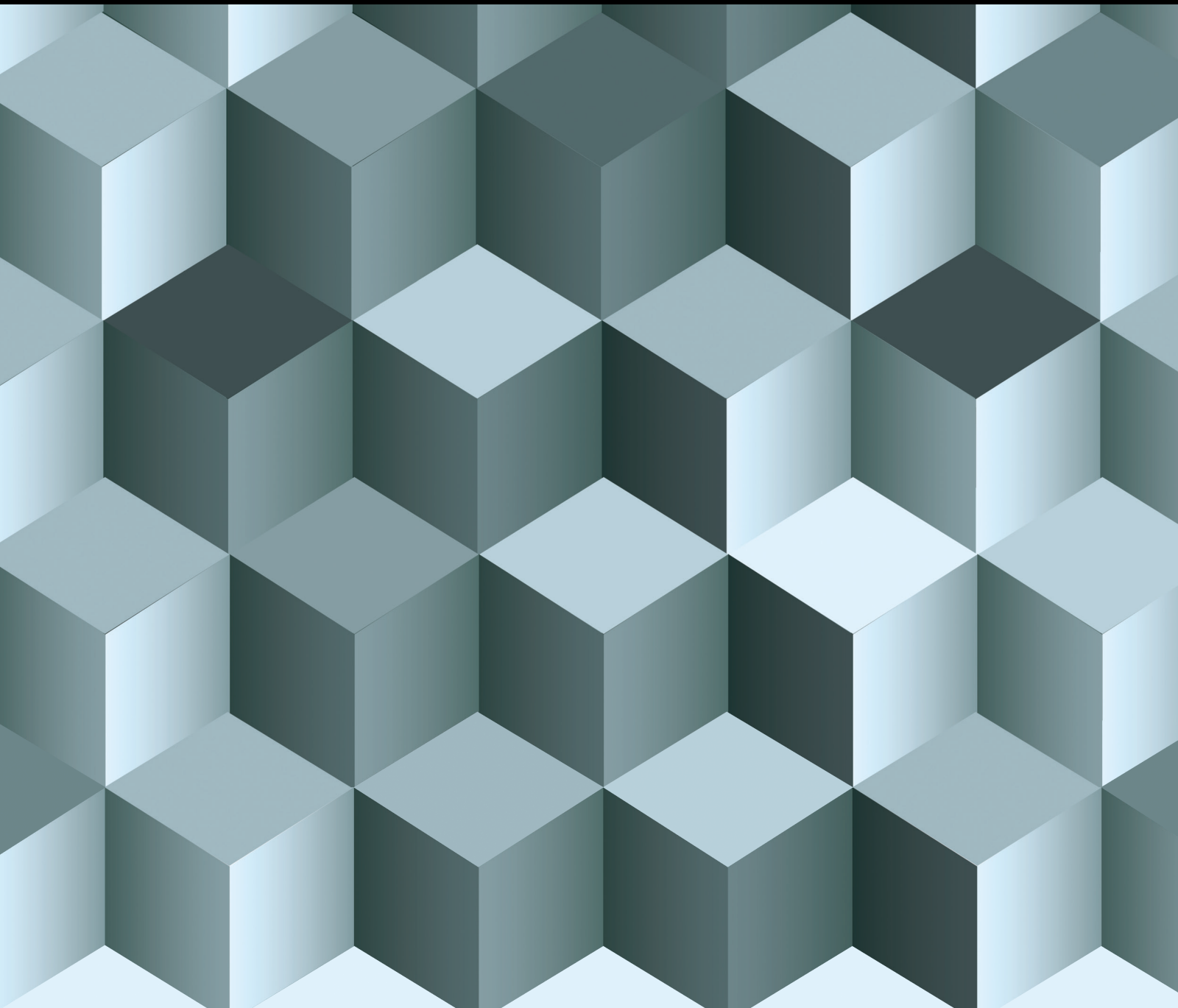


Nonlinear and Variational Analysis and their Applications

Lead Guest Editor: Fanglei Wang

Guest Editors: Mustafa Avci, Yuanfang Ru, and Zisen Mao



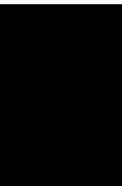


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Research Article

Existence Results for a Class of the Quasilinear Elliptic Equations with the Logarithmic Nonlinearity

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In this paper, the nonlinear quasilinear elliptic problem with the logarithmic nonlinearity $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = a(x)\varphi_p(u) \log |u| + h(x)\psi_p(u)$ in $\Omega \subset \mathbb{R}^n$ was studied. By means of a double perturbation argument and Nehari manifold, the authors obtain the existence results.

1. Introduction

In this paper, we consider the existence of solution to the following problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = a(x)\varphi_p(u) \log |u| + h(x)\psi_p(u), \text{ in } \Omega, \quad (1)$$

where $\Omega \subset \mathbb{R}^n$, $\varphi_p(z) = |z|^{p-2}z$, $\psi_p(z) = |z|^{p-1}z$, $p > 2$, and $n \geq 1$. We always suppose that $a(x)$ is a sign-changing function; $h(x) \geq 0$ is a C^1 function.

Equations of the above form are mathematical models occurring in studies of the p -Laplace equation, generalized reaction-diffusion theory [1], non-Newtonian fluid theory [2, 3], non-Newtonian filtration theory [4, 5], and the turbulent flow of a gas in porous medium [6]. In the non-Newtonian fluid theory, the pair p is a characteristic quantity of the medium. Media with $p > 2$ are called dilatant fluids, and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. When

$p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lost or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be founded in [7, 8].

In recent years, logarithmic nonlinearity is widely used in pseudo-parabolic equations which describe the mathematical and physical phenomena. Equations of the type (1) have been studied by many authors when $p = 2$ (see, for example, [9–12] and the reference therein). To do so, the authors always use the nice properties of Δ , such that, maximum principle and comparison principle and so on. Meanwhile, existence and structure of solutions for such equations with $p > 1$ in bounded domains have also attracted much interest (see [13, 14]).

In the following discussion, we consider two different situations. Firstly, we consider the existence of positive solution for problem (1) with Neumann boundary conditions. In this case, suppose that $\Omega = B_R = B_R(0) \subset \mathbb{R}^n$, $a(x) > 0$, $h(x) \geq 0$ are also radial functions, $a(x) = a(|x|)$, $h(x) = h(|x|)$ in B_R . Our strategy in the study of problem (1) is to adopt a double perturbation argument. First, following [15, 16] (see also [17]), for each $0 < \varepsilon < 1$, we consider

a family of approximate problems

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = a(x)\varphi_p(u) \log\left(\frac{u^2 + \varepsilon u + \varepsilon}{u + \varepsilon}\right) + h(x)\psi_p(u) & \text{in } B_R, \\ u > 0, & \text{in } B_R, \\ \partial_\nu u = 0, & \text{on } \partial B_R. \end{cases} \quad (2)$$

Then, it is natural to look for a family of solutions of (2) and then to pass the limit as $\varepsilon \rightarrow 0$ to obtain a solution to (1).

For each $0 < r < R$, define $A_{rR} := B_R \setminus \bar{B}_r$. Consider the second family of problems

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = a(x)\varphi_p(u) \log\left(\frac{u^2 + \varepsilon u + \varepsilon}{u + \varepsilon}\right) + h(x)\psi_p(u), & \text{in } A_{rR}, \\ u > 0, & \text{in } A_{rR}, \\ u = \theta, & \text{on } \partial B_r, \\ \partial_\nu u = 0, & \text{on } \partial B_R. \end{cases} \quad (3)$$

Here, $\theta > 0$ is an appropriate constant. When $r \rightarrow 0^+$, we get a solution to (2). The role of problem (3) is that we cannot use Poincaré inequality to solve (2) directly by variational methods.

Secondly, we consider the multiple solutions for problem (1) with Dirichlet boundary conditions. In this case, we consider $a(x)$ is a sign-changing function, $h(x) = 0$. The method is based on Nehari manifold and logarithmic Sobolev inequality.

By modification of the methods given in [18–22], we obtain the following results.

Theorem 1. *Let $a(x) > 0$, $h(x) \geq 0$ be any radial C^1 function. Then, problem (1) has a positive radial solution $u \in C^1(\bar{B}_R \setminus \{0\}) \cap C(\bar{B}_R)$.*

Remark 2. Theorem 1 is valid even if we change the logarithm by a more general singular function. In fact, suppose $g : (0, 1) \rightarrow \mathbb{R}$ is a smooth function such that

$$\begin{aligned} \lim_{s \rightarrow 0^+} g(s) &= -\infty, \\ \lim_{s \rightarrow 0^+} \frac{g(s)}{s^m} &= 1, \end{aligned} \quad (4)$$

for some $m \in (0, 1)$. Then, we can perturb g by a family g_ε of smooth functions decreasing in ε , such that $g_\varepsilon(0) = 0$ and $g_\varepsilon(s) \rightarrow g(s)$ pointwise in $s \in (0, \infty)$ as $\varepsilon \rightarrow 0$. This perturbation can be done in such a way that $g_{\varepsilon_0} \geq 0$ for some $\varepsilon_0 > 0$, and then, all the results in Section 2 hold with little modification.

Theorem 3. *Let $h(x) = 0$, $a(x) \in C(\bar{\Omega})$ and changes sign in $\bar{\Omega}$, satisfying*

$$\max_{\bar{\Omega}} |a(x)| \leq \frac{1}{\mu}, \quad (5)$$

where $\mu = (nL_p/pe) \exp((mp^2|\Omega|_n)/ne)$, $|\Omega|_n$ is the volume of Ω in \mathbb{R}^n . Then, (1) possesses at least two nontrivial solutions.

The paper is organized as follows. In Section 2, we construct a sub- and a supersolution for 3 and finish the proof of Theorem 1. In Section 3, we prove Theorem 3 by the method of Nehari manifold and logarithmic Sobolev inequality.

2. Proof of Theorem 1

2.1. Sub- and Supersolution for 3

Lemma 4. *Suppose that $\theta > 1$. Then, the function $u \equiv 1$ is a subsolution for 3 which does not depend on $0 < \varepsilon \leq 1$ and θ .*

Proof. We just need to see that, since $a(x) > 0$, $h(x) \geq 0$ in B_R , the following inequality holds independently of $0 < \varepsilon \leq 1$ and $\theta > 1$:

$$a(x) \log\left(\frac{1 + \varepsilon + \varepsilon}{1 + \varepsilon}\right) + h(x) \geq \log 1 = 0, \quad (6)$$

We proceed to find a supersolution for 3. Denote by X_r , the following subspace of $H^1(A_{rR})$:

$$X_r := \{u \in H^1(A_{rR}) \mid u = 0 \text{ on } \partial B_r\}. \quad (7)$$

For $v \in X_r$, we define the expression:

$$|v|_r := \left(\int_{A_{rR}} |\nabla v|^2 dx \right)^{1/2}. \quad (8)$$

Remark. The expression $|\cdot|_r$ defines a norm on X_r , and $(X_r, |\cdot|_r)$ is a reflexive Banach space. Furthermore, by ([23], (7.44)), the Poincaré inequality holds on X_r , that is, there exists $\eta > 0$ such that

$$\int_{A_{rR}} v^p dx \leq \eta \int_{A_{rR}} |\nabla v|^p dx. \quad (9)$$

Next, we work with the radial formulation for $E_{\varepsilon,r}$ in the specific case that $\varepsilon = 1$,

$$\begin{cases} -\left(s^{n-1}o_p(u')\right)' = s^{n-1}a(s)o_p(u) \log\left(\frac{u^2 + u + 1}{u + 1}\right) + s^{n-1}h(s)\psi_p(u), & \text{in } r < s < R, \\ u > 0, & \text{in } r < s < R, \\ u(r) = \theta, & u'(R) = 0, \end{cases} \quad (10)$$

where $\phi_p(s) = |s|^{p-2}s$. Notice that

$$\log \left(\frac{u^2 + u + 1}{u + 1} \right) \geq 0 \text{ for } u \geq 0. \quad (11)$$

For simplicity, denote

$$f(s, z) = a(s) o_p(z) \log \left(\frac{z^2 + z + 1}{z + 1} \right) + h(s) \psi_p(z). \quad (12)$$

Then, if v solves

$$\begin{cases} -\left(s^{n-1} o_p(v')\right)' = s^{n-1} f(s, v + \theta), & \text{in } r < s < R, \\ v > 0, & \text{in } r < s < R, \\ v(r) = \theta, & v'(R) = 0, \end{cases} \quad (13)$$

we will have that $v + \theta$ is a solution of Eq. (10). In order to prove existence of such v , we find a minimum of the functional in the sequel. Let $S \subset X_r$ denote the set of symmetric functions with respect to the origin. We define $\Phi : S \rightarrow \mathbb{R}$ by

$$\Phi(v) = \frac{1}{p} \int_r^R s^{n-1} |v'|^p ds + \int_r^R s^{n-1} F(s, v(s)) s^{n-1} ds, \quad (14)$$

where $F(s, v(s)) = \int_0^t f(s, (z + \theta)^+) dz$ and $z^+ := \max\{z, 0\}$.

Lemma 5. *The functional Φ is C^1 , weakly lower semicontinuous and coercive so that there exist $v \in X_r$ such that*

$$\Phi(v) = \min_{u \in X_r} \Phi(u) \text{ and } \Phi'(v) \equiv 0. \quad (15)$$

The proof is standard by (9). Also, since v is a weak solution of (13), we have

$$v(s) = \int_r^s o_p^{-1} \left[t^{1-n} \int_t^R z^{n-1} f(z, v(z) + \theta) dz \right] dt, \quad (16)$$

in which

$$o_p^{-1}(u) = \begin{cases} u^{1/(p-1)}, & \text{if } u \geq 0, \\ -(-u)^{1/(p-1)}, & \text{if } u < 0. \end{cases} \quad (17)$$

Then, we define

$$u_r := v + 0. \quad (18)$$

Lemma 6. *Suppose that $\theta > 1$. Then, the function $\bar{u} \equiv u_r$ is a supersolution for (3) which does not depend on $0 < \varepsilon \leq 1$.*

Lemma 7. *There exists a constant $M > 0$ such that $|u_r|_\infty \leq M$ and the constant M does not depend on $r \in (0, R)$. Moreover, for each $\rho \in (0, R)$, there exist a constant C_ρ and $r_\rho \in (0, R)$*

such that we have the following estimates:

$$|u_r| c^0[\rho, R], |u_r| c^1[\rho, R], \left| o_p(u_r') \right| c^1[\rho, R] \leq C\rho. \quad (19)$$

Proof of Lemmas 6 and 7 can be found in [18], we omit them here.

2.2. Existence of Solution for 3. In this section, we use the sub- and supersolution from Section 2.1 (u and u_r , respectively) to obtain a solution for the problem 3. Define the function

$$\begin{aligned} g_\varepsilon(s, u) &:= s^{n-1} a(s) o_p(u) \log \left(\frac{u^2 + \varepsilon u + \varepsilon}{u + \varepsilon} \right) + s^{n-1} h(s) \psi_p(u) \\ &\quad + bu, \quad s \in [r, R], u \geq 0, \end{aligned} \quad (20)$$

where we choose b in such a way that the function $u \rightarrow g_\varepsilon(s, u)$ is increasing in u for all $s \in [r, R]$. Now, starting with $u_0 = u$, we define a sequence u_n such that each u_n satisfies

$$\begin{cases} -\left(s^{n-1} o_p(u_{n+1}')\right)' + bu_{n+1} = g_\varepsilon(s, u_n), & \text{in } r < s < R, \\ u_{n+1} > 0, & \text{in } r < s < R, \\ u_{n+1}'(r) = 0, & u_{n+1}'(R) = 0. \end{cases} \quad (21)$$

Let us now recall Lemma 2.1 in [24],

Lemma 8 (weak comparison principle). *Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 2$) with smooth boundary $\partial\Omega$ and $\theta : (0, +\infty) \rightarrow (0, +\infty)$ is continuous and nondecreasing, let $u_1, u_2 \in W^{1,p}(\Omega)$ satisfy*

$$\begin{aligned} &\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla v dx \\ &\quad + \int_\Omega \theta(u_1) v dx \leq \int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla v dx \\ &\quad + \int_\Omega \theta(u_2) v dx. \end{aligned} \quad (22)$$

For all nonnegative $v \in W_0^{1,m}(\Omega)$. Then, the inequality

$$u_1 \leq u_2, \text{ on } \partial\Omega, \quad (23)$$

implies that

$$u_1 \leq u_2, \text{ in } \Omega. \quad (24)$$

Lemma 9. *The sequence $\{u_n\}$ is nondecreasing and satisfies $u_0(s) \leq u_n(s) \leq u_{n+1}(s) \leq u_r(s)$ for all $s \in [r, R]$ and all $n \in \mathbb{N}$.*

Proof. We just need to see that $u_0 \leq u_1 \leq u_r$ and the general case follows by induction in an analogous way. We have

$$\begin{cases} -(s^{n-1}o_p(u'_0))' + b(u_0) \leq -(s^{n-1}o_p(u'_1))' + b(u_1), & \text{in } r < s < R, \\ (u_0 - u_1)(r) \leq 0, & (u_0 - u_1)'(R) = 0. \end{cases} \quad (25)$$

So, we can apply Lemma 8 and obtain that $u_0 \leq u_1$ in $[r, R]$. On the other hand,

$$\begin{cases} -(s^{n-1}o_p(u'_1))' + b(u_1) \leq -(s^{n-1}o_p(u'_r))' + b(u_r), & \text{in } r < s < R, \\ (u_1 - u_r)(r) \leq 0, & (u_1 - u_r)'(R) = 0. \end{cases} \quad (26)$$

Again, Lemma 8 implies $u_1 \leq u_r$ in $[r, R]$.

By Lemma 9, we define the pointwise limit

$$u_r^\varepsilon(s) := \lim_{n \rightarrow \infty} u_n(s), \quad s \in [r, R], \quad (27)$$

and we see that

$$1 \leq u_r^\varepsilon \leq u_r(s), \quad s \in [r, R]. \quad (28)$$

The function u_r^ε is in fact a solution of 3.

Lemma 10. *The function u_r^ε is a solution of 3, and it belongs to $C^1[r, R]$.*

Proof. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} u_n(s) = \theta + \int_r^s o_p^{-1} \left[t^{1-n} \int_t^R z^{n-1} \left(a(z) o_p(u_n) \log \left(\frac{u_n^2 + \varepsilon u_n + \varepsilon}{u_n + \varepsilon} \right) \right. \right. \\ \left. \left. + h(z) v_p(u_n) \right) dz \right] dt. \end{aligned} \quad (29)$$

Since we have

$$1 \leq u_n \leq M, \quad \text{for all } n \in \mathbb{N}, \quad (30)$$

we obtain, as in Lemma 7, that $|\phi_p(u'_p)|C^1[\rho, R]$ is bounded. Then, for a subsequence that we still denote by u_n , we have the convergence

$$u_n \rightarrow u_r^\varepsilon \text{ in } C^1[\rho, R]. \quad (31)$$

2.3. Obtaining a Solution for E_ε . In this section, we pass the limit as $r \rightarrow 0^+$ and then obtain a solution for 2.

Lemma 11. *For a fixed $0 < \varepsilon \leq 1$, the problem (2) has a solution u^ε which is obtained as the limit of u_r^ε as $r \rightarrow 0^+$.*

Proof. For simplicity, we omit the dependence on $\varepsilon > 0$ for u_r^ε . We know that

$$\begin{aligned} u_r(s) = \theta + \int_r^s o_p^{-1} \left[t^{1-n} \int_t^R z^{n-1} \left(a(z) o_p(u_r) \log \left(\frac{u_r^2 + \varepsilon u_r + \varepsilon}{u_r + \varepsilon} \right) \right. \right. \\ \left. \left. + h(z) v_p(u_r) \right) dz \right] dt. \end{aligned} \quad (32)$$

Also, we have

$$\begin{aligned} 1 \leq u_r \leq M, & \quad \text{in } [r, R], \\ 1 \leq u_r \leq M, & \quad \text{in } [r, R]. \end{aligned} \quad (33)$$

As in Lemma 7, we can prove, for each $\rho \in (0, R)$, there exist a constant $C_\rho > 0$ and $r_\rho \in (0, R)$ such that we have the following estimates:

$$|u_r|C^0[\rho, R], |u_r|C^1[\rho, R], |o_p(u'_r)|C^1[\rho, R] \leq C_\rho. \quad (34)$$

Then, from the compact imbedding $C^1[\rho, R] \rightarrow C^0[\rho, R]$, we see that there exist a sequence r_n and u^ε defined on $(0, R]$ such that, if we define $w_n := u_{r_n}$, then

$$\begin{aligned} w_n &\rightarrow u^\varepsilon \quad \text{in } C_{loc}^1(0, R), \\ w_n &\rightarrow u^\varepsilon \quad \text{in } C^1(\rho, R). \end{aligned} \quad (35)$$

2.4. Concluding the Proof of Theorem 1. Now, we would like to pass the limit in the family u^ε obtained in Section 2.3 and get a solution to (1). In order to do that, we need some estimates like the ones in Lemma 7 independently of ε .

First, we observe that the following estimate holds in $(0, R]$

$$1 \leq u^\varepsilon \leq M. \quad (36)$$

Notice that the family $(u^\varepsilon)_{0 < \varepsilon \leq 1}$ satisfies $\varepsilon > 0$ for u_r^ε . We know that

$$\begin{aligned} u^\varepsilon(s) = u^\varepsilon\left(\frac{R}{2}\right) + \int_{R/2}^s o_p^{-1} \left[t^{1-n} \int_t^R z^{n-1} \right. \\ \left. \cdot \left(a(z) o_p(u^\varepsilon) \log \left(\frac{u^{\varepsilon 2} + \varepsilon u^\varepsilon + \varepsilon}{u^\varepsilon + \varepsilon} \right) \right. \right. \\ \left. \left. + h(z) v_p(u^\varepsilon) \right) dz \right] dt, \end{aligned} \quad (37)$$

if $s \in [R/2, R]$, and

$$\begin{aligned} u^\varepsilon(s) = u^\varepsilon\left(\frac{R}{2}\right) - \int_s^{R/2} o_p^{-1} \left[t^{1-n} \int_t^R z^{n-1} \right. \\ \left. \cdot \left(a(z) o_p(u^\varepsilon) \log \left(\frac{u^{\varepsilon 2} + \varepsilon u^\varepsilon + \varepsilon}{u^\varepsilon + \varepsilon} \right) \right. \right. \\ \left. \left. + h(z) v_p(u^\varepsilon) \right) dz \right] dt, \end{aligned} \quad (38)$$

if $s \in (0, R/2]$.

From Eqs. (36)–(38) we see, as in Lemma 7 that, for each for each $\rho \in (0, R)$, there exist a constant $C_\rho > 0$ and $\varepsilon_\rho \in (0, R)$ such that we have the following estimates:

$$|u^\varepsilon|c^0[\rho, R], |u^\varepsilon|c^1[\rho, R], \left|o_p(u^\varepsilon)\right|c^1[\rho, R] \leq C_\rho \quad \text{for all } \varepsilon \in (0, \varepsilon_\rho). \quad (39)$$

Now, arguing as in Section 2.4, we can find a function u which satisfies

$$\begin{cases} -\left(s^{n-1}o_p(u')\right)' = s^{n-1} \log u + s^{n-1}h(s)u^q, & \text{in } r < s < R, \\ u > 0, \\ u'(R) = 0. \end{cases} \quad \text{in } r < s < R, \quad (40)$$

that is, u is a radial solution for the problem (1).

We see that $u \in C^1(0, R) \cap C(0, R]$. Now, extend continuously u to the whole interval $(0, R]$. Indeed, let r_i be a sequence in $(0, R/2)$ with $r_i \rightarrow 0$ as $i \rightarrow \infty$. From Eq. (13) (after we have passed the limit in ε)

$$u(r_i) = u\left(\frac{R}{2}\right) - \int_{r_i}^{R/2} o_p^{-1} \left[t^{1-n} \int_t^R z^{n-1} (a(z)o_p(u) \log u + h(z)v_p(u)) dz \right] dt. \quad (41)$$

Then, if $r_j > r_i$, we get

$$|u(r_j) - u(r_i)| = \left| \int_{r_i}^{r_j} o_p^{-1} \left[t^{1-n} \int_t^R z^{n-1} (a(z)o_p(u) \log u + h(z)v_p(u)) dz \right] dt \right|. \quad (42)$$

From Eq. (36), we obtain that there exists a constant $C > 0$ such that

$$|u(r_j) - u(r_i)| \leq C|r_j - r_i|, \quad (43)$$

so $u(r_i)$ is a Cauchy sequence in R . Let L be the limit of such sequence. By a similar argument, we conclude that if s_i is another sequence in $(0, R/2)$ converging to 0, then we necessarily have $u(s_i) \rightarrow L$. So, we have proved that

$$\lim_{r \rightarrow 0} u(r) = L, \quad (44)$$

finishing the proof of Theorem 1.

3. Proof of Theorem 2

3.1. Preliminaries. In this section, we consider the multiple solutions for problem (1) with Dirichlet boundary conditions. In this case, we consider $a(x)$ is a sign-changing function, $h(x) = 0$. Moreover, it is necessary to note that the presence of the logarithmic nonlinearity leads to some difficulties in deploying the potential well method. In order to

handle this situation, we need the following logarithmic Sobolev inequality which was introduced by [25].

Proposition 12. Let $p > 1$, $\mu > 0$, and $u \in W^{1,p}(\Omega) \setminus \{0\}$. Then, we have

$$\begin{aligned} p \int_{R^n} |u(x)|^p \log \left(\frac{|u(x)|}{\|u\|_{L^p(R^n)}} \right) dx + \frac{n}{p} \log \\ \cdot \left(\frac{p\mu e}{nL_p} \right) \int_{R^n} |u(x)|^p dx \leq \mu \int_{R^n} |\nabla u(x)|^p dx, \end{aligned} \quad (45)$$

where

$$L_p = \frac{p}{n} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-(p/2)} \left[\frac{\Gamma((n/2)+1)}{\Gamma(n(p-1/p)+1)} \right]^{p/n}. \quad (46)$$

Remark. If $u \in W_0^{1,p}(\Omega)$ then, by defining $u(x) = 0$ for $x \in R^n \setminus \{\Omega\}$, we derive

$$\begin{aligned} p \int_{\Omega} |u(x)|^p \log \left(\frac{|u(x)|}{\|u\|_p} \right) dx + \frac{n}{p} \log \\ \cdot \left(\frac{p\mu e}{nL_p} \right) \int_{\Omega} |u(x)|^p dx \leq \mu \int_{\Omega} |\nabla u(x)|^p dx, \end{aligned} \quad (47)$$

for any real number $\mu > 0$.

We start by considering the energy functional J by

$$J(u) = \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} \int_{\Omega} a(x)|u|^p \log |u| dx + \frac{1}{p^2} \int_{\Omega} a(x)|u|^p dx, \quad (48)$$

in which $\|u\|_p = \|u\|_{L^p(\Omega)}$.

Lemma 13. For $u \in H_0^1(\Omega)$ and $\int_{\Omega} a(x)|u|^p dx = 0$, let $M = \max_{\Omega} |a(x)|$, then it holds

$$J(u) \geq \left(\frac{1}{p} - \frac{M\mu}{p} \right) \|\nabla u\|_p^p, \quad (49)$$

in which $\mu = (nL_p/p) \exp((mp^2|\Omega|n)/ne)$.

Proof. Using the fact $\int_{\Omega} a(x)|u|^p dx = 0$, we have

$$J(u) = \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} \int_{\Omega} a(x)|u|^p \log \left(\frac{|u(x)|}{\|u\|_p} \right) dx. \quad (50)$$

Let $\bar{u}(x) = u(x)/\|u\|_p$, then

$$\begin{aligned} & \int_{\Omega} a(x)|u|^p \log \left(\frac{|u(x)|}{\|u\|_p} \right) dx \\ &= \int_{\Omega_1} a(x)|u|^p \log |\bar{u}| dx + \int_{\Omega_2} a(x)|u|^p \log |\bar{u}| dx, \end{aligned} \quad (51)$$

where

$$\Omega_1 = \{x \in \Omega, |\bar{u}(x)| < 1\}, \text{ and } \Omega_2 = \{x \in \Omega, |\bar{u}(x)| \geq 1\}. \quad (52)$$

By direct calculations, we know

$$\int_{\Omega_1} a(x)|u|^p \log |\bar{u}| dx \leq \frac{M|\Omega|_n}{2e} \|u\|_p^p. \quad (53)$$

Also, by logarithmic Sobolev inequality (47) and (51), we have

$$\begin{aligned} & \int_{\Omega_2} a(x)|u|^p \log |\bar{u}| dx \leq M \left(\int_{\Omega} |u|^p \log |\bar{u}| dx + \frac{|\Omega|_n}{2e} \|u\|_p^p \right) \\ & \leq M \left[\frac{\mu}{p} \int_{\Omega} |\nabla u(x)|^p dx - \left(\frac{n}{p^2} \log \left(\frac{p\mu e}{nL_p} \right) - \frac{M|\Omega|_n}{2e} \right) \|u\|_p^p \right]. \end{aligned} \quad (54)$$

Then, combining (50), (51), (53), and (54), we have

$$J(u) \geq \left(\frac{1}{p} - \frac{M\mu}{p} \right) \|\nabla u\|_p^p + \left(\frac{n}{p^3} \log \left(\frac{p\mu e}{nL_p} \right) - \frac{|\Omega|_n}{pe} \right) M \|u\|_p^p. \quad (55)$$

Taking $\mu = (nL_p/pe) \exp((mp^2|\Omega|_n)/ne)$ in (55), then

$$\frac{n}{p^3} \log \left(\frac{p\mu e}{nL_p} \right) - \frac{|\Omega|_n}{pe} = 0, \quad (56)$$

we know (49).

Lemma 14. [19] Let $\{u_m\}$ be a sequence in $W_0^{1,p}(\Omega)$. If $u_m \rightarrow u_0$ and $u_m \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$, then

$$J(u_0) < \lim_{m \rightarrow \infty} J(u_m). \quad (57)$$

If $u_m \rightarrow u_0$ in $W_0^{1,p}(\Omega)$, then

$$J(u_0) = \lim_{m \rightarrow \infty} J(u_m). \quad (58)$$

3.2. Multiple Solutions. Inspired by [19], we seek the weak solutions of (E) by Nehari manifold. First, a simple calculation shows that $J(u) \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$, and its derivative is

given by

$$\langle J'(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \int_{\Omega} a(x) \varphi_p(u) v \log |u| dx, \quad (59)$$

for all $u, v \in W_0^{1,p}(\Omega)$.

From (49), $J(u)$ is not bounded on $W_0^{1,p}(\Omega)$, but we can prove that $J(u)$ is bounded from below on Nehari manifold

$$N = \left\{ u \in W^{1,p}(\Omega) \setminus \{0\} : \langle J'(u), u \rangle = 0 \right\}, \quad (60)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality.

It is clear that all nontrivial critical points of J must lie on N , and as we will see below, local minimizers on N are usually critical points of J . Also, we can see that

$$u \in N \Leftrightarrow \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} a(x)|u|^p \log |u| dx = 0. \quad (61)$$

Let $u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}$ and consider the real function j : $\lambda \rightarrow J(\lambda u)$ for $\lambda > 0$ defined by

$j(\lambda) := J(\lambda u) = \lambda^{-p/p} \|\nabla u\|_p^p - \lambda^{p/p} \int_{\Omega} a(x)|u|^p \log |u| dx - (\lambda^{p/p}) \log \lambda \int_{\Omega} a(x)|u|^p dx + (\lambda^{p/p^2}) \int_{\Omega} a(x)|u|^p dx$. Such maps are known as fibering maps which were introduced by Drabek and Pohozaev [26].

Then, by direct calculations, we have

$$j'(\lambda) = \frac{\lambda^{p-1}}{p-1} \|\nabla u\|_p^p - \frac{\lambda^{p-1}}{p-1} \int_{\Omega} a(x)|u|^p \log |\lambda u| dx, \quad (62)$$

$$\begin{aligned} j''(\lambda) &= \frac{\lambda^{p-2}}{p-2} \|\nabla u\|_p^p - \frac{\lambda^{p-2}}{p-2} \int_{\Omega} a(x)|u|^p \log |\lambda u| dx \\ &\quad - \frac{\lambda^{p-2}}{p-1} \int_{\Omega} a(x)|u|^p dx. \end{aligned} \quad (63)$$

Lemma 15. Let $u \in W^{1,p}(\Omega) \setminus \{0\}$ and $\lambda > 0$. Then, $\lambda u \in N$ if and only if $j'(\lambda) = 0$.

Proof. First, by direct calculations, we know

$$\begin{aligned} \lambda u \in N &\Leftrightarrow \frac{\lambda^{p-1}}{p-1} \left(\|\nabla u\|_p^p - \int_{\Omega} a(x)|u|^p \log |\lambda u| dx \right) \\ &= 0 \Leftrightarrow \lambda j'(\lambda) = 0. \end{aligned} \quad (64)$$

Since $\lambda > 0$, then $\lambda u \in N$ if and only if $j'(\lambda) = 0$.

Then, if $u \in N$, we have $j'(1) = 0$ and $j''(1) = -(1/(p-1)) \int_{\Omega} a(x)|u|^p dx$.

Thus, we can divide N into three subsets N^+ , N^- , and N^0 , where

$$\begin{aligned} N^+ &= \left\{ u \in N : \int_{\Omega} a(x)|u|^p dx > 0 \right\}, \\ N^- &= \left\{ u \in N : \int_{\Omega} a(x)|u|^p dx < 0 \right\}, \\ N^0 &= \left\{ u \in N : \int_{\Omega} a(x)|u|^p dx = 0 \right\}. \end{aligned} \quad (65)$$

Lemma 16. *If u_0 is a local minimizer for J on N and $u_0 \notin N^0$. Then, $J'(u_0) = 0$.*

Proof. If u_0 is a local minimizer for J on N , by Lagrange multipliers, there exists $\kappa \in \mathbb{R}$ such that

$$J'(u_0) = \kappa \chi'(u_0), \quad (66)$$

where $\chi(u) = \|\nabla u\|_p^p - \int_{\Omega} a(x)|u|^p \log |u| dx$.

Since $u_0 \in N$, then

$$\langle J'(u_0), u_0 \rangle = 0, \text{ and } \kappa \langle \chi'(u_0), u_0 \rangle = 0. \quad (67)$$

On the other hand, from $u_0 \notin N^0$, we can see

$$\langle \chi'(u_0), u_0 \rangle = j''_{u_0}(1) = -\frac{\lambda^{p-2}}{p-1} \int_{\Omega} a(x)|u|^p dx \neq 0. \quad (68)$$

Then, $\kappa = 0$ and $J'(u_0) = 0$.

Proposition 17. *Both N^+ and N^- are nonempty.*

Proof. From (62), $j'(\lambda)$ has a unique turning point at

$$\lambda(u) = \exp \left(\frac{\|\nabla u\|_p^p - \int_{\Omega} a(x)|u|^p \log |u| dx}{\int_{\Omega} a(x)|u|^p dx} \right). \quad (69)$$

Since $a(x)$ is sign-changing, then we can take u_1 such that

$$\int_{\Omega} a(x)|u_1|^p dx < 0, \text{ and then } \lambda(u_1)u_1 \in N^+. \quad (70)$$

Also, we can take u_2 such that

$$\int_{\Omega} a(x)|u_2|^p dx > 0, \text{ and then } \lambda(u_2)u_2 \in N^-. \quad (71)$$

Then, both N^+ and N^- are nonempty.

Just like [19], by Lemmas 13–16, we can get the following results.

Lemma 18. [19] *N^+ is bounded; J is bounded below on N^+ .*

Lemma 19. [19] *Every minimizing sequence for J on N^- is bounded, $0 \notin \overline{N^-}$, $\inf_{u \in N^-} J(u) > 0$.*

Proposition 20. *J has a minimizer on N^+ .*

Proof. Let $\{u_m\} \subseteq N^+$ be a minimizing sequence, i.e., $\lim_{m \rightarrow \infty} J(u_m) = \inf_{u \in N^+} J(u) < 0$.

By Lemma 18, N^+ is bounded; we may assume that

$$u_m \rightharpoonup u_0, \text{ in } W_0^{1,p}(\Omega), \text{ and so } u_m \rightarrow u_0 \text{ in } L^p(\Omega). \quad (72)$$

Since $\{u_m\} \subseteq N^+$, we can get

$$\begin{aligned} \int_{\Omega} a(x)|u_m|^p dx &< 0, \int_{\Omega} a(x)|u_0|^p dx \\ &= \lim_{m \rightarrow \infty} \int_{\Omega} a(x)|u_m|^p dx = \lim_{m \rightarrow \infty} p^2 \int_{\Omega} a(x)|u_m|^p dx < 0, \\ \|\nabla u_m\|_p^p - \int_{\Omega} a(x)|u_m|^p \log |u_m| dx &= 0. \end{aligned} \quad (73)$$

Suppose $u_m \not\rightarrow u_0$ in $W_0^{1,p}(\Omega)$, then

$$\begin{aligned} \|\nabla u_0\|_p^p - \int_{\Omega} a(x)|u_0|^p \log |u_0| dx \\ < \lim_{m \rightarrow \infty} \left(\|\nabla u_m\|_p^p - \int_{\Omega} a(x)|u_m|^p \log |u_m| dx \right) = 0. \end{aligned} \quad (74)$$

Then, there exists

$$\lambda(u_0) = \exp \left(\frac{\|\nabla u_0\|_p^p - \int_{\Omega} a(x)|u_0|^p \log |u_0| dx}{\int_{\Omega} a(x)|u_0|^p dx} \right) > 1, \quad (75)$$

such that $\lambda(u_0)u_0 \in N^+$, and then, J attains minimum at $\lambda(u_0)u_0$.

Hence

$$J(\lambda(u_0)u_0) < J(u_0) \leq \lim_{m \rightarrow \infty} J(u_m) = \inf_{u \in N^+} J(u), \quad (76)$$

which is impossible. Hence, $u_m \rightarrow u_0$ in $W_0^{1,p}(\Omega)$, $u_0 \in N^+$, and $J(u_0) = \inf_{u \in N^+} J(u) < 0$, this means that u_0 is a minimizer for J on N^+ .

Proposition 21. *There exists a minimizer of J on N^- .*

Proof. Let $\{u_m\} \subseteq N^-$ be a minimizing sequence. By Lemma 19, $\{u_m\}$ is bounded; we may assume that

$$u_m \rightharpoonup u_0, \text{ in } W_0^{1,p}(\Omega), \text{ and so } u_m \rightarrow u_0 \text{ in } L^p(\Omega). \quad (77)$$

Since $J(u_m) = 1/p^2 \int_{\Omega} a(x)|u_m|^p dx$, by Lemma 19, we can get

$$\int_{\Omega} a(x)|u_0|^p dx = \lim_{m \rightarrow \infty} \int_{\Omega} a(x)|u_m|^p dx > 0. \quad (78)$$

Suppose $u_m \rightarrow u_0$ in $W_0^{1,p}(\Omega)$, then

$$\begin{aligned} \|\nabla u_0\|_p^p - \int_{\Omega} a(x)|u_0|^p \log |u_0| dx \\ < \lim_{m \rightarrow \infty} \left(\|\nabla u_m\|_p^p - \int_{\Omega} a(x)|u_m|^p \log |u_m| dx \right) = 0. \end{aligned} \quad (79)$$

Then, there exists

$$\lambda(u_0) = \exp \left(\frac{\|\nabla u_0\|_p^p - \int_{\Omega} a(x)|u_0|^p \log |u_0| dx}{\int_{\Omega} a(x)|u_0|^p dx} \right) < 1, \quad (80)$$

such that $\lambda(u_0)u_0 \in N^-$, $\lambda(u_0)u_m \rightarrow \lambda(u_0)u_0$, but $\lambda(u_0)u_m \not\rightarrow \lambda(u_0)u_0$ in $W_0^{1,p}(\Omega)$.

Hence

$$J(\lambda(u_0)u_0) < \lim_{m \rightarrow \infty} J(\lambda(u_0)u_m). \quad (81)$$

Since the map $\lambda \rightarrow J(\lambda u_m)$ attains its maximum at $t = 1$,

$$\lim_{m \rightarrow \infty} J(\lambda(u_0)u_m) \leq \lim_{m \rightarrow \infty} J(u_m) = \inf_{u \in N^-} J(u). \quad (82)$$

This means $J(\lambda(u_0)u_0) < \inf_{u \in N^-} J(u)$ is impossible.

Hence, $u_m \rightarrow u_0$ in $W_0^{1,p}(\Omega)$, and this means that u_0 is a minimizer for J on N^- .

Proof of Theorem 3. Propositions 20 and 21 show that the energy functional J has two minimizers u_1 on N^+ and u_2 on N^- . Next, by Lemma 16, J has two critical points u_1 and u_2 on $W_0^{1,p}(\Omega)$, which means that the problem (1) has at least two nontrivial solutions under the condition $h(x) = 0$.

Data Availability

All the data in our manuscript are available.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

All authors carried out the proof and conceived of the study. All authors read and approved the final manuscript.

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Research Article

V-Prox-Regular Functions in Smooth Banach Spaces

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In this paper, we continue the study of the V -prox-regularity that we have started recently for sets. We define an appropriate concept of the V -prox-regularity for functions in reflexive smooth Banach spaces by adapting the one given in Hilbert spaces. Our main goal is to study the relationship between the V -prox-regularity of a given l.s.c. f and the V -prox-regularity of its epigraph.

1. Introduction and Preliminaries

Throughout this work, X will denote a reflexive smooth Banach space unless otherwise specified. We recall from [1] the concept of V -proximal subdifferential.

Definition 1. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (l.s.c.) function and $x \in X$, where f is finite. We recall that the V -proximal subdifferential of f at x is defined as $x^* \in \partial^\pi f(x)$ if and only if there exists $\sigma > 0$ such that

$$\langle x^*, x' - x \rangle \leq f(x') - f(x) + \sigma V(J(x), x'), \forall x' \text{ near } x. \quad (1)$$

Here, J is the normalised duality mapping on X , and V is the functional defined from $X^* \times X$ to $[0, \infty)$ by

$$V(x^*, x) = \|x^*\|^2 - 2\langle x^*, x \rangle + \|x\|^2, \text{ for any } (x^*, x) \in X^* \times X. \quad (2)$$

The V -proximal normal cone of a nonempty closed subset S in X at $x \in S$ is defined as the V -proximal subdifferential of the indicator function of S , that is, $N^\pi(S; x) = \partial^\pi \psi_S(x)$.

Another proximal subdifferential $\partial_G^\pi f(x)$ is defined (see [5]) geometrically via the V -proximal normal cone of the epi-

graph as follows:

$$\partial_G^\pi f(x) = \{x^* \in X^* : (x^*, -1) \in N^\pi(\text{epi } f; (x, f(x)))\}. \quad (3)$$

It has been proved in [2] that in general, we have the inclusion $\partial^\pi f(x) \subset \partial_G^\pi f(x)$. The generalized projection on a closed nonempty set S is defined as follows: $\bar{x} \in \pi_S(x^*)$ if and only if $V(x^*, \bar{x}) = \inf_{x \in S} V(x^*, x)$ (see [3] for convex sets and see [1] for nonconvex sets). We also recall (see for instance [2]) the definition of the Fréchet subdifferential and Fréchet normal cone as follows: $x^* \in \partial^F f(\bar{x})$ if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \varepsilon \|x - \bar{x}\|, \forall x \in \bar{x} + \delta \mathbb{B}. \quad (4)$$

The Fréchet normal cone $N^F(S; x)$ of a nonempty closed subset S in X at $\bar{x} \in S$ is defined as $N^F(S; \bar{x}) = \partial^F \psi_S(\bar{x})$.

2. Generalized V-Prox-Regular Sets

In this section, we recall the concept of the generalized V -prox-regularity of sets introduced and studied in [4] and other results needed in our work.

Definition 2. Let S be a nonempty closed set in a reflexive Banach space X and let $\bar{x} \in S$. We will say that S is generalized V -prox-regular at \bar{x} if and only if there exist $r > 0$ and $\varepsilon > 0$ such that for all $x \in S \cap (\bar{x} + \varepsilon \mathbb{B})$ and for any $x^* \in N^\pi(S; x)$

$\cap \varepsilon \mathbb{B}_*$, the point x is a generalized projection of $Jx + rx^*$ on $S \cap (\bar{x} + \varepsilon \mathbb{B})$, that is, $x \in \pi_{S \cap (\bar{x} + \varepsilon \mathbb{B})}(Jx + rx^*)$.

We quote from [5, 6] the following two results.

Lemma 3. *For any $x \in \text{dom } f$ and any $(x, \alpha) \in \text{epi } f$ (i.e., $f(x) \leq \alpha$), we have*

$$(x^*, 0) \in N^\pi(\text{epi } f; (x, \alpha)) \Rightarrow (x^*, 0) \in N^\pi(\text{epi } f; (x, f(x))). \quad (5)$$

Proposition 4. *Let X be an Asplund space and let $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper l.s.c. function around $\bar{x} \in \text{dom } f$. Then, for any $x^* \in X^*$ with $(x^*, 0) \in N^F(\text{epi } f; (\bar{x}, f(\bar{x})))$, there exist sequences $x_k \rightarrow \bar{x}$ with $f(x_k) \rightarrow f(\bar{x})$, $\lambda_k \rightarrow 0^+$, and $x_k^* \in \lambda_k \partial^F f(x_k)$ such that $\|x_k^* - x^*\|_* \rightarrow 0$.*

We recall from [2] that X is V -proximal trustworthy provided that for any $\varepsilon > 0$, any two functions $f_1, f_2 : X \rightarrow \mathbb{R} \cup \{\infty\}$ and any $u \in X$ such that f_1 is lower semicontinuous and f_2 is Lipschitz around u the following fuzzy sum rule holds:

$$\partial_A^\pi(f_1 + f_2)(u) \subset \bigcup \left\{ \partial_A^\pi f_1(u_1) + \partial_A^\pi f_2(u_2) : u_i \in U_{f_i}(u, \varepsilon), i = 1, 2 \right\} + \varepsilon \mathbb{B}_*. \quad (6)$$

Here, $U_{f_i}(u, \varepsilon) := \{x \in u + \varepsilon \mathbb{B} \text{ such that } |f_i(x) - f_i(u)| < \varepsilon\}$, and \mathbb{B}_* denotes the closed unit ball in X^* . The proof of the following proposition can be found in [2].

Proposition 5. *Let X be a V -proximal trustworthy space. Let S be a closed subset of X with $x \in S$ and let $x^* \in N^F(S; x)$. Then, for any $\varepsilon > 0$, there exists $x_\varepsilon \in (\bar{x} + \varepsilon \mathbb{B}) \cap S$ such that $x^* \in N^\pi(S; x_\varepsilon) + \varepsilon \mathbb{B}_*$.*

3. V-Prox-Regular Functions

In Hilbert spaces setting, the authors in [7] extended and studied the concept of prox-regularity for functions which has been introduced for finite dimensional spaces in [8]. Many papers studied this concept in finite dimensional spaces and its applications to nonsmooth optimization (see for instance [9] and the references therein). Our aim here is to extend the concept of prox-regularity to reflexive smooth Banach spaces by using the concept of the V -proximal normal cone and the functional V . Another way to extend this concept has been proposed in [10].

Definition 6. Assume that X is a reflexive smooth Banach space. A l.s.c. function $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ is said to be a V -prox-regular at $x_0 \in \text{dom } f$ for $x_0^* \in \partial^\pi f(x_0)$ if and only if there exist $\varepsilon > 0, r > 0$ such that $\forall (x, x^*) \in \partial_G^\pi f$ with $\|x - x_0\| \leq \varepsilon, |f(x) - f(x_0)| < \varepsilon_0$ and $\|x^* - x_0^*\| < \varepsilon_0$ such that

$$f(x') \geq f(x) + \langle x^*, x' - x \rangle - \frac{r}{2} V(Jx, x'), \forall x' \in x_0 + \varepsilon_0 \mathbb{B}. \quad (7)$$

We say that f is V -prox-regular at x_0 if it is V -prox-regular for any $x_0^* \in \partial^\pi f(x_0)$. Obviously, this concept coincides with the one studied in [7] in Hilbert spaces. Using this definition for functions, we can associate to it another definition of the V -prox-regularity for sets via the indicator function. We have the following definition:

Definition 7. Let X be a reflexive smooth Banach space. A closed nonempty set S is said to be V -prox-regular at $x_0 \in S$ for $x_0^* \in N^\pi(S; x_0)$ if and only if the indicator function ψ_S is V -prox-regular at x_0 for $x_0^* \in \partial^\pi \psi_S(x_0)$.

Proposition 8. *Let X be a reflexive smooth Banach space. A closed set S is generalized V -prox-regular at $x_0 \in S$ in the sense of Definition 2, if and only if the indicator function ψ_S is V -prox-regular at x_0 for $x_0^* = 0$ in the sense of Definition 6.*

Proof. Assume that S is generalized V -prox-regular at $x_0 \in S$, then there exist $r > 0$ and $\varepsilon > 0$ such that for all $x \in S \cap (x_0 + \varepsilon \mathbb{B})$ and for any $x^* \in N^\pi(S; x) \cap \varepsilon \mathbb{B}_*$, the point x is a generalized projection of $Jx + rx^*$ on $S \cap (x_0 + \varepsilon \mathbb{B})$, that is, $x \in \pi_{S \cap (x_0 + \varepsilon \mathbb{B})}(Jx + rx^*)$. By definition of generalized projection, we have

$$V(Jx + rx^*; x) \leq V(Jx + rx^*; x'). \forall x' \in S \cap (x_0 + \varepsilon \mathbb{B}). \quad (8)$$

So, for any $x' \in S \cap (x_0 + \varepsilon \mathbb{B})$, we have

$$\|x\|^2 - 2\langle Jx + rx^*; x \rangle \leq \|x'\|^2 - 2\langle Jx + rx^*; x' \rangle. \quad (9)$$

Thus,

$$\begin{aligned} 2\langle rx^*; x' - x \rangle &= 2\langle Jx + rx^*; x' - x \rangle - 2\langle Jx; x' - x \rangle \\ &= 2\langle Jx + rx^*; x' - x \rangle - 2\langle Jx; x' \rangle + 2\|x\|^2 \\ &\leq \|x'\|^2 - 2\langle Jx; x' \rangle + \|x\|^2 = V(Jx, x'), \end{aligned} \quad (10)$$

for any $x' \in S \cap (x_0 + \varepsilon \mathbb{B})$. Hence,

$$\langle x^*; x' - x \rangle \leq \frac{1}{2r} V(Jx, x'), \text{ for all } x' \in S \cap (x_0 + \varepsilon \mathbb{B}) \quad (11)$$

and so

$$\langle x^*; x' - x \rangle \leq \psi_S(x') - \psi_S(x) + \frac{1}{2r} V(Jx, x'), \text{ for all } x' \in x_0 + \varepsilon \mathbb{B}. \quad (12)$$

On the other hand, we have $x^* \in N^\pi(S; x) \cap \varepsilon \mathbb{B}_*$ which implies that $x^* \in \partial_G^\pi \psi_S(x)$ with $\|x^* - x_0^*\| < \varepsilon$ and $x_0^* = 0$. Thus, the inequality (12) ensures that ψ_S is V -prox-regular at x_0 for $x_0^* = 0$. For the opposite direction, we follow the same lines as in the direct one.

In the next theorem, we study the relationship between the V -prox-regularity of a function f and the V -prox-regularity of its epigraph.

Theorem 9. *Let $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ be a function that is l.s.c. at $x_0 \in \text{dom } f$. If f is V -prox-regular at x_0 for $x_0^* \in \partial^\pi f(x_0)$, then the epigraph $\text{epi } f$ is V -prox-regular at $(x_0, f(x_0))$ for $(x_0^*, -1) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$. If, in addition, the space X is 2-uniformly convex, then the previous implication becomes an equivalence, that is, f is V -prox-regular at x_0 for $x_0^* \in \partial^\pi f(x_0)$ if and only if the epigraph $\text{epi } f$ is V -prox-regular at $(x_0, f(x_0))$ for $(x_0^*, -1) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$.*

Proof. Assume that f is V -prox-regular at x_0 for $x_0^* \in \partial^\pi f(x_0)$, fix $\varepsilon_0 \in (0, 1)$ and $r > 0$ such that $\forall(x, x^*) \in \partial_G^\pi f$ with $\|x - x_0\| \leq \varepsilon$, $|f(x) - f(x_0)| < \varepsilon_0$ and $\|x^* - x_0^*\| < \varepsilon_0$ such that

$$f(x') \geq f(x) + \langle x^*, x' - x \rangle - \frac{r}{2} V(Jx, x'), \forall x' \in x_0 + \varepsilon_0 \mathbb{B}. \quad (13)$$

By l.s.c. of f , we choose $\varepsilon \in (0, 1)$ with $(\varepsilon/1 - \varepsilon)(1 + \|x_0^*\|) < \varepsilon_0$ and such that for any $x' \in x_0 + \varepsilon \mathbb{B}$, we have $f(x_0) - \varepsilon < f(x')$. Fix any $(x, \alpha) \in \text{epi } f$, with $\|(x, \alpha) - (x_0, f(x_0))\| < \varepsilon$ and $(x^*, -\lambda) \in N^\pi(\text{epi } f; (x, \alpha))$ with $\|(x^*, -\lambda) - (x_0^*, -1)\| < \varepsilon$. Clearly, $0 < 1 - \varepsilon < \lambda < 1 + \varepsilon$, which ensures that $\alpha = f(x)$ and so $x^*/\lambda \in \partial_G^\pi f(x)$. Take now any $(x', \alpha') \in \text{epi } f$ with $\|(x', \alpha') - (x_0, f(x_0))\| < \varepsilon$. Notice that

$$\begin{aligned} f(x_0) - \varepsilon &< f(x') \leq \alpha' < f(x_0) + \varepsilon_0, \\ \left\| \frac{x^*}{\lambda} - x_0^* \right\| &\leq \frac{1}{\lambda} \|x^* - \lambda x_0^*\| = \frac{1}{\lambda} \|x^* - x_0^* - \lambda x_0^* + x_0^*\| \leq \frac{1}{\lambda} (\|x^*\| \\ &\quad - x_0^* \| + \|\lambda x_0^* - x_0^*\|) \leq \frac{1}{\lambda} [\varepsilon + |1 - \lambda| \|x_0^*\|] \leq \frac{\varepsilon}{\lambda} \\ &\quad + \frac{\varepsilon}{1 - \varepsilon} \|x_0^*\| \leq \frac{\varepsilon}{1 - \varepsilon} (1 + \|x_0^*\|) < \varepsilon_0, \end{aligned} \quad (14)$$

which yields by (13)

$$\begin{aligned} \left\langle \frac{x^*}{\lambda}, x' - x \right\rangle - (\alpha' - f(x)) &\leq \left\langle \frac{x^*}{\lambda}, x' - x \right\rangle \\ &\quad - (f(x') - f(x)) \leq \frac{r}{2} V(Jx, x'), \end{aligned} \quad (15)$$

and hence,

$$\langle x^*, x' - x \rangle - \lambda(\alpha' - \alpha) \leq \frac{r\lambda}{2} V(Jx, x'). \quad (16)$$

On the other hand, we have

$$V_{X \times \mathbb{B}}((Jx, \alpha); (x' - \alpha')) = V(Jx, x') + (\alpha' - \alpha)^2. \quad (17)$$

So,

$$\langle (x^*, -\lambda); (x', \alpha') - (x, \alpha) \rangle \leq \frac{r\lambda}{2} V_{X \times \mathbb{B}}((Jx, \alpha); (x' - \alpha')). \quad (18)$$

Thus, for any $(x, \alpha) \in \text{epi } f$, with $\|(x, \alpha) - (x_0, f(x_0))\| < \varepsilon$ and $(x^*, -\lambda) \in N^\pi(\text{epi } f; (x, \alpha))$ with $\|(x^*, -\lambda) - (x_0^*, -1)\| < \varepsilon$, we have

$$\langle (x^*, -\lambda); (x', \alpha') - (x, \alpha) \rangle \leq \frac{r(1 + \varepsilon_0)}{2} V_{X \times \mathbb{B}}((Jx, \alpha); (x' - \alpha')), \quad (19)$$

for any $(x', \alpha') \in \text{epi } f$ with $\|(x', \alpha') - (x_0, f(x_0))\| < \varepsilon$. This means that the epigraph $\text{epi } f$ is V -prox-regular at $(x_0, f(x_0))$ for $(x_0^*, -1) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$.

Assume now that the space X is a 2-uniformly convex space. Assume that the epigraph $\text{epi } f$ is V -prox-regular at $(x_0, f(x_0))$ for $(x_0^*, -1) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$, choose $\varepsilon_0 \in (0, 1)$ and $r > 0$ such that for any $(x, \alpha) \in \text{epi } f$ and any $(x^*, -\lambda) \in N^\pi(\text{epi } f; (x, \alpha))$ with $\|(x, \alpha) - (x_0, f(x_0))\| < \varepsilon_0$ and with $\|(x^*, -\lambda) - (x_0^*, -1)\| < \varepsilon_0$, we have

$$\langle (x^*, -\lambda); (x', \alpha') - (x, \alpha) \rangle \leq \frac{r}{2} \left[V(Jx; x') + (\alpha' - \alpha)^2 \right], \quad (20)$$

for all $(x', \alpha') \in \text{epi } f$ with $\|(x', \alpha') - (x_0, f(x_0))\| < \varepsilon_0$. Using the fact that space X is a 2-uniformly convex space, we can find a positive constant $\bar{\beta} > 0$ depending only on the space X and the constant $M := (\|x_0\| + 4\varepsilon_0)$ such that

$$V(Jx, x') \geq \bar{\beta} \|x - x'\|^2, \forall x', x \in M\mathbb{B}. \quad (21)$$

Choose now a positive number $\varepsilon \in (0, \min \{\varepsilon_0, (1/3r(1 + \|x_0^*\|), 1/(9\beta r)^{1/p})\})$ such that

$$[2\varepsilon(1 + \|x_0^*\|) + \beta\varepsilon^p] \left[1 + \bar{\beta}^{-1} \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \right] < \min \left\{ \frac{1}{r}, \frac{1}{r^2} \right\} \quad (22)$$

and such that (according to the l.s.c. of f)

$$f(x) > f(x_0) - \frac{1}{2r}, \forall x \in x_0 + \varepsilon \mathbb{B}. \quad (23)$$

Here, β depends only on the space X and M . Then, for any $x \in x_0 + \varepsilon \mathbb{B}$, we have $-\varepsilon < f(x) - f(x_0) < \varepsilon$ and so

$$f(x) - f(x_0) + \frac{1}{r} > f(x_0) - \frac{1}{2r} - f(x_0) + \frac{1}{r} - \varepsilon = \frac{1}{2r} - \varepsilon > 0. \quad (24)$$

Also, we have by (23) one that has

$$r \left[1 + \bar{\beta}^{-1} \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \right] < \frac{1}{r[2\varepsilon(1+\|x_0^*\|) + \beta\varepsilon^p]}, \quad (25)$$

and hence since $1/r[2\varepsilon(1+\|x_0^*\|) + \beta\varepsilon^p] > 1$ (by (23) again), we may fix some positive number σ such that

$$\max \left\{ 1, r \left[1 + \bar{\beta}^{-1} \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \right] \right\} < \sigma < \frac{1}{r[2\varepsilon(1+\|x_0^*\|) + \beta\varepsilon^p]}. \quad (26)$$

Take now any $x, x' \in x_0 + (\varepsilon/2)\mathbb{B}$ with $|f(x) - f(x_0)| < \varepsilon/2$ and $|f(x') - f(x_0)| < \varepsilon/2$ and $x^* \in \partial^\pi f(x)$ with $x^* \in x_0^* + (\varepsilon/2)\mathbb{B}_*$. Then, applying (20) with $(x^*, -1)$ instead of $(x^*, -\lambda)$, $f(x)$ instead of α , and $f(x')$ instead of α' , we obtain

$$\begin{aligned} \langle x^*; x' - x \rangle - (f(x') - f(x)) &\leq \frac{r}{2} V(Jx; x') \\ &\quad + \frac{r}{2} (f(x') - f(x))^2, \end{aligned} \quad (27)$$

which is equivalent to (with $\rho := 1/r$)

$$\begin{aligned} 2\rho \langle x^*; x' - x \rangle - V(Jx; x') &\leq (f(x') - f(x))^2 \\ &\quad + 2\rho (f(x') - f(x)) = [f(x') - f(x) + \rho]^2 - \rho^2 \end{aligned} \quad (28)$$

and so

$$\rho^2 + 2\rho \langle x^*; x' - x \rangle - V(Jx; x') \leq [f(x') - f(x) + \rho]^2. \quad (29)$$

Note that

$$\begin{aligned} \rho^2 + 2\rho \langle x^*; x' - x \rangle - V(Jx; x') \\ &\geq \rho^2 - 2\rho \|x^*\| \|x' - x\| - \beta\varepsilon^p \geq \rho^2 - 2\rho\varepsilon(1+\|x_0^*\|) - \beta\varepsilon^p \\ &\geq \rho^2 - 2\rho \frac{\rho}{3} - \frac{\rho^2}{9} = \frac{2\rho^2}{9} > 0. \end{aligned} \quad (30)$$

Thus,

$$f(x') - f(x) + \rho \geq [\rho^2 + 2\rho \langle x^*; x' - x \rangle - V(Jx; x')]^{1/2}. \quad (31)$$

Now, we develop the expression

$$\begin{aligned} E &:= [\rho^2 + 2\rho \langle x^*; x' - x \rangle - V(Jx; x')] \\ &\quad - [\rho + \langle x^*; x' - x \rangle - \sigma V(Jx; x')]^2, \\ E &= -V(Jx; x') + 2\rho\sigma V(Jx; x') \\ &\quad - [\langle x^*; x' - x \rangle - \sigma V(Jx; x')]^2 = (2\rho\sigma - 1)V(Jx; x') \\ &\quad - [\langle x^*; x' - x \rangle]^2 + 2\sigma V(Jx; x') \langle x^*; x' - x \rangle \\ &\quad - \sigma^2 [V(Jx; x')]^2. \end{aligned} \quad (32)$$

Observe that

$$\begin{aligned} \langle x^*; x' - x \rangle &\leq \|x^*\| \|x' - x\| \leq \left(\frac{\varepsilon}{2} + \|x_0^*\| \right) \varepsilon, \\ V(Jx; x') &\leq \beta\varepsilon^p, \\ [\langle x^*; x' - x \rangle]^2 &\leq \|x^*\|^2 \|x' - x\|^2 \leq \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \bar{\beta}^{-1} V(Jx; x'). \end{aligned} \quad (33)$$

Thus,

$$\begin{aligned} E &\geq (2\rho\sigma - 1)V(Jx; x') - \left[\frac{\varepsilon}{2} + \|x_0^*\| \right]^2 \bar{\beta}^{-1} V(Jx; x') \\ &\quad + 2\sigma V(Jx; x') \langle x^*; x' - x \rangle - \sigma^2 [V(Jx; x')]^2 = V(Jx; x') \\ &\quad \cdot \left[2\rho\sigma - 1 - \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \bar{\beta}^{-1} - 2 \left(\frac{\varepsilon}{2} + \|x_0^*\| \right) \varepsilon \sigma - \sigma^2 \beta\varepsilon^p \right]. \end{aligned} \quad (34)$$

But as $\sigma > 1$, one has

$$\begin{aligned} 2\rho\sigma - 1 - 2\sigma\varepsilon(1+\|x_0^*\|) - \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \bar{\beta}^{-1} - \sigma^2 \beta\varepsilon^p \\ &\geq \left[-1 - \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \bar{\beta}^{-1} + \rho\sigma \right] \\ &\quad + \sigma[\rho - 2\sigma\varepsilon(1+\|x_0^*\|) - \sigma\beta\varepsilon^p]. \end{aligned} \quad (35)$$

Using the choice of σ in the inequality (26), we have

$$\begin{aligned} \sigma &> r \left(1 + \bar{\beta}^{-1} \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \right), \\ \sigma &< \frac{1}{r[2\varepsilon(1+\|x_0^*\|) + \beta\varepsilon^p]}. \end{aligned} \quad (36)$$

which ensure, respectively,

$$\begin{aligned} -1 - \left(\frac{\varepsilon}{2} + \|x_0^*\|\right)^2 \bar{\beta}^{-1} + \frac{\sigma}{r} &> 0, \\ \frac{1}{r} - 2\sigma\varepsilon[1 + \|x_0^*\|] - \sigma\beta\varepsilon^p &> 0. \end{aligned} \quad (37)$$

Since $\rho = \sigma/r$, we obtain

$$\left[-1 - \left(\frac{\varepsilon}{2} + \|x_0^*\|\right)^2 \bar{\beta}^{-1} + \rho\sigma\right] + \sigma[\rho - 2\sigma\varepsilon[1 + \|x_0^*\|] - \sigma\beta\varepsilon^p] > 0. \quad (38)$$

Therefore, it follows from (35) that $E > 0$ and so

$$\begin{aligned} &[\rho^2 + 2\rho\langle x^*; x' - x \rangle - V(Jx; x')] \\ &> [\rho + \langle x^*, x' - x \rangle - \sigma V(Jx, x')]^2 > 0, \end{aligned} \quad (39)$$

which ensures that

$$\begin{aligned} &[\rho^2 + 2\rho\langle x^*; x' - x \rangle - V(Jx; x')]^{1/2} \\ &> \rho + \langle x^*, x' - x \rangle - \sigma V(Jx, x') > 0. \end{aligned} \quad (40)$$

Thus, by (31) and (35), we obtain

$$f(x') - f(x) + \rho \geq \rho + \langle x^*, x' - x \rangle - \sigma V(Jx, x'). \quad (41)$$

Now, it remains to show the conclusion for x' satisfying $|f(x') - f(x_0)| \geq \varepsilon$. First choose, by the l.s.c. of f , some positive number

$$\eta < \frac{\varepsilon}{4(1 + \|x_0^*\|)}, \quad (42)$$

such that

$$f(x) - f(x_0) > -\varepsilon, \quad \forall x \in x_0 + \eta\mathbb{B}. \quad (43)$$

Now, fix any $x, x' \in x_0 + \eta\mathbb{B}$ with only $f(x) - f(x_0) < \eta$, and fix any $x^* \in x_0^* + \eta\mathbb{B}_*$. Thus, if $|f(x') - f(x_0)| \geq \varepsilon$, then

$$f(x') - f(x_0) \geq \varepsilon \text{ or } f(x) - f(x_0) \leq -\varepsilon. \quad (44)$$

But since $f(x') - f(x_0) > -\varepsilon$ (by (43)), we get only

$$f(x') \geq \varepsilon + f(x_0). \quad (45)$$

Therefore,

$$\begin{aligned} f(x') &= f(x) + [f(x_0) - f(x)] + [f(x') - f(x_0)] \\ &\geq f(x) - \eta + \varepsilon \geq f(x) + \frac{\varepsilon}{2}. \end{aligned} \quad (46)$$

Observe that

$$\langle x^*, x' - x \rangle - \sigma V(Jx, x') \leq \|x^*\| \|x' - x\| \leq (1 + \|x_0^*\|)\eta < \frac{\varepsilon}{2}, \quad (47)$$

and hence, we get

$$f(x') - f(x) + \rho \geq \rho + \langle x^*, x' - x \rangle - \sigma V(Jx, x'). \quad (48)$$

Finally, the relation (41) together with the last inequality conclude that f is V -prox-regular at \bar{x} for x_0^* .

Remark 10. In the previous theorem, we proved the following: if f is V -prox-regular at x_0 for $x_0^* \in \partial^\pi f(x_0)$, then we have the V -prox-regularity of $\text{epi } f$ at $(x_0, f(x_0))$ for $(x_0^*, -1) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$. Unfortunately, we do not obtain the generalized V -prox-regularity of the epigraph in the sense of Definition 2 since $\text{epi } f$ is a generalized V -prox-regularity at $(x_0, f(x_0))$ if and only if it is V -prox-regularity at $(x_0, f(x_0))$ for $(0, 0) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$ which cannot be the case (since $(x_0^*, -1) \neq (0, 0)$) under the V -prox-regularity of f at x_0 . To ensure the generalized V -prox-regularity of the epigraph, we need another kind of regularity called the V -primal lower nice function introduced and studied in [5].

Definition 11. Let $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be a l.s.c. function. We will say that f is V -primal lower nice at $\bar{x} \in \text{dom } f$, if there exist $\lambda_1, \lambda_2 > 0, c > 0, T > 0$ such that

$$f(y) \geq f(x) + \langle x^*, y - x \rangle - \frac{t}{2} V(Jx, y), \quad (49)$$

whenever $t \geq T, x \in \mathcal{N}_V(\bar{x}, \lambda_1, \lambda_2), y \in \bar{x} + \lambda_1\mathbb{B}, x^* \in \partial_G^\pi f(x)$, and $\|x^*\| \leq ct$. Here,

$$\mathcal{N}_V(\bar{x}, \lambda_1, \lambda_2) := \{x \in \bar{x} + \lambda_1\mathbb{B} \text{ with } V(J\bar{x}, x) \leq \lambda_2\}. \quad (50)$$

This concept is stronger than the V -prox-regularity. Indeed, as mentioned in [8] for finite dimensional spaces, obviously, any function that is a V -p.l.n. function is V -prox-regular. However, the inverse is not true. For a V -p.l.n. function, the condition (49) must hold for all V -proximal gradients with the *linear growth* condition, whereas for V -prox-regular functions, the condition (7) only has to hold for V -proximal subgradients close to a fixed $x_0^* \in \partial^\pi f(x_0)$ and just in a neighborhood of x_0 with $f(x)$ close to $f(x_0)$. For instance (see [8]), the function f on \mathbb{R} with $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$ is easily seen to be V -prox-regular at $x_0 = 0$ for $x^* = 0$ but not V -p.l.n. for this x_0 . We have to notice that this definition of V -p.l.n. functions extends, to reflexive smooth Banach spaces, the definition of primal lower nice functions defined in finite dimension spaces in [11], and in Hilbert spaces in [12]. Our next proposition shows the generalized V -prox-regularity of the epigraph in the sense of Definition 2 of any V -primal lower nice functions.

Proposition 12. *Let X be a V -proximal trustworthy space. If f is V -primal lower nice at $\bar{x} \in \text{dom } f$, then $\text{epi } f$ is generalized V -prox-regular at $(\bar{x}, f(\bar{x}))$.*

Proof. By definition of V -p.l.n. at \bar{x} , we have positive numbers $\varepsilon_1 > 0, \varepsilon_2 > 0, r > 0, T > 0$ such that for any $t \geq T$, any $x \in \mathcal{N}(\bar{x}, \varepsilon_1, \varepsilon_2)$, and any $x^* \in \partial_G^\pi f(x) \cap rt\mathbb{B}_*$, we have

$$\langle x^*, y - x \rangle \leq f(y) - f(x) + \frac{t}{2} V(Jx, y), \forall y \in \bar{x} + \varepsilon_1 \mathbb{B}. \quad (51)$$

Take $(x, \alpha) \in \text{epi } f$ and $(x^*, -\lambda) \in N^\pi(\text{epi } f; (x, \alpha)) \cap \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}$ and $x \in \bar{x} + \varepsilon_0 \mathbb{B}$ with $|f(\bar{x}) - \alpha| < \varepsilon_0$, where ε_0 is taken in $(0, \varepsilon_1)$ such that

$$V(J\bar{x}, z) < \varepsilon_2, \quad \forall z \in \bar{x} + \varepsilon_0 \mathbb{B}. \quad (52)$$

Fix now any $(x', \alpha') \in \text{epi } f$ with $\|x' - \bar{x}\| < \varepsilon_0$ and $|\alpha' - f(\bar{x})| < \varepsilon_0$. Clearly, $x' \in \mathcal{N}_V(\bar{x}; \varepsilon_1, \varepsilon_2)$.

Case 1. $\lambda > 0$. In this case, we necessarily have $\alpha = f(x)$. Then, $(x^*/\lambda, -1) \in N^\pi(\text{epi } f; (x, f(x)))$, so $x^*/\lambda \in \partial_G^\pi f(x)$ and $\|x^*/\lambda\| \leq \varepsilon_0/\lambda \leq rt$ for every $t \geq \max\{T, (1/r\lambda)\}$. Hence, by (3.18), we obtain

$$\langle \lambda^{-1}x^*, x' - x \rangle \leq f(x') - f(x) + \frac{t}{2} V(Jx, x'), \quad (53)$$

which entails

$$\begin{aligned} \langle x^*, x' - x \rangle &\leq \lambda [f(x') - f(x)] + \frac{t\lambda}{2} V(Jx, x') \\ &\leq \lambda [f(x') - f(x)] + \frac{t\lambda}{2} V(Jx, x') + \frac{t\lambda}{2} [\alpha' - \alpha]^2. \end{aligned} \quad (54)$$

Since $f(x') \leq \alpha'$, we have $\forall t' \geq \max\{(T/2), (1/2r)\}$, and we have $2\lambda^{-1}t' \geq 1/r\lambda$ and $2\lambda^{-1}t' \geq T/\lambda \geq T$ and hence (54) entails with $t = 2\lambda^{-1}t'$

$$\begin{aligned} \langle x^*, x' - x \rangle &\leq \lambda [f(x') - f(x)] + (2\lambda^{-1}t') \frac{\lambda}{2} V(Jx, x') \\ &\quad + (2\lambda^{-1}t') \frac{\lambda}{2} [\alpha' - \alpha]^2 \leq \lambda [f(x') - f(x)] \\ &\quad + t' V(Jx, x') + t' [\alpha' - \alpha]^2 \leq t' \|x\|^2 + t' \|x'\|^2 \\ &\quad - 2t' \langle Jx, x' - x \rangle + \lambda [\alpha' - \alpha] + t' [\alpha' - \alpha]^2. \end{aligned} \quad (55)$$

Hence,

$$\langle x^* + 2t' Jx, x' - x \rangle \leq t' \|x'\|^2 - t' \|x\|^2 + \lambda [\alpha' - \alpha] + t' [\alpha' - \alpha]^2. \quad (56)$$

So,

$$\begin{aligned} t' \|x\|^2 - 2t' \left\langle Jx + \frac{x^*}{2t'}, x \right\rangle &\leq t' \|x'\|^2 - 2t' \left\langle Jx + \frac{x^*}{2t'}, x' \right\rangle \\ &\quad + \lambda [\alpha' - \alpha] + t' [\alpha' - \alpha]^2. \end{aligned} \quad (57)$$

Dividing by $t' > 0$ yields

$$V\left(Jx + \frac{x^*}{2t'}; x\right) \leq V\left(Jx + \frac{x^*}{2t'}; x'\right) + \frac{\lambda}{t'} [\alpha' - \alpha] + [\alpha' - \alpha]^2. \quad (58)$$

On the other hand, we have

$$\begin{aligned} \left[\alpha' - \alpha + \frac{1}{2t'} \lambda\right]^2 &= [\alpha' - \alpha]^2 + \frac{\lambda}{t'} (\alpha' - \alpha) + \frac{\lambda^2}{4t'^2} \\ &= \frac{1}{t'} \left[t' [\alpha' - \alpha]^2 + \lambda (\alpha' - \alpha) + \frac{\lambda^2}{4t'}\right]. \end{aligned} \quad (59)$$

So,

$$t' \left[\alpha' - \alpha + \frac{1}{2t'} \lambda\right]^2 - \frac{\lambda^2}{4t'} = t' [\alpha' - \alpha]^2 + \lambda (\alpha' - \alpha). \quad (60)$$

Thus, (58) becomes

$$\begin{aligned} V\left(Jx + \frac{x^*}{2t'}; x\right) &\leq V\left(Jx + \frac{x^*}{2t'}; x'\right) \\ &\quad + \frac{1}{t'} \left[t' [\alpha' - \alpha]^2 + \lambda [\alpha' - \alpha]\right] \\ &\leq V\left(Jx + \frac{x^*}{2t'}; x'\right) + \left[\alpha' - \alpha + \frac{1}{2t'} \lambda\right]^2 - \frac{\lambda^2}{4t'^2} \end{aligned} \quad (61)$$

and so

$$V\left(Jx + \frac{x^*}{2t'}; x\right) + \frac{\lambda^2}{4t'^2} \leq V\left(Jx + \frac{x^*}{2t'}; x'\right) + \left[\alpha' - \alpha + \frac{1}{2t'} \lambda\right]^2. \quad (62)$$

Observe that

$$\begin{aligned} V_{X \times \mathbb{R}}\left(\left(Jx + \frac{x^*}{2t'}, \left(\alpha + \frac{1}{2t'} (-\lambda)\right)\right); (x, \alpha)\right) &= V\left(Jx + \frac{x^*}{2t'}; x\right) \\ &\quad + \frac{\lambda^2}{4t'^2}, \text{ and } V_{X \times \mathbb{R}}\left(\left(Jx + \frac{x^*}{2t'}, \left(\alpha + \frac{1}{2t'} (-\lambda)\right)\right); (x', \alpha')\right) \\ &= V\left(Jx + \frac{x^*}{2t'}; x'\right) + \left[\alpha' - \alpha + \frac{1}{2t'} \lambda\right]^2. \end{aligned} \quad (63)$$

Therefore,

$$\begin{aligned} V_{X \times \mathbb{R}} \left(\left(Jx + \frac{x^*}{2t'}, \left(\alpha + \frac{1}{2t'}(-\lambda) \right) \right); (x, \alpha) \right) \\ \leq V_{X \times \mathbb{R}} \left(\left(Jx + \frac{x^*}{2t'}, \left(\alpha + \frac{1}{2t'}(-\lambda) \right) \right); (x', \alpha') \right), \end{aligned} \quad (64)$$

for any $(x', \alpha') \in \text{epi } f \cap [(\bar{x}, f(\bar{x})) + \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}]$, that is,

$$(x, \alpha) \in \pi_{\text{epi } f \cap [(\bar{x}, f(\bar{x})) + \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}]} Jx + \frac{x^*}{2t'}, \left(\alpha + \frac{1}{2t'}(-\lambda) \right). \quad (65)$$

Case 2. $\lambda = 0$. In this case, we have $(x^*; 0) \in N^\pi(\text{epi } f; (x, \alpha))$ and so by Lemma 3, we obtain $(x^*, 0) \in N^\pi(\text{epi } f; (x, f(x)))$. Using the fact that $N^\pi(\text{epi } f; (x, f(x))) \subset N^F(\text{epi } f; (x, f(x)))$, we get $(x^*, 0) \in N^F(\text{epi } f; (x, f(x)))$ and so by Proposition 4, there exists a sequence $u_n \rightarrow x$ with $f(u_n) \rightarrow f(x)$, $\lambda_n \rightarrow 0^+$, and $u_n^* \in \lambda_n \partial^F f(u_n)$ (i.e., $(u_n^*/\lambda_n, -1) \in N^F \text{epi } f; (u_n, f(u_n))$) such that $\|u_n^* - x^*\|_* \rightarrow 0$. Using now Proposition 5, we choose for each $n \geq 1$

$$\begin{aligned} (x_n, \alpha_n) &\in (u_n, f(u_n)) + \frac{\lambda_n}{2} \mathbb{B}_{X \times \mathbb{R}}, (x_n, \alpha_n) \\ &\in \text{epi } f, (y_n^*, -\mu_n) \in N^\pi(\text{epi } f; (x_n, \alpha_n)), \quad (66) \\ \|y_n^* - u_n^*\| &< \frac{\lambda_n}{2} \text{ and } |\lambda_n - \mu_n| < \frac{\lambda_n}{2}. \end{aligned}$$

This ensures that $\mu_n > 0$ and so $\alpha_n = f(x_n)$. Consequently, $y_n^* \in \mu_n \partial_G^\pi f(x_n)$ with

$$\begin{aligned} \|x_n - x\| &\leq \|x_n - u_n\| + \|u_n - x\| \\ &\leq \frac{\lambda_n}{2} + \|u_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty, \\ |f(x_n) - f(x)| &\leq |\alpha_n - f(u_n)| + |f(u_n) - f(x)| \leq \frac{\lambda_n}{2} \\ &+ |f(u_n) - f(x)| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (67)$$

Also, we have

$$0 < \mu_n < \frac{3\lambda_n}{2} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (68)$$

Let $x_n^* := \mu_n^{-1} y_n^*$. Then, $\|\mu_n x_n^* - x^*\| \leq \|y_n^* - u_n^*\| + \|u_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$, that is, $\mu_n x_n^* \rightarrow x^*$. For n large enough, (i.e., $\forall n \geq N_0$), we have $\|x_n - \bar{x}\| < \varepsilon_0$ and $\mu_n < 1$. Assume for a moment that $x^* \neq 0$. Let $t_n := \max \{T/2\mu_n, (\|x_n^*\|/2r\|x^*\|)\}$, $\forall n \geq N_0$, we see that

$$2t_n \geq \max \left\{ T, \frac{\|x_n^*\|}{r} \right\}, \left(\text{since } \frac{1}{\mu_n} > 1 \text{ and } \frac{1}{\|x^*\|} > 1 \right) \quad (69)$$

and hence by definition of V -p.l.n. functions with $y := x'$ and $t := 2t_n$, we obtain

$$\langle x_n^*, x' - x_n \rangle \leq f(x') - f(x_n) + t_n V(Jx_n, x'). \quad (70)$$

Multiplying this inequality by μ_n , we get

$$\langle \mu_n x_n^*, x' - x_n \rangle \leq \mu_n [f(x') - f(x_n)] + \mu_n t_n V(Jx_n, x'). \quad (71)$$

Let $\rho_n := \max \{(T/2), (\mu_n \|x_n^*\|/2r\|x^*\|)\}$. Clearly, $\rho_n \rightarrow \rho := \max \{(T/2), (1/2r)\}$ and $t_n \mu_n = \rho_n$. So, for any $n \geq N_0$, we have

$$\langle \mu_n x_n^*, x' - x_n \rangle \leq \mu_n [f(x') - f(x_n)] + \rho_n V(Jx_n, x'). \quad (72)$$

Now, taking the limit as $n \rightarrow \infty$ that yields by continuity of V and J

$$\langle x^*, x' - x \rangle \leq \rho V(Jx, x') \leq \rho V(Jx, x') + \rho [\alpha' - \alpha]^2 \quad (73)$$

and so

$$\langle x^*, x' \rangle \leq \langle x^*, x \rangle + \rho \|x\|^2 + \rho \|x'\|^2 - 2\rho \langle Jx, x' \rangle + \rho [\alpha' - \alpha]^2. \quad (74)$$

Thus,

$$\rho \|x\|^2 - \langle x^* + 2\rho Jx, x \rangle \leq \rho \|x'\|^2 - \langle x^* + 2\rho Jx, x' \rangle + \rho [\alpha' - \alpha]^2. \quad (75)$$

Dividing by $\rho > 0$ gives

$$\|x\|^2 - 2 \left\langle Jx + \frac{1}{2\rho} x^*, x \right\rangle \leq \|x'\|^2 - 2 \left\langle Jx + \frac{1}{2\rho} x^*, x' \right\rangle + [\alpha' - \alpha]^2 \quad (76)$$

and so

$$V \left(Jx + \frac{1}{2\rho} x^*, x \right) \leq V \left(Jx + \frac{1}{2\rho} x^*, x' \right) + [\alpha' - \alpha]^2. \quad (77)$$

This ensures for $\lambda = 0$

$$\begin{aligned} V_{X \times \mathbb{R}} \left(\left(Jx + \frac{x^*}{2\rho}, \left(\alpha + \frac{1}{2\rho}(-\lambda) \right) \right); (x, \alpha) \right) \\ \leq V_{X \times \mathbb{R}} \left(\left(Jx + \frac{x^*}{2\rho}, \left(\alpha + \frac{1}{2\rho}(-\lambda) \right) \right); (x', \alpha') \right), \end{aligned} \quad (78)$$

for any $(x', \alpha') \in \text{epi } f \cap [(\bar{x}, f(\bar{x})) + \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}]$, that is,

$$(x, \alpha) \in \pi_{\text{epi } f \cap [(\bar{x}, f(\bar{x})) + \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}]} \left(Jx + \frac{x^*}{2\rho}, \left(\alpha + \frac{1}{2\rho}(-\lambda) \right) \right). \quad (79)$$

Finally, we obtained from Case 1 and Case 2 two positive numbers ε_0 and $r_0 := 1/2\rho$ such that for any $(x, \alpha) \in \text{epi } f$ and $(x^*, -\lambda) \in N^\pi(\text{epi } f; (x, \alpha)) \cap \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}$ and any $x \in \bar{x} + \varepsilon_0 \mathbb{B}$ and $|f(\bar{x}) - \alpha| < \varepsilon_0$, we have

$$(x, \alpha) \in \pi_{\text{epi } f \cap [(\bar{x}, f(\bar{x})) + \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}]}((Jx, \alpha) + r_0(x^*, -\lambda)). \quad (80)$$

This means by Definition 2 that the epigraph $\text{epi } f$ is generalized V -prox-regular at $(\bar{x}, f(\bar{x}))$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

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Research Article

Multiple Solution Results for Perturbed Fractional Differential Equations with Impulses

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The multiplicity of classical solutions for impulsive fractional differential equations has been studied by many scholars. Using Morse theory, Brezis and Nirenberg's Linking Theorem, and Clark theorem, we aim to solve this kind of problems. By this way, we obtain the existence of at least three classical solutions and k distinct pairs of classical solutions. Finally, an example is presented to illustrate the feasibility of the main results in this paper.

1. Introduction

Consider the multiple solutions of fractional order impulsive systems as follows:

$$\begin{cases} -\frac{1}{2} \frac{d}{dt} ({}_0D_t^{-\beta} + {}_tD_T^{-\beta}) u'(t) = a(t)u(t) + \nabla F(t, u(t)), & t \neq t_j, \text{ a.e. } t \in [0, T], \\ \Delta({}_tD_t^\alpha u)(t_j) = I_j(u(t_j)), & t_j \in [0, T], j = 1, 2, \dots, l, \\ u(0) = u(T) = 0, \end{cases} \quad (1)$$

where $\beta \in [0, 1)$, $\alpha = 1 - \beta/2 \in (1/2, 1]$; ${}_0D_t^{-\beta}$, ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order β , ${}_0D_t^\alpha$, ${}_tD_T^\alpha$ are used to denote the left and right Caputo fractional derivatives of order α , $0 = t_0 < t_1 < \dots < t_l < t_{l+1} = T$, $a \in C[0, T]$, $F: [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a given function, $\nabla F(t, x)$ is the gradient of F at x , there are constants a_0, a_1 with $0 < a_0 \leq a(t) \leq a_1$,

$$({}_tD_t^\alpha u)(t) = \frac{1}{2} \{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t),$$

$$\begin{aligned} \Delta({}_tD_t^\alpha u)(t_j) &= \frac{1}{2} \{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t_j^+) \\ &\quad - \frac{1}{2} \{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t_j^-), \\ &= \lim_{t \rightarrow t_j^+} \{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t), \\ &= \lim_{t \rightarrow t_j^-} \{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t), \end{aligned} \quad (2)$$

for $j = 1, 2, \dots, l$.

The problem (1) arises from the phenomena of advection dispersion and was first scrutinized by Erwin and Roop in [1]. From then on, more and more scholars began to pay attention to the problem in [1] and the related problems.

Fractional calculus is different from integral calculus in nature. It has nonlocal characteristics and is very suitable for describing materials and processes with memory effect and genetic properties. Therefore, fractional differential equations are widely used in many domains, for instance,

biomedicine, economic mathematics, and technology science [2, 3]. In recent years, the variational methods and critical point theory have been widely used to study fractional differential equations [4–8].

In [8], the authors discussed the following fractional order differential systems:

$$\begin{cases} -\frac{1}{2} \frac{d}{dt} \left({}_0 D_t^{-\beta} + {}_t D_T^{-\beta} \right) u'(t) = \nabla F(t, u(t)), & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \quad (3)$$

They used the critical point theory and other tools to verify the existence of solutions. From then on, a number of scholars began to use such methods for research, as shown in [9–11].

In [12], the authors discussed the following problems:

$$\begin{cases} {}_t D_T^\alpha (a(t) {}_0 D_t^\alpha u(x)) = \lambda u(t) \nabla F(t, u(t)), & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \quad (4)$$

They proved that there are at least k pairs of weak solutions and two weak solutions by using the Clark Theorem and other methods.

An impulsive phenomenon is a common phenomenon in nature and engineering applications. The models reflected in mathematics are impulsive differential equations. The most prominent feature of impulsive differential equation is that it can fully consider the impact of instantaneous mutation on the state. Therefore, in recent decades, impulsive differential equation theory has been widely used in biological mathematics, theoretical mechanics, biomedicine, and economic mathematics (see [13–18]).

For the past few years, very few scholars used the variational method and critical point theory to discuss impulsive fractional differential equations and their boundary value problems. Moreover, few papers discuss the fractional order system by using Morse theory (see [19–24]).

In [23], the authors discussed the following problems:

$$\begin{cases} {}_t D_T^\alpha ({}_0^c D_t^\alpha u(t)) + k(t)u(t) = f(t, u(t)), & 0 < t < T, t \neq t_j, \\ \Delta({}_t D_T^{\alpha-1} ({}_0^c D_t^\alpha u)) (t_j) = I_j(u(t_j)), & j = 1, 2, \dots, m, \\ u(0) = u(T) = 0. \end{cases} \quad (5)$$

The multiple solutions of this problem are verified with Morse theory and the Clark theorem by the authors.

In [25], the sufficient conditions for the existence of infinite solutions to the system (1) are obtained by using the variational method.

Based on the above literatures, in the present paper, we will discuss the existence of multiple classical solutions for (1) by using Morse theory, Clark theorem, and Brezis and Nirenberg's Linking Theorem.

First of all, we give some assumptions.

(H1) $I_j \in C([0, T], R)$, $I_j(0) = 0$, there exist some constants $e_j, \gamma_j \in [0, 1)$, $a_j, b_j > 0$, such that $\lim_{|u| \rightarrow 0} (|I_j(u)|/|u|^{e_j}) = b_j$, $|I_j(u)| \leq a_j |u|^{\gamma_j}$, $j = 1, \dots, l$, and $\int_0^u I_j(s) ds \geq 0$, for $\forall u \in R$

(H2) $F \in C([0, T], R)$ and $F(t, 0) = 0$, $\limsup_{|u| \rightarrow \infty} (F(t, u)/|u|^2) < 1/2(I^2(\alpha + 1)|\cos(\pi\alpha)/T^{2\alpha}| - a_1)$ uniformly on $t \in [0, T]$

(H3) $\limsup_{|u| \rightarrow 0} (|F(t, u)|/|u|^2) < I^2(\alpha + 1)/2T^{2\alpha}$ uniformly for $t \in [0, T]$. There exist four constants $C > 0, r, r_0, \gamma \in (1,$

$\max_{j \in \{1, 2, \dots, l\}} \{\gamma_j + 1\})$ such that $F(t, u) \geq C|u|^\gamma$, $r \leq |u| \leq r_0$, a.e. $t \in [0, T]$

(H4) $F(t, -u) = F(t, u)$ and $I_j(-u) = -I_j(u)$ ($j = 1, 2, \dots, l$), for $u \in R$

The key outcomes are as follows.

Theorem 1. *Let (H1)–(H3) hold. Then, the problem (1) has at least three classical solutions.*

Theorem 2. *Let (H1)–(H4) hold. Then, the problem (1) has at least k distinct pairs of classical solutions.*

Note that the methods in this article are distinct from [25] and our results are richer. The problems in this paper we studied are different from the problems in [23]. Compared with [23], classical solutions are investigated in this paper.

The structure of this article is as below. In Section 2, we provide some preliminary knowledge, which are helpful to the proof of the key outcomes. We prove the key outcomes in Section 3. Finally, an example is given to illustrate the main results.

2. Preliminaries

Similar to [25], we first convert system (1) into a new format as follows:

$$\begin{cases} \frac{d}{dt} \left\{ \frac{1}{2} {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u(t)) - \frac{1}{2} {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u(t)) \right\} + a(t)u(t) + \nabla F(t, u(t)) = 0, & t \neq t_j, \text{ a.e. } t \in [0, T], \\ \Delta({}_t D_T^\alpha u)(t_j) = I_j(u(t_j)), & t_j \in (0, T), j = 1, 2, \dots, l, \\ u(0) = u(T) = 0. \end{cases} \quad (6)$$

Remark 3. Because of the equivalence of system (1) and system (6), we know that the solutions of system (6) are the solutions of system (1).

We first build the function spaces as below, the goal of which is to establish the variational framework of system (6).

Let us recall that for any fixed $1 \leq p \leq \infty$ and $t \in [0, T]$, $\|u\|_\infty = \max_{t \in [0, T]} |u(t)|$, $\|u\|_{L^p} = (\int_0^T |u(s)|^p ds)^{1/p}$.

Definition 4 (see [22]). Let $0 < \alpha \leq 1$, we define the fractional derivative space E_0^α by the closure of $C_0^\infty([0, T], R)$ with $u(0) = u(T)$ under the norm

$$\|u\|_\alpha = \left(\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt \right)^{1/2}, \quad \forall u \in E_0^\alpha. \quad (7)$$

Lemma 5 (see [25]). Let $0 < \alpha \leq 1$, and E_0^α is a Banach space with reflexive and separable.

Definition 6 (see [25]). We define that the function $u \in E_0^\alpha$ is a weak solution of the system (6) if the following holds:

$$\begin{aligned} & \int_0^T \left\{ -\frac{1}{2} [{}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha v(t) + {}_t^c D_T^\alpha u(t) {}_0^c D_t^\alpha v(t)] - a(t)u(t)v(t) \right\} dt \\ & + \sum_{j=1}^l I_j(u(t_j))v(t_j) - \int_0^T \nabla F(t, u(t))v(t) dt = 0 \end{aligned} \quad (8)$$

for $\forall v \in E_0^\alpha$.

We define $\Phi : E_0^\alpha \longrightarrow R$ as

$$\begin{aligned} \Phi(u) = & \int_0^T -\frac{1}{2} {}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha u(t) dt - \frac{1}{2} \int_0^T a(t)u^2(t) dt \\ & - \int_0^T F(t, u(t)) dt + \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds. \end{aligned} \quad (9)$$

From (H1), (H2), we know the functional Φ is continuously differentiable. So $\forall u, v \in E_0^\alpha$, we have

$$\begin{aligned} \langle \Phi'(u), v \rangle = & \int_0^T -\frac{1}{2} [{}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha v(t) + {}_t^c D_T^\alpha u(t) {}_0^c D_t^\alpha v(t)] dt \\ & - \int_0^T a(t)u(t)v(t) dt + \sum_{j=1}^l I_j(u(t_j))v(t_j) \\ & - \int_0^T \nabla F(t, u(t))v(t) dt. \end{aligned} \quad (10)$$

Remark 7. Obviously, from (10), we know that the critical points of functional Φ are the weak solutions of system (6).

Definition 8 (see [25]). We define

$$u \in \left\{ u \in AC([0, T]): \int_{t_j}^{t_{j+1}} \left(\int_0^T |{}_0^c D_t^\alpha u(t)|^2 + \int_0^T |u(t)|^2 \right)^{1/2} dt < \infty, j=0, \dots, l \right\}, \quad (11)$$

as a classic solution of system (6) if it satisfies the following conditions:

$$(i) \quad 1/2 \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \} (t_j^+) = \lim_{t \rightarrow t_j^+} 1/2 \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \} (t),$$

$$\begin{aligned} & \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \} (t_j^-) \\ & = \lim_{t \rightarrow t_j^-} \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \} (t), \end{aligned}$$

$$\Delta(D_t^\alpha u)(t_j) = I_j(u(t_j)),$$

$$u(0) = u(T) = 0 \quad (12)$$

$$(ii) \quad u \text{ content system (1) a.e. on } t \in [0, T] \setminus \{t_1, t_2, \dots, t_l\}$$

Lemma 9 (see [25]). The function $u \in E_0^\alpha$ is a classical solution of system (6) when u is a weak solution of system (6).

Remark 10. Combine Remarks 3 and 7 and Lemma 9, we know that the critical point of functional Φ is the classical solution of the system (1). Therefore, we will directly discuss the critical point of Φ as below.

Lemma 11 (see [8]). Let $1/2 < \alpha \leq 1$ and $1 < p < \infty$, for all $u \in E_0^\alpha$, one has

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^c D_t^\alpha u\|_{L^p}. \quad (13)$$

Moreover, if $\alpha > 1/p$, $1/p + 1/q = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0^c D_t^\alpha u\|_{L^p}. \quad (14)$$

In particular, if $p = 2$, then

$$\begin{aligned} \|u\|_{L^2}^2 & \leq \frac{T^{2\alpha}}{\Gamma^2(\alpha+1)} \|u\|_\alpha^2, \\ \|u\|_\infty^2 & \leq \frac{T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \|u\|_\alpha^2. \end{aligned} \quad (15)$$

It is easy to prove that the norm $\|u\|_\alpha = (\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt)^{1/2}$ is equivalent to $\|u\|_\alpha =$

$(\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt)^{1/2}$, $\forall u \in E_0^\alpha$. Next, we will use $\|u\|_\alpha = (\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt)^{1/2}$ as the norm in E_0^α .

Lemma 12 (see [8]). Let $1/2 < \alpha \leq 1$, $\forall u \in E_0^\alpha$, have

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \leq - \int_0^T {}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha u(t) dt \leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2. \quad (16)$$

Lemma 13 (see [8]). Let $1/2 < \alpha \leq 1$. Assume the sequence $\{u_n\}$ converges weakly to u in E_0^α . Then, $u_n \rightarrow u$ strongly in $C([0, T], R)$, i.e., $\|u_n - u\|_\infty \rightarrow 0$, as $n \rightarrow \infty$.

Definition 14 (see [23]). We say that Φ satisfies the (PS) condition in E_0^α , if any $\{u_n\}_{n \in \mathbb{N}} \subset E_0^\alpha$, for which $\{\Phi(u_n)\}_{n \in \mathbb{N}}$ is bounded and $\Phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ owns a strongly convergent subsequence in E_0^α .

Lemma 15 (see [26]). Let E have a direct sum decomposition $E = V \oplus W$, and $k = \dim V < \infty$. Let 0 be a critical point of Φ with $\Phi(0) = 0$, Φ is bounded below and satisfying (PS) condition. Suppose that, for some $\rho > 0$,

$$\begin{aligned} \Phi(u) &\leq 0, \quad \forall u \in V, \|u\| \leq \rho, \\ \Phi(u) &> 0, \quad \forall u \in W, \|u\| \leq \rho. \end{aligned} \quad (17)$$

Also, assume that $\inf_E \Phi < 0$. Then, Φ has at least two nonzero critical points and $C_k(\Phi, 0) \equiv 0$.

Lemma 16 (see [27]). Let E be a real Banach space, $\Phi \in C^1(E, R)$; assume that Φ is even, bounded from below, and satisfying (PS) condition. Assume $\Phi(0) = 0$, there exists a set $E' \subset E$ such that E' is homeomorphic to S^{k-1} by an odd map, and $\sup_{E'} \Phi < 0$. Then, Φ has at least k distinct pairs of critical points.

3. Proofs of Main Results

Lemma 17. Suppose (H1), (H2) hold, if $\{u_n\}$ is a (PS) sequence, then $\{u_n\}$ is bounded.

Proof. If $\{u_n\}$ is a (PS) sequence, that is,

$$\Phi(u_n) \text{ is bounded, } \Phi'(u_n) \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (18)$$

From (H2), for some $\xi > 0$ small enough, there is a constant $C_\xi > 0$, for any $u \in R$, $t \in [0, T]$ such that

$$|F(t, u)| \leq \frac{1}{2} \left(\frac{\Gamma^2(\alpha+1)}{T^{2\alpha}} |\cos(\pi\alpha)| - a_1 - \xi \right) |u|^2 + C_\xi. \quad (19)$$

According to (19), for $u \in E_0^\alpha$, $u \neq 0$, one has

$$\begin{aligned} \Phi(u) &= \int_0^T -\frac{1}{2} {}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha u(t) dt - \frac{1}{2} \int_0^T a(t) u^2(t) dt \\ &\quad - \int_0^T F(t, u(t)) dt + \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 \\ &\quad - \frac{1}{2} \left(\frac{\Gamma^2(\alpha+1)}{T^{2\alpha}} |\cos(\pi\alpha)| - a_1 - \xi \right) \int_0^T |u(t)|^2 dt \\ &\quad - \frac{a_1}{2} \int_0^T u^2(t) dt - C_\xi T = \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 \\ &\quad - \frac{1}{2} \left(\frac{\Gamma^2(\alpha+1)}{T^{2\alpha}} |\cos(\pi\alpha)| - \xi \right) \|u\|_\alpha^2 - C_\xi T \\ &\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \frac{1}{2} \left(|\cos(\pi\alpha)| - \frac{T^{2\alpha}}{\Gamma^2(\alpha+1)} \xi \right) \|u\|_\alpha^2 \\ &\quad - C_\xi T \geq \frac{\xi T^{2\alpha}}{2\Gamma^2(\alpha+1)} \|u\|_\alpha^2 - C_\xi T. \end{aligned} \quad (20)$$

Because $\Phi(u)$ is bounded, by (20), we can get $\{u_n\}$ is bounded in E_0^α and Φ is bounded from below. The proof is completed.

Lemma 18. Assume (H1), (H2) hold, then Φ satisfies the (PS) condition.

Proof. If $\{u_n\}$ is a (PS) sequence, from Lemma 17, we get $\{u_n\}$ is a bounded sequence in E_0^α . By Lemma 5, we get $\{u_n\}$ has a weakly convergent subsequence. Without loss of generality, we also assume that u_n converges weakly to u_0 in E_0^α , then from (9) and (18), we know

$$\begin{aligned} \langle \Phi'(u_n) - \Phi'(u_0), u_n - u_0 \rangle &\leq \|\Phi'(u_n)\| \cdot \|u_n - u_0\| \\ &\rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (21)$$

By Lemma 13, we can obtain that $u_n \rightarrow u_0$ in $C([0, T], R)$, as $n \rightarrow \infty$, i.e.,

$$\|u_n - u_0\|_\infty \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (22)$$

From (10), we have

$$\begin{aligned} &\langle \Phi'(u_n) - \Phi'(u_0), u_n - u_0 \rangle \\ &= \int_0^T -\frac{1}{2} [{}_0^c D_t^\alpha (u_n - u_0) {}_t^c D_T^\alpha (u_n - u_0) + {}_t^c D_T^\alpha (u_n - u_0) {}_0^c D_t^\alpha (u_n - u_0)] \\ &\quad - \int_0^T (a(t)(u_n - u_0))(u_n - u_0) dt \\ &\quad - \int_0^T [\nabla F(t, u_n) - \nabla F(t, u_0)](u_n - u_0) dt \\ &\quad + \sum_{j=1}^l [I_j(u_n(t_j)) - I_j(u_0(t_j))](u_n(t_j) - u_0(t_j)) \\ &\geq \frac{1}{2} |\cos(\pi\alpha)| \|u_n - u_0\|_\alpha^2 - \left| \int_0^T [\nabla F(t, u_n) - \nabla F(t, u_0)] \right| \|u_n - u_0\|_\infty \\ &\quad + \left| \sum_{j=1}^l [I_j(u_n(t_j)) - I_j(u_0(t_j))] \right| \|u_n - u_0\|_\infty. \end{aligned} \quad (23)$$

By (21), (22), and (23), we can infer that $\|u_n - u_0\|_\alpha^2 \rightarrow 0$, as $n \rightarrow \infty$, i.e., u_n strongly converges to u_0 . Therefore, Φ satisfies the (PS) condition.

By Lemma 5, we can obtain that there is an orthogonal basis $\{e_i\}$ of E_0^α such that $E_0^\alpha = \overline{\text{span}\{e_i : i = 1, 2, \dots\}}$. We define $X_i := \text{span}\{e_i\}$, $V_k = \bigoplus_{i=1}^k X_i$, $Y_k = \bigoplus_{i=k}^\infty X_i$ ($k = 1, 2, \dots$). Then, $E_0^\alpha = V_k \oplus Y_k$.

Proof of Theorem 1. From (H1), (H2), one knows $F(t, 0) = 0$ and $I_j(0) = 0$, $j = 1, \dots, l$. We find out Φ has a critical point at 0. Therefore, we can get the linking $E_0^\alpha = V_k \oplus Y_k$ of Φ at 0.

According to the equivalence of norm of normed space in finite dimension, there exist positive constants M_1, M_2, M'_1, M'_2 such that

$$\begin{aligned} M_1 \|u\|_\alpha &\leq \|u\|_\infty \leq M_2 \|u\|_\alpha, \\ M'_1 \|u\|_\alpha &\leq \|u\|_{L^p} \leq M'_2 \|u\|_\alpha, \end{aligned} \quad (24)$$

$$u \in V_k.$$

First, let $u \in V_k$. Because V_k is finite dimensional, there exists $0 < \rho_1 < 1$ small for $r_0 > 0$, such that

$$|u(t)| \leq \|u\|_\infty \leq M_2 \|u\|_\alpha < M_2 \rho_1 < r_0, \quad u \in V_k, \|u\|_\alpha < \rho_1. \quad (25)$$

For any $r \in (0, r_0)$, we set $\Omega_1 = \{t \in [0, T] : |u| \leq r\}$, $\Omega_2 = \{t \in [0, T] : r \leq |u| \leq r_0\}$, $\Omega_3 = \{t \in [0, T] : r_0 \leq |u|\}$, where $[0, T] = \bigcup_{i=1}^3 \Omega_i$ and Ω_i ($i = 1, 2, 3$) are pairwise disjoint.

Set $F^*(t, u) = F(t, u) - C|u|^\gamma$, for $\|u\|_\alpha < \rho_1$, $u \in V_k$, combine (H1), (H3) and Lemma 12, we have

$$\begin{aligned} \Phi(u) &= \int_0^T -\frac{1}{2} {}^c D_t^\alpha u(t) {}^c D_T^\alpha u(t) dt - \frac{1}{2} \int_0^T a(t) u^2(t) dt \\ &\quad - \int_0^T F(t, u(t)) dt + \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \leq \frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^2 \\ &\quad + \sum_{j=1}^l \frac{a_j}{\gamma_j + 1} |u(t_j)|^{\gamma_j+1} - \int_0^T C|u|^\gamma dt - \int_{\Omega_1} F^*(t, u(t)) dt \\ &\quad - \int_{\Omega_2} F^*(t, u(t)) dt - \int_{\Omega_3} F^*(t, u(t)) dt \leq \frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^2 \\ &\quad + \sum_{j=1}^l \frac{a_j A_0^{\gamma_j+1}}{\gamma_j + 1} \|u\|_\alpha^{\gamma_j+1} - C\|u\|_{L^p}^\gamma - \int_{\Omega_1} F^*(t, u(t)) dt \\ &\quad - \int_{\Omega_2} F^*(t, u(t)) dt - \int_{\Omega_3} F^*(t, u(t)) dt, \end{aligned} \quad (26)$$

where $A_0 = T^{\alpha-1/2}/\Gamma(\alpha)\sqrt{2\alpha-1}$.

According to (25) and the definition of Ω_3 is empty set, we have $\int_{\Omega_3} F^*(t, u(t)) dt = 0$, for any $u \in V_k$. By (H3), one has $\int_{\Omega_2} F^*(t, u(t)) dt \geq C|u|^\gamma - C|u|^\gamma = 0$. On Ω_1 , $|u| < r$. From (H3), we can get $\int_{\Omega_1} F^*(t, u(t)) dt \rightarrow 0$, as $r \rightarrow 0$.

Then, $\forall u \in V_k, r \in (0, r_0)$, $\|u\|_\alpha \leq \rho \leq 1$, $1 < \gamma < \max\{\gamma_j + 1\} < 2$, according to (26), we can get

$$\begin{aligned} \Phi(u) &\leq \frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^2 + \sum_{j=1}^l \frac{a_j A_0^{\gamma_j+1}}{\gamma_j + 1} \|u\|_\alpha^{\gamma_j+1} - M_1^\gamma C \|u\|_\alpha^\gamma \\ &\leq \|u\|_\alpha^\gamma \left(\frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^{2-\gamma} + \sum_{j=1}^l \frac{a_j A_0^{\gamma_j+1}}{\gamma_j + 1} \|u\|_\alpha^{\gamma_j+1-\gamma} - M_1^\gamma C \right) \leq 0. \end{aligned} \quad (27)$$

Hence,

$$\Phi(u) \leq 0, \quad \forall u \in V_k, \|u\|_\alpha \leq \rho_1. \quad (28)$$

Next, set $u \in Y_k$. Because $E_0^\alpha \rightarrow C_0^\infty([0, T, R])$ is continuous compact embedding. Hence, for $u \in Y_k, \varepsilon > 0$, there exists $0 < \rho_2 < 1$ small such that $|u| \leq \|u\|_\infty \leq T^{2\alpha-1} \|u\|_\alpha / \Gamma^2(\alpha) (2\alpha-1) < T^{2\alpha-1} \rho_2 / \Gamma^2(\alpha) (2\alpha-1) < \varepsilon$, for $\|u\|_\alpha < \rho_2$.

From (H3), $\forall |u| < \varepsilon, u \in Y_k, \|u\|_\alpha \leq \rho_2, t \in [0, T]$, there is $\xi \in (0, |\cos(\pi\alpha)|)$, one has

$$|F(t, u)| \leq (|\cos(\pi\alpha)| - \xi) \frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}} |u|^2. \quad (29)$$

From (H1), $\forall |u| < \varepsilon, u \in Y_k, \|u\|_\alpha \leq \rho_2$, one has

$$\frac{|I_j(u)|}{|u|^{e_j}} > \frac{1}{2} b_j. \quad (30)$$

Let $b = \min_{j=1}^l b_j, e = \max_{j=1}^l e_j, \forall |u| < \varepsilon, u \in Y_k, \|u\|_\alpha \leq \rho_2 < 1$, by Lemmas 11 and 12 and (29) and (30), we obtain

$$\begin{aligned} \Phi(u) &= \int_0^T -\frac{1}{2} {}^c D_t^\alpha u(t) {}^c D_T^\alpha u(t) dt - \frac{1}{2} \int_0^T a(t) u^2(t) dt \\ &\quad - \int_0^T F(t, u(t)) dt + \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 \\ &\quad - (|\cos(\pi\alpha)| - \xi) \frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}} \|u\|_{L^2}^2 - \frac{1}{2} a_1 T |u|^2 + \frac{1}{2} \frac{bl}{e+1} |u|^{e+1} \\ &\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \frac{1}{2} (|\cos(\pi\alpha)| - \xi) \|u\|_\alpha^2 - \frac{1}{2} a_1 T |u|^2 \\ &\quad + \frac{1}{2} \frac{bl}{e+1} |u|^{e+1} \geq \frac{1}{2} \left(\xi \|u\|_\alpha^2 + |u|^{e+1} \left(\frac{bl}{e+1} - a_1 T |u|^{1-e} \right) \right) > 0. \end{aligned} \quad (31)$$

Hence,

$$\Phi(u) > 0, \forall u \in Y_k, \|u\|_\alpha \leq \rho_2. \quad (32)$$

Let $\rho = \min\{\rho_1, \rho_2\}$, from (28) and (32), we obtain

$$\begin{aligned} \Phi(u) &\leq 0, \quad \forall u \in V_k, \|u\|_\alpha \leq \rho, \\ \Phi(u) &> 0, \quad \forall u \in Y_k, \|u\|_\alpha \leq \rho. \end{aligned} \quad (33)$$

It follows from Lemmas 17 and 18 that Φ is bounded from below and satisfies the (PS) condition. Then, from Lemma 15,

we can get Φ has at least two nonzero critical points, and $C_k(\Phi, 0) \cong 0$, so $u = 0$ is a homological nontrivial point of Φ . Hence, the system (1) has at least three classical solutions.

Proof of Theorem 2. According to (H4), we can deduce that Φ is even. By Lemmas 17 and 18, we know Φ is bounded from below and satisfies the (PS) condition. For given $\rho > 0$, set $E' = S_\rho = \{u \in V_k : \|u\| = \rho\}$. By (27), if ρ is small enough, one has $\sup_{E'} \Phi(u) < 0$. Clearly, $\dim V_k = k$. Then, we can conclude that Φ has at least k distinct pairs of critical points from Lemma 16. Hence, the system (1) has at least k distinct pairs of classical solutions. We complete the proof.

Example 19.

$$\begin{cases} -\frac{1}{2} \frac{d}{dt} ({}_0 D_t^{-0.6} + {}_t D_T^{0.6}) u'(t) = \frac{2}{5} u(t) + \nabla F(t, u(t)), & t \neq t_j, \text{ a.e. } t \in [0, 1], \\ \Delta({}_0 D_t^{0.7} u)(t_j) = I_j(u(t_j)), & t_j \in [0, 1], j = 1, 2, \dots, l, \\ u(0) = u(T) = 0. \end{cases} \quad (34)$$

According to (34), we can see that $\beta = 0.6$, $\alpha = 0.7$, $a(t) = 2/5$, $T = 1$.

Let $I_j(u) = (4/3)u^{1/3}$, then the condition (H1) holds with $a_j = 3/2$, $\gamma_j = 1/3$, $b_j = 4/3$, $e_j = 1/3$.

Let $F(t, u(t)) = 1/15(1 + \sin^2 t)|u|^2(1/\ln(|u|^{2/3} + 1.5))$, $a_1 = 9/20$. By simple calculations, we can get $\Gamma^2(\alpha + 1)/T^{2\alpha} \approx 0.84474$, $|\cos(0.7\pi)| \approx 0.58778$,

$$\lim_{|u| \rightarrow \infty} \sup \frac{F(t, u)}{|u|^2} \rightarrow 0 < \frac{1}{2} \left(\frac{\Gamma^2(\alpha + 1)}{T^{2\alpha}} |\cos(\pi\alpha)| - a_1 \right) \approx 0.02326; \quad (35)$$

then, the condition (H2) holds.

By (H3), we know

$$\lim_{|u| \rightarrow 0} \sup \frac{|F(t, u)|}{|u|^2} \leq 0.32884 < \frac{\Gamma^2(\alpha + 1)}{2T^{2\alpha}} \approx 0.42237. \quad (36)$$

Let $r = e - 1.7$, $r_0 = e - 1.5$, $C = 2/25$, $\gamma = 6/5$; then, for $e - 1.7 \leq |u| \leq e - 1.5$, we have

$$F(t, u) > 0.14471|u|^2 \geq \frac{2}{25}|u|^{6/5}; \quad (37)$$

then, the condition (H3) is satisfied.

It easy to see that the condition (H4) holds.

According to Theorem 1, the system (1) exists at least three classical solutions. According to Theorem 2, the system (1) possesses at least k distinct pairs of classical solutions.

4. Conclusions

In this work, we study perturbed fractional differential equation with impulses. We give sufficient conditions of the

existence of at least three classical solutions and at least k distinct pairs of classical solutions for problems (1), where k is the dimension of V_k .

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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Research Article

Application of Quasisubordination to Certain Classes of Meromorphic Functions

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Inequalities play a fundamental role in many branches of mathematics and particularly in real analysis. By using inequalities, we can find extrema, point of inflection, and monotonic behavior of real functions. Subordination and quasisubordination are important tools used in complex analysis as an alternate of inequalities. In this article, we introduce and systematically study certain new classes of meromorphic functions using quasisubordination and Bessel function. We explore various inequalities related with the famous Fekete-Szegő inequality. We also point out a number of important corollaries.

1. Introduction

A complex valued function is said to be meromorphic if it has poles as its only singularities. Let Σ_1 denotes the class of all of meromorphic functions which has a simple pole at $\omega = 0$ and has Laurent series expansion of the form:

$$\lambda(\omega) = \frac{1}{\omega} + \sum_{t=0}^{\infty} a_t \omega^t, \quad (1)$$

which are analytic in the punctured open unit disc $U^* = \{\omega : \omega \in \mathbb{C} \text{ and } 0 < |\omega| < 1\} = U - \{0\}$, as open unit disc $U = U^* \cup \{0\}$.

Here, we are listing some important subclasses of meromorphic functions which will be used in our subequal work. In 1936, Robertson [1] introduced the classes of meromorphic starlike and meromorphic convex functions of order α . By $\Sigma^{MS}(\alpha)$, we mean the subclass of Σ_1 consisting of all meromorphic starlike functions of order α . Analytically,

$$\lambda(\omega) \in \Sigma^{MS}(\alpha) \Leftrightarrow \operatorname{Re} \left(\frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right) < -\alpha, \quad (0 \leq \alpha < 1; \omega \in U^*). \quad (2)$$

A closely related class of meromorphic convex functions of order α is denoted by $\Sigma^{MC}(\alpha)$ and defined as

$$\lambda(\omega) \in \Sigma^{MC}(\alpha) \Leftrightarrow -\omega \lambda'(\omega) \in \Sigma^{MS}(\alpha). \quad (3)$$

In 1952, W. Kaplan [2] introduced and studied an important class of analytic functions known as close-to-convex functions in the open unit disc U . A function λ belongs to Σ_1 , is in class $\Sigma^{MC}(\alpha, \beta)$, of meromorphic close-to-convex functions of order α and type β if there exist $\delta(\omega) \in \Sigma^{MS}(\beta)$, and

$$\operatorname{Re} \left(\frac{\omega \lambda'(\omega)}{\delta(\omega)} \right) < -\alpha. \quad (4)$$

Let $\delta(\omega) \in \Sigma_1$ and having series representation of the form

$$\delta(\omega) = \frac{1}{\omega} + \sum_{t=0}^{\infty} b_t \omega^t. \quad (5)$$

Then, the convolution of λ and δ as denoted by $\lambda * \delta$ is defined as

$$(\lambda * \delta)(\omega) = \frac{1}{\omega} + \sum_{t=0}^{\infty} a_t b_t \omega^t = (\delta * \lambda)(\omega), \quad (6)$$

where λ is given by (1).

A function λ is subordinate to δ in U^* and written as $\lambda(\omega) \prec \delta(\omega)$, if there exists a Schwarz function $k(\omega)$, which is holomorphic in U with $k(0) = 0$, such that $\lambda(\omega) = \delta(k(\omega))$. Let $\phi(\omega)$ be an analytic function with positive real part on U satisfies $\phi(0) = 1$ and $\phi'(0) > 0$ which maps U which is star shape with respect to $\omega = 1$, also symmetric with respect to the real axis. We denote $\Sigma(\phi)$ be the class of function $\lambda \in \Sigma_1$ for which $-\omega \lambda'(\omega)/\lambda(\omega) \prec \phi(\omega)$, ($\omega \in U^*$). The class $\Sigma(\phi)$ was introduced and studied by Silverman et al. [3] (see also [4]). The class $\Sigma(\alpha)$ is a special case of the class $\Sigma(\phi)$ when $\phi(\omega) = 1 + (1 - 2\alpha)\omega/1 - \omega$ ($0 \leq \alpha < 1$).

Robertson [5] gave the idea of quasisubordination. For any two functions $\lambda(\omega)$ and $\delta(\omega)$, holomorphic in U , the function $\lambda(\omega)$ is said to be quasi-subordinate to the function $\delta(\omega)$ written as $\lambda(\omega) \prec_q \delta(\omega)$, if there exists two holomorphic functions $\lambda(\omega)$ and $\varphi(\omega)$, with $|\varphi(\omega)| \leq 1$, $\lambda(\omega)/\varphi(\omega)$ is holomorphic in U^* and such that $\lambda(\omega) = \varphi(\omega)\delta(k(\omega))$. In particular, if $\varphi(\omega) = 1$, then quasisubordination reduces to subordination. Furthermore, if $k(\omega) = 1$, the quasisubordination becomes the majorization, (see [6]), which implies

$$\lambda(\omega) \prec_q \delta(\omega) \Rightarrow \lambda(\omega) = \varphi(\omega)\delta(\omega) \Rightarrow \lambda(\omega) \prec \delta(\omega) (\omega \in U^*). \quad (7)$$

For recent work on meromorphic functions, we refer [7–18].

Motivated from the above cited work, we introduce the following subclasses of meromorphic functions. Throughout in this paper, we shall assume $0 \leq \gamma < 1$, $\gamma \neq 1/2$, $0 \leq \eta < 1$, $\omega \in U^*$, $\lambda, \delta \in \Sigma_1$, and $\phi(\omega)$ be an analytic function with positive real part on U that satisfies $\phi(0) = 1$ and $\phi'(0) > 0$ which maps U which is star shape with respect to $\omega = 1$, and also symmetric with respect to the real axis unless otherwise mentioned.

Definition 1. Let $\Sigma_q^{MC}(\phi, \gamma)$ be the class of functions $\lambda(\omega) \in \Sigma_1$ and satisfy

$$-\left[(1-\gamma)\left(\frac{\omega \lambda'(\omega)}{\lambda(\omega)}\right) + \gamma\left(1 + \frac{\omega \lambda''(\omega)}{\lambda'(\omega)}\right)\right] - 1 \prec_q \phi(\omega) - 1, \quad (\omega \in U^*). \quad (8)$$

The abovementioned class $\Sigma_q^{MC}(\phi, \gamma)$ is the meromorphic analogue of the class $S_q(\phi)$ introduced and studied by

Mohd and Darus [19]. For $\gamma = 0$, the class $\Sigma_q^{MS}(\phi)$ was studied by Zayed et al. [20].

Definition 2. Let $\Sigma_q^{MK}(\phi, \eta)$ be the subclass of Σ_1 consisting of all functions $\lambda(\omega)$ for which there exist $\delta(\omega) \in \Sigma^{MS}(\alpha)$ and satisfy

$$\frac{-\omega \lambda'(\omega)}{(1-\eta)\delta(\omega) + \eta \omega \delta'(\omega)} - 1 \prec_q \phi(\omega) - 1, \quad (\omega \in U^*). \quad (9)$$

For $\eta = 0$ and $\delta(\omega) = \lambda(\omega)$, the class $\Sigma_q^{MS}(\phi)$ was studied by Zayed et al. [20].

In this paper, we obtain the Fekete-Szegő inequality for meromorphic functions belonging to above defined classes. Let $k \in \Omega$ denote the class of functions of the form $k(\omega) = k_1\omega + k_2\omega^2 + k_3\omega^3 + \dots$, satisfying $|k(\omega)| < 1$, for $\omega \in U^*$. For more details, see [21–23]. To prove our main results, we need the following lemma.

Lemma 3 [24]. *If $k \in \Omega$, then for any complex number u , $|k_2 - uk_1^2| \leq \max\{1; u\}$, the result is sharp for the functions given by $k(\omega) = \omega$ or $k(\omega) = \omega^2$.*

2. Main Results

In this section, we explore certain Fekete-Szegő-related inequalities for the class $\Sigma_q^{MC}(\phi, \gamma)$ and $\Sigma_q^{MK}(\phi, \eta)$.

Theorem 4. *Let $\phi(\omega) = 1 + B_1\omega + B_2\omega^2 + \dots$, $B_1 > 0$, and $\varphi(\omega) = c_0 + c_1\omega + c_2\omega^2 + \dots$ if $\lambda(\omega)$ given by (1) be in the class $\Sigma_q^{MC}(\phi, \gamma)$, and μ is a complex number, then*

$$|a_1 - \mu a_0^2| \leq \frac{B_1}{2(1-2\gamma)} \left[1 + \max \left\{ 1, \left| \frac{B_2}{B_1} \right| + B_1 \left| \frac{1}{1-\gamma} - 2\mu \frac{(1-2\gamma)}{(1-\gamma)^2} \right| \right\} \right]. \quad (10)$$

The inequalities are sharp for $k(\omega) = \omega$ or $k(\omega) = \omega^2$.

Proof. Let $\lambda(\omega) \in \Sigma_q^{MC}(\phi, \gamma)$, then there exist analytic functions $\varphi(\omega)$ and $k(\omega)$, with $|\varphi(\omega)| < 1$, $k(0) = 0$, and $k(\omega) < 1$ such that

$$-\left[(1-\gamma)\left(\frac{\omega \lambda'(\omega)}{\lambda(\omega)}\right) + \gamma\left(1 + \frac{\omega \lambda''(\omega)}{\lambda'(\omega)}\right)\right] - 1 = \varphi(\omega)[\phi(k(\omega)) - 1], \quad (\omega \in U^*). \quad (11)$$

Taking first and second derivative of (1), and use in the left hand side of above equation, we obtain

$$\begin{aligned}
& - \left[(1-\gamma) \left(\frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right) + \gamma \left(1 + \frac{\omega \lambda''(\omega)}{\lambda'(\omega)} \right) \right] \\
& = (1-\gamma) [1 - a_0 \omega + (a_0^2 - 2a_1) \omega^2] + \gamma (1 + 2a_1 \omega^2) + \dots,
\end{aligned} \tag{12}$$

then implies

$$\begin{aligned}
& - \left[(1-\gamma) \left(\frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right) + \gamma \left(1 + \frac{\omega \lambda''(\omega)}{\lambda'(\omega)} \right) \right] - 1 \\
& = -a_0(\omega)(1-\gamma) + [(a_0^2 - 2a_1 + \gamma(-a_0^2 + 4a_1)) \omega^2 + \dots,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\phi(k(\omega)) & = \phi(k_1 \omega + k_2 \omega^2 + k_3 \omega^3 + \dots) \\
& = 1 + k_1 B_1 \omega + (k_1^2 B_2 + k_2 B_1) \omega^2 \\
& \quad + (k_3 B_1 + 2k_1 k_2 B_2 + k_1^3 B_3) \omega^3 + \dots,
\end{aligned} \tag{14}$$

then

$$\begin{aligned}
\varphi(\omega)[\phi(k(\omega)) - 1] & = [c_0 + c_1 \omega + c_2 \omega^2 + \dots] \\
& \quad \cdot [1 + k_1 B_1 \omega + (k_1^2 B_2 + k_2 B_1) \omega^2 \\
& \quad + (k_3 B_1 + 2k_1 k_2 B_2 + k_1^3 B_3) \omega^3 + \dots],
\end{aligned} \tag{15}$$

which implies

$$\varphi(\omega)[\phi(k(\omega)) - 1] = c_0 k_1 B_1 \omega + (c_0 k_1^2 B_2 + c_0 k_2 B_1 + c_1 k_1 B_1) \omega^2 + \dots. \tag{16}$$

Comparing (13) and (16), we get

$$\begin{aligned}
a_0 & = -\frac{c_0 k_1 B_1}{1-\gamma}, \\
a_1 & = -\frac{B_1 c_0}{2(1-2\gamma)} \left[k_2 + k_1 \frac{c_1}{c_0} + k_1^2 \left(\frac{B_2}{B_1} - \frac{B_1 c_0}{(1-\gamma)} \right) \right].
\end{aligned} \tag{17}$$

Thus,

$$\begin{aligned}
a_1 - \mu a_0^2 & = -\frac{B_1 c_0}{2(1-2\gamma)} \left[k_2 + k_1 \frac{c_1}{c_0} + k_1^2 \left(\frac{B_2}{B_1} - \frac{B_1 c_0}{(1-\gamma)} + 2\mu c_0 B_1 \frac{(1-2\gamma)}{(1-\gamma)^2} \right) \right], \\
|a_1 - \mu a_0^2| & \leq \left| -\frac{B_1 c_0}{2(1-2\gamma)} \right| \\
& \quad \cdot \left[\left| k_1 \frac{c_1}{c_0} \right| + \left| k_2 + k_1^2 \left(\frac{B_2}{B_1} - \frac{B_1 c_0}{(1-\gamma)} + 2\mu c_0 B_1 \frac{(1-2\gamma)}{(1-\gamma)^2} \right) \right| \right].
\end{aligned} \tag{18}$$

Since $\varphi(\omega)$ is analytic and bounded in U^* (see [25]), so we have

$$|c_n| \leq 1 - |c_0|^2 \leq 1, \quad (n > 0). \tag{19}$$

By using this fact and the well-known inequality $|k_1| \leq 1$, we get

$$\begin{aligned}
|a_1 - \mu a_0^2| & \leq \frac{B_1}{2(1-2\gamma)} \\
& \quad \cdot \left[1 + \left| k_2 + k_1^2 \left(\frac{B_2}{B_1} - \frac{B_1 c_0}{(1-\gamma)} + 2\mu c_0 B_1 \frac{(1-2\gamma)}{(1-\gamma)^2} \right) \right| \right].
\end{aligned} \tag{20}$$

Corollary 5. For $\varphi(\omega) = 1$ and $\gamma = 0$ in Theorem 4, we obtain the result by Silverman et al. [3] (see Theorem 4).

For $k(\omega) = \omega$ and repeating steps of Theorem 4, we obtain the following corollary.

Corollary 6. Let $\lambda(\omega) \in \Sigma_I$ satisfies

$$- \left[(1-\gamma) \left(\frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right) + \gamma \left(1 + \frac{\omega \lambda''(\omega)}{\lambda'(\omega)} \right) \right] - 1 < \phi(\omega) - 1, \quad (\omega \in U^*), \tag{21}$$

then for any complex number μ ,

$$|a_1 - \mu a_0^2| \leq \frac{B_1}{2(1-2\gamma)} \left[1 + \left| \frac{B_2}{B_1} \right| + B_1 \left| \frac{1}{1-\gamma} - 2\mu \frac{(1-2\gamma)}{(1-\gamma)^2} \right| \right]. \tag{22}$$

Theorem 7. Let $\omega = 1 + B_1 \omega + B_2 \omega^2 + \dots, B_1 > 0$, and $\varphi(\omega) = c_0 + c_1 \omega + c_2 \omega^2 + \dots$, if $\lambda(\omega)$, $\delta(\omega)$, given by (1) and (5) be in the class $\Sigma_q^{MK}(\phi, \eta)$, and μ is a complex number, then

$$\begin{aligned}
& \left| \frac{b_1 + a_1}{2} - a_1 \eta - \mu b_0^2 \right| \\
& \leq \frac{B_1}{2} \left[1 + \max \left\{ 1, \left| \frac{B_2}{B_1} \right| + B_1 \left| \frac{1}{(-1-\eta)^2} - \frac{2\mu}{(1-\eta)^2(-1-4\eta)^2} \right| \right\} \right].
\end{aligned} \tag{23}$$

The inequalities is sharp for $k(\omega) = \omega$ or $k(\omega) = \omega^2$.

Proof. Let $\lambda(\omega)$, $\delta(\omega) \in \Sigma_q^{MK}(\phi, \eta)$, then there exist analytic functions $\varphi(\omega)$ and $k(\omega)$, with $|\varphi(\omega)| < 1$, $k(0) = 0$, and $k(\omega) < 1$ such that

$$\frac{-\omega \lambda'(\omega)}{(1-\eta)\delta(\omega) + \eta \omega \delta'(\omega)} - 1 = \varphi(\omega)[\phi(k(\omega)) - 1], \quad (\omega \in U^*). \tag{24}$$

Taking first derivative of (1) and (5), and use in the left hand side of above equation, we obtain

$$\frac{-\omega\lambda'(\omega)}{(1-\eta)\delta(\omega)+\eta\omega\delta'(\omega)} = 1 + 2\eta(1+2\eta) + b_0(1-\eta)(-1-4\eta)\omega + [b_0^2(1-\eta)^2 - b_1 + 2a_1\eta - a_1]\omega^2 + \dots, \quad (25)$$

then implies

$$\frac{-\omega\lambda'(\omega)}{(1-\eta)\delta(\omega)+\eta\omega\delta'(\omega)} - 1 = 2\eta(1+2\eta) + b_0(1-\eta)(-1-4\eta)\omega + [b_0^2(1-\eta)^2 - b_1 + 2a_1\eta - a_1]\omega^2 + \dots, \quad (26)$$

$$[\phi(k(\omega))] = 1 + k_1B_1\omega + (k_1^2B_2 + k_2B_1)\omega^2 + (k_3B_1 + 2k_1k_2B_2 + k_1^3B_3)\omega^3 + \dots, \quad (27)$$

which implies

$$\varphi(\omega)[\phi(k(\omega)) - 1] = c_0k_1B_1\omega + (c_0k_1^2B_2 + c_0k_2B_1 + c_1k_1B_1)\omega^2 + \dots \quad (28)$$

Comparing (26) and (28), we get

$$b_0 = \frac{c_0k_1B_1}{(1-\eta)(-1-4\eta)},$$

$$\frac{b_1 + a_1}{2} - a_1\eta = -\frac{B_1c_0}{2} \left[k_2 + k_1 \frac{c_1}{c_0} + k_1^2 \left(\frac{B_2}{B_1} - \frac{B_1c_0}{(-1-4\eta)^2} \right) \right]. \quad (29)$$

Thus,

$$\frac{b_1 + a_1}{2} - a_1\eta - \mu b_0^2 = -\frac{B_1c_0}{2} \cdot \left[k_2 + k_1 \frac{c_1}{c_0} + k_1^2 \left(\frac{B_2}{B_1} - \frac{B_1c_0}{(-1-4\eta)^2} + 2\mu \frac{c_0B_1}{(1-\eta)^2(-1-4\eta)^2} \right) \right]. \quad (30)$$

Since $\varphi(\omega)$ is analytic and bounded in U^* (see [25]), we have

$$|c_n| \leq 1 - |c_0|^2 \leq 1, \quad (n > 0). \quad (31)$$

By using this fact and the well-known inequality $|k_1| \leq 1$, we get

$$\left| \frac{b_1 + a_1}{2} - a_1\eta - \mu b_0^2 \right| \leq \frac{B_1}{2} \left[1 + \left| k_2 + k_1^2 \left(\frac{B_2}{B_1} - \frac{B_1c_0}{(-1-4\eta)^2} + 2\mu \frac{c_0B_1}{(1-\eta)^2(-1-4\eta)^2} \right) \right| \right]. \quad (32)$$

Corollary 8. For $\varphi(\omega) = 1$, $\delta(\omega) = \lambda(\omega)$, and $\eta = 0$ in Theorem 7, we obtain the result by Silverman et al. [3] (see Theorem 7). For $k(\omega) = \omega$ and repeating steps of Theorem 7, we obtain the following corollary.

Corollary 9. Let $\lambda(\omega)$ and $\delta(\omega) \in \Sigma_I$ satisfy

$$\frac{-\omega\lambda'(\omega)}{(1-\eta)\delta(\omega)+\eta\omega\delta'(\omega)} - 1 < \phi(\omega) - 1, \quad (\omega \in U^*), \quad (33)$$

then for any complex number μ ,

$$\left| \frac{b_1 + a_1}{2} - a_1\eta - \mu b_0^2 \right| \leq \frac{B_1}{2} \left[1 + \left| \frac{B_2}{B_1} \right| + B_1 \left| \frac{1}{(-1-4\eta)^2} - \frac{2\mu}{(1-\eta)^2(-1-4\eta)^2} \right| \right]. \quad (34)$$

3. Meromorphic Functions Related with the Bessel Function

Let us consider the second order linear homogenous differential equation (see, Baricz [26])

$$\omega^2 k''(\omega) + \alpha \omega k'(\omega) + [\beta \omega^2 - \nu^2 + (1-\alpha)] k(\omega) = 0. \quad (35)$$

The function

$$k_{\nu, \alpha, \beta}(\omega) = \sum_{t=0}^{\infty} \frac{(-\beta)^t}{\Gamma(t+1)\Gamma(t+\nu+1+(\alpha+1/2))} \left(\frac{\omega}{2} \right)^{2t+\nu} \quad (36)$$

is known as generalized Bessel's function of first kind and is the solution of differential equation given in (35). If we denote

$$\zeta_{\nu, \alpha, \beta} \lambda(\omega) = \frac{2^\nu \Gamma(\nu + (\alpha + 1/2))}{\omega^{(\nu/2)+1}} k_{\nu, \alpha, \beta}(\omega^{1/2}),$$

$$= \frac{1}{\omega} + \sum_{t=0}^{\infty} \frac{(-\beta)^{t+1} \Gamma(\nu + (\alpha + 1/2))}{4^{t+1} \Gamma(t+2) \Gamma(t+\nu+1+(\alpha+1/2))} (\omega^t), \quad (37)$$

where ν , α , and β are positive real numbers. The operator $\zeta_{\nu, \alpha, \beta}$ is a meromorphic analogue introduced by Deniz [27] (see also Baricz et al. [28]) for analytic functions. In terms of convolution, $\zeta_{\nu, \alpha, \beta}$ is given by

$$(\zeta_{\nu, \alpha, \beta} \lambda)(\omega) = k_{\nu, \alpha, \beta}(\omega) * \lambda(\omega)$$

$$= \frac{1}{\omega} + \sum_{t=0}^{\infty} \frac{(-\beta)^{t+1} \Gamma(\nu + (\alpha + 1/2))}{4^{t+1} \Gamma(t+2) \Gamma(t+\nu+1+(\alpha+1/2))} a_t(\omega)^t. \quad (38)$$

The operator $\zeta_{\nu, \alpha, \beta}$ was introduced and studied by Mostafa et al. [29]. For more details, see [30, 31] and references cited therein. Motivated from the above cited work, we introduce the following classes of meromorphic functions.

Definition 10. A function $\lambda(\omega) \in \Sigma_1$ given by (1) is said to belong to the class $\Sigma_{\nu, \alpha, \beta}^{MC(q)}(\phi, \gamma)$ if

$$-\left[(1-\gamma) \left(\frac{\omega(\zeta_{\nu, \alpha, \beta} \lambda)'(\omega)}{(\zeta_{\nu, \alpha, \beta} \lambda)(\omega)} \right) + \gamma \left(1 + \frac{\omega(\zeta_{\nu, \alpha, \beta} \lambda)''(\omega)}{(\zeta_{\nu, \alpha, \beta} \lambda)'(\omega)} \right) \right] - 1 <_q \phi(\omega) - 1. \quad (39)$$

For $\gamma = 0$, the class $\sum_{\nu, \alpha, \beta}^{MS(q)}(\phi)$ was studied by Zayed et al. [20].

Definition 11. Let $\sum_{\nu, \alpha, \beta}^{MK(q)}(\phi, \eta)$ be the subclasses of \sum_1 consisting of all functions $\lambda(\omega) \in \sum_1$ for which there exist $\delta(\omega) \in \sum^{MS}(\alpha)$, and

$$\frac{-\omega(\zeta_{\nu, \alpha, \beta} \lambda)'(\omega)}{(1-\eta)(\zeta_{\nu, \alpha, \beta} \delta)(\omega) + \eta \omega(\zeta_{\nu, \alpha, \beta} \delta)'(\omega)} - 1 <_q \phi(\omega) - 1 \quad (\omega \in U^*). \quad (40)$$

For $\eta = 0$ and $\delta(\omega) = \lambda(\omega)$, the class $\sum_{\nu, \alpha, \beta}^{MS(q)}(\phi)$ was studied by Zayed et al. [20].

Theorem 12. Let $\omega = 1 + B_1\omega + B_2\omega^2 + \dots, B_1 > 0$, and $\varphi(\omega) = c_0 + c_1\omega + c_2\omega^2 + \dots$, if $\lambda(\omega)$ given by (1) be in the class $\sum_{\nu, \alpha, \beta}^{MC(q)}(\phi, \gamma)$, and μ is a complex number, then

$$|a_1 - \mu a_0^2| \leq \frac{4^2(\nu + (\alpha + 1/2))(\nu + 1 + (\alpha + 1/2))B_1}{\beta^2(1-2\gamma)} \times \left[1 + \max \left\{ 1, \left| \frac{B_2}{B_1} \right| + B_1 \left| 1 - \mu \left(\frac{(\nu + (\alpha + 1/2))(1-2\gamma)}{(\nu + 1 + (\alpha + 1/2))(1-\gamma)} \right) \right| \right\} \right]. \quad (41)$$

Proof. Let $\lambda(\omega) \in \sum_{\nu, \alpha, \beta}^{MC(q)}(\phi, \gamma)$, then there exist analytic functions $\varphi(\omega)$ and $k(\omega)$, with $|\varphi(\omega)| < 1$, $k(0) = 0$, and $k(\omega) < 1$ such that

$$-\left[(1-\gamma) \left(\frac{\omega(\zeta_{\nu, \alpha, \beta} \lambda)'(\omega)}{(\zeta_{\nu, \alpha, \beta} \lambda)(\omega)} \right) + \gamma \left(1 + \frac{\omega(\zeta_{\nu, \alpha, \beta} \lambda)''(\omega)}{(\zeta_{\nu, \alpha, \beta} \lambda)'(\omega)} \right) \right] - 1 = \varphi(\omega)[\phi(k(\omega)) - 1]. \quad (42)$$

Taking first and second derivative of (38), in use of the left side of the above equation, we obtain

$$\begin{aligned} & -\left[(1-\gamma) \left(\frac{\omega(\zeta_{\nu, \alpha, \beta} \lambda)'(\omega)}{(\zeta_{\nu, \alpha, \beta} \lambda)(\omega)} \right) + \gamma \left(1 + \frac{\omega(\zeta_{\nu, \alpha, \beta} \lambda)''(\omega)}{(\zeta_{\nu, \alpha, \beta} \lambda)'(\omega)} \right) \right] \\ & = (1-\gamma) \left[1 + \frac{\beta a_0 \omega}{4(\nu + (\alpha + 1/2))} \right. \\ & \quad - \frac{2\beta^2 a_1 \omega^2}{4^2 \times 2(\nu + 1 + (\alpha + 1/2))(\nu + (\alpha + 1/2))} \\ & \quad \left. + \frac{\beta^2 a_0^2 \omega^2}{4^2(\nu + (\alpha + 1/2))^2} + \dots \right] \\ & + \gamma \left[1 + \frac{2\beta^2 a_1 \omega^2}{4^2 \times 2(\nu + 1 + (\alpha + 1/2))(\nu + (\alpha + 1/2))} + \dots \right], \end{aligned} \quad (43)$$

thus implies

$$\begin{aligned} & -\left[(1-\gamma) \left(\frac{\omega(\zeta_{\nu, \alpha, \beta} \lambda)'(\omega)}{(\zeta_{\nu, \alpha, \beta} \lambda)(\omega)} \right) + \gamma \left(1 + \frac{\omega(\zeta_{\nu, \alpha, \beta} \lambda)''(\omega)}{(\zeta_{\nu, \alpha, \beta} \lambda)'(\omega)} \right) \right] - 1 \\ & = \frac{\beta a_0}{4(\nu + (\alpha + 1/2))} (1-\gamma) \omega \\ & + \left[\frac{\beta^2 a_0^2}{4^2(\nu + (\alpha + 1/2))^2} (1-\gamma) \right. \\ & \quad \left. - \frac{2\beta^2 a_1}{4^2 \times 2(\nu + 1 + (\alpha + 1/2))(\nu + (\alpha + 1/2))} (1-2\gamma) \right] \omega^2 + \dots, \\ & [\phi(k(\omega))] = 1 + k_1 B_1 \omega + (k_1^2 B_2 + k_2 B_1) \omega^2 \\ & + (k_3 B_1 + 2k_1 k_2 B_2 + k_1^3 B_3) \omega^3 + \dots, \end{aligned} \quad (44)$$

$$[\phi(k(\omega))] = 1 + k_1 B_1 \omega + (k_1^2 B_2 + k_2 B_1) \omega^2 + (k_3 B_1 + 2k_1 k_2 B_2 + k_1^3 B_3) \omega^3 + \dots, \quad (45)$$

which implies

$$\varphi(\omega)[\phi(k(\omega)) - 1] = c_0 k_1 B_1 \omega + (c_0 k_1^2 B_2 + c_0 k_2 B_1 + c_1 k_1 B_1) \omega^2 + \dots \quad (46)$$

Comparing (44) and (46), we get

$$\begin{aligned} a_0 &= \frac{4(\nu + (\alpha + 1/2))c_0 k_1 B_1}{\beta(1-\gamma)}, \\ a_1 &= -\frac{4^2(\nu + 1 + (\alpha + 1/2))(\nu + (\alpha + 1/2))B_1}{\beta^2(1-2\gamma)} \\ &\quad \cdot \left[c_0 k_2 + c_1 k_1 + k_1^2 \left(\frac{c_0 B_2}{B_1} - \frac{c_0 B_1}{(1-\gamma)} \right) \right]. \end{aligned} \quad (47)$$

Thus,

$$\begin{aligned} a_1 - \mu a_0^2 &= -\frac{4^2(\nu + 1 + (\alpha + 1/2))(\nu + (\alpha + 1/2))B_1}{\beta^2(1-2\gamma)} \\ &\times \left[c_0 k_2 + k_1^2 \left\{ \frac{c_1}{k_1} + \frac{c_0 B_2}{B_1} - B_1 \left(\frac{c_0}{(1-\gamma)} - \frac{(\nu + (\alpha + 1/2))(1-2\gamma)c_0}{(\nu + 1 + (\alpha + 1/2))(1-\gamma)} \right) \right\} \right], \\ |a_1 - \mu a_0^2| &\leq \left| -\frac{4^2(\nu + 1 + (\alpha + 1/2))(\nu + (\alpha + 1/2))B_1}{\beta^2(1-2\gamma)} \right| \\ &\times \left[|c_0 k_2| + \left| k_1^2 \left\{ \frac{c_1}{k_1} + \frac{c_0 B_2}{B_1} - B_1 \left(\frac{c_0}{(1-\gamma)} - \frac{(\nu + (\alpha + 1/2))(1-2\gamma)c_0}{(\nu + 1 + (\alpha + 1/2))(1-\gamma)} \right) \right\} \right| \right]. \end{aligned} \quad (48)$$

Since $\varphi(\omega)$ is analytic and bounded in U^* (see [25]), we have $|c_n| \leq 1 - |c_0|^2 \leq 1$, ($n > 0$).

By using this fact and the well-known inequality $|k_1| \leq 1$, we get

$$\begin{aligned} |a_1 - \mu a_0^2| &\leq \frac{4^2(\nu + (\alpha + 1/2))(\nu + 1 + (\alpha + 1/2))B_1}{\beta^2(1-2\gamma)} \\ &\times \left[1 + \max \left\{ 1, \left| \frac{B_2}{B_1} \right| + B_1 \left| 1 - \mu \left(\frac{(\nu + (\alpha + 1/2))(1-2\gamma)}{(\nu + 1 + (\alpha + 1/2))(1-\gamma)} \right) \right| \right\} \right]. \end{aligned} \quad (49)$$

We have thus completed the proof of Theorem 12.

For $k(\omega) = \omega$ and repeating steps of Theorem 12, we obtain the following corollary.

Corollary 13. Let $\lambda(\omega) \in \Sigma_I$ satisfies

$$-\left[(1-\gamma)\left(\frac{\omega(\zeta_{v,\alpha,\beta}\lambda)'(\omega)}{(\zeta_{v,\alpha,\beta}\lambda)(\omega)}\right) + \gamma\left(1 + \frac{\omega(\zeta_{v,\alpha,\beta}\lambda)''(\omega)}{(\zeta_{v,\alpha,\beta}\lambda)'(\omega)}\right)\right] - 1 \ll \phi \cdot (\omega) - 1 \quad (\omega \in U^*). \quad (50)$$

Then, for any complex number μ ,

$$|a_1 - \mu a_0^2| \leq \frac{4^2(v + (\alpha + 1/2))(v + 1 + (\alpha + 1/2))B_1}{\beta^2(1-2\gamma)} \cdot \left[1 + \left|\frac{B_2}{B_1}\right| + B_1 \left|\frac{1}{(1-\gamma)} - \mu \left(\frac{(v + (\alpha + 1/2))(1-2\gamma)}{(v + 1 + (\alpha + 1/2))}\right)\right|\right]. \quad (51)$$

Theorem 14. Let $\phi(\omega) = 1 + B_1\omega + B_2\omega^2 + \dots, B_1 > 0$, and $\varphi(\omega) = c_0 + c_1\omega + c_2\omega^2 + \dots$, if $\lambda(\omega)$ and $\delta(\omega)$ given by (1) and (5) be in the class $\Sigma_{v,\alpha,\beta}^{MK(q)}(\phi, \eta)$ and μ is a complex number, then

$$\left|\frac{b_1 + a_1}{2} - b_1\eta - \mu b_0^2\right| \leq \frac{4^2(v + (\alpha + 1/2))(v + 1 + (\alpha + 1/2))B_1}{\beta^2} \times \left[1 + \max\left\{1, \left|\frac{B_2}{B_1}\right| + B_1 \left|1 - \mu \left(\frac{(v + (\alpha + 1/2))}{(v + 1 + (\alpha + 1/2))(1-\eta)^2}\right)\right|\right\}\right]. \quad (52)$$

Proof. Let $\lambda(\omega)$ and $\delta(\omega) \in \Sigma_{v,\alpha,\beta}^{MK(q)}(\phi, \eta)$ then there exist analytic functions $\varphi(\omega)$ and $k(\omega)$, with $|\varphi(\omega)| < 1$, $k(0) = 0$, and $k(\omega) < 1$ such that

$$\frac{-\omega(\zeta_{v,\alpha,\beta}\lambda)'(\omega)}{(1-\eta)(\zeta_{v,\alpha,\beta}\delta)(\omega) + \eta\omega(\zeta_{v,\alpha,\beta}\delta)'(\omega)} - 1 = \varphi(\omega)[\phi(k(\omega)) - 1]. \quad (53)$$

Taking first derivative of (5) and (38) in use of the left side of above equation, we have

$$\begin{aligned} & \frac{-\omega(\zeta_{v,\alpha,\beta}\lambda)'(\omega)}{(1-\eta)(\zeta_{v,\alpha,\beta}\delta)(\omega) + \eta\omega(\zeta_{v,\alpha,\beta}\delta)'(\omega)} \\ &= 1 + \frac{\beta b_0(1-\eta)}{4(v + (\alpha + 1/2))}\omega \\ &+ \left(\frac{\beta^2 b_0^2(1-\eta)^2}{4^2(v + (\alpha + 1/2))^2} - \frac{\beta^2 b_1(1-2\eta)}{4^2 \times 2(v + 1 + (\alpha + 1/2))(v + (\alpha + 1/2))}\right. \\ &\quad \left.- \frac{\beta^2 a_1}{4^2 \times 2(v + 1 + (\alpha + 1/2))(v + (\alpha + 1/2))}\right)\omega^2 + \dots, \end{aligned} \quad (54)$$

and this implies

$$\begin{aligned} & \frac{-\omega(\zeta_{v,\alpha,\beta}\lambda)'(\omega)}{(1-\eta)(\zeta_{v,\alpha,\beta}\delta)(\omega) + \eta\omega(\zeta_{v,\alpha,\beta}\delta)'(\omega)} - 1 \\ &= \frac{\beta b_0(1-\eta)}{4(v + (\alpha + 1/2))}\omega \\ &+ \left(\frac{\beta^2 b_0^2(1-\eta)^2}{4^2(v + (\alpha + 1/2))^2} - \frac{\beta^2 b_1(1-2\eta)}{4^2 \times 2(v + 1 + (\alpha + 1/2))(v + (\alpha + 1/2))}\right. \\ &\quad \left.- \frac{\beta^2 a_1}{4^2 \times 2(v + 1 + (\alpha + 1/2))(v + (\alpha + 1/2))}\right)\omega^2 + \dots, \end{aligned} \quad (55)$$

$$[\phi(k(\omega))] = 1 + k_1 B_1 \omega + (k_1^2 B_2 + k_2 B_1) \omega^2 + (k_3 B_1 + 2k_1 k_2 B_2 + k_1^3 B_3) \omega^3 + \dots, \quad (56)$$

which implies

$$\varphi(\omega)[\phi(k(\omega)) - 1] = c_0 k_1 B_1 \omega + (c_0 k_1^2 B_2 + c_0 k_2 B_1 + c_1 k_1 B_1) \omega^2 + \dots \quad (57)$$

Comparing (55) and (57), we get

$$\begin{aligned} b_0 &= \frac{4(v + (\alpha + 1/2))c_0 k_1 B_1}{\beta(1-\eta)}\omega, \\ \frac{b_1 + a_1}{2} - b_1\eta &= -\frac{4^2(v + (\alpha + 1/2))(v + 1 + (\alpha + 1/2))B_1}{\beta^2} \\ &\cdot \left[c_0 k_2 + c_1 k_1 + k_1^2 \left(\frac{c_0 B_2}{B_1} - c_0^2 B_1\right)\right]. \end{aligned} \quad (58)$$

Thus,

$$\begin{aligned} \frac{b_1 + a_1}{2} - b_1\eta - \mu b_0^2 &= -\frac{4^2(v + (\alpha + 1/2))(v + 1 + (\alpha + 1/2))B_1}{\beta^2} \\ &\times \left[c_0 k_2 + c_1 k_1 + k_1^2 \left(\frac{c_0 B_2}{B_1} - c_0^2 B_1 + \mu \frac{(v + (\alpha + 1/2))c_0^2 B_1}{(v + 1 + (\alpha + 1/2))(1-\eta)^2}\right)\right], \\ \left|\frac{b_1 + a_1}{2} - b_1\eta - \mu b_0^2\right| &\leq \left|-\frac{4^2(v + (\alpha + 1/2))(v + 1 + (\alpha + 1/2))B_1}{\beta^2}\right| \\ &\times \left[|c_0 k_2| + \left|k_1^2 \left\{c_0 k_2 + \frac{c_0 B_2}{B_1} - B_1 \left(c_0^2 - \mu \frac{(v + (\alpha + 1/2))c_0^2}{(v + 1 + (\alpha + 1/2))(1-\eta)^2}\right)\right\}\right|\right]. \end{aligned} \quad (59)$$

Since $\varphi(\omega)$ is analytic and bounded in U^* (see [25]), we have

$$|c_n| \leq 1 - |c_0|^2 \leq 1, \quad (n > 0). \quad (60)$$

By using this fact and the well-known inequality, $|k_1| \leq 1$, we get

$$\left| \frac{b_1 + a_1}{2} - b_1 \eta - \mu b_0^2 \right| \leq \frac{4^2(\nu + (\alpha + 1/2))(\nu + 1 + (\alpha + 1/2))B_1}{\beta^2} \\ \times \left[1 + \max \left\{ 1, \left| \frac{B_2}{B_1} \right| + B_1 \right\} \left| 1 - \mu \left(\frac{(\nu + (\alpha + 1/2))}{(\nu + 1 + (\alpha + 1/2))(1 - \eta)^2} \right) \right| \right]. \quad (61)$$

We have thus completed the proof of Theorem 14.

For $k(\omega) = \omega$ and repeating steps of Theorem 14, we obtain the following corollary.

Corollary 15. Let $\lambda(\omega)$ and $\delta(\omega)$ satisfy

$$\frac{-\omega(\zeta_{\nu, \alpha, \beta} \lambda)'(\omega)}{(1 - \eta)(\zeta_{\nu, \alpha, \beta} \delta)(\omega) + \eta \omega(\zeta_{\nu, \alpha, \beta} \delta)''(\omega)} - 1 < \phi(\omega) - 1, \quad (\omega \in U^*), \quad (62)$$

then for any complex number μ ,

$$\left| \frac{b_1 + a_1}{2} - b_1 \eta - \mu b_0^2 \right| \leq \frac{4^2(\nu + (\alpha + 1/2))(\nu + 1 + (\alpha + 1/2))B_1}{\beta^2} \\ \times \left[1 + \left| \frac{B_2}{B_1} \right| + B_1 \right] \left| 1 - \mu \left(\frac{(\nu + (\alpha + 1/2))}{(\nu + 1 + (\alpha + 1/2))(1 - \eta)^2} \right) \right|. \quad (63)$$

4. Conclusion

In our present investigation, we have defined and systematically studied the famous Fekete-szegő inequality for some subclass of meromorphic functions by using quasubordination. It is important to mention that certain results in the literature, for example [3, 19, 20], are special cases of the results obtained by us.

Data Availability

No data is used.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

SH came with the main thoughts and helped to draft the manuscript. SGAS and AR proved the main theorems. ZS and MD revised the paper. All authors read and approved the final manuscript.

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Research Article

Weak Solutions and Optimal Control of Hemivariational Evolutionary Navier-Stokes Equations under Rauch Condition

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In this paper, we consider the evolutionary Navier-Stokes equations subject to the nonslip boundary condition together with a Clarke subdifferential relation between the dynamic pressure and the normal component of the velocity. Under the Rauch condition, we use the Galerkin approximation method and a weak precompactness criterion to ensure the convergence to a desired solution. Moreover, a control problem associated with such system of equations is studied with the help of a stability result with respect to the external forces. At the end of this paper, a more general condition due to Z. Naniewicz, namely the directional growth condition, is considered and all the results are reexamined.

1. Introduction

In many engineering situations, one deals with fluid flow problems in tubes or channels, or for semipermeable walls and membranes. In practice, hydraulic control devices are used as a mechanism allowing the adjustment of orifice dimensions so that the normal velocity on the boundary of the tube is regulated to reduce the dynamic pressure. The model that usually describes this situation is represented by the Navier-Stokes equations for incompressible viscous fluids with the nonslip boundary conditions together with a Clarke subdifferential relation between the dynamic pressure and the normal component of the velocity. The resulting multivalued subdifferential boundary condition leads, after a standard variational transformation, to the so-called hemivariational inequality.

The theory of hemivariational inequalities was introduced for the first time by Panagiotopoulos [1–5] for the sake of generalization of the classical convex variational theory to a nonconvex one. The main tool in this effort is the generalized gradient of Clarke and Rockafellar [6–8]. From this perspective, the literature has seen a fast emergence of applications in a mathematical and mechanical point of view, see

[3, 4, 9–13] for more details. Among the main applications of this theory, we mention the Newtonian and non-Newtonian Navier-Stokes equations and their variants (the Oseen model, heat-conducting fluids, miscible liquids, etc.) with nonstandard boundary conditions ensuing from the multivalued nonmonotone friction law with leak, slip, or nonslip conditions. For recent directions on the hemivariational theory, we refer to [14–17].

Over the last two decades, intensive research has been conducted on hemivariational inequalities for the stationary and nonstationary Navier-Stokes equations. For convex functionals, the problem has been studied essentially by Chebotarev [18–20]. We mention also [21] for stationary Boussinesq equations and [22] by Konovalova for nonstationary Boussinesq equations. In all these papers, the considered problems were formulated as variational inequalities. In the nonconvex case, the stationary case was considered by Migórski and Ochal [23] and Migórski [24], and the nonstationary case was considered by Migórski and Ochal in [25]; see also [26]. For an equilibrium approach, one can see for example [27]. On the other hand, the optimal control problem involving hemivariational inequalities attracts more and more attention from researchers in recent years. We

refer to the introductions of [28, 29] for a short review on the subject.

There are two main conditions that one can impose on the locally Lipschitz function under a subdifferential effect, namely the classical growth condition or the Rauch condition due to J. Rauch [30]. The last one is less popular even if it was the main assumption in the beginning of the theory of hemivariational inequalities. The Rauch condition expresses actually the ultimate increase of the graph of a certain locally bounded function and is, in fact, a special case of another unpopular condition, namely the directional growth condition due to Naniewicz [31]. An advantage of the Rauch condition is that it allows avoiding smallness conditions (i.e., the relationship between the constants of the problem) brought by the classical growth condition. In the case of the Navier-Stokes equations, the smallness condition links the growth condition constant, the coercivity constant, and the norm of the trace operator. It is, however, not clear how it can be checked in a concrete situation. Another advantage is that it allows us to consider the “Stanger” functions at infinity. In fact, the only thing we require from the function is for the essential supremum of the function on the left side to be greater than the essential infimum on the right side.

Among the disadvantages of the Rauch condition is that although it ensures the existence of a solution, it does not allow the conclusion that the nonconvex functional is locally Lipschitz or even finite on the whole space. The Aubin-Clarke formula cannot be used, and a slight change in the definition of a solution has to be made. On the other hand, we are looking for the dynamical pressure in a larger space, which makes the question of uniqueness more difficult without a classical growth condition even if a monotonicity type assumption is acquired [32]. Finally, it is worth mentioning that there is no direct link between the Rauch condition and the classical growth condition, and the choice depends mainly on the concrete situation.

The present paper represents a continuation of our previous paper [32], where existence and optimal control questions involving the stationary Navier-Stokes problem with the multivalued nonmonotone boundary condition are studied. In this paper, we tackle the nonstationary problem. Always under the Rauch condition, we use the Faedo-Galerkin approximation to regularize the system at the level of the multivalued boundary condition and we use the fact that the approximation sequence so obtained is weakly precompact in the space of integrable functions. We also take advantage of the techniques used in [25] at the level of the nonlinear term to ensure the convergence of the approximate sequence to the desired solution. This study can be also done with the directional growth condition as a generalization. The question of the existence of an optimal control is important in applications. We tackle this subject in the spirit of the works of Barbu [33] and Migórski [34].

The outline of this paper is as follows. In section 2, we state the problem and give its hemivariational form by using the Lamb formulation. In section 3, we regularize our problem by using the Faedo-Galerkin approximation method and prove the existence of solutions to the regularized problem. By combining techniques from [25, 32], we will provide

an existence result in section 4. Section 5 is devoted to the optimal control problem subjected to our evolutionary hemivariational inequality, while section 6 is dedicated to the directional growth condition as a generalization of the Rauch condition.

2. Problem Statement

Let Ω be a bounded simply connected domain in \mathbb{R}^d with $d = 2, 3$ with connected boundary $\partial\Omega$ of class C^2 and $\Omega_T = (0, T) \times \Omega$ where $T > 0$. We consider the following evolution Navier-Stokes system:

$$\begin{aligned} u' - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f_{\text{ext}}, & \text{in } \Omega_T, \\ \operatorname{div} u &= 0, & \text{in } \Omega_T, \\ u(0) &= u_0, & \text{in } \Omega. \end{aligned} \quad (1)$$

This system describes the flux of an incompressible viscous fluid in a domain Ω subjected to an external force $f_{\text{ext}} = \{f_{\text{ext},k}\}_{k=1}^d$. $u = \{u_k\}_{k=1}^d$, p and ν denote, respectively, the velocity, the pressure, and the kinematic viscosity of the fluid. The nonlinear term $(u \cdot \nabla)u$ (called the convective term) is the symbolic notation of the vector $\sum_{j=1}^d u_j (\partial u_i / \partial x_j)$. As usual, we use the Lamb formulation ([35], chapter I) to rewrite the evolution Navier-Stokes system as follows:

$$u' + \nu \operatorname{rot} \operatorname{rot} u + \operatorname{rot} u \times u + \nabla \tilde{p} = f_{\text{ext}}, \quad \text{in } \Omega_T, \quad (2)$$

$$\operatorname{div} u = 0, \quad \text{in } \Omega_T, \quad (3)$$

$$u(0) = u_0, \quad \text{in } \Omega, \quad (4)$$

where $\tilde{p} = p + (1/2)|u|^2$ is the total head of the fluid, or “total pressure.”

We suppose that on boundary $\partial\Omega$, the tangential components of the velocity vector are known, and without loss of generality, we put them equal to zero (the nonslip condition):

$$u_\tau(t, x) := u(t, x) - u_N(t, x)n, \quad \text{on } \partial\Omega_T := (0, T) \times \partial\Omega, \quad (5)$$

where $n = \{n_k\}_{k=1}^d$ is the unit outward normal on the boundary $\partial\Omega$ and $u_N(t, x) = u(t, x) \cdot n = \sum_{i=1}^d u_i(t, x)n_i$ denotes the normal component of the vector u . Moreover, we assume the following subdifferential boundary condition:

$$\tilde{p}(t, x) \in \partial j(u_N(t, x)) \text{ on } \partial\Omega_T, \quad (6)$$

where $\partial j(\xi)$ is the Clarke subdifferential of j at ξ and is given by

$$\partial j(\xi) = \{\xi^* \in V^* : j^0(\xi; h) \geq \langle \xi^*, h \rangle_{V^* \times V} \text{ for all } h \in V\}, \quad (7)$$

and $j^0(\xi; h)$ is the generalized derivative of a locally Lipschitz function j at $\xi \in V$ in the direction $h \in V$ defined by

$$j^0(\xi; v) = \limsup_{v \rightarrow \xi, \lambda \downarrow 0} \frac{j(v + \lambda v) - j(v)}{\lambda}. \quad (8)$$

To work conveniently on problems (2), (3), (4), (5), and (6), we need the following functional spaces:

$$\begin{aligned} C &= \left\{ u \in \mathcal{C}^\infty(\Omega; \mathbb{R}^d) : \operatorname{div} u = 0 \text{ in } \Omega, u_\tau = 0 \text{ on } \partial\Omega \right\}, \\ V &= \text{the closure of } C \text{ in the norm of } H^1(\Omega; \mathbb{R}^d), \\ H &= \text{the closure of } C \text{ in the norm of } L^2(\Omega; \mathbb{R}^d). \end{aligned} \quad (9)$$

Then, we have $V \subset H \simeq H^* \subset V^*$, with all the embedding being continuous and compact. Moreover, for an interval time $[0, T]$, we introduce the following spaces:

$$\begin{aligned} \mathcal{V} &= L^2(0, T; V), \\ \mathcal{H} &= L^2(0, T; H), \\ \mathcal{W} &= \left\{ u \in \mathcal{V} : u' \in \mathcal{V}^* \right\}. \end{aligned} \quad (10)$$

Then, we also have the following continuous embedding, $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$.

We consider the operators $\mathcal{A} : V \longrightarrow V^*$ and $\mathcal{B} : V \times V \longrightarrow V^*$ defined by

$$\begin{aligned} \langle \mathcal{A}u, v \rangle &= \int_{\Omega} \operatorname{rot} u \cdot \operatorname{rot} v \, dx, \\ \langle \mathcal{B}(u, v), w \rangle &= \int_{\Omega} (\operatorname{rot} u \times v) \cdot w \, dx, \end{aligned} \quad (11)$$

for all $u, v, w \in V$. As usual, we will use the notation $\mathcal{B}[\cdot] = \mathcal{B}(\cdot, \cdot)$. It is well known (cf. [36]) that if the domain Ω is simply connected, the bilinear form

$$((u, v))_V = \int_{\Omega} \operatorname{rot} u \cdot \operatorname{rot} v \, dx, \quad (12)$$

generates a norm in V , $\|u\|_V = ((u, v))_V^{(1/2)}$, which is equivalent to the $H^1(\Omega, \mathbb{R}^d)$ -norm. Hence, it is clear that the operator \mathcal{A} is coercive.

In order to give the weak formulation to problems (2), (3), (4), (5), and (6), we multiply it by a certain $v \in V$ and apply the Green formula. We obtain

$$\left\langle u'(t) + \mathcal{A}u(t) + \mathcal{B}[u(t)], v \right\rangle + \int_{\partial\Omega} \tilde{p}(t, x) v_N \, d\sigma(x) = \langle f(t), v \rangle, \quad (13)$$

where $\langle f(t), v \rangle = \int_{\Omega} f_{\text{ext}}(t) \cdot v \, dx$. From relation (6), by using the definition of the Clarke subdifferential, we have

$$\int_{\partial\Omega} \tilde{p}(t, x) v_N(x) \, d\sigma(x) \leq \int_{\partial\Omega} j^0(x, u_N(x); v_N(x)) \, d\sigma(x). \quad (14)$$

The relations (13) and (14) yield to the following weak formulation:

$$\text{(EHVI)} \begin{cases} \text{Find } u \in \mathcal{W} \text{ such that for all } v \in V \text{ and a.e. } t \in (0, T), \\ \left\langle u'(t) + \mathcal{A}u(t) + \mathcal{B}[u(t)], v \right\rangle + \int_{\partial\Omega} j^0(t, x, u_N(t, x); v_N(x)) \, d\sigma(x) \geq \langle f(t), v \rangle, \\ u(0) = u_0. \end{cases} \quad (15)$$

The equation above is called an hemivariational inequality.

We have already mentioned in Introduction that the Rauch assumption is not sufficient to make the functional $J(u) = \int_{\partial\Omega} j(u) \, d\sigma$ locally lipschitz or even finite in the whole space \mathcal{V} . Because of this reason, a slight modified definition of being a solution should be adopted. Define the functional space

$$L_N^\infty(\partial\Omega) = \{u, u_N = \gamma(u) \cdot n \in L^\infty(\partial\Omega; \mathbb{R})\}, \quad (16)$$

where γ is the trace operator from V in $L^2(\partial\Omega; \mathbb{R}^d)$. Now, we are able to give what we mean by a solution to the problem (EHVI).

Definition 1. A function $u \in \mathcal{W}$ is said to be the solution of (EHVI) if there exists $\kappa \in L^1((0, T) \times \partial\Omega, \mathbb{R})$ such that for a.e. $t \in (0, T)$

$$\begin{cases} \left\langle u'(t) + \mathcal{A}u(t) + \mathcal{B}[u(t)], v \right\rangle + \int_{\partial\Omega} \kappa(t, z) v_N(z) \, d\sigma(z) = \langle f(t), v \rangle, & \forall v \in V \cap L_N^\infty(\partial\Omega), \\ \kappa(t, z) \in \partial j(t, z; u_N(t, z)), & \text{for a.e. } (t, z) \in (0, T) \times \partial\Omega, \\ u(0) = u_0. \end{cases} \quad (17)$$

Note that since $\mathcal{W} \subset C(0, T; H)$ continuously, the initial condition $u(0) = u_0$ makes sense in H . To justify the above definition, we refer to [31, 32].

3. Regularized Problem

In what follows, we restrict our study to superpotentials j , which are independent of x and which subdifferential is obtained by “filling in the gaps” procedure (cf. [30]). Let $\theta \in L_{\text{loc}}^\infty(\mathbb{R})$, for $\varepsilon > 0$ and $t \in \mathbb{R}$, we define

$$\begin{aligned} \underline{\theta}_\varepsilon(t) &= \operatorname{ess\,inf}_{|t-s| \leq \varepsilon} \theta(s), \\ \bar{\theta}_\varepsilon(t) &= \operatorname{ess\,sup}_{|t-s| \leq \varepsilon} \theta(s). \end{aligned} \quad (18)$$

For a fixed $t \in \mathbb{R}$, the functions $\underline{\theta}_\varepsilon, \bar{\theta}_\varepsilon$ are decreasing and increasing in ε , respectively. Let

$$\begin{aligned}\underline{\theta}(t) &= \lim_{\varepsilon \rightarrow 0^+} \underline{\theta}_\varepsilon(t), \\ \bar{\theta}(t) &= \lim_{\varepsilon \rightarrow 0^+} \bar{\theta}_\varepsilon(t),\end{aligned}\quad (19)$$

and let $\widehat{\theta}(t): \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ be a multifunction defined by

$$\widehat{\theta}(t) = [\underline{\theta}(t), \bar{\theta}(t)]. \quad (20)$$

From Chang [37], we know that a locally Lipschitz function $j: \mathbb{R} \longrightarrow \mathbb{R}$ can be determined up to an additive constant by the relation

$$j(t) = \int_0^t \theta(s) ds, \quad (21)$$

such that $\partial j(t) \subset \widehat{\theta}(t)$ for all $t \in \mathbb{R}$. If moreover, the limits $\theta(t \pm 0)$ exist for every $t \in \mathbb{R}$, then $\partial j(t) = \widehat{\theta}(t)$.

In order to define the regularized problem, we consider the mollifier

$$\begin{aligned}\mathfrak{h} &\in C_0^\infty(-1, 1), \\ \mathfrak{h} &\geq 0 \text{ with } \int_{-\infty}^{+\infty} \mathfrak{h}(s) ds = 1,\end{aligned}\quad (22)$$

and let

$$\theta_\varepsilon = \mathfrak{h}_\varepsilon * \theta \text{ with } \mathfrak{h}_\varepsilon(s) = \frac{1}{\varepsilon} \mathfrak{h}\left(\frac{s}{\varepsilon}\right), \quad (23)$$

where $*$ denotes the convolution product.

Consider the following auxiliary problem associated to (EHVI):

$$(\mathcal{P}_\varepsilon) \begin{cases} \text{Find } u \in \mathcal{W} \text{ such that for all } v \in V \cap L_N^\infty(\partial\Omega), \quad \text{a.e. } t \in (0, T), \\ \left\langle u'(t) + \mathcal{A}u(t) + \mathcal{B}[u(t)], v \right\rangle + \int_{\partial\Omega} \theta_\varepsilon(u_N(t, x)) v_N d\sigma = \langle f(t), v \rangle, \\ u(0) = u_0. \end{cases} \quad (24)$$

Now and in order to define the corresponding finite dimensional problem, we shall use the Faedo-Galerkin approximation approach. Let us consider a Galerkin basis $\{z_1, z_2, \dots\}$ in $V \cap L_N^\infty(\partial\Omega)$, i.e., $\{z_1, z_2, \dots\}$ forms at most a countable sequence of elements of $V \cap L_N^\infty(\partial\Omega)$, finitely $\{z_1, z_2, \dots, z_m\}$ are linearly independent. Consider $V_m = \text{span}\{z_1, z_2, \dots, z_m\}$, we have $V_m \subset V_{m+1}$ and $\cup_{m \geq 0} V_m = V \cap L_N^\infty(\partial\Omega)$. Moreover, the family $\{V_m\}_m$ satisfies

$$\begin{aligned}\forall v &\in V \cap L_N^\infty(\partial\Omega), \\ \exists (v_m)_m, \\ v_m &\in V_m,\end{aligned}\quad (25)$$

such that $v_m \longrightarrow v$ in $V \cap L_N^\infty(\partial\Omega)$, as $n \longrightarrow +\infty$.

Let $\{u_m(0)\}$ be an approximation of the given initial value u_0 such that $u_m(0) \in V_m$ for $m \in \mathbb{N}$ and suppose that

$$\begin{aligned}u_{m0} &\longrightarrow u_0 \text{ in } H, \quad \text{as } m \longrightarrow +\infty, \\ (u_{m0})_m &\text{ is bounded in } V \cap L_N^\infty(\partial\Omega).\end{aligned}\quad (26)$$

We consider the following regularized Galerkin system of finite dimensional differential equations associated to (EHVI):

$$(\mathcal{P}_\varepsilon^m) \begin{cases} \text{Find } u_m \in \mathcal{W}_m \text{ such that for all } v \in V_m, \quad \text{a.e. } t \in (0, T), \\ \left\langle u'_m(t) + \mathcal{A}u_m(t) + \mathcal{B}[u_m(t)], v \right\rangle + \int_{\partial\Omega} \theta_\varepsilon(u_{mN}(t)) v d\sigma = \langle f(t), v \rangle, \\ u_m(0) = u_{m0}, \end{cases} \quad (27)$$

where $\mathcal{W}_m = \{u \in L^2(0, T; V_m): u' \in L^2(0, T; V_m)\}$.

The generalized derivative $\mathcal{L}u = u'$ restricted to the subset $D(\mathcal{L}) = \{u \in \mathcal{V}: u' \in \mathcal{V}^* \text{ and } u(0) = u_0\} = \{u \in \mathcal{W}: u(0) = u_0\}$ defines a linear operator $\mathcal{L}: D(\mathcal{L}) \subset \mathcal{V} \longrightarrow *$ given by

$$\ll \mathcal{L}u, v \gg = \int_0^T \langle u'(t), v(t) \rangle dt \text{ for all } v \in \mathcal{V}. \quad (28)$$

For the existence of solutions, we will need the following hypothesis $H(\theta)$:

- (1) (Chang assumption) $\theta \in L_{\text{loc}}^\infty(\mathbb{R})$, $\theta(t \pm 0)$ exists for any $t \in \mathbb{R}$
- (2) (Rauch assumption) There is $\delta_0 > 0$ such that

$$\text{ess sup}_{]-\infty, -\delta_0[} \theta(t) \leq 0 \leq \text{ess inf}_{] \delta_0, +\infty[} \theta(t). \quad (29)$$

Remark 2. If one assumes more generally that

$$\text{ess sup}_{]-\infty, -\delta_0[} \theta(t) \leq \alpha \leq \text{ess inf}_{] \delta_0, +\infty[} \theta(t), \quad (30)$$

for some real number α , it is possible to come back to the situation where the Rauch assumption is imposed by simply replacing θ by $\theta - \alpha$ and f by $f - \alpha$.

Remark 3. We point out that the Rauch and growth conditions are completely independent. Indeed, by taking examples, we show that neither of both conditions implies the other. In fact, consider the function $\beta: \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\beta(t) = \begin{cases} \lfloor t^3 \rfloor, & \text{if } |t| \geq 1, \\ -t, & \text{if } |t| < 1, \end{cases} \quad (31)$$

where $\lfloor t \rfloor$ stands for the integer part of t . One can prove easily (eventually by a contradiction argument) that the function β satisfies the Rauch condition while the growth condition cannot be satisfied. Conversely, one can take a function

$\beta : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\beta(t) = 1 + \sqrt{|t|}$; it is clear that it satisfies the growth condition but not the Rauch condition as β is positive for negative values.

Lemma 4. Suppose that $H(\theta)$ holds. Then we can determine $\rho_1, \rho_2 > 0$, such that for every $u \in V_m$

$$\int_{(0,T) \times \partial\Omega} \theta_\varepsilon(u_N(t,z))u_N(t,z)d\sigma(z)dt \geq -\rho_1\rho_2T \cdot \sigma(\partial\Omega). \quad (32)$$

Proof. This is a classical result in the stationary case (cf. [32], Lemma 3.2). It suffices to integrate over $t \in (0, T)$ to obtain the result.

Proposition 5. The sequence $(\theta_{\varepsilon_m}(u_m))_{m \in \mathbb{N}}$ is weakly pre-compact in $L^1((0, T) \times \partial\Omega)$.

Proof. The proof is similar to ([32], Proposition 3.7) with minor changes consisting mainly in replacing $\partial\Omega$ by $(0, T) \times \partial\Omega$ and remarking that $\langle u'_m(t), u_m(t) \rangle \geq 0$ for a.e. $t \in [0, T]$.

Proposition 6. The regularized problem $(\mathcal{P}_\varepsilon^m)$ has at least one solution u_m .

Proof. We substitute $u_m(t) = \sum_{k=1}^m c_{km}(t)z_k$ in $(\mathcal{P}_\varepsilon^m)$ to obtain

$$\begin{aligned} \sum_{k=1}^m c'_{km}(t)\langle z_k, z_i \rangle + \sum_{k=1}^m c_{km}(t)\langle Az_k, z_i \rangle + \sum_{k,j=1}^m c_{km}(t)c_{jm}(t)b(z_k, z_j, z_i) \\ + \int_{\partial\Omega} \theta_{\varepsilon_m} \left(\sum_{k=1}^m c_{km}(t)z_k \cdot n \right) z_i \cdot n d\sigma \\ = \langle f(t), z_i \rangle, \quad i = 1, \dots, m, \text{ a.e. } t \in [0, T], \end{aligned} \quad (33)$$

$$c_{im}(0) = \alpha_{i0}, \quad i = 1, \dots, n. \quad (34)$$

The matrix with elements $\langle z_k, z_i \rangle$, $1 \leq i, k \leq m$ is nonsingular (i.e., $\det \{\langle z_k, z_i \rangle\}_{k,i=1}^m \neq 0$), we invert the matrix, then equation (33) can be written in the usual form:

$$\begin{aligned} c'_{km}(t) = \sum_{i=1}^m \beta_{ki} \langle f(t), z_i \rangle - \sum_{i=1}^m \alpha_{ki} c_{im}(t) - \sum_{i,j=1}^m \xi_{kij} c_{km}(t) c_{jm}(t) \\ - \sum_{i=1}^m q_{ki} \int_{\partial\Omega} \theta_{\varepsilon_m} \left(\sum_{k=1}^m c_{km}(t)z_k \cdot n \right) z_i \cdot n d\sigma, \end{aligned} \quad (35)$$

where the initial values $c_{km}(0)$, $k = 1, \dots, m$ are given, i.e., $u_{m0} = \sum_{k=1}^m c_{km}(0)z_k$:

$$c_{km}(0) \text{ is the } k^{\text{th}} \text{ component of } u_{m0}. \quad (36)$$

The differential system (35) with the initial condition (36) define uniquely the scalar c_{km} on the interval $[0, t_m)$.

Then, the solution u_m exists on $[0, t_m)$, and we can extend it on the closed interval $[0, T]$ by using a priori estimates in Lemma 7. Since the scalar function $t \longrightarrow \langle f(t), z_i \rangle$ in equation (33) are square integrable, so are the functions c_{km} ; therefore, for each m we have:

$$\begin{aligned} u_m &\in L^2(0, T; V), \\ u'_m &\in L^2(0, T; V^*). \end{aligned} \quad (37)$$

4. Existence Result

In this section, we will prove the existence of solutions to the problem (EHVI) by analysing the convergence of the sequence $(u_m)_m$ solutions to $(\mathcal{P}_\varepsilon^m)$. To do so, we need some a priori estimates.

Lemma 7. The solution $\{u_m\}_m$ is bounded in $L^2(0, T; V) \cap L^\infty(0, T; H)$.

Proof. From Proposition 5, the regularized problem $(\mathcal{P}_\varepsilon^m)$ has at least one solution $\{u_m\}_m$. By replacing v by $u_m(t)$ in $(\mathcal{P}_\varepsilon^m)$, we get for a.e. $t \in [0, T]$

$$\begin{aligned} \langle u'_m(t) + \mathcal{A}u_m(t) + \mathcal{B}[u_m(t)], u_m(t) \rangle \\ + \int_{\partial\Omega} \theta_{\varepsilon_m}(u_{mN}(t, x))u_{mN}(t, x)d\sigma = \langle f(t), u_m(t) \rangle. \end{aligned} \quad (38)$$

Because of (37) we have

$$\langle u'_m(t), u_m(t) \rangle = \frac{1}{2} \frac{d}{dt} |u_m(t)|^2. \quad (39)$$

Then, equation (38) becomes

$$\begin{aligned} \frac{d}{dt} |u_m(t)|^2 + 2 \langle \mathcal{A}u_m(t), u_m(t) \rangle + 2 \int_{\partial\Omega} \theta_{\varepsilon_m}(u_{mN}(t, x))u_{mN}(t, x)d\sigma \\ = 2 \langle f(t), u_m(t) \rangle_{V^*} \text{ for a.e. } t \in [0, T]. \end{aligned} \quad (40)$$

By the coerciveness of \mathcal{A} , the Cauchy-Schwartz inequality, and the Young inequality, we obtain

$$\begin{aligned} \frac{d}{dt} |u_m(t)|^2 + M \|u_m(t)\|^2 + 2 \int_{\partial\Omega} \theta_{\varepsilon_m}(u_{mN}(t, x))u_{mN}(t, x)d\sigma \\ \leq \frac{2}{M} \|f(t)\|_{V^*}^2, \end{aligned} \quad (41)$$

for a.e. $t \in (0, T)$ (M is the constant of coercivity). Integrating equation (40) from 0 to s , $0 \leq s \leq T$, and using Lemma 4, one has

$$\begin{aligned} |u_m(s)|^2 + M \int_0^s \|u_m(\tau)\|_{V^*}^2 d\tau \leq \frac{2}{M} \int_0^s \|f(\tau)\|_{V^*}^2 d\tau + 2\rho_1\rho_2s \cdot \sigma(\partial\Omega) + |u_m(0)|^2. \end{aligned} \quad (42)$$

Hence

$$\sup_{s \in [0, T]} |u_m(s)|^2 \leq \frac{2}{M} f_{\mathcal{V}^*}^2 + 2\rho_1 \rho_2 T \cdot \sigma(\partial\Omega) + |u_m(0)|^2. \quad (43)$$

The right-hand side of the previous inequality is finite and independent of m . We deduce that $\{u_m\}_m$ is bounded in $L^\infty(0, T; H)$.

Again, from (42) we have

$$M \int_0^T \|u_m(t)\|_V^2 dt \leq \frac{2}{M} \|f\|_{\mathcal{V}^*}^2 + 2\rho_1 \rho_2 T \cdot \sigma(\partial\Omega) + |u_m(0)|^2. \quad (44)$$

Then

$$\|u_m\|_{\mathcal{V}}^2 \leq \frac{2}{M^2} \|f\|_{\mathcal{V}^*}^2 + \frac{2}{M} \rho_1 \rho_2 T \cdot \sigma(\partial\Omega) + \frac{1}{M} |u_m(0)|^2. \quad (45)$$

Then, $\{u_m\}_m$ remains in a bounded subset of \mathcal{V} .

Theorem 8. *Under assumption $H(\theta)$, the problem (EHVI) has at least one solution.*

Proof. From Proposition 5 and Proposition 6, we get

$$\begin{aligned} u_m &\longrightarrow u \text{ weakly in } \mathcal{V}, \\ u_m &\longrightarrow u \text{ weakly} - \text{star in } L^\infty(0, T; H), \\ \theta_{\varepsilon_m}(u_{mN}) &\longrightarrow \kappa \text{ weakly in } L^1((0, T) \times \partial\Omega). \end{aligned} \quad (46)$$

Now, we focus on the weak convergence of the nonlinear term $\mathcal{B}[u_m]$ by using exactly the same procedure as in [25]. For the case $d = 2$, we obtain from Temam [38] the following:

$$\|\mathcal{B}[u_m]\|_{\mathcal{V}^*} \leq c \|u_m\|_{L^\infty(0, T; H)} \|u_m\|_{\mathcal{V}} \text{ with } c > 0. \quad (47)$$

Moreover, operator \mathcal{A} is continuous. Hence $\{u'_m\}$ is bounded in \mathcal{V}^* . Thus, by passing to a next subsequence, if necessary, we have the following:

$$u'_m \longrightarrow u' \text{ weakly in } \mathcal{W}. \quad (48)$$

Using the facts that $\mathcal{W} \subset C(0, T; H)$ continuously, $\mathcal{W} \subset \mathcal{H}$ compactly, and $\mathcal{W} \subset L^2(0, T; L^2(\Gamma; \mathbb{R}^n))$ compactly, we have $u \in C(0, T; H)$ and

$$\begin{aligned} u_m &\longrightarrow u \text{ in } \mathcal{H}, \\ \gamma(u_m) &\longrightarrow \gamma(u) \text{ in } L^2\left(0, T; L^2\left(\Gamma; \mathbb{R}^d\right)\right). \end{aligned} \quad (49)$$

Since $u_m \longrightarrow u$ weakly in \mathcal{V} and in \mathcal{H} , analogously as in Ahmed [39], we have $B[u_m] \longrightarrow B[u]$ weakly in \mathcal{V}^* . We remark that if $d = 3$, we also have the convergence of $B[u_m] \longrightarrow B[u]$ weakly in \mathcal{V}^* by a compactness embedding theorem as in [39].

Let $\phi \in C_0^\infty(0, T)$ and $v \in V \cap L_N^\infty(\partial\Omega)$. Then, there exists $\{v_m\}_{m \in \mathbb{N}}$ such that $v_m \in V_m$ and $v_m \longrightarrow v$ in \mathcal{V} , as $m \longrightarrow \infty$. Denoting $\psi_m(x, t) = \phi(t)v_m(x)$ and $\psi(x, t) = \phi(t)v(x)$, we have $\psi_m \longrightarrow \psi$ in \mathcal{W} . From $(\mathcal{S}_\varepsilon^m)$, we have the following:

$$\begin{aligned} &\int_0^T \left\langle u'_m(t) + \mathcal{A}u_m(t) + \mathcal{B}[u_m(t)], \psi_m(t) \right\rangle \\ &+ \int_{(0, T) \times \partial\Omega} \theta_{\varepsilon_m}(u_{mN}(t)) \psi_{mN}(t) d\sigma dt = \int_0^T \langle f(t), \psi_m(t) \rangle dt. \end{aligned} \quad (50)$$

Using the above convergences, letting $m \longrightarrow +\infty$, we obtain

$$\begin{aligned} &\int_0^T \left\langle u'(t) + \mathcal{A}u(t) + \mathcal{B}[u(t)], v \right\rangle \phi(t) dt \\ &+ \int_{(0, T) \times \partial\Omega} \kappa \cdot v_N \phi(t) d\sigma dt = \int_0^T \langle f(t), v \rangle \phi(t) dt. \end{aligned} \quad (51)$$

Since ϕ is arbitrary, we deduce that

$$\left\langle u'(t) + \mathcal{A}u(t) + \mathcal{B}[u(t)], v \right\rangle + \int_{\partial\Omega} \kappa \cdot v_N d\sigma = \langle f(t), v \rangle, \quad (52)$$

for a.e. $t \in (0, T)$ and for all $v \in V \cap L_N^\infty(\partial\Omega)$.

In order to complete the proof, it will be shown that

$$\kappa(t, z) \in \widehat{\Theta}(u_N(t, z)) = \partial^c j(u_N(t, z)), \quad \text{for a.e. } (t, z) \in [0, T] \times \partial\Omega. \quad (53)$$

Since $\gamma(u_m) \longrightarrow \gamma(u)$ in $L^2(0, T; L^2(\partial\Omega))$, we obtain $u_{mN} \longrightarrow u_N$ in $L^2(0, T; L^2(\partial\Omega))$, and consequently, $u_{mN}(t, x) \longrightarrow u_N(t, x)$ for a.e. $(t, x) \in [0, T] \times \partial\Omega$; then, by applying Egoroff's theorem, we can find that for any $\alpha > 0$, we can determine $\omega \subset [0, T] \times \partial\Omega$ with $\sigma(\omega) < \alpha$, such that

$$u_{mN} \longrightarrow u_N, \quad \text{uniformly on } [0, T] \times \partial\Omega \setminus \omega, \quad (54)$$

with $u_N \in L^\infty([0, T] \times \partial\Omega \setminus \omega)$. Thus, for any $\alpha > 0$, we can find $\omega \subset [0, T] \times \partial\Omega$ with $\sigma(\omega) < \alpha$, such that for any $\mu > 0$ and for $\varepsilon < \varepsilon_0 < \mu/2$ and $n > n_0 > 2/\mu$, we have

$$|u_{mN} - u_N| < \frac{\mu}{2}, \quad \text{on } [0, T] \times \partial\Omega \setminus \omega. \quad (55)$$

Consequently, one obtains

$$\theta_\varepsilon(u_{mN}) \leq \operatorname{ess\,sup}_{|u_{mN} - \xi| \leq \varepsilon} \theta(\xi) \leq \operatorname{ess\,sup}_{|u_{mN} - \xi| \leq \frac{\mu}{2}} \theta(\xi) \leq \operatorname{ess\,sup}_{|u_{mN} - \xi| \leq \mu} \theta(\xi) = \bar{\theta}_\mu(u_N). \quad (56)$$

Analogously, we prove the inequality

$$\underline{\theta}_\mu(u_N) = \operatorname{ess\,inf}_{|u_N - \xi| \leq \mu} \theta(\xi) \leq \theta_\varepsilon(u_{mN}). \quad (57)$$

We now take $v \geq 0$ a.e. on $[0, T] \times \partial\Omega \setminus \omega$ with $v \in L^\infty([0, T] \times \partial\Omega \setminus \omega)$. This implies

$$\int_{[0, T] \times \partial\Omega \setminus \omega} \underline{\theta}_\mu(u_N) v d\sigma \leq \int_{[0, T] \times \partial\Omega \setminus \omega} \theta_\varepsilon(u_{mN}) v d\sigma \leq \int_{[0, T] \times \partial\Omega \setminus \omega} \bar{\theta}_\mu(u_N) v d\sigma. \quad (58)$$

Taking the limits as $\varepsilon \rightarrow 0$ and $m \rightarrow \infty$, we obtain

$$\int_{[0, T] \times \partial\Omega \setminus \omega} \underline{\theta}_\mu(u_N) v d\sigma \leq \int_{[0, T] \times \partial\Omega \setminus \omega} \kappa v d\sigma \leq \int_{\partial[0, T] \times \Omega \setminus \omega} \bar{\theta}_\mu(u_N) v d\sigma, \quad (59)$$

and as $\mu \rightarrow 0^+$, we obtain

$$\int_{[0, T] \times \partial\Omega \setminus \omega} \underline{\theta}(u_N) v d\sigma \leq \int_{[0, T] \times \partial\Omega \setminus \omega} \kappa v d\sigma \leq \int_{[0, T] \times \partial\Omega \setminus \omega} \bar{\theta}(u_N) v d\sigma. \quad (60)$$

Since v is arbitrary, we have

$$\kappa \in [\underline{\theta}(u_N), \bar{\theta}(u_N)] = \widehat{\theta}(u_N), \quad (61)$$

where $\sigma(\partial\Omega) < \alpha$. For α as small as possible, we obtain the result.

5. Optimal Control

In this section, we provide a result on the dependence of solutions with respect to the density of the external forces and use it to study the distributed parameter optimal control problem corresponding to it.

Let $f \in L^2(0, T; V^*)$. Under $H(\theta)$, we denote by $S_{u_0}^\theta(f) \subset \mathcal{V}$ the solution set corresponding to f of the problem (EHVI). That is, $u \in \mathcal{W}$ and there exists $\kappa \in L^1([0, T] \times \partial\Omega)$ such that $u(0) = u_0 \in H$, $\kappa \in \partial j(u_N) = \widehat{\theta}(u_N)$ and

$$\left\langle u'(t) + \mathcal{A}u(t) + \mathcal{B}[u(t)], v \right\rangle + \int_{\partial\Omega} \kappa \cdot v_N(z) d\sigma(z) = \langle f(t), v \rangle, \quad (62)$$

for a.e. $t \in [0, T]$ and all $v \in V \cap L^\infty(\partial\Omega)$.

Lemma 9. *Let $f \in L^2(0, T; V^*)$. For every $u \in S_{u_0}^\theta(f)$, there exist $\delta_0, \delta_1 > 0$ such that*

$$\int_{[0, T] \times \partial\Omega} \kappa \cdot u_N d\sigma dt \geq -\delta_0 \delta_1 T \sigma(\partial\Omega), \quad (63)$$

where δ_0 and δ_1 are independent from u .

Proof. By definition of $\underline{\theta}(u_N)$ and $\bar{\theta}(u_N)$, we have for every $\varepsilon > 0$, there exists $\underline{\delta}$ with $|\mu| < \underline{\delta}$ such that

$$\underline{\theta}_\mu(u_N) - \varepsilon \leq \underline{\theta}(u_N) \leq \underline{\theta}_\mu(u_N) + \varepsilon, \quad (64)$$

and there exists $\bar{\delta}$ with $|\mu| < \bar{\delta}$, such that

$$\bar{\theta}_\mu(u_N) - \varepsilon \leq \bar{\theta}(u_N) \leq \bar{\theta}_\mu(u_N) + \varepsilon. \quad (65)$$

It follows that

$$\underline{\theta}_\mu(u_N) - \varepsilon \leq \kappa \leq \bar{\theta}_\mu(u_N) + \varepsilon. \quad (66)$$

That is,

$$\operatorname{ess\,inf}_{|u_N - s| < \mu} \theta(s) - \varepsilon \leq \kappa \leq \operatorname{ess\,sup}_{|u_N - s| < \mu} \theta(s) + \varepsilon. \quad (67)$$

Enlarging the bounds, we obtain

$$\operatorname{ess\,inf}_{u_N - \mu \leq s < +\infty} \theta(s) - \varepsilon \leq \kappa \leq \operatorname{ess\,sup}_{-\infty < s \leq u_N + \mu} \theta(s) + \varepsilon. \quad (68)$$

For ε small enough and $|\mu| < \underline{\delta}, \bar{\delta}$, one has

$$\operatorname{ess\,inf}_{u_N - \underline{\delta} \wedge \bar{\delta} \leq s < +\infty} \theta(s) - \varepsilon \leq \kappa \leq \operatorname{ess\,sup}_{-\infty < s \leq u_N + \underline{\delta} \wedge \bar{\delta}} \theta(s) + \varepsilon. \quad (69)$$

Consequently,

$$\begin{aligned} \sup_{u_N \in]-\infty, -\delta_0 - \underline{\delta} \wedge \bar{\delta}[} \kappa &\leq \operatorname{ess\,sup} \theta(s) + \varepsilon, \\ \inf_{u_N \in]\delta_0 + \underline{\delta} \wedge \bar{\delta}, +\infty[} \kappa &\geq \operatorname{ess\,inf} \theta(s) - \varepsilon, \end{aligned} \quad (70)$$

where δ_0 is defined in $H(\theta)$. Thus, from $H(\theta)$ we obtain

$$\begin{aligned} \sup_{u_N \in]-\infty, -\delta_0 - \underline{\delta} \wedge \bar{\delta}[} \kappa &\leq \varepsilon, \\ \inf_{u_N \in]\delta_0 + \underline{\delta} \wedge \bar{\delta}, +\infty[} \kappa &\geq -\varepsilon. \end{aligned} \quad (71)$$

It results in $\kappa \leq \varepsilon$ if $u_N < -\delta_0 - \underline{\delta} \wedge \bar{\delta} < -\delta_0$ and $\kappa \geq -\varepsilon$ if $u_N > \delta_0 + \underline{\delta} \wedge \bar{\delta} > \delta_0$. We let $\varepsilon \rightarrow 0^+$ to arrive at $\kappa \leq 0$ if $u_N < -\delta_0$ and $\kappa \geq 0$ if $u_N > \delta_0$. Consequently, as $\theta \in L_{\text{loc}}^\infty(\mathbb{R})$ and $\kappa \in \widehat{\theta}(u_N)$, then in the case of $|u_N| \leq \delta_0$, we have

$$\sup_{|u_N| \leq \delta_0} |\kappa| \leq \operatorname{ess\,sup}_{|s| \leq \delta_0} |\theta(s)| := \delta_1. \quad (72)$$

It follows that

$$\begin{aligned} \int_{[0, T] \times \partial\Omega} \kappa \cdot u_N d\sigma dt &= \int_{\{u_N < -\delta_0\}} \kappa \cdot u_N d\sigma dt + \int_{\{u_N > \delta_0\}} \kappa \cdot u_N d\sigma dt \\ &\quad + \int_{\{|u_N| \leq \delta_0\}} \kappa \cdot u_N d\sigma dt \geq 0 - \delta_0 \int_{\{|u_N| \leq \delta_0\}} \kappa \cdot u_N d\sigma dt \\ &\quad \cdot |\kappa| d\sigma dt \geq -\delta_0 \delta_1 T \sigma(\partial\Omega). \end{aligned} \quad (73)$$

Theorem 10. *Under $H(\theta)$, assume that $f_m, f \in L^2(0, T; V^*)$ such that $f_m \rightarrow f$ weakly in \mathcal{V}^* . Let $\{u_m\}_m \subset \mathcal{W}$ be a*

sequence such that $u_m \in S_{u_0}^\theta(f_m)$ for each $m \in \mathbb{N}$; then, we can find a subsequence (still denoted with the same symbol) such that $u_m \rightharpoonup u$ weakly in \mathcal{V} and $u \in S_{u_0}^\theta(f)$.

Proof. Let $f_m, f \in \mathcal{V}^*$ with $f_m \rightharpoonup f$ weakly in \mathcal{V}^* . Let $\{u_m\}_m$ be a sequence such that $u_m \in S_{u_0}^\theta(f_m)$ for each $m \in \mathbb{N}$; then, by Theorem 8, there exists $\kappa_m \in L^1([0, T] \times \partial\Omega)$, such that $\kappa_m(t, x) \in \partial j(u_{mN}(t, x))$ for a.e. $(t, x) \in [0, T] \times \partial\Omega$ and

$$\left\langle u'_m(t) + \mathcal{A}u_m(t) + \mathcal{B}[u_m(t)], v \right\rangle + \int_{\partial\Omega} \kappa_m v_N d\sigma = \langle f_m(t), v \rangle, \quad (74)$$

for a.e. $t \in [0, T]$ and all $v \in V \cap L_N^\infty(\partial\Omega)$. With the same calculations as in the last section, one obtains

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u_m(t)|^2 + M \|u_m(t)\|_V^2 &\leq \frac{2}{M} \|f_m(t)\|_{V^*}^2 \\ &+ \frac{M}{2} \|u(t)\|_V^2 - 2 \int_{\partial\Omega} \kappa_m u_{mN} d\sigma. \end{aligned} \quad (75)$$

It follows that

$$\frac{d}{dt} |u_m(t)|^2 + M \|u_m(t)\|_V^2 \leq \frac{4}{M} \|f_m(t)\|_{V^*}^2 - 2 \int_{\partial\Omega} \kappa_m u_{mN} d\sigma. \quad (76)$$

Integrating over $(0, t)$, we get

$$|u_m(t)|^2 + M \int_0^t \|u_m(s)\|_V^2 ds \leq \frac{4}{M} \|f_m\|_{\mathcal{V}^*}^2 + |u(0)|^2 + 2\delta_0 \delta_1 T \sigma(\partial\Omega). \quad (77)$$

It follows that $\{u_m\}_m$ is bounded in $L^\infty(0, T; H) \cap L^2(0, T; V)$. Hence, by passing to a subsequence if necessary, there exists u such that $\{u_m\}_m$ converges to u weakly in $L^2(0, T; V)$ and weakly * in $L^\infty(0, T; V)$. Using the compactness of the trace operator γ , we may assume that $\gamma u_m \rightharpoonup \gamma u$ in $L^2(0, T; H)$ and then $\gamma u_m(t, z) \rightharpoonup \gamma u(t, z)$ for a.e. $(t, z) \in [0, T] \times \partial\Omega$. Consequently, $u_{mN}(t, z) \rightharpoonup u_N(t, z)$ for a.e. $(t, z) \in [0, T] \times \partial\Omega$. Let us show that there exists $m_0 \in \mathbb{N}$ such that

$$\partial j(u_{mN}) \subset \partial j(u_N), \quad \text{for all } m \geq m_0. \quad (78)$$

As $u_{mN} \rightharpoonup u_N$ for a.e. $(t, z) \in [0, T] \times \partial\Omega$, we can find by Egoroff's theorem that for any $\alpha > 0$, we can determine $\omega \subset [0, T] \times \partial\Omega$ with $(dt \times \sigma)(\omega) < \alpha$, such that

$$u_{mN} \rightharpoonup u_N, \quad \text{uniformly on } [0, T] \times \partial\Omega \setminus \omega, \quad (79)$$

with $u_N \in L^\infty([0, T] \times \partial\Omega \setminus \omega)$. Thus for any $\mu > 0$, there exists m_0 , such that for all $m > m_0$, we have

$$|u_{mN} - u_N| < \frac{\mu}{2}, \quad \text{a.e. on } [0, T] \times \partial\Omega \setminus \omega. \quad (80)$$

By using the triangle inequality, we have

$$\bar{\theta}_\mu(u_{mN}) = \text{ess sup}_{|u_{mN} - \xi| \leq \frac{\mu}{2}} \theta(\xi) \leq \text{ess sup}_{|u_N - \xi| \leq \mu} \theta(\xi) = \bar{\theta}_\mu(u_N). \quad (81)$$

Analogously, we prove the inequality

$$\underline{\theta}_\mu(u_N) \leq \underline{\theta}_\mu(u_{mN}). \quad (82)$$

Taking the limit $\mu \rightarrow 0^+$, we obtain for each $m \geq m_0$

$$\kappa_m \in \widehat{\theta}(u_{mN}) \subset \widehat{\theta}(u_N), \quad \text{a.e. on } [0, T] \times \partial\Omega \setminus \omega, \quad (83)$$

where $(dt \times \sigma)(\omega) < \alpha$. For α as small as possible, we obtain

$$\kappa_m \in \widehat{\theta}(u_{mN}) \subset \widehat{\theta}(u_N), \quad \text{a.e. on } [0, T] \times \partial\Omega, \quad (84)$$

from which we can conclude that the $\{\kappa_m\}_m$ that follows is bounded. By the Dunford-Pettis theorem ([40], p. 239), we will show that the sequence $\{\kappa_m\}_m$ is weakly precompact in $L^1([0, T] \times \partial\Omega)$. For this end, we show that for each $\mu > 0$, there exists $\delta > 0$, such that for $\omega \subset [0, T] \times \partial\Omega$, $(dt \times \sigma)(\omega) < \delta$:

$$\int_\omega |\kappa_m| d\sigma dt < \mu. \quad (85)$$

For some $a > 0$ and remarking that in $\{|u_{mN}| > a\}$, $1 < (|u_{mN}|/a)$, one has

$$\begin{aligned} \int_\omega |\kappa_m| d\sigma dt &= \int_\omega |\kappa_m| 1_{\{|u_{mN}| > a\}} d\sigma dt + \int_\omega |\kappa_m| 1_{\{|u_{mN}| \leq a\}} d\sigma dt \\ &\leq \frac{1}{a} \int_{[0, T] \times \partial\Omega} |\kappa_m u_{mN}| d\sigma dt \\ &\quad + \int_\omega |\kappa_m| 1_{\{|u_{mN}| \leq a\}} d\sigma dt. \end{aligned} \quad (86)$$

From one hand, one has

$$\begin{aligned} \int_{[0, T] \times \partial\Omega} |\kappa_m u_{mN}| d\sigma dt &= \int_{\{|u_{mN}| > \delta_0\}} |\kappa_m u_{mN}| d\sigma dt + \int_{\{|u_{mN}| \leq \delta_0\}} |\kappa_m u_{mN}| d\sigma dt \\ &= \int_{\{|u_{mN}| > \delta_0\}} |\kappa_m u_{mN}| d\sigma dt - \int_{\{|u_{mN}| \leq \delta_0\}} |\kappa_m u_{mN}| d\sigma \\ &\quad + 2 \int_{\{|u_{mN}| \leq \delta_0\}} |\kappa_m u_{mN}| d\sigma dt \\ &\leq \int_{\{|u_{mN}| > \delta_0\}} |\kappa_m u_{mN}| d\sigma dt + \int_{\{|u_{mN}| \leq \delta_0\}} \kappa_m u_{mN} d\sigma dt \\ &\quad + 2 \int_{\{|u_{mN}| \leq \delta_0\}} |\kappa_m u_{mN}| d\sigma dt \\ &= \int_{[0, T] \times \partial\Omega} \kappa_m u_{mN} d\sigma dt + 2 \int_{\{|u_{mN}| \leq \delta_0\}} |\kappa_m u_{mN}| d\sigma dt. \end{aligned} \quad (87)$$

From equation (74) with $v = u_m(t)$

$$\begin{aligned}
\int_{[0,T] \times \partial\Omega} |\kappa_m u_{mN}| \sigma dt &\leq \int_0^T \langle f_m(t), u_m(t) \rangle dt - \int_0^T \langle u'_m(t), u_m(t) \rangle dt \\
&\quad - \int_0^T \langle \mathcal{A}u_m(t), u_m(t) \rangle dt \\
&\quad + 2 \int_{\{|u_{mN}| \leq \delta_0\}} |\kappa_m u_{mN}| d\sigma dt \\
&\leq \int_0^T \|f_m(t)\|_{V^*} \|u_m(t)\|_V dt - \frac{1}{2} |u(T)|^2 \\
&\quad + \frac{1}{2} |u(0)|^2 - M \int_0^T \|u_m(t)\|_V^2 dt 2\delta_0 \int_{\{|u_{mN}| \leq \delta_0\}} |\kappa_m| d\sigma dt \\
&\leq \frac{1}{2} \int_0^T \|f_m(t)\|_{V^*}^2 dt \\
&\quad + \frac{1}{2} \int_0^T \|u_m\|_V^2 dt + \frac{1}{2} |u_0|^2 \\
&\quad + 2\delta_0 (dt \times \sigma)(\{|u_{mN}| \leq \delta_0\}) \sup_{|u_{mN}| \leq \delta_0} |\kappa_m| \\
&\leq c + 2\delta_0 \delta_1 T \sigma(\partial\Omega).
\end{aligned} \tag{88}$$

On the other hand, for each $\varepsilon > 0$ there is $a_\varepsilon > 0$, such that for $|v| < a_\varepsilon$ one has

$$|\kappa_m| \leq \text{ess sup}_{|s-u_{mN}| < v} |\theta(s)| + \varepsilon \leq \text{ess sup}_{|s-u_{mN}| < a_\varepsilon} |\theta(s)| + \varepsilon. \tag{89}$$

This implies

$$\sup_{|u_{mN}| \leq a} |\kappa_m| \leq \text{ess sup}_{|s| < a+a_\varepsilon} |\theta(s)| + \varepsilon. \tag{90}$$

We choose for example $\varepsilon = 1$, which leads to

$$\sup_{|u_{mN}| \leq a} |\kappa_m| \leq \text{ess sup}_{|s| < a+a_1} |\theta(s)| + 1. \tag{91}$$

Now we choose a such that

$$\frac{1}{a} \int_{[0,T] \times \partial\Omega} |\kappa_m u_{mN}| d\sigma dt \leq \frac{1}{a} (c + 2\delta_0 \delta_1 T \sigma(\partial\Omega)) < \frac{\mu}{2}, \tag{92}$$

and δ such that

$$\text{ess sup}_{|s| < a+a_1} |\theta(s)| + 1 < \frac{\mu}{2\delta}. \tag{93}$$

With this choice of δ one have

$$\begin{aligned}
\int_{\omega} |\kappa_m| 1_{\{|u_{mN}| \leq a\}} d\sigma dt &\leq \sup_{|u_{mN}| \leq a} |\kappa_m| (dt \times \sigma)(\omega) \\
&\leq \left(\text{ess sup}_{|s| < a+a_1} |\theta(s)| + 1 \right) (dt \times \sigma)(\omega) \\
&< \frac{\mu}{2\delta} \delta = \frac{\mu}{2}.
\end{aligned} \tag{94}$$

It follows

$$\int_{\omega} |\kappa_m| d\sigma \leq \frac{1}{a} \int_{\partial\Omega} |\kappa_m u_{mN}| d\sigma + \int_{\omega} |\kappa_m| 1_{\{|u_{mN}| \leq a\}} d\sigma < \frac{\mu}{2} + \frac{\mu}{2} = \mu. \tag{95}$$

Consequently, we can extract from $\{\kappa_m\}_m$ a subsequence (denoted with the same symbol) that converges in $L^1((0, T) \times \partial\Omega)$ to some $\kappa \in L^1((0, T) \times \partial\Omega)$. By passing the limit in (74), we get

$$\left\langle u'(t) + \mathcal{A}u(t) + \mathcal{B}[u(t)], v \right\rangle + \int_{\partial\Omega} \kappa v_N d\sigma = \langle f(t), v \rangle, \tag{96}$$

with

$$\kappa \in \text{co}\bar{\text{nv}} \hat{\theta}(u_N) = \hat{\theta}(u_N), \quad \text{a.e. on } [0, T] \times \partial\Omega. \tag{97}$$

Remark 11. We will need Theorem 10 just for external forces in $L^2(0, T; H)$. As in this situation, the duality between V and V^* coincides with the one on H , and this will bring no more difficulties.

Remark 12. One can prove in the same way as in ([32], Theorem 5.1) that the solutions of (EHVI) are stable under the perturbation of θ .

In the remaining of this section, we will use the notation $S(f)$ instead of $S_{u_0}^\theta(f)$. We follow Migórski [34], and we let $\mathcal{U} = L^2(0, T; H)$ be the space of controls and \mathcal{U}_{ad} a non-empty subset of \mathcal{U} consisting of admissible controls. Let $\mathcal{F} : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$ be the objective functional we want to minimize. The control problem reads as follows:

$$\begin{cases} \text{Find a control } \hat{f} \in \mathcal{U}_{ad} \text{ and a state } \hat{u} \in S(\hat{f}) \text{ such that :} \\ \mathcal{F}(\hat{f}, \hat{u}) = \inf \{ \mathcal{F}(f, u) : f \in \mathcal{U}_{ad}, u \in S(f) \}. \end{cases} \tag{98}$$

A pair (\hat{f}, \hat{u}) which solves (98) is called an optimal solution. The existence of such optimal solutions can be proved by using Theorem 10. To do so, we need the following additional hypotheses:

- (1) $H(\mathcal{U}_{ad})$: \mathcal{U}_{ad} is a bounded and weakly closed subset of \mathcal{U}
- (2) $H(\mathcal{F})$: \mathcal{F} is lower semicontinuous with respect to $\mathcal{U} \times \mathcal{V}$ endowed with weak topology

Theorem 13. Assume that $H(\theta)$, $H(\mathcal{U}_{ad})$, and $H(\mathcal{F})$ are fulfilled. Then problem (98) has an optimal solution.

Proof. Let (f_m, u_m) be a minimizing sequence for problem (98), i.e., $f_m \in \mathcal{U}_{ad}$ and $u_m \in S(f_m)$, such that

$$\lim_{m \rightarrow \infty} \mathcal{F}(f_m, u_m) = \inf \{ \mathcal{F}(f, u) : f \in \mathcal{U}_{ad}, u \in S(f) \} =: \vartheta. \quad (99)$$

It follows that the sequence f_m belongs to a bounded subset of the reflexive Banach space \mathcal{V} . We may then assume that $f_m \rightharpoonup \hat{f}$ weakly in \mathcal{V} (by passing to a subsequence if necessary). By $H(\mathcal{U}_{ad})$, we have $\hat{f} \in \mathcal{U}_{ad}$. From Theorem 10, we obtain, by again passing to a subsequence if necessary, that $u_m \rightharpoonup \hat{u}$ weakly in \mathcal{V} with $\hat{u} \in S(\hat{f})$. By $H(\mathcal{F})$, we have $\vartheta \leq \mathcal{F}(\hat{f}, \hat{u}) \leq \liminf_{m \rightarrow \infty} \mathcal{F}(f_m, u_m) = \vartheta$. Which completes the proof.

Next we apply Theorem 13 in a concrete example. Let X be another Hilbert space, $\mathcal{X} := L^2(0, T; X)$, $\mathcal{X}_{ad} \subset \mathcal{X}$ the set of admissible controls, and $C \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ a bounded linear operator from \mathcal{X} to \mathcal{U} . Let $f \in L^2(0, T; V^*)$, as we aim to study the following optimal control problem

$$\begin{cases} \text{Find a control } w \in \mathcal{X}_{ad} \text{ and a state } \hat{u} \in S(C\hat{w} + f) \text{ such that :} \\ \mathcal{J}(\hat{w}, \hat{u}) = \inf \{ \mathcal{J}(w, u) : w \in \mathcal{X}_{ad}, u \in S(Cw + f) \}, \end{cases} \quad (100)$$

where the objective functional is given by

$$\mathcal{J}(w, u) = \int_0^T \int_{\Omega} (u(t, x) - z(t, x))^2 dx dt + \int_0^T h(w(t)) dt, \quad (101)$$

for some function $h : X \rightarrow \mathbb{R}$ and $z \in L^2(0, T; H)$. Such optimal control problems arise in a wide range of applications, particularly in fluid flow control. More specifically, one tries to act on the flow in such a way that a certain flow profile is stabilized or enforced by devices like actuators. Also sensors are used to provide necessary information for the actuation measured here by the control input operator C . Our goal is to minimize the discrepancy between the ideal velocity profile z and the actual flow u . Moreover, the cost related to the actuators and the sensors should be also minimized. A more sophisticated example of this framework is the blood flow in an artificial heart. The goal will be to avoid, among other things, the stagnation causing some serious hemodynamic problems.

Let us first announce the following corollaries of Theorem 10.

Corollary 14. Under $H(\theta)$ assume that φ_m , φ , and $f \in \mathcal{V}^*$, such that $\varphi_m \rightharpoonup \varphi$ weakly in \mathcal{V}^* . Then, for every $u_m \in S(\varphi_m + f)$, we can find a subsequence (still denoted with the same symbol), such that $u_m \rightharpoonup u$ in \mathcal{V} and $u \in S(\varphi + f)$.

Proof. It suffices to take $f_m = \varphi_m + f$ in Theorem 10.

Corollary 15. Under $H(\theta)$, assume that $f \in \mathcal{V}^*$ and $w_m, w \in \mathcal{X}$ are such that w_m converges weakly to w in \mathcal{X} . Then, for every sequence $\{u_m\}_m$, such that $u_m \in S(Cw_m + f)$, we can find a subsequence that converges weakly in $L^2(0, T; V)$ to $u \in S(Cw + f)$.

Proof. It suffices to take $\varphi_m = Cw_m$ in Corollary 14.

Assume the following:

- (i) $f \in L^2(0, T, V^*)$ and $z \in L^2(0, T; H)$
- (ii) \mathcal{X}_{ad} is a weakly compact subset of \mathcal{X}
- (iii) The function $h : X \rightarrow \mathbb{R}$ is convex, lower semicontinuous, and satisfies the coercivity condition

$$|h(w)| \geq \alpha |w|_X^2 + \beta, \quad (102)$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$. $|\cdot|_X$ stands for the norm of the Hilbert space X

Theorem 16. If hypotheses (i)–(iii) and $H(\theta)$ hold, then problem (100) has an optimal solution.

Proof. Let (w_n, u_n) be a minimizing sequence to problem (100), i.e., $w_n \in \mathcal{X}_{ad}$ and $u_n \in S(Cw_n + f)$ such that

$$\lim_{n \rightarrow \infty} \mathcal{J}(w_n, u_n) = \inf \{ \mathcal{J}(w, u) : w \in \mathcal{X}_{ad}, u \in S(Cw + f) \}. \quad (103)$$

Denote $f_m = Cw_m + f$, $\mathcal{F}(f_m, u_m) = \mathcal{J}(w_m, u_m)$, and $\mathcal{U}_{ad} = C\mathcal{X}_{ad}$. It suffices now to apply Theorem 13 for \mathcal{U}_{ad} and \mathcal{F} .

6. Directional Growth Condition

As mentioned in the Introduction, the Rauch condition is a particular case of the directional growth condition due to Naniwicz [31]. It is of common knowledge that the foregoing mentioned conditions are sufficient to establish the existence of solution without any additional growth hypothesis on j . The notion of being a solution needs only to be modified. Here, we will reconsider the same problem of the evolutionary hemivariational Navier-Stokes equations but with the more general condition of directional growth.

Let $j : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function with respect to the first argument and locally Lipschitz with respect to the second argument. We assume the following:

- (1) $H(j)$: there exists $\beta : \partial\Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ integrable with respect to the first argument and nondecreasing with respect to the second argument such that

$$|j(x, \xi) - j(x, \eta)| \leq \beta(x, r)|\xi - \eta|, \quad \forall \xi, \eta \in B(0, r), \quad r \geq 0. \quad (104)$$

- (2) $H(j^0)$: there exists a function $\alpha : \partial\Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ square-integrable with respect to the first argument and nondecreasing with respect to the second argument such that the following estimate holds

$$j^0(x; \xi, \eta - \xi) \leq \alpha(x, r)(1 + |\eta|), \quad (105)$$

for almost every $x \in \partial\Omega$ and for any $\xi, \eta \in \mathbb{R}$ with $-r \leq \eta \leq r$, $r \geq 0$.

Remark that if j does not depend on $x \in \partial\Omega$, then it satisfies $H(j)$ automatically. The hypothesis $H(j^0)$ is called the directional growth condition.

For $x \in \partial\Omega$ and $\xi \in \mathbb{R}$, define $j_\varepsilon(x, \xi) = \mathbf{h}_\varepsilon \star j(x, \cdot)(\xi)$ and denote by j'_ε the derivative of j_ε with respect to the second argument. As usual, let $\{\varphi_1, \varphi_2, \dots\}$ be a basis in $V \cap L^\infty(\partial\Omega)$ and $V_m = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$. We then consider the following regularized problem of the Galerkin type associated with (EHVI), noted $(\mathcal{P}_\varepsilon^m)$: find $u_m \in \mathcal{W}_m$ such that $u_m(0) = u_{0m}$ and

$$\left\langle j'_m(t) + \mathcal{A}u_m(t) + \mathcal{B}[u_m(t)], v \right\rangle + \int_{\partial\Omega} j'_{\varepsilon_m}(u_{mN}) \cdot \nu_N d\sigma = \langle f(t), v \rangle, \quad (106)$$

for all $t \in (0, T)$ and all $v \in V_m$.

Note that due to the integrability of β with respect to the first argument and $H(j)$, the integral above is finite for each $u_m, v \in V_m$. In fact, we have

$$|j'_{\varepsilon_m}(x; u_{mN(x)}) \cdot \nu_N(x)| \leq \beta(x, \|u_N\|_{L^\infty(\partial\Omega)} + 1) \|\nu_N\|_{L^\infty(\partial\Omega)}. \quad (107)$$

Since $\beta(\cdot, \|u_N\|_{L^\infty(\partial\Omega)} + 1) \in L^1(\partial\Omega)$, the integrability of $j'_{\varepsilon_m}(u_{mN}) \cdot \nu_N$ over $\partial\Omega$ follows immediately for any $v \in V_m$. We have the following lemma (cf. [28], Lemma 3.1).

Lemma 17. *Suppose that $H(j^0)$ holds. Then the estimate*

$$j'_\varepsilon(x; \xi)(\eta - \xi) \leq \bar{\alpha}(x, r)(1 + |\xi|), \quad 0 < \varepsilon < 1, \quad (108)$$

is valid for any $\xi, \eta \in \mathbb{R}$ with $-r \leq \eta \leq r$, $r \geq 0$, and almost all $x \in \partial\Omega$, where $\bar{\alpha}(x, r) = 2\alpha(x, r + 1)$.

The problem $(\mathcal{P}_\varepsilon^m)$ has at least one solution in V_m . In fact, substitution of $u_m(t) = \sum_{k=1}^m c_{km}(t)\varphi_k$ gives an initial value problem for a system of first order ordinary differential equations for $c_{km}(\cdot)$, $k = 1, 2, \dots, m$. Its solvability on some

interval $[0, t_m)$ follows from the Carathéodory theorem. This solution can be extended on the closed interval $[0, T]$ by using the a priori estimates below.

Using the coercivity of \mathcal{A} , the properties of $\mathcal{B}[\cdot]$, and the Young inequality, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u_m(t)|_H + M \|u_m(t)\|_V^2 + \int_{\partial\Omega} j'_{\varepsilon_m}(u_{mN}(t)) \cdot u_{mN}(t) d\sigma \\ \leq \frac{M}{2} \|u_m(t)\|_V^2 + \frac{2}{M} \|f(t)\|_{V^*}^2. \end{aligned} \quad (109)$$

From Lemma 17, we have

$$j'_{\varepsilon_m}(u_{mN}) \cdot u_{mN} \geq -\bar{\alpha}(x, 0)(1 + |u_{mN}|). \quad (110)$$

Then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u_m(t)|_H + M \|u_m(t)\|_V^2 &\leq \frac{2}{M} \|f(t)\|_{V^*}^2 \\ &+ \int_{\partial\Omega} \bar{\alpha}(x, 0)(1 + |u_{mN}|) d\sigma \leq \frac{2}{M} \|f(t)\|_{V^*}^2 \\ &+ \sigma(\partial\Omega)^{(1/2)} \|\bar{\alpha}(0)\|_{L^2(\partial\Omega)} \\ &+ \|\bar{\alpha}(0)\|_{L^2(\partial\Omega)} \|u_{mN}(t)\|_{L^2(\partial\Omega)} \\ &\leq \frac{2}{M} \|f(t)\|_{V^*}^2 + \sigma(\partial\Omega)^{(1/2)} \|\bar{\alpha}(0)\|_{L^2(\partial\Omega)} \\ &+ \|\bar{\alpha}(0)\|_{L^2(\partial\Omega)} \|\gamma\| \|u_m(t)\|_V. \end{aligned} \quad (111)$$

Integrating over $(0, t)$ we get

$$\begin{aligned} \frac{1}{2} |u_m(t)|_H - \frac{1}{2} |u_{0m}|_H^2 + \frac{M}{2} \int_0^t \|u_m(s)\|_V^2 ds &\leq \frac{2}{M} \|f\|_{\mathcal{V}^*}^2 \\ &+ \sigma(\partial\Omega)^{(1/2)} \|\bar{\alpha}(0)\|_{L^2(\partial\Omega)} t \\ &+ \|\bar{\alpha}(0)\|_{L^2(\partial\Omega)} \|\gamma\| \int_0^t \|u_{mN}(s)\|_V ds. \end{aligned} \quad (112)$$

It follows

$$\begin{aligned} \frac{1}{2} |u_m(t)|_H + \frac{M}{2} \|u_m\|_V^2 &\leq \frac{1}{2} |u_{0m}|_H^2 + \frac{2}{M} \|f\|_{\mathcal{V}^*}^2 \\ &+ \sigma(\partial\Omega)^{(1/2)} T \|\bar{\alpha}(0)\|_{L^2(\partial\Omega)} \\ &+ \sqrt{T} \|\bar{\alpha}(0)\|_{L^2(\partial\Omega)} \|\gamma\| \|u_m\|_V. \end{aligned} \quad (113)$$

It follows that $\{u_m\}_m$ is a bounded subset of \mathcal{V} and $L^\infty(0, T; H)$, so passing to subsequence, if necessary, we have

$$u_m \longrightarrow u \text{ weakly in } \mathcal{V} \text{ and weakly } -^* \text{ in } L^\infty(0, T; H). \quad (114)$$

Following the same procedure as in section 4, see also [25], we have $u'_m \longrightarrow u'$ weakly in \mathcal{W} and $\mathcal{B}[u_m] \longrightarrow \mathcal{B}[u]$ weakly in \mathcal{V}^* . Using the same proof as in ([31], Lemma

3.3), one can prove that the sequence $\{j'_{\varepsilon_m}(u_{mN})\}$ is weakly precompact in $L^1((0, T) \times \partial\Omega)$. This means that

$$j'_{\varepsilon_m}(u_{mN}) \longrightarrow \kappa \text{ weakly in } L^1((0, T) \times \partial\Omega). \quad (115)$$

Moreover, the following equality holds

$$\left\langle u'_m(t) + \mathcal{A}u_m(t) + \mathcal{B}[u_m(t)], v \right\rangle + \int_{\partial\Omega} j'_{\varepsilon_m}(u_{mN}) \cdot v_N d\sigma = \langle f(t), v \rangle, \quad (116)$$

for almost every $t \in (0, T)$ and $v \in V \cap L^\infty_N(\partial\Omega)$. We pass to the limit as usual to obtain

$$\left\langle u'(t) + \mathcal{A}u(t) + \mathcal{B}[u(t)], v \right\rangle + \int_{\partial\Omega} \kappa(t) \cdot v_N d\sigma = \langle f(t), v \rangle, \quad (117)$$

for almost every $t \in [0, T]$ and $v \in V \cap L^\infty_N(\partial\Omega)$.

We still need to prove that $\kappa(t, x) \in \partial j(x, u_N(t, x))$ for almost every $t \in (0, T)$ and $x \in \partial\Omega$. Since $\gamma u_n \longrightarrow \gamma u$ in $L^2(0, T; L^2(\partial\Omega; \mathbb{R}^d))$, we obtain that $u_{mN} \longrightarrow u_N$ in $L^2(0, T; L^2(\partial\Omega))$ and consequently for almost every $t \in (0, T)$:

$$u_{mN}(t, x) \longrightarrow u_N(t, x) \quad \text{a.e. } x \in \partial\Omega. \quad (118)$$

By Egoroff's theorem, with respect to $x \in \partial\Omega$, we have for any $\rho > 0$, a subset ω of Γ with

$$u_{mN} \longrightarrow u_N \text{ uniformly on } \partial\Omega \setminus \omega, \quad (119)$$

with $u_N \in L^\infty(\partial\Omega \setminus \omega)$. Let $v \in L^\infty_N(\partial\Omega \setminus \omega)$ be arbitrarily given. Due to Fatou's lemma, for any positive $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ and N_ε such that

$$\begin{aligned} & \int_{\partial\Omega \setminus \omega} \frac{j(x, u_{mN}(x) - \tau + v v_N(x)) - j(x, u_m(x) - \tau)}{v} d\sigma \\ & \leq \int_{\partial\Omega \setminus \omega} j^0(x, u_N(x); v_N(x)) d\sigma + \varepsilon, \end{aligned} \quad (120)$$

provided $m > N_\varepsilon$, $|\tau| < \delta_\varepsilon$, and $0 < v < \delta_\varepsilon$. This inequality multiplied by $\mathbf{h}_{\varepsilon_m}$ and integrated over \mathbb{R} yields

$$\begin{aligned} & \int_{\partial\Omega \setminus \omega} \frac{j_{\varepsilon_m}(x, u_{mN}(x) + v v_N(x)) - j_{\varepsilon_m}(x, u_{mN}(x))}{v} d\sigma(x) \\ & = \int_{\mathbb{R}} \mathbf{h}_{\varepsilon_m}(\tau) \int_{\partial\Omega \setminus \omega} \frac{j(x, u_{mN}(x) - \tau + v v_N(x)) - j(x, u_m(x) - \tau)}{v} d\sigma d\tau \\ & \leq \int_{\partial\Omega \setminus \omega} j^0(x, u_N(x); v_N(x)) d\sigma + \varepsilon. \end{aligned} \quad (121)$$

But as $v \longrightarrow 0$, we get

$$\int_{\partial\Omega \setminus \omega} j'_{\varepsilon_m}(x, u_N(x)) \cdot v_N(x) d\sigma \leq \int_{\partial\Omega \setminus \omega} j^0(x, u_N(x); v_N(x)) d\sigma + \varepsilon, \quad (122)$$

which is valid for $m > N_\varepsilon$. Now letting $m \longrightarrow \infty$, we are led to

$$\int_{\partial\Omega \setminus \omega} \kappa \cdot v_N d\sigma \leq \int_{\partial\Omega \setminus \omega} j^0(x, u_N(x); v_N(x)) d\sigma + \varepsilon. \quad (123)$$

Since $\varepsilon > 0$ was chosen arbitrarily

$$\int_{\partial\Omega \setminus \omega} \kappa \cdot v_N d\sigma \leq \int_{\partial\Omega \setminus \omega} j^0(x, u_N(x); v_N(x)) d\sigma, \quad \text{for all } v \in L^\infty_N(\partial\Omega \setminus \omega). \quad (124)$$

But the last inequality easily implies that

$$\kappa(x) \in \partial j(x, u_N(x)), \quad \text{for a.e. } x \in \partial\Omega \setminus \omega, \quad (125)$$

where $\sigma(\omega) < \rho$. Now since ρ was chosen arbitrarily

$$\kappa(x) \in \partial j(x, u_N(x)), \quad \text{for a.e. } x \in \partial\Omega, \quad (126)$$

which completes the proof.

Remark 18. The directional growth condition is meant to study problems involving vector valued functions, i.e., functions on \mathbb{R}^N . Our situation is simpler as $N = 1$. In this case, the direction growth condition can be simplified to the following condition

$$j^0(x, \xi, -\xi) \leq k(x)|\xi|, \quad \forall \xi \in \mathbb{R} \text{ for a.e. } x \in \partial\Omega, \quad (127)$$

for some nonnegative function $k \in L^2(\partial\Omega)$ (cf. [31], Remark 4.1). Moreover, the Rauch condition and sign condition also fulfil the estimate (127) (cf. [31], Remark 4.7).

Remark 19. It is an easy task to check that the results in section 5, regarding optimal solution, are also valid if one replaces the assumption $H(\theta)$ by the more general assumption $H(j^0)$.

Data Availability

There is no data needed in our manuscript.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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Research Article

The Regularity Criteria and the A Priori Estimate on the 3D Incompressible Navier-Stokes Equations in Orthogonal Curvilinear Coordinate Systems

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The paper considers the regularity problem on three-dimensional incompressible Navier-Stokes equations in general orthogonal curvilinear coordinate systems. We establish one regularity criteria of the weak solutions involving only in a vorticity component ω^3 and one a priori estimate on the solution that $\|H_3 u^3\|_{L^\infty(0,T;L^p(\mathbb{R}^3))}$ is bounded for $1 \leq p \leq \infty$ to three-dimensional incompressible Navier-Stokes equations in orthogonal curvilinear coordinate systems. These extent greatly the corresponding results on axisymmetric cylindrical flow.

1. Introduction

In this paper, we investigate the regularity problem on the following three-dimensional (3D) incompressible Navier-Stokes equations

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \Delta \mathbf{u}, \mathbf{x} \in \mathbb{R}^3, t > 0, \\ \operatorname{div} \mathbf{u} = 0, \mathbf{x} \in \mathbb{R}^3, t > 0, \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \mathbf{x} \in \mathbb{R}^3 \end{cases} \quad (1)$$

in general orthogonal curvilinear coordinate systems. Here, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\mathbf{u} = (u_1, u_2, u_3)(t, \mathbf{x})$ denotes the velocity fields, $P = P(t, \mathbf{x})$ is the scalar pressure, and \mathbf{u}_0 is a given initial velocity with $\operatorname{div} \mathbf{u}_0 = 0$.

The existence of global weak solutions to (1) is known since the famous work of Leray [1] (see also Hopf [2] for the bounded domain case) for initial data $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$. The uniqueness and global regularity of Leray-Hopf weak solutions is still one of the most challenging open problems in the mathematical fluid dynamics [1–6]. Many researchers are devoted to looking for certain sufficient con-

ditions to ensure the smoothness of solutions, called the regularity criterion or Serrin-type criterion. Thanks to the pioneering work by [6–8], we have known that the weak solution \mathbf{u} will be smooth as long as

$$\mathbf{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \frac{2}{p} + \frac{3}{q} \leq 1, 2 \leq p \leq \infty, 3 \leq q \leq \infty. \quad (2)$$

Afterwards, there are many progresses on the regularity criteria involving only one component of the velocity fields, one can refer to [9–12] for details.

An interesting problem is to study the globally stabilizing effects of the geometry structures of the domain or/and solutions on the evolution of solution in time to the 3D incompressible Navier-Stokes equations. For example, the axisymmetric flow makes the 3D flow close to 2D flow, that is, all velocity components (radial, angular (or swirl) and x_3 -component) and the pressure are independent of the angular variable in the cylindrical coordinates. It is well known that the 3D incompressible axisymmetric Navier-Stokes equations without swirl have the unique global smooth solution [13–16]. However, it is still open for the global regularity with swirl ([17–20] and therein). These

results indicate that the swirl of the fluid plays a crucial role in the issue of global regularity. Subsequently, to understand this problem better, many efforts have been devoted to looking for suitable regularity criteria, see [21–26] for details.

The paper is motivated by the studies on the axisymmetric flow (see [5, 13, 14, 16]) and the helical flow (see [27] and references therein) of the 3D incompressible flows and on the absence of simple hyperbolic blow-up for the 3D incompressible Euler and quasigeostrophic equations [28], we investigate the regularity criteria of the weak solutions to the 3D incompressible Navier-Stokes equations in general orthogonal curvilinear coordinate systems. Recently, global well-posedness results on the smooth solution for 3D incompressible Navier-Stokes equations in spherical coordinates are obtained in [29–31]. The main purpose of this paper is to establish the a priori estimate and the regularity criteria for the 3D incompressible Navier-Stokes equations in general orthogonal curvilinear coordinate systems.

To state our main results in this paper, let us begin with some notations, see [32]. A point in \mathbb{R}^3 is denoted by $\mathbf{x} = (\xi_1, \xi_2, \xi_3)$, where (ξ_1, ξ_2, ξ_3) is general orthogonal curvilinear coordinates with a line element ds given by

$$ds = H_1 d\xi_1 \mathbf{e}_{\xi_1} + H_2 d\xi_2 \mathbf{e}_{\xi_2} + H_3 d\xi_3 \mathbf{e}_{\xi_3}. \quad (3)$$

Here, $\mathbf{e}_{\xi_i} = \mathbf{e}_{\xi_i}(\xi)(i = 1, 2, 3)$ are orthogonal, of unit length and parallel to the coordinate lines with ξ_i increasing; the nonnegative functions $H_i (i = 1, 2, 3)$ are the G.Lamé coefficients corresponding to the vectors $\mathbf{e}_{\xi_1}, \mathbf{e}_{\xi_2}, \mathbf{e}_{\xi_3}$, respectively. Throughout this paper, we assume $H_i (i = 1, 2, 3)$ are independent of ξ_3 , i.e., $H_i = H_i(\xi_1, \xi_2) (i = 1, 2, 3)$, and the measure of the set $\{(\xi_1, \xi_2) \in D : (H_1 H_2 H_3)(\xi_1, \xi_2) = 0\}$ is zero in the sense of Lebesgue measure in \mathbb{R}^2 .

In this paper, we consider the solution (u, P) to the 3D incompressible Navier-Stokes (1) of having the form

$$\begin{aligned} \mathbf{u}(t, \mathbf{x}) &= u^1(t, \xi_1, \xi_2) \mathbf{e}_{\xi_1} + u^2(t, \xi_1, \xi_2) \mathbf{e}_{\xi_2} \\ &\quad + u^3(t, \xi_1, \xi_2) \mathbf{e}_{\xi_3}, \quad P(t, \mathbf{x}) = P(t, \xi_1, \xi_2), \end{aligned} \quad (4)$$

with the initial data

$$\mathbf{u}_0(\mathbf{x}) = u_0^1(\xi_1, \xi_2) \mathbf{e}_{\xi_1} + u_0^2(\xi_1, \xi_2) \mathbf{e}_{\xi_2} + u_0^3(\xi_1, \xi_2) \mathbf{e}_{\xi_3}. \quad (5)$$

Our main results are the a priori estimates and the regularity criteria involving only in a vorticity component ω^3 on 3D incompressible Navier-Stokes equations in general orthogonal curvilinear coordinate systems.

Theorem 1 (the a priori estimate of $\|H_3 u^3\|_{L^\infty(0, T; L^p(\mathbb{R}^3))}$). Suppose that \mathbf{u} be a smooth solution of system (1) with the form (4) and the initial data (5) satisfying $\operatorname{div} \mathbf{u}_0 = 0$ and $H_3 u_0^3 \in L^p(\mathbb{R}^3)$ for $p \in [1, \infty]$. Then, we have for any $T > 0$,

$$H_3 u^3 \in L^\infty(0, T; L^2(\mathbb{R}^3)) \text{ and } |H_3 u^3| \in L^2(0, T; H^1(\mathbb{R}^3)), \quad (6)$$

and, moreover, if assume that the G.Lamé coefficient $H_3(\xi_1, \xi_2)$ satisfies

$$|\Delta(\ln H_3)| \leq C < \infty, \quad (7)$$

it holds, for any $T > 0$ and any $p \in [1, 2) \cup (2, \infty]$, that

$$H_3 u^3 \in L^\infty(0, T; L^p(\mathbb{R}^3)) \text{ and } |H_3 u^3|^{p/2} \in L^2(0, T; H^1(\mathbb{R}^3)). \quad (8)$$

Theorem 2 (the regularity criteria involving ω^3). Let \mathbf{u} be a weak solution of system (1) with the form (4) and the initial data (5) satisfying $\mathbf{u}_0 \in H^2(\mathbb{R}^3)$ and $\operatorname{div} \mathbf{u}_0 = 0$. Then, the solution \mathbf{u} is smooth in $(0, T] \times \mathbb{R}^3$, if

$$\omega^3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad (9)$$

where $(2/p) + (3/q) \leq 2$, $1 \leq p \leq \infty$, $3/2 \leq q \leq \infty$.

Remark 3. The assumption (7) comes from the geometry on the harmonic mapping in some sense. It is easy to see that

$$\Delta(\ln H_3) = \tilde{\Delta}(\ln H_3) \quad (10)$$

in (7), based on the notation $\tilde{\Delta}$ introduced in Section 2, because H_3 is independent of ξ_3 . Thus, if H_3 is a radial function in \mathbb{R}^2 , i.e.,

$$H_3 = H_3(r) \quad (11)$$

with

$$r = \sqrt{x_1^2 + x_2^2} \text{ or } r = \sqrt{x_2^2 + x_3^2} \text{ or } r = \sqrt{x_1^2 + x_3^2}, \quad (12)$$

then

$$\Delta(\ln H_3) \equiv 0. \quad (13)$$

In this case, the assumption (7) is naturally true because the function $\ln H_3$ is the harmonic one.

Remark 4. The assumption (7) of Theorem 1 is satisfied in the cases of cylindrical coordinates. More precisely, we have the known results on 3D problem in the case of cylindrical coordinate system in \mathbb{R}^3 are covered in Theorem 1 and Theorem 2, i.e., let

$$\xi_1 = r = \sqrt{x_1^2 + x_2^2}, \xi_2 = x_2, \xi_3 = \theta = \arctan \frac{x_3}{x_1};$$

$$\begin{aligned} u^1(t, \xi_1, \xi_2) &= u^r(t, r, x_2), u^2(t, \xi_1, \xi_2) \\ &= u_3(t, r, x_2), u^3(t, \xi_1, \xi_2) = u^\theta(t, r, x_2); \end{aligned}$$

$$\begin{aligned} \mathbf{e}_{\xi_1} &= \mathbf{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \mathbf{e}_{\xi_2} = \mathbf{e}_3 = (0, 0, 1), \mathbf{e}_{\xi_3} \\ &= \mathbf{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0 \right); \end{aligned} \quad (14)$$

we consider an axisymmetric solution of the Navier-Stokes equations of the form (4) in the cylindrical coordinate system, where the mapping is taken as $x = x(\xi_1, \xi_2, \xi_3) = (\xi_1 \cos \xi_3, \xi_2, \xi_1 \sin \xi_3)$, and G.Lamé coefficients are $H_1 = H_2 = 1, H_3 = \xi_1$, satisfying the assumption in Theorem 1. Then, Theorem 1 is equivalent to Proposition 1 in [9], Theorem 2 is equivalent to Theorem 1.3 in [21].

Remark 5. This difference from the case of curvilinear cylindrical coordinates may be imply that one should care about the advantage or overcome the difficulty brought by the choice of curvilinear coordinates, including nonorthogonal curvilinear coordinates, which will be discussed in the future.

The remaining of this paper is organized as follows. In Section 2, we will derive the Navier-Stokes equations in orthogonal curvilinear coordinate systems. In Section 3, we introduce some basic lemmas and one estimate used for the proof of main theorems. In Section 4 and Section 5, we prove Theorem 1 and Theorem 2 separately.

2. Navier-Stokes Equations in Orthogonal Curvilinear Coordinate Systems

In this section, we will first derive the incompressible Navier-Stokes equations in orthogonal curvilinear coordinate systems ξ_1, ξ_2, ξ_3 , given by Section 1.

We assume that $\mathbf{x} = \mathbf{x}(\xi)$, being one-to-one and onto mapping, transforms $\xi \in \tilde{\Omega} \subset \mathbb{R}^3$ into $\mathbf{x} \in \Omega \subset \mathbb{R}^3$ with $\tilde{\Omega} = D \times [\alpha, \beta] \subset \mathbb{R}^3$ and

$$\Omega = \{(x_1, x_2, x_3) = (x_1, x_2, x_3)(\xi) \in \mathbb{R}^3 : (\xi_1, \xi_2) \in D \subset \mathbb{R}^2, -\infty < \alpha \leq \xi_3 \leq \beta < \infty\}, \quad (15)$$

where the domain D is the bounded or unbounded domain of \mathbb{R}^2 with the smooth boundary ∂D if D is bounded, and the constants α and β satisfy $-\infty < \alpha \leq \beta < \infty$.

Since $H_i = H_i(\xi_1, \xi_2)$ ($i = 1, 2, 3$), by the derivatives of the unit vectors $\mathbf{e}_{\xi_1}, \mathbf{e}_{\xi_2}$ and \mathbf{e}_{ξ_3} , we have

$$\begin{aligned} \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_1} &= -\frac{\mathbf{e}_{\xi_2}}{H_2} \frac{\partial H_1}{\partial \xi_2}, \quad \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_2} = \frac{\mathbf{e}_{\xi_2}}{H_1} \frac{\partial H_2}{\partial \xi_1}, \quad \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_3} = \frac{\mathbf{e}_{\xi_3}}{H_1} \frac{\partial H_3}{\partial \xi_1}, \\ \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_1} &= \frac{\mathbf{e}_{\xi_1}}{H_2} \frac{\partial H_1}{\partial \xi_2}, \quad \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_2} = -\frac{\mathbf{e}_{\xi_1}}{H_1} \frac{\partial H_2}{\partial \xi_1}, \quad \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_3} = \frac{\mathbf{e}_{\xi_3}}{H_2} \frac{\partial H_3}{\partial \xi_2}, \\ \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_1} &= \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_2} = 0, \quad \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_3} = -\frac{\mathbf{e}_{\xi_1}}{H_1} \frac{\partial H_3}{\partial \xi_1} - \frac{\mathbf{e}_{\xi_2}}{H_2} \frac{\partial H_3}{\partial \xi_2}. \end{aligned} \quad (16)$$

Using the definition of gradient, we get the expression of the gradient operator ∇ in orthogonal curvilinear coordinate systems:

$$\nabla = \mathbf{e}_{\xi_1} \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{1}{H_2} \frac{\partial}{\partial \xi_2} + \mathbf{e}_{\xi_3} \frac{1}{H_3} \frac{\partial}{\partial \xi_3}, \quad (17)$$

we also obtain the expression of Laplacian operator Δ in orthogonal curvilinear coordinate systems:

$$\Delta = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial}{\partial \xi_2} \right) + \frac{\partial}{\partial \xi_3} \left(\frac{H_1 H_2}{H_3} \frac{\partial}{\partial \xi_3} \right) \right]. \quad (18)$$

Furthermore, for a vector field $\mathbf{V} = \mathbf{V}(\xi_1, \xi_2, \xi_3) = V_1 \mathbf{e}_{\xi_1} + V_2 \mathbf{e}_{\xi_2} + V_3 \mathbf{e}_{\xi_3}$, we get the expressions of $\text{div } \mathbf{V}$ and $\text{rot } \mathbf{V}$ in orthogonal curvilinear coordinate systems:

$$\text{div } \mathbf{V} = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} (H_2 H_3 V_1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 V_2) + \frac{\partial}{\partial \xi_3} (H_1 H_2 V_3) \right], \quad (19)$$

$$\text{rot } \mathbf{V} = \frac{1}{H_1 H_2 H_3} \begin{vmatrix} H_1 \mathbf{e}_{\xi_1} & H_2 \mathbf{e}_{\xi_2} & H_3 \mathbf{e}_{\xi_3} \\ \frac{\partial}{\partial \xi_1} & \frac{\partial}{\partial \xi_2} & \frac{\partial}{\partial \xi_3} \\ H_1 V_1 & H_2 V_2 & H_3 V_3 \end{vmatrix}. \quad (20)$$

By the above expressions (17)–(19), then taking the inner product of equation (1)₁ with $\mathbf{e}_{\xi_1}, \mathbf{e}_{\xi_2}, \mathbf{e}_{\xi_3}$, respectively, we can derive the Navier-Stokes equations in orthogonal curvilinear coordinate systems as follows:

$$\begin{cases} \frac{\partial u^1}{\partial t} + \frac{u^1}{H_1} \frac{\partial u^1}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial u^1}{\partial \xi_2} + \frac{1}{H_1} \frac{\partial P}{\partial \xi_1} = \left[L_1 u^1 - \frac{1}{(H_1 H_3)^2} \left(\frac{\partial H_3}{\partial \xi_1} \right)^2 u^1 + L_2 u^2 - \frac{1}{H_1 H_2 (H_3)^2} \frac{\partial H_3}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_2} u^2 \right] - \frac{u^1 u^2}{H_1 H_2} \frac{\partial H_1}{\partial \xi_2} + \frac{(u^2)^2}{H_1 H_2} \frac{\partial H_2}{\partial \xi_1} + \frac{(u^3)^2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_2}, \\ \frac{\partial u^2}{\partial t} + \frac{u^1}{H_1} \frac{\partial u^2}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial u^2}{\partial \xi_2} + \frac{1}{H_2} \frac{\partial P}{\partial \xi_2} = \left[L_1 u^2 - \frac{1}{(H_2 H_3)^2} \left(\frac{\partial H_3}{\partial \xi_2} \right)^2 u^2 - L_2 u^1 - \frac{1}{H_1 H_2 (H_3)^2} \frac{\partial H_3}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_2} u^1 \right] - \frac{u^1 u^2}{H_1 H_2} \frac{\partial H_2}{\partial \xi_1} + \frac{(u^1)^2}{H_1 H_2} \frac{\partial H_1}{\partial \xi_2} + \frac{(u^3)^2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1}, \\ \frac{\partial u^3}{\partial t} + \frac{u^1}{H_1} \frac{\partial u^3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial u^3}{\partial \xi_2} = L_3 u^3 - \frac{u^3}{H_3} \left(\frac{u^1}{H_1} \frac{\partial H_3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_3}{\partial \xi_2} \right), \end{cases} \quad (21)$$

where $(\xi_1, \xi_2) \in D \subset \mathbb{R}^2$, $t > 0$ and

$$\begin{aligned}\nabla_{\xi_1, \xi_2} &= \mathbf{e}_{\xi_1} \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{1}{H_2} \frac{\partial}{\partial \xi_2}, \\ \tilde{\Delta} &= \Delta_{\xi_1, \xi_2} = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial}{\partial \xi_2} \right) \right], \\ L_1 &= \Delta_{\xi_1, \xi_2} - \frac{1}{(H_1 H_2)^2} \left(\left(\frac{\partial H_1}{\partial \xi_2} \right)^2 + \left(\frac{\partial H_2}{\partial \xi_1} \right)^2 \right), \\ L_2 &= \frac{1}{H_1 H_2 H_3} \left(\frac{2 H_3}{H_1} \frac{\partial H_1}{\partial \xi_2} \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_1} \left(\frac{H_3}{H_1} \frac{\partial H_1}{\partial \xi_2} \right) \right. \\ &\quad \left. - \frac{2 H_3}{H_2} \frac{\partial H_2}{\partial \xi_1} \frac{\partial}{\partial \xi_2} - \frac{\partial}{\partial \xi_2} \left(\frac{H_3}{H_2} \frac{\partial H_2}{\partial \xi_1} \right) \right), \\ L_3 &= \Delta_{\xi_1, \xi_2} - \frac{1}{(H_3)^2} \left(\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right).\end{aligned}\quad (22)$$

The incompressible constraint is

$$\frac{\partial}{\partial \xi_1} (H_2 H_3 u^1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 u^2) = 0. \quad (23)$$

It is clear that equations (21) and (23) completely determine the evolution of 3D Navier-Stokes equations in orthogonal curvilinear coordinate systems, respectively, once the initial value and/or boundary conditions are given.

We take initial condition for the system (21) as follows:

$$(u^1, u^2, u^3)(t=0, \xi_1, \xi_2) = (u_0^1, u_0^2, u_0^3)(\xi_1, \xi_2). \quad (24)$$

Moreover, the boundary condition $\mathbf{u} \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty, t \geq 0$ is equivalent to the following condition

$$(u^1, u^2, u^3)|_{\partial D} = 0, t \geq 0, \quad (25)$$

if the domain D is bounded or of having partially bounded boundary.

By the expressions (4) and (20), using the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, in orthogonal curvilinear coordinate systems, we have

$$\boldsymbol{\omega}(t, \mathbf{x}) = \omega^1(t, \xi_1, \xi_2) \mathbf{e}_{\xi_1} + \omega^2(t, \xi_1, \xi_2) \mathbf{e}_{\xi_2} + \omega^3(t, \xi_1, \xi_2) \mathbf{e}_{\xi_3}, \quad (26)$$

with the initial vorticity

$$\boldsymbol{\omega}_0 = \boldsymbol{\omega}(t=0, \mathbf{x}) = \omega_0^1(\xi_1, \xi_2) \mathbf{e}_{\xi_1} + \omega_0^2(\xi_1, \xi_2) \mathbf{e}_{\xi_2} + \omega_0^3(\xi_1, \xi_2) \mathbf{e}_{\xi_3}, \quad (27)$$

Where

$$\begin{aligned}\omega^1 &= \frac{1}{H_2 H_3} \frac{\partial}{\partial \xi_2} (H_3 u^3), \omega^2 = -\frac{1}{H_1 H_3} \frac{\partial}{\partial \xi_1} (H_3 u^3), \omega^3 \\ &= \frac{1}{H_1 H_2} \left(\frac{\partial}{\partial \xi_1} (H_2 u^2) - \frac{\partial}{\partial \xi_2} (H_1 u^1) \right).\end{aligned}\quad (28)$$

Moreover, with the help of (28), we can get the equation of ω^3 from (21) as follows:

$$\begin{aligned}\frac{\partial \omega^3}{\partial t} + \frac{u^1}{H_1} \frac{\partial \omega^3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial \omega^3}{\partial \xi_2} &= L_3 \omega^3 + \frac{\omega^3}{H_3} \left(\frac{u^1}{H_1} \frac{\partial H_3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_3}{\partial \xi_2} \right) \\ &+ \frac{1}{H_1 H_2 (H_3)^2} \left[\frac{\partial H_3}{\partial \xi_2} \frac{\partial (H_3 u^3)^2}{\partial \xi_1} - \frac{\partial H_3}{\partial \xi_1} \frac{\partial (H_3 u^3)^2}{\partial \xi_2} \right].\end{aligned}\quad (29)$$

3. Some Useful Estimates

To study the main estimates of Theorem 1 and Theorem 2, we need to introduce two basic lemmas and one estimate relates to ω^3 in orthogonal curvilinear coordinate systems.

Lemma 6 (see [6–8]). *Suppose that the initial data $\mathbf{u}_0 \in H^2(\mathbb{R}^3)$ in (1), then any Leray-Hopf weak solution \mathbf{u} of 3D incompressible Navier-Stokes equations (1) is also a smooth solution in $(0, T] \times \mathbb{R}^3$, if there holds that*

$$\mathbf{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad (30)$$

in which p and q satisfy the conditions

$$\frac{2}{p} + \frac{3}{q} \leq 1 \text{ with } 3 < q \leq \infty, 2 \leq p < \infty \text{ or } 3 \leq q < \infty, p = \infty. \quad (31)$$

And we would like to recall the well-known relation between the velocity and vorticity of 3D flow.

Lemma 7 (see [33]). *Let $\mathbf{u} \in W^{1,p}(\mathbb{R}^3)$ be a velocity field with its vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, then the inequality*

$$\|\nabla \mathbf{u}\|_{L^p(\mathbb{R}^3)} \leq C_p \left(\|\boldsymbol{\omega}\|_{L^p(\mathbb{R}^3)} + \|\operatorname{div} \mathbf{u}\|_{L^p(\mathbb{R}^3)} \right) \quad (32)$$

holds for any $p \in (1, \infty)$, where the constant C_p depends only on p .

As one kind of fluid with the special geometry structure of form (4), the incompressible 3D flow also has one particular property, which is shown as follows.

Proposition 8. *Suppose that $\mathbf{u} \in W^{1,p}(\mathbb{R}^3)$ with the form (4) be a field with zero divergence, then the estimate*

$$\|\nabla \tilde{\mathbf{u}}\|_{L^p(\mathbb{R}^3)} \leq \tilde{C}_p \|\omega^3\|_{L^p(\mathbb{R}^3)} \quad (33)$$

holds for any $p \in (1, \infty)$, where

$$\tilde{\mathbf{u}} = u^1(t, \xi_1, \xi_2) \mathbf{e}_{\xi_1} + u^2(t, \xi_1, \xi_2) \mathbf{e}_{\xi_2} \quad (34)$$

and the constant \tilde{C}_p depends only on p .

Proof. Since $\operatorname{div} \mathbf{u} = 0$ and we have

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{u} = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} (H_2 H_3 u^1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 u^2) + \frac{\partial}{\partial \xi_3} (H_1 H_2 u^3) \right] \\ &= \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} (H_2 H_3 u^1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 u^2) \right]. \end{aligned} \quad (35)$$

Thus, we obtain

$$\operatorname{div} \tilde{\mathbf{u}} = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} (H_2 H_3 u^1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 u^2) \right] = 0. \quad (36)$$

On the other hand, by the expressions (20), (34), and (28), we get

$$\begin{aligned} \nabla \times \tilde{\mathbf{u}} &= \frac{1}{H_2 H_3} \left[-\frac{\partial}{\partial \xi_3} (H_2 u^2) \right] \mathbf{e}_{\xi_1} + \frac{1}{H_1 H_3} \left[\frac{\partial}{\partial \xi_3} (H_1 u^1) \right] \mathbf{e}_{\xi_2} \\ &\quad + \frac{1}{H_1 H_2} \left[\frac{\partial}{\partial \xi_1} (H_2 u^2) - \frac{\partial}{\partial \xi_2} (H_1 u^1) \right] \mathbf{e}_{\xi_3} \\ &= \left[\frac{1}{H_1 H_2} \left(\frac{\partial}{\partial \xi_1} (H_2 u^2) - \frac{\partial}{\partial \xi_2} (H_1 u^1) \right) \right] \mathbf{e}_{\xi_3} = \omega^3 \mathbf{e}_{\xi_3}. \end{aligned} \quad (37)$$

Consequently, by Lemma 7 and using (36) and (37), one has

$$\|\nabla \tilde{\mathbf{u}}\|_{L^p(\mathbb{R}^3)} \leq \tilde{C}_p \|\omega^3 \mathbf{e}_{\xi_3}\|_{L^p(\mathbb{R}^3)} = \tilde{C}_p \|\omega^3\|_{L^p(\mathbb{R}^3)}. \quad (38)$$

This finishes the proof of the proposition.

4. Proof of Theorem 1

In this section, we prove Theorem 1.

Proof of Theorem 1. Let $F(t, \xi_1, \xi_2) = H_3(\xi_1, \xi_2) u^3(t, \xi_1, \xi_2)$, then putting $u^3(t, \xi_1, \xi_2) = F(t, \xi_1, \xi_2)/H_3(\xi_1, \xi_2)$ into (21)₃, we can obtain the equation for F

$$\begin{aligned} \partial_t F &+ \frac{u^1}{H_1} \frac{\partial F}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial F}{\partial \xi_2} \\ &= \Delta_{\xi_1, \xi_2} F - \frac{1}{H_1 H_2} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) F + \frac{2H_2}{H_1 H_3} \frac{\partial F}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_1} \right. \\ &\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) F + \frac{2H_1}{H_2 H_3} \frac{\partial F}{\partial \xi_2} \frac{\partial H_3}{\partial \xi_2} \right] \\ &\quad - \frac{F}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right], \end{aligned} \quad (39)$$

with the help of the following calculations

$$\begin{aligned} \partial_t u^3 &= \frac{1}{H_3} \partial_t F, \\ \frac{u^1}{H_1} \frac{\partial u^3}{\partial \xi_1} &+ \frac{u^2}{H_2} \frac{\partial u^3}{\partial \xi_2} \\ &= \frac{u^1}{H_1} \left(\frac{1}{H_3} \frac{\partial F}{\partial \xi_1} - \frac{F}{H_3^2} \frac{\partial H_3}{\partial \xi_1} \right) + \frac{u^2}{H_2} \left(\frac{1}{H_3} \frac{\partial F}{\partial \xi_2} - \frac{F}{H_3^2} \frac{\partial H_3}{\partial \xi_2} \right) \\ &= \frac{1}{H_3} \left(\frac{u^1}{H_1} \frac{\partial F}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial F}{\partial \xi_2} \right) - \frac{F}{H_3^2} \left(\frac{u^1}{H_1} \frac{\partial H_3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_3}{\partial \xi_2} \right), \\ L_3 u^3 &= \frac{1}{H_3} \left\{ \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial F}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial F}{\partial \xi_2} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{H_2 H_3}{H_1} \frac{\partial^2 F}{\partial \xi_1^2} \right) + \left(\frac{H_1 H_3}{H_2} \frac{\partial^2 F}{\partial \xi_2^2} \right) \right] \right. \\ &\quad \left. - \frac{F}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] \right\} \\ &\quad + \frac{1}{H_1 H_2 H_3} \left[F \frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial}{\partial \xi_1} \right) + \frac{2H_2 H_3}{H_1} \frac{\partial F}{\partial \xi_1} \frac{\partial}{\partial \xi_1} \right. \\ &\quad \left. + F \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial}{\partial \xi_2} \right) + \frac{2H_1 H_3}{H_2} \frac{\partial F}{\partial \xi_2} \frac{\partial}{\partial \xi_2} \right] \frac{1}{H_3} \\ &= \frac{1}{H_3} \Delta_{\xi_1, \xi_2} F - \frac{1}{H_3} \frac{1}{H_1 H_2 H_3} \\ &\quad \cdot \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) F + \frac{2H_2}{H_1 H_3} \frac{\partial F}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_1} \right. \\ &\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) F + \frac{2H_1}{H_2 H_3} \frac{\partial F}{\partial \xi_2} \frac{\partial H_3}{\partial \xi_2} \right] \\ &\quad - \frac{F}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right]. \end{aligned} \quad (40)$$

Multiplying the both sides of equation (39) by $|F|^{p-2} F$ and integrating over \mathbb{R}^3 , we have

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|F\|_{L^p(\mathbb{R}^3)}^p &+ \frac{4(p-1)}{p^2} \|\nabla^- |F|^p\|_{L^2(\mathbb{R}^3)}^2 \\ &+ \int_{\mathbb{R}^3} |F|^{p-1} \left(\frac{u^1}{H_1} \frac{\partial F}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial F}{\partial \xi_2} \right) dx \\ &= - \int_{\mathbb{R}^3} \frac{1}{H_1 H_2} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) F^p + \frac{2}{p} \frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \frac{\partial F^p}{\partial \xi_1} \right. \\ &\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) F^p + \frac{2}{p} \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] dx \\ &\quad - \int_{\mathbb{R}^3} \frac{F^p}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] dx \\ &= - \int_{\mathbb{R}^3} \frac{1}{H_1 H_2} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) F^p + \frac{H_2}{H_1 H_3} \frac{\partial F^p}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_1} \right. \\ &\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) F^p + \frac{H_1}{H_2 H_3} \frac{\partial F^p}{\partial \xi_2} \frac{\partial H_3}{\partial \xi_2} \right] dx \\ &\quad + \frac{F^p}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] dx \\ &\quad - \int_{\mathbb{R}^3} \left(\frac{2}{p} - 1 \right) \frac{1}{H_1 H_2} \left[\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \frac{\partial F^p}{\partial \xi_1} + \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] dx = I_1 + I_2. \end{aligned} \quad (41)$$

Here, thanks to (15) and (25) and the incompressibility condition (23), by the fact of $d\mathbf{x} = H_1 H_2 H_3 d\xi_1 d\xi_2 d\xi_3$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^3} |F|^{p-1} \left(\frac{u^1}{H_1} \frac{\partial F}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial F}{\partial \xi_2} \right) d\mathbf{x} \\
&= \frac{1}{p} (\beta - \alpha) \int_D \left(\frac{u^1}{H_1} \frac{\partial}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial}{\partial \xi_2} \right) |F|^p(t, \xi_1, \xi_2) H_1 H_2 H_3 d\xi_1 d\xi_2 \\
&= \frac{1}{p} (\beta - \alpha) \int_D \left[(H_2 H_3 u^1) \frac{\partial}{\partial \xi_1} + (H_1 H_3 u^2) \frac{\partial}{\partial \xi_2} \right] |F|^p(t, \xi_1, \xi_2) d\xi_1 d\xi_2 \\
&= -\frac{1}{p} (\beta - \alpha) \int_D \left[\frac{\partial}{\partial \xi_1} (H_2 H_3 u^1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 u^2) \right] |F|^p(t, \xi_1, \xi_2) d\xi_1 d\xi_2 \equiv 0
\end{aligned} \tag{42}$$

For the I_1 and I_2 on the right of (41), by simple integration, one has

$$\begin{aligned}
I_1 &= - \int_{\mathbb{R}^3} \frac{1}{H_1 H_2} \left[F^p \frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) + \frac{H_2}{H_1 H_3} \frac{\partial F^p}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_1} \right. \\
&\quad \left. + F^p \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) + \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] \\
&\quad + \frac{F^p}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] d\mathbf{x} \\
&= -(\beta - \alpha) \int_D H_3 \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) F^p + \frac{H_2}{H_1 H_3} \frac{\partial F^p}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_1} \right. \\
&\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) F^p + \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] d\xi_1 d\xi_2 \\
&\quad - (\beta - \alpha) \int_D \frac{H_1 H_2}{H_3} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] F^p d\xi_1 d\xi_2 \\
&= -(\beta - \alpha) \int_D H_3 \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} F^p \right) \right. \\
&\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} F^p \right) \right] d\xi_1 d\xi_2 \\
&\quad - (\beta - \alpha) \int_D \frac{H_1 H_2}{H_3} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] F^p d\xi_1 d\xi_2 \\
&= (\beta - \alpha) \int_D \left[\frac{H_2}{H_1 H_3} \left(\frac{\partial H_3}{\partial \xi_1} \right)^2 + \frac{H_1}{H_2 H_3} \left(\frac{\partial H_3}{\partial \xi_2} \right)^2 \right] F^p d\xi_1 d\xi_2 \\
&\quad - (\beta - \alpha) \int_D \left[\frac{H_2}{H_1 H_3} \left(\frac{\partial H_3}{\partial \xi_1} \right)^2 + \frac{H_1}{H_2 H_3} \left(\frac{\partial H_3}{\partial \xi_2} \right)^2 \right] F^p d\xi_1 d\xi_2 \equiv 0
\end{aligned} \tag{43}$$

$$\begin{aligned}
I_2 &= - \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} \frac{1}{H_1 H_2} \left[\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \frac{\partial F^p}{\partial \xi_1} + \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] d\mathbf{x} \\
&= - \left(\frac{2}{p} - 1 \right) (\beta - \alpha) \int_D \left[\frac{H_2}{H_1} \frac{\partial H_3}{\partial \xi_1} \frac{\partial F^p}{\partial \xi_1} + \frac{H_1}{H_2} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] d\xi_1 d\xi_2 \\
&= \left(\frac{2}{p} - 1 \right) (\beta - \alpha) \int_D F^p \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1} \frac{\partial H_3}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right) \right] d\xi_1 d\xi_2 \\
&= \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} \frac{F^p}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1} \frac{\partial H_3}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right) \right] d\mathbf{x} \\
&= \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} \frac{F^p}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{1}{H_3} \frac{\partial H_3}{\partial \xi_1} \right) \right. \\
&\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{1}{H_3} \frac{\partial H_3}{\partial \xi_2} \right) \right] d\mathbf{x} \\
&= \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} \frac{F^p}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial(\ln H_3)}{\partial \xi_1} \right) \right. \\
&\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial(\ln H_3)}{\partial \xi_2} \right) \right] d\mathbf{x} \\
&= \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} F^p \Delta(\ln H_3) d\mathbf{x}.
\end{aligned} \tag{44}$$

Combining (42)–(44) with (41), we have

$$\begin{aligned}
& \frac{1}{p} \frac{d}{dt} \|F\|_{L^p(\mathbb{R}^3)}^p + \frac{4(p-1)}{p^2} \left\| \nabla_{\xi_1, \xi_2} |F|^{\frac{p}{2}} \right\|_{L^2(\mathbb{R}^3)}^2 \\
&= \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} F^p \Delta(\ln H_3) d\mathbf{x}.
\end{aligned} \tag{45}$$

Note that $\|\nabla_{\xi_1, \xi_2} |F|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^3)} = \|\nabla |F|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^3)}$.

If $p = 2$, it has, immediately, by Gronwall's inequality,

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |H_3 u^3|^p d\mathbf{x} + C \int_0^T \int_{\mathbb{R}^3} |\nabla |H_3 u^3|^{\frac{p}{2}}|^2 d\mathbf{x} dt \\
&\leq |H_3 u_0^3|_{L^p(\mathbb{R}^3)}^p.
\end{aligned} \tag{46}$$

If $p \in [1, 2) \cup (2, \infty]$, from (45) and (7), the right hand of (45) can be estimated by

$$\begin{aligned}
& \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} F^p \Delta(\ln H_3) d\mathbf{x} \\
&\leq C \int_{\mathbb{R}^3} F^p |\Delta(\ln H_3)| d\mathbf{x} \leq C \|F\|_{L^p(\mathbb{R}^3)}.
\end{aligned} \tag{47}$$

and, then, by Gronwall's inequality again, one also has

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |H_3 u^3|^p d\mathbf{x} + C \int_0^T \int_{\mathbb{R}^3} |\nabla |H_3 u^3|^{\frac{p}{2}}|^2 d\mathbf{x} dt \\
&\leq C \left(T, \|H_3 u_0^3\|_{L^p(\mathbb{R}^3)}^p \right).
\end{aligned} \tag{48}$$

The case $p = \infty$ is immediate if we let $p \rightarrow \infty$ in (48). Thus, we finish the proof of Theorem 1.

5. Proof of Theorem 2

In this section, we prove Theorem 2.

Proof of Theorem 2. From (1:1)₁, we have the basic energy inequality, for any $T > 0$,

$$\sup_{0 \leq t \leq T} \|u\|_{L^2(\mathbb{R}^3)}^2 + 2 \int_0^T \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \leq \|u_0\|_{L^2(\mathbb{R}^3)}^2. \tag{49}$$

It is known that the vorticity equation for the vorticity $\omega = \nabla \times \mathbf{u}$ for 3D incompressible Navier-Stokes equation is the following:

$$\partial_t \omega + (\mathbf{u} \cdot \nabla) \omega - \Delta \omega = (\omega \cdot \nabla) \mathbf{u}. \quad (50)$$

Using the expressions (4), (17), and (26), we have

$$\begin{aligned} (\omega \cdot \nabla) \mathbf{u} &= (\omega^1 \mathbf{e}_{\xi_1} + \omega^2 \mathbf{e}_{\xi_2} + \omega^3 \mathbf{e}_{\xi_3}) \\ &\quad \cdot \left(\mathbf{e}_{\xi_1} \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{1}{H_2} \frac{\partial}{\partial \xi_2} + \mathbf{e}_{\xi_3} \frac{1}{H_3} \frac{\partial}{\partial \xi_3} \right) (u^1 \mathbf{e}_{\xi_1} + u^2 \mathbf{e}_{\xi_2} + u^3 \mathbf{e}_{\xi_3}) \\ &= \left(\omega^1 \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \omega^2 \frac{1}{H_2} \frac{\partial}{\partial \xi_2} + \omega^3 \frac{1}{H_3} \frac{\partial}{\partial \xi_3} \right) (u^1 \mathbf{e}_{\xi_1} + u^2 \mathbf{e}_{\xi_2} + u^3 \mathbf{e}_{\xi_3}) \\ &= \omega^1 \frac{1}{H_1} \left[\left(u^1 \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_1} + \mathbf{e}_{\xi_1} \frac{\partial u^1}{\partial \xi_1} \right) + \left(u^2 \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{\partial u^2}{\partial \xi_1} \right) + \left(u^3 \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_1} + \mathbf{e}_{\xi_3} \frac{\partial u^3}{\partial \xi_1} \right) \right] \\ &\quad + \omega^2 \frac{1}{H_2} \left[\left(u^1 \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_2} + \mathbf{e}_{\xi_1} \frac{\partial u^1}{\partial \xi_2} \right) + \left(u^2 \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_2} + \mathbf{e}_{\xi_2} \frac{\partial u^2}{\partial \xi_2} \right) + \left(u^3 \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_2} + \mathbf{e}_{\xi_3} \frac{\partial u^3}{\partial \xi_2} \right) \right] \\ &\quad + \omega^3 \frac{1}{H_3} \left[\left(u^1 \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_3} + \mathbf{e}_{\xi_1} \frac{\partial u^1}{\partial \xi_3} \right) + \left(u^2 \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_3} + \mathbf{e}_{\xi_2} \frac{\partial u^2}{\partial \xi_3} \right) + \left(u^3 \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_3} + \mathbf{e}_{\xi_3} \frac{\partial u^3}{\partial \xi_3} \right) \right] \\ &= \left(\frac{\omega^1 \partial u^1}{H_1 \partial \xi_1} + \frac{\omega^2 \partial u^1}{H_2 \partial \xi_2} + \frac{\omega^1 u^2 \partial H_1}{H_1 H_2 \partial \xi_2} - \frac{\omega^2 u^2 \partial H_2}{H_1 H_2 \partial \xi_1} - \frac{\omega^3 u^3 \partial H_3}{H_1 H_3 \partial \xi_1} \right) \mathbf{e}_{\xi_1} \\ &\quad + \left(\frac{\omega^1 \partial u^2}{H_1 \partial \xi_1} + \frac{\omega^2 \partial u^2}{H_2 \partial \xi_2} - \frac{\omega^1 u^1 \partial H_1}{H_1 H_2 \partial \xi_2} - \frac{\omega^2 u^1 \partial H_2}{H_1 H_2 \partial \xi_1} - \frac{\omega^3 u^3 \partial H_3}{H_2 H_3 \partial \xi_2} \right) \mathbf{e}_{\xi_2} \\ &\quad + \left(\frac{\omega^1 \partial u^3}{H_1 \partial \xi_1} + \frac{\omega^2 \partial u^3}{H_2 \partial \xi_2} + \frac{\omega^3 u^1 \partial H_3}{H_1 H_3 \partial \xi_1} + \frac{\omega^3 u^2 \partial H_3}{H_2 H_3 \partial \xi_2} \right) \mathbf{e}_{\xi_3}. \end{aligned} \quad (51)$$

Then, by multiplying ω on the both sides of equation (50) and integrating over \mathbb{R}^3 , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2 &= \int_{\mathbb{R}^3} (\omega \cdot \nabla) \mathbf{u} \cdot \omega \, dx \\ &= \int_{\mathbb{R}^3} \left[\left(\frac{\omega^1 \partial u^1}{H_1 \partial \xi_1} + \frac{\omega^2 \partial u^1}{H_2 \partial \xi_2} + \frac{\omega^1 u^2 \partial H_1}{H_1 H_2 \partial \xi_2} \right. \right. \\ &\quad \left. \left. - \frac{\omega^2 u^2 \partial H_2}{H_1 H_2 \partial \xi_1} - \frac{\omega^3 u^3 \partial H_3}{H_1 H_3 \partial \xi_1} \right) \omega^1 \right. \\ &\quad \left. + \left(\frac{\omega^1 \partial u^2}{H_1 \partial \xi_1} + \frac{\omega^2 \partial u^2}{H_2 \partial \xi_2} - \frac{\omega^1 u^1 \partial H_1}{H_1 H_2 \partial \xi_2} \right. \right. \\ &\quad \left. \left. - \frac{\omega^2 u^1 \partial H_2}{H_1 H_2 \partial \xi_1} - \frac{\omega^3 u^3 \partial H_3}{H_2 H_3 \partial \xi_2} \right) \omega^2 \right. \\ &\quad \left. + \left(\frac{\omega^1 \partial u^3}{H_1 \partial \xi_1} + \frac{\omega^2 \partial u^3}{H_2 \partial \xi_2} + \frac{\omega^3 u^1 \partial H_3}{H_1 H_3 \partial \xi_1} + \frac{\omega^3 u^2 \partial H_3}{H_2 H_3 \partial \xi_2} \right) \omega^3 \right] dx \\ &= \int_{\mathbb{R}^3} \frac{\omega^1 \omega^1}{H_1} \left(\frac{\partial u^1}{\partial \xi_1} + \frac{u^2 \partial H_1}{H_2 \partial \xi_2} \right) dx + \int_{\mathbb{R}^3} \frac{\omega^1 \omega^2}{H_1} \left(\frac{\partial u^2}{\partial \xi_1} - \frac{u^1 \partial H_1}{H_2 \partial \xi_2} \right) dx \\ &\quad + \int_{\mathbb{R}^3} \frac{\omega^1 \omega^3}{H_1} \frac{\partial u^3}{\partial \xi_1} dx + \int_{\mathbb{R}^3} \frac{\omega^1 \omega^2}{H_2} \left(\frac{\partial u^1}{\partial \xi_2} - \frac{u^2 \partial H_2}{H_1 \partial \xi_1} \right) dx \\ &\quad + \int_{\mathbb{R}^3} \frac{\omega^2 \omega^2}{H_2} \left(\frac{\partial u^2}{\partial \xi_2} + \frac{u^1 \partial H_2}{H_1 \partial \xi_1} \right) dx + \int_{\mathbb{R}^3} \frac{\omega^2 \omega^3}{H_2} \frac{\partial u^3}{\partial \xi_2} dx \\ &\quad - \int_{\mathbb{R}^3} \frac{\omega^2 \omega^3 u^3 \partial H_3}{H_2 H_3 \partial \xi_2} dx - \int_{\mathbb{R}^3} \frac{\omega^1 \omega^3 u^3 \partial H_3}{H_1 H_3 \partial \xi_1} dx \\ &\quad + \int_{\mathbb{R}^3} \frac{\omega^3 \omega^3}{H_3} \left(\frac{u^1 \partial H_3}{H_1 \partial \xi_1} + \frac{u^2 \partial H_3}{H_2 \partial \xi_2} \right) dx = \sum_{i=1}^3 J_i. \end{aligned} \quad (52)$$

Just for the sake of clarity, we give the expressions as follows:

$$\begin{aligned} \nabla \mathbf{u} &= \left(\mathbf{e}_{\xi_1} \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{1}{H_2} \frac{\partial}{\partial \xi_2} + \mathbf{e}_{\xi_3} \frac{1}{H_3} \frac{\partial}{\partial \xi_3} \right) (u^1 \mathbf{e}_{\xi_1} + u^2 \mathbf{e}_{\xi_2} + u^3 \mathbf{e}_{\xi_3}) \\ &= \frac{\mathbf{e}_{\xi_1} \otimes \mathbf{e}_{\xi_1}}{H_1} \left(\frac{\partial u^1}{\partial \xi_1} + \frac{u^2 \partial H_1}{H_2 \partial \xi_2} \right) + \frac{\mathbf{e}_{\xi_1} \otimes \mathbf{e}_{\xi_2}}{H_1} \left(\frac{\partial u^2}{\partial \xi_1} - \frac{u^1 \partial H_1}{H_2 \partial \xi_2} \right) \\ &\quad + \frac{\mathbf{e}_{\xi_1} \otimes \mathbf{e}_{\xi_3}}{H_1} \frac{\partial u^3}{\partial \xi_1} + \frac{\mathbf{e}_{\xi_2} \otimes \mathbf{e}_{\xi_1}}{H_2} \left(\frac{\partial u^1}{\partial \xi_2} - \frac{u^2 \partial H_2}{H_1 \partial \xi_1} \right) \\ &\quad + \frac{\mathbf{e}_{\xi_2} \otimes \mathbf{e}_{\xi_2}}{H_2} \left(\frac{\partial u^2}{\partial \xi_2} + \frac{u^1 \partial H_2}{H_1 \partial \xi_1} \right) + \frac{\mathbf{e}_{\xi_2} \otimes \mathbf{e}_{\xi_3}}{H_2} \frac{\partial u^3}{\partial \xi_2} \\ &\quad + \frac{\mathbf{e}_{\xi_3} \otimes \mathbf{e}_{\xi_1}}{H_3} \left(-\frac{u^3 \partial H_3}{H_1 \partial \xi_1} \right) + \frac{\mathbf{e}_{\xi_3} \otimes \mathbf{e}_{\xi_2}}{H_3} \left(-\frac{u^3 \partial H_3}{H_2 \partial \xi_2} \right) \\ &\quad + \frac{\mathbf{e}_{\xi_3} \otimes \mathbf{e}_{\xi_3}}{H_3} \left(\frac{u^1 \partial H_3}{H_1 \partial \xi_1} + \frac{u^2 \partial H_3}{H_2 \partial \xi_2} \right) \end{aligned} \quad (53)$$

$$\begin{aligned} \nabla \tilde{\mathbf{u}} &= \left(\mathbf{e}_{\xi_1} \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{1}{H_2} \frac{\partial}{\partial \xi_2} + \mathbf{e}_{\xi_3} \frac{1}{H_3} \frac{\partial}{\partial \xi_3} \right) (u^1 \mathbf{e}_{\xi_1} + u^2 \mathbf{e}_{\xi_2}) \\ &= \frac{\mathbf{e}_{\xi_1} \otimes \mathbf{e}_{\xi_1}}{H_1} \left(\frac{\partial u^1}{\partial \xi_1} + \frac{u^2 \partial H_1}{H_2 \partial \xi_2} \right) + \frac{\mathbf{e}_{\xi_1} \otimes \mathbf{e}_{\xi_2}}{H_1} \left(\frac{\partial u^2}{\partial \xi_1} - \frac{u^1 \partial H_1}{H_2 \partial \xi_2} \right) \\ &\quad + \frac{\mathbf{e}_{\xi_2} \otimes \mathbf{e}_{\xi_1}}{H_2} \left(\frac{\partial u^1}{\partial \xi_2} - \frac{u^2 \partial H_2}{H_1 \partial \xi_1} \right) + \frac{\mathbf{e}_{\xi_2} \otimes \mathbf{e}_{\xi_2}}{H_2} \left(\frac{\partial u^2}{\partial \xi_2} + \frac{u^1 \partial H_2}{H_1 \partial \xi_1} \right). \end{aligned} \quad (54)$$

Then, we estimate each integral $J_i (i = 1, \dots, 9)$ in the right-hand side of (52) by using the relation (53) and the relation (54), respectively.

On the one hand, by applying Proposition 8, Hölder inequality, Sobolev's imbedding inequality, and Young inequality, it follows, with the help of the relation (54), that

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^3} \frac{\omega^1 \omega^1}{H_1} \left(\frac{\partial u^1}{\partial \xi_1} + \frac{u^2 \partial H_1}{H_2 \partial \xi_2} \right) dx \leq \int_{\mathbb{R}^3} |\omega| |\omega| |\nabla \tilde{\mathbf{u}}| dx \\ &\leq \|\omega\|_{L^{\frac{2q}{q-1}}(\mathbb{R}^3)}^2 \|\nabla \tilde{\mathbf{u}}\|_{L^q(\mathbb{R}^3)} \leq C \|\omega\|_{L^{\frac{2q}{q-1}}(\mathbb{R}^3)}^2 \|\omega^3\|_{L^q(\mathbb{R}^3)} \\ &\leq C \|\omega\|_{L^2(\mathbb{R}^3)}^{\frac{2q-3}{q}} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^{\frac{3}{q}} \|\omega^3\|_{L^q(\mathbb{R}^3)} \\ &\leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{18} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2. \end{aligned} \quad (55)$$

The term J_2 can be estimated similarly to J_1 as

$$\begin{aligned} J_2 &= \int_{\mathbb{R}^3} \frac{\omega^1 \omega^2}{H_1} \left(\frac{\partial u^2}{\partial \xi_1} - \frac{u^1 \partial H_1}{H_2 \partial \xi_2} \right) dx \\ &\leq \int_{\mathbb{R}^3} |\omega| |\omega| |\nabla \tilde{\mathbf{u}}| dx \leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 \\ &\quad + \frac{1}{18} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2. \end{aligned} \quad (56)$$

Similarly,

$$\begin{aligned}
 J_4 &= \int_{\mathbb{R}^3} \frac{\omega^1 \omega^2}{H_2} \left(\frac{\partial u^1}{\partial \xi_2} - \frac{u^2}{H_1} \frac{\partial H_2}{\partial \xi_1} \right) d\mathbf{x} \leq \int_{\mathbb{R}^3} |\omega| |\omega| |\nabla \tilde{u}| d\mathbf{x} \\
 &\leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{18} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2 \\
 J_5 &= \int_{\mathbb{R}^3} \frac{\omega^2 \omega^2}{H_2} \left(\frac{\partial u^2}{\partial \xi_2} + \frac{u^1}{H_1} \frac{\partial H_2}{\partial \xi_1} \right) d\mathbf{x} \leq C \int_{\mathbb{R}^3} |\omega| |\omega| |\nabla \tilde{u}| d\mathbf{x} \\
 &\leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{18} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2.
 \end{aligned} \tag{57}$$

On the other hand, by using Lemma 7, Hölder inequality, Sobolev imbedding inequality, and Young inequality, it yields, with the help of the relation (53), that

$$\begin{aligned}
 J_3 &= \int_{\mathbb{R}^3} \frac{\omega^1 \omega^3}{H_1} \frac{\partial u^3}{\partial \xi_1} d\mathbf{x} \leq \int_{\mathbb{R}^3} |\omega| |\nabla u| |\omega^3| d\mathbf{x} \\
 &\leq \|\omega\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{q-1}} \|\nabla u\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{q-1}} \|\omega^3\|_{L^q(\mathbb{R}^3)} \\
 &\leq C \|\omega\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{q-1}} \|\omega^3\|_{L^q(\mathbb{R}^3)} \\
 &\leq C \|\omega\|_{L^2(\mathbb{R}^3)}^{\frac{2q-3}{q}} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^{\frac{3}{q}} \|\omega^3\|_{L^q(\mathbb{R}^3)} \\
 &\leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{18} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2.
 \end{aligned} \tag{58}$$

The terms J_6, J_7, J_8 and J_9 can be estimated similarly to J_3 as

$$\begin{aligned}
 J_6 + J_7 + J_8 + J_9 &\leq C \int_{\mathbb{R}^3} |\omega| |\nabla u| |\omega^3| d\mathbf{x} \\
 &\leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{2}{9} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2.
 \end{aligned} \tag{59}$$

Combining all above estimates about $J_i (i=1, \dots, 9)$, from (52), we obtain

$$\frac{d}{dt} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2 \leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2. \tag{60}$$

Applying Gronwall's inequality, it yields, for any $0 \leq t \leq T$

$$\omega \in L^\infty(0, T; L^2(\mathbb{R}^3)). \tag{61}$$

Together with Lemma 7 and (49), it implies that, for any $0 \leq t \leq T$

$$u \in L^\infty(0, T; H^1(\mathbb{R}^3)). \tag{62}$$

By Sobolev's imbedding Theorem, we obtain that, for any $0 \leq t \leq T$

$$u \in L^\infty(0, T; L^6(\mathbb{R}^3)). \tag{63}$$

Hence, with the help of Lemma 6, we finish the proof of Theorem 2.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

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Research Article

An Existence Result for a Generalized Quasilinear Schrödinger Equation with Nonlocal Term

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In this paper, we consider the following generalized quasilinear Schrödinger equation with nonlocal term $-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \lambda[|x|^{-\mu} * |u|^p]|u|^{p-2}u$, $x \in \mathbb{R}^N$, where $N \geq 3$, $g: \mathbb{R} \rightarrow \mathbb{R}^+$ is a C^1 even function, $g(0) = 1$, $g'(s) \geq 0$ is for all $s \geq 0$, $\lim_{|s| \rightarrow +\infty} g(s)/|s|^{\alpha-1} = \beta > 0$ is for some $\alpha > 1$, and $(\alpha - 1)g(s) \geq g'(s)s$ is for all $s \geq 0$, $2\alpha \leq p \leq 2\alpha(N - \mu)/N - 2$, and $0 < \mu < N$.

We prove that the equation admits a solution by using a constrained minimization argument.

1. Introduction and Preliminaries

The main purpose of this paper is to investigate the existence of solutions for the following generalized quasilinear Schrödinger equation with nonlocal term

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \lambda[|x|^{-\mu} * |u|^p]|u|^{p-2}u, x \in \mathbb{R}^N, \quad (1)$$

where $N \geq 3$, $g: \mathbb{R} \rightarrow \mathbb{R}^+$ is a C^1 even function, $g(0) = 1$, $g'(s) \geq 0$ is for all $s \geq 0$, $\lim_{|s| \rightarrow +\infty} g(s)/|s|^{\alpha-1} = \beta > 0$ is for some $\alpha > 1$, and $(\alpha - 1)g(s) \geq g'(s)s$ is for all $s \geq 0$, $2\alpha \leq p \leq 2\alpha(N - \mu)/N - 2$, and $0 < \mu < N$.

When $g(u) = 1$, (1) boils down to the so-called nonlinear Choquard or Choquard-Pekar equation

$$-\Delta u + V(x)u = \lambda[|x|^{-\mu} * |u|^p]|u|^{p-2}u, x \in \mathbb{R}^N. \quad (2)$$

Such like equation has several physical origins. The problem

$$-\Delta u + u = [|x|^{-1} * |u|^2]u, x \in \mathbb{R}^3 \quad (3)$$

appeared at least as early as in 1954, in a work by Pekar describing the quantum mechanics of a polaron at rest [1]. In 1976, Choquard used (3) to describe an electron trapped in its own hole and in a certain approximation to Hartree-Fock theory of one component plasma [2]. In 1996, Penrose proposed (3) as a model of self-gravitating matter, in a program in which quantum state reduction is understood as a gravitational phenomenon [3]. In this context, equation of type (3) is usually called the nonlinear Schrödinger-Newton equation. The first investigations for existence and symmetry of the solutions to (3) go back to the works of Lieb [2] and Lions [4]. In [2], by using symmetric decreasing rearrangement inequalities, Lieb proved that the ground state solution of equation (3) is radial and unique up to translations. Lions

[4] showed the existence of a sequence of radially symmetric solutions. Since then, many efforts have been made to study the existence of nontrivial solutions for nonlinear Choquard equations. Wei and Winter [5] showed that the ground state solution is nondegenerate. Ma and Zhao [6] considered the generalized Choquard equation

$$-\Delta u + u = [|x|^{-\mu} * |u|^q] |u|^{q-2} u \quad (q \geq 2) \quad (4)$$

and proved that every positive solution of it is radially symmetric and monotone decreasing about some fixed point, under the assumption that a certain set of real numbers, defined in terms of N, μ , and q , is nonempty. Under the same assumption, Cingolani, Clapp, and Secchi [7] gave some existence and multiplicity results in the electromagnetic case and established the regularity and some decay asymptotically at infinity of the ground states. In [8], Moroz and Van Schaftingen eliminated this restriction and showed the regularity, positivity, and radial symmetry of the ground states for the optimal range of parameters and derived decay asymptotically at infinity for them as well. Moreover, they [9] also obtained a similar conclusion under the assumption of Berestycki-Lions type nonlinearity. We point out that the existence, multiplicity, and concentration of such like equation have been established by many authors. We refer the readers to [10, 11] for the existence of sign-changing solutions, [5, 12] for the existence and concentration behavior of the semiclassical solutions and [13] for the critical nonlocal part with respect to the Hardy-Littlewood-Sobolev inequality. For more details associated with the Choquard equation, please refer to [14–16] and the references therein.

In the past, even the research on the existence of solitary wave solutions for the Schrödinger equation with local term

$$-\operatorname{div} (g^2(u) \nabla u) + g(u) g'(u) |\nabla u|^2 + V(x) u = f(x, u), \quad x \in \mathbb{R}^N \quad (5)$$

is for some given special function $g(\cdot)$, see [17–19]. However, related to the nonlocal equation (1), as far as we know, there is no result in this direction. In this paper, with the aid of the new variable replacement developed by Shen and Wang in [18] and inspired by [20, 21], existence of solutions for equation (1) have been established. Problem (1) has a variational structure, and the corresponding energy functional is defined by

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx \\ &\quad - \frac{\lambda}{2p} \int_{\mathbb{R}^N} [|x|^{-\mu} * |u|^p] |u|^p dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx \\ &\quad - \frac{\lambda}{2p} \int_{\mathbb{R}^{2N}} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\mu} dx dy. \end{aligned} \quad (6)$$

However, I is not well defined in $H^1(\mathbb{R}^N)$ because of the term $\int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx$. To overcome this difficulty, we make a change of variable constructed by Shen and Wang in [18]: $v := G(u) := \int_0^u g(t) dt$. Then, we obtain

$$\begin{aligned} J(v) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) G^{-1}(v)^2 dx \\ &\quad - \frac{\lambda}{2p} \int_{\mathbb{R}^N} [|x|^{-\mu} * |G^{-1}(v)|^p] |G^{-1}(v)|^p dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) G^{-1}(v)^2 dx \\ &\quad - \frac{\lambda}{2p} \int_{\mathbb{R}^{2N}} \frac{|G^{-1}(v(x))|^p |G^{-1}(v(y))|^p}{|x-y|^\mu} dx dy. \end{aligned} \quad (7)$$

We say that u is a weak solution of (1), if

$$\begin{aligned} \langle I'(u), \varphi \rangle &= \int_{\mathbb{R}^N} \left\{ g^2(u) \nabla u \nabla \varphi + g(u) g'(u) |\nabla u|^2 \varphi \right. \\ &\quad \left. + V(x) u \varphi - \lambda [|x|^{-\mu} * |u|^p] |u|^{p-2} u \varphi \right\} dx = 0 \end{aligned} \quad (8)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Let $\varphi = (1/g(u))\psi$. By [18], we know that the above formula is equivalent to

$$\begin{aligned} \langle J'(v), \psi \rangle &= \int_{\mathbb{R}^N} \left\{ \nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi \right. \\ &\quad \left. - \lambda \frac{[|x|^{-\mu} * |G^{-1}(v)|^p] |G^{-1}(v)|^{p-2} G^{-1}(v) \psi}{g(G^{-1}(v))} \right\} dx = 0 \end{aligned} \quad (9)$$

for all $\psi \in C_0^\infty(\mathbb{R}^N)$. Therefore, in order to find the solution of (1), it suffices to study the solution of the following equation:

$$-\Delta v + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} - \lambda \frac{[|x|^{-\mu} * |G^{-1}(v)|^p] |G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))} = 0. \quad (10)$$

In this paper, we assume that the following condition holds.

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $0 < V_0 := \inf_{x \in \mathbb{R}^N} V(x)$, and $\lim_{|x| \rightarrow \infty} V(x) = +\infty$.

Set $H_V^1(\mathbb{R}^N) = \{v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} [|\nabla v|^2 + V(x) v^2] dx < +\infty\}$ with the norm

$$\|v\|_{H_V^1}^2 = \int_{\mathbb{R}^N} [|\nabla v|^2 + V(x) v^2] dx. \quad (11)$$

Then, by the proof of Lemma 4 in [22], the embedding

$H_V^1(\mathbb{R}^N)^\circ L^t(\mathbb{R}^N)$ is compact for all $t \in [2, 2^*)$. Moreover, for any $a > 0$, we define $m_a := \inf_{v \in M_a} E(v)$, where

$$M_a := \left\{ v \in H_V^1(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|G^{-1}(v(x))|^p |G^{-1}(v(y))|^p}{|x-y|^\mu} dx dy = a \right\}$$

$$E(v) := \int_{\mathbb{R}^N} [|\nabla v|^2 + V(x)G^{-1}(v)^2] dx. \quad (12)$$

Our main result is the following:

Theorem 1. *Suppose that (V) is satisfied, then, there exists $\lambda_n \rightarrow +\infty$ such that equation (1) with $\lambda = \lambda_n$ has a solution.*

2. Proof of Theorem 1

To begin with, we give some lemmas.

Lemma 2 (see [23, 24]). *The functions g , G and G^{-1} possess the following properties:*

- (1) $G(s) \leq g(s)s \leq \alpha G(s)$ for all $s \geq 0$; $G(s) \geq g(s)s \geq \alpha G(s)$ for all $s \leq 0$
- (2) $G^{-1}(s)s/g(G^{-1}(s)) \leq |G^{-1}(s)|^2 \leq \alpha(G^{-1}(s)s/g(G^{-1}(s)))$ for all $s \in \mathbb{R}$
- (3) $|s|^\alpha \leq (\alpha/\beta) |G(s)|$ for all $s \in \mathbb{R}$

Proposition 3 [25] (Hardy-Littlewood-Sobolev inequality). *Let $r, t > 1$ and $0 < \mu < N$ with $(1/r) + (\mu/N) + (1/t) = 2$. Let $g \in L^t(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. Then, there exists a sharp constant $C_{r,N,\mu,t}$ independent of g and h such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^\mu} dx dy \right| \leq C_{r,N,\mu,t} \|g\|_r \|h\|_t. \quad (13)$$

Proof of Theorem 1. The proof consists of two steps.

Step 1: we prove that for each $a > 0$, m_a is achieved at some $v_a \in M_a$, which is a weak solution of equation (10) with $\lambda = \lambda_a$ satisfying $\lambda_a \in [(m_a/\alpha a), (\alpha m_a/a)]$.

For fixed $a > 0$, let $\{v_n\} \subset M_a$ be a minimizing sequence for m_a , i.e., $v_n \in H_V^1(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^{2N}} (|G^{-1}(v_n(x))|^p |G^{-1}(v_n(y))|^p / |x-y|^\mu) dx dy = a$ such that $E(v_n) \rightarrow m_a$ as $n \rightarrow \infty$. We assert that there exists a constants $C_1 > 0$ such that $E(v_n) \geq C_1 \int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x)v_n^2] dx$. Indeed, we may assume that $v_n \neq 0$ (otherwise, the conclusion is trivial). If the conclusion is false, then for any positive integer n , we may assume that

$$E(v_n) = \int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x)G^{-1}(v_n)^2] dx < \frac{1}{n} \|v_n\|_{H_V^1}^2 \quad (14)$$

$$= \frac{1}{n} \int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x)v_n^2] dx.$$

Set $w_n = v_n / \|v_n\|_{H_V^1}$ and $g_n = G^{-1}(v_n)^2 / \|v_n\|_{H_V^1}^2$. Then,

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(x)g_n(x) dx \rightarrow 0, \quad (15)$$

which implies that

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx \rightarrow 0, \int_{\mathbb{R}^N} V(x)g_n(x) dx \rightarrow 0, \int_{\mathbb{R}^N} V(x)w_n^2 dx \rightarrow 1 \quad (16)$$

as $n \rightarrow \infty$. Then for each $\varepsilon > 0$, there exists a constant $C_2 > 0$ independent of n such that $meas(\Omega_n) < \varepsilon$, where $\Omega_n := \{x \in \mathbb{R}^N : |G^{-1}(v_n(x))| \geq C_2\}$. Otherwise, there exist $\varepsilon_0 > 0$ and a subsequence $\{G^{-1}(v_{n_k})\}$ of $\{G^{-1}(v_n)\}$ such that for any positive integer k ,

$$meas(\Omega_{n_k}) \geq \varepsilon_0 > 0, \quad (17)$$

where $\Omega_{n_k} = \{x \in \mathbb{R}^N : |G^{-1}(v_{n_k}(x))| \geq k\}$. By (V), one has

$$C \geq E(v_{n_k}) \geq \int_{\mathbb{R}^N} V(x)G^{-1}(v_{n_k}(x))^2 dx$$

$$\geq \int_{\Omega_{n_k}} V(x)G^{-1}(v_{n_k}(x))^2 dx \geq V_0 k^2 \varepsilon_0 \rightarrow +\infty \quad (18)$$

as $k \rightarrow +\infty$, a contradiction. Noting that as $|G^{-1}(v_n)| < C_2$, by Lemma 2 (1) and monotonicity of g , we have

$$v_n^2 \leq g^2(G^{-1}(v_n))G^{-1}(v_n)^2 \leq g^2(C_2)G^{-1}(v_n)^2. \quad (19)$$

Hence,

$$\int_{\mathbb{R}^N \setminus \Omega_n} V(x)w_n^2 dx \leq g^2(C_2) \int_{\mathbb{R}^N} V(x)g_n(x) dx \rightarrow 0. \quad (20)$$

By the integral absolutely continuity, there exists $\varepsilon > 0$ such that whenever $\Omega \subset \mathbb{R}^N$ and $meas(\Omega) < \varepsilon$, $\int_{\Omega} V(x)w_n^2 dx < 1/2$. For this ε , one has

$$\int_{\mathbb{R}^N} V(x)w_n^2 dx = \int_{\Omega_n} V(x)w_n^2 dx + \int_{\mathbb{R}^N \setminus \Omega_n} V(x)w_n^2 dx \leq \frac{1}{2}$$

$$+ \int_{\mathbb{R}^N \setminus \Omega_n} V(x)w_n^2 dx, \quad (21)$$

which implies $1 \leq 1/2$, a contradiction. Therefore, up to a subsequence, there exists $v_a \in H_V^1(\mathbb{R}^N)$ such that $v_n \rightarrow v_a$ in $H_V^1(\mathbb{R}^N)$, $v_n \rightarrow v_a$ in $L^t(\mathbb{R}^N)$ for $2 \leq t < 2^*$, and $v_n(x) \rightarrow v_a(x)$, a.e., on \mathbb{R}^N . By means of the definition of weak

convergence, we know

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^N} |\nabla v_n - \nabla v_a|^2 dx &= \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \\ &\quad + \int_{\mathbb{R}^N} |\nabla v_a|^2 dx - 2 \int_{\mathbb{R}^N} \nabla v_n \cdot \nabla v_a dx, \end{aligned} \quad (22)$$

which implies that

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \geq \liminf_{n \rightarrow \infty} \left[2 \int_{\mathbb{R}^N} \nabla v_n \cdot \nabla v_a dx - \int_{\mathbb{R}^N} |\nabla v_a|^2 dx \right] = \int_{\mathbb{R}^N} |\nabla v_a|^2 dx. \quad (23)$$

By Fatou Lemma, we have

$$\int_{\mathbb{R}^N} V(x) G^{-1}(v_a)^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(x) G^{-1}(v_n)^2 dx. \quad (24)$$

Consequently, $E(v_a) \leq \liminf_{n \rightarrow \infty} [\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x) G^{-1}(v_n)^2] = \liminf_{n \rightarrow \infty} E(v_n)$. Moreover, by the Hardy-Littlewood-Sobolev inequality and Lemma 2 (3), one has

$$\begin{aligned} a &= \int_{\mathbb{R}^{2N}} \frac{|G^{-1}(v_n(x))|^p |G^{-1}(v_n(y))|^p}{|x-y|^\mu} dx dy \\ &\leq C \|G^{-1}(v_n)\|_{2N/(2N-\mu)}^2 \leq C \left(\int_{\mathbb{R}^N} |v_n|^{2Np/\alpha(2N-\mu)} dx \right)^{2N-\mu/N}. \end{aligned} \quad (25)$$

Since $2 < 2Np/\alpha(2N-\mu) < 2^*$, by Lemma A.1 in [26] and Lebesgue's dominated convergence theorem, we can easily infer that $\int_{\mathbb{R}^{2N}} (|G^{-1}(v_n(x))|^p |G^{-1}(v_n(y))|^p / |x-y|^\mu) dx dy = a$, and so $v_a \in M_a$. Hence, $m_a \leq E(v_a) \leq \liminf_{n \rightarrow \infty} E(v_n) = m_a$, which means that m_a is achieved at some $v_a \in M_a$. Moreover, by a standard argument, we can conclude that v_a is a weak solution of

$$-\Delta v + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} = \lambda_a \frac{[|x|^{-\mu} * |G^{-1}(v)|^p] |G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))}. \quad (26)$$

Multiplying the above equation by v_a and integrating over \mathbb{R}^N , one has

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_a) v_a}{g(G^{-1}(v_a))} dx \\ = \lambda_a \int_{\mathbb{R}^N} \frac{[|x|^{-\mu} * |G^{-1}(v_a)|^p] |G^{-1}(v_a)|^{p-2} G^{-1}(v_a) v_a}{g(G^{-1}(v_a))} dx. \end{aligned} \quad (27)$$

By Lemma 2 (2), we obtain $m_a/a \leq \lambda_a \leq \alpha m_a/a$. Indeed, by Lemma 2 (2), we have

$$\begin{aligned} \frac{1}{\alpha} m_a &= \frac{1}{\alpha} E(v_a) = \frac{1}{\alpha} \left[\int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \int_{\mathbb{R}^N} V(x) G^{-1}(v_a)^2 dx \right] \\ &\leq \int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \frac{1}{\alpha} \int_{\mathbb{R}^N} V(x) G^{-1}(v_a)^2 dx \\ &\leq \int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_a) v_a}{g(G^{-1}(v_a))} dx \\ &\leq \lambda_a \int_{\mathbb{R}^N} [|x|^{-\mu} * |G^{-1}(v_a)|^p] |G^{-1}(v_a)|^p dx = \lambda_a \cdot a, \end{aligned} \quad (28)$$

i.e., $\lambda_a \geq m_a/a$. Furthermore,

$$\begin{aligned} m_a &= E(v_a) = \int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \int_{\mathbb{R}^N} V(x) G^{-1}(v_a)^2 dx \\ &\geq \int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_a) v_a}{g(G^{-1}(v_a))} dx \\ &\geq \frac{\lambda_a}{\alpha} \int_{\mathbb{R}^N} [|x|^{-\mu} * |G^{-1}(v_a)|^p] |G^{-1}(v_a)|^p dx = \frac{\lambda_a}{\alpha} \cdot a, \end{aligned} \quad (29)$$

i.e., $\lambda_a \leq \alpha m_a/a$.

Step 2: we prove that $\lambda_a \rightarrow +\infty$ as $a \rightarrow 0$.

If the conclusion is false, then there exists a constant $G_0 > 0$ and $a_n \rightarrow 0$ ($n \rightarrow \infty$) such that $\lambda_n := \lambda_{a_n} \leq G_0$. Set $v_n := v_{a_n}$, by Lemma 2 (2) and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_n) v_n}{g(G^{-1}(v_n))} dx \\ = \lambda_n \int_{\mathbb{R}^N} \frac{[|x|^{-\mu} * |G^{-1}(v_n)|^p] |G^{-1}(v_n)|^{p-2} G^{-1}(v_n) v_n}{g(G^{-1}(v_n))} dx \\ \leq \lambda_n \int_{\mathbb{R}^N} [|x|^{-\mu} * |G^{-1}(v_n)|^p] |G^{-1}(v_n)|^p dx = \lambda_n a_n \\ \leq G_0 \cdot a_n \rightarrow 0 \end{aligned} \quad (30)$$

as $n \rightarrow \infty$. Since $2 < 2Np/\alpha(2N-\mu) < 2^*$, there exists a constant $\theta \in (0, 1)$ such that $1/2Np/\alpha(2N-\mu) = (\theta/2) + (1-\theta/2^*)$. Consequently, by Lemma 2 (3), (V), Hölder inequality,

and Young inequality, one has

$$\begin{aligned}
 \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{2Np/2N-\mu} dx &\leq C \int_{\mathbb{R}^N} |v_n|^{2Np/\alpha(2N-\mu)} dx \\
 &= C \int_{\mathbb{R}^N} |v_n|^{\theta 2Np/\alpha(2N-\mu)} |v_n|^{(1-\theta)2Np/\alpha(2N-\mu)} dx \\
 &\leq C \|v_n\|_2^{\theta 2Np/\alpha(2N-\mu)} \|v_n\|_{2^*}^{(1-\theta)2Np/\alpha(2N-\mu)} \\
 &\leq C \theta \|v_n\|_2^{2Np/\alpha(2N-\mu)} \\
 &\quad + C(1-\theta) \|v_n\|_{2^*}^{2Np/\alpha(2N-\mu)} \\
 &\leq C \left(\int_{\mathbb{R}^N} V(x) v_n^2 dx \right)^{Np/\alpha(2N-\mu)} \\
 &\quad + C \left(\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \right)^{Np/\alpha(2N-\mu)} \\
 &\leq C \left(\int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x) v_n^2] dx \right)^{Np/\alpha(2N-\mu)} \\
 &\leq C \left(\int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x) G^{-1}(v_n)^2] dx \right)^{Np/\alpha(2N-\mu)}.
 \end{aligned} \tag{31}$$

Hence, again, by Lemma 2 (2) and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
 \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_n) v_n}{g(G^{-1}(v_n))} dx \\
 \leq \lambda_n \int_{\mathbb{R}^N} \left[|x|^{-\mu} * |G^{-1}(v_n)|^p \right] |G^{-1}(v_n)|^p dx \\
 \leq C \lambda_n \|G^{-1}(v_n)\|_{2N/2N-\mu}^p \\
 \leq C \cdot G_0 \left(\int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x) G^{-1}(v_n)^2] dx \right)^{p/\alpha} \\
 \leq C \cdot G_0 \left(\int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x) \frac{G^{-1}(v_n) v_n}{g(G^{-1}(v_n))}] dx \right)^{p/\alpha},
 \end{aligned} \tag{32}$$

and so $\int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x)(G^{-1}(v_n) v_n / g(G^{-1}(v_n)))] dx \geq C$ since $p/\alpha \geq 2 > 1$, a contradiction. By steps 1 and 2, we complete the proof of Theorem 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Approximating Fixed Points of Operators Satisfying (RCSC) Condition in Banach Spaces

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Let K be a nonempty subset of a Banach space E . A mapping $T : K \rightarrow K$ is said to satisfy (RCSC) condition if each $a, b \in K$, $(1/2)\|a - Fa\| \leq \|a - b\| \Rightarrow \|Fa - Fb\| \leq (1/3)(\|a - b\| + \|a - Fb\| + \|b - Fa\|)$. In this paper, we study, under some appropriate conditions, weak and strong convergence for this class of maps through M iterates in uniformly convex Banach space. We also present a new example of mappings with condition (RCSC). We connect M iteration and other well-known processes with this example to show the numerical efficiency of our results. The presented results improve and extend the corresponding results of the literature.

1. Introduction

\mathbb{N} will denote the set of all natural numbers throughout. In 2008, Suzuki [1] introduced a new class of mappings as follows. A self-map F on a subset K of a Banach space E is said to satisfy (C) condition if for all $a, b \in K$, we have

$$\frac{1}{2}\|a - Fa\| \leq \|a - b\| \Rightarrow \|Fa - Fb\| \leq \|a - b\|. \quad (1)$$

Obviously, when F is nonexpansive mapping, that is, $\|Fa - Fb\| \leq \|a - b\|$ holds for all $a, b \in K$, then F satisfies the (C) condition. However, an example in [1] shows that there exists mappings, which satisfy the (C) condition but not nonexpansive. A mapping with (C) condition is often called Suzuki-type nonexpansive mapping. The class of Suzuki-type nonexpansive mappings is extensively studied by many authors (cf. [2–12] and others).

In 2012, motivated by Suzuki (C) condition, Karapinar [13] suggested a new condition on mappings, the so-called (RCSC) condition (or Reich-Chatterjea-Suzuki (C) condition). A self-map F on a subset K of a Banach space is said to satisfy the (RCSC) condition if for all $a, b \in K$, we have

$$\begin{aligned} \frac{1}{2}\|a - Fa\| \leq \|a - b\| &\Rightarrow \|Fa - Fb\| \\ &\leq \frac{1}{3}(\|a - b\| + \|b - Fa\| + \|a - Fb\|). \end{aligned} \quad (2)$$

The purpose of this work is to prove some weak and strong convergence results for this class of mappings through the M iteration process [12] in the context of Banach spaces. We also give a numerical example to show the usefulness of our results. In this way, we extend and improve many well-known corresponding results of the current literature.

Approximating fixed points of nonlinear mappings played an important role and solved many problems [14–20]. It is now well known that if F is nonexpansive, then the sequence of Picard iterates $w_{n+1} = Fw_n$ may not converge to a fixed point of F . To overcome such problems and to get better a rate of convergence, many iterative processes are available in the literature. The well-known iterative processes are the Mann [21], Ishikawa [22], Noor [23], Agarwal et al. [24], Abbas and Nazir [25], Thakur et al. [7], Ullah and Arshad [12], and so on. Let $\alpha_n, \beta_n, \gamma_n \in (0, 1)$, $n \in \mathbb{N}$, and F be a self-map on a nonempty convex subset K of a Banach space.

The Mann iteration process [21] is a sequence $\{w_n\}$ defined as follows:

$$\left. \begin{aligned} w_1 &= w \in K, \\ w_{n+1} &= (1 - \alpha_n)w_n + \alpha_n Fw_n. \end{aligned} \right\} \quad (3)$$

The Ishikawa iteration process [22] is a sequence $\{w_n\}$ defined as follows:

$$\left. \begin{aligned} w_1 &= w \in K, \\ s_n &= (1 - \beta_n)w_n + \beta_n Fw_n, \\ w_{n+1} &= (1 - \alpha_n)w_n + \alpha_n Fs_n. \end{aligned} \right\} \quad (4)$$

The Noor iteration process [23] is a sequence $\{w_n\}$ defined as follows:

$$\left. \begin{aligned} w_1 &= w \in K, \\ v_n &= (1 - \gamma_n)w_n + \gamma_n Fw_n, \\ s_n &= (1 - \beta_n)w_n + \beta_n Fv_n, \\ w_{n+1} &= (1 - \alpha_n)w_n + \alpha_n Fs_n. \end{aligned} \right\} \quad (5)$$

The S iteration process [24] is a sequence $\{w_n\}$ defined as follows:

$$\left. \begin{aligned} w_1 &= w \in K, \\ s_n &= (1 - \beta_n)w_n + \beta_n Fw_n, \\ w_{n+1} &= (1 - \alpha_n)Fw_n + \alpha_n Fs_n. \end{aligned} \right\} \quad (6)$$

The Abbas and Nazir iteration process [25] is a sequence $\{w_n\}$ defined as follows:

$$\left. \begin{aligned} w_1 &= w \in K, \\ v_n &= (1 - \gamma_n)w_n + \gamma_n Fw_n, \\ s_n &= (1 - \beta_n)Fw_n + \beta_n Fv_n, \\ w_{n+1} &= (1 - \alpha_n)Fs_n + \alpha_n Fv_n. \end{aligned} \right\} \quad (7)$$

The Thakur et al. iteration process [7] is a sequence $\{w_n\}$ defined as follows:

$$\left. \begin{aligned} w_1 &= w \in K, \\ v_n &= (1 - \beta_n)w_n + \beta_n Fw_n, \\ s_n &= F((1 - \alpha_n)w_n + \alpha_n v_n), \\ w_{n+1} &= Fs_n. \end{aligned} \right\} \quad (8)$$

The M iteration process [12] is a sequence $\{w_n\}$ defined as follows:

$$\left. \begin{aligned} w_1 &= w \in K, \\ v_n &= (1 - \alpha_n)w_n + \alpha_n Fw_n, \\ s_n &= Fv_n, \\ w_{n+1} &= Fs_n. \end{aligned} \right\} \quad (9)$$

In this paper, we will present some weak and strong convergence results using the M iteration process (9) for mappings with (RCSC) condition. Similar results for the processes (3)–(8) can be proved on the same line of proofs.

2. Preliminaries

$p \in K$ is called a fixed point of a self-map F on K if $p = Fp$. We will denote by $\text{fix}(F)$ throughout the set of all fixed points of F . A Banach space E is said to satisfy Opial condition [26] if and only if for each weakly convergent sequence $\{w_n\} \subseteq E$ with a weak limit $w \in E$, we have the following property:

$$\liminf_{n \rightarrow \infty} \|w_n - w\| < \liminf_{n \rightarrow \infty} \|w_n - z\| \quad \text{for all } z \in E - \{w\}. \quad (10)$$

A self-map F on a subset K of a Banach space is said to satisfy the condition I [27] if there is nondecreasing function $g : [0, \infty) \rightarrow [0, \infty)$ with the properties $g(0) = 0$, $g(z) > 0$ for every $z > 0$, and $\|a - Fa\| \geq g(\text{dist}(a, \text{fix}(F)))$ for all $a \in K$.

Let K be a nonempty subset of a Banach space E and $\{w_n\}$ a bounded sequence in E . For each $x \in E$, define

- (i) asymptotic radius of $\{w_n\}$ at x by $r(x, \{w_n\}) := \limsup_{n \rightarrow \infty} \|x - w_n\|$
- (ii) asymptotic radius of $\{w_n\}$ relative to K by $r(K, \{w_n\}) = \inf \{r(x, \{w_n\}) : x \in K\}$
- (iii) asymptotic center of $\{w_n\}$ relative to K by $A(K, \{w_n\}) = \{x \in K : r(x, \{w_n\}) = r(K, \{w_n\})\}$

When the space E is uniformly convex [28], then the set $A(K, \{w_n\})$ is always singleton. Notice also that the set $A(K, \{w_n\})$ is convex as well as nonempty provided that K is weakly compact convex (see, e.g., [29, 30]).

Lemma 1. [13].

Let F be a self-map on a subset K of a Banach space. If F satisfies the (RCSC) condition, then for all $a, b \in K$, the following holds:

$$\|a - Fb\| \leq 9\|a - Fa\| + \|a - b\|. \quad (11)$$

The following facts are also needed.

Lemma 2. [13].

Let E be a Banach space having Opial's property, $\emptyset \neq K \subseteq E$ and $F : K \rightarrow K$. If F satisfies the condition (RCSC), then the following condition holds:

$$\{w_n\} \subseteq K, w_n \rightarrow w, \|w_n - Fw_n\| \rightarrow 0 \Rightarrow Fw = w. \quad (12)$$

The following lemma gives the structure of the fixed point set associated with a mapping satisfying (RCSC) condition.

Lemma 3. [13].

Let F be a self-map on a subset $\emptyset \neq K$ of a Banach space. If F satisfies the (RCSC) condition, then $\text{fix}(F)$ is closed. Moreover, if E is strictly convex and K is convex, then $\text{fix}(F)$ is also convex.

Lemma 4. [13].

Let F be a self-map on a subset $\emptyset \neq K$ of a Banach space. If F satisfies (RCSC) condition, then for all $a \in K$ and $p \in \text{fix}(F)$, $\|Fa - Fp\| \leq \|a - p\|$ holds.

Lemma 5. [31].

Let $0 < x \leq \eta_n \leq y < 1$ for each $n \in \mathbb{N}$ and $\{v_n\}$ and $\{w_n\}$ be any two sequences in a uniformly convex Banach space E such that $\limsup_{n \rightarrow \infty} \|v_n\| \leq \zeta$, $\limsup_{n \rightarrow \infty} \|w_n\| \leq \zeta$, and $\lim_{n \rightarrow \infty} \|\eta_n v_n + (1 - \eta_n)w_n\| = \zeta$ for some $\zeta \geq 0$; then, $\lim_{n \rightarrow \infty} \|v_n - w_n\| = 0$.

3. Main Results

We begin this section by proving a crucial lemma.

Lemma 6. Let F be a self-map on a subset $\emptyset \neq K$ of a Banach space. Assume that F satisfies the (RCSC) condition and let $\{w_n\}$ be a sequence generated by (9). If $\text{fix}(F) \neq \emptyset$, then $\lim_{n \rightarrow \infty} \|w_n - p\|$ exists for each $p \in \text{fix}(F)$.

Proof. Let $p \in \text{fix}(F)$ and $n \in \mathbb{N}$. By Lemma 4, we have

$$\begin{aligned} \|v_n - p\| &= \|(1 - \alpha_n)w_n + \alpha_n Fw_n - p\| \\ &\leq (1 - \alpha_n)\|w_n - p\| + \alpha_n \|Fw_n - p\| \\ &\leq (1 - \alpha_n)\|w_n - p\| + \alpha_n \|w_n - p\| = \|w_n - p\|, \end{aligned} \quad (13)$$

which implies that

$$\begin{aligned} \|w_{n+1} - p\| &= \|s_n - p\| \leq \|s_n - p\| = \|Fv_n - p\| \\ &\leq \|v_n - p\| \leq \|w_n - p\|. \end{aligned} \quad (14)$$

Hence, $\|w_{n+1} - p\| \leq \|w_n - p\|$ for all $n \in \mathbb{N}$ and $p \in \text{fix}(F)$. Thus, $\{\|w_n - p\|\}$ is bounded and nonincreasing, which implies that $\lim_{n \rightarrow \infty} \|w_n - p\|$ exists for each $p \in \text{fix}(F)$.

Now we give the necessary and sufficient condition for the existence of a fixed point for mapping with (RCSC) condition defined on a nonempty closed convex subset of a complete uniformly convex Banach space.

Theorem 7. Let F be a self-map on a closed convex subset $\emptyset \neq K$ of a uniformly convex Banach space. Assume that F satisfies the (RCSC) condition and let $\{w_n\}$ be a sequence generated by (9). Then, $\text{fix}(F) \neq \emptyset$ if and only if $\{w_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|Fw_n - w_n\| = 0$.

Proof. Let $\{w_n\}$ be bounded and $\lim_{n \rightarrow \infty} \|Fw_n - w_n\| = 0$. Let $p \in A(K, \{w_n\})$. By Lemma 1, we have

$$\begin{aligned} r(Fp, \{w_n\}) &= \limsup_{n \rightarrow \infty} \|w_n - Fp\| \\ &\leq 9 \limsup_{n \rightarrow \infty} \|Fw_n - w_n\| + \limsup_{n \rightarrow \infty} \|w_n - p\| \\ &= \limsup_{n \rightarrow \infty} \|w_n - p\| = r(p, \{w_n\}). \end{aligned} \quad (15)$$

Hence, we conclude that $Fp \in A(K, \{w_n\})$. Since E is uniformly convex, $A(K, \{w_n\})$ consists of a unique element. Thus, we have $Fp = p$.

Conversely, suppose that $\text{fix}(F) \neq \emptyset$ and $p \in \text{fix}(F)$. By Lemma 6, $\lim_{n \rightarrow \infty} \|w_n - p\|$ exists and $\{w_n\}$ is bounded. Put

$$\lim_{n \rightarrow \infty} \|w_n - p\| = \zeta. \quad (16)$$

From (13), we have

$$\begin{aligned} \|v_n - p\| &\leq \|w_n - p\| \Rightarrow \limsup_{n \rightarrow \infty} \|v_n - p\| \\ &\leq \limsup_{n \rightarrow \infty} \|w_n - p\| = \zeta. \end{aligned} \quad (17)$$

By Lemma 4, we have

$$\begin{aligned} \|Fw_n - p\| &\leq \|w_n - p\| \Rightarrow \limsup_{n \rightarrow \infty} \|Fw_n - p\| \\ &\leq \limsup_{n \rightarrow \infty} \|w_n - p\| = \zeta. \end{aligned} \quad (18)$$

From (14), we have

$$\begin{aligned} \|w_{n+1} - p\| &\leq \|v_n - p\| \Rightarrow \zeta = \liminf_{n \rightarrow \infty} \|w_{n+1} - p\| \\ &\leq \liminf_{n \rightarrow \infty} \|v_n - p\|. \end{aligned} \quad (19)$$

From (17) and (19), we have

$$\zeta = \lim_{n \rightarrow \infty} \|v_n - p\|. \quad (20)$$

From (20), we have

$$\zeta = \lim_{n \rightarrow \infty} \|v_n - p\| = \lim_{n \rightarrow \infty} \|(1 - \alpha_n)(w_n - p) + \alpha_n(Fw_n - p)\|. \quad (21)$$

Hence,

$$\zeta = \lim_{n \rightarrow \infty} \|(1 - \alpha_n)(w_n - p) + \alpha_n(Fw_n - p)\|. \quad (22)$$

By Lemma 5, we have

$$\lim_{n \rightarrow \infty} \|Fw_n - w_n\| = 0. \quad (23)$$

Now we can prove the following weak convergence theorem.

Theorem 8. *Let F be a self-map on a closed convex subset $\emptyset \neq K$ of a uniformly convex Banach space E having Opial's property. Assume that F satisfies the (RCSC) condition with $\text{fix}(F) \neq \emptyset$ and let $\{w_n\}$ be a sequence generated by (9). Then, $\{w_n\}$ converges weakly to a fixed point of F .*

Proof. Since E is uniformly convex, E is reflexive. By Theorem 7, $\{w_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|Fw_n - w_n\| = 0$ for all $n \in \mathbb{N}$. By the reflexivity, one can find a weakly convergent subsequence $\{w_{n_i}\}$ of $\{w_n\}$ with a weak limit say $w \in K$. By Lemma 2, we have $Fw = w$. It is suffice to show that w is the weak limit of $\{w_n\}$. w is not the weak limit of $\{w_n\}$. Then, one can find another weakly convergent subsequence $\{w_{n_j}\}$ of $\{w_n\}$ with a weak limit w' such that $w' \neq w$. Again by Lemma 2, $Fw' = w'$. By Lemma 6 together with Opial's property, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|w_n - w\| &= \lim_{i \rightarrow \infty} \|w_{n_i} - w\| < \lim_{i \rightarrow \infty} \|w_{n_i} - w'\| \\ &= \lim_{n \rightarrow \infty} \|w_n - w'\| = \lim_{j \rightarrow \infty} \|w_{n_j} - w'\| \\ &< \lim_{j \rightarrow \infty} \|w_{n_j} - w\| = \lim_{n \rightarrow \infty} \|w_n - w\|. \end{aligned} \quad (24)$$

This is a contradiction. So, we have $w = w'$. Hence, w is the weak limit of $\{w_n\}$

Now we prove a strong convergence theorem as follows.

Theorem 9. *Let F be a self-map on a compact convex subset $\emptyset \neq K$ of a uniformly convex Banach space. Assume that F satisfies the (RCSC) condition with $\text{fix}(F) \neq \emptyset$ and let $\{w_n\}$ be a sequence generated by (9). Then, $\{w_n\}$ converges strongly to a fixed point of F .*

Proof. By Theorem 7, $\lim_{n \rightarrow \infty} \|Fw_n - w_n\| = 0$ for all $n \in \mathbb{N}$. Since K is compact and convex, we can find a strongly convergent subsequence $\{w_{n_j}\}$ of $\{w_n\}$ with a strong limit say q . By Lemma 1, we have

$$\|w_{n_j} - Fq\| \leq 9\|w_{n_j} - Fw_{n_j}\| + \|w_{n_j} - q\| \longrightarrow 0. \quad (25)$$

By the uniqueness of limits in Banach spaces, $Fq = q$. By Lemma 6, $\lim_{n \rightarrow \infty} \|w_n - q\|$ exists and hence q is the strong limit of $\{w_n\}$.

Now we state the following theorem. Since the proof is elementary, we will not include the details.

Theorem 10. *Let F be a self-map on a closed convex subset $\emptyset \neq K$ of a uniformly convex Banach space. Assume that F satisfies the (RCSC) condition with $\text{fix}(F) \neq \emptyset$ and let $\{w_n\}$ be a sequence generated by (9). Then, $\{w_n\}$ converges strongly to a fixed point of F provided that $\liminf_{n \rightarrow \infty} \text{dist}(w_n, \text{fix}(F)) = 0$.*

The following convergence theorem is based on condition I.

Theorem 11. *Let F be a self-map on a closed convex subset $\emptyset \neq K$ of a uniformly convex Banach space. Assume that F satisfies the (RCSC) condition with $\text{fix}(F) \neq \emptyset$ and let $\{w_n\}$ be a sequence generated by (9). Then, $\{w_n\}$ converges strongly to a fixed point of F provided that F satisfies the condition I.*

Proof. By Theorem 7, it follows that $\liminf_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$. By the condition I, we have $\liminf_{n \rightarrow \infty} \text{dist}(w_n, \text{fix}(F)) = 0$. The conclusion follows from Theorem 10.

4. Example

In this section, we compare the rate of convergence of the M iteration process with other iterations in the setting of mappings with (RCSC) condition.

Example 1. Let $K = [2, 5]$ be endowed with the usual norm. Set $Fa = 2$ if $a = 5$ and $Fa = (2 + a)/2$ if $a \neq 5$. We shall prove that F satisfies the (RCSC) condition. The case when $a, b \in \{5\}$ is trivial. We consider only the following three nontrivial cases.

When $a, b \in [2, 5)$, then $Fa = (2 + a)/2$ and $Fb = (2 + b)/2$. Using triangle inequality, we have

$$\begin{aligned} |Fa - Fb| &= \frac{1}{2}|a - b| \leq \frac{1}{3}|a - b| + \frac{1}{2}|a - b| \\ &= \frac{1}{3}|a - b| + \frac{1}{3}\left|\frac{3a}{2} - \frac{3b}{2}\right| \\ &= \frac{1}{3}|a - b| + \frac{1}{3}\left|a - \left(\frac{2+b}{2}\right) - \left(b - \left(\frac{2+a}{2}\right)\right)\right| \\ &\leq \frac{1}{3}|a - b| + \frac{1}{3}\left|a - \left(\frac{2+b}{2}\right)\right| + \frac{1}{3}\left|b - \left(\frac{2+a}{2}\right)\right| \\ &= \frac{1}{3}(|a - b| + |a - Fb| + |b - Fa|). \end{aligned} \quad (26)$$

TABLE 1: Computation table obtained from the M , Thakur et al., Abbas and Nazir, S , Noor, Ishikawa, and Mann iterates for mapping F defined in Example 1.

n	M	Thakur et al.	Abbas and Nazir	S	Noor	Ishikawa	Mann
1	3	3	3	3	3	3	3
2	2.1625	2.1931	2.2456	2.3863	2.4851	2.5363	2.6500
3	2.0264	2.0373	2.0603	2.1492	2.2353	2.2876	2.4225
4	2.0043	2.0072	2.0148	2.0576	2.1141	2.1542	2.2746
5	2.0007	2.0014	2.0036	2.0223	2.0554	2.0827	2.1785
6	2.0001	2.0003	2.0009	2.0086	2.0269	2.0443	2.1160
7	2	2.0001	2.0002	2.0033	2.0130	2.0238	2.0754
8	2	2	2.0001	2.0013	2.0063	2.0123	2.0490
9	2	2	2	2.0005	2.0031	2.0068	2.0318
10	2	2	2	2.0002	2.0015	2.0037	2.0207
11	2	2	2	2.0001	2.0007	2.0020	2.0134
12	2	2	2	2	2.0003	2.0011	2.0088
13	2	2	2	2	2.0002	2.0006	2.0057
14	2	2	2	2	2.0001	2.0003	2.0037
15	2	2	2	2	2	2.0002	2.0024
16	2	2	2	2	2	2.0001	2.0016
17	2	2	2	2	2	2	2.0010
18	2	2	2	2	2	2	2.0007
19	2	2	2	2	2	2	2.0004
20	2	2	2	2	2	2	2.0003
21	2	2	2	2	2	2	2.0002
22	2	2	2	2	2	2	2.0001
23	2	2	2	2	2	2	2

When $a \in [2, 5)$ and $b \in \{5\}$, then $Fa = (2 + a)/2$ and $Fb = 2$. Now

$$\begin{aligned}
 |Fa - Fb| &= \left| \left(\frac{a-2}{2} \right) - 2 \right| = \left| \frac{a-2}{2} \right| = \frac{1}{3} \left(\left| \frac{3a-6}{2} \right| \right) \\
 &= \frac{1}{3} \left(\left| \frac{a-2}{2} + (a-2) \right| \right) \leq \frac{1}{3} \left| \frac{a-2}{2} \right| + \frac{1}{3} |a-2| \\
 &= \frac{1}{3} \left| (a-b) + \left(b - \left(\frac{2+a}{2} \right) \right) \right| + \frac{1}{3} |a-2| \\
 &\leq \frac{1}{3} |a-b| + \frac{1}{3} \left| b - \left(\frac{2+a}{2} \right) \right| + \frac{1}{3} |a-2| \\
 &= \frac{1}{3} (|a-b| + |a-Fb| + |b-Fa|).
 \end{aligned}
 \tag{27}$$

Finally, when $a \in \{5\}$ and $b \in [2, 5)$, then $Fa = 2$ and $Fb = (2 + b)/2$. Now

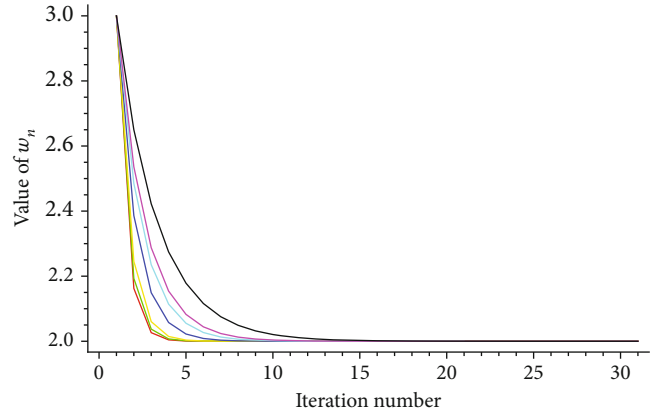


FIGURE 1: Convergence behavior of the M (red line), Thakur et al. (green line), Abbas and Nazir (yellow line), Agarwal et al. (blue line), Noor (cyan line), Ishikawa (magenta line), and Mann (black line) iterates for mapping F defined in Example 1 where $w_1 = 3$.

TABLE 2: $\alpha_n = 2n/\sqrt{7n+11}$ and $\beta_n = 1/\sqrt{3n+9}$. Number of iterations required to obtain the fixed point.

Initial points	S (6)	Thakur et al. (8)	M (9)
2.2	23	13	8
2.7	24	14	8
3.3	25	14	8
3.8	25	15	8
4.3	26	15	8
4.8	26	15	8

$$\begin{aligned}
 |Fa - Fb| &= \left| 2 - \left(\frac{b-2}{2} \right) \right| = \left| \frac{b-2}{2} \right| = \frac{1}{3} \left(\left| \frac{3b-6}{2} \right| \right) \\
 &= \frac{1}{3} \left(\left| \frac{b-2}{2} + (b-2) \right| \right) \leq \frac{1}{3} \left| \frac{b-2}{2} \right| + \frac{1}{3} |b-2| \\
 &= \frac{1}{3} \left| (b-a) + \left(a - \left(\frac{2+b}{2} \right) \right) \right| + \frac{1}{3} |b-2| \\
 &\leq \frac{1}{3} |b-a| + \frac{1}{3} \left| a - \left(\frac{2+b}{2} \right) \right| + \frac{1}{3} |b-2| \\
 &= \frac{1}{3} (|a-b| + |a-Fb| + |b-Fa|).
 \end{aligned}
 \tag{28}$$

Next, for $a = 4.2$ and $b = 5$, $(1/2) |a - Fa| < |a - b|$ but $|Fa - Fb| > |a - b|$. Hence, F does not satisfy the (C) condition. Let $\alpha_n = 0.70$, $\beta_n = 0.65$, and $\gamma_n = 0.90$. The strong convergence of the M (9), Thakur et al. (8), Abbas and Nazir (7), S (6), Noor (5), Ishikawa (4), and Mann (3) iterates to a fixed point $p = 2$ is given in Table 1.

Remark 12. From Table 1 and Figure 1, we see that the M iteration process converges faster to $p = 2$ than the others.

Now using the above example, we make different choices of parameters α_n and β_n and initial points and also we get $\|w_n - p\| < 10^{-10}$ as our stopping criterion where $p = 2$ is a

TABLE 3: $\alpha_n = n/(n+5)^{10/9}$ and $\beta_n = 1/(n+5)^{2/3}$. Number of iterations required to obtain the fixed point.

Initial points	S (6)	Thakur et al. (8)	M (9)
2.2	28	14	13
2.7	29	15	14
3.3	30	16	14
3.8	31	16	14
4.3	31	16	14
4.8	31	16	14

TABLE 4: $\alpha_n = ((n+3)/(5n+2))^{1/15}$ and $\beta_n = 2n/(5n+100)^{1/4}$. Number of iterations required to obtain the fixed point.

Initial points	S (6)	Thakur et al. (8)	M (9)
2.2	–	11	10
2.7	–	12	11
3.3	–	13	11
3.8	–	14	11
4.3	–	15	12
4.8	–	15	12

fixed point of F . The number of iterations for M (9) to reach $p = 2$ is compared with the leading three steps of Thakur et al. (8) and leading two steps of S (6) iterations. The numbers in italic in Tables 2–4 show that M iteration is better than the others. The “–” represents that the number of iterations exceeds 50.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Blow up of Coupled Nonlinear Klein-Gordon System with Distributed Delay, Strong Damping, and Source Terms

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This present work deals with the blow up of the coupled Klein-Gordon system with strong damping, distributed delay, and source terms, under suitable conditions.

1. Introduction

In the present paper, we consider the following system:

$$\begin{cases} u_{tt} + m_1 u^2 - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t-s) \Delta u(s) ds + \mu_1 u_t + \int_{\tau_2}^{\tau_1} |\mu_2(\mathbf{q})| u_t(x, t-\mathbf{q}) d\mathbf{q} = f_1(u, v), & (x, t) \in \Omega \times \mathbb{R}_+, v_{tt} m_2 v^2 - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t-s) \Delta v(s) ds + \mu_3 v_t + \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{q})| v_t(x, t-\mathbf{q}) d\mathbf{q} = f_2(u, v), & (x, t) \in \Omega \times \mathbb{R}_+, \\ u(x, t) = 0, \quad v(x, t) = 0, & x \in \partial\Omega, \\ u_t(x, -t) = f_0(x, t), \quad v_t(x, -t) = k_0(x, t) & (x, t) \in \Omega \times (0, \tau_2), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & x \in \Omega, \end{cases} \quad (1)$$

where

$$\begin{cases} f_1(u, v) = a_1 |u + v|^{2(p+1)} (u + v) + b_1 |u|^p \cdot u \cdot |v|^{p+2}, \\ f_2(u, v) = a_1 |u + v|^{2(p+1)} (u + v) + b_1 |v|^p \cdot v \cdot |u|^{p+2}, \end{cases} \quad (2)$$

and $m_1, m_2, \omega_1, \omega_2, \mu_1, \mu_3, a_1, b_1 > 0$, and τ_1, τ_2 are the time delay with $0 \leq \tau_1 < \tau_2$, and μ_2, μ_4 are a L^∞ functions, and g, h are differentiable functions.

Viscous materials are the opposite of flexible materials that have dissipate mechanical energy and the ability to store.

The mechanical properties of viscous materials are so important that we find them in many applications of natural sciences. Many authors have been concerned with this problem in recent decades.

If there is only one equation and if $\omega_1 = 0$, that is, for absence of Δu_t , and $\mu_1 = \mu_2 = 0$. Our problem (1) has been studied by Berrimi and Messaoudi [1]. Using Galerkin's method they proved the result of local existence. They also made it clear that the local solution is global in time under suitable conditions and at the same rate of decaying (exponential or polynomial) of the kernel g . In addition, the authors themselves demonstrated that the dissipation can

be deduced by the term viscous integral and that it is strong enough to stabilize the solution oscillations. Their results were also obtained under weaker conditions than those used by Cavalcanti et al. [2].

In [3], the authors considered the following problem:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^\gamma, \quad u = 0, \quad (3)$$

where the authors proved the exponential decay result. This subsequent result was improved by Berrimi et al. in [1], as they showed that the viscosity elastic dissipation alone is strong enough to stabilize the problem even with the exponential rate with respect to the kernel g assumptions. In the case $\mu_1 \neq 0$, in problem (1), Kafini and Mes-saudi in [4] proved a blow up result for the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(s)ds + u_t = |u|^{p-2} \cdot u, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \end{cases} \quad (4)$$

where g satisfies

$$\int_0^\infty g(s)ds < (2p-4)/(2p-3). \quad (5)$$

The initial data was backed by negative energy as

$$\int u_0 u_1 dx > 0. \quad (6)$$

In [5], Song and Xue considered the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(s)ds - \Delta u_t |u|^{p-2} \cdot u, & (x, t) \in \Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \end{cases} \quad (7)$$

where the authors showed that there were solutions of (7) with initial energy according to suitable assumptions on g . Moreover, they showed the blow up in a finite time. Then,

the same authors in [6] continued to prove that there were solutions of (7) with positive initial energy that blow up in finite time. In [7], the author studied the following problem:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m-2} \cdot u_t = |u|^{p-2} \cdot u, & (x, t) \in \Omega(0, \infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ u(x, t) = 0, \quad x \in \partial\Omega, \end{cases} \quad (8)$$

where they proved the exponential growth result under suitable assumptions. The authors in [8] studied the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)ds - \varepsilon_1 \Delta u_t + \varepsilon_2 u_t |u_t|^{m-2} = \varepsilon_3 u |u|^{p-2}, \\ u(x, t) = 0, \quad x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \end{cases} \quad (9)$$

where they showed a blow up result if $p > m$ and established the global existence. In the coupled equation case, the authors in [9] studied the following system:

$$\begin{cases} u_{tt} - \Delta u + u_t |u_t|^{m-2} = f_1(u, v), \\ v_{tt} - \Delta v + v_t |v_t|^{r-2} = f_2(u, v), \end{cases} \quad (10)$$

with f_1 and f_2 nonlinear functions satisfying appropriate conditions. According to certain restrictions imposed on the initial data and parameters, they obtained numerous

results on the existence of weak solutions. They obtained many results on the presence of weak solutions. In addition, by using the same techniques similar to that in [10] with negative initial, energy blows up for a finite period of time.

In [11], the authors have proved the solution of the problem:

$$\begin{cases} u_{tt} - \Delta u + (a|u|^k + b|v|^l)u_t|u_t|^{m-2} = f_1(u, v), \\ v_{tt} - \Delta v + (a|u|^\theta + b|v|^\vartheta)v_t|v_t|^{r-2} = f_2(u, v), \end{cases} \quad (11)$$

where under some restrictions on positive initial energy for certain conditions on the functions f_1 and f_2 , the authors proved the blows up in finite time of solution.

The result of [11] has been extended by the authors in [12], where they studied the following system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)ds + (a|u|^k + b|v|^l)u_t|u_t|^{m-2} = f_1(u, v), \\ v_{tt} - \Delta v + \int_0^\infty h(s)\Delta v(t-s)ds + (a|u|^\theta + b|v|^\vartheta)v_t|v_t|^{r-2} = f_2(u, v), \end{cases} \quad (12)$$

they proved that the solutions of a system of wave equations with degenerate damping, viscoelastic term and strong non-linear sources acting in both equations at the same time are globally nonexisting provided that the initial data are sufficiently large in a bounded domain of Ω .

As complement to these works, we are working to prove the blow up result with distributed delay of problem (1), under appropriate assumptions, and we prove these results using the energy method. In the following, let $c, c_i > 0, i = 1, \dots, 12$.

The present paper is organized as follows. In Section 2, we give some necessarily assumptions for the main result. In Section 3, we prove the blow up result.

2. Assumptions

We consider the following suitable assumptions.

(A1) $g, h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are differentiable and decreasing functions such that

$$g(t) \geq 0, \quad 1 - \int_0^\infty g(s)ds = l_1 > 0, \quad (13)$$

$$h(t) \geq 0, \quad 1 - \int_0^\infty h(s)ds = l_2 > 0. \quad (14)$$

(A2) There exists a constants $\xi_1, \xi_2 > 0$ such that

$$g'(t) \leq -\xi_1 g(t), \quad t \geq 0, \quad (15)$$

$$h'(t) \leq -\xi_2 h(t), \quad t \geq 0. \quad (16)$$

(A3) $\mu_2, \mu_4 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ are a L^∞ functions so that

$$\left(\frac{2\delta-1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})|d\mathbf{Q} < \mu_1, \quad \delta > \frac{1}{2}, \quad (17)$$

$$\left(\frac{2\delta-1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})|d\mathbf{Q} < \mu_3, \quad \delta > \frac{1}{2}. \quad (18)$$

3. Blow up

In this section, we obtain the proof of the blow up result of the solution of problem (1). First, of all in [13], we introduce the new variables

$$\begin{aligned} y(x, \rho, \mathbf{Q}, t) &= u_t(x, t - \mathbf{Q}\rho), \\ z(x, \rho, \mathbf{Q}, t) &= v_t(x, t - \mathbf{Q}\rho), \end{aligned} \quad (19)$$

then, we obtain

$$\begin{cases} \mathbf{Q}y_t(x, \rho, \mathbf{Q}, t) + y_\rho(x, \rho, \mathbf{Q}, t) = 0, \\ y(x, 0, \mathbf{Q}, t) = u_t(x, t), \\ \mathbf{Q}z_t(x, \rho, \mathbf{Q}, t) + z_\rho(x, \rho, \mathbf{Q}, t) = 0, \\ z(x, 0, \mathbf{Q}, t) = v_t(x, t). \end{cases} \quad (20)$$

Let us denote by

$$gou = \int_\Omega \int_0^t g(t-s)|u(t) - u(s)|^2 ds dx. \quad (21)$$

Therefore, problem (1) get the following form:

$$\begin{cases} u_{tt} + m_1 u^2 - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})|y(x, 1, \mathbf{Q}, t)d\mathbf{Q} = f_1(u, v), & x \in \Omega, t \geq 0, \\ v_{tt} + m_2 v^2 - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t-s)\Delta v(s)ds + \mu_3 v_t + \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})|z(x, 1, \mathbf{Q}, t)d\mathbf{Q} = f_2(u, v), & x \in \Omega, t \geq 0, \\ \mathbf{Q}y_t(x, \rho, \mathbf{Q}, t) + y_\rho(x, \rho, \mathbf{Q}, t) = 0, \\ \mathbf{Q}z_t(x, \rho, \mathbf{Q}, t) + z_\rho(x, \rho, \mathbf{Q}, t) = 0, \end{cases} \quad (22)$$

with initial and boundary conditions

$$\begin{cases} u(x, t) = 0, & v(x, t) = 0, & x \in \partial\Omega, \\ y(x, \rho, \mathbf{Q}, 0) = f_0(x, \mathbf{Q}\rho), & z(x, \rho, \mathbf{Q}, 0) = k_0(x, \mathbf{Q}\rho), \\ u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), \\ v(x, 0) = v_0(x), & v_t(x, 0) = v_1(x), \end{cases} \quad (23)$$

where

$$(x, \rho, \mathbf{Q}, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty). \quad (24)$$

Theorem 1. Assume (14), (16), and (17) hold. Let

$$\begin{cases} -1 < p < \frac{4-n}{n-2}, & n \geq 3, \\ p \geq -1, & n = 1, 2. \end{cases} \quad (25)$$

For any initial data,

$$(u_0, u_1, v_0, v_1, f_0, k_0) \in \mathcal{H}, \quad (26)$$

where

$$\begin{aligned} \mathcal{H} = & H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \\ & \times L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2)) \times L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2)), \end{aligned} \quad (27)$$

then, problem (22) has a unique solution

$$u \in C([0, T]; \mathcal{H}), \quad (28)$$

for some $T > 0$.

Lemma 2. There exists a function $F(u, v)$ such that

$$\begin{aligned} F(u, v) &= \frac{1}{2(\rho+2)} [uf_1(u, v) + vf_2(u, v)], \\ &= \frac{1}{2(\rho+2)} [a_1|u+v|^{2(p+2)} + 2b_1|uv|^{p+2}] \geq 0, \end{aligned} \quad (29)$$

where

$$\begin{aligned} \frac{\partial F}{\partial u} &= f_1(u, v), \\ \frac{\partial F}{\partial v} &= f_2(u, v), \end{aligned} \quad (30)$$

we take $a_1 = b_1 = 1$ for convenience.

Lemma 3. (see [12]). There exist two positive constants c_0 and c_1 such that

$$\begin{aligned} & \frac{c_0}{2(\rho+2)} (|u|^{2(p+2)} + |v|^{2(p+2)}) \\ & \leq F(u, v) \leq \frac{c_1}{2(\rho+2)} (|u|^{2(p+2)} + |v|^{2(p+2)}). \end{aligned} \quad (31)$$

We define the energy functional (see, e.g., [14–16] and reference therein).

Lemma 4. Assume (14), (16), (17), and (25) hold, let (u, v, y, z) be a solution of (22), then $E(t)$ is nonincreasing, that is,

$$\begin{aligned} E(t) = & \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \|v_t\|_2^2 + \frac{m_1}{2} \|u\|_2^2 + \frac{m_2}{2} \|v\|_2^2 \\ & + \frac{1}{2} l_1 \|\nabla u\|_2^2 + \frac{1}{2} l_2 \|\nabla v\|_2^2 + \frac{1}{2} (go \nabla u) + \frac{1}{2} (ho \nabla u) \\ & + \frac{1}{2} K(y, z) - \int_{\Omega} F(u, v) dx, \end{aligned} \quad (32)$$

satisfies

$$\begin{aligned} E'(t) \leq & -c_3 \left\{ \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 \right. \\ & + \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| y^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \\ & \left. + \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| z^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \right\} \leq 0, \end{aligned} \quad (33)$$

where

$$\begin{aligned} K(y, z) = & \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \mathbf{Q} \{ |\mu_2(\mathbf{Q})| y^2(x, \rho, \mathbf{Q}, t) \\ & + |\mu_4(\mathbf{Q})| z^2(x, \rho, \mathbf{Q}, t) d\mathbf{Q} d\rho dx \}. \end{aligned} \quad (34)$$

Proof. By multiplying (3.4)₁, (3.4)₂ by u_t, v_t and integrating over Ω , we get

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \|v_t\|_2^2 + \frac{m_1}{2} \|u\|_2^2 + \frac{m_2}{2} \|v\|_2^2 + \frac{1}{2} l_1 \|\nabla u\|_2^2 \right. \\ & + \frac{1}{2} l_2 \|\nabla v\|_2^2 + \frac{1}{2} (go \nabla u) + \frac{1}{2} (ho \nabla u) - \int_{\Omega} F(u, v) dx \Big\} \\ = & -\mu_1 \|u_t\|_2^2 - m_1 \|u\|_2^2 - \int_{\Omega} u_t \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| y(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \\ & - \mu_3 \|v_t\|_2^2 - m_2 \|v\|_2^2 - \int_{\Omega} v_t \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| z(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \\ & + \frac{1}{2} (g' o \nabla u) - \frac{1}{2} g(t) \|\nabla u\|_2^2 - \omega_1 \|\nabla u_t\|_2^2 + \frac{1}{2} (h' o \nabla u) \\ & - \frac{1}{2} h(t) \|\nabla v\|_2^2 - \omega_2 \|\nabla v_t\|_2^2, \end{aligned} \quad (35)$$

and, from (3.4)₃, (3.4)₄, we have

$$\begin{aligned}
& \frac{d}{dt} \frac{1}{2} \int_{\Omega} \int_0^{\tau_2} \int_{\tau_1}^1 \mathbf{Q} |\mu_2(\mathbf{Q})| y^2(x, \rho, \mathbf{Q}, t) d\mathbf{Q} d\rho dx \\
&= -\frac{1}{2} \int_{\Omega} \int_0^{\tau_2} \int_{\tau_1}^1 2 |\mu_2(\mathbf{Q})| y y_{\rho} d\mathbf{Q} d\rho dx, \\
&+ \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| y^2(x, 0, \mathbf{Q}, t) d\mathbf{Q} dx \\
&- \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| y^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx, \\
&= \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| d\mathbf{Q} \right) \|u_t\|_2^2 \\
&- \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| y^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx,
\end{aligned} \tag{36}$$

$$\begin{aligned}
& \frac{d}{dt} \frac{1}{2} \int_{\Omega} \int_0^{\tau_2} \int_{\tau_1}^1 \mathbf{Q} |\mu_4(\mathbf{Q})| z^2(x, \rho, \mathbf{Q}, t) d\mathbf{Q} d\rho dx \\
&= -\frac{1}{2} \int_{\Omega} \int_0^{\tau_2} \int_{\tau_1}^1 2 |\mu_4(\mathbf{Q})| z z_{\rho} d\mathbf{Q} d\rho dx, \\
&+ \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| z^2(x, 0, \mathbf{Q}, t) d\mathbf{Q} dx \\
&- \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| z^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx, \\
&= \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| d\mathbf{Q} \right) \|v_t\|_2^2 \\
&- \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| z^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx,
\end{aligned} \tag{37}$$

then, we get

$$\begin{aligned}
& \frac{d}{dt} E(t) \\
&= -\mu_1 \|u_t\|_2^2 - m_1 \|u\|_2^2 - \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| u_t y(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \\
&+ \frac{1}{2} (g' \circ \nabla u) - \frac{1}{2} g(t) \|\nabla u\|_2^2 - \omega_1 \|\nabla u_t\|_2^2 \\
&+ \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| d\mathbf{Q} \|u_t\|_2^2 - \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| y^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \right. \\
&- \mu_3 \|v_t\|_2^2 - m_2 \|v\|_2^2 - \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| v_t z(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \\
&+ \frac{1}{2} (h' \circ \nabla v) - \frac{1}{2} h(t) \|\nabla v\|_2^2 - \omega_2 \|\nabla v_t\|_2^2 \\
&+ \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| d\mathbf{Q} \|v_t\|_2^2 - \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| z^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \right).
\end{aligned} \tag{38}$$

By (35)–(37), we get (32).

And by using Young's inequality, (14), (16), and (17) in (38), we obtain (33).

Now, we define the functional

$$\begin{aligned}
\mathbb{H}(t) &= -E(t) = -\frac{1}{2} \|u_t\|_2^2 - \frac{1}{2} \|v_t\|_2^2 - \frac{m_1}{2} \|u\|_2^2 - \frac{m_2}{2} \|v\|_2^2 \\
&- \frac{1}{2} l_1 \|\nabla u\|_2^2 - \frac{1}{2} l_2 \|\nabla v\|_2^2 - \frac{1}{2} (g \circ \nabla u) - \frac{1}{2} (h \circ \nabla v) \\
&- \frac{1}{2} K(y, z) + \frac{1}{2(p+2)} \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2 \|uv\|_{(p+2)}^{(p+2)} \right].
\end{aligned} \tag{39}$$

Theorem 5. Assume (14)–(17) and (25) hold. Assume further that $E(0) < 0$, then the solution of problem (22) blow up in finite time.

Proof. From (32), we have

$$E(t) \leq E(0) \leq 0. \tag{40}$$

Therefore,

$$\begin{aligned}
\mathbb{H}'(t) &= -E'(t) \\
&\geq c_3 \left(\|u_t\|_2^2 + \|v_t\|_2^2 + \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| y^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \right. \\
&\quad \left. + \|v_t\|_2^2 + \|v\|_2^2 + \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| z^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \right),
\end{aligned} \tag{41}$$

hence,

$$\mathbb{H}'(t) \geq c_3 \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| y^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \geq 0, \tag{42}$$

$$\mathbb{H}'(t) \geq c_3 \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| z^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \geq 0, \tag{43}$$

$$\begin{aligned}
0 &\leq \mathbb{H}(0) \mathbb{H}(t) \leq \frac{1}{2(p+2)} \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2 \|uv\|_{p+2}^{p+2} \right], \\
&\leq \frac{c_1}{2(p+2)} \left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right].
\end{aligned} \tag{44}$$

We set

$$\begin{aligned}
\mathcal{K}(t) &= \mathbb{H}^{1-\alpha} + \varepsilon \int_{\Omega} (uu_t + vv_t) dx + \frac{3}{2} \int_{\Omega} (\mu_1 u^2 + \mu_3 v^2) dx \\
&+ \frac{\varepsilon}{2} \int_{\Omega} (\omega_1 (\nabla u)^2 + \omega_2 (\nabla v)^2) dx,
\end{aligned} \tag{45}$$

where $\varepsilon > 0$ to be assigned later and

$$0 < \alpha < \frac{2p+2}{4(p+2)} < 1. \tag{46}$$

By multiplying (3.4)₁, (3.4)₂ by u, v and with a derivative of (45), we get

$$\begin{aligned} \mathcal{K}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2) \\ &\quad - \varepsilon(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \varepsilon \int_{\Omega} \nabla u \int_0^1 g(t-s) \nabla u(s) ds dx \\ &\quad + \varepsilon \int_{\Omega} \nabla v \int_0^1 h(t-s) \nabla v(s) ds dx \\ &\quad - \varepsilon \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| uy(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \\ &\quad - \varepsilon \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| vz(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \\ &\quad + \varepsilon \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \right]. \end{aligned} \quad (47)$$

Using Young's inequality, we get

$$\begin{aligned} &\varepsilon \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| uy(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \\ &\leq \varepsilon \left\{ \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| d\mathbf{Q} \right) \|u\|_2^2 \right. \\ &\quad \left. + \frac{1}{4\delta_1} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| y^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \right\}, \\ &\varepsilon \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| vz(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \\ &\leq \varepsilon \left\{ \delta_2 \left(\int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| d\mathbf{Q} \right) \|u\|_2^2 \right. \\ &\quad \left. + \frac{1}{4\delta_2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| z^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \right\}, \end{aligned} \quad (48)$$

and we have

$$\begin{aligned} &\varepsilon \int_0^t g(t-s) ds \int_{\Omega} \nabla u \cdot \nabla u(s) dx ds \\ &= \varepsilon \int_0^t g(t-s) ds \int_{\Omega} \nabla u \cdot (\nabla u(s) - \nabla u(t)) dx ds \\ &\quad + \varepsilon \int_0^t g(s) ds \|\nabla u\|_2^2 \geq \frac{\varepsilon}{2} \int_0^t g(s) ds \|\nabla u\|_2^2 - \frac{\varepsilon}{2} (go\nabla u), \\ &\varepsilon \int_0^t h(t-s) ds \int_{\Omega} \nabla v \cdot \nabla v(s) dx ds \\ &= \varepsilon \int_0^t h(t-s) ds \int_{\Omega} \nabla v \cdot (\nabla v(s) - \nabla v(t)) dx ds \\ &\quad + \varepsilon \int_0^t h(s) ds \|\nabla v\|_2^2 \geq \frac{\varepsilon}{2} \int_0^t h(s) ds \|\nabla v\|_2^2 - \frac{\varepsilon}{2} (ho\nabla v). \end{aligned} \quad (49)$$

We obtain, from (47),

$$\begin{aligned} \mathcal{K}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2) \\ &\quad - \varepsilon \left(\left(1 - \frac{1}{2} \int_0^t g(s) ds \right) \|\nabla u\|_2^2 \right. \\ &\quad \left. + \left(1 - \frac{1}{2} \int_0^t h(s) ds \right) \|\nabla v\|_2^2 \right) \\ &\quad - \varepsilon \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| d\mathbf{Q} \right) \|u\|_2^2 \\ &\quad - \varepsilon \delta_2 \left(\int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| d\mathbf{Q} \right) \|v\|_2^2 - \frac{\varepsilon}{2} (go\nabla u) \\ &\quad - \frac{\varepsilon}{4\delta_1} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| y^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx - \frac{\varepsilon}{2} (ho\nabla v) \\ &\quad - \frac{\varepsilon}{4\delta_2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| z^2(x, 1, \mathbf{Q}, t) d\mathbf{Q} dx \\ &\quad + \varepsilon \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \right]. \end{aligned} \quad (50)$$

Therefore, using (43) and by setting δ_1, δ_2 so that, $1/4\delta_1 c_3 = \kappa \mathbb{H}^{-\alpha}(t)/2$ and $1/4\delta_2 c_3 = \kappa \mathbb{H}^{-\alpha}(t)/2$, substituting in (50), we get

$$\begin{aligned} \mathcal{K}'(t) &\geq [(1-\alpha) - \varepsilon\kappa] \mathbb{H}^{-\alpha}\mathbb{H}'(t) \\ &\quad + \varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2) \\ &\quad - \varepsilon \left[\left(1 - \frac{1}{2} \int_0^t g(s) ds \right) \|\nabla u\|_2^2 \right. \\ &\quad \left. - \varepsilon \left[\left(1 - \frac{1}{2} \int_0^t h(s) ds \right) \|\nabla v\|_2^2 \right] \right] \\ &\quad - \varepsilon \frac{\mathbb{H}^{\alpha}(t)}{2c_3\kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| d\mathbf{Q} \right) \|u\|_2^2 - \frac{\varepsilon}{2} (go\nabla u) \\ &\quad - \varepsilon \frac{\mathbb{H}^{\alpha}(t)}{2c_3\kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| d\mathbf{Q} \right) \|v\|_2^2 - \frac{\varepsilon}{2} (ho\nabla v) \\ &\quad + \varepsilon \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \right]. \end{aligned} \quad (51)$$

For $0 < a < 1$, from (39),

$$\begin{aligned} &\varepsilon \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \right] \\ &= \varepsilon a \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \right] + \varepsilon 2(p+2)(1-a)\mathbb{H}(t) \\ &\quad + \varepsilon(p+2)(1-a)(\|u_t\|_2^2 + \|v_t\|_2^2) + \varepsilon(p+2)(1-a) \\ &\quad \cdot \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + \varepsilon(p+2)(1-a) \\ &\quad \cdot \left(1 - \int_0^t h(s) ds \right) \|\nabla v\|_2^2 - \varepsilon(p+2)(1-a)(go\nabla u) \\ &\quad - \varepsilon(p+2)(1-a)(ho\nabla v) + \varepsilon(p+2)(1-a)K(y, z), \end{aligned} \quad (52)$$

substituting in (51), we get

$$\begin{aligned}
\mathcal{K}'(t) \geq & [(1-\alpha) - \varepsilon\kappa] \mathbb{H}^{-\alpha} \mathbb{H}'(t) \\
& + \varepsilon[(p+2)(1-\alpha) + 1] (\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2) \\
& + \varepsilon \left[(p+2)(1-\alpha) \left(1 - \int_0^t g(s) ds \right) \right. \\
& \left. - \left(1 - \frac{1}{2} \int_0^t g(s) ds \right) \right] \|\nabla u\|_2^2 \\
& + \varepsilon \left[(p+2)(1-\alpha) \left(1 - \int_0^t h(s) ds \right) \right. \\
& \left. - \left(1 - \frac{1}{2} \int_0^t h(s) ds \right) \right] \|\nabla v\|_2^2 \\
& - \varepsilon \frac{\mathbb{H}^\alpha(t)}{2c_3\kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| d\mathbf{Q} \right) \|u\|_2^2 \\
& - \varepsilon \frac{\mathbb{H}^\alpha(t)}{2c_3\kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| d\mathbf{Q} \right) \|v\|_2^2 \\
& + \varepsilon(p+2)(1-\alpha)K(y, z) \\
& + \varepsilon \left[(p+2)(1-\alpha) - \frac{1}{2} \right] (go\nabla u + ho\nabla v) \\
& + \varepsilon a \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \right] \\
& + \varepsilon 2(p+2)(1-\alpha)\mathbb{H}(t).
\end{aligned} \tag{53}$$

Since (25) hold, we obtain by using (44) and (46)

$$\mathbb{H}^\alpha(t) \|u\|_2^2 \leq c_4 \left(\|u\|_{2(p+2)}^{2\alpha(p+2)+2} + \|v\|_{2(p+2)}^{2\alpha(p+2)+2} \|u\|_2^2 \right), \tag{54}$$

$$\mathbb{H}^\alpha(t) \|v\|_2^2 \leq c_5 \left(\|v\|_{2(p+2)}^{2\alpha(p+2)+2} + \|u\|_{2(p+2)}^{2\alpha(p+2)+2} \|v\|_2^2 \right), \tag{55}$$

for some positive constants c_4, c_5 . By using (46) and the algebraic inequality,

$$B^\theta \leq (B+1) \leq \left(1 + \frac{1}{b} \right) (B+b), \quad \forall B > 0, \quad 0 < \theta < 1, \quad b > 0, \tag{56}$$

we have, $\forall t > 0$

$$\|u\|_{2(p+2)}^{2\alpha(p+2)+2} \leq d \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0) \right) \leq d \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \tag{57}$$

$$\|v\|_{2(p+2)}^{2\alpha(p+2)+2} \leq d \left(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right) \leq d \left(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \tag{58}$$

where $d = 1 + (1/\mathbb{H}(0))$. Also, since

$$(x+y)^\gamma \leq C(x^\gamma + y^\gamma), \quad \forall x, y > 0, \quad \gamma > 0, \tag{59}$$

we conclude

$$\begin{aligned}
\|v\|_{2(p+2)}^{2\alpha(p+2)} \|u\|_2^2 & \leq c_6 \left(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_2^{2(p+2)} \right) \\
& \leq c_7 \left(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} \right),
\end{aligned} \tag{60}$$

$$\begin{aligned}
\|u\|_{2(p+2)}^{2\alpha(p+2)} \|v\|_2^2 & \leq c_8 \left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_2^{2(p+2)} \right) \\
& \leq c_9 \left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right),
\end{aligned} \tag{61}$$

substituting (58) and (61) in (55), we get

$$\mathbb{H}^\alpha(t) \|u\|_2^2 \leq c_{10} \left(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} \right) + c_{10}\mathbb{H}(t), \tag{62}$$

$$\mathbb{H}^\alpha(t) \|v\|_2^2 \leq c_{11} \left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right) + c_{11}\mathbb{H}(t), \tag{63}$$

Combining (53) and (63), using (31), we get

$$\begin{aligned}
\mathcal{K}'(t) \geq & [(1-\alpha) - \varepsilon\kappa] \mathbb{H}^{-\alpha} \mathbb{H}'(t) \\
& + \varepsilon[(p+2)(1-\alpha) + 1] (\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2) \\
& + \varepsilon \left\{ [(p+2)(1-\alpha) - 1] - \left(\int_0^t g(s) ds \right) \right. \\
& \cdot \left[(p+2)(1-\alpha) - \frac{1}{2} \right] \left. \right\} \|\nabla u\|_2^2 \\
& + \varepsilon \left\{ [(p+2)(1-\alpha) - 1] - \left(\int_0^t h(s) ds \right) \right. \\
& \cdot \left[(p+2)(1-\alpha) - \frac{1}{2} \right] \left. \right\} \|\nabla v\|_2^2 \\
& + \varepsilon(p+2)(1-\alpha)K(y, z) \\
& + \varepsilon \left[(p+2)(1-\alpha) - \frac{1}{2} \right] (go\nabla u + ho\nabla v) \\
& + \varepsilon \left(c_0 a - \frac{\lambda_1 + \lambda_2}{2c_3\kappa} \right) \left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right] \\
& + \varepsilon \left(2(p+2)(1-\alpha) - \frac{\lambda_1 + \lambda_2}{2c_3\kappa} \right) \mathbb{H}(t),
\end{aligned} \tag{64}$$

where $\lambda_1 = c_{10} \int_{\tau_1}^{\tau_2} |\mu_2(\mathbf{Q})| d\mathbf{Q}$, $\lambda_2 = c_{11} \int_{\tau_1}^{\tau_2} |\mu_4(\mathbf{Q})| d\mathbf{Q}$.

In this case, we take $a > 0$ small enough, then

$$\alpha_1 = (p+2)(1-a) - 1 > 0, \tag{65}$$

assuming

$$\begin{aligned}
\max \left\{ \int_0^\infty g(s) ds, \int_0^\infty h(s) ds \right\} & < \frac{(p+2)(1-a) - 1}{((p+2)(1-a) - (1/2))} \\
& = \frac{2\alpha_1}{2\alpha_1 + 1},
\end{aligned} \tag{66}$$

we have

$$\begin{aligned}\alpha_2 &= \left\{ (p+2)(1-a) - 1 - \int_0^t g(s) ds \left((p+2)(1-a) - \frac{1}{2} \right) \right\} > 0, \\ \alpha_3 &= \left\{ (p+2)(1-a) - 1 - \int_0^t h(s) ds \left((p+2)(1-a) - \frac{1}{2} \right) \right\} > 0,\end{aligned}\quad (67)$$

choose κ so large that

$$\begin{aligned}\alpha_4 &= ac_0 - \frac{\lambda_1 + \lambda_2}{2c_3\kappa} > 0, \\ \alpha_5 &= 2(p+2)(1-a) - \frac{\lambda_1 + \lambda_2}{2c_3\kappa} > 0,\end{aligned}\quad (68)$$

fix κ and a , we appoint ε small enough so that

$$\alpha_6 = (1-\alpha) - \varepsilon\kappa > 0. \quad (69)$$

Then, for $\beta > 0$, we estimate (64) and it becomes

$$\begin{aligned}\mathcal{K}'(t) &\geq \beta \left\{ \mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 \right. \\ &\quad + \|\nabla v\|_2^2 + (go\nabla u) + (ho\nabla v) + K(y, z) \\ &\quad \left. + \left[\|u\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} \right] \right\}.\end{aligned}\quad (70)$$

By (31), for $\beta_1 > 0$, we get

$$\begin{aligned}\mathcal{K}'(t) &\geq \beta_1 \left\{ \mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 \right. \\ &\quad + \|\nabla v\|_2^2 + (go\nabla u) + (ho\nabla v) + K(y, z) \\ &\quad \left. + \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \right] \right\},\end{aligned}$$

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0, t > 0. \quad (71)$$

Using Holder's and Young's inequalities, we have

$$\begin{aligned}\left| \int_{\Omega} (uu_t + vv_t) dx \right|^{1/(1-\alpha)} &\geq C \left[\|u\|_{2(p+2)}^{\theta/(1-\alpha)} + \|u_t\|_2^{\mu/(1-\alpha)} \right. \\ &\quad \left. + \|v\|_{2(p+2)}^{\theta/(1-\alpha)} + \|v_t\|_2^{\mu/(1-\alpha)} \right],\end{aligned}\quad (72)$$

where $(1/\mu) + (1/\theta) = 1$ put $\theta = 2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq 2(p+2). \quad (73)$$

Subsequently, for $s = 2/(1-2\alpha)$ and by using (39), we get

$$\begin{aligned}\|u\|_{2(p+2)}^{2/(1-2\alpha)} &\leq d \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \\ \|v\|_{2(p+2)}^{2/(1-2\alpha)} &\leq d \left(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \quad \forall t \geq 0.\end{aligned}\quad (74)$$

Therefore,

$$\begin{aligned}\left| \int_{\Omega} (uu_t + vv_t) dx \right|^{1/(1-\alpha)} &\geq c_{12} \left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_2^2 + \|v_t\|_2^2 + \mathbb{H}(t) \right].\end{aligned}\quad (75)$$

Subsequently,

$$\begin{aligned}\mathcal{K}^{1/(1-\alpha)}(t) &= \left(\mathbb{H}^{1-\alpha} + \varepsilon \int_{\Omega} (uu_t + vv_t) dx \right. \\ &\quad + \frac{\varepsilon}{2} \int_{\Omega} (\mu_1 u^2 + \mu_3 v^2) dx \\ &\quad \left. + \frac{\varepsilon}{2} \int_{\Omega} (\omega_1 \nabla u^2 + \omega_2 \nabla v^2) dx \right)^{1/(1-\alpha)} \\ &\leq c \left\{ \mathbb{H}(t) \left| \int_{\Omega} (uu_t + vv_t) dx \right|^{1/(1-\alpha)} + \|u\|_2^{2/(1-\alpha)} \right. \\ &\quad \left. + \|\nabla u\|_2^{2/(1-\alpha)} + \|v\|_2^{2/(1-\alpha)} + \|\nabla v\|_2^{2/(1-\alpha)} \right\} \\ &\leq c \left[\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 \right. \\ &\quad + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (go\nabla u) + (ho\nabla v) \\ &\quad \left. + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right].\end{aligned}\quad (76)$$

From (70) and (76), gives

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}^{\frac{1}{1-\alpha}}(t), \quad (77)$$

with $\lambda > 0$, this quantity depends on β and c . By simple integration of (77), we obtain

$$\mathcal{K}^{\alpha/(1-\alpha)}(t) \geq \frac{1}{\mathcal{K}^{-\alpha/(1-\alpha)}(0) - \lambda(\alpha/(1-\alpha))t}, \quad (78)$$

Hence, $\mathcal{K}(t)$ in a situation of blow up in time, when

$$T \leq T^* = \frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}, \quad (79)$$

Then, this completes the proof of the theorem.

4. Conclusion

In this work, we have studied the blow up of the coupled Klein-Gordon system with strong damping, distributed delay, and source terms, under suitable conditions which are so important that we find them in many applications of natural sciences. Many authors have been concerned with this problem in recent decades (see, for example, [17–19]). In the next work, we will try to apply the same technique with a new class of Boussinesq equations which are nonlinear partial differential equation that arises in hydrodynamics and some physical applications. It was subsequently applied to problems in the percolation of water in porous subsurface strata (see, for example, [20, 21]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally in this article. They have all read and approved the final manuscript.

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Research Article

Global Existence and Decay for a System of Two Singular Nonlinear Viscoelastic Equations with General Source and Localized Frictional Damping Terms

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The current paper deals with the proof of a global solution of a viscoelasticity singular one-dimensional system with localized frictional damping and general source terms, taking into consideration nonlocal boundary condition. Moreover, similar to that in Boulaaras' recent studies by constructing a Lyapunov functional and use it together with the perturbed energy method in order to prove a general decay result.

1. Introduction

The evolution problem with integral conditions is related with many branches of sciences ([1–6]). Cause of this, interest in it occurs naturally in inflation cosmology, nuclear physics, supersymmetric field theories, and quantum mechanics (see

for example [2, 7]). Later, by the motivation of this work, some authors gave necessary and sufficient conditions for the hyperbolic equation with source term (see, e.g., [8–10]).

This manuscript is devoted to the study of the global existence and decay for a system of two singular one-dimensional nonlinear viscoelastic equations with general source terms

$$\left\{ \begin{array}{l} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s) \frac{1}{x}(xu_x(x,s))_x ds + \mu(x)u_t = f_1(u, v), \text{ in } (0, L) \times (0, T), \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s) \frac{1}{x}(xv_x(x,s))_x ds + \mu(x)v_t = f_2(u, v), \text{ in } (0, L) \times (0, T), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in (0, L), \\ v(x, 0) = v_0(x), v_t(x, 0) = v_1(x), x \in (0, L), \\ u(L, t) = v(L, t) = 0, \int_0^L xu(x, t)dx = \int_0^L xv(x, t)dx = 0, \end{array} \right. \quad (1)$$

where $L < \infty$, $T < \infty$, $\mu \in C^1((0, \alpha))$, $g_1(\cdot), g_2(\cdot): \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $f_1(\cdot, \cdot), f_2(\cdot, \cdot): \mathbb{R}^2 \longrightarrow \mathbb{R}$ are given functions, which will be specified later.

The problems related with localized frictional damping have extensively studied by many teams as [11], where the authors obtained an exponential rate of decay for the solution of the viscoelastic nonlinear wave equation:

$$\begin{aligned} u_{tt} - \Delta u + f(x, t, u) + \int_0^t g_1(t-s)\Delta u(s)ds + a(x)u_t \\ = 0, \text{ in } (0, L) \times (0, T), \end{aligned} \quad (2)$$

for damping term $a(x)u_t$ may be null for some part of the domain.

We used the techniques in [11]; we have proven in [8] the existence of a global solution using the potential well theory for the following viscoelastic system with nonlocal boundary condition and localized frictional damping

$$\left\{ \begin{array}{l} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s) \frac{1}{x}(xu_x(x,s))_x ds + a(x)u_t = |v|^{q+1}|u|^{p-1}u, \text{ in } (0, L) \times (0, T), \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s) \frac{1}{x}(xv_x(x,s))_x ds + a(x)v_t = |v|^{q+1}|u|^{p-1}v, \text{ in } (0, L) \times (0, T), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in (0, \alpha), \\ v(x, 0) = v_0(x), v_t(x, 0) = v_1(x), x \in (0, \alpha), \\ u(\alpha, t) = v(\alpha, t) = 0, \int_0^\alpha xu(x, t)dx = \int_0^\alpha xv(x, t)dx = 0. \end{array} \right. \quad (3)$$

Very recently, in ([9]), we study the following singular one-dimensional nonlinear equations that arise in generalized viscoelasticity with long-term memory:

$$\left\{ \begin{array}{l} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s) \frac{1}{x}(xu_x(x,s))_x ds = f_1(u, v), \text{ in } (0, L) \times (0, T), \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s) \frac{1}{x}(xv_x(x,s))_x ds = f_2(u, v), \text{ in } (0, L) \times (0, T), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in (0, L), \\ v(x, 0) = v_0(x), v_t(x, 0) = v_1(x), x \in (0, L), \\ u(L, t) = v(L, t) = 0, \int_0^L xu(x, t)dx = \int_0^L xv(x, t)dx = 0. \end{array} \right. \quad (4)$$

In view of the articles mentioned above in ([8, 9, 11]), much less effort has been devoted to the existence of a global solution to the system of two singular nonlinear equations which arise in generalized viscoelasticity with localized frictional damping terms using the potential-well theory. Moreover, we prove a general decay result by constructing a Lyapunov functional and use it together with the perturbed energy method.

The structure of the work is as follows: To facilitate the description, firstly in Section 2, we give the fundamental definitions and theorems on function spaces that will be needed in the body of the paper and state the local existence theorem. In Section 3, the energy function $E(t)$ is defined and proved to be a nonincreasing function of time. Finally, the main result is obtained, which gives the general decay conditions:

$$g'_i(t) \leq -\xi(t)g_i^r(t), i = 1, 2. \quad (5)$$

2. Preliminaries

Let $L_x^p = L_x^p((0, L))$ be the weighted Banach space equipped with the norm

$$\|u\|_{L_x^p} = \left(\int_0^L x|u|^p dx \right)^{1/p}, \quad (6)$$

when $p = 2$, we get a Hilbert space, and we denote by $H = L_x^2$, it provided with the finite norm

$$\|u\|_H = \left(\int_0^L xu^2 dx \right)^{1/2}. \quad (7)$$

$V = V_x^1((0, L))$ be the Hilbert space equipped with the norm

$$\|u\|_V = (\|u\|_H^2 + \|u_x\|_H^2)^{1/2}. \quad (8)$$

We get the following lemma by combining the Poincare inequality and (see [8]).

Lemma 1. Let V_0 space defined as follows

$$V_0 = \{u \in V \text{ such that } u(L) = 0\}. \quad (9)$$

Then, for $2 \leq p < 4$, we have

$$\int_0^L x|v|^p dx \leq C_* \|v_x\|_{H=L_x^2(0,L)}^p, \forall u \in V_0, \quad (10)$$

where C_* is a constant depending on L and p only and for $p = 2$, $C_* = C_p$ is the Poincare constant.

Remark 2. It is clear that $\|u\|_{V_0} = \|u_x\|_H$ defines an equivalent norm on V_0 .

The next theorem confirms that our problem has a local solution under some condition on p and the relaxation func-

tions g_i , the proof can be established by following the argument of [12].

Theorem 3. We take $(u_0, v_0) \in V_0^2$ and $(v_1, v_2) \in H^2$. If $p < 3$ and

$$g_i(0) > 0, \left(1 - \int_0^\infty g_i(s) ds\right) = l > 0, \text{ for } i = 1, 2, \quad (11)$$

then, there exists $t_* > 0$ small enough such that the problem (1) has a unique local solution

$$u \in C(0, t_*; V_0) \cap C^1(0, t_*; H). \quad (12)$$

Remark 4. The condition on p is needed so that the embedding of V_0 in L_x^2 is Lipchitz.

We need the following assumptions to get our results.

(G₁) For $i = 1, 2$, $g_i(t): \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nonincreasing C_2 function such that

$$\begin{cases} g_i(s) \geq 0, g'_i(s) \leq 0 \text{ and,} \\ g_i(0) \geq 0, 1 - \int_0^\infty g_i(s) ds = l_1 > 0, \end{cases} \quad (13)$$

and

(G₂)

$$g'_i(t) \leq -\xi(t)g_i^\sigma(t), \quad t \geq 0, 1 \leq \sigma < \frac{3}{2}, \quad (14)$$

where $\xi(t)$ is a positive differentiable function. It satisfies for some positive constant l

$$\left| \frac{\xi'(t)}{\xi(t)} \right| \leq l, \xi'(t) \leq 0, \quad \int_0^\infty \xi(s) ds = +\infty, \forall t > 0. \quad (15)$$

Furthermore, for any $t_0 > 0$ and $1 < \sigma < 3/2$, there exists a positive constant C_σ such that

$$\frac{t}{\left(1 + \int_{t_0}^t \xi(s) ds\right)^{1/2(\sigma-1)}} \leq C_\sigma, \forall t \geq t_0. \quad (16)$$

(G₃) We take

$$\begin{aligned} f_1(u, v) &= a|u + v|^{2(r+1)}(u + v) + b|u|^r |v|^{r+2}, \\ f_2(u, v) &= a|u + v|^{2(r+1)}(u + v) + b|v|^r |u|^{r+2}, \end{aligned} \quad (17)$$

where $a, b > 0$ are constants and $r > -1$.

One can easily verify that

$$uf_1(u, v) + vf_2(u, v) = 2(r+2)F(u, v), \forall (u, v) \in \mathbb{R}^2, \quad (18)$$

where

$$F(u, v) = \frac{1}{2(r+2)} \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right]. \quad (19)$$

(G_4) $\mu \geq 0, \mu > 0$ in $(L_0, L]$, where $0 \leq L_0 \leq L$.

Lemma 5. For $r > -1$, there exist $\eta > 0$ such that for any $u, v \in V \cap V_0(0, L)$, we have

$$\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{(r+2)} \leq \eta(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2)^{r+2}. \quad (20)$$

Proof. From Minkowski inequality, we have

$$\|u+v\|_{L_x^{2(r+2)}}^2 \leq 2\left(\|u\|_{L_x^{2(r+2)}}^2 + \|v\|_{L_x^{2(r+2)}}^2\right). \quad (21)$$

We apply successively Holder's and Young's inequalities we obtain

$$\|uv\|_{L_x^{(r+2)}}^{(r+2)} \leq \|u\|_{L_x^{2(r+2)}}^{2(r+2)} \|v\|_{L_x^{2(r+2)}}^{2(r+2)} \leq c(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2)^{r+2}. \quad (22)$$

We combine the two previous inequalities and the embedding $V \cap V_0(0, L) \hookrightarrow L_x^{2(r+2)}(0, L)$, we get (20).

Lemma 6. There exist two positive constants Λ_1 and Λ_2 such that

$$\begin{aligned} x|f_i(u, v)|^2 dx \\ \leq \Lambda_i \left(l_1 \int_0^L xu_x^2 dx + l_2 \int_0^L xv_x^2 dx \right)^{2r+3}, \forall x \in (0, L), i = 1, 2. \end{aligned} \quad (23)$$

Proof. It is clear that

$$\begin{aligned} |f_1(u, v)| &\leq C(|u+v|^{2r+3} + |u|^{r+1}|v|^{r+2}) \\ &\leq C[|u|^{2r+3}|v|^{2r+3} + |u|^{r+1}|v|^{r+2}], \end{aligned} \quad (24)$$

Applying Young's inequality with exponents $q = (2r+3)/(r+1)$, $q' = (2r+3)/(r+2)$, in the last term in the above inequality, we get

$$|f_1(u, v)| \leq C[|u|^{2r+3} + |v|^{2r+3}]. \quad (25)$$

Consequently, by using Poincaré's inequality and (20), we obtain

$$\begin{aligned} \int_0^L x|f_i(u, v)|^2 dx &\leq C\left(\|u_x\|_H^{2(2r+3)} + \|v_x\|_H^{2(2r+3)}\right) \\ &\leq \Lambda_1(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2)^{(2r+3)}. \end{aligned} \quad (26)$$

Similarly, we get the inequality for f_2 . The proof is completed.

We define the energy function as

$$\begin{aligned} E(t) &= \frac{1}{2} \int_0^L xu_t^2 dx + \frac{1}{2} \int_0^L xv_t^2 dx + \frac{1}{2} \left(1 - \int_0^t g_1(s) ds \right) \int_0^L xu_x^2 dx \\ &\quad + \frac{1}{2} \left(1 - \int_0^t g_1(s) ds \right) \int_0^L xv_x^2 dx - \int_0^L F(u, v) dx \\ &\quad + \frac{1}{2} (g_1 \circ u_x)(t) + \frac{1}{2} (g_2 \circ v_x)(t), \end{aligned} \quad (27)$$

where

$$(g_2 \circ u_x)(t) = \int_0^L \int_0^t xg(t-s)|u_x(x, t) - u_x(x, s)|^2 ds dx. \quad (28)$$

Lemma 7. Let (u, v) be the solution of system (1) then for all $t \geq 0$

$$\begin{aligned} \frac{d}{dt}[E(t)] &= - \int_0^L x\mu(x)u_t^2 dx - \int_0^L x\mu(x)v_t^2 dx + \frac{1}{2} (g' \circ u_x)(t) \\ &\quad - \frac{1}{2} g_1(t) \int_0^L xu_x^2 dx + \frac{1}{2} (g' \circ v_x)(t) - \frac{1}{2} g_2 \int_0^L xv_x^2 dx. \end{aligned} \quad (29)$$

Hence, $E(t)$ is a nonincreasing function.

Proof. Multiplying the first and the second equations in (1) by xu_t and xv_t , respectively, integrating over $(0, L)$, summing up, we obtain (30)

$$\begin{aligned} &\int_0^L xu_{tt}u_t dx - \int_0^L (xu_x)_x u_t dx + \int_0^L \int_0^t g_1(t-s)(xu_x(x, s))_x ds u_t dx \\ &\quad + \int_0^L xv_{tt}v_t dx - \int_0^L (xv)_x v_t + \int_0^L \int_0^t g_2(t-s)(xv_x(x, s))_x ds v_t dx \\ &= - \int_0^L x\mu(x)u_t^2 dx - \int_0^L x\mu(x)v_t^2 dx \\ &\quad + \int_0^L [a|u+v|^{2(r+1)}(u+v) + b|u|^r|v|^{r+2}] xu_t dx \\ &\quad + \int_0^L [a|u+v|^{2(r+1)}(u+v) + b|v|^r|u|^{r+2}] xv_t dx. \end{aligned} \quad (30)$$

By integration by parts, we obtain

$$\int_0^L xu_{tt}u_t dx = \frac{1}{2} \frac{d}{dt} \left[\int_0^L xu_t^2 dx \right], \quad (31)$$

$$\int_0^L xv_{tt}v_t dx = \frac{1}{2} \frac{d}{dt} \left[\int_0^L xv_t^2 dx \right], \quad (32)$$

$$- \int_0^L (xu_x)_x u_t dx = \frac{1}{2} \frac{d}{dt} \left[\int_0^L xu_x^2 dx \right], \quad (33)$$

$$-\int_0^L (xv_x)_x v_t dx = \frac{1}{2} \frac{d}{dt} \left[\int_0^L x v_x^2 dx \right], \quad (34)$$

$$\begin{aligned} & \frac{1}{2(r+2)} \int_0^L x f_1(u, v) u u_t dx + \frac{1}{2(r+t)} \int_0^L x f_2(u, v) v v_t dx \\ &= \frac{1}{2(r+2)} \frac{d}{dt} \int_0^L \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] x dx, \end{aligned} \quad (35)$$

$$\begin{aligned} & \int_0^L \int_0^t g_1(t-s) (xu_x(s))_x ds u_t(t) dx \\ &= \frac{1}{2} \frac{d}{dt} \left[(g_1 \circ u_x)(t) - \int_0^t g_1(s) ds \int_0^L x u_x^2 dx \right] \\ & \quad - \frac{1}{2} (g_1' \circ u_x)(t) + \frac{1}{2} g_1(t) \int_0^L x u_x^2 dx, \end{aligned} \quad (36)$$

$$\begin{aligned} & \int_0^L \int_0^t g_2(t-s) (xv_x(s))_x ds v_t(t) dx \\ &= \frac{1}{2} \frac{d}{dt} \left[(g_2 \circ u_x)(t) - \int_0^t g_2(s) ds \int_0^L x v_x^2 dx \right] \\ & \quad - \frac{1}{2} (g_2' \circ u_x)(t) + \frac{1}{2} g_2(t) \int_0^L x v_x^2 dx, \end{aligned} \quad (37)$$

Combining (32)–(2.22) in (31), we get (30).

3. Global Existence

In order to state and prove the global existence, we set the following notation

$$\begin{aligned} I(t) := I(u(t), v(t)) &= \left(1 - \int_0^t g_1(s) ds \right) \int_0^L x u_x^2 dx \\ &+ \left(1 - \int_0^t g_2(s) ds \right) \int_0^L x v_x^2 dx + (g_1 \circ u_x)(t) \\ &+ (g_2 \circ v_x)(t) - 2(r+2) \int_0^L x \left[a|u+v|^{2(r+2)} \right. \\ & \quad \left. + 2b|uv|^{r+2} \right] dx, \end{aligned} \quad (38)$$

$$\begin{aligned} J(t) := J(u(t), v(t)) &= \frac{1}{2} \left(1 - \int_0^t g_1(s) ds \right) \int_0^L x u_x^2 dx \\ &+ \frac{1}{2} \left(1 - \int_0^t g_2(s) ds \right) \int_0^L x v_x^2 dx + \frac{1}{2} (g_1 \circ u_x)(t) \\ &+ \frac{1}{2} (g_2 \circ v_x)(t) - \int_0^L x \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx, \end{aligned} \quad (39)$$

we remark that

$$E(t) = J(t) \frac{1}{2} \int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx. \quad (40)$$

Lemma 8. Assume that (G_1) , (G_2) , and (20) hold also for any $(u_0, v_0) \in V_0^2$ and $(u_1, v_1) \in H_2$ satisfying

$$I(0) > 0, \beta := \eta \left(\frac{2(r+2)}{(r+1)} E(0) \right)^{r+1} < 1, \quad (41)$$

where

$$E(0) = J(0) + \frac{1}{2} \int_0^L x u_1^2 dx + \frac{1}{2} \int_0^L x v_1^2 dx. \quad (42)$$

Then, there exists $t_* > 0$ such that

$$I(t) > 0, \forall t \in [0, t_*]. \quad (43)$$

Proof. Since $I(0) > 0$, then from the continuity of $I(t)$, there exist $t_m \leq t_*$ such that $I(t) \geq 0$ for all $t \in [0, t_m]$; this implies that we have a maximum time value noting T_m such that

$$\{I(T_m) = 0 \quad \text{and} \quad I(t) > 0, \quad \text{for all} \quad 0 \leq t < T_m\}. \quad (44)$$

From formulas of $J(t)$ and $I(t)$ together with (G_1) , we have

$$\begin{aligned} J(t) &= \frac{r+1}{2(r+2)} \left[\left(1 - \int_0^t g_1(s) ds \right) \int_0^L x u_x^2 ds \right. \\ & \quad \left. + \left(1 - \int_0^t g_2(s) ds \right) \int_0^L x v_x^2 ds \right] \\ &+ \frac{r+1}{2(r+2)} \left[(g_1 \circ u_x)(t) + \frac{1}{2} (g_2 \circ v_x)(t) \right] \\ &+ \frac{1}{2(r+2)} I(t) \geq \frac{r+1}{2(r+2)} \left[\left(l_1 \int_0^L x u_x^2 dx + l_2 \int_0^L x v_x^2 dx \right) \right. \\ & \quad \left. + (g_1 \circ u_x)(t) + \frac{1}{2} (g_2 \circ v_x)(t) \right], \end{aligned} \quad (45)$$

hence,

$$\begin{aligned} & l_1 \int_0^L x u_x^2 dx + l_2 \int_0^L x v_x^2 dx \\ & \leq \frac{2(r+2)}{r+1} J(t) \leq \frac{2(r+2)}{r+1} E(t) \\ & \leq \frac{2(r+2)}{r+1} E(0), \forall t \in [0, T_m], \end{aligned} \quad (46)$$

Recalling Lemma 5 and (41), we get

$$\begin{aligned}
& 2(r+2) \int_0^L F(u(T_m), v(T_m)) dx \\
& \leq \eta \left(l_1 \int_0^L xu_x^2 dx + l_2 \int_0^L xv_x^2 dx \right)^{r+2} \\
& \leq \eta \left(\frac{2(r+2)}{r+1} E(0) \right)^{r+1} \left(l_1 \int_0^L xu_x^2 dx + l_2 \int_0^L xv_x^2 dx \right) \\
& = \beta \left(l_1 \int_0^L xu_x^2 dx + l_2 \int_0^L xv_x^2 dx \right) \\
& < \left(1 - \int_0^t g_1(s) ds \right) \left(\int_0^L xu_x^2 dx \right) \\
& \quad + \left(1 - \int_0^t g_2(s) ds \right) \left(\int_0^L xv_x^2 dx \right) \\
& \quad + (g_1 \circ u_x)(t) + (g_2 \circ v_x)(t),
\end{aligned} \tag{47}$$

consequently

$$\begin{aligned}
& \left(1 - \int_0^t g_1(s) ds \right) \left(\int_0^L xu_x^2 dx \right) \\
& \quad + \left(1 - \int_0^t g_2(s) ds \right) \left(\int_0^L xv_x^2 dx \right) + (g_1 \circ u_x)(t) \\
& \quad + (g_2 \circ v_x)(t) - 2(r+2) \int_0^L xF(u, v) dx > 0,
\end{aligned} \tag{48}$$

we deduce that $I(t) > 0, \forall t \in [0, T_m]$. By repeating the procedure, T_m is extended to t_* .

Theorem 9. Suppose that (G_1) , (G_2) , and (20) hold. Then, for any $(u_0, v_0) \in V_0^2$ and $(u_1, v_1) \in H^2$ satisfying (41), the solution of system (1) is a bounded and globally in time.

Proof. To achieve the proof of this theorem, it suffices to show that $\|u_x\|_H^2 + \|v_x\|_H^2 + \|u_t\|_H^2 + \|v_t\|_H^2$ is bounded independently of t . As $E(t)$ is a nonincreasing function, we have

$$E(0) \geq E(t), \tag{49}$$

in the other hand and for the definition of $I(t)$, we have

$$\begin{aligned}
& x \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx \\
& = I(t) - \left(1 - \int_0^t g_1(s) ds \right) \int_0^L xu_x^2 dx \\
& \quad - \left(1 - \int_0^t g_2(s) ds \right) \int_0^L xv_x^2 dx - (g_1 \circ u_x)(t) \\
& \quad - (g_2 \circ v_x)(t),
\end{aligned} \tag{50}$$

we introduce (49) into (50), we get

$$\begin{aligned}
E(0) \geq E(t) & = \frac{1}{2} \int_0^L xu_t^2 dx + \frac{1}{2} \int_0^L xv_t^2 dx \\
& \quad + \frac{1}{2} \left(1 - \int_0^t g_1(s) ds \right) \int_0^L xu_x^2 dx \\
& \quad + \frac{1}{2} \left(1 - \int_0^t g_2(s) ds \right) \int_0^L xv_x^2(x, t) dx \\
& \quad + \frac{1}{2} (g_1 \circ u_x)(t) + \frac{1}{2} (g_2 \circ v_x)(t) + I(t),
\end{aligned} \tag{51}$$

by using (14), (15), and (41), (51) yields

$$\begin{aligned}
E(0) \geq E(t) & \geq \frac{1}{2} \int_0^L xu_t^2 dx + \frac{1}{2} \int_0^L xv_t^2 dx \\
& \quad + \left(\frac{r+1}{2(r+2)} \right) l_1 \int_0^L xu_x^2 dx + \left(\frac{r+1}{2(r+2)} \right) l_2 \int_0^L xv_x^2 dx \\
& \geq \mu_0 \left(\int_0^L xu_t^2 dx + \int_0^L xv_t^2 dx + \int_0^L xu_x^2 dx + \int_0^L xv_x^2 dx \right).
\end{aligned} \tag{52}$$

So

$$\|u_x\|_H^2 + \|v_x\|_H^2 + \|u_t\|_H^2 + \|v_t\|_H^2 \leq \tau E(0), \tag{53}$$

where

$$\tau := \max \left\{ 2, \frac{2(r+2)}{(r+1)l_1}, \frac{2(r+2)}{(r+1)l_2} \right\}. \tag{54}$$

The proof is completed.

4. Decay of Solutions

Throughout this section, we will study the asymptotic behavior of solutions' decay by constructing a suitable Lyapunov function; to do so, for N , ε_1 , and ε_2 are positive constants, we define the following function as

$$F(t) := E(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \chi(t) + \psi(t), \tag{55}$$

where

$$\begin{aligned}
\Phi(t) & := \xi(t) \int_0^L xu_t u dx + \xi(t) \int_0^L xv_t v dx \\
& \quad + \frac{\xi(t)}{2} \int_0^L x \mu(x) (u^2 + v^2) dx,
\end{aligned} \tag{56}$$

$$\begin{aligned}
\chi(t) & := -\xi(t) \int_0^L xu_t \int_0^t g_1(t-s)(u(t)-u(s)) ds dx \\
& \quad - \xi(t) \int_0^L xv_t \int_0^t g_2(t-s)(v(t)-v(s)) ds dx,
\end{aligned} \tag{57}$$

and

$$\psi(t) := \xi(t) \int_0^L x u_t h(x) u_x dx + \xi(t) \int_0^L x v_t h(x) v_x dx, \quad (58)$$

with $h \in C^1([0, L])$, $h(0) = h(L) = 0$, $(xh(x))' \leq x$,

In the first step, we prove the equivalence between $F(t)$ and $E(t)$ given in the following lemma.

Lemma 10. For a choice of ε_1 and ε_2 small enough, we find two positive constants α_1 and α_2 such that

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t). \quad (59)$$

Proof. By using Young inequality, follow by recalling Lemma 1 and the fact that $0 < \xi(t) \leq \xi(0)$, we get

$$\left| \varepsilon_1 \xi(t) \int_0^L x u_t u dx \right| \leq \frac{\varepsilon_1}{2} \xi(0) \left(\int_0^L x u_t^2 dx + C_p \int_0^L x u_x^2 dx \right), \quad (60)$$

$$\left| \varepsilon_1 \xi(t) \int_0^L x v_t v dx \right| \leq \frac{\varepsilon_1}{2} \xi(0) \left(\int_0^L x v_t^2 dx + C_p \int_0^L x v_x^2 dx \right), \quad (61)$$

$$\begin{aligned} & \left| -\varepsilon_2 \xi(t) \int_0^L x u_t \int_0^t g_1(t-s)(u(t) - u(s)) ds dx \right| \\ & \leq \frac{\varepsilon_2}{2} \xi(0) \left(\int_0^L x u_t^2 dx + C_p (1 - l_1)(g_1 \circ u_x)(t) \right), \end{aligned} \quad (62)$$

$$\begin{aligned} & \left| -\varepsilon_2 \xi(t) \int_0^L x v_t \int_0^t g_2(t-s)(v(t) - v(s)) ds dx \right| \\ & \leq \frac{\varepsilon_2}{2} \xi(0) \left(\int_0^L x v_t^2 dx + C_p (1 - l_2)(g_2 \circ v_x)(t) \right), \end{aligned} \quad (63)$$

$$\begin{aligned} & \left| \xi(t) \int_0^L x u_t h(x) u_x dx \right| \\ & \leq \frac{\xi(0)}{2} \|h\|_\infty \left(\int_0^L x u_x^2 dx + \int_0^L x u_t^2 dx \right), \end{aligned} \quad (64)$$

$$\begin{aligned} & \left| \xi(t) \int_0^L x v_t h(x) v_x dx \right| \\ & \leq \frac{\xi(0)}{2} \|h\|_\infty \left(\int_0^L x v_x^2 dx + \int_0^L x v_t^2 dx \right), \end{aligned} \quad (65)$$

$$\begin{aligned} & \left| \frac{\xi(t)}{2} \int_0^L x \mu(x) (u^2 + v^2) dx \right| \\ & \leq \frac{\xi(0)}{2} \|\mu\|_\infty C_p \left(\int_0^L x u_x^2 dx + \int_0^L x v_x^2 dx \right), \end{aligned} \quad (66)$$

combining (60)–(66) in (55), we get

$$\begin{aligned} & |F(t) - NE(t)| \\ & \leq \left(\left(\frac{\varepsilon_1 + \varepsilon_2}{2} \right) \xi(0) + \frac{\xi(0)}{2} \|h\|_\infty \right) \int_0^L x u_t^2 dx \\ & \quad + \left(\left(\frac{\varepsilon_1 + \varepsilon_2}{2} \right) \xi(0) + \frac{\xi(0)}{2} \|h\|_\infty \right) \int_0^L x v_t^2 dx \\ & \quad + \left(\frac{1 + \|\mu\|_\infty}{2} \varepsilon_1 C_p \xi(0) + \frac{\xi(0)}{2} \|h\|_\infty (p+1) \right) \int_0^L x u_x^2 dx \\ & \quad + \left(\frac{1 + \|\mu\|_\infty}{2} \varepsilon_1 C_p \xi(0) + \frac{\xi(0)}{2} \|h\|_\infty (q+1) \right) \int_0^L x v_x^2 dx \\ & \quad + \frac{\varepsilon_2}{2} C_p \xi(0) ((1 - l_1)(g_1 \circ u_x)(t) + (1 - l_2)(g_2 \circ v_x)(t)). \end{aligned} \quad (67)$$

If we choose $\varepsilon_1, \varepsilon_2$ small enough, and N large enough we find $\alpha_1, \alpha_2 > 0$ such that (59) holds true.

Now, we state a Lemma corresponding to the boundness of $(\text{gov}_x)(t)$. It will be used in the calculus.

Lemma 11. Let $w \in L^\infty((0, T); H)$ be such that $w_x \in L^\infty((0, t); H)$ and g be a continuous function on $[0, T]$ and suppose that. Then, there exists a constant $C > 0$ such that

$$\begin{aligned} (\text{gow}_x)(t) & \leq C \left(\sup_{0 \leq s \leq T} \|w(\cdot, s)\|_H^2 \int_0^t g^{1-\theta}(s) ds \right)^{\rho-1/\rho-1+\theta} \\ & \quad \times \left(\int_0^t g^\rho(t-s) \|w_x(\cdot, s)\|_H^2 ds \right)^{\theta/\rho-1+\theta}, \\ & \quad \forall 0 < \theta < 1 \text{ and } \rho > 1. \end{aligned} \quad (68)$$

$$\begin{aligned} (\text{gow}_x)(t) & \leq c \left(t \|w_x(\cdot, t)\|_H^2 + \int_0^t \|w_x(\cdot, s)\|_H^2 ds \right)^{\rho-1/\rho} \\ & \quad \times \left(\int_0^t g^\rho(t-s) \|w_x(\cdot, t) - w_x(\cdot, s)\|_H^2 ds \right)^{1/2} \\ & \quad \text{for all } \rho > 1. \end{aligned} \quad (69)$$

Proof.

(1) For any $\sigma > 1$, we have

$$\begin{aligned} (\text{gow}_x)(t) & = \int_0^t (g(t-s))^{(1-\theta)/\sigma} \|w_x(\cdot, t) - w_x(\cdot, s)\|^{2/\sigma} \\ & \quad \cdot (g(t-s))^{\sigma-1+\theta/\sigma} \|w_x(\cdot, t) - w_x(\cdot, s)\|^{2(\sigma-1)/\sigma} ds. \end{aligned} \quad (70)$$

Applying Holder's inequality with exponents σ and $\sigma/\sigma - 1$, we get

$$(gow_x)(t) \leq \left(\int_0^t g^{1-\theta}(t-s) \|w_x(\cdot, t) - w_x(\cdot, s)\|^2 ds \right)^{1/\sigma} \times \left(\int_0^t g^{\sigma-1+\theta/\sigma-1}(t-s) \|w_x(\cdot, t) - w_x(\cdot, s)\|^2 ds \right)^{\sigma-1/\sigma}, \quad (71)$$

We set $\sigma = (\rho - 1 + \theta)/(\rho - 1)$, (71) yields

$$(gow_x)(t) \leq \left(\int_0^t g^{1-\theta}(t-s) \|w_x(\cdot, t) - w_x(\cdot, s)\|_H^2 ds \right)^{\rho-1/\rho-1+\theta} \times \left(\int_0^t g^\rho(t-s) \|w_x(\cdot, t) - w_x(\cdot, s)\|_H^2 ds \right)^{\theta/\rho-1+\theta}. \quad (72)$$

It is easy to see that

$$\int_0^t g^{1-\theta}(t-s) \|w_x(\cdot, t) - w_x(\cdot, s)\|_H^2 ds \leq C \sup_{0 < s < T} \|w_x(\cdot, s)\|_H^2 \int_0^t g^{1-\theta}(s) ds. \quad (73)$$

We obtain (68) by combining (72) and (73).

(2) We set $\theta = 1$ in (72) and it suffices to note that

$$\int_0^t \|w_x(\cdot, t) - w_x(\cdot, s)\|_H^2 ds \leq 2t \|w_x(\cdot, t)\|_H^2 + 2 \int_0^t \|w_x(\cdot, s)\|_H^2 ds, \quad (74)$$

to arrive at (69).

In the next, we present three lemmas in which we give an upper bound of each derivative's functions in $F(t)$.

Lemma 12. Suppose that $r > -1$ and (39) hold. Then

$$\begin{aligned} \Phi'(t) &\leq \left(1 + \frac{l}{2\delta}\right) \xi(t) \int_0^L x u_t^2 dx + \left(1 + \frac{l}{2\delta}\right) \xi(t) \int_0^L x v_t^2 dx \\ &\quad - \xi(t) \left(\frac{l_1 - \delta C_p l}{2} - \frac{C_p \|\mu\|_\infty}{2} \right) \int_0^L x u_x^2 dx \\ &\quad - \xi(t) \left(\frac{l_2 - \delta C_p l}{2} - \frac{C_p \|\mu\|_\infty}{2} \right) \int_0^L x v_x^2 dx - \xi(t) \end{aligned}$$

$$\begin{aligned} &+ \frac{\xi(t)}{2l_1} \left(\int_0^t g_1^{2-\sigma}(s) ds \right) (g_1^\sigma \circ u_x)(t) \\ &+ \frac{\xi(t)}{2l_2} \left(\int_0^t g_2^{2-\sigma}(s) ds \right) (g_2^\sigma \circ v_x)(t) \\ &+ \frac{\xi(t)}{2(r+2)} \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx. \end{aligned} \quad (75)$$

For any $\delta > 0$.

Proof. After derivation of (56), we recall the differential equations in (1), we get

$$\begin{aligned} \Phi'(t) &= \xi'(t) \int_0^L x u_t u dx + \xi(t) \int_0^L x u_t^2 dx + (t) \int_0^L x u_{tt} u dx \\ &\quad + \xi'(t) \int_0^L x v_t v dx + \xi(t) \int_0^L x v_t^2 dx \\ &\quad + \xi(t) \int_0^L x v_{tt} v dx + \frac{\xi'(t)}{2} \int_0^L x \mu(x) (u^2 + v^2) dx \\ &\quad + \xi(t) \left(\int_0^L x \mu(x) (u_t u + v_t v) dx \right) \\ &= \xi'(t) \int_0^L x u_t u dx + \xi(t) \int_0^L x u_t^2 dx - \xi(t) \int_0^L x u_x^2 dx \\ &\quad + \xi(t) \int_0^L x u_x \int_0^t g_1(t-s) u_x(s) ds dx \\ &\quad + \xi'(t) \int_0^L x v_t v dx + \xi(t) \int_0^L x v_t^2 dx - \xi(t) \int_0^L x v_x^2 dx \\ &\quad + \xi(t) \int_0^L x v_x \int_0^t g_2(t-s) v_x(s) ds dx \\ &\quad + \frac{\xi(t)}{2(r+2)} \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx. \end{aligned} \quad (76)$$

By Young's inequality and from, (G_1) , (G_2) , and Lemma 1, we arrive at

$$\begin{aligned} &\xi(t) \int_0^L x u_x(t) \left(\int_0^t g_1(t-s) u_x(s) ds \right) dx \\ &\leq \frac{\xi(t)}{2} \int_0^L x u_x^2 dx + \frac{\xi(t)}{2} \int_0^L x \left(\int_0^t g_1(t-s) \cdot (|u_x(s) - u_x(t)| + |u_x(t)|) ds \right)^2 dx \\ &\leq \frac{\xi(t)}{2} \int_0^L x u_x^2 dx + \frac{\xi(t)}{2} (1 + \eta_1)(1 + l_1)^2 \int_0^L x u_x^2(t) dx \\ &\quad + \frac{\xi(t)}{2} \left(1 + \frac{1}{\eta_1} \right) \left(\int_0^t g_1^{2-\sigma}(s) ds \right) (g_1^\sigma \circ u_x)(t) \end{aligned}$$

$$\begin{aligned}
&= \xi(t) \left(\frac{1 + (1 + \eta_1)(1 - l_1)^2}{2} \right) \int_0^L x u_x^2 dx \\
&\quad + \frac{\xi(t)}{2} \left(1 + \frac{1}{\eta_1} \right) \left(\int_0^t g_1^{2-\sigma}(s) ds \right) (g_1^\sigma \circ u_x)(t) \\
&\quad + \frac{\xi(t)}{r+2} \int_0^L \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx, \\
&\quad \cdot \left(\int_0^t g_1^{2-\sigma}(s) ds \right) (g_1^\sigma \circ u_x)(t) + \frac{\xi(t)}{2} \left(1 + \frac{1}{\eta_2} \right) \\
&\quad \cdot \left(\int_0^t g_2^{2-\sigma}(s) ds \right) (g_2^\sigma \circ v_x)(t) \\
&\quad + \frac{\xi(t)}{2(r+2)} \int_0^L \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx,
\end{aligned} \tag{77}$$

similarly, we get

$$\begin{aligned}
&\int_0^L x v_x(t) \left(\int_0^t g_1(t-s) v_x(s) ds \right) dx \\
&\leq \xi(t) \left(\frac{1 + (1 + \eta_2)(1 - l_2)^2}{2} \right) \int_0^L x v_x^2 dx \\
&\quad + \frac{\xi(t)}{2} \left(1 + \frac{1}{\eta_2} \right) \left(\int_0^t g_2^{2-\sigma}(s) ds \right) (g_2^\sigma \circ v_x)(t).
\end{aligned} \tag{78}$$

For any η_1 and $\eta_2 > 0$. We also have

$$\begin{aligned}
\xi'(t) \int_0^L x u_t u dx &\leq \frac{\xi(t)}{2} \left| \frac{\xi'(t)}{\xi(t)} \right| \left(C_p \delta \int_0^L x u_x^2 dx + \frac{1}{\delta} \int_0^L x u_t^2 dx \right) \\
&\leq \frac{\xi(t)}{2} \left(C_p l \delta \int_0^L x u_x^2 dx + \frac{l}{\delta} \int_0^L x u_t^2 dx \right), \\
&\quad \forall \delta > 0,
\end{aligned} \tag{79}$$

and similarly, we get

$$\xi'(t) \int_0^L x v_t v dx \leq \frac{\xi(t)}{2} \left(C_p l \delta \int_0^L x v_x^2 dx + \frac{l}{\delta} \int_0^L x v_t^2 dx \right). \tag{80}$$

Also, by Lemma 1, we have

$$\begin{aligned}
&\frac{\xi'(t)}{2} \int_0^L x \mu(x) (u^2 + v^2) dx \\
&\leq \|\mu\|_\infty \frac{\xi(t)}{2} \left[C_p \left(\int_0^L x (u_x^2 + v_x^2) dx \right) \right].
\end{aligned} \tag{81}$$

Combining (77)–(81) in (76) leads to

$$\begin{aligned}
\Phi'(t) &\leq \left(1 + \frac{l}{2\delta} \right) \xi(t) \int_0^L x u_t^2 dx + \left(1 + \frac{l}{2\delta} \right) \xi(t) \int_0^L x v_t^2 dx \\
&\quad - \frac{\xi(t)}{2} [1 - (1 + \eta_1)(1 - l_1)^2 - \delta C_p l - C_p \|u\|_\infty] \\
&\quad \cdot \int_0^L x u_t^2 dx - \frac{\xi(t)}{2} [1 - (1 + \eta_2)(1 - l_2)^2 - \delta C_p l \\
&\quad - C_p \|u\|_\infty] \int_0^L x u_t^2 dx + \frac{\xi(t)}{2} \left(1 + \frac{1}{\eta_1} \right)
\end{aligned}$$

we choose $\eta_1 = l_1/1 - l_1$ and $\eta_2 = l_2/1 - l_2$, hence (75) is established.

Lemma 13. Suppose that $r > -1$, (G_1) , (G_2) , and (41) hold. Then

$$\begin{aligned}
\chi'(t) &\leq \xi(t) \theta \left[2 + c_1 + c'_1 + 2(1 - l_1)^2 \right] \left(\int_0^L x u_x^2 dx \right) \\
&\quad + \xi(t) \theta \left[2 + c_2 + c'_2 + 2(1 - l_2)^2 \right] \left(\int_0^L x v_x^2 dx \right) \\
&\quad + \xi(t) \left[\theta - \left(\int_0^t g_1(s) ds \right) + \theta l \right] \left(\int_0^L x u_t^2 dx \right) \\
&\quad + \xi(t) \left[\theta - \left(\int_0^t g_2(s) ds \right) + \theta l \right] \left(\int_0^L x v_t^2 dx \right) \\
&\quad + \left[\frac{1}{2\theta} + 2\theta + \frac{C_p(2+l)}{4\theta} \right] \times \xi(t) \left(\int_0^t g_1^{2-\sigma}(s) ds \right) \\
&\quad \cdot (g_1^\sigma \circ u_x)(t) + \left[\frac{1}{2\theta} + 2\theta + \frac{C_p(2+l)}{4\theta} \right] \\
&\quad \times \xi(t) \left(\int_0^t g_2^{2-\sigma}(s) ds \right) (g_2^\sigma \circ v_x)(t) \\
&\quad - \frac{C_p}{4\theta} \xi(t) g_1(0) (g_1^\sigma \circ u_x)(t) \\
&\quad - \frac{C_p}{4\theta} \xi(t) g_2(0) (g_2^\sigma \circ v_x)(t),
\end{aligned} \tag{83}$$

for any $\theta > 0$.

Proof. A derivation of (57) gives

$$\begin{aligned}
\chi'(t) &= -\xi'(t) \int_0^L x u_t \int_0^t g_1(t-s) (u(t) - u(s)) ds dx \\
&\quad - \xi(t) \int_0^L x u_{tt} \int_0^t g_1(t-s) (u(t) - u(s)) ds dx \\
&\quad - \xi(t) \int_0^L x u_t \frac{d}{dt} \left(\int_0^t g_1(t-s) (u(t) - u(s)) ds \right) dx \\
&\quad - \xi'(t) \int_0^L x v_t \int_0^t g_2(t-s) (v(t) - v(s)) ds dx \\
&\quad - \xi(t) \int_0^L x v_{tt} \int_0^t g_2(t-s) (v(t) - v(s)) ds dx \\
&\quad - \xi(t) \int_0^L x v_t \frac{d}{dt} \left(\int_0^t g_2(t-s) (v(t) - v(s)) ds \right) dx,
\end{aligned} \tag{84}$$

by using Liebniz's formula, we get

$$\begin{aligned}
 \chi'(t) = & -\xi'(t) \int_0^L x u_t \int_0^t g_1(t-s)(u(t)-u(s)) ds dx \\
 & -\xi(t) \int_0^L x u_{tt} \int_0^t g_1(t-s)(u(t)-u(s)) ds dx \\
 & -\xi(t) \int_0^L x u_t \int_0^t g_1'(t-s)(u(t)-u(s)) ds dx \\
 & -\xi(t) \left(\int_0^t g_1(s) ds \right) \int_0^L x u_t^2 dx \\
 & -\xi'(t) \int_0^L x v_t \int_0^t g_2(t-s)(v(t)-v(s)) ds dx \\
 & -\xi(t) \int_0^L x v_{tt} \int_0^t g_2(t-s)(v(t)-v(s)) ds dx \\
 & -\xi(t) \int_0^L x v_t \int_0^t g_2'(t-s)(v(t)-v(s)) ds dx \\
 & -\xi(t) \left(\int_0^t g_2(s) ds \right) \int_0^L x v_t^2 dx.
 \end{aligned} \tag{85}$$

Recalling the differentials equation in (1), we get

$$\begin{aligned}
 \chi'(t) = & -\xi'(t) \int_0^L x u_t \left(\int_0^t g_1(t-s)(u(t)-u(s)) ds \right) dx \\
 & + \xi(t) \int_0^L x \mu(x) u_t \left(\int_0^t g_1(t-s)(u(t)-u(s)) ds \right) dx \\
 & + \xi(t) \int_0^L x u_x \left(\int_0^t g_1(t-s)(u_x(t)-u_x(s)) ds \right) dx \\
 & - \xi(t) \int_0^L x \left(\int_0^t g_1(t-s) u_x(s) ds \right) \\
 & \cdot \left(\int_0^t g_1(t-s)(u_x(t)-u_x(s)) ds \right) dx \\
 & - \xi(t) \int_0^L (x f_1(u, v)) \left(\int_0^t g_1(t-s)(u(t)-u(s)) ds \right) dx \\
 & - \xi(t) \int_0^L x u_t \left(\int_0^t g_1'(t-s)(u(t)-u(s)) ds \right) dx \\
 & - \xi(t) \left(\int_0^t g_1(s) ds \right) \int_0^L x u_t^2 dx \\
 & - \xi'(t) \int_0^L x v_t \left(\int_0^t g_2(t-s)(v(t)-v(s)) ds \right) dx \\
 & + \xi(t) \int_0^L x \mu(x) v_t \left(\int_0^t g_2(t-s)(v(t)-v(s)) ds \right) dx \\
 & + \xi(t) \int_0^L x v_x \left(\int_0^t g_2(t-s)(v_x(t)-v_x(s)) ds \right) dx \\
 & - \xi(t) \int_0^L x \left(\int_0^t g_2(t-s)(v_x(s)) ds \right) \\
 & \cdot \left(\int_0^t g_2(t-s)(v_x(t)-v_x(s)) ds \right) dx
 \end{aligned}$$

$$\begin{aligned}
 & -\xi(t) \int_0^L (x f_2(u, v)) \left(\int_0^t g_2(t-s)(v(t)-v(s)) ds \right) dx \\
 & - \xi(t) \int_0^L x v_t \left(\int_0^t g_2'(t-s)(v(t)-v(s)) ds \right) dx \\
 & - \xi(t) \left(\int_0^t g_2(s) ds \right) \int_0^L x v_t^2 dx.
 \end{aligned} \tag{86}$$

We will estimate all term in (86) by Young's inequality, Lemma 1, (\mathbf{G}_1) , and (\mathbf{G}_2) .

$$\begin{aligned}
 & -\xi'(t) \int_0^L x u_t \left(\int_0^t g_1(t-s)(u(t)-u(s)) ds \right) dx \\
 & \leq \xi(t) \left| \frac{\xi'(t)}{\xi(t)} \right| \left[\theta \int_0^L x u_t^2 dx + \frac{C_p}{4\theta} \left(\int_0^t g_1^{2-\sigma}(s) ds \right) (g_1^\sigma \circ u_x)(t) \right] \\
 & \leq \theta l \xi(t) \int_0^L x u_t^2 dx + \frac{C_p}{4\theta} \xi(t) \left(\int_0^t g_1^{2-\sigma}(s) ds \right) (g_1^\sigma \circ u_x)(t),
 \end{aligned} \tag{87}$$

$$\begin{aligned}
 & -\xi(t) \int_0^L x \mu(x) u_t \left(\int_0^t g_1(t-s)(u(t)-u(s)) ds \right) dx \\
 & \leq \xi(t) \|\mu\|_\infty \left[\theta \int_0^L x u_t^2 dx + \frac{C_p}{4\theta} \left(\int_0^t g_1^{2-r}(s) ds \right) (g_1^r \circ u_x)(t) \right],
 \end{aligned} \tag{88}$$

and

$$\begin{aligned}
 & -\xi(t) \int_0^L x u_x \left(\int_0^t g_1(t-s)(u_x(t)-u_x(s)) ds \right) dx \\
 & \leq \theta \xi(t) \int_0^L x u_x^2 dx + \frac{1}{4\theta} \xi(t) \left(\int_0^t g_1^{2-\sigma}(s) ds \right) (g_1^\sigma \circ u_x)(t), \\
 & -\xi(t) \int_0^L x \left(\int_0^t g_1(t-s) u_x(s) ds \right) \left(\int_0^t g_1(t-s)(u_x(t)-u_x(s)) ds \right) dx \\
 & \leq 2\theta(1-l_1)^2 \xi(t) \int_0^L x u_x^2 dx + \left(2\theta + \frac{1}{4\theta} \right) \xi(t) \\
 & \cdot \left(\int_0^t g_1^{2-\sigma}(s) ds \right) (g_1^\sigma \circ u_x)(t), \\
 & -\xi(t) \int_0^L x u_t \left(\int_0^t g_1'(t-s)(u(t)-u(s)) ds \right) dx \\
 & \leq \theta \xi(t) \int_0^L x u_t^2 dx - \frac{g_1(0)}{4\theta} C_p \xi(t) (g_1' \circ u_x)(t),
 \end{aligned} \tag{89}$$

and

$$\begin{aligned}
 & -\xi(t) \int_0^L x f_1(u, v) \left(\int_0^t g_1(t-s)(u(t)-u(s)) ds \right) dx \\
 & \leq \frac{C_p}{4\theta} \xi(t) \left(\int_0^t g_1^{2-\sigma}(s) ds \right) (g_1^\sigma \circ u_x)(t) \\
 & + c_1 \theta \xi(t) \int_0^L x u_x^2 dx + c_2 \theta \xi(t) \int_0^L x v_x^2 dx,
 \end{aligned} \tag{90}$$

where

$$\begin{cases} c_1 := \Lambda_1 \left(\frac{2(r+2)}{(r+1)} E(0) \right)^{2(r+1)}, \\ c_2 := \Lambda_2 \left(\frac{2(r+2)}{(r+1)} E(0) \right)^{2(r+1)}. \end{cases} \quad (91)$$

By the same technique, we obtain estimations on integrals corresponding to v , g_2 , and f_2

$$\begin{aligned} & -\xi'(t) \int_0^L x v_t \left(\int_0^t g_2(t-s)(v(t)-v(s))ds \right) dx \\ & \leq \theta l \xi(t) \int_0^L x v_t^2 dx + \frac{C_p l}{4\theta} \xi(t) \left(\int_0^t g_2^{2-\sigma}(s)ds \right) (g_2^\sigma \circ v_x)(t), \\ & -\xi(t) \int_0^L x \mu(x) v_t \left(\int_0^t g_2(t-s)(v(t)-v(s))ds \right) dx \\ & \leq \xi(t) \|\mu\|_\infty \left[\theta \int_0^L x v_t^2 dx + \frac{C_p}{4\theta} \left(\int_0^t g_2^{2-\sigma}(s)ds \right) (g_2^\sigma \circ v_x)(t) \right], \\ & \xi(t) \int_0^L x v_x \left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds \right) dx \\ & \leq \theta \xi(t) \int_0^L x v_x^2 dx + \frac{1}{4\theta} \xi(t) \left(\int_0^t g_2^{2-\sigma}(s)ds \right) (g_2^\sigma \circ v_x)(t), \\ & -\xi(t) \int_0^L x \left(\int_0^t g_2(t-s) v_x(s) ds \right) \\ & \quad \cdot \left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds \right) dx \\ & \leq 2\theta(1-l_2)^2 \xi(t) \int_0^L x v_x^2 dx + \left(2\theta + \frac{1}{4\theta} \right) \xi(t) \\ & \quad \cdot \left(\int_0^t g_2^{2-\sigma}(s)ds \right) (g_2^\sigma \circ u_x)(t), \\ & -\xi(t) \int_0^L x v_t \left(\int_0^t g_2'(t-s)(v(t)-v(s))ds \right) dx \\ & \leq \theta \xi(t) \int_0^L x v_t^2 dx - \frac{g_2(0)}{4\theta} C_p \xi(t) (g_2' \circ v_x)(t), \end{aligned} \quad (92)$$

and

$$\begin{aligned} & -\frac{\xi(t)}{2(r+2)} \int_0^L x f_2(u, v) \left(\int_0^t g_2(t-s)(v(t)-v(s))ds \right) dx \\ & \leq \frac{C_p}{4\theta} \xi(t) \left(\int_0^t g_2^{2-\sigma}(s)ds \right) (g_2^\sigma \circ v_x)(t) \\ & \quad + c_1' \theta \xi(t) \int_0^L x u_x^2 dx + c_2' \theta \xi(t) \int_0^L x v_x^2 dx, \end{aligned} \quad (93)$$

where

$$\begin{cases} c_1' := \Lambda_1' \left(\frac{2(r+2)}{(r+1)} E(0) \right)^{2(r+1)}, \\ c_2' := \Lambda_2' \left(\frac{2(r+2)}{(r+1)} E(0) \right)^{2(r+1)}. \end{cases} \quad (94)$$

A combination of (87)–(93) into (86) yields (83).

Lemma 14. Suppose that $r > -1$, (G_1) , (G_2) , (G_4) , and (41) hold. Then

$$\begin{aligned} \psi(t)' & \leq \xi(t) [1 + \|h\|_\infty \theta (2 + \|\mu\|_\infty)] \left(\int_0^L x u_x^2 dx \right) \\ & \quad + \xi(t) [1 + \|h\|_\infty \theta (2 + \|\mu\|_\infty)] \left(\int_0^L x v_x^2 dx \right) \\ & \quad + \xi(t) \int_0^L \left[1 + \|h\|_\infty \theta \left(\frac{1}{2\theta} + \frac{1}{4\theta} \mu(x) \right) \right] x u_t^2 dx \\ & \quad + \xi(t) \int_0^L \left[1 + \|h\|_\infty \theta \left(\frac{1}{2\theta} + \frac{1}{4\theta} \mu(x) \right) \right] x v_t^2 dx \\ & \quad + \xi(t) \left[\|h\|_\infty \frac{C_p}{4\theta} \right] \left(\int_0^t g_1^{2-\sigma}(s)ds \right) (g_1^\sigma \circ u_x)(t) \\ & \quad + \xi(t) \left[\|h\|_\infty \frac{C_p}{4\theta} \right] \left(\int_0^t g_2^{2-\sigma}(s)ds \right) (g_2^\sigma \circ v_x)(t) \\ & \quad + \frac{\xi(t)}{2(r+2)} \int_0^L x \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx, \end{aligned} \quad (95)$$

For any $\theta > 0$.

Proof. We derivate (58) and integrate by part and we finish by using the differential equations in (1), we get

$$\begin{aligned} \psi'(t) & = \xi'(t) \int_0^L u_t x h(x) u_x dx + \xi'(t) \int_0^L v_t x h(x) v_x dx \\ & \quad + \xi(t) \int_0^L u_t x h(x) u_{tx} dx + (q+1) \xi'(t) \int_0^L v_t x h(x) v_{tx} dx \\ & \quad + \xi(t) \int_0^L u_{tt} x h(x) u_x dx + \xi(t) \int_0^L v_{tt} x h(x) v_{tx} dx \\ & = \xi'(t) \int_0^L u_t x h(x) u_x dx + \xi'(t) \int_0^L v_t x h(x) v_x dx \\ & \quad - \frac{1}{2} \xi(t) \int_0^L (x h(x))' u_t^2 dx - \frac{1}{2} \xi(t) \int_0^L (x h(x))' v_t^2 dx \\ & \quad - \frac{1}{2} \xi(t) \int_0^L (x h(x))' u_x^2 dx - \frac{1}{2} \xi(t) \int_0^L (x h(x))' v_x^2 dx \\ & \quad - \xi(t) \int_0^L x \mu(x) h(x) u_t u_x dx - \xi(t) \int_0^L x \mu(x) h(x) v_t v_x dx \end{aligned}$$

$$\begin{aligned}
& -\xi(t) \int_0^L xh(x)u_x \left(\int_0^t g_1(t-s)(u(t)-u(s))ds \right) dx \\
& -\xi(t) \int_0^L xh(x)v_x \left(\int_0^t g_2(t-s)(v(t)-v(s))ds \right) dx \\
& + \frac{\xi(t)}{2(r+2)} \int_0^L (xh(x))' \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx,
\end{aligned} \tag{96}$$

Applying Young's and Poincaré's inequalities, we get

$$\begin{aligned}
& \xi'(t) \int_0^L u_t xh(x)u_x dx \\
& \leq \xi(t) \|h\|_\infty \left(\theta \int_0^L xu_x^2 dx + \frac{1}{4\theta} \int_0^L xu_t^2 dx \right),
\end{aligned} \tag{97}$$

$$\begin{aligned}
& \int_0^L x\mu(x)h(x)u_t u_x dx \\
& \leq \|h\|_\infty \left(\theta \int_0^L x\mu(x)u_x^2 dx + \frac{1}{4\theta} \int_0^L x\mu(x)u_t^2 dx \right),
\end{aligned} \tag{98}$$

finally

$$\begin{aligned}
& \int_0^L xh(x)u_x \left(\int_0^t g_1(t-s)(u(t)-u(s))ds \right) dx \\
& \leq \|h\|_\infty \left(\theta \int_0^L xu_x^2 dx + \frac{C_p}{4\theta} \left(\int_0^t g_1^{2-r}(s)ds \right) (g_1' \circ u_x)(t) \right),
\end{aligned} \tag{99}$$

similarly

$$\begin{aligned}
& \xi'(t) \int_0^L v_t xh(x)v_x dx \\
& \leq \xi(t) \|h\|_\infty \left(\theta \int_0^L xv_x^2 dx + \frac{1}{4\theta} \int_0^L xv_t^2 dx \right),
\end{aligned} \tag{100}$$

$$\begin{aligned}
& \int_0^L x\mu(x)h(x)v_t v_x dx \\
& \leq \|h\|_\infty \left(\theta \int_0^L x\mu(x)v_x^2 dx + \frac{1}{4\theta} \int_0^L x\mu(x)v_t^2 dx \right),
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^L xh(x)v_x \left(\int_0^t g_2(t-s)(v(t)-v(s))ds \right) dx \\
& \leq \|h\|_\infty \left(\theta \int_0^L xv_x^2 dx + \frac{C_p}{4\theta} \left(\int_0^t g_2^{2-r}(s)ds \right) (g_2' \circ v_x)(t) \right),
\end{aligned} \tag{101}$$

Using (97)–(101) and $(xh(x))' \leq x$, we obtain (95).

Next Theorem show that solutions decreases exponentially with respect to $\xi(t)$ and σ .

Theorem 15. Suppose that $r > -1$, (G_1) , and (G_2) hold and taking $u_0, v_0 \in V_{\theta}^2$, and $(u_1, v_1) \in H^2$ such that (41) hold true. Then, for each $t_0 > 0$, there exist positive constants K and k such that

$$E(t) \leq \begin{cases} Ke^{-k \int_{t_0}^t \xi(s)ds}, & \sigma = 1, \\ K \left(1 + \int_{t_0}^t \xi(s)ds \right)^{-1/\sigma-1}, & 1 < \sigma < \frac{3}{2}, \forall t \geq t_0. \end{cases} \tag{102}$$

Proof. Since g_1 and g_2 is continuous and $g_1(0) > 0$, $g_2(0) > 0$ then for any $t_0 > 0$, we have

$$\left\{ \int_0^t g_i(s)ds \geq \int_{t_0}^t g_i(s)ds = g_{i,0} > 0, \quad \forall t \geq t_0, i = 1, 2. \right. \tag{103}$$

As ξ is a positive decreasing function hence $1 < \xi(t)/\xi(0)$ and by recalling Lemmas 7, 12, 13, 14, and (101), we get

$$\begin{aligned}
F'(t) &= E'(t) + \varepsilon_1 \Phi'(t) + \varepsilon_2 \chi' + \psi'(t) \\
&\leq - \int_0^L \left[\left(N - \frac{\|h\|_\infty}{4\theta} \right) \mu(x) - \left(1 + \frac{\|h\|_\infty}{4\theta} \right) \right. \\
&\quad - \varepsilon_1 \left(1 + \frac{1}{2\delta} \right) + \varepsilon_2 (g_{1,0} - \theta - \theta l) \left. \right] \xi(t) x u_t^2 dx \\
&\quad - \int_0^L \left[\left(N - \frac{\|h\|_\infty}{4\theta} \right) \mu(x) - \left(1 + \frac{\|h\|_\infty}{4\theta} \right) \right. \\
&\quad - \varepsilon_1 \left(1 + \frac{1}{2\delta} \right) + \varepsilon_2 (g_{2,0} - \theta - \theta l) \left. \right] \xi(t) x v_t^2 dx \\
&\quad + (2\varepsilon_1 + 1) \xi(t) \int_0^L x \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx \\
&\quad + \left(\frac{1}{2} - \frac{\varepsilon_2 \xi(0)}{4\theta} C_p g_1(0) \right) (g_1' \circ u_x)(t) \\
&\quad + \left(\frac{1}{2} - \frac{\varepsilon_2 \xi(0)}{4\theta} C_p g_2(0) \right) (g_2' \circ v_x)(t) \\
&\quad - \left[\frac{\varepsilon_1}{2} (l_2 - C_p (\delta l + \|\mu\|_\infty)) - \varepsilon_2 \theta (1 + c_2 \right. \\
&\quad + c_2' + 2(1 - l_2)^2) - (1 + \|h\|_\infty) \theta (2 + \|\mu\|_\infty) \left. \right] \\
&\quad \times \xi(t) \left(\int_0^L x v_x^2 dx \right) + \left[\frac{\varepsilon_1}{2l_1} \right. \\
&\quad + \varepsilon_2 \left(\frac{1}{2\theta} + 2\theta + \frac{2 + l + C_p \|h\|_\infty + 4\theta}{4\theta} \right) \left. \right] \\
&\quad \times \xi(t) \left(\int_0^L g_1^{2-\sigma}(s)ds \right) (g_1^\sigma \circ u_x)(t) \\
&\quad + \left[\frac{\varepsilon_1}{2l_2} + \varepsilon_2 \left(\frac{1}{2\theta} + 2\theta + \frac{2 + l + C_p \|h\|_\infty + 4\theta}{4\theta} \right) \right] \\
&\quad \times \xi(t) \left(\int_0^L g_2^{2-\sigma}(s)ds \right) (g_2^\sigma \circ u_x)(t).
\end{aligned} \tag{104}$$

We take $\delta < (1/2C_p l) \min \{l_1, l_2\}$, hence

$$\begin{cases} (l_1 - \delta C_p l) > \frac{l_1}{2}, \\ (l_2 - \delta C_p l) > \frac{l_2}{2}, \end{cases} \quad (105)$$

Also, we take $\theta > \max \{g_{1,0}/2(1+l), g_{2,0}/2(1+l)\}$, hence

$$\begin{cases} (g_{1,0} - (1+l)\theta) < \frac{1}{2}g_{1,0}, \\ (g_{2,0} - (1+l)\theta) < \frac{1}{2}g_{2,0}. \end{cases} \quad (106)$$

Now, we choose ε_2 small enough such

$$\begin{aligned} k_1 &:= \left(\frac{1}{2} - \frac{\varepsilon_2 \xi(0)}{4\theta} C_p g_1(0) \right) > 0, \\ k_1 &:= \left(\frac{1}{2} - \frac{\varepsilon_2 \xi(0)}{4\theta} C_p g_2(0) \right) > 0, \end{aligned} \quad (107)$$

As far as δ , θ , and ε_2 are fixed, we then pick ε_1 so small that

$$\begin{aligned} &\begin{cases} k_3 := \frac{\varepsilon_1}{4} (l_1 - 2C_p \|\mu\|_\infty) - \varepsilon_2 \theta (1 + c_1 + c'_1 + 2(1-l_1)^2) - (1 + \|h\|_\infty \theta (2 + \|\mu\|_\infty)) > 0, \\ k_4 := \frac{\varepsilon_1}{4} (l_2 - 2C_p \|\mu\|_\infty) - \varepsilon_2 \theta (1 + c_2 + c'_2 + 2(1-l_2)^2) - (1 + \|h\|_\infty \theta (2 + \|\mu\|_\infty)) > 0, \end{cases} \\ &k_5 := \left[-\varepsilon_1 \left(1 + \frac{1}{2\delta} \right) + \varepsilon_2 (g_{1,0} - (1+l)\theta) \right] > 0, \\ &k_6 := \left[-\varepsilon_1 \left(1 + \frac{1}{2\delta} \right) + \varepsilon_2 (g_{2,0} - (1+l)\theta) \right] > 0, \\ &-\left\{ \left[\frac{\varepsilon_1}{2l_1} + \varepsilon_2 \left(\frac{1}{2\theta} + 2\theta + \frac{2+l+C_p\|h\|_\infty+4\theta}{4\theta} \right) \right] \left(\int_0^t g_1^{2-\sigma}(s) ds \right) \right\} > 0, \end{aligned} \quad (108)$$

and

$$\begin{aligned} &-\left\{ \left[\frac{\varepsilon_1}{2l_3} + \varepsilon_2 \left(\frac{1}{2\theta} + 2\theta + \frac{2+l+C_p\|h\|_\infty+4\theta}{4\theta} \right) \right] \right. \\ &\quad \cdot \left. \left(\int_0^t g_2^{2-\sigma}(s) ds \right) \right\} > 0, \end{aligned} \quad (109)$$

Finally we choose $N > \|h\|_\infty/4\theta$ large enough such that

$$\left(N - \frac{\|h\|_\infty}{4\theta} \right) \mu(x) - \left(1 + \frac{\|h\|_\infty}{4\theta} \right) > 0. \quad (110)$$

After fixed all this choices then from (14) and (27), we obtain for some $\sigma > 0$,

$$\begin{aligned} F'(t) &\leq -\sigma \xi(t) \left[\int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx \right. \\ &\quad - \int_0^L x \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx \\ &\quad + \int_0^L x u_x^2 dx + \int_0^L x v_x^2 dx + (g_1^\sigma \circ u_x)(t) \\ &\quad \left. + (g_2^\sigma \circ v_x)(t) \right] \leq -\gamma \xi(t) E(t), \forall t \geq t_0. \end{aligned} \quad (111)$$

We distingue two cases according σ

Case 16. $\sigma = 1$

We recall lemma 10, and the estimate (111) gives

$$F'(t) \leq -\gamma \alpha_1 \xi(t) F(t), \forall t \geq t_0. \quad (112)$$

A simple integration of the above inequality over (t_0, t) leads to

$$F'(t) \leq F(t_0) e^{(-\gamma \alpha_1) \int_{t_0}^t \xi(s) ds}, \forall t \geq t_0. \quad (113)$$

Hence, (102) is established.

Case 17. $1 < \sigma < 3/2$

By using (16), we get

$$g_i(t)^{1-\sigma} \geq (\sigma-1) \left(\int_{t_0}^t \xi(s) ds \right) + g_i(t_0)^{1-\sigma}, \quad i=1,2. \quad (114)$$

So, for $\forall 0 < \tau < 2 - \sigma < 1$, (hence $(1 - \tau/\sigma - 1) > 1$), we have

$$\int_0^\infty g_i^{1-\tau}(s) ds \leq \int_0^\infty \frac{1}{\left[(\sigma-1) \left(\int_{t_0}^t \xi(s) ds \right) + g_i(t_0)^{1-\sigma} \right]^{1-\tau/\sigma-1}} ds. \quad (115)$$

In the other hand, by using the fact that $\int_0^\infty \xi(s)ds = +\infty$, we obtain

$$\int_0^\infty g_i^{1-\tau}(s)ds < \infty, \forall 0 < \tau < 2 - \sigma, \text{ for } i = 1, 2. \quad (116)$$

So form (i) of Lemma 11 with $\theta = \tau$ and $\rho = \sigma$ and (41) yield

$$\begin{aligned} (g_i \circ w_x)(t) &\leq C_i \left(E(0) \int_0^\infty g_i^{1-\tau}(s)ds \right)^{\sigma-1/\sigma-1+\tau} \\ &\quad \cdot ((g_i^\sigma \circ w_x)(t))^{\tau/\sigma-1+\tau} \\ &\leq C'_i ((g_i^\sigma \circ w_x)(t))^{\tau/\sigma-1+\tau}, \end{aligned} \quad (117)$$

for $i = 1, 2$ and $w = u, v$, respectively, with C'_i are positive constants.

Therefore, for any $\sigma_1 > 1$, we arrive at

$$\begin{aligned} E^{\sigma_1}(t) &\leq C'' E^{\sigma_1-1}(0) \left(\int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx \right. \\ &\quad \left. + \int_0^L x v_x^2 dx - \int_0^L x [a|u+v|^{2(r+2)} + 2b|uv|^{r+2}] dx \right) \\ &\quad + C_1'' ((g_1 \circ u_x)(t))^{\sigma_1} + C_2'' ((g_2 \circ v_x)(t))^{\sigma_1} \\ &\leq C'' E^{\sigma_1-1}(0) \left(\int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx \right. \\ &\quad \left. + \int_0^L x v_x^2 dx - \int_0^L x [a|u+v|^{2(r+2)} + 2b|uv|^{r+2}] dx \right) \\ &\quad + C_1''' ((g_1^\sigma \circ u_x)(t))^{\tau\sigma_1/\sigma-1+\tau} \\ &\quad + C_2''' ((g_2^\sigma \circ v_x)(t))^{\tau\sigma_1/\sigma-1+\tau}. \end{aligned} \quad (118)$$

By setting $\tau = 1/2$ and $\sigma_1 = 2\sigma - 1$ which give $\tau\sigma_1/\sigma-1+\tau=1$, the estimate (119) becomes,

$$\begin{aligned} E^{\sigma_1}(t) &\leq \Gamma \left[\int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx + \int_0^L x v_x^2 dx \right] \\ &\quad - \int_0^L x [a|u+v|^{2(r+2)} + 2b|uv|^{r+2}] dx \\ &\quad + (g_1^\sigma \circ u_x)(t) + (g_2^\sigma \circ v_x)(t), \end{aligned} \quad (119)$$

for some $\Gamma > 0$. Form the equivalence between F and E and by combining the above inequality with (111), we obtain

$$F'(t) \leq -\frac{\sigma}{\Gamma} \xi(t) E^{\sigma_1}(t) \leq -\frac{\sigma}{\Gamma} \alpha_1^{\sigma_1} F^{\sigma_1}(t), \quad \forall t \geq t_0. \quad (120)$$

By twice integration over (t_0, t) and (t_0, ∞) of (119) successively leads to

$$F(t) \leq C_1^* \left(1 + \int_{t_0}^t \xi(s)ds \right)^{-1/\sigma_1-1}, \quad \forall t \geq t_0, \quad (121)$$

$$\int_{t_0}^\infty F(t)dt \leq C_1^* \int_{t_0}^\infty \frac{1}{\left(1 + \int_{t_0}^t \xi(s)ds \right)^{1/\sigma_1-1}} dt. \quad (122)$$

Since $1/\sigma_1-1 > 0$ and $(1 + \int_{t_0}^t \xi(s)ds) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, we get

$$\int_{t_0}^\infty F(t)dt < \infty. \quad (123)$$

Again from (121), we have

$$tF(t) \leq \frac{C_1^* t}{\left(1 + \int_{t_0}^t \xi(s)ds \right)^{1/\sigma_1-1}} \leq C_\sigma, \quad \forall t \geq t_0, \quad (124)$$

which implies that

$$\sup_{t \geq t_0} tF(t) < \infty. \quad (125)$$

Summing (123) and (125), we get

$$\int_{t_0}^\infty F(t)dt + \sup_{t \geq t_0} (tF(t)) < \infty. \quad (126)$$

By recalling (ii) of Lemma 11 with $\rho = \sigma$, Lemma 10, and by using (27) and the above bounded, we have

$$\begin{aligned} (g_i \circ w_x)(t) &\leq C_{i2}^* \left(t \|w_x(x, t)\|_H^2 + \int_0^t \|w_x(x, s)\|_H^2 ds \right)^{\sigma-1/\sigma} \\ &\quad \times \left(\int_0^t g_i^\sigma(t-s) \|w_x(x, t) - w_x(x, s)\|_H^2 ds \right)^{1/\sigma} \\ &\leq C_{i2}^* \left(tF(t) + \int_{t_0}^t F(s)ds \right)^{\sigma-1/\sigma} ((g_i^\sigma \circ w_x)(t))^{1/\sigma} \\ &\leq C_{i3}^* ((g_i^\sigma \circ w_x)(t))^{1/\sigma}, \end{aligned} \quad (127)$$

Hence,

$$(g_i^\sigma \circ w_x)(t) \geq (C_{i3}^*)^{-\sigma} ((g_i \circ w_x)(t))^\sigma, \quad (128)$$

for $i = 1, 2$, $w = u, v$, respectively, and some positive constants C_{i3}^* .

Consequently, introduce (128) in (111) and in, (118), we find

$$\begin{aligned} F'(t) \leq & -C_4 \xi(t) \left[\int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx \right. \\ & + \int_0^L x v_x^2 dx - \int_0^L x \left[a|u + v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx \\ & \left. + ((g_1 \circ u_x)(t))^\sigma + ((g_2 \circ v_x)(t))^\sigma \right], \end{aligned} \quad (129)$$

and

$$\begin{aligned} E^\sigma(t) \leq & C_5 \xi(t) \left[\int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx + \int_0^L x v_x^2 dx \right. \\ & - \int_0^L x \left[a|u + v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx + ((g_1 \circ u_x)(t))^\sigma \\ & \left. + ((g_2 \circ v_x)(t))^\sigma \right], \end{aligned} \quad (130)$$

for all $t \geq 0$ and some positive constant C_4, C_5 .

By combining the last two inequalities and along the equivalence between F and E , we obtain

$$F'(t) \leq -C_6 \xi(t) F^\sigma(t), \quad \forall t \geq t_0, \quad (131)$$

for some constant $C_6 > 0$.

A simple integration of (131) over (t_0, t) gives

$$F(t) \leq C_9 \left(1 + \int_{t_0}^t \xi(s) ds \right)^{-1/\sigma-1}, \quad \forall t \geq t_0. \quad (132)$$

The proof is completed by using (59).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally in this article. They have all read and approved the final manuscript.

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Research Article

Analysis on Existence of Positive Solutions for a Class Second Order Semipositone Differential Equations

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In this paper, we study the existence of positive solutions of the following second-order semipositone system (see equation 1). By applying a well-known fixed-point theorem, we prove that the problem admits at least one positive solution, if f is bounded below.

1. Introduction

This paper is focused on the existence of positive solutions of a second-order semipositone system

$$\begin{cases} -u'' + \rho u = \phi u + f(t, u, \phi), & t \in (0, 1), \\ -\phi'' = \mu u, & t \in (0, 1), \\ u(0) = u(1) = \phi(0) = \phi(1) = 0, \end{cases} \quad (1)$$

where μ is a positive constant and f satisfies the following assumption: $(F_0)f : [0, 1] \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is continuous, and

$$f(t, u, \phi) \geq -e(t), \quad \text{for } (t, u, \phi) \in [0, 1] \times \mathbb{R}_+^2, \quad (2)$$

where $e : [0, 1] \rightarrow \mathbb{R}_+$ is continuous and $e(t) \not\equiv 0$ on $[0, 1]$.

The second-order elliptic systems

$$\begin{cases} -\Delta u + \rho u = \phi u + f(u), & x \in \Omega, \\ -\Delta \phi = \mu u, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \quad (3)$$

have a strong physical meaning in quantum mechanics models [1, 2], in semiconductor theory [3], or in a time- and space-dependent mathematical model of nuclear reactors in a closed container [4]. To the best of our knowledge, existence and multiplicity of nontrivial solutions of BVP(1) have been widely studied by using the variational method [5], bifurcation techniques [6, 7], or fixed-point theorems [8–11]. In general, in order to ensure the positivity of the solutions of Equation (1), one of the crucial assumptions is that the nonlinearity f is nonnegative. Of course, the natural question is whether Equation (1) has a positive solution or not if f satisfies the assumption (F_0) .

On the other hand, many authors have been interested in finding the relations between the positivity of solutions and the changing sign of the nonlinearity in order to prove the existence of the positive solutions. We refer the readers to [12–16] and the references.

Inspired by these references, the purpose of this paper is to find some new conditions, which are used to study the existence and multiplicity of positive solutions of the semipositone Equation (1). The main tool is the following well-known fixed-point theorem.

Lemma 1 [17]. *Let E be a Banach space and K be a cone in E . Assume Ω_r and Ω_R are open bounded subsets of E with $\Omega_r \cap K \neq \emptyset$, $\overline{\Omega_r \cap K} \subset \Omega_R \cap K$. Let $T : \overline{\Omega_R \cap K} \rightarrow K$ be a completely continuous operator such that*

- (a) $\|Tu\| \leq \|u\|$, for $u \in \partial(\Omega_r \cap K)$, and
- (b) there exists a $\eta(t) \in K \setminus \{0\}$ such that

$$u \neq Tu + \lambda\eta(t), \quad \text{for } u \in \partial(\Omega_R \cap K), \lambda > 0. \quad (4)$$

Then, T has a fixed point in $\overline{\Omega_R \cap K} \setminus \Omega_r \cap K$. The same conclusion remains valid if (a) holds on $\partial(\Omega_R \cap K)$ and (b) holds on $\partial(\Omega_r \cap K)$.

The paper is organized as follows: in Section 2, we give some preliminaries, which are about the properties of the Green functions, the notations of some sets, etc.; in Section 3, we give the main results and the corresponding proof. In Section 4, some examples are given to illustrate the main results.

2. Preliminary

Let $G(t, s)$ be the Green function of linear boundary value problem

$$-u'' + \rho u = 0, \quad u(0) = u(1) = 0, \quad (5)$$

where $\rho > -\pi^2$.

Lemma 2 [18]. *Let $\omega = \sqrt{|\rho|}$, then $G(t, s)$ can be expressed by*

- (i) $G(t, s) = \begin{cases} \sinh \omega t \sinh \omega(1-s)/\omega \sinh \omega, & 0 \leq t \leq s \leq 1, \\ \sinh \omega s \sinh \omega(1-t)/\omega \sinh \omega, & 0 \leq s \leq t \leq 1, \text{ if } \rho > 0 \end{cases}$
- (ii) $G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \text{ if } \rho = 0 \end{cases}$
- (iii) $G(t, s) = \begin{cases} \sin \omega t \sin \omega(1-s)/\omega \sin \omega, & 0 \leq t \leq s \leq 1, \\ \sin \omega s \sin \omega(1-t)/\omega \sin \omega, & 0 \leq s \leq t \leq 1, \text{ if } -\pi^2 < \rho < 0. \end{cases}$

Lemma 3 [18]. *The function $G(t, s)$ has the following properties:*

- (i) $G(t, s) > 0, \forall t, s \in (0, 1)$
- (ii) $G(t, s) \leq CG(s, s), \forall t, s \in [0, 1]$
- (iii) $G(t, s) \geq \delta G(t, t)G(s, s), \forall t, s \in [0, 1]$

where $C = 1, \delta = \omega/\sinh \omega$, if $\rho > 0$; $C = 1, \delta = 1$, if $\rho = 0$; and $C = 1/\sin \omega, \delta = \omega \sin \omega$, if $-\pi^2 < \rho < 0$.

Lemma 4. *For the function $G(t, s)$, there exists a $\xi > 0$ such that*

$$G(t, t) \geq \xi \int_0^1 G(t, s) ds. \quad (6)$$

Proof.

- (i) For $\rho > 0$, we have

$$\int_0^1 G(t, s) ds = \frac{1}{\omega^2 \sinh \omega} \{ \sinh \omega t [\cosh \omega(1-t) - 1] + \sinh \omega(1-t) [\cosh \omega t - 1] \}. \quad (7)$$

Let

$$J_1(t) = \begin{cases} \frac{\cosh \omega(1-t) - 1}{\sinh \omega(1-t)}, & 0 \leq t < 1, \\ 0, & t = 1, \end{cases} \quad (8)$$

$$J_2(t) = \begin{cases} \frac{\cosh \omega t - 1}{\sinh \omega t}, & 0 < t \leq 1, \\ 0, & t = 0. \end{cases}$$

Since $J_1(t)$ is positive and continuous on $[0, 1]$ and $J_1(1) = 0$, we have

$$J_1^* = \max_{t \in [0, 1]} J_1(t) > 0. \quad (9)$$

In the similar way, we also have

$$J_2^* = \max_{t \in [0, 1]} J_2(t) > 0. \quad (10)$$

Choosing $\xi < \omega/(J_1^* + J_2^*)$. Then, for any $t \in (0, 1)$, we have

$$\begin{aligned} \xi \int_0^1 G(t, s) ds &= \xi \frac{1}{\omega^2 \sinh \omega} \{ \sinh \omega t [\cosh \omega(1-t) - 1] \\ &\quad + \sinh \omega(1-t) [\cosh \omega t - 1] \} \\ &= \xi \frac{1}{\omega^2 \sinh \omega} \left\{ \sinh \omega t \sinh \omega(1-t) \frac{\cosh \omega(1-t) - 1}{\sinh \omega(1-t)} \right. \\ &\quad \left. + \sinh \omega t \sinh \omega(1-t) \frac{\cosh \omega t - 1}{\sinh \omega t} \right\} \\ &= \xi \frac{1}{\omega \sinh \omega} \{ \sinh \omega t \sinh \omega(1-t) J_1^* \\ &\quad + \sinh \omega t \sinh \omega(1-t) J_2^* \} \\ &= \frac{\xi(J_1^* + J_2^*)}{\omega} G(t, t) \leq G(t, t). \end{aligned} \quad (11)$$

Since $G(0, 0) = \int_0^1 G(0, s) ds = G(1, 1) = \int_0^1 G(1, s) ds = 0$, then for any $\xi > 0$, we have

$$G(t, t) = \xi \int_0^1 G(t, s) ds, \quad \text{for } t = 0, 1. \quad (12)$$

Therefore, there exists a $\xi > 0$ such that

$$G(t, t) \geq \xi \int_0^1 G(t, s) ds. \quad (13)$$

(ii) For $\rho = 0$, it is obvious that

$$\int_0^1 G(t, s) ds = \frac{1}{2} t^2 (1-t) + \frac{1}{2} t (1-t)^2 \leq t(1-t)^2 = G(t, t). \quad (14)$$

(iii) For $-\pi^2 < \rho < 0$, we have

$$\int_0^1 G(t, s) ds = \frac{1}{\omega^2 \sin \omega} \{ \sin \omega t [1 - \cos \omega(1-t)] + \sinh \omega(1-t) [1 - \cos \omega t] \}. \quad (15)$$

Using the similar discussion of Case (i), it follows that there exists a $\xi > 0$ such that

$$G(t, t) \geq \xi \int_0^1 G(t, s) ds. \quad (16)$$

For convenience, let $K(t, s)$ denote the Green function for $\rho = 0$. Then, Equation (1) can be rewritten as

$$\begin{cases} -u'' + \rho u = \mu u \int_0^1 K(t, s) u(s) ds + f\left(t, u \int_0^1 K(t, s) u(s) ds\right), \\ u(0) = u(1) = 0. \end{cases} \quad (17)$$

Furthermore, let $x = u + \omega$, where $\omega(t) = \int_0^1 G(t, s) e(s) ds$ is the unique solution of the linear boundary value problem

$$\begin{cases} -x'' + \rho x = e(t), \\ x(0) = x(1) = 0. \end{cases} \quad (18)$$

Then, we rewrite (17) as

$$\begin{cases} -x'' + \rho x = F(t, x - \omega), \\ x(0) = x(1) = 0, \end{cases} \quad (19)$$

where

$$\begin{aligned} F(t, x - \omega) &= \mu(x - \omega) \int_0^1 K(t, s) [x(s) - \omega(s)] ds \\ &\quad + \left\{ f\left(t, x - \omega, \int_0^1 K(t, s) [x(s) - \omega(s)] ds\right) + e(t) \right\}. \end{aligned} \quad (20)$$

From the above discussion, then we have the following lemma.

Lemma 5. Assume that (F_0) holds. Then, $u(t)$ is a positive solution of (1) if only if $x(t)$ is a positive solution of the following problem:

$$-x'' + \rho x = F(t, H(x - \omega)(x - \omega)), \quad (21)$$

with $x(t) \geq \omega(t)$. Here, $H(t)$ denotes the Heaviside function of a single real variable:

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (22)$$

Let E denote the Banach space $C[0, 1]$ with the norm $\|x\| = \max_{t \in [0, 1]} |x(t)|$.

Define a cone $K \subset E$ by

$$K = \left\{ x(t) \in C[0, 1] : \min_{t \in [\theta, 1-\theta]} x(t) \geq \sigma \|x\| \right\}, \quad (23)$$

where $\theta \in (0, 1/2)$, $\sigma = \min_{t \in [\theta, 1-\theta]} (\sigma/C) G(t, t) \in (0, 1)$. Define an operator T by

$$T(x)(t) = \int_0^1 G(t, s) F(s, H(x - \omega)(x - \omega)) ds. \quad (24)$$

Lemma 6. Assume that (F_0) holds. Then, $T(K) \subseteq K$, and $T : K \rightarrow K$ is completely continuous.

Proof. For any $x(t) \in K$, from Lemma 3, it follows that

$$\begin{aligned} T(x)(t) &= \int_0^1 G(t, s) F(s, H(x - \omega)(x - \omega)) ds \\ &\geq \delta \int_0^1 G(t, t) G(s, s) F(s, H(x - \omega)(x - \omega)) ds \\ &= \frac{\delta}{C} G(t, t) \int_0^1 C G(s, s) F(s, H(x - \omega)(x - \omega)) ds \\ &\geq \frac{\delta}{C} G(t, t) \int_0^1 G(t, s) F(s, H(x - \omega)(x - \omega)) ds, \end{aligned} \quad (25)$$

which implies that $T(K) \subseteq K$.

Now, we show that $T : K \rightarrow K$ is completely continuous.

First, we show that T maps the bounded set into itself. Since e and f are continuous, for any given $c > 0$, let

$$L = \max \{F(t, H(x - \omega)(x - \omega)) : 0 \leq t \leq 1, 0 \leq x \leq c\}. \quad (26)$$

Then, for $x \in \bar{K}_c$, we have

$$|T(x)(t)|_\infty = \left| \int_0^1 G(t, s) F(s, H(x - \omega)(x - \omega)) ds \right|_\infty \leq L \max_{0 \leq t, s \leq 1} G(t, s), \quad (27)$$

which implies that $T(\bar{K}_c)$ is uniformly bounded.

Second, for $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned} |Tx(t_2) - Tx(t_1)| &= \left| \int_0^1 [G(t_2, s) - G(t_1, s)] F(s, H(x - \omega)(x - \omega)) ds \right| \\ &= \left| \int_0^1 [G_i(t_2, s) - G_i(t_1, s)] \tilde{f}_i(s, u(s) - \omega(s)) ds \right| \\ &\leq L \max_{0 \leq t, s \leq 1} \left| \frac{\partial G(t, s)}{\partial t} \right| |t_2 - t_1|, \end{aligned} \quad (28)$$

which implies that the operator T is equicontinuous.

Thus, by applying the Arzela-Ascoli theorem [17], we obtain that $T(\bar{K}_c)$ is relatively compact, namely, the operator T is compact.

Finally, we claim that $T : \bar{K}_c \rightarrow K$ is continuous. Assume that $\{x_n\}_{n=1}^\infty \subset \bar{K}_c$ which converges to $x(t)$ uniformly on $[0, 1]$. By Lebesgue's dominated convergence theorem and letting $n \rightarrow \infty$, we have

$$\begin{aligned} &\|Tx_n(t) - Tx(t)\| \\ &= \left\| \int_0^1 G(t, s) [F(s, H(x_n - \omega)(x_n - \omega)) - F(s, H(x - \omega)(x - \omega))] ds \right\| \\ &\leq C \int_0^1 G(s, s) [F(s, H(x_n - \omega)(x_n - \omega)) - F(s, H(x - \omega)(x - \omega))] ds \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \end{aligned} \quad (29)$$

So, T is continuous on K_c . The proof is completed.

At the end of this section, let

$$e^* = \max_{t \in [0, 1]} e(t) > 0, \quad \omega^* = \max_{t \in [0, 1]} \omega(t). \quad (30)$$

Define the height functions

$$\begin{aligned} \Phi_*(r) &= \min \left\{ f(t, u, \phi) : (t, u, \phi) \in [0, 1] \times [0, r] \times \left[0, \frac{r}{6}\right] \right\}, \\ \Phi_*(t, r) &= \max \left\{ f(t, u, \phi) : (u, \phi) \in [0, r] \times \left[0, \frac{r}{6}\right] \right\}. \end{aligned} \quad (31)$$

In addition, we need to select some suitable open bounded sets. For any $\gamma > 0$, let

$$\begin{aligned} \Omega^\gamma &= \left\{ x \in E : \min_{t \in [\theta, 1-\theta]} x(t) < \sigma\gamma \right\}, \quad B^\gamma = \{x \in E : \|x\| < \gamma\}, \\ \Omega_K^\gamma &= \Omega_\gamma \cap K, \quad \partial\Omega_K^\gamma = \partial\Omega_\gamma \cap K, \\ B_K^\gamma &= B_\gamma \cap K, \quad \partial B_K^\gamma = \partial B_\gamma \cap K. \end{aligned} \quad (32)$$

From [19, 20], we can conclude the lemma below.

Lemma 7.

- (i) $\Omega_K^\gamma, B_K^\gamma$ are open relative to K
- (ii) $B_K^{\sigma\gamma} \subset \Omega_K^\gamma \subset B_K^\gamma$
- (iii) $x \in \partial\Omega_K^\gamma$ if and only if $\min_{[\theta, 1-\theta]} x(t) = \sigma\gamma$
- (iv) If $x \in \partial\Omega_K^\gamma$, then $\sigma\gamma \leq x(t) \leq \gamma$, for $t \in [\theta, 1 - \theta]$

3. Main Results

Theorem 8. Assume that (F_0) holds. In addition, the function f satisfies the following assumption:

(F_1) There exists a $\alpha > 0$ such that $\Phi_*(\alpha) \geq 0$ and

$$\mu \frac{\alpha^2}{6} C \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds + C \int_0^1 G(s, s) \Phi^*(s, \alpha) ds < \alpha. \quad (33)$$

Then, we have

- (i) If $\sigma\alpha > \omega^*$, then (1) has at least one partly positive solution (u, ϕ) , namely,

$$u(t) > 0, \quad \text{for } t \in [\theta, 1 - \theta] \quad (34)$$

- (ii) If $\alpha\delta\xi > Ce^*$, then (1) has at least one positive solution (u, ϕ) , which satisfies

$$u(t) > 0, \quad \text{for } t \in [0, 1] \quad (35)$$

Proof. For any $x \in \partial B_K^\alpha$, it is obvious that

$$\begin{aligned} H(x - \omega)(x - \omega) &\leq \|x\| = \alpha, \\ \int_0^1 K(t, s) H(x - \omega)[x(s) - \omega(s)] ds \\ &\leq \int_0^1 K(s, s) H(x - \omega)[x(s) - \omega(s)] ds \leq \frac{1}{6} \alpha. \end{aligned} \quad (36)$$

Then, from (F_1) it follows that

$$\begin{aligned}
 T(x)(t) &= \int_0^1 G(t, s) F(s, H(x - \omega)(x - \omega)) ds \\
 &= \mu \int_0^1 G(t, s) \left[H(x - \omega)(x - \omega) \int_0^1 K(s, \tau) H(x - \omega) \right. \\
 &\quad \cdot [x(\tau) - \omega(\tau)] d\tau \Big] ds + \int_0^1 G(t, s) [f(s, H(x - \omega)) \\
 &\quad \cdot (x - \omega) \int_0^1 K(s, \tau) H(x - \omega) [x(\tau) - \omega(\tau)] d\tau \\
 &\quad + e(s)] ds \leq \mu \frac{\alpha^2}{6} C \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds \\
 &\quad + C \int_0^1 G(s, s) f(s, H(x - \omega)(x - \omega)) \\
 &\quad \int_0^1 K(s, \tau) H(x - \omega) [x(\tau) - \omega(\tau)] d\tau ds \\
 &\leq \mu \frac{\alpha^2}{6} C \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds \\
 &\quad + C \int_0^1 G(s, s) \Phi^*(s, \alpha) ds < \alpha = \|x\|,
 \end{aligned} \tag{37}$$

which implies that (a) of Lemma 1 holds.

Let

$$\Psi(\rho) = \max_{0 \leq s \leq \rho} \left\{ H(x - \omega)(x - \omega) \cdot \int_{\theta}^{1-\theta} K(\tau, \tau) H(x - \omega) [x(\tau) - \omega(\tau)] d\tau \right\}. \tag{38}$$

From [21], we have that

$$\lim_{\rho \rightarrow +\infty} \frac{\Psi(\rho)}{\rho} = +\infty. \tag{39}$$

Then, there exists a $\beta >$ with $\sigma\beta > \alpha$ such that

$$H(\sigma\beta - \omega)(\sigma\beta - \omega) \cdot \int_{\theta}^{1-\theta} K(\tau, \tau) H(\sigma\beta - \omega) [\sigma\beta - \omega(\tau)] d\tau > \Lambda\beta, \tag{40}$$

where Λ satisfies

$$\Lambda\mu\sigma C \int_{\theta}^{1-\theta} G(s, s) K(s, s) ds > 1. \tag{41}$$

Let $\eta(t) = 1$; now we prove that $x \neq Tx + \lambda$, for $x \in \partial\Omega_K^\beta$ and $\lambda > 0$. On the contrary, if there exists a pair of $x_0 \in \partial\Omega_K^\beta$ and $\lambda_0 > 0$ such that $x_0(t) = T(x_0)(t) + \lambda_0$, then from (iv) of Lemma 7, it follows that

$$\sigma\beta = \sigma\|x_0\| \leq x_0(t) \leq \beta, \quad \text{for } t \in [\theta, 1 - \theta]. \tag{42}$$

Furthermore, for $t \in [\theta, 1 - \theta]$, we have

$$\begin{aligned}
 \|x_0\| &\geq \min_{\theta \leq t \leq 1-\theta} x_0(t) = \min_{\theta \leq t \leq 1-\theta} Tx_0(t) + \lambda_0 \\
 &= \min_{\theta \leq t \leq 1-\theta} \mu \int_0^1 G(t, s) H(x_0 - \omega)(x_0 - \omega) \int_0^1 K(s, \tau) H(x - \omega) \\
 &\quad \cdot [x(\tau) - \omega(\tau)] d\tau ds + \min_{\theta \leq t \leq 1-\theta} \int_0^1 G(t, s) \\
 &\quad \cdot \left[f\left(s, H(x_0 - \omega)(x - \omega), \int_0^1 K(s, \tau) H(x_0 - \omega) \right. \right. \\
 &\quad \cdot [x(\tau) - \omega(\tau)] d\tau \Big) + e(s) \Big] ds + \lambda_0 \\
 &\geq \min_{\theta \leq t \leq 1-\theta} \mu \int_0^1 G(t, s) \left[H(x_0 - \omega)(x_0 - \omega) \int_0^1 K(s, \tau) H \right. \\
 &\quad \cdot (x - \omega) [x(\tau) - \omega(\tau)] d\tau \Big] ds \geq \min_{\theta \leq t \leq 1-\theta} \mu \delta G(t, t) \\
 &\quad \cdot \int_0^1 G(s, s) \left[H(x_0 - \omega)(x_0 - \omega) \int_0^1 K(s, \tau) H(x - \omega) \right. \\
 &\quad \cdot [x(\tau) - \omega(\tau)] d\tau \Big] ds \geq \mu\sigma C \int_0^1 G(s, s) \\
 &\quad \cdot \left[H(x_0 - \omega)(x_0 - \omega) \int_0^1 K(s, s) K(\tau, \tau) H(x - \omega) \right. \\
 &\quad \cdot [x(\tau) - \omega(\tau)] d\tau \Big] ds = \mu\sigma C \int_0^1 G(s, s) K(s, s) \\
 &\quad \cdot \left[H(x_0 - \omega)(x_0 - \omega) \int_0^1 K(\tau, \tau) H(x - \omega) [x(\tau) - \omega(\tau)] d\tau \right] ds \\
 &= \mu\sigma C \int_0^1 G(s, s) K(s, s) \left[H(x_0 - \omega)(x_0 - \omega) \int_{\theta}^{1-\theta} K(\tau, \tau) H(x - \omega) \right. \\
 &\quad \cdot [x(\tau) - \omega(\tau)] d\tau \Big] ds = \mu\sigma C \int_0^1 G(s, s) K(s, s) \\
 &\quad \cdot \left[H(\sigma\|x_0\| - \omega)(\sigma\|x_0\| - \omega) \int_{\theta}^{1-\theta} K(\tau, \tau) H(\sigma\|x_0\| - \omega) \right. \\
 &\quad \cdot [\sigma\|x_0\| - \omega(\tau)] d\tau \Big] ds \geq \mu\sigma C \int_0^1 G(s, s) K(s, s) ds \cdot \Lambda\beta > \beta = \|x_0\|,
 \end{aligned} \tag{43}$$

which contradicts with the statement (iii) of Lemma 7. So (b) holds.

Since $\alpha < \sigma\beta$, from Lemma 7, we have $\overline{B_K^\alpha} \subset B_K^{\sigma\beta} \subset \Omega_K^\beta$. Therefore, by Lemma 1, we can get that T has at least one positive fixed-point $x(t) \in \Omega_K^\beta \setminus B_K^\alpha$. Hence, the inequalities hold,

$$\|x\| \geq \alpha, \sigma\alpha \leq \min_{t \in [\theta, 1-\theta]} x(t) \leq \sigma\beta. \tag{44}$$

On the other hand, since $\sigma\|x\| \leq \min_{t \in [\theta, 1-\theta]} x(t) \leq \sigma\beta$, we have $\|x\| \leq \beta$.

(i) Since

$$\min_{t \in [\theta, 1-\theta]} x(t) \geq \sigma\alpha > \omega^* > \omega(t) = \int_0^1 G(t, s) e(s) ds, \tag{45}$$

we have

$$u(t) = x(t) - \omega(t) > 0, \quad t \in [\theta, 1 - \theta] \quad (46)$$

(ii) From Lemmas 3 and 4, we have

$$\begin{aligned} x(t) &\geq \frac{\delta}{C} G(t, t) \|t\| = \frac{\delta}{C} G(t, t) \alpha \geq \alpha \frac{\delta}{Ce^*} \xi e^* \int_0^1 G(t, s) ds \\ &\geq \alpha \frac{\delta}{Ce^*} \xi \int_0^1 G(t, s) e(s) ds = \alpha \frac{\delta}{Ce^*} \xi \cdot \omega(t) > \omega(t), \end{aligned} \quad (47)$$

which implies that $u(t) = x(t) - \omega(t) > 0$

Therefore, (1) has one positive solution $(u, \phi) = (u, \int_0^1 K(t, s) u(s) ds)$.

Theorem 9. Assume that (F_0) holds. In addition, the function f satisfies the following assumptions:

(F_2) There exists $\alpha > \max \{\omega^*/\sigma, Ce^*/\delta\xi\}$ such that $\Phi_*(\alpha) \leq 0$ and

$$\mu \frac{\alpha^2}{6} C \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds + C \int_0^1 G(s, s) \Phi^*(s, \alpha) ds < \alpha. \quad (48)$$

(F_3) There exists a $r^* \in (0, \sigma\alpha - \omega^*)$ such that

$$\begin{aligned} \mu \frac{r^{*2}}{6} C \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds \\ + C \int_0^1 G(s, s) \Phi^*(s, r^*) ds < r^*. \end{aligned} \quad (49)$$

(F_4) $\lim_{u+\phi \rightarrow 0^+} (f(t, u, \phi)/(u + \phi)) = +\infty$, uniformly for $t \in [0, 1]$.

Then, (1) has at least two positive solution (u_i, ϕ_i) ($i = 1, 2$), which satisfies

$$0 \leq u_1(t) < r^*, \quad \min_{t \in [\theta, 1-\theta]} u_2(t) > \sigma\alpha - \omega^*. \quad (50)$$

Proof. From (F_2) and Theorem 8, it follows that there exists a solution $u_2(t) \geq 0$ and

$$\min_{t \in [\theta, 1-\theta]} u_2(t) > \sigma\alpha - \omega^*. \quad (51)$$

Now, we apply Lemma 1 to prove the existence of another solution $u_1(t)$.

Since $r^* < \sigma\alpha - \omega^* < \alpha$ and $\Phi_*(\alpha)$, then we can define the operator

$$\bar{T}(u)(t) = \int_0^1 G(t, s) F(s, u(s)) ds. \quad (52)$$

For any $u \in \partial B_K^{r^*}$, it is obvious that

$$\int_0^1 K(t, s) u(s) ds \leq \int_0^1 K(s, s) u(s) ds \leq \frac{1}{6} r^*. \quad (53)$$

Then, we have

$$\begin{aligned} \bar{T}(u)(t) &= \int_0^1 G(t, s) F(s, u(s)) ds \\ &\leq \mu \frac{r^{*2}}{6} C \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds + C \int_0^1 G(s, s) f \\ &\quad \cdot \left(s, H(x - \omega)(x - \omega), \int_0^1 K(s, \tau) H(x - \omega)[x(\tau) - \omega(\tau)] d\tau \right) ds \\ &\leq \mu \frac{r^{*2}}{6} C \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds \\ &\quad + C \int_0^1 G(s, s) \Phi^*(s, r^*) ds < r^* = \|u\|, \end{aligned} \quad (54)$$

which implies that (a) of Lemma 1 holds.

Since $\lim_{u+\phi \rightarrow 0^+} (f(t, u, \phi)/(u + \phi)) = +\infty$, uniformly for $t \in [0, 1]$, there exists a $r_* < r^*$ such that

$$f(t, u, \phi) > M(u + \phi), \quad \text{for } 0 < u + \phi \leq r_*, \quad (55)$$

where M satisfies

$$M\delta\sigma \cdot \max_{0 \leq t \leq 1} G(t, t) \cdot \int_\theta^{1-\theta} G(s, s) \left[1 + \int_\theta^{1-\theta} K(s, \tau) d\tau \right] ds > 1. \quad (56)$$

Let $\eta(t) = 1$, now we prove that $u \neq \bar{T}u + \lambda$, for $u \in \partial \Omega_K^{r_*/2}$ and $\lambda > 0$. On the contrary, if there exists a pair of $u_0 \in \partial \Omega_K^{r_*/2}$ and $\lambda_0 > 0$ such that $u_0(t) = \bar{T}(u_0)(t) + \lambda_0$, then from (iv) of Lemma 7, it follows that

$$u_0(t) \leq \frac{r_*}{2}, \quad \phi_0(t) = \int_0^1 K(t, s) u_0(s) ds < \frac{r_*}{12}. \quad (57)$$

Furthermore, we have

$$\begin{aligned} \|u_0\| &= \|\bar{T}(u_0)(t)\| + \lambda_0 \geq \left\| \int_0^1 G(t, s) f\left(s, u(s), \int_0^1 K(s, \tau) u(\tau) d\tau\right) ds \right\| \\ &\geq \delta \max_{0 \leq t \leq 1} G(t, t) \int_0^1 G(s, s) f\left(s, u(s), \int_0^1 K(s, \tau) u(\tau) d\tau\right) ds \\ &\geq \delta \max_{0 \leq t \leq 1} G(t, t) \int_0^1 G(s, s) M\left[u, (s) + \int_0^1 K(s, \tau) u(\tau) d\tau\right] ds \\ &\geq \delta \max_{0 \leq t \leq 1} G(t, t) \int_0^{1-\theta} G(s, s) M\left[u, (s) + \int_0^{1-\theta} K(s, \tau) u(\tau) d\tau\right] ds \\ &\geq M\delta\sigma \cdot \max_{0 \leq t \leq 1} G(t, t) \cdot \int_0^{1-\theta} G(s, s) \left[1 + \int_0^{1-\theta} K(s, \tau) d\tau \right] ds \\ &\quad \cdot \|u_0\| > \|u_0\|, \end{aligned} \quad (58)$$

which contradicts with the statement (iii) of Lemma 7. So (b) holds.

Therefore, from Lemma 1, we can get that \tilde{T} has at least one positive fixed-point $u_1(t) \in \overline{\Omega_K^*} \setminus B_K^{r^*/2}$. Hence, the inequalities hold

$$\|u_1\| \geq \frac{r^*}{2}, \sigma \frac{r^*}{2} \leq \min_{t \in [\theta, 1-\theta]} u_1(t) \leq \sigma r^*. \quad (59)$$

On the other hand, since $\sigma \|u_1\| \leq \min_{t \in [\theta, 1-\theta]} u_1(t) \leq \sigma r^*$, we have $\|u_1\| \leq r^*$.

Finally, since

$$\min_{t \in [\theta, 1-\theta]} u_2(t) \leq \sigma \alpha - \omega^* > r^*, \|u_1\| \leq r^*, \quad (60)$$

(1) has at least two positive solutions.

4. Examples

Example 10. Let us consider the following system:

$$\begin{cases} -u'' + u = \phi u + \frac{1}{4}e^u + \frac{1}{4}\cos \pi \phi - \frac{1}{4}t^2, 0 < t < 1, \\ -\phi'' = \frac{1}{4}u, \\ \phi(0) = \phi(1) = 0, \phi''(0) = \phi''(1) = 0, \end{cases} \quad (61)$$

where $\mu = 1, \rho = 1, f : [0, 1] \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is continuous, and

$$f = \frac{1}{4}(e^u + \cos \pi \phi - t^2) \geq -\frac{1}{4}t^2 = -e(t) \quad \text{for } (t, u, \phi) \in [0, 1] \times \mathbb{R}_+^2. \quad (62)$$

It is obvious that (F_0) holds. Via some computations, we have

$$e^* = \frac{1}{4}, C = 1, \delta = \frac{1}{\sinh 1}, J_1^* = J_2^* = \frac{e^{1/2} - e^{-(1/2)}}{e^{1/2} + e^{-(1/2)}} < 1. \quad (63)$$

Choosing $\theta = 1/4 \in (0, 1/2)$, $\xi = 1/2$. Then, we have

$$\begin{aligned} \sigma &= \min_{t \in [1/4, 3/4]} \frac{\delta}{C} G(t, t) = \frac{\sinh(3/4) \sinh(1/4)}{(\sinh 1)^2}, \\ \omega^* &= \frac{\sinh 1 - 2 \sinh(1/2)}{\sinh 1}. \end{aligned} \quad (64)$$

Furthermore, we have

$$\begin{aligned} \frac{\omega^*}{\sigma} &= \frac{(\sinh 1 - 2 \sinh(1/2)) \sinh 1}{\sinh(3/4) \sinh(1/4)} < 3, \\ \frac{Ce^*}{\delta \xi} &< \sin 1 (J_1^* + J_2^*) < 3. \end{aligned} \quad (65)$$

Choosing $\alpha = 3 > \max \{\omega^*/\sigma, Ce^*/\delta \xi\}$, $r^* = 3/10 \in (0, \sigma \alpha - \omega^*)$. Then,

$$\begin{aligned} \Phi_*(3) &\geq 0, \Phi^*(t, 3) = \frac{1}{4}(e^3 + 1), \\ \Phi_*\left(\frac{3}{10}\right) &\geq 0, \Phi^*\left(t, \frac{3}{10}\right) = \frac{1}{4}(e^{3/10} + 1). \end{aligned} \quad (66)$$

It is easy to get

$$\begin{aligned} \mu \frac{\alpha^2}{6} C \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds + C \int_0^1 G(s, s) \Phi^*(s, \alpha) ds \\ \leq \frac{3}{2} \frac{e^{-1}}{2 \sinh 1} + \frac{e^{-1}}{2 \sinh 1} + \frac{e^{-1}}{2 \sinh 1} \left[\frac{1}{4}(e^3 + 1) \right] \leq 3, \end{aligned} \quad (67)$$

$$\begin{aligned} \mu \frac{(r^*)^2}{6} C \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds + C \int_0^1 G(s, s) \Phi^*(s, r^*) ds \\ \leq \frac{3}{200} \frac{e^{-1}}{2 \sinh 1} + \frac{e^{-1}}{2 \sinh 1} + \frac{e^{-1}}{2 \sinh 1} \left[\frac{1}{4}(e^{3/10} + 1) \right] \leq \frac{3}{10}, \end{aligned} \quad (68)$$

which implies that (F_2) and (F_3) hold.

Finally, it is obvious that

$$\lim_{u+\phi \rightarrow 0+} \frac{f(t, u, \phi)}{u + \phi} = +\infty, \quad \text{uniformly for } t \in [0, 1]. \quad (69)$$

So (F_4) holds.

Therefore, by Theorem 9, (61) has two positive solutions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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Research Article

Speckle Noise Removal by Energy Models with New Regularization Setting

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In this paper, we introduce two novel total variation models to deal with speckle noise in ultrasound image in order to retain the fine details more effectively and to improve the speed of energy diffusion during the process. Firstly, two new convex functions are introduced as regularization term in the adaptive total variation model, and then, the diffusion performances of Hypersurface Total Variation (HYPTV) model and Logarithmic Total Variation (LOGTV) model are analyzed mathematically through the physical characteristics of local coordinates. We have shown that the larger positive parameter in the model is set, the greater energy diffusion speed appears to be, but it will cause the image to be too smooth that required adequate attention. Numerical experimental results show that our proposed LOGTV model for speckle noise removal is superior to traditional models, not only in visual effect but also in quantitative measures.

1. Introduction

With the development of digital image technology, a large number of digital images are transmitted and compressed through various channels. However, image corruption usually is unavoidable during transmission and storage, and the resultant noise quite often seriously affects the visual effect of the image. Clearly, high-quality images are desirable in many areas, such as in medical imaging and pattern recognition. Thus, image denoising is a critical step in image processing and computer vision, which plays an important role in various applied areas, especially in medical imaging, video processing, and remote sensing. On the other hand, it is also an important preprocessing process for other image processing that relies on subsequent processing.

Today, image denoising becomes research of focus, and many image denoising methods have been proposed such as Lee filter [1], Kuan filter [2], locally adaptive statistic filters [3–5], PDE-based and curvature-based methods [6, 7], wavelet transform based thresholding methods [8], and total vari-

ational [9–11]. In addition, the method based on machine learning [12–15] has received wide attention in recent years, such as deep learning [12, 13], linear regression [14], and Bayesian learning [13, 15].

In 1992, Rudin et al. [10] proposed a denoising model based on the total variation:

$$E_{\lambda_1}(u) = \int_{\Omega} |Du| dx + \frac{\lambda_1}{2} \int_{\Omega} |u - u^0|^2 dx, \quad (1)$$

where $\int_{\Omega} |Du| dx = \sup \{ \int_{\Omega} u \operatorname{div}(\varphi) \mid \varphi \in C_c^1(\Omega, \mathbb{R}^n), \|\varphi\|_{\infty} \leq 1 \}$ represents the TV regularization term, $u : \Omega \rightarrow \mathbb{R}$ is an original image that is without noise, $u^0 = u + n$ is noisy image, and n represents the Gaussian random noise with mean zero and standard deviation σ . $\lambda_1 > 0$ represents the regularization parameter which can be used to balance the fidelity terms and regularized terms in TV model, and $|Du|$ represents the L^1 norm of the image gradient.

It is well known that medical ultrasonic images may have many noises of speckle, which will bring a significant problem in terms of the quality of ultrasonic images and cover up the lesions of certain important tissues. Further, it brings great difficulties to the accurate diagnosis and identification of certain specific diseases and can create the potential risk of missed diagnosis and misdiagnosis. Thus, it is very desirable to eliminate the speckle noise in ultrasound image and simultaneously retain the important features in practices. As mentioned in article [3], the speckle noise in medical ultrasonic images can be formulated in the following form:

$$f = u + \sqrt{n}n, \quad (2)$$

where f is a noisy image, and n represents the Gaussian random noise with zeros mean and standard deviation σ .

In this paper, we focus on the image denoising form by using variation method, where the noisy image is ultrasound speckle noise. Based on the model (2) and the characteristics of the Gaussian distribution, Krissian et al. in the article [16] derived a data fidelity term:

$$F(u, f) = \int_{\Omega} \frac{(f - u)^2}{u} dx. \quad (3)$$

Within the variational framework, the data fidelity function is derived from the degradation model (2). One of the technical approaches to solve the variational model is the regularization technique, which minimizes the cost function to obtain stable and accurate solutions. In general, the image denoising variation method is to consider:

$$\min_{u \in \Omega} E(u) = TV(u) + \lambda F(u, f), \quad (4)$$

where $TV(u)$ is a regularization term that represents a prior information about the object to be restored, and $F(u, f)$ is a fidelity term to ensure that the restoration u is not far from the original observation f . $\lambda > 0$ represents the regularization parameter which can balance the fidelity term and the regularization term.

In [11], motivated by the classical ROF model [10], the authors proposed a convex variational model for removing the speckle noise in ultrasound image. The convex variational model involves the TV regularization term and convex fidelity term (see Equation (5)):

$$\min_u \left[\int_{\Omega} |Du| dx + \lambda \int_{\Omega} \frac{(f - u)^2}{u} dx \right], \quad (5)$$

where $\int_{\Omega} |Du| dx$ represents the TV regularization term, and Du represents the directional gradient of u . The existence and uniqueness of the solution of model (5) is proved in [11]. In this paper, we call the model in (5) the “JIN’s model.” In [17], the authors proposed a well-balanced speckle noise reduction (WBSN) model that can detect edges.

Although TV regularization is effective for image denoising, it also leads to some staircase effects that is undesirable. In order to solve this problem, many methods based on improved TV regularization are proposed, such as high-order TV regularization [18–20], several hybrid TV regularization [21], the improved infimal convolution [22, 23], non-local TV model [24, 25], fractional order TV model [26], and anisotropic TV model [27, 28]. Fractional theory [29, 30], wavelet [31], and statistical information [32–34] are also employed to deal with intensity inhomogeneity or noise. Although these denoising methods can reduce the staircase effects in the restoration of additive noise images, however, there are more staircase effects that appeared in the restoration of speckle noise images. In this paper, we introduce HYPTV model and LOGTV model to reduce speckle noise and staircase effects in ultrasound images in an effective way.

In numerical algorithms, most of the energy function minimization problems can be transformed to an Euler-Lagrange equation and then be solved by using the finite difference method. However, the choice of adequate regularization terms is critical in terms of solution accuracy. Moreover, when solving the Euler-Lagrange equation, the energy diffusion form of the noise image in different regions is required to handle differently based on the physical characteristics of local coordinates, since the diffusion velocity of different parameters to the energy function is different in the two models. The new proposed HYPTV model and the LOGTV model, as shown in this paper, not only can preserve the edges of the restored images well when restoring the ultrasonic image with speckle noise but also better reduce the staircase effect generated during the recovery process.

The rest of this paper is as follows: in Section 2, we review some background knowledge. In Section 3, we propose two new models based on variation; meanwhile, we not only analyze diffusion performance of the proposed models but also give the corresponding numerical algorithms. Section 4 shows five different experiments and results. The paper ends with concluding remarks in Section 5.

2. Preliminary

2.1. Some Theoretical Background. The Euler-Lagrange equation was developed in the 1750s by Euler and Lagrange in connection with their studies of the tautochrone problem.

Theorem 1. *A multidimensional generalization comes from considering a function on m variables. If Ω is open, bounded Lipschitz domain in R^n , then*

$$A[h] = \int_{\Omega} L(x_1, x_2, \dots, x_m, h, h_1, h_2, \dots, h_m) dx, \quad (6)$$

is extremized only if h satisfies the partial differential equation:

$$\frac{\partial L}{\partial h} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial h_i} \right) = 0, \quad (7)$$

For model (5), the corresponding Euler-Lagrange equation is as follow:

$$\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) + \lambda \left(\frac{f^2}{u^2} - 1 \right) = 0, \quad (8)$$

where ∇ and div , respectively, represent gradient operators and divergence operators. Using gradient descent method, we can get the model as follows:

$$\begin{cases} u_t = \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) + \lambda \left(\frac{f^2}{u^2} - 1 \right), & t > 0 \quad x, y \text{ in } \Omega, \\ \frac{\partial u}{\partial \vec{n}} = 0 & \text{on the boundar of } \Omega, \\ u|_{t=0} = u_0 & \text{in } \bar{\Omega}, \end{cases} \quad (9)$$

where \vec{n} is the unit out normal vector of $\partial\Omega$.

Without losing generality, in the following, we consider the grayscale images as $M \times N$ matrices.

Definition 2. Let $u \in U = C_2^2(\Omega, R)$, $g = (g_1, g_2) \in G = C_2^2(\Omega, R^2)$, the gradient operators on the space U and the divergence operators on the space G are defined as:

$$\nabla : U \longrightarrow G, \quad \nabla u = \left(\partial_x^+ u, \partial_y^+ u \right), \quad (10)$$

$\operatorname{div} : G \longrightarrow U$, $\operatorname{div} g = \partial_x^- g_1 + \partial_y^- g_2$, where ∂_x^+ , ∂_y^+ , ∂_x^- , and ∂_y^- are the first-order forward and backward discrete derivation operators in the x -direction and y -direction, respectively, which are defined as:

$$\begin{aligned} (\partial_x^+ u) &= \begin{cases} u_{i,j+1} - u_{i,j}, & \text{if } 1 \leq j \leq N-1, \\ 0, & \text{if } j = N, \end{cases} \\ (\partial_y^+ u) &= \begin{cases} u_{i+1,j} - u_{i,j}, & \text{if } 1 \leq i \leq M-1, \\ 0, & \text{if } i = M, \end{cases} \\ (\partial_x^- u) &= \begin{cases} u_{i,j} - u_{i,j-1}, & \text{if } 2 \leq j \leq N, \\ 0, & \text{if } j = 1, \end{cases} \\ (\partial_y^- u) &= \begin{cases} u_{i,j} - u_{i-1,j}, & \text{if } 2 \leq i \leq M, \\ 0, & \text{if } i = 1. \end{cases} \end{aligned} \quad (11)$$

Applying the above gradient operators and divergence operators to model (9), we can obtain the equivalent minimization problem.

Definition 3. Let C be a convex subset of R . A function $\phi : C \longrightarrow R$ is called convex if

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y), \forall x, y \in C, \forall t \in [0, 1]. \quad (12)$$

The following facts are easily checked:

Theorem 4. If functions ϕ_1 and ϕ_2 are convex and have the same domain definition, then $\phi = \phi_1 + \phi_2$ is also convex.

Proof. Let functions ϕ_1 and ϕ_2 are convex. According to the Definition 3, for any $x, y \in R$ and $t \in [0, 1]$, we have

$$\begin{aligned} \phi_1(tx + (1-t)y) &\leq t\phi_1(x) + (1-t)\phi_1(y), \\ \phi_2(tx + (1-t)y) &\leq t\phi_2(x) + (1-t)\phi_2(y), \\ \phi(tx + (1-t)y) &= \phi_1(tx + (1-t)y) + \phi_2(tx + (1-t)y) \\ &\leq t\phi_1(x) + (1-t)\phi_1(y) + t\phi_2(x) + (1-t)\phi_2(y) \\ &\leq t(\phi_1(x) + \phi_2(x)) + (1-t)(\phi_1(y) + \phi_2(y)) \\ &= t\phi(x) + (1-t)\phi(y). \end{aligned} \quad (13)$$

Hence $\phi = \phi_1 + \phi_2$ is convex.

Theorem 5. If a differentiable function $\phi : C \longrightarrow R$ satisfied

$$\phi''(x) \geq 0, \quad \forall x \in C, \quad (14)$$

then ϕ is convex.

2.2. The Condition of TV Regularization Term. Although TV regularization is very effective in image restoration, it usually generates some staircase effects. Thus, it is suggested in literature to use general variational methods, i.e., to consider:

$$J_1(u) = \int_{\Omega} \varphi(|\nabla u|) dx, \quad (15)$$

where $\varphi(s)$ represents a potential function, and the case $\varphi(s) = s$ leads to the total variation regularization term. In the literature [35], the author Costanzino chooses $\varphi(s) = s^2$ that leads to the well-known harmonic model.

In practice, we prefer good smoothing in some domain where the intensity of variations is relatively weak. This can be achieved by requiring a function $\varphi(s)$ to satisfy the following conditions:

$$\varphi'(0) = 0, \lim_{s \rightarrow 0^+} \varphi''(s) = \lim_{s \rightarrow 0^+} \frac{\varphi'(s)}{s} = c > 0. \quad (16)$$

Near the edge of the image, the intensity of variations is strong. If we would like to preserve the edge, then the function $\varphi(s)$ should satisfy the following conditions:

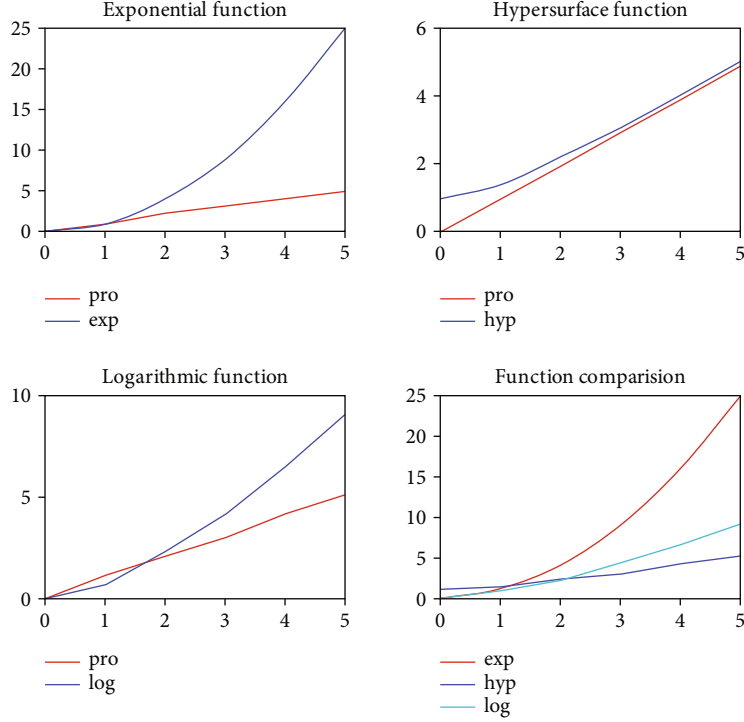


FIGURE 1: The plots of convex and nondecreasing functions.

$$\lim_{s \rightarrow \infty} \varphi''(s) = \lim_{s \rightarrow \infty} \frac{\varphi'(s)}{s} = 0, \quad \lim_{s \rightarrow \infty} \frac{s\varphi''(s)}{\varphi'(s)} = 0. \quad (17)$$

With the conditions (16) and (17), the function $\varphi(s)$ is convex and nondecreasing function, such as:

$$\begin{aligned} \text{proportional function : } \varphi(s) &= s, \quad s \geq 0, \\ \text{exponential function : } \varphi(s) &= s^2, \quad s \geq 0, \\ \text{hypersurface function : } \varphi(s) &= \sqrt{1 + \alpha_1 s^2}, \quad s \geq 0, \alpha_1 > 0, \\ \text{logarithmic function : } \varphi(s) &= s \ln(\alpha_2 + s), \quad s \geq 0, \alpha_2 > 0. \end{aligned} \quad (18)$$

These functions are convex and nondecreasing on $s \geq 0$, as shown in Figure 1.

In this paper, we will use two new functions $\varphi_1(s) = \sqrt{1 + \alpha_1 s^2}$ and $\varphi_2(s) = s \ln(\alpha_2 + s)$, which appear to be quite effective for image processing, in particular, for ultrasound image denoising. Obviously, the functions $\varphi_1(s)$ and $\varphi_2(s)$ are convex and nondecreasing.

3. The Proposed Restoration Model

In this section, we propose adaptive total variation model for image restoration. We use the finite difference method to solve the Euler-Lagrange equation directly, and then find the minimum value of the energy function.

3.1. The Adaptive Total Variation Model. Apply the selected function to model (15), we propose adaptive total variation model,

$$\min_u \left[E(u) = \int_{\Omega} \varphi_i(|\nabla u|) dx + \lambda \int_{\Omega} \frac{(f - u)^2}{u} dx \right], \quad (19)$$

where $\int_{\Omega} \varphi_i(|\nabla u|) dx$ is a regularization term, and $i = 1, 2$; $\int_{\Omega} ((f - u)^2/u) dx$ is a fidelity term; λ represents the regularization parameter which can balance fidelity term and regularization term.

Firstly, the energy functional $E(u)$ is convex, which guarantees the existence of the minimal solution of the model (19).

Theorem 6. *The energy functional $E(u)$ is convex. That is to say, for any $u_1, u_2 \in \Omega$, and $t \in [0, 1]$, we have:*

$$E(tu_1 + (1 - t)u_2) \leq tE(u_1) + (1 - t)E(u_2), \quad (20)$$

where $E(u) = \int_{\Omega} \varphi_i(|\nabla u|) dx + \lambda \int_{\Omega} ((f - u)^2/u) dx$.

Proof. Firstly, the function $\varphi_i(s)$ is convex; according to Definition 3, for any $u_1, u_2 \in \Omega$, and $t \in [0, 1]$, we have

$$\begin{aligned} \int_{\Omega} \varphi_i(|\nabla(tu_1 + (1 - t)u_2)|) dx &\leq t \int_{\Omega} \varphi_i(|\nabla u_1|) dx \\ &+ (1 - t) \int_{\Omega} \varphi_i(|\nabla u_2|) dx. \end{aligned} \quad (21)$$

Meanwhile, we have (The proof step of the inequality is in the appendix.)

$$\begin{aligned}
 \int_{\Omega} \frac{(f - (tu_1 + (1-t)u_2))^2}{tu_1 + (1-t)u_2} dx &= \int_{\Omega} \left(\frac{f^2}{tu_1 + (1-t)u_2} - 2f + tu_1 + (1-t)u_2 \right) dx \\
 &\leq \int_{\Omega} \left(\frac{f^2(tu_2 + (1-t)u_1)}{u_1 u_2} - 2f + tu_1 + (1-t)u_2 \right) dx \\
 &= \int_{\Omega} \left(\frac{tf^2}{u_1} - 2tf + tu_1 + \frac{(1-t)f^2}{u_2} - 2(1-t)f + (1-t)u_2 \right) dx \\
 &= \int_{\Omega} t \left(\frac{(f - u_1)^2}{u_1} \right) + (1-t) \left(\frac{(f - u_2)^2}{u_2} \right) dx.
 \end{aligned} \tag{22}$$

Therefore,

$$E(tu_1 + (1-t)u_2) \leq tE(u_1) + (1-t)E(u_2). \tag{23}$$

This proof is established.

Secondly, the uniqueness of the minimum solution of the model (19) can also be proved.

Theorem 7. If u_1 and u_2 are two minimize solutions of model (19), then we have $u_1 = u_2$.

Proof. According to Theorem 6, we have

$$\begin{aligned}
 E\left(\frac{u_1 + u_2}{2}\right) &= \int_{\Omega} \varphi_i \left(\left| \nabla \left(\frac{1}{2}u_1 + \frac{1}{2}u_2 \right) \right| \right) dx \\
 &\quad + \lambda \int_{\Omega} \frac{(f - ((u_1/2) + (u_2/2)))^2}{u_1 + u_2/2} dx \\
 &\leq \int_{\Omega} \left(\frac{1}{2} \varphi_i(|\nabla u_1|) + \frac{1}{2} \varphi_i(|\nabla u_2|) \right) dx \\
 &\quad + \lambda \int_{\Omega} \left(\frac{f^2(1/4(u_1 + u_2)^2 - 1/4(u_1 - u_2)^2)}{(1/2u_1 + 1/2u_2)u_1 u_2} \right. \\
 &\quad \left. - 2f + \frac{1}{2}u_1 + \frac{1}{2}u_2 \right) dx \\
 &\leq \frac{1}{2} \int_{\Omega} \varphi_i(|\nabla u_1|) dx + \frac{1}{2} \int_{\Omega} \varphi_i(|\nabla u_2|) dx \\
 &\quad + \lambda \int_{\Omega} \left(\frac{(f - u_1)^2}{2u_1} + \frac{(f - u_2)^2}{2u_2} - \frac{f^2(u_1 - u_2)^2}{2(u_1 + u_2)u_1 u_2} \right) dx \\
 &= \frac{1}{2}E(u_1) + \frac{1}{2}E(u_2) - \lambda \int_{\Omega} \frac{f^2(u_1 - u_2)^2}{2(u_1 + u_2)u_1 u_2} dx \\
 &= E(u_1) - \lambda \int_{\Omega} \frac{f^2(u_1 - u_2)^2}{2(u_1 + u_2)u_1 u_2} dx.
 \end{aligned} \tag{24}$$

If $u_1 \neq u_2$, then the above assumption gives a contradiction that u_1 is not a minimize solution.

3.2. Diffusion Performance. In this subsection, we mainly analyze the diffusion performance and diffusion speed of the energy function of HYPTV model and LOGTV model.

3.2.1. Diffusion Performance of HYPTV Model. Firstly, we use the finite difference method to solve the HYPTV model, associated with the potential function $\varphi_1(s) = \sqrt{1 + \alpha_1 s^2}$.

From Definition 2, we can obtain the corresponding Euler-Lagrange equation HYPTV model that as follows:

$$\operatorname{div} \left[\left(\frac{\alpha_1}{\sqrt{1 + \alpha_1 |\nabla u|^2}} \right) \nabla u \right] + \lambda \left(\frac{f^2}{u^2} - 1 \right) = 0. \tag{25}$$

Using gradient descent method, Equation (25) can be transformed to:

$$\begin{cases} u_t = \operatorname{div} \left[\left(\frac{\alpha_1}{\sqrt{1 + \alpha_1 |\nabla u|^2}} \right) \nabla u \right] + \lambda \left(\frac{f^2}{u^2} - 1 \right) & \text{in } \Omega, \\ \frac{\partial u}{\partial \vec{n}} = 0 & \text{on the boundar of } \Omega, \\ u|_{t=0} = u_0 & \text{in } \bar{\Omega}, \end{cases} \tag{26}$$

where \vec{n} is the unit out normal vector of $\partial\Omega$.

In order to analyze the diffusion performance, local image coordinate system ξ - η is established. As shown in Figure 2, the η -axis represents the direction parallel to the image gradient at the pixel level, and the ξ -axis is the corresponding vertical direction.

According to Figure 2, we can know:

$$\begin{cases} \xi = \frac{1}{|\nabla u|} (-u_y, u_x), \\ \eta = \frac{1}{|\nabla u|} (u_x, u_y). \end{cases} \tag{27}$$

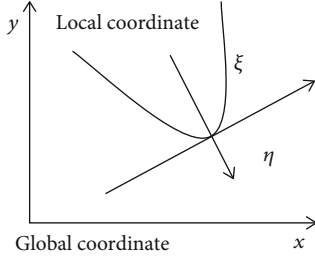


FIGURE 2: Global and local coordinate schematic diagram.

So, Equation (26) can be rewritten as:

$$u_t = \psi_1^1(|\nabla u|)u_{\xi\xi} + \psi_2^1(|\nabla u|)u_{\eta\eta} + \lambda\left(\frac{f^2}{u^2} - 1\right), \quad (28)$$

where

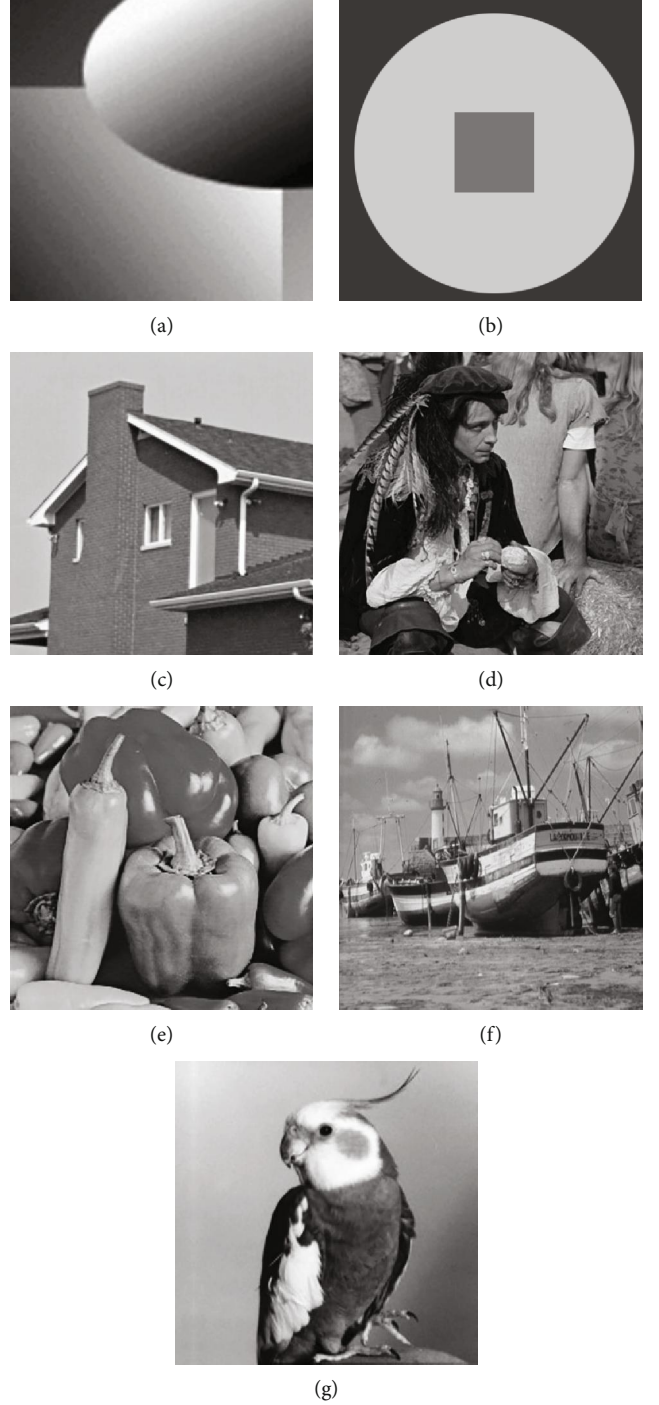
$$\begin{cases} \psi_1^1(|\nabla u|) = \frac{\alpha_1}{\sqrt{(1 + \alpha_1|\nabla u|^2)}}, \\ \psi_2^1(|\nabla u|) = \frac{\alpha_1}{(1 + \alpha_1|\nabla u|^2)^{3/2}}, \\ u_{\xi\xi} = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{|\nabla u|^2}, \\ u_{\eta\eta} = \frac{u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}}{|\nabla u|^2}. \end{cases} \quad (29)$$

The $\psi_1^1(|\nabla u|)$ and $\psi_2^1(|\nabla u|)$ are control functions of the diffusion along the ξ -direction and η -direction, respectively. Now, we consider the diffusion of image restoration. Some test images are shown in Figure 3.

(1) *Smooth area.* When $|\nabla u| \rightarrow 0$, $\lim_{|\nabla u| \rightarrow 0} \psi_1^1(|\nabla u|) = \lim_{|\nabla u| \rightarrow 0} \psi_2^1(|\nabla u|) = \alpha_1$. This shows that the diffusion form of the energy Equation (19) is isotropic. In other words, the energy diffusion rate along direction ξ and direction η is very close in the process of image restoration in the smooth region. And the rate of energy diffusion is obviously positively correlated with the parameter α_1 .

(2) *Sharp area.* When $|\nabla u| \rightarrow \infty$, we obtain $\lim_{|\nabla u| \rightarrow \infty} (\psi_2^1(|\nabla u|)/\psi_1^1(|\nabla u|)) = 0$. This shows that the diffusion form of the energy Equation (19) is anisotropic. In other words, the energy diffusion rate in ξ -direction in Equation (28) is much larger than that in the η -direction in the sharp region. But the gradient $|\nabla u|$ does not exceed 255, so $\lim_{|\nabla u| \rightarrow 255} (\psi_2^1(|\nabla u|)/\psi_1^1(|\nabla u|)) = \lim_{|\nabla u| \rightarrow 255} (1/\sqrt{1 + \alpha_1|\nabla u|^2}) = 1/\sqrt{1 + \alpha_1 \times 255^2}$.

One can see that the larger the parameter α_1 is set, the smaller

FIGURE 3: Test images: (a) image 1 (256×256); (b) image 2 (256×256); (c) house (256×256); (d) pirate (512×512); (e) peppers (256×256); (f) boat (512×512); (g) bird (256×256).

the limit becomes. And the rate of energy diffusion is obviously positively correlated with the parameter α_1 .

3.2.2. Diffusion Performance of LOGTV Model. Secondly, we use the finite difference method to solve the LOGTV model, which the potential function is $\phi_2(s) = s \ln(\alpha_2 + s)$.


```

1:Initialize  $\lambda^0 = 0, u^0 = f, k = 0$ 
2:Repeat
3:While  $NDR > \text{limit}$  do
4:  Update  $u_k$  by (35)
5:  Update  $\lambda_k$  by (39)
6:  Computer NDR
7:  Set  $k = k + 1$ 
8: End While
9: Final Input:  $u$ 

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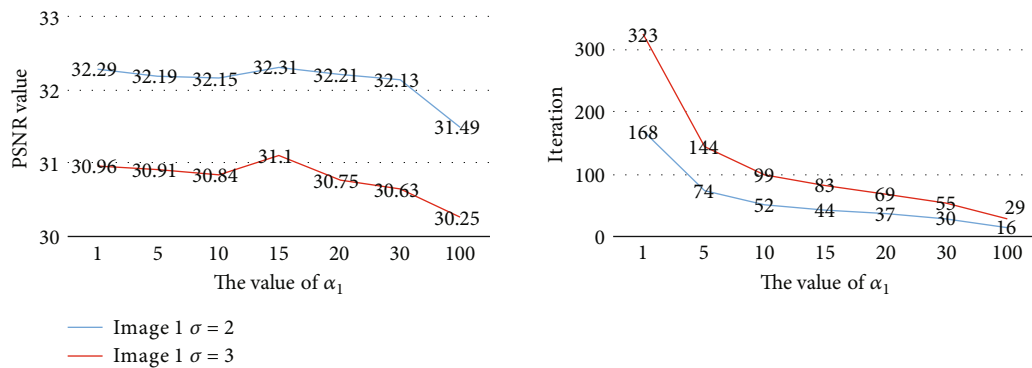
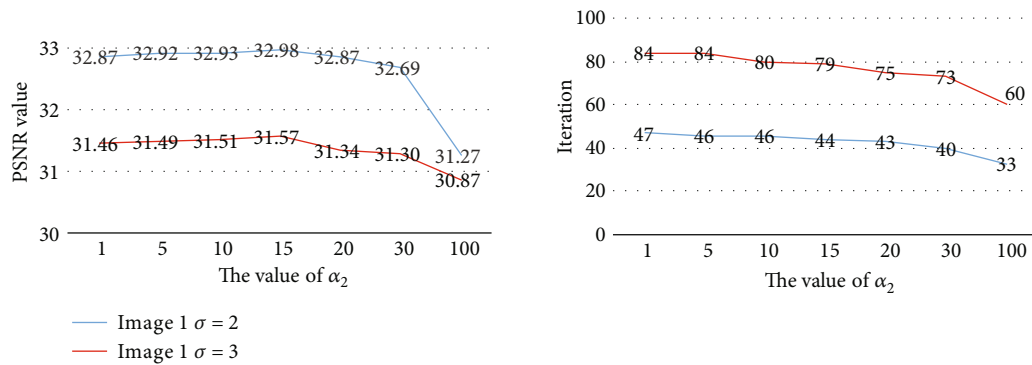
ALGORITHM 1

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1:Initialize  $\lambda^0 = 0, u^0 = f, k = 0$ 
2:Repeat
3:While  $NDR > \text{limit}$  do
4:  Update  $u_k$  by (41)
5:  Update  $\lambda_k$  by (43)
6:  Computer NDR
7:  Set  $k = k + 1$ 
8: End While
9: Final Input:  $u$ 

```

ALGORITHM 2

FIGURE 4: The PSNR values and iteration times of different parameter α_2 (HYPTV model).FIGURE 5: The PSNR values and iteration times of different parameter α_2 (LOGTV model).

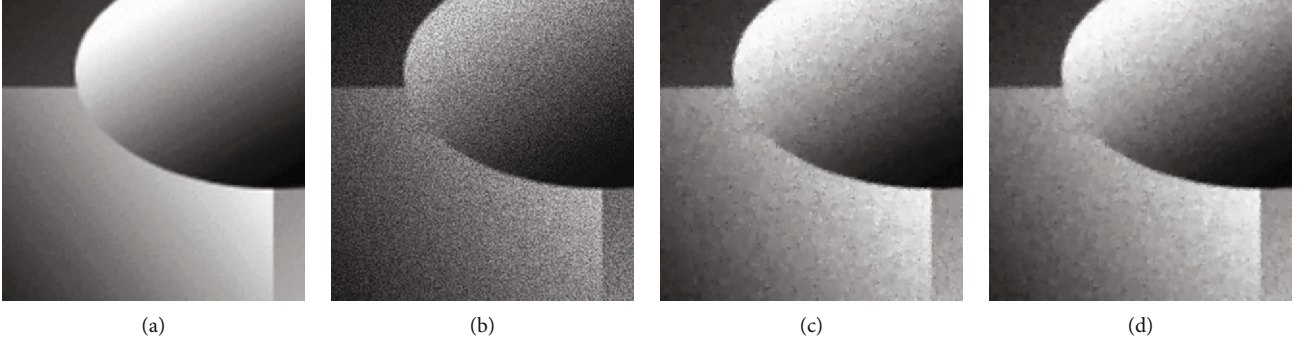


FIGURE 6: Numerical result of the “image 1” image with noise standard deviation $\sigma = 2$. (a) Original image (image 1); (b) noisy image; (c) restored image by HYPTV model $\alpha_1 = 15$; (d) restored image by LOGTV model $\alpha_2 = 15$.

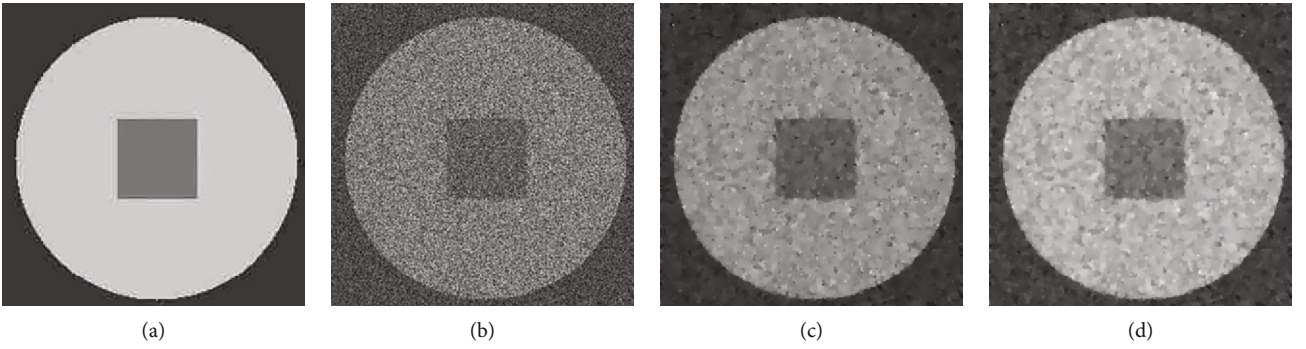


FIGURE 7: Numerical result of the “image 2” image with noise standard deviation $\sigma = 3$. (a) Original image (image 2); (b) noisy image; (c) restored image by HYPTV model $\alpha_1 = 15$; (d) restored image by LOGTV model $\alpha_2 = 15$.

TABLE 1: Numerical result of the “image 1” and “image 2” images by the HYPTV and LOGTV model.

Image	σ	Noise image PSNR	HYPTV (PSNR/SSIM)	LOGTV (PSNR/SSIM)
Image 1	2	23.76	32.31/0.7786	32.98/0.8025
Image 2	2	23.26	34.78/0.8360	35.31/0.8562
Image 1	3	20.24	31.10/0.7806	31.57/0.7948
Image 2	3	19.76	33.47/0.8300	34.07/0.8568
Image 1	4	17.77	29.97/0.7920	30.45/0.8113
Image 2	4	17.51	32.51/0.8649	32.63/0.8667

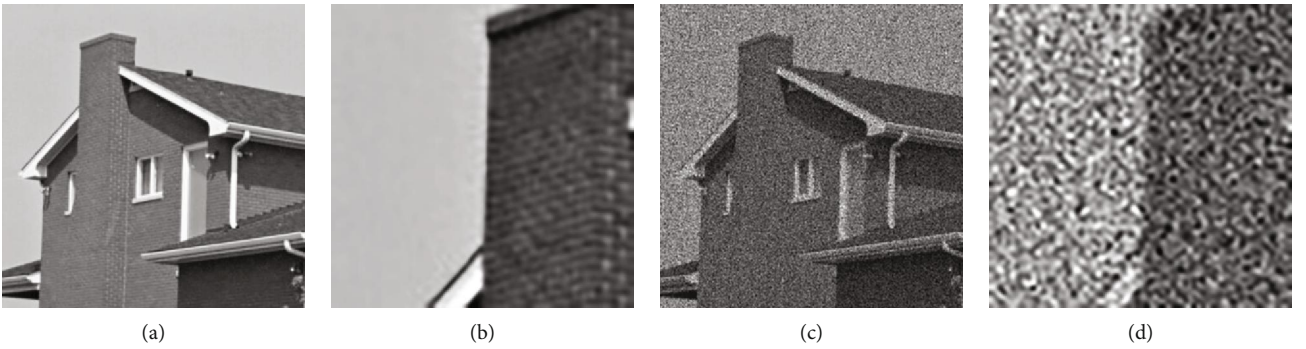


FIGURE 8: Numerical result of the “lena” and “house” images with noise standard deviation $\sigma = 3$. (a) Original images (house); (b) about the detailed image of (a); (c) noisy images; (d) about the detailed image of (c).

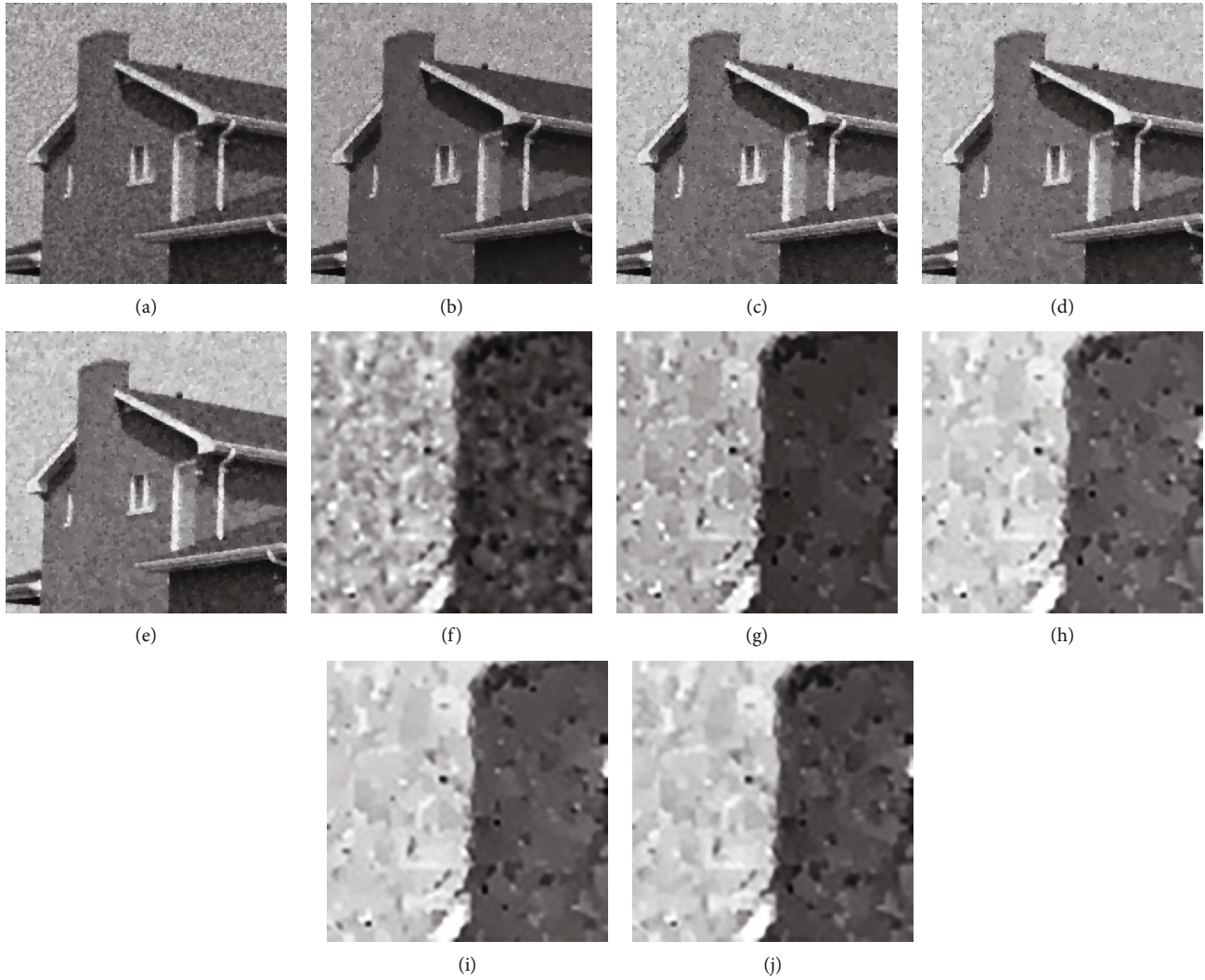


FIGURE 9: Numerical result of the “house” images with noise standard deviation $\sigma = 3$. (a) Restored image by the ROF model; (b) restored image by the ATV model; (c) restored image by the JIN’s model; (d) restored image by HYPTV model; (e) restored image by LOGTV model; (f), (g), (h), (i), and (j) about the detailed image of (a), (b), (c), (d), and (e), respectively.

From the Definition 2, we can obtain the corresponding Euler-Lagrange equation LOGTV model that as follows:

$$\operatorname{div} \left[\left(\frac{\ln(\alpha_2 + |\nabla u|)}{|\nabla u|} + \frac{1}{\alpha_2 + |\nabla u|} \right) \nabla u \right] + \lambda \left(\frac{f^2}{u^2} - 1 \right) = 0. \quad (30)$$

Using gradient descent method, Equation (30) can be transformed to:

$$u_t = \operatorname{div} \left[\left(\frac{\ln(\alpha_2 + |\nabla u|)}{|\nabla u|} + \frac{1}{\alpha_2 + |\nabla u|} \right) \nabla u \right] + \alpha \left(\frac{f^2}{u^2} - 1 \right), \quad (31)$$

where \vec{n} is the unit out normal vector of $\partial\Omega$.

Hence, Equation (31) can be rewritten as:

$$u_t = \psi_1^2(|\nabla u|) u_{\xi\xi} + \psi_2^2(|\nabla u|) u_{\eta\eta} + \lambda \left(\frac{f^2}{u^2} - 1 \right), \quad (32)$$

where

$$\begin{cases} \psi_1^2(|\nabla u|) = \frac{\ln(\alpha_2 + |\nabla u|)}{|\nabla u|} + \frac{1}{\alpha_2 + |\nabla u|}, \\ \psi_2^2(|\nabla u|) = \frac{1}{\alpha_2 + |\nabla u|} + \frac{\alpha_2}{(\alpha_2 + |\nabla u|)^2}. \end{cases} \quad (33)$$

$\psi_1^2(|\nabla u|)$ and $\psi_2^2(|\nabla u|)$ are control functions of the diffusion along the ξ -direction and η -direction, respectively. Now, we consider the diffusion of image restoration.

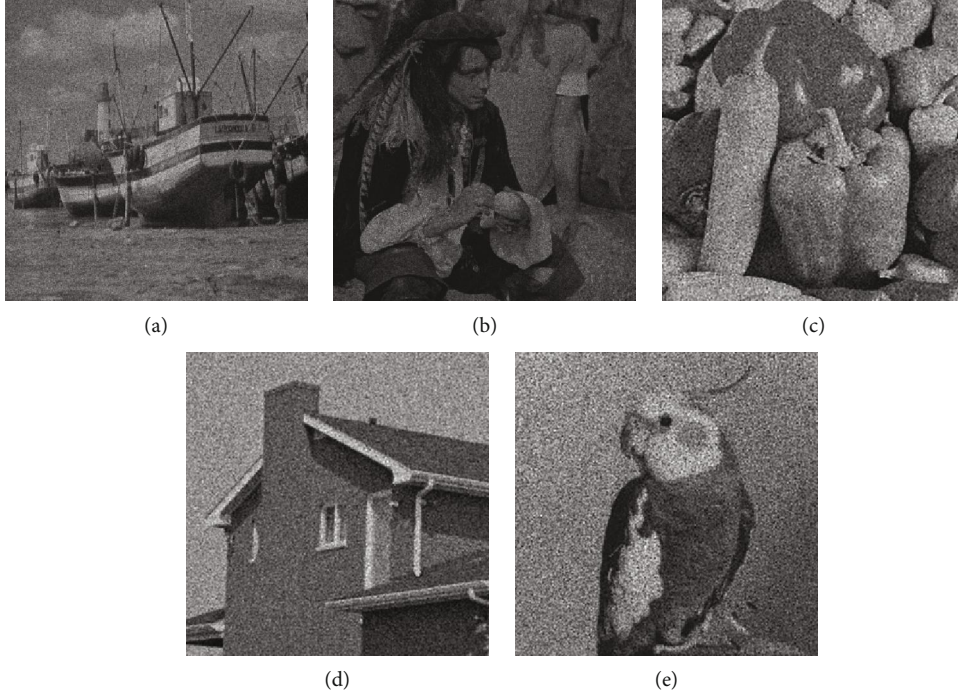


FIGURE 10: Noisy images. (a) Boat ($\sigma = 3$); (b) pirate ($\sigma = 4$); (c) peppers ($\sigma = 2$); (d) house ($\sigma = 3$); (e) bird ($\sigma = 4$).

(1) *Smooth area.* When $\alpha_2 = 1$, and $|\nabla u| \rightarrow 0$, we can obtain $\lim_{|\nabla u| \rightarrow 0} \psi_1^2(|\nabla u|) = 2$ and $\lim_{|\nabla u| \rightarrow 0} \psi_2^2(|\nabla u|) = 2$. This shows that the diffusion form of the energy Equation (19) is isotropic. In other words, the energy diffusion rate along direction ξ and direction η is very close in the process of image restoration in the smooth region. When $\alpha_2 \neq 1$, and $|\nabla u| \rightarrow 0$, we can obtain $\lim_{|\nabla u| \rightarrow 0} \psi_1(|\nabla u|) = \infty$ and $\lim_{|\nabla u| \rightarrow 0} \psi_2(|\nabla u|) = 2/\alpha_2$. This shows that the diffusion form of the energy Equation (19) is anisotropic. However, the gradient of noise image is relatively large, so in the smooth region, whatever the value of α_2 , it has little effect on the model.

(2) *Sharp area.* When $|\nabla u| \rightarrow \infty$, we can obtain $\lim_{|\nabla u| \rightarrow \infty} (\psi_1^2(|\nabla u|)/\psi_2^2(|\nabla u|)) = 0$. This shows that the diffusion form of the energy Equation (19) is anisotropic. In other words, the energy diffusion rate in ξ -direction in Equation (28) is much larger than that in the η -direction in the sharp region. But the gradient $|\nabla u|$ does not exceed 255, so $\lim_{|\nabla u| \rightarrow 255} (\psi_2(|\nabla u|)/\psi_1(|\nabla u|)) = (510\alpha_2 + 255^2)/(225\alpha_2 + 255^2 + (\alpha_2 + 255)^2 * \ln(\alpha_2 + 255))$. One can see that the larger the parameter α_2 is set, the smaller the limit becomes. And the rate of energy diffusion is clearly positively correlated with the parameter α_2 .

3.3. Numerical Implementation. We will describe the corresponding numerical algorithm in this section. Firstly, the HYPTV model can be solved by discretization as follows:

$$u_{i,j}^{k+1} = u_{i,j}^k + \Delta t \left[\operatorname{div} \left(T_1(|\nabla u^k|) \nabla u^k \right)_{i,j} + \lambda^k \left(\frac{f^2}{(u^k)^2} - 1 \right)_{i,j} \right], \quad (34)$$

where $T_1(|\nabla u^k|) = \alpha_1 / \sqrt{1 + \alpha_1 |\nabla u^k|^2}$, and Δt represents time step. Furthermore, the iterative formula can approximate as:

$$u_{i,j}^{k+1} = u_{i,j}^k + \Delta t \left[A_1(\nabla u^k)_{i,j} + \lambda^k \left(\frac{f^2}{(u^k)^2} - 1 \right)_{i,j} \right], \quad (35)$$

for $i = 1, \dots, M$; $j = 1, \dots, N$, and $M \times N$ represent the size of the image. Here:

$$A_1(\nabla u^k)_{i,j} = \partial_x^- \left(T_1(|\nabla_x u^k|) \partial_x^+ u^k \right)_{i,j} + \partial_y^- \left(T_1(|\nabla_y u^k|) \partial_y^+ u^k \right)_{i,j},$$

$$\begin{cases} |\nabla_x(u_{i,j})| = \sqrt{(\partial_x^+(u_{i,j}))^2 + (m[\partial_y^+(u_{i,j}), \partial_y^-(u_{i,j})])^2} + \delta, \\ |\nabla_y(u_{i,j})| = \sqrt{(\partial_y^+(u_{i,j}))^2 + (m[\partial_x^+(u_{i,j}), \partial_x^-(u_{i,j})])^2} + \delta, \end{cases} \quad (36)$$

where $m[a, b] = ((\operatorname{sign} a + \operatorname{sign} b)/2) \cdot \min(|a|, |b|)$, and $\delta > 0$ is a positive parameter that is close to zero. With boundary conditions:

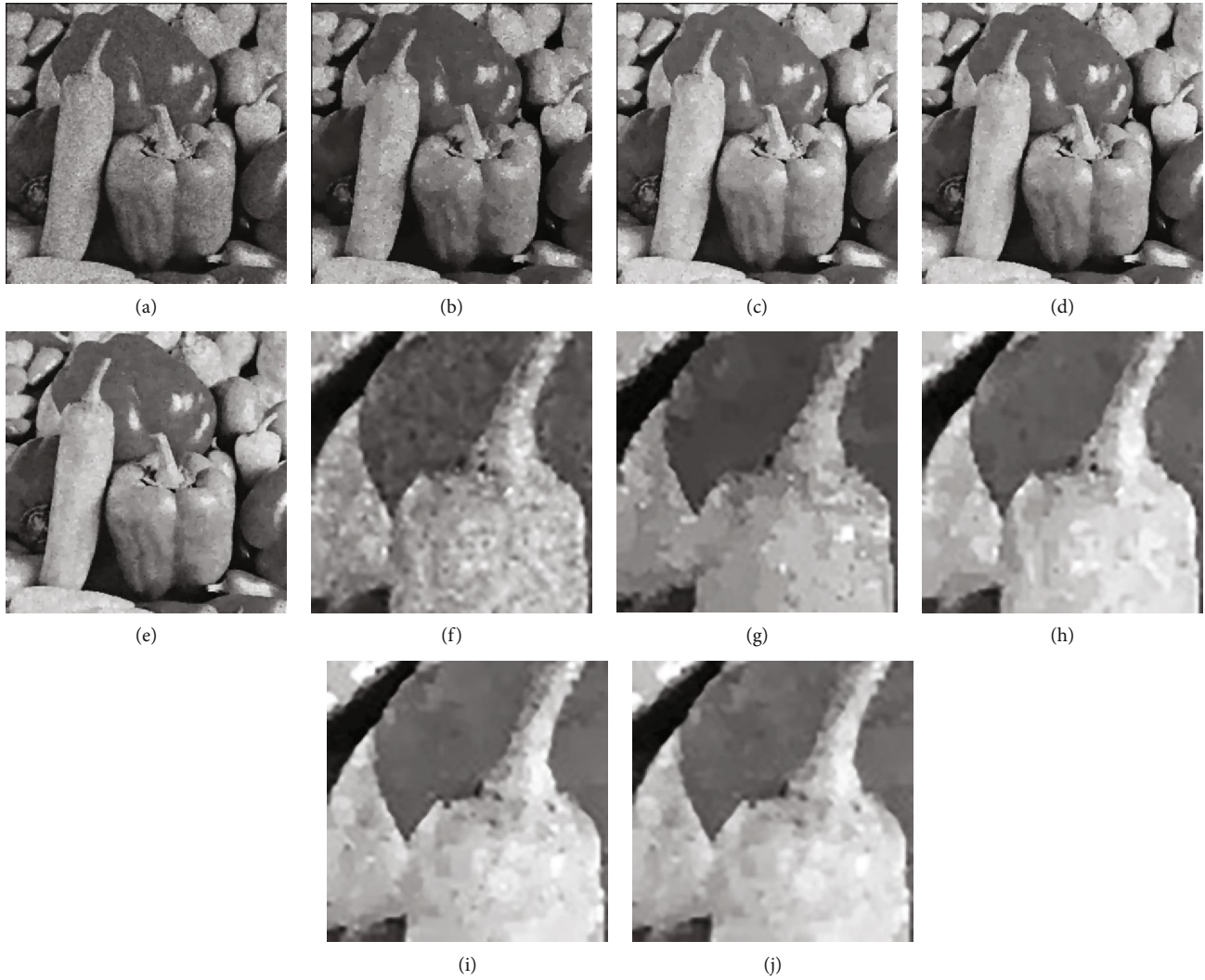


FIGURE 11: Numerical result of the “peppers” image with noise standard deviation $\sigma = 2$; (a) restored image by the ROF model; (b) restored image by the ATV model; (c) restored image by the JIN’s model; (d) restored image by HYPTV model; (e) restored image by LOGTV model; (f), (g), (h), (i), and (j) about the detailed image of (a), (b), (c), (d) and (e), respectively.

$$\begin{cases} u_{0,j}^k = u_{1,j}^k; & u_{N+1,j}^k = u_{N,j}^k, \\ u_{i,0}^k = u_{i,1}^k; & u_{i,N+1}^k = u_{i,N}^k. \end{cases} \quad (37)$$

Now, note Equation (25), the two sides are multiplied by $(f - u)u/f + u$, and then, the integral on the domain Ω can be obtained:

$$\lambda \int_{\Omega} \frac{(f - u)^2}{u} = \int_{\Omega} \operatorname{div} \left[\left(\frac{\alpha_1}{\sqrt{1 + \alpha_1 |\nabla u|^2}} \right) \nabla u \right] \frac{(u - f)u}{u + f}, \quad (38)$$

Because the Gaussian noise n have mean 0 and variance σ^2 , we can obtain:

$$\lambda^k = \frac{1}{\sigma^2 |\Omega|} \sum_{i,j} \left(A_1(\nabla u^k) \right) \frac{(u^k - f)u^k}{u^k + f}, \quad (39)$$

In the process of iteration, we always use the previous solution to calculate the next solution. The optimization algorithm for HYPTV model is given in the following (Algorithm 1).

Secondly, the LOGTV model can be solve by discretization as follows:

$$u_{i,j}^{k+1} = u_{i,j}^k + \Delta t \left[\operatorname{div} \left(T_2(|\nabla u^k|) \nabla u^k \right)_{i,j} + \lambda^k \left(\frac{f^2}{(u^k)^2} - 1 \right)_{i,j} \right], \quad (40)$$

where $T_2(|\nabla u^k|) = ((\ln(\alpha_2 + |\nabla u^k|))/|\nabla u^k|) + (1/(\alpha_2 + |\nabla u^k|))$. Furthermore, the iterative formula can approximate as:



FIGURE 12: Numerical result of the “boat” image with noise standard deviation $\sigma = 3$. (a) Restored image by the ROF model; (b) restored image by the ATV model; (c) restored image by the JIN’s model; (d) restored image by HYPTV model; (e) restored image by LOGTV model; (f), (g), (h), (i), and (j) about the detailed image of (a), (b), (c), (d), and (e), respectively.

$$u_{i,j}^{k+1} = u_{i,j}^k + \Delta t \left[A_2 \left(\nabla u^k \right)_{i,j} + \lambda^k \left(\frac{f^2}{(u^k)^2} - 1 \right)_{i,j} \right], \quad (41)$$

$$A_2 \left(\nabla u^k \right)_{i,j} = \partial_x^- \left(T_2 \left(\left| \nabla_x^+ u^k \right| \right) \partial_x^+ u^k \right)_{i,j} + \partial_y^- \left(T_2 \left(\left| \nabla_y^+ u^k \right| \right) \partial_y^+ u^k \right)_{i,j}, \quad (42)$$

$$\lambda^k = \frac{1}{\sigma^2 |\Omega|} \sum_{i,j} \left(A_2 \left(\nabla u^k \right) \right) \frac{(u^k - f)}{u^k + f}. \quad (43)$$

Similar to the HYPTV model, the optimization algorithm for LOGTV model is given in the following (Algorithm 2).

4. Experimental Results

In this section, we present numerical results to demonstrate the effectiveness of the HYPTV and LOGTV model

in image restoration. Firstly, to evaluate the quality of restored images, we use the peak signal-to-noise ratio (PSNR) value and the structure similarity (SSIM) index, which are defined as follows:

$$\text{PSNR}(u, \bar{u}) = 10 \log_{10} \left(\frac{255^2 mn}{\|u - \bar{u}\|_2^2} \right), \quad (44)$$

$$\text{SSIM}(u, \bar{u}) = \frac{(2\mu_u \mu_{\bar{u}} + c_1)(\sigma_{\bar{u}u} + c_2)}{(\mu_u^2 + \mu_{\bar{u}}^2 + c_1)(\sigma_u^2 + \sigma_{\bar{u}}^2 + c_2)},$$

where $u \in \mathbb{R}^{m \times n}$ is the clean image, and $\bar{u} \in \mathbb{R}^{m \times n}$ is the restored image. μ_a is the average of a , σ_a is the standard deviation of a , and c_1 and c_2 are some constants for stability. Secondly, we calculated the noise deviation reduction (NDR) at each iteration as a convergence condition;

$$\text{NDR} = \text{mean} \frac{(f - u_k)^2}{u_k}, \quad (45)$$

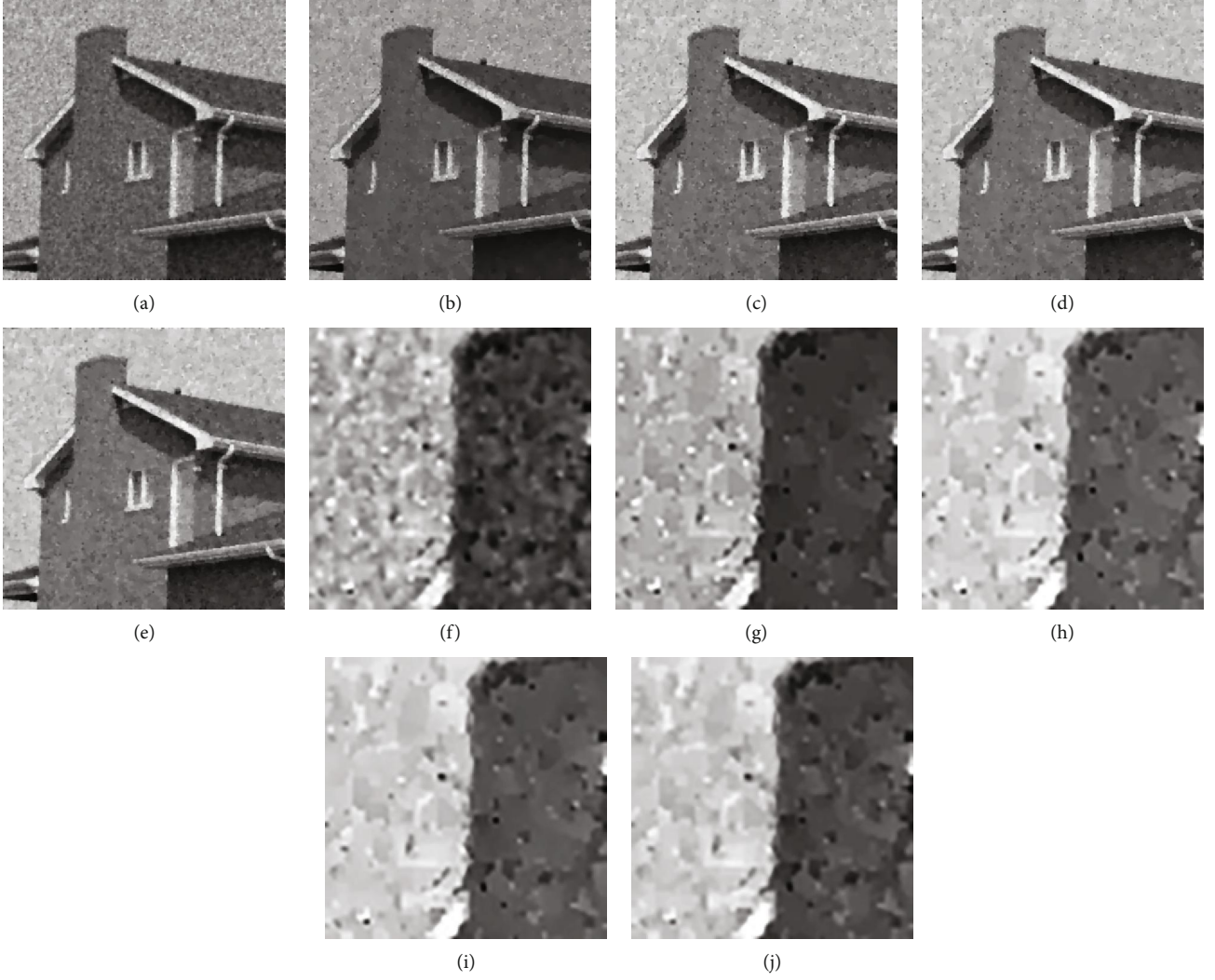


FIGURE 13: Numerical result of the “house” image with noise standard deviation $\sigma = 3$. (a) Restored image by the ROF model; (b) restored image by the ATV model; (c) restored image by the JIN’s model; (d) restored image by HYPTV model; (e) restored image by LOGTV model; (f), (g), (h), (i), and (j) about the detailed image of (a), (b), (c), (d), and (e), respectively.

where u_k represents the results of the k th iterations, respectively. Finally, the stopping condition (NDR) for the HYPTV and LOGTV models is as follows:

$$\left| \sqrt{\text{NDR}} - \sigma \right| \leq 10^{-1}, \quad (46)$$

In the numerical experiment, we will use the noise image as the initial value, that is, $f = u_0$. Moreover, the gray values of all original images are in range $[0, 255]$.

4.1. Denoising Effect of Different Parameters of HYPTV Model and LOGTV Model. In this example, we use different parameter $\alpha_i (i = 1, 2)$ values in the algorithm to test the effect of HYPTV and LOGTV models. The test images is shown in Figure 3(a), and the noise levels are $\sigma = 2$ and $\sigma = 3$. Figure 4 shows the different PSNR values and iteration numbers when different β values are used in the HYPTV model algorithm. Figure 5 shows the different PSNR values and iteration numbers when different β values are used in the

LOGTV model algorithm. Firstly, we can also see that PSNR is the largest in $\alpha_i (i = 1, 2) = 15$. Secondly, with the increase of parameters, the speed of image restoration is faster. Based on the above analysis, when $\alpha_i (i = 1, 2) = 15$, the denoising performance of HPYTV and LOGTV models is close to the best. Therefore, in the following experiment, we choose $\alpha_i (i = 1, 2) = 15$ in HPYTV and LOGTV models.

4.2. Denoising Effect of the HYPTV and LOGTV Model. In this subsection, we use the algorithm to test the effect of the HYPTV and LOGTV model on image denoising. In Figures 6 and 7, we show the original images, the noise images, and the restored images by HYPTV and LOGTV models. Figures 6(a) and 7(a) are original images, Figures 6(b) and 7(b) are noisy images, which the noise levels are $\sigma = 2$ and $\sigma = 3$, respectively. Figures 6(c) and 7(c) are corresponding restored images by HYPTV model. Figures 6(d) and 7(d) are corresponding restored images by LOGTV model. Table 1 shows that the PSNR values, SSIM values, and the numbers of iteration for the different test



FIGURE 14: Numerical result of the “pirate” image with noise standard deviation $\sigma = 4$. (a) Restored image by the ROF model; (b) restored image by the ATV model; (c) restored image by the JIN’s model; (d) restored image by HYPTV model; (e) restored image by LOGTV model; (f), (g), (h), (i), and (j) about the detailed image of (a), (b), (c), (d), and (e), respectively.

images can be got by using the HYPTV model and the LOGTV model. “Noise image PSNR” is the peak signal-to-noise ratio of noisy images and original images. “Denoising image PSNR” is the peak signal-to-noise ratio of restored images and original images. “Iter” is the number of iterations of the algorithm. From the results, it is obvious that the HYPTV model and the LOGTV model are fairly effective in reducing the speckle noise and edge-preserving.

4.3. Reduction of Staircase. In this subsection, we test the reduction of the staircase effect in HYPTV model and LOGTV model by image in Figure 8. Figure 8(a) shows original images (“house”). Figure 8(b) shows detailed images of Figure 8(a), respectively. Figure 8(c) shows noise images which noise standard deviation $\sigma = 3$. Figure 8(d) shows detailed images of Figure 8(c). The original images contain a lot of details information, such as textures edges and in homogeneous regions.

Figure 9 displays the restoration results of the noisy “house” image, respectively. Figures 9(a)–9(e) was restored ROF model [10], ATV model [36], JIN’s model [11], HYPTV model, and LOGTV model, respectively. Figures 9(f)–9(j) is corresponding details images.

According to the results, we can see that JIN’s model, HYPTV model, and LOGTV model had a good effect in removing image noise and preserving image edges. In addition, we can clearly see staircase effect in the detailed images obtained by ROF model, ATV model, and JIN’s model. At the same time, the staircase effect of HYPTV model and LOGTV model restored images is reduced. Although both the HYPTV model and the LOGTV model can reduce the ladder effect, the LOGTV model is better than the HYPTV model.

4.4. Comparison with ROF Model, ATV Model, and JIN’s Model. In this subsection, we compare the effect of the ROF model, ATV model, and JIN’s model with HYPTV and LOGTV models for some images. The test original images

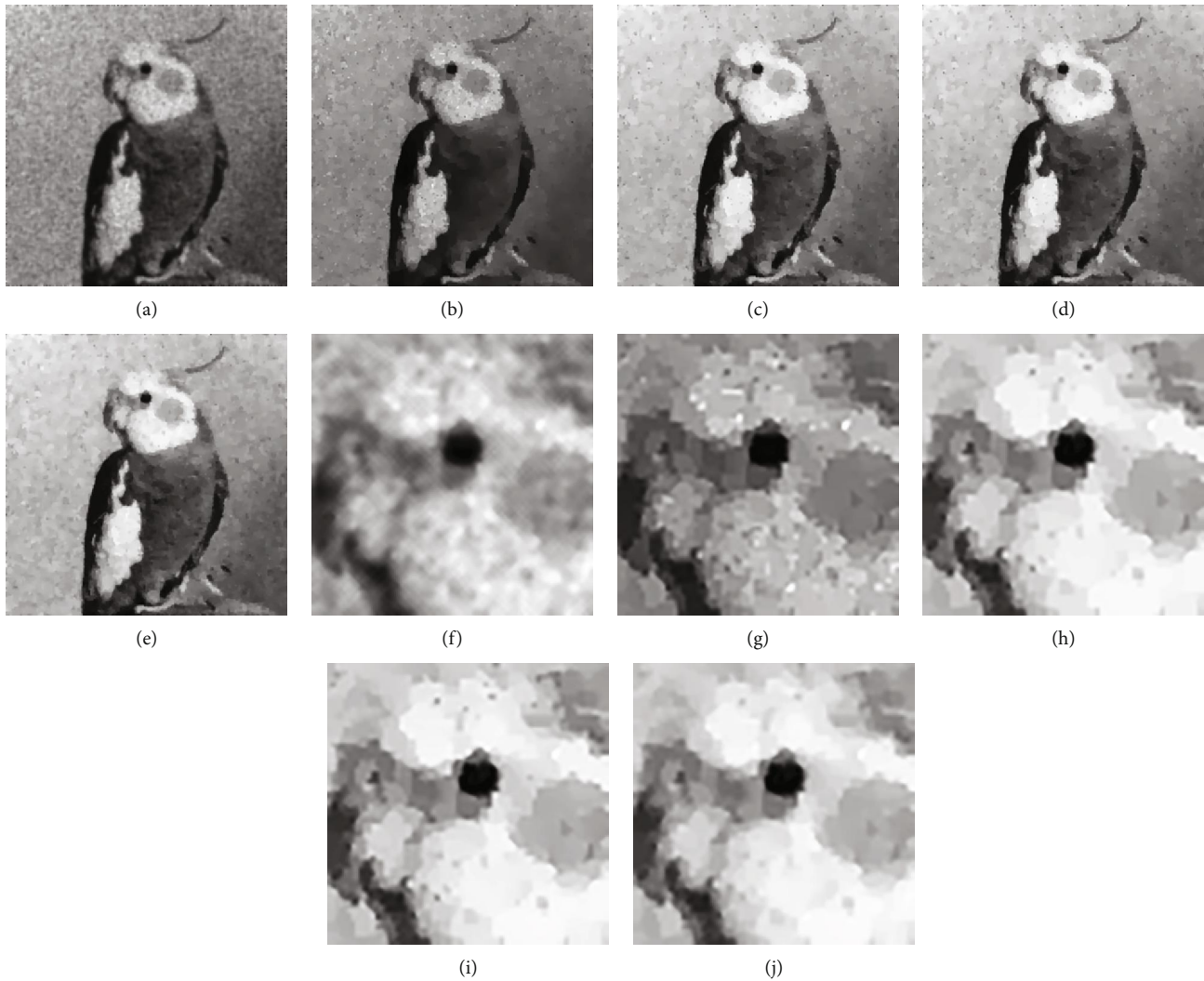


FIGURE 15: Numerical result of the “bird” image with noise standard deviation $\sigma = 4$. (a) Restored image by the ROF model; (b) restored image by the ATV model; (c) restored image by the JIN’s model; (d) restored image by HYPTV model; (e) restored image by LOGTV model; (f), (g), (h), (i), and (j) about the detailed image of (a), (b), (c), (d), and (e), respectively.

are shown in Figure 3 (“house,” “peppers,” “boat,” “pirate,” and “bird”), with two sizes of 512×512 and three sizes of 256×256 . Figure 10 shows some noise images with different standard deviation.

Figures 11–15 display the restoration results for images (“peppers,” “boat,” “house,” “pirate,” and “bird”) through ROF model, ATV model, JIN’s model, HYPTV model, and LOGTV model. The noise versions of “peppers,” “boat” and “house,” and “pirate” and “bird” are obtained by model (2) with standard deviations 2, 3, and 4, respectively. In addition, the detailed images of the restored images are also displayed. Table 2 shows the PSNR and SSIM values for different test images by using the ROF model, ATV model, JIN’s model, HYPTV model, and LOGTV model. From the results of Figures 11–15 and Table 2, the noise standard deviation $\sigma = 2$. We can observe that although the four models can effectively remove the noises while preserving the edges and details, the restored images by HYPTV model and LOGTV model have better visual effect with less staircase effects than by the ROF model,

ATV model, and JIN’s model. The noise standard deviation $\sigma = 3, 4$. The visual effect of restored images by the ROF model and ATV model is particularly poor, but JIN’s model, HYPTV model, and LOGTV model can effectively remove the noises. Finally, Table 2 shows that LOGTV model has higher PSNR and SSIM values than other four models. This means that our proposed LOGTV model is available in reducing the speckle noise in some images.

4.5. Denoising Results of Real Ultrasound Images. In this subsection, we test some real ultrasound images. Figure 16 shows the experimental results of real ultrasound images by applying JIN’s model, HYPTV model, and LOGTV model. Table 3 shows the different iteration for the different test images by using the JIN’s model, HYPTV model, and LOGTV model. We find that LOGTV model is much effective than JIN’s model and HYPTV model in obtaining the satisfactory restored images.

TABLE 2: The PSNR of the restored images by the different model.

Image	σ	ROF (PSNR/SSIM)	ATV (PSNR/SSIM)	JIN's (PSNR/SSIM)	HYPTV (PSNR/SSIM)	LOGTV (PSNR/SSIM)
Peppers	2	27.60/0.7183	28.66/0.8377	29.46/0.8195	29.43/0.8250	29.53/0.8274
Boat	2	27.26/0.8240	27.90/0.8574	28.35/0.8622	28.49/0.8659	28.64/0.8661
House	2	27.45/0.6185	29.06/0.8101	28.77/0.6876	29.55/0.7374	29.74/0.7447
Pirate	2	28.33/0.8975	27.22/0.8530	28.51/0.9042	28.55/0.9047	28.74/0.9060
Bird	2	28.99/0.6907	30.28/0.7849	30.36/0.7774	30.88/0.8137	31.07/0.8173
Peppers	3	25.93/0.6758	26.70/0.7409	27.13/0.7446	27.19/0.7492	27.37/0.7516
Boat	3	25.79/0.7481	26.47/0.7986	26.68/0.8029	26.64/0.7985	26.82/0.7989
House	3	26.08/0.5752	27.33/0.6611	27.24/0.6551	27.62/0.6866	27.82/0.6896
Pirate	3	26.35/0.8392	26.38/0.8332	26.81/0.8578	26.85/0.8580	26.98/0.8583
Bird	3	27.36/0.6451	27.92/0.6930	28.48/0.7307	28.92/0.7686	29.13/0.7701
Peppers	4	24.49/0.6153	25.59/0.7153	25.63/0.6800	25.65/0.6981	25.83/0.7011
Boat	4	24.47/0.6807	24.95/0.7338	25.24/0.7370	25.31/0.7382	25.51/0.7388
House	4	24.63/0.5130	25.30/0.5760	26.18/0.6399	26.19/0.6419	26.43/0.6437
Pirate	4	24.03/0.7652	24.33/0.7205	24.75/0.7787	25.64/0.8118	25.77/0.8159
Bird	4	25.66/0.5679	26.49/0.6584	27.25/0.7263	27.25/0.7270	27.51/0.7282

Best denoising performance are given in bold.

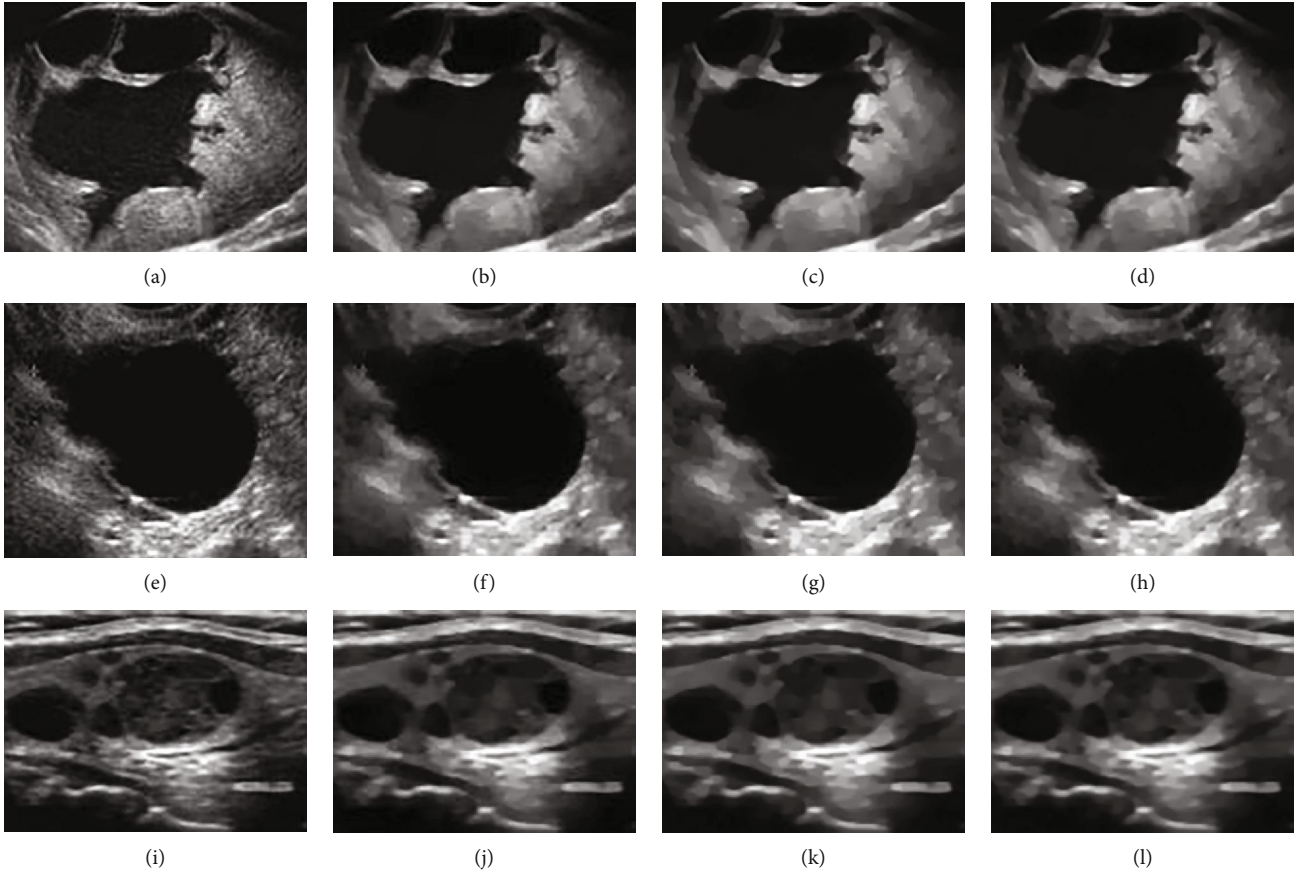


FIGURE 16: Numerical result of the real ultrasound image (the real ultrasound image from [18]). (a, e, i) Noisy image; (b, f, j) restored image by the JIN's model; (c, g, k) restored image by HYPTV model; (d, h, l) restored image by LOGTV model.

5. Concluding Remarks

In this paper, we propose a new speckle noise restoration model based on adaptive TV method. Two new convex func-

tions are introduced as the TV regularization term. By analyzing the diffusion performance of the proposed two models, one can see that the LOGTV model has faster diffusion speed than the HYPTV model. Moreover, we introduced

TABLE 3: Iteration times and time of different models.

Image	JIN's (iter/time)	HYPTV (iter/time)	LOGTV (iter/time)
Ultra 1	81/0.46 s	26/0.31 s	22/0.24 s
Ultra 2	76/0.49 s	22/0.27 s	20/0.23 s
Ultra 3	88/0.45 s	27/0.27 s	24/0.23 s

Best performance are given in bold.

two iterative numerical algorithms to solve the proposed models. The experiment results show the affectivity of our proposed model and the similarity between JIN's model and HYPTV model. In addition, we compared the effect of the ROF model, ATV model, and JIN's model with the LOGTV model, and the experiment results show high efficiency of LOGTV model in image restoration.

Appendix

The Proof of the Inequality of Theorem 6

Proof. For any $u_1, u_2 \in \Omega$, and $t \in [0, 1]$, $(tu_2 + (1-t)u_1)(tu_1 + (1-t)u_2) \geq u_1u_2$.

Here, we know $t(1-t)(u_1^2 + u_2^2) + 2t^2u_1u_2 - 2tu_1u_2 + u_1u_2 \geq u_1u_2$,

so, we have $(u_1^2 + u_2^2) \geq u_1u_2$. Therefore, the proof holds to be true.

Data Availability

The experimental data are obtained by MATLAB R2017a, 2.93 GHz cup, 4 G RAM, and Windows 7.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors typed, read, and approved the final manuscript.

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Research Article

Global Well Posedness for the Thermally Radiative Magnetohydrodynamic Equations in 3D

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In this paper, we study the thermally radiative magnetohydrodynamic equations in 3D, which describe the dynamical behaviors of magnetized fluids that have nonignorable energy and momentum exchange with radiation under the nonlocal thermal equilibrium case. By using exquisite energy estimate, global existence and uniqueness of classical solutions to Cauchy problem in \mathbb{R}^3 or \mathbb{T}^3 are established when initial data is a small perturbation of some given equilibrium. We can further prove that the rates of convergence of solution toward the equilibrium state are algebraic in \mathbb{R}^3 and exponential in \mathbb{T}^3 under some additional conditions on initial data. The proof is based on the Fourier multiplier technique.

1. Introduction

In the study of plasma physics, due to the high temperature and high pressure environment, the motion of charged particles flow is usually regarded as compressible fluids, and their dynamics is very often shaped and controlled by magnetic fields and high temperature radiation effects. Meanwhile, it is known that the radiation energy is carried by photons. When the distribution of photon is almost isotropic, based on the standard hydrodynamics, such dynamics can be described by the following 3D thermally radiative magnetohydrodynamic equations (cf. [1, 2]):

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u) &= 0, \\ \rho(u_t + u \cdot \nabla u) + \nabla P &= \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u + H \cdot \nabla H - \frac{1}{2} \nabla(|H|^2), \\ H_t - \nu \Delta H + u \cdot \nabla H - H \cdot \nabla u + H \operatorname{div} u &= 0, \quad \operatorname{div} H = 0, \\ c_v \rho(\theta_t + u \cdot \nabla \theta) &= \kappa \Delta \theta - P \operatorname{div} u + \lambda(\operatorname{div} u)^2 + 2\mu D \cdot D \\ &\quad + \nu |\nabla \times H|^2 - \theta^4 + n, \\ n_t - \Delta n &= \theta^4 - n, \end{aligned} \quad (1)$$

where $\rho = \rho(x, t) > 0$, $u = u(x, t) \in \mathbb{R}^3$, $\theta = \theta(x, t) > 0$, $H = H(x, t) \in \mathbb{R}^3$, $n = n(x, t) \geq 0$ for $x \in \Omega$, $t \geq 0$ denote the mass density, velocity field of the fluid, mass temperature, magnetic field, and radiation field, respectively, and $P = R\rho\theta$ denotes the material pressure. The spatial domain $\Omega = \mathbb{R}^3$ or \mathbb{T}^3 . The parameter $R > 0$ is the perfect gas constant, $c_v > 0$ is the specific heat at constant volume, and κ is the heat conductivity coefficient; λ and μ are the viscosity coefficients of the flow satisfying $\mu > 0$ and $3\lambda + 2\mu \geq 0$ and $\nu > 0$ is the magnetic diffusion coefficient. Throughout this paper, we assume that $\lambda, \mu, \nu, \kappa$ are all positive constants. $D = D(u)$ is the deformation tensor

$$\begin{aligned} D_{ij} &:= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \\ D \cdot D &:= \sum_{i,j=1}^3 D_{ij}^2. \end{aligned} \quad (2)$$

When the magnetic is ignored (i.e., $H = 0$ in (1)), system (1) can be reduced to the nonequilibrium diffusion approximation model in radiation hydrodynamics. This model describes the energy flow due to radiative process in a

semiquantitative sense and is particularly accurate if the specific intensity of radiation is almost isotropic (cf. [3–5]). There are some mathematical results on this model. For the global existence of smooth solution for one-dimensional case, see [6]; for the global well-posedness and large time behavior of classical solutions for multidimensional case, see [7]. For the inviscid case, in [8], the authors considered a 1D model and showed the existence of shock profiles for inviscid nonequilibrium gases provided that the initial strength is suitably small. For the local existence of smooth solutions for multidimensional system, see [9]. System ((1)) is the compressible MHD equations coupled with the radiative transport equation with nonlocal terms and is very difficult to solve both numerically and analytically. Ducomet and Feireisl consider the thermally radiative MHD system first and show the existence of global weak solutions for the multidimensional case in [10] (also see, for instance, Li and Guo [11]). For the one-dimensional case, this model has been studied by many authors under the various growth constraints on the heat conductivity κ [12–14].

In this paper, we are focused on the asymptotic and global existence of classical solutions of system (1) with the initial data:

$$(\rho, u, H, \theta, n)|_{t=0} = (\rho_0, u_0, H_0, \theta_0, n_0)(x), \quad x \in \Omega. \quad (3)$$

It is easy to check that $(\rho, u, H, \theta, n) \equiv (1, 0, 0, 1, 1)$ is an equilibrium state of (1). Therefore, it is natural to introduce the transforms

$$\rho = 1 + \mathbf{q}, \quad \theta = 1 + \Theta, \quad n = 1 + \eta. \quad (4)$$

Without loss of generality, we assume the positive constants $R = c_v = \kappa = \mu = \lambda = \nu \equiv 1$. Then we can rewrite the system (1) as

$$\mathbf{q}_t + (1 + \mathbf{q})\operatorname{div} u + \nabla \mathbf{q} \cdot u = 0, \quad (5)$$

$$\begin{aligned} u_t + u \cdot \nabla u + \frac{1 + \Theta}{1 + \mathbf{q}} \nabla \rho + \nabla \Theta = & \frac{\Delta u}{1 + \mathbf{q}} + \frac{2 \nabla \operatorname{div} u}{1 + \mathbf{q}} \\ & + \frac{H \cdot \nabla H - (1/2) \nabla(|H|^2)}{1 + \mathbf{q}}, \end{aligned} \quad (6)$$

$$H_t + u \cdot \nabla H - H \cdot \nabla u + H \operatorname{div} u = \Delta H, \quad \operatorname{div} H = 0, \quad (7)$$

$$\begin{aligned} \Theta_t + u \cdot \nabla \Theta = & \frac{\Delta \Theta}{1 + \mathbf{q}} - (1 + \Theta) \operatorname{div} u + \frac{(\operatorname{div} u)^2}{1 + \mathbf{q}} + \frac{2D \cdot D}{1 + \mathbf{q}} \\ & - \frac{(1 + \Theta)^4}{1 + \mathbf{q}} + \frac{1 + \eta}{1 + \mathbf{q}} + \frac{|\nabla \times H|^2}{1 + \mathbf{q}}, \end{aligned} \quad (8)$$

$$\eta_t - \Delta \eta = (1 + \Theta)^4 - (1 + \eta). \quad (9)$$

with initial data

$$\begin{aligned} (\mathbf{q}, u, H, \Theta, \eta)|_{t=0} &= (\mathbf{q}_0, u_0, H_0, \Theta_0, \eta_0)(x) \\ &= (\rho_0 - 1, u_0, H_0, \theta_0 - 1, n_0 - 1)(x). \end{aligned} \quad (10)$$

Then, the main results in this paper read as follows:

Theorem 1. *Let $\Omega = \mathbb{R}^3$. Suppose that $\|(\mathbf{q}_0, u_0, H_0, \Theta_0, \eta_0)\|_{H^4}$ is small enough. Then, the Cauchy problem (5)–(10) admits a unique global classical solution $(\mathbf{q}, u, H, \Theta, \eta)$ satisfying*

$$\begin{aligned} \mathbf{q}, u, H, \Theta, \eta &\in C([0, \infty]; H^4), \\ \sup_{t \geq 0} \|(\mathbf{q}, u, H, \Theta, \eta)\|_{H^4} &\leq C \|(\mathbf{q}_0, u_0, H_0, \Theta_0, \eta_0)\|_{H^4}. \end{aligned} \quad (11)$$

Theorem 2. *Under the conditions of Theorem 1, if we further assume that $\|(\rho_0, u_0, H_0, \Theta_0, \eta_0)\|_{L^1}$ is sufficiently small, then*

$$\|(\mathbf{q}, u, H, \Theta, \eta)\|_{L^2} \leq C(1 + t)^{-(3/4)}, \quad (12)$$

$$\|\nabla(\mathbf{q}, u, H, \Theta, \eta)\|_{H^3} \leq C(1 + t)^{-(5/4)}, \quad (13)$$

for all $t \geq 0$.

Theorem 3. *Let $\Omega = \mathbb{T}^3$. Suppose that $\|(\mathbf{q}_0, u_0, H_0, \Theta_0, \eta_0)\|_{H^4}$ is small enough, and*

$$\begin{aligned} \int_{\mathbb{T}^3} \mathbf{q}_0 dx = 0, \quad \int_{\mathbb{T}^3} (1 + \mathbf{q}_0) u_0 dx = 0, \quad \int_{\mathbb{T}^3} H_0 dx = 0, \\ \int_{\mathbb{T}^3} \left(\frac{1}{2} (1 + \mathbf{q}_0) |u_0|^2 + \frac{1}{2} |H_0|^2 + \mathbf{q}_0 + \Theta_0 + \rho_0 \Theta_0 + \eta_0 \right) dx = 0. \end{aligned} \quad (14)$$

Then, the problem (5)–(10) admits a unique global solution $(\mathbf{q}, u, H, \Theta, \eta)$ satisfying

$$\begin{aligned} \mathbf{q}, u, H, \Theta, \eta &\in C([0, \infty]; H^4), \\ \sup_{t \geq 0} e^{\gamma t} \|(\rho, u, H, \Theta, \eta)\|_{H^4} &\leq C \|(\rho_0, u_0, H_0, \Theta_0, \eta_0)\|_{H^4}, \end{aligned} \quad (15)$$

where $\gamma > 0$ is a constant.

The proof of the global existence of classical solution to (5)–(10) relies on the global a priori estimates together with the local existence of classical solutions and continuum argument. The main difficulty in establishing prior estimates in high-order Sobolev spaces is how to control the linear term in (5)–(9), such as $4\Theta/(1 + \mathbf{q})$, $\eta/(1 + \mathbf{q})$ in (8) and $4\Theta, \eta$ in (1.6). We develop the method in [15] and use the structure of system (5)–(9) itself to construct novel dissipation term

$4\Theta - \eta$ to overcome this difficulty. To prove Theorem 2, we first use a Fourier multiplier technique to establish the $L^p - L^q$ time-decay property of linearized system (47)–(52). Then, the time decay rate can be given by combining the global a priori estimate obtained in Theorem 1 and the above $L^p - L^q$ property and applying the energy estimate technique to the nonlinear problem (5)–(10), whose solutions can be represented by the solution-semigroup operator for the linearized system (47)–(52) by using the Duhamel principle. Here, some nonlinear terms of magnetic field H involved in (6) and (8) may lead to difficulties to gain the desired rate of convergence of solutions. Thus, we will construct some novel functionals such as (79) and (80) and adopt with modification some techniques motivated by [16–18] combined to some vector analysis formula to obtain expected decay rates.

The remainder of this paper is organized as follows. In Section 2, we derive the uniform-in-time a priori estimates and then establish the existence of a global classical solution. In Section 3, we investigate the decay rates of solutions. In Section 4, we adapt our proof to the periodic domain case. Throughout this paper C denotes a positive (generally large) constant and γ a positive (generally small) constant, where both C and γ may take different values in different places. The symbol $A \sim B$ means $CA \leq B \leq (1/C)A$ for a generic constant $C > 0$. For simplicity, we shall use $\|\cdot\|$ to denote norm $L^2(\mathbb{R}^3)$.

2. Global Existence

In what follows, our analysis is based on the Cauchy problem (5)–(10). To obtain the global existence, the most important point is to establish the uniform-in-time a priori estimates.

2.1. A Priori Estimates. Now, we begin to establish the global a priori estimates in the case of the whole space \mathbb{R}^3 under the assumption

$$\sup_{t \geq 0} \|(\mathbf{Q}, u, H, \Theta, \eta)\|_{H^4} \leq \delta, \quad (16)$$

where $0 < \delta < 1$ is a generic constant small enough and $(\rho, u, H, \Theta, \eta)$ is the smooth solution to the Cauchy problem (5)–(10) on $0 \leq t < T$ for $T > 0$. Firstly, we list two important lemmas in Sobolev space.

Lemma 4 (see [16, 19]). *There exist a positive constant C , such that for any $f, g \in H^4(\mathbb{R}^3)$ and any multi-index α with $1 \leq |\alpha| \leq 4$,*

$$\begin{aligned} \|f\|_{L^\infty(\mathbb{R}^3)} &\leq C \|\nabla f\|_{L^2(\mathbb{R}^3)}^{1/2} \|\nabla^2 f\|_{L^2(\mathbb{R}^3)}^{1/2}, \\ \|f\|_{L^p(\mathbb{R}^3)} &\leq C \|\nabla f\|_{H^1(\mathbb{R}^3)}, \quad \text{where } 2 \leq p \leq 6, \\ \|fg\|_{H^3(\mathbb{R}^3)} &\leq C \|f\|_{H^3(\mathbb{R}^3)} \|\nabla g\|_{H^3(\mathbb{R}^3)}, \end{aligned} \quad (17)$$

$$\|\partial^\alpha(fg)\|_{L^2(\mathbb{R}^3)} \leq C \|\nabla f\|_{H^3(\mathbb{R}^3)} \|\nabla g\|_{H^3(\mathbb{R}^3)}.$$

Lemma 5 (Moser-type calculus inequalities) (see [20]). *Let $s \geq 1$ be an integer. Suppose $f \in H^s(\mathbb{R}^3)$, $\nabla f \in L^\infty(\mathbb{R}^3)$ and $g \in H^{s-1}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Then, for all multi-index α with $|\alpha| \leq s$, we have*

$$\begin{aligned} \|\partial^\alpha(fg) - f\partial^\alpha g\| &\leq C_s (\|\nabla f\|_{L^\infty} \|D^{s-1}g\| + \|D^s f\| \|g\|_{L^\infty}), \\ \|D^s f\| &= \sum_{|\alpha|=s} \|\partial^\alpha f\|. \end{aligned} \quad (18)$$

Then, we begin to give the priori estimate of $\mathbf{Q}, u, H, \Theta, \eta$.

Lemma 6. *Suppose that $(\mathbf{Q}, u, H, \Theta, \eta)$ be a smooth solution to (5)–(10). Then, for all $0 \leq t \leq T$ with any fixed $T > 0$, it holds*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|2\rho, 2u, H, 2\Theta, \eta\|^2) &+ \gamma (\|\nabla(u, H, \Theta, \eta)\|^2 \\ &+ \|\operatorname{div} u\|^2 + \|4\Theta - \eta\|^2) \\ &\leq C (\|(\mathbf{Q}, u, H, \Theta, \eta)\|_{H^2}^2 \\ &+ \|(\mathbf{Q}, u, H, \Theta, \eta)\|_{H^2}^2 (\|\nabla(\mathbf{Q}, u, H, \Theta, \eta)\|^2 \\ &+ \|\operatorname{div} u\|^2 + \|4\Theta - \eta\|^2). \end{aligned} \quad (19)$$

Proof. Multiplying (5)–(9) by $4\rho, 4u, H, 4\Theta$, and η and then taking integration and summation, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|2\mathbf{Q}\|^2 + \|2u\|^2 + \|H\|^2 + \|2\Theta\|^2 + \|\eta\|^2) \\ + \int \frac{4|\nabla u|^2}{1+\rho} dx + \int \frac{8(\operatorname{div} u)^2}{1+\mathbf{Q}} dx + \int \frac{4|\nabla \Theta|^2}{1+\mathbf{Q}} dx \\ + \|\nabla H\|^2 + \|\nabla \eta\|^2 + \|4\Theta - \eta\|^2 \\ = - \int 2\rho^2 \operatorname{div} u dx + \int \frac{4(\rho - \Theta)}{1+\mathbf{Q}} \nabla \rho \cdot u dx \\ - \int 4(u \cdot \nabla u) \cdot u dx - \int 4 \left(\nabla \frac{1}{1+\mathbf{Q}} \cdot \nabla u \right) \cdot u dx \\ - \int 8 \nabla \frac{1}{1+\mathbf{Q}} \cdot u \operatorname{div} u dx - \int 4\Theta^2 \operatorname{div} u dx \\ - \int 4\Theta \nabla \Theta \cdot u dx + \int \frac{4\Theta(\operatorname{div} u)^2}{1+\mathbf{Q}} dx + \int \frac{8\Theta D \cdot D}{1+\mathbf{Q}} dx \\ - \int 4\Theta \nabla \frac{1}{1+\mathbf{Q}} \cdot \nabla \Theta dx + \int \frac{4\Theta |\nabla \times H|^2}{1+\mathbf{Q}} dx \\ + \int \frac{4\rho\Theta(4\Theta - \eta)}{1+\mathbf{Q}} dx - \int \frac{(4\Theta - \eta)(6\Theta^2 + 4\Theta^3 + \Theta^4)}{1+\mathbf{Q}} dx \\ + \int \frac{\rho\eta(6\Theta^2 + 4\Theta^3 + \Theta^4)}{1+\mathbf{Q}} dx \\ + \int \left(H \cdot \nabla H - \frac{1}{2} \nabla(|H|^2) \right) \frac{4u}{1+\mathbf{Q}} dx \\ + \int (-u \cdot \nabla H + H \cdot \nabla H - H \operatorname{div} u) \cdot H dx \\ \equiv \sum_{j=1}^{16} I_j. \end{aligned} \quad (20)$$

For I_1 to I_{12} , using Hölder's and Sobolev's inequalities, we have

$$\begin{aligned}
I_1 &\leq C\|\varrho\|_{L^3}\|\operatorname{div} u\|\|\varrho\|_{L^6} \leq C\|\rho\|_{H^1}\|\nabla\varrho\|\|\operatorname{div} u\|, \\
I_2 &\leq C(\|\varrho\|_{L^3} + \|\Theta\|_{L^3})\|\nabla\varrho\|\|u\|_{L^6} \\
&\leq C(\|\varrho\|_{H^1} + \|\Theta\|_{H^1})\|\nabla\varrho\|\|\nabla u\|, \\
I_3 &\leq C\|u\|_{L^3}\|\nabla u\|\|u\|_{L^6} \leq C\|u\|_{H^1}\|\nabla u\|^2, \\
I_4 + I_5 &\leq C\|u\|_{L^6}(\|\nabla u\|_{L^3} + \|\operatorname{div} u\|_{L^3})\left\|\nabla\frac{1}{1+\varrho}\right\| \\
&\leq C\|u\|_{H^2}\|\nabla\varrho\|\|\nabla u\|, \\
I_6 &\leq C\|\Theta\|_{L^3}\|\operatorname{div} u\|\|\Theta\|_{L^6} \leq C\|\Theta\|_{H^1}\|\operatorname{div} u\|\|\nabla\Theta\|, \\
I_7 &\leq C\|u\|_{L^3}\|\nabla\Theta\|\|\Theta\|_{L^6} \leq C\|u\|_{H^1}\|\nabla\Theta\|^2, \\
I_8 &\leq C\|\Theta\|_{L^\infty}\|\operatorname{div} u\|^2 \leq C\|\Theta\|_{H^2}\|\operatorname{div} u\|^2, \\
I_9 &= \sum_{i,j=1}^3 \int \frac{2\Theta(u_{x_j}^i + u_{x_i}^j)^2}{1+\varrho} dx \leq C \sum_{i,j=1}^3 \|\Theta\|_{L^\infty}\|u_{x_j}^i\|\|u_{x_i}^j\| \\
&\leq C\|\Theta\|_{H^2}\|\nabla u\|^2, \\
I_{10} &\leq C\left\|\nabla\frac{1}{1+\varrho}\right\|\|\nabla\Theta\|_{L^3}\|\Theta\|_{L^6} \leq C\|\Theta\|_{H^2}\|\nabla\varrho\|\|\nabla\Theta\|, \\
I_{11} &\leq C\|\Theta\|_{L^\infty}\|\nabla \times H\|^2 \leq C\|\Theta\|_{H^2}\|\nabla H\|^2, \\
I_{12} &\leq C\|\varrho\|_{L^3}\|4\Theta - \eta\|\|\Theta\|_{L^6} \leq C\|\varrho\|_{H^1}\|4\Theta - \eta\|\|\nabla\Theta\|.
\end{aligned} \tag{21}$$

For I_{13} , I_{14} , under the assumption (16), one also has

$$\begin{aligned}
I_{13} &\leq C\|4\Theta - \eta\|(\|\Theta\|_{L^3}\|\Theta\|_{L^6} + \|\Theta^2\|_{L^3}\|\Theta\|_{L^6} + \|\Theta\|_{L^\infty}\|\Theta^2\|_{L^3}\|\Theta\|_{L^6}) \\
&\leq C\|4\Theta - \eta\|(\|\Theta\|_{H^1}\|\nabla\Theta\| + \|\nabla\Theta\|^3 + \|\Theta\|_{H^2}\|\nabla\Theta\|^3) \\
&\leq C(\|\Theta\|_{H^2} + \|\Theta\|_{H^2}^2)(\|4\Theta - \eta\|^2 + \|\nabla\Theta\|^2), \\
I_{14} &\leq C\|\varrho\|_{H^1}\|\eta\|_{H^1}(\|\nabla\Theta\|^2 + \|\nabla\Theta\|^3 + \|\nabla\Theta\|^4) \\
&\leq C(\|\varrho\|_{H^1}^2 + \|\Theta\|_{H^1}^2 + \|\eta\|_{H^1}^2)\|\nabla\Theta\|^2.
\end{aligned} \tag{22}$$

At last, for I_{15} and I_{16} , we have

$$\begin{aligned}
I_{15} &\leq C\|u\|_{L^3}\|\nabla H\|\|H\|_{L^6} + C\|H\|_{L^3}\|\operatorname{div} u\|\|H\|_{L^6} \\
&\leq C\|u\|_{H^1}\|\nabla H\|^2 + C\|H\|_{H^1}\|\nabla H\|\|\nabla u\|, \\
I_{16} &\leq C\|u\|_{L^3}\|\nabla H\|\|H\|_{L^6} + C\|H\|_{L^3}\|\nabla u\|\|H\|_{L^6} \\
&\quad + C\|H\|_{L^3}\|\operatorname{div} u\|\|H\|_{L^6} \\
&\leq C\|u\|_{H^1}\|\nabla H\|^2 + C\|H\|_{H^1}\|\nabla H\|\|\nabla u\|.
\end{aligned} \tag{23}$$

Plugging all the above estimates into (20), we obtain (19).

Lemma 7. Suppose that $(\varrho, u, H, \Theta, \eta)$ be a smooth solution to (5)–(10). Then, for all $0 \leq t \leq T$ with any fixed $T > 0$, it holds

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \sum_{1 \leq |\alpha| \leq 4} (\|2\partial^\alpha \varrho\|^2 + \|2\partial^\alpha u\|^2 + \|\partial^\alpha H\|^2 + \|2\partial^\alpha \Theta\|^2 + \|\partial^\alpha \eta\|^2) \\
&\quad + \gamma \sum_{1 \leq |\alpha| \leq 4} (\|\nabla \partial^\alpha u\|^2 + \|\operatorname{div} \partial^\alpha u\|^2 + \|\nabla \partial^\alpha H\|^2 + \|\nabla \partial^\alpha \Theta\|^2 \\
&\quad + \|\nabla \partial^\alpha \eta\|^2 + \|\partial^\alpha (4\Theta - \eta)\|^2) \\
&\leq C(\|(\varrho, u, H, \Theta, \eta)\|_{H^4} + \|(\varrho, u, H, \Theta, \eta)\|_{H^4}^2) \\
&\quad \cdot (\|\nabla(\varrho, u, H, \Theta, \eta)\|_{H^3}^2 + \|\operatorname{div} u\|_{H^3}^2).
\end{aligned} \tag{24}$$

Proof. Applying ∂^α with $1 \leq |\alpha| \leq 4$ to (5)–(9) and multiplying by $4\partial^\alpha \varrho$, $4\partial^\alpha u$, $\partial^\alpha H$, $4\partial^\alpha \Theta$, and $\partial^\alpha \eta$, respectively, then taking integration and summation, we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} (\|2\partial^\alpha \varrho\|^2 + \|2\partial^\alpha u\|^2 + \|\partial^\alpha H\|^2 + \|2\partial^\alpha \Theta\|^2 + \|\partial^\alpha \eta\|^2) \\
&\quad + \int \frac{4|\nabla \partial^\alpha u|^2}{1+\varrho} dx + \int \frac{8(\operatorname{div} \partial^\alpha u)^2}{1+\varrho} dx + \int \frac{4|\nabla \partial^\alpha \Theta|^2}{1+\varrho} dx \\
&\quad + \|\nabla \partial^\alpha H\|^2 + \|\nabla \partial^\alpha \eta\|^2 + \|\partial^\alpha (4\Theta - \eta)\|^2 \\
&= \int 4[-\partial^\alpha, \varrho \operatorname{div}] u \partial^\alpha \varrho dx + \int 4[-\partial^\alpha, u \cdot \nabla] \varrho \partial^\alpha \rho dx \\
&\quad + \int 2|\partial^\alpha \varrho|^2 \operatorname{div} u dx - \int 4\varrho \partial^\alpha \rho \operatorname{div} \partial^\alpha u dx \\
&\quad + \int 4[-\partial^\alpha, u \cdot \nabla] u \partial^\alpha u dx + \int 4\left[-\partial^\alpha, \frac{1+\Theta}{1+\varrho}\right] \varrho \partial^\alpha u dx \\
&\quad + \int 2|\partial^\alpha u|^2 \operatorname{div} u dx + \int \frac{4(\Theta - \rho)}{1+\varrho} \partial^\alpha \varrho \operatorname{div} \partial^\alpha u dx \\
&\quad + \int 4\partial^\alpha \varrho \nabla \frac{1+\Theta}{1+\varrho} \cdot \partial^\alpha u dx - \int 4\nabla \left(\frac{1}{1+\varrho}\right) \cdot \nabla \partial^\alpha u \cdot \partial^\alpha u dx + \sum_{0 \leq \beta < \alpha} C_{\alpha, \beta} \int 4\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho}\right) \partial^\beta \Delta u \\
&\quad \cdot \partial^\alpha u dx - \int 8\nabla \left(\frac{1}{1+\varrho}\right) \cdot \partial^\alpha u \operatorname{div} \partial^\alpha u dx \\
&\quad + \sum_{0 \leq \beta < \alpha} C_{\alpha, \beta} \int 8\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho}\right) \partial^\beta \nabla \operatorname{div} u \cdot \partial^\alpha u dx \\
&\quad - \int \partial^\alpha (u \cdot \nabla H - H \cdot \nabla u + H \operatorname{div} u) \partial^\alpha H dx \\
&\quad + \int 4\partial^\alpha \left(\frac{1}{1+\varrho} \left(H \cdot \nabla H - \frac{1}{2} \nabla(|H|^2)\right)\right) \partial^\alpha u dx \\
&\quad + \int 4[-\partial^\alpha, u \cdot \nabla] \Theta \partial^\alpha \Theta dx + \int 2|\partial^\alpha \Theta|^2 \operatorname{div} u dx \\
&\quad + \int 4[-\partial^\alpha, \Theta \operatorname{div}] u \partial^\alpha \Theta dx - \int 4\nabla \left(\frac{1}{1+\varrho}\right) \cdot \nabla \partial^\alpha \Theta \partial^\alpha \Theta dx \\
&\quad - \int 4\Theta \operatorname{div} \partial^\alpha u \partial^\alpha \Theta dx \\
&\quad + \sum_{0 \leq \beta < \alpha} C_{\alpha, \beta} \int 4\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho}\right) \partial^\beta \Delta \Theta \partial^\alpha \Theta dx
\end{aligned}$$

$$\begin{aligned}
& + \int \frac{16\rho|\partial^\alpha\Theta|^2}{1+\varrho} dx - \int \frac{4\rho\partial^\alpha\eta\partial^\alpha\Theta}{1+\varrho} dx \\
& + \sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta} \int 4\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right) \partial^\beta ((\operatorname{div} u)^2) \\
& + 2D \cdot D + |\nabla \times H|^2 \partial^\alpha \Theta dx \\
& - \int 4\partial^\alpha \left(\frac{6\Theta^2 + 4\Theta^3 + \Theta^4}{1+\varrho} \right) \partial^\alpha \Theta dx \\
& + \sum_{0\leq\beta<\alpha} C_{\alpha,\beta} \int 4\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right) \partial^\beta \eta \partial^\alpha \Theta dx \\
& - \sum_{0\leq\beta<\alpha} C_{\alpha,\beta} \int 16\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right) \partial^\beta \Theta \partial^\alpha \Theta dx \\
& + \int \partial^\alpha (6\Theta^2 + 4\Theta^3 + \Theta^4) \partial^\alpha \eta dx \equiv \sum_{j=1}^{28} I_j, \quad (25)
\end{aligned}$$

where $[A, B]$ denotes the commutator $AB - BA$ for two operators A and B ; $C_{\alpha,\beta}$ is constant depending only on α and β . We now bound each term on the right-hand side of (25). Utilizing Lemma 5, we get

$$\begin{aligned}
I_1 & \leq C\|\varrho\|_{H^4}\|\nabla\varrho\|_{H^3}\|\operatorname{div} u\|_{H^3}, \\
I_2 & \leq C\|\varrho\|_{H^4}\|\nabla\varrho\|_{H^3}\|\nabla u\|_{H^3}, \\
I_5 & \leq C\|u\|_{H^4}\|\nabla u\|_{H^3}^2, \\
I_6 & \leq C\|u\|_{H^4}(\|\nabla\varrho\|_{H^3}\|\nabla\Theta\|_{H^3} + \|\nabla\varrho\|_{H^3}^2) \\
& \quad + C\|\varrho\|_{H^4}\|u\|_{H^4}\|\nabla\varrho\|_{H^3}\|\nabla\Theta\|_{H^3}, \\
I_{16} & \leq C\|\Theta\|_{H^4}\|\nabla u\|_{H^3}\|\nabla\Theta\|_{H^3}, \\
I_{18} & \leq C\|\Theta\|_{H^4}\|\operatorname{div} u\|_{H^3}\|\nabla\Theta\|_{H^3}.
\end{aligned} \quad (26)$$

For I_{11} , we have

$$I_{11} \leq C\|u\|_{H^4}\|\nabla\varrho\|_{H^3}\|\nabla\partial^\alpha u\| \leq \varepsilon\|\nabla\partial^\alpha u\|^2 + C_\varepsilon\|u\|_{H^4}^2\|\nabla\varrho\|_{H^3}^2, \quad (27)$$

with $\varepsilon > 0$ a small constant, where the first inequality follows that for $\beta < \alpha$,

$$\begin{aligned}
& \int \partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right) \partial^\beta \Delta u \partial^\alpha u \\
& \leq \begin{cases} \|\partial^\alpha \left(\frac{1}{1+\varrho} \right)\| \|\Delta u\|_{L^\infty} \|\partial^\alpha u\| & (|\beta| = 0), \\ \|\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right)\|_{L^3} \|\partial^\beta \Delta u\|_{L^6} \|\partial^\alpha u\| & (|\beta| = 1), \\ \|\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right)\|_{L^\infty} \|\partial^\beta \Delta u\| \|\partial^\alpha u\| & (|\beta| \geq 2), \end{cases} \quad (28)
\end{aligned}$$

and Sobolev's and Young's inequalities were further used.

Similarly, we have

$$\begin{aligned}
I_{13} & \leq \varepsilon\|\nabla\partial^\alpha u\|^2 + C_\varepsilon\|u\|_{H^4}^2\|\nabla\varrho\|_{H^3}^2, \\
I_{21} & \leq \varepsilon\|\nabla\partial^\alpha \Theta\|^2 + C_\varepsilon\|\Theta\|_{H^4}^2\|\nabla\varrho\|_{H^3}^2, \\
I_{24} & \leq \|u\|_{H^4}\|\Theta\|_{H^4}\|\nabla\varrho\|_{H^3}(\|\nabla u\|_{H^3} + \|\operatorname{div} u\|_{H^3} + \|\nabla H\|_{H^3}) \\
& \quad + \varepsilon(\|\nabla\partial^\alpha u\|^2 + \|\operatorname{div} \partial^\alpha u\|^2 + \|\nabla\partial^\alpha H\|^2) \\
& \quad + C_\varepsilon\|\Theta\|_{H^4}^2(\|\nabla u\|_{H^2}^2 + \|\operatorname{div} u\|_{H^2}^2 + \|\nabla H\|_{H^2}^2), \\
I_{26} + I_{27} & \leq C\|\Theta\|_{H^4}\|\nabla\rho\|_{H^3}(\|\nabla\eta\|_{H^2} + \|\nabla\Theta\|_{H^2}).
\end{aligned} \quad (29)$$

For I_{15} , we have

$$\begin{aligned}
I_{15} & = \int \frac{4}{1+\varrho} \partial^\alpha (H \cdot \nabla H) \partial^\alpha u dx \\
& \quad + \sum_{0\leq\beta<\alpha} C_{\alpha,\beta} \int 4\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right) \partial^\beta (H \cdot \nabla H) \partial^\alpha u dx \\
& \quad - \int \frac{2}{1+\varrho} \partial^\alpha \nabla (|H|^2) \partial^\alpha u dx \\
& \quad - \sum_{0\leq\beta<\alpha} C_{\alpha,\beta} \int 2\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right) \partial^\beta \nabla (|H|^2) \partial^\alpha u dx \\
& \equiv \sum_{j=1}^4 I_{15}^j.
\end{aligned} \quad (30)$$

By Lemma 4, we obtain

$$\begin{aligned}
I_{15}^1 & \leq C\|\nabla H\|_{H^3}\|\nabla\partial^\alpha H\|\|\partial^\alpha u\| \leq \varepsilon\|\nabla\partial^\alpha H\|^2 + C_\varepsilon\|u\|_{H^4}^2\|\nabla H\|_{H^3}^2, \\
I_{15}^2 & \leq C\|\varrho\|_{H^4}\|u\|_{H^4}\|\nabla H\|_{H^3}^2,
\end{aligned} \quad (31)$$

since

$$\begin{aligned}
& \int \partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right) \partial^\beta (H \cdot \nabla H) \partial^\alpha u \\
& \leq \begin{cases} \|\partial^\alpha \left(\frac{1}{1+\varrho} \right)\| \|H \cdot \nabla H\|_{L^\infty} \|\partial^\alpha u\| & (|\beta| = 0), \\ \|\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right)\|_{L^3} \|\partial^\beta (H \cdot \nabla H)\|_{L^6} \|\partial^\alpha u\| & (|\beta| = 1), \\ \|\partial^{\alpha-\beta} \left(\frac{1}{1+\varrho} \right)\|_{L^\infty} \|\partial^\beta (H \cdot \nabla H)\| \|\partial^\alpha u\| & (|\beta| \geq 2). \end{cases}
\end{aligned} \quad (32)$$

Similarly, we can deduce that

$$\begin{aligned}
I_{15}^3 & \leq \varepsilon\|\nabla\partial^\alpha H\|^2 + C_\varepsilon\|u\|_{H^4}^2\|\nabla H\|_{H^3}^2, \\
I_{15}^4 & \leq C\|\rho\|_{H^4}\|u\|_{H^4}\|\nabla H\|_{H^3}^2.
\end{aligned} \quad (33)$$

Therefore,

$$I_{15} \leq \varepsilon \|\nabla \partial^\alpha H\|^2 + C(1 + \|\mathbf{Q}\|_{H^4}) \|u\|_{H^4} \|\nabla H\|_{H^3}^2. \quad (34)$$

Using Hölder's, Sobolev's, and Young's inequalities and Lemma 4, we can get following bounds:

$$\begin{aligned} I_3 + I_7 + I_{17} &\leq C \|\operatorname{div} u\|_{L^\infty} (\|\partial^\alpha \mathbf{Q}\|^2 + \|\partial^\alpha u\|^2 + \|\partial^\alpha \Theta\|^2) \\ &\leq C \|u\|_{H^3} (\|\nabla \mathbf{Q}\|_{H^3}^2 + \|\nabla u\|_{H^3}^2 + \|\nabla \Theta\|_{H^3}^2), \\ I_4 &\leq \varepsilon \|\operatorname{div} \partial^\alpha u\|^2 + C_\varepsilon \|\mathbf{Q}\|_{H^4}^2 \|\nabla \rho\|_{H^1}^2, \\ I_8 &\leq \varepsilon \|\operatorname{div} \partial^\alpha u\|^2 + C_\varepsilon \|\mathbf{Q}\|_{H^4}^2 (\|\nabla \mathbf{Q}\|_{H^1}^2 + \|\nabla \Theta\|_{H^1}^2), \\ I_9 &\leq C \|\mathbf{Q}\|_{H^4} \|u\|_{H^4} \|\nabla \mathbf{Q}\|_{H^2} \|\nabla \Theta\|_{H^1} \\ &\quad + C \|\mathbf{Q}\|_{H^4} \|\nabla u\|_{H^3} (\|\nabla \mathbf{Q}\|_{H^2} + \|\nabla u\|_{H^2}), \\ I_{10} &\leq \varepsilon \|\nabla \partial^\alpha u\|^2 + C_\varepsilon \|u\|_{H^4}^2 \|\nabla \mathbf{Q}\|_{H^2}^2, \\ I_{12} &\leq \varepsilon \|\operatorname{div} \partial^\alpha u\|^2 + C_\varepsilon \|u\|_{H^4}^2 \|\nabla \mathbf{Q}\|_{H^2}^2, \\ I_{14} &\leq \varepsilon (\|\nabla \partial^\alpha u\|^2 + \|\operatorname{div} \partial^\alpha u\|^2 + \|\nabla \partial^\alpha H\|^2) \\ &\quad + C_\varepsilon \|H\|_{H^4}^2 (\|\nabla u\|_{H^3}^2 + \|\nabla H\|_{H^3}^2), \\ I_{19} &\leq \varepsilon \|\nabla \partial^\alpha \Theta\|^2 + C_\varepsilon \|\Theta\|_{H^4}^2 \|\nabla \mathbf{Q}\|_{H^2}^2, \\ I_{20} &\leq \varepsilon \|\operatorname{div} \partial^\alpha u\|^2 + C_\varepsilon \|\Theta\|_{H^4}^2 \|\nabla \Theta\|_{H^1}^2, \\ I_{22} + I_{23} &\leq C \|\mathbf{Q}\|_{H^2} \|\nabla \Theta\|_{H^3} (\|\nabla \Theta\|_{H^3} + \|\nabla \eta\|_{H^3}). \end{aligned} \quad (35)$$

For the remaining terms, under the assumption (16), one also has

$$\begin{aligned} I_{25} &\leq C \|\Theta\|_{H^4} \|\nabla \mathbf{Q}\|_{H^3} (\|\nabla \Theta\|_{H^3} + \|\nabla \Theta\|_{H^3}^2 + \|\nabla \Theta\|_{H^3}^2) \\ &\leq C \|\Theta\|_{H^4} \|\nabla \mathbf{Q}\|_{H^3} \|\nabla \Theta\|_{H^3}, \\ I_{28} &\leq C \|\eta\|_{H^4} (\|\Theta\|_{H^3}^2 + \|\Theta\|_{H^3}^3 + \|\Theta\|_{H^3}^4) \\ &\leq C \|\eta\|_{H^4} \|\nabla \Theta\|_{H^3}^2. \end{aligned} \quad (36)$$

Putting all the above estimates into (25) and taking the sum over $1 \leq |\alpha| \leq 4$, then (24) follows, and thus, Lemma 7 is proven.

Next, we will give the dissipation rate of ρ .

Lemma 8. Suppose that $(\mathbf{Q}, u, H, \Theta, \eta)$ be a smooth solution to (5)–(10). Then, for all $0 \leq t \leq T$ with any fixed $T > 0$, it holds

$$\begin{aligned} &\frac{d}{dt} \sum_{|\alpha| \leq 3} \int \nabla \partial^\alpha \mathbf{Q} \cdot \partial^\alpha u dx + \gamma \|\nabla \mathbf{Q}\|_{H^3}^2 \\ &\leq C (\|\nabla u\|_{H^4}^2 + \|\operatorname{div} u\|_{H^3}^2 + \|\nabla \Theta\|_{H^3}^2) \\ &\quad + C (\|(\mathbf{Q}, u, H, \Theta)\|_{H^4} + \|(\mathbf{Q}, u, H, \Theta)\|_{H^4}^2) \|\nabla(\mathbf{Q}, u, H, \Theta)\|_{H^3}^2. \end{aligned} \quad (37)$$

Proof. Taking differentiation $\partial^\alpha (|\alpha| \leq 3)$ to (8) and multiplying by $\nabla \partial^\alpha \mathbf{Q}$, then taking integration, one can get

$$\begin{aligned} \int |\nabla \partial^\alpha \mathbf{Q}|^2 dx &= - \int \nabla \partial^\alpha \mathbf{Q} \cdot \partial^\alpha u_t dx - \int \nabla \partial^\alpha \mathbf{Q} \partial^\alpha (u \cdot \nabla u) dx \\ &\quad - \int \nabla \partial^\alpha \mathbf{Q} \cdot \partial^\alpha \left(\left(\frac{\Theta - \mathbf{Q}}{1 + \mathbf{Q}} \right) \nabla \mathbf{Q} \right) dx \\ &\quad - \int \nabla \partial^\alpha \mathbf{Q} \cdot \nabla \partial^\alpha \Theta dx + \int \nabla \partial^\alpha \mathbf{Q} \cdot \partial^\alpha \left(\frac{\Delta u}{1 + \mathbf{Q}} \right) dx \\ &\quad + \int 2 \nabla \partial^\alpha \mathbf{Q} \cdot \partial^\alpha \left(\frac{\nabla \operatorname{div} u}{1 + \mathbf{Q}} \right) dx \\ &\quad + \int \nabla \partial^\alpha \mathbf{Q} \cdot \partial^\alpha \left(\frac{1}{1 + \mathbf{Q}} \left(H \cdot \nabla H - \frac{1}{2} \nabla(|H|^2) \right) \right) dx \\ &\equiv \sum_{j=1}^7 I_j. \end{aligned} \quad (38)$$

For I_1 , applying (5), we have

$$\begin{aligned} I_1 &= - \frac{d}{dt} \int \nabla \partial^\alpha \mathbf{Q} \cdot \partial^\alpha u dx + \int \partial^\alpha \operatorname{div} u \partial^\alpha ((1 + \mathbf{Q}) \operatorname{div} u + \nabla \mathbf{Q} \cdot u) dx \\ &\leq - \frac{d}{dt} \int \nabla \partial^\alpha \mathbf{Q} \cdot \partial^\alpha u dx + C \|\operatorname{div} u\|_{H^3}^2 + C \|\mathbf{Q}\|_{H^3} \|\nabla u\|_{H^3}^2. \end{aligned} \quad (39)$$

Using Hölder's, Sobolev's, and Young's inequalities, we obtain

$$\begin{aligned} I_2 &\leq C \|\mathbf{Q}\|_{H^4} \|\nabla u\|_{H^3}^2, \\ I_3 &\leq C \|\mathbf{Q}\|_{H^4}^2 \|\nabla \mathbf{Q}\|_{H^3} (\|\nabla \mathbf{Q}\|_{H^3} + \|\nabla \Theta\|_{H^3}), \\ I_4 &\leq \frac{1}{4} \|\nabla \partial^\alpha \mathbf{Q}\|^2 + C \|\nabla \Theta\|_{H^3}^2, \\ I_5 + I_6 &\leq C \|\mathbf{Q}\|_{H^4} \|\nabla \mathbf{Q}\|_{H^3} \|\nabla u\|_{H^3} + C \|\nabla \partial^\alpha \mathbf{Q}\| \|\nabla u\|_{H^4} \\ &\leq \frac{1}{4} \|\nabla \partial^\alpha \mathbf{Q}\|^2 + C \|\mathbf{Q}\|_{H^4} \|\nabla \mathbf{Q}\|_{H^3} \|\nabla u\|_{H^3} + C \|\nabla u\|_{H^4}^2, \\ I_7 &\leq C \|\mathbf{Q}\|_{H^4} \|\nabla H\|_{H^3}^2 + C \|\mathbf{Q}\|_{H^4}^2 \|\nabla H\|_{H^3}^2. \end{aligned} \quad (40)$$

Putting these estimates into (38) and taking the sum over $|\alpha| \leq 3$ gives (37), and Lemma 8 is proven.

2.2. Proof of Global Existence. In this section, we will show there exists a unique global in time solution to the problem (5)–(10). Firstly, combining estimates obtained in Lemmas 6–8, one can finish the proof of uniform-in-time a priori estimates as follows. Define a total temporal energy functional

$\mathcal{E}(t)$ and the corresponding dissipation rate functional $\mathcal{D}(t)$ by

$$\begin{aligned} \mathcal{E}(t) = & \|2\mathbf{Q}\|^2 + \|2u\|^2 + \|H\|^2 + \|2\Theta\|^2 + \|\eta\|^2 \\ & + \sum_{1 \leq |\alpha| \leq 4} (\|2\partial^\alpha \mathbf{Q}\|^2 + \|2\partial^\alpha u\|^2 + \|2\partial^\alpha \Theta\|^2 + \|\partial^\alpha H\|^2 + \|\partial^\alpha \eta\|^2) \\ & + \tau_1 \sum_{|\alpha| \leq 3} \int \nabla \partial^\alpha \mathbf{Q} \cdot \partial^\alpha u dx, \end{aligned} \quad (41)$$

$$\mathcal{D}(t) = \|\nabla \rho\|_{H^3}^2 + \|\nabla(u, H, \Theta, \eta)\|_{H^4}^2 + \|\operatorname{div} u\|_{H^4}^2 + \|4\Theta - \eta\|_{H^4}^2, \quad (42)$$

where $0 < \tau_1 \ll 1$ is a small constant. Under the assumption (16), then

$$\mathcal{E}(t) \sim \|(\mathbf{Q}, u, H, \Theta, \eta)(t)\|_{H^4}^2, \quad (43)$$

holds true uniformly for all $0 \leq t < T$. Furthermore, summing ((19)) and ((24)) and $\tau_1 \times ((37))$ and noticing that τ_1 is sufficiently small, we have

$$\frac{d}{dt} \mathcal{E}(t) + \gamma \mathcal{D}(t) \leq C[\mathcal{E}^{1/2}(t) + \mathcal{E}(t)]\mathcal{D}(t), \quad (44)$$

for all $0 \leq t < T$. With the help of (16), one has $\mathcal{E}^{1/2}(t) + \mathcal{E}(t) \leq C(\delta + \delta^2)$ with $0 < \delta < 1$ being small enough. Then, the time integration of (44) yields

$$\mathcal{E}(t) + \gamma \int_0^t \mathcal{D}(s) ds \leq \mathcal{E}(0), \quad (45)$$

for all $0 \leq t < T$. Besides, (16) can be justified by choosing

$$\mathcal{E}(0) \sim \|(\mathbf{Q}_0, u_0, H_0, \Theta_0, \eta_0)\|_{H^4}^2, \quad (46)$$

which is sufficiently small. For brevity, the proof for local existence of smooth solutions is omitted. Then, the global existence and uniqueness of solutions follow from (45) together with the local existence as well as application of the continuity argument.

3. Convergence Rates

In this section, we consider the convergence rate of solutions obtained in Theorem 1. In order to obtain the desired decay rate estimates in Theorem 2, we firstly consider the linearized system corresponding to (5)–(9):

$$\mathbf{Q}_t + \operatorname{div} u = 0, \quad (47)$$

$$u_t + \nabla \rho + \nabla \Theta - \Delta u - 2\nabla \operatorname{div} u = 0, \quad (48)$$

$$H_t - \Delta H = 0, \quad \operatorname{div} H = 0, \quad (49)$$

$$\Theta_t - \Delta \Theta + \operatorname{div} u + 4\Theta - \eta = 0, \quad (50)$$

$$\eta_t - \Delta \eta + \eta - 4\Theta = 0, \quad (51)$$

with initial data

$$(\mathbf{Q}, u, H, \Theta, \eta)|_{t=0} = (\mathbf{Q}_0, u_0, H_0, \Theta_0, \eta_0)(x), \quad x \in \mathbb{R}^3. \quad (52)$$

Denote by $U(t) = (\mathbf{Q}, u, H, \Theta, \eta)(t)$ to be the solution of the Cauchy problem (47)–(52), then $U(t)$ can be presented as

$$U(t) = \mathbb{A}(t)U_0, \quad (53)$$

where $\mathbb{A}(t)$ is named as the solution operator of (47)–(52) and $U_0 = U|_{t=0}$. Then, we utilize the energy method to Cauchy problem (47)–(52) in the Fourier space to present that there is a time-frequency Lyapunov functional which is equivalent to $|U^\wedge(t, k)|^2$. This estimate can help us to establish the $L^p - L^q$ time decay property of $U(t)$ as follows.

Theorem 9. Let $1 \leq q \leq 2$. For any α, α' with $\alpha' \leq \alpha$ and $m = |\alpha - \alpha'|$,

$$\begin{aligned} \|\partial^\alpha \mathbb{A}(t)U_0\|_{L^2} & \leq C(1+t)^{-(3/2)((1/q)-(1/2))-(m/2)} \\ & \cdot \left(\|\partial^{\alpha'} U_0\|_{L^q} + \|\partial^\alpha U_0\|_{L^2} \right), \end{aligned} \quad (54)$$

hold for all $t \geq 0$.

Proof. By taking Fourier transforming in x for (47)–(51), one has

$$\widehat{\mathbf{Q}}_t + ik \cdot \widehat{u} = 0, \quad (55)$$

$$\widehat{u}_t + ik\widehat{\rho} + ik\widehat{\Theta} + |k|^2\widehat{u} + 2k(k \cdot \widehat{u}) = 0, \quad (56)$$

$$\widehat{H}_t + |k|^2\widehat{H} = 0, \quad (57)$$

$$\widehat{\Theta}_t + |k|^2\widehat{\Theta} + ik \cdot \widehat{u} + 4\widehat{\Theta} - \widehat{\eta} = 0, \quad (58)$$

$$\widehat{\eta}_t + |k|^2\widehat{\eta} + \widehat{\eta} - 4\widehat{\Theta} = 0, \quad (59)$$

where $k \in \mathbb{R}^3$, $i = \sqrt{-1} \in \mathbb{C}$ is the imaginary unit.

Multiplying (55)–(59) by $4\widehat{\rho}$, $4\widehat{u}$, \widehat{H} , $4\widehat{\Theta}$, $\widehat{\eta}$, respectively, its real part gives

$$\begin{aligned} \partial_t \left| \left(\sqrt{2}\widehat{\mathbf{Q}}^\wedge, \sqrt{2}\widehat{u}^\wedge, \widehat{H}^\wedge, \sqrt{2}\widehat{\Theta}^\wedge, \widehat{\eta}^\wedge \right) \right|^2 & + 4|k|^2|\widehat{u}^\wedge|^2 + |k|^2|\widehat{H}^\wedge|^2 \\ & + 8|k \cdot \widehat{u}^\wedge|^2 + 4|k|^2|\widehat{\Theta}^\wedge|^2 + |k|^2|\widehat{\eta}^\wedge|^2 + |4\widehat{\Theta}^\wedge - \widehat{\eta}^\wedge|^2 = 0. \end{aligned} \quad (60)$$

Multiplying (56) by $ik\widehat{\rho}$, utilizing integration by parts in t , and replacing $\partial_t \widehat{\rho}$ by (55), one has

$$\partial_t (\widehat{u} | ik\widehat{\mathbf{Q}}) + |k|^2|\widehat{\mathbf{Q}}^\wedge|^2 = |k \cdot \widehat{u}^\wedge|^2 + 3|k|^2 ik \cdot \widehat{u}^\wedge \widehat{\mathbf{Q}} - |k|^2 \widehat{\Theta}^\wedge \widehat{\mathbf{Q}}, \quad (61)$$

here $(\cdot | \cdot)$ means the complex inner product. For the real part of (61) and with the help of Cauchy-Schwarz inequality, one has

$$\begin{aligned} \partial_t \operatorname{Re}(\hat{u} | ik\hat{Q}) + |k|^2 |Q\Lambda|^2 &\leq |k \cdot u\Lambda|^2 + \varepsilon |k|^2 |Q\Lambda|^2 \\ &\quad + C_\varepsilon |k|^2 |k \cdot u\Lambda|^2 + \varepsilon |k|^2 |Q\Lambda|^2 \\ &\quad + C_\varepsilon |k|^2 |\Theta\Lambda|^2, \end{aligned} \quad (62)$$

with $\varepsilon > 0$ being a small constant. Multiplying it by $1/(1 + |k|^2)$, we conclude that there exists $\gamma > 0$ such that

$$\partial_t \frac{\operatorname{Re}(\hat{u} | ik\hat{Q})}{1 + |k|^2} + \frac{\gamma |k|^2 |Q\Lambda|^2}{1 + |k|^2} \leq C |u\Lambda|^2 + \frac{C |k|^2 |\Theta\Lambda|^2}{1 + |k|^2}. \quad (63)$$

Now, we define the time-frequency Lyapunov functional as

$$\begin{aligned} \mathcal{E}(\hat{U}(t, k)) &= \left| \left(\sqrt{2}Q\Lambda, \sqrt{2}u\Lambda, H\Lambda, \sqrt{2}\Theta\Lambda, \eta\Lambda \right) \right|^2 \\ &\quad + \tau_2 \frac{\operatorname{Re}(\hat{u} | ik\hat{Q})}{1 + |k|^2}, \end{aligned} \quad (64)$$

where $0 < \tau_2 \ll 1$ is sufficiently small. It also holds that $\mathcal{E}(\hat{U}) \sim |U\Lambda|^2$. Moreover, by suitably choosing constants τ_1 , the sum of equations (19), (24), $\tau_1 \times (37)$ gives the linear combination (60) + $\tau_1 \times (63)$ which gives (60)

$$\partial_t \mathcal{E}(\hat{U}(t, k)) + \frac{\gamma |k|^2}{1 + |k|^2} \mathcal{E}(\hat{U}(t, k)) \leq 0. \quad (65)$$

As in [15, 21], the desired time decay estimates (54) and directly follows from the above estimate, and the detailed proof is omitted for brevity.

Now, we prove the rate of convergence (12). We quote a technical lemma in [19] for later proofs.

Lemma 10. *Given any $0 < \beta_1 \neq 1$ and $\beta_2 > 1$,*

$$\int_0^t (1+t-s)^{-\beta_1} (1+s)^{-\beta_2} ds \leq C(1+t)^{-\min\{\beta_1, \beta_2\}} \quad (66)$$

for all $t \geq 0$.

By the Duhamel principle, the solution of nonlinear Cauchy problem (5)–(10) can be formally written as

$$U(t) = \mathbb{A}(t)U_0 + \int_0^t \mathbb{A}((t-s)(Y_1, Y_2, Y_3, Y_4, Y_5))ds, \quad (67)$$

with

$$Y_1 = -Q \operatorname{div} u - \nabla Q \cdot u,$$

$$\begin{aligned} Y_2 &= -u \cdot \nabla u - \frac{\Theta - Q}{1+Q} \nabla Q - \frac{Q}{1+Q} \Delta u - \frac{2Q}{1+Q} \nabla \operatorname{div} u \\ &\quad + \frac{1}{1+Q} \left(H \cdot \nabla H - \frac{1}{2} \nabla(|H|^2) \right), \end{aligned}$$

$$Y_3 = u \cdot \nabla H - H \cdot \nabla u + H \operatorname{div} u,$$

$$\begin{aligned} Y_4 &= -u \cdot \nabla \Theta - \frac{Q}{1+Q} \Delta \Theta - \Theta \operatorname{div} u + \frac{(\operatorname{div} u)^2}{1+Q} + \frac{2D \cdot D}{1+Q} \\ &\quad - \frac{Q\eta}{1+Q} - \frac{4\rho\Theta}{1+Q} - \frac{6\Theta^2 + 4\Theta^3 + \Theta^4}{1+Q} + \frac{|\nabla \times H|^2}{1+Q}, \end{aligned}$$

$$Y_5 = 6\Theta^2 + 4\Theta^3 + \Theta^4.$$

(68)

By the definitions of $\mathcal{E}(t)$ and $\mathcal{D}(t)$ in (41) and (42), respectively, we have

$$\mathcal{E}(t) \leq C(\mathcal{D}(t) + \|U\|^2). \quad (69)$$

From (44), we have

$$\frac{d}{dt} \mathcal{E}(t) + \frac{\gamma}{C} (\mathcal{E}(t) - \|U\|^2) \leq \frac{d}{dt} \mathcal{E}(t) + \gamma \mathcal{D}(t) \leq 0, \quad (70)$$

which implies

$$\frac{d}{dt} \mathcal{E}(t) + \gamma \mathcal{E}(t) \leq C\|U\|^2. \quad (71)$$

Gronwall's inequality gives

$$\mathcal{E}(t) \leq e^{-\gamma t} \mathcal{E}(0) + C \int_0^t e^{-\lambda(t-s)} \|U\|^2 ds. \quad (72)$$

Next, we give estimate of $\|U(t)\|$. Firstly, we further rewrite (67) as

$$\begin{aligned} U(t) &= \mathbb{A}(t)U_0 + \int_0^t \mathbb{A}(t-s)(Y_1, Y_2, Y_3, 0, 0)ds \\ &\quad + \int_0^t \mathbb{A}(t-s)(0, 0, 0, Y_4, Y_5)ds \\ &\equiv \sum_{i=1}^3 J_i(t). \end{aligned} \quad (73)$$

Define

$$\mathcal{E}_\infty(t) = \sup_{0 \leq s \leq t} (1+s)^{3/2} \mathcal{E}(s). \quad (74)$$

One has that from (54) and Lemma 10,

$$\begin{aligned}\|J_1(t)\| &\leq C(1+t)^{-(3/4)}\|U_0\|_{L^1\cap L^2}, \\ \|J_2(t)\| &\leq C\int_0^t (1+t-s)^{-(3/4)}\|(Y_1, Y_2, Y_3)\|_{L^1\cap L^2}ds \\ &\leq C\int_0^t (1+t-s)^{-(3/4)}\mathcal{E}(s)ds \\ &\leq C\int_0^t (1+t-s)^{-(3/4)}(1+s)^{-(3/2)}ds\mathcal{E}_\infty(t) \\ &\leq C(1+t)^{-(3/4)}\mathcal{E}_\infty(t),\end{aligned}\quad (75)$$

$$\begin{aligned}\|J_3(t)\| &\leq C\int_0^t (1+t-s)^{-(3/4)}\|(Y_4, Y_5)\|_{L^1\cap L^2}dsds \\ &\leq C\int_0^t (1+t-s)^{-(3/4)}(\mathcal{E}(s) + \mathcal{E}^2(s))ds \\ &\leq C(1+t)^{-(3/4)}(\mathcal{E}_\infty(t) + \mathcal{E}_\infty^2(t)).\end{aligned}$$

Therefore, it follows that

$$\|U\|^2 \leq C(1+t)^{-(3/2)}\{\|U_0\|_{L^1\cap L^2}^2 + \mathcal{E}_\infty^2(t) + \mathcal{E}_\infty^4(t)\}. \quad (76)$$

Substituting (76) into (72), we get

$$\mathcal{E}_\infty(t) \leq C\{\|U_0\|_{H^4}^2\bigcap_{L^1} + \mathcal{E}_\infty^2(t) + \mathcal{E}_\infty^4(t)\}, \quad (77)$$

which implies that $\mathcal{E}_\infty(t) \leq C\|U_0\|_{H^4}^2\bigcap_{L^1}$ for all $t \geq 0$, provided that $\|U_0\|_{H^4}^2\bigcap_{L^1}$ is sufficiently small. Thus,

$$\mathcal{E}(t) \leq C(1+t)^{-(3/2)}\|U_0\|_{H^4}^2\bigcap_{L^1}. \quad (78)$$

This can deduce (12).

We continue to prove the rate of convergence (13). Firstly, we define the new energy functional and dissipation rate functional by

$$\begin{aligned}\mathcal{M}(t) &= \sum_{1 \leq |\alpha| \leq 4} (\|2\partial^\alpha \mathbf{Q}\|^2 + \|2\partial^\alpha u\|^2 + \|\partial^\alpha H\|^2 + \|2\partial^\alpha \Theta\|^2 + \|\partial^\alpha \eta\|^2) \\ &\quad + \tau_1 \sum_{|\alpha| \leq 3} \int \nabla \partial^\alpha \mathbf{Q} \cdot \partial^\alpha u dx,\end{aligned}\quad (79)$$

$$\begin{aligned}\mathcal{N}(t) &= \sum_{1 \leq |\alpha| \leq 3} (\|\partial^\alpha \nabla \mathbf{Q}\|^2 + \|\partial^\alpha \nabla (4\Theta - \eta)\|^2) \\ &\quad + \sum_{1 \leq |\alpha| \leq 4} (\|\partial^\alpha \nabla(u, H, \Theta, \eta)\|^2 + \|\partial^\alpha \operatorname{div} u\|^2).\end{aligned}\quad (80)$$

By using Lemma 4 and similar arguments to those in the proof of Lemmas 6–8, we obtained

$$\frac{d}{dt}\mathcal{M}(t) + \gamma\mathcal{N}(t) \leq C(\mathcal{M}^{1/2}(t) + \mathcal{M}(t))\mathcal{N}(t). \quad (81)$$

Adding the term $\|\nabla U\|^2$ to both sides of (81) gives

$$\frac{d}{dt}\mathcal{M}(t) + \gamma\mathcal{M}(t) \leq C\|\nabla U\|^2, \quad (82)$$

if $\mathcal{M}(t)$ is small enough. Being similar to the proof of $\|U\|$ and defining $\mathcal{M}_\infty(t) = \sup_{0 \leq s \leq t} (1+s)^{5/2}\mathcal{H}(s)$ and by (73), we can deduce that

$$\|\nabla U\|^2 \leq C(1+t)^{-(5/2)}\{\|U_0\|_{L^1}^2 + \|\nabla U_0\|^2 + \mathcal{M}_\infty^2(t) + \mathcal{M}_\infty^4(t)\}. \quad (83)$$

From (81), (83), and Gronwall's inequality, we have

$$\begin{aligned}\mathcal{M}(t) &\leq e^{-\gamma t}\mathcal{M}(0) + C(1+t)^{-(5/2)}\{\|\nabla U_0\|^2 + \|U_0\|_{L^1}^2 + \mathcal{M}_\infty^2(t) \\ &\quad + \mathcal{M}_\infty^4(t)\},\end{aligned}\quad (84)$$

and hence

$$\mathcal{M}_\infty(t) \leq C\{\|\nabla U_0\|_{H^3}^2 + \|U_0\|_{L^1}^2 + \mathcal{M}_\infty^2(t) + \mathcal{M}_\infty^4(t)\}. \quad (85)$$

Thus, since $\|\nabla U_0\|_{H^3}^2 + \|U_0\|_{L^1}^2$ can be small enough, this implies

$$\mathcal{M}_\infty(t) \leq C\left(\|\nabla U_0\|_{H^3}^2 + \|U_0\|_{L^1}^2\right), \quad (86)$$

for all $t \geq 0$, that is,

$$\mathcal{M}(t) \leq C(1+t)^{-(5/2)}\left(\|\nabla U_0\|_{H^3}^2 + \|U_0\|_{L^1}^2\right), \quad (87)$$

which means

$$\|\nabla(\rho, u, H, \Theta, \eta)\|_{H^3} \leq C(1+t)^{-(5/4)}, \quad (88)$$

for all $t \geq 0$, this completes the proof of Theorem 2.

4. The Periodic Case

In this section, we deal with the spatial domain $\Omega = \mathbb{T}^3$. For smooth solution of the system (1), it is not hard to get the following conservation laws in the case of torus,

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{T}^3} \rho dx &= 0, \\ \frac{d}{dt} \int_{\mathbb{T}^3} \rho u dx &= 0, \quad \frac{d}{dt} \int_{\mathbb{T}^3} H dx = 0, \\ \frac{d}{dt} \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho |u|^2 + \frac{1}{2} |H|^2 + \rho \theta + n \right) dx &= 0,\end{aligned}\quad (89)$$

and by the assumption (14), it follows that

$$\begin{aligned} \int_{\mathbb{T}^3} \rho dx &= 0, \int_{\mathbb{T}^3} (1 + \rho) u dx = 0, \int_{\mathbb{T}^3} H dx = 0, \\ \int_{\mathbb{T}^3} \left(\frac{1}{2} (1 + \rho) |u|^2 + \frac{1}{2} |H|^2 + \rho + \Theta + \rho \Theta + \eta \right) dx &= 0, \end{aligned} \quad (90)$$

for all $t \geq 0$.

Proof of Theorem 11. We only give the proof of the global a priori estimates. Firstly, let the temporal energy functional $\mathcal{E}(t)$ and the corresponding dissipation rate functional $\mathcal{D}(t)$ be defined in the same way as in (41) and (42), respectively, for the case of the whole space $\Omega = \mathbb{R}^3$. Similarly, we conclude that

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) + \gamma \mathcal{D}(t) &\leq C (\|(\rho, u, H, \Theta, \eta)\|_{H^4} + \|(\rho, u, H, \Theta, \eta)\|_{H^4}^2) \\ &\quad \cdot (\|\nabla(\rho, u, H, \Theta, \eta)\|_{H^3}^2 + \|\operatorname{div} u\|_{H^3}^2). \end{aligned} \quad (91)$$

Thanks to Poincaré's inequality and the conservation laws (90), we have

$$\|\rho\| \leq C \|\nabla \rho\|, \quad \|H\| \leq C \|\nabla H\|, \quad (92)$$

$$\begin{aligned} \|u\| &\leq \|u + \rho u\| + \|\rho u\| \leq C \|\nabla u\| + C \|\nabla(\rho u)\| + \|\rho\| \|u\|_{L^\infty} \\ &\leq C \|\nabla u\| + C \|u\|_{H^2} \|\nabla \rho\| + C \|\rho\|_{H^2} \|\nabla u\|, \end{aligned} \quad (93)$$

$$\begin{aligned} \|\Theta + \eta\| &= \left\| \frac{1}{2} (1 + \rho) |u|^2 + \frac{1}{2} |H|^2 + \rho + \Theta + \rho \Theta + \eta \right\| \\ &\quad + \left\| \frac{1}{2} (1 + \rho) |u|^2 + \frac{1}{2} |H|^2 + \rho + \rho \Theta \right\| \\ &\leq C \|u\|_{H^2} \|\nabla u\| + C \|\rho\|_{H^2} \|u\|_{H^2} \|\nabla u\| + C \|u\|_{H^2}^2 \|\nabla \rho\| \\ &\quad + C \|H\|_{H^2} \|\nabla H\| + C \|\Theta\|_{H^2} \|\nabla \rho\| + C \|\rho\|_{H^2} \|\nabla \Theta\| \\ &\quad + C (\|\nabla \rho\| + \|\nabla \Theta\| + \|\nabla \eta\|). \end{aligned} \quad (94)$$

Then, define

$$\mathcal{D}_{\mathbb{T}}(t) = \mathcal{D}(t) + \tau_3 (\|\rho\|^2 + \|u\|^2 + \|H\|^2 + \|\Theta + \eta\|^2), \quad (95)$$

where $0 < \tau_3 \ll 1$ is sufficiently small. Notice

$$\mathcal{D}_{\mathbb{T}}(t) \sim \|\rho\|_{H^4}^2 + \|(u, H, \Theta, \eta)\|_{H^5}^2, \quad (96)$$

uniformly for all $t \geq 0$. Combining (91)–(94) together, we have

$$\frac{d}{dt} \mathcal{E}(t) + \gamma \mathcal{D}_{\mathbb{T}}(t) \leq C [\mathcal{E}^{1/2}(t) + \mathcal{E}(t) + \mathcal{E}^2(t)] \mathcal{D}_{\mathbb{T}}(t). \quad (97)$$

Using the fact that $\mathcal{E}(t)$ is small enough and uniform in time, and $\mathcal{E}(t) \leq C \mathcal{D}_{\mathbb{T}}(t)$, we then obtain

$$\frac{d}{dt} \mathcal{E}(t) + \gamma \mathcal{E}(t) \leq 0, \quad (98)$$

for all $t \geq 0$. Applying Gronwall's inequality to (98), one has

$$\mathcal{E}(t) \leq e^{-\gamma t} \mathcal{E}(0). \quad (99)$$

This gives the desired exponential decay of $\mathcal{E}(t) \sim \|(\rho, u, H, \Theta, \eta)\|_{H^4}^2$, and hence completes the proof of Theorem 11.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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