3	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$
Z_4	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$

TABLE 2: 2nd spherical cubic fuzzy decision-making.

	A_1	A_2	A_3	A_4
Z_1	$\langle [0.3, 0.4], 0.4 \rangle,$ $\langle [0.2, 0.3], 0.5 \rangle,$ $\langle [0.6, 0.7], 0.6 \rangle.$	$\langle [0.2, 0.4], 0.4 \rangle,$ $\langle [0.5, 0.6], 0.3 \rangle,$ $\langle [0.5, 0.6], 0.3 \rangle.$	$\langle [0.7, 0.8], 0.5 \rangle,$ $\langle [0.6, 0.4], 0.2 \rangle,$ $\langle [0.2, 0.3], 0.6 \rangle.$	$\langle [0.5, 0.6], 0.5 \rangle,$ $\langle [0.5, 0.3], 0.1 \rangle,$ $\langle [0.2, 0.3], 0.2 \rangle.$
Z_2	$\langle [0.3, 0.4], 0.7 \rangle,$ $\langle [0.5, 0.7], 0.2 \rangle,$ $\langle [0.6, 0.7], 0.4 \rangle.$	$\langle [0.1, 0.2], 0.6 \rangle,$ $\langle [0.5, 0.4], 0.2 \rangle,$ $\langle [0.5, 0.6], 0.3 \rangle.$	$\langle [0.4, 0.7], 0.6 \rangle,$ $\langle [0.8, 0.5], 0.2 \rangle,$ $\langle [0.5, 0.6], 0.6 \rangle.$	$\langle [0.2, 0.3], 0.1 \rangle,$ $\langle [0.4, 0.5], 0.4 \rangle,$ $\langle [0.3, 0.4], 0.3 \rangle.$
Z_3	$\langle [0.6, 0.7], 0.3 \rangle,$ $\langle [0.2, 0.4], 0.1 \rangle,$ $\langle [0.3, 0.4], 0.6 \rangle.$	$\langle [0.2, 0.3], 0.5 \rangle,$ $\langle [0.6, 0.2], 0.1 \rangle,$ $\langle [0.5, 0.6], 0.3 \rangle.$	$\langle [0.4, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.4], 0.6 \rangle,$ $\langle [0.5, 0.6], 0.3 \rangle.$	$\langle [0.2, 0.3], 0.7 \rangle,$ $\langle [0.4, 0.5], 0.4 \rangle,$ $\langle [0.3, 0.4], 0.2 \rangle.$
Z_4	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.8], 0.2 \rangle,$ $\langle [0.3, 0.4], 0.5 \rangle.$	$\langle [0.4, 0.6], 0.3 \rangle,$ $\langle [0.1, 0.2], 0.8 \rangle,$ $\langle [0.3, 0.4], 0.4 \rangle.$	$\langle [0.5, 0.7], 0.7 \rangle,$ $\langle [0.1, 0.6], 0.2 \rangle,$ $\langle [0.4, 0.6], 0.3 \rangle.$	$\langle [0.2, 0.3], 0.6 \rangle,$ $\langle [0.3, 0.6], 0.2 \rangle,$ $\langle [0.5, 0.6], 0.1 \rangle.$

TABLE 3: 3rd spherical cubic fuzzy decision-making.

	A_1	A_2	A_3	A_4
Z_1	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.4 \rangle.$	$\langle [0.4, 0.5], 0.4 \rangle,$ $\langle [0.2, 0.5], 0.1 \rangle,$ $\langle [0.3, 0.4], 0.3 \rangle.$	$\langle [0.5, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.1], 0.6 \rangle,$ $\langle [0.4, 0.5], 0.4 \rangle.$	$\langle [0.4, 0.5], 0.6 \rangle,$ $\langle [0.2, 0.5], 0.3 \rangle,$ $\langle [0.3, 0.5], 0.2 \rangle.$
Z_2	$\langle [0.4, 0.7], 0.1 \rangle,$ $\langle [0.5, 0.2], 0.2 \rangle,$ $\langle [0.4, 0.5], 0.8 \rangle.$	$\langle [0.5, 0.6], 0.2 \rangle,$ $\langle [0.2, 0.5], 0.4 \rangle,$ $\langle [0.2, 0.3], 0.6 \rangle.$	$\langle [0.5, 0.6], 0.7 \rangle,$ $\langle [0.5, 0.3], 0.2 \rangle,$ $\langle [0.4, 0.5], 0.3 \rangle.$	$\langle [0.4, 0.5], 0.7 \rangle,$ $\langle [0.5, 0.2], 0.1 \rangle,$ $\langle [0.3, 0.4], 0.1 \rangle.$
Z_3	$\langle [0.6, 0.7], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.3 \rangle,$ $\langle [0.3, 0.4], 0.6 \rangle.$	$\langle [0.4, 0.5], 0.6 \rangle,$ $\langle [0.2, 0.1], 0.5 \rangle,$ $\langle [0.3, 0.4], 0.3 \rangle.$	$\langle [0.6, 0.7], 0.7 \rangle,$ $\langle [0.2, 0.1], 0.4 \rangle,$ $\langle [0.3, 0.5], 0.3 \rangle.$	$\langle [0.4, 0.5], 0.2 \rangle,$ $\langle [0.2, 0.1], 0.4 \rangle,$ $\langle [0.2, 0.4], 0.6 \rangle.$
Z_4	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.4 \rangle,$ $\langle [0.3, 0.4], 0.5 \rangle.$	$\langle [0.5, 0.6], 0.4 \rangle,$ $\langle [0.5, 0.6], 0.4 \rangle,$ $\langle [0.2, 0.3], 0.5 \rangle.$	$\langle [0.6, 0.7], 0.6 \rangle,$ $\langle [0.2, 0.4], 0.1 \rangle,$ $\langle [0.3, 0.4], 0.3 \rangle.$	$\langle [0.5, 0.6], 0.7 \rangle,$ $\langle [0.2, 0.5], 0.6 \rangle,$ $\langle [0.2, 0.3], 0.1 \rangle.$

$$\begin{aligned}
d_1^+ &= 0.0818, \\
d_2^+ &= 0.0987, \\
d_3^+ &= 0.0925, \\
d_4^+ &= 0.1108, \\
d_1^- &= 0.1273, \\
d_2^- &= 0.0799, \\
d_3^- &= 0.1074, \\
d_4^- &= 0.1326.
\end{aligned} \tag{38}$$

Step 6: calculate \tilde{C}_i by using equation (35):

$$\begin{aligned}
\tilde{C}_1 &= 0.6090, \\
\tilde{C}_2 &= 0.4473, \\
\tilde{C}_3 &= 0.5372, \\
\tilde{C}_4 &= 0.5448.
\end{aligned} \tag{39}$$

Step 7: rank all the alternatives Z_i ($i = 1, 2, 3, 4$) according to \tilde{C}_i :

$$Z_1 > Z_4 > Z_3 > Z_2. \tag{40}$$

5. Comparison

The proposed method is compared and shown to be more general while achieving the same results as existing technique. We convert the more general SCFN to IVSFN to do this. In order to achieve these nonmembership values, spherical fuzzy numbers (SFNs) are omitted. Examples of this are given in the following sections.

5.1. Comparison with Interval-Valued Spherical Fuzzy Sets. By removing the nonmembership degree, SCFNs can be converted to IVSFNs. Table 5 presents the interval-valued spherical fuzzy information.

By using IVSF TOPSIS methodology, IVSF \check{P} IS \check{p}^+ and IVSF \check{N} IS \check{p}^- for IVSF are as follows.

The IVSF \check{P} IS \check{p}^+ and IVSF \check{N} IS \check{p}^- are given by equations (31) and (32):

$$p^+ = \left\{ \begin{array}{l} \left(\begin{array}{l} [0.6000, 0.7000], \\ [0.2000, 0.4026], \\ [0.3000, 0.4000], \end{array} \right), \left(\begin{array}{l} [0.4676, 0.6000], \\ [0.3085, 0.4315], \\ [0.2351, 0.3341], \end{array} \right) \\ \left(\begin{array}{l} [0.5681, 0.7000], \\ [0.1625, 0.4517], \\ [0.3341, 0.4752], \end{array} \right), \left(\begin{array}{l} [0.4277, 0.5281], \\ [0.2633, 0.4290], \\ [0.2744, 0.4530], \end{array} \right) \end{array} \right\},$$

$$p^- = \left\{ \begin{array}{l} \left(\begin{array}{l} [0.3669, 0.5918], \\ [0.5000, 0.2912], \\ [0.4752, 0.5761], \end{array} \right), \left(\begin{array}{l} [0.3085, 0.4315], \\ [0.2633, 0.4676], \\ [0.3294, 0.4275], \end{array} \right) \\ \left(\begin{array}{l} [0.4676, 0.6284], \\ [0.5757, 0.3497], \\ [0.4337, 0.5337], \end{array} \right), \left(\begin{array}{l} [0.3249, 0.4290], \\ [0.2624, 0.1621], \\ [0.2351, 0.4000], \end{array} \right) \end{array} \right\}, \quad (41)$$

The distance measures d_i^+ and d_i^- IVSF \check{P} IS \check{p}^+ and the IVSF \check{N} IS \check{p}^- are as follows:

$$\begin{aligned} d_1^+ &= 0.0411, \\ d_2^+ &= 0.0703, \\ d_3^+ &= 0.0489, \\ d_4^+ &= 0.0633, \\ d_1^- &= 0.0693, \\ d_2^- &= 0.0464, \\ d_3^- &= 0.0666, \\ d_4^- &= 0.0649. \end{aligned} \quad (42)$$

Calculate \tilde{C}_i by using equation (35):

$$\begin{aligned} \tilde{C}_1 &= 0.6277, \\ \tilde{C}_2 &= 0.3974, \\ \tilde{C}_3 &= 0.5768, \\ \tilde{C}_4 &= 0.5065. \end{aligned} \quad (43)$$

Rank all the alternatives Z_i ($i = 1, 2, 3, 4$) according to \tilde{C}_i :

$$Z_1 > Z_3 > Z_4 > Z_2. \quad (44)$$

Hence, Z_1 is the best option.

This paper applied the TOPSIS approach to spherical cubic fuzzy sets. This approach has also been shown to provide more general knowledge than previous techniques. If several contradictory and/or unknown variables characterize the information needed for decision-making, this approach is able to decide the best decision and handle some uncertainty which other methods cannot, thus enabling decision-makers to take more informed decisions.

5.2. Comparative Study with Spherical Fuzzy Sets. SFNs are special types of SCFNs in which decision-makers only determine the roles of membership, neutral, and nonmembership. Table 6 demonstrates the membership, neutral, and nonmembership of an SCFN converted to SFN by removing the interval portion of SCFN.

Based on Table 6, spherical fuzzy TOPSIS is utilized to calculate SF (PIS p^+) and SF (NIS p^-) as follows:

$$p^+ = \left\{ \begin{array}{l} (0.6000, 0.3249, 0.5000), (0.3669, 0.4925, 0.4734) \\ (0.6284, 0.1231, 0.3000), (0.5681, 0.2158, 0.2000) \end{array} \right\},$$

$$p^- = \left\{ \begin{array}{l} (0.1793, 0.2000, 0.7320), (0.2781, 0.3249, 0.5373) \\ (0.6684, 0.2000, 0.4257), (0.2912, 0.4000, 0.5265) \end{array} \right\} \quad (45)$$

The distance measures d_i^+ and d_i^- SF p^+ and the SF p^- are as follows:

$$\begin{aligned} d_1^+ &= 0.3657, \\ d_2^+ &= 0.0284, \\ d_3^+ &= 0.0532, \\ d_4^+ &= 0.0426, \\ d_1^- &= 0.0545, \\ d_2^- &= 0.0336, \\ d_3^- &= 0.0425, \\ d_4^- &= 0.0659. \end{aligned} \quad (46)$$

Calculate \tilde{C}_i by using equation (35):

$$\begin{aligned} \tilde{C}_1 &= 0.1298, \\ \tilde{C}_2 &= 0.5413, \\ \tilde{C}_3 &= 0.4437, \\ \tilde{C}_4 &= 0.6075. \end{aligned} \quad (47)$$

Rank all the alternatives Z_i ($i = 1, 2, 3, 4$) according to \tilde{C}_i :

$$Z_4 > Z_2 > Z_3 > Z_1. \quad (48)$$

Hence, Z_4 is the best option.

This approach varies from the previous method in this paper in the order of the list of decisions. In particular, all alternatives have switched positions. Since SFN does not

TABLE 4: 4th spherical cubic fuzzy decision-making.

	A_1	A_2	A_3	A_4
Z_1	$\langle [0.4874, 0.5918], 0.5313 \rangle,$ $\langle [0.2000, 0.3000], 0.4277 \rangle,$ $\langle [0.3872, 0.5261], 0.4752 \rangle.$	$\langle [0.3249, 0.4676], 0.4000 \rangle,$ $\langle [0.2633, 0.5281], 0.1390 \rangle,$ $\langle [0.3776, 0.4752], 0.3000 \rangle.$	$\langle [0.5531, 0.7286], 0.5681 \rangle,$ $\langle [0.2781, 0.1516], 0.4315 \rangle,$ $\langle [0.3545, 0.4530], 0.4752 \rangle.$	$\langle [0.4277, 0.5281], 0.5681 \rangle,$ $\langle [0.2633, 0.4290], 0.2158 \rangle,$ $\langle [0.2744, 0.4530], 0.2000 \rangle.$
Z_2	$\langle [0.3669, 0.5918], 0.1793 \rangle,$ $\langle [0.5000, 0.2912], 0.2000 \rangle,$ $\langle [0.4752, 0.5761], 0.7320 \rangle.$	$\langle [0.3085, 0.4315], 0.2781 \rangle,$ $\langle [0.2633, 0.4676], 0.3249 \rangle,$ $\langle [0.3294, 0.4275], 0.5373 \rangle.$	$\langle [0.4676, 0.6284], 0.6684 \rangle,$ $\langle [0.5757, 0.3497], 0.2000 \rangle,$ $\langle [0.4337, 0.5337], 0.4257 \rangle.$	$\langle [0.3249, 0.4290], 0.3905 \rangle,$ $\langle [0.4676, 0.2633], 0.1516 \rangle,$ $\langle [0.3000, 0.4000], 0.1863 \rangle.$
Z_3	$\langle [0.6000, 0.7000], 0.3669 \rangle,$ $\langle [0.2000, 0.4000], 0.2158 \rangle,$ $\langle [0.3000, 0.4000], 0.6000 \rangle.$	$\langle [0.3249, 0.4290], 0.5681 \rangle,$ $\langle [0.2781, 0.1231], 0.3085 \rangle,$ $\langle [0.3759, 0.4752], 0.3000 \rangle.$	$\langle [0.5313, 0.7000], 0.6684 \rangle,$ $\langle [0.2000, 0.1516], 0.4517 \rangle,$ $\langle [0.3759, 0.5337], 0.3000 \rangle.$	$\langle [0.3249, 0.4290], 0.2912 \rangle,$ $\langle [0.2624, 0.1621], 0.4000 \rangle,$ $\langle [0.2351, 0.4000], 0.5265 \rangle.$
Z_4	$\langle [0.6000, 0.7000], 0.6000 \rangle,$ $\langle [0.2000, 0.4026], 0.3249 \rangle,$ $\langle [0.3000, 0.4000], 0.5000 \rangle.$	$\langle [0.4676, 0.6000], 0.3669 \rangle,$ $\langle [0.3085, 0.4315], 0.4925 \rangle,$ $\langle [0.2351, 0.3341], 0.4734 \rangle.$	$\langle [0.5681, 0.7000], 0.6284 \rangle,$ $\langle [0.1625, 0.4517], 0.1231 \rangle,$ $\langle [0.3341, 0.4752], 0.3000 \rangle.$	$\langle [0.3789, 0.4874], 0.6684 \rangle,$ $\langle [0.2259, 0.5281], 0.4315 \rangle,$ $\langle [0.3294, 0.4257], 0.1000 \rangle.$

TABLE 5: IVSFD matrix.

	A_1	A_2	A_3	A_4
Z_1	$\begin{pmatrix} [0.4874, 0.5918], \\ [0.2000, 0.3000], \\ [0.3872, 0.5261]. \end{pmatrix}$	$\begin{pmatrix} [0.3249, 0.4676], \\ [0.2633, 0.5281], \\ [0.3776, 0.4752]. \end{pmatrix}$	$\begin{pmatrix} [0.5531, 0.7286], \\ [0.2781, 0.1516], \\ [0.3545, 0.4530]. \end{pmatrix}$	$\begin{pmatrix} [0.4277, 0.5281], \\ [0.2633, 0.4290], \\ [0.2744, 0.4530]. \end{pmatrix}$
Z_2	$\begin{pmatrix} [0.3669, 0.5918], \\ [0.5000, 0.2912], \\ [0.4752, 0.5761]. \end{pmatrix}$	$\begin{pmatrix} [0.3085, 0.4315], \\ [0.2633, 0.4676], \\ [0.3294, 0.4275]. \end{pmatrix}$	$\begin{pmatrix} [0.4676, 0.6284], \\ [0.5757, 0.3497], \\ [0.4337, 0.5337]. \end{pmatrix}$	$\begin{pmatrix} [0.3249, 0.4290], \\ [0.4676, 0.2633], \\ [0.3000, 0.4000]. \end{pmatrix}$
Z_3	$\begin{pmatrix} [0.6000, 0.7000], \\ [0.2000, 0.4000], \\ [0.3000, 0.4000]. \end{pmatrix}$	$\begin{pmatrix} [0.3249, 0.4290], \\ [0.2781, 0.1231], \\ [0.3759, 0.4752]. \end{pmatrix}$	$\begin{pmatrix} [0.5313, 0.7000], \\ [0.2000, 0.1516], \\ [0.3759, 0.5337]. \end{pmatrix}$	$\begin{pmatrix} [0.3249, 0.4290], \\ [0.2624, 0.1621], \\ [0.2351, 0.4000]. \end{pmatrix}$
Z_4	$\begin{pmatrix} [0.6000, 0.7000], \\ [0.2000, 0.4026], \\ [0.3000, 0.4000]. \end{pmatrix}$	$\begin{pmatrix} [0.4676, 0.6000], \\ [0.3085, 0.4315], \\ [0.2351, 0.3341]. \end{pmatrix}$	$\begin{pmatrix} [0.5681, 0.7000], \\ [0.1625, 0.4517], \\ [0.3341, 0.4752]. \end{pmatrix}$	$\begin{pmatrix} [0.3789, 0.4874], \\ [0.2259, 0.5281], \\ [0.3294, 0.4257]. \end{pmatrix}$

TABLE 6: Spherical cubic fuzzy decision-making.

	A_1	A_2	A_3	A_4
Z_1	(0.5313, 0.4277, 0.4752)	(0.4000, 0.1390, 0.3000)	(0.5681, 0.4315, 0.4752)	(0.5681, 0.2158, 0.2000)
Z_2	(0.1793, 0.2000, 0.7320)	(0.2781, 0.3249, 0.5373)	(0.6684, 0.2000, 0.4257)	(0.3905, 0.1516, 0.1863)
Z_3	(0.3669, 0.2158, 0.6000)	(0.5681, 0.3085, 0.3000)	(0.6684, 0.4517, 0.3000)	(0.2912, 0.4000, 0.5265)
Z_4	(0.6000, 0.3249, 0.5000)	(0.3669, 0.4925, 0.4734)	(0.6284, 0.1231, 0.3000)	(0.6684, 0.4315, 0.1000)

contain as much details as just membership, neutral, and nonmembership that can result in loss of data, this results in a different result.

We have the following benefits from the above analysis:

- (1) SCFNs can convey uncertainty in the MCDM more accurately than IVSFNs. In other words, SCFNs are the IVSFN extension. We should, therefore, know that the SCFNs have a greater prospect of application than the IVSFNs.
- (2) The proposed approach combines decision-maker expectations and intuition, decreasing the possibility of MCDM problems.
- (3) The method presented in this paper is a modern extension of an existing technique that can solve a greater variety of MCDM problems compared to previously defined TOPSIS methods.

6. Conclusions

Many realistic MCGDM problems arise in a dynamic setting and frequently conform to incomplete data and ambiguity. The SCFS is a very powerful method to tackle the fuzziness of the experts' decisions on alternative parameters. In this paper, we have first developed a method called the maximizing deviation method to determine the optimal relative weights of criteria based on a spherical cubic fuzzy setting. An important advantage of the proposed method is its ability to relieve the influence of subjectivity of the experts and at the same time keep the original decision information sufficiently. Then we suggested an expanded TOPSIS-based approach for solving the spherical cubic fuzzy knowledge MCGDM problems. The graphical representation of comparison analysis by using SCFS, IVSFS, and SFS is given in Figure 2.

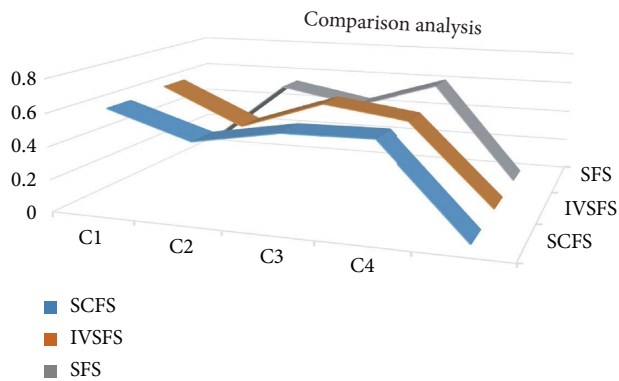


FIGURE 2: Comparison of extended TOPSIS SCFS with IVSFS and SFS.

The method is based on the relative similarity of each alternative for determining the ranking order of all alternatives, which avoids the loss of too much information in the process of aggregating information. Finally, an illustration shows the efficiency and applicability of the proposed process. Our solution tends to be straightforward and to have less knowledge loss and can easily be extended to other management decisions in a hesitant spherical environment. TOPSIS technique is useful in finding the unknown weights. For this method, we find the accurate weight vector.

In the future, under spherical cubic fuzzy set, we will implement the principle of TODIM methods. We will also describe the spherical cubic fuzzy linguistic sets and propose the TOPSIS and TODIM MCGDM-based methods in a spherical cubic fuzzy linguistic environment. Furthermore, we can extend the TOPSIS for confidence interval using the spherical cubic fuzzy set for multicriteria group decision-making.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of the research article.

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Retraction

Retracted: A New Method for the Aggregate Proportion Calculation and Gradation Optimization of Asphalt-Treated Base (ATB-25)

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

A New Method for the Aggregate Proportion Calculation and Gradation Optimization of Asphalt-Treated Base (ATB-25)

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Asphalt-treated base (ATB-25) is a widely used flexible base material. The composition and gradation of mineral aggregate are important factors affecting pavement performance of asphalt treated base. In this study, two new methods were proposed to address the problems of existing aggregate proportion calculation for asphalt mixtures: (1) the combination of generalized inverse solution of the normal equation and spreadsheet trial and (2) quadratic programming. Both methods can calculate mass ratios of various aggregates in a quick and accurate manner. The orthogonal test was used to design nine aggregate gradations within the range of asphalt treated base (ATB-25) stated in the industrial standard. The aggregate proportion was calculated by two new methods. The Marshall test, water weight test, rutting test, and water-soaked Marshall test were carried out on the asphalt mixture specimens. The pavement performance test results were fuzzified using the fuzzy mathematics method, and the weights of pavement performance evaluation indexes were determined through the analytic hierarchy process. Taking the fuzzy comprehensive evaluation values as the objective function, test results were analyzed and evaluated. Finally, the optimal aggregate gradation was determined considering factors of compactness, high-temperature rutting resistance, and water stability.

1. Introduction

Asphalt-treated base (ATB-25) is a widely used flexible base material, which has the characteristics of small stiffness, high shear strength, flexural tensile strength, fatigue resistance, and not easy to produce shrinkage cracking and water damage. The composition design of the asphalt mixture determines the optimal mixing ratio of the coarse aggregate, fine aggregate, mineral powder, and asphalt to be used in the mixture to meet the performance requirements of the road. Because the use of waste material can reduce the cost of construction and increase the strength, steel slag and coconut shell were used in the asphalt mixture [1–3]. Owing to different gradations of the aggregates added, the asphalt mixtures have different composition and structures and thus exhibit different physical and mechanical properties and qualities during use. The characteristics of the arrangement and aggregate gradation have a significant effect on the structure and performance of the asphalt mixture. Previous

studies [4–6] indicated that aggregate gradation has a significant effect on the rutting resistance of the asphalt mixture at a high temperature. It was also concluded that aggregate gradation has a significant impact on volumetric indicators such as the voids in mineral aggregates (VMA) and voids filled with asphalt (VFA) of the asphalt mixture [7]. Further, it was reported that aggregate gradation significantly affects the resistance of the asphalt mixture to permanent deformation [8]. In the literature [9, 10], it was indicated that the tensile strength, shear strength, and horizontal tensile strain of the asphalt mixture are significantly affected by the aggregate gradation. Husain et al. studied semiflexible pavements and concluded that different aggregate gradations have a significant impact on the pavement performance of semiflexible pavements [11]. Roberts et al. concluded that aggregate gradation is the main factor that affects the stiffness, stability, durability, permeability, fatigue resistance, and water damage resistance of the asphalt mixture [12]. These studies indicate that aggregate gradation can

significantly influence the engineering properties of the asphalt mixture. However, very few studies have comprehensively analyzed gradation optimization design of the asphalt mixture.

The gradation of aggregate materials can be classified as continuous or discontinuous based on the shape of the gradation curve. Particles of various sizes, ranging from large to small, exist in aggregates with continuous gradation. The particles of each grade are mixed in a certain proportion, and the gradation curve is smooth and uninterrupted. For aggregates with discontinuous gradation, there is a lack of particles of one or several grades, causing a relatively large "break" between large and small particles in the gradation curve, which is thus discontinuous and intermittent. With regard to continuous gradation, various design methods for aggregate gradation have been proposed, including the n , i , and k methods based on the maximum density curves proposed by Fuller [13]. With regard to discontinuous gradation, the stone matrix asphalt and open-graded friction course have been proposed [14–18] for the design of aggregate gradation. Based on the aforementioned gradation theory and design methods, the Chinese industrial standard [19] determines the ranges of aggregate gradations for different types of asphalt mixtures. However, when adopting the recommended aggregate gradations to design the asphalt mixing ratio, the following two key issues need to be resolved. First, the table-based trial method and the common equation method can be used for the calculation of the aggregate composition of the asphalt mixture; the former is time-consuming and labor-intensive, yet the latter does not consider the requirements for the range of the design gradation curve and therefore cannot ensure that the calculation result is a nonnegative number. Second, the range of the aggregate gradation recommended by the standard is relatively uncertain; thus, it is difficult to ensure that the asphalt mixture meets the comprehensive performance requirements of the pavement. Therefore, it is necessary to optimize the aggregate gradation of the asphalt mixture and select the best gradation.

To address these issues, this paper proposes new methods for calculating the aggregate proportion. Asphalt treated base (ATB-25) is considered as an example in this study. The aggregate gradation of nine different asphalt mixtures is designed within the gradation range, specified by the standard, based on the orthogonal design. The aggregate composition is calculated using the normal equation, table-based trial, and quadratic programming methods. The advantages and disadvantages of the different calculation methods are analyzed and demonstrated. The membership function from fuzzy mathematics is introduced to establish a fuzzy matrix, and the analytic hierarchy process is used to determine the weights of three evaluation indicators for pavement performance. Taking the fuzzy comprehensive evaluation value as the objective function, the test results indicating the pavement performance of the asphalt mixtures of the nine gradations are evaluated comprehensively using the fuzzy matrix, and a new method for optimizing the gradation of asphalt mixtures is finally proposed.

2. Raw Material Test and Aggregate Gradation Design

2.1. Raw Material Test. The asphalt was produced by Panjin Northern Asphalt Co., Ltd. The test was conducted according to regulation [20], whereas the specification values were based on the industrial standard [19]. The test results are presented in Table 1.

The aggregate was limestone, whereas the filler was limestone powder. The test was conducted based on regulation [21], whereas the specification values were based on the industrial standard [19]. The test results are presented in Table 2.

2.2. Aggregate Gradation of Orthogonal Design. In this study, within the range for aggregate gradation specified by the industrial standard [19], the impact of aggregate size (if the diameter of the aggregate exceeds 4.75 mm, it is considered coarse; otherwise, it is considered fine) and three positions of the gradation curve within the particular range for the ATB-25 were investigated in terms of compactness, high-temperature rutting resistance, and water stability. The three positions of the gradation curve are such that the upper position is the bisector between the upper limit and median lines, the medium position is the median line of the gradation range, and the lower position is the bisector between the median and lower limit lines; these positions are detailed in Table 3.

The cumulative passing rate of each gradation conforms to the L_9 (3^4) orthogonal array, and the nine gradations obtained in the design are depicted in Figure 1.

3. Calculation Method for Aggregate Proportion

3.1. Establishment of Mathematical Model. Assuming that the number of hole types for sieving the aggregates of ATB-25 is m , the upper, lower, and median values of the designed gradation curve are expressed, respectively, as the following matrices:

$$\begin{aligned} G &= [g_1 \ g_2 \ \cdots \ g_m]^T, \\ L &= [l_1 \ l_2 \ \cdots \ l_m]^T, \\ B &= [b_1 \ b_2 \ \cdots \ b_m]^T, \end{aligned} \quad (1)$$

where g_i , l_i , and b_i ($i = 1, 2, \dots, m$) are the respective cumulative passing rates of the upper, lower, and median values of the design gradation curve in the i^{th} sieve hole type.

Assuming that the aggregate mixture has n types of raw materials, the results for each raw material after sieving are expressed in matrix form as follows:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{bmatrix}, \quad (2)$$

TABLE 1: Main technical indicators of asphalt.

Technical indicators	Units	Measured values	Specification values
Penetration (25°C)	0.1 mm	67.1	60 ~ 80
Penetration index	—	-0.65	-1.5 ~ +1.0
Softening point	°C	51.7	≥45
Ductility (15°C)	cm	67.5	≥40
Density (15°C)	g/cm ³	1.0066	Measured value
Mass loss after aging	%	0.05	-0.8 ~ +0.8
Penetration ratio after aging	%	62.5	61
Residual ductility (10°C)	cm	23	8

TABLE 2: Main technical indicators of aggregate.

Technical indicators	Units	Measured values	Specification values
Crushing value of coarse aggregate	%	20.96	≤28
Los Angeles abrasion value of coarse aggregate	%	24.32	≤30
Adhesion between asphalt and aggregate	Level	4	—
Relative density of 19–26.5 mm limestone	—	2.661	—
Relative density of 9.5–19 mm limestone	—	2.641	—
Relative density of 4.75–9.5 mm limestone	—	2.636	—
Relative density of 2.36–4.75 mm limestone	—	2.571	—
Relative density of 0–2.36 mm limestone	—	2.685	-
Relative density of limestone mineral powder	—	2.70	—

TABLE 3: Design factors and levels of aggregate gradation.

Sieve size (mm)	Cumulative passing rate of coarse aggregate (%)						Cumulative passing rate of fine aggregate (%)						
	31.5	26.5	19	16	13.2	9.5	2.36	1.18	0.6	0.3	0.15	0.075	
Level 1	100	95	70	58	52	42	23.5	17.5	13	9.5	6.5	4	
Level 2	100	97.5	75	63	57	47	27.75	21.25	15.5	11.75	8.25	5	
Level 3	100	92.5	65	53	47	37	19.25	13.75	10.5	7.25	4.75	3	

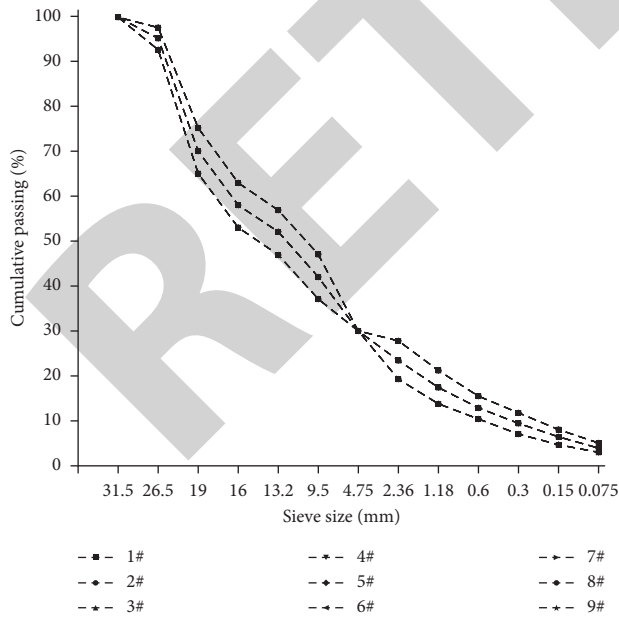


FIGURE 1: Gradations used for asphalt-treated base (ATB-25).

where p_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) are the cumulative passing rates of the j^{th} raw material in the i^{th} sieve hole type.

The n known types of raw materials are used to prepare a mixture that meets the design gradation. The mass ratio of the various raw materials in the mixture is calculated and expressed in a matrix as follows:

$$X = [x_1 \ x_2 \ \dots \ x_n]^T, \quad (3)$$

where x_j ($j = 1, 2, \dots, n$) is the mass ratio of the j^{th} raw material in the aggregate asphalt mixture.

The mass ratio of the various raw materials should satisfy the following equation:

$$P \cdot X \leq G, \quad (4)$$

$$P \cdot X \geq L, \quad (5)$$

$$\sum_{j=1}^n x_j = 1, \quad j = 1, 2, \dots, n, \quad (6)$$

$$x_j \geq 0, \quad j = 1, 2, \dots, n. \quad (7)$$

Formulas (4) and (5) represent the upper and lower bounds of the gradation of the mixture. Formula (6) indicates that the sum of the mass ratios of all the raw materials is equal to 1. Formula (7) restricts the result of the calculation to a nonnegative number.

3.2. Problem-Solving Methods

3.2.1. Normal Equation Method. The fundamentals of normal equation method are the principle of least squares. The synthetic aggregate gradations do not meet the conditions of equations (4) and (5). However, the quadratic sum of the deviation between the sieve weight value of each sieve hole and the target design grading value should be minimized. The mass ratios of raw materials in synthetic aggregates should satisfy the following equation:

$$P \cdot X = B. \quad (8)$$

The passing percentage of aggregates at any sieve hole is the theoretical design grading value b_i , which is equal to the sum of the passing percentage of various aggregates at certain sieve hole multiplied by the amount of various aggregates in the mixture; that is,

$$\sum_{j=1}^n p_{ij}x_j = 1. \quad (9)$$

The least square principle was used to minimize the quadratic sum of the deviation between the sieve weight value of each sieve hole and the theoretical design grading value; that is,

$$\min F(x) = \sum_{i=1}^m \left(\sum_{j=1}^n p_{ij}x_j - b_i \right)^2. \quad (10)$$

Under the condition that the sieving results of various aggregates were linearly independent, the extreme value condition was used:

$$\frac{\partial F(x)}{\partial x_j} = 0, \quad j = 1, 2, \dots, n. \quad (11)$$

The normal equations of x_j were obtained as follows:

$$\sum_{j=1}^n \left(\sum_{k=1}^m p_{ki}p_{kj} \right) x_j = \sum_{k=1}^m p_{ki}b_k, \quad i = 1, 2, \dots, n. \quad (12)$$

Equation (12) is expressed in matrix as

$$P^T P X = P^T B, \quad (13)$$

where

$$P^T P = \begin{bmatrix} \sum_{k=1}^m p_{k1}p_{k1} & \sum_{k=1}^m p_{k1}p_{k2} & \cdots & \sum_{k=1}^m p_{k1}p_{kn} \\ \sum_{k=1}^m p_{k2}p_{k1} & \sum_{k=1}^m p_{k2}p_{k2} & \cdots & \sum_{k=1}^m p_{k2}p_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^m p_{kn}p_{k1} & \sum_{k=1}^m p_{kn}p_{k2} & \cdots & \sum_{k=1}^m p_{kn}p_{kn} \end{bmatrix}, \quad (14)$$

$$P^T B = \left[\sum_{k=1}^m p_{k1}b_k \quad \sum_{k=1}^m p_{k2}b_k \quad \cdots \quad \sum_{k=1}^m p_{kn}b_k \right]^T.$$

When $P^T P$ is invertible, equation (13) has a unique solution.

$$X^0 = (P^T P)^{-1} P^T B. \quad (15)$$

When $P^T P$ is not invertible, the generalized inverse function is used for the solution.

$$X^0 = \text{pinv}(P) \cdot B. \quad (16)$$

3.2.2. Spreadsheet Trial Method. The retained percentage a_i is the percentage of residual mass on the i^{th} sieve in the total mass of the sample, which can be calculated by the following equation:

$$a_i = \frac{m_i}{m_0}, \quad (17)$$

where m_i is the mass retained on the i^{th} sieve (g); m_0 is the total mass of the specimen (g).

The cumulative retained percentage A_i is the sum of the retained percentages of the i^{th} sieve hole and sieve holes with a larger size than that of the i^{th} sieve hole. It can be calculated as follows:

$$A_i = a_1 + a_2 + \dots + a_i, \quad (18)$$

where a_1, a_2, \dots, a_i are the retained percentage (%) for each sieve.

The passing rate p_i represents the percentage of the mass passing the i^{th} sieve hole in the total mass of the specimen, which is the difference between 100 and the cumulative percentage of sieve residue on the i^{th} sieve hole. It can be obtained using the following equation:

$$p_i = 100 - A_i. \quad (19)$$

The results obtained from equation (15) or equation (16) were substituted into the spreadsheet prepared using equations (17)–(19) for trial. Only minor adjustments were required to obtain optimal mixing mass ratios for various raw materials.

3.2.3. Quadratic Programming Method. Let the passing rates of the synthetic aggregate gradations at the i^{th} sieve hole be $f_i(X)$, and let $P_i = [p_{i1}, p_{i2}, \dots, p_{in}]$. Replace b_i with $f_i(X)$. Therefore, equation (8) was transformed into the following form:

$$f_i(X) = P_i X. \quad (20)$$

A fuzzy mathematical membership function $A_i(f_i(X))$ was introduced to fuzzify the passing rates of synthetic aggregate gradations at various sieve holes with the following equation:

$$\begin{cases} A_i(f_i(X)) = \frac{(f_i(X) - g_i)(f_i(X) - l_i)}{(b_i - g_i)(b_i - l_i)}, & l_i \leq f_i(X) \leq g_i, \\ A_i(f_i(X)) = 0, & \text{else.} \end{cases} \quad (21)$$

Let $U_i(X) = A_i(f_i(X))$ be substituted into equation (15), and then substitute equation (13) into equation (16) to obtain

$$\begin{cases} U_i(X) = \frac{(P_i X - g_i)(P_i X - l_i)}{(b_i - g_i)(b_i - l_i)} & l_i \leq P_i X \leq g_i, \\ U_i(X) = 0, & \text{else.} \end{cases} \quad (22)$$

Let $U(X) = \sum_{i=1}^m (P_i X - g_i)(P_i X - l_i) / (b_i - g_i)(b_i - l_i)$, which can be further modified into a standard quadratic form for the following programming problem:

$$-U(X) = \frac{1}{2} X^T H X + F X + C, \quad (23)$$

where $H = \sum_{i=1}^m -2P_i^T P_i / (b_i - g_i)(b_i - l_i)$, $F = \sum_{i=1}^m (l_i + g_i)P_i / (b_i - g_i)(b_i - l_i)$, and $C = \sum_{i=1}^m -l_i g_i / (b_i - g_i)(b_i - l_i)$.

4. Fuzzy Comprehensive Evaluation Method

Since Zadeh published the paper on fuzzy mathematics [22], it has been widely used in many fields to solve engineering problems. In this study, different aggregate gradations were obtained through the orthogonal design, and then the corresponding pavement performance test results were fuzzified. The experimental data were processed using the fuzzy mathematics method. Taking the fuzzy comprehensive evaluation value as the objective function, the test results were analyzed and evaluated to obtain the optimal aggregate gradation. The void ratio was used as the index to evaluate the compactness of the asphalt mixtures. The dynamic stability was used as the index to assess the high-temperature rutting resistance of the asphalt mixtures. The residual stability of the water-soaked Marshall test was used to evaluate the water stability of the asphalt mixtures. The evaluation index set was established based on the above three indexes. The pavement performance test results of the nine aggregate gradations obtained through the orthogonal test design were used to determine the evaluation object set.

4.1. Establishment of Membership Functions and Fuzzy Matrix. From the industrial standard [19], it is known that the recommended void ratio range is 3%–6%. The void ratio is an intermediate indicator, and the degree of membership can be calculated using the following equation.

$$r_{ij} = \frac{\min(u_{ij}, 4.5\%)}{\max(u_{ij}, 4.5\%)}, \quad (24)$$

where $i = 1, 2, \dots, 9$ and $j = 1$.

It is known from the industrial standard [19] that the dynamic stability is not less than 800 times/mm and that the residual stability of the water-soaked Marshall test is not less than 80%. Both of them are partial large indexes. The degree of membership can be calculated based on the following equation:

$$r_{ij} = \frac{u_{ij}}{\max(u_{ij})}, \quad (25)$$

where $i = 1, 2, \dots, 9$ and $j = 2, 3$.

The degree of membership values calculated from equation (24) and equation (25) form the fuzzy relationship matrix.

$$R = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ \dots & \dots & \dots \\ r_{9,1} & r_{9,2} & r_{9,3} \end{bmatrix}. \quad (26)$$

4.2. The Determination of Evaluation Index Weights. The comparison matrix was constructed according to the scales of pairwise comparisons (see Table 4) and judgment principles proposed by the literature [23]. The scale values of 2, 4, 6, and 8 denoted the median of two adjacent scale value comparisons, respectively. D_{ij} ($i = 1, 2, 3; j = 1, 2, 3$) denotes the importance comparison of D_i and D_j , and $1/D_{ij}$ denotes the importance comparison of D_j and D_i .

Let the comparison matrix constructed by the pairwise comparison of three pavement performance evaluation indexes be D . By definition,

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{23} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} = \begin{bmatrix} \frac{D_1}{D_1} & \frac{D_1}{D_2} & \frac{D_1}{D_3} \\ \frac{D_2}{D_1} & \frac{D_2}{D_2} & \frac{D_2}{D_3} \\ \frac{D_3}{D_1} & \frac{D_3}{D_2} & \frac{D_3}{D_3} \end{bmatrix}. \quad (27)$$

The square root method was used to solve for the maximum eigenvalue λ_{max} and the eigenvector W of the comparison matrix D . The elements of the comparison matrix D were multiplied by rows to obtain the product M_i ($i = 1, 2, 3$) of each element.

$$M_i = \prod_{j=1}^3 W_{ij}. \quad (28)$$

Calculate the square root of M_i .

$$\bar{W}_i = \sqrt[3]{M_i}. \quad (29)$$

Normalize the vector \bar{W} .

$$W_i = \frac{\bar{W}_i}{\sum_{j=1}^3 \bar{W}_j}. \quad (30)$$

Calculate the maximum eigenvalue of the comparison matrix.

$$\lambda_{max} = \sum_{i=1}^3 \frac{(DW)_i}{3 \cdot W_i}. \quad (31)$$

TABLE 4: The value rule of comparison matrix element D_{ij} .

Scale values	Definition	Explanation
1	Equally important	D_i is as important as D_j
3	Slightly important	The importance of D_i is slightly higher than that of D_j
5	Clearly important	The importance of index D_i is obviously higher than that of D_j
7	Strongly important	The importance of D_i is strongly higher than that of D_j
9	Extremely important	The importance of D_i is absolutely higher than that of D_j

The consistency test of the comparison matrix is as follows:

$$C_R = \frac{C_I}{R_I}, \quad (32)$$

where $C_I = \lambda_{\max} - 1/3 - 1$ is the consistency test indicator and R_I is the average random consistency indicator proposed by the literature [23]. According to the comparison matrix D with an order of 3 ($n=3$), $R_I = 0.58$.

When $C_R < 0.1$, the consistency of the comparison matrix D is generally acceptable. Otherwise, the comparison matrix D requires modification until the consistency test is passed.

With the comparison matrix satisfying the consistency condition, the set of weights assigned to the evaluation indicators is obtained as follows:

$$W = [w_1, w_2, w_3]. \quad (33)$$

4.3. Calculation of Fuzzy Evaluation Values. A fuzzy subset E , called the evaluation set, was introduced based on the evaluation index set. Its fuzzy evaluation values were obtained based on the fuzzy matrix R and the weight allocation set W according to the following equation:

$$E = R \times W^T = [e_1, e_2, \dots, e_9]^T. \quad (34)$$

According to the principle of maximum degree of membership, the combination of coarse and fine aggregates is more reasonable, and the pavement performance of corresponding aggregate gradation improves as e_i ($i=1, 2, \dots, 9$) becomes larger.

5. Engineering Examples

Asphalt treated base (ATB-25) was used for the lower surface course of the Ji-Cao Expressway. The total number of aggregate and mineral powder types used was 6 ($n=6$), and the sieving results of the raw materials are presented in Table 5. The aggregate proportion calculation was carried out using 1# gradation as the target gradation (Table 6).

5.1. Results of the Aggregate Proportion Calculation. The normal equation, table trial, and quadratic programming methods were used to calculate the mass ratios of various raw materials in the mineral aggregates. The results are presented in Table 6.

Mineral aggregates were prepared according to the mixing mass ratios of each raw material as presented in

Table 6 and the synthetic aggregate gradations were calculated. The results are presented in Table 7.

Methods 1, 2, and 3 in Tables 6 and 7 were the normal equation, table trial, and quadratic programming methods, respectively. As can be seen from Table 6, the calculation results of the normal equation method cannot satisfy the condition that the sum of raw material mixing mass ratios should be equal to 1 (equation (7)). The calculation results satisfying equation (7) and other constraints can be obtained quickly after minor adjustments using the table trial method. As can be seen from Table 7, the quadratic sum of deviation between synthetic aggregate gradation and design aggregate gradation obtained from the formal equation method was the smallest. However, the calculated cumulative passing percentage for the sieve size of 31.5 mm was 100.75%, which substantially exceeded the upper limit of the design gradation curve.

From Table 6, it can be seen that the calculated results of the quadratic programming method satisfied the constraints. According to Table 7, the calculated cumulative passing percentage for the sieve size of 31.5 mm was 100.01%, which exceeded the upper limit of the design gradation curve of 100%. However, the deviation was quite small and thus acceptable for practical application. The calculation results of quadratic programming method and table trial method showed that the aggregate gradations obtained by both methods are in good agreement with the target gradations and, thus, meet the design requirements.

5.2. Fuzzy Comprehensive Evaluation of Pavement Performance. Marshall tests were carried out according to the specification [20]. The optimum asphalt aggregate ratio for each gradation was determined by plotting the Marshall stability, flow value, relative density, VFA, and voids in mineral aggregate of the specimens against the asphalt aggregate ratio. Results are presented in Table 8. With the optimum asphalt aggregate ratio, the Marshall test, water weight test, rutting test, and water-soaked Marshall test were carried out, and the results are presented in Table 8. The test results in Table 8 were substituted into equation (24) and equation (25) to calculate the degree of membership of each index r_{ij} . The results are presented in Table 8.

According to the fundamentals of analytic hierarchy process, literature review, and communications with experts in the road industry, the comparison matrix D was constructed, and corresponding weights were calculated according to equation (27) to equation (30). The results are presented in Table 9.

TABLE 5: The sieving results of the raw materials.

Sieve size (mm)	Cumulative passing rate (%)					
	Aggregate 1	Aggregate 2	Aggregate 3	Aggregate 4	Aggregate 5	Mineral power
31.5	100	100	100	100	100	100
26.5	84.33	100	100	100	100	100
19.0	19.75	100	100	100	100	100
16.0	5.05	88.57	100	100	100	100
13.2	0.88	6.77	100	100	100	100
9.5	0	0.1	93.11	100	100	100
4.75	0	0	11.69	99.62	100	100
2.36	0	0	0.09	8.2	92.74	100
1.18	0	0	0	0.96	72.35	100
0.6	0	0	0	0	51.22	100
0.3	0	0	0	0	35.94	100
0.15	0	0	0	0	20.13	100
0.075	0	0	0	0	0	79.12

TABLE 6: Calculation results of the mass ratios.

Calculation methods	The mass ratios of various raw materials					
	Aggregate 1	Aggregate 2	Aggregate 3	Aggregate 4	Aggregate 5	Mineral power
Method 1	0.4055	0.1864	0.1298	0.0512	0.2046	0.0300
Method 2	0.4055	0.1864	0.1290	0.0412	0.2046	0.0333
Method 3	0.3925	0.1946	0.1236	0.0751	0.1708	0.0435

TABLE 7: Calculation results of the synthetic aggregate gradation.

Sieve size (mm)	1# gradation (%)	The synthetic aggregate gradation (%)		
		Method 1	Method 2	Method 3
31.5	100	100.75	100	100.01
26.5	95	94.40	93.65	93.86
19.0	70	68.21	67.46	68.51
16.0	58	60.12	59.37	60.52
13.2	52	51.53	50.78	51.68
9.5	42	41.97	41.23	41.81
4.75	30	30.10	29.42	30.38
2.36	23.5	22.41	22.66	20.82
1.18	17.5	17.85	18.17	16.78
0.6	13	13.48	13.81	13.10
0.3	9.5	10.35	10.68	10.49
0.15	6.5	7.12	7.45	7.79
0.075	4	2.38	2.64	3.44
The quadratic sum of deviation		14.15	18.57	20.79

TABLE 8: Results of pavement performance test and fuzzy comprehensive evaluation value.

Gradation	Optimum asphalt aggregate ratio (%)	Void ratio (%)	Dynamic stability (times/mm)	Marshall remnant stability ratio (%)				
					r_{i1}	r_{i2}	r_{i3}	e_i
1#	3.8	4.0	1909	81.3	0.889	0.814	0.767	0.840
2#	3.9	4.2	1350	83.4	0.933	0.575	0.787	0.807
3#	4.0	4.4	1536	86.5	0.978	0.655	0.816	0.857
4#	3.9	4.5	2298	106.0	1.000	0.980	1.000	0.995
5#	3.9	4.7	1853	82.9	0.957	0.790	0.782	0.872
6#	4.0	5.8	2346	83.9	0.776	1.000	0.792	0.836
7#	3.9	3.7	1813	80.1	0.822	0.773	0.756	0.793
8#	3.9	4.4	1468	97.3	0.978	0.626	0.918	0.875
9#	4.0	5.2	1654	81.6	0.865	0.705	0.770	0.801

TABLE 9: The comparison matrix D and weights.

	D_1	D_2	D_3	Weights
D_1	1	2	2	0.50
D_2	0.5	1	1	0.25
D_3	0.5	1	1	0.25

According to the weights of each evaluation index, the maximum eigenvalue was calculated ($\lambda_{\max} = 3.00$) from equation (31), which was then substituted into equation (32). It was found that $C_R = 0$. Because $C_R < 0.1$, the comparison matrix passed the consistency test. In other words, the weight set W of the evaluation indexes was acceptable. According to equation (34), the fuzzy evaluation values e_i were calculated for the pavement performance of nine aggregate gradations, and the results are presented in Table 8. As presented in Table 8, the comprehensive pavement performance of asphalt mixtures prepared with 1#–9# aggregate gradations was ranked as follows: 4# > 8# > 5# > 3# > 1# > 6# > 2# > 9# > 7# according to the principle of maximum degree of membership. The ATB-25 formed by 4# aggregate gradation had the optimal overall performance in terms of compactness, high-temperature rutting resistance, and water stability.

6. Conclusions

New methods for the aggregate proportion calculation and gradation optimization were proposed in this study and then verified by example calculations using ATB-25. The main conclusions are as follows:

- (1) Using the generalized inverse solution of the normal equation, raw material mass ratios of the synthetic mineral aggregates were calculated, which were substituted into the spreadsheet as the initial values. Minor adjustments were required to obtain the calculation results satisfying the constraint conditions. The equation of this method was simple and did not require programming. Therefore, it has great potential for engineering practice.
- (2) The degree of membership function in fuzzy mathematics was introduced into the aggregate proportion calculation of asphalt mixtures. The range of design gradations was taken as the domain, and the values of synthetic gradations were fuzzified on the domain. The linear programming was transformed into nonlinear quadratic programming. Through the programming module, satisfactory raw material mass ratios and synthetic aggregate gradations were calculated. From the example calculations, it was found that the quadratic programming method has the advantages of high calculation efficiency and accurate calculation results. Also, it can be used to analyze the problem from different angles. Therefore, it has the potential to be widely used.
- (3) Fuzzy mathematics and analytic hierarchy process were applied to evaluate the pavement performance of ATB-25. Through the establishment of fuzzy

matrix and the determination of corresponding weights, the fuzzy comprehensive evaluation values were calculated. Finally, the 4# aggregate gradation was determined as the optimal choice considering factors of compactness, high-temperature rutting resistance, and water stability was selected.

- (4) In the design of asphalt mixtures, the influence of various factors was fully considered using the orthogonal experimental design and fuzzy mathematics. The analytic hierarchy process was employed to determine the weight distribution set. In this way, the aggregate gradation was optimized.

Data Availability

All the data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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Retraction

Retracted: Some Improved Correlation Coefficients for q-Rung Orthopair Fuzzy Sets and Their Applications in Cluster Analysis

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Some Improved Correlation Coefficients for q-Rung Orthopair Fuzzy Sets and Their Applications in Cluster Analysis

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The structure of q-rung orthopair fuzzy sets (q-ROFSs) is a generalization of fuzzy sets (FSs), intuitionistic FSs (IFSs), and Pythagorean FSs (PFSs). The notion of q-ROFSs has the proficiency of coping with uncertainty without any restrictions. In addition, the structure of q-ROFSs can effectively cope with the situations involving dual opinions without any restrictions, instead of dealing with only single opinion or dual opinions under certain restrictions. In clustering problems, the correlation coefficients are worthwhile because they provide the degree of similarity or correlation between two elements or sets. The theme of this study is to formulate the correlation coefficients for q-ROFSs that are basically the generalization of correlation coefficients of IFSs and PFSs. Moreover, an application of these correlation coefficients to a clustering problem is proposed. Also, an analysis of the outcomes is carried out. Furthermore, a comparison is carried out among the correlation coefficients for q-ROFSs and the existing ones. Finally, the downsides of the existing works and benefits of the correlation coefficients for q-ROFSs are discussed.

1. Introduction

Zadeh [1] initiated the notion of fuzzy set theory and logic in 1965. The fuzzy set (FS) is characterized by a function known as membership grade that attains values from a unit interval. This innovative theory was a nice tool for handling the uncertainties in practical life. Adlassing [2] applied the FS theory in medical diagnosis, Bezdek and Douglas Harris [3] defined the fuzzy partitions and relations, and Kandel [4] proposed a fuzzy technique in pattern recognition. The downside of FSs is that they do not describe the nonmembership grade, in spite of the fact that the nonmembership grade can be acquired in the fuzzy environment by subtracting the membership grade from 1. Henceforth, Atanassov [5] defined the intuitionistic FS (IFS) which

describes both the membership and nonmembership grades independently. The addition of IFS contributed greatly in the FS theory. Chaira [6] proposed a novel concept of the IF C means clustering algorithm and applied it to medical images, Dengfeng and Chuntian [7] discussed the similarity measures of IFSs and applied them in pattern recognitions, and Hung and Yang [8] also worked on the similarity measures of IFSs based on Hausdorff distance. Eventhough IFSs discuss both the membership and nonmembership grades, still they have restrictions in the structure that the sum of both the grades must not exceed 1. To overcome this barrier, Yager [9] developed the Pythagorean FS (PFS) which relaxes the restrictions on the membership and nonmembership grades by keeping the sum of the squares of both the grades within the unit interval. Li and Zeng [10] formulated the

distance measures of PFSs, Li and Lu [11] offered the similarity and distance measures of PFSs and their applications, and Ejegwa and Awolola [12] applied the distance measures for PFSs to pattern recognition problems. Yet again, the structure of PFSs has certain limitations that affected the decision-making abilities of the professionals. So Yager [13] gave the concept of the q -rung orthopair fuzzy set (q -ROFS) that not only discusses the membership and nonmembership grades but also provides the largest possible domain for better decision-makings. A comparison among the domains of IFs, PFSs, and q -ROFSs is portrayed in Figures 1–4. Liu et al. [14] devised the multiple-attribute decision-making based on q -ROF power Maclaurin symmetric mean operators, Liu and Wang [15] defined some q -ROF aggregation operators and applied them in multiple-attribute decision-making, and Wang et al. [16] discussed the similarity measures of q -ROFSs with their applications.

The notion of correlation coefficient is often used in the statistical problem. Bonizzoni et al. [17] worked on the correlation clustering and consensus clustering, Cheung and Li [18] proposed a quantitative correlation coefficient method for business intelligence, and Kumar et al. [19] conceived the method for ranking of L-R type generalized fuzzy numbers. Actually, the intention of correlation coefficient is to determine the strength of correlation or similarity between two objects or sets. These correlation coefficients have momentous use and applications in the theories of FSs, IFs, and PFSs. Yang and Lin [20] proposed the similarity and inclusion measures for type-2 FSs and used these measures in the clustering, Chen et al. [21] defined the correlation coefficients of hesitant FSs and used these notions for analysis of clustering, Xu et al. [22] presented the clustering algorithm for IFs, Hox et al. [23] discussed the techniques and applications of multilevel analysis, Garg [24] came up with a new correlation coefficient among PFSs and applied them in decision-making, Park et al. [25] devised the correlation coefficient of interval-valued IFs and illustrated their application by using them in the problems of multiple-attribute group decision-making, Nguyen [26] concocted the similarity or dissimilarity measure for IFs with its applications in pattern recognition, and Du [27] developed the correlation and correlation coefficients of q -ROFSs. Garg and Kumar [28, 29] studied the similarity measures of IFs and thought up of the aggregation operators for linguistic IFs with their applications in decision-making processes. In 2017, Garg [30] proposed a new method for IF decision-making founded on the improved operation laws with applications. Singh and Garg [31] gave distance measures for type-2 IFs with their application to multicriterion decision-making. Garg [32] formulated the distance and similarity measures for intuitionistic multiplicative preference relation and its applications, and Jamkhaneh and Garg [33] perceived some new operations over the generalized IFs and applied them in the decision-making process.

The progression in the theory as well as the applications of correlation coefficients to practical problems drove us to study these notions. Hereafter, this study presents the correlation coefficients for q -ROFSs and their clustering

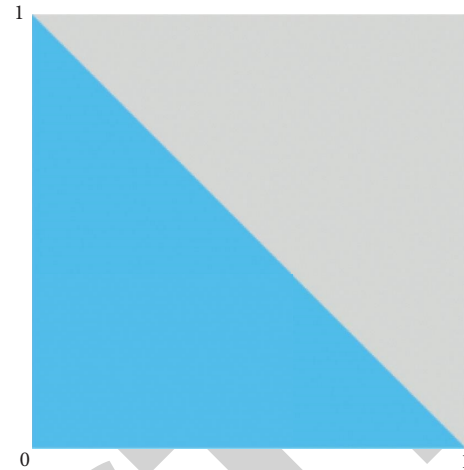


FIGURE 1: Range of IFs.

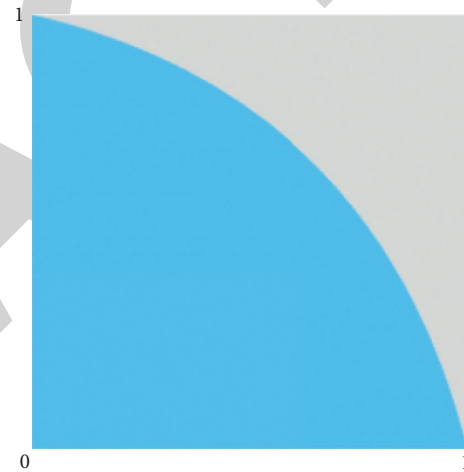
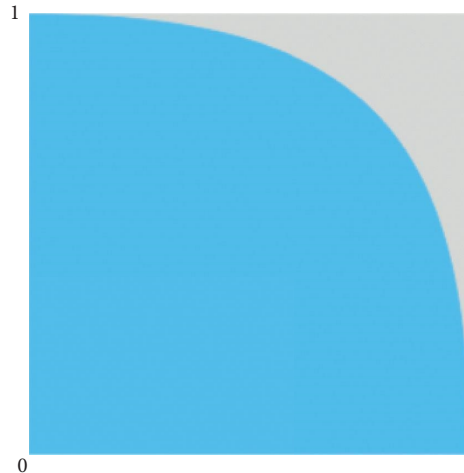
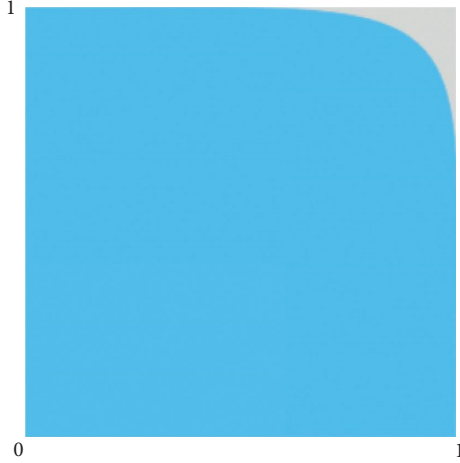


FIGURE 2: Range of PFS.

FIGURE 3: Range q -ROFS for $q = 5$.

algorithm. Unlike fuzzy sets, the q -ROFSs involve both the membership grade and nonmembership grade. The IFs and PFSs also talk about the membership and nonmembership

FIGURE 4: Range q-ROFS for $q = 10$.

grades, but they have certain limitations and constraints on the selection of these grades, while q-ROFSs do not have those restrictions. For example, we cannot assign 0.5 and 0.6 as membership and nonmembership grades in the framework of IFSs because their sum exceeds 1. Similarly, we cannot choose 0.7 and 0.8 as membership and nonmembership grades in the environment of PFSs due to the restriction that the sum of their squares exceeds 1. But the range of numbers to be assigned as membership and nonmembership grades is so vast in q-ROFS, i.e., we can assign any values from $[0, 1]$ to membership and nonmembership grades. Thus, the structure of q-ROFSs is superior as compared to other existing frameworks, and it generalizes all the predecessors such as FS, IFS, and PFS. Since the proposed idea is the generalization of the correlation coefficients of IFSs and PFSs, therefore, it can muddle through the information that the aforementioned structures could not handle. Moreover, some interesting properties of new correlation coefficients for q-ROFSs are studied along with a practical clustering problem in the environment of q-ROFSs.

This research article is arranged in such a way that the first section briefly describes the history of fuzzy set theory and reviews its literature. The second section provides an explanation of fundamental concepts such as FS, IFS, PFS, q-ROFS, information energy, correlation, and the correlation coefficients for IFSs. The correlation coefficients for q-ROFSs and their results are presented in section three. The fourth section establishes an algorithm for clustering. In section five, the established algorithm is used to solve an example. Section six carries out the comparison among the new and previously existing concepts. Finally, the research is concluded in section eight.

2. Preliminaries

This section defines some of the fundamental concepts such as FS, IFS, PFS, q-ROFS, information energy, correlation, and the correlation coefficients for IFSs.

Definition 1 (see [1]). For a nonempty set U , a fuzzy set (FS) is defined as $F = \{x, m(x) : x \in U\}$, where the membership grade $m(x)$ maps each $x \in U$ into $[0, 1]$. Additionally, $1 - m(x)$ represents the uncertainty of $x \in U$ and m is known as the fuzzy number (FN).

Definition 2 (see [5]). For a nonempty set U , an intuitionistic FS (IFS) is defined as $F = \{x, m(x), n(x) : x \in U \wedge 0 \leq m(x) + n(x) \leq 1\}$, where the membership grade $m(x)$ and the nonmembership grade $n(x)$ map each $x \in U$ into $[0, 1]$. Additionally, $1 - (m(x) + n(x))$ represents the uncertainty of $x \in U$ and (m, n) is known as the intuitionistic FN (IFN).

Definition 3 (see [9]). For a nonempty set U , a Pythagorean FS (PFS) is defined as $F = \{x, m(x), n(x) : x \in U \wedge 0 \leq (m(x))^2 + (n(x))^2 \leq 1\}$, where the membership grade $m(x)$ and the nonmembership grade $n(x)$ map each $x \in U$ into $[0, 1]$. Additionally, $\sqrt{1 - ((m(x))^2 + (n(x))^2)}$ represents the uncertainty of $x \in U$ and (m, n) is known as the Pythagorean FN (PFN).

Definition 4 (see [13]). For a nonempty set U , a q-rung orthopair FS (q-ROFS) is defined as $F = \{x, m(x), n(x) : x \in U \wedge 0 \leq (m(x))^q + (n(x))^q \leq 1\}$, where q is a non-negative integer, and the membership grade $m(x)$ and the nonmembership grade $n(x)$ map each $x \in U$ into $[0, 1]$. Additionally, $\sqrt[q]{1 - ((m(x))^q + (n(x))^q)}$ represents the uncertainty of $x \in U$ and (m, n) is known as the q-rung orthopair FN (q-ROFN).

In the light of above definitions, a summary of generalizations of FSs is given in Figure 5.

Definition 5 (see [22]). For an IFS F on U , the information energy E_{IFS} is given by

$$E_{\text{IFS}}(F) = \sum_{i=1}^n [(m_F(x_i))^2 + (n_F(x_i))^2]. \quad (1)$$

Definition 6 (see [22]). The correlation C_{IFS} of two IFSs F and G is given by

$$C_{\text{IFS}}(F, G) = \sum_{i=1}^n [m_F(x_i) \cdot m_G(x_i) + n_F(x_i) \cdot n_G(x_i)]. \quad (2)$$

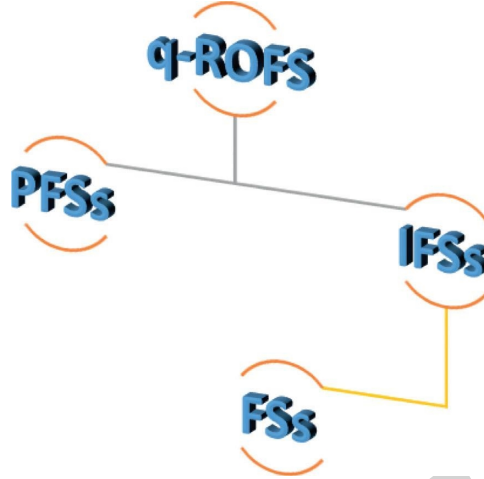


FIGURE 5: Generalizations of FSs.

Definition 7 (see [22]). The correlation coefficient \mathcal{K}_{IFS} of two IFSs F and G is given by

$$\begin{aligned}\mathcal{K}_{\text{IFS}}(F, G) &= \frac{C_{\text{IFS}}(F, G)}{\sqrt{[E_{\text{IFS}}(F) \cdot E_{\text{IFS}}(G)]}} \\ &= \frac{\sum_{i=1}^n [m_F(x_i) \cdot m_G(x_i) + n_F(x_i) \cdot n_G(x_i)]}{\sqrt{[\sum_{i=1}^n [(m_F(x_i))^2 + (n_F(x_i))^2] \cdot \sum_{i=1}^n [(m_G(x_i))^2 + (n_G(x_i))^2] \sum_{i=1}^n 1]}}.\end{aligned}\quad (3)$$

The notions of information energy, correlation, and correlation coefficients in Definitions 5–7 are suitable for intuitionistic fuzzy environment [22]. But these notions flop when the information is of q-rung orthopair fuzzy type. Hence, in the following section, the generalization of existing correlation coefficients is presented.

3. Correlation Coefficient for q-Rung Orthopair Fuzzy Sets

This section generalizes the correlation coefficients of IFSs and PFSs in order to formulate the correlation coefficients for q-ROFSs.

Definition 8 (see [27]). For a q-ROFS F on U , the information energy E_{qROFS} is given by

$$E_{\text{qROFS}}(F) = \sum_{i=1}^n [(m_F(x_i))^{2q} + (n_F(x_i))^{2q}]. \quad (4)$$

Remark 1. For $q = 2$, equation (4) gives the information energy of a PFS.

Definition 9 (see [27]). The correlation C_{qROFS} of two q-ROFSs F and G is given by

$$\begin{aligned}C_{\text{qROFS}}(F, G) &= \sum_{i=1}^n [(m_F(x_i))^q \cdot (m_G(x_i))^q \\ &\quad + (n_F(x_i))^q \cdot (n_G(x_i))^q].\end{aligned}\quad (5)$$

Definition 10 (see [27]). The correlation coefficient $\mathcal{K}_{\text{qROFS}}$ of two q-ROFSs F and G is given by

$$\mathcal{K}_{\text{qROFS}}(F, G) = \frac{C_{\text{qROFS}}(F, G)}{\sqrt{[E_{\text{qROFS}}(F) \cdot E_{\text{qROFS}}(G)]}}, \quad (6)$$

$$= \frac{\sum_{i=1}^n [(m_F(x_i))^q \cdot (m_G(x_i))^q + (n_F(x_i))^q \cdot (n_G(x_i))^q]}{\sqrt{[\sum_{i=1}^n [(m_F(x_i))^{2q} + (n_F(x_i))^{2q}] \cdot \sum_{i=1}^n [(m_G(x_i))^{2q} + (n_G(x_i))^{2q}]}}, \quad (7)$$

Remark 2. For $q = 2$, equation (7) gives the correlation coefficient of a PFS.

The correlation coefficients of PFSs and q-ROFSs are established through the membership grade and nonmembership grade.

Theorem 1. A $\mathcal{K}_{qROFS}(F, G)$ fulfils the following:

- (i) $\mathcal{K}_{qROFS}(F, G) = \mathcal{K}_{qROFS}(G, F)$
- (ii) $0 \leq \mathcal{K}_{qROFS}(F, G) \leq 1$

$$(iii) \mathcal{K}_{qROFS}(F, G) = 1 \Leftrightarrow F = G$$

Proof

(i) The proof is straight forward

(ii) Since $\mathcal{K}_{qROFS}(F, G)$ is based on the membership grades and nonmembership grades of F and G , thus $0 \leq \mathcal{K}_{qROFS}(F, G)$. Now, to prove that $\mathcal{K}_{qROFS}(F, G) \leq 1$, consider the correlation of F and G :

$$\begin{aligned} C_{qROFS}(F, G) &= \sum_{i=1}^n [(m_F(x_i))^q \cdot (m_G(x_i))^q + (n_F(x_i))^q \cdot (n_G(x_i))^q] \\ &= [(m_F(x_1))^q \cdot (m_G(x_1))^q + (n_F(x_1))^q \cdot (n_G(x_1))^q] \\ &\quad + [(m_F(x_2))^q \cdot (m_G(x_2))^q + (n_F(x_2))^q \cdot (n_G(x_2))^q] \\ &\quad + \dots + [(m_F(x_n))^q \cdot (m_G(x_n))^q + (n_F(x_n))^q \cdot (n_G(x_n))^q]. \end{aligned} \quad (8)$$

(iii) The Cauchy-Schwarz inequality states that for $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$,

$$\sqrt{(x_1 y_1 + x_2 y_2 + x_3 y_3 + \dots + x_n y_n)} \leq \sqrt{(x_1^2 + x_2^2 + \dots + x_n^2)} \cdot \sqrt{(y_1^2 + y_2^2 + \dots + y_n^2)}. \quad (9)$$

(iv) Applying Cauchy-Schwarz inequality to $C_{qROFS}(F, G)$ implies

$$\begin{aligned} (C_{qROFS}(F, G))^2 &\leq \left(\begin{aligned} &((m_F(x_1))^q + (n_F(x_1))^q) + (m_F(x_2))^q \\ &+ (n_F(x_2))^q + (m_F(x_3))^q + (n_F(x_3))^q + \dots + (m_F(x_n))^q + (n_F(x_n))^q \end{aligned} \right) \\ &\quad \times \left(\begin{aligned} &((m_G(x_1))^q + (n_G(x_1))^q) + (m_G(x_2))^q \\ &+ (n_G(x_2))^q + (m_G(x_3))^q + (n_G(x_3))^q + \dots + (m_G(x_n))^q + (n_G(x_n))^q \end{aligned} \right) \\ &= \left(\sum_{i=1}^n [(m_F(x_i))^q + (n_F(x_i))^q] \right) \times \left(\sum_{i=1}^n [(m_G(x_i))^q + (n_G(x_i))^q] \right) \\ &= E_{qROFS}(F) \cdot E_{qROFS}(G). \end{aligned} \quad (10)$$

(v) Therefore, $(C_{qROFS}(F, G))^2 \leq E_{qROFS}(F) \cdot E_{qROFS}(G)$. Thus, $0 \leq \mathcal{K}_{qROFS}(F, G) \leq 1$.

(vi) Since, $F = G \cdot (m_F(x_i))^q = (m_G(x_i))^q$ and $(n_F(x_i))^q = (n_G(x_i))^q$, where $x_i \in U$. Therefore, $\mathcal{K}_{qROFS}(F, G) = 1$. \square

Remark 3. For $q = 2$, the above theorem reduces to the correlation coefficient of PFS.

Example 1. Consider two q-ROFSs F and G on $U = \{x_1, x_2, x_3\}$ with $q = 7$, such that

$$\begin{aligned} F &= \{(x_1, 0.7, 0.9), (x_2, 0.7, 0.4), (x_3, 0.5, 0.5)\}, \\ G &= \{(x_1, 0.7, 0.5), (x_2, 0.9, 0.9), (x_3, 0.4, 0.5)\}. \end{aligned} \quad (11)$$

The information energies of F and G are calculated as

$$\begin{aligned} E_{qROFS}(F) &= 0.24, \\ E_{qROFS}(G) &= 0.46. \end{aligned} \quad (12)$$

The correlations of F and G are calculated as

$$C_{qROFS}(F, G) = 0.05. \quad (13)$$

The correlations of F and G are calculated as

$$\mathcal{K}_{qROFS}(F, G) = 0.15. \quad (14)$$

Another definition of correlation coefficient for q-ROFSs is stated.

$$\begin{aligned} \mathcal{K}'_{qROFS}(F, G) &= \frac{C_{qROFS}(F, G)}{\max\{C_{qROFS}(F, F), C_{qROFS}(G, G)\}} \\ &= \frac{\sum_{i=1}^n [(m_F(x_i))^q \cdot (m_G(x_i))^q + (n_F(x_i))^q \cdot (n_G(x_i))^q]}{\max\{\sum_{i=1}^n [(m_F(x_i))^{2q} + (n_F(x_i))^{2q}], \sum_{i=1}^n [(m_G(x_i))^{2q} + (n_G(x_i))^{2q}]\}}. \end{aligned} \quad (15)$$

Theorem 2. A $\mathcal{K}'_{qROFS}(F, G)$ fulfils the following:

- (i) $\mathcal{K}'_{qROFS}(F, G) = \mathcal{K}'_{qROFS}(G, F)$
- (ii) $0 \leq \mathcal{K}'_{qROFS}(F, G) \leq 1$
- (iii) $\mathcal{K}'_{qROFS}(F, G) = 1 \Leftrightarrow F = G$

Proof

- (i) The proof is straight forward
- (ii) Clearly, $0 \leq \mathcal{K}'_{qROFS}(F, G)$. Theorem 1 implies the following:

$$C_{qROFS}(F, G) \leq \sqrt{C_{qROFS}(F, F) \cdot C_{qROFS}(G, G)}. \quad (16)$$

(iii) Hence,

Definition 11. The correlation coefficient \mathcal{K}_{qROFS} of two q-ROFSs F and G is given by

$$C_{qROFS}(F, G) \leq \max\{C_{qROFS}(F, F), C_{qROFS}(G, G)\}. \quad (17)$$

(iv) Thus, $\mathcal{K}'_{qROFS}(F, G) \leq 1$

(v) The proof is straight forward

Practically speaking, the weight of a specialist's view has a vital role in multi-attribute decision-making problems, and these attributes have certain weights. Therefore, some weighted correlation coefficients have been developed. A weight vector is denoted as $w = (w_1, w_2, \dots, w_n)^T$, such that $0 \leq w_i$ and $\sum_{i=1}^n w_i = 1$, where $i, j \in \{1, 2, 3, \dots, n\}$. \square

Definition 12. The weighted correlation coefficient $\mathcal{W}\mathcal{K}_{qROFS}$ of two q-ROFSs F and G is given by

$$\mathcal{W}\mathcal{K}_{qROFS}(F, G) = \frac{\sum_{i=1}^n (w_i) [(m_F(x_i))^q \cdot (m_G(x_i))^q + (n_F(x_i))^q \cdot (n_G(x_i))^q]}{\sqrt{\sum_{i=1}^n (w_i) [(m_F(x_i))^{2q} + (n_F(x_i))^{2q}] \cdot \sum_{i=1}^n (w_i) [(m_G(x_i))^{2q} + (n_G(x_i))^{2q}]}}. \quad (18)$$

Definition 13. The weighted correlation coefficient $\mathcal{W}\mathcal{K}'_{qROFS}$ of two q-ROFSs F and G is given by

$$\mathcal{W}\mathcal{K}'_{qROFS}(F, G) = \frac{\sum_{i=1}^n (w_i) [(m_F(x_i))^q \cdot (m_G(x_i))^q + (n_F(x_i))^q \cdot (n_G(x_i))^q]}{\max\{\sum_{i=1}^n (w_i) [(m_F(x_i))^{2q} + (n_F(x_i))^{2q}], \sum_{i=1}^n (w_i) [(m_G(x_i))^{2q} + (n_G(x_i))^{2q}]\}}. \quad (19)$$

Remark 4. For $q = 2$, equations (18) and (19) give the weighted correlation coefficient of a PFS.

Remark 5. For $w = ((1/n), (1/n), (1/n), \dots, (1/n))^T$, the weighted correlation coefficient reduces to correlation coefficient, i.e., $\mathcal{W}\mathcal{K}_{qROFS}(F, G) = \mathcal{K}_{qROFS}(F, G)$ and $\mathcal{W}\mathcal{K}'_{qROFS}(F, G) = \mathcal{K}'_{qROFS}(F, G)$.

Theorem 3. A $\mathcal{W}\mathcal{K}_{qROFS}(F, G)$ fulfils the following:

- (i) $\mathcal{K}_{qROFS}(F, G) = \mathcal{K}_{qROFS}(G, F)$

(ii) $0 \leq \mathcal{K}_{qROFS}(F, G) \leq 1$

(iii) $\mathcal{K}_{qROFS}(F, G) = 1 \Leftrightarrow F = G$

Proof: The proofs are straight forward. \square

Theorem 4. $\mathcal{W}\mathcal{K}'_{qROFS}(F, G)$ fulfils the following:

(i) $\mathcal{K}'_{qROFS}(F, G) = \mathcal{K}'_{qROFS}(G, F)$

(ii) $0 \leq \mathcal{K}'_{qROFS}(F, G) \leq 1$

(iii) $\mathcal{K}'_{qROFS}(F, G) = 1 \Leftrightarrow F = G$

Proof: The proofs are straight forward. \square

4. Clustering Algorithm for q-Rung Orthopair Fuzzy Numbers

This section extends the clustering algorithms proposed for IFS in [22] to the environment of q-ROFSs. Additionally, a solution to a clustering problem involving the q-rung orthopair fuzzy information is given.

Definition 14. For a set of q-ROFNs F_i , define a matrix of correlation coefficients $M = (\mathcal{K}_{ij})_{m \times m}$, where $\mathcal{K}_{ij} = \mathcal{K}(F_i, F_j)$ is a correlation coefficient among (F_i, F_j) , such that

- (i) $0 \leq \mathcal{K}(F_i, F_j) \leq 1$
- (ii) $\mathcal{K}(F_i, F_j) = 1$
- (iii) $\mathcal{K}(F_i, F_j) = \mathcal{K}(F_j, F_i)$

Definition 15. If $M^2 = MOM = (\tilde{\mathcal{K}}_{ij})_{m \times m}$, where $M = (\mathcal{K}_{ij})_{m \times m}$ is a matrix of correlation coefficients, then such M^2 is known as composite matrix that is symbolized by $(\tilde{\mathcal{K}}_{ij})_{m \times m}$ and defined as

$$(\tilde{\mathcal{K}}_{ij})_{m \times m} = \max_k \{ \min(\mathcal{K}_{ik}, \mathcal{K}_{kj}) \}. \quad (20)$$

Theorem 5. For $h_1, h_2 \in \mathbb{Z}$, the composition of two correlation matrices M^{h_1} and M^{h_2} is also a correlation matrix, i.e., $M^{h_1 h_2} = M^{h_1} \circ M^{h_2}$ is a correlation matrix.

Definition 16. If $M^2 \subseteq M$, where $M = (\mathcal{K}_{ij})_{m \times m}$ is a matrix of correlation coefficients that is known as an equivalent correlation matrix, $M^2 \subseteq M \Rightarrow \max_k \{ \min(\mathcal{K}_{ik}, \mathcal{K}_{kj}) \} \leq \mathcal{K}_{ij}$

Theorem 6. For a correlation matrix $M = (\mathcal{K}_{ij})_{m \times m}$, $\exists h \in \mathbb{Z}$, such that the finite repeated compositions of M , i.e., $M \longrightarrow M^2 \longrightarrow M^4 \longrightarrow \dots M^{2h} \longrightarrow \dots$ imply that $M^{2h} = M^{2(h+1)}$, where M^{2h} is an equivalent correlation matrix.

Definition 17. The α -cutting matrix of a matrix of a correlation coefficients $M = (\mathcal{K}_{ij})_{m \times m}$ is symbolized and defined as

$$M_\alpha = (\alpha \mathcal{K}_{ij})_{m \times m}, \quad (21)$$

where $0 \leq \alpha \leq 1$ and $\alpha \mathcal{K}(F_i, F_j) = \begin{cases} 0, & \text{if } \mathcal{K}_{ij} \leq \alpha, \\ 1, & \text{if } \mathcal{K}_{ij} \geq \alpha. \end{cases}$

The comprehensive clustering algorithm for q-ROFNs is explained as follows (Algorithm 1)

Algorithm 1:

Step 1. The initial step is to construct a matrix of correlation coefficients, i.e., $M = (\mathcal{K}_{ij})_{m \times m}$ for the set of q-ROFNs F_i

Step 2. If the matrix of correlation M is not an equivalent matrix, then an equivalent matrix M^{2h} is constructed through the finite repeated compositions till $M^{2h} = M^{2(h+1)}$

Step 3. In the final step, the classification of q-ROFNs is carried out by forming the α -cutting matrix. The following principle is used to classify each q-ROFN:

“The q-ROFNs are said to be of the same type if every element of i^{th} line and the corresponding element of j^{th} line belonging to M_α are same.”

A visual representation of the algorithm is portrayed in Figure 6 through a flowchart.

5. Illustrated Example

This section demonstrates the application of the correlation coefficients defined for q-rung orthopair fuzzy information through a numerical example.

Example 2. Consider a situation in which an automobile company wants to classify their vehicles on the basis of some features. This example is set to classify four vehicles on the basis of three features. Suppose that the set $F = \{f_1, f_2, f_3, f_4\}$ be the representative of the collection of four vehicles and the set $G = \{g_1, g_2, g_3\}$ be the representative of the collection of three features, such that g_1, g_2 , and g_3 symbolize the fuel efficiency, the safety, and the selling cost of the vehicle, respectively. Assume that the weighted vector is $w = \{0.5, 0.3, 0.2\}^T$. Furthermore, the assessments of the professionals on each vehicle are listed in Table 1. Each value provided in the table is absolutely q-ROFN for $q = 3$.

By following the steps for the clustering discussed in the previous section, the stepwise calculations are carried out.

Step 1. The matrix of correlation coefficients is constructed by computing the correlation coefficients from Table 1:

$$M = \begin{pmatrix} 1 & 0.5284 & 0.8062 & 0.5423 \\ 0.5284 & 1 & 0.4465 & 0.8373 \\ 0.8062 & 0.4465 & 1 & 0.4375 \\ 0.5423 & 0.8373 & 0.4375 & 1 \end{pmatrix}. \quad (22)$$

Step 2. The equivalent correlation matrix is constructed by finite repeated compositions.

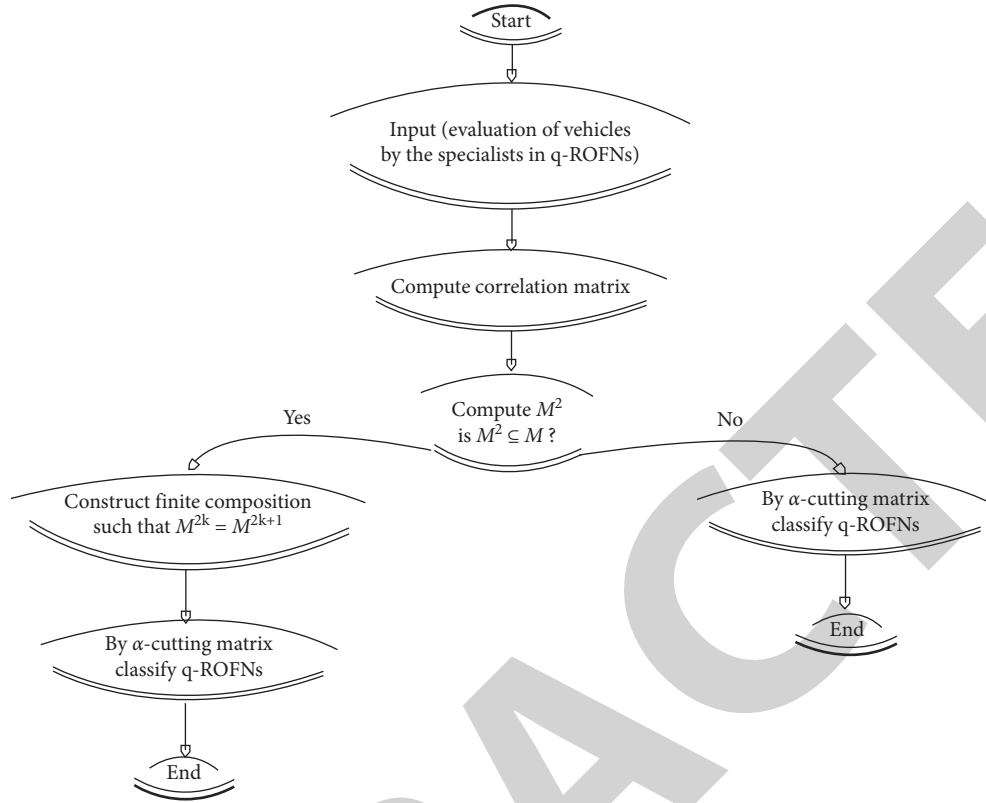


FIGURE 6: Flow chart of the proposed clustering algorithm.

$$\begin{aligned}
 M^2 = MOM &= \begin{pmatrix} 1 & 0.8373 & 0.8062 & 0.8373 \\ 0.8373 & 1 & 0.8062 & 0.8373 \\ 0.8062 & 0.8062 & 1 & 0.8373 \\ 0.8373 & 0.8373 & 0.8373 & 1 \end{pmatrix}, \\
 M^4 = M^2 OM^2 &= \begin{pmatrix} 1 & 0.8373 & 0.8373 & 0.8373 \\ 0.8373 & 1 & 0.8373 & 0.8373 \\ 0.8373 & 0.8373 & 1 & 0.8373 \\ 0.8373 & 0.8373 & 0.8373 & 1 \end{pmatrix}, \\
 M^8 = M^4 OM^4 &= \begin{pmatrix} 1 & 0.8373 & 0.8373 & 0.8373 \\ 0.8373 & 1 & 0.8373 & 0.8373 \\ 0.8373 & 0.8373 & 1 & 0.8373 \\ 0.8373 & 0.8373 & 0.8373 & 1 \end{pmatrix}.
 \end{aligned} \tag{23}$$

Since $M^8 = M^4$, therefore, M^4 is an equivalent correlation matrix

Step 3. The classifications are worked out by forming the α -cutting matrix as

- (1) If $0 \leq \alpha \leq 0.5384$, then all of f_1, f_2, f_3 , and f_4 are of the same type, i.e., $\{f_1, f_2, f_3, f_4\}$
- (2) If $0.5384 \leq \alpha \leq 0.8062$, then the vehicles are classified into two types as $\{f_2\}$ and $\{f_1, f_3, f_4\}$
- (3) If $0.8062 \leq \alpha \leq 0.8373$, then the vehicles are classified into three types as $\{f_1, f_3\}$, $\{f_2\}$, and $\{f_4\}$
- (4) If $0.8373 \leq \alpha \leq 1$, then the vehicles are classified into four types as $\{f_1\}$, $\{f_2\}$, $\{f_3\}$, and $\{f_4\}$

Clearly, the outcomes achieved specify the efficiency of correlation of q-ROFNs, since every vehicle is classified into a different type, which is infrequent in clustering analysis.

6. Comparative Analysis

This section studies the comparison among applications of the proposed correlation coefficients in the environment of IFSs, PFSs, and q-ROFNs. The data provided in the Example 2 cannot be modeled by using the correlations of IFSs and PFSs.

Remark 7. For $q = 1$, equation (7) gives the correlation coefficient for an IFS.

Remark 6. Verify the generalization of correlation of q-ROFNs. The following example is presented in order to compare the proposed correlation coefficients with the existing ones.

Example 3. This example solves the problem discussed in the previous example in the environment of intuitionistic fuzzy information. Table 2 contains the similar information to Table 1. Step by step calculation is carried out. Moreover, for $q = 1$, an IFS is a special case of q-ROFNs. Therefore, in this example, we consider $q = 1$ in the solution.

By following the steps for the clustering, the stepwise calculations are carried out.

TABLE 1: q-rung orthopair fuzzy information.

	g_1	g_2	g_3
f_1	(0.67, 0.46)	(0.93, 0.46)	(0.46, 0.71)
f_2	(0.74, 0.50)	(0.10, 0.46)	(0.67, 0.74)
f_3	(0.59, 0.46)	(0.90, 0.10)	(0.89, 0.00)
f_4	(0.90, 0.20)	(0.50, 0.40)	(0.60, 0.40)

- (i) Step 1. The matrix of correlation coefficients is constructed by computing the correlation coefficients from Table 2:

$$M = \begin{pmatrix} 1 & 0.8536 & 0.9244 & 0.9079 \\ 0.8536 & 1 & 0.7316 & 0.7521 \\ 0.9244 & 0.7316 & 1 & 0.8463 \\ 0.9079 & 0.7521 & 0.5463 & 1 \end{pmatrix}. \quad (24)$$

- (ii) Step 2. The equivalent correlation matrix is constructed by finite repeated compositions.

$$\begin{aligned} M^2 = M \circ M &= \begin{pmatrix} 1 & 0.8536 & 0.9244 & 0.8042 \\ 0.8536 & 1 & 0.9244 & 0.9079 \\ 0.9244 & 0.9244 & 1 & 0.9079 \\ 0.8042 & 0.9079 & 0.9079 & 1 \end{pmatrix}, \\ M^4 = M^2 \circ M^2 &= \begin{pmatrix} 1 & 0.9244 & 0.9244 & 0.9079 \\ 0.9244 & 1 & 0.9244 & 0.9079 \\ 0.9244 & 0.9244 & 1 & 0.9079 \\ 0.9079 & 0.9079 & 0.9079 & 1 \end{pmatrix}, \\ M^8 = M^4 \circ M^4 &= \begin{pmatrix} 1 & 0.9244 & 0.9244 & 0.9079 \\ 0.9244 & 1 & 0.9244 & 0.9079 \\ 0.9244 & 0.9244 & 1 & 0.9079 \\ 0.9079 & 0.9079 & 0.9079 & 1 \end{pmatrix}. \end{aligned} \quad (25)$$

- (iii) Since $M^8 = M^4$, therefore, M^4 is an equivalent correlation matrix.

- (iv) Step 3. The classifications are worked out by forming the α -cutting matrix as

- (5) If $0 \leq \alpha \leq 0.7521$, then all of f_1, f_2, f_3 , and f_4 are of the same type, i.e., $\{f_1, f_2, f_3, f_4\}$
- (6) If $0.7521 \leq \alpha \leq 0.8536$, then the vehicles are classified into two types as $\{f_2\}$ and $\{f_1, f_3, f_4\}$
- (7) If $0.8536 \leq \alpha \leq 0.9244$, then the vehicles are classified into three types as $\{f_1, f_3\}$, $\{f_2\}$, and $\{f_4\}$
- (8) If $0.9244 \leq \alpha \leq 1$, then the vehicles are classified into four types as $\{f_1\}$, $\{f_2\}$, $\{f_3\}$, and $\{f_4\}$

The outcomes achieved specify that the correlation of q-ROFSs generalizes the correlation of IFSs. Because these outcomes achieved by substituting $q = 1$ in the proposed method are identical to the outcomes through the

correlation of IFS. Thus, the proposed correlation coefficient for q-ROFSs can be applied to the problems of an intuitionistic fuzzy nature. These examples verify the superiority and dominance of the correlations of q-ROFSs.

7. Downsides of Current Structures and Benefits of the Proposed Methods

This section discusses the shortcomings of the current structures, and the benefits of the proposed methods over the existing ones are also talked over.

7.1. Downsides

- (1) When speaking of the problems with dual opinions, i.e., the membership and the nonmembership grades, the concept of FSs fails to model them
- (2) Despite the fact that IFSs can model problems with dual opinions, they also flop because of the strong constraints on its characteristic functions. These restrictions bound the decision makers to a limited set of choices.
- (3) Likewise, PFSs also have constraints on making the choice of membership and nonmembership grades which bound the decision makers to a certain domain
- (4) Because of these restrictions in the structures of FSs, IFSs, and PFSs, their correlation coefficients are inoperable at dealing with the information in q-ROFSs

7.2. Benefits

- (1) The notion of q-ROFSs generalizes the structures of FSs, IFSs, and PFSs which implies that the structure of q-ROFSs is capable of dealing with the information provided in the article
- (2) The proposed correlation coefficients of q-ROFSs also generalize the correlation coefficients of IFSs and PFSs. Therefore, these correlation coefficients of q-ROFSs are capable of coping with the intuitionistic fuzzy information and Pythagorean fuzzy information, as discussed in Example 3 in the previous reaction.
- (3) With some modifications, the correlation coefficients of q-ROFSs can be applied to the intuitionistic fuzzy information and Pythagorean fuzzy information

TABLE 2: Intuitionistic fuzzy information.

	g_1	g_2	g_3
f_1	(0.67, 0.46)	(0.93, 0.46)	(0.46, 0.71)
f_2	(0.74, 0.50)	(0.10, 0.46)	(0.67, 0.74)
f_3	(0.59, 0.46)	(0.90, 0.10)	(0.89, 0.00)
f_4	(0.90, 0.20)	(0.50, 0.40)	(0.60, 0.40)

8. Conclusion

This research work discusses some fundamental concepts such as fuzzy sets (FSs), intuitionistic FSs (IFSs), and Pythagorean FSs (PFSs). Furthermore, the structure without any restrictions called the q-rung orthopair FSs (q-ROFSs) is discussed, that is, the generalization of aforementioned structures. In addition, the correlation coefficients for IFSs and PFSs are discussed. Furthermore, the shortcomings of these correlations are also identified. Moreover, some innovative correlation coefficients for q-ROFSs are introduced, and their generalizations are proved through examples and remarks. Also, the properties and results of the proposed correlation coefficients are presented. Furthermore, an algorithm for clustering via the proposed correlation coefficients is given along with an application to the practical clustering problem. Finally, a comparative analysis is carried out among the proposed correlation coefficients and the existing conceptions. The benefits of the proposed generalization and the downsides to the other available theories are argued. In future, these concepts can be introduced for other generalizations of fuzzy theory, which will develop many interesting structures, results, and applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: A Solution of Fredholm Integral Inclusions via Suzuki-Type Fuzzy Contractions

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] I. Uddin, A. Perveen, H. Işık, and R. Bhardwaj, "A Solution of Fredholm Integral Inclusions via Suzuki-Type Fuzzy Contractions," *Mathematical Problems in Engineering*, vol. 2021, Article ID 6579405, 8 pages, 2021.

Research Article

A Solution of Fredholm Integral Inclusions via Suzuki-Type Fuzzy Contractions

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In this study, we introduce fuzzy weak ϕ -contraction and Suzuki-type fuzzy weak ϕ -contraction and employ these to prove some fuzzy fixed point results for fuzzy mappings in the setting of metric spaces, which is followed by an example to support our claim. Next, we deduce some corollaries and fixed point results for multivalued mappings from our main result. Finally, as an application of our result, we provide the existence of a solution for a Fredholm integral inclusion.

1. Introduction and Preliminaries

The idea of fuzzy mapping was inspired by the fuzzy set theory given by Zadeh [1]. It was initiated by Heilpern [2] in 1981, defined to be a mapping from an arbitrary set to a subfamily of fuzzy sets in metric linear spaces. He established a fuzzy expansion of Banach contraction principle. It broadens and develops the concept of fuzzy fixed point theory, and several authors worked in this field afterward ([3–7] and references therein).

We describe some related concepts in short in the successive lines.

Here, (M, d) depicts a metric space. A fuzzy set in M is a function with domain M and codomain $[0, 1]$. If F is a fuzzy set and $\mu \in M$, the function value $F(\mu)$ is called grade of membership of μ in F . The collection of all fuzzy set in M is denoted by $\mathfrak{F}(M)$.

Let $F \in \mathfrak{F}(M)$ and $\alpha \in [0, 1]$. The α -level set of F , which we denote here by F_α , is defined by

$$\begin{aligned} F_\alpha &= \{\mu: F(\mu) \geq \alpha\}, \quad \alpha \in (0, 1], \\ F_0 &= \overline{\{\mu: F(\mu) > 0\}}, \end{aligned} \quad (1)$$

where \overline{B} denotes closure of set B .

Definition 1 (see [2]). A fuzzy subset F on M is said to be an approximate quantity if and only if its α -level set is a compact convex subset of M , for each $\alpha \in [0, 1]$ and $\sup_{\mu \in M} F(\mu) = 1$.

We denote by $\mathscr{W}(M) \subseteq \mathfrak{F}(M)$, the subcollection of approximate quantities. We also denote $\mathscr{W}_\alpha(M) = \{F \in \mathfrak{F}(M): F_\alpha \text{ is nonempty compact convex subset of a metric}$

space (M, d) . If $F \in \mathcal{W}(M)$ and $F(\mu_0) = 1$, for some $\mu_0 \in M$, then F is an approximation of μ_0 .

$$\begin{aligned} p_\alpha(F_1, F_2) &= \inf_{\mu \in (F_1)_\alpha, \nu \in (F_2)_\alpha} d(\mu, \nu); \\ p(F_1, F_2) &= \sup_\alpha p_\alpha(F_1, F_2); \\ D_\alpha(F_1, F_2) &= H((F_1)_\alpha, (F_2)_\alpha) = \max \left\{ \sup_{a \in (F_1)_\alpha} d(a, (F_2)_\alpha), \sup_{b \in (F_2)_\alpha} d(b, (F_1)_\alpha) \right\}; \\ D(F_1, F_2) &= \sup_\alpha D_\alpha(F_1, F_2). \end{aligned} \quad (2)$$

Remark 1 (see [2]). D is a metric on $\mathcal{W}(M)$. Let $F_1, F_2 \in \mathcal{W}(M)$. Then, F_1 is more accurate than F_2 , denoted by $F_1 \subset F_2$ iff $F_1(\mu) \leq F_2(\mu)$, for each $\mu \in M$.

Definition 3 (see [2]). Let M be a nonempty set and N any metric linear space. A mapping S is called fuzzy mapping if and only if S is a mapping from M into $\mathcal{W}(N)$ (or $\mathcal{W}_\alpha(N)$), i.e., $S\mu \in \mathcal{W}(N)$ (or $\mathcal{W}_\alpha(N)$), for each $\mu \in M$.

Lemma 1 (see [2]). *The following conditions hold for a metric space (M, d) :*

- (a) If $p_\alpha(\mu, F) = 0$, then $\mu_\alpha \subset F$
- (b) $p_\alpha(\mu, F) \leq d(\mu, \nu) + p_\alpha(\nu, F)$
- (c) If $\mu_\alpha \subset F_1$, then $p_\alpha(\mu, F_2) \leq D_\alpha(F_1, F_2)$

For all $z, \nu \in M$ and $F, F_1, F_2 \in \mathcal{W}(M)$.

A fuzzy mapping S is a fuzzy subset on $M \times N$ with membership function $S(\mu, \nu)$. The function value $S(\mu, \nu)$ is the grade of membership of ν in $S(\mu)$.

Definition 4 (see [8]). Let $\alpha \in [0, 1]$ and $\mu \in M$. The fuzzy point μ_α of M is the fuzzy set $\mu_\alpha: M \rightarrow [0, 1]$ given by

$$\mu_\alpha(\nu) = \begin{cases} \alpha, & \text{if } \mu = \nu, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

For $\alpha = 1$, we have

$$\mu_1(\nu) = \{\mu\} = \begin{cases} 1, & \text{if } \mu = \nu, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Definition 5 (see [9]). A fuzzy point μ_α in M is a fixed fuzzy point of the mapping S over M if $\mu_\alpha \subset S\mu$, i.e., $(S\mu)\mu \geq \alpha$ or $\mu \in (S\mu)_\alpha$.

Remark 2 (see [2]). If $\{\mu\} \subset S\mu$, then μ is a fixed point of fuzzy mapping S .

Also, the generalization of Banach contraction principle has been done in many ways along with providing their applications in different fields. One of them is by

Definition 2 (see [2]). [2] For $F_1, F_2 \in \mathcal{W}(M)$ and $\alpha \in [0, 1]$, define

generalizing the contraction condition, specially using nonlinear contractions, e.g., Suzuki-type contraction, F -contraction, and θ -contraction [10–16]. One of such generalizations, namely, ϕ -weak contraction was done by Alber and Guerre-Delabriere [17] in 1997 to prove fixed point result in the setting of Hilbert space, which was further utilized by Rhoades [18] in metric fixed point theory. Recently, a generalization of the same was furnished by Xue [19]. He used the class of mappings $\Gamma = \{\text{class of all continuous nondecreasing functions } \phi: [0, \infty) \rightarrow [0, \infty) \text{ with } \phi(0) = 0\}$ and defined generalized ϕ -weak contraction as follows:

$$d(S\mu, S\nu) \leq d(\mu, \nu) - \phi(d(S\mu, S\nu)), \quad \forall \mu, \nu \in M, \quad (5)$$

where S is a self-mapping on a metric space (M, d) and $\phi \in \Gamma$. Also, he showed that this contraction condition is more weaker than ϕ -weak contraction condition (viz. $d(S\mu, S\nu) \leq d(\mu, \nu) - \phi(d(\mu, \nu))$). After that, Perveen et al. [20, 21] used his idea and proved results using weaker conditions.

In this study, we utilize the above ideas and define fuzzy weak ϕ -contraction and Suzuki-type fuzzy weak ϕ -contraction, which we use to prove the existence of fuzzy fixed point supported by an example. Last, we furnish an application of our result to prove the existence of a solution of integral inclusion of Fredholm type.

2. Main Result

First, we define the same class of mappings used in [20, 21].

Let Φ denotes the set of all mappings $\phi: [0, \infty) \rightarrow [0, \infty)$ satisfying the following:

- (a) $(\Phi 1)$ ϕ is nondecreasing
- (b) $(\Phi 2)$ $\phi(\tau) = 0$ iff $\tau = 0$ and $\liminf_{n \rightarrow \infty} \phi(\tau_n) > 0$, whenever $\lim_{n \rightarrow \infty} \tau_n > 0$

We have noticed that [19] used the continuity of ϕ . Inspired by [22], we dropped the continuity condition and use a weaker condition, which is given in $(\Phi 2)$. In fact, $(\Phi 2)$ is also weaker than the condition that ϕ is lower semicontinuous. Indeed, if ϕ is lower semicontinuous, then for a sequence $\{\tau_n\}$ with $\lim_{n \rightarrow \infty} \tau_n = \tau > 0$, we have $\liminf_{n \rightarrow \infty} \phi(\tau_n) \geq \phi(\tau) > 0$.

Using the class defined above, we define the following contraction for fuzzy mapping.

Definition 6. Let (M, d) be a metric space. A fuzzy mapping $S: M \longrightarrow \mathcal{W}_\alpha(M)$ is

(a) a fuzzy weak ϕ -contraction mapping if

$$D_\alpha(S\mu, S\nu) \leq d(\mu, \nu) - \phi(D_\alpha(S\mu, S\nu)). \quad (6)$$

(b) a Suzuki-type fuzzy weak ϕ -contraction mapping if the following condition is satisfied:

$$\frac{1}{2}p_\alpha(\mu, S\mu) \leq d(\mu, \nu) \Rightarrow D_\alpha(S\mu, S\nu) \leq d(\mu, \nu) - \phi(D_\alpha(S\mu, S\nu)), \quad (7)$$

for all $\mu, \nu \in M$, where $\phi \in \Phi$

Remark 3. If S is fuzzy weak ϕ -contraction, then S is Suzuki-type fuzzy weak ϕ -contraction.

Now, we are ready to commence our main theorem.

Theorem 1. Let a complete metric space (M, d) and $S: M \longrightarrow \mathcal{W}_\alpha(M)$ be a Suzuki-type fuzzy weak ϕ -contraction, such that for every $\mu \in M$, $(S\mu)_\alpha$ is closed. Then, there exists $\mu^* \in M$, such that μ_α^* is a fuzzy fixed point of S , i.e., $\mu_\alpha^* \subset S\mu^*$.

Proof. Let $\mu_1 \in M$ be any arbitrary point. Since $S\mu_1 \in \mathcal{W}_\alpha(M)$, we can choose $\mu_2 \in (S\mu_1)_\alpha$, such that $d(\mu_1, \mu_2) = p_\alpha(\mu_1, S\mu_1)$. If $\mu_1 = \mu_2 \in (S\mu_1)_\alpha$, then we are done. Suppose that $\mu_1 \neq \mu_2$. Since $S\mu_2 \in \mathcal{W}_\alpha(M)$, there exists $\mu_3 \in (S\mu_2)_\alpha$, such that

$$d(\mu_2, \mu_3) = p_\alpha(\mu_2, S\mu_2) \leq D_\alpha(S\mu_1, S\mu_2). \quad (8)$$

Again, if $\mu_2 = \mu_3$, we are done. Otherwise, we continue this process and obtain a sequence $\{\mu_n\}$ satisfying the following conditions:

$$\begin{aligned} \mu_{n+1} &\in (S\mu_n)_\alpha, \\ \mu_{n+2} &\in (S\mu_{n+1})_\alpha, \\ d(\mu_{n+1}, \mu_{n+2}) &= p_\alpha(\mu_{n+1}, S\mu_{n+1}) \\ &\leq D_\alpha(S\mu_n, S\mu_{n+1}), \quad \forall n \in \mathbb{N}. \end{aligned} \quad (9)$$

Thus, we easily obtain

$$\frac{1}{2}p_\alpha(\mu_n, S\mu_n) < d(\mu_n, \mu_{n+1}), \quad (10)$$

which implies that

$$\begin{aligned} d(\mu_{n+1}, \mu_{n+2}) &\leq D_\alpha(S\mu_n, S\mu_{n+1}) \\ &\leq d(\mu_n, \mu_{n+1}) - \phi(D_\alpha(S\mu_n, S\mu_{n+1})) \\ &\leq d(\mu_n, \mu_{n+1}) - \phi(d(\mu_{n+1}, \mu_{n+2})), \end{aligned} \quad (11)$$

which further implies

$$d(\mu_{n+1}, \mu_{n+2}) \leq d(\mu_n, \mu_{n+1}). \quad (12)$$

Thus, we see that $\{\mu_n\}$ is a nonincreasing sequence of positive real number bounded below by 0. Hence, $\{\mu_n\}$ converges to a point $r \geq 0$. We assert that $r = 0$. Suppose it is not so, then taking limit in (11), we obtain

$$r \leq r - \liminf_{n \rightarrow \infty} \phi(d(\mu_{n+1}, \mu_{n+2})), \quad (13)$$

which is a contradiction. Therefore, we have

$$\lim_{n \rightarrow \infty} d(\mu_n, \mu_{n+1}) = 0. \quad (14)$$

Next, we prove that $\{\mu_n\}$ is a Cauchy sequence. Suppose on contrary that it is not so, then there exist two subsequences $\{\mu_{m_k}\}$ and $\{\mu_{n_k}\}$ of $\{\mu_n\}$, such that n_k is the smallest positive integer for which

$$n_k > m_k > k,$$

$$d(\mu_{m_k}, \mu_{n_k}) \geq \varepsilon, \quad (15)$$

$$d(\mu_{m_k}, \mu_{n_k-1}) < \varepsilon.$$

Now, utilizing triangular inequality, we obtain

$$\begin{aligned} \varepsilon &\leq d(\mu_{m_k}, \mu_{n_k}) \leq d(\mu_{m_k}, \mu_{n_k-1}) + d(\mu_{n_k-1}, \mu_{n_k}) \\ &< \varepsilon + d(\mu_{n_k-1}, \mu_{n_k}). \end{aligned} \quad (16)$$

Letting $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} d(\mu_{m_k}, \mu_{n_k}) = \varepsilon. \quad (17)$$

Again, by triangular inequality,

$$\begin{aligned} d(\mu_{m_k}, \mu_{n_k}) &\leq d(\mu_{m_k}, \mu_{m_k+1}) + d(\mu_{m_k+1}, \mu_{n_k+1}) + d(\mu_{n_k+1}, \mu_{n_k}), \\ d(\mu_{m_k+1}, \mu_{n_k+1}) &\leq d(\mu_{m_k+1}, \mu_{m_k}) + d(\mu_{m_k}, \mu_{n_k}) + d(\mu_{n_k}, \mu_{n_k+1}), \end{aligned} \quad (18)$$

which on letting $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} d(\mu_{m_k+1}, \mu_{n_k+1}) = \varepsilon. \quad (19)$$

Next, by (14) and (17), there exists $n_0 \geq 1$, such that

$$\frac{1}{2}p_\alpha(\mu_{m_k}, S\mu_{m_k}) < \frac{1}{2}\varepsilon < d(\mu_{m_k}, \mu_{n_k}), \quad \forall k \geq n_0. \quad (20)$$

Thus, for $\mu = \mu_{m_k}$ and $\nu = \mu_{n_k}$, by (7), we obtain

$$D_\alpha(S\mu_{m_k}, S\mu_{n_k}) \leq d(\mu_{m_k}, \mu_{n_k}) - \phi(D_\alpha(S\mu_{m_k}, S\mu_{n_k})), \quad (21)$$

which on letting $n \rightarrow \infty$ yields $\varepsilon \leq \varepsilon - \liminf_{n \rightarrow \infty} \phi(D_\alpha(S\mu_{m_k}, S\mu_{n_k}))$, a contradiction. Thus, $\{\mu_n\}$ is Cauchy in M . The completeness of (M, d) implies that $\mu_n \rightarrow \mu^*$, for some $\mu^* \in M$.

Next, we show that $\mu_\alpha^* \subset S\mu^*$. As $\mu_n \rightarrow \mu^*$, there exists $n_1 \in \mathbb{N}$, such that for all $n \geq n_1$,

$$d(\mu_n, \mu^*) \leq \frac{1}{3}d(\mu, \mu^*), \quad \forall \mu \in M. \quad (22)$$

Using the above inequality, we obtain (for all $n \geq n_1$)

$$\begin{aligned}
\frac{1}{2}p_\alpha(\mu_n, S\mu_n) &\leq p_\alpha(\mu_n, S\mu_n) \\
&\leq d(\mu_n, \mu_{n+1}) \\
&\leq d(\mu_n, \mu^*) + d(\mu^*, \mu_{n+1}) \\
&\leq d(\mu_n, \mu^*) + p_\alpha(\mu^*, \mu_{n+1}) \\
&\leq \frac{1}{3}d(\mu, \mu^*) + \frac{1}{3}d(\mu, \mu^*) \\
&= d(\mu, \mu^*) - \frac{1}{3}d(\mu, \mu^*) \\
&\leq d(\mu, \mu^*) - d(\mu^*, \mu_n) \\
&\leq d(\mu_n, \mu),
\end{aligned} \tag{23}$$

i.e.,

$$\frac{1}{2}p_\alpha(\mu_n, S\mu_n) \leq d(\mu_n, \mu), \quad \forall n \geq n_1. \tag{24}$$

Thus, by (7), we obtain

$$\begin{aligned}
p_\alpha(\mu_{n+1}, S\mu) &\leq D_\alpha(S\mu_n, S\mu) \\
&\leq d(\mu_n, \mu) - \phi(D_\alpha(S\mu_n, S\mu)),
\end{aligned} \tag{25}$$

which implies

$$p_\alpha(\mu_{n+1}, S\mu) \leq d(\mu_n, S\mu). \tag{26}$$

Taking limit $n \rightarrow \infty$, we obtain

$$p_\alpha(\mu^*, S\mu) \leq d(\mu^*, \mu). \tag{27}$$

Furthermore, we prove that

$$D_\alpha(S\mu, S\mu^*) \leq d(\mu, \mu^*) - \phi(D_\alpha(S\mu, S\mu^*)), \quad \forall \mu \in M. \tag{28}$$

The above equation holds trivially for $\mu = \mu^*$. Suppose $\mu \neq \mu^*$. Then, for every $n \in \mathbb{N}$, there exists $\nu_n \in (S\mu)_\alpha$, such that

$$d(\mu^*, \nu_n) \leq p_\alpha(\mu^*, S\mu) + \frac{1}{n}d(\mu, \mu^*). \tag{29}$$

Thus, with the help of the above inequality and (27),

$$\begin{aligned}
p_\alpha(\mu, S\mu) &\leq d(\mu, \nu_n) \\
&\leq d(\mu, \mu^*) + d(\mu^*, \nu_n) \\
&\leq d(\mu, \mu^*) + p_\alpha(\mu^*, S\mu) + \frac{1}{n}d(\mu, \mu^*) \\
&\leq d(\mu, \mu^*) + d(\mu, \mu^*) + \frac{1}{n}d(\mu, \mu^*) \\
&= \left(2 + \frac{1}{n}\right)d(\mu, \mu^*).
\end{aligned} \tag{30}$$

On taking limit $n \rightarrow \infty$, we obtain

$$\frac{1}{2}p_\alpha(\mu, S\mu) \leq d(\mu, \mu^*). \tag{31}$$

So, (28) holds true for all $\mu \in M$. Now, if $\lim_{n \rightarrow \infty} p_\alpha(\mu_{n+1}, S\mu^*) = 0$, then we are done. Assume that it is not so, then there exists $\varepsilon_0 > 0$, such that for every $k \in \mathbb{N}$, we can choose $n_k \in \mathbb{N}$, such that $p_\alpha(\mu_{n_k+1}, S\mu^*) > \varepsilon_0 > 0$ for all $n_k \geq k$. For $\mu = \mu_{n_k}$, (28) reduces to

$$\begin{aligned}
p_\alpha(\mu_{n_k+1}, S\mu^*) &\leq D_\alpha(S\mu_{n_k}, S\mu^*) \\
&\leq d(\mu_{n_k}, \mu^*) - \phi(D_\alpha(S\mu_{n_k}, S\mu^*)).
\end{aligned} \tag{32}$$

Taking $k \rightarrow \infty$, we obtain

$$\begin{aligned}
\lim_{k \rightarrow \infty} p_\alpha(\mu_{n_k+1}, S\mu^*) &\leq \lim_{k \rightarrow \infty} D_\alpha(S\mu_{n_k}, S\mu^*) \\
&\leq \lim_{k \rightarrow \infty} [d(\mu_{n_k}, \mu^*) - \phi(D_\alpha(S\mu_{n_k}, S\mu^*))] \\
&\leq \lim_{k \rightarrow \infty} d(\mu_{n_k}, \mu^*),
\end{aligned} \tag{33}$$

i.e., $p_\alpha(\mu^*, S\mu^*) \leq 0$, a contradiction. So, $\mu_\alpha^* \subset S\mu^*$, and the proof is completed. \square

We present the following example to illustrate the utility of our proven result.

Example 1. Let $M = \{1, 2, 3\}$, and $d: M \times M \rightarrow [0, \infty)$ is defined by

$$\begin{aligned}
d(1, 3) &= \frac{3}{8}, \\
d(1, 2) &= \frac{1}{2}, \\
d(2, 3) &= \frac{3}{2}, \\
d(\mu, \mu) &= 0, \quad \forall \mu \in M, \\
d(\mu, \nu) &= d(\nu, \mu), \quad \forall \mu, \nu \in M.
\end{aligned} \tag{34}$$

We define $\phi: [0, \infty) \rightarrow [0, \infty)$ by

$$\phi(\tau) = \frac{\tau}{2}, \quad \forall \tau \in [0, \infty), \tag{35}$$

and a fuzzy mapping by

$$\begin{aligned}
 S_1(\mu) &= \begin{cases} 0, & \text{if } \mu = 1, \\ \alpha, & \text{if } \mu = 2, \\ \frac{\alpha}{2}, & \text{if } \mu = 3, \end{cases} \\
 S_2(\mu) &= \begin{cases} \frac{\alpha}{2}, & \text{if } \mu = 1, \\ 2\alpha, & \text{if } \mu = 2, \\ \frac{\alpha}{4}, & \text{if } \mu = 3, \end{cases} \\
 S_3(\mu) &= \begin{cases} 2\alpha, & \text{if } \mu = 1, \\ \frac{\alpha}{3}, & \text{if } \mu = 2, \\ 0, & \text{if } \mu = 3. \end{cases}
 \end{aligned} \quad (36)$$

Then, $(S_1)_\alpha = (S_2)_\alpha = \{2\}$ and $(S_3)_\alpha = \{1\}$, and

$$\begin{aligned}
 \frac{1}{2}p_\alpha(1, S_1) &\leq d(1, \nu), \quad \nu = 2, 3; \\
 \frac{1}{2}p_\alpha(2, S_2) &\leq d(2, \nu), \quad \forall \nu \in M, \\
 \frac{1}{2}p_\alpha(3, S_3) &\leq d(3, \nu), \quad \nu = 1, 2.
 \end{aligned} \quad (37)$$

We consider three cases.

Case 1. If $\mu, \nu \in \{1, 2\}$, then we have

$$D_\alpha(S\mu, S\nu) = 0, \quad \forall \mu, \nu \in M. \quad (38)$$

Hence, (7) is satisfied for $\mu, \nu \in \{1, 2\}$ trivially.

Case 2. If $\mu = 3$ and $\nu = 1$, then we have

$$\begin{aligned}
 D_\alpha(S3, S1) &= d(1, 2) = \frac{1}{2}, \\
 d(3, 1) &= \frac{3}{8}.
 \end{aligned} \quad (39)$$

Then,

$$\begin{aligned}
 D_\alpha(S3, S1) &= \frac{1}{2} \\
 &\leq \frac{3}{8} - \frac{11}{19} \\
 &= \frac{3}{8} - \phi(D_\alpha(S3, S1)).
 \end{aligned} \quad (40)$$

So condition (7) is satisfied.

Case 3. If $\mu = 3$ and $\nu = 2$, then we have

$$\begin{aligned}
 D_\alpha(S3, S2) &= d(1, 2) = \frac{1}{2}, \\
 d(3, 2) &= \frac{3}{2}.
 \end{aligned} \quad (41)$$

Thus, we get

$$\begin{aligned}
 D_\alpha(S3, S2) &= \frac{1}{2} \\
 &\leq \frac{3}{2} - \frac{11}{19} \\
 &= \frac{3}{2} - \phi(D_\alpha(S3, S1)).
 \end{aligned} \quad (42)$$

We see that the assumptions of Theorem 1 are fulfilled in all cases, and hence, S has a fuzzy fixed point which is 2.

In view of Remark 3, we deduce the underlying result.

Theorem 2. Let (M, d) be a complete metric space and $S: M \rightarrow \mathcal{W}_\alpha(M)$ a fuzzy weak ϕ -contraction, such that for every $\mu \in M$, $(S\mu)_\alpha$ is closed. Then, there exists $\mu^* \in M$, such that μ_α^* is a fuzzy fixed point of S , i.e., $\mu_\alpha^* \subset S\mu^*$.

If the fuzzy mapping S is a Suzuki-type fuzzy weak ϕ -contraction, then it immediately satisfies the following contraction condition:

$$\frac{1}{2}p_\alpha(\mu, S\mu) \leq d(\mu, \nu) \Rightarrow D_\alpha(S\mu, S\nu) \leq d(\mu, \nu) - \phi(d(\mu, \nu)), \quad (43)$$

where $\mu, \nu \in M$ and $\phi \in \Phi$. But the converse need not be true. We justify this claim by showing that the condition $D_\alpha(S\mu, S\nu) \leq d(\mu, \nu) - \phi(D_\alpha(\mu, \nu))$ is weaker than $D_\alpha(S\mu, S\nu) \leq d(\mu, \nu) - \phi(d(\mu, \nu))$. For this, we consider the following example.

Let $M = \{a, b, c\}$ with the metric $d: M \times M \rightarrow [0, \infty)$ defined by $d(a, b) = d(a, c) = 5$, $d(b, c) = 3$; $d(\mu, \mu) = 0$, and $d(\mu, \nu) = d(\nu, \mu)$, $\forall \mu, \nu \in M$. Define

$$\phi(\tau) = \frac{\tau}{2}, \quad \forall \tau \in [0, \infty), \quad (44)$$

and a fuzzy mapping S by

$$\begin{aligned}
 S_a(\mu) &= \begin{cases} \alpha, & \text{if } \mu = a, \\ 2\alpha, & \text{if } \mu \in \{b, c\}, \end{cases} \\
 S_b(\mu) &= \begin{cases} \alpha, & \text{if } \mu \in \{a, b\}, \\ \frac{\alpha}{2}, & \text{if } \mu = c, \end{cases} \\
 S_c(\mu) &= \begin{cases} 2\alpha, & \text{if } \mu = a, \\ \alpha, & \text{if } \mu = b, \\ 0, & \text{if } \mu = c. \end{cases}
 \end{aligned} \quad (45)$$

So, we get $(S_a)_\alpha = \{a, b, c\}$ and $(S_b)_\alpha = (S_c)_\alpha = \{a, b\}$, and

$$\begin{aligned} D_\alpha(S_a, S_b) &= D_\alpha(S_a, S_c) = 3, \\ D_\alpha(S_b, S_c) &= 0. \end{aligned} \quad (46)$$

We observe that the condition $D_\alpha(S\mu, S\nu) \leq d(\mu, \nu) - \phi(D_\alpha(\mu, \nu))$ is satisfied for all $\mu, \nu \in M$, but for $\mu = a$ and $\nu = b$, the condition $D_\alpha(S\mu, S\nu) \leq d(\mu, \nu) - \phi(d(\mu, \nu))$ is not fulfilled.

Hence, we obtain the following result.

Theorem 3. Let (M, d) be a complete metric space and $S: M \rightarrow \mathcal{W}_\alpha(M)$ a fuzzy mapping satisfying (43). Then, there exists $\mu^* \in M$, such that $\mu_\alpha^* \subset S\mu^*$.

Taking $\phi(\tau) = (1 - k)\tau$, $k \in [0, 1)$ in the above result (viz. (43)), we obtain the next result.

Theorem 4. Let (M, d) be a complete metric space and $S: M \rightarrow \mathcal{W}_\alpha(M)$ a fuzzy mapping satisfying the condition

$$\frac{1}{2}p_\alpha(\mu, S\mu) \leq d(\mu, \nu) \Rightarrow D_\alpha(S\mu, S\nu) \leq k d(\mu, \nu), \quad (47)$$

where $\mu, \nu \in M$. Then, there exists $\mu^* \in M$, such that $\mu_\alpha^* \subset S\mu^*$.

Remark 4. Let S be a fuzzy mapping from M to $\mathcal{W}_\alpha(M)$ and $T: M \rightarrow K(M)$ a closed mapping (where $K(M)$ denotes the set of all compact subsets of M). Define

$$(S\mu)(\nu) = \begin{cases} \alpha, & \text{if } \nu \in T\mu, \\ 0, & \text{otherwise,} \end{cases} \quad (48)$$

for each $\mu \in M$. Note that,

$$(S\mu)_\alpha = \{\nu: (S\mu)\nu \geq \alpha\} = T\mu. \quad (49)$$

In view of above remark, we obtain the fixed point results for multivalued mapping T (defined above) from Theorems 1-4.

Theorem 5. Let (M, d) be a complete metric space and $T: M \rightarrow K(M)$ a multivalued closed mapping satisfying

$$\frac{1}{2}d(\mu, T\mu) \leq d(\mu, \nu) \Rightarrow H(T\mu, T\nu) \leq d(\mu, \nu) - \phi(H(T\mu, T\nu)), \quad (50)$$

$\forall \mu, \nu \in M$ and $\phi \in \Phi$. Then, T has a fixed point.

Theorem 6. Let (M, d) be a complete metric space and $T: M \rightarrow K(M)$ a multivalued closed mapping satisfying

$$\frac{1}{2}d(\mu, T\mu) \leq d(\mu, \nu) \Rightarrow H(T\mu, T\nu) \leq d(\mu, \nu) - \phi(d(\mu, \nu)), \quad (51)$$

$\forall \mu, \nu \in M$ and $\phi \in \Phi$. Then, T has a fixed point.

Theorem 7. Let (M, d) be a complete metric space and $T: M \rightarrow K(M)$ a multivalued closed mapping satisfying

$$\frac{1}{2}d(\mu, T\mu) \leq d(\mu, \nu) \Rightarrow H(T\mu, T\nu) \leq k d(\mu, \nu), \quad (52)$$

$\forall \mu, \nu \in M$. Then, T has a fixed point.

Similarly, we can obtain the results corresponding to Theorem 2.

3. An Application to the Fredholm Integral Inclusion

Consider the following Fredholm integral inclusion:

$$\mu(\tau) \in f(\tau) + \int_a^b K(\tau, s, \mu(s))ds, \quad \tau \in [a, b], \quad (53)$$

where $f \in C[a, b]$ and $K: [a, b] \times [a, b] \times \mathbb{R} \rightarrow P_{CV}(\mathbb{R})$ ($P_{CV}(\mathbb{R})$ denotes the class of all nonempty compact and convex subsets of \mathbb{R}), and $\mu \in C[0, 1]$ is an unknown function. Consider $M = C[a, b]$ and take the complete metric space (M, d) , where

$$d(\mu, \nu) = \max_{\tau \in [a, b]} |\mu(\tau) - \nu(\tau)|, \quad \forall \mu, \nu \in C[a, b]. \quad (54)$$

Before proving our claim, we note down the following lemma.

Lemma 2 (see [23, 24]). Let (M, d) be a metric space and $P, Q \in P(M)$. If there exists $\eta \in \mathbb{R}$ ($\eta > 0$), such that

(a) For each $p \in P$, there exists $q \in Q$, such that $d(p, q) \leq \eta$

(b) For each $q \in Q$, there exists $p \in P$, such that $d(q, p) \leq \eta$

Then, $H(P, Q) \leq \eta$.

Theorem 8. Under the conditions given as follows:

(A₁) for all $\mu \in C[a, b]$, the operator $K: [a, b] \times [a, b] \times \mathbb{R} \rightarrow P_{CV}(\mathbb{R})$ is such that $K_\mu(\tau, s) = K(\tau, s, \mu(s))$ is lower semicontinuous on $[a, b] \times [a, b]$

(A₂) there exists a continuous function $\lambda: [a, b] \times [a, b] \rightarrow [0, \infty)$, such that

$$H(K_\mu(\tau, s), K_\nu(\tau, s)) \leq \lambda(\tau, s)|\mu(\tau) - \nu(\tau)|, \quad (55)$$

for all $\tau, s \in [a, b]$ and $\mu, \nu \in C[a, b]$ with $\int_a^b \lambda(\tau, s)ds \leq 2/3$

The Fredholm integral inclusion (53) has a solution in $C[a, b]$.

Proof. Define the fuzzy mapping $S: M \rightarrow \mathfrak{F}(M)$ in such a way that

$$(S\mu)_\alpha = \left\{ \nu \in M: \nu(\tau) \in f(\tau) + \int_a^b K_\mu(\tau, s)ds, \tau \in [a, b] \right\}. \quad (56)$$

It is very obvious that the set of solutions of $(S\mu)_\alpha$ coincides with the set of fixed points of (53). So, we need to prove that $(S\mu)_\alpha$ has at least one fixed point.

For this, we consider an arbitrary fixed point $\mu \in M$ and the set-valued operator $K_\mu: [a, b] \times [a, b] \longrightarrow P_{CV}(\mathbb{R})$. Using Michael's theorem, we obtain a continuous function, such that $k_\mu(\tau, s) \in K_\mu(\tau, s)$, for each $\tau, s \in [a, b]$. Thus, $f(\tau) + k_\mu(\tau, s) \in (S\mu)_\alpha$, and so, $(S\mu)_\alpha \neq \emptyset$. Clearly, $(S\mu)_\alpha$ is closed (hence compact) and convex. So, $S \in W_\alpha(M)$.

Now, we will check that

$$D_\alpha(S\mu_1, S\mu_2) \leq d(\mu_1, \mu_2) - \phi(D_\alpha(S\mu_1, S\mu_2)), \quad \forall \mu_1, \mu_2 \in M. \quad (57)$$

Let $\nu_1 \in (S\mu_1)_\alpha$ (arbitrary), such that $\nu_1(\tau) \in f(\tau) + \int_a^b K(\tau, s, \mu_1(s))ds$, for $\tau \in [a, b]$. This means for all $\tau, s \in [a, b]$, there exists $k_{\mu_1}(\tau, s) \in K_{\mu_1}(\tau, s)$, such that $\nu_1(\tau) = f(\tau) + \int_a^b k_{\mu_1}(\tau, s)ds$. Now, from (A_2) , we have

$$H(K_{\mu_1}(\tau, s), K_{\mu_2}(\tau, s)) \leq \lambda(\tau, s)|\mu_1(\tau) - \mu_2(\tau)|. \quad (58)$$

Then, there exists $\mu(\tau, s) \in K_{\mu_2}(\tau, s)$, such that

$$|k_{\mu_1}(\tau, s) - \mu(\tau, s)| \leq \lambda(\tau, s)|\mu_1(\tau) - \mu_2(\tau)|. \quad (59)$$

Now, we consider the multivalued operator U defined by

$$U(\tau, s) = K_{\mu_2}(\tau, s) \cap \left\{ u \in \mathbb{R} : |k_{\mu_1}(\tau, s) - u| \leq \lambda(\tau, s)|\mu_1(\tau) - \mu_2(\tau)| \right\} \quad (60)$$

Hence, by (A_1) , U is lower semicontinuous which ensures the existence of a continuous operator $k_{\mu_2}(\tau, s) \in U(\tau, s)$, implying that

$$\nu_2(\tau) = f(\tau) + \int_a^b k_{\mu_2}(\tau, s)ds, \quad (61)$$

and hence,

$$\nu_2(\tau) \in f(\tau) + \int_a^b K_{\mu_2}(\tau, s)ds. \quad (62)$$

So, we get $\nu_2 \in (S\mu_2)_\alpha$, and

$$\begin{aligned} |\nu_2(\tau) - \nu_1(\tau)| &\leq \int_a^b |k_{\mu_2}(\tau, s) - k_{\mu_1}(\tau, s)|ds \\ &\leq \int_a^b |\lambda(\tau, s)|\mu_2(s) - \mu_1(s)||ds \\ &\leq \max_{t \in [a, b]} |\mu_2(\tau) - \mu_1(\tau)| \int_a^b |\lambda(\tau, s)|ds \\ &< \frac{2}{3}d(\mu_2, \mu_1). \end{aligned} \quad (63)$$

After interchanging the roles of μ_1 and μ_2 and using Lemma 2, we obtain (for each $\mu_1, \mu_2 \in M$)

$$H((S\mu_1)_\alpha, (S\mu_2)_\alpha) \leq \frac{2}{3}d(\mu_1, \mu_2), \quad (64)$$

and by considering $\phi(\tau) = \tau/2$ (for all $\tau \in [0, \infty)$), all the assumptions of Theorem 1 as well as Theorem 2 are satisfied. Hence, the inclusion problem (53) has a solution. \square

4. Conclusion

In this study, inspired by the work of Suzuki [10] and Xue [19], we define two new contractions, i.e., fuzzy weak ϕ -contraction and Suzuki-type fuzzy weak ϕ -contraction and use them to prove the existence of fuzzy fixed point and well exemplify them. Also, we provide an application of our proven result to show the existence of solution of Fredholm integral inclusion problem.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Retraction

Retracted: On Strongly $b - \theta$ -Continuous Mappings in Fuzzifying Topology

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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- [1] T. Yang and A. M. Khalil, "On Strongly $b - \theta$ -Continuous Mappings in Fuzzifying Topology," *Mathematical Problems in Engineering*, vol. 2021, Article ID 3244618, 15 pages, 2021.

Review Article

On Strongly $b - \theta$ -Continuous Mappings in Fuzzifying Topology

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In this article, we will define the new notions (e.g., $b - \theta$ -neighborhood system of point, $b - \theta$ -closure (interior) of a set, and $b - \theta$ -closed (open) set) based on fuzzy logic (i.e., fuzzifying topology). Then, we will explain the interesting properties of the above five notions in detail. Several basic results (for instance, Definition 7, Theorem 3 (iii), (v), and (vi), Theorem 5, Theorem 9, and Theorem 4.6) in classical topology are generalized in fuzzy logic. In addition to, we will show that every fuzzifying $b - \theta$ -closed set is fuzzifying γ -closed set (by Theorem 3 (vi)). Further, we will study the notion of fuzzifying $b - \theta$ -derived set and fuzzifying $b - \theta$ -boundary set and discuss several of their fundamental basic relations and properties. Also, we will present a new type of fuzzifying strongly $b - \theta$ -continuous mapping between two fuzzifying topological spaces. Finally, several characterizations of fuzzifying strongly $b - \theta$ -continuous mapping, fuzzifying strongly $b - \theta$ -irresolute mapping, and fuzzifying weakly $b - \theta$ -irresolute mapping along with different conditions for their existence are obtained.

1. Introduction and Preliminaries

In classical topology, the notions of b -open set, b -closed set, and strongly $\theta - b$ -continuous mapping are presented in [1, 2]. After that, Hanafe [3] used the term γ -open sets instead of b -open sets and studied the notions of γ -open sets and γ -continuous mapping in fuzzy topology [4]. Benchalli and Karna [5] presented a novel form of fuzzy subset named fuzzy b -open (closed) set, and some basic properties are proved and also their relations with different fuzzy sets in fuzzy topological spaces are investigated. In 2017, Dutta and Tripathy [6] introduced a new kind of open set named fuzzy $b - \theta$ open set (i.e., which is a generalization of $b - \theta$ open set). Ying [7] extended the basic notions in classical topology to fuzzifying topology based on fuzzy logic (i.e., as considered a novel approach of fuzzy topology, which depends on the various basic relations of topological spaces and the logical analysis of topological axioms). Many researchers are interested in fuzzifying topology (such as fuzzifying semi-open sets [8], fuzzifying preopen sets [9], fuzzifying α -open sets [10], fuzzifying β -open sets [11], and fuzzifying γ -open sets [12]). Therefore, in this article, we will extend the

notions of $b - \theta$ -neighborhood system of a point, $b - \theta$ -closure (interior) of a set, $b - \theta$ -open (closed) set, $b - \theta$ -derived sets, and $b - \theta$ -boundary sets in fuzzifying topology. Also, we introduce the notion of fuzzifying strongly $b - \theta$ -continuous mapping, fuzzifying strongly $b - \theta$ -irresolute mapping, and fuzzifying weakly $b - \theta$ -irresolute mapping between two fuzzifying topological spaces.

The rest of this article is arranged as follows. In this section, we briefly recall several notions: closed (open) set, closure (interior) of a set, neighborhood system of point, γ -closed (open) set, γ -closure (interior) of a set, γ -neighborhood system of point, continuous mapping, and γ -continuous mapping in fuzzifying topology which are used in the sequel. In Section 2, we define the notions of $b - \theta$ -neighborhood system of a point, $b - \theta$ -closure (interior) of a set, and $b - \theta$ -open (closed) set in fuzzifying topology. The interesting relation properties of the above notions are explained in detail. In Section 3, we present the notions $b - \theta$ -derived set and $b - \theta$ -boundary set in fuzzifying topology and introduce the characterizations of interesting properties between fuzzifying $b - \theta$ -derived set and fuzzifying $b - \theta$ -closure of a set. In Section 4, we define the

fuzzifying strongly $b - \theta$ -continuous mapping, fuzzifying strongly $b - \theta$ -irresolute mapping, and fuzzifying weakly $b - \theta$ -irresolute mapping between two fuzzifying topological spaces and investigate some properties of them.

Firstly, we give the notions of a fuzzy logical [7, 13] as follows:

$$\begin{aligned}
 [\vartheta] &= 1 - [\vartheta], \\
 [\vartheta_1 \wedge \vartheta_2] &= \min([\vartheta_1], [\vartheta_2]), \\
 [\vartheta_1 \longrightarrow \vartheta_2] &= \min(1, 1 - [\vartheta_1] + [\vartheta_2]), \\
 [\forall x \vartheta(x)] &= \inf_{x \in X} [\vartheta(x)], \quad [x \in \Phi] = \Phi(x), \quad [\exists x \vartheta(x)] := [(\forall x \vartheta(x))], \\
 [\vartheta_1 \vee \vartheta_2] &:= [(\vartheta_1 \wedge \vartheta_2)], \quad [\vartheta_1 \leftrightarrow \vartheta_2] := [(\vartheta_1 \longrightarrow \vartheta_2) \wedge (\vartheta_2 \longrightarrow \vartheta_1)], \\
 [\Phi \sqsubseteq \Psi] &:= [\forall x (x \in \Phi \longrightarrow x \in \Psi)] = \inf_{x \in X} \min(1, 1 - \Phi(x) + \Psi(x)), \\
 [\Phi \equiv \Psi] &:= [(\Phi \sqsubseteq \Psi) \wedge (\Psi \sqsubseteq \Phi)], \\
 [\vartheta_1 \dot{\vee} \vartheta_2] &:= [(\vartheta_1 \longrightarrow \vartheta_2)] = \min(1, [\vartheta_1] + [\vartheta_2]), \\
 [\vartheta_1 \dot{\wedge} \vartheta_2] &:= [(\vartheta_1 \longrightarrow \vartheta_2)] = \max(0, [\vartheta_1] + [\vartheta_2] - 1).
 \end{aligned} \tag{1}$$

Secondly, we present the basic notions related to fuzzifying topological space as follows.

Definition 1 (see [7]). τ (i.e., $\tau \in \widehat{\mathcal{F}}(2^X)$, 2^X the set of all subsets of a set X) is called a fuzzifying topological space (for short, $\mathcal{FTS}(X, \tau)$) if we have the following three conditions:

- (i) $\tau(X) = \tau(\emptyset) = 1$
- (ii) $\forall \Phi, \Psi, \tau(\Phi \cap \Psi) \geq \tau(\Phi) \wedge \tau(\Psi)$
- (iii) $\forall \{\Phi_\lambda : \lambda \in \Lambda\}, \tau(\bigcup_{\lambda \in \Lambda} \Phi_\lambda) \geq \inf_{\lambda \in \Lambda} \tau(\Phi_\lambda)$

Definition 2 (see [7, 13]). The several notions of \mathcal{FTS} (X, τ) are given as follows ($\forall \Phi, \Psi \in 2^X$):

- (i) $\overline{\mathcal{F}}$ (i.e., $\overline{\mathcal{F}} \in \widehat{\mathcal{F}}(2^X)$) is called the set of all fuzzifying closed sets if $\Phi \in \overline{\mathcal{F}} := \Phi^c \in \tau$, where $\Phi^c = X - \Phi$ is the complement of Φ
- (ii) \mathcal{N}_x (i.e., $\mathcal{N}_x \in \widehat{\mathcal{F}}(2^X)$, $x \in X$) is called a fuzzifying neighborhood system of x if $\mathcal{N}_x(\Phi) = \sup_{x \in \Psi \sqsubseteq \Phi} \tau(\Psi)$
- (iii) $\overline{\mathcal{C}}(\Phi)$ is called a fuzzifying closure of Φ if $\overline{\mathcal{C}}(\Phi)(x) = 1 - \mathcal{N}_x(\Phi^c)$
- (iv) $\overline{\mathcal{I}}(\Phi)$ is called a fuzzifying interior of Φ if $\overline{\mathcal{I}}(\Phi)(x) = \mathcal{N}_x(\Phi)$

Noiri and Sayed [12] presented and studied the following notions in $\mathcal{FTS}(X, \tau)$ as indicated below.

Definition 3 (see [12]).

- (i) τ_γ (i.e., $\tau_\gamma \in \widehat{\mathcal{F}}(2^X)$) is called the set of all fuzzifying γ -open sets if

- (1) We give the symbol $[\vartheta]$ which is the truth value of ϑ (i.e., the set of truth values means the unit interval $[0, 1]$).
- (2) We can write $\models \varphi \Leftrightarrow [\varphi] = 1$ (i.e., φ is valid).
- (3) For $\Phi, \Psi \in [0, 1]^X$ (i.e., $[0, 1]^X$ mean the whole of fuzzy subsets of a set X) have the following for $x \in X$:

$$\Phi \in \tau_\gamma := \forall x (x \in \Phi \longrightarrow x \in \overline{\mathcal{C}}(\overline{\mathcal{F}}(\Phi)) \sqcup \overline{\mathcal{I}}(\overline{\mathcal{C}}(\Phi))), \tag{2}$$

i.e.,

$$\tau_\gamma(\Phi) = \inf_{x \in \Phi} \max(\overline{\mathcal{C}}(\overline{\mathcal{F}}(\Phi))(x), \overline{\mathcal{I}}(\overline{\mathcal{C}}(\Phi))(x)). \tag{3}$$

- (ii) $\overline{\mathcal{F}}_\gamma$ (i.e., $\overline{\mathcal{F}}_\gamma \in \widehat{\mathcal{F}}(2^X)$) is called the set of all fuzzifying γ -closed sets if $\Phi \in \overline{\mathcal{F}}_\gamma := \Phi^c \in \tau_\gamma$, where Φ^c is the complement of Φ .
- (iii) \mathcal{N}_x^γ (i.e., $\mathcal{N}_x^\gamma \in \widehat{\mathcal{F}}(2^X)$, $x \in X$) is called a fuzzifying γ -neighborhood system of x if $\mathcal{N}_x^\gamma(\Phi) = \sup_{x \in \Psi \sqsubseteq \Phi} \tau_\gamma(\Psi)$.
- (iv) $\overline{\mathcal{C}}_\gamma(\Phi)$ is called a fuzzifying γ -closure of Φ if $\overline{\mathcal{C}}_\gamma(\Phi)(x) = 1 - \mathcal{N}_x^\gamma(\Phi^c)$.
- (v) $\overline{\mathcal{I}}_\gamma(\Phi)$ is called a fuzzifying γ -interior of Φ if $\overline{\mathcal{I}}_\gamma(\Phi)(x) = \mathcal{N}_x^\gamma(\Phi)$.

Definition 4 (see [14]). C_o (i.e., $C_o \in \widehat{\mathcal{F}}(Y^X)$) (a unary fuzzy predicate) is called fuzzifying continuous mappings between $\mathcal{FTS}(X, \tau)$ and $\mathcal{FTS}(Y, \varphi)$ if

$$C_o(\psi) := (\forall O) \quad (O \in \varphi \longrightarrow \psi^{-1}(O) \in \tau), \tag{4}$$

i.e.,

$$[C_o(\psi)] = \inf_{O \in 2^Y} \min(1, 1 - \varphi(O) + \tau(\psi^{-1}(O))). \tag{5}$$

Definition 5 (see [12]). C_{oy} (i.e., $C_{oy} \in \widehat{\mathcal{F}}(Y^X)$) (a unary fuzzy predicate) is called fuzzifying γ -continuous mappings between $\mathcal{FTS}(X, \tau)$ and $\mathcal{FTS}(Y, \varphi)$ if

$$C_{oy}(\psi) := (\forall O) \quad (O \in \varphi \longrightarrow \psi^{-1}(O) \in \tau), \quad (6) \quad \text{Proof}$$

i.e.,

$$[C_{oy}(\psi)] = \inf_{O \in 2^Y} \min(1, 1 - \varphi(O) + \tau_y(\psi^{-1}(O))). \quad (7)$$

2. On Fuzzifying $b - \theta$ -Neighborhood System

Definition 6. $\mathcal{N}_x^{b\theta}$ (i.e., $\mathcal{N}_x^{b\theta} \in \widehat{\mathcal{F}}(2^X)$, $x \in X$) is called a fuzzifying $b - \theta$ -neighborhood system of x if

$$\mathcal{N}_x^{b\theta}(\Phi) = \sup_{\Psi \in 2^X} \max\left(0, \mathcal{N}_x^\gamma(\Psi) + \inf_{y \notin \Phi} \mathcal{N}_y^\gamma(\Psi^c) - 1\right). \quad (8)$$

Theorem 1. Let $\mathcal{N}^{b\theta}$ be a mapping from X to $\widehat{\mathcal{F}}^N(2^X)$ s.t. $x \mapsto \mathcal{N}_x^{b\theta}$ and $\widehat{\mathcal{F}}^N$ is a family of N -fuzzy (Normal fuzzy) subsets of 2^X having the following three properties ($\forall x \in X, \forall \Phi, \Psi \in 2^X$):

- (i) $\models \Phi \in \mathcal{N}_x^{b\theta} \longrightarrow x \in \Phi$
- (ii) $\models \Phi \sqsubseteq \Psi \longrightarrow (\Phi \in \mathcal{N}_x^{b\theta} \longrightarrow \Psi \in \mathcal{N}_x^{b\theta})$
- (iii) $\models (\Phi \in \mathcal{N}_x^{b\theta}) \wedge (\Psi \in \mathcal{N}_x^{b\theta}) \leftrightarrow \Phi \cap \Psi \in \mathcal{N}_x^{b\theta}$

(i) Case 1: if $[\Phi \in \mathcal{N}_x^{b\theta}] = 0$, then $[\Phi \in \mathcal{N}_x^{b\theta}] \leq [x \in \Phi]$.

Case 2: if $[\Phi \in \mathcal{N}_x^{b\theta}] > 0$, then $\sup_{\Psi \in 2^X} \max(0, \mathcal{N}_x^\gamma(\Psi) + [\overline{\mathcal{C}}_\gamma(\Psi) \sqsubseteq \Phi] - 1) > 0$. There exists $\Omega \in 2^X$ such that $\mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi] - 1 > 0$. From Theorems 4.2 (1) and 5.2 (3) in [12], we obtain

Consequently, $[\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi] = \inf_{y \notin \Phi} \Phi(1 - \overline{\mathcal{C}}_\gamma(\Omega)(y)) > 1 - \overline{\mathcal{C}}_\gamma(\Omega)(x)$, and so $x \in \Phi$. If $x \notin \Phi$, then we have $\inf_{y \notin \Phi} (1 - \overline{\mathcal{C}}_\gamma(\Omega)(y)) > 1 - \overline{\mathcal{C}}_\gamma(\Omega)(x)$; it is contradiction. Thus, $[x \in \Phi] = 1$, and hence, $[\Phi \in \mathcal{N}_x^{b\theta}] \leq [x \in \Phi]$.

(ii) Case 1: if $[\Phi \sqsubseteq \Psi] = 0$, then $\mathcal{N}_x^{b\theta}(\Phi) \leq \mathcal{N}_x^{b\theta}(\Psi)$.

Case 2: if $[\Phi \sqsubseteq \Psi] = 1$ and $\Omega \in 2^X$, then

and so $\mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Psi] - 1 \geq \mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi] - 1$. Thus, $\sup_{\Omega \in 2^X} \max(0, \mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Psi] - 1) \geq \sup_{\Omega \in 2^X} \max(0, \mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi] - 1)$. Hence, $\mathcal{N}_x^{b\theta}(\Phi) \leq \mathcal{N}_x^{b\theta}(\Psi)$.

(iii) $[\Phi \cap \Psi \in \mathcal{N}_x^{b\theta}] = \sup_{\Omega \in 2^X} \max(0, \mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi \cap \Psi] - 1)$. Since

$$\begin{aligned} [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi \cap \Psi] &= \forall x (x \in \overline{\mathcal{C}}_\gamma(\Omega) \longrightarrow x \in \Phi \cap \Psi) = \inf_{x \in X} \min(1, 1 - \overline{\mathcal{C}}_\gamma(\Omega)(x) + [(\Phi \cap \Psi)(x)]) \\ &= \inf_{x \notin \Phi \cap \Psi} (1 - \overline{\mathcal{C}}_\gamma(\Omega)(x)) = \inf_{x \in (\Phi^c) \cup (\Psi^c)} (1 - \overline{\mathcal{C}}_\gamma(\Omega)(x)) = \inf_{x \in \Phi^c} (1 - \overline{\mathcal{C}}_\gamma(\Omega)(x)) \wedge \inf_{x \in \Psi^c} (1 - \overline{\mathcal{C}}_\gamma(\Omega)(x)) \\ &= [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi] \wedge [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Psi]. \end{aligned} \quad (9)$$

Thus,

$$\begin{aligned} [\mathcal{N}_x^{b\theta}(\Phi \cap \Psi)] &= \sup_{\Omega \in 2^X} \max(0, \mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi \cap \Psi] - 1) \\ &= \sup_{\Omega \in 2^X} \max(0, \mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi] \wedge [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Psi] - 1) \\ &= \sup_{\Omega \in 2^X} \max(0, (\mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi] - 1) \wedge (\mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Psi] - 1)) \\ &= \sup_{\Omega \in 2^X} (\max(0, \mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi] - 1) \wedge \max(0, \mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Psi] - 1)) \\ &= \left(\sup_{\Omega \in 2^X} \max(0, \mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Phi] - 1) \right) \wedge \left(\sup_{\Omega \in 2^X} \max(0, \mathcal{N}_x^\gamma(\Omega) + [\overline{\mathcal{C}}_\gamma(\Omega) \sqsubseteq \Psi] - 1) \right) \\ &= [\mathcal{N}_x^{b\theta}(\Phi)] \wedge [\mathcal{N}_x^{b\theta}(\Psi)]. \end{aligned} \quad (10)$$

□

Next, we will generalize the notion of $b - \theta$ -closure [2] in $\mathcal{FSS}(X, \tau)$ as follows.

Definition 7. $\overline{\mathcal{C}}_{b\theta}(\Phi)$ ($\Phi \in 2^X$) is called a fuzzifying $b - \theta$ -closure of Φ if

$$x \in \overline{\mathcal{C}}_{b\theta}(\Phi) := (\forall \Psi \in 2^X) (\Psi \in \mathcal{N}_x^\gamma \longrightarrow (\Phi \cap \overline{\mathcal{C}}_\gamma(\Psi) \equiv \emptyset)). \quad (11)$$

Theorem 2. The following relation is holding in $\mathcal{FSS}(X, \tau)$:

$$\overline{\mathcal{C}}_{b\theta}(\Phi)(x) := \inf_{\Psi \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Psi) + \sup_{y \in \Phi} \overline{\mathcal{C}}_\gamma(\Psi)(y) \right). \quad (12)$$

Proof.

$$\begin{aligned} & [(\forall \Psi) (\mathcal{N}_x^\gamma(\Psi) \longrightarrow (\Phi \cap \overline{\mathcal{C}}_\gamma(\Psi) \equiv \emptyset))] \\ &= \inf_{\Psi \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Psi) + [(\Phi \cap \overline{\mathcal{C}}_\gamma(\Psi) \equiv \emptyset)] \right) \\ &= \inf_{\Psi \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Psi) + 1 - [\Phi \cap \overline{\mathcal{C}}_\gamma(\Psi) \equiv \emptyset] \right) \\ &= \inf_{\Psi \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Psi) + t1n - d[\Phi \cap \overline{\mathcal{C}}_\gamma(\Psi) \equiv \emptyset] \wedge [\emptyset \subseteq \Phi \cap \overline{\mathcal{C}}_\gamma(\Psi)] \right) \\ &= \inf_{\Psi \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Psi) + 1 - \inf_{y \in \Phi} \min(1, 1 - (\Phi \cap \overline{\mathcal{C}}_\gamma(\Psi)(y) + 0)) \right) \\ &= \inf_{\Psi \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Psi) + \sup_{y \in \Phi} \max(0, (\Phi \cap \overline{\mathcal{C}}_\gamma(\Psi)(y))) \right) \\ &= \inf_{\Psi \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Psi) + \sup_{y \in \Phi} \min(\Phi(y), \overline{\mathcal{C}}_\gamma(\Psi)(y)) \right) \\ &= \inf_{\Psi \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Psi) + \sup_{y \in \Phi} \overline{\mathcal{C}}_\gamma(\Psi)(y) \right). \end{aligned} \quad (13)$$

Theorem 3. The following relations are holding in $\mathcal{FSS}(X, \tau)$:

$$(i) \overline{\mathcal{C}}_{b\theta}(\Phi)(x) = 1 - \mathcal{N}_x^{b\theta}(\Phi^c)$$

$$(ii) \models \overline{\mathcal{C}}_{b\theta}(\emptyset) \equiv \emptyset$$

$$(iii) \models \Phi \subseteq \overline{\mathcal{C}}_{b\theta}(\Phi)$$

$$(iv) \models x \in \overline{\mathcal{C}}_{b\theta}(\Phi) \leftrightarrow (\forall \Psi) (\Psi \in \mathcal{N}_x^{b\theta} \longrightarrow \Phi \cap \Psi \neq \emptyset)$$

$$(v) \models \Phi \subseteq \Psi \longrightarrow \overline{\mathcal{C}}_{b\theta}(\Phi) \subseteq \overline{\mathcal{C}}_{b\theta}(\Psi)$$

$$(vi) \models \overline{\mathcal{C}}_\gamma(\Phi) \subseteq \overline{\mathcal{C}}_{b\theta}(\Phi)$$

$$(vii) \models \overline{\mathcal{C}}_{b\theta}(\Phi \sqcup \Psi) \equiv \overline{\mathcal{C}}_{b\theta}(\Phi) \sqcup \overline{\mathcal{C}}_{b\theta}(\Psi)$$

Proof. (i)

$$\begin{aligned} \overline{\mathcal{C}}_{b\theta}(\Phi)(x) &= \inf_{\Psi \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Psi) + \sup_{y \in \Phi} \overline{\mathcal{C}}_\gamma(\Psi)(y) \right) \\ &= 1 - \sup_{\Psi \in 2^X} \max \left(0, \mathcal{N}_x^\gamma(\Psi) - \sup_{y \in \Phi} \overline{\mathcal{C}}_\gamma(\Psi)(y) \right) \\ &= 1 - \sup_{\Psi \in 2^X} \max \left(0, \mathcal{N}_x^\gamma(\Psi) + 1 - \sup_{y \in \Phi} \overline{\mathcal{C}}_\gamma(\Psi)(y) - 1 \right) \\ &= 1 - \sup_{\Psi \in 2^X} \max \left(0, \mathcal{N}_x^\gamma(\Psi) + \inf_{y \in \Phi} (1 - \overline{\mathcal{C}}_\gamma(\Psi)(y)) - 1 \right) \\ &= 1 - [(\exists \Psi) ((\Psi \in \mathcal{N}_x^\gamma) \wedge (\overline{\mathcal{C}}_\gamma(\Psi) \subseteq t\Phi^c))] \\ &= 1 - \mathcal{N}_x^{b\theta}(\Phi^c). \end{aligned} \quad (14)$$

- (ii) From (i) above and since $\mathcal{N}_x^{b\theta}$ is normal, we have $\overline{\mathcal{E}}_{b\theta}(\emptyset)(x) = 1 - \mathcal{N}_x^{b\theta}(\emptyset^c) = 0$.
- (iii) Case 1: if $x \notin \Phi$, then $\mathcal{N}_x^{b\theta}(\Phi) = 0$.
 Case 2: if $x \in \Phi$, then $\overline{\mathcal{E}}_{b\theta}(\Phi)(x) = 1 - \mathcal{N}_x^{b\theta}(\Phi^c) = 1 - 0 = 1$. Thus, $[\Phi \sqsubseteq \overline{\mathcal{E}}_{b\theta}(\Phi)] = 1$.
- (iv) $[(\forall \Psi)(\Psi \in \mathcal{N}_x^{b\theta} \longrightarrow \Phi \sqcap \Psi \neq \emptyset)] = \inf_{\Psi \sqsubseteq \Phi^c} (1 - \mathcal{N}_x^{b\theta}(\Psi)) = 1 - \mathcal{N}_x^{b\theta}(\Phi^c) = [x \in \overline{\mathcal{E}}_{b\theta}(\Phi)]$.
- (v) Case 1: if $[\Phi \sqsubseteq \Psi] = 0$, then $\overline{\mathcal{E}}_{b\theta}(\Phi)(x) \leq \overline{\mathcal{E}}_{b\theta}(\Psi)(x)$.

Case 2: if $[\Phi \sqsubseteq \Psi] = 1$, then $\overline{\mathcal{E}}_{b\theta}(\Phi)(x) = 1 - \mathcal{N}_x^{b\theta}(\Phi^c) \leq 1 - \mathcal{N}_x^{b\theta}(\Psi^c) = \overline{\mathcal{E}}_{b\theta}(\Psi)(x)$.

- (vi) From Theorems 5.2 (3) and 5.3 (2) in [12], we have $\overline{\mathcal{E}}_{b\theta}(\Phi)(x) = \inf_{\Psi \in 2^X} \min(1, 1 - \mathcal{N}_x^\gamma(\Psi) + \sup_{y \in A} \mathcal{E}_\gamma(\Psi)(y)) \leq \inf_{\Psi \in 2^X} \min(1, 1 - \mathcal{N}_x^\gamma(\Psi) + \sup_{y \in \Phi} \mathcal{E}_\gamma(\Psi)(y)) = \overline{\mathcal{E}}_{b\theta}(\Phi)(x)$.
- (vii) It is easy to get $\overline{\mathcal{E}}_{b\theta}(\Phi) \sqcup \overline{\mathcal{E}}_{b\theta}(\Psi) \sqsubseteq \overline{\mathcal{E}}_{b\theta}(\Phi \sqcup \Psi)$.
 Conversely, for every $x \in X$,

$$\begin{aligned}
 \overline{\mathcal{E}}_{b\theta}(\Phi \sqcup \Psi)(x) &= \inf_{\Omega \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Omega) + \sup_{y \in (\Phi \sqcup \Psi)} \overline{\mathcal{E}}_\gamma(\Omega)(y) \right) \\
 &= \inf_{\Omega \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Omega) + \left(\sup_{y \in \Phi} \overline{\mathcal{E}}_\gamma(\Omega)(y) \vee \sup_{y \in \Psi} \overline{\mathcal{E}}_\gamma(\Omega)(y) \right) \right) \\
 &\leq \inf_{\Omega \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Omega) + \sup_{y \in \Phi} \overline{\mathcal{E}}_\gamma(\Omega)(y) \right) \vee \inf_{\Omega \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Omega) + \sup_{y \in \Psi} \overline{\mathcal{E}}_\gamma(\Omega)(y) \right) \\
 &= \overline{\mathcal{E}}_{b\theta}(\Phi)(x) \vee \overline{\mathcal{E}}_{b\theta}(\Psi)(x).
 \end{aligned} \tag{15}$$

Lemma 1. $\models \Phi \in \mathcal{N}_x^{b\theta} \longrightarrow \Phi \in \mathcal{N}_x^\gamma$.

$$x \in \overline{\mathcal{F}}_{b\theta}(\Phi) := \Phi \in \mathcal{N}_x^{b\theta}, \tag{18}$$

i.e.,

Proof. It follows from Theorem 3 (vi). \square

$$\overline{\mathcal{F}}_{b\theta}(\Phi)(x) = \mathcal{N}_x^{b\theta}(\Phi). \tag{19}$$

Clearly, $\mathcal{N}_u^\gamma(\{v\}) = 0 \leq \mathcal{N}_u^{b\theta}(\{v\}) = 1$ if we consider $X = \{u, v\}$ and $\tau(X) = \tau(\emptyset) = \tau(\{u\}) = 1$, $\tau(\{v\}) = 0$ in $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$.

Definition 8

- (i) $\overline{\mathcal{F}}_{b\theta}$ (i.e., $\overline{\mathcal{F}}_{b\theta} \in \widehat{\mathcal{F}}(2^X)$) is called the set of all fuzzifying $b - \theta$ -closed sets if

$$\Phi \in \overline{\mathcal{F}}_{b\theta} := \Phi \equiv \overline{\mathcal{E}}_{b\theta}(\Phi), \tag{16}$$

i.e.,

$$\overline{\mathcal{F}}_{b\theta}(\Phi) = \inf_{x \in \Phi^c} (1 - \overline{\mathcal{E}}_{b\theta}(\Phi)(x)). \tag{17}$$

- (ii) $\tau_{b\theta}$ (i.e., $\tau_{b\theta} \in \widehat{\mathcal{F}}(2^X)$) is called the set of all fuzzifying $b - \theta$ -open sets if $\Phi \in \tau_{b\theta} := \Phi^c \in \overline{\mathcal{F}}_{b\theta}$, where Φ^c is the complement of Φ .

By Definition 8, we can conclude $\mathcal{N}_x^{b\theta}(\Phi) = \sup_{x \in \Psi \sqsubseteq \Phi} \tau_{b\theta}(\Psi)$.

Definition 9. $\overline{\mathcal{F}}_{b\theta}(\Phi)$ ($\Phi \in 2^X$) is called a fuzzifying $b - \theta$ -interior of Φ if

Theorem 4. The following relations are holding in $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$:

- (i) $\models \overline{\mathcal{F}}_{b\theta}(\Phi) \equiv (\overline{\mathcal{E}}_{b\theta}(\Phi^c))^c$
- (ii) $\models \overline{\mathcal{F}}_{b\theta}(X) \equiv X$
- (iii) $\models \overline{\mathcal{F}}_{b\theta}(\Phi) \sqsubseteq \Phi$
- (iv) $\models \Phi \in \tau_{b\theta} \leftrightarrow \forall x (x \in \Phi \longrightarrow \Phi \in \mathcal{N}_x^{b\theta})$
- (v) $\models (\Phi \in \tau_{b\theta}) \wedge (\Phi \sqsubseteq \Psi) \longrightarrow \Phi \sqsubseteq \overline{\mathcal{F}}_{b\theta}(\Psi)$
- (vi) $\models \Phi \sqsubseteq \Psi \longrightarrow \overline{\mathcal{F}}_{b\theta}(\Phi) \sqsubseteq \overline{\mathcal{F}}_{b\theta}(\Psi)$
- (vii) $\models \overline{\mathcal{F}}_{b\theta}(\Phi) \sqsubseteq \overline{\mathcal{F}}_\gamma(\Phi)$
- (viii) $\models \overline{\mathcal{F}}_{b\theta}(\Phi \sqcap \Psi) \equiv \overline{\mathcal{F}}_{b\theta}(\Phi) \sqcap \overline{\mathcal{F}}_{b\theta}(\Psi)$

Proof

- (i) By Theorem 3 (i), we obtain $\overline{\mathcal{E}}_{b\theta}(\Phi^c)(x) = 1 - \mathcal{N}_x^{b\theta}(\Phi) = 1 - \overline{\mathcal{F}}_{b\theta}(\Phi)(x)$. Hence, $[\overline{\mathcal{F}}_{b\theta}(\Phi) \equiv (\overline{\mathcal{E}}_{b\theta}(\Phi^c))^c] = 1$.
- (ii) By (i) above, we obtain $\overline{\mathcal{F}}_{b\theta}(X) = (\overline{\mathcal{E}}_{b\theta}(X^c))^c = (\overline{\mathcal{E}}_{b\theta}(\emptyset))^c = \emptyset^c = X$.
- (iii) By Theorem 3 (iii), we obtain $\overline{\mathcal{F}}_{b\theta}(\Phi) \equiv (\overline{\mathcal{E}}_{b\theta}(\Phi^c))^c \sqsubseteq (\Phi^c)^c \equiv \Phi$.

(iv)

$$\begin{aligned}
[\Phi \in \tau_{b\theta}] &= \inf_{x \in \Phi} (1 - \overline{\mathcal{C}}_{b\theta}(\Phi^c)(x)) \\
&= \inf_{x \in \Phi} \left(1 - \inf_{\Psi \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Psi) + \sup_{y \in \Phi^c} \overline{\mathcal{C}}_\gamma(\Psi)(y) \right) \right) \\
&= \inf_{x \in \Phi} \sup_{\Psi \in 2^X} \max \left(0, \mathcal{N}_x^\gamma(\Psi) - \sup_{y \in \Phi^c} \overline{\mathcal{C}}_\gamma(\Psi)(y) \right) \\
&= \inf_{x \in \Phi} \sup_{\Psi \in 2^X} \max \left(0, \mathcal{N}_x^\gamma(\Psi) + 1 - \sup_{y \in \Phi^c} \overline{\mathcal{C}}_\gamma(\Psi)(y) - 1 \right) \\
&= \inf_{x \in \Phi} \sup_{\Psi \in 2^X} \max \left(0, \mathcal{N}_x^\gamma(\Psi) + \inf_{y \in \Phi^c} (1 - \overline{\mathcal{C}}_\gamma(\Psi)(y)) - 1 \right) \\
&= \inf_{x \in \Phi} \sup_{\Psi \in 2^X} \left(\mathcal{N}_x^\gamma(\Phi) \wedge \inf_{y \in \Phi^c} (1 - \overline{\mathcal{C}}_\gamma(\Psi)(y)) \right) \\
&= \left[\forall x (x \in \Phi \longrightarrow \exists \Psi ((\Psi \in \mathcal{N}_x^\gamma) \wedge (\overline{\mathcal{C}}_\gamma(\Psi) \sqsubseteq \Phi))) \right] \\
&= \left[\forall x (x \in \Phi \longrightarrow \Phi \in \mathcal{N}_x^{b\theta}) \right].
\end{aligned} \tag{20}$$

(v) Case 1: if $[\Phi \sqsubseteq \Psi] = 0$, then $[\Phi \sqsubseteq \overline{\mathcal{F}}_{b\theta}(\Psi)] \geq [(\Phi \in \tau_{b\theta}) \wedge (\Phi \sqsubseteq \Psi)]$.

Case 2: if $[\Phi \sqsubseteq \Psi] = 1$, then $[\Phi \sqsubseteq \overline{\mathcal{F}}_{b\theta}(\Psi)] = \inf_{x \in \Phi} \overline{\mathcal{F}}_{b\theta}(\Psi)(x) = \inf_{x \in \Phi} \mathcal{N}_x^{b\theta}(\Psi) \geq \inf_{x \in \Phi} \mathcal{N}_x^{b\theta}(\Phi) = \tau_{b\theta}(\Phi) = [(\Phi \in \tau_{b\theta}) \wedge (\Phi \sqsubseteq \Psi)]$ from Theorem 1 (ii) and by (v) above.

(vi) Similar to Theorem 3 (vi).

(vii) By Lemma 1, we obtain $\overline{\mathcal{F}}_{b\theta}(\Phi)(x) = \mathcal{N}_x^{b\theta}(\Phi) \leq \mathcal{N}_x^\gamma(\Phi) = \overline{\mathcal{F}}_\gamma(\Phi)(x)$.

(viii) By Theorem 3 (vii), we obtain

$$\begin{aligned}
\overline{\mathcal{F}}_{b\theta}(\Phi \sqcap \Psi)(x) &= 1 - \overline{\mathcal{C}}_{b\theta}(\Phi \sqcap \Psi)^c(x) = 1 - \overline{\mathcal{C}}_{b\theta}((\Phi^c) \sqcup (\Psi^c))(x) = 1 - (\overline{\mathcal{C}}_{b\theta}(\Phi^c)(x) \sqcup \overline{\mathcal{C}}_{b\theta}(\Psi^c)(x)) \\
&= (1 - \overline{\mathcal{C}}_{b\theta}(\Phi^c)(x)) \sqcap (1 - \overline{\mathcal{C}}_{b\theta}(\Psi^c)(x)) = \overline{\mathcal{F}}_{b\theta}(\Phi)(x) \sqcap \overline{\mathcal{F}}_{b\theta}(\Psi)(x).
\end{aligned} \tag{21}$$

□

Theorem 5. The following two relations are holding in $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$:

- (i) $\models \Phi \in \tau_{b\theta} \leftrightarrow \Phi \equiv \overline{\mathcal{F}}_{b\theta}(\Phi)$
- (ii) $\models \Phi \in \overline{\mathcal{F}}_{b\theta} \leftrightarrow \Phi \equiv \overline{\mathcal{C}}_{b\theta}(\Phi)$

Proof

(i)

$$\begin{aligned}
[\Phi \in \tau_{b\theta}] &= [\Phi^c \in \overline{\mathcal{F}}_{b\theta}] = \inf_{x \in X \setminus (\Phi^c)} (1 - \overline{\mathcal{C}}_{b\theta}(\Phi^c)(x)) \\
&= \inf_{x \in \Phi} (\overline{\mathcal{F}}_{b\theta}(\Phi)(x)) = [\Phi \sqsubseteq \overline{\mathcal{F}}_{b\theta}(\Phi)] \\
&= ([\Phi \sqsubseteq \overline{\mathcal{F}}_{b\theta}(\Phi)] \wedge [\overline{\mathcal{F}}_{b\theta}(\Phi) \sqsubseteq \Phi]) \\
&= [\Phi \equiv \overline{\mathcal{F}}_{b\theta}(\Phi)].
\end{aligned} \tag{22}$$

(ii) Similar to (i).

□

Theorem 6. $\tau_{b\theta}$ (i.e., $\tau_{b\theta} \in \widehat{\mathcal{F}}(2^X)$, X is a set) is a $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ if we have the following three conditions:

- (i) $\tau_{b\theta}(X) = \tau_{b\theta}(\emptyset) = 1$
- (ii) $\forall \Phi, \Psi, \tau_{b\theta}(\Phi \sqcap \Psi) \geq \tau_{b\theta}(\Phi) \wedge \tau_{b\theta}(\Psi)$
- (iii) $\forall \{\Phi_\lambda : \lambda \in \Lambda\}, \tau_{b\theta}(\sqcup_{\lambda \in \Lambda} \Phi_\lambda) \geq \inf_{\lambda \in \Lambda} \tau_{b\theta}(\Phi_\lambda)$

Proof

(i) Clear.

(ii) By Theorem 4 (v) and Theorem 1 (iii), we obtain

$$\begin{aligned}
\tau_{b\theta}(\Phi) \wedge \tau_{b\theta}(\Psi) &= \inf_{x \in \Phi} \overline{\mathcal{F}}_{b\theta}(\Phi)(x) \wedge \inf_{x \in B} \overline{\mathcal{F}}_{b\theta}(\Psi)(x) \\
&\leq \inf_{x \in \Phi \sqcap \Psi} (\overline{\mathcal{F}}_{b\theta}(\Phi)(x) \wedge \overline{\mathcal{F}}_{b\theta}(\Psi)(x)) \\
&= \inf_{x \in \Phi \sqcap \Psi} \overline{\mathcal{F}}_{b\theta}(\Phi \sqcap \Psi)(x) = \tau_{b\theta}(\Phi \sqcap \Psi).
\end{aligned} \tag{23}$$

□

(iii) By Theorem 4 (v), we obtain

$$\begin{aligned}\tau_{b\theta}\left(\bigsqcup_{\lambda \in \Lambda} \Phi_\lambda\right) &= \inf_{x \in \bigsqcup_{\lambda \in \Lambda} \Phi_\lambda} \overline{\mathcal{F}}_{b\theta}\left(\bigsqcup_{\lambda \in \Lambda} \Phi_\lambda\right)(x) \\ &= \inf_{\lambda \in \Lambda} \inf_{x \in \Phi_\lambda} \overline{\mathcal{F}}_{b\theta}\left(\bigsqcup_{\lambda \in \Lambda} \Phi_\lambda\right)(x) \\ &\geq \inf_{\lambda \in \Lambda} \inf_{x \in \Phi_\lambda} \overline{\mathcal{F}}_{b\theta}(\Phi_\lambda)(x) = \inf_{\lambda \in \Lambda} \tau_{b\theta}(\Phi_\lambda).\end{aligned}\quad (24)$$

This completes the proof. \square

Theorem 7. The following two relations are holding in $\mathcal{FSS}(X, \tau)$:

- (i) $\models \Phi \in \tau_{b\theta} \leftrightarrow \forall x (x \in \Phi \longrightarrow \exists \Psi (\Psi \in \tau_{b\theta} \wedge x \in \Psi \sqsubseteq \Phi))$
- (ii) $\models \Phi \in \tau_{b\theta} \leftrightarrow \forall x (x \in \Phi \longrightarrow \exists \Psi (\Psi \in \mathcal{N}_x^{b\theta} \wedge \Psi \sqsubseteq \Phi))$

Proof

- (i) $[\forall x (x \in \Phi \longrightarrow \exists \Psi (\Psi \in \tau_{b\theta} \wedge x \in \Psi \sqsubseteq \Phi))] = \inf_{x \in \Phi} \sup_{x \in \Psi \sqsubseteq \Phi} \tau_{b\theta}(\Psi)$. Firstly, we obtain $\inf_{x \in \Phi} \sup_{x \in \Psi \sqsubseteq \Phi} \tau_{b\theta}(\Psi) \geq \tau_{b\theta}(\Phi)$. Further, assume that $\Psi_x = \{\Psi : x \in \Psi \sqsubseteq \Phi\}$. Hence, $f \in \prod_{x \in A} \Psi_x$, we obtain $\bigsqcup_{x \in A} f(x) = \Phi$, $\tau_{b\theta}(\Phi) = \tau_{b\theta}(\bigsqcup_{x \in \Phi} f(x)) \geq \inf_{x \in \Phi} \tau_{b\theta}(f(x))$ and $\tau_{b\theta}(\Phi) \geq \sup_f \inf_{x \in \Phi} \tau_{b\theta}(f(x)) = \inf_{x \in \Phi} \sup_{x \in \Psi \sqsubseteq \Phi} \tau_{b\theta}(\Psi)$.

(ii) By (i) we obtain

$$\begin{aligned}[\forall x (x \in \Phi \longrightarrow \exists \Psi (\Psi \in \mathcal{N}_x^{b\theta} \wedge \Psi \sqsubseteq \Phi))] &= \inf_{x \in \Phi} \sup_{\Psi \sqsubseteq \Phi} \mathcal{N}_x^{b\theta}(\Psi) \\ &= \inf_{x \in \Phi} \sup_{\Psi \sqsubseteq \Phi} \sup_{x \in \Omega \sqsubseteq \Psi} \tau_{b\theta}(\Omega) = \inf_{x \in \Phi} \sup_{x \in \Omega \sqsubseteq \Psi} \tau_{b\theta}(\Omega) \\ &= [\Phi \in \tau_{b\theta}].\end{aligned}\quad (25)$$

\square

Theorem 8. The following two relations are holding in $\mathcal{FSS}(X, \tau)$:

- (i) $\models \tau_{b\theta} \sqsubseteq \tau_\gamma$
- (ii) $\models \overline{\mathcal{F}}_{b\theta} \sqsubseteq \overline{\mathcal{F}}_\gamma$

Proof

- (i) By Corollary 4.1 in [12] and Theorem 3 (vi), we obtain $[\Phi \in \tau_{b\theta}] = \inf_{x \in \Phi} (1 - \overline{\mathcal{E}}_{b\theta}(\Phi^c)(x)) \leq \inf_{x \in \Phi} (1 - \overline{\mathcal{E}}_\gamma(\Phi^c)(x)) = \inf_{x \in \Phi} (1 - 1 + \mathcal{N}_x^\gamma(\Phi)) = \inf_{x \in \Phi} \mathcal{N}_x^\gamma(\Phi) = [\Phi \in \tau_\gamma]$.

(ii) Follows from (i) above. \square

Next, we will generalize Theorem 3.8 (a) in [2] in $\mathcal{FSS}(X, \tau)$ by the following theorem.

Theorem 9. The following relation is holding in $\mathcal{FSS}(X, \tau)$:

$$\models \Phi \in \tau_\gamma \longrightarrow (\overline{\mathcal{E}}_{b\theta}(\Phi) \equiv \overline{\mathcal{E}}_\gamma(\Phi)). \quad (26)$$

Proof. Firstly, we prove that $\models [\Phi \in \tau_\gamma] \longrightarrow [(\Phi \sqcap \overline{\mathcal{E}}_\gamma(\Psi) \equiv \emptyset) \longrightarrow (\Phi \sqcap \Psi \equiv \emptyset)]$. By Corollary 4.1 and Theorem 5.3 (1) in [12], we obtain

$$\begin{aligned}\tau_\gamma(\Phi) \wedge [(\Phi \sqcap \overline{\mathcal{E}}_\gamma(\Psi) \equiv \emptyset)] &= \max\left(0, \inf_{x \in \Phi} \mathcal{N}_x^\gamma(\Phi) + \sup_{y \in \Phi} \overline{\mathcal{E}}_\gamma(\Psi)(y) - 1\right) \\ &= \max\left(0, \sup_{y \in \Phi} \overline{\mathcal{E}}_\gamma(\Psi)(y) - \sup_{x \in \Phi} (1 - \mathcal{N}_x^\gamma(\Phi))\right) \\ &\leq \max\left(0, \sup_{x \in \Phi} (\overline{\mathcal{E}}_\gamma(\Psi)(x) - 1 + \mathcal{N}_x^\gamma(\Phi))\right) \\ &= \max\left(0, \sup_{x \in \Phi} ((\Psi \sqcup D_\gamma(\Psi))(x) - 1 + \mathcal{N}_x^\gamma(\Phi))\right) \\ &= \max\left(0, \sup_{x \in \Phi} (\max(\Psi(x), D_\gamma(\Psi)(x)) - 1 + \mathcal{N}_x^\gamma(\Phi))\right).\end{aligned}\quad (27)$$

Case 1: if $x \in \Psi$, then

$$\begin{aligned} \tau_\gamma(\Phi) \wedge [(\Phi \sqcap \overline{\mathcal{E}}_\gamma(\Psi) \equiv \emptyset)] &= \max \left(0, \sup_{x \in \Phi} (\Psi(x) - 1 + \mathcal{N}_x^\gamma(\Phi)) \right) \\ &\leq \sup_{x \in \Phi} \mathcal{N}_x^\gamma(\Phi)(x) \leq \sup_{x \in \Phi} \Phi(x) \leq \sup_{x \in \Phi} (\Phi(x) \wedge \Psi(x)) = [(\Phi \sqcap \Psi \equiv \emptyset)]. \end{aligned} \quad (28)$$

Case 2: if $x \notin \Psi$, then

$$\begin{aligned} \tau_\gamma(\Phi) \wedge [(\Phi \sqcap \overline{\mathcal{E}}_\gamma(\Psi) \equiv \emptyset)] &= \max \left(0, \sup_{x \in \Phi} (D_\gamma(\Psi)(x) - 1 + \mathcal{N}_x^\gamma(\Phi)) \right) \\ &= \max \left(0, \sup_{x \in \Phi} \left(\inf_{\Omega \in 2^X} \min \left(1, 1 - \mathcal{N}_x^\gamma(\Omega) + \sup_{y \in \Omega} (\Omega - \{x\})(x) - 1 + \mathcal{N}_x^\gamma(\Phi) \right) \right) \right) \\ &\leq \max \left(0, \sup_{x \in \Phi} \left(1 - \mathcal{N}_x^\gamma(\Omega) + \sup_{y \in \Phi} (\Psi - \{x\})(y) - 1 + \mathcal{N}_x^\gamma(\Phi) \right) \right) \\ &= \sup_{y \in \Phi} (\Psi - \{x\})(y) = \sup_{y \in \Phi} \Psi(y) = [(\Phi \sqcap \Psi \equiv \emptyset)]. \end{aligned} \quad (29)$$

Consequently, $[(\Phi \sqcap \overline{\mathcal{E}}_\gamma(\Psi) \equiv \emptyset) \longrightarrow (\Phi \sqcap \Psi \equiv \emptyset)] \geq [\tau_\gamma(\Phi)]$. Thus,

$$\begin{aligned} [\overline{\mathcal{E}}_{b\theta}(\Phi) \equiv \overline{\mathcal{E}}_\gamma(\Phi)] &= [\overline{\mathcal{E}}_{b\theta}(\Phi) \sqsubseteq \overline{\mathcal{E}}_\gamma(\Phi)] \wedge [\overline{\mathcal{E}}_\gamma(\Phi) \sqsubseteq \overline{\mathcal{E}}_{b\theta}(\Phi)] \\ &= [\overline{\mathcal{E}}_{b\theta}(\Phi) \sqsubseteq \overline{\mathcal{E}}_\gamma(\Phi)] = (\forall \Psi) (\Psi \in \mathcal{N}_x^\gamma \longrightarrow (\Phi \sqcap \overline{\mathcal{E}}_\gamma(\Psi) \equiv \emptyset)) \longrightarrow (\forall \Omega) (\Omega \in \mathcal{N}_x^\gamma \longrightarrow (\Phi \sqcap \Omega \equiv \emptyset)) \\ &\geq (\forall \Psi) (\Psi \in \mathcal{N}_x^\gamma \longrightarrow (\Phi \sqcap \overline{\mathcal{E}}_\gamma(\Psi) \equiv \emptyset) \longrightarrow (\Psi \in \mathcal{N}_x^\gamma \longrightarrow (\Phi \sqcap \Psi \equiv \emptyset))) \\ &= (\forall \Psi) ((\Phi \sqcap \overline{\mathcal{E}}_\gamma(\Psi) \equiv \emptyset) \longrightarrow (\Phi \sqcap \Psi \equiv \emptyset)) \geq \inf_{\Psi \in 2^X} \tau_\gamma(\Phi) = \tau_\gamma(\Phi). \end{aligned} \quad (30)$$

This completes the proof. \square

Corollary 1. The following relation is holding in $\mathcal{FSS}(X, \tau)$:

$$\models \Phi \in \tau \longrightarrow (\overline{\mathcal{E}}_{b\theta}(\Phi) \equiv \overline{\mathcal{E}}_\gamma(\Phi)). \quad (31)$$

Theorem 10. The following relation is holding in $\mathcal{FSS}(X, \tau)$:

$$\models \Phi \in \overline{\mathcal{F}}_{b\theta} \longrightarrow (\overline{\mathcal{E}}_{b\theta}(\Phi) \equiv \overline{\mathcal{E}}_\gamma(\Phi)). \quad (32)$$

Proof.

$$\begin{aligned} [\overline{\mathcal{E}}_{b\theta}(\Phi) \equiv \overline{\mathcal{E}}_\gamma(\Phi)] &= \min(1, 1 - \overline{\mathcal{E}}_{b\theta}(\Phi)(x) + \overline{\mathcal{E}}_\gamma(\Phi)(x)) = \inf_{x \in X} \min(1, 1 - \overline{\mathcal{E}}_{b\theta}(\Phi)(x) + 1 - \mathcal{N}_x^\gamma(\Phi^c)) \\ &= \inf_{x \in \Phi^c} \min(1, 1 - \overline{\mathcal{E}}_{b\theta}(\Phi)(x) + 1 - \mathcal{N}_x^\gamma(\Phi^c)) \\ &\geq \inf_{x \in \Phi^c} \min(1, 1 - \overline{\mathcal{E}}_{b\theta}(\Phi)(x)) = \inf_{x \in \Phi^c} (1 - \overline{\mathcal{E}}_{b\theta}(\Phi)(x)) = [\Phi \in \overline{\mathcal{F}}_{b\theta}]. \end{aligned} \quad (33)$$

\square

3. Fuzzifying $b - \theta$ -Derived Sets and Fuzzifying $b - \theta$ -Boundary Sets

Definition 10. $\mathcal{D}_{b\theta}(\Phi)$ ($\Phi \in 2^X$) is called a fuzzifying $b - \theta$ -derived set of Φ if

$$\begin{aligned} x \in \mathcal{D}_{b\theta}(\Phi) \\ \equiv (\forall \Psi) (\Psi \in \mathcal{N}_x^{b\theta} \longrightarrow \Psi \sqcap (\Phi - \{x\}) \neq \emptyset), \end{aligned} \quad (34)$$

i.e.,

$$\mathcal{D}_{b\theta}(\Phi)(x) = \inf_{\Psi \cap (\Phi \setminus \{x\}) = \emptyset} (1 - \mathcal{N}_x^{b\theta}(\Psi)). \quad (35)$$

Theorem 11. The following relations are holding in $\mathcal{FSS}(X, \tau)$:

- (i) $\models \mathcal{D}_{b\theta}(\Phi)(x) = 1 - \mathcal{N}_x^{b\theta}(\Phi^c \sqcup \{x\})$
- (ii) $\models \mathcal{D}_{b\theta}(\emptyset) \equiv \emptyset$
- (iii) $\models \Phi \sqsubseteq \Psi \longrightarrow \mathcal{D}_{b\theta}(\Phi) \sqsubseteq \mathcal{D}_{b\theta}(\Psi)$

Proof

- (i) $\mathcal{D}_{b\theta}(\Phi)(x) = \inf_{\Psi \cap (\Phi \setminus \{x\}) = \emptyset} \inf_{\Psi \cap (\Phi \setminus \{x\}) = \emptyset} (1 - \mathcal{N}_x^{b\theta}(\Psi)) = 1 - \sup_{\Psi \cap (\Phi \setminus \{x\}) = \emptyset} \mathcal{N}_x^{b\theta}(\Psi) = 1 - \sup_{\Psi \sqsubseteq (\Phi^c) \sqcup \{x\}} \mathcal{N}_x^{b\theta}(\Psi) = 1 - \mathcal{N}_x^{b\theta}(\Phi^c \sqcup \{x\})$.
- (ii) By (i) above, we obtain $\mathcal{D}_{b\theta}(\emptyset)(x) = 1 - \mathcal{N}_x^{b\theta}((\emptyset^c) \sqcup \{x\}) = 1 - \mathcal{N}_x^{b\theta}(X) = 1 - 1 = 0$ (i.e., $\mathcal{N}_x^{b\theta}$ is normal).
- (iii) Case 1: $[\Phi \sqsubseteq \Psi] = 0$, then $[\mathcal{D}_{b\theta}(\Phi)] \leq [\mathcal{D}_{b\theta}(\Psi)]$.
Case 2: if $[\Phi \sqsubseteq \Psi] = 1$ and by Theorem 1 (ii), we obtain $\mathcal{D}_{b\theta}(\Phi)(x) = 1 - \mathcal{N}_x^{b\theta}((\Phi^c) \sqcup \{x\}) \leq 1 - \mathcal{N}_x^{b\theta}((\Psi^c) \sqcup \{x\}) = \mathcal{D}_{b\theta}(\Psi)(x)$. \square

Theorem 12. The following relation is holding in $\mathcal{FSS}(X, \tau)$:

$$\models \Phi \in \overline{\mathcal{F}}_{b\theta} \leftrightarrow \mathcal{D}_{b\theta}(\Phi) \sqsubseteq \Phi. \quad (36)$$

Proof. By Theorem 7, we obtain

$$\begin{aligned} [\mathcal{D}_{b\theta}(\Phi) \sqsubseteq \Phi] &= \inf_{x \in X} \min(1, 1 - \mathcal{D}_{b\theta}(\Phi)(x) + [x \in \Phi]) \\ &= \inf_{x \in \Phi^c} (1 - \mathcal{D}_{b\theta}(\Phi)(x)) \\ &= \inf_{x \in \Phi^c} (1 - (1 - \mathcal{N}_x^{b\theta}((\Phi^c) \sqcup \{x\}))) \\ &= \inf_{x \in \Phi^c} \mathcal{N}_x^{b\theta}((\Phi^c) \sqcup \{x\}) = \inf_{x \in \Phi^c} \mathcal{N}_x^{b\theta}(\Phi^c) \\ &= \inf_{x \in \Phi^c} \sup_{x \in \Psi \sqsubseteq \Phi^c} \tau_{b\theta}(\Psi) = \tau_{b\theta}(\Phi^c) = [\Phi \in \overline{\mathcal{F}}_{b\theta}]. \end{aligned} \quad (37)$$

Theorem 13. The following two relations are holding in $\mathcal{FSS}(X, \tau)$:

- (i) $\models \mathcal{D}_{b\theta}(\Phi) \sqcup \mathcal{D}_{b\theta}(\Psi) \equiv \mathcal{D}_{b\theta}(\Phi \sqcup \Psi)$
- (ii) $\models x \in \overline{\mathcal{F}}_{b\theta}(\Phi) \leftrightarrow (x \in \Phi) \wedge (x \in (\mathcal{D}_{b\theta}(\Phi^c))^c)$

Proof

- (i) By Theorem 11 (i) and Theorem 1 (iii), we obtain

$$\begin{aligned} \mathcal{D}_{b\theta}(\Phi \sqcup \Psi)(x) &= 1 - \mathcal{N}_x^{b\theta}(((\Phi \sqcup \Psi)^c) \sqcup \{x\}) \\ &= 1 - \mathcal{N}_x^{b\theta}(((\Phi^c) \cap (\Psi^c)) \sqcup \{x\}) \\ &= 1 - \mathcal{N}_x^{b\theta}(((\Phi^c) \sqcup \{x\}) \cap ((\Psi^c) \sqcup \{x\})) \\ &= 1 - (\mathcal{N}_x^{b\theta}((\Phi^c) \sqcup \{x\}) \wedge \mathcal{N}_x^{b\theta}((\Psi^c) \sqcup \{x\})) \\ &= (1 - \mathcal{N}_x^{b\theta}((\Phi^c) \sqcup \{x\})) \\ &\quad \vee (1 - \mathcal{N}_x^{b\theta}((\Psi^c) \sqcup \{x\})) \\ &= \mathcal{D}_{b\theta}(\Phi)(x) \vee \mathcal{D}_{b\theta}(\Psi)(x). \end{aligned} \quad (38)$$

- (ii) Case 1: if $x \notin \Phi$, then by Theorem 1 (i), $\mathcal{N}_x^{b\theta}(\Phi) = 0$. Hence, $[x \in \overline{\mathcal{F}}_{b\theta}(\Phi)] = 0 = [(x \in \Phi) \wedge (x \in (\mathcal{D}_{b\theta}(\Phi^c))^c)]$.

Case 2: if $x \in \Phi$, then $[x \in (\mathcal{D}_{b\theta}(\Phi^c))^c] = [1 - \mathcal{D}_{b\theta}(\Phi^c)(x)] = [1 - (1 - \mathcal{N}_x^{b\theta}(\Phi \sqcup \{x\}))] = [\mathcal{N}_x^{b\theta}(\Phi)] = [x \in \overline{\mathcal{F}}_{b\theta}(\Phi)]$. \square

Theorem 14. The following two relations are holding in $\mathcal{FSS}(X, \tau)$:

- (i) $\models \overline{\mathcal{C}}_{b\theta}(\Phi) \equiv \Phi \sqcup \mathcal{D}_{b\theta}(\Phi)$
- (ii) $\models \Psi \equiv \overline{\mathcal{C}}_{b\theta}(\Phi) \longrightarrow \Psi \in \overline{\mathcal{F}}_{b\theta}$

Proof

- (i) Case 1: if $x \in \Phi$, $(\Phi \sqcup \mathcal{D}_{b\theta}(\Phi))(x) = \overline{\mathcal{C}}_{b\theta}(\Phi)(x)$.

Case 2: if $x \notin \Phi$, then

- (ii) If $[\Phi \sqsubseteq \Psi] = 0$, then $[\Psi \equiv \overline{\mathcal{C}}_{b\theta}(\Phi)] = 0$. Now, we suppose that $[\Phi \sqsubseteq \Psi] = 1$ and have $[\Psi \sqsubseteq \overline{\mathcal{C}}_{b\theta}(\Phi)] = 1 - \sup_{x \in \Psi - \Phi} \mathcal{N}_x^{b\theta}(\Phi^c)$, $[\overline{\mathcal{C}}_{b\theta}(\Phi) \sqsubseteq \Psi] = \inf_{x \in \Psi^c} \mathcal{N}_x^{b\theta}(\Phi^c)$. So $[\Psi \equiv \overline{\mathcal{C}}_{b\theta}(\Phi)] = \max(0, \inf_{x \in \Psi^c} \mathcal{N}_x^{b\theta}(\Phi^c) - \sup_{x \in \Psi - \Phi} \mathcal{N}_x^{b\theta}(\Phi^c))$. If $[\Psi \equiv \overline{\mathcal{C}}_{b\theta}(\Phi)] > t$, then $\inf_{x \in \Psi^c} \mathcal{N}_x^{b\theta}(\Phi^c) > t + \sup_{x \in \Psi - \Phi} \mathcal{N}_x^{b\theta}(\Phi^c)$. For any $x \in \Psi^c$, we get $\sup_{x \in \Omega \sqsubseteq \Phi^c} \tau_{b\theta}(\Omega) > t + \sup_{x \in \Psi - \Phi} \mathcal{N}_x^{b\theta}(\Phi^c)$, i.e., there exists Ω_x such that $x \in \Omega_x \sqsubseteq \Phi^c$ and $\tau_{b\theta}(\Omega_x) > t + \sup_{x \in \Psi - \Phi} \mathcal{N}_x^{b\theta}(\Phi^c)$. Next, we will show that $\Omega_x \sqsubseteq \Psi^c$. If not, then there is $x' \in \Psi - \Phi$ with $x' \in \Omega_x$. Thus, we get $\sup_{x \in \Psi - \Phi} \mathcal{N}_x^{b\theta}(\Phi^c) \geq \mathcal{N}_{x'}^{b\theta}(\Phi^c) \geq \tau_{b\theta}(\Omega_x) \geq t + \sup_{x \in \Psi - \Phi} \mathcal{N}_x^{b\theta}(\Phi^c)$; it is contradiction. Therefore, $\overline{\mathcal{F}}_{b\theta}(\Psi) = \tau_{b\theta}(\Psi^c) = \inf_{x \in \Psi^c} \mathcal{N}_x^{b\theta}(\Psi^c) \geq \inf_{x \in \Psi^c} \tau_{b\theta}(\Omega_x) > t + \sup_{x \in \Psi - \Phi} \mathcal{N}_x^{b\theta}(\Phi^c) > t$. Since t is arbitrary, it holds that $[\Psi \equiv \overline{\mathcal{C}}_{b\theta}(\Phi)] \leq [\Psi \in \overline{\mathcal{F}}_{b\theta}]$. \square

Definition 11. $\mathcal{B}_{b\theta}(\Phi)$ ($\Phi \in 2^X$) is called a fuzzifying $b - \theta$ -boundary set of Φ if

$$x \in \mathcal{B}_{b\theta}(\Phi) := x \in \overline{\mathcal{C}}_{b\theta}(\Phi) \wedge x \in \overline{\mathcal{C}}_{b\theta}(\Phi^c), \quad (39)$$

i.e.,

$$\mathcal{B}_{b\theta}(\Phi)(x) = \min(\overline{\mathcal{C}}_{b\theta}(\Phi)(x), \overline{\mathcal{C}}_{b\theta}(\Phi^c)(x)). \quad (40)$$

Theorem 15. The following relation is holding in $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$:

$$\begin{aligned} \models x \in \mathcal{B}_{b\theta}(\Phi) &\leftrightarrow \\ &\cdot (\forall \Psi)(\Psi \in \mathcal{N}_x^{b\theta} \longrightarrow (\Psi \sqcap \Phi \neq \emptyset) \wedge (\Psi \sqcap (\Phi^c) \neq \emptyset)). \end{aligned} \quad (41)$$

Proof. Similar to Lemma 2.1 in [13]. \square

Theorem 16. The following two relations are holding in $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$:

- (i) $\models \overline{\mathcal{C}}_{b\theta}(\Phi) \equiv \Phi \sqcup \mathcal{B}_{b\theta}(\Phi)$ and so $\models \Phi \in \overline{\mathcal{F}}_{b\theta} \leftrightarrow \mathcal{B}_{b\theta}(\Phi) \sqsubseteq \Phi$
- (ii) $\models \overline{\mathcal{F}}_{b\theta}(\Phi) \equiv \Phi \sqcap (\mathcal{B}_{b\theta}^c(\Phi))$ and so $\models \Phi \in \tau_{b\theta} \leftrightarrow \mathcal{B}_{b\theta}(\Phi) \sqcap \Phi \equiv \emptyset$

Proof

- (i) Case 1: if $x \in \Phi$, then $[\overline{\mathcal{C}}_{b\theta}(\Phi)(x)] = [(\Phi \sqcup \mathcal{B}_{b\theta}(\Phi))(x)] = 1$ (by Theorem 3 (iii)).
Case 2: if $x \notin \Phi$, then $[(\Phi \sqcup \mathcal{B}_{b\theta}(\Phi))(x)] = [\mathcal{B}_{b\theta}(\Phi)(x)] = \min(\overline{\mathcal{C}}_{b\theta}(\Phi)(x), \overline{\mathcal{C}}_{b\theta}(\Phi^c)(x)) = [\overline{\mathcal{C}}_{b\theta}(\Phi)(x)]$. Therefore, by Theorem 12 and Theorem 14 (i), we get
- (ii) By Theorem 4 (i) and (i) above, we get $\overline{\mathcal{F}}_{b\theta}(\Phi) = (\Phi^c \sqcup \mathcal{B}_{b\theta}(\Phi^c))^c = \Phi \sqcap (\mathcal{B}_{b\theta}^c(\Phi))$. Also, from Theorem 5 (i), we obtain $\models \Phi \in \tau_{b\theta} \leftrightarrow \overline{\mathcal{F}}_{b\theta}(\Phi) \equiv \Phi \leftrightarrow \Phi \sqcap (\mathcal{B}_{b\theta}^c(\Phi)) \equiv \Phi \leftrightarrow \Phi \sqsubseteq (\mathcal{B}_{b\theta}(\Phi))^c \leftrightarrow \mathcal{B}_{b\theta}(\Phi) \sqcap \Phi \equiv \emptyset$. \square

4. Fuzzifying Strongly $b - \theta$ -Continuous Mapping, Fuzzifying Strongly $b - \theta$ -Irresolute Mapping, and Fuzzifying Weakly $b - \theta$ -Irresolute Mapping

In the following section, we define a fuzzifying strongly $b - \theta$ -continuous mapping, fuzzifying strongly $b - \theta$ -irresolute mapping, and fuzzifying weakly $b - \theta$ -irresolute mapping between $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ and $\mathcal{F}\mathcal{T}\mathcal{S}(Y, \varphi)$.

Definition 12. $SC_o^{b\theta}$ (i.e., $SC_o^{b\theta} \in \widehat{\mathcal{F}}(Y^X)$ (a unary fuzzy predicate) is called fuzzifying strongly $b - \theta$ -continuous mappings between $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ and $\mathcal{F}\mathcal{T}\mathcal{S}(Y, \varphi)$ if

$$SC_o^{b\theta}(\psi) := (\forall O)(O \in \varphi \longrightarrow \psi^{-1}(O) \in \tau_{b\theta}), \quad (42)$$

i.e.,

$$[SC_o^{b\theta}(\psi)] = \inf_{O \in 2^Y} \min(1, 1 - \varphi(O) + \tau_{b\theta}(\psi^{-1}(O))). \quad (43)$$

Definition 13. Assuming that $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ and $\mathcal{F}\mathcal{T}\mathcal{S}(Y, \varphi)$ and $\forall \psi \in Y^X$, we put

- (i) $\zeta_1(\psi) := (\forall \Psi)(\Psi \in \overline{\mathcal{F}}^Y \longrightarrow \psi^{-1}(\Psi) \in \overline{\mathcal{F}}_{b\theta}^X)$, where $\overline{\mathcal{F}}^Y$ is a family of fuzzifying closed subset of Y and also $\overline{\mathcal{F}}_{b\theta}^X$ is a family of fuzzifying $b - \theta$ -closed subset of X
- (ii) $\zeta_2(\psi) := (\forall x)(\forall O)(O \in \mathcal{N}_{\psi(x)} \longrightarrow \psi^{-1}(O) \in \mathcal{N}_x^{b\theta})$, where $\mathcal{N}_{\psi(x)}$ is a fuzzifying neighborhood system of $\psi(x)$ of Y and $\mathcal{N}_x^{b\theta}$ is a fuzzifying $b - \theta$ -neighborhood system of x of X
- (iii) $\zeta_3(\psi) := (\forall x)(\forall O)(O \in \mathcal{N}_{\psi}(x) \longrightarrow (\exists P)((\psi(P) \sqsubseteq O) \wedge (P \in \mathcal{N}_x^{b\theta})))$
- (iv) $\zeta_4(\psi) := (\forall \Phi)(\psi(\overline{\mathcal{C}}_{b\theta}^X(\Phi)) \sqsubseteq \overline{\mathcal{C}}^Y(\psi(\Phi)))$
- (v) $\zeta_5(\psi) := (\forall \Psi)(\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi)) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}^Y(\Psi)))$
- (vi) $\zeta_6(\psi) := (\forall \Phi)(\psi^{-1}(\overline{\mathcal{F}}(\Phi)) \sqsubseteq \overline{\mathcal{F}}_{b\theta}(\psi^{-1}(\Phi)))$

Theorem 17. $\models \psi \in SC_o^{b\theta} \leftrightarrow \psi \in \zeta_k, k = i, \dots, 6$.

Proof

- (i) We show that $\models \psi \in SC_o^{b\theta} \leftrightarrow \psi \in \zeta_6$.

$$\begin{aligned} [\psi \in \zeta_1] &= \inf_{\Psi \in 2^Y} \min(1, 1 - \overline{\mathcal{F}}^Y(\Psi) + \overline{\mathcal{F}}_{b\theta}^X(\psi^{-1}(\Psi))) \\ &= \inf_{\Psi \in 2^Y} \min(1, 1 - \varphi(Y - \Psi) + \overline{\mathcal{F}}_{b\theta}(X - \psi^{-1}(\Psi))) \\ &= \inf_{\Psi \in 2^Y} \min(1, 1 - \varphi(Y - \Psi) + \tau_{b\theta}(\psi^{-1}(Y - \Psi))) \\ &= \inf_{O \in 2^Y} \min(1, 1 - \varphi(O) + \tau_{b\theta}(\psi^{-1}(O))) \\ &= [SC_o^{b\theta}(\psi)]. \end{aligned} \quad (44)$$

- (ii) We prove that $\models \psi \in SC_o^{b\theta} \leftrightarrow \psi \in \zeta_2$. Firstly, we show that $[\psi \in \zeta_2] \geq [\psi \in SC_o^{b\theta}]$. If $\mathcal{N}_{\psi(x)}(O) \leq \mathcal{N}_x^{b\theta}(\psi^{-1}(O))$, then $\min(1, 1 - \mathcal{N}_{\psi(x)}(O) + \mathcal{N}_x^{b\theta}(\psi^{-1}(O))) = 1 \geq [\psi \in SC_o^{b\theta}]$. Assume that $\mathcal{N}_{\psi(x)}(O) > \mathcal{N}_x^{b\theta}(\psi^{-1}(O))$. Thus, if $\psi(x) \in \Phi \sqsubseteq O$, then $x \in \psi^{-1}(\Phi) \sqsubseteq \psi^{-1}(O)$. Hence,

$$\begin{aligned} \mathcal{N}_{\psi(x)}(O) - \mathcal{N}_x^{b\theta}(\psi^{-1}(O)) &= \sup_{\psi(x) \in \Phi \sqsubseteq O} \varphi(\Phi) - \sup_{x \in \Psi \sqsubseteq \psi^{-1}(O)} \tau_{b\theta}(\Phi) \\ &\leq \sup_{\psi(x) \in \Phi \sqsubseteq O} \varphi(\Phi) - \sup_{\psi(x) \in \Phi \sqsubseteq O} \tau_{b\theta}(\psi^{-1}(\Phi)) \leq \sup_{\psi(x) \in \Phi \sqsubseteq O} (\varphi(\Phi) - \tau_{b\theta}(\psi^{-1}(\Phi))). \end{aligned} \quad (45)$$

Consequently, $1 - \mathcal{N}_{\psi(x)}(O) + \mathcal{N}_x^{b\theta}(\psi^{-1}(O)) \geq \sup_{\psi(x) \in \Phi \sqsubseteq O} (1 - \varphi(\Phi) + \tau_{b\theta}(\psi^{-1}(\Phi)))$. Thus,

$$\begin{aligned} \min(1, 1 - \mathcal{N}_{\psi(x)}(O) + \mathcal{N}_x^{b\theta}(\psi^{-1}(O))) &\geq \inf_{\psi(x) \in \Phi \sqsubseteq O} (1 - \varphi(\Phi) + \tau_{b\theta}(\psi^{-1}(\Phi))) \\ &\geq \inf_{P \in 2^Y} \min(1 - \varphi(P) + \tau_{b\theta}(\psi^{-1}(P))) = [\psi \in SC_o^{b\theta}]. \end{aligned} \quad (46)$$

Therefore, $\inf_{x \in X} \inf_{O \in 2^Y} \min(1 - \varphi(O) + \tau_{b\theta}(\psi^{-1}(O))) \geq [\psi \in SC_o^{b\theta}]$.

Secondly, we show that $[\psi \in SC_o^{b\theta}] \geq [\psi \in \alpha_2]$. From Corollary 4.1 in [12] and Theorem 4 (iv), we have

$$\begin{aligned} [\psi \in SC_o^{b\theta}] &= \inf_{O \in 2^Y} \min(1, 1 - \varphi(O) + \tau_{b\theta}(\psi^{-1}(O))) \\ &\geq \inf_{O \in 2^Y} \min\left(1, 1 - \inf_{\psi(x) \in O} \mathcal{N}_{\psi(x)}(O) + \inf_{x \in \psi^{-1}(O)} \mathcal{N}_x^{b\theta}(\psi^{-1}(O))\right) \\ &\geq \inf_{O \in 2^Y} \min\left(1, 1 - \inf_{x \in \psi^{-1}(O)} \mathcal{N}_{\psi(x)}(O) + \inf_{x \in \psi^{-1}(O)} \mathcal{N}_x^{b\theta}(\psi^{-1}(O))\right) \\ &\geq \inf_{x \in X} \inf_{O \in 2^Y} \min(1 - \mathcal{N}_{\psi(x)}(O) + \mathcal{N}_x^{b\theta}(\psi^{-1}(O))) \\ &= [\psi \in \zeta_2]. \end{aligned} \quad (47)$$

(iii) We show that $\models \psi \in \zeta_2 \leftrightarrow \psi \in \zeta_3$. By Theorem 1 (ii), we obtain

$$\sup_{P \in 2^X, \psi(P) \sqsubseteq O} \mathcal{N}_x^{b\theta}(P) = \sup_{P \in 2^X, P \sqsubseteq \psi^{-1}(O)} \mathcal{N}_x^{b\theta}(P) = \mathcal{N}_x^{b\theta}(\psi^{-1}(O)). \quad (48)$$

Thus,

$$\begin{aligned} [\psi \in \alpha_3] &= \inf_{x \in X} \inf_{O \in 2^Y} \min\left(1 - \mathcal{N}_{\psi(x)}(O) + \sup_{P \in 2^X, \psi(P) \sqsubseteq O} \mathcal{N}_x^{b\theta}(P)\right) \\ &= \inf_{x \in X} \inf_{O \in 2^Y} \min(1 - \mathcal{N}_{\psi(x)}(O) + \mathcal{N}_x^{b\theta}(\psi^{-1}(O))) \\ &= [\psi \in \zeta_2]. \end{aligned} \quad (49)$$

(iv) We show that $\models \psi \in \zeta_4 \leftrightarrow \psi \in \zeta_5$. Firstly, we prove that $[\psi \in \zeta_4] \leq [\psi \in \zeta_5]$. For every $\Psi \in 2^Y$, we have that $[\psi^{-1}(\psi(\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi))) \sqsubseteq \overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi)))] = 1$,

$[\overline{\mathcal{C}}^Y(\psi(\psi^{-1}(\Psi))) \sqsubseteq \overline{\mathcal{C}}^Y(\Psi)] = 1$ and $[\psi^{-1}(\overline{\mathcal{C}}^Y(\psi(\psi^{-1}(\Psi)))) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}^Y(\Psi))] = 1$. By Lemma 1.2 (2) in [14], we obtain

$$\begin{aligned} [\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi)) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}^Y(\Psi))] &\geq [\psi^{-1}(\psi(\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi)))) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}^Y(\Psi))] \\ &\geq [\psi^{-1}(\psi(\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi)))) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}^Y(\psi(\psi^{-1}(\Psi))))] \\ &\geq [\psi(\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi))) \sqsubseteq \overline{\mathcal{C}}^Y(\psi(\psi^{-1}(\Psi)))]. \end{aligned} \quad (50)$$

Thus,

$$\begin{aligned}
 [\psi \in \zeta_5] &= \inf_{\Psi \in 2^Y} \left[\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi)) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}^Y(\Psi)) \right] \\
 &\geq \inf_{\Psi \in 2^Y} \left[\psi(\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi))) \sqsubseteq \overline{\mathcal{C}}^Y(\psi(\psi^{-1}(\Psi))) \right] \\
 &\geq \inf_{\Phi \in 2^Y} \left[\psi(\overline{\mathcal{C}}_{b\theta}^X(\Phi)) \sqsubseteq \overline{\mathcal{C}}^Y(\psi(\Phi)) \right] = [\psi \in \zeta_4].
 \end{aligned} \tag{51}$$

Secondly, $\forall \Phi \in 2^X$, there exists $\Psi \in 2^Y$ s.t. $\psi(\Phi) = \Psi$ and $\psi^{-1}(\Psi) \supseteq \Phi$. By Lemma 1.2 (1) in [14], we obtain

$$\begin{aligned}
 [\psi \in \alpha_4] &= \inf_{\Phi \in 2^Y} \left[\psi(\overline{\mathcal{C}}_{b\theta}^X(\Phi)) \sqsubseteq \overline{\mathcal{C}}^Y(\psi(\Phi)) \right] \geq \inf_{\Phi \in 2^Y} \left[\psi(\overline{\mathcal{C}}_{b\theta}^X(\Phi)) \sqsubseteq \psi(\psi^{-1}(\overline{\mathcal{C}}^Y(\psi(\Phi)))) \right] \\
 &\geq \inf_{\Phi \in 2^Y} \left[\overline{\mathcal{C}}_{b\theta}^X(\Phi) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}^Y(\psi(\Phi))) \right] \\
 &\geq \inf_{\Psi \in 2^Y, \Psi = \psi(\Phi)} \left[\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi)) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}^Y(\Psi)) \right] \geq \inf_{\Psi \in 2^Y} \left[\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi)) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}^Y(\Psi)) \right] = [\psi \in \zeta_5].
 \end{aligned} \tag{52}$$

(v) We show that $\models \psi \in \zeta_2 \leftrightarrow \psi \in \zeta_5$. Thus,

$$\begin{aligned}
 [\psi \in \zeta_5] &= \inf_{\Psi \in 2^Y} \left[\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi)) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}^Y(\Psi)) \right] \\
 &= \inf_{\Psi \in 2^Y} \inf_{x \in X} \min(1, 1 - (1 - \mathcal{N}_x^{b\theta}(X - \psi^{-1}(\Psi))) + (1 - \mathcal{N}_{\psi(x)}(Y - \Psi))) \\
 &= \inf_{\Psi \in 2^Y} \inf_{x \in X} \min(1, 1 - \mathcal{N}_{\psi(x)}(Y - \Psi) + \mathcal{N}_x^{b\theta}(\psi^{-1}(Y - \Psi))) \\
 &= \inf_{O \in 2^Y} \inf_{x \in X} \min(1, 1 - \mathcal{N}_{\psi(x)}(O) + \mathcal{N}_x^{b\theta}(\psi^{-1}(O))) \\
 &= [\psi \in \zeta_2].
 \end{aligned} \tag{53}$$

(vi) We prove that $\models \psi \in \zeta_6 \leftrightarrow \psi \in \zeta_2$. Thus,

$$\begin{aligned}
 [\psi \in \zeta_6] &= \inf_{\Phi \subseteq Y} \inf_{x \in X} \min(1, 1 - \overline{\mathcal{F}}(\Phi)(\psi(x)) + \overline{\mathcal{F}}_{b\theta}(\psi^{-1}(\Phi))(x)) \\
 &= \inf_{\Phi \subseteq Y} \inf_{x \in X} \min(1, 1 - \mathcal{N}_{\psi(x)}(\Phi) + \mathcal{N}_x^{b\theta}(\psi^{-1}(\Phi))) \\
 &= [\psi \in \zeta_2].
 \end{aligned} \tag{54}$$

This completes the proof. \square

Remark 1. The following relation $\models \psi \in SC_{b\theta} \longrightarrow \psi \in C_{o\gamma}$ is holding in crisp setting, while in $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ does not hold by the next example.

Example 1. If we consider $X = \{u, v, w\}$ and $\tau(X) = \tau(\emptyset) = \tau(\{u\}) = \tau(\{u, w\}) = 1$, $\tau(\{v\}) = \tau(\{u, v\}) = 0$, and $\tau(\{w\}) = \tau(\{v, w\}) = 1/6$ in $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ and defined the identity mapping between $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ and $\mathcal{F}\mathcal{T}\mathcal{S}(X, \varphi)$ by

$$\varphi(\Phi) = \begin{cases} 1, & \Phi \in \{X, \emptyset, \{u, v\}\}, \\ 0, & \text{otherwise.} \end{cases} \quad (55)$$

Thus, $[SC_o^{b\theta}(\psi)] = 1 > (5/6) = [C_{o\gamma}(\psi)]$.

Definition 14. $SI_r^{b\theta}$ (i.e., $SI_r^{b\theta} \in \widehat{\mathcal{F}}(Y^X)$ (a unary fuzzy predicate) is called fuzzifying strongly $b - \theta$ -irresolute mapping between $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ and $\mathcal{F}\mathcal{T}\mathcal{S}(Y, \varphi)$ if

$$SI_r^{b\theta}(\psi) := (\forall O)(O \in \varphi_\gamma \longrightarrow \psi^{-1}(O) \in \tau_{b\theta}), \quad (56)$$

i.e.,

$$[SI_r^{b\theta}(\psi)] = \inf_{O \in 2^Y} \min(1, 1 - \varphi_\gamma(O) + \tau_{b\theta}(\psi^{-1}(O))). \quad (57)$$

Definition 15. Assuming that $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ and $\mathcal{F}\mathcal{T}\mathcal{S}(Y, \varphi)$ and $\forall \psi \in Y^X$, we put

- (i) $\epsilon_1(\psi) := (\forall \Psi)(\Psi \in \overline{\mathcal{F}}_\gamma^Y \longrightarrow \psi^{-1}(\Psi) \in \overline{\mathcal{F}}_{b\theta}^X)$, where $\overline{\mathcal{F}}_\gamma^Y$ is a family of fuzzifying γ -closed subset of Y and $\overline{\mathcal{F}}_{b\theta}^X$ is a family of fuzzifying $b - \theta$ -closed subset of X
- (ii) $\epsilon_2(\psi) := (\forall x)(\forall O)(O \in \mathcal{N}_{\psi(x)}^\gamma \longrightarrow \psi^{-1}(O) \in \mathcal{N}_x^{b\theta})$, where $\mathcal{N}_{\psi(x)}^\gamma$ is a fuzzifying γ -neighborhood system of $\psi(x)$ of Y and $\mathcal{N}_x^{b\theta}$ is a fuzzifying $b - \theta$ -neighborhood system of x of X
- (iii) $\epsilon_3(\psi) := (\forall x)(\forall O)(O \in \mathcal{N}_{\psi(x)}^\gamma \longrightarrow (\exists P)((\psi(P) \sqsubseteq O) \wedge (P \in \mathcal{N}_x^{b\theta})))$
- (iv) $\epsilon_4(\psi) := (\forall \Phi)(\psi(\overline{\mathcal{C}}_{b\theta}^X(\Phi)) \sqsubseteq \overline{\mathcal{C}}_\gamma^Y(\psi(\Phi)))$
- (v) $\epsilon_5(\psi) := (\forall \Psi)(\overline{\mathcal{C}}_{b\theta}^X(\psi^{-1}(\Psi)) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}_\gamma^Y(\Psi)))$
- (vi) $\epsilon_6(\psi) := (\forall \Phi)(\psi^{-1}(\overline{\mathcal{F}}_\gamma^Y(\Phi)) \sqsubseteq \overline{\mathcal{F}}_{b\theta}^X(\psi^{-1}(\Phi)))$

Theorem 18. $\models \psi \in SI_r^{b\theta} \leftrightarrow \psi \in \epsilon_k$, $k = i, \dots, 6$.

Proof. Similar to Theorem 17. \square

Definition 16. $WI_r^{b\theta}$ (i.e., $WI_r^{b\theta} \in \widehat{\mathcal{F}}(Y^X)$) (a unary fuzzy predicate) is called fuzzifying weakly $e - \theta$ -irresolute mapping between $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ and $\mathcal{F}\mathcal{T}\mathcal{S}(Y, \varphi)$ if

$$WI_r^{b\theta}(\psi) := (\forall O)(O \in \varphi_{b\theta} \longrightarrow \psi^{-1}(O) \in \tau_\gamma), \quad (58)$$

i.e.,

$$[WI_r^{b\theta}(\psi)] = \inf_{O \in 2^Y} \min(1, 1 - \varphi_{b\theta}(O) + \tau_\gamma(\psi^{-1}(O))). \quad (59)$$

Definition 17. Assuming that $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ and $\mathcal{F}\mathcal{T}\mathcal{S}(Y, \varphi)$ and $\forall \psi \in Y^X$, we put

- (i) $\eta_1(\psi) := (\forall \Psi)(\Psi \in \overline{\mathcal{F}}_{b\theta}^Y \longrightarrow \psi^{-1}(\Psi) \in \overline{\mathcal{F}}_\gamma^X)$, where $\overline{\mathcal{F}}_{b\theta}^Y$ is a family of fuzzifying $b - \theta$ -closed subset of Y and $\overline{\mathcal{F}}_\gamma^X$ is a family of fuzzifying $b - \theta$ -closed subset of X
- (ii) $\eta_2(\psi) := (\forall x)(\forall O)(O \in \mathcal{N}_{\psi(x)}^{b\theta} \longrightarrow \psi^{-1}(O) \in \mathcal{N}_x^\gamma)$, where $\mathcal{N}_{\psi(x)}^{b\theta}$ is a fuzzifying $b - \theta$ -neighborhood system of $\psi(x)$ of Y and \mathcal{N}_x^γ is a fuzzifying γ -neighborhood system of x of X
- (iii) $\eta_3(\psi) := (\forall x)(\forall O)(O \in \mathcal{N}_{\psi(x)}^{b\theta} \longrightarrow (\exists P)((\psi(P) \sqsubseteq O) \wedge (P \in \mathcal{N}_x^\gamma)))$
- (iv) $\eta_4(\psi) := (\forall \Phi)(\psi(\overline{\mathcal{C}}_\gamma^X(\Phi)) \sqsubseteq \overline{\mathcal{C}}_{b\theta}^Y(\psi(\Phi)))$
- (v) $\eta_5(\psi) := (\forall \Psi)(\overline{\mathcal{C}}_\gamma^X(\psi^{-1}(\Psi)) \sqsubseteq \psi^{-1}(\overline{\mathcal{C}}_{b\theta}^Y(\Psi)))$
- (vi) $\eta_6(\psi) := (\forall \Phi)(\psi^{-1}(\overline{\mathcal{F}}_{b\theta}^Y(\Phi)) \sqsubseteq \overline{\mathcal{F}}_\gamma^X(\psi^{-1}(\Phi)))$

Theorem 19. $\models \psi \in WI_r^{b\theta} \leftrightarrow \psi \in \eta_k$, $k = i, \dots, 6$.

Proof. Similar to Theorem 17. \square

Next, we will generalize Theorem 5.8 (a) in [2] in $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$ by the following theorem.

Theorem 20. Assume that $\mathcal{F}\mathcal{T}\mathcal{S}(X, \tau)$, $\mathcal{F}\mathcal{T}\mathcal{S}(Y, \varphi)$, $\mathcal{F}\mathcal{T}\mathcal{S}(Z, \vartheta)$, $\psi \in Y^X$, and $\chi \in Z^Y$. The following two relations are holding:

- (i) $\models SC_o^{b\theta}(\psi) \longrightarrow (C_o(\chi) \longrightarrow SC_o^{b\theta}(\chi \circ \psi))$
- (ii) $\models C_o(\chi) \longrightarrow (SC_o^{b\theta}(\psi) \longrightarrow SC_o^{b\theta}(\chi \circ \psi))$

Proof

- (i) We will show that $[SC_o^{b\theta}(\psi)] \leq [C_o(\chi) \longrightarrow SC_o^{b\theta}(\chi \circ \psi)]$.

Case 1: if $[C_o(\chi)] \leq [SC_o^{b\theta}(\chi \circ \psi)]$, the results holds.
Case 2: if $[C_o(\chi)] \geq [SC_o^{b\theta}(\chi \circ \psi)]$, then

$$\begin{aligned} [C_o(\chi)] - [SC_o^{b\theta}(\chi \circ \psi)] &= \inf_{P \in 2^Z} \min(1, 1 - \vartheta(P) + \varphi(\chi^{-1}(P))) - \inf_{P \in 2^Z} \min(1, 1 - \vartheta(P) + \tau_{b\theta}((\chi \circ \psi)^{-1}(P))) \\ &\leq \sup_{P \in 2^Z} (\varphi(\chi^{-1}(P)) - \tau_{b\theta}((\chi \circ \psi)^{-1}(P))) \\ &= \sup_{P \in 2^Z} (\varphi(\chi^{-1}(P)) - t\tau_{b\theta}n(\psi^{-1}(\chi^{-1}(P)))) \\ &\leq \sup_{O \in 2^Y} (\varphi(O) - \tau_{b\theta}(\psi^{-1}(O))). \end{aligned} \quad (60)$$

Therefore, $[C_o(\chi) \longrightarrow SC_o^{b\theta}(\chi o \psi)] = \min(1, 1 - [C_o(\chi)] + [SC_o^{b\theta}(\chi o \psi)]) \geq \inf_{O \in 2^Y} \min(1, 1 - \varphi(O) + \tau_{b\theta}(\psi^{-1}(O))) = [SC_o^{b\theta}(\psi)]$. (ii)

$$\begin{aligned}
 [C_o(\chi) \longrightarrow (SC_o^{b\theta}(\psi) \longrightarrow SC_o^{b\theta}(\chi o \psi))] &= [C_o(\chi) \longrightarrow (SC_o^{b\theta}(\psi) \wedge tn(SC_o^{b\theta}(\chi o \psi)))] \\
 &= [(C_o(\chi) \wedge tnq(SC_o^{b\theta}(\psi) \wedge (SC_o^{b\theta}(\chi o \psi))))] \\
 &= [(C_o(\chi) \wedge tSnC_o^{b\theta}q(\psi)h \wedge x(SC_o^{b\theta}(\chi o \psi)))] \\
 &= [(SC_o^{b\theta}(\psi) \wedge tC_on(\chi)q \wedge h(SC_o^{b\theta}(\chi o \psi)))] \quad (61) \\
 &= [(SC_o^{b\theta}(\psi) \wedge tnq(C_o(\chi) \wedge (SC_o^{b\theta}(\chi o \psi))))] \\
 &= [SC_o^{b\theta}(\psi) \longrightarrow (C_o(\chi) \wedge tn(SC_o^{b\theta}(\chi o \psi)))] \\
 &= [SC_o^{b\theta}(\psi) \longrightarrow (C_o(\chi) \longrightarrow SC_o^{b\theta}(\chi o \psi))].
 \end{aligned}$$

□

5. Conclusions

The present paper investigates topological notions when these are planted into the framework of Ying's fuzzifying topological spaces (in the semantic method of continuous-valued logic). It continues various investigations into fuzzy topology in a legitimate way and extends some fundamental results in general topology to fuzzifying topology. An important virtue of our approach (in which we follow Ying) is that we define topological notions as fuzzy predicates (by formulae of Łukasiewicz fuzzy logic) and prove the validity of fuzzy implications (or equivalences). Unlike the (more widespread) style of defining notions in fuzzy mathematics as crisp predicates of fuzzy sets, fuzzy predicates of fuzzy sets provide a more genuine fuzzification; furthermore, the theorems in the form of valid fuzzy implications are more general than the corresponding theorems on crisp predicates of fuzzy sets. The main contributions of the present paper are to define fuzzifying $b - \theta$ -neighborhood system of a point, fuzzifying $b - \theta$ -closure of a set, fuzzifying $b - \theta$ -interior of a set, fuzzifying $b - \theta$ -open sets, fuzzifying $b - \theta$ -closed sets, fuzzifying $b - \theta$ -derived sets, and fuzzifying $b - \theta$ -boundary sets in the setting fuzzifying topological space. Also, we define the concepts of fuzzifying strongly $b - \theta$ -continuous mapping, fuzzifying strongly $b - \theta$ -irresolute mapping, and fuzzifying weakly $b - \theta$ -irresolute mapping of fuzzifying topological spaces and obtain some basic properties of such spaces. There are some problems for further study:

- (1) One obvious problem is our results are derived in the Łukasiewicz continuous logic. It is possible to generalize them to a more general logic setting, like resituated lattice-valued logic considered in [13, 14].
- (2) What is the justification for fuzzifying strongly $b - \theta$ -continuous functions in the setting of $(2, L)$ topologies?

(3) Obviously, fuzzifying topological spaces in [15] form a fuzzy category. Perhaps, this will become a motivation for further study of the fuzzy category.

(4) What is the justification for fuzzifying strongly $b - \theta$ -continuous functions in (M, L) -topologies, etc.?

Furthermore, the future possible research of the authors will be to give several new results (e.g., fuzzifying semi- θ -open sets and fuzzifying $\alpha - \theta$ -open sets), which are similar to the results of fuzzifying $b - \theta$ -open sets, fuzzifying semipre- θ -open sets [16] and fuzzifying pre- θ -open sets [17] based on fuzzifying topology.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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