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Geometry of Warped Product Manifolds: Theory and Applications 2022

Lead Guest Editor: Meraj Ali Khan
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Contents

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Research Article

Position Vectors of the Natural Mate and Conjugate of a Space Curve

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The concept of the natural mate and the conjugate curves associated to a smooth curve in Euclidian 3-space were introduced initially by Dashmukh and others. In this paper, we give some extra results that add more properties of the natural mate and the conjugate curves associated with a smooth space curve in $\mathbb{E}^3$. The position vectors of the natural mate and the conjugate of a given smooth curve are investigated. Also, using the position vector of the natural mate, the necessary and sufficient condition for a smooth given curve to be a Bertrand curve is introduced. Moreover, a new characterization of a general helix is introduced.

1. Introduction

The differential geometry of curves and surfaces is an ancient topic in differential geometry, but it is still an active area of research. This is because of its applications in several fields such as computer graphics, computer vision, medical imaging, physics, and aerospace. A helix plays a crucial role in many applications in engineering and also, in DNA structures. In fact, a DNA molecule can be described by double helix. Also, it has been observed that in a molecular model of the DNA there are two side-by-side in opposing direction helices linked by hydrogen bonds (cf. [1]). The rectifying curves are used to analyze joint kinematics (cf. [2, 3]). The Salkowski curves are useful in constructing closed curves with constant curvature and continuous torsion such as knotted curves (cf. [4]). The Salkowski curves are examples of slant helice with constant curvatures.

In differential geometry, curves and their Frenet frames play central roles for creating special surfaces (c.f [5–10]). The Frenet frame associated with a regular curve in $\mathbb{E}^3$, which is a moving frame along the curve, forms an orthonormal basis for the Euclidean space $\mathbb{E}^3$ at each point of the given curve. This allows geometers to analyze a curve and to study the position vector of the given curve and other curves. The terminologies of natural mate and conjugate associated with a smooth curve were introduced and studied in [11]. Mainly, some relationships between a given curve and its natural mates were investigated in [11] as well as the necessary and sufficient conditions for the natural mate associated with a given Frenet curve to be a spherical curve, a helix, or a curve with a constant curvature. The most natural geometric object in differential geometry of curves in Euclidian 3-space is a position vector. The position vector is very important, owing to its applications in mathematics, engineering, physics, and other natural sciences.

In this paper, we investigate the position vectors of the natural mate and the conjugate of a given space curve using the Frenet frame of the given curve as a basis for $\mathbb{E}^3$. The position vectors of the natural mate and conjugate curves will be useful for studying the surfaces generated by these curves such as ruled and translation surfaces. In Section 2 of this paper, we review some basic concepts of space curves which will be used in the rest of this study. In Section 3, we study the position vectors of natural mate and conjugate of a unit speed curve with nonvanishing curvature and torsion. Using the position vector of the natural mate, we give the necessary and sufficient condition for a given curve with nonvanishing curvature and torsion to be a Bertrand curve. Also, a new characterization for a general helix is proven.

2. Preliminaries

In this section, we review some basic concepts of the differential geometry of curves in Euclidean 3-space, and for more detail, we refer the reader to [2, 3, 11–16]. First, we start with the definition of a smooth space curve. A parametrized curve α in $\mathbb{E}^3$ is a map $\alpha : I \longrightarrow \mathbb{E}^3$, where I is a real interval, given by $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ such that $\alpha_1(t), \alpha_2(t)$, and $\alpha_3(t)$ are smooth functions on I. α is a regular curve if $\alpha'(t) \neq 0$ for all $t \in I$. The unit tangent vector, the unit principal normal vector, and the unit binormal vector of a regular curve α are defined by $T = \alpha'/\|\alpha'\|$, $N = (\alpha' \wedge \alpha''/\|\alpha' \wedge \alpha''\|) \wedge (\alpha'/\|\alpha'\|)$, and $B = \alpha' \wedge \alpha''/\|\alpha' \wedge \alpha''\|$, respectively. At any point of α, there are three planes spanned by the vectors N, B, T, B, and T, N, these planes are the normal plane, the rectifying plane, and the osculating plane, respectively. The curvature and the torsion of a regular curve α are given by $\kappa = \|\alpha' \wedge \alpha''\|/\|\alpha'\|^3$ and $\tau = \det(\alpha', \alpha'', \alpha''')/\|\alpha' \wedge \alpha''\|^2$, respectively, and α is called a Frenet curve if $\kappa > 0$ and $\tau \neq 0$. The Serret Frenet apparatus associated to α is given by $\{\kappa, \tau, T, N, B\}$.

If $\alpha'(t) = 1$ for all $t \in I$, then α is called a unit speed curve and the Frenet- Serret equations are given by

$$\begin{cases} T' = \kappa N, \\ N' = -\kappa T + \tau B, \\ B' = -\tau N. \end{cases} \quad (1)$$

In the rest of this section, we state definitions of some special curves.

Definition 1. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a smooth space curve. Then, α is called a helix if its tangent makes a fixed angle with a fixed direction.

Definition 2. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a smooth space curve. Then, α is called a slant helix if its principal normal makes a fixed angle with a fixed direction.

Definition 3. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a smooth space curve. Then, α is called a rectifying curve if it lies in the rectifying plane at each point.

It has been obtained by Chen in [3] that the distance squared function of a rectifying curve is a quadratic polynomial in its arc length.

Definition 4. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a smooth space curve. Then, α is called a spherical curve if it lies in a sphere.

For a spherical curve, it is obvious to obtain that its distance from the center of the sphere, which the curve lies on, is equal to the radius of the sphere. This will play a role in the proof of Theorem 10.

Definition 5. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a smooth space curve. Then, α is called a Salkowski curve if it has constant curvature and nonconstant torsion.

Definition 6. Given a unit speed curve α with nonvanishing curvature and torsion. The natural mate of α is defined by $\beta = \int (N) ds$. If α has negative torsion, then its conjugate is given by $\bar{\alpha} = \int (B) ds$.

3. Natural and Conjugate Mates Associated with a Smooth Space Curve

In this section, we give the position vectors of natural and conjugate curves. Using position vectors of the mentioned curves, we give more brand results which carry interesting relationship between a given smooth curve and its associated natural and conjugate curves.

Theorem 7. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed Frenet curve with Serret-Frenet apparatus $\{\kappa, \tau, T, N, B\}$. The natural mate β of α is given by*

$$\beta = \left(\int (\kappa h) ds \right) T + hN - \left(\int (\tau h) ds \right) B, \quad (2)$$

where $h = dd'$, d is the distance function of β, and d' is the derivative of d with respect to s.

Proof. Since the unit tangent of β is given by $T_\beta = N$, we have

$$\beta = \int (T_\beta) ds = \int (N) ds = gT + hN + lB. \quad (3)$$

Now, differentiating equation (3), we get

$$N = \left(g' - \kappa h \right) T + \left(h' + \kappa g - \tau l \right) N + \left(l' + \tau h \right) B. \quad (4)$$

Thus, from equation (4), we have the following equations:

$$\begin{cases} g' - \kappa h = 0, \\ h' + \kappa g - \tau l = 1, \\ l' + \tau h = 0. \end{cases} \quad (5)$$

Therefore, we have

$$\begin{cases} g = \int (\kappa h) ds, \\ l = - \int (\tau h) ds. \end{cases} \quad (6)$$

Our task now is to find h. The distance squared function, d^2, of β is given by

$$d^2 = g^2 + h^2 + l^2. \tag{7}$$

Now, differentiating equation (7), we get

$$dd' = gg' + hh' + ll'. \tag{8}$$

Hence, using equations (5) and (6) in equation (8), we obtain $h = dd'$ which completes the proof. □

Now, as an application of Theorem 7, we give a criterion for a Bertrand curve with a neat proof. First, we state the following definition and a well-known result regarding the Bertrand curve.

Definition 8. A curve $\gamma : I \longrightarrow \mathbb{E}^3$ is called a Bertrand curve if there is another curve $\bar{\gamma}$, different from γ, and a bijection η between γ and $\bar{\gamma}$ such that γ and $\bar{\gamma}$ have the same principal normal at each pair of corresponding points under η.

The following theorem is a well-know result, and it can be found in many books of elementary differential geometry of curves and surfaces.

Theorem 9. *A curve $\gamma : I \longrightarrow \mathbb{E}^3$ with $\kappa \neq 0$ and $\tau \neq 0$ is called a Bertrand curve if and only if it satisfies the condition*

$$a\kappa + b\tau = 1, \tag{9}$$

where a and b are constants.

Now, we give a criterion for a Bertrand curve in term of natural mate.

Theorem 10. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with nonzero curvature and nonzero torsion and Serret-Frenet apparatus $\{\kappa, \tau, T, N, B\}$ and β be its natural mate. Then, the following assertions are equivalent:*

(1) α is a Bertrand curve

(2) β is a spherical curve

(3) β lies in the rectifying plane of α

Proof. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with $\kappa \neq 0$ and $\tau \neq 0$. Then, from Theorem 7, the naturel mate of α is given by

$$\beta = \left(\int (\kappa h) ds\right) T + hN - \left(\int (\tau h) ds\right) B, \tag{10}$$

where $h = dd'$ and d is the distance function of β. Now, β is a spherical curve if and only if d is a positive nonzero constant if and only if

$$\beta = c_1 T - c_2 B, \tag{11}$$

where c_1 and c_2 are constants (i.e., β lies in the rectifying plane of α), if and only if

$$N = (c_1\kappa + c_2\tau)N, \tag{12}$$

if and only if $c_1\kappa + c_2\tau = 1$ if and only if α is a Bertrand curve. □

Remark 11. Theorem 10 gives a method to create a spherical curve using a Bertrand curve. In [9], a method to create a Bertrand curve using a spherical curve with Sabban frame was introduced. Also, in [17], some methods to create special curves such as helix, slant helix, Bertrand curves, and Mannheim curves were introduced.

Corollary 12. *If α is a Salkowski curve, then its natural mate is given by*

$$\beta = \frac{1}{\kappa} T. \tag{13}$$

In what follows, we give an example of the natural mate of a Salkowski curve, and for more detail in Salkowski curve, we refer the reader to [4].

Example 13. Let $\alpha : I \longrightarrow \mathbb{E}^3$, be a Salkowski curve given by $\alpha = (\alpha_1, \alpha_2, \alpha_2)$ where,

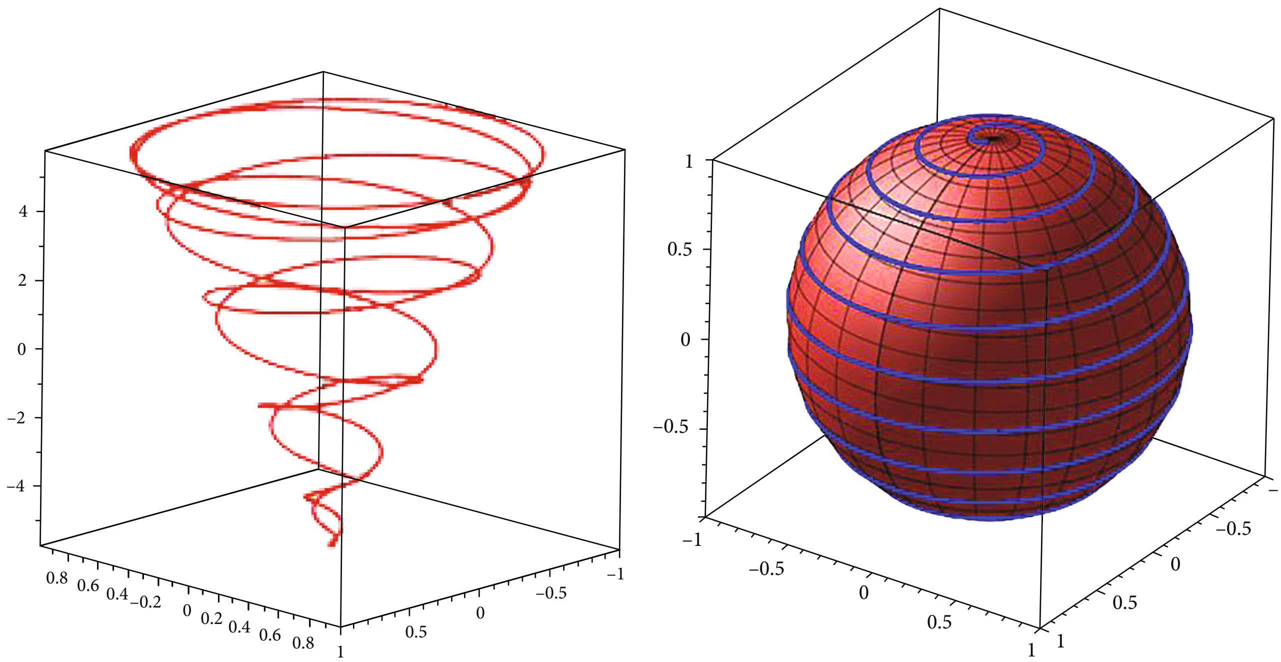
$$\begin{aligned}
\alpha_1 &= \frac{1}{\sqrt{1+m^2}}\left(-\frac{1-n}{4+8m}\sin(1+2t)\right.\\
&\quad\left.-\frac{1+n}{4-8m}\sin(1-2t)-\frac{1}{2}\sin(t)\right),\\
\alpha_2 &= \frac{1}{\sqrt{1+m^2}}\left(\frac{1-n}{4+8m}\cos(1+2t)\right.\\
&\quad\left.+\frac{1+n}{4-8m}\cos(1-2t)+\frac{1}{2}\cos(t)\right),\\
\alpha_3 &= \frac{1}{\sqrt{1+m^2}}\left(\frac{1}{4m}\cos(2nt)\right),
\end{aligned} \tag{14}$$

where $n = m/\sqrt{1+m^2}$ and $m \neq 0, \pm 1/\sqrt{3}$. This curve was investigated by Salkowski in 1909.

This curve has $\kappa = 1$ and torsion $\tau = -\tan(mt/\sqrt{m^2+1})$. The unit tangent of α is given by $T = (T_1, T_2, T_3)$ where

$$\begin{aligned}
T_1 &= -\cos(t)\cos(nt) - n\sin(t)\sin(nt),\\
T_2 &= n\cos(t)\sin(nt) - \cos(nt)\sin(t),\\
T_3 &= \frac{n}{m}\sin(nt).
\end{aligned} \tag{15}$$

Now, using Theorem 10, the natural mate β of α is given by $\beta = T$. Now, we draw pictures for α and its natural mate when $m = 1/23$ as shown in Figure 1. In this, Figure 1(a) is the curve α and (b) is the natural mate of α. It can be



(a) The curve α

(b) The blue curve is β lying on the unit sphere centered at the origin

Figure 1: The curve α and its natural mate β when $m = 1/23$.

observed from (b) that the natural mate of α lies on a unit sphere.

It can be easily observed from Theorem 10 that if α is a circular helix or a Salkowski curve, then its natural mate always lies on the rectifying planes of α.

In the coming theorem, we give a new criterion for a general helix.

Theorem 14. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with nonvanishing curvature and torsion. Then, α is a general helix if and only if there exists a fixed direction orthogonal to its natural mate.*

Proof. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with nonvanishing curvature and torsion, and β be its natural mate. First, assume that α is a general helix, then there exists a fixed direction makes a constant angle with its tangent. Let $\mathscr{U}$ be a unit constant vector lies on that direction, then $T \cdot \mathscr{U} = \cos\theta = \text{constant}$, and $B \cdot \mathscr{U} = \sin\theta = \text{constant}$. Now, using Theorem 7, we have

$$\beta \cdot \mathscr{U} = \cos\theta \int (\kappa h) ds - \sin\theta \int (\tau h) ds. \tag{16}$$

Since α is a helix, then $\cos\theta \int \kappa h ds - \sin\theta \int (\tau h) ds = 0$. Therefore, $\beta \cdot \mathscr{U} = 0$, which means that $\mathscr{U}$ is orthogonal to β.

Conversely, assume that there exists a fixed direction orthogonal to β. Let $\mathscr{U}$ be a unit constant vector lies on that direction, then $\beta \cdot \mathscr{U} = 0$; therefore, $N \cdot \mathscr{U} = 0$ which implies that $T \cdot \mathscr{U} = \text{constant}$, which means that α is a general helix. □

Remark 15. In [7, 11], it has been proved that a smooth curve with nonvanishing curvature and torsion is a general helix if and only if its natural mate is a plane curve. Using this fact and the result in Theorem 14, it can be concluded that the axis of a helix is always normal to the plane containing its natural mate.

Now, we give the position vector for conjugate.

Theorem 16. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a Frenet curve with Serret-Frenet apparatus $\{\kappa, \tau, T, N, B\}$ with negative torsion. The conjugate mate $\bar{\alpha}$ of α is given by*

$$\bar{\alpha} = \left(\int (\kappa h_1) ds \right) T + h_1 N + \left(s - \int (\tau h_1) ds \right) B, \tag{17}$$

where $h_1 = (1/\tau)(1 - d_1'^2 - d_1 d_1'')$ and d_1 is the distance function of $\bar{\alpha}$.

Proof. Since the unit tangent of $\bar{\alpha}$ is given by $T_{\bar{\alpha}} = B$, we have

$$\bar{\alpha} = \int (T_{\bar{\alpha}}) ds = \int (B) ds = g_1 T + h_1 N + l_1 B. \tag{18}$$

Now, differentiating equation (18) w.r.t the arc length, we get

$$B = \left(g_1' - \kappa h_1 \right) T + \left(h_1' + \kappa g_1 - \tau l_1 \right) N + \left(l_1' + \tau h_1 \right) B. \tag{19}$$

Thus, from equation (19), we have the following equations:

$$\begin{cases} g_1' - \kappa h_1 = 0, \\ h_1' + \kappa g_1 - \tau l_1 = 0, \\ l_1' + \tau h_1 = 1. \end{cases} \tag{20}$$

Therefore, we have

$$\begin{cases} g_1 = \displaystyle\int (\kappa h_1) ds, \\ l_1 = s - \displaystyle\int (\tau h_1) ds. \end{cases} \tag{21}$$

Our task now is to find h_1. The distance squared function, d_1^2, of $\bar{\alpha}$ is given by

$$d_1^2 = g_1^2 + h_1^2 + l_1^2. \tag{22}$$

Now, differentiating equation (22), we get

$$dd' = g_1 g_1' + h_1 h_1' + l_1 l_1'. \tag{23}$$

Hence, using equations (20) and (21) in equation (23), we obtain $dd' = s - \int (\tau h_1)$ which implies that $h_1 = (1/\tau)(1 - d_1'^2 - d_1 d_1'')$ which completes the proof. □

From Theorem 16, we have the following corollary.

Corollary 17. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with Serret-Frenet apparatus $\{\kappa, \tau, T, N, B\}$ with negative torsion. Then,*

(1) If $\bar{\alpha}$ is a spherical curve, then $h_1 = 1/\tau$

(2) If $\bar{\alpha}$ is a rectifying curve, then $h_1 = 0$

Proof. If $\bar{\alpha}$ is a spherical curve, then its distance function d is a constant. Thus, $h_1 = 1/\tau$. If $\bar{\alpha}$ is a rectifying curve, then it has been proved in [3] that its distance function d_1 satisfies $d_1^2(s) = s^2 + c_1 s + c_2$ for some constants c_1 and c_2. Therefore, $h_1 = 0$. □

4. Conclusion

The position vector of a curve in the Euclidian 3-space is the most natural geometric object. It is important in many applications in several areas such as mathematics, engineering, and natural sciences. Throughout this paper, we study the position vectors of the natural and conjugate mates associated with a given smooth space curve in Euclidian 3-space. The position vectors of the natural and the conjugate mates associated with a given smooth curve are very useful for studying these curves. Also, the position vector of the natural mate associated to a given curve is used to prove a new criterion for a helix and Bertrand curves. Moreover, it can be easy to study the surfaces generated by the natural and conjugate mates associated with a smooth curve using their position vectors.

Data Availability

No external data has been used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Inequalities for the Class of Warped Product Submanifold of Para-Cosymplectic Manifolds

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The aim of this paper is to study the warped product pointwise semislant submanifolds in the para-cosymplectic manifold with the semi-Riemannian metric. For which, firstly we provide the more generalized definition of pointwise slant submanifolds and related characterization results followed by the definition of pointwise slant distributions and pointwise semislant submanifolds. We also derive some results for different foliations on distribution, and lastly, we defined pointwise semislant warped product submanifold, given existence and nonexistence results, basic lemmas, theorems, and optimal inequalities for the ambient manifold.

1. Introduction

To generalize the Riemannian product manifolds, Bishop and O'Neil [1] introduced the concept of warped product for the manifolds with negative curvature and showed the surface of revolution as the simplest example of warped product manifold. The authors of [2–5] studied the warped product submanifolds for different manifolds. Warped product plays the beneficial role in encoding the universe, and the inequalities related to the second fundamental form with the warping function cover the wide as well as important section of it. These were firstly formulated by Chen in [6, 7]. Warped product for lightlike manifolds for the first time was studied in [8] and for semi-Riemannian manifold under the name PR-warped product on para-Kähler manifold in [9], where he derived the aforesaid inequalities for the case of semi-Riemannian metric. From there, the study on warped product escalates among geometers with also in view that the same has so many applications in the physics mainly in general relativity and black hole theory [10].

Beside this, the name slant submanifolds were introduced as the generalized version of holomorphic and totally real cases of submanifolds by Chen in [11]. Further, the theory extended to various manifolds with Riemannian as well as semi-Riemannian metric by many geometers. Later, in 2017, the authors in [12] defined the slant submanifolds irrespective of the writinger angle for the semi-Riemannian manifold and formulated three cases which are separately explained and achieved some effective results with bunch of examples. They defined it in terms of quotient $g(tX, tX)/g(JX, JX)$ which is constant for the case of slant submanifolds for every vector field X (spacelike or timelike) on the submanifold M of manifold $(\bar{M}, J, g)$. As slant and semislant submanifolds generalized to pointwise slant submanifolds (former called quasi slant) by Etayo in [13], Chen and Garay studied the same for the almost Hermitian case [14]. Sahin [15] defined pointwise semislant notion of submanifolds with an example. Recently, there are many interesting papers related with submanifold theory, singularity theory, classical differential geometry, etc. The readers can find more details about those techniques and theories in a series of papers [16–29]. Moreover, interdisciplinary research is one of the hottest trends in science; in the future work, we intend to apply and combine the techniques and results presented in [16–25] alongside with the methods in this paper to obtain more new results.

The paper is structured as follows: Section 2 contains the preliminary knowledge about ambient manifold, submanifold, and warped product with some important lemmas. Section 3 defines the pointwise slant submanifold, characterization lemma, and an example. Section 4 and Section 5 deal with the study of the pointwise slant distributions and pointwise semislant submanifolds, respectively. Section 6 includes the definition of warped product, some nonexistence results, lemmas, and theorems provided with an example. Finally, inequalities for the same submanifold are given in Section 7.

2. Preliminaries

Definition 1. A $(2m+1)$-dimensional smooth manifold $\widehat{M}$ admits (φ, ξ, η, g) structure with φ as a $(1,1)$-tensor field, ξ as a characteristic vector field, η as a globally differential 1-form, and g as a semi-Riemannian metric named as an almost paracontact semi-Riemannian manifold $(\widehat{M}, \varphi, \xi, \eta, g)$ which satisfies

$$\varphi^2 = I - \eta \otimes \xi, \eta(\xi) = 1, \tag{1}$$

$$g(\cdot, \cdot) = -g(\varphi \cdot, \varphi \cdot) + \eta(\cdot)\eta(\cdot), \tag{2}$$

where I represents an identity transformation of tangent space of $\widehat{M}$ and $\otimes$ represents a tensor product. A structure compatible semi-Riemannian metric "g" relates to η as [30].

$$g(\cdot, \xi) = \eta(\cdot). \tag{3}$$

Equations (1) and (2) easily ensure the following:

$$\begin{aligned} \text{rank } (\varphi) &= 2\text{m}, \\ \varphi\xi &= 0, \\ \eta \circ \phi &= 0, \end{aligned} \tag{4}$$

$$g(\varphi \cdot, \cdot) + g(\cdot, \varphi \cdot) = 0. \tag{5}$$

Let Φ be the fundamental 2-form on $\widehat{M}$; then,

$$\Phi(\cdot, \cdot) = d\eta(\cdot, \cdot) = g(\cdot, \varphi \cdot). \tag{6}$$

Basis. An almost paracontact semi-Riemannian manifold always exists with a $\varphi -$ basis$\{E_i, E_i^{\mathring{a}}, \xi\}$, a certain type of local pseudoorthonormal basis which includes E_i, ξ as space-like, and $E_i^{\mathring{a}} = \varphi E_i$ as timelike vector fields.

Definition 2 (see [31]). An almost paracontact semi-Riemannian manifold $\widehat{M}$ is termed as para-cosymplectic if the forms η and Φ are parallel with respect to the Levi-Civita connection $\widehat{\nabla}$ by

$$\begin{aligned} \widehat{\nabla}\eta &= 0, \\ \widehat{\nabla}\Phi &= 0. \end{aligned} \tag{7}$$

Lemma 3. *Let $\widehat{M}$ be a para-cosymplectic manifold with structure vector field $\xi \in \Gamma(T\widehat{M})$; then,*

$$\widehat{\nabla}_X \xi = 0, \tag{8}$$

$\forall X \in \Gamma(T\widehat{M})$.

Proof. Directly follow with the help of Equations (4) and (7) and covariant differentiation. □

2.1. Submanifold. Let M be an isometrically immersed submanifold of a para-cosymplectic manifold $\widehat{M}$ with an induced nondegenerate metric g (denoted metric by same symbol as on $\widehat{M}$), denoting ∇ as Levi-Civita connection and h as the second fundamental form on M. Thus, the Gauss-Weingarten formulas are

$$\widehat{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{9}$$

$$\widehat{\nabla}_X \zeta = -A_\zeta X + \nabla_X^\perp \zeta, \tag{10}$$

for $X, Y \in \Gamma(TM)$ (tangent bundle), and $\zeta \in \Gamma(TM^\perp)$ (normal bundle); $\nabla^\perp$ denotes normal connection, and A_ζ denotes shape operator associated with the normal section on M. The metric relation of A_ζ and h is given as

$$g(A_\zeta X, Y) = g(h(X, Y), \zeta). \tag{11}$$

Every $X \in \Gamma(TM)$ is split as

$$\varphi X = tX + nX. \tag{12}$$

Similarly, every $\zeta \in \Gamma(TM^\perp)$ is split as

$$\varphi\zeta = t^\perp\zeta + n^\perp\zeta, \tag{13}$$

where tX and $t^\perp\zeta$ (nX and $n^\perp\zeta$) are the tangential parts (normal parts) of φX and $\varphi\zeta$, respectively. Based on Equation (12), the submanifold M classifies as *anti-invariant* if $t = 0$ or *invariant* if $n = 0$ on M. After using Equation (12) in Equation (5), we get

$$g(X, tY) = -g(tX, Y). \tag{14}$$

Now, from Lemma 3 and Equation (11), we have our next result.

Lemma 4. *If M is a submanifold immersed in a paracosymplectic manifold $\widehat{M}$ with structure vector field $\xi \in \Gamma(TM)$, then*

$$\begin{aligned} &\nabla_X \xi = \nabla_\xi X = \nabla_\xi \xi = 0, \\ &h(X, \xi) = 0, \\ &A_\zeta \xi = 0, A_\zeta X \perp \xi, \end{aligned} \tag{15}$$

for every $X \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$.

Next, let us take two semi-Riemannian manifolds (M_B, g_1) and (M_F, g_2) and a positive smooth function f on M_B. Taking $M_B \times M_F$ as the product manifold along with canonical projections,

$$\begin{aligned} \sigma_1 &: M_B \times M_F \longrightarrow M_B, \\ \sigma_2 &: M_B \times M_F \longrightarrow M_F, \end{aligned} \tag{16}$$

such that $\sigma_1(p_B, p_F) = p_B$ and $\sigma_2(p_B, p_F) = p_F$ for any point $p = (p_B, p_F) \in M_B \times M_F$. Then, the product manifold $M_W = M_B \times_f M_F$ is called warped product if metric g called the warped metric on M_W can be formulated as

$$\begin{aligned} g(X, Y) = g_1(\sigma_1 * (X), \sigma_1 * (Y)) \\ + (f \circ \sigma_1)^2 g_2(\sigma_2 * (X), \sigma_2 * (Y)). \end{aligned} \tag{17}$$

For every $X, Y \in \Gamma(TM_W)$, "$*$" represents the derivation map, and we call f as a *warping function*. Abstractly, the metric can be written as

$$g = g_1 + f^2 g_2, \tag{18}$$

where the warped product $M_W = M_B \times_f M_F$ is split into a product of the base space M_B and the fiber space M_F, except that the fiber M_F is warped [1, 32].

Proposition 5 (see [32]). *The warped product submanifold $M_W = M_B \times_f M_F$ satisfies*

(i) $\nabla_X Y = \Gamma(TM_B)$

(ii) $\nabla_X U = \nabla_U X = (Xf/f)U$

(iii) $\nabla_U V = \nabla'_V U = (g(U, V)/f)\nabla f$

for $X, Y \in \Gamma(TM_B)$ and $U, V \in \Gamma(TM_F)$, where ∇ is the Levi-Civita connection on M_W, ∇' is the connection on M_F, and ∇f is the gradient of f defined as $g(\nabla f, X) = Xf$.

Further, let $\{e_1, \cdots, e_K, e_{K+1}, \cdots, e_{2m+1}\}$ be a local orthonormal basis on $T\widehat{M}$ among which $\{e_1, \cdots, e_{K+1}\}$ are tangent to M and $\{e_{K+2}, \cdots, e_{2m+1}\}$ are normal to M. If we set

$$\begin{aligned} &h^k_{xy} = g\left(h\left(e_x, e_y\right), e_k\right), \\ &x, y \in \{e_1, \cdots, e_{K+1}\}, \\ &k \in \{e_{K+2}, \cdots, e_{2m+1}\}, \end{aligned} \tag{19}$$

then we get

$$h\left(e_x, e_y\right) = \sum_{k=K+1}^{2m+1} \varepsilon_k h^k_{xy} e_r, \varepsilon_k = g\left(e_k, e_k\right), \tag{20}$$

where h^k_{xy} are the coefficients of h. Accordingly, squared norm of the second fundamental form h is defined as

$$\|h\|^2 = \sum_{x,y=1}^{K+1} \varepsilon_x \varepsilon_y g\left(h\left(e_x, e_y\right), h\left(e_x, e_y\right)\right). \tag{21}$$

3. Pointwise Slant Submanifolds

The semi-Riemannian manifold has difficulty of defining the Writinger angle as the vector fields may be timelike. Thus, the next definition is in the view of [12], generalizing the slant submanifold in our ambient semi-Riemannian manifold.

Definition 6. An isometrically immersed submanifold M of an almost paracontact manifold $\widehat{M}$ is termed as pointwise slant if at every point $p \in M$, the quotient $g(tX, tX)/g(\varphi X, \varphi X) = \lambda(\theta)$ for $\theta \geq 0$ is independent of the choice of any nonzero spacelike or timelike vector $X \in M_p$, where $M_p = \{X \in T_p M : g(X, \xi) = 0\}$. For slant angle θ, we say $\lambda(\theta)$ a slant coefficient.

Remark 7. The value of $\lambda(\theta)$ can be

(i) $\lambda(\theta) = \cosh^2(\theta) \in [1, \infty)$ for $|tX|/|\varphi X| > 1$; tX is timelike or spacelike of each spacelike or timelike vector field X adding $\theta > 0$

(ii) $\lambda(\theta) = \cos^2(\theta) \in [0, 1]$ for $|tX|/|\varphi X| < 1$; tX is timelike or spacelike of each spacelike or timelike vector field X adding $0 \leq \theta \leq 2\pi$

(iii) $\lambda(\theta) = -\sinh^2(\theta) \in (-\infty, 0]$ for tX is timelike or spacelike for any timelike or spacelike vector field X adding $\theta > 0$

Remark 8. The special cases are as follows:

(i) The constant value of $\lambda(\theta)$ throughout M implies M is *slant submanifold* [11, 12]

(ii) The point $p \in M$ is called a *complex point* if $t \equiv \varphi$, which means that the slant coefficient $\lambda(\theta)$ is equal to 1. The submanifold with every point as complex point is *complex or holomorphic submanifold*

(iii) The point $p \in M$ is called a *totally real point* if $t \equiv 0$, which means that the slant coefficient $\lambda(\theta)$ is equal to 0. The submanifold with every point as totally real point is *totally real submanifold*

Furthermore, let us take the union of all M_p's and denoting the same by

$$T^*M = \bigcup_{p\in M} \{X \in M_p | g(X, \xi) = 0\}. \tag{22}$$

Lemma 9. *The submanifold M isometrically immersed in para-cosymplectic manifold $\widehat{M}$ is a pointwise slant submanifold if and only if on every point $p \in M$; there exists $\lambda \in (-\infty, \infty)$ for some $\theta \geq 0$ such that $t^2X = \lambda(\theta)X$ for each spacelike (or timelike) vector field $X \in M_p$.*

Proof. For each point p of a pointwise slant submanifold M, the definition (22) follow as

$$g(tX, tX) = \lambda(\theta) g(\varphi X, \varphi X), \tag{23}$$

for $X \in T_pM$. With the use of Equations (5) and (14) and the condition that $X \in T_pM$ in Equation (23), we get the desired result. □

Proceeding further with some results which are not hard to prove, any pointwise slant submanifold M satisfies

$$g(tX, tY) = \lambda(\theta) g(\varphi X, \varphi Y) = -\lambda(\theta) g(X, Y),$$
$$g(nX, nY) = (1-\lambda(\theta)) g(\varphi X, \varphi Y) = -(1-\lambda(\theta)) g(X, Y), \tag{24}$$

for $X, Y \in T^*M$.

Proposition 10. *The submanifold M of a para-cosymplectic manifold $\widehat{M}$ is pointwise slant submanifold if and only if*

(i) $t^\perp nX = (1-\lambda(\theta))X$ and $n\,tX = -n^\perp nX$ for any spacelike (or timelike) vector field $X \in \Gamma(TM)$

(ii) $(n^\perp)^2\zeta = \lambda(\theta)\zeta$ for nonlightlike normal vector field ζ, where $\lambda(\theta)$ is the slant coefficient of M

Proof. Assume M as a pointwise slant submanifold.

(i) Then for every $X \in T^*M$, $\varphi^2 X = X$. On other way,

$$\varphi^2 X = t^2X + n\,tX + t^\perp nX + n^\perp nX. \tag{25}$$

Equating tangential and normal parts and using Lemma 9, we can attain the result

(ii) Since $\zeta \in \Gamma(TM^\perp)$, thus there exists $X \in \Gamma(T^*M)$ as M is pointwise slant submanifold such that $nX = \zeta$. Now, $(n')^2\zeta = n^\perp n^\perp nX = -n^\perp n\,tX = n\,t^2X = \lambda(\theta)\zeta$. The converse can be easily derived using same equations

□

Theorem 11 (see [33]). *A totally geodesic and connected pointwise slant submanifold M of a para-cosymplectic manifold $\widehat{M}$ is a slant submanifold.*

4. Pointwise Slant Distributions

Analogous to [34], we generalize slant distributions by defining pointwise slant distributions in $\widehat{M}$. Furthermore, we study some basic characterizations for the distributions on our ambient manifold.

Definition 12. A pointwise slant distribution $\mathfrak{D}$ on M is a differentiable distribution for which the quotient $g(t_{\mathfrak{D}}X, t_{\mathfrak{D}}X)/g(\varphi X, \varphi X) = \lambda_{\mathfrak{D}}(\theta)$ is independent of the choice of any spacelike or timelike vector field $X \in \mathfrak{D}_p$. Here,

(i) $\mathfrak{D}_p$ is the distribution at point $p \in M$

(ii) $t_{\mathfrak{D}}X$ is the projection of φX on the distribution $\mathfrak{D}$

(iii) $\lambda_{\mathfrak{D}}(\theta)$ is the slant coefficient corresponding to the distribution $\mathfrak{D}$ on M for $\theta \geq 0$, and the value of $\lambda_{\mathfrak{D}}(\theta)$ may be $\cos h^2\theta$, $\cos^2\theta$, or $\sin h^2\theta$

Remark 13.

(1) A pointwise slant distribution $\mathfrak{D}$ is *invariant* if $t_{\mathfrak{D}}X \equiv \varphi X$ with $\lambda_{\mathfrak{D}}(\theta) = 1$ or *anti-invariant* for $t_{\mathfrak{D}}X \equiv 0$ with $\lambda_{\mathfrak{D}}(\theta) = 0$. Other than these two cases, we call the distribution to be *proper pointwise slant distribution* [12]

(2) The distribution $\mathfrak{D}$ on M is as follows [9, 31]:

(i) totally geodesic: if $h(X, Y) = 0$

(ii) involutive: if $[X, Y] \in \mathfrak{D}$

for every $X, Y \in \mathfrak{D}$.

Corollary 14. *The distribution $\mathfrak{D}$ on the submanifold M is pointwise slant distribution if and only if there exists $\lambda_{\mathfrak{D}}(\theta)$ for $\theta \geq 0$ such that $(t_{\mathfrak{D}})^2X = \lambda_{\mathfrak{D}}(\theta)X$ for any nonlightlike vector field $X \in \mathfrak{D}_p \subset T_pM$.*

Proof. The result follows similar to Lemma 9. □

5. Pointwise Semislant Submanifold

Definition 15. A submanifold M of a para-cosymplectic $\widehat{M}$ is named as pointwise semislant submanifold if the set of complementary orthogonal distributions $\{\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda\}$ exists on M and fulfills the listed conditions:

(i) $TM = \mathfrak{D}_{\mathfrak{T}} \oplus \mathfrak{D}_\lambda$

(ii) $\mathfrak{D}_{\mathfrak{T}}$ is φ-invariant distribution, i.e., $\mathfrak{D}_{\mathfrak{T}} \subseteq \mathfrak{D}_T$

(iii) $\mathfrak{D}_\lambda$ is a pointwise slant distribution having $\lambda(\theta)$ as a slant coefficient for $\theta \geq 0$

Remark 16. Further, submanifold M is

(i) *proper pointwise semislant* when $\mathfrak{D}_{\mathfrak{T}} \neq 0, \mathfrak{D}_\lambda \neq \{0\}$ with nonconstant $\lambda(\theta)$

(ii) *proper slant submanifold* when $\mathfrak{D}_{\mathfrak{T}} = \{0\}$ and $\mathfrak{D}_\lambda \neq \{0\}$ with $\lambda(\theta)$ globally constant for θ [35]

(iii) *proper semi-invariant* when $\mathfrak{D}_{\mathfrak{T}} \neq \{0\}$ and $\mathfrak{D}_\lambda \neq \{0\}$ such that $tX \equiv 0$ for any $X \in \Gamma(\mathfrak{D}_\lambda)$ [12]

(iv) *invariant submanifold* when $\mathfrak{D}_\lambda = \{0\}$ [35]

(v) *anti-invariant submanifold* when $\mathfrak{D}_{\mathfrak{T}} = \{0\}$ and $tX \equiv 0$ for every $X \in \Gamma(\mathfrak{D}_\lambda)$ [35]

Remark 17. The decomposition of the tangent space can be expressed in two ways:

(i) If $\xi \in \Gamma(TM)$, the $TM = \langle\xi\rangle \oplus \mathfrak{D}_{\mathfrak{T}}{}' \oplus \mathfrak{D}_\lambda$

(ii) If $\xi \in \Gamma(TM^\perp)$, the $TM = \mathfrak{D}_{\mathfrak{T}}{}' \oplus \mathfrak{D}_\lambda$. Here, $\mathfrak{D}_{\mathfrak{T}}{}' = \{X \in \mathfrak{D}_{\mathfrak{T}} : g(X, \xi) = 0\} \subseteq \mathfrak{D}_{\mathfrak{T}}$. Thus, we have either $\mathfrak{D}_{\mathfrak{T}} = \mathfrak{D}_{\mathfrak{T}}{}'$ or $\mathfrak{D}_{\mathfrak{T}} = \langle\xi\rangle \oplus \mathfrak{D}_{\mathfrak{T}}{}'$ [4]

Denote $\mathscr{P}_T$ and $\mathscr{P}_\lambda$ as the projections, respectively, on the distributions $\mathfrak{D}_{\mathfrak{T}}$ and $\mathfrak{D}_\lambda$. Then, any $X \in \Gamma(TM)$ is split as

$$X = \mathscr{P}_T X + \mathscr{P}_\lambda X. \tag{26}$$

Operating φ, using Equation (12) and the case distribution $\mathfrak{D}_{\mathfrak{T}}$ which is φ-invariant on the previous equation, we concluded that

$$tX = t\mathscr{P}_T X + t\mathscr{P}_\lambda X \in \Gamma(TM), nX = n\mathscr{P}_\lambda X \in \Gamma\left(TM^\perp\right). \tag{27}$$

As $\mathfrak{D}_\lambda$ is pointwise slant distribution, by the consequences of Corollary 14, we obtain that

$$t^2 X = \lambda(\theta) X, \tag{28}$$

for $X \in \Gamma(\mathfrak{D}_\lambda)$ with $\lambda(\theta)$ as the slant coefficient. Clearly, for any point $p \in M$, if $\xi \in T_p M$, then

$$\varphi X = t\mathscr{P}_{\mathscr{T}}{}' X + t\mathscr{P}_\lambda X + n\mathscr{P}_\lambda X, \tag{29}$$

where $\mathscr{P}_{\mathscr{T}}{}'$ is the projection on the distribution $\mathfrak{D}_{\mathfrak{T}}{}'$. But this does not affect our result as ξ disappears when φ operates on Z.

However, the normal bundle denoted as $TM^\perp$ may be written as

$$TM^\perp = n\mathfrak{D}_\lambda \oplus \nu, \tag{30}$$

where ν represents the subspace of normal bundle that is invariant under φ.

Lemma 18 (see [31]). *The shape operator A of a proper pointwise semislant submanifold M of para-cosymplectic manifold $\widehat{M}$ ensures the listed conditions:*

$$g\left(\varphi A_{\varphi U} S, X\right) = g(\nabla_S U, X), \tag{31}$$

$$A_{\varphi U} V = A_{\varphi V} U, \tag{32}$$

$$A_\zeta X = \varphi A_{\varphi\zeta} X, \tag{33}$$

$$g\left(A_\zeta X, U\right) = -g\left(A_{\varphi\zeta}\varphi X, U\right), \tag{34}$$

for $S \in \Gamma(TM)$, $X \in \Gamma(T\mathfrak{D}_{\mathfrak{T}})$, $U, V \in \Gamma(T\mathfrak{D}_\lambda)$, and $\zeta \in \Gamma(TM^\perp)$.

Both when ξ is normal or tangent to M, the integrability and geodesic conditions brought out to be same after calculations for both the distributions, thus denoting them as common $\mathfrak{D}_{\mathfrak{T}}$.

Lemma 19. *If M is a proper pointwise semislant submanifold of para-cosymplectic manifold $\widehat{M}$, for $\xi \in \Gamma(TM)$ or $\xi \in \Gamma(TM^\perp)$, the invariant distribution $\mathfrak{D}_{\mathfrak{T}}$ on M is*

(i) integrable if and only if $h(tX, Y) = h(X, tY)$

(ii) totally geodesic if and only if $A_{ntU} Y, = A_{nU} tY$

for $X, Y \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$ and $U \in \Gamma(\mathfrak{D}_\lambda)$.

Proof. Equation (2) expands as

$$g([X, Y], U) = -g\left(\varphi\left(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X\right), \varphi U\right), \tag{35}$$

for every nonzero vector fields $X, Y \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$ and $U \in \Gamma(\mathfrak{D}_\lambda)$. Using Equation (12) for the φU in Equation (35) and followed by using Equations (5), (7), and (9) and Lemma 9, we arrive at

$$(1 - \lambda(\theta)) g([X, Y], U) = g(h(X, tY) - h(Y, tX), nU). \tag{36}$$

Result (i) is clear using remark (28) as $\lambda(\theta)$ is nonconstant in Equation (36). Again, from Gauss formula and Equation (2),

$$g(\nabla_X Y, U) = g\left(\widehat{\nabla}_X Y, U\right) = -g\left(\varphi\widehat{\nabla}_X Y, \varphi U\right). \tag{37}$$

Employing Equations (7), (9), (11), (12), and (28) and Remark 16 in Equation (37), result (ii) follows. □

Lemma 20. *If M is a proper pointwise semislant submanifold of para-cosymplectic manifold $\widehat{M}$, for $\xi \in \Gamma(TM)$ or $\xi \in \Gamma(TM^{\perp})$, the pointwise slant distribution $\mathfrak{D}_\lambda$ on M is*

(i) involutive if and only if

$$g(A_{nV}U - A_{nU}V, tX) = g(A_{ntU}V - A_{ntV}U, X) \tag{38}$$

(ii) totally geodesic if and only if

$$g(A_{nV}tX, U) = g(A_{ntV}X, U) \tag{39}$$

for $X \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$ and $U, V \in \Gamma(\mathfrak{D}_\lambda)$.

Proof. Equation (2) implies

$$g([U, V], X) = -g(\varphi[U, V], \varphi X) + \eta([U, V])\eta(X), \tag{40}$$

for every nonzero vector fields $X \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$ and $U, V \in \Gamma(\mathfrak{D}_\lambda)$. Solving separately the term $\{g(\varphi[U, V], \varphi X)\}$ using Equations (5), (7), (10), (12), and (28), we receive

$$\begin{aligned} g(\varphi[U, V], \varphi X) = \lambda(\theta) g([U, V], X) &+ g\left(\lambda'(\theta)V(\theta)U - \lambda'(\theta)U(\theta)V, X\right) \\ &- g(A_{nV}U - A_{nU}V, \varphi X) \\ &+ g(A_{ntV}U - A_{ntU}V, X), \end{aligned} \tag{41}$$

where $\lambda'(\theta)$ is the first derivative of $\lambda(\theta)$. Surely, U, V are orthogonal to X after using this fact in Equation (41), and substituting in Equation (40), we get

$$\begin{aligned} (1-\lambda(\theta)) g([U, V], X) = g(-A_{ntV}U + A_{ntU}V, X) &+ g(A_{nV}U - A_{nU}V, tX) \\ &+ \eta([U, V])\eta(X). \end{aligned} \tag{42}$$

For $\xi \in \Gamma(TM)$, one can replace X by ξ in Equation (42), and consequently, we get

$$-\lambda(\theta) g([U, V], \xi) = g(h(U, \xi), ntV) - g(h(V, \xi), ntU). \tag{43}$$

Using Lemma 4 in Equation (43) and for reason that $\lambda(\theta)$ a nonconstant, we get

$$\eta([U, V]) = 0. \tag{44}$$

Therefore, in Equation (42) using Equation (44) along with the facts that M is proper, we arrived at the desired result (i).

Further, using Gauss formula and employing Equations (2), (7), (9), (11), (12), and (28) give

$$\begin{aligned} g(\nabla_U V, X) = \lambda(\theta) g(\nabla_X Y, U) &+ g\left(\lambda'(\theta)U(\theta)V, X\right) \\ &+ g(A_{ntV}X, U) - g(A_{nV}tX, U)\eta\left(\widehat{\nabla}_U V\right)\eta(X). \end{aligned} \tag{45}$$

Since $\xi \in \Gamma(TM)$, we can replace X by ξ in Equation (45), and consequently, we get

$$\begin{aligned} (1-\lambda(\theta))\eta(\nabla_V U) &= -g(A_{ntU}\xi, V) + \eta(\nabla_V U), \\ (-\lambda(\theta))\eta(\nabla_V U) &= -g(A_{ntU}\xi, V). \end{aligned} \tag{46}$$

Using Lemma 4 in above expression, we get

$$\eta(\nabla_V U) = \eta\left(\widehat{\nabla}_V U\right) = 0. \tag{47}$$

Hence, Equation (45) implies that

$$(1-\lambda(\theta))(\nabla_U V, X) = g(A_{ntV}X, U) - g(A_{nV}tX, U). \tag{48}$$

Thus, from (48) and M as proper, X, Y, U as nonnull vector fields, the proof of the (ii) directly follows. □

6. Pointwise Semislant Warped Product Submanifold

Definition 21. A pointwise semislant warped product submanifold M of a para-cosymplectic manifold $\widehat{M}$ is a warped product of an invariant submanifold $M_{\mathfrak{T}}$ and a proper pointwise slant submanifold M_λ either in the form $M_{\mathfrak{T}} \times_f M_\lambda$ or $M_\lambda \times_f M_{\mathfrak{T}}$, where f is a positive smooth function taken on first submanifold in the product and slant coefficient of M_λ is $\lambda(\theta)$. A trivial product is the case of such submanifold for which warping function f is constant.

Proposition 22 (see [33]). *A nontrivial pointwise semislant warped product submanifold M of the form $M_\lambda \times_f M_{\mathfrak{T}}$ with $\xi \in \Gamma(TM^{\perp})$ does not exist on a para-cosymplectic manifold $\widehat{M}$.*

Proposition 23 (see [33]). *A nontrivial pointwise semislant warped product submanifold M of the form $M_\lambda \times_f M_{\mathfrak{T}}$ with $\xi \in \Gamma(TM)$ does not exist on a para-cosymplectic manifold $\widehat{M}$.*

Proposition 24. *A nontrivial pointwise semislant warped product submanifold M of the form $M_{\mathfrak{T}} \times_f M_\lambda$ with $\xi \in \Gamma(TM_\lambda)$ does not exist on a para-cosymplectic manifold $\widehat{M}$.*

Proof. Directly follow from Lemma 4 and Proposition 5. □

Lemma 25. *For a nontrivial pointwise semislant warped product submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ of a para-cosymplectic manifold $\widehat{M}$,*

$$g(h(X, U), ntU) - g(h(X, tU), nU) = 0, \tag{49}$$

$$g(h(X, tU), nU) = \lambda(\theta)(X \ln f) g(U, U), \tag{50}$$

$\forall X \in \Gamma(D_{\mathfrak{T}})$ *and* $U \in \Gamma(D_\lambda)$.

Proof.

(1) Since $g(h(X, U), ntU) = g(\widehat{\nabla}_X U, ntU)$ for $X \in \Gamma(TM_{\mathfrak{T}})$ and $U \in \Gamma(M_\lambda)$, on right side, using Equations (5), (7), and (12), Proposition 5, and Lemma 9, we have

$$\begin{aligned} g(h(X, U), ntU) = &-(X \ln f) g(tU, tU) \\ &- g\left(\widehat{\nabla}_X nU, tU\right) \\ &- \lambda(\theta)(X \ln f) g(U, U). \end{aligned} \tag{51}$$

Equations (10), (11), and (14) and Lemma 9 further help to achieve (49)

(2) As $g(h(X, tU), nU) = g(\widehat{\nabla}_X tU, nU)$, next substituting $nU = \varphi U - tU$ applying Equations (5) and (7), Proposition 5, and Lemma 9 and the facts that $M_{\mathfrak{T}}$ is invariant, we get (50)

□

Proposition 26. *If $M = M_{\mathfrak{T}} \times_f M_\lambda$ is a nontrivial pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$, then*

$$(1 - \lambda(\theta)) g\left(\widehat{\nabla}_X U, Y\right) = g(h(X, tY), nU) - g(h(X, Y), ntU), \tag{52}$$

$$(1 - \lambda(\theta)) g\left(\widehat{\nabla}_U V, X\right) = g(h(U, tX), nV) - g(h(U, X), ntV), \tag{53}$$

$\forall X, Y \in \Gamma(D_{\mathfrak{T}})$ *and* $U, V \in \Gamma(D_\lambda)$.

Proof. As $g(\widehat{\nabla}_X U, Y) = -g(\varphi \widehat{\nabla}_X U, \varphi Y), \eta(\widehat{\nabla}_X U) = -g(\widehat{\nabla}_X \xi, U) = 0$. Using Equations (7), (9), and (12) and Lemma 9 in above expression gives (52). In similar way, we can prove (53). □

Lemma 27. *A nontrivial proper pointwise semislant submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ of a para-cosymplectic manifold $\widehat{M}$ satisfies*

$$g(h(X, Y), nU) = 0, \tag{54}$$

$$g(h(X, V), nU) = -\varphi X(lnf) g(V, U) - X(\ln f) g(tV, U), \tag{55}$$

$$\begin{aligned} g(h(X, tV), nU) = &-\varphi X(\ln f) g(tV, U) \\ &- \lambda(\theta) X(\ln f) g(V, U), \end{aligned} \tag{56}$$

$$\begin{aligned} g(h(X, V), ntU) = &-\varphi X(\ln f) g(V, tU) \\ &+ \lambda(\theta) X(\ln f) g(V, U), \end{aligned} \tag{57}$$

$\forall X, Y \in \Gamma(D_{\mathfrak{T}})$ *and* $U, V \in \Gamma(D_\lambda)$.

Proof. Result (54) is not hard to prove using Equations (7), (12), and (14) and Proposition 5. Substituting $tU = V$ in Equation (49) gives $g(h(X, V), nU) = g(h(X, U), nV)$; one can replace $nV = \varphi V - tV$, and using Equations (5) and (7) and Proposition 5 gives (55). Putting $V = tV$ and $U = tU$, respectively, in Equation (55) gives results (56) and (57). □

Lemma 28. *If $M = M_{\mathfrak{T}} \times_f M_\lambda$ is a nontrivial pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$, then*

(i) for $\xi \in \Gamma(TM_{\mathfrak{T}})$,

$$\begin{aligned} g(h(tX, V), nU) = &-(X - \eta(X)\xi)(\ln f) g(V, U) \\ &- \varphi X(\ln f) g(tV, U) \end{aligned} \tag{58}$$

(ii) for $\xi \in \Gamma(TM^\perp)$,

$$\begin{aligned} g(h(tX, V), nU) = &-X(\ln f) g(V, U) \\ &- \varphi X(\ln f) g(tV, U) \end{aligned} \tag{59}$$

$\forall X \in \Gamma(D_{\mathfrak{T}})$ *and* $U, V \in \Gamma(D_\lambda)$.

Proof. Replacing $X = \varphi X$ in Equation (55) and having the fact that submanifold $M_{\mathfrak{T}}$ is invariant, both results directly follow. □

Proposition 29. *Let $M = M_{\mathfrak{T}} \times_f M_\lambda$ be nontrivial pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$, then*

$$g(h(X, V), ntU) - g(h(X, tU), nV) = \lambda'(\theta) X(\theta) g(U, V), \tag{60}$$

$\forall X \in \Gamma(D_{\mathfrak{T}})$ *and* $U, V \in \Gamma(D_\lambda)$, *and $\lambda'(\theta)$ is the first derivative of slant coefficient.*

Proof. Using metric and para-cosymplectic condition

$$g\left(\widehat{\nabla}_X U, V\right) = -g\left(\widehat{\nabla}_X \varphi U, \varphi V\right), \tag{61}$$

this expression under the effect of Equations (5), (9), (12), and (14) turns as

$$g\left(\widehat{\nabla}_X U, V\right) = -g\left(\widehat{\nabla}_X tU, tV\right) - g(h(X, tU), nV) + g\left(\widehat{\nabla}_X t^{\perp} nU + n^{\perp} nU, V\right). \tag{62}$$

Further, using Propositions 10 and (5) ended with the desired result. □

Proposition 30. *A nontrivial pointwise semislant warped product submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ of a para-cosymplectic manifold $\widehat{M}$ satisfies the following:*

(i) For $\xi \in \Gamma(TM)$,

$$g(h(tX, V), nU) + g(h(X, V), ntU) = -[(1 - \lambda(\theta))X - \eta(X)\xi](\ln f)g(V, U) \tag{63}$$

(ii) For $\xi \in \Gamma(TM^{\perp})$,

$$g(h(tX, V), nU) + g(h(X, V), ntU) = -[(1 - \lambda(\theta))X](\ln f)g(V, U) \tag{64}$$

$\forall X \in \Gamma(TM_{\mathfrak{T}})$ and $U, V \in \Gamma(TM_\lambda)$.

Proof. Lemma 28 and Equation (57) of Proposition 26 directly give the results. □

Definition 31. The submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ is named as mixed totally geodesic if for every $X \in \Gamma(D_{\mathfrak{T}})$ and $U \in \Gamma(D_\lambda)$,

$$h(X, U) = 0. \tag{65}$$

Theorem 32. *If $M = M_{\mathfrak{T}} \times_f M_\lambda$ is a mixed totally geodesic pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$, following cases arise:*

(i) If $\xi \in \Gamma(TM)$, then M is either a trivial product or a warped product of a holomorphic (complex) submanifold and a totally real submanifold

(ii) If $\xi \in \Gamma(TM^{\perp})$, then M is either a trivial product or a warped product of two complex submanifolds

Proof. Using definition (59), M satisfies $h(X, U) = 0$ as well as $h(\varphi X, U) = 0$ (as $M_{\mathfrak{T}}$ is φ-invariant) for $X \in \Gamma(TM_{\mathfrak{T}})$ and $U \in \Gamma(TM_\lambda)$. Using this condition in proposition (58) when $\xi \in \Gamma(TM)$, we get

$$[(1 - \lambda(\theta))]X - \eta(X)\xi](\ln f)g(V, U) = 0. \tag{66}$$

Indicate either $\ln f = 0$ implies the trivial case or $[(1 - \lambda(\theta)]X = \eta(X)\xi$, after taking inner product with ξ, and in the view of Remark 8, the condition for the totally real holds for the submanifold M_λ. Following similar way for the second case, we ended up with $\lambda(\theta) = 1$ which is the condition for complex submanifold. □

Theorem 33. *A mixed totally geodesic pointwise semislant warped product submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ of a para-cosymplectic manifold $\widehat{M}$ satisfies*

$$\varepsilon(ntU \ln f)\|X\|^2 = g\left(\left(\widehat{\nabla}_X t\right)X, tU\right), \tag{67}$$

$\forall$ spacelike (or timelike) vector fields $X \in \Gamma(TM_{\mathfrak{T}})$ and $U \in \Gamma(TM_\lambda)$.

Proof. As $g(\widehat{\nabla}_X tX, tU) = g(\widehat{\nabla}_X t)X, tU) + g(t\widehat{\nabla}_X X, tU)$, under the effects of Equation (14), Proposition 5, and Lemma 9, it turns

$$g\left(\widehat{\nabla}_X tX, tU\right) = g\left(\widehat{\nabla}_X t\right)X, tU\right) + \lambda(\theta)(U \ln f)g(X, X). \tag{68}$$

Other way, $g(\widehat{\nabla}_X tX, tU) + g(\widehat{\nabla}_X nX, tU) = g(\widehat{\nabla}_X \varphi X, tU)$; with this expression under the use of Equations (5), (7), and (12), Proposition 5, and Lemma 9, $g(\widehat{\nabla}_X nX, tU) = 0$ and M as mixed totally geodesic, we have

$$g\left(\widehat{\nabla}_X tX, tU\right) = \lambda(\theta)(U \ln f)g(X, X) + \varepsilon(ntU \ln f)\|X\|^2. \tag{69}$$

This expression with the use of Equation (68) and for the reason vector field X can be spacelike or timelike yields the result. □

Example 34. Consider a 7-dimensional smooth manifold $\widehat{M} = \mathbb{R}^6 \times \mathbb{R}^+ \subset \mathbb{R}^7$ having standard Cartesian coordinates as $(y_1, y_2, y_3, y_4, y_5, y_6, t)$ and defining a structure (φ, ξ, η, g) as

$$\begin{aligned} &\varphi e_1 = e_4, \varphi e_2 = e_5, \varphi e_3 = e_6, \varphi e_4 = e_1, \varphi e_5 = e_2, \\ &\varphi e_6 = e_3, \varphi e_7 = 0, \\ &\xi = e_7, \eta = dt \text{ and } g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) \\ &= g(e_7, e_7) = 1, \\ &g(e_4, e_4) = g(e_5, e_5) = g(e_6, e_6) = -1, \end{aligned} \tag{70}$$

for $\{e_1, \cdots, e_7\}$ as a local orthonormal frame on $\Gamma(T\widehat{M})$. Obviously, $\widehat{M}$ over (φ, ξ, η, g) fufills the condition of para-cosymplectic manifold. Let M be a submanifold of $\widehat{M}$ with

ξ tangent to it and defined for $x, y \in \mathbb{R}$, $u, v \geq 0$ and some k(constant) as

$$M(x, y, u, v, z) = ((x+y)\cosh u, (x+y)\sinh v, u, (x+y) \cdot \sinh u, (x+y)\cosh v, v, kx - ky + z). \tag{71}$$

The vector fields that generates the tangent bundle TM are

$$\begin{aligned} X &= \cosh u e_1 + \sinh v e_2 + \sinh u e_4 + \cosh v e_5 + k e_7, \\ Y &= \cosh u e_1 + \sinh v e_2 + \sinh u e_4 + \cosh v e_5 - k e_7, \\ U &= (x+y)\sinh u e_1 + e_3 + (x+y)\cosh u e_4, \\ V &= (x+y)\cosh v e_2 + (x+y)\sinh v e_5 + e_6, \\ Z &= e_7. \end{aligned} \tag{72}$$

After calculations, it is found that the invariant distributions $(\mathfrak{D}_{\mathfrak{T}} \oplus \lhd\xi\rhd)$ is the span of subspace $\{X, Y, Z\}$ and pointwise slant distribution $\mathfrak{D}_\lambda$ is the span of subspace $\{U, V\}$ with $t^2 = (1/(1 - x^2 - y^2)^2)Id$ such that $x^2 + y^2 \neq 1$; then, the slant coefficient is

(i) $\lambda(\theta) = \cosh^2\theta$ for $x^2 + y^2 < 1$

(ii) $\lambda(\theta) = \cos^2\theta$ for $x^2 + y^2 > 1$

As the distributions $(\mathfrak{D}_{\mathfrak{T}} \oplus \lhd\xi\rhd)$ and $\mathfrak{D}_\lambda$ are integrable, let M_T and M_λ be their respective integral manifolds such that $M = M_{\mathfrak{T}} \times_f M_\lambda$ turns a nontrivial 5-dimensional pointwise semislant warped product submanifold of $\widehat{M}$ with induced metric g(semi-Riemannian) as

$$g = k^2 dX^2 + k^2 dY^2 + dZ^2 + \left(1 - x^2 - y^2\right)\left\{dU^2 - dV^2\right\}, \tag{73}$$

with warping function $f = \sqrt{(1 - x^2 - y^2)}$.

Let M be an another submanifold of $\widehat{M}$ with ξ normal to it and defined for $x, y, u, v \in \mathbb{R}$ as

$$M(x, y, u, v) = (xu, x, u + yv, yu, u + v, xv, 0). \tag{74}$$

Then, the vector fields that generates TM are

$$\begin{aligned} X &= u e_1 + e_2 + v e_6, \\ Y &= v e_3 + u e_4 + e_5, \\ U &= x e_1 + e_3 + y e_4 + e_5, \\ V &= y e_3 + x e_6). \end{aligned} \tag{75}$$

The invariant distribution $\mathfrak{D}_{\mathfrak{T}}$ is the span of subspace $\{X, Y\}$, and pointwise slant distribution $\mathfrak{D}_\lambda$ is the span of subspace $\{U, V\}$ with $t^2 = (x^2/(x^2 - y^2)^2)I$ such that $x^2 \neq y^2$; then, the slant coefficient is

(i) $\lambda(\theta) = \cosh^2\theta$ for $x^2 > y^2$

(ii) $\lambda(\theta) = \cos^2\theta$ for $y^2 > x^2$

As the distributions $\mathfrak{D}_{\mathfrak{T}}$ and $\mathfrak{D}_\lambda$ are integrable, let M_T and M_λ be their respective integral manifolds such that $M = M_{\mathfrak{T}} \times_f M_\lambda$ turns a nontrivial 4-dimensional pointwise semislant warped product submanifold of $\widehat{M}$ with induced metric g(semi-Riemannian) as

$$g = \left(1 + u^2 - v^2\right)dX^2 - \left(1 + u^2 - v^2\right)dY^2 + \left(x^2 - y^2\right)\left\{dU^2 - dV^2\right\}, \tag{76}$$

with warping function $f = \sqrt{(x^2 - y^2)}$.

7. Inequalities

This section includes the geometric sharp inequalities for the aforesaid submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ for the case ξ tangent and normal to M.

Lemma 35 (see [31]). *Let $M = M_{\mathfrak{T}} \times_f M_\lambda$ be a pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$. Then, M ensures*

$$g(h(X, U), \zeta) = -g(h(X, \varphi U), \varphi\zeta), \tag{77}$$

$$g(h(X, U), \varphi\zeta) = -g\left(\nabla^\perp_X \varphi U, \zeta\right), \tag{78}$$

$$g(h(X, U), \varphi\zeta) = -g\left(\nabla^\perp_Z \varphi U, \zeta\right), \tag{79}$$

$\forall X \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$, $U \in \Gamma(\mathfrak{D}_\lambda)$, and $\zeta \in \Gamma(\nu)$.

Theorem 36. *Let $M = M_{\mathfrak{T}} \times_f M_\lambda$ be a pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$ with $\xi \in \Gamma(TM)$. If $M_{\mathfrak{T}}$ is an invariant submanifold of $(2n_1 + 1)$-dimension and M_λ is a proper pointwise slant submanifold $2n_2$-dimension satisfying $\nabla^\perp \varphi\mathfrak{D}_\lambda \subseteq \varphi\mathfrak{D}_\lambda$, the succeeding inequalities holds for h*

$$\begin{aligned} \|h\|^2 &\geq n_2\left(1 + \lambda^2(\theta)\|\nabla \ln f\|^2 + \left\|h_\nu^{\mathfrak{D}_{\mathfrak{T}}}\right\|^2 \text{for } S_1 \geq S_2, \\ \|h\|^2 &\leq n_2\left(1 + \lambda^2(\theta)\|\nabla \ln f\|^2 + \left\|h_\nu^{\mathfrak{D}_{\mathfrak{T}}}\right\|^2 \text{for } S_1 \leq S_2, \end{aligned} \tag{80}$$

where $\lambda(\theta)$ is the slant coefficient corresponding to M_λ, $\nabla(\ln f)$ is the gradient of $\ln f$, $\|h_\nu^{\mathfrak{D}_{\mathfrak{T}}}\|^2 = g(h_\nu(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}}), h_\nu(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}}))$ with its ν component and invariant distribution $\mathfrak{D}_{\mathfrak{T}}$, $S_1 = (h^s_{rt})^2 + (h^s_{r't'})^2$, and $S_2 = (h^s_{r't})^2 + (h^s_{rt'})^2$.

Proof. For $\xi \in \Gamma(TM)$, choose the local orthonormal frame on the following:

(a) $M_{\mathfrak{T}}$ by $\{e_i, e_{i'} = \varphi e_i\}$ for $i = \{1, \cdots, n_1$ and $e_i = \xi$ for $i = 2n_1 + 1$ such that $g(e_i, e_i) = \varepsilon_i = 1$ implies $g(e_{i'}, e_{i'}) = \varepsilon_{i'} = -1$ and $g(\xi, \xi) = \varepsilon_0 = 1$

(b) M_λ by $\{\bar{e}_r, \bar{e}_{r'} = (1/\sqrt{|-\lambda(\theta)|}) t\bar{e}_r\}$ for $r = \{1, \cdots, n_2\}$ and such that $g(\bar{e}_r, \bar{e}_r) = \bar{\varepsilon}_r = 1$ implies $g(\bar{e}_{r'}, \bar{e}_{r'}) = \bar{\varepsilon}_{r'} = -1$

(c) (nM_λ) by $\bar{e}_s = (1/\sqrt{|-(1-\lambda(\theta))|}) n\bar{e}_r$ for $r = \{1, \cdots, n_2\}$, having $g(\bar{e}_s, \bar{e}_s) = \bar{\varepsilon}_s = -1$ and on ν by $\{\zeta_l, \zeta_{l'} = \varphi\zeta\}$ such that $g(\zeta_l, \zeta_l) = \varepsilon_l = 1$ implies $g(\zeta_{l'}, \zeta_{l'}) = \varepsilon_{l'} = -1$

Compute $\|h\|^2$ which is given as

$$\|h\|^2 = \|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})\|^2 + 2\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda)\|^2 + \|h(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda)\|^2. \tag{81}$$

The first term $\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})\|^2$ can be expanded as

$$\begin{aligned}
\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})\|^2 &= g(h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}}), h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})) \\
&= \sum_{i,j=0}^{n_1} \left[\varepsilon_i \varepsilon_j g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right)\right. \\
&\quad + \varepsilon_{i'} \varepsilon_j g\left(h\left(e_{i'}, e_j\right), h\left(e_{i'}, e_j\right)\right) \\
&\quad + \varepsilon_i \varepsilon_{j'} g\left(h\left(e_i, e_{j'}\right), h\left(e_i, e_{j'}\right)\right) \\
&\quad \left.+ \varepsilon_{i'} \varepsilon_{j'} g\left(h\left(e_{i'}, e_{j'}\right), h\left(e_{i'}, e_{j'}\right)\right)\right] \\
&\quad + \sum_{i=0}^{n_1} [\varepsilon_0 \varepsilon_i g(h(e_0, e_i), h(e_0, e_i)) \\
&\quad + \varepsilon_i \varepsilon_0 g(h(e_i, e_0), h(e_i, e_o))].
\end{aligned} \tag{82}$$

As $\mathfrak{D}_{\mathfrak{T}}$ is totally geodesic and Equation (54) of Lemma 27 directs that $h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}}) \in \nu$ using which, we can write

$$\begin{aligned}
&h\left(e_i, e_j\right) = h_{ij}^l \zeta_l + h_{ij}^{l'} \zeta_{l'}, h\left(e_{i'}, e_j\right) = h_{i'j}^l \zeta_l + h_{i'j}^{l'} \zeta_{l'}, \\
&h\left(e_i, e_{j'}\right) = h_{ij'}^l \zeta_l + h_{ij'}^{l'} \zeta_{l'}, h\left(e_{i'}, e_{j'}\right) = h_{i'j'}^l \zeta_l + h_{i'j'}^{l'} \zeta_{l'}, \\
&h(e_0, e_i) = h_{0i}^l \zeta_l + h_{0i}^{l'} \zeta_{l'}, h(e_i, e_0) = h_{i0}^l \zeta_l + h_{i0}^{l'} \zeta_{l'}, \\
&h(e_0, e_0) = h_{00}^l \zeta_l + h_{00}^{l'} \zeta_{l'}.
\end{aligned} \tag{83}$$

Simplifying these expressions in Equation (82) and using Equation (19) and Lemma 4 and in view of orthonormal frame, we get

$$\begin{aligned}
\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})\|^2 = \sum_{i,j=1}^{n_1} \sum_{l=1}^{n_3} &\left[\left\{\left(h_{ij}^l\right)^2 - \left(h_{ij}^{l'}\right)^2\right\} - \left\{\left(h_{ij'}^l\right)^2 - \left(h_{i'j}^{l'}\right)^2\right\}\right. \\
&\left. - \left\{\left(h_{ij'}^l\right)^2 - \left(h_{ij'}^{l'}\right)^2\right\} + \left(h_{i'j'}^l\right)^2 - \left(h_{i'j'}^{l'}\right)^2\right\}\right].
\end{aligned} \tag{84}$$

The integrable condition of the $\mathfrak{D}_{\mathfrak{T}}$ and Equation (77) of the Lemma 35 implies that

$$\begin{aligned}
&\left(h_{i'j}^l\right)^2 = \left(h_{ij'}^l\right)^2, \left(h_{i'j}^{l'}\right)^2 = \left(h_{ij'}^{l'}\right)^2, \\
&\left(h_{i'j'}^l\right)^2 = \left(h_{i'j'}^l\right)^2, \left(h_{i'j'}^{l'}\right)^2 = \left(h_{ij}^{l'}\right)^2,
\end{aligned} \tag{85}$$

$$\left(h_{i'j}^l\right)^2 = \left(h_{i'j}^{l'}\right)^2, \left(h_{ij}^{l'}\right)^2 = \left(h_{i'j}^l\right)^2. \tag{86}$$

After substitution of Equation (84) in (86), we get

$$\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})\|^2 = 4 \sum_{i,j=1}^{n_1} \sum_{l=1}^{n_2} \left[\left(h_{ij}^l\right)^2 - \left(h_{ij}^{l'}\right)^2\right] = \left\|h_\nu^{\mathfrak{D}_{\mathfrak{T}}}\right\|^2. \tag{87}$$

For the second part, we have

$$\begin{aligned}
\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda)\|^2 &= g(h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda), h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda)) \\
&= \sum_{i=1}^{n_1} \sum_{r=1}^{n_2} [\varepsilon_i \bar{\varepsilon}_r g(h(e_i, \bar{e}_r), h(e_i, \bar{e}_r)) \\
&\quad + \varepsilon_{i'} \bar{\varepsilon}_r g(h(e_{i'}, \bar{e}_r), h(e_{i'}, \bar{e}_r)) \\
&\quad + \varepsilon_i \bar{\varepsilon}_{r'} g(h(e_i, \bar{e}_{r'}), h(e_i, \bar{e}_{r'})) \\
&\quad + \varepsilon_{i'} \bar{\varepsilon}_{r'} g(h(e_{i'}, \bar{e}_{r'}), h(e_{i'}, \bar{e}_{r'})) \\
&\quad + \varepsilon_0 \bar{\varepsilon}_r g(h(e_0, \bar{e}_r), h(e_i, \bar{e}_r)) \\
&\quad + \varepsilon_0 \bar{\varepsilon}_{r'} g(h(e_0, \bar{e}_{r'}), h(e_0, \bar{e}_{r'}))],
\end{aligned} \tag{88}$$

where

$$h(e_i, \bar{e}_r) = h_{ir}^s \bar{e}_s + h_{ir}^l \zeta_l + h_{ir}^{l'} \zeta_{l'}, h(e_{i'}, \bar{e}_r) = h_{i'r}^s \bar{e}_s + h_{i'r}^l \zeta_l + h_{i'r}^{l'} \zeta_{l'}, \tag{89}$$

$$\begin{aligned}
h(e_i, \bar{e}_{r'}) &= h_{ir'}^s \bar{e}_s + h_{ir'}^l \zeta_l + h_{ir'}^{l'} \zeta_{l'}, h(e_{i'}, \bar{e}_{r'}) \\
&= h_{i'r'}^s \bar{e}_s + h_{i'r'}^l \zeta_l + h_{i'r'}^{l'} \zeta_{l'},
\end{aligned} \tag{90}$$

$$\begin{aligned}
h(e_0, \bar{e}_r) &= h_{0r}^s \bar{e}_s + h_{0r}^l \zeta_l + h_{0r}^{l'} \zeta_{l'}, h(e_0, \bar{e}_{r'}) \\
&= h_{0r'}^s \bar{e}_s + h_{0r'}^l \zeta_l + h_{0r'}^{l'} \zeta_{l'}.
\end{aligned} \tag{91}$$

After simplifying Equation (88) using expressions in Equation (91), we get

$$\begin{aligned}
\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda)\|^2 = \sum_{i=1}^{n_1} \sum_{r,s=1}^{n_2} &\left[(h_{ir}^s)^2 + \left(h_{i'r'}^s\right)^2 - \left(h_{ir'}^s\right)^2 - \left(h_{ir'}^s\right)^2\right. \\
&+ \left\{\left(h_{ir}^l\right)^2 + \left(h_{i'r'}^l\right)^2 - \left(h_{i'r}^l\right)^2 - \left(h_{ir'}^l\right)^2\right\} \\
&- \left\{\left(h_{ir}^{l'}\right)^2 + \left(h_{i'r'}^{l'}\right)^2 - \left(h_{i'r}^{l'}\right)^2 - \left(h_{ir'}^{l'}\right)^2\right\}.
\end{aligned} \tag{92}$$

Using Equations (55), (56), (58), (77), and (78), we have

$$\begin{aligned} &h^s_{ir} = -e_{i'}(\ln f)g(\bar{e}_r, \bar{e}_s), h^s_{i'r} = -e_i(\ln f)g(\bar{e}_r, \bar{e}_s),\\ &h^s_{ir'} = -\lambda(\theta)e_i(\ln f)g(\bar{e}_r, \bar{e}_s),\\ &h^{l'}_{i'r'} = -\lambda(\theta)e_{i'}(\ln f)g(\bar{e}_r, \bar{e}_s), \left(h^l_{ir}\right)^2 = \left(h^{l'}_{ir'}\right)^2,\\ &\left(h^l_{i'r}\right)^2 = \left(h^{l'}_{i'r'}\right)^2 \text{ and } \left(h^{l'}_{ir}\right)^2 = \left(h^l_{ir'}\right)^2. \end{aligned} \tag{93}$$

Substituting above values in Equation (92), we have

$$\begin{aligned} \|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda)\|^2 = {} & n_2\left(1+\lambda^2(\theta)\right)\sum_{i=1}^{n_1}\left[(e_i(\ln f))^2 - (e_{i'}(\ln f))^2\right]\\ & + 2\left[\left\{\left(h^l_{i'r}\right)^2 - \left(h^{l'}_{i'r}\right)^2\right\} - \left\{\left(h^l_{ir}\right)^2 - \left(h^{l'}_{ir}\right)^2\right\}\right]. \end{aligned} \tag{94}$$

Since $\sum_{i=1}^{n_1}[(e_i(\ln f))^2 - (e_{i'}(\ln f))^2] = g(\nabla(\ln f),\nabla(\ln f)) = \|\nabla(\ln f)\|^2$ and using the condition that $\nabla^\perp\varphi(D_\lambda) \subseteq \varphi(D_\lambda)$ in formula (78), we concluded that $h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda) \subseteq \varphi(\mathfrak{D}_\lambda)$, above equation leads to

$$\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda)\|^2 = n_2\left(1+\lambda^2(\theta)\right)\|\nabla(\ln f)\|^2. \tag{95}$$

Lastly,

$$\begin{aligned} \left\|h\left(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda\right)\right\|^2 &= g(h(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda), h(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda))\\ &= \sum_{r,t=1}^{n_2}[\bar{\varepsilon}_r\bar{\varepsilon}_t g(h(\bar{e}_r, \bar{e}_t), h(\bar{e}_r, \bar{e}_t))\\ &\quad + \bar{\varepsilon}_{r'}\bar{\varepsilon}_t g(h(\bar{e}_{r'}, \bar{e}_t), h(\bar{e}_{r'}, \bar{e}_t))\\ &\quad + \bar{\varepsilon}_r\bar{\varepsilon}_{t'} g(h(\bar{e}_r, \bar{e}_{t'}), h(\bar{e}_r, \bar{e}_{t'}))\\ &\quad + \bar{\varepsilon}_{r'}\bar{\varepsilon}_{t'} g(h(\bar{e}_{r'}, \bar{e}_{t'}), h(\bar{e}_{r'}, \bar{e}_{t'})), \end{aligned} \tag{96}$$

where the included expressions are as below:

$$\begin{aligned} h(\bar{e}_r, \bar{e}_t) &= h^s_{rt}\bar{e}_s + h^l_{rt}\zeta_l + h^{l'}_{rt}\zeta_{l'},\\ h(\bar{e}_{r'}, \bar{e}_t) &= h^s_{r't}\bar{e}_s + h^l_{r't}\zeta_l + h^{l'}_{r't}\zeta_{l'}\\ h(\bar{e}_r, \bar{e}_{t'}) &= h^s_{rt'}\bar{e}_s + h^l_{rt'}\zeta_l + h^{l'}_{rt'}\zeta_{l'},\\ h(\bar{e}_{r'}, \bar{e}_{t'}) &= h^s_{r't'}\bar{e}_s + h^l_{r't'}\zeta_l + h^{l'}_{r't'}\zeta_{l'}. \end{aligned} \tag{97}$$

Employing these expressions in Equation (96) in view of the chosen frame and simplifying, we get

$$\begin{aligned} \left\|h\left(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda\right)\right\|^2 = \sum_{r,s,t=1}^{n_2} & \left[\left\{\left(h^s_{rt}\right)^2 + \left(h^l_{rt}\right)^2 - \left(h^{l'}_{rt}\right)^2\right\}\right.\\ & - \left\{\left(h^s_{r't}\right)^2 + \left(h^l_{r't}\right)^2 - \left(h^{l'}_{r't}\right)^2\right\}\\ & - \left\{\left(h^s_{rt'}\right)^2 + \left(h^l_{rt'}\right)^2 - \left(h^{l'}_{rt'}\right)^2\right\}\\ & \left. + \left\{\left(h^s_{r't'}\right)^2 + \left(h^l_{r't'}\right)^2 - \left(h^{l'}_{r't'}\right)^2\right\}\right]. \end{aligned} \tag{98}$$

Using the condition that $\nabla^\perp\varphi(D_\lambda) \subseteq \varphi(D_\lambda)$ in formula (79), we concluded that $h(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda) \subseteq \varphi(\mathfrak{D}_\lambda)$, which implies the Equation (98) with

$$\begin{aligned} \left\|h\left(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda\right)\right\|^2 = \sum_{r,s,t=1}^{n_2} & \left[\left\{\left(h^s_{rt}\right)^2 + \left(h^s_{r't'}\right)^2\right\}\right.\\ & \left. - \left\{\left(h^s_{r't}\right)^2 + \left(h^s_{rt'}\right)^2\right\}\right]. \end{aligned} \tag{99}$$

Result directly follows by letting $S_1 = (h^s_{rt})^2 + (h^s_{r't'})^2$ and $S_2 = (h^s_{r't})^2 + (h^s_{rt'})^2$. □

Remark 37. Equality holds if $S_1 = S_2$.

Theorem 38. *Let $M = M_{\mathfrak{T}} \times_f M_\lambda$ be a pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$ with ξ normal to M such that $\xi \in \Gamma(\nu)$. If $M_{\mathfrak{T}}$ is an invariant submanifold of $2n_1$-dimension and M_λ is a proper pointwise slant submanifold of $2n_2$-dimension satisfying $\nabla^\perp\varphi\mathfrak{D}_\lambda \subseteq \varphi\mathfrak{D}_\lambda$, the succeeding inequalities holds for h*

$$\begin{aligned} \|h\|^2 &\geq n_2\left(1+\lambda^2(\theta)\|\nabla\ln f\|^2 + \left\|h_\nu^{\mathfrak{D}_{\mathfrak{T}}}\right\|^2 \text{ for } S_1 \geq S_2,\\ \|h\|^2 &\leq n_2\left(1+\lambda^2(\theta)\|\nabla\ln f\|^2 + \left\|h_\nu^{\mathfrak{D}_{\mathfrak{T}}}\right\|^2 \text{ for } S_1 \leq S_2, \end{aligned} \tag{100}$$

where $\lambda(\theta)$ is the slant coefficient corresponding to M_λ, $\nabla(\ln f)$ is the gradient of $\ln f$, $\|h_\nu^{\mathfrak{D}_{\mathfrak{T}}}\|^2 = g(h_\nu(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}}), h_\nu(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}}))$ with its ν component and invariant distribution $\mathfrak{D}_{\mathfrak{T}}$, $S_1 = (h^s_{rt})^2 + (h^s_{r't'})^2$, and $S_2 = (h^s_{r't})^2 + (h^s_{rt'})^2$.

Proof. For $\xi \in \Gamma(TM^\perp)$, choose the local orthonormal frame on the following:

(a) $M_{\mathfrak{T}}$ by $\{e_i, e_{i'} = \varphi e_i\}$ for $i = \{1, \cdots, n_1\}$ such that $g(e_i, e_i) = \varepsilon_i = 1$ implies $g(e_{i'}, e_{i'}) = \varepsilon_{i'} = -1$

(b) M_λ by $\{\bar{e}_r, \bar{e}_{r'} = (1/\sqrt{|-\lambda(\theta)|})t\bar{e}_r\}$ for $r = \{1, \cdots, n_2\}$ and such that $g(\bar{e}_r, \bar{e}_r) = \bar{\varepsilon}_r = 1$ implies $g(\bar{e}_{r'}, \bar{e}_{r'}) = \bar{\varepsilon}_{r'} = -1$

(c) (nM_λ) by $\bar{e}_s = (1/\sqrt{|-(1-\lambda(\theta))|})n\bar{e}_r$ for $r = \{1, \cdots, n_2\}$, having $g(\bar{e}_s, \bar{e}_s) = \bar{\varepsilon}_s = -1$ and on ν by $\{\zeta_l, \zeta_{l'} = \varphi\zeta\}$ for $l = \{1, \cdots, n_3\}$ and $\zeta_l = \xi$ for $l = 2n_3 + 1$ such that $g(\zeta_l, \zeta_l) = \varepsilon_l = 1$ implies $g(\zeta_{l'}, \zeta_{l'}) = \varepsilon_{l'} = -1$ and $g(\xi, \xi) = \varepsilon_0 = 1$

Further, result can be acquired carrying the same steps as above proof and using Equations (33) and (34) of Lemma 18. □

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors declare no competing interest.

Authors' Contributions

M.D. and S.K.S conceptualized the study. A.A. was responsible for the methodology. F.M. was responsible for the software. W.A.M.O, F.M., and M.D. were responsible for the validation. A.A. was responsible for the formal analysis. S.K.S. was responsible for the investigation. S.K.S was responsible for the resources. A.A. was responsible for the data curation. S.K.S and M.D wrote the original draft. A.A. and F.M. wrote, reviewed, and edited the manuscript. F.M. was responsible for the visualization. S.K.S. supervised the study. F.M. was responsible for the project administration. A.A was responsible for the funding acquisition. All authors have read and agreed to the published version of the manuscript.

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Research Article

Asymmetric Bidirectional Controlled Quantum Teleportation of Three- and Four-Qubit States

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In this paper, we theoretically realize bidirectional controlled quantum teleportation by using ten-qubit entangled state method. This paper uses a case to introduce the specific process of realizing quantum teleportation: Alice sends an unknown four-qubit GHZ state to Bob, and Bob sends an arbitrary three-qubit GHZ state to Alice. In addition, Charlie controls the transfer to ensure the integrity of the protocol. A ten-qubit quantum channel is constructed and used in this paper. Then, the unitary matrix transformation is used to complete the communication protocol. The research results show that the communication protocol constructed in this paper is more efficient than most communication protocols.

1. Introduction

Quantum information has become increasingly popular in recent years. Quantum communication is a new communication method which uses quantum superposition state and quantum entanglement effect to transmit information. Quantum communication is based on three principles, along with uncertainty, measurement collapse, and no-cloning theorem in quantum mechanics. Quantum communication is an absolutely secure means of communication that cannot be eavesdropped or cracked. Quantum communication is mainly divided into quantum teleportation and quantum key distribution. This paper studies the communication mode of quantum teleportation.

In this paper, BQCT by using ten-qubit entangled state is devised. Alice has unknown qubit state A, B, C, D, a, b, c, d; Bob has unknown qubit state E, F, G, e, f, g, h, i; and Charlie has unknown qubit state e. Alice sends arbitrary four-qubit GHZ state to Bob, Bob transmits unknown three-qubit GHZ state to Alice, and ten-qubit entangled state is used as quantum channel. Alice performs a five-qubit GHZ-state measurement on qubits A, B, C, D, a; and Bob operates a four-qubit GHZ-state measurement on qubits E, F, G, f. Both Alice and Bob tells Charlie to the basis of measurement, and Charlie controls the process of the protocol. If Charlie believes the protocol is safety, Charlie measures the remaining quantum state using single-qubit basis and tells Alice and Bob about information of the used basis. Alice and Bob can obtain the initial state by appropriate unitary operations. In contrast, this protocol efficiency is relatively high.

2. Literature Review

In 1935, Einstein et al. proposed a paradox to prove the incompleteness of quantum mechanics, which is referred to as "EPR paradox" [1]. In 1964, Bell presented Bell inequality to support localized realism and can prove the completeness of quantum mechanics in mathematics [2].

In the field of quantum information, quantum teleportation is very important. In 1993, quantum teleportation was first proposed [3]. In 2013, Zha et al. present the first bidirectional quantum controlled teleportation (BQCT) protocol [4]. In 2016, the scheme which has three controllers was proposed for BCQT via seven-qubit entangled state to convey one-qubit each other [5]. In 2017, Zadeh et al. presented bidirectional quantum teleportation (BQT) without controller to teleport an arbitrary two-qubit state to each other simultaneously via an eight-qubit entangled state [6]. In 2018, Sarvaghad-Moghaddam et al. used five-qubit entangled states as a quantum channel to teleport one-qubit each other under permission of controller [7]. In 2019, Zhou et al. used six-qubit cluster state to send single-qubit and three-qubit GHZ state to each other [8]. In 2020, Zhou et al. proposed BQCT of two-qubit states through seven-qubit entangled state [9]. Protocol which transmits two-qubit each other and two-qubit and three-qubit each other about six-qubit quantum channel was reported as well [10]. In 2021, Jiang et al. presented BQCT of three-qubit GHZ state through an entangled eleven-qubit quantum channel [11] and Huo et al. presented asymmetric BCQT of two- and three-qubit states via an entangled eleven-qubit quantum channel [12]. In 2022, Kazemikhah et al. present asymmetric bidirectional controlled quantum teleportation protocol of two-qubit and three-qubit unknown states using eight-qubit cluster state [13].

3. Construction of Quantum Channel

Quantum communication is a new communication method which uses quantum superposition state and quantum entanglement effect to transmit information. Quantum communication is an absolutely secure means of communication that cannot be eavesdropped or cracked. Therefore, in this paper, the quantum channel adopted is

$$|\Psi\rangle_{abccdefjhij} = \frac{1}{2}\Big(|0000000000\rangle_{abcdefjhij} + |0000011111\rangle_{abcdefjhij} \cdot + |1111100000\rangle_{abcdefjhij} + |1111111111\rangle_{abcdefjhij}\Big). \tag{1}$$

This quantum channel can not only be theoretically proposed but also constructed. The step method is as follows.

Step 1. The ten-qubit initial state is prepared like

$$|\Psi_0\rangle_{abccdefjhij} = |0\rangle_a \otimes |0\rangle_b \otimes |0\rangle_c \otimes |0\rangle_d \otimes |0\rangle_e \otimes |0\rangle_f \otimes |0\rangle_g \otimes |0\rangle_h \otimes |0\rangle_i \otimes |0\rangle_j. \tag{2}$$

Step 2. Two Hadamard gates are implemented to qubits a and f. Then, the state $|\psi\rangle_{abcdefghij}$ changes into

$$|\Psi_1\rangle_{abccdefjhij} = \frac{\left(|0\rangle_a + |1\rangle_a\right)}{\sqrt{2}} \otimes |0\rangle_b \otimes |0\rangle_c \otimes |0\rangle_d \otimes |0\rangle_e \otimes \frac{\left(|0\rangle_f + |1\rangle_f\right)}{\sqrt{2}} \otimes |0\rangle_g \otimes |0\rangle_h \otimes |0\rangle_i \otimes |0\rangle_j. \tag{3}$$

Step 3. When qubit a can be control qubits and qubits b, c, d, e are target qubits, CNOT gates operate on $|\Psi_1\rangle_{abccdefjhij}$. In the same way, CNOT gates operate on $|\Psi_1\rangle_{abccdefjhij}$ when qubits f can be control qubits and qubits g, h, i, j are target qubits. We can obtain the quantum channel $|\Psi_1\rangle_{abccdefjhij}$.

4. Bidirectional Quantum Controlled Teleportation

4.1. Quantum Teleportation. Suppose Alice has an arbitrary four-qubit GHZ state

$$|\Psi\rangle_{ABCD} = \alpha|0000\rangle_{ABCD} + \beta|1111\rangle_{ABCD}. \tag{4}$$

And Bob has an arbitrary three-qubit GHZ state

$$|\Psi\rangle_{EFG} = \upsilon|000\rangle_{EFG} + \mu|111\rangle_{EFG}, \tag{5}$$

where $|\alpha|^2 + |\beta|^2 = 1, |\nu|^2 + |\mu|^2 = 1$. Alice and Bob do not know what α, β, ν, and μ are. Alice wants to transmit A, B, C, D to Bob who wants to transmit E, F, G to Alice through ten-qubit quantum channel. Supervisor Charlie who has qubit e controls whether or not the protocol continues. We have ten-qubit state quantum channel

$$|\Psi\rangle_{abccdefjhij} = \frac{1}{2}\Big[|0000000000\rangle_{abcdefjhij} + |0000011111\rangle_{abcdefjhij} + |1111100000\rangle_{abcdefjhij} + |1111111111\rangle_{abcdefjhij}\Big]. \tag{6}$$

Here, qubits a, b, c, d belong to Alice, qubits f, g, h, i belong to Bob, and qubit j belongs to Charlie, respectively. The initial state of the total system is

$$|\Psi\rangle_{ABCDEFGabcdefghij} = |\Psi\rangle_{ABCD} \otimes |\Psi\rangle_{EFG} \otimes |\Psi\rangle_{abcdefghij}. \quad (7)$$

Four-qubit GHZ states which form a set of basis can be described as

$$\begin{aligned}
|\xi_1^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle \pm |1111\rangle), |\xi_2^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0001\rangle \pm |1110\rangle),\\
|\xi_3^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|0011\rangle \pm |1100\rangle), |\xi_4^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0111\rangle \pm |1000\rangle),\\
|\xi_5^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|0101\rangle \pm |1010\rangle), |\xi_6^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0110\rangle \pm |1001\rangle),\\
|\xi_7^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|0100\rangle \pm |0100\rangle), |\xi_8^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0010\rangle \pm |1111\rangle).
\end{aligned} \quad (8)$$

Five-qubit GHZ states which form a set of basis can be described as

$$\begin{aligned}
|\gamma_1^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|00000\rangle \pm |11111\rangle), |\gamma_2^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00001\rangle \pm |11110\rangle),\\
|\gamma_3^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|00010\rangle \pm |11101\rangle), |\gamma_4^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00100\rangle \pm |11011\rangle),\\
|\gamma_5^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|01000\rangle \pm |10111\rangle), |\gamma_6^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00011\rangle \pm |11100\rangle),\\
|\gamma_7^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|00110\rangle \pm |11001\rangle), |\gamma_8^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01100\rangle \pm |10011\rangle),\\
|\gamma_9^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|00111\rangle \pm |11000\rangle), |\gamma_{10}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01110\rangle \pm |10001\rangle),\\
|\gamma_{11}^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|01101\rangle \pm |10010\rangle), |\gamma_{12}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01011\rangle \pm |10100\rangle),\\
|\gamma_{13}^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|01110\rangle \pm |10001\rangle), |\gamma_{14}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01111\rangle \pm |10000\rangle),\\
|\gamma_{15}^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|01010\rangle \pm |10101\rangle), |\gamma_{16}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00101\rangle \pm |11010\rangle).
\end{aligned} \quad (9)$$

Alice can carry out a five-qubit GHZ-state measurement on qubits A, B, C, D, a, and Bob can carry out a four-qubit GHZ-state measurement on qubits E, F, G, f. Then, quantum state $|\Psi\rangle_{ABCDEFGabcdefghij}$ can be expressed as

$$\begin{aligned}
|\Psi\rangle_{ABCDEFGabcdefghij} = {}& \alpha\nu\Big[(|\gamma_1^+\rangle + |\gamma_1^-\rangle)_{ABCDa}\left(|\xi_1^+\rangle + |\xi_1^-\rangle\right)_{EFGf}\Big|00000000\rangle_{bcdeghij} + (|\gamma_2^+\rangle + |\gamma_2^-\rangle)_{ABCDa}\left(|\xi_1^+\rangle + |\xi_1^-\rangle\right)_{EFGf}\Big|11100000\rangle_{bcdeghij}\\
& + (|\gamma_1^+\rangle \quad + |\gamma_1^-\rangle)_{ABCDa}\left(|\xi_2^+\rangle + |\xi_2^-\rangle\right)_{EFGf}\Big|00001111\rangle_{bcdeghij} + (|\gamma_2^+\rangle + |\gamma_2^-\rangle)_{ABCDa}\left(|\xi_2^+\rangle + |\xi_2^-\rangle\right)_{EFGf}\Big|11111111\rangle_{bcdeghij}\Big]\\
& + \beta\nu\Big[(|\gamma_2^+\rangle - |\gamma_2^-\rangle)_{ABCDa}\left(|\xi_1^+\rangle + |\xi_1^-\rangle\right)_{EFGf}\Big|00000000\rangle_{bcdeghij} + (|\gamma_1^+\rangle - |\gamma_1^-\rangle)_{ABCDa}\left(|\xi_1^+\rangle + |\xi_1^-\rangle\right)_{EFGf}\Big|11100000\rangle_{bcdeghij}\\
& + (|\gamma_2^+\rangle - |\gamma_2^-\rangle)_{ABCDa}\left(|\xi_2^+\rangle + |\xi_2^-\rangle\right)_{EFGf}\Big|00001111\rangle_{bcdeghij} + (|\gamma_1^+\rangle - |\gamma_1^-\rangle)_{ABCDa}\left(|\xi_2^+\rangle + |\xi_2^-\rangle\right)_{EFGf}\Big|11111111\rangle_{bcdeghij}\Big]\\
& + \alpha\mu\Big[(|\gamma_1^+\rangle + |\gamma_1^-\rangle)_{ABCDa}\left(|\xi_2^+\rangle - |\xi_2^-\rangle\right)_{EFGf}\Big|00000000\rangle_{bcdeghij} + (|\gamma_2^+\rangle - |\gamma_2^-\rangle)_{ABCDa}\left(|\xi_2^+\rangle + |\xi_2^-\rangle\right)_{EFGf}\Big|11100000\rangle_{bcdeghij}\\
& + (|\gamma_1^+\rangle + |\gamma_1^-\rangle)_{ABCDa}\left(|\xi_1^+\rangle + |\xi_1^-\rangle\right)_{EFGf}\Big|00001111\rangle_{bcdeghij} + (|\gamma_2^+\rangle + |\gamma_2^-\rangle)_{ABCDa}\left(|\xi_2^+\rangle - |\xi_2^-\rangle\right)_{EFGf}\Big|11111111\rangle_{bcdeghij}\Big]\\
& + \beta\mu\Big[(|\gamma_2^+\rangle - |\gamma_2^-\rangle)_{ABCDa}\left(|\xi_2^+\rangle - |\xi_2^-\rangle\right)_{EFGf}\Big|00000000\rangle_{bcdeghij} + (|\gamma_1^+\rangle - |\gamma_1^-\rangle)_{ABCDa}\left(|\xi_2^+\rangle - |\xi_2^-\rangle\right)_{EFGf}\Big|11100000\rangle_{bcdeghij}\\
& + (|\gamma_2^+\rangle - |\gamma_2^-\rangle)_{ABCDa}\left(|\xi_1^+\rangle - |\xi_1^-\rangle\right)_{EFGf}\Big|00001111\rangle_{bcdeghij} + (|\gamma_1^+\rangle - |\gamma_1^-\rangle)_{ABCDa}\left(|\xi_1^+\rangle - |\xi_1^-\rangle\right)_{EFGf}\Big|11111111\rangle_{bcdeghij}\Big]\\
& \cdot \beta\mu\Big[(|\gamma_2^+\rangle - |\gamma_2^-\rangle)_{ABCDa}\left(|\xi_2^+\rangle - |\xi_2^-\rangle\right)_{EFGf}\Big|00000000\rangle_{bcdeghij}\Big] + .
\end{aligned} \quad (10)$$

4.2. Quantum Teleportation Results. As mentioned above, both Alice and Bob tell each other the measurement basis by the classical channel and different basis vectors which Alice and Bob choose and the corresponding collapse state is as Table 1. Then, Charlie is told the measurement results by the classical communication channel. And Charlie can perform single-qubit Von Neumann measurement on $|+\rangle$ or $|-\rangle$ and

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \quad (11)$$

Then, if Charlie wants to continue the protocol, he needs to deliver his result to both Alice and Bob. Finally, Alice and Bob use correct unitary operations on their state to obtain the state teleported by the other party. The different collapse states and unitary operations in $|+\rangle$ or $|-\rangle$ are as Tables 2 and 3. In

TABLE 1: The collapsed states of qubits b, c, d, e, g, h, i, j under Alice's and Bob's GHZ-state measurement.

Alice's results	Bob's results	Collapsed state of qubits b, c, d, e, g, h, i, j
$\vert\gamma_1^+\rangle$	$\vert\xi_1^+\rangle$	$\alpha\nu\vert 00000000\rangle + \beta\nu\vert 11110000\rangle + \alpha\mu\vert 00001111\rangle + \beta\mu\vert 11111111\rangle$
$\vert\gamma_1^+\rangle$	$\vert\xi_1^-\rangle$	$\alpha\nu\vert 00000000\rangle + \beta\nu\vert 11110000\rangle - \alpha\mu\vert 00001111\rangle - \beta\mu\vert 11111111\rangle$
$\vert\gamma_1^+\rangle$	$\vert\xi_2^+\rangle$	$\alpha\nu\vert 00001111\rangle + \beta\nu\vert 11111111\rangle + \alpha\mu\vert 00000000\rangle + \beta\mu\vert 11110000\rangle$
$\vert\gamma_1^+\rangle$	$\vert\xi_2^-\rangle$	$\alpha\nu\vert 00001111\rangle + \beta\nu\vert 11111111\rangle - \alpha\mu\vert 00000000\rangle - \beta\mu\vert 00000000\rangle$
$\vert\gamma_1^-\rangle$	$\vert\xi_1^+\rangle$	$\alpha\nu\vert 00000000\rangle - \beta\nu\vert 11110000\rangle + \alpha\mu\vert 00001111\rangle - \beta\mu\vert 11111111\rangle$
$\vert\gamma_1^-\rangle$	$\vert\xi_1^-\rangle$	$\alpha\nu\vert 00000000\rangle - \beta\nu\vert 11110000\rangle - \alpha\mu\vert 00001111\rangle + \beta\mu\vert 00000000\rangle$
$\vert\gamma_1^-\rangle$	$\vert\xi_1^-\rangle$	$\alpha\nu\vert 00001111\rangle - \beta\nu\vert 11111111\rangle + \alpha\mu\vert 00000000\rangle - \beta\mu\vert 11110000\rangle$
$\vert\gamma_1^-\rangle$	$\vert\xi_1^-\rangle$	$\alpha\nu\vert 00001111\rangle - \beta\nu\vert 11111111\rangle - \alpha\mu\vert 00000000\rangle + \beta\mu\vert 11110000\rangle$
$\vert\gamma_2^+\rangle$	$\vert\xi_1^+\rangle$	$\alpha\nu\vert 11110000\rangle + \beta\nu\vert 00000000\rangle + \alpha\mu\vert 11111111\rangle + \beta\mu\vert 00001111\rangle$
$\vert\gamma_2^+\rangle$	$\vert\xi_1^-\rangle$	$\alpha\nu\vert 11110000\rangle + \beta\nu\vert 00000000\rangle - \alpha\mu\vert 11111111\rangle - \beta\mu\vert 00001111\rangle$
$\vert\gamma_2^+\rangle$	$\vert\xi_2^+\rangle$	$\alpha\nu\vert 11111111\rangle + \beta\nu\vert 00001111\rangle + \alpha\mu\vert 11110000\rangle + \beta\mu\vert 00000000\rangle$
$\vert\gamma_2^+\rangle$	$\vert\xi_2^-\rangle$	$\alpha\nu\vert 11111111\rangle + \beta\nu\vert 00001111\rangle - \alpha\mu\vert 11110000\rangle - \beta\mu\vert 00000000\rangle$
$\vert\gamma_2^-\rangle$	$\vert\xi_1^+\rangle$	$\alpha\nu\vert 11110000\rangle - \beta\nu\vert 00000000\rangle + \alpha\mu\vert 11111111\rangle - \beta\mu\vert 00001111\rangle$
$\vert\gamma_2^-\rangle$	$\vert\xi_1^-\rangle$	$\alpha\nu\vert 11110000\rangle - \beta\nu\vert 11110000\rangle - \alpha\mu\vert 00001111\rangle + \beta\mu\vert 00000000\rangle$
$\vert\gamma_2^-\rangle$	$\vert\xi_2^-\rangle$	$\alpha\nu\vert 11111111\rangle - \beta\nu\vert 00001111\rangle - \alpha\mu\vert 11110000\rangle - \beta\mu\vert 00000000\rangle$
$\vert\gamma_2^-\rangle$	$\vert\xi_2^-\rangle$	$\alpha\nu\vert 11111111\rangle - \beta\nu\vert 00001111\rangle - \alpha\mu\vert 11110000\rangle + \beta\mu\vert 00000000\rangle$

TABLE 2: The specific unitary transformation and collapsed states correspond to Alice's, Bob's, and Charlie's measurement results.

Alice's results	Bob's results	Charlie's results	Collapsed state of qubits b, c, d, g, h, i, j	Alice's unitary operator	Bob's unitary operator
$\vert\gamma_1^+\rangle$	$\vert\xi_1^+\rangle$	$\vert+\rangle$	$(\alpha\vert 000\rangle + \beta\vert 111\rangle) \otimes (\nu\vert 0000\rangle + \mu\vert 1111\rangle)$	$I \otimes I \otimes I$	$I \otimes I \otimes I \otimes I$
$\vert\gamma_1^+\rangle$	$\vert\xi_1^+\rangle$	$\vert-\rangle$	$(\alpha\vert 000\rangle - \beta\vert 111\rangle) \otimes (\nu\vert 0000\rangle + \mu\vert 1111\rangle)$	$Z \otimes I \otimes I$	$I \otimes I \otimes I \otimes I$
$\vert\gamma_1^+\rangle$	$\vert\xi_1^-\rangle$	$\vert+\rangle$	$(\alpha\vert 000\rangle + \beta\vert 111\rangle) \otimes (\nu\vert 0000\rangle - \mu\vert 1111\rangle)$	$I \otimes I \otimes I$	$Z \otimes I \otimes I \otimes I$
$\vert\gamma_1^+\rangle$	$\vert\xi_1^-\rangle$	$\vert-\rangle$	$(\alpha\vert 000\rangle - \beta\vert 111\rangle) \otimes (\nu\vert 0000\rangle - \mu\vert 1111\rangle)$	$Z \otimes I \otimes I$	$Z \otimes I \otimes I \otimes I$
$\vert\gamma_1^+\rangle$	$\vert\xi_2^+\rangle$	$\vert+\rangle$	$(\alpha\vert 000\rangle + \beta\vert 111\rangle) \otimes (\nu\vert 0000\rangle + \mu\vert 1111\rangle)$	$I \otimes I \otimes I$	$I \otimes I \otimes I \otimes I$
$\vert\gamma_1^+\rangle$	$\vert\xi_2^+\rangle$	$\vert-\rangle$	$(\alpha\vert 000\rangle - \beta\vert 111\rangle) \otimes (\mu\vert 0000\rangle + \nu\vert 1111\rangle)$	$Z \otimes I \otimes I$	$X \otimes X \otimes X \otimes X$
$\vert\gamma_1^+\rangle$	$\vert\xi_2^-\rangle$	$\vert+\rangle$	$(\alpha\vert 000\rangle + \beta\vert 111\rangle) \otimes (\mu\vert 0000\rangle + \nu\vert 1111\rangle)$	$I \otimes I \otimes I$	$X \otimes X \otimes X \otimes X$
$\vert\gamma_1^+\rangle$	$\vert\xi_2^-\rangle$	$\vert-\rangle$	$(\alpha\vert 000\rangle - \beta\vert 111\rangle) \otimes (-\mu\vert 0000\rangle + \nu\vert 1111\rangle)$	$Z \otimes I \otimes I$	$iY \otimes I \otimes I \otimes I$
$\vert\gamma_1^-\rangle$	$\vert\xi_1^+\rangle$	$\vert+\rangle$	$(\alpha\vert 000\rangle - \beta\vert 111\rangle) \otimes (\nu\vert 0000\rangle + \mu\vert 1111\rangle)$	$Z \otimes I \otimes I$	$I \otimes I \otimes I \otimes I$
$\vert\gamma_1^-\rangle$	$\vert\xi_1^+\rangle$	$\vert-\rangle$	$(\alpha\vert 000\rangle + \beta\vert 111\rangle) \otimes (\nu\vert 0000\rangle + \mu\vert 1111\rangle)$	$I \otimes I \otimes I$	$I \otimes I \otimes I \otimes I$
$\vert\gamma_1^-\rangle$	$\vert\xi_1^-\rangle$	$\vert+\rangle$	$(\alpha\vert 000\rangle - \beta\vert 111\rangle) \otimes (\nu\vert 0000\rangle - \mu\vert 1111\rangle)$	$Z \otimes I \otimes I$	$Z \otimes I \otimes I \otimes I$
$\vert\gamma_1^-\rangle$	$\vert\xi_1^-\rangle$	$\vert-\rangle$	$(\alpha\vert 000\rangle + \beta\vert 111\rangle) \otimes (\nu\vert 0000\rangle - \mu\vert 1111\rangle)$	$I \otimes I \otimes I$	$Z \otimes I \otimes I \otimes I$
$\vert\gamma_1^-\rangle$	$\vert\xi_2^+\rangle$	$\vert+\rangle$	$(\alpha\vert 000\rangle - \beta\vert 111\rangle) \otimes (\mu\vert 0000\rangle + \nu\vert 1111\rangle)$	$I \otimes I \otimes I$	$X \otimes X \otimes X \otimes X$
$\vert\gamma_1^-\rangle$	$\vert\xi_2^+\rangle$	$\vert-\rangle$	$(\alpha\vert 000\rangle + \beta\vert 111\rangle) \otimes (\mu\vert 0000\rangle + \nu\vert 1111\rangle)$	$I \otimes I \otimes I$	$X \otimes X \otimes X \otimes X$
$\vert\gamma_1^-\rangle$	$\vert\xi_2^-\rangle$	$\vert+\rangle$	$(\alpha\vert 000\rangle - \beta\vert 111\rangle) \otimes (-\mu\vert 0000\rangle + \nu\vert 1111\rangle)$	$Z \otimes I \otimes I$	$iY \otimes X \otimes X \otimes X$
$\vert\gamma_1^-\rangle$	$\vert\xi_2^-\rangle$	$\vert-\rangle$	$(\alpha\vert 000\rangle + \beta\vert 111\rangle) \otimes (-\mu\vert 0000\rangle + \nu\vert 1111\rangle)$	$I \otimes I \otimes I$	$iY \otimes I \otimes I \otimes I$

Tables 2 and 3, i is an imaginary unit, X, Y, and Z are Pauli matrices, and I is the identity matrix. These matrices have the form

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (12)$$

Table 3: Following the above table.

Alice's results	Bob's results	Charlie's results	Collapsed state of qubits b, c, d, g, h, i, j	Alice's unitary operator	Bob's unitary operator
$\lvert\gamma_2^+\rangle$	$\lvert\xi_1^+\rangle$	$\lvert+\rangle$	$(\beta\lvert 000\rangle + \alpha\lvert 111\rangle) \otimes (\nu\lvert 0000\rangle + \mu\lvert 1111\rangle)$	$X \otimes X \otimes X$	$I \otimes I \otimes I \otimes I$
$\lvert\gamma_2^+\rangle$	$\lvert\xi_1^+\rangle$	$\lvert-\rangle$	$(\beta\lvert 000\rangle - \alpha\lvert 111\rangle) \otimes (\nu\lvert 0000\rangle + \mu\lvert 1111\rangle)$	$-iY \otimes X \otimes X$	$I \otimes I \otimes I \otimes I$
$\lvert\gamma_2^+\rangle$	$\lvert\xi_1^-\rangle$	$\lvert+\rangle$	$(\beta\lvert 000\rangle + \alpha\lvert 111\rangle) \otimes (\nu\lvert 0000\rangle - \mu\lvert 1111\rangle)$	$X \otimes X \otimes X$	$Z \otimes I \otimes I \otimes I$
$\lvert\gamma_2^+\rangle$	$\lvert\xi_1^-\rangle$	$\lvert-\rangle$	$(\beta\lvert 000\rangle - \alpha\lvert 111\rangle) \otimes (-\nu\lvert 0000\rangle + \mu\lvert 1111\rangle)$	$iY \otimes X \otimes X$	$-iY \otimes I \otimes I \otimes I$
$\lvert\gamma_2^+\rangle$	$\lvert\xi_2^+\rangle$	$\lvert+\rangle$	$(\alpha\lvert 000\rangle + \beta\lvert 111\rangle) \otimes (\mu\lvert 0000\rangle + \nu\lvert 1111\rangle)$	$X \otimes X \otimes X$	$X \otimes X \otimes X \otimes X$
$\lvert\gamma_2^+\rangle$	$\lvert\xi_2^+\rangle$	$\lvert-\rangle$	$(-\beta\lvert 000\rangle + \alpha\lvert 111\rangle) \otimes (\mu\lvert 0000\rangle + \nu\lvert 1111\rangle)$	$-iY \otimes X \otimes X$	$X \otimes X \otimes X \otimes X$
$\lvert\gamma_2^+\rangle$	$\lvert\xi_2^-\rangle$	$\lvert+\rangle$	$(\beta\lvert 000\rangle + \alpha\lvert 111\rangle) \otimes (-\mu\lvert 0000\rangle + \nu\lvert 1111\rangle)$	$X \otimes X \otimes X$	$-iY \otimes X \otimes X \otimes X$
$\lvert\gamma_2^+\rangle$	$\lvert\xi_2^-\rangle$	$\lvert-\rangle$	$(-\beta\lvert 000\rangle + \alpha\lvert 111\rangle) \otimes (-\mu\lvert 0000\rangle + \nu\lvert 1111\rangle)$	$-iY \otimes X \otimes X$	$iY \otimes X \otimes X \otimes X$
$\lvert\gamma_2^-\rangle$	$\lvert\xi_1^+\rangle$	$\lvert+\rangle$	$(-\beta\lvert 000\rangle + \alpha\lvert 111\rangle) \otimes (\nu\lvert 0000\rangle + \mu\lvert 1111\rangle)$	$-iY \otimes X \otimes X$	$I \otimes I \otimes I \otimes I$
$\lvert\gamma_2^-\rangle$	$\lvert\xi_1^+\rangle$	$\lvert-\rangle$	$(-\beta\lvert 000\rangle - \alpha\lvert 111\rangle) \otimes (\nu\lvert 0000\rangle + \mu\lvert 1111\rangle)$	$-X \otimes X \otimes X$	$I \otimes I \otimes I \otimes I$
$\lvert\gamma_2^-\rangle$	$\lvert\xi_1^-\rangle$	$\lvert+\rangle$	$(-\beta\lvert 000\rangle + \alpha\lvert 111\rangle) \otimes (\nu\lvert 0000\rangle - \mu\lvert 1111\rangle)$	$iY \otimes X \otimes X$	$Z \otimes I \otimes I \otimes I$
$\lvert\gamma_2^-\rangle$	$\lvert\xi_1^-\rangle$	$\lvert-\rangle$	$(\beta\lvert 000\rangle - \alpha\lvert 111\rangle) \otimes (\mu\lvert 0000\rangle + \nu\lvert 1111\rangle)$	$iY \otimes X \otimes X$	$X \otimes X \otimes X \otimes X$
$\lvert\gamma_2^-\rangle$	$\lvert\xi_2^+\rangle$	$\lvert+\rangle$	$(\alpha\lvert 000\rangle - \beta\lvert 111\rangle) \otimes (\mu\lvert 0000\rangle + \nu\lvert 1111\rangle)$	$-X \otimes X \otimes X$	$X \otimes X \otimes X \otimes X$
$\lvert\gamma_2^-\rangle$	$\lvert\xi_2^+\rangle$	$\lvert-\rangle$	$(-\beta\lvert 000\rangle - \alpha\lvert 111\rangle) \otimes (\mu\lvert 0000\rangle + \nu\lvert 1111\rangle)$	$-X \otimes X \otimes X$	$X \otimes X \otimes X \otimes X$
$\lvert\gamma_2^-\rangle$	$\lvert\xi_2^-\rangle$	$\lvert+\rangle$	$(\beta\lvert 000\rangle - \alpha\lvert 111\rangle) \otimes (-\mu\lvert 0000\rangle + \nu\lvert 1111\rangle)$	$iY \otimes X \otimes X$	$iY \otimes X \otimes X \otimes X$
$\lvert\gamma_2^-\rangle$	$\lvert\xi_2^-\rangle$	$\lvert-\rangle$	$(\beta\lvert 000\rangle + \alpha\lvert 111\rangle) \otimes (\mu\lvert 0000\rangle - \nu\lvert 1111\rangle)$	$X \otimes X \otimes X$	$-iY \otimes X \otimes X \otimes X$

Table 4: Comparing the efficiency of different protocols.

Year and reference	The number of Alice's transmitted qubits	The number of Bob's transmitted qubits	The number of quantum channel	The efficiency of protocol
2019 [14]	3	3	6	54.6%
2020 [10]	2	2	6	40%
2020 [10]	2	3	6	45.5%
2021 [11]	3	3	11	30%
2022 [13]	2	3	8	38.5%
This paper	4	3	10	46.7%

5. Comparison of Efficiency

The protocol efficiency of bidirectional quantum controlled teleportation can be defined as

$$\eta = \frac{c}{q + p}. \tag{13}$$

Here, c represent the total number of qubits to be transmitted by both parties and q is the total number of quantum channel in the protocol. In this paper, the total number of qubits to be transmitted is seven and the total number of quantum channel is ten. The efficiency of this bidirectional quantum controlled teleportation η is equal to 46.7%. The other protocols are as Table 4, and the efficiency of this scheme is relatively high.

6. Conclusion

In conclusion, this paper proves that the implementation of BQCT protocol using quantum channel constructed by entanglement of ten-qubit is more efficient than traditional methods. In addition, quantum communication is an absolutely safe means of communication because it cannot be eavesdropped or cracked. Therefore, the quantum channel constructed in this paper can be used for communication with better security and confidentiality than the existing communication means. However, at present, the research results of this paper only verify its feasibility in theory, and future empirical research is needed to verify its feasibility in practice.

Data Availability

The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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Research Article

Conformal η-Ricci-Yamabe Solitons within the Framework of ϵ-LP-Sasakian 3-Manifolds

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In the present note, we study ϵ-LP-Sasakian 3-manifolds $M^3(\epsilon)$ whose metrics are conformal η-Ricci-Yamabe solitons (in short, CERYS), and it is proven that if an $M^3(\epsilon)$ with a constant scalar curvature admits a CERYS, then $\pounds_U\zeta$ is orthogonal to ζ if and only if $\Lambda - \epsilon\sigma = -2\epsilon l + (mr/2) + (1/2)(p + (2/3))$. Further, we study gradient CERYS in $M^3(\epsilon)$ and proved that an $M^3(\epsilon)$ admitting gradient CERYS is a generalized conformal η-Einstein manifold; moreover, the gradient of the potential function is pointwise collinear with the Reeb vector field ζ. Finally, the existence of CERYS in an $M^3(\epsilon)$ has been drawn by a concrete example.

1. Introduction

The index of a metric generates variety of vector fields such as space-like, time-like, and light-like vector fields. Therefore, the study of manifolds with indefinite metrics becomes of great importance in physics and relativity. About three decades ago, the concept of ϵ-Sasakian manifolds was intoduced by Bejancu and Duggal [1]. Later, Xufeng and Xiaoli [2] have shown that these manifolds are real hypersurfaces of indefinite Kaehlerian manifolds. Recently, the manifolds with indefinite structures have also been studied by several authors such as [3–7].

The concept of conformal Ricci flow was introduced by Fischer [8] as a generalization of the classical Ricci flow equation, which is defined on an n-dimensional Riemannian manifold M by the equations

$$\frac{\partial g}{\partial t} = -2\left(S + \frac{g}{n}\right) - pg, \quad r(g) = -1, \tag{1}$$

where p defines a time dependent nondynamical scalar field (also called the conformal pressure), g is the Riemannian metric, and r and S represent the scalar curvature and the Ricci tensor of M, respectively. The term $-pg$ plays a role of constraint force to maintain r in the above equation.

In 2015, Basu and Bhattacharya [9] proposed the concept of conformal Ricci soliton on M and is defined by

$$\pounds_U g + 2S = \left\{\frac{1}{n}(pn + 2) - 2\Lambda\right\} g, \tag{2}$$

where $\pounds_U$ represents the Lie derivative operator along the smooth vector field U on M and $\Lambda \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers).

In [10], Guler and Crasmareanu established a scalar combination of Ricci and Yamabe flows; this new class of geometric flows called Ricci-Yamabe flow of type (l, m) and is defined by

$$\frac{\partial}{\partial t} g(t) = 2lS(g(t)) - mr(t)g(t), \quad g(0) = g_0, \tag{3}$$

for some scalars l and m.

A solution to the Ricci-Yamabe flow is called Ricci-Yamabe soliton if it depends only on one parameter group of diffeomorphism and scaling. A Riemannian manifold is said to have a Ricci-Yamabe solitons (RYS) if [11]

$$\pounds_U g + (2\Lambda - mr)g + 2lS = 0, \tag{4}$$

where $l, m, \Lambda \in \mathbb{R}$.

In [12], Zhang et al. studied conformal Ricci-Yamabe soliton (CRYS), which is defined on an n-dimensional Riemannian manifold by

$$\pounds_U g + 2lS + \left\{2\Lambda - mr - \frac{1}{n}(pn+2)\right\}g = 0. \tag{5}$$

Motivated by the above studies, we introduce the notion of conformal η-Ricci-Yamabe soliton (CERYS). A Riemannian manifold M of dimension n is said to have CERYS if

$$\pounds_U g + 2lS + \left\{2\Lambda - mr - \frac{1}{n}(pn+2)\right\}g + 2\sigma\,\eta\otimes\eta = 0, \tag{6}$$

where $l, m, \Lambda, \sigma \in \mathbb{R}$ and η is a 1-form on M.

If U is the gradient of a smooth function f on M, then equation (6) is called the gradient conformal η-Ricci-Yamabe soliton (gradient CERYS) and takes the form

$$\nabla^2 f + lS + \left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{n}\right)\right\}g + \sigma\,\eta\otimes\eta = 0, \tag{7}$$

where $\nabla^2 f$ is said to be the Hessian of f. A CRYS (or gradient CRYS) is said to be shrinking, steady or expanding if $\Lambda < 0$, $= 0$ or >0, respectively. A CERYS (or gradient CERYS) reduces to

(i) conformal η – Ricci soliton if $l = 1, m = 0$,

(ii) conformal η – Yamabe soliton if $l = 0, m = 1$,

(iii) conformal η – Einstein soliton if $l = 1, m = -1$.

If $S(V_1, V_2) = \{\Lambda - (mr/2) - (1/2)(p + (2/n))\}g(V_1, V_2) + \sigma\,\eta(V_1)\eta(V_2)$ for all vector fields V_1, V_2 on M, then we call the manifold as a conformal η-Einstein manifold. Further, if $\sigma = 0$, that is, $S(V_1, V_2) = \{\Lambda - (mr/2) - (1/2)(p + (2/n))\}g(V_1, V_2)$, then M is called a conformal Einstein manifold. If an ϵ-LP-Sasakian 3-manifold $M^3(\epsilon)$ satisfies (6) (resp., (7)), then we say that $M^3(\epsilon)$ admits a CERYS (resp., gradient CERYS).

The study of indefinite structures of the manifolds admitting various types of solitons is of high interest of researchers from different fields due to its wide applications in general relativity, cosmology, quantum field theory, string theory, thermodynamics, etc. This is why, the researchers from various fields are attracted by this study. For more details about the related studies, we recommend the papers ([13–25]) and the references therein.

In this paper, we handle the study of $M^3(\epsilon)$ admitting CERYS. The article is unfolded as follows: Preliminaries on $M^3(\epsilon)$ are the focus of Section 2. Sections 3 and 4 are dedicated to conferring the CERYS and gradient CERYS in $M^3(\epsilon)$, respectively. At last, we model an example of $M^3(\epsilon)$ which helps to examine the existence of CERYS on $M^3(\epsilon)$.

2. Preliminaries

A differentiable manifold of dimension n is called an ϵ-Lorentzian para-Sasakian (in short, $M^3(\epsilon)$), in case it admits a $(1,1)$ tensor field φ, a contravariant vector field ζ, a 1-form η, and a Lorentzian metric g fulfilling [6]

$$\varphi^2 V_1 = V_1 + \eta(V_1)\zeta,\ \eta(\zeta) = -1, \tag{8}$$

$$g(\zeta,\zeta) = -\epsilon, \quad \eta(V_1) = \epsilon g(V_1,\zeta), \quad \varphi\zeta = 0, \quad \eta(\varphi V_1) = 0, \tag{9}$$

$$g(\varphi V_1, \varphi V_2) = g(V_1, V_2) - \epsilon\eta(V_1)\eta(V_2), \tag{10}$$

$$(\nabla_{V_1}\varphi)V_2 = g(V_1, V_2)\zeta + \epsilon\eta(V_2)V_1 + 2\epsilon\eta(V_1)\eta(V_2)\zeta, \tag{11}$$

$$\nabla_{V_1}\zeta = \epsilon\varphi V_1, \tag{12}$$

for all vector fields V_1, V_2 on $M^3(\epsilon)$, where ϵ is -1 or 1 according as ζ is space-like or time-like vector field, and ∇ represents the Levi-Civita connection with respect to g.

Moreover, in an $M^3(\epsilon)$, we have [6, 22]

$$(\nabla_{V_1}\eta)V_2 = \Phi(V_1, V_2) = g(\varphi V_1, V_2), \tag{13}$$

$$R(V_1, V_2)\zeta = \eta(V_2)V_1 - \eta(V_1)V_2, \tag{14}$$

$$R(\zeta, V_1)V_2 = \epsilon g(V_1, V_2)\zeta - \eta(V_2)V_1, \tag{15}$$

$$R(\zeta, V_1)\zeta = -R(V_1,\zeta)\zeta = V_1 + \eta(V_1)\zeta, \tag{16}$$

$$S(V_1,\zeta) = 2\eta(V_1) \Longleftrightarrow Q\zeta = 2\epsilon\zeta, \tag{17}$$

where Φ is a symmetric $(0,2)$ tensor field, R is the curvature tensor, and Q is the Ricci operator related by $g(QV_1, V_2) = S(V_1, V_2)$.

We note that if $\epsilon = 1$ and ζ is time-like vector field, then an $M^3(\epsilon)$ is usual LP-Sasakian manifold of dimension 3.

Definition 1. An $M^3(\epsilon)$ is called a generalized η-Einstein manifold if its Ricci tensor $S(\neq 0)$ satisfies

$$S(V_1, V_2) = ag(V_1, V_2) + b\eta(V_1)\eta(V_2) + cg(\varphi V_1, V_2), \tag{18}$$

where a, b, and c are scalar functions of ϵ. If $c = 0$ (resp., $b = c = 0$), then $M^3(\epsilon)$ is called η-Einstein (resp., Einstein) manifold.

Proposition 2. *In an $M^3(\epsilon)$, the Ricci tensor S is expressed as*

$$S(V_1, V_2) = \left(\frac{r}{2} - \epsilon\right) g(V_1, V_2) + \left(\frac{\epsilon r}{2} - 3\right)\eta(V_1)\eta(V_2), \tag{19}$$

for any V_1, V_2 on $M^3(\epsilon)$.

Proof. Since in an $M^3(\epsilon)$, the conformal curvature tensor vanishes, therefore, we have

$$\begin{aligned} R(V_1, V_2)V_3 &= S(V_2, V_3)V_1 - S(V_1, V_3)V_2 \\ &+ g(V_2, V_3)QV_1 - g(V_1, V_3)QV_2 \\ &- \frac{r}{2}(g(V_2, V_3)V_1 - g(V_1, V_3)V_2), \end{aligned} \tag{20}$$

which by putting $V_3 = \zeta$ then using (9), (14), and (17) leads to

$$\eta(V_2)QV_1 - \eta(V_1)QV_2 = \left(\epsilon - \frac{r}{2}\right)(\eta(V_1)V_2 - \eta(V_2)V_1). \tag{21}$$

Again, putting $V_2 = \zeta$ in (21) then using (8) and (17), we find

$$QV_1 = \left(\frac{r}{2} - \epsilon\right)V_1 + \left(\frac{r}{2} - 3\epsilon\right)\eta(V_1)\zeta. \tag{22}$$

The inner product of (22) with V_2 gives (19). □

3. $M^3(\epsilon)$ Admitting CERYS

First, we prove the following theorem.

Theorem 3. *If an $M^3(\epsilon)$ with the constant scalar curvature admits a CERYS, then*

$$\Lambda - \epsilon\sigma = -2\epsilon l + \frac{mr}{2} + \frac{1}{2}\left(p + \frac{2}{3}\right). \tag{23}$$

Moreover, $\pounds_U\zeta$ is orthogonal to ζ if and only if (23) holds.

Proof. Let an $M^3(\epsilon)$ admit a CERYS, then by using (19) in (6), we have

$$\begin{aligned} (\pounds_U g)(V_1, V_2) &= -\left\{(l-m)r + 2\Lambda - 2\epsilon l - \left(p + \frac{2}{3}\right)\right\} g(V_1, V_2) \\ &- \{l(\epsilon r - 6) + 2\sigma\}\eta(V_1)\eta(V_2). \end{aligned} \tag{24}$$

The covariant differentiation of (24) with respect to V_3 leads to

$$\begin{aligned} \left(\nabla_{V_3}\pounds_U g\right)(V_1, V_2) &= -l(V_3 r)\, g(\varphi V_1, \varphi V_2) + m(V_3 r)g(V_1, V_2) \\ &- \{l(\epsilon r - 6) + 2\sigma\}(g(\varphi V_3, V_1)\eta(V_2) + g(\varphi V_3, V_2)\eta(V_1)). \end{aligned} \tag{25}$$

As g is parallel with respect to ∇, then the relation [26].

$$\begin{aligned} &\left(\pounds_U\nabla_{V_1}g - \nabla_{V_1}\pounds_U g - \nabla_{[U,V_1]}g\right)(V_2, V_3) \\ &= -g((\pounds_U\nabla)(V_1, V_3), V_2) - g((\pounds_U\nabla)(V_1, V_2), V_3), \end{aligned} \tag{26}$$

turns to

$$\begin{aligned} \left(\nabla_{V_1}\pounds_U g\right)(V_2, V_3) &= g((\pounds_U\nabla)(V_1, V_3), V_2) \\ &+ g((\pounds_U\nabla)(V_1, V_2), V_3). \end{aligned} \tag{27}$$

Due to symmetric property of $\pounds_U\nabla$, equation (27) takes the form

$$\begin{aligned} 2g((\pounds_U\nabla)(V_1, V_2), V_3) &= \left(\nabla_{V_1}\pounds_U g\right)(V_2, V_3) \\ &+ \left(\nabla_{V_2}\pounds_U g\right)(V_1, V_3) \\ &- \left(\nabla_{V_3}\pounds_U g\right)(V_1, V_2). \end{aligned} \tag{28}$$

Using (25) in (28), we have

$$\begin{aligned} &2g((\pounds_U\nabla)(V_1, V_2), V_3) = \\ &-l\{(V_1 r)g(\varphi V_2, \varphi V_3) + (V_2 r)g(\varphi V_1, \varphi V_3) - (V_3 r)g(\varphi V_1, \varphi V_2)\} \\ &+ m\{(V_1 r)g(V_2, V_3) + (V_2 r)g(V_1, V_3) - (V_3 r)g(V_1, V_2)\} \\ &- 2\{l(\epsilon r - 6) + 2\sigma\}g(\varphi V_1, V_2)\eta(V_3). \end{aligned} \tag{29}$$

By eliminating V_3 from the foregoing equation, it follows that

$$\begin{aligned} &2(\pounds_U\nabla)(V_1, V_2) = \\ &-l\{(V_1 r)(V_2 + \eta(V_2)\zeta) + (V_2 r)(V_1 + \eta(V_1)\zeta) - (Dr)g(\varphi V_1, \varphi V_2)\} \\ &- 2\epsilon\{l(\epsilon r - 6) + 2\sigma\}g(\varphi V_1, V_2)\zeta + m\{(V_1 r)V_2 + (V_2 r)V_1 - (Dr)g(V_1, V_2)\}, \end{aligned} \tag{30}$$

where $V_1 l = g(Dl, V_1)$, D stands for the gradient operator with respect to g. Taking $V_2 = \zeta$ and using r constant (hence $(Dr = 0)$ and $(\zeta r = 0)$), (30) turns to

$$(\pounds_U\nabla)(V_1, \zeta) = 0. \tag{31}$$

The covariant derivative of (31) with respect to V_2 leads to

$$\left(\nabla_{V_2}\pounds_U\nabla\right)(V_1, \zeta) = -\epsilon(\pounds_U\nabla)(V_1, \varphi V_2), \tag{32}$$

which by using in $(\nabla_U R)(V_1, V_2)V_3 = (\nabla_{V_1}\pounds_U\nabla)(V_2, V_3) - (\nabla_{V_2}\pounds_U\nabla)(V_1, V_3)$, we deduce

$$(\nabla_U R)(V_1, \zeta)\zeta = 0. \tag{33}$$

The Lie derivative of $R(V_1, \zeta)\zeta = -V_1 - \eta(V_1)\zeta$ along U

yields

$$(\nabla_U R)(V_1,\zeta)\zeta + 2\eta(\pounds_U\zeta)V_1 - \epsilon g(V_1,\pounds_U\zeta)\zeta = -(\pounds_U\eta)(V_1)\zeta, \tag{34}$$

which by using (33) reduces to

$$(\pounds_U\eta)(V_1)\zeta = -2\eta(\pounds_U\zeta)V_1 + \epsilon g(V_1,\pounds_U\zeta)\zeta. \tag{35}$$

Now, taking the Lie derivative of $\eta(V_1)=\epsilon g(V_1,\zeta)$, it follows that

$$(\pounds_U\eta)V_1 = \epsilon(\pounds_U g)(V_1,\zeta) + \epsilon g(V_1,\pounds_U\zeta). \tag{36}$$

Taking $V_2=\zeta$ in (24), we find

$$(\pounds_U g)(V_1,\zeta) = \left\{-2\epsilon\Lambda + \epsilon mr - 4l + 2\sigma + \epsilon\left(p+\frac{2}{3}\right)\right\}\eta(V_1). \tag{37}$$

Again, taking the Lie-derivative of $g(\zeta,\zeta)=-\epsilon$, we have

$$(\pounds_U g)(\zeta,\zeta) = -2\epsilon\eta(\pounds_U\zeta). \tag{38}$$

Now, by combining the equations (35)–(38), we have

$$\left\{2\epsilon\Lambda - \epsilon mr + 4l - 2\sigma - \epsilon\left(p+\frac{2}{3}\right)\right\}\varphi^2 V_1 = 0. \tag{39}$$

From the foregoing equation, it follows that

$$\Lambda - \epsilon\sigma = -2\epsilon l + \frac{mr}{2} + \frac{1}{2}\left(p+\frac{2}{3}\right) = 0, \tag{40}$$

where $\varphi^2 V_1 \neq 0$.

Next, from the equations (37)–(40), we observe that $\eta(\pounds_U\zeta)=0$, i.e., $\pounds_U\zeta$ is orthogonal to ζ. Conversely, from (37) and (38), one can see that if $\pounds_U\zeta$ is orthogonal to ζ, then (40) immediately follows. This completes the proof. □

In particular, if $l=1, m=\sigma=0$, then (40) reduces to $\Lambda = -2\epsilon + (1/2)(p+(2/3))$. Thus, we have the following.

Corollary 4. *If an $M^3(\epsilon)$ with the constant scalar curvature admits a conformal Ricci soliton, then the soliton on $M^3(\epsilon)$ is concluded as follows:*

(i) if $\epsilon=1$, (i.e., ζ is time-like), then the soliton on $M^3(\epsilon)$ is expanding, steady, or shrinking according to $p>(10/3)$, $=(10/3)$, or $<(10/3)$

(ii) if $\epsilon=-1$, (i.e., ζ is space-like), then the soliton on $M^3(\epsilon)$ is expanding, steady or shrinking according to $p>(-14/3)$, $=(-14/3)$, or $<(-14/3)$

Next, if $m=1, l=\sigma=0$, then (40) reduces to $\Lambda=(r/2)+(1/2)(p+(2/3))$. Thus, we have the following.

Corollary 5. *If an $M^3(\epsilon)$ with the constant scalar curvature admits a conformal Yamabe soliton, then the soliton on $M^3(\epsilon)$ is expanding, steady or shrinking according to $p>-(r+(2/3))$, $=-(r+(2/3))$ or $<-(r+(2/3))$.*

Again, if $l=1, m=-1, \sigma=0$, then (40) reduces to $\Lambda=-2\epsilon-(r/2)+(1/2)(p+(2/3))$. Thus, we have the following.

Corollary 6. *If an $M^3(\epsilon)$ with the constant scalar curvature admits a conformal Einstein soliton, then the soliton on $M^3(\epsilon)$ is concluded as follows:*

(i) if $\epsilon=1$, (i.e., ζ is time-like), then the soliton on $M^3(\epsilon)$ is expanding, steady, or shrinking according to $p>(10/3)+r$, $=(10/3)+r$ or $<(10/3)+r$

(ii) if $\epsilon=-1$, (i.e., ζ is space-like), then the soliton on $M^3(\epsilon)$ is expanding, steady or shrinking according to $p>(-14/3)+r$, $=(-14/3)+r$ or $<(-14/3)+r$.

Furthermore, let an $M^3(\epsilon)$ admit a CERYS at $U=\zeta$, then from (6), we have

$$(\pounds_\zeta g)(V_1,V_2) + 2lS(V_1,V_2) + \left\{2\Lambda - mr - \left(p+\frac{2}{3}\right)\right\}g(V_1,V_2) + 2\sigma\eta(V_1)\eta(V_2) = 0, \tag{41}$$

which by using the value $(\pounds_\zeta g)(V_1,V_2) = g(\nabla_{V_1}\zeta,V_2) + g(V_1,\nabla_{V_2}\zeta) = 2\epsilon g(\varphi V_1,V_2)$, we arrive

$$S(V_1,V_2) = -\frac{1}{l}\left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p+\frac{2}{3}\right)\right\}g(V_1,V_2) - \frac{\sigma}{l}\eta(V_1)\eta(V_2) - \frac{\epsilon}{l}g(\varphi V_1,V_2), \quad \text{where} \quad l\neq 0. \tag{42}$$

By putting $V_2=\zeta$ in (42) and using (17), we find

$$\Lambda - \epsilon\sigma = -2\epsilon l + \frac{mr}{2} + \frac{1}{2}\left(p+\frac{2}{3}\right). \tag{43}$$

Thus, we have the following.

Corollary 7. *If an $M^3(\epsilon)$ admits a CERYS at $U=\zeta$, then $M^3(\epsilon)$ is a generalized conformal η-Einstein manifold and the scalars Λ and σ are related by (43). Moreover, the nature of the soliton on $M^3(\epsilon)$ is concluded as Corollaries 4 and 6.*

Definition 8. A vector field U on an $M^3(\epsilon)$ is called torse forming vector field in case [27].

$$\nabla_{V_1}U = fV_1 + \gamma(V_1)U, \tag{44}$$

where f and γ are smooth function and 1-form, respectively.

Let us consider an $M^3(\epsilon)$ admitting a CERYS, further considering the Reeb vector field ζ as a torse-forming vector field. Thus, from (44), we have

$$\nabla_{V_1}\zeta = fV_1 + \gamma(V_1)\zeta, \tag{45}$$

for all V_1 on $M^3(\epsilon)$. Taking the inner product of (45) with ζ, we find

$$g(\nabla_{V_1}\zeta, \zeta) = \epsilon f\eta(V_1) - \epsilon\gamma(V_1). \tag{46}$$

Also, from (12), we find

$$g(\nabla_{V_1}\zeta, \zeta) = 0. \tag{47}$$

Thus, the last two equations give $\gamma = f\eta$ (where $\epsilon \neq 0$), and hence (45) turns to

$$\nabla_{V_1}\zeta = f(V_1 + \eta(V_1)\zeta). \tag{48}$$

Now, in view of (48), we have

$$(\pounds_\zeta g)(V_1, V_2) = 2f\{g(V_1, V_2) + \eta(V_1)\eta(V_2)\}. \tag{49}$$

By virtue of (49), (42) turns to

$$\begin{aligned} S(V_1, V_2) = &-\frac{1}{l}\left\{\Lambda + f - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}g(V_1, V_2) \\ &- \frac{1}{l}(\epsilon f + \sigma)\,\eta(V_1)\eta(V_2), l \neq 0. \end{aligned} \tag{50}$$

Thus, we state the following.

Theorem 9. *If an $M^3(\epsilon)$ admits a CERYS at $U = \zeta$ with torse-forming vector field ζ. Then, $M^3(\epsilon)$ is a conformal η-Einstein manifold.*

In particular, if $\sigma = -\epsilon f$, then (50) takes the form $S(V_1, V_2) = -(1/l)\{\Lambda + f - (mr/2) - (1/2)(p + (2/3))\}g(V_1, V_2)$, $l \neq 0$. Thus, we have the following.

Corollary 10. *An $M^3(\epsilon)$ admitting a CERYS with torse-forming vector field ζ is a conformal Einstein manifold if $\sigma = f$ for space-like vector field (or $\sigma = -f$ for time-like vector field).*

4. Gradient CERYS on $M^3(\epsilon)$

Let the metric g on $M^3(\epsilon)$ be a gradient CERYS. Then, equation (7) can be expressed as

$$\nabla_{V_2}Df + lQV_2 + \left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}V_2 + \epsilon\sigma\eta(V_2)\zeta = 0, \tag{51}$$

for all V_2 on $M^3(\epsilon)$, where D stands for the gradient operator of g.

The covariant derivative (51) with respect to V_1 leads to

$$\begin{aligned} \nabla_{V_1}\nabla_{V_2}Df = &-l\{(\nabla_{V_1}Q)V_2 + Q(\nabla_{V_1}V_2)\} \\ &- \left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}\nabla_{V_1}V_2 + m\frac{V_1(r)}{2}V_2 \\ &- \epsilon\sigma\{g(\varphi V_1, V_2)\zeta + \eta(\nabla_{V_1}V_2)\zeta + \epsilon\eta(V_2)\varphi V_1\}. \end{aligned} \tag{52}$$

Interchanging the role of V_1 and V_2 in (52), we have

$$\begin{aligned} \nabla_{V_2}\nabla_{V_1}Df = &-l\{(\nabla_{V_2}Q)V_1 + Q(\nabla_{V_2}V_1)\} \\ &- \left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}\nabla_{V_2}V_1 + m\frac{V_2(r)}{2}V_1 \\ &- \epsilon\sigma\{g(\varphi V_2, V_1)\zeta + \eta(\nabla_{V_2}V_1)\zeta + \epsilon\eta(V_1)\varphi V_2\}. \end{aligned} \tag{53}$$

By using (51)–(53), the well-known relation $R(V_1, V_2)Df = \nabla_{V_1}\nabla_{V_2}Df - \nabla_{V_2}\nabla_{V_1}Df - \nabla_{[V_1,V_2]}Df$ takes the form

$$\begin{aligned} R(V_1, V_2)Df = &\,l\{(\nabla_{V_2}Q)V_1 - (\nabla_{V_1}Q)V_2\} \\ &+ \frac{m}{2}\{V_1(r)V_2 - V_2(r)V_1\} \\ &+ \sigma\{\eta(V_1)\varphi V_2 - \eta(V_2)\varphi V_1\}. \end{aligned} \tag{54}$$

The covariant differentiation of (22) with respect to V_2 gives

$$\begin{aligned} (\nabla_{V_2}Q)V_1 = &\,\frac{V_2(r)}{2}(V_1 + \eta(V_1)\zeta) \\ &+ \left(\frac{r}{2} - 3\epsilon\right)(g(\varphi V_1, V_2)\zeta + \epsilon\eta(V_1)\varphi V_2), \end{aligned} \tag{55}$$

which by replacing $V_1 = \zeta$ then using (8) and (9) reduces to

$$(\nabla_{V_2}Q)\zeta = -\left(\frac{\epsilon r}{2} - 3\right)\varphi V_2. \tag{56}$$

Again, replacing V_2 by ζ in (55) and using (9), we find

$$(\nabla_\zeta Q)V_1 = \frac{(\zeta r)}{2}(V_1 + \eta(V_1)\zeta). \tag{57}$$

Subtracting (57) from (56), we find

$$(\nabla_{V_2}Q)\zeta - (\nabla_\zeta Q)V_1 = -\left(\frac{\epsilon r}{2} - 3\right)\varphi V_2 - \frac{(\zeta r)}{2}(V_1 + \eta(V_1)\zeta). \tag{58}$$

Now, putting $V_1 = \zeta$ in (54) then using (8) and (9), we have

$$R(\zeta, V_2)Df = l\{(\nabla_{V_2}Q)\zeta - (\nabla_\zeta Q)V_2\} + \frac{m}{2}\{\zeta(r)V_2 - V_2(r)\zeta\} - \sigma\varphi V_2. \tag{59}$$

Taking the inner product of foregoing equation with ζ and using (58), we infer

$$g(R(\zeta, V_2)Df, \zeta) = \frac{\epsilon m}{2}\{\zeta(r)\eta(V_2) + V_2(r)\}. \tag{60}$$

From relation (15), we have

$$g(R(\zeta, V_2)Df, \zeta) = -(V_2 f) - \zeta(f)\eta(V_2). \tag{61}$$

By combining equations (60) and (61), it follows that $(V_2 f) + \{\zeta f + (\epsilon m\zeta(r)/2)\}\eta(V_2) + (\epsilon m/2)V_2(r) = 0$ for any V_2 on $M^3(\epsilon)$. Therefore, for r constant, we have

$$U = Df = -\epsilon(\zeta f)\zeta. \tag{62}$$

This informs that the vector field U is pointwise collinear with ζ.

Now, taking the covariant derivative of (62) with respect to V_1, we have

$$\nabla_{V_1} Df = -\epsilon\{V_1(\zeta f)\zeta\} - (\zeta f)\varphi V_1. \tag{63}$$

The inner product of (63) with ζ gives

$$g\left(\nabla_{V_1} Df, \zeta\right) = V_1(\zeta f). \tag{64}$$

From (63) and (64), we arrive

$$\nabla_{V_1} Df = -\epsilon g\left(\nabla_{V_1} Df, \zeta\right)\zeta - (\zeta f)\varphi V_1. \tag{65}$$

The inner product of (51) with ζ leads to $g(\nabla_{V_1} Df, \zeta) = \{-2l - \epsilon\Lambda + \sigma + (\epsilon m r/2) + (\epsilon/2)(p + (2/3))\}\eta(V_1)$, which in view of (40) reduces to

$$g\left(\nabla_{V_1} Df, \zeta\right) = 0. \tag{66}$$

Thus, (51) together with (65) and (66) takes the form

$$QV_1 = -\frac{1}{l}\left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}V_1 - \frac{\epsilon\sigma}{l}\eta(V_1)\zeta + \frac{1}{l}(\zeta f)\varphi V_1, \quad l \neq 0. \tag{67}$$

This informs that $M^3(\epsilon)$ is a generalized conformal η-Einstein manifold.

Next, from (51) and (63), we have

$$lQV_1 + \left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}V_1 + \epsilon\sigma\eta(V_1)\zeta = \epsilon\{V_1(\zeta f)\zeta\} + \epsilon(\zeta f)\varphi V_1. \tag{68}$$

By putting $V_1 = \zeta$ in (68) then using (8), (9), and (17), we find

$$\left\{2\epsilon l + \Lambda - \epsilon\sigma - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}\zeta = \epsilon\{\zeta(\zeta f)\zeta\}. \tag{69}$$

The inner product of (69) with ζ and the use of (9) and (40) leads to $\zeta(\zeta f) = 0$.

If possible, we suppose that $\zeta = \partial/\partial t$ then the above equation takes the form

$$\frac{\partial^2 f}{\partial t^2} = 0. \tag{70}$$

It is noticed that the potential function $f = d_1 + td_2$ where d_1 and d_2 are independent of t, satisfies equation (70). By considering the above facts, we can state the following.

Theorem 11. *Let an $M^3(\epsilon)$ admit a gradient CERYS. Then,*

(i) $M^3(\epsilon)$ is a generalized conformal η-Einstein manifold

(ii) the gradient of the potential function f is pointwise collinear with the Reeb vector field ζ and f satisfies equation (70) and it is governed by $f = d_1 + td_2$.

Example 1. We consider the manifold $M^3 = \{(u_1, u_2, u_3) \in R^3\}$, where (u_1, u_2, u_3) are the usual coordinates in R^3. Let κ_1, κ_2, and κ_3 be the vector fields on M^3 given by

$$\begin{aligned} \kappa_1 &= \cosh u_3 \frac{\partial}{\partial u_1} + \sinh u_3 \frac{\partial}{\partial u_2}, \kappa_2 \\ &= \sinh u_3 \frac{\partial}{\partial u_1} + \cosh u_3 \frac{\partial}{\partial u_2}, \kappa_3 = \epsilon \frac{\partial}{\partial u_3} = \zeta, \end{aligned} \tag{71}$$

and these are linearly independent at each point of M^3. Let g be the Lorentzian metric defined by

$$g\left(\kappa_i, \kappa_j\right) = \begin{cases} 1, & \text{for } 1 \le i \le 2, \\ -\epsilon, & \text{for } i = j = 3, \\ 0, & \text{otherwise.} \end{cases} \tag{72}$$

We define η, a 1-form as $\eta(V_1) = \epsilon g(V_1, \kappa_3)$ for all V_1 on M^3. Let φ be the $(1, 1)$ tensor field defined by

$$\varphi\kappa_1 = -\kappa_2, \quad \varphi\kappa_2 = -\kappa_1, \quad \varphi\kappa_3 = 0. \tag{73}$$

Using the linearity of φ and g, we yield

$$\begin{aligned} \eta(\kappa_3) = -1, \varphi^2 V_1 = V_1 + \eta(V_1)\zeta, g(\varphi V_1, \varphi V_2) \\ = g(V_1, V_2) - \epsilon\eta(V_1)\eta(V_2), \end{aligned} \tag{74}$$

for all V_1, V_2 on M^3

Now, by direct computations, we obtain

$$[\kappa_1, \kappa_2] = 0, \quad [\kappa_2, \kappa_3] = -\epsilon\kappa_1, \quad [\kappa_1, \kappa_3] = -\epsilon\kappa_2. \tag{75}$$

By using well-known Koszul's formula, we find

$$\nabla_{\kappa_1}\kappa_1 = 0, \quad \nabla_{\kappa_2}\kappa_1 = -\kappa_3, \quad \nabla_{\kappa_3}\kappa_1 = 0, \quad \nabla_{\kappa_1}\kappa_2 = -\kappa_3, \quad \nabla_{\kappa_2}\kappa_2 = 0,$$
$$\nabla_{\kappa_3}\kappa_2 = 0, \quad \nabla_{\kappa_1}\kappa_3 = -\epsilon\kappa_2, \quad \nabla_{\kappa_2}\kappa_3 = -\epsilon\kappa_1, \quad \nabla_{\kappa_3}\kappa_3 = 0. \tag{76}$$

Let $V_1 = V_1{}^1\kappa_1 + V_1{}^2\kappa_2 + V_1{}^3\kappa_3$ and $V_2 = V_2{}^1\kappa_1 + V_2{}^2\kappa_2 + V_2{}^3\kappa_3$ be the vector fields on M^3. Then, for $\kappa_3 = \zeta$ one can easily verify that

$$\nabla_{V_1}\zeta = \epsilon\varphi V_1 \quad \text{and} \quad \left(\nabla_{V_1}\varphi\right) V_2 = g(V_1, V_2)\zeta + \epsilon\eta(V_2)V_1 + 2\epsilon\eta(V_1)\eta(V_2)\zeta. \tag{77}$$

Thus, the manifold M^3 is an ϵ-LP-Sasakian 3-manifold.

By using the above results, we can easily obtain the following components of the curvature tensor R:

$$R(\kappa_1, \kappa_2)\kappa_1 = \epsilon\kappa_2, \quad R(\kappa_1, \kappa_2)\kappa_2 = -\epsilon\kappa_1, \quad R(\kappa_1, \kappa_2)\kappa_3 = 0,$$
$$R(\kappa_2, \kappa_3)\kappa_1 = 0, \quad R(\kappa_2, \kappa_3)\kappa_2 = -\epsilon\kappa_3, \quad R(\kappa_2, \kappa_3)\kappa_3 = -\kappa_2,$$
$$R(\kappa_1, \kappa_3)\kappa_1 = -\epsilon\kappa_3, \quad R(\kappa_1, \kappa_3)\kappa_2 = 0, \quad R(\kappa_1, \kappa_3)\kappa_3 = -\kappa_1. \tag{78}$$

We calculate the Ricci tensors as follows:

$$S(\kappa_1, \kappa_1) = S(\kappa_2, \kappa_2) = 0, \quad S(\kappa_3, \kappa_3) = -2 \Longrightarrow r = 2. \tag{79}$$

By putting $V_1 = V_2 = \kappa_3$ in (42) and using $S(\kappa_3, \kappa_3) = -2$, it follows that

$$\Lambda - \epsilon\sigma = -2\epsilon l + \frac{mr}{2} + \frac{1}{2}\left(p + \frac{2}{3}\right). \tag{80}$$

Again putting $V_1 = V_2 = \kappa_1$ in (42) and using $S(\kappa_1, \kappa_1) = 0$, we obtain $\Lambda = (mr/2) + (1/2)(p + (2/3))$. Thus, from (80), we find $\sigma = 2l$. Hence, we can say that for $\Lambda = (mr/2) + (1/2)(p + (2/3))$ and $\sigma = 2l$, the data $(g, \zeta, l, m, \Lambda, \sigma)$ defines a CERYS on the manifold $(M^3, \varphi, \zeta, \eta, g, \epsilon)$.

Data Availability

No data is used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Numerical Calculation of Three-Dimensional Ground State Potential Energy Function of Na_2F System

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We present a new three-dimensional global potential energy surface (PES) for the ground state of Na_2F system. A total of about 1460 points were generated for the PES. All of the points have been carried out by using the coupled-cluster single-, double-, and perturbative triple-excitations [CCSD(T)]. Two Jacobi coordinates, R and θ, and the frozen molecular equilibrium geometries were used. We mixed the basis sets of aug-cc-pCVQZ for the sodium atom and the basis sets of aug-cc-pCVDZ for the fluorine atom with an additional (3s3p2d) set of midbond functions; the energies obtained were extrapolated to the complete basis set limit. The whole calculation adopted supramolecular approximation approach. We divided the potential energy surface into three regions, the peak region, the well region, and the long range region, and calculate the single point energy, respectively. Our ab initio calculations will be useful for future studies of the collision-induced absorption for the Na_2-F dimer, and it can be used for modeling the dynamical behavior in Na_2F system too.

1. Introduction

Because the alkali atoms are small electron affinity, the excess electron in the alkali anion is loosely bound in space. Recently, Alkali metal diatomic molecules are found to be form stoichiometric system with various new elements. On the contrary, sodium fluoride phosphate is the core of the electrolyte material NaF, and other electronic injection material introductions of organic optoelectronic devices have become a good luminescent material [1–4]. Na_2F system belongs to super valence compounds containing odd electronic; it has good nonlinear optical properties, so the scientists study on super molecular structure of alkali metal fluoride which has always maintained a strong interest in Na_2F system [5–7].

The first thing we should do is to build precise PES when we studied reaction kinetics characteristics. In the past ten years, some studies on polarization molecular science of the system offer Na_2F system structure and the dynamic response process [8–14]. Through investigation, we learned that most of the potential energy surface of Na_2F system before is studied using semiempirical fitting.

In our calculations, there were 1460 adiabatic energy points chosen from previous 3D diabatic PES. In this paper, our calculations covered a wide range of interaction energy of the potential energy surface including the peak area, the well area, and the long-range area. We considered this system is vibrational weakly bound van der Waals complexes and the good performance on similar optimization, then we used the CCSD (T) calculation method for single point of interaction energy. By fitting, we gave the algebraic analytic function of the Na_2F system. Finally, we analyzed the three-dimensional characteristics of the potential energy surface.

2. Methodology

The electronic related functions must be considered when we do calculation, because the single-point energy calculation and geometric optimization (including optimization to transition states) are the most common types of tasks. The sensitivity of geometric optimization to the basis group is much lower than the calculation of single point energy, and the time of geometric optimization is ten times, dozens of times, or even hundreds of times of the single point calculation, so the geometric optimization absolutely does not need large basis group; using medium basis group is enough.

In consideration of computational efficiency, we have chosen the basis sets of aug-cc-pCVQZ for the sodium atom and the basis sets of aug-cc-pCVDZ for the fluorine atom. In order to improve the convergence of basis set, we added an additional (3s3p2d) set of midbond functions (mf) at the midpoint of R. We used quantum analysis framework in the process of computing the Jacobi coordinates system (r, R, θ). As shown in Figure 1, r is the distance of Na-Na, R is the length of the vector connecting the Na-Na center of mass and the F atom, and θ is the angle between R and the x-axis. For a given value of R, the angle θ changes from 0° to 360° in steps of 10°. We calculated 1460 geometries for the whole interaction energy, and the ground state of the spacing is $r_e = 3.228a_0$ [15].

The whole ab initio calculations have been calculated with Gaussian 09 W perform packet [16]. We considered all electronic correlation calculation processes. When we calculated the interaction between alkali metal pairs to the atom fluoride for the supramolecular systems described here, they are only weakly adsorbed on a substrate, so the method of supramolecular was used.

In order to avoid the fluorine atom to be too close to the geometric center of Na-Na set, in the process of calculation, we added diffuse augmentation functions to ensure that the basis permits polarization by Na-Na. In the peak area (the short range) $0a_0 < R < 4a_0$ and $\theta = -60^\circ \sim 60^\circ$ and$120^\circ \sim 240^\circ$, we used the interval equal step way $\Delta R = 0.1a_0$. In the well area$0a_0 < R < 4a_0$and$\theta = -70^\circ \sim 110^\circ$ and $250^\circ \sim 290^\circ$, we used the interval equal step way $\Delta R = 0.2a_0$. In the long-range area $4a_0 < R < 12a_0$ and $\theta = -0^\circ \sim 360^\circ$, we used the interval equal step way $\Delta R = 1a_0$. The aim is to hope that it describes the characteristics of the peak value and potential well more clearly.

We calculated the freeze the nuclear energy (E) as follows:

$$E(\mathrm{r}, R, \theta) = E_{\mathrm{p}}(\mathrm{r}, R, \theta) + E_w(\mathrm{r}, R, \theta) + E_l(\mathrm{r}, R, \theta) \tag{1}$$

where $E(..)$ represents the total electronic energy of respective species including zero point correction. The function contains the location of the potential peak range Ep, the well area Ew, and the long range El. The peak range and the well range include a damped dispersion expansion.

The exponential functional form is as follows:

$$E(r, R, \theta) = \sum_{n=4}^{8} \sum_{l=0,2\cdots} f_4(A(\theta)R) \times \frac{B(\theta)}{R^n} P_l^0(\cos\theta), \tag{2}$$

where the term $f_n(x)$ is defined by

$$f_n(x) = 1 - e^{-x} \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{3}$$

$A(r, \theta)$ and $B(r, \theta)$ denote expansions in Legendre polynomials $P_l(\cos\theta)$:

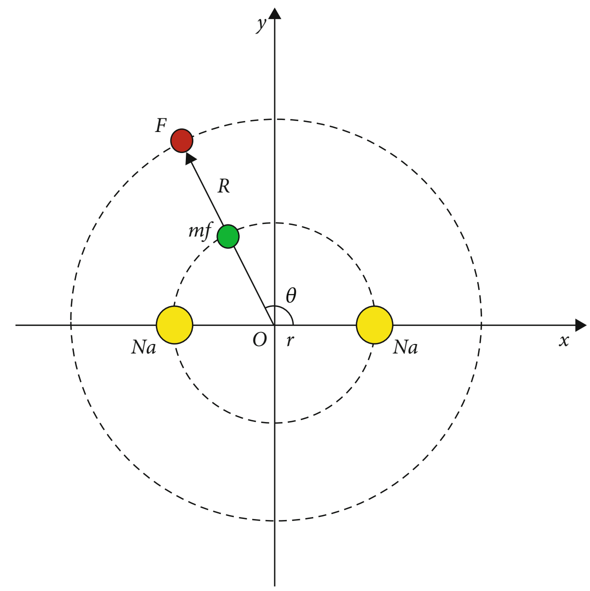

Figure 1: The coordinate system for calculation.

Table 1: Parameters for the analytic PES of the Na_2F system.

l	a_l	b_l
0	5.272×10^{-7}	4.011×10^{-6}
2	1.411×10^{-5}	8.643×10^{-7}
4	9.565×10^{-5}	6.153×10^{-7}
6	3.099×10^{-8}	2.225×10^{-7}
8	5.057×10^{-6}	2.457×10^{-5}

$$\begin{aligned} A(r, \theta) &= \sum_{l=0}^{L_1} a^l(r) P_l(\cos\theta), \\ B(r, \theta) &= \sum_{l=0}^{L_1} b^l(r) P_l(\cos\theta). \end{aligned} \tag{4}$$

We present all the fitting parameters for the analytic PES in Table 1; the 1460 ab initio points on the PES are fitted to a 10-parameter algebraic form. The maximum error is 0.0565%, and the average absolute error is less than 0.00483%.

3. Results and Discussion

We show the behavior of the potential energy surface from ten different anglers as we can see in Figure 2. From the picture, we can analyze that the peak appears in the region of $0a_0 < R < 3a_0$, with the increase of R ten different points of view of potential energy are gradually increasing. An obvious the potential barrier appears at $\theta = 0^\circ$. After reaching different peaks, the potential energy reduces with the increase of R. In the scope of $R > 5a_0$, the potential energy changes flatten. Potential energy curve appearing in the overall trend is consistent; there are differences between the local

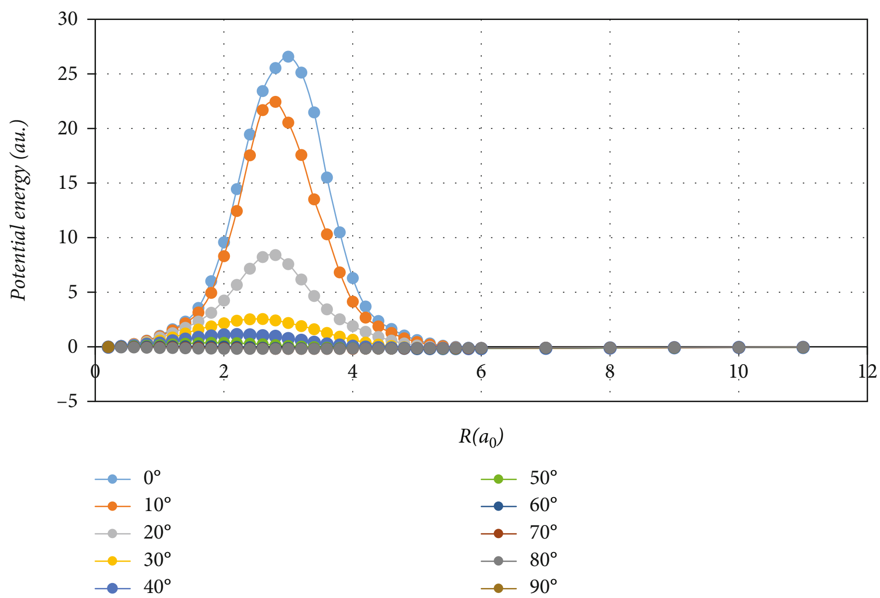

Figure 2: The analysis of the peak from 0 to 90 degrees for the potential curve.

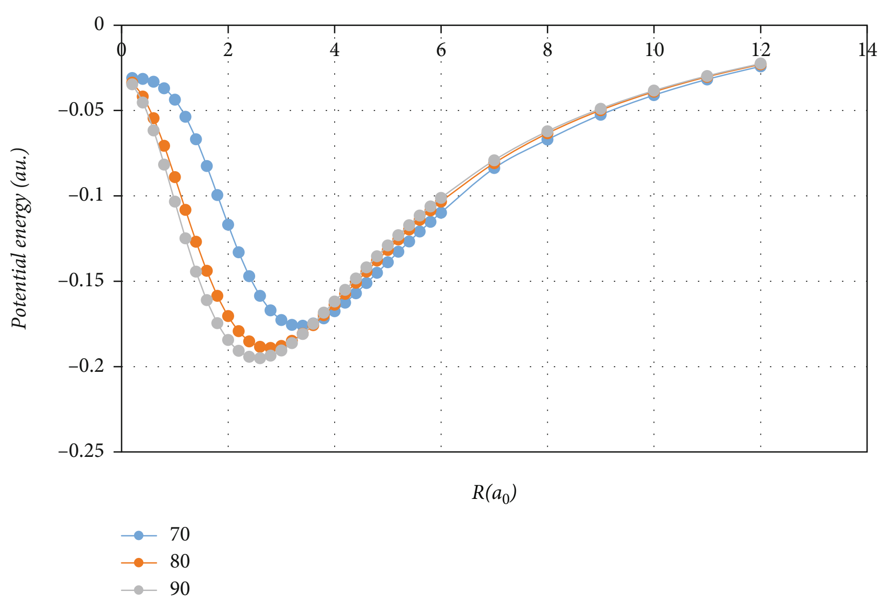

Figure 3: The analysis of the potential well from 70, 80, and 90 degrees for the potential curve.

phenomena. The calculation results show that the highest peak to linearity of Na-Na-F angle of 0° the height of the barrier is 721 eV at $R = 2.6a_0$.

Figure 3 shows the details of Figure 2 when we discuss R in the potential well area. In Figure 3, we can clearly see that an obvious potential well appears at $\theta = 90^{\circ}$. When the angle changes from 70 to 90 degrees by the interval equal step way $\Delta\theta = 10^{\circ}$, the position of the potential well also decreases with the increase of R coordinates, 90 degrees at the minimum, that is, the potential energy surface potential well position. The shallow potential well appears as the Na-F-Na configuration angle of 90°; the depth of potential well is -5.3061 eV at $R = 3a_0$.

In Figure 4, we can see clearly that as the$R$increases in the large area of the long range, the interaction converges to the same asymptotic value. The shape of a "T" backwards (Na-F-Na) is the lowest energy configuration of -5.3061 eV at $R = 3a_0$ which is close to that obtained from the experiment [17].

In Figure 5, we show the 3D-PES for angles $\theta = -0^{\circ} \sim 360^{\circ}$. The figure shows that the potential energy changes the present strong anisotropy. The highest peak to linearity of Na-Na-F angle of 0° is very clear. Also we can see that a shallow well appears at $\theta = 90^{\circ}$.

There are two obvious peaks on the ground state potential energy surface in Figure 5. The peak corresponds to the

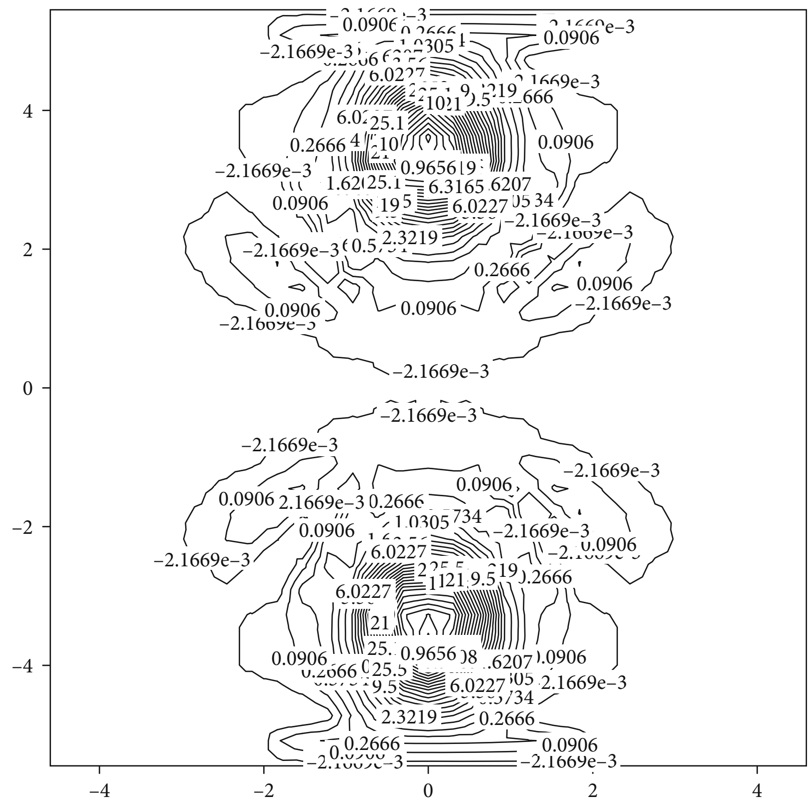

Figure 4: Contours of the V00 PES for Na_2F system.

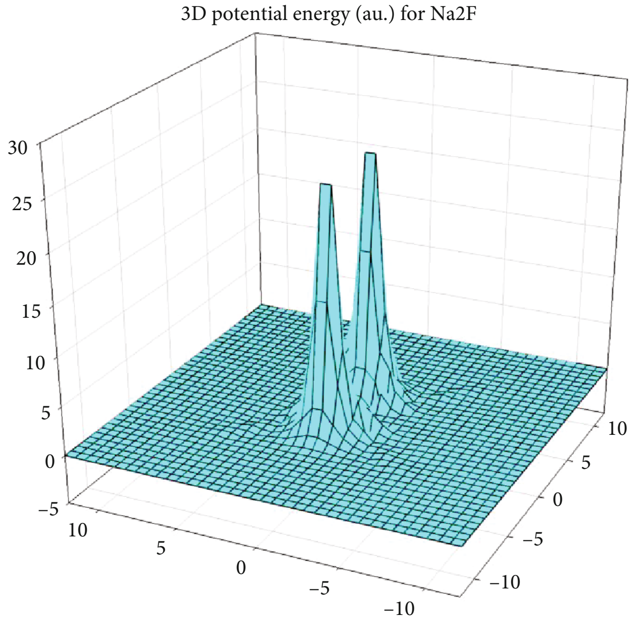

Figure 5: PES for the Na-Na-F (angle $\theta = -0° \sim 360°$).

left Na_2+F, and the right peak corresponds to the Na-F-Na reactants. We can easily see that the whole potential energy changes in large angle are anisotropic. By analytic potential energy function, we can know that whether there are two symmetric saddle points on the static potential energy surface, reaction for the threshold. Such features, reflects the alkali metal diatomic molecules interact with the fluorine atoms, in short range has the strong exclusive but in the long-range attract each other.

Table 2: Comparison of the barriers with experimental values.

Parameters	Experimental data (Ref. [17])	Ref. [12]	Relative error	Ours	Relative error
$R_{\text{Na-Nai}}(a_0)$	3.30	3.224	2.3%	3.228	2.18%
D(eV)	5.3	5.72	7.9%	5.306	0.1%

In Table 2, we compared the calculation results with the experimental data and analyzed the previous calculation results of others. Because the basis group used in our calculation is appropriate, there is not much difference with the experimental results, so our model is reasonable and the calculation is reliable.

4. Conclusion

We adopted ab initio calculation method to calculate the ground state potential energy of Na_2F system and r_e fixed at $3.228a_0$. We draw out the potential energy surface in the whole process of the three-dimensional space, by the continental scientific drilling CCSD (T) method and aug-cc-pCVQZ/aug-cc-pCVDZ+332 basis set for the sodium atom and the fluorine atom, respectively. Compared with previous experience and semiempirical potential curves earlier, our theoretical results agree well with the experimental data.

Data Availability

The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Levi-Civita Ricci-Flat Doubly Warped Product Hermitian Manifolds

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Let (M_1, g) and (M_2, h) be two Hermitian manifolds. The doubly warped product (abbreviated as DWP) Hermitian manifold of (M_1, g) and (M_2, h) is the product manifold $M_1 \times M_2$ endowed with the warped product Hermitian metric $G = f_2^2 g + f_1^2 h$, where f_1 and f_2 are positive smooth functions on M_1 and M_2, respectively. In this paper, the formulae of Levi-Civita connection, Levi-Civita curvature, the first Levi-Civita Ricci curvature, and Levi-Civita scalar curvature of the DWP-Hermitian manifold are derived in terms of the corresponding objects of its components. We also prove that if the warped function f_1 and f_2 are holomorphic, then the DWP-Hermitian manifold is Levi-Civita Ricci-flat if and only if (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat manifolds. Thus, we give an effective way to construct Levi-Civita Ricci-flat DWP-Hermitian manifold.

1. Introduction

It is well-known that the classification of various Ricci-flat manifolds are important topics in differential geometry. In 1967, Tani [1] first proposed the concept of Ricci-flat space in Riemannian geometry. Alvarez-Gaume and Freedman [2] showed that Ricci-flat space is a kind of space with great significance in theoretical physics, which attracted many scholars' research [3, 4]. In 1988, Bando and Kobayashi [5] characterized the Ricci-flat metric on Einstein-K*ä*hler manifold. In 2014, Liu and Yang [6] gave a sufficient and necessary condition for Hopf manifolds to be Levi-Civita Ricci-flat.

Levi-Civita connection is one of the most natural and effective tools for studying Riemannian manifolds [7]. In the complex case, Hsiung et al. [8] studied the general sectional curvature, the holomorphic sectional curvature, and holomorphic bisectional curvature of almost Hermitian manifolds by Levi-Civita connection and showed the relevance of above sectional curvatures. In 2012, Liu and Yang [8] gave Ricci-type curvatures and scalar curvatures of Hermitian manifolds by Levi-Civita connection (resp. Chern connection and Bismut connection) and obtained the relevance of these curvatures.

Warped product and twisted product are important methods used to construct manifold with special curvature properties in Riemann geometry and Finsler geometry. In Riemann geometry, Bishop and O'Neill [9] constructed Riemannian manifolds with negative curvature by warped product. Then, Brozos-Va'zquez et al. [10] used the warped product metrics to construct new examples of complete locally conformally flat manifolds with nonpositive curvature. After that, Leandro et al. [11] proved that an Einstein warped product manifold is a compact Riemannian manifold and its fibre is a Ricci-flat semi-Riemannian manifold.

On the other hand, warped product was extended to real Finsler geometry by the work of Asanov [12, 13]. In 2016, He and Zhong [14] generalized the warped product to complex Finsler geometry and proved that if complex Finsler manifold (M_1, F_1) and (M_2, F_2) are projectively flat, then the DWP-complex Finsler manifold is projectively flat if and only if the warped functions are positive constants. Moreover, He and Zhang [15] extended the doubly warped product to Hermitian case and got the Chern curvature, Chern Ricci curvature, and Chern Ricci scalar curvature of DWP-Hermitian manifold. They also gave the necessary and sufficient condition for a compact nontrivial DWP-Hermitian manifold to be of constant holomorphic sectional curvature. Recently, Xiao et al. [16] systematically studied holomorphic curvatures of doubly twisted product complex Finsler manifolds, and they [17] gave the necessary and sufficient condition for doubly twisted product complex Finsler manifold to be locally dually flat.

Thus, it is natural and interesting to ask the following question. Let (M_1, g) and (M_2, h) be two Levi-Civita Ricci-flat Hermitian manifolds, whether the DWP-Hermitian manifold is also a Levi-Civita Ricci-flat Hermitian manifold. Our purpose of doing this is to study the possibility of constructing Levi-Civita Ricci-flat manifold.

The structure of this paper is as follows. In Section 2, we briefly recall some basic concepts and notations which we need in this paper. In Section 3, we derive formulae of Levi-Civita connection, Levi-Civita curvature, the first Levi-Civita Ricci curvature, and Levi-Civita scalar curvature of DWP-Hermitian manifolds. In Section 4, we show that if the warped function f_1 and f_2 are holomorphic, then the DWP-Hermitian manifold is Levi-Civita Ricci-flat if and only if (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat manifolds.

2. Preliminary

Let (M, J, G) be a Hermitian manifold with $\dim_{\mathbb{C}} M = n$; here, J is the complex structure, and G is a Hermitian metric. For a point $p \in M$, the complexified tangent bundle $T_p^{\mathbb{C}} M = T_p M \otimes C$ is decomposed as

$$T_p^{\mathbb{C}} M = T_p^{1,0} M \oplus T_p^{0,1} M, \tag{1}$$

where $T_p^{1,0} M$ and $T_p^{0,1} M$ are the eigenspaces of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

In this paper, we set $\partial_\alpha = \partial / \partial z^\alpha$ and $\partial_{\bar{\alpha}} = \partial / \partial \bar{z}^\alpha$. Let $z = (z^1, \cdots, z^n)$ be the local holomorphic coordinates on M; then, the vector fields $(\partial_1, \cdots, \partial_n)$ form a basis for $T_p^{1,0} M$. Levi-Civita connection ∇^{LC} on the holomorphic tangent bundle $T_p^{1,0} M$ is defined by [18]

$$\nabla^{LC} = \pi \circ \nabla : \Gamma\left(M, T^{1,0} M\right) \xrightarrow{\nabla} \Gamma\left(M, T_p M \otimes T_p M\right) \xrightarrow{\pi} \Gamma\left(M, T_p M \otimes T^{1,0} M\right). \tag{2}$$

In local coordinate system, its connection is as follows [18]:

$$\begin{aligned} \nabla^{LC}_{\partial/\partial z^\alpha} \frac{\partial}{\partial z^\beta} \pounds^\circ &= \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial z^\gamma}, \\ \nabla^{LC}_{\partial/\partial \bar{z}^\varepsilon} \frac{\partial}{\partial z^\beta} \pounds^\circ &= \Gamma^\gamma_{\bar{\varepsilon}\beta} \frac{\partial}{\partial z^\gamma}, \end{aligned} \tag{3}$$

where the Levi-Civita connection coefficients $\Gamma^\gamma_{\alpha\beta}$ and $\Gamma^\gamma_{\bar{\alpha}\beta}$ are given by [18]

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} G^{\gamma\bar{\varepsilon}} \left(\partial_\alpha G_{\beta\bar{\varepsilon}} + \partial_\beta G_{\alpha\bar{\varepsilon}}\right), \tag{4}$$

$$\Gamma^\gamma_{\bar{\alpha}\beta} = \frac{1}{2} G^{\gamma\bar{\varepsilon}} \left(\partial_{\bar{\alpha}} G_{\beta\bar{\varepsilon}} - \partial_{\bar{\varepsilon}} G_{\beta\bar{\alpha}}\right). \tag{5}$$

Let $K \in \Gamma(M, \Lambda^2 T_p M \otimes T^{*1,0} M \otimes T^{1,0} M)$ be the Levi-Civita curvature tensor such as

$$K(X, Y)s = \nabla^{LC}_X \nabla^{LC}_Y s - \nabla^{LC}_Y \nabla^{LC}_X s - \nabla^{LC}_{[X,Y]} s, \tag{6}$$

where $X, Y \in T_p M, s \in T^{1,0} M$. In the local coordinate system, the coefficients of K are given by

$$K^\varepsilon_{\alpha\bar{\beta}\gamma} = -\left[\partial_{\bar{\beta}} \Gamma^\varepsilon_{\alpha\gamma} - \partial_\alpha \Gamma^\varepsilon_{\bar{\beta}\gamma} + \Gamma^\lambda_{\alpha\gamma} \Gamma^\varepsilon_{\bar{\beta}\lambda} - \Gamma^\lambda_{\bar{\beta}\gamma} \Gamma^\varepsilon_{\lambda\alpha}\right]. \tag{7}$$

Definition 1 (see [6]). The first Levi-Civita Ricci curvature $K^{(1)}$ on the Hermitian manifold (M, J, G) is defined by

$$K^{(1)} = \sqrt{-1} K^{(1)}_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta, \tag{8}$$

where

$$K^{(1)}_{\alpha\bar{\beta}} = G^{\gamma\bar{\delta}} K_{\alpha\bar{\beta}\gamma\bar{\delta}}, \tag{9}$$

$$K_{\alpha\bar{\beta}\gamma\bar{\delta}} = G_{\varepsilon\bar{\delta}} K^\varepsilon_{\alpha\bar{\beta}\gamma}. \tag{10}$$

Levi-Civita Ricci scalar curvature S_{LC} on $T^{1,0} M$ is given by

$$S_{LC} = G^{\alpha\bar{\beta}} K^{(1)}_{\alpha\bar{\beta}}. \tag{11}$$

Definition 2 (see [6]). Hermitian metric G on M is called Levi-Civita Ricci-flat if

$$K^{(1)}(G) = 0. \tag{12}$$

Let (M_1, g) and (M_2, h) be two Hermitian manifolds with $\dim_{\mathbb{C}} M_1 = m$ and $\dim_{\mathbb{C}} M_2 = n$; then, $M = M_1 \times M_2$ is a Hermitian manifold with $\dim_{\mathbb{C}} M = m + n$.

Denote $\pi_1 : M \longrightarrow M_1$ and $\pi_2 : M \longrightarrow M_2$ the natural projections. Note that $\pi_1(z) = z_1$ and $\pi_2(z) = z_2$ for every $z = (z_1, z_2) \in M$ with $z_1 = (z^1, \cdots, z^m) \in M_1$ and $z_2 = (z^{m+1}, \cdots, z^{m+n}) \in M_2$.

Denote $d\pi_1 : T^{1,0}(M) \longrightarrow T^{1,0} M_1, d\pi_2 : T^{1,0}(M) \longrightarrow T^{1,0} M_2$ the holomorphic tangent maps induced by π_1 and π_2, respectively. Note that $d\pi_1(z, v) = (z_1, v_1)$ and $d\pi_2(z, v) = (z_2, v_2)$ for every $v = (v_1, v_2) \in T_z^{1,0}(M)$ with $v_1 = (v^1, \cdots, v^m) \in T_{z_1}^{1,0} M_1$ and $v_2 = (v^{m+1}, \cdots, v^{m+n}) \in T_{z_2}^{1,0} M_2$.

Definition 3 (see [15]). Let (M_1, g) and (M_2, h) be two Hermitian manifolds. $f_1 : M_1 \longrightarrow (0, +\infty)$ and $f_2 : M_2 \longrightarrow (0, +\infty)$ be two positive smooth functions. The doubly warped product (abbreviated as DWP) Hermitian manifold $({}_{f_2}M_1 \times_{f_1} M_2, G)$ is the product Hermitian manifold $M = M_1 \times M_2$ endowed with the Hermitian metric $G : M \longrightarrow \mathbb{R}^+$

defined by

$$G(z,v)=(f_2\circ\pi_2)^2(z)g(\pi_1(z),d\pi_1(v))+(f_1\circ\pi_1)^2(z)h(\pi_2(z),d\pi_2(v)), \tag{13}$$

for $z=(z_1,z_2)\in M$ and $v=(v_1,v_2)\in T_z^{1,0}M$. f_1 and f_2 are warped functions; the DWP-Hermitian manifold of (M_1,g) and (M_2,h) is denoted by $({}_{f_2}M_1\times_{f_1}M_2,G)$.

If either $f_1=1$ or $f_2=1$, then $({}_{f_2}M_1\times_{f_1}M_2,G)$ becomes a warped product of Hermitian manifolds (M_1,g) and (M_2,h). If $f_1\equiv 1$ and $f_2\equiv 1$, then $({}_{f_2}M_1\times_{f_1}M_2,G)$ becomes a product of Hermitian manifolds (M_1,g) and (M_2,h). If neither f_1 nor f_2 is constant, then we call $({}_{f_2}M_1\times_{f_1}M_2,G)$ a nontrivial DWP-Hermitian manifolds of (M_1,g) and (M_2,h).

Notation 4. Lowercase Greek indices such as α, β, and γ will run from 1 to $m+n$, lowercase Latin indices such as i, j, and k will run from 1 to m, and lowercase Latin indices with a prime, such as i', j', and k', will run from $m+1$ to $m+n$. Quantities associated to (M_1,g) and (M_2,h) are denoted with upper indices 1 and 2, respectively, such as $\overset{1}{\Gamma^i_{jk}}$ and $\overset{2}{\Gamma^{i'}_{j'k'}}$ are Levi-Civita connection coefficients of (M_1,g) and (M_2,h), respectively.

Denote

$$\begin{aligned} g_{i\bar{j}} &= \frac{\partial^2 g}{\partial v^i\partial\bar{v}^j},\\ h_{i'\bar{j}'} &= \frac{\partial^2 h}{\partial v^{i'}\partial\bar{v}^{j'}}. \end{aligned} \tag{14}$$

The fundamental tensor matrix of G is given by

$$\left(G_{\alpha\bar{\beta}}\right)=\left(\frac{\partial^2 G}{\partial v^\alpha\partial\bar{v}^\beta}\right)=\begin{pmatrix} f_2^2 g_{i\bar{j}} & 0\\ 0 & f_1^2 h_{i'\bar{j}'}\end{pmatrix}, \tag{15}$$

and its inverse matrix $(G^{\bar{\beta}\alpha})$ is given by

$$\left(G^{\bar{\beta}\alpha}\right)=\begin{pmatrix} f_2^{-2} g^{\bar{j}i} & 0\\ 0 & f_1^{-2} h^{\bar{j}'i'}\end{pmatrix}. \tag{16}$$

Proposition 5. *Let $({}_{f_2}M_1\times_{f_1}M_2,G)$ be a DWP-Hermitian manifold of (M_1,g) and (M_2,h). Then, the Levi-Civita connection coefficients $\Gamma^\gamma_{\alpha\beta}$ associated to G are given by*

$$\begin{aligned} \Gamma^k_{ij} &= \overset{1}{\Gamma^k_{ij}},\\ \Gamma^k_{i'j} &= f_2^{-1}\frac{\partial f_2}{\partial z^{i'}}\delta^k_j,\\ \Gamma^k_{ij'} &= f_2^{-1}\frac{\partial f_2}{\partial z^{j'}}\delta^k_i,\\ \Gamma^{k'}_{i'j'} &= \overset{2}{\Gamma^{k'}_{i'j'}},\\ \Gamma^{k'}_{ij'} &= f_1^{-1}\frac{\partial f_1}{\partial z^i}\delta^{k'}_{j'},\\ \Gamma^{k'}_{i'j} &= f_1^{-1}\frac{\partial f_1}{\partial z^j}\delta^{k'}_{i'},\\ \Gamma^k_{i'j'} &= \Gamma^{k'}_{ij}=0. \end{aligned} \tag{17}$$

Proof. Substituting (15) and (16) into (4), we obtain

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2}G^{k\bar{\varepsilon}}\left(\partial_i G_{j\bar{\varepsilon}}+\partial_j G_{i\bar{\varepsilon}}\right)+\frac{1}{2}G^{k\overline{\varepsilon'}}\left(\partial_i G_{j\overline{\varepsilon'}}+\partial_j G_{i\overline{\varepsilon'}}\right)\\ &= \frac{1}{2}G^{k\bar{l}}\left(\frac{\partial G_{j\bar{l}}}{\partial z^i}+\frac{\partial G_{i\bar{l}}}{\partial z^j}\right)+\frac{1}{2}G^{k\overline{l'}}\left(\frac{\partial G_{j\overline{l'}}}{\partial z^i}+\frac{\partial G_{i\overline{l'}}}{\partial z^j}\right)\\ &= \frac{1}{2}f_2^{-2}g^{k\bar{l}}\left(2f_2\frac{\partial f_2}{\partial z^i}g_{j\bar{l}}+f_2^2\frac{\partial g_{j\bar{l}}}{\partial z^i}+2f_2\frac{\partial f_2}{\partial z^j}g_{i\bar{l}}+f_2^2\frac{\partial g_{i\bar{l}}}{\partial z^j}\right)\\ &= \frac{1}{2}g^{k\bar{l}}\left(\frac{\partial g_{j\bar{l}}}{\partial z^i}+\frac{\partial g_{i\bar{l}}}{\partial z^j}\right)=\overset{1}{\Gamma^k_{ij}}. \end{aligned} \tag{18}$$

Similarly, we can obtain other equations of Proposition 5. □

Plugging (15) and (16) into (5), we have the following proposition.

Proposition 6. *Let $({}_{f_2}M_1\times_{f_1}M_2,G)$ be a DWP-Hermitian manifold of (M_1,g) and (M_2,h). Then, the Levi-Civita connection coefficients $\Gamma^\gamma_{\bar{\alpha}\beta}$ associated to G are given by*

$$\begin{aligned} \Gamma^k_{\bar{i}j} &= \overset{1}{\Gamma^k_{\bar{i}j}},\\ \Gamma^k_{\bar{i}'j'} &= -f_2^{-2}f_1 g^{k\bar{l}}\frac{\partial f_1}{\partial\bar{z}^l}h_{j'\bar{i}'},\\ \Gamma^k_{\bar{i}'j} &= f_2^{-1}\frac{\partial f_2}{\partial\bar{z}^{i'}}\delta^k_j,\\ \Gamma^{k'}_{\bar{i}'j'} &= \overset{2}{\Gamma^{k'}_{\bar{i}'j'}},\\ \Gamma^{k'}_{\bar{i}j} &= -f_1^{-2}f_2 h^{k'\overline{l'}}\frac{\partial f_2}{\partial\bar{z}^{l'}}g_{j\bar{i}},\\ \Gamma^{k'}_{\bar{i}j'} &= f_1^{-1}\frac{\partial f_1}{\partial\bar{z}^i}\delta^{k'}_{j'},\\ \Gamma^k_{\bar{i}j'} &= \Gamma^{k'}_{\bar{i}'j}=0. \end{aligned} \tag{19}$$

3. Levi-Civita Ricci Scalar Curvature of Doubly Warped Product Hermitian Manifolds

In this section, we derive formulae of Levi-Civita curvature, Levi-Civita Ricci curvature, and Levi-Civita Ricci scalar curvature of DWP-Hermitian manifold.

Proposition 7. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). Then, the coefficients of Levi-Civita curvature tensor $K^{\varepsilon}_{\alpha\bar\beta\gamma}$ are given by*

$$K^t_{k\bar j s} = \overset{1}{K}{}^t_{k\bar j s} + f_1^{-2} h^{i'\bar l'} \frac{\partial f_2}{\partial \bar z^{l'}} \frac{\partial f_2}{\partial z^{i'}} g_{s\bar j}\delta^t_k, \tag{20}$$

$$K^{t'}_{k'\bar j' s'} = \overset{2}{K}{}^{t'}_{k'\bar j' s'} + f_2^{-2} g^{i\bar l} \frac{\partial f_1}{\partial \bar z^{l}} \frac{\partial f_1}{\partial z^{i}} h_{s'\bar j'}\delta^{t'}_{k'}, \tag{21}$$

$$K^t_{k'\bar j' s} = f_2^{-2} g^{t\bar l} \frac{\partial f_1}{\partial \bar z^{l}} \frac{\partial f_1}{\partial z^{s}} h_{k'\bar j'}, \tag{22}$$

$$K^{t'}_{k\bar j s'} = f_1^{-2} h^{t'\bar l'} \frac{\partial f_2}{\partial \bar z^{l'}} \frac{\partial f_2}{\partial z^{s'}} g_{k\bar j}, \tag{23}$$

$$K^t_{k\bar j s'} = f_1^{-1} f_2^{-1} \frac{\partial f_1}{\partial \bar z^{j}} \frac{\partial f_2}{\partial z^{i'}} \delta^{i'}_{s'}\delta^t_k - \frac{1}{2} f_2^{-1} \frac{\partial f_2}{\partial z^{s'}} g^{t\bar l}\left(\frac{\partial g_{i\bar l}}{\partial \bar z^{j}} - \frac{\partial g_{i\bar j}}{\partial \bar z^{l}}\right)\delta^i_k, \tag{24}$$

$$K^{t'}_{k'\bar j' s} = f_1^{-1} f_2^{-1} \frac{\partial f_1}{\partial z^{i}} \frac{\partial f_2}{\partial \bar z^{j'}} \delta^{i}_{s}\delta^{t'}_{k'} - \frac{1}{2} f_1^{-1} \frac{\partial f_1}{\partial z^{s}} h^{i'\bar l'}\left(\frac{\partial h_{i'\bar l'}}{\partial \bar z^{j'}} - \frac{\partial h_{i'\bar l'}}{\partial \bar z^{l'}}\right)\delta^{i'}_{k'}, \tag{25}$$

$$K^t_{k'\bar j' s'} = f_2^{-3} f_1 g^{t\bar l} \frac{\partial f_1}{\partial \bar z^{l}} \frac{\partial f_2}{\partial z^{k'}} h_{s'\bar j'} + f_2^{-2} f_1 g^{t\bar l} \frac{\partial f_1}{\partial \bar z^{l}} \left[\frac{1}{2}\left(\frac{\partial h_{k'\bar l'}}{\partial z^{s'}} + \frac{\partial h_{s'\bar l'}}{\partial z^{k'}}\right)\delta^{\bar l'}_{\bar j'} + \frac{\partial h_{s'\bar j'}}{\partial z^{k'}}\right]. \tag{26}$$

$$K^{t'}_{k\bar j s} = f_1^{-3} f_2 h^{t'\bar l'} \frac{\partial f_2}{\partial \bar z^{l'}} \frac{\partial f_1}{\partial z^{k}} g_{s\bar j} + f_1^{-2} f_2 h^{t'\bar l'} \frac{\partial f_2}{\partial \bar z^{l'}} \left[\frac{1}{2}\left(\frac{\partial g_{k\bar l}}{\partial z^{s}} + \frac{\partial g_{s\bar l}}{\partial z^{k}}\right)\delta^{\bar l}_{\bar j} + \frac{\partial g_{s\bar j}}{\partial z^{k}}\right], \tag{27}$$

$$\begin{aligned} K^t_{k\bar j' s'} = {} & \frac{\partial^2 \ln f_2}{\partial z^{s'}\partial \bar z^{j'}}\delta^t_k + \frac{1}{2} f_2^{-1} h^{i'\bar l'} \frac{\partial f_2}{\partial z^{i'}}\left(\frac{\partial h_{s'\bar l'}}{\partial \bar z^{j'}} - \frac{\partial h_{s'\bar j'}}{\partial \bar z^{l'}}\right)\delta^t_k \\ & + f_2^{-2} f_1 h_{s'\bar j'}\left[\frac{\partial g^{t\bar l}}{\partial z^k}\frac{\partial f_1}{\partial \bar z^l} - g^{t\bar l}\frac{\partial^2 f_1}{\partial z^k \partial \bar z^l} - \frac{1}{2} g^{i\bar l} g^{t\bar l}\frac{\partial f_1}{\partial \bar z^l}\left(\frac{\partial g_{k\bar l}}{\partial z^i} + \frac{\partial g_{i\bar l}}{\partial z^k}\right)\right], \end{aligned} \tag{28}$$

$$\begin{aligned} K^{t'}_{k'\bar j s} = {} & \frac{\partial^2 \ln f_1}{\partial z^{s}\partial \bar z^{j}}\delta^{t'}_{k'} + \frac{1}{2} f_1^{-1} g^{i\bar l} \frac{\partial f_1}{\partial z^{i}}\left(\frac{\partial g_{s\bar l}}{\partial \bar z^{j}} - \frac{\partial g_{s\bar j}}{\partial \bar z^{l}}\right)\delta^{t'}_{k'} \\ & + f_1^{-2} f_2 g_{s\bar j}\left[\frac{\partial h^{t'\bar l'}}{\partial z^{k'}}\frac{\partial f_2}{\partial \bar z^{l'}} - h^{t'\bar l'}\frac{\partial^2 f_2}{\partial z^{k'} \partial \bar z^{l'}} - \frac{1}{2} h^{i'\bar l'} h^{t'\bar l'}\frac{\partial f_2}{\partial \bar z^{l'}}\left(\frac{\partial h_{k'\bar l'}}{\partial z^{i'}} + \frac{\partial h_{i'\bar l'}}{\partial z^{k'}}\right)\right]. \end{aligned} \tag{29}$$

$$K^t_{k'\bar j s} = K^t_{k'\bar j s'} = K^t_{k\bar j' s} = K^{t'}_{k\bar j' s} = K^{t'}_{k'\bar j s'} = K^{t'}_{k\bar j' s'} = 0. \tag{30}$$

Proof. Using (7), we have

$$K^t_{k\bar j s} = -\left[\partial_{\bar j}\Gamma^t_{ks} - \partial_k \Gamma^t_{\bar j s} + \Gamma^\lambda_{ks}\Gamma^t_{\bar j\lambda} - \Gamma^\lambda_{\bar j s}\Gamma^t_{\lambda k} + \Gamma^{\lambda'}_{ks}\Gamma^t_{\bar j\lambda'} - \Gamma^{\lambda'}_{\bar j s}\Gamma^t_{\lambda' k}\right]. \tag{31}$$

Taking the formulae of Proposition 5 and Proposition 6 into (31), we obtain

$$\begin{aligned} K^t_{k\bar j s} & = -\left[\frac{\partial \overset{1}{\Gamma}{}^t_{ks}}{\partial \bar z^j} - \frac{\partial \overset{1}{\Gamma}{}^t_{\bar j s}}{\partial z^k} + \overset{1}{\Gamma}{}^i_{ks}\overset{1}{\Gamma}{}^t_{\bar j i} - \overset{1}{\Gamma}{}^i_{\bar j s}\overset{1}{\Gamma}{}^t_{ik} + \Gamma^{i'}_{ks}\Gamma^t_{\bar j i'} - \Gamma^{i'}_{\bar j s}\Gamma^t_{i' k}\right] \\ & = \overset{1}{K}{}^t_{k\bar j s} + f_1^{-2} h^{i'\bar l'} \frac{\partial f_2}{\partial \bar z^{l'}} \frac{\partial f_2}{\partial z^{i'}} g_{s\bar j}\delta^t_k. \end{aligned} \tag{32}$$

Similarly, we can obtain other equations of Proposition 7. □

Proposition 8. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). Then,*

$$K_{k\bar j s\bar p} = f_2^2 \overset{1}{K}_{k\bar j s\bar p} + f_1^{-2} f_2^2 h^{i'\bar l'} \frac{\partial f_2}{\partial \bar z^{l'}} \frac{\partial f_2}{\partial z^{i'}} g_{s\bar j} g_{k\bar p},$$

$$K_{k'\bar j' s'\bar p'} = f_1^2 \overset{2}{K}_{k'\bar j' s'\bar p'} + f_2^{-2} f_1^2 g^{i\bar l} \frac{\partial f_1}{\partial \bar z^{l}} \frac{\partial f_1}{\partial z^{i}} h_{s'\bar j'} h_{k'\bar p'},$$

$$K_{k\bar j s'\bar p'} = \frac{\partial f_2}{\partial \bar z^{l'}} \frac{\partial f_2}{\partial z^{s'}} g_{k\bar j}\delta^{\bar l'}_{\bar p'},$$

$$K_{k'\bar j' s\bar p} = \frac{\partial f_1}{\partial \bar z^{l}} \frac{\partial f_1}{\partial z^{s}} h_{k'\bar j'}\delta^{\bar l}_{\bar p},$$

$$K_{k\bar j s'\bar p} = f_1^{-1} f_2 g_{k\bar p} \frac{\partial f_1}{\partial \bar z^{j}} \frac{\partial f_2}{\partial z^{i'}} \delta^{i'}_{s'} - \frac{1}{2} f_2 \frac{\partial f_2}{\partial z^{s'}}\left(\frac{\partial g_{i\bar l}}{\partial \bar z^{j}} - \frac{\partial g_{i\bar j}}{\partial \bar z^{l}}\right)\delta^i_k \delta^{\bar l}_{\bar p},$$

$$K_{k'\bar j' s\bar p'} = f_1 f_2^{-1} h_{k'\bar p'} \frac{\partial f_1}{\partial z^{i}} \frac{\partial f_2}{\partial \bar z^{j'}} \delta^{i}_{s} - \frac{1}{2} f_1 \frac{\partial f_1}{\partial z^{s}}\left(\frac{\partial h_{i'\bar l'}}{\partial \bar z^{j'}} - \frac{\partial h_{i'\bar j'}}{\partial \bar z^{l'}}\right)\delta^{i'}_{k'} \delta^{\bar l'}_{\bar p'},$$

$$K_{k\bar j s\bar p'} = f_1^{-3} f_2 \frac{\partial f_2}{\partial \bar z^{l'}} \frac{\partial f_1}{\partial z^{k}} g_{s\bar j}\delta^{\bar l'}_{\bar p'} + f_2 \frac{\partial f_2}{\partial \bar z^{l'}}\left[\frac{1}{2}\left(\frac{\partial g_{k\bar l}}{\partial z^{s}} + \frac{\partial g_{s\bar l}}{\partial z^{k}}\right)\delta^{\bar l}_{\bar j} + \frac{\partial g_{s\bar j}}{\partial z^{k}}\right]\delta^{\bar l'}_{\bar p'},$$

$$K_{k'\bar j' s'\bar p} = f_2^{-3} f_1 \frac{\partial f_1}{\partial \bar z^{l}} \frac{\partial f_2}{\partial z^{k'}} h_{s'\bar j'}\delta^{\bar l}_{\bar p} + f_1 \frac{\partial f_1}{\partial \bar z^{l}}\left[\frac{1}{2}\left(\frac{\partial h_{k'\bar l'}}{\partial z^{s'}} + \frac{\partial h_{s'\bar l'}}{\partial z^{k'}}\right)\delta^{\bar l'}_{\bar j'} + \frac{\partial h_{s'\bar j'}}{\partial z^{k'}}\right]\delta^{\bar l}_{\bar p},$$

$$\begin{aligned} K_{k'\bar j s\bar p'} = {} & f_1^2 h_{k'\bar p'}\frac{\partial^2 \ln f_1}{\partial z^s \partial \bar z^j} + \frac{1}{2} f_1 h_{k'\bar p'} g^{i\bar l}\frac{\partial f_1}{\partial z^i}\left(\frac{\partial g_{s\bar l}}{\partial \bar z^j} - \frac{\partial g_{s\bar j}}{\partial \bar z^l}\right) \\ & + f_2 g_{s\bar j}\left[\frac{\partial h^{t'\bar l'}}{\partial z^{k'}}\frac{\partial f_2}{\partial \bar z^{l'}} h_{t'\bar p'} + \delta^{\bar l'}_{\bar p'}\frac{\partial^2 f_2}{\partial z^{k'}\partial \bar z^{l'}} \right. \\ & \left. + \frac{1}{2} h^{i'\bar l'}\delta^{\bar l'}_{\bar p'}\frac{\partial f_2}{\partial \bar z^{l'}}\left(\frac{\partial h_{k'\bar l'}}{\partial z^{i'}} + \frac{\partial h_{i'\bar l'}}{\partial z^{k'}}\right)\right], \end{aligned}$$

$$\begin{aligned} K_{k\bar j' s'\bar p} = {} & f_2^2 g_{k\bar p}\frac{\partial^2 \ln f_2}{\partial z^{s'} \partial \bar z^{j'}} + \frac{1}{2} f_2 g_{k\bar p} h^{i'\bar l'}\frac{\partial f_2}{\partial z^{i'}}\left(\frac{\partial h_{s'\bar l'}}{\partial \bar z^{j'}} - \frac{\partial h_{s'\bar j'}}{\partial \bar z^{l'}}\right) \\ & + f_1 h_{s'\bar j'}\left[\frac{\partial g^{t\bar l}}{\partial z^{k}}\frac{\partial f_1}{\partial \bar z^{l}} g_{t\bar p} + \delta^{\bar l}_{\bar p}\frac{\partial^2 f_1}{\partial z^{k}\partial \bar z^{l}} + \frac{1}{2} g^{i\bar l}\delta^{\bar l}_{\bar p}\frac{\partial f_1}{\partial \bar z^{l}}\left(\frac{\partial g_{k\bar l}}{\partial z^{i}} + \frac{\partial g_{i\bar l}}{\partial z^{k}}\right)\right]. \end{aligned}$$

$$K_{k'\bar j s'\bar p} = K_{k'\bar j s\bar p} = K_{k'\bar j s'\bar p'} = K_{k\bar j' s\bar p} = K_{k\bar j' s\bar p'} = K_{k\bar j' s'\bar p'} = 0. \tag{33}$$

Proof. According to (10), we get

$$K_{k\bar{j}s\bar{p}} = G_{\varepsilon\bar{p}}K^{\varepsilon}_{k\bar{j}s} = G_{t\bar{p}}K^{t}_{k\bar{j}s} + G_{t'\bar{p}}K^{t'}_{k\bar{j}s}. \tag{34}$$

Substituting (20), (27), and (15) into (34), we have

$$K_{k\bar{j}s\bar{p}} = G_{t\bar{p}}\left(\overset{1}{K}{}^{t}_{k\bar{j}s} + f_1^{-2}h^{i'\overline{l'}}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{i'}}g_{s\bar{j}}\delta^t_k\right) = f_2^2\overset{1}{K}_{k\bar{j}s\bar{p}} + f_1^{-2}f_2^2h^{i'\overline{l'}}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{i'}}g_{s\bar{j}}g_{k\bar{p}}. \tag{35}$$

Similarly, we can obtain other equations of Proposition 8. □

Proposition 9. *Let $\left({}_{f_2}M_1 \times {}_{f_1}M_2, G\right)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). Then, the coefficients of the first Levi-Civita Ricci curvature $K^{(1)}_{\alpha\bar{\beta}}$ are given by*

$$\begin{cases} K^{(1)}_{k\bar{j}} = \overset{1}{K}{}^{(1)}_{k\bar{j}} + 2f_1^{-2}\dfrac{\partial f_2}{\partial \bar{z}^{l'}}\dfrac{\partial f_2}{\partial z^{i'}}g_{k\bar{j}}h^{i'\overline{l'}}, \\ K^{(1)}_{k'\overline{j'}} = \overset{2}{K}{}^{(1)}_{k'\overline{j'}} + 2f_2^{-2}\dfrac{\partial f_1}{\partial \bar{z}^{l}}\dfrac{\partial f_1}{\partial z^{s}}h_{k'\overline{j'}}g^{s\bar{l}}, \\ K^{(1)}_{k'\bar{j}} = 0, \\ K^{(1)}_{k\overline{j'}} = 0. \end{cases} \tag{36}$$

where $\overset{1}{K}{}^{(1)}_{k\bar{j}}$ and $\overset{2}{K}{}^{(1)}_{k'\overline{j'}}$ are coefficients of the first Levi-Civita Ricci curvature of g and h, respectively.

Proof. From (9) and (16), we get

$$K^{(1)}_{k\bar{j}} = G^{\gamma\bar{\delta}}K_{k\bar{j}\gamma\bar{\delta}} = G^{s\bar{p}}K_{k\bar{j}s\bar{p}} + G^{s'\overline{p'}}K_{k\bar{j}s'\overline{p'}}. \tag{37}$$

According to (16) and the first equation of proposition 8, we have

$$G^{s\bar{p}}K_{k\bar{j}s\bar{p}} = f_2^{-2}g^{s\bar{p}}\left(f_2^2\overset{1}{K}_{k\bar{j}s\bar{p}} + f_1^{-2}f_2^2h^{i'\overline{l'}}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{i'}}g_{s\bar{j}}g_{k\bar{p}}\right) = \overset{1}{K}{}^{(1)}_{k\bar{j}} + f_1^{-2}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{i'}}g_{k\bar{j}}h^{i'\overline{l'}}. \tag{38}$$

Similarly, by using (16) and the third equation of proposition 8, we can get

$$G^{s'\overline{p'}}K_{k\bar{j}s'\overline{p'}} = f_1^{-2}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}}g_{k\bar{j}}h^{s'\overline{l'}}. \tag{39}$$

Replacing the summation index i' on the right side of (38) with s' and then taking it and (39) into (37), we can obtain

$$K^{(1)}_{k\bar{j}} = \overset{1}{K}{}^{(1)}_{k\bar{j}} + 2f_1^{-2}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}}g_{k\bar{j}}h^{s'\overline{l'}}. \tag{40}$$

Similarly, we can obtain

$$\begin{aligned} K^{(1)}_{k'\overline{j'}} &= \overset{2}{K}{}^{(1)}_{k'\overline{j'}} + 2f_2^{-2}\frac{\partial f_1}{\partial \bar{z}^{l}}\frac{\partial f_1}{\partial z^{s}}h_{k'\overline{j'}}g^{s\bar{l}}, \\ K^{(1)}_{k'\bar{j}} &= 0, \\ K^{(1)}_{k\overline{j'}} &= 0. \end{aligned} \tag{41}$$

This completes the proof. □

Theorem 10. *Let $\left({}_{f_2}M_1 \times {}_{f_1}M_2, G\right)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). Then, the Levi-Civita Ricci scalar curvature of G along a nonzero vector $v = (v^i, v^{i'}) \in T^{1,0}_zM$ is given by*

$$\begin{aligned} S_{LC}(v) = {} & f_2^{-2}S_g(v_1) + f_1^{-2}S_h(v_2) + 2f_1^{-2}f_2^{-2}g^{s\bar{l}}\frac{\partial f_1}{\partial \bar{z}^{l}}\frac{\partial f_1}{\partial z^{s}} \\ & + 2f_1^{-2}f_2^{-2}h^{s'\overline{l'}}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}}, \end{aligned} \tag{42}$$

where $S_g(v_1)$ and $S_h(v_2)$ are Levi-Civita Ricci scalar curvatures of g and h, respectively.

Proof. According to (11), the Levi-Civita Ricci scalar curvature of G is given by

$$S_{LC} = G^{\alpha\bar{\beta}}K^{(1)}_{\alpha\bar{\beta}} = G^{k\bar{j}}K^{(1)}_{k\bar{j}} + G^{k'\overline{j'}}K^{(1)}_{k'\overline{j'}} + G^{k'\bar{j}}K^{(1)}_{k'\bar{j}} + G^{k\overline{j'}}K^{(1)}_{k\overline{j'}}. \tag{43}$$

Combining (16) and (40), we have

$$\begin{aligned} G^{k\bar{j}}K^{(1)}_{k\bar{j}} &= f_2^{-2}g^{k\bar{j}}\left(\overset{1}{K}{}^{(1)}_{k\bar{j}} + 2f_1^{-2}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}}g_{k\bar{j}}h^{s'\overline{l'}}\right) \\ &= f_2^{-2}S_g(v_1) + 2f_1^{-2}f_2^{-2}h^{s'\overline{l'}}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}}. \end{aligned} \tag{44}$$

Similarly, we can get

$$G^{k'\overline{j'}}K^{(1)}_{k'\overline{j'}} = f_1^{-2}S_h(v_2) + 2f_1^{-2}f_2^{-2}g^{s\bar{l}}\frac{\partial f_1}{\partial \bar{z}^{l}}\frac{\partial f_1}{\partial z^{s}}, \tag{45}$$

$$G^{k'\bar{j}}K^{(1)}_{k'\bar{j}} = 0, \tag{46}$$

$$G^{k\overline{j'}}K^{(1)}_{k\overline{j'}} = 0. \tag{47}$$

Taking (44)–(47) into (43), we obtain (42). □

Theorem 11. *Let $\left({}_{f_2}M_1 \times {}_{f_1}M_2, G\right)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). If f_1 and f_2 are holomorphic functions on M_1 and M_2, respectively, then $S_{LC}(v) = f_2^{-2}S_g(v_1) + f_1^{-2}S_h(v_2)$.*

Proof. If f_1 and f_2 are holomorphic functions on M_1 and M_2, respectively, i.e.,

$$\begin{aligned} \frac{\partial f_1}{\partial \bar{z}^l} &= 0, \\ \frac{\partial f_2}{\partial \bar{z}^{l'}} &= 0. \end{aligned} \tag{48}$$

Thus,

$$2 f_1^{-2} f_2^{-2} g^{s\bar{l}} \frac{\partial f_1}{\partial \bar{z}^l} \frac{\partial f_1}{\partial z^s} = 0, \tag{49}$$

$$2 f_1^{-2} f_2^{-2} h^{s'\bar{l}'} \frac{\partial f_2}{\partial \bar{z}^{l'}} \frac{\partial f_2}{\partial z^{s'}} = 0. \tag{50}$$

Substituting (49) into (42), we have $S_{LC}(v) = f_2^{-2} S_g(v_1) + f_1^{-2} S_h(v_2)$. □

4. Levi-Civita Ricci-Flat Doubly Warped Product Hermitian Manifolds

Let (M_1, g) and (M_2, h) be two Levi-Civita Ricci-flat Hermitian manifolds; one may want to know whether the DWP-Hermitian manifold $({}_{f_2}M_1 \times_{f_1} M_2, G)$ is also a Levi-Civita Ricci-flat Hermitian manifold. We shall give an answer to this question in this section.

Theorem 12. *Let $({}_{f_2}M_1 \times_{f_1} M_2, G)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). If f_1 and f_2 are holomorphic functions on M_1 and M_2, respectively, then $({}_{f_2}M_1 \times_{f_1} M_2, G)$ is Levi-Civita Ricci-flat if and only if (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat.*

Proof. If f_1 and f_2 are holomorphic functions on M_1 and M_2, respectively, i.e.,

$$\begin{aligned} \frac{\partial f_1}{\partial \bar{z}^l} &= 0, \\ \frac{\partial f_2}{\partial \bar{z}^{l'}} &= 0. \end{aligned} \tag{51}$$

Taking above equations into the first formula and second formula of (36), we get

$$2 f_1^{-2} \frac{\partial f_2}{\partial \bar{z}^{l'}} \frac{\partial f_2}{\partial z^{i'}} g_{k\bar{j}} h^{i'\bar{l}'} = 0, \tag{52}$$

$$2 f_2^{-2} \frac{\partial f_1}{\partial \bar{z}^{l}} \frac{\partial f_1}{\partial z^{s}} h_{k'\bar{j}'} g^{s\bar{l}} = 0. \tag{53}$$

Firstly, we assume $({}_{f_2}M_1 \times_{f_1} M_2, G)$ be Levi-Civita Ricci-flat; using Definition 2 and (36), we have

$$\begin{cases} K^{(1)}_{k\bar{j}} = \overset{1}{K}{}^{(1)}_{k\bar{j}} + 2 f_1^{-2} \dfrac{\partial f_2}{\partial \bar{z}^{l'}} \dfrac{\partial f_2}{\partial z^{i'}} g_{k\bar{j}} h^{i'\bar{l}'} = 0, \\ K^{(1)}_{k'\bar{j}'} = \overset{2}{K}{}^{(1)}_{k'\bar{j}'} + 2 f_2^{-2} \dfrac{\partial f_1}{\partial \bar{z}^{l}} \dfrac{\partial f_1}{\partial z^{s}} h_{k'\bar{j}'} g^{s\bar{l}} = 0, \\ K^{(1)}_{k'\bar{j}} = 0, \\ K^{(1)}_{k\bar{j}'} = 0. \end{cases} \tag{54}$$

Substituting (52) and (53) into the first formula and second formula of (54), respectively, we get

$$\begin{cases} K^{(1)}_{k\bar{j}} = \overset{1}{K}{}^{(1)}_{k\bar{j}} = 0, \\ K^{(1)}_{k'\bar{j}'} = \overset{2}{K}{}^{(1)}_{k'\bar{j}'} = 0, \\ K^{(1)}_{k'\bar{j}} = 0, \\ K^{(1)}_{k\bar{j}'} = 0. \end{cases} \tag{55}$$

Obviously,

$$\begin{aligned} \overset{1}{K}{}^{(1)}_{k\bar{j}} &= 0, \\ \overset{2}{K}{}^{(1)}_{k'\bar{j}'} &= 0. \end{aligned} \tag{56}$$

According to Definition 2, these mean that (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat.

Conversely, we assume (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat; according to Definition 2, we know that

$$\overset{1}{K}{}^{(1)}_{k\bar{j}} = 0, \tag{57}$$

$$\overset{2}{K}{}^{(1)}_{k'\bar{j}'} = 0. \tag{58}$$

Since f_1 and f_2 are holomorphic, thus (52) and (53) are established. Then, taking (52), (53), (57), and (58) into (36), we obtain

$$\begin{cases} K^{(1)}_{k\bar{j}} = 0, \\ K^{(1)}_{k'\bar{j}'} = 0, \\ K^{(1)}_{k'\bar{j}} = 0, \\ K^{(1)}_{k\bar{j}'} = 0. \end{cases} \tag{59}$$

By Definition 2, (59) indicates that $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ is Levi-Civita Ricci-flat. □

Notation 13. Theorem 12 implies that when warped functions to be holomorphic, then the DWP-Hermitian manifold is a Levi-Civita Ricci-flat Hermitian manifold if and only if its component manifolds are Levi-Civita Ricci-flat. Thus, this theorem provides us an effective way to construct Levi-Civita Ricci-flat DWP-Hermitian manifold.

5. Conclusions

In this paper, we derived formulae of Levi-Civita connection, Levi-Civita curvature, the first Levi-Civita Ricci curvature, and Levi-Civita scalar curvature of the DWP-Hermitian manifold and proved that if the warped function f_1 and f_2 are holomorphic, then the DWP-Hermitian manifold is Levi-Civita Ricci-flat if and only if (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat manifolds. Thus, we gave an effective way to construct Levi-Civita Ricci-flat DWP-Hermitian manifold.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Turbulence Characteristics of Cavitating Flows Downstream of Triangular Multiorifice Plates

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Cavitating flow fields downstream of triangular multiorifice plates with different geometrical parameters were measured by PIV technique, and effects of orifice size, orifice number, and orifice layout on turbulence intensity and Reynolds stress were analyzed. The experimental results showed that the turbulence intensity and Reynolds stress downstream of the different multiorifice plates exhibited sawtooth-like profiles. Decrease in orifice size, increase in orifice number, and taking staggered layout could contribute to intensification of turbulence mixing and shear effects of multiple cavitating jets downstream of the multiorifice plates and thus reaching the expected cavitation effects.

1. Introduction

Once flow velocity reaches a certain point where the pressure of flow lowers below the saturated vapor pressure at the corresponding temperature, a cavitation phenomenon will occur in liquids. The collapse of cavitation bubbles in the zone where pressure rises can generate super high pressure and temperature and forms microjets and shock waves in a micro second interval and thus will cause severe damage to ship propellers, hydraulic release structures, hydraulic components, and hydraulic machinery. Conversely, Pandit and Joshi [1] applied hydrodynamic cavitation into the hydrolysis of fatty oil. Since then, hydrodynamic cavitation has been studied for the potential application in water treatment. Many studies have found that the hydraulic condition of cavitation reactor was the major factor and played an important role in effective wastewater treatment [2–5]. Dong et al. [6] and Yao et al. [7] studied the cavitational characteristics due to circular and triangular multiorifice plates, and their results revealed that the multiorifice plates with larger and more orifices incurred stronger cavitation and hence improving the degradation rate. And Dong et al. [8] studied the degradation of hydrophilic and hydrophobic mixture due to the combination of the Venturi tube with the multiorifice plates. Wang et al. [9], Geng et al. [10], and Dong and Qin [11], respectively, used hydrodynamic cavitation due to the Venturi tube to kill Escherichia coli in raw water. They focused on the effects of variable diffusion angle, varying throat lengths, throat velocity, treatment time, cavitation number, and initial concentration of Escherichia coli on the killing rate. Also, killing rate of pathogenic microorganisms in raw water by hydrodynamic cavitation due to triangular and square multiorifice plates was, respectively, studied by the references [12–15]. As mentioned above, the characteristics of cavitating flow directly affect the degradation rates of refractory pollutants and the killing rates of pathogenic microorganisms. Dong et al. [16, 17] and Zhang et al. [18], respectively, reported time-averaged velocity and pressure distributions of the Venturi tube and triangular multiorifice plates and their combinations. In fact, characteristics of cavitating filed flow behind cavitation reactors such as multiorifice plates contribute further to understanding the mechanisms of degrading refractory pollutant and of killing pathogenic microorganism by hydrodynamic cavitation and to optimizing the design of cavitation reactor. However, less study of turbulence characteristics downstream of the multi-orifice plates was reported. In this paper, cavitating flow fields downstream of 5 triangular multiorifice plates with different orifice sizes, numbers and layouts were measured by PIV technique, and the effects of the geometric parameters on turbulence intensity and Reynolds stress were analyzed.

2. Experimental Facility and Methodology

The experimental apparatus is shown in Figure 1. Five types of triangular multiorifice plates were designed in the experiment as shown in Figure 2. The size of each plate was 50 mm × 50 mm. The total orifice area of each plate remained the same. The orifice numbers of these multiorifice plates were $n = 9$, 16, 25, and 64, and the orifice sizes were $a = 2.6$, 4.0, 5.1, and 6.7 mm, respectively. The checkerboard-type and staggered layouts of orifices were arranged on the plates. The geometric parameters of the multiorifice plates are shown in Table 1. The size of cross-section of the working section was 50 mm × 50 mm, and the length of the working section was 200 mm. The top and two sides of the working section were installed by polymethyl methacrylate for observation window. The multiorifice plates and working section were made of stainless steel plate, which were fabricated by computer-controlled machine tool.

The basic principle of PIV technique is that the trace particles can be of good reflectivity and tracking features, whose relative density is equivalent to the fluid evenly spread in the measured flow field. Then, the moving images of these particles will be captured by a camera at certain interval before matching the particles in adjacent images. In this way, the parameters of movements can be worked out through calculating the velocity vector at each point in the flow filed. In this paper, lots of cavitation bubbles existed in the high-velocity flow, and since the tiny bubble and the moving fluid have better following characteristics, the speed of moving bubble was almost equal to that of fluid particle. So the motion of bubble can be used to reflect the motion of fluid particle. The Dantec 3D-PIV was used to measure the instantaneous velocity field downstream of the triangular multiorifice plates. The sampling frequency was 15 Hz, and the time interval of instantaneous flow field was 0.02 s. Considering that the number of instantaneous flow field should statistically meet the need of steady turbulent flow, 500 groups of instantaneous flow fields verified by the preexperiment were chosen to analyze the turbulence intensity and Reynolds stress of flow fields.

The measuring position was along the center line on the top surface in the working section, ranging within 50 mm × 200 mm. In order to analyze and compare turbulence characteristics in detail, the working section was divided into 8 cross-sections as shown in Figure 3 and the positions of cross-sections are listed in Table 2. For the sake of comparison, the position of cross-section was nondimensionalized by the length of working section ($L = 200$ mm).

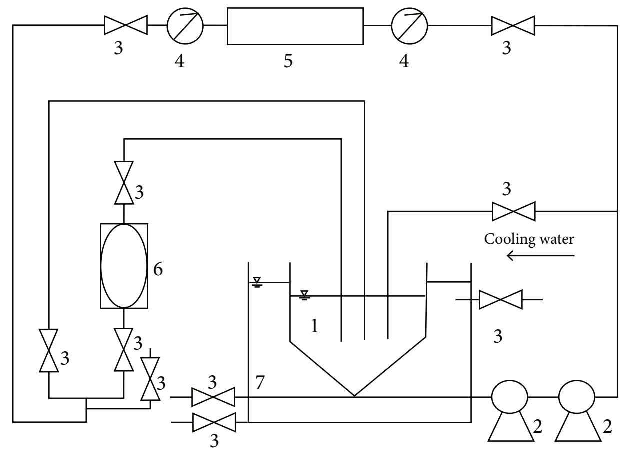

Figure 1: Sketch of experimental setup. 1: inner solution tank, 2: centrifugal pump, 3: control valves, 4: pressure gauge, 5: working section, 6: rotator flow meter, and 7: water cooling tank.

3. Results and Discussion

The relative longitudinal turbulence intensity can be expressed as

$$T_X = \frac{\sqrt{\acute{u}2}}{U}. \tag{1}$$

And the relative Reynolds stress can be defined as

$$\eta = \frac{\overline{\acute{u}\acute{v}}}{U^2}, \tag{2}$$

where $\acute{u}$ and $\acute{v}$ denote longitudinal and vertical fluctuating velocities and U means velocity of orifice.

3.1. Effect of Orifice Size on Turbulence Intensity and Reynolds Stress. The distribution of relative turbulence intensity at cross-section 1-1 was taken for an example, and variation in relative turbulence intensity for the 4 multi-orifice plates, namely, side length of orifice $a = 2.6$, 4.0, 5.1, and 6.7 mm, is shown in Figure 4. It follows from the Figure that there exist different extents of turbulence as a result of shearing and mixing effects of multiple jets downstream of multiorifice plates. The turbulence intensity for $a = 4.0$ mm plate exhibits obvious sawtooth-like distribution. Intense turbulence means that the fluctuating velocity is larger, which can induce cavitation. Variation of turbulence intensity T_x with vertical height y/H is smaller, the turbulence intensity for $a = 6.7$ mm plate is the weakest among the 4 multiorifice plates, and the turbulence intensities for $a = 2.6$ mm and 5.1 mm plates are in between for $a = 4.0$ mm and $a = 6.7$ mm plates.

Distribution of Reynolds stress for the 4 multiorifice plates with different orifice sizes is shown in Figure 5. It follows that variation in Reynolds stress downstream of multiorifice plates also exhibits a sawtooth-like distribution, which means strong shearing effect took place among the

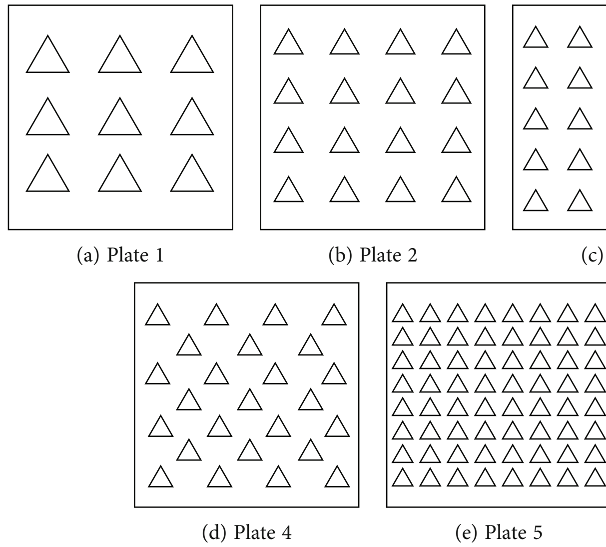

Figure 2: Triangular multiorifice plates.

Table 1: Geometric parameters of triangular multiorifice plates.

Orifice number	Orifice arrangement	Side length of orifice, mm
9	Checkerboard-type	6.7
16	Checkerboard-type	5.1
25	Checkerboard-type	4.0
64	Checkerboard-type	2.6
25	Staggered	4.0

Table 2: Dimensionless cross-section positions behind multiorifice plates.

Cross-section	x/L	Cross-section	x/L
1-1	0.05	5-5	0.25
2-2	0.1	6-6	0.35
3-3	0.15	7-7	0.5
4-4	0.2	8-8	0.75

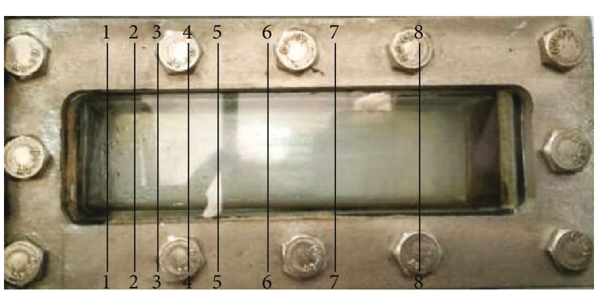

Figure 3: Cross-section positions downstream of multiorifice plate.

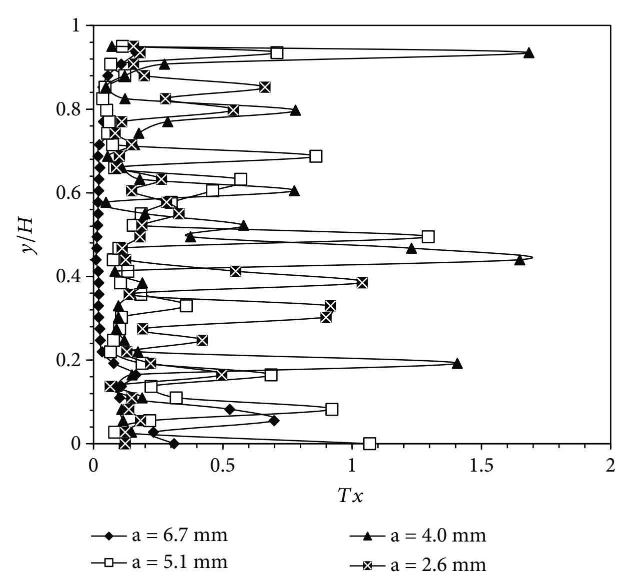

Figure 4: Effect of orifice size on turbulence intensity.

high-velocity multiple jets. The sawtooth-like distribution of Reynolds stress for $a = 5.1$ mm plate is denser than that for $a = 6.7$ mm plate and exhibits alternative fluctuations between positive and negative directions. Reynolds stress due to $a = 4.0$ mm plate is larger than that due to $a = 5.1$ mm plate along the vertical line, implying that the shearing effect among multiple jets downstream of the plate with smaller orifice is stronger. The reason is that velocity gradient between multijet and ambient fluid became larger due to the smaller orifice, so more intense entrainment and mixing occur, thus producing more eddies and increasing internal disturbance and turbulence energy in cavitating flow field.

It was found through further comparison between $a = 4.0$ mm and 2.6 mm plates that the sawtooth-like distribution of Reynolds stress due to the latter was denser, but variation in the values was within a smaller range. Also,

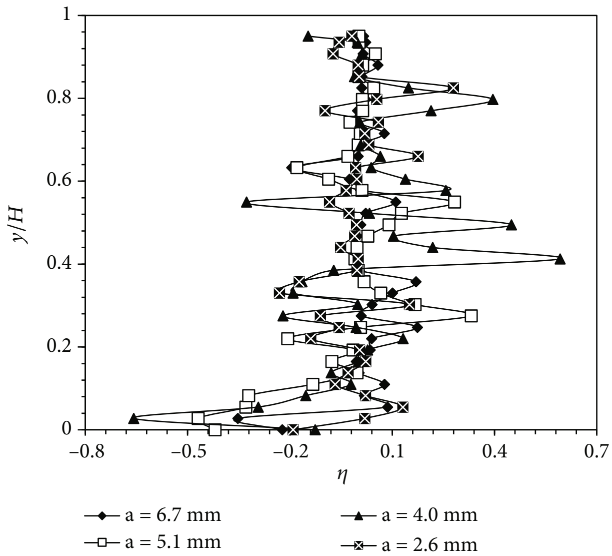

Figure 5: Effect of orifice size on Reynolds stress.

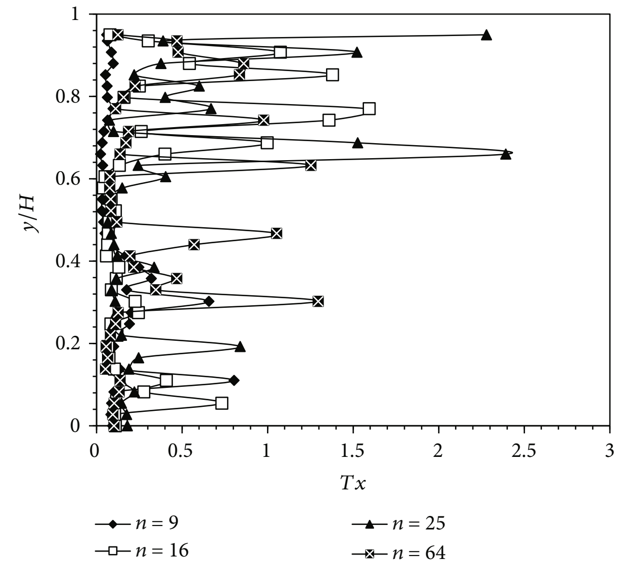

Figure 6: Effect of orifice number on turbulence intensity.

Reynolds stress for the latter was basically smaller than that for the former. It means the intense shearing effect could not occur for the minimum orifice in this experiment.

3.2. Effect of Orifice Number on Turbulence Intensity and Reynolds Stress. The variation of relative turbulence intensity for 4 multiorifice plates with different orifice numbers is shown in Figure 6. It can be seen from Figure 6 that the turbulence intensity for 9-orifice plate approximates a straight-line, and for 16-orifice and 64-orifice plates, the intensities exhibit sawtooth-like distributions. However, there are some larger values of turbulence intensity for 25-orifice plate, it means that a more intense turbulent shearing field occurred, and that numerous high-frequency and small-size eddies were generated in the field, which contributed to transfer of turbulence energy and increased pressure fluctuation. Therefore, an appropriate increase in orifice number could contribute to intensifying interjet mixing and to prompting formation, growth, and collapse of cavitation bubble, leading to intense cavitation effect.

Distribution of Reynolds stress due to multiorifice plates with different orifice numbers is shown in Figure 7. As can been seen in Figure 7, Reynolds stress downstream of 16-orifice plate changes between positive and negative range, which is more intense than that downstream of 9-orifice plate. In addition, the value of Reynolds stress due to 25-orifice plate is larger than that due to 16-orifice plate. All of these implied that the more the orifice number, the stronger the turbulence exchange and shearing effect, which could prompt the formation, growth, and collapse of cavitation bubbles. It was found based on the further comparison between 25-orifice and 64-orifice plates that the sawtooth-like distribution of Reynolds stress due to the latter was denser. However, the value of Reynolds stress due to 64-orifice plate only fluctuated within a smaller range, which was probably the weakening effect of combined jets due to more orifice numbers.

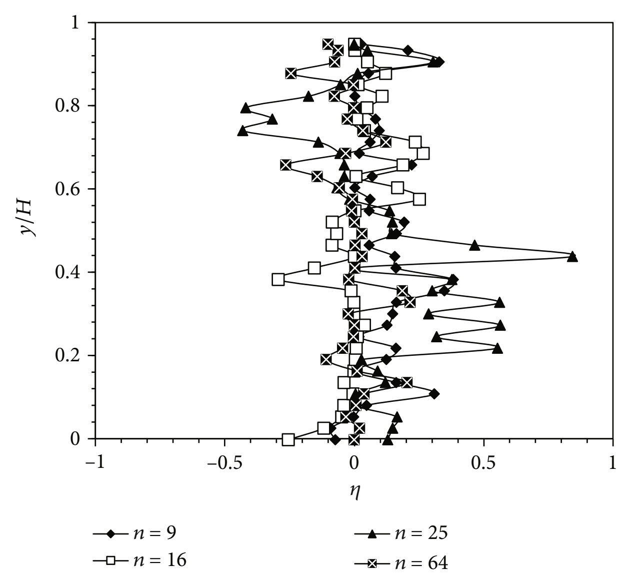

Figure 7: Effect of orifice number of Reynolds stress.

3.3. Effect of Orifice Layout on Turbulence Intensity and Reynolds Stress. Two multiorifice plates with the same size and number of orifice but different layouts of checkerboard-type and staggered orifices were taken for the effect on layout. The distribution of turbulence intensity is shown in Figure 8. It follows from the figure that variation in turbulence intensity for the checkerboard-type layout is relatively mild; however, the variation for the staggered layout is steeper. The reason is that the flow field due to the staggered layout was of more intense entraining and mixing effects of multiorifice plates.

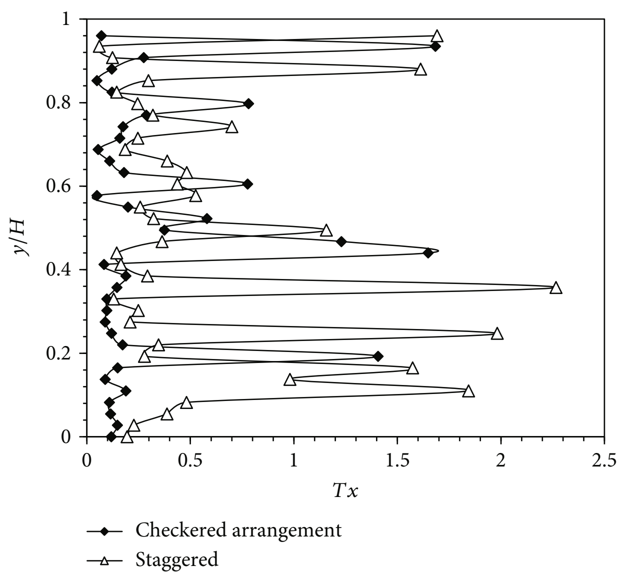

Figure 8: Effect of orifice layout on turbulence intensity.

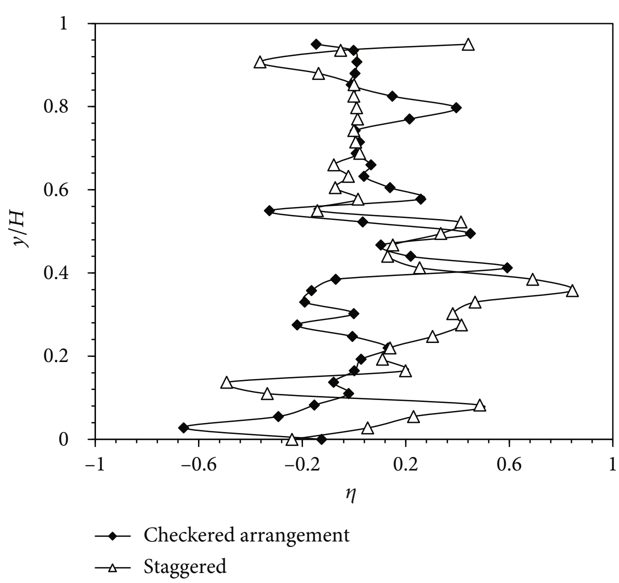

Figure 9: Effect of orifice layout on Reynolds stress.

The distribution of Reynolds stress due to two plates with the same size and number of orifice but different layouts of the checkerboard-type and staggered orifices is shown in Figure 9. We can easily see the variation in Reynolds stress due to the two plates is similar.

4. Conclusion

There existed different turbulence due to shearing and mixing effects of multiple jets downstream of different multiorifice plates. The turbulence intensity for $a = 4.0$ mm and 25-orifice plates exhibited apparent sawtooth-like distribution, which contributed to transfer of turbulence energy and increase in pressure fluctuation. Also, the variation in turbulence intensity for the staggered layout plate was steeper than that for the checkerboard-type one; Reynolds stress downstream of different multiorifice plates also exhibited sawtooth-like profiles. Appropriately decreasing the orifice size ($a = 4.0$ mm), increasing the orifice number ($n = 25$), and taking staggered layout could contribute to intensifying turbulence mixing and shearing effects of multiple jets downstream of multiorifice plates, thus resulting in an expected cavitation effect. Improving the velocity of orifice and designing the various shapes of orifice will be considered in the following study.

Data Availability

The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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