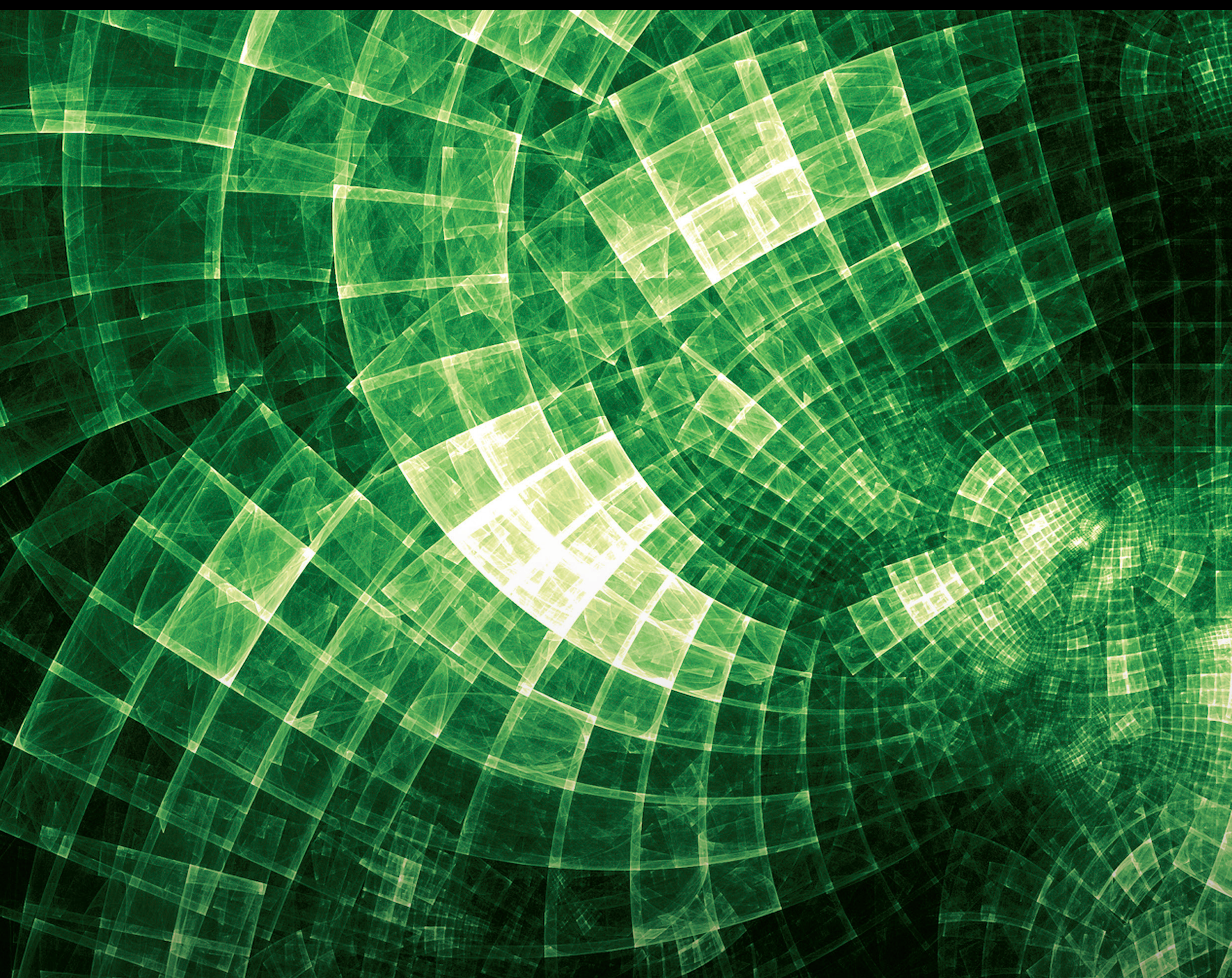
Journal of Mathematics

Theory, Algorithms, and Applications within Neutrosophic Modelling and Optimisation

Lead Guest Editor: Broumi Said
Guest Editors: Jun Ye and Mani Parimala



Theory, Algorithms, and Applications within Neutrosophic Modelling and Optimisation

Journal of Mathematics

Theory, Algorithms, and Applications within Neutrosophic Modelling and Optimisation

Lead Guest Editor: Broumi Said
Guest Editors: Jun Ye and Mani Parimala

Contents

Matrix Theory for Neutrosophic Hypersoft Set and Applications in Multiattributive Multicriteria Decision-Making Problems
Muhammad Naveed Jafar and Muhammad Saeed
Research Article (15 pages), Article ID 6666408, Volume 2022 (2022)

Uncertainty-Based Trimmed Coefficient of Variation with Application
Ali Hussein Al-Marshadi, Muhammad Aslam, and Alharbey Abdullah
Research Article (6 pages), Article ID 5511904, Volume 2021 (2021)

A Novel Method for Solving Multiobjective Linear Programming Problems with Triangular Neutrosophic Numbers
Qing Wang, Yi Huang, Shiming Kong, Xinqiang Ma, Youyuan Liu, S. K. Das, and S. A. Edalatpanah
Research Article (8 pages), Article ID 6631762, Volume 2021 (2021)

Single-Valued Neutro Hyper BCK-Subalgebras
M. Hamidi and F. Smarandache
Research Article (11 pages), Article ID 6656739, Volume 2021 (2021)

Neutrosophic Logic-Based Document Summarization
O. G. El Barbary and Radwan Abu Gdairi
Research Article (7 pages), Article ID 9938693, Volume 2021 (2021)

Optimal Solutions for Constrained Bimatrix Games with Payoffs Represented by Single-Valued Trapezoidal Neutrosophic Numbers
Mohamed Gaber, Majed G. Alharbi, Abd Alwahed Dagestani, and El-Saeed Ammar
Research Article (13 pages), Article ID 5594623, Volume 2021 (2021)

Neutrosophic Semiopen Hypersoft Sets with an Application to MAGDM under the COVID-19 Scenario
D. Ajay, J. Joseline Charisma, N. Boonsatit, P. Hammachukiattikul, and G. Rajchakit
Research Article (16 pages), Article ID 5583218, Volume 2021 (2021)

Neutrosophic Number Optimization Models and Their Application in the Practical Production Process
Angyan Tu, Jun Ye, and Bing Wang
Research Article (8 pages), Article ID 6668711, Volume 2021 (2021)

Single-Valued Neutrosophic DEMATEL for Segregating Types of Criteria: A Case of Subcontractors' Selection
Lazim Abdullah, Zheeching Ong, and Shalela Mohd Mahali
Research Article (12 pages), Article ID 6636029, Volume 2021 (2021)

Applications of (Neutro/Anti)sophications to Semihypergroups
A. Rezaei, F. Smarandache, and S. Mirvakili
Research Article (7 pages), Article ID 6649349, Volume 2021 (2021)

Study of Two Kinds of Quasi AG-Neutrosophic Extended Triplet Loops
Xiaogang An and Mingming Chen
Research Article (10 pages), Article ID 6649751, Volume 2021 (2021)

Multicriteria Decision-Making Method and Application in the Setting of Trapezoidal Neutrosophic Z-Numbers
Rui Yong, Jun Ye, and Shigui Du
Research Article (13 pages), Article ID 6664330, Volume 2021 (2021)

On the Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations
Mohammad Abobala
Research Article (5 pages), Article ID 5591576, Volume 2021 (2021)

Neutrosophic D'Agostino Test of Normality: An Application to Water Data
Mohammed Albassam, Nasrullah Khan, and Muhammad Aslam
Research Article (5 pages), Article ID 5582102, Volume 2021 (2021)

On Refined Neutrosophic Matrices and Their Application in Refined Neutrosophic Algebraic Equations
Mohammad Abobala
Research Article (5 pages), Article ID 5531093, Volume 2021 (2021)

Hindawi
Journal of Mathematics
Volume 2022, Article ID 6666408, 15 pages
https://doi.org/10.1155/2022/6666408

Research Article

Matrix Theory for Neutrosophic Hypersoft Set and Applications in Multiattributive Multicriteria Decision-Making Problems

Muhammad Naveed Jafar and Muhammad Saeed

Department of Mathematics, University of Management and Technology, Johar Town C Block, Lahore 54000, Pakistan

Correspondence should be addressed to Muhammad Saeed; muhammad.saeed@umt.edu.pk

Received 16 December 2020; Revised 29 April 2021; Accepted 27 May 2021; Published 25 June 2022

Academic Editor: Parimala Mani

Decision-making is a complex issue due to the vague, imprecise, and indeterminate environment especially when attributes are more than one and further bifurcated. To solve such types of problems, the concept of neutrosophic hypersoft set is proposed by Smarandache. In this paper, the primary focus is to extend the concept of neutrosophic hypersoft sets (NHSs) to the neutrosophic hypersoft matrices (NHSMs) with the essential study of matrices with suitable examples. Then, the analytical study of some common operations for NHSM has been created. Lastly, decision-making issues have been presented by establishing a new algorithm based on a score function, and it has been interpreted with the help of numerical example for the selection of teachers at the college level. In this study, NHSM algorithm is elaborated efficiently and conveniently for optimal choice selection to solve decision-making problems.

1. Introduction

In decision-making, among the multiattributive and multiobjective problems, in uncertain and vague environments, it is difficult to differentiate valid from invalid and logical from illogical. In these cases, decision makers get more confused and uncertain. Zadeh developed fuzzy sets [1] to deal with such type of information. Another issue in information is vagueness. Likewise, it is the type of uncertainty where the investigators cannot' separate between two unique things, and to deal with vagueness, intuitionistic fuzzy sets [2] are used. Later, Molodtsov [3] presents soft sets to manage uncertainties and vagueness, and this research was effectively applied in numerous applications such as game theory, activity research, and probability [4]. Maji et al. [5, 6] exhibited a logical study of the soft sets, which incorporates every essential operators and property. The study was extended to fuzzy soft set [7] and intuitionistic softsets [8] to deal uncertainity and vagueness. As a result, Smarandache [9, 10] has presented the idea of neutrosophic sets, which is a generalization of the crisp set, fuzzy set, and intuitionistic fuzzy set.

In any case, from the philosophical perspective, truthness, indeterminacy, and falsity of neutrosophic set always lies in [0,1]. Maji [11] has extended the concept of a soft set to neutrosophic soft set. The matrix representation and aggregate operators of this idea were presented by Deli and Broumi in [12]. Multicriteria decision-making MCDM problems were solved by utilizing a neutrosophic soft set, and many mathematicians have proposed their examination work in various scientific fields by proposing TOPSIS, VIKOR, etc. techniques, and this idea is likewise utilized in advancing decision-making theories along with application in the neutrosophic environment [13–17]. Akram et al. [18–20] established group decision-making methods based on hesitant N-soft sets, Pythagorean fuzzy TOPSIS, and ELECTRIC I method in Pythagorean fuzzy information. Garg [21, 22] had carried out lot of work related to decision-making problems using different tools relating to fuzzy, intuitionistic, and neutrosophic theories. Mehmood et al. [23, 24] used bipolar soft sets and spherical fuzzy sets for decision-making problems. Sabbir and Naz [25] also worked on bipolar soft sets.

Smarandache [26] displayed another strategy to manage uncertainty by providing the extension of the soft set to the hypersoft set and its hybrids, such as a fuzzy hypersoft set, intuitionistic hypersoft set, and neutrosophic hypersoft set, by changing the function into a multiargument function.

1.1. Motivation

(1) Multicriteria decision problems (MCDM) consist of several attributes and indeterminacy. To deal with such types, neutrosophic sets (NSs) are used because (NSs) fully deal with indeterminacy, whereas to deal with vagueness and uncertainty, neutrosophic soft sets (NS's) are used. However, when attributes are more than one and further bifurcated, the concept of neutrosophic soft set (NSs) cannot be used to tackle such issues. There was a dire need to define the new environment. For this purpose, the concept of neutrosophic hypersoft set (NHSS) was proposed by [27]. Matrices are more reliable, logical, and practical for the decision makers and play an important role in understanding, modeling, and solving the MCDM problems.

(2) how MCDM problems can be represented in the matrices' form consisting of more than one attribute, which is further bifurcated? The answer to this question leads us to develop the matrix theory by combining the concept of NHSS and soft matrix theory and, hence, the motivation of the present study.

(3) In this exploration, the primary focus is to extend the neutrosophic hypersoft set (NHSS) concept to the neutrosophic hypersoft matrices (NHSM) by the essential study of matrices. This study helps us apply all the definitions, operators, and properties of matrices to NHSS and decision-making problems, especially when attributes are more than one and further subdivided.

Section 1 contains an introduction about soft set, neutrosophic soft set, hypersoft set, and neutrosophic hypersoft sets. Section 2 deals with mathematical preliminaries, which will be used in the rest of the paper. In Section 3 the concept of NHSM has been discussed broadly with definitions and suitable examples. In Section 4 basic operators of NHSM are proposed along with their properties. In Section 5, a decision-making algorithm has been developed with the help of score function and it is applied in the selection for the hiring of teachers. This algorithm is briefer and more accurate rather than others, and Section 6 contains some comparison in Table 7 with the existing techniques of Hashmi et al. [28], and finally, we will discuss the conclusion of the research paper.

2. Preliminaries

In this section, we present some definitions which will help understand the rest of the article.

2.1. Soft Set [6]. Let $\mathscr{U}$ be the universal set and $\mathscr{E}$ be the set of attributes with respect to $\mathscr{U}$. Let $\mathscr{P}(\mathscr{U})$ be the power set of $\mathscr{U}$ and $A \subseteq \mathscr{E}$. A pair $(\mathscr{F}, \mathscr{A})$ is called a soft set over $\mathscr{U}$, and its mapping is given as

$$\mathscr{F}\colon \mathscr{A} \longrightarrow \mathscr{P}(\mathscr{U}). \tag{1}$$

It is also defined as

$$(\mathscr{F}, \mathscr{A}) = \left\{ \begin{array}{c} \mathscr{F}(e) \in \mathscr{P}(\mathscr{U}) \\ e \in \mathscr{E}, \mathscr{F}(e) = \varnothing \text{ if } e \neq \mathscr{A} \end{array} \right\}. \tag{2}$$

2.2. Neutrosophic Soft Set [11]. Let $\mathscr{U}$ be the universal set and $\mathscr{E}$ be the set of attributes with respect to $\mathscr{U}$. Let $\mathscr{P}(\mathscr{U})$ be the set of neutrosophic values of $\mathscr{U}$ and $\mathscr{A} \subseteq \mathscr{E}$. A pair $(\mathscr{F}, \mathscr{A})$ is called a neutrosophic soft set over $\mathscr{U}$, and its mapping is given as

$$\mathscr{F}\colon \mathscr{A} \longrightarrow \mathscr{P}(\mathscr{U}). \tag{3}$$

2.3. Hypersoft Set [21]. Let $\mathscr{U}$ be the universal set and $\mathscr{P}(\mathscr{U})$be the power set of $\mathscr{U}$. Consider $\ell^1, \ell^2, \ell^3, \ldots, \ell^n$, for $n \geq 1$, and let n be well-defined attributes, whose corresponding attributive values are, respectively, the set $\mathscr{L}^1, \mathscr{L}^2, \mathscr{L}^3, \ldots, \mathscr{L}^n$ with $\mathscr{L}^i \cap \mathscr{L}^j = \varnothing$, for $i \neq j$ and $i, j \,\varepsilon \{1, 2, 3, \ldots, n\}$; then, the pair $(\mathscr{F}, \mathscr{L}^1 \times \mathscr{L}^2 \times \mathscr{L}^3, \ldots, \mathscr{L}^n)$ is said to be hypersoft set over $\mathscr{U}$, where

$$\mathscr{F}\colon \mathscr{L}^1 \times \mathscr{L}^2 \times \mathscr{L}^3, \ldots, \mathscr{L}^n \longrightarrow \mathscr{P}(\mathscr{U}). \tag{4}$$

2.4. Neutrosophic Hypersoft Set [23]. Let $\mathscr{U}$ be the universal set and $\mathscr{P}(\mathscr{U})$ be the power set of $\mathscr{U}$. Consider $\ell^1, \ell^2, \ell^3, \ldots, \ell^n$, for $n \geq 1$; let n be well-defined attributes, whose corresponding attributive values are, respectively, the set $\mathscr{L}^1, \mathscr{L}^2, \mathscr{L}^3, \ldots, \mathscr{L}^n$ with $\mathscr{L}^i \cap \mathscr{L}^j = \varnothing$, for $i \neq j$ and $i, j \,\varepsilon \{1, 2, 3, \ldots, n\}$, and their relation $\mathscr{L}^1 \times \mathscr{L}^2 \times \mathscr{L}^3, \ldots, \mathscr{L}^n = \mathscr{S}$; then, the pair $(\mathscr{F}, \mathscr{S})$ is said to be neutrosophic hypersoft set (NHSS) over $\mathscr{U}$, where

$$\begin{aligned} &\mathscr{F}\colon \mathscr{L}^1 \times \mathscr{L}^2 \times \mathscr{L}^3 \ldots, \mathscr{L}^n \longrightarrow \mathscr{P}(\mathscr{U}), \\ &\mathscr{F}\left(\mathscr{L}^1 \times \mathscr{L}^2 \times \mathscr{L}^3, \ldots, \mathscr{L}^n\right) \\ &= \{\langle x, \mathscr{T}(\mathscr{F}(\mathscr{S})), \mathscr{I}(\mathscr{F}(\mathscr{S})), \mathscr{F}(\mathscr{F}(\mathscr{S}))\rangle, x \in \mathscr{U}\}, \end{aligned} \tag{5}$$

where $\mathscr{T}$ is the membership value of truthiness, $\mathscr{J}$ is the membership value of indeterminacy, and $\mathscr{F}$ is the membership value of falsity such that $\mathscr{T}, \mathscr{J}, \mathscr{F}\colon \mathscr{U} \longrightarrow [0, 1]$ also $0 \leq \mathscr{T}(\mathscr{F}(\mathscr{S})) + \mathscr{J}(\mathscr{F}(\mathscr{S})) + \mathscr{F}(\mathscr{F}(\mathscr{S})) \leq 3$.

3. Neutrosophic Hypersoft Matrix (NHSM)

In this section, we have introduced some definition with suitable examples.

3.1. NHSM. Let $\mathscr{U}=\{u^1,u^2,\ldots,u^\alpha\}$ and $\mathscr{P}(\mathscr{U})$ be the universal set and power set of universal set, respectively; also, consider $\mathscr{L}_1,\mathscr{L}_2,\ldots,\mathscr{L}_\beta$, for $\beta\geq 1$, where β is well-defined attributes, whose corresponding attributive values are, respectively, the set $\mathscr{L}_1^a,\mathscr{L}_2^b,\ldots,\mathscr{L}_\beta^z$ and their relation $\mathscr{L}_1^a\times\mathscr{L}_2^b\times\cdots\times\mathscr{L}_\beta^z$, where $a,b,c,\ldots,z=1,2,\ldots,n$; then, the pair $(\mathscr{F},\mathscr{L}_1^a\times\mathscr{L}_2^b\times\cdots\times\mathscr{L}_\beta^z)$ is said to be neutrosophic hypersoft set over $\mathscr{U}$, where $\mathscr{F}\colon(\mathscr{L}_1^a\times\mathscr{L}_2^b\times\cdots\times\mathscr{L}_\beta^z)\longrightarrow\mathscr{P}(\mathscr{U})$ and it is defined as $\mathscr{F}(\mathscr{L}_1^a\times\mathscr{L}_2^b\times\cdots\times\mathscr{L}_\beta^z)=\{\langle u,T_{\mathfrak{L}}(u),I_{\mathfrak{L}}(u),F_{\mathfrak{L}}(u)\rangle u\in\mathscr{U},\mathfrak{L}\in(\mathscr{L}_1^a\times\mathscr{L}_2^b\times\cdots\times\mathscr{L}_\beta^z)\}$. Table 1 represents the tabular form of NHSS $\mathscr{R}_{\mathfrak{L}}$.

If $O_{ij}=\mathscr{X}_{\mathscr{R}_{\mathfrak{L}}}(u^i,\mathscr{L}_j^k)$, where $i=1,2,3,\ldots,\alpha$, $j=1,2,3,\ldots,\beta$, and $k=a,b,c,\ldots,z$, then a matrix is defined as

$$\left[O_{ij}\right]_{\alpha\times\beta}=\begin{pmatrix} O_{11} & O_{12} & \ldots & O_{1\beta}\\ O_{21} & O_{22} & \ldots & O_{2\beta}\\ \vdots & \vdots & \ddots & \vdots\\ O_{\alpha 1} & O_{\alpha 2} & \ldots & O_{\alpha\beta}\end{pmatrix}, \tag{6}$$

where $O_{ij}=(T_{\mathscr{L}_j^k}(u_i),I_{\mathscr{L}_j^k}(u_i),F_{\mathscr{L}_j^k}(u_i),u_i\in\mathscr{U},,\mathscr{L}_j^k\in(\mathscr{L}_1^a\times\mathscr{L}_2^b\times\cdots\times\mathscr{L}_\beta^z))=(\mathscr{T}_{ijk}^o,\mathscr{I}_{ijk}^o,\mathscr{F}_{ijk}^o)$.

Thus, we can represent any neutrosophic hypersoft set in terms of a neutrosophic hypersoft matrix (NHSM), and it means that they are interchangeable.

Example 1. Teachers' recruitment problem (TRP) is the most complex and absurd task. There is no fixed and fabricated design to know their subject knowledge or pedagogical skills. Therefore, decision makers find themselves in a blind alley. Consequently, based on their own knowledge and experience, they select a person who does not meet the institutional requirement. Thus, TRP is typically a multicriteria decision-making MCDM problem.

Assumptions:

(i) Independent attributes are considered
(ii) Everyone attends the interview
(iii) Hesitant environment is not yet considered

Formulation of the Problem. Let us consider an institute that wants to hire a teacher appropriate to its requirements, and they received the following statistics-based CVs. Let $\mathscr{U}$ be the set of candidates for the teaching at the college level:

$$\mathscr{U}=\left\{\mathscr{T}^1,\mathscr{T}^2,\mathscr{T}^3,\mathscr{T}^4,\mathscr{T}^5\right\}. \tag{7}$$

Also, consider the set of attributes as

$$\begin{aligned}\mathscr{A}_1&=\text{Qualification},\\ \mathscr{A}_2&=\text{Experience},\\ \mathscr{A}_3&=\text{Gender},\\ \mathscr{A}_4&=\text{Publications}.\end{aligned} \tag{8}$$

Parameters:

(i) $\mathbf{T_i}$ = universal set of teachers, where $i=1,2,3,4,5$
(ii) $\mathbf{A_i}$ = attributes, where $i=1,2,3,4$ that are further categorized into the following:
(iii) $\mathscr{A}_1^a$ = qualification
(iv) $\mathscr{A}_1^a$ = {BS Hons., MS/Mphill, Phd, Post Doctorate}
(v) $\mathscr{A}_2^b$ = experience = {5yr, 8yr, 10yr, 15yr}
(vi) $\mathscr{A}_3^c$ = gender = {Male, Female}
(vii) $\mathscr{A}_4^d$ = publications = {3, 5, 8, 10+}

Let the function be $\mathscr{F}\colon\mathscr{A}_1^a\times\mathscr{A}_2^b\times\mathscr{A}_3^c\times\mathscr{A}_4^d\longrightarrow P(\mathscr{U})$

Below are Tables 2–5 of their neutrosophic values assigned by different decision makers.

The neutrosophic hypersoft set is defined as

$$\mathscr{F}\colon\left(\mathscr{A}_1^a\times\mathscr{A}_2^b\times\mathscr{A}_3^c\times\mathscr{A}_4^d\right)\longrightarrow P(\mathscr{U}). \tag{9}$$

Let us assume

$$\begin{aligned}\mathscr{F}\left(\left(\mathscr{A}_1^a\times\mathscr{A}_2^b\times\mathscr{A}_3^c\times\mathscr{A}_4^d\right)\right.&=\mathscr{F}(\text{Mphill, 5yr, male, 3})=\left\{\mathscr{T}^1,\mathscr{T}^2,\mathscr{T}^4,\mathscr{T}^5\right\},\\ \mathscr{F}\left(\left(\mathscr{A}_1^a\times\mathscr{A}_2^b\times\mathscr{A}_3^c\times\mathscr{A}_4^d\right)\right.&=\mathscr{F}(\text{Mphill, 5yr, male, 3})\\ &=\left\{\ll\mathscr{T}^1,\ (\text{Mphill}\{0.5,0.3,0.6\},\text{5yr}\{0.3,0.4,0.7\},\text{male}\{0.5,0.6,0.9\},3\{0.6,0.4,0.5\})\gg,\right.\\ &\quad\ll\mathscr{T}^2,\ (\text{Mphill}\{0.3,0.2,0.1\},\text{5yr}\{0.6,0.5,0.3\},\text{male}\{0.7,0.8,0.3\},3\{0.7,0.5,0.3\})\gg,\\ &\quad\ll\mathscr{T}^4\ (\text{Mphill}\{0.7,0.3,0.6\},\text{5yr}\{0.6,0.4,0.8\},\text{male}\{0.8,0.5,0.4\},3\{0.6,0.2,0.1\})\gg,\\ &\quad\left.\ll\mathscr{T}^5\ (\text{Mphill}\{0.5,0.4,0.5\},\text{5yr}\{0.3,0.6,0.7\},\text{male}\{0.9,0.2,0.1\},3\{0.4,0.5,0.3\})\gg,\right\}.\end{aligned} \tag{10}$$

Then, a neutrosophic hypersoft set of above-assumed relation in the tabular form is represented in Table 6.

And, its matrix is defined as

Table 1: Matrix representation of NHSS.

	$\mathscr{L}_1^a$	$\mathscr{L}_2^b$	$\cdots$	$\mathscr{L}_\beta^z$
u^1	$\mathscr{X}_{\mathscr{R}_{\mathfrak{Q}}}(u^1,\mathscr{L}_1^a)$	$\mathscr{X}_{\mathscr{R}_{\mathfrak{Q}}}(u^1,\mathscr{L}_2^b)$	$\cdots$	$\mathscr{X}_{\mathscr{R}_{\mathfrak{Q}}}(u^1,\mathscr{L}_\beta^z)$
u^2	$\mathscr{X}_{\mathscr{R}_{\mathfrak{Q}}}(u^2,\mathscr{L}_1^a)$	$\mathscr{X}_{\mathscr{R}_{\mathfrak{Q}}}(u^2,\mathscr{L}_2^b)$	$\cdots$	$\mathscr{X}_{\mathscr{R}_{\mathfrak{Q}}}(u^2,\mathscr{L}_\beta^z)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u^α	$\mathscr{X}_{\mathscr{R}_{\mathfrak{Q}}}(u^\alpha,\mathscr{L}_1^a)$	$\mathscr{X}_{\mathscr{R}_{\mathfrak{Q}}}(u^\alpha,\mathscr{L}_2^b)$	$\cdots$	$\mathscr{X}_{\mathscr{R}_{\mathfrak{Q}}}(u^\alpha,\mathscr{L}_\beta^z)$

Table 2: Decision makers will assign neutrosophic numbers to each candidate $\mathbf{T_i}$ against qualification.

$\mathscr{A}_1^a$ (qualification)	$\mathscr{T}^1$	$\mathscr{T}^2$	$\mathscr{T}^3$	$\mathscr{T}^4$	$\mathscr{T}^5$
BS Hons.	(0.4,0.5,0.8)	(0.7,0.6,0.4)	(0.4,0.5,0.7)	(0.5,0.3,0.7)	(0.5,0.3,0.8)
MS/MPhil.	(0.5,0.3,0.6)	(0.3,0.2,0.1)	(0.3,0.6,0.2)	(0.7,0.3,0.6)	(0.5,0.4,0.5)
Ph.D.	(0.8,0.2,0.4)	(0.9,0.5,0.3)	(0.9,0.4,0.1)	(0.6,0.3,0.2)	(0.6,0.1,0.2)
Post doctorate	(0.9,0.3,0.1)	(0.5,0.2,0.1)	(0.8,0.5,0.2)	(0.8,0.2,0.1)	(0.7,0.4,0.2)

Table 3: Decision makers will assign neutrosophic numbers to each candidate $\mathbf{T_i}$ against experience.

$\mathscr{A}_2^b$ (experience)	$\mathscr{T}^1$	$\mathscr{T}^2$	$\mathscr{T}^3$	$\mathscr{T}^4$	$\mathscr{T}^5$
5 yr.	(0.3,0.4,0.7)	(0.6,0.5,0.3)	(0.5,0.6,0.8)	(0.6,0.4,0.8)	(0.3,0.6,0.7)
8 yr.	(0.4,0.2,0.5)	(0.8,0.1,0.2)	(0.4,0.7,0.3)	(0.4,0.8,0.7)	(0.7,0.5,0.6)
10 yr.	(0.7,0.2,0.3)	(0.9,0.3,0.1)	(0.8,0.3,0.2)	(0.5,0.4,0.3)	(0.5,0.2,0.1)
15 yr.	(0.8,0.2,0.1)	(0.6,0.4,0.3)	(0.9,0.4,0.1)	(0.6,0.2,0.3)	(0.5,0.3,0.2)

Table 4: Decision makers will assign neutrosophic numbers to each candidate $\mathbf{T_i}$ against gender.

$\mathscr{A}_3^c$ (Gen de r)	$\mathscr{T}^1$	$\mathscr{T}^2$	$\mathscr{T}^3$	$\mathscr{T}^4$	$\mathscr{T}^5$
Male	(0.5, 0.6, 0.9)	(0.7, 0.8, 0.3)	(0.6, 0.4, 0.3)	(0.8, 0.5, 0.4)	(0.9, 0.2, 0.1)
Female	(0.6, 0.4, 0.7)	(0.3, 0.6, 0.4)	(0.8, 0.2, 0.1)	(0.4, 0.5, 0.6)	(0.8, 0.4, 0.2)

Table 5: Decision makers will assign neutrosophic numbers to each candidate $\mathbf{T_i}$ against publication.

$\mathscr{A}_4^d$ (publication)	z	$\mathscr{T}^1$	$\mathscr{T}^2$	$\mathscr{T}^3$	$\mathscr{T}^4$	$\mathscr{T}^5$
3	—	(0.6, 0.4, 0.5)	(0.7, 0.5, 0.3)	(0.6, 0.4, 0.3)	(0.6, 0.2, 0.1)	(0.4, 0.5, 0.3)
5	—	(0.8, 0.2, 0.4)	(0.7, 0.3, 0.2)	(0.8, 0.3, 0.1)	(0.3, 0.4, 0.5)	(0.3, 0.5, 0.8)
8	—	(0.5, 0.3, 0.4)	(0.6, 0.3, 0.4)	(0.5, 0.7, 0.2)	(0.8, 0.4, 0.1)	(0.7, 0.4, 0.3)
10+	—	(0.4, 0.9, 0.6)	(0.8, 0.4, 0.2)	(0.2, 0.6, 0.5)	(0.7, 0.5, 0.2)	(0.6, 0.4, 0.7)

Table 6: The tabular form of the above relation.

	$\mathscr{A}_1^a$	$\mathscr{A}_2^b$	$\mathscr{A}_3^c$	$\mathscr{A}_4^d$
$\boldsymbol{\mathscr{T}}^1$	(Mphill, (0.5, 0.3, 0.6))	(5yr, (0.3, 0.4, 0.7))	(male, (0.5, 0.6, 0.9))	(3, (0.6, 0.4, 0.5))
$\boldsymbol{\mathscr{T}}^2$	(Mphill, (0.3, 0.2, 0.1))	(5yr, (0.6, 0.5, 0.3))	(male, (0.7, 0.8, 0.3))	(3, (0.7, 0.5, 0.3))
$\boldsymbol{\mathscr{T}}^4$	(Mphill, (0.7, 0.3, 0.6))	(5yr, (0.6, 0.4, 0.8))	(male, (0.8, 0.5, 0.4))	(3, (0.6, 0.2, 0.1))
$\boldsymbol{\mathscr{T}}^5$	(Mphill, (0.5, 0.4, 0.5))	(5yr, (0.3, 0.6, 0.7))	(male, (0.9, 0.2, 0.1))	(3, (0.4, 0.5, 0.3))

$$[O]_{4\times 4} = \begin{bmatrix} (\text{Mphill}, (0.5, 0.3, 0.6)) & (5\text{yr}, (0.3, 0.4, 0.7)) & (\text{male}, (0.5, 0.6, 0.9)) & (3, (0.6, 0.4, 0.5)) \\ (\text{Mphill}, (0.3, 0.2, 0.1)) & (5\text{yr}, (0.6, 0.5, 0.3)) & (\text{male}, (0.7, 0.8, 0.3)) & (3, (0.7, 0.5, 0.3)) \\ (\text{Mphill}, (0.7, 0.3, 0.6)) & (5\text{yr}, (0.6, 0.4, 0.8)) & (\text{male}, (0.8, 0.5, 0.4)) & (3, (0.6, 0.2, 0.1)) \\ (\text{Mphill}, (0.5, 0.4, 0.5)) & (5\text{yr}, (0.3, 0.6, 0.7)) & (\text{male}, (0.9, 0.2, 0.1)) & (3, (0.4, 0.5, 0.3)) \end{bmatrix}. \tag{11}$$

3.2. Square NHSM. Let $O = [O_{ij}]$ be the NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$. Then, O is said to be square NHSM if $\alpha = \beta$. It means that if an NHSM has the same number of rows (attributes) and columns (alternatives), it is a square NHSM.

Example 2. Above defined Example 1 is also the example of square NHSM.

3.3. Transpose of Square NHSM. Let $O = [O_{ij}]$ be the square NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$; then, O^t is said to be transpose of square NHSM if rows and columns of O are interchanged. It is denoted as

$$O^t = \left[O_{ij}\right]^t = \left(\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk}\right)^t = \left(\mathcal{T}^o_{jki}, \mathcal{J}^o_{jki}, \mathcal{F}^o_{jki}\right) = \left[O_{ji}\right]. \tag{12}$$

Example 3. Transpose of the matrix define in Example 1 is given as

$$[O]^t_{4\times 4} = \begin{bmatrix} (\text{Mphill}, (0.5, 0.3, 0.6)) & (\text{Mphill}, (0.3, 0.2, 0.1)) & (\text{Mphill}, (0.7, 0.3, 0.6)) & (\text{Mphill}, (0.5, 0.4, 0.5)) \\ (\text{5yr}, (0.3, 0.4, 0.7)) & (\text{5yr}, (0.6, 0.5, 0.3)) & (\text{5yr}, (0.6, 0.4, 0.8)) & (\text{5yr}, (0.3, 0.6, 0.7)) \\ (\text{male}, (0.5, 0.6, 0.9)) & (\text{male}, (0.7, 0.8, 0.3)) & (\text{male}, (0.8, 0.5, 0.4)) & (\text{male}, (0.9, 0.2, 0.1)) \\ (3, (0.6, 0.4, 0.5)) & (3, (0.7, 0.5, 0.3)) & (3, (0.6, 0.2, 0.1)) & (3, (0.4, 0.5, 0.3)) \end{bmatrix}. \tag{13}$$

3.4. Symmetric NHSM. Let $O = [O_{ij}]$ be the square NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$; then, O is said to be symmetric NHSM if $O^t = O$, i.e., $(\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk}) = (\mathcal{T}^o_{jki}, \mathcal{J}^o_{jki}, \mathcal{F}^o_{jki})$.

3.5. Scalar Multiplication of NHSM. Let $O = [O_{ij}]$ be the NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$ and s be any scalar then the product of matrix O and a scalar s is a matrix formed by multiplying each element of matrix O by s. It is denoted as $sO = [sO_{ij}]$, where $0 \le s \le 1$.

Example 4. Let us consider a NHSM $[O]_{4\times 4}$:

$$[O]_{4\times 4} = \begin{bmatrix} (\text{Mphill}, (0.5, 0.3, 0.6)) & (\text{5yr}, (0.3, 0.4, 0.7)) & (\text{male}, (0.5, 0.6, 0.9)) & (3, (0.6, 0.4, 0.5)) \\ (\text{Mphill}, (0.3, 0.2, 0.1)) & (\text{5yr}, (0.6, 0.5, 0.3)) & (\text{male}, (0.7, 0.8, 0.3)) & (3, (0.7, 0.5, 0.3)) \\ (\text{Mphill}, (0.7, 0.3, 0.6)) & (\text{5yr}, (0.6, 0.4, 0.8)) & (\text{male}, (0.8, 0.5, 0.4)) & (3, (0.6, 0.2, 0.1)) \\ (\text{Mphill}, (0.5, 0.4, 0.5)) & (\text{5yr}, (0.3, 0.6, 0.7)) & (\text{male}, (0.9, 0.2, 0.1)) & (3, (0.4, 0.5, 0.3)) \end{bmatrix}. \tag{14}$$

And, **0.1** is the scalar; then, scalar multiplication of NHSM $[O]_{4\times 4}$ is given as

$$[(0.1)O]_{4\times 4} = \begin{bmatrix} (\text{Mphill}, (0.05, 0.03, 0.06)) & (\text{5yr}, (0.03, 0.04, 0.07)) & (\text{male}, (0.05, 0.06, 0.09)) & (3, (0.06, 0.04, 0.05)) \\ (\text{Mphill}, (0.03, 0.02, 0.01)) & (\text{5yr}, (0.06, 0.05, 0.03)) & (\text{male}, (0.07, 0.08, 0.03)) & (3, (0.07, 0.05, 0.03)) \\ (\text{Mphill}, (0.07, 0.03, 0.06)) & (\text{5yr}, (0.06, 0.04, 0.08)) & (\text{male}, (0.08, 0.05, 0.04)) & (3, (0.06, 0.02, 0.01)) \\ (\text{Mphill}, (0.05, 0.04, 0.05)) & (\text{5yr}, (0.03, 0.06, 0.07)) & (\text{male}, (0.09, 0.02, 0.01)) & (3, (0.04, 0.05, 0.03)) \end{bmatrix}. \tag{15}$$

Proposition 1. *Let $O = [O_{ij}]$ and $\mathcal{M} = [\mathcal{M}_{ij}]$ be two NHSM, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$ and $\mathcal{M}_{ij} = (\mathcal{T}^{\mathcal{M}}_{ijk}, \mathcal{J}^{\mathcal{M}}_{ijk}, \mathcal{F}^{\mathcal{M}}_{ijk})$.*

For two scalars $s, t \in [0, 1]$, then

(i) $s(tO) = (st)O$

(ii) If $s < t$, then $sO < tO$

(iii) If $O \subseteq \mathcal{M}$, then $sO \subseteq s\mathcal{M}$

Proof

(i) $s(tO) = s[tO_{ij}] = s[(t\mathcal{T}^o_{ijk}, t\mathcal{J}^o_{ijk}, t\mathcal{F}^o_{ijk})] = [(st\mathcal{T}^o_{ijk}, st\mathcal{J}^o_{ijk}, st\mathcal{F}^o_{ijk})] = st[(\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})] = st[O_{ij}] = (st)O$

(ii) Since $\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk} \in [0, 1]$, so $s\mathcal{T}^o_{ijk} \le t\mathcal{T}^o_{ijk}$, $s\mathcal{J}^o_{ijk} \le t\mathcal{J}^o_{ijk}, s\mathcal{F}^o_{ijk} \le t\mathcal{F}^o_{ijk}$

(iii) Now, $sO = [sO_{ij}] = [(s\mathcal{T}^o_{ijk}, s\mathcal{J}^o_{ijk}, s\mathcal{F}^o_{ijk})] \le [(t\mathcal{T}^o_{ijk}, t\mathcal{J}^o_{ijk}, t\mathcal{F}^o_{ijk})] = [tO_{ij}] = tO$

(iv) $O \subseteq \mathscr{M} \Rightarrow [O_{ij}] \subseteq [\mathscr{M}_{ij}]$

$$\begin{aligned}&\Rightarrow \mathscr{T}^{o}_{ijk} \leq \mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{o}_{ijk} \leq \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{o}_{ijk} \geq \mathscr{F}^{\mathscr{M}}_{ijk}\\&\Rightarrow s\mathscr{T}^{o}_{ijk} \leq s\mathscr{T}^{\mathscr{M}}_{ijk}, s\mathscr{I}^{o}_{ijk} \leq s\mathscr{I}^{\mathscr{M}}_{ijk}, s\mathscr{F}^{o}_{ijk} \geq s\mathscr{F}^{\mathscr{M}}_{ijk}\\&\Rightarrow s\left[O_{ij}\right] \subseteq s\left[\mathscr{M}_{ij}\right]\\&\Rightarrow sO \subseteq s\mathscr{M}.\end{aligned} \tag{16}$$

□

Theorem 1. *Let $O = [O_{ij}]$ be the NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$. Then,*

(i) $(sO)^t = sO^t$, where $s \in [0,1]$

(ii) $(O^t)^t = O$

(iii) If $O = [O_{ij}]$ is the upper triangular NHSM, then O^t is lower triangular NHSM and vice versa

Proof

(i) Here, $(sO)^t, sO^t \in \text{NHSM}_{\alpha\times\beta}$, so

$$\begin{aligned}(sO)^t &= \left[\left(s\mathscr{T}^{o}_{ijk}, s\mathscr{I}^{o}_{ijk}, s\mathscr{F}^{o}_{ijk}\right)\right]^t\\&= \left[\left(s\mathscr{T}^{o}_{jki}, s\mathscr{I}^{o}_{jki}, s\mathscr{F}^{o}_{jki}\right)\right]\\&= s\left[\left(\mathscr{T}^{o}_{jki}, \mathscr{I}^{o}_{jki}, \mathscr{F}^{o}_{jki}\right)\right]\\&= s\left[\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right)\right]^t = sO^t.\end{aligned} \tag{17}$$

(ii) Since $O^t \in \text{NHSM}_{\alpha\times\beta}$, so $(O^t)^t \in \text{NHSM}_{\alpha\times\beta}$. Now,

$$\begin{aligned}\left(O^t\right)^t &= \left(\left[\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right)\right]^t\right)^t\\&= \left(\left[\left(\mathscr{T}^{o}_{jki}, \mathscr{I}^{o}_{jki}, \mathscr{F}^{o}_{jki}\right)\right]\right)^t\\&= \left[\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right)\right] = O.\end{aligned} \tag{18}$$

(iii) proved with the help of example. □

3.6. Trace of NHSM. Let $O = [O_{ij}]$ be the square NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$ and $\alpha = \beta$. Then, trace of NHSM is denoted as $tr(O)$ and is defined as $tr(O) = \sum_{i=1,k=a}^{\alpha,z} [\mathscr{T}^{o}_{iik} - (\mathscr{I}^{o}_{iik} + \mathscr{F}^{o}_{iik})]$.

Example 5. Let us consider a NHSM $[O]_{4\times4}$:

$$[O]_{4\times4} = \begin{bmatrix}(\text{Mphill}, (0.5, 0.3, 0.6)) & (5\text{yr}, (0.3, 0.4, 0.7)) & (\text{male}, (0.5, 0.6, 0.9)) & (3, (0.6, 0.4, 0.5))\\(\text{Mphill}, (0.3, 0.2, 0.1)) & (5\text{yr}, (0.6, 0.5, 0.3)) & (\text{male}, (0.7, 0.8, 0.3)) & (3, (0.7, 0.5, 0.3))\\(\text{Mphill}, (0.7, 0.3, 0.6)) & (5\text{yr}, (0.6, 0.4, 0.8)) & (\text{male}, (0.8, 0.5, 0.4)) & (3, (0.6, 0.2, 0.1))\\(\text{Mphill}, (0.5, 0.4, 0.5)) & (5\text{yr}, (0.3, 0.6, 0.7)) & (\text{male}, (0.9, 0.2, 0.1)) & (3, (0.4, 0.5, 0.3))\end{bmatrix}. \tag{19}$$

Then, $tr(O) = (0.5 - 0.3 - 0.6) + (0.6 - 0.5 - 0.3) + (0.8 - 0.5 - 0.4) + (0.4 - 0.5 - 0.3) = -1.1.$

Proposition 2. *Let $O = [O_{ij}]$ be the square NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$ and$\alpha = \beta$. s be any scalar then $tr(sO) = str(O)$.*

Proof.

$$tr(sO) = \sum_{i=1,k=a}^{\alpha,z} \left[s\mathscr{T}^{o}_{iik} - \left(s\mathscr{I}^{o}_{iik} + s\mathscr{F}^{o}_{iik}\right)\right] = s \sum_{i=1,k=a}^{\alpha,z} \left[\mathscr{T}^{o}_{iik} - \left(\mathscr{I}^{o}_{iik} + \mathscr{F}^{o}_{iik}\right)\right] = str(O). \tag{20}$$

□

3.7. Max-Min Product of NHSM. Let $O = [O_{ij}]$ and $\mathscr{M} = [\mathscr{M}_{jm}]$ be two NHSM, where $O_{ij} = (\mathscr{T}^o_{ijk}, \mathscr{J}^o_{ijk}, \mathscr{F}^o_{ijk})$ and $\mathscr{M}_{jm} = (\mathscr{T}^{\mathscr{M}}_{jkm}, \mathscr{J}^{\mathscr{M}}_{jkm}, \mathscr{F}^{\mathscr{M}}_{jkm})$. Then, O and $\mathscr{M}$ are said to be conformable if their dimensions are equal to each other (number of columns of O is equal to number of rows of $\mathscr{M}$). If $O = [O_{ij}]_{\alpha\times\beta}$ and $\mathscr{M} = [\mathscr{M}_{jm}]_{\beta\times\gamma}$, then $O\otimes\mathscr{M} = [\mathscr{S}_{im}]_{\alpha\times\gamma}$, where

$$[\mathscr{S}_{im}] = \begin{pmatrix} \max_{jk}\min\left(\mathscr{T}^o_{ijk}, J^{\mathscr{M}}_{jkm}\right), \min_{jk}\max\left(\mathscr{J}^o_{ijk}, J^{\mathscr{M}}_{jkm}\right), \\ \min_{jk}\max\left(\mathscr{F}^o_{ijk}, \mathscr{F}^{\mathscr{M}}_{jkm}\right) \end{pmatrix}. \tag{21}$$

Theorem 2. *Let $O = [O_{ij}]_{\alpha\times\beta}$and $\mathscr{M} = [\mathscr{M}_{jm}]_{\beta\times\gamma}$ be two NHSM, where $O_{ij} = (\mathscr{T}^o_{ijk}, \mathscr{J}^o_{ijk}, \mathscr{F}^o_{ijk})$ and $\mathscr{M}_{jm} = (\mathscr{T}^{\mathscr{M}}_{jkm}, \mathscr{J}^{\mathscr{M}}_{jkm}, \mathscr{F}^{\mathscr{M}}_{jkm})$. Then,*

$$(O\otimes\mathscr{M})^t = \mathscr{M}^t\otimes O^t. \tag{22}$$

Proof. Let $O\otimes\mathscr{M} = [\mathscr{S}_{im}]_{\alpha\times\gamma}$; then, $(O\otimes\mathscr{M})^t = [\mathscr{S}_{mi}]_{\gamma\times\alpha}, O^t = [O_{ji}]_{\beta\times\alpha}$, and $\mathscr{M}^t = [\mathscr{M}_{mj}]_{\gamma\times\beta}$.

Now,

$$\begin{aligned}(O\otimes\mathscr{M})^t &= \left(\mathscr{T}^{\mathscr{S}}_{kmi}, \mathscr{J}^{\mathscr{S}}_{kmi}, \mathscr{F}^{\mathscr{S}}_{kmi}\right)_{\gamma\times\alpha}\\ &= \begin{pmatrix} \max_{jk}\min\left(\mathscr{T}^{\mathscr{M}}_{mjk}, \mathscr{T}^o_{jki}\right), \min_{jk}\max\left(\mathscr{J}^{\mathscr{M}}_{mjk}, \mathscr{J}^o_{jki}\right), \\ \min_{jk}\max\left(\mathscr{F}^{\mathscr{M}}_{mjk}, \mathscr{F}^0_{jki}\right) \end{pmatrix}_{\gamma\times\alpha}\\ &= \left(\mathscr{T}^{\mathscr{M}}_{mjk}, \mathscr{J}^{\mathscr{M}}_{mjk}, \mathscr{F}^{\mathscr{M}}_{mjk}\right)_{\gamma\times\beta}\otimes\left(\mathscr{T}^o_{jki}, \mathscr{J}^o_{jki}, \mathscr{F}^0_{jki}\right)_{\beta\times\alpha} = \mathscr{M}^t\otimes O^t.\end{aligned} \tag{23}$$

□

3.8. Operators of NHSMs. Let $O = [O_{ij}]$ and $\mathscr{M} = [\mathscr{M}_{ij}]$ be two NHSM, where $O_{ij} = (\mathscr{T}^o_{ijk}, \mathscr{J}^o_{ijk}, \mathscr{F}^o_{ijk})$ and $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{J}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$. Then,

(i) Union:

$$O\cup\mathscr{M} = \mathscr{S}, \tag{24}$$

where $\mathscr{T}^s_{ijk} = \max(\mathscr{T}^o_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk})$, $\mathscr{J}^s_{ijk} = ((\mathscr{J}^o_{ijk} + \mathscr{J}^{\mathscr{M}}_{ijk})/2)$, and $\mathscr{F}^s_{ijk} = \min(\mathscr{F}^o_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$.

(ii) Intersection:

$$O\cap\mathscr{M} = \mathscr{S}, \tag{25}$$

where

$$\begin{aligned}\mathscr{T}^s_{ijk} &= \min\left(\mathscr{T}^o_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right),\\ \mathscr{J}^s_{ijk} &= \frac{\left(\mathscr{J}^o_{ijk} + \mathscr{J}^{\mathscr{M}}_{ijk}\right)}{2},\\ \mathscr{F}^s_{ijk} &= \max\left(\mathscr{F}^o_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right).\end{aligned} \tag{26}$$

(iii) Arithmetic mean:

$$O\oplus\mathscr{M} = \mathscr{S}, \tag{27}$$

where

$$\begin{aligned}\mathscr{T}^s_{ijk} &= \frac{\left(\mathscr{T}^o_{ijk} + \mathscr{T}^{\mathscr{M}}_{ijk}\right)}{2},\\ \mathscr{J}^s_{ijk} &= \frac{\left(\mathscr{J}^o_{ijk} + \mathscr{J}^{\mathscr{M}}_{ijk}\right)}{2},\\ \mathscr{F}^s_{ijk} &= \frac{\left(\mathscr{F}^o_{ijk} + \mathscr{F}^{\mathscr{M}}_{ijk}\right)}{2}.\end{aligned} \tag{28}$$

(iv) Weighted arithmetic mean:

$$O\odot^w\mathscr{M} = \mathscr{S}, \tag{29}$$

where

$$\begin{aligned}\mathscr{T}^s_{ijk} &= \frac{\left(w^1\mathscr{T}^o_{ijk} + w^2\mathscr{T}^{\mathscr{M}}_{ijk}\right)}{w^1 + w^2},\\ \mathscr{J}^s_{ijk} &= \frac{\left(w^1\mathscr{J}^o_{ijk} + w^2\mathscr{J}^{\mathscr{M}}_{ijk}\right)}{w^1 + w^2},\\ \mathscr{F}^s_{ijk} &= \frac{\left(w^1\mathscr{F}^o_{ijk} + w^2\mathscr{F}^{\mathscr{M}}_{ijk}\right)}{w^1 + w^2}\cdot w^1, w^2 > 0.\end{aligned} \tag{30}$$

(v) Geometric mean:

$$O\odot\mathscr{M} = \mathscr{S}, \tag{31}$$

where

$$\mathcal{T}^s_{ijk} = \sqrt{\mathcal{T}^o_{ijk} \cdot \mathcal{T}^{\mathcal{M}}_{ijk}},$$
$$\mathcal{I}^s_{ijk} = \sqrt{\mathcal{I}^o_{ijk} \cdot \mathcal{I}^{\mathcal{M}}_{ijk}}, \quad (32)$$
$$\mathcal{F}^s_{ijk} = \sqrt{\mathcal{F}^o_{ijk} \cdot \mathcal{F}^{\mathcal{M}}_{ijk}}.$$

(vi) Weighted geometric mean:

$$O \odot^w \mathcal{M} = \mathcal{S}, \quad (33)$$

where

$$\mathcal{T}^s_{ijk} = \sqrt[w1+w2]{\left(\mathcal{T}^o_{ijk}\right)^{w^1} \cdot \left(\mathcal{T}^{\mathcal{M}}_{ijk}\right)^{w^2}},$$
$$\mathcal{I}^s_{ijk} = \sqrt[w1+w2]{\left(\mathcal{I}^o_{ijk}\right)^{w^1} \cdot \left(\mathcal{I}^{\mathcal{M}}_{ijk}\right)^{w^2}}, \quad (34)$$
$$\mathcal{F}^s_{ijk} = \sqrt[w1+w2]{\left(\mathcal{F}^o_{ijk}\right)^{w^1} \cdot \left(\mathcal{F}^{\mathcal{M}}_{ijk}\right)^{w^2}},$$
$$w^1, w^2 > 0.$$

(vii) Harmonic mean:

$$O \oslash \mathcal{M} = \mathcal{S}, \quad (35)$$

where

$$\mathcal{T}^s_{ijk} = \frac{2\mathcal{T}^o_{ijk}\mathcal{T}^{\mathcal{M}}_{ijk}}{\mathcal{T}^o_{ijk} + \mathcal{T}^{\mathcal{M}}_{ijk}},$$
$$\mathcal{I}^s_{ijk} = \frac{2\mathcal{I}^o_{ijk}\mathcal{I}^{\mathcal{M}}_{ijk}}{\mathcal{I}^o_{ijk} + \mathcal{I}^{\mathcal{M}}_{ijk}}, \quad (36)$$
$$\mathcal{F}^s_{ijk} = \frac{2\mathcal{F}^o_{ijk}\mathcal{F}^{\mathcal{M}}_{ijk}}{\mathcal{F}^o_{ijk} + \mathcal{F}^{\mathcal{M}}_{ijk}}.$$

(viii) Weighted harmonic mean:

$$O \oslash^w \mathcal{M} = \mathcal{S}, \quad (37)$$

where

$$\mathcal{T}^s_{ijk} = \frac{w^1 + w^2}{\left(w^1/\mathcal{T}^o_{ijk}\right) + \left(w^2/\mathcal{T}^{\mathcal{M}}_{ijk}\right)},$$
$$\mathcal{I}^s_{ijk} = \frac{w^1 + w^2}{\left(w^1/\mathcal{I}^o_{ijk}\right) + \left(w^2/\mathcal{I}^{\mathcal{M}}_{ijk}\right)}, \quad (38)$$
$$\mathcal{F}^s_{ijk} = \frac{w^1 + w^2}{\left(w^1/\mathcal{F}^o_{ijk}\right) + \left(w^2/\mathcal{F}^{\mathcal{M}}_{ijk}\right)},$$

Proposition 3. *Let $O = [O_{ij}]$ and $\mathcal{M} = [\mathcal{M}_{ij}]$ be two NHSM, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{I}^o_{ijk}, \mathcal{F}^o_{ijk})$ and $\mathcal{M}_{ij} = (\mathcal{T}^{\mathcal{M}}_{ijk}, \mathcal{I}^{\mathcal{M}}_{ijk}, \mathcal{F}^{\mathcal{M}}_{ijk})$.*
Then,

(i) $(O \cup \mathcal{M})^t = O^t \cup \mathcal{M}^t$
(ii) $(O \cap \mathcal{M})^t = O^t \cap \mathcal{M}^t$
(iii) $(O \oplus \mathcal{M})^t = O^t \oplus \mathcal{M}^t$
(iv) $(O \oplus^w \mathcal{M})^t = O^t \oplus \mathcal{M}^t$
(v) $(O \odot \mathcal{M})^t = O^t \odot \mathcal{M}^t$
(vi) $(O \odot^w \mathcal{M})^t = O^t \odot^w \mathcal{M}^t$
(vii) $(O \oslash \mathcal{M})^t = O^t \oslash \mathcal{M}^t$
(viii) $(O \oslash^w \mathcal{M})^t = O^t \oslash^w \mathcal{M}^t$

Proof. (i)

$$\begin{aligned}(O \cup \mathcal{M})^t &= \left[\left(\max\left(\mathcal{T}^o_{ijk}, \mathcal{T}^{\mathcal{M}}_{ijk}\right), \frac{\left(\mathcal{I}^o_{ijk} + \mathcal{I}^{\mathcal{M}}_{ijk}\right)}{2}, \min\left(\mathcal{F}^o_{ijk}\mathcal{F}^{\mathcal{M}}_{ijk}\right)\right)\right]^t \\ &= \left[\left(\max\left(\mathcal{T}^o_{jki}, \mathcal{T}^{\mathcal{M}}_{jki}\right), \frac{\left(\mathcal{I}^o_{jki} + \mathcal{I}^{\mathcal{M}}_{jki}\right)}{2}, \min\left(\mathcal{F}^o_{jki}, \mathcal{F}^{\mathcal{M}}_{jki}\right)\right)\right] \\ &= \left[\left(\mathcal{T}^o_{jki}, \mathcal{I}^o_{jki}, \mathcal{F}^o_{jki}\right)\right] \cup \left[\left(\mathcal{T}^{\mathcal{M}}_{jki}, \mathcal{I}^{\mathcal{M}}_{jki}, \mathcal{F}^{\mathcal{M}}_{jki}\right)\right] \\ &= \left[\left(\mathcal{T}^o_{ijk}, \mathcal{I}^o_{ijk}, \mathcal{F}^o_{ijk}\right)\right]^t \cup \left[\left(\mathcal{T}^{\mathcal{M}}_{ijk}, \mathcal{I}^{\mathcal{M}}_{ijk}, \mathcal{F}^{\mathcal{M}}_{ijk}\right)\right]^t \\ &= O^t \cup \mathcal{M}^t.\end{aligned} \quad (39)$$

Remaining parts are proved in a similar way. □

Proposition 4. *Let $O = [O_{ij}]$ and $\mathscr{M} = [\mathscr{M}_{ij}]$ be two upper triangular NHSM, where $O_{ij} = (\mathscr{T}^o_{ijk}, \mathscr{I}^o_{ijk}, \mathscr{F}^o_{ijk})$ and $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$. Then, $(O \cup \mathscr{M})$, $(O \cap \mathscr{M})$, $(O \oplus \mathscr{M})$, $(O \oplus^w \mathscr{M})$, $(O \odot \mathscr{M})$, and $(O \odot^w \mathscr{M})$ are all upper triangular NHSM and vice versa.*

Theorem 3. *Let $O = [O_{ij}]$ and $\mathscr{M} = [\mathscr{M}_{ij}]$ be two NHSM, where $O_{ij} = (\mathscr{T}^o_{ijk}, \mathscr{I}^o_{ijk}, \mathscr{F}^o_{ijk})$ and $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$. Then,*

(i) $(O \cup \mathscr{M})^\diamond = O^\diamond \cap \mathscr{M}^\diamond$
(ii) $(O \cap \mathscr{M})^\diamond = O^\diamond \cup \mathscr{M}^\diamond$
(iii) $(O \oplus \mathscr{M})^\diamond = O^\diamond \oplus \mathscr{M}^\diamond$
(iv) $(O \oplus^w \mathscr{M})^\diamond = O^\diamond \oplus \mathscr{M}^\diamond$
(v) $(O \odot \mathscr{M})^\diamond = O^\diamond \odot \mathscr{M}^\diamond$
(vi) $(O \odot^w \mathscr{M})^\diamond = O^\diamond \odot^w \mathscr{M}^\diamond$
(vii) $(O \oslash \mathscr{M})^\diamond = O^\diamond \oslash \mathscr{M}^\diamond$
(viii) $(O \oslash^w \mathscr{M})^\diamond = O^\diamond \oslash^w \mathscr{M}^\diamond$

Proof. (i)

$$\begin{aligned}(O \cup \mathscr{M})^\diamond &= \left[\left(\max\left(\mathscr{T}^o_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right), \frac{\left(\mathscr{I}^o_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk}\right)}{2}, \min\left(\mathscr{F}^o_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)\right)\right]^\diamond \\ &= \left[\left(\min\left(\mathscr{F}^o_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right), \frac{\left(\mathscr{I}^o_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk}\right)}{2}, \max\left(\mathscr{T}^o_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right)\right)\right] \\ &= \left(\mathscr{F}^o_{ijk}, \mathscr{I}^o_{ijk}, \mathscr{T}^o_{ijk}\right) \cap \left(\mathscr{F}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right) \\ &= \left(\mathscr{T}^o_{ijk}, \mathscr{I}^o_{ijk}, \mathscr{F}^o_{ijk}\right)^\diamond \cap \left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)^\diamond \\ &= O^\diamond \cap \mathscr{M}^\diamond.\end{aligned} \tag{40}$$

Remaining parts are proved in a similar way. □

Theorem 4. *Let $O = [O_{ij}]$ and $\mathscr{M} = [\mathscr{M}_{ij}]$ be two NHSM, where $O_{ij} = (\mathscr{T}^o_{ijk}, \mathscr{I}^o_{ijk}, \mathscr{F}^o_{ijk})$ and $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$. Then,*

(i) $(O \cup \mathscr{M}) = (\mathscr{M} \cup O)$
(ii) $(O \cap \mathscr{M}) = (\mathscr{M} \cap O)$
(iii) $(O \oplus \mathscr{M}) = (\mathscr{M} \oplus O)$
(iv) $(O \oplus^w \mathscr{M}) = (\mathscr{M} \oplus^w O)$
(v) $(O \odot \mathscr{M}) = (\mathscr{M} \odot O)$
(vi) $(O \odot^w \mathscr{M}) = (\mathscr{M} \odot^w O)$
(vii) $(O \oslash \mathscr{M}) = (\mathscr{M} \oslash O)$
(viii) $(O \oslash^w \mathscr{M}) = (\mathscr{M} \oslash^w O)$

Proof. (i)

$$\begin{aligned}(O \cup \mathscr{M}) &= \left[\left(\max\left(\mathscr{T}^o_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right), \frac{\left(\mathscr{I}^o_{ijk} + J^{\mathscr{M}}_{ijk}\right)}{2}, \min\left(\mathscr{F}^o_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)\right)\right] \\ &\quad \left[\left(\max\left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{T}^o_{ijk}\right), \frac{\left(\mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^o_{ijk}\right)}{2}, \min\left(\mathscr{F}^{\mathscr{M}}_{ijk}, \mathscr{F}^o_{ijk}\right)\right)\right] \\ &= \left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right) \cup \left(\mathscr{T}^o_{ijk}, \mathscr{I}^o_{ijk}, \mathscr{F}^o_{ijk}\right) \\ &= (\mathscr{M} \cup O).\end{aligned} \tag{41}$$

Remaining parts are proved in a similar way. □

Theorem 5. *Let* $= [O_{ij}]$, $\mathscr{M} = [\mathscr{M}_{ij}]$, *and* $\mathscr{N} = [\mathscr{M}_{ij}]$*be NHSM, where* $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$, $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$, *and* $\mathscr{N}_{ij} = (\mathscr{T}^{\mathscr{N}}_{ijk}, \mathscr{I}^{\mathscr{N}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk})$. *Then,*

(i) $(O \cup \mathscr{M}) \cup \mathscr{N} = O \cup (\mathscr{M} \cup \mathscr{N})$

(ii) $(O \cap \mathscr{M}) \cap \mathscr{N} = O \cap (\mathscr{M} \cap \mathscr{N})$

(iii) $((O \oplus \mathscr{M}) \oplus \mathscr{N} \neq O \oplus (\mathscr{M} \oplus \mathscr{N})$

(iv) $(O \odot \mathscr{M}) \odot \mathscr{N} \neq O \odot (\mathscr{M} \odot \mathscr{N})$

(v) $(O \oslash \mathscr{M}) \oslash \mathscr{N} \neq O \oslash (\mathscr{M} \oslash \mathscr{N})$

Proof. (i)

$$\begin{aligned}
(O \cup \mathscr{M}) \cup \mathscr{N} &= \left[\left(\max\left(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right), \frac{\left(\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk}\right)}{2}, \min\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)\right)\right] \cup \left[\left(\mathscr{T}^{\mathscr{N}}_{ijk}, \mathscr{I}^{\mathscr{N}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right] \\
&= \left[\left(\max\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{T}^{\mathscr{N}}_{ijk}\right), \frac{\left(\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)}{3}, \min\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right)\right] \\
&= \left[\left(\max\left(\mathscr{T}^{O}_{ijk}, \mathscr{T}^{M}_{ijk}, \mathscr{T}^{\mathscr{N}}_{ijk}\right), \frac{\left(\mathscr{I}^{O}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)}{3}, \min\left(\mathscr{F}^{O}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right)\right] \\
&= \left(\mathscr{T}^{O}_{ijk}, \mathscr{I}^{O}_{ijk}, \mathscr{F}^{O}_{ijk}\right) \cup \left[\left(\max\left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{T}^{\mathscr{N}}_{ijk}\right), \frac{\left(\mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)}{2}, \min\left(\mathscr{F}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right)\right] \\
&= \left(\mathscr{T}^{O}_{ijk}, \mathscr{I}^{O}_{ijk}, \mathscr{F}^{O}_{ijk}\right) \cup \left(\left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right) \cup \left(\mathscr{T}^{\mathscr{N}}_{ijk}, \mathscr{I}^{\mathscr{N}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right) \\
&= O \cup (\mathscr{M} \cup \mathscr{N}).
\end{aligned} \tag{42}$$

Remaining parts are proved in a similar way. □

Theorem 6. *Let* $= [O_{ij}]$, $\mathscr{M} = [\mathscr{M}_{ij}]$, *and* $\mathscr{N} = [\mathscr{M}_{ij}]$*be NHSM, where* $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$, $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$, *and* $\mathscr{N}_{ij} = (\mathscr{T}^{\mathscr{N}}_{ijk}, \mathscr{I}^{\mathscr{N}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk})$. *Then,*

(i) $O \cap (\mathscr{M} \oplus \mathscr{N}) = (O \cap \mathscr{M}) \oplus (O \cap \mathscr{N})$

(ii) $(O \oplus \mathscr{M}) \cap \mathscr{N} = (O \cap \mathscr{N}) \oplus (\mathscr{M} \cap \mathscr{N})$

(iii) $O \cup (\mathscr{M} \oplus \mathscr{N}) = (O \cup \mathscr{M}) \oplus (O \cup \mathscr{N})$

(iv) $(O \oplus \mathscr{M}) \cup \mathscr{N} = (O \cup \mathscr{N}) \oplus (\mathscr{M} \cup \mathscr{N})$

Proof. (i)

$$\begin{aligned}
O \cap (\mathscr{M} \oplus \mathscr{N}) &= \left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right) \cap \left[\left(\frac{\left(\mathscr{T}^{\mathscr{M}}_{ijk} + \mathscr{T}^{\mathscr{N}}_{ijk}\right)}{2}, \frac{\left(\mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)}{2}, \frac{\left(\mathscr{F}^{\mathscr{M}}_{ijk} + \mathscr{F}^{\mathscr{N}}_{ijk}\right)}{2}\right)\right] \\
&= \left[\left(\min\left(\mathscr{T}^{o}_{ijk}, \frac{\left(\mathscr{T}^{\mathscr{M}}_{ijk} + \mathscr{T}^{\mathscr{N}}_{ijk}\right)}{2}\right), \frac{\left(\mathscr{I}^{o}_{ijk} + \left(\left(\left(\mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)/2\right)\right)\right)}{2}, \max\left(\mathscr{F}^{o}_{ijk}, \frac{\left(\mathscr{F}^{\mathscr{M}}_{ijk} + \mathscr{F}^{\mathscr{N}}_{ijk}\right)}{2}\right)\right)\right] \\
&= \left[\left(\min\left(\frac{\left(\mathscr{T}^{O}_{ijk} + \mathscr{T}^{\mathscr{M}}_{ijk}\right)}{2}, \frac{\left(\mathscr{T}^{O}_{ijk} + \mathscr{T}^{\mathscr{N}}_{ijk}\right)}{2}\right), \frac{\left(\left(\left(\mathscr{I}^{O}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk}\right)/2\right) + \left(\left(\mathscr{I}^{O}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)/2\right)\right)}{2},\right.\right. \\
&\qquad \left.\left.\max\left(\frac{\left(\mathscr{F}^{O}_{ijk} + \mathscr{F}^{\mathscr{M}}_{ijk}\right)}{2}, \frac{\left(\mathscr{F}^{O}_{ijk} + \mathscr{F}^{\mathscr{N}}_{ijk}\right)}{2}\right)\right)\right] \\
&= \left[\left(\min\left(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right), \frac{\left(\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk}\right)}{2}, \max\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)\right)\right] \\
&\quad \oplus \left[\left(\min\left(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{N}}_{ijk}\right), \frac{\left(\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)}{2}, \max\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right)\right] \\
&= \left[\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right) \cap \left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)\right] \oplus \left[\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right) \cap \left(\mathscr{T}^{\mathscr{N}}_{ijk}, \mathscr{I}^{\mathscr{N}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right] \\
&= (O \cap \mathscr{M}) \oplus (O \cap \mathscr{N}).
\end{aligned} \tag{43}$$

The remaining parts are proved in a similar way. □

4. Neutrosophic Hypersoft Matrix (NHSM) in Decision-Making Using Score Function

Suppose that some decision makers wish to select from α number of objects. Each object is further characterized by β number of attributes, whose respective attributes form a relation just like NHSM. Each decision makes different neutrosophic values to these respective attributes. Corresponding to these neutrosophic values for the required relation, we get a NHSM of order $\alpha \times \beta$. From this NHSM, we calculate values' matrices, which help to obtain a score matrix. And, finally, we calculate the total score of each object from the score matrix.

Value matrices are the real matrices that obey all the properties of real matrices. Score function is also a real matrix which is obtained from two or more value matrices.

Definition 1. Let $O = [O_{ij}]$ be the NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$; then, the value of matrix O is denoted as $\mathcal{V}(O)$, and it is defined as $\mathcal{V}(O) = [\mathcal{V}^O_{ij}]$ of order $\alpha \times \beta$, where$\mathcal{V}^O_{ij} = \mathcal{T}^o_{ijk} - \mathcal{J}^o_{ijk}, -\mathcal{F}^o_{ijk}$. The score of two NHSM $O = [O_{ij}]$ and $\mathcal{M} = [\mathcal{M}_{ij}]$ of order $\alpha \times \beta$ is given as $\mathcal{S}(O, \mathcal{M}) = \mathcal{V}(O) + \mathcal{V}(\mathcal{M})$ and $\mathcal{S}(O, \mathcal{M}) = [\mathcal{S}_{ij}]$, where $\mathcal{S}_{ij} = \mathcal{V}^O_{ij} + \mathcal{V}^{\mathcal{M}}_{ij}$. The total score of each object in universal set is $|\sum_{j=1}^{h} \mathcal{S}_{ij}|$.

Algorithm is graphically represented with Figure 1.

Step 1: construct a NHSM as defined in Section 3.1.

Step 2: calculate the value matrix from NHSM. Let $O = [O_{ij}]$ be the NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$; then, the value of matrix O is denoted as $\mathcal{V}(O)$, and it is defined as $\mathcal{V}(O) = [\mathcal{V}^O_{ij}]$ of order $\alpha \times \beta$, where$\mathcal{V}^O_{ij} = \mathcal{T}^o_{ijk} - \mathcal{J}^o_{ijk}, -\mathcal{F}^o_{ijk}$.

Step 3: compute the score matrix with the help of value matrices. The score of two NHSM $O = [O_{ij}]$ and $\mathcal{M} = [\mathcal{M}_{ij}]$ of order and $\alpha \times \beta$ is given as $\mathcal{S}(O, \mathcal{M}) = \mathcal{V}(O) + \mathcal{V}(\mathcal{M})$ and $\mathcal{S}(O, \mathcal{M}) = [\mathcal{S}_{ij}]$, where $\mathcal{S}_{ij} = \mathcal{V}^O_{ij} + \mathcal{V}^{\mathcal{M}}_{ij}$.

Step 4: compute the total score from the score matrix. The total score of each object in the universal set is $|\sum_{j=1}^{j=1} \mathcal{S}_{ij}|$.

Step 5: find the optimal solution by selecting an object of maximum score from the total score matrix.

4.1. Numerical Example. Teachers' recruitment problem (TRP) is the most complex and absurd task. There is no fixed and fabricated design to know their subject knowledge or pedagogical skills. Therefore, decision makers find themselves in a blind alley. Consequently, based on their own knowledge and experience, they select a person who does not meet the institutional requirement; thus, TRP is typically a multicriteria decision-making MCDM problem.

Assumptions:

(i) Independent attributes are considered

(ii) Everyone attends the interview

(iii) Hesitant environment is not yet considered

Formulation of the Problem. Let us consider an institute that wants to hire a teacher appropriate to its requirements, and he received the following statistics-based CVs. Let $\mathcal{U}$ be the set of candidates for the teaching at the college level:

$$\mathcal{U} = \left\{\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^3, \mathcal{T}^4, \mathcal{T}^5, \mathcal{T}^6, \mathcal{T}^7, \mathcal{T}^8, \mathcal{T}^9, \mathcal{T}^{10}, \mathcal{T}^{11}, \mathcal{T}^{12}, \mathcal{T}^{13}, \mathcal{T}^{14}, \mathcal{T}^{15}\right\}. \tag{44}$$

Also, consider the set of attributes as

$$\begin{aligned} \mathcal{A}_1 &= \text{Qualification}, \\ \mathcal{A}_2 &= \text{Experience}, \\ \mathcal{A}_3 &= \text{Gender}, \\ \mathcal{A}_4 &= \text{Publications}. \end{aligned} \tag{45}$$

Parameters:

$\mathbf{T_i}$ = *universal set of teachers,* where $i = 1, 2, 3, 4, 5$

$\mathbf{A_i}$ = attributes, where $i = 1, 2, 3, 4$ that are further categorized into the following:

(i) $\mathcal{A}^a_1$ = Qualification

(ii) $\mathcal{A}^a_1 = \{\text{BS Hons., MS/Mphill, Phd, Post Doctorate}\}$

(iii) $\mathcal{A}^b_2$ = Experience = $\{5\text{yr}, 8\text{yr}, 10\text{yr}, 15\text{yr}\}$

(iv) $\mathcal{A}^c_3$ = Gender = $\{\text{Male, Female}\}$

(v) $\mathcal{A}^d_4$ = Publications = $\{3, 5, 8, 10+\}$

The function $\mathcal{F}\colon \mathcal{A}^a_1 \times \mathcal{A}^b_2 \times \mathcal{A}^c_3 \times \mathcal{A}^d_4 \longrightarrow P(\mathcal{U})$.

Let us assume the relation $\mathcal{F}((\mathcal{A}^a_1 \times \mathcal{A}^b_2 \times \mathcal{A}^c_3 \times \mathcal{A}^d_4) = \mathcal{F}(\text{Mphill, 5yr, male, 3})$ which is the actual requirement of college for the selection of candidates.

Four candidates $\{\mathcal{T}^2, \mathcal{T}^6, \mathcal{T}^8, \mathcal{T}^{14}\}$ are shortlisted on the basis of assumed relation, i.e., (Mphill, 5yr, male, 3).

A jury of two members $\{\mathbb{A}, \mathbb{B}\}$ is set for the selection of shortlisted candidates. These jury members give their valuable opinion in the form of NHSSs separately as

Figure 1: Flowchart of the proposed algorithm.

$$
\begin{aligned}
\mathbb{A} &= \mathscr{F}\,(\text{Mphill}, 5\text{yr}, \text{male}, 3)\\
&= \Big\{ \ll \mathscr{T}^{2}, (\text{Mphill}\{0.5, 0.3, 0.6\}, 5\text{yr}\{0.3, 0.4, 0.7\}, \text{male}\{0.5, 0.6, 0.9\}, 3\{0.6, 0.4, 0.5\}) \gg,\\
&\quad \ll \mathscr{T}^{6}, (\text{Mphill}\{0.3, 0.2, 0.1\}, 5\text{yr}\{0.6, 0.5, 0.3\}, \text{male}\{0.7, 0.8, 0.3\}, 3\{0.7, 0.5, 0.3\}) \gg,\\
&\quad \ll \mathscr{T}^{8} (\text{Mphill}\{0.7, 0.3, 0.6\}, 5\text{yr}\{0.6, 0.4, 0.8\}, \text{male}\{0.8, 0.5, 0.4\}, 3\{0.6, 0.2, 0.1\}) \gg,\\
&\quad \ll \mathscr{T}^{14}, (\text{Mphill}\{0.5, 0.4, 0.5\}, 5\text{yr}\{0.3, 0.6, 0.7\}, \text{male}\{0.9, 0.2, 0.1\}, 3\{0.4, 0.5, 0.3\}) \gg, \Big\},\\
\mathbb{B} &= \mathscr{F}\,(\text{Mphill}, 5\text{yr}, \text{male}, 3)\\
&= \Big\{ \ll \mathscr{T}^{2}, (\text{Mphill}\{0.8, 0.1, 0.2\}, 5\text{yr}\{0.7, 0.4, 0.3\}, \text{male}\{0.4, 0.6, 0.3\}, 3\{0.5, 0.3, 0.5\}) \gg,\\
&\quad \ll \mathscr{T}^{6}, (\text{Mphill}\{0.8, 0.2, 0.1\}, 5\text{yr}\{0.7, 0.4, 0.3\}, \text{male}\{0.8, 0.2, 0.1\}, 3\{0.9, 0.3, 0.2\}) \gg,\\
&\quad \ll \mathscr{T}^{8} (\text{Mphill}\{0.5, 0.3, 0.4\}, 5\text{yr}\{0.7, 0.3, 0.2\}, \text{male}\{0.9, 0.2, 0.1\}, 3\{0.4, 0.2, 0.7\}) \gg,\\
&\quad \ll \mathscr{T}^{14}, (\text{Mphill}\{0.7, 0.4, 0.2\}, 5\text{yr}\{0.2, 0.4, 0.7\}, \text{male}\{0.7, 0.2, 0.1\}, 3\{0.6, 0.3, 0.4\}) \gg, \Big\}.
\end{aligned} \tag{46}
$$

Let us apply the above define algorithm for the calculation of total score.

Step I (construction of NHSM):the above two NHSSs are given in the form of NHSMs as

$$
\begin{aligned}
[\mathbb{A}] &= \begin{bmatrix}
(\text{Mphill}, (0.5, 0.3, 0.6)) & (5\text{yr}, (0.3, 0.4, 0.7)) & (\text{male}, (0.5, 0.6, 0.9)) & (3, (0.6, 0.4, 0.5))\\
(\text{Mphill}, (0.3, 0.2, 0.1)) & (5\text{yr}, (0.6, 0.5, 0.3)) & (\text{male}, (0.7, 0.8, 0.3)) & (3, (0.7, 0.5, 0.3))\\
(\text{Mphill}, (0.7, 0.3, 0.6)) & (5\text{yr}, (0.6, 0.4, 0.8)) & (\text{male}, (0.8, 0.5, 0.4)) & (3, (0.6, 0.2, 0.1))\\
(\text{Mphill}, (0.5, 0.4, 0.5)) & (5\text{yr}, (0.3, 0.6, 0.7)) & (\text{male}, (0.9, 0.2, 0.1)) & (3, (0.4, 0.5, 0.3))
\end{bmatrix},\\
[\mathbb{B}] &= \begin{bmatrix}
(\text{Mphill}, (0.8, 0.1, 0.2)) & (5\text{yr}, (0.7, 0.4, 0.3)) & (\text{male}, (0.4, 0.6, 0.3)) & (3, (0.5, 0.3, 0.5))\\
(\text{Mphill}, (0.8, 0.2, 0.1)) & (5\text{yr}, (0.7, 0.4, 0.3)) & (\text{male}, (0.8, 0.2, 0.1)) & (3, (0.9, 0.3, 0.2))\\
(\text{Mphill}, (0.5, 0.3, 0.4)) & (5\text{yr}, (0.7, 0.3, 0.2)) & (\text{male}, (0.9, 0.2, 0.1)) & (3, (0.4, 0.2, 0.7))\\
(\text{Mphill}, (0.7, 0.4, 0.2)) & (5\text{yr}, (0.2, 0.4, 0.7)) & (\text{male}, (0.7, 0.2, 0.1)) & (3, (0.6, 0.3, 0.4))
\end{bmatrix}.
\end{aligned} \tag{47}
$$

Step II: calculation of the value matrices of NHSMs defined in Step I:

Table 7: Alternative rank comparison using NHSM and NSM techniques.

Method	Alternative final ranking	Optimal choice
Proposed in this paper	$\mathcal{T}^2 > \mathcal{T}^6 > \mathcal{T}^8 > \mathcal{T}^{14}$	$\mathcal{T}^2$
Hashmi et al. [28]	$\mathcal{T}^2 > \mathcal{T}^8 > \mathcal{T}^{14} > \mathcal{T}^6$	$\mathcal{T}^2$

$$[\mathscr{V}(\mathbb{A})] = \begin{bmatrix} (\text{Mphill}, (-0.4)) & (5\text{yr}, (-0.8)) & (\text{male}, (-1)) & (3, (-0.3)) \\ (\text{Mphill}, (0)) & (5\text{yr}, (-0.2)) & (\text{male}, (-0.4)) & (3, (-0.1)) \\ (\text{Mphill}, (-0.2)) & (5\text{yr}, (-0.6)) & (\text{male}, (-0.1)) & (3, (0.3)) \\ (\text{Mphill}, (-0.4)) & (5\text{yr}, (-1)) & (\text{male}, (0.6)) & (3, (-0.4)) \end{bmatrix},$$
$$[\mathscr{V}(\mathbb{B})] = \begin{bmatrix} (\text{Mphill}, (0.5)) & (5\text{yr}, (0)) & (\text{male}, (-0.5)) & (3, (-0.3)) \\ (\text{Mphill}, (0.5)) & (5\text{yr}, (0)) & (\text{male}, (0.5)) & (3, (0.4)) \\ (\text{Mphill}, (-0.2)) & (5\text{yr}, (0.2)) & (\text{male}, (0.6)) & (3, (-0.5)) \\ (\text{Mphill}, (0.1)) & (5\text{yr}, (-0.9)) & (\text{male}, (0.4)) & (3, (-0.1)) \end{bmatrix}. \tag{48}$$

Step III: computation of the score matrix by adding value matrices obtained in Step II:

$$[\mathcal{S}(\mathbb{A}, \mathbb{B})] = \begin{bmatrix} (\text{Mphill}, (0.1)) & (5\text{yr}, (-0.8)) & (\text{male}, (-1.5)) & (3, (-0.6)) \\ (\text{Mphill}, (0.5)) & (5\text{yr}, (-0.2)) & (\text{male}, (0.1)) & (3, (0.3)) \\ (\text{Mphill}, (-0.4)) & (5\text{yr}, (-0.4)) & (\text{male}, (0.5)) & (3, (-0.2)) \\ (\text{Mphill}, (-0.3)) & (5\text{yr}, (-1.9)) & (\text{male}, (1)) & (3, (-0.5)) \end{bmatrix}. \tag{49}$$

Step IV: calculation of the score matrix:

$$\text{Total score} = \begin{bmatrix} 2.8 \\ 0.1 \\ 0.5 \\ 1.7 \end{bmatrix}. \tag{50}$$

Step V: the candidate $\mathcal{T}^2$ will be selected for teaching at the college level as the total score of $\mathcal{T}^2$ is highest among the rest of the total score of candidates.

5. Result and Comparison Analysis

We propose an algorithm for NHSM of the real-world problems, and results are compared with the algorithms on NSM already established. Graphical representations of the ranking of the proposed algorithm are given in Figure 1. The proposed algorithm is valid and practical. As it could be observed in the comparison Table 7, the proposed method's best selection is comparable with the already established method, which is expressive in itself and approve the reliability and validity of the proposed method. According to the refinement of the philosophy of neutrosophy, it could be a more efficient technique.

5.1. Limitations and Advantages of Proposed Matrix Theory.

The neutrosophic soft set theory is not very efficient in selecting the optimal object of a decision-making problem that possesses some attributes which are further divided, whereas neutrosophic hypersoft matrix theory can be applied.

The advantages of the proposed theory are

(1) Firstly, this new method's specialty is that it may solve any MCDM problem involving a huge number of decision makers very easily along with a simple computational procedure

(2) Secondly, when compared with existing methods for MCDM problems under a neutrosophic environment, the proposed operators are consistent and accurate, which illustrate their application's practicability

(3) Thirdly, the proposed method considers the interrelationships of attributes in practical application, while existing approaches cannot

(4) Lastly, the proposed algorithm for MCDM problems in this paper can further consider more correlations between attributes, which means that they have higher accuracy and greater reference value

(5) The matrix is useful for storing (neutrosophic hypersoft set) in the computer memory, which is very useful and applicable

6. Conclusion

This paper has first defined NHSM theory and then introduced some aggregate operators that are more functional to make theoretical studies in the neutrosophic soft set arena. Moreover, we have proposed the concept of the score function. Additionally, the utilization of NHSM in the decision-making problem (teacher recruitment problem (TRP)) has been made with the score matrix's assistance. At the end, we compared the result with existing techniques and showed that the purposed technique is more efficient and refined. We expect, this paper will advance the future investigation on various calculations such as TOPSIS, VIKOR, and AHP in other decision-making problems. Also, in future, it can be linked with Pythagorean fuzzy interactive Hamacher power aggression operators, interval-valued q-rung orthopair fuzzy sets in decision-making, CN-q-ROFS, connection number-based q-rung orthopair fuzzy set and their application to the decision-making process, and average operators based on the spherical cubic fuzzy number.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[3] D. Molodtsov, "Soft set theory-first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[4] N. Cagman and S. Enginoglu, "Soft set theory and uni-int decision making," *European Journal of Operational Research*, vol. 207, no. 2, pp. 848–855, 2010.

[5] P. K. Maji, A. R. Biswas, and A. R. Roy, "An application of soft sets in a decision making problem," *Computers & Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[6] P. K. Maji, R. Biswas, and A. R. Roy, "Soft set theory," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 555–562, 2003.

[7] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 589–602, 2001.

[8] P. K. Maji, R. Biswas, and A. R. Roy, "Intuitionistic fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 677–692, 2001.

[9] F. Smarandache, "Neutrosophic set, a generalization of the intuitionistic fuzzy sets," *International Journal of Pure and Applied Mathematics*, vol. 24, no. 3, pp. 287–297, 2004.

[10] F. Smarandache, *Neutrosophy, Neutrosophic Probability, Set and Logic*, http://fs.gallup.unm.edu/eBook-neutrosophics4.pdffourth version, p. 105, American Research Press, Rehoboth, DE, USA, 1998, .

[11] P. K. Maji, "Neutrosophic soft set," *Analysis of Fuzzy Mathematics And Informatics*, vol. 5, no. 1, pp. 157–168, 2013.

[12] I. Deli and S. Broumi, "Neutrosophic soft matrices and NSM-decision making," *Journal of Intelligent & Fuzzy Systems*, vol. 28, no. 5, pp. 2233–2241, 2015.

[13] S. Das, S. Kumar, S. Kar, and T. Pal, "Group decision making using neutrosophic soft matrix: an algorithmic approach," *Journal of King Saud University - Computer and Information Sciences*, vol. 31, no. 4, pp. 459–468, 2019.

[14] M. Riaz, M. Saqlain, and M. Saeed, "Application of generalized fuzzy TOPSIS in decision making for neutrosophic soft set to predict the champion of FIFA 2018: a mathematical analysis," *Punjab University Journal of Mathematics*, vol. 51, no. 8, pp. 111–126, 2019.

[15] M. Abdel-Baset, V. Chang and A. Gamal, Evaluation of the green supply chain management practices: a novel neutrosophic approach," *Computers in Industry*, vol. 108, pp. 210–220, 2019.

[16] M. Abdel-Basset, M. Saleh, A. Gamal, and F. Smarandache, "An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number," *Applied Soft Computing*, vol. 77, pp. 438–452, 2019.

[17] M. Abdel-Basset, G. Manogaran, A. Gamal, and F. Smarandache, "A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection," *Journal of Medical Systems*, vol. 43, no. 2, p. 38, 2019.

[18] M. Akram, A. Adeel, and J. C. R. Alcantud, "Group decision-making methods based on hesitant N-soft sets," *Expert Systems with Applications*, vol. 115, pp. 95–105, 2019.

[19] M. Akram, W. A. Dudek, and F. Ilyas, "Group decision-making based on pythagorean fuzzy TOPSIS method," *International Journal of Intelligent Systems*, vol. 34, no. 7, pp. 1455–1475, 2019.

[20] M. Akram, F. Ilyas, and H. Garg, "Multi-criteria group decision making based on ELECTRE I method in Pythagorean fuzzy information," *Soft Computing*, vol. 24, no. 5, pp. 3425–3453, 2020.

[21] H. Garg, "Novel neutrality aggregation operator-based multiattribute group decision-making method for single-valued neutrosophic numbers," *Soft Computing*, vol. 24, no. 14, pp. 10327–10349, 2020.

[22] H. Garg, "New exponential operation laws and operators for interval-valued q-rung orthopair fuzzy sets in group decision-making process," *Neural Computing and Applications*, 2021.

[23] T. A. Mehmood, "Novel approach towards bipolar soft sets and their applications," *Journal of Mathematics*, vol. 2020, Article ID 4690808, 2020.

[24] T. Mehmood, K. Ullah, Q. Khan, and N. Jan, "An approach towards decision making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Computing and Applications*, vol. 31, pp. 7041–7053, 2019.

[25] M. Shabbir and M. Naz, "On bipolar soft sets," 2013, https://arxiv.org/abs/1303.1344.

[26] F. Smarandache, "Extension of soft set to hypersoft set, and then to plithogenic hypersoft set," *Neutrosophic Sets and System*, vol. 22, pp. 168–170, 2018.

[27] M. Saqlain, S. Moin, M. N. Jafar, M. Saeed, and F. Smarandache, "Aggregate operators of neutrosophic

hypersoft set," *Neutrosophic Sets And Systems*, vol. 32, pp. 294–306, 2020.

[28] M. R. Hashmi, M. Riaz, and F. Smarandache, "m-polar neutrosophic topology with applications to multi-criteria decision-making in medical diagnosis and clustering Analysis," *International Journal of Fuzzy Systems*, vol. 22, no. 1, pp. 273–292, 2020.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 5511904, 6 pages
https://doi.org/10.1155/2021/5511904

Research Article

Uncertainty-Based Trimmed Coefficient of Variation with Application

Ali Hussein Al-Marshadi, Muhammad Aslam , and Alharbey Abdullah

Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21551, Saudi Arabia

Correspondence should be addressed to Muhammad Aslam; aslam_ravian@hotmail.com

Received 12 January 2021; Accepted 24 September 2021; Published 4 October 2021

Academic Editor: Parimala Mani

In this paper, the neutrosophic trimmed average, neutrosophic trimmed standard deviation, and neutrosophic trimmed coefficient of variation (NTCV) are introduced. The application of the proposed neutrosophic trimmed descriptive statistics is given with the help of measurement data. The comparisons of the proposed NTCV are compared with the existing neutrosophic coefficient of variation (NCV). From the comparisons, it is concluded that the proposed NTCV is more efficient than NCV in terms of consistency and measures of indeterminacy. Based on the study, it is recommended to apply the proposed NTCV in the industry when there is a need to make decisions on the basis of measurement data.

1. Introduction

The statistical methods and techniques are playing an important role in decision-making in all fields of social sciences, medical sciences, and industries. Among them, the average and coefficient of variation (CV) have been widely used in decision-making in the presence of more than one characteristic. The average is used to select the variable of interest which is better on average, and CV is applied to check the consistency of that characteristic. For example, industrials are interested to make the decision about the product on the basis of measurements recorded by different operators. To make the decision, a more consistent operator is selected using the CV. It is important to note that better on the average does not mean more consistent than the others. In other words, the CV tells about the variation in the data. Less the variation means better the data for the decision-making. A CV of less than 10% is considered very good and larger than 30% is not acceptable. The variation in the data can be reduced by omitting the outliers from the data. The outlier of the data can be removed from the data using the idea of the trimmed mean. In this method, a preselected percentage of the values are removed from the starting and ending of the ordered data. The use of the trimmed average is helpful to reduce the variation by removing extreme observations from the data. Wu and Zuo [1] proposed trimmed measures using the scale deviation method. Alkhazaleh and Razali [2] worked on estimation using the trimmed average. Yusof et al. [3] discussed various trimmed methods. Wang et al. [4] introduced the mean approach in medical science. Lugosi and Mendelson [5] introduced heavy-tailed distribution. More information about the application of trimmed measures can be seen in [6–10].

Uncertainty is defined as the lack of sureness about measurement, parameters, and observations. For example, measuring the water level, measuring rock joint roughness, and measuring the lifetime of a virus is done under uncertain environment. According to [11], "different sources of uncertainty may affect the quality of measurement results: environment, measurement setup, measuring instrument, appraiser, measuring object, measuring procedure, physical constants, the definition of the characteristic, software, and calculations." In case, when uncertainty is presented in the data, the fuzzy logic can be applied for the analysis of the data. The trimmed average under fuzzy logic can be applied to remove the extreme observations from the fuzzy data. The authors of [12–15] discussed the applications of trimmed average using fuzzy logic. More applications can be seen in [16, 17].

Fuzzy logic is based on membership and nonmembership values. Neutrosophic logic is a general form of logic that deals with three measures, namely, the measure of truth (membership), the measure of falsehood (nonmembership), and the measure of indeterminacy. The fuzzy logic is a special case of neutrosophic logic, see [18]. The information about the measure of indeterminacy can be obtained from the neutrosophic logic. The neutrosophic logic has been applied in a variety of fields, see [19–21]. Using the idea of neutrosophic logic, neutrosophic statistics which is the extension of classical statistics was introduced by [22]. The methods to analyze the neutrosophic data were discussed in [23, 24]. Aslam [25] introduced the neutrosophic coefficient of variation. Aslam and Bantan [26] introduced a measurement system using neutrosophic statistics. More information on dealing neutrosophic numbers can be seen in [27–29].

As mentioned earlier, the trimmed average is a useful technique to reduce the variation by removing the extreme observations from the data. In this method, a small percentage of values are removed to minimize the variation in the data. The trimmed average helps to remove the outliers from the data before calculating the traditional average. The coefficient of variation under neutrosophic statistics is known as the neutrosophic coefficient of variation (NCV) and the coefficient of variation using the trimmed average under neutrosophic statistics is known as the neutrosophic trimmed standard deviation (NTSD). The coefficient of variation using trimmed average under classical is called the trimmed coefficient of variation. Aslam [25] introduced NCV. By exploring the literature and best of our knowledge, there is no work on neutrosophic trimmed average, neutrosophic trimmed standard deviation (NTSD), and NTCV. In this paper, the introduction of average, standard deviation, and coefficient of variation using the neutrosophic statistics will be given. In addition, we will give the application of the proposed NTCV using the measurement data from the industry. It is expected that the proposed NTCV will be helpful to increase the consistency as compared to NCV. Furthermore, the proposed NTCV will be helpful to minimize the measure of indeterminacy.

2. Methodology

Let $I_N \epsilon [I_L, I_U]$ be an indeterminacy interval associated with neutrosophic random number $X_{Ni} = X_{Li} + X_{Ui} I_N \, (i = 1, 2, 3, \ldots, n_N)$ of size $n_N \epsilon [n_L, n_U]$, where X_L, n_L and X_U, n_U are the lower and upper values, respectively. The basic operations such as multiplication, division, and inverse of these neutrosophic numbers can be seen in [23, 24]. Suppose a data analyst has a neutrosophic sample $n_N \epsilon [n_L, n_U]$ and he is interested to find α% neutrosophic trimmed average (NTA). Suppose that X_L and X_U denote the lower and upper values of an indeterminate interval of measurement parts. The trimmed observation is denoted by$k_N = n_N \alpha$, where α is the percentage of values trimmed from the data. Suppose that $R_N = n_N - n_N \alpha$ shows the difference between the total observation and trimmed observations. The following process can be adopted to calculate α% neutrosophic trimmed average.

Step 1: arrange X_L and X_U observations in the ascending order

Step 2: trim $k_N \epsilon [k_L, k_U]$ observations at both ends of arranged data, where $k_N = n_N \alpha$

Step 3: compute NTA of remaining observations, $R_N = n_N - n_N \alpha$, $R_N \epsilon [R_L, R_U]$

The neutrosophic trimmed average, say $\overline{T}_L$ of values X_L, is calculated as

$$\overline{T}_L = \frac{1}{R_L} \sum_{i=k_L+1}^{n_L-k_L} X_{Li}, \tag{1}$$

where index of summation runs from the lower value of k_N to the lower value of R_N.

The neutrosophic trimmed average, say $\overline{T}_U$ of values X_U, is calculated as

$$\overline{T}_U = \frac{1}{R_U} \sum_{i=k_U+1}^{n_U-k_U} X_{Ui}, \tag{2}$$

where index of summation runs from the upper value of k_N to the upper value of R_N.

The neutrosophic trimmed average, say $\overline{X}_{Ni}$, using equations (1) and (2), is calculated by

$$\overline{X}_{Ni} = \frac{1}{R_L} \sum_{i=k_L+1}^{n_L-k_L} X_{Li} + \frac{1}{R_U} \sum_{i=k_U+1}^{n_U-k_U} X_{Ui} I_{\overline{X}N}; \; I_{\overline{X}N} \epsilon \left[I_{\overline{X}L}, I_{\overline{X}U} \right]. \tag{3}$$

The neutrosophic trimmed sum of the square of observations from $\overline{X}_{Ni}$ is calculated by

$$\sum_{i=k_U+1}^{n_U-k_U} \left(X_{Ni} - \overline{X}_{Ni}\right)^2 = \sum_{i=k_U+1}^{n_U-k_U} \left[\begin{matrix} \min \begin{pmatrix} \left(X_{Li} + X_{Ui} I_L\right)\left(\overline{X}_L + \overline{X}_U I_L\right), \left(X_{Li} + X_{Ui} I_L\right)\left(\overline{X}_L + \overline{X}_U I_U\right), \\ \left(X_{Li} + X_{Ui} I_U\right)\left(\overline{X}_L + \overline{X}_U I_L\right), \left(X_{Li} + X_{Ui} I_U\right)\left(\overline{X}_L + \overline{X}_U I_U\right) \end{pmatrix} \\ \max \begin{pmatrix} \left(X_{Li} + X_{Ui} I_L\right)\left(\overline{X}_L + \overline{X}_U I_L\right), \left(X_{Li} + X_{Ui} I_L\right)\left(\overline{X}_L + \overline{X}_U I_U\right), \\ \left(X_{Li} + X_{Ui} I_U\right)\left(\overline{X}_L + \overline{X}_U I_L\right), \left(X_{Li} + X_{Ui} I_U\right)\left(\overline{X}_L + \overline{X}_U I_U\right) \end{pmatrix} \end{matrix} \right], \; I_N \epsilon [I_L, I_U], \tag{4}$$

where $\overline{X}_L = 1/n_L \sum_{i=1}^{n_L} X_{Li}$ and $\overline{X}_U = 1/n_U \sum_{i=1}^{n_U} X_{Ui}$

The neutrosophic trimmed standard deviation (NTSD), say s_{NT}, is given by

$$s_{NT} = \sqrt{\frac{1}{R_N} \sum_{i=k_N+1}^{n_N-k_N} \left(X_{Ni} - \overline{X}_{Ni}\right)^2}. \quad (5)$$

The neutrosophic trimmed coefficient of variation (NTCV) tells about the consistency and is computed by

$$CV_{NT} = \frac{s_{NT}}{\overline{X}_{Ni}} \times 100. \quad (6)$$

3. Application Using Measurement Data

Now, we present the case study from the automotive industry in Kachiran Company in Asia, see [30], for more details. The company is a manufacturing housing clutch used as automobile parts. To make a better decision about the performance of these parts, the company needs the measurements of these parts. The decision about the performance depends on the consistency of the operators. The operators working in the company have the instruction to measure the length of the parts. The measurements of these parts cannot be recorded completely; therefore, the measurement observations are neutrosophic. The measurements in mm by three operators are shown in Table 1.

From Table 1, it can be seen that the use of classical statistics may mislead the managers in decision-making. Therefore, the consistency of the operators in measuring will be discussed with the help of the proposed methods. Let α = 1% and $n_N \epsilon [10, 10]$. The application of the proposed method to find NTCV for operator 1 is stated as follows (Tables 2 and 3).

Step 1: arrange X_L and X_U observations of operator 1 in the ascending order as shown in Table 2.

Step 2: trim $k_N = 1$ observations at both ends of arranged data, where $k_N = 10 \times 0.1$. The remaining data are given in Table 3.

Step 3: compute NTA of remaining observations, $R_N = 10 - 2 = 8$.

The neutrosophic trimmed average of values X_L is calculated as

$$\overline{T}_L = \frac{1}{8} \sum_{i=2}^{8} X_{Li} = 62.12. \quad (7)$$

The neutrosophic trimmed average of values X_U is calculated as

$$\overline{T}_U = \frac{1}{8} \sum_{i=2}^{8} X_{Ui} = 62.24. \quad (8)$$

The neutrosophic trimmed average is defined by

$$\overline{X}_{Ni} = 62.12 + 62.24 I_N;\ I_N \epsilon [0, 0.0019]. \quad (9)$$

The neutrosophic trimmed sum of the square is calculated by

$$\sum_{i=2}^{8} \left(X_{Ni} - \overline{X}_{Ni}\right)^2 = [0.0264, 0.1059]. \quad (10)$$

The neutrosophic trimmed standard deviation (NTSD), say s_{NT}, is given by

$$s_{NT} = \sqrt{\frac{1}{8} \sum_{i=2}^{8} \left(X_{Ni} - \overline{X}_{Ni}\right)^2} = [0.0575, 0.1150]. \quad (11)$$

The neutrosophic trimmed coefficient of variation (NTCV) tells about the consistency and computed by

$$CV_{NT} = \frac{[0.0575, 0.1150]}{[62.12, 62.24]} \times 100 = [0.0924, 0.1851]. \quad (12)$$

The values of NTCV for other operators can be calculated in the same way as for operator 1. The neutrosophic descriptive statistics for three operators are shown in Table 4. From the first column of Table 4, it can be seen that, on average in measurement, operator 2 is better than other operators. We also note that the indeterminacy interval of operator 3 is smaller than other operators. Therefore, operator 3 is more consistent in measuring the length of housing clutch parts. Based on this study, it is concluded that the management can make the decision about the product on the basis of measurement recorded by operator 3.

4. Comparative Study

Aslam [25] introduced the neutrosophic coefficient of variation (NCV) under the neutrosophic statistics. In this section, we will discuss the advantages of the proposed NTCV with NCV. Note here that the proposed NTCV reduces to the existing NCV when no observation is trimmed from the data ($\alpha = 0\%$). To show the efficiency of the proposed NTCV over NCV, we will consider the same descriptive neutrosophic statistics of measurement data are presented in the last section. The NCV and NTCV for three operators are shown in Table 5. From column four of Table 5, it can be noted that the values of NTCV from the proposed method are smaller than the existing NCV which indicates that the proposed NTCV is more consistent in measurement as compared to NCV. For example, for the measurement data given by operator 3, the indeterminate interval is from 0.0826 to 0.1695. On the contrary, this interval from the existing NCV is from 0.1029 to 0.2045. From Table 5, it can also be noted that the use of the proposed method increases the efficiency of the values of the coefficient of variation. From this study, it is concluded that the proposed NTCV is smaller than the existing NCV. We conclude that the proposed method is helpful to increase the consistency of measurement. The neutrosophic forms of NCV and NTCV along with the measures of indeterminacy are placed in Table 6. The first values in neutrosophic form denote the determined values under classical statistics and the second

Table 1: The real example data.

Part no.	Operators		
	1	2	3
1	[62.14, 62.26]	[62.09, 62.21]	[62.09, 62.21]
2	[62.13, 62.25]	[62.13, 62.25]	[62.13, 62.25]
3	[62.05, 62.17]	[62.05, 62.17]	[62.04, 62.16]
4	[62.11, 62.23]	[62.11, 62.23]	[62.11, 62.23]
5	[62.19, 62.31]	[62.19, 62.31]	[62.19, 62.31]
6	[62.06, 62.18]	[62.06, 62.18]	[62.06, 62.18]
7	[62.07, 62.19]	[62.08, 62.20]	[62.07, 62.19]
8	[62.14, 62.26]	[62.14, 62.26]	[62.14, 62.26]
9	[62.24, 62.36]	[62.24, 62.36]	[62.23, 62.35]
10	[62.22, 62.34]	[62.22, 62.34]	[62.22, 62.34]

Table 2: Observations of operator 1.

X_L	62.05	62.06	62.07	62.11	62.13	62.14	62.14	62.19	62.22	62.24
X_U	62.17	62.18	62.19	62.23	62.25	62.26	62.26	62.31	62.34	62.36

Table 3: Trimmed observations of operator 1.

X_L	62.06	62.07	62.11	62.13	62.14	62.14	62.19	62.22
X_U	62.18	62.19	62.23	62.25	62.26	62.26	62.31	62.34

Table 4: Neutrosophic descriptive statistics.

Operators	$\overline{X}_{Ni}$	s_{NT}	CV_{NT}	Range
1	$[62.12, 62.24]$	$[0.0575, 0.1150]$	$[0.0924, 0.1851]$	0.0927
2	$[62.12, 62.26]$	$[0.0514, 0.1078]$	$[0.0826, 0.1731]$	0.0905
3	$[62.12, 62.24]$	$[0.0526, 0.1053]$	$[0.0845, 0.1695]$	0.085

Table 5: The comparison in NCV and NTCV.

Operators	NCV	Status	NTCV	Status
1	[0.1003, 0.2011]	Good	$[0.0924, 0.1851]$	Very good
2	[0.1013, 0.5628]	Not acceptable	$[0.0826, 0.1731]$	Very good
3	[0.1020, 0.2045]	Good	$[0.0845, 0.1695]$	Very good

Table 6: The comparison in NCV and NTCV.

Operators	Neutrosophic form of NCV	Neutrosophic form of NTCV
1	$0.1003 + 0.2011\ I_N, I_N \epsilon [0, 0.5012]$	$0.0924 + 0.1851 I_N, I_N \epsilon [0, 0.5]$
2	$0.1013 + 0.5628\ I_N, I_N \epsilon [0, 0.82]$	$0.0826 + 0.1731\ I_N, I_N \epsilon [0, 0.5228]$
3	$0.1020 + 0.2045\ I_N, I_N \epsilon [0, 0.5012]$	$0.0845 + 0.1695 I_N, I_N \epsilon [0, 0.5]$

part is indeterminate parts. For example, in neutrosophic form $0.0826 + 0.1731\ I_N, I_N \epsilon [0, 0.5228]$, the value 0.0826 presents the value of the coefficient of variation (CV) for classical statistics. The value $0.1731\ I_N, I_N \epsilon [0, 0.5228]$, is the indeterminate part with the measure of indeterminacy $(0.1731 - 0.0826)/0.1731 = 0.5228$. We note that the measure of indeterminacy from the existing method is given by [25] is 0.82. From this study, it is concluded that the proposed method is helpful to minimize the measure of indeterminacy. We also compared the results of the proposed study with interval statistics. The interval statistics used intervals in order to capture the data inside the intervals. Therefore, the interval statistics tells the values of NTCV from 0.0826 to 0.173 without giving any information about the measure of indeterminacy. Therefore, it is concluded that the proposed NTCV is more efficient in measure of indeterminacy than the existing CV proposed by [25] and interval statistics.

5. Concluding Remarks

In this paper, the neutrosophic trimmed average, neutrosophic trimmed standard deviation, and neutrosophic trimmed coefficient of variation (NTCV) were introduced. The application of the proposed neutrosophic trimmed descriptive statistics was given with the help of measurement data. The comparisons of the proposed NTCV are compared with the existing neutrosophic coefficient of variation (NCV). From this study, it is can be seen that the proposed NTCV is more efficient than NCV in terms of measures of indeterminacy. In addition, it can be seen that the proposed NTCV reduces the variation in the measurement data. The proposed NTCV can be applied for the decision-making in the industry when the data are obtained from the measurement having the neutrosophy. The other trimmed statistical methods under neutrosophic statistics can be considered as future research.

Data Availability

The data used to support the findings of the study are given within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant no. D-104-130-1441. The author, therefore, gratefully acknowledge the DSR technical and financial support.

References

[1] M. Wu and Y. Zuo, "Trimmed and Winsorized means based on a scaled deviation," *Journal of Statistical Planning and Inference*, vol. 139, no. 2, pp. 350–365, 2009.

[2] A. Alkhazaleh and A. Razali, "New technique to estimate the asymmetric trimming mean," *Journal of Probability and Statistics*, vol. 2010, Article ID 739154, 2010.

[3] Z. M. Yusof, S. Abdullah, S. S. S. Yahaya, and A. R. Othman, "Testing the equality of central tendency measures using various trimming strategies," *African Journal of Mathematics and Computer Science Research*, vol. 4, pp. 32–38, 2011.

[4] M.-D. Wang, J. Liu, G. Molenberghs, and C. Mallinckrodt, "An evaluation of the trimmed mean approach in clinical trials with dropout," *Pharmaceutical Statistics*, vol. 17, no. 3, pp. 278–289, 2018.

[5] G. Lugosi and S. Mendelson, "Mean estimation and regression under heavy-tailed distributions: a survey," *Foundations of Computational Mathematics*, vol. 19, no. 5, pp. 1145–1190, 2019.

[6] R. Oten and R. J. P. De Figueiredo, "Adaptive alpha-trimmed mean filters under deviations from assumed noise model," *IEEE Transactions on Image Processing*, vol. 13, no. 5, pp. 627–639, 2004.

[7] A. Brik, M. Goddi, J. Dhahri, and N. B. Fredj, "Assessing process capability index using sampling plan in the presence of measurement system errors," *The International Journal of Advanced Manufacturing Technology*, vol. 102, no. 9–12, pp. 3301–3313, 2019.

[8] Z. Zhao, D. Ding, Y. Fu, J. Xu, and J. Han, "A hybrid approach for measurement thickness of complex structural parts using ultrasonic inspection and on-machine probing," *The International Journal of Advanced Manufacturing Technology*, vol. 103, no. 9–12, pp. 4777–4785, 2019.

[9] A. García-Pérez, "An approximation to the small sample distribution of the trimmed mean for Gaussian mixture models," in *Strengthening Links Between Data Analysis and Soft Computing*, pp. 115–122, Springer, Berlin, Germany, 2015.

[10] H. Liu, R. Yang, P. Wang, J. Chen, H. Xiang, and G. Chen, "Measurement point selection and compensation of geometric error of NC machine tools," *The International Journal of Advanced Manufacturing Technology*, vol. 108, no. 11-12, pp. 3537–3546, 2020.

[11] S. F. Beckert and W. S. Paim, "Critical analysis of the acceptance criteria used in measurement systems evaluation," *International Journal of Metrology and Quality Engineering*, vol. 8, p. 23, 2017.

[12] A. Hussain, Q. Javaid, and M. Siddique, "Impulse noise removal using fuzzy logic and alpha-trimmed mean," in *Proceedings of the 2011 IEEE International Conference on Signal and Image Processing Applications (ICSIPA)*, pp. 216–220, Kuala Lumpur, Malaysia, November 2011.

[13] T. Veerakumar, S. Esakkirajan, and I. Vennila, "Combined fuzzy logic and unsymmetric trimmed median filter approach for the removal of high density impulse noise," *WSEAS Transactions on Signal Processing*, vol. 8, pp. 32–42, 2012.

[14] S. Vijaya Kumar and C. Nagaraju, "A fast adaptive fuzzy unsymmetric trimmed mean filter for removal of impulse noise from digital images," in *Proceedings of the International Conference on Computing and Communication Systems*, pp. 137–146, Shillong, India, April 2018.

[15] V. V. Vela-Rincón, D. Mújica-Vargas, M. Mejía Lavalle, and A. Magadán Salazar, "Spatial $$\alpha $$-trimmed fuzzy C-means algorithm to image segmentation," in *Proceedings of the Mexican Conference on Pattern Recognition*, pp. 118–128, Morelia, Mexico, June 2020.

[16] M. Asplund and J. Lin, "Evaluating the measurement capability of a wheel profile measurement system by using GR&R," *Measurement*, vol. 92, pp. 19–27, 2016.

[17] K. Mittal, P. C. Tewari, and D. Khanduja, "On the fuzzy evaluation of measurement system analysis in a manufacturing and process industry environment: a comparative study," *Management Science Letters*, vol. 8, pp. 201–216, 2018.

[18] F. Smarandache, *Neutrosophy. Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning*, vol. 105, pp. 118–123, American Research Press, Ann Arbor, MI, USA, 1998.

[19] R. Alhabib and A. Salama, "Using moving averages to pave the neutrosophic time series," *International Journal of Neutrosophic Science*, vol. 3, pp. 14–20, 2020.

[20] S. K. Das and S. Edalatpanah, "A new ranking function of triangular neutrosophic number and its application in integer programming," *International Journal of Neutrosophic Science*, vol. 4, 2020.

[21] P. Schweizer, "Uncertainty: two probabilities for the three states of neutrosophy," *International Journal of Neutrosophic Science*, vol. 2, pp. 18–26, 2020.

[22] F. Smarandache, "Introduction to neutrosophic statistics: infinite study," 2014, https://arxiv.org/ftp/arxiv/papers/1406/1406.2000.

[23] J. Chen, J. Ye, and S. Du, "Scale effect and anisotropy analyzed for neutrosophic numbers of rock joint roughness coefficient based on neutrosophic statistics," *Symmetry*, vol. 9, no. 10, p. 208, 2017.

[24] J. Chen, J. Ye, S. Du, and R. Yong, "Expressions of rock joint roughness coefficient using neutrosophic interval statistical numbers," *Symmetry*, vol. 9, no. 7, p. 123, 2017.

[25] M. Aslam, "A new method to analyze rock joint roughness coefficient based on neutrosophic statistics," *Measurement*, vol. 146, pp. 65–71, 2019.

[26] M. Aslam and R. A. Bantan, "A study on measurement system analysis in the presence of indeterminacy," *Measurement*, vol. 166, Article ID 108201, 2020.

[27] A. Kumar, B. Sah, A. R. Singh et al., "A review of multi criteria decision making (MCDM) towards sustainable renewable energy development," *Renewable and Sustainable Energy Reviews*, vol. 69, pp. 596–609, 2017.

[28] A. Hussain and H.-M. Kim, "Goal-programming-based multi-objective optimization in off-grid microgrids," *Sustainability*, vol. 12, no. 19, p. 8119, 2020.

[29] T. Yang, Y. Kuo, D. Parker, and K. H. Chen, "A multiple attribute group decision making approach for solving problems with the assessment of preference relations," *Mathematical Problems in Engineering*, vol. 2015, Article ID 849897, 2015.

[30] A. Kazemi and V. Hajipour, "A fuzzy approach to increase accuracy and precision in measurement system analysis," *Economic Computation & Economic Cybernetics Studies & Research*, vol. 47, pp. 171–187, 2013.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6631762, 8 pages
https://doi.org/10.1155/2021/6631762

Research Article

A Novel Method for Solving Multiobjective Linear Programming Problems with Triangular Neutrosophic Numbers

Qing Wang[1] **Yi Huang,**[2,3] **Shiming Kong,**[3] **Xinqiang Ma**[2,3] **Youyuan Liu,**[3] **S. K. Das,**[4] **and S. A. Edalatpanah**[5]

[1]*Department of Mathematics and Physics, Suzhou Vocational University, Suzhou 215104, China*
[2]*College of Computer Science and Technology, Guizhou University, Guiyang, China*
[3]*Institute of Intelligent Computing and Visualization Based on Big Data, Chongqing University of Arts and Sciences, Chongqing, China*
[4]*Department of Revenue, Ministry of Finance, Government of India, New Delhi, India*
[5]*Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran*

Correspondence should be addressed to Qing Wang; wqing@jssvc.edu.cn and S. A. Edalatpanah; saedalatpanah@gmail.com

Received 6 November 2020; Revised 17 December 2020; Accepted 21 July 2021; Published 23 September 2021

Academic Editor: M M Bhatti

In the field of operation research, linear programming (LP) is the most utilized apparatus for genuine application in various scales. In our genuine circumstances, the manager/decision-makers (DM) face problems to get the optimal solutions and it even sometimes becomes impossible. To overcome these limitations, neutrosophic set theory is presented, which can handle all types of decision, that is, concur, not certain, and differ, which is common in real-world situations. By thinking about these conditions, in this work, we introduced a method for solving neutrosophic multiobjective LP (NMOLP) problems having triangular neutrosophic numbers. In the literature study, there is no method for solving NMOLP problem. Therefore, here we consider a NMOLP problem with mixed constraints, where the parameters are assumed to be triangular neutrosophic numbers (TNNs). So, we propose a method for solving NMOLP problem with the help of linear membership function. After utilizing membership function, the problem is converted into equivalent crisp LP (CrLP) problem and solved by any suitable method which is readily available. To demonstrate the efficiency and accuracy of the proposed method, we consider one classical MOLP problem and solve it. Finally, we conclude that the proposed approach also helps decision-makers to not only know and optimize the most likely situation but also realize the outcomes in the optimistic and pessimistic business situations, so that decision-makers can prepare and take necessary actions for future uncertainty.

1. Introduction

Linear programming (LP) problem has an important application in various sectors of our daily life. The major drawback faced by manager or decision-makers (DM) in daily-life application is to determine the parameters. Because of several factors in real-life problems, the real-life problems are very complex. Due to uncertainty, the decision-makers cannot always formulate the problem in a well-defined and exact manner, nor can they always precisely predict the outcome of viable decisions. To overcome these uncertainty complex problems, we take more realistic descriptive knowledge of experts, which can be represented as fuzzy data. Firstly, the basic concept of fuzzy set theory was proposed by Zadeh [1]. Further, the basic concept of fuzzy decision-making was proposed by Bellman and Zadeh [2]. Thus, linear programming (LP) problem with fuzzy environment would be very effective in solving real-life problems. If the parameters of LP problem are considered as fuzzy, then it is called fuzzy linear programming (FLP) problem. The concepts of the feasible solution and αefficient solutions of the FLP problem were proposed by Ramik [3]. Maleki et al. introduced the idea of using the ranking function for solving the FLP problem [4]. The concept of sensitivity analysis for solving the FLP problem was proposed by Ebrahimnejad [5]. A trapezoidal fuzzy number was considered by Wan and Dong [6] for solving LP problems using multiobjective programming and membership function. Ganesan and Veeramani [7] also considered a new fuzzy symmetric trapezoidal fuzzy number and solved it. Another type of problem was considered by Lotfi et al. [8], where all the parameters, variables, and constraints are chosen as fully fuzzy LP (FFLP) problem and solved by lexicographic method. Kumar et al. [9] also proposed a method for solving FFLP problem with equality constraints by using ranking function. Najafi and Edalatpanah [10] have suggested some modifications of paper [9]. Many researchers [11–13] considered the lexicographic technique to apply in various problems like FFLP problem with triangular numbers and FFLP problem with trapezoidal fuzzy numbers.

After successful application of fuzzy sets in real-life application, decision-makers (DM) want a more realistic approach to handle the uncertainty in real-world problems. Thus, Atanassov [14] proposed the concept of a new set which is combined with both membership functions and nonmembership functions and the set was called intuitionistic fuzzy set (IFS). IFS is an extension version of fuzzy set. Singh and Yadav [15] proposed intuitionistic fuzzy multiobjective linear programming problem with various membership functions. Singh and Yadav [16] proposed transportation problem with intuitionistic fuzzy type-2 problem. Some of researchers [17–22] focused on solving multiobjective LP (MOLP) problem and LP problems with intuitionistic fuzzy numbers. Till now, several works have been pioneered in both FS and IFS. Afterwards, Smarandache [23] introduced the structure of neutrosophic set (NS) for developing the solution of any kind of real-world problem in a reasonable way. After Smarandache, Wang et al. [24] disclosed the establishment of single-typed neutrosophic set, which demands a crucial position in NS theory. The notion of single-valued neutrosophic set was more suitable for solving many real-life problems like image processing, medical diagnosis, decision-making, water resource management, and supply chain management. To reflect the decision-making information in an objective way, the triangular neutrosophic numbers (TNN) can be used in real-life problems to express the attribute value. This can not only maintain the variables but also highlight the possibility of various values within this interval. Of late, Abdel-Basset et al. [25] solved LP problems under neutrosophic triangular numbers by using ranking functions. An integer programming problem with triangular neutrosophic numbers was developed by Das and Edalatpanah [26]. For the first time in neutrosophic sets, Das and Chakraborty [27] proposed a model for solving LP problem in pentagonal numbers. Ye et al. [28] introduced the idea of finding the optimal solution of the LP problem in NNs environment. Das [29] also used pentagonal neutrosophic numbers in transportation problem. Pythagorean fuzzy numbers also can handle the uncertainly problem. Wang and Li [30] proposed a Pythagorean fuzzy number in decision-making problem; for more details about the applications of fuzzy extension sets, see [31–34].

Our contribution, motivation, and novelties are as follows.

1.1. Contribution. In this article, we mainly focused on neutrosophic multiobjective linear programming (NMOLP) problem with mixed constraints under triangular neutrosophic numbers. Several factors are also involved in our day-to-day life; therefore, DM choose the neutrosophic numbers for better results. In neutrosophic numbers, DM always choose any membership function as per the problem. Some basic operational laws of triangular neutrosophic numbers are demonstrated to enhance the pertinence of our proposed theory. With the progression of the study, a newly conceptualized ranking function is established under triangular neutrosophic number background. Utilizing this constructive tool, the NMOLP problem is transformed into crisp MOLP problem. Notably, the well-known various membership functions are used for conversion into an equivalent crisp convex programming problem.

1.2. Motivation. Neutrosophic sets play an important role in uncertainty modelling. The development of uncertainty theory plays a fundamental role in formulation of real-life scientific mathematical model, structural modelling in engineering field, medical diagnosis problem, and so forth. How can we solve multiobjective linear programming based triangular neutrosophic numbers? Is it possible to apply in real-life problem? Still there is no method for application in multiobjective linear programming problem having triangular neutrosophic numbers. From this aspect, we try to extend this research paper.

1.3. Novelties. A linear membership function is usually very comfortable in real-life situations. It is defined by two points, that is, the upper levels and lower levels of acceptability. A numerical problem related real mathematical problem is set forth to validate our anticipated hypothesis. Lastly, the comparison work involving the ranking system of the alternatives uplifted the superiority of our proposed supposition. To the best of our knowledge, no method is available for solving NMOLP problem. Therefore, we attempt to establish a new strategy to solve this problem.

The rest of the paper is organized as follows: Some basic definitions and preliminaries are presented in Section 2. In Section 3, the classical MOLP problem and membership functions are presented. The proposed method is discussed in Section 4. In Section 5, we present a numerical example, and a real-life problem is discussed. The analysis of the result is also discussed is Section 6. Finally, the conclusion is discussed in Section 7.

2. Preliminaries

In this segment, we establish some fundamental mathematical operations and definition which is required throughout the paper.

Definition 1 (see [31]). Consider that $\widetilde{V}$ in all-inclusive discourse X, which is meant conventionally by x, is supposed to be a single-valued neutrosophic (SVN) set if $\widetilde{V} = \{\langle x\colon [\alpha^p(x), \alpha^q(x), \alpha^r(x)]\rangle\colon x \in X\}$. The set is described by a reality enrollment work, level of certainty: $\alpha^p(x)\colon X \longrightarrow [0,1]$; an indeterminacy enrollment work, level of vulnerability: $\alpha^q(x)\colon X \longrightarrow [0,1]$; and a false enrollment work, level of falsity: $\alpha^r(x)\colon X \longrightarrow [0,1]$. Also, a SVN set satisfies the condition $0 \le \alpha^p(x) + \alpha^q(x) + \alpha^r(x) \le 3$.

Definition 2 (see [30]). A triangular neutrosophic number (TNN) is denoted by $\widetilde{V} = \langle (p^l, p^m p^n), (\alpha^p, \alpha^q, \alpha^r)\rangle$ whose three membership functions for the truth, indeterminacy, and falsity of x can be defined as follows:

$$\alpha^p(x) = \begin{cases} \dfrac{(x - p^l)}{(p^m - p^n)}\alpha^p, & p^l \le x < p^m, \\ \alpha^p, & x = p^l, \\ \dfrac{(p^n - x)}{(p^n - p^m)}\alpha^p, & p^m \le x < p^n, \\ 0, & \text{otherwise}, \end{cases}$$

$$\alpha^q(x) = \begin{cases} \dfrac{(p^m - x)}{(p^m - p^n)}\alpha^q, & p^l \le x < p^m, \\ \alpha^q & x = p^m, \\ \dfrac{(x - p^n)}{(p^n - p^m)}\alpha^q, & p^m \le x < p^n, \\ 1, & \text{otherwise}, \end{cases} \tag{1}$$

$$\alpha^r(x) = \begin{cases} \dfrac{(p^l - x)}{(p^m - p^n)}\alpha^r, & p^l \le x < p^m, \\ \alpha^r, & x = p^m, \\ \dfrac{(x - p^n)}{(p^n - p^m)}\alpha^r, & p^m \le x < p^n, \\ 1 & \text{otherwise}, \end{cases}$$

where $0 \le \alpha^p(x) + \alpha^q(x) + \alpha^r(x) \le 3$, $x \in R$. Additionally, when $p^l \ge 0$, $R$is called a nonnegative TNN. Similarly, when $p^l < 0$,R becomes a negative TNN.

Definition 3 (see[30]). Let$A_1 = \langle (p^l, p^m, p^n), (\alpha^p, \alpha^q, \alpha^r)\rangle$ and $A_2 = \langle (q^l, q^m, q^n), (\beta^p, \beta^q, \beta^r)\rangle$be two TNNs. Then the arithmetic relations are defined as follows:

(i) $A_1 \oplus A_2 = \langle (p^l + q^l, p^m + q^m, p^n + p^n), (\alpha^p \wedge \beta^p, \alpha^q \wedge \beta^q, \alpha^r \wedge \beta^r)\rangle$.

(ii) $A_1 - A_2 = \langle (p^l - q^n, p^m - q^m, p^n - q^l), (\alpha^p \wedge \beta^p, \alpha^q \wedge \beta^q, \alpha^r \wedge \beta^r)\rangle$.

(iii) $A_1 \otimes A_2 = \langle (p^l q^l, p^m q^m, p^n q^n), (\alpha^p \wedge \beta^p, \alpha^q \wedge \beta^q, \alpha^r \wedge \beta^r)\rangle$, if $p^l > 0$, $q^l > 0$, $\langle (p^l q^n, p^m q^m, p^n q^l), (\alpha^p \wedge \beta^p, \alpha^q \wedge \beta^q, \alpha^r \wedge \beta^r)\rangle$, if $p^l < 0$, $q^l > 0$, $\langle (p^n q^n, p^m q^m, p^l q^l), (\alpha^p \wedge \beta^p, \alpha^q \wedge \beta^q, \alpha^r \wedge \beta^r)\rangle$, if $p^l < 0$, $q^l < 0$.

(iv) $\lambda A_1 = \{ \langle (\lambda p^l, \lambda p^m, \lambda p^n), (\alpha^p, \alpha^q, \alpha^r)\rangle$, if $\lambda > 0$, $\langle (\lambda p^n, \lambda p^m, \lambda p^l), (\alpha^p, \alpha^q, \alpha^r)\rangle$, if $\lambda < 0$.

(v) $A_1 / A_2 = \{ \langle (p^l/q^n, p^m/q^m, p^n/q^l); \alpha^p \wedge \beta^p, \alpha^q \wedge \beta^q, \alpha^r \wedge \beta^r\rangle (p^n > 0, q^n > 0)$, $\langle (p^n/q^n, p^m/q^m, p^l/q^l)$; $\alpha^p \wedge \beta^p, \alpha^q \wedge \beta^q, \alpha^r \wedge \beta^r\rangle (p^n < 0, q^n > 0)$, $\langle (p^n/q^l, p^m/q^m, p^l/q^n)$;
$\alpha^p \wedge \beta^p, \alpha^q \wedge \beta^q, \alpha^r \wedge \beta^r\rangle (p^n < 0, q^n < 0)$.

Definition 4 (see [30]). Ranking neutrosophic numbers is consistently assumed as a fundamental function in phonetic dynamic and some other neutrosophic application frameworks, which has been concentrated by numerous mathematicians. Separation measure between two neutrosophic numbers is firmly identified with the idea of neutrosophic numbers which is closely related to the concept of ranking neutrosophic numbers. Let $A_1 = (p^l, p^m, p^n); (\alpha^p, \alpha^q, \alpha^r)$ be a TNN. The ranking function for triangular neutrosophic number A_1 is denoted by $\mathfrak{R}(A_1)$ and defined by $\mathfrak{R}(A_1) = (p^l + p^m + p^n/9)(\alpha^p + (1 - \alpha^q) + (1 - \alpha^r))$.

Definition 5 (see [29]). Let A_1 and A_2 be two TNNs. Let $A_1 = \langle (p^l, p^m, p^n), (\alpha^p, \alpha^q, \alpha^r)\rangle$ and$A_2 = \langle (q^l, q^m, q^n), (\beta^p, \beta^q, \beta^r)\rangle$ be two TNNs. Then, we have the following:

(i) $\mathfrak{R}(A_1) \le \mathfrak{R}(A_2) \Longleftrightarrow A_1 \le A_2$

(ii) $\mathfrak{R}(A_1) \ge \mathfrak{R}(A_2) \Longleftrightarrow A_1 \ge A_2$

(iii) $\mathfrak{R}(A_1) = \mathfrak{R}(A_2) \Longleftrightarrow A_1 = A_2$

(iv) $\min(A_1, A_2) = A_1$, if $A_1 \le A_2$ or $A_2 \ge A_1$

Definition 6. Let $A_1 = \langle (p^l, p^m, p^n), (\alpha^p, \alpha^q, \alpha^r)\rangle$ and $A_2 = \langle (q^l, q^m, q^n), (\beta^p, \beta^q, \beta^r)\rangle$be two TNNs; then $A_1 = A_2$ if and only if $p^l = q^l$, $p^m = q^m$, $p^r = q^r$, $\alpha^p = \beta^p$, $\alpha^q = \beta^q$, and $\alpha^r = \beta^r$.

3. Classical Problem of MOLP

The popular multiobjective linear programming (MOLP) issue with blended requirements is portrayed by

$$\begin{aligned}
&\text{Min}\, Z = \left[Z^1, Z^2, \ldots, Z^L\right]\\
&\text{s.t.}\quad \begin{cases}\sum_{h=1}^{o} b_{gh} y_h \geq d_g,\ g = 1, 2, \ldots, n_1,\\ \sum_{h=1}^{o} b_{gh} y_h \leq d_g,\ g = 1, 2, \ldots, n_2,\\ \sum_{h=1}^{o} b_{gh} y_h = d_g,\ g = 1, 2, \ldots, n_3\\ y_h \geq 0,\ h = 1, 2, \ldots\ldots, o.\end{cases}
\end{aligned}\tag{2}$$

Definition 7. Let T_q be the doable district for (2). A point $\overline{y}$ is supposed to be productive or pareto ideal arrangement solution of (2) if there does not survive any $y \in T_q$ with end goal that$Z_w(\overline{y}) \geq Z_w(y) \forall w$ and $Z_w(\overline{y}) \geq Z_w(y)$ for any w.

Definition 8. A point $\widetilde{y} \in T_q$ is supposed to be feeble pareto ideal arrangement of (2) if there does not survive any $y \in T_q$ with end goal $Z_w(\widetilde{y}) \geq Z_w(y)\, \forall w$, where $w = 1, 2, \ldots\ldots, L$.

3.1. Membership Functions. There are different techniques to take care of a MOLP issue. These strategies are arranged into two general classes: scalarization techniques and non-scalarization techniques. These methodologies convert the MOLP issue into a solitary target programming problem. In the above literature study, we found that two types of membership functions are used for solving MOLP problem. Linear membership function is an emerging technique to solve fuzzy linear programming problem. Linear function is based on two points only, that is, upper level and lower level of acceptability of the decision variable. In uncertainty circumstances, this type of function is not fixed for all conditions. Therefore, here we considered both linear and nonlinear membership functions.

3.1.1. Linear Membership Functions. A linear membership function μ_{LM} can be defined as follows:

$$\mu_{\text{LM}}(Z_D(x)) = \begin{cases} 1, & \text{if } Z_D \leq L_D,\\ \dfrac{U_D - Z_D}{U_D - L_D}, & \text{if } L_D \leq Z_D < U_D,\\ 0, & \text{if } Z_D \geq U_D.\end{cases}\tag{3}$$

3.1.2. Parabolic Membership Function. The parabolic membership function μ_{PM} can be defined as follows:

$$\mu_{\text{PM}}(Z_D(x)) = \begin{cases} 1, & \text{if } Z_D \leq L_D,\\ 1 - \dfrac{(Z_D - L_D)^2}{(U_D - L_D)^2}, & \text{if } L_D \leq Z_D < U_D,\\ 0, & \text{if } Z_D \geq U_D.\end{cases}\tag{4}$$

Corollary 1 (see [17]). *The following sets where $\lambda \geq 0$ are convex sets:*

$$\begin{aligned}
&\left\{X\colon \mu_L\left(Z_p'(X)\right) \geq \lambda\right\},\\
&\left\{X\colon \mu_H\left(Z_p'(X)\right) \geq \lambda, Z_p'(X) \leq \frac{1}{2}\left(U_p + L_p\right)\right\},\\
&\left\{X\colon \mu_p\left(Z_p'(X)\right) \geq \lambda\right\}.
\end{aligned}\tag{5}$$

Proof. The proof is straightforward. □

4. Proposed Method

In this section, by using a new ranking function, we suggest a new method for solving NMOLP problems. The main work will be presented as follows:

Step 1: consider problem (2) of classical MOLP problem.

Step 2: on the off chance that the coefficients of the goal capacities, choice factors, and right-hand sides of requirements are dubious, which are spoken with by three-sided neutrosophic numbers, at that point problem (2) becomes NMOLP problem and the problem might be composed as

$$\begin{aligned}
&\text{Min}\, \widetilde{Z}_N = \left[\widetilde{Z}_N^1, \widetilde{Z}_N^2, \ldots, \widetilde{Z}_N^L\right]\\
&\text{s.t.}\quad \begin{cases}\sum_{h=1}^{o} \widetilde{b}_{gh}^N y_j \geq \widetilde{d}_g^N,\ g = 1, 2, \ldots\ldots, n_1,\\ \sum_{h=1}^{o} \widetilde{b}_{gh}^N y_h \leq \widetilde{d}_g^N,\ g = 1, 2, \ldots\ldots, n_2,\\ \sum_{h=1}^{o} \widetilde{b}_{gh}^N y_h = \widetilde{d}_g^N,\ g = 1, 2, \ldots\ldots, n_3,\\ y_h \geq 0,\ h = 1, 2, \ldots\ldots, o.\end{cases}
\end{aligned}\tag{6}$$

Step 3: utilizing the ranking function which is linear, problem (7) is changed to the accompanying crisp MOLP problem.

$$\begin{aligned}
&\text{Min } \widetilde{Z} = \left[\widetilde{Z}^1, \widetilde{Z}^2, \ldots, \widetilde{Z}^L\right] \\
&\text{s.t.} \quad
\begin{cases}
\sum_{h=1}^{o} \widetilde{b}_{gh} y_h \geq \widetilde{d}_g, \; g = 1, 2, \ldots\ldots, n_1, \\
\sum_{h=1}^{o} \widetilde{b}_{gh} y_h \leq \widetilde{d}_g, \; g = 1, 2, \ldots\ldots, n_2, \\
\sum_{h=1}^{o} \widetilde{b}_{gh} y_h = \widetilde{d}_g, \; g = 1, 2, \ldots\ldots, n_3, \\
y_h \geq 0, \; h = 1, 2, \ldots\ldots, o.
\end{cases}
\end{aligned} \tag{7}$$

Step 4: determine the goal programming as follows:

$$\begin{aligned}
&\text{Find } \{y_h, \; h = 1, 2, \ldots, o\} \\
&\text{s.t.} \quad
\begin{cases}
\widetilde{Z}_D \approx L_D, \; D = 1, 2, \ldots, L, \\
\sum_{h=1}^{o} \widetilde{b}_{gh} y_h \geq \widetilde{d}_g, g = 1, 2, \ldots, n_1, \\
\sum_{h=1}^{o} \widetilde{b}_{gh} y_h \leq \widetilde{d}_g, \; g = 1, 2, \ldots, n_2, \\
\sum_{h=1}^{o} \widetilde{b}_{gh} y_h = \widetilde{d}_g, \; g = 1, 2, \ldots, n_3, \\
y_h \geq 0, \; h = 1, 2, \ldots\ldots, o.
\end{cases}
\end{aligned} \tag{8}$$

In problem (8), the symbol " $\approx$ " is used to denote that some deviation ought to be permitted while exacting objective. To change the model in (8) into a crisp programming model, we already discussed the above linear and nonlinear membership functions.

Step 5: use appropriate enrollment works and change the GP model into crisp programming model.

Step 6: solve the crisp programming problem using any suitable technique or LINGO or MATLAB.

Theorem 1. *An effective solution for (7) is a proficient solution for (6).*

Proof. The proof is straightforward.

So, from the above theorem, NMOLP problem (6) is equal to settling crisp model (7). □

5. Numerical Experiments

In this section, some numerical examples are given below to illustrate the new model.

Example 1. Let us consider the following NMOLP problem:

$$\begin{aligned}
&\text{Min } \widetilde{Z}_N^1 = \widetilde{5}_N y_1 + \widetilde{3}_N y_2 \\
&\text{Min } \widetilde{Z}_N^2 = \widetilde{2}_N y_1 + \widetilde{7}_N y_2 \\
&\text{s.t.} \quad \widetilde{2}^N y_1 + \widetilde{4}^N y_2 \geq 2\widetilde{5}^N, \widetilde{4}^N y_1 + \widetilde{5}^N y_2 \leq \widetilde{50}^N, \widetilde{1}^N y_1 \\
&\quad + \widetilde{1}^N y_2 \geq 1\widetilde{0}^N, y_1, y_2 \geq 0,
\end{aligned} \tag{9}$$

where the parameters are as follows:

$$\begin{aligned}
\widetilde{5} &= (4, 5, 6; 0.8, 0.6, 0.4), \\
\widetilde{3} &= (2.5, 3, 4; 0.75, 0.5, 0.3), \\
\widetilde{2} &= (2, 2, 3; 1, 0.5, 0), \\
\widetilde{7} &= (7, 7, 7.5; 0.8, 0.6, 0.4), \\
\widetilde{2} &= (1.5, 2, 2; 0.9, 0.6, 0.2), \\
\widetilde{4} &= (3, 4, 4; 1, 0.5, 0), \\
\widetilde{1} &= (0.5, 1, 1; 1, 0.2, 0.2), \\
\widetilde{1} &= (1, 1, 1; 1, 0, 0.5), \\
2\widetilde{5} &= (22, 25, 25; 0.8, 0.6, 0.4), \\
1\widetilde{0} &= (9, 10, 10; 1, 0.5, 0), \\
5\widetilde{0} &= (50, 50, 55; 0.75, 0.5, 0.3).
\end{aligned} \tag{10}$$

As Step 3, we use ranking function of Definition 5; the above problem is equivalent to the following crisp MOLP problem:

$$\begin{aligned}
&\text{Min } \widetilde{Z}^1 = 3.001 y_1 + 2.058 y_2 \\
&\text{Min } \widetilde{Z}^2 = 1.94 y_1 + 4.3 y_2 \\
&\text{s.t.} \quad
\begin{cases}
1.283 y_1 + 3.05 y_2 \geq 14.4, \\
3.05 y_1 + 3.001 y_2 \leq 30.14, \\
0.723 y_1 + 0.86 y_2 \geq 8.05, \\
y_1, y_2 \geq 0.
\end{cases}
\end{aligned} \tag{11}$$

Solving problem (11) as per Step 4, we have the following solutions:

$$\begin{aligned}
y_1 &= (0, 8.721), \\
y_2 &= (11.53, 0.08), \\
L_1 &= 30.06, \\
U_1 &= 63.29, \\
L_2 &= 31.26, \\
U_2 &= 51.14,
\end{aligned} \tag{12}$$

where L_i, U_i are deviation points of membership functions. Now, applying Step 5, problem (11) is equal to the accompanying GP model as follows:

$$\begin{aligned}
&\text{Find } \{y_h : h = 1, 2\} \\
&\text{s.t.} \quad
\begin{cases}
3.001 y_1 + 2.058 y_2 \approx 30.06, \\
1.94 y_1 + 4.3 y_2 \approx 31.26, \\
1.283 y_1 + 3.05 y_2 \geq 14.4, \\
3.05 y_1 + 3.001 y_2 \leq 30.14, \\
0.723 y_1 + 0.86 y_2 \geq 8.05, \\
y_1, y_2 \geq 0.
\end{cases}
\end{aligned} \tag{13}$$

Applying the membership functions and solving by LINGO 18.0, the solution of (13) is reported in Table 1.

Table 1: Result discussion of Example 2.

Membership functions	Solutions	Objective values
Linear	$y_1 = 4.24$, $y_2 = 3.97$	(20.89, 25.29; 0.8, 0.6, 0.4)
Parabolic	$y_1 = 5.32$, $y_2 = 4.07$	(24.14, 27.82; 0.8, 0.4, 0.2)

Example 2. Let us consider the following NMOLP problem:

$$\begin{aligned} &\operatorname{Max} \widetilde{Z}_N^1 = \widetilde{4}_N y_1 + \widetilde{10}_N y_2 \\ &\operatorname{Max} \widetilde{Z}_N^2 = \widetilde{2}_N y_1 + \widetilde{5}_N y_2 \\ &\quad \widetilde{2}^N y_1 + y_2 \leq \widetilde{5}^N, \\ &\text{s.t.} \quad \widetilde{2}^N y_1 + \widetilde{5}^N y_2 \leq \widetilde{10}^N, \\ &\quad y_1, y_2 \geq 0. \end{aligned} \tag{14}$$

Problem (14) can be modelled as a multiobjective neutrosophic linear programming problem with single-valued triangular neutrosophic numbers.

$$\begin{aligned} &\operatorname{Max} \widetilde{Z}_N^1 = (3,4,5;0.5,0.5,0.6)y_1 + (9,10,11;0.5,0.7,0.4)y_2 \\ &\operatorname{Max} \widetilde{Z}_N^2 = (1,2,3;0.5,0.5,0.5)y_1 + (4,5,6;0.5,0.7,0.4)y_2 \\ &\quad (1,2,3;0.5,0.4,0.8)y_1 + (1,1,1;0.5,0.3,0.5)y_2 \leq (4,5,6;0.5,0.6,0.5)^N, \\ &\text{s.t.} \quad (1,2,3;0.5,0.4,0.8)y_1 + (4,5,6;0.5,0.7,0.4)y_2 \leq (9,10,11;0.5,0.7,0.4)^N, \\ &\quad y_1, y_2 \geq 0. \end{aligned} \tag{15}$$

By utilizing our Step 3 to Step 6, the optimal solution of the above problem is reported in Table 2.

5.1. Real-Life Application: Diet Problem. In this section, to show the application of the proposed method, the real-life problem is solved by the proposed method, and it is concluded that the proposed method can be applied in any real-life problem. For a very simple diet problem in which the nutrients are starch and protein as a group, the two types of foods with data are given in Table 3.

The activities and their levels in the model are given as follows: activity j: to include 1 kg of food type j in the diet, associated level y_j, for $j = 1, 2$. The various nutrients in the model lead to different constraints. For example, the amount of starch contained in the diet is $5y_1 + 2y_2$, which must be ≥ 5 for feasibility. Similarly, $y_1 + 2y_2 \geq 6$. In this diet problem, the total cost of food and the procurement cost of food should be minimized. Since the cost coefficients and all other coefficients are indecisive and also contain the indeterminacy part, the problem is modelled as a bilevel multiobjective linear programming problem.

The formulation of the above problem is given as follows:

$$\begin{aligned} &\operatorname{Min} \widetilde{Z}_N^1 = (2,3,4,;0.6,0.5,0.5)y_1 + (1,1,1;0.5,0.7,0.5)y_2 \\ &\operatorname{Min} \widetilde{Z}_N^2 = (1,2,3;0.6,0.5,0.5)y_1 + (2,3,4;0.6,0.5,0.5)y_2 \\ &\quad (4,5,6;0.6,0.5,0.5)y_1 + (1,2,3;0.5,0.7,0.5)y_2 \geq (4,5,6;0.6,0.5,0.5)^N, \\ &\text{s.t.} \quad (1,1,1;0.5,0.7,0.5)y_1 + (1,2,3;0.5,0.7,0.5)y_2 \geq (5,6,7;0.5,0.6,0.5)^N, \\ &\quad y_1, y_2 \geq 0. \end{aligned} \tag{16}$$

Table 2: Result discussion of Example 2.

Membership functions	Solutions	Objective values
Linear	$y_1 = 1.87, y_2 = 1.25$	(18.26, 19.98, 25.43; 0.5, 0.7, 0.6)
Parabolic	$y_1 = 1.58, y_2 = 1.52$	(5.4, 8.5, 13; 0.5, 0.7, 0.6)

Table 3: Data considered by the manager.

Nutrients	Nutrient units/kg of food type		
	Food 1	Food 2	Minimum requirements
Starch	5	1	5
Protein	2	2	6
Cost/kg	3	1	
Procurement cost/kg	2	3	

Table 4: Results of diet problem.

Membership functions	Solutions	Objective values
Linear	$y_1 = 0.5, y_2 = 2.5$	(3.5, 4, 4.5; 0.6, 0.5, 0.5)
Parabolic	$y_1 = 3, y_2 = 0$	(3, 6, 9; 0.5, 0.7, 0.5)

Table 5: Comparison of the proposed method with the existing method [17].

Membership functions	Proposed method	Existing method [17]
Linear	4	4.8
Parabolic	6	7

Using our method, the above problem is solved using our Step 3 to Step 5, and the above optimal solution is obtained in Table 4.

6. Result Analysis

In the above literature study, we found that there is no method for solving multiobjective linear programming problem in neutrosophic environment. Therefore, for rationality and effectiveness of the proposed method, we consider another uncertainty problem, that is, intuitionistic fuzzy numbers. Singh and Yadav [17] considered the same problem and solved it with various membership functions. Here, we consider the diet problem for comparison of our proposed method with the existing method.

From Table 5, we get that our result is better than the existing results. Since the object of the problem is minimization, so based on this point of view, our results are better than the existing approach under both linear and parabolic membership functions. Therefore, we can conclude that our proposed algorithm is a new way to handle the uncertainty in real-life problems.

7. Conclusions

In this paper, we develop a new method for solving neutrosophic multiobjective LP (NMOLP) problems, and the model is transformed into a MOLP problem by using ranking function. After successful application in ranking function, we use scalarization technique to convert the goal programming (GP) problem. We also investigated various membership functions to solve the GP model. As per our discussion, the DM choose the membership functions independently which fit the model. From the obtained results, we conclude that the nonlinear membership functions, that is, parabolic functions, are always better than linear membership functions (parabolic > linear). To the best of our knowledge, there is no method in literature for solving NMOLP problem by using membership functions. We also used our proposed method to demonstrate a numerical example. Our proposed method is a new way in neutrosophic environment to handle multiobjective programming problem. There are various scopes in the future to develop our algorithm, like application in real-life problem from industrial sector, transportation problem, and assignment problem.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

The authors contributed equally to writing this article. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

This research was funded by the Key Industrial Technology Development Project of Chongqing Development and Reform Commission, China (Grant no. 2018148208), Key Technological Innovation and Application Development Project of Chongqing, China (Grant no. cstc2019jscx-fxydX0094), Innovation and Entrepreneurship Demonstration Team of Yingcai Program of Chongqing, China (Grant no. CQYC201903167), and Science and Technology Innovation Project of Yongchuan District (Grant no. Ycstc, 2020cc0501). The authors express their gratitude for the support.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] R. E. Bellman and L. A. Zadeh, "Decision-making in a fuzzy environment," *Management Science*, vol. 17, no. 4, 141 pages, 1970.

[3] J. Ramík, "Duality in fuzzy linear programming: some new concepts and results," *Fuzzy Optimization and Decision Making*, vol. 4, no. 1, pp. 25–39, 2005.

[4] H. R. Maleki, M. Tata, and M. Mashinchi, "Linear programming with fuzzy variables," *Fuzzy Sets and Systems*, vol. 109, no. 1, pp. 21–33, 2000.

[5] A. Ebrahimnejad and M. Tavana, "A novel method for solving linear programming problems with symmetric trapezoidal fuzzy numbers," *Applied Mathematical Modelling*, vol. 38, no. 17-18, pp. 4388–4395, 2014.

[6] S. P. Wan and J. Y. Dong, "Possibility linear programming with trapezoidal fuzzy numbers," *Applied Mathematical Modelling*, vol. 38, no. 5-6, pp. 1660–1672, 2014.

[7] K. Ganesan and P. Veeramani, "Fuzzy linear programs with trapezoidal fuzzy numbers," *Annals of Operations Research*, vol. 143, no. 1, pp. 305–315, 2006.

[8] F. Hosseinzadeh Lotfi, T. Allahviranloo, M. Alimardani Jondabeh, and L. Alizadeh, "Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution," *Applied Mathematical Modelling*, vol. 33, no. 7, pp. 3151–3156, 2009.

[9] A. Kumar, J. Kaur, and P. Singh, "A new method for solving fully fuzzy linear programming problems," *Applied Mathematical Modelling*, vol. 35, no. 2, pp. 817–823, 2011.

[10] H. S. Najafi and S. A. Edalatpanah, "A note on "A new method for solving fully fuzzy linear programming problems"" *Applied Mathematical Modelling*, vol. 37, no. 14-15, pp. 7865–7867, 2013.

[11] M. Dehghan, B. Hashemi, and M. Ghatee, "Computational methods for solving fully fuzzy linear systems," *Applied Mathematics and Computation*, vol. 179, no. 1, pp. 328–343, 2006.

[12] R. Ezzati, E. Khorram, and R. Enayati, "A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem," *Applied Mathematical Modelling*, vol. 39, no. 12, pp. 3183–3193, 2015.

[13] B. Pérez-Cañedo and E. R. Concepción-Morales, "A method to find the unique optimal fuzzy value of fully fuzzy linear programming problems with inequality constraints having unrestricted L-R fuzzy parameters and decision variables," *Expert Systems with Applications*, vol. 123, pp. 256–269, 2019.

[14] K. T. Atanssov, "Intuitionistic fuzzy set," *Fuzzy Sets and Systems*, vol. 20, pp. 87–96, 1986.

[15] S. K. Singh and S. P. Yadav, "Intuitionistic fuzzy multi-objective linear programming problem with various membership functions," *Annals of Operations Research*, vol. 269, no. 1-2, pp. 693–707, 2018.

[16] S. K. Singh and S. P. Yadav, "A new approach for solving intuitionistic fuzzy transportation problem of type-2," *Annals of Operations Research*, vol. 243, no. 1-2, pp. 349–363, 2016.

[17] B. Jana and T. K. Roy, "Multi-objective intuitionistic fuzzy linear programming and its application in transportation model," *Notes on Intuitionistic Fuzzy Sets*, vol. 13, no. 1, pp. 34–51, 2007.

[18] B. Jana and T. K. Roy, "Multi-objective fuzzy linear programming and its application in transportation model," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 21, no. 2, pp. 243–269, 2005.

[19] R. Parvathi and C. Malathi, "Intuitionistic fuzzy linear programming problems," *World Applied Sciences Journal*, vol. 17, no. 12, pp. 1802–1807, 2012.

[20] D. Dubey and A. Mehra, "Linear programming with triangular intuitionistic fuzzy number," in *Proceedings of the 7th Conference of the European Society for Fuzzy Logic and Technology*, pp. 563–569, Atlantis Press, August 2011.

[21] S. K.Bharati and S. R. Singh, "A note on solving a fully intuitionistic fuzzy linear programming problem based on sign distance," *International Journal of Computer Applications*, vol. 119, no. 23, pp. 30–35, 2015.

[22] S. K. Bharati and S. R. Singh, "Solving multi objective linear programming problems using intuitionistic fuzzy optimization method: a comparative study," *International Journal of Modeling and Optimization*, vol. 4, no. 1, pp. 10–16, 2014.

[23] F. Smarandache, "Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis," 1998.

[24] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, "Single valued neutrosophic sets," *Multispace and Multistruct*, vol. 4, pp. 410–413, 2010.

[25] M. Abdel-Basset, M. Gunasekaran, M. Mohamed, and F. Smarandache, "A novel method for solving the fully neutrosophic linear programming problems," *Neural Computing and Applications*, vol. 31, no. 5, pp. 1595–1605, 2019.

[26] S. K. Das and S. A. Edalatpanah, "A new ranking function of triangular neutrosophic number and its application in integer programming," *International Journal of Neutrosophic Science*, vol. 4, no. 2, pp. 82–92, 2020.

[27] S. K. Das and A. Chakraborty, "A new approach to evaluate linear programming problem in pentagonal neutrosophic environment," *Complex & Intelligent Systems*, vol. 7, no. 1, pp. 101–110, 2020.

[28] J. Ye, "Neutrosophic number linear programming method and its application under neutrosophic number environments," *Soft Computing*, vol. 22, no. 14, pp. 4639–4646, 2018.

[29] S. Kumar Das, "Application of transportation problem under pentagonal Neutrosophic environment," *Journal of Fuzzy Extension and Applications*, vol. 1, no. 1, pp. 27–41, 2020.

[30] L. Wang and N. Li, "Pythagorean fuzzy interaction power Bonferroni mean aggregation operators in multiple attribute decision making," *International Journal of Intelligent Systems*, vol. 35, no. 1, pp. 150–183, 2020.

[31] İ. Deli, "A novel defuzzification method of SV-trapezoidal neutrosophic numbers and multi-attribute decision making: a comparative analysis," *Soft Computing*, vol. 23, no. 23, pp. 12529–12545, 2019.

[32] A. Saha, I. Deli, and S. Broumi, "Hesitant triangular neutrosophic numbers and their applications to MADM," *Neutrosophic Sets and Systems*, vol. 35, pp. 269–298, 2020.

[33] M. Riaz and S. T. Tehrim, "On bipolar fuzzy soft topology with decision-making," *Soft Computing*, vol. 24, no. 24, pp. 18259–18272, 2020.

[34] N. Martin, R. Priya, and F. Smarandache, "New plithogenic sub cognitive maps approach with mediating effects of factors in COVID-19 diagnostic model," *Journal of Fuzzy Extension and Applications*, vol. 2, no. 1, pp. 1–14, 2021.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6656739, 11 pages
https://doi.org/10.1155/2021/6656739

Research Article

Single-Valued Neutro Hyper BCK-Subalgebras

M. Hamidi [1] and F. Smarandache [2]

[1]Department of Mathematics, University of Payame Noor, Tehran, Iran
[2]Mathematics & Science, University of New Mexico, Gallup Campus, Gallup, USA

Correspondence should be addressed to M. Hamidi; m.hamidi@pnu.ac.ir

Received 25 November 2020; Revised 24 December 2020; Accepted 27 August 2021; Published 22 September 2021

Academic Editor: Firdous A. Shah

The purpose of this paper is to introduce the notation of single-valued neutrosophic hyper BCK-subalgebras and a novel concept of neutro hyper BCK-algebras as a generalization and alternative of hyper BCK-algebras, that have a larger applicable field. In order to realize the article's goals, we construct single-valued neutrosophic hyper BCK-subalgebras and neutro hyper BCK-algebras on a given nonempty set. The result of the research is the generalization of single-valued neutrosophic BCK-subalgebras and neutro BCK-algebras to single-valued neutrosophic hyper BCK-subalgebras and neutro hyper BCK-algebras, respectively. Also, some results are obtained between extended (extendable) single-valued neutrosophic BCK-subalgebras and single-valued neutrosophic hyper BCK-subalgebras via fundamental relation. The paper includes implications for the development of single-valued neutrosophic BCK-subalgebras and neutro BCK-algebras and for modelling the uncertainty problems by single-valued neutrosophic hyper BCK-subalgebras and neutro hyper BCK-algebras. The new conception of single-valued neutrosophic hyper BCK-subalgebras and neutro hyper BCK-algebras was given for the first time in this paper. We find a method that can apply these concepts in some complex networks.

1. Introduction

The theory of logical (hyper) algebra is related to the study of certain propositional calculi and tries to solve logical problems using (hyper) algebraic methods. Jun et al. [1] has introduced a logical (hyper) algebra named hyper BCK-algebras as development of BCK-algebras, which were initiated by Imai and Iseki [2] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. The theory of neutrosophic set as an extension of classical set and (intuitionistic) fuzzy set [3], and interval-valued (intuitionistic) fuzzy set, is introduced by Smarandache for the first time in 1998 [4] and mentioned second time in 2005 [5]. This concept handles problems involving imprecise, indeterminacy, and inconsistent data and describes an important role in the modelling of unsure hypernetworks in all sciences. Recently, due to the importance of these subjects, by combining the neutrosophic sets and (hyper) BCK-algebras, some researchers worked in more branches of neutrosophic (hyper) BCK-algebras such as MBJ-neutrosophic hyper BCK-ideals in hyper BCK-algebras, an approach to BMBJ-neutrosophic hyper BCK-ideals of hyper BCK-algebras, structures on doubt neutrosophic ideals of (BCK/BCI)-algebras under (S, T)-norms, BMBJ-neutrosophic subalgebras in (BCI/BCK)-algebras, MBJ-neutrosophic ideals of (BCK/BCI)-algebras, implicative neutrosophic quadruple BCK-algebras and ideals, neutrosophic hyper BCK-ideals, implicative neutrosophic quadruple BCK-algebras and ideals, bipolar-valued fuzzy soft hyper BCK ideals in hyper BCK-algebras, single-valued neutrosophic ideals in Sostak's sense, and multipolar intuitionistic fuzzy hyper BCK-ideals in hyper BCK-algebras [6–16]. Recently, a novel concept of neutrosophy theory titled neutro (hyper) algebra as development of classical (hyper) algebra and partial (hyper) algebra is introduced by Smarandache [17].

A neutro (hyper) algebra is a system that has at least one neutro (hyper) operation or one neutro axiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements), while a partial (hyper) algebra is a (hyper) algebra that has at least one partial (hyper) operation, and all its axioms are classical (i.e., axioms true for all elements). Smarandache proved that a neutron (hyper) algebra is a generalization of a partial (hyper) algebra and showed that neutro (hyper) algebras are not partial (hyper) algebras, necessarily. Hamidi and Smarandache [18] introduced the concept of neutro BCK-subalgebras as a generalization of BCK-algebras and presented main results in neutro BCK-subalgebras as an extension of BCK-algebras structures and their applications. In addition, the concept of neutro (hyper) algebra is studied in different branches such as neutro algebra structures and neutro (hyper) graph [19, 20].

Regarding these points, one of the aims of this paper is to introduce the concept of single-valued neutrosophic hyper BCK-subalgebras and extendable single-valued neutrosophic BCK-subalgebras and generalize the notion of single-valued neutrosophic hyper BCK-subalgebras by considering the notion of single-valued neutrosophic BCK-subalgebras. Also, we want to establish the relationship between single-valued neutrosophic BCK-algebras and single-valued neutrosophic hyper $\text{BCK}-\text{algebras}$. So a strongly regular relation is applied on any hyper BCK-algebras using the concept of single-valued neutrosophic hyper BCK-subalgebras, and a quotient hyper BCK-algebras ($\text{BCK}-\text{algebras}$) can be obtained. The main aim of this study is to introduce the notation of neutro hyper BCK-algebras as a generalization of neutro BCK-algebras in regard to single-valued neutrosophic hyper BCK-subalgebras. In the study of neutro hyper BCK-algebra, despite having key mathematical tools, there are some limitations. The union of two neutro hyper BCK-algebra is not necessarily a neutro hyper BCK-algebra so the class of neutro hyper BCK-algebra is not closed under any given algebraic operation. In addition, neutro hyper BCK-algebras are different with (intuitionistic fuzzy) hyper BCK-algebras and single-valued neutrosophic hyper BCK-algebras so could not generalize the capabilities of (intuitionistic fuzzy) single-valued neutrosophic hyper BCK-algebras to neutro hyper BCK-algebras.

2. Preliminaries

Definition 1 (see [2]) Let $X \neq \varnothing$. Then a universal algebra $(X, \vartheta, 0)$ of type $(2, 0)$ is called a BCK-algebra if, for all, $x, y, z \in X$:

$(\text{BCI}-1)\,((x\varrho y)\varrho(x\varrho z))\varrho(z\varrho y) = 0$,

$(\text{BCI}-2)\,(x\varrho(x\varrho y))\varrho y = 0$,

$(\text{BCI}-3)\,x\varrho x = 0$,

$(\text{BCI}-4)\,x\varrho y = 0$ and $y\varrho x = 0$ imply $x = y$,

$(\text{BCK}-5)\,0\varrho x = 0$, where $\varrho(x, y)$ is denoted by $x\varrho y$.

Definition 2 (see [1]). Let $X \neq \varnothing$ and $P^*(X) = \{Y \mid \varnothing \neq Y \subseteq X\}$. Then for a map $\vartheta\colon X^2 \longrightarrow P^*(X)$, a hyperalgebraic system $(X, \vartheta, 0)$ is called a hyper BCK-algebra if, for all, $x, y, z \in X$:

$(H1)\,(x\,\vartheta\, z)\,\vartheta\,(y\,\vartheta\, z) \ll x\,\vartheta\, y$,

$(H2)\,(x\,\vartheta\, y)\,\vartheta\, z = (x\,\vartheta\, z)\,\vartheta\, y$,

$(H3) x\,\vartheta\, X \ll x$,

$(H4) x \ll y$ and $y \ll x$ imply $x = y$,

where $x \ll y$ is defined by $0 \in x\,\vartheta\, y$, $\forall A, B \subseteq H$, $A \ll B \Longleftrightarrow \forall a \in A\ \exists b \in B$ s.t $a \ll b$, $(A\,\vartheta\, B) = \cup_{a\in A, b\in B}(a\,\vartheta\, b)$, and $\vartheta(x, y)$ is denoted by $x\,\vartheta\, y$.

We will call X is a weak commutative hyper BCK-algebra if $\forall x, y \in X$, $(x\,\vartheta\,(x\,\vartheta\, y)) \cap (y\,\vartheta\,(y\,\vartheta\, x)) \neq \varnothing$ [21].

Theorem 1 (see [1]). *Let $(X, \vartheta, 0)$ be a hyper BCK-algebra. Then $\forall x, y, z \in X$ and $A, B \subseteq X$:*

(i) $(0\,\vartheta\, 0) = 0$, $0 \ll x$, $(0\,\vartheta\, x) = 0$, $x \in (x\,\vartheta\, 0)$ and $A \ll 0 \Rightarrow A = 0$

(ii) $x \ll x$, $x\,\vartheta\, y \ll x$ and $y \ll z$ implies that $x\,\vartheta\, z \ll x\,\vartheta\, y$

(iii) $A\,\vartheta\, B \ll A$, $A \ll A$ and $A \subseteq B$ implies $A \ll B$

Definition 3 (see [22]). Let $(X, \vartheta, 0)$ be a hyper BCK-algebra. A fuzzy set $\mu\colon X \longrightarrow [0, 1]$ is called a fuzzy hyper BCK-subalgebra if $\forall x, y \in X$, $\wedge(\mu(x\,\vartheta\, y)) \geq T_{\min}(\mu(x), \mu(y))$.

Definition 4 (see [5]). Let V be a universal set. A neutrosophic subset (NS) X in V is an object having the following form: $X = \{(x, T_X(x), I_X(x), F_X(x)) \mid x \in V\}$, or $X\colon V \longrightarrow [0, 1] \times [0, 1] \times [0, 1]$, which is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X, and a falsity-membership function F_X. There is no restriction on the sum of $T_X(x), I_X(x)$, and $F_X(x)$.

3. Single-Valued Neutrosophic Hyper BCK-Subalgebras

In this section, the concept of single-valued neutrosophic hyper BCK-subalgebras will be considered as a generalization of single-valued neutrosophic BCK-subalgebras, and some of its properties will be investigated. We will also prove that single-valued neutrosophic hyper BCK-subalgebras and single-valued neutrosophic BCK-subalgebras are related, and single-valued neutrosophic hyper BCK-subalgebras and single-valued neutrosophic BCK-subalgebras can be constructed from single-valued neutrosophic hyper BCK-subalgebras via a fundamental relation. We will define the concept of extendable single-valued neutrosophic BCK-subalgebras and will show that any infinite set is an extended single-valued neutrosophic BCK-subalgebra.

Throughout this section, we denote hyper BCK-algebra $(X, \vartheta, 0)$ by X. From now on, for all, $x, y \in [0, 1]$, $T_{\min}(x, y) = \min\{x, y\}$ and $S_{\max}(x, y) = \max\{x, y\}$ are considered as triangular norm and triangular conorm, respectively. In the following definition, the notation of single-valued neutrosophic hyper BCK-subalgebra of any given nonempty is defined.

Definition 5. A single-valued neutrosophic set $A = (T_A, I_A, F_A)$ in an X is called a single-valued neutrosophic hyper BCK-subalgebra of X, if

(i) $\wedge (T_A(x \vartheta y)) \geq T_{\min}(T_A(x), T_A(y))$

(ii) $\vee (I_A(x \vartheta y)) \leq S_{\max}(I_A(x), I_A(y))$

(iii) $\vee (F_A(x \vartheta y)) \leq S_{\max}(F_A(x), F_A(y))$

The importance of the following theorems is to determine the role and the effect of truth-membership function T_A, indeterminacy-membership function I_A, and falsity-membership function F_A on the element $0 \in A$.

Theorem 2. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(i) $T_A(0) \geq T_A(x)$

(ii) $\wedge (T_A(x \vartheta 0)) = T_A(x)$

(iii) $\wedge (T_A(0 \vartheta x)) = T_A(0)$

Proof

(i) Let $x \in X$. Since $0 \in x \vartheta x$, we get that $T_A(0) \geq \wedge (T_A(x \vartheta x)) \geq T_{\min}(T_A(x), T_A(x)) = T_A(x)$.

(ii) Let $x \in X$. Since $x \in x \vartheta 0$, we get that $T_A(x) \geq \wedge (T_A(x \vartheta 0)) \geq T_{\min}(T_A(x), T_A(0)) = T_A(x)$. So $\wedge (T_A(x \vartheta 0)) = T_A(x)$.

(iii) Immediate by Theorem 1. □

Theorem 3. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(i) $I_A(0) \leq I_A(x)$

(ii) $\vee (I_A(x \vartheta 0)) = I_A(x)$

(iii) $\vee (I_A(0 \vartheta x)) = I_A(0)$

Proof

(i) Let $x \in X$. Since $0 \in x \vartheta x$, we get that $I_A(0) \leq \vee (I_A(x \vartheta x)) \leq S_{\max}(I_A(x), I_A(x)) = I_A(x)$.

(ii) Let $x \in X$. Since $x \in x \vartheta 0$, we get that $I_A(x) \leq \vee (I_A(x \vartheta 0)) \leq S_{\max}(I_A(x), I_A(0)) = I_A(x)$. So $\vee (I_A(x \vartheta 0)) = I_A(x)$.

(iii) Immediate by Theorem 1. □

Corollary 1. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(i) $F_A(0) \leq F_A(x)$

(ii) $\vee (F_A(x \vartheta 0)) = F_A(x)$

(iii) $\vee (F_A(0 \vartheta x)) = F_A(0)$

(iv) $T_{\min}(T_A(x), I_A(0), F_A(0)) \leq T_{\min}(T_A(0), I_A(x), F_A(x))$

In the following theorem, we construct single-valued neutrosophic subset on any nonempty set.

Theorem 4. *Let $0 \notin X \neq \emptyset$. Then there exist a hyper-operation "ϑ," a single-valued neutrosophic subset $A = (T_A, I_A, F_A)$ of $X' = X \cup \{0\}$ such that $(X', \vartheta, 0)$ is a hyper BCK-algebra and A is a single-valued neutrosophic hyper BCK-subalgebra of X'.*

Proof. Let $x, y \in X'$. Define "ϑ" on X' by $x \vartheta y = \begin{cases} 0, & \text{if } x = 0, \\ \{0, x\}, & \text{if } x = y, x \neq 0, \\ x, & \text{otherwise} \end{cases}$. Clearly, $(X', \vartheta, 0)$ is a hyper BCK-algebra. Now, it is easy to see that every single-valued neutrosophic set $A = (T_A, I_A, F_A)$ that $T_A(0) = 1$, $I_A(0) = F_A(0) = 0$ is a single-valued neutrosophic hyper BCK-subalgebra of X'.

Let $\mathrm{SVN}h = \{A = (T_A, I_A, F_A) \mid A \text{ is a single–valued neutrosophic hyper BCK – subalgebra of } X\}$, whence X is a hyper BCK-algebra and $|X| \geq 1$. □

Corollary 2. *Let $X \neq \emptyset$. Then X can be extended to a hyper BCK-algebra that $|\mathrm{SVN}h| = |\mathbb{R}|$.*

Proof. Let $X = \{x\}$. Then (X, ϑ, x) is a hyper BCK-algebra such that $x \vartheta x = \{x\}$. Then for a single-valued neutrosophic set, $A = (T_A, I_A, F_A)$ by $T_A(x) = I_A(x) = F_A(x) = \alpha$ is a single-valued neutrosophic hyper BCK-subalgebra of X, where $\alpha \in [0, 1]$. If $|X| \geq 2$; then by Theorem 4, we can construct at least a hyper BCK-subalgebra on X. Now, $\forall \alpha \in [0, 1]$ define $A = (T_{A_\alpha}, I_{A_\alpha}, F_{A_\alpha})$ by

$$T_{A_\alpha}(x) = \begin{cases} 1, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0, \end{cases}$$
$$I_{A_\alpha}(x) = \begin{cases} 0, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0, \end{cases} \quad (1)$$
$$F_{A_\alpha}(x) = \begin{cases} 0, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0. \end{cases}$$

Obviously, $A = (T_{A_\alpha}, I_{A_\alpha}, F_{A_\alpha})$ a single-valued neutrosophic hyper BCK-subalgebra of X and so $|\mathrm{SVN}h| = |[0, 1]|$.

Let X be a hyper BCK-algebra, $A = (T_A, I_A, F_A)$ a single-valued neutrosophic hyper BCK-subalgebra of X and $\alpha, \beta, \gamma \in [0, 1]$. Define $T_A^\alpha = \{x \in X \mid T_A(x) \geq \alpha\}$, $I_A^\beta = \{x \in X \mid I_A(x) \leq \beta\}$, $F_A^\gamma = \{x \in X \mid F_A(x) \leq \gamma\}$, and $A^{(\alpha,\beta,\gamma)} = \{x \in X \mid T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}$.

Considering the relation between single-valued neutrosophic hyper BCK-subalgebras and (fuzzy) hyper BCK-subalgebra is the main aim of the following results via the level subsets. □

Theorem 5. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(i) $0 \in A^{(\alpha,\beta,\gamma)} = T_A^\alpha \cap I_A^\beta \cap F_A^\gamma$

(ii) $A^{(\alpha,\beta,\gamma)}$ *is a hyper BCK-subalgebra of X*

(iii) If $0 \leq \alpha \leq \alpha' \leq 1$, *then* $T_A^{\alpha'} \subseteq T_A^\alpha$, $I_A^{\alpha'} \supseteq I_A^\alpha$ *and* $F_A^{\alpha'} \supseteq F_A^\alpha$

Proof

(i) Clearly, $A^{(\alpha,\beta,\gamma)} = A^\alpha \cap A^\beta \cap A^\gamma$ and by Theorems 2 and 3, and Corollary 1, we get that $0 \in A^{(\alpha,\beta,\gamma)}$.

(ii) Let $x, y \in T_A^{\alpha}$. Then $T_{\min}(T_A(x), T_A(y)) \geq \alpha$. Now, for any, $z \in x \vartheta y, T_A(z) \geq \inf(T_A(x \vartheta y)) \geq T_{\min}(T_A(x), T_A(y)) \geq \alpha$. Hence, $z \in T_A^{\alpha}$, and so $x \vartheta y \subseteq T_A^{\alpha}$. In similar a way, $x, y \in I_A^{\beta} \cap F_A^{\gamma}$ implies that $x \vartheta y \subseteq (I_A^{\beta} \cap F_A^{\gamma})$. Then $A^{(\alpha,\beta,\gamma)}$ is a hyper BCK-subalgebra of X.

(iii) Immediate. □

Corollary 3. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. If $0 \leq \alpha \leq \alpha' \leq 1$, then $A^{(\alpha',\alpha,\alpha)}$ is a hyper BCK-subalgebra of $A^{(\alpha,\alpha',\alpha')}$.*

Let X be a hyper BCK-algebra, S be a hyper BCK-subalgebra of X and $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in [0,1]$. Define

$$T_A^{[\alpha,\alpha']}(x) = \begin{cases} \alpha', & \text{if } x \in S, \\ \alpha, & \text{if } x \notin S, \end{cases}$$
$$I_A^{[\beta,\beta']}(x) = \begin{cases} \beta', & \text{if } x \in S, \\ \beta, & \text{if } x \notin S, \end{cases} \quad (2)$$
$$F_A^{[\gamma,\gamma']}(x) = \begin{cases} \gamma', & \text{if } x \in S, \\ \gamma, & \text{if } x \notin S. \end{cases}$$

Thus, we have the following theorem.

Theorem 6. *Let X be a hyper BCK-algebra and S be a hyper BCK-subalgebra of X. Then*

(i) $T_A^{[\alpha,\alpha']}$ is a fuzzy hyper BCK-subalgebra of X
(ii) $I_A^{[\beta,\beta']}$ is a fuzzy hyper BCK-subalgebra of X
(iii) $F_A^{[\gamma,\gamma']}$ is a fuzzy hyper BCK-subalgebra of X
(iv) $A = (T_A^{[\alpha,\alpha']}, I_A^{[\beta,\beta']}, F_A^{[\gamma,\gamma']})$ is a single-valued neutrosophic hyper BCK-subalgebra of X

Proof

(i) Let $x, y \in X$. If $x, y \in S$, since S is a hyper subalgebra of X, we get that $x \vartheta y \subseteq S$ and so

$$\wedge T_A^{[\alpha,\alpha']}(x \vartheta y) \geq \wedge T_A^{[\alpha,\alpha']}(S) = \alpha' \geq T_{\min}\left(T_A^{[\alpha,\alpha']}(x), T_A^{[\alpha,\alpha']}(y)\right). \quad (3)$$

If ($x \in S$ and $y \notin S$) or ($x \notin S$ and $y \in S$) or ($x \notin S$ and $y \notin S$), then $\wedge T_A^{[\alpha,\alpha']}(x \vartheta y) \in \{\alpha, \alpha'\}$. Thus, $\wedge T_A^{[\alpha,\alpha']}(x \vartheta y) \geq T_{\min}(T_A^{[\alpha,\alpha']}(x), T_A^{[\alpha,\alpha']}(y))$, and so $T_A^{[\alpha,\alpha']}$ is a fuzzy hyper BCK-subalgebra of X.

(ii) and (iii) They are similar to (i).

(iv) Let $x, y \in X$. If $x, y \in S$, since S is a hyper BCK-subalgebra of X, we get that $x \vartheta y \subseteq S$, and so $\vee I_A^{[\beta,\beta']}(x \vartheta y) \leq \vee I_A^{[\beta,\beta']}(S) = \alpha' \leq S_{\max}(I_A^{[\beta,\beta']}(x), I_A^{[\beta,\beta']}(y))$. If ($x \in S$ and $y \notin S$) or ($x \notin S$ and $y \in S$) or ($x \notin S$ and $y \notin S$), then $\vee I_A^{[\beta,\beta']}(x \vartheta y) \in \{\beta, \beta'\}$. Thus, $\vee T_A^{[\beta,\beta']}(x \vartheta y) \leq S_{\max}(I_A^{[\beta,\beta']}(x), I_A^{[\beta,\beta']}(y))$. In a similar way, we can see that $\vee F_A^{[\gamma,\gamma']}(x \vartheta y) \leq S_{\max}(F_A^{[\gamma,\gamma']}(x), F_A^{[\gamma,\gamma']}(y))$ an by item (i), $A = (T_A^{[\alpha,\alpha']}, I_A^{[\beta,\beta']}, F_A\ [\gamma,\gamma'])$ is a single-valued neutrosophic hyper BCK-subalgebra of X.

Let X be a hyper BCK-algebra and $x, y \in X$. Then $x\beta y \Longleftrightarrow \exists n \in \mathbb{N}, (a_1, \ldots, a_n) \in X^n$ and $\exists u \in \vartheta(a_1, \ldots, a_n)$ such that $\{x, y\} \subseteq u$. The relation β is a reflexive and symmetric relation but not transitive relation. Let $C(\beta)$ be the transitive closure of β (the smallest transitive relation such that contains β). Borzooei et al. in [21], proved that for any given weak commutative hyper BCK-algebra X, $C(\beta)$ is a strongly regular relation on X, and $((X/C(\beta)), \varrho, \overline{0})$ is a BCK-algebra, where $C(\beta)(x)\varrho C(\beta)(y) = C(\beta)(x \vartheta y)$ and $\overline{0} = C(\beta)(0)$.

Considering the relation between single-valued neutrosophic hyper BCK-subalgebras and single-valued neutrosophic BCK-subalgebras has very important, especially in extension of single-valued neutrosophic BCK-subalgebras. So we prove the following theorems and corollaries. □

Theorem 7. *Let X be a weak commutative hyper BCK-subalgebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then there exists a single-valued neutrosophic set $\overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A})$ of BCK-algebra $((X/C(\beta)), \vartheta, \overline{0})$ that $\forall x, y \in X$,*

(i) $\overline{T_A}(C(\beta)(0)) \geq \overline{T_A}(C(\beta)(x))$
(ii) if $yC(\beta)x$, then $\overline{T_A}(C(\beta)(x)) = \overline{T_A}(C(\beta)(y))$
(iii) $\overline{I_A}(C(\beta)(0)) \leq \overline{I_A}(C(\beta)(x))$
(iv) if $yC(\beta)x$, then $\overline{I_A}(C(\beta)(x)) = \overline{I_A}(C(\beta)(y))$
(v) $\overline{F_A}(C(\beta)(0)) \leq \overline{F_A}(C(\beta)(x))$
(vi) if $yC(\beta)x$, then $\overline{F_A}(C(\beta)(x)) = \overline{F_A}(C(\beta)(y))$

Proof. Let $x, y, t \in X$. Then on $(X/C(\beta))$, define $\overline{T_A}(C(\beta)(t)) = \begin{cases} T_A(0), & \text{if } 0 \in C(\beta)(x), \\ \wedge_{tC(\beta)x} T_A(x), & \text{otherwise}, \end{cases}$, $\overline{I_A}(C(\beta)(t)) = \begin{cases} I_A(0), & \text{if } 0 \in C(\beta)(x), \\ \vee_{tC(\beta)x} I_A(x), & \text{otherwise}, \end{cases}$, and $\overline{F_A}(C(\beta)(t)) = \begin{cases} F_A(0), & \text{if } 0 \in C(\beta)(x), \\ \vee_{tC(\beta)x} F_A(x), & \text{otherwise}, \end{cases}$. Using Theorems 2 and 3, we get that:

(i) $\overline{T_A}(C(\beta)(0)) = T_A(0) \geq \wedge_{t'C(\beta)x} T_A(t') = \overline{T_A}(C(\beta)(x))$

(ii) Since $xC(\beta)y$ and $C(\beta)$ is transitive, we get that $\overline{T_A}(C(\beta)(x)) = \wedge_{tC(\beta)x} T_A(t) \geq \wedge_{tC(\beta)y} T_A(t) = \overline{T_A}(C(\beta)(y))$

(iii) $\overline{I_A}(C(\beta)(0)) = I_A(0) \leq \vee_{t'C(\beta)x} I_A(t') = \overline{I_A}(C(\beta)(x))$

(iv) Since $xC(\beta)y$ and $C(\beta)$ is transitive, we get that $\overline{I_A}(C(\beta)(x)) = \vee_{tC(\beta)x} I_A(t) = \vee_{tC(\beta)y} I_A(t) = \overline{I_A}(C(\beta)(y))$

(v) and (vi) They are similar to (iii) and (iv), respectively. □

Theorem 8. *Let X be a weak commutative hyper BCK-subalgebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then there exists a single-valued neutrosophic subset $\overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A})$ of BCK-algebra $((X/C(\beta)), \vartheta, \overline{0})$ that $\forall x, y \in X$:*

(i) There exists $t \in x \vartheta y$ such that $\overline{T_A}(C(\beta)\ (x \vartheta y)) = T_A(t)$

(ii) *There exists* $t\prime \in x\vartheta y$ *such that* $\overline{I_A}(C(\beta)(x\vartheta y)) = I_A\ (t)$

(iii) *There exists* $t\prime\prime \in x\vartheta y$ *such that* $\overline{F_A}(C(\beta)(x\vartheta y)) = F_A\ (t)$

Proof

(i) Let $x, y \in X$. Applying Theorem 7,

$$\begin{aligned}&\overline{T_A}(C(\beta)(x)\varrho C(\beta)(y)) = \overline{T_A}(C(\beta)(x\vartheta y))\\ &= \overline{T_A}\{C(\beta)(m) \mid m \in x\vartheta y\} = \bigwedge_{\substack{sC(\beta)m\\ m\in x\vartheta y}} T_A(s).\end{aligned} \tag{4}$$

Now, since $sC(\beta)m$ and $m \in x\vartheta y$, then $s \in x\vartheta y$, and so there exists $t \in x\vartheta y$ such that $T_A(t) = \wedge_{\substack{sC(\beta)m\\ m\in x\vartheta y}} T_A(s)$.

(ii) Let $x, y \in X$. Then

$$\begin{aligned}&\overline{I_A}(C(\beta)(x)\varrho C(\beta)(y)) = \overline{I_A}(C(\beta)(x\vartheta y))\\ &= \overline{I_A}\{C(\beta)(n) \mid n \in x\vartheta y\} = \bigvee_{\substack{tC(\beta)n\\ n\in x\vartheta y}} I_A(t).\end{aligned} \tag{5}$$

Now, since $tC(\beta)n$ and $n \in x\vartheta y$, then $t \in x\vartheta y$, and so there exists $t' \in x\vartheta y$ such that $I_A(t') = \wedge_{\substack{tC(\beta)n\\ n\in x\vartheta y}} I_A(t)$.

(iii) It is similar to item (ii).

Some categorical properties of single-valued neutrosophic BCK-subalgebras is investigated in the following theorem based on the categorical properties of single-valued neutrosophic hyper BCK-subalgebras. □

Theorem 9. *Let X be a weak commutative hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then there exists a single-valued neutrosophic BCK-subalgebra $B = (T_B, I_B, F_B)$ of $((X/C(\beta)), F_B, C(\beta)(0))$ that $((T_B\vartheta\pi) \le T_A, (I_B\vartheta\pi) \ge I_A$ and $(I_B\vartheta F_B) \ge F_A)$ or the following diagrams are quasi commutative:*

$$X \quad \longrightarrow^{T_A} [0\ \ 1]_{\pi}\downarrow\nearrow_{T_B}\frac{X}{C(\beta)},\ X \quad \longrightarrow^{I_A} [0\ \ 1]_{\pi}\downarrow\nearrow_{I_B}\frac{X}{C(\beta)},\ X \quad \longrightarrow^{F_A} [0\ \ 1]_{\pi}\downarrow\nearrow_{F_B}\frac{X}{C(\beta)}. \tag{6}$$

Proof. Choice $T_B = \overline{T_A}$, $I_B = \overline{I_A}$ and $F_B = \overline{F_A}$. Then by Theorem 7, (i) $\forall x \in X$,

$$\begin{aligned}T_B(C(\beta)(0)) &\ge T_B(C(\beta)(x)),\\ I_B(C(\beta)(0)) &\le I_B(C(\beta)(x)),\\ F_B(C(\beta)(0)) &\le F_B(C(\beta)(x)).\end{aligned} \tag{7}$$

(ii) By Theorem 8, $\forall x, y \in X$; there exists $\{t, t', t''\} \subseteq x\vartheta y$ that

$$\begin{aligned}T_B(C(\beta)(x\vartheta y)) &= T_A(t),\\ I_B(C(\beta)(x\vartheta y)) &= I_A(t\prime),\\ F_B(C(\beta)(x\vartheta y)) &= F_A(t\prime\prime).\end{aligned} \tag{8}$$

So

$$\begin{aligned}T_B(C(\beta)(x)\varrho C(\beta)(y)) &= T_B(C(\beta)(x\vartheta y)) = T_A(t) \ge \wedge\left(T_A(x\vartheta y)\right)\\ &\ge T_{\min}(T_A(x), T_A(y)) \ge T_{\min}(T_B(C(\beta)(x)), T_B(C(\beta)(y))),\\ I_B(C(\beta)(x)\varrho C(\beta)(y)) &= I_B(C(\beta)(x\vartheta y)) = I_A(t\prime) \le \vee\left(I_A(x\vartheta y)\right)\\ &\le S_{\max}(I_A(x), I_A(y)) \le S_{\max}(I_B(C(\beta)(x)), I_B(C(\beta)(y))),\\ F_B(C(\beta)(x)\varrho C(\beta)(y)) &= F_B(C(\beta)(x\vartheta y)) = F_A(t\prime\prime) \le \vee\left(F_A(x\vartheta y)\right)\\ &\le S_{\max}(F_A(x), F_A(y)) \le S_{\max}(F_B(C(\beta)(x)), F_B(C(\beta)(y))).\end{aligned} \tag{9}$$

Therefore, $B = (T_B, I_B, F_B)$ is a single-valued neutrosophic BCK-subalgebra of $(X/C(\beta))$, $(T_B\vartheta\pi) \le T_A$, $(I_B\vartheta\pi) \ge I_A$, and $(I_B\vartheta\pi) \ge F_A$.

Based on the fundamental relation, we can obtain the single-valued neutrosophic BCK-subalgebras, and single-valued neutrosophic BCK-subalgebras are derived from some single-valued neutrosophic hyper BCK-subalgebras. In this regard, it is important that single-valued neutrosophic BCK-subalgebras are derived from single-valued neutrosophic hyper BCK-subalgebra with minimal order. So the concepts of (extended) extendable single-valued neutrosophic BCK-subalgebra are introduced as follows. □

Definition 6

(i) Let $(X, \varrho, 0)$ be a BCK-algebra and $(Y, \vartheta, 0)$ be a hyper BCK-algebra. We say that the BCK-algebra X is derived from the hyper BCK-algebra Y if X is isomorphic to a nontrivial quotient of Y $(X \cong (Y/C(\beta)))$.

(ii) A single-valued neutrosophic BCK-subalgebra $A = (T_A, I_A, F_A)$ of X is called an extendable single-valued neutrosophic BCK-subalgebra, if there exist a hyper BCK-algebra $(Y, \vartheta, 0)$, a single-valued neutrosophic hyper BCK-subalgebra $B = (T_B, I_B, F_B)$ of Y, and $n \in \mathbb{N}$ such that $|(X, \vartheta, A)| = |(Y, \vartheta, B)| - n$, and BCK-algebra X is derived of hyper BCK-algebra Y. If $X = Y$ and almost everywhere $(T_A, I_A, F_A) = (T_B, I_B, F_B)$ $((T_A, I_A, F_A) = (T_B, I_B, F_B)$a.e that means $|\{x; T_A(x) \neq T_B(x), I_A(x) \neq I_B(x), F_A(x) \neq F_B(x)\}| = 1)$, we will say that it is an extended single-valued neutrosophic BCK-subalgebra.

The following example introduces an extendable single-valued neutrosophic BCK-subalgebra.

Example 1. Let $X = \{-1, -2, -3, -4\}$. Then $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic BCK-subalgebra of BCK-algebra $(X, \vartheta, -1)$ (see Table 1).

Now, set $Y = \{0, -1, -2, -3, -4\} = X \cup \{0\}$. Then $B = (T_B, I_B, F_B)$ is a single-valued neutrosophic hyper BCK-subalgebra of $(Y, \vartheta, 0)$ (see Table 2).

Clearly, $(Y/C(\beta)) \cong X$, $|Y| = |X| + 1$, and so $A = (T_A, I_A, F_A)$ is an extendable single-valued neutrosophic BCK-subalgebra of $(X, \vartheta, -1)$.

In the following theorem, we try to generate BCK-algebras based on single-valued neutrosophic hyper BCK-subalgebras.

Theorem 10. *Let $(X, \vartheta, 0)$ be a hyper BCK-algebra, $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X, and $\overline{X} = \{(T_A(x), I_A(x), F_A(x)) \mid x \in X\}$. If A is one to one map, then:*

(i) There exists a hyperoperation "ϑ'" on $\overline{X}$ such that $(\overline{X}, \vartheta', (T_A(0), I_A(0), F_A(0)))$ is a hyper BCK-algebra

(ii) There exists a single-valued neutrosophic hyper BCK-subalgebra $\overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A})$ of $\overline{X}$ related to $A = (T_A, I_A, F_A)$

(iii) There exists an operation "ϱ" (related to ϑ) on $\overline{X}$ that $(\overline{X}, \varrho, (T_A(0), I_A(0), F_A(0)))$ is a BCK-algebra

Proof

(i) Let $x, y \in X$. Define a hyperoperation ϑ' on $\overline{X}$, by

$$(T_A(x), I_A(x), F_A(x))\, \vartheta'\, (T_A(y), I_A(y), F_A(y)) = (T_A(x \vartheta y), I_A(x \vartheta y), F_A(x \vartheta y)). \tag{10}$$

It can be easily seen that $(T_A(x), I_A(x), F_A(x)) \ll' (T_A(y), I_A(y), F_A(y)) \Longleftrightarrow x \ll y$. It is easy to see that $(\overline{X}, \vartheta', (T_A(0), I_A(0), F_A(0)))$ is a hyper BCK-algebra.

(ii) Let $x \in X$. Define $\overline{A}(A(x)) = A(x)$. Clearly, $\overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A})$ is a single-valued neutrosophic hyper BCK-subalgebra of $(\overline{X}, \vartheta')$.

(iii) Assume $x, y \in X$. Define an operation ϱ on $\overline{X}$ by

$$(T_A(x), I_A(x), F_A(x))\varrho(T_A(y), I_A(y), F_A(y)) = \begin{cases} (T_A(x), I_A(x), F_A(x)), & \text{if } y = 0, \\ (\vee T_A(x \vartheta y), \wedge I_A(x \vartheta y), \wedge F_A(x \vartheta y)) & \text{otherwise.} \end{cases} \tag{11}$$

We just prove BCI-4. Let $x, y \in X$ and

$$\begin{aligned} &(T_A(x), I_A(x), F_A(x))\varrho(T_A(y), I_A(y), F_A(y)) \\ &\quad = (T_A(x), I_A(x), F_A(x))\varrho(T_A(y), I_A(y), F_A(y)) \\ &\quad = (T_A(0), I_A(0), F_A(0)). \end{aligned} \tag{12}$$

Since A is a one to one map, $0 \in x \vartheta y$ and $0 \in y \vartheta x$. It follows that $(T_A(x), I_A(x), F_A(x)) = (T_A(y), I_A(y), F_A(y))$. It is easy to see that BCI-1, BCI-2, BCI-3, and BCK-5 are valid, and so $(\overline{X}, \varrho, (T_A(0), I_A(0), F_A(0)))$ is a BCK-algebra. □

Corollary 4. *Let $(\overline{X}, \vartheta, (T_A(0), I_A(0), F_A(0)))$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of $\overline{X}$. Then there exists a binary operation "ϱ" on $\overline{X}$, such that $(\overline{X}, \varrho, (T_A(0), I_A(0), F_A(0)))$ is a BCK-algebra.*

In the following theorem, we try to generate hyper BCK-algebras based on single-valued neutrosophic hyper BCK-subalgebras.

Theorem 11. *Let X be a nonempty set, $0 \notin X$ and $X' = X \cup \{0\}$. Then there exist a hyperoperation "ϑ" on X', a hyperoperation "ϑ'" on X', a binary operation "ϱ" on X', a single-valued neutrosophic subset $A = (T_A, I_A, F_A)$ of X', and a single-valued neutrosophic subset $B = (T_B, I_B, F_B)$ of X' that:*

(i) $(X', \vartheta, 0)$ is a hyper BCK-algebra, and $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-subalgebra of X'

Table 1

ϱ	-1	-2	-3	-4
-1	-1	-1	-1	-1
-2	-2	-1	-2	-2
-3	-3	-3	-1	-3
-4	-4	-4	-4	-1
	-1	-2	-3	-4
T_A	1	0.2	0.4	0.6
I_A	0.1	0.3	0.7	0.9
F_A	0.05	0.25	0.45	0.65

Table 2

ϑ	0	-1	-2	-3	-4
e	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
-1	$\{-1\}$	$\{0,-1\}$	$\{0,-1\}$	$\{e,-1\}$	$\{0,-1\}$
-2	$\{-2\}$	$\{-2\}$	$\{0,-1\}$	$\{-2\}$	$\{-2\}$
-3	$\{-3\}$	$\{-3\}$	$\{-3\}$	$\{0,-1\}$	$\{-3\}$
-4	$\{-4\}$	$\{-4\}$	$\{-4\}$	$\{-4\}$	$\{0,-1\}$
	0	-1	-2	-3	-4
T_B	1	1	0.2	0.4	0.6
I_B	0.1	0.1	0.3	0.7	0.9
F_B	0.05	0.05	0.25	0.45	0.65

(ii) $(\overline{X'}, \vartheta', (T_A(0), I_A(0), F_A(0)))$ *is a hyper BCK-algebra, and* $A = (T_A, I_A, F_A)$ *is a single-valued neutrosophic hyper BCK-subalgebra of* $\overline{X'}$

(iii) $(\overline{X'}, \varrho, (T_A(0), I_A(0), F_A(0)))$ *is a BCK-algebra, and* $B = (T_B, I_B, F_B)$ *is a single-valued neutrosophic BCK-subalgebra of* $\overline{X'}$

(iv) $|X'| = |\overline{X'}| + 1$

Proof. Let $|X| \geq 2$ and $b \in X$ be fixed. For any $x, y \in X'$, define a binary hyperoperation ϑ on X' as follows:

$$x \vartheta y = \begin{cases} 0, & \text{if } x = 0, \\ \{0, b\}, & \text{if } x = y \text{ and } x \neq 0, \\ \{b\}, & \text{if } x = b \text{ and } y = 0, \\ \{0, b\}, & \text{if } x = b \text{ and } y \neq 0, \\ x, & \text{otherwise.} \end{cases} \tag{13}$$

Now, we show that $(X', \vartheta, 0)$ is a hyper BCK-algebra. We just check that conditions (H1) and (H2) are valid.

(H1): Let $x, y, z \in X'$. If $x = 0$, then $(x \vartheta z) \vartheta (y \vartheta z) = \{0\} \vartheta (y \vartheta z) = \{0\} \ll x \vartheta y$. If $x = b$, then $(x \vartheta z) \vartheta (y \vartheta z) \subseteq \{0, b\} \vartheta (y \vartheta z) \subseteq \{0, b\} \ll x \vartheta y$. If $x \notin \{0, b\}$, we consider the following cases:

Case 1: $x = y \neq z$. Then $(x \vartheta z) \vartheta (y \vartheta z) = x \vartheta y = x \vartheta x = \{0, b\} \ll \{0, b\} = x \vartheta y$.

Case 2: $x = z \neq y$. Then $(x \vartheta z) \vartheta (y \vartheta z) = \{0, b\} \vartheta (y \vartheta z) = \{0, b\} \ll x = x \vartheta y$.

Case 3: $y = z \neq x$. Then $(x \vartheta z) \vartheta (y \vartheta z) \subseteq x \vartheta \{0, b\} = \{0, b\} \ll x = x \vartheta y$.

Case 4: $x \neq y \neq z$. Then $(x \vartheta z) \vartheta (y \vartheta z) = x \vartheta y = x \ll x = x \vartheta y$.

Case 5: $x = y = z$. Then $(x \vartheta z) \vartheta (y \vartheta z) = \{0, b\} \ll \{0, b\} = x \vartheta y$.

(H2): Let $x, y, z \in X$. The proof of $(x \vartheta y) \vartheta z = (x \vartheta z) \vartheta y$ is similar to that of (H1), and then it is easy to see that $(X', \vartheta, 0)$ is a hyper BCK-algebra. Consider a single-valued neutrosophic subset $A = (T_A, I_A, F_A)$ of X' such that $T_A(0) = T_A(b) = 1$, $I_A(0) = I_A(b) = F_A(0) = F_A(b) = 0$; by equation (2) and some modifications, we get that

$$\begin{aligned} &\wedge (T_A(x \vartheta y)) \geq T_{\min}(T_A(x), T_A(y)), \\ &\vee (I_A(x \vartheta y)) \leq S_{\max}(I_A(x), I_A(y)), \\ &\vee (F_A(x \vartheta y)) \leq S_{\max}(F_A(x), F_A(y)). \end{aligned} \tag{14}$$

Hence, $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-subalgebra of $(\overline{X'}, \vartheta, 0)$. Now, $\forall x, y \in X$; define a hyperoperation ϑ' on $\overline{X'}$ by

$$\begin{aligned} A(x) \vartheta' A(y) &= (T_A(x), I_A(x), F_A(x)) \vartheta' (T_A(y), I_A(y), F_A(y)) \\ &= (T_A(x \vartheta y), I_A(x \vartheta y), F_A(x \vartheta y)). \end{aligned} \tag{15}$$

Define a single-valued neutrosophic subset $B = (T_B, I_B, F_B)$ of $\overline{X'}$ by

$$B(A(x)) = A(x),$$
$$\text{or } (T_B(T_A(x)), I_B(I_A(x)), F_B(F_A(x))) = (T_A(x), I_A(x), F_A(x)), \tag{16}$$

and an operation ϱ on $\overline{X'}$ by

$$\begin{aligned} &(T_A(x), I_A(x), F_A(x)) \varrho (T_A(y), I_A(y), F_A(y)) \\ &= (\vee (T_A(x) \vartheta' T_A(y)), \wedge (I_A(x) \vartheta' I_A(y)), \wedge (F_A(x) \vartheta' F_A(y))). \end{aligned} \tag{17}$$

It can be easily seen that $(T_A(x), I_A(x), F_A(x)) \ll' (T_A(y), I_A(y), F_A(y)) \Longleftrightarrow x \ll y$, $(\overline{X'}, \vartheta', (T_A(0), I_A(0), F_A(0)))$ is a hyper BCK-algebra, $A = (T_A(x), I_A(x), F_A(x))$ is a single-valued neutrosophic hyper BCK-subalgebra of $\overline{X'}$, $(\overline{X'}, \vartheta, (T_A(0), I_A(0), F_A(0)))$ is a BCK-algebra, and $B = (T_B(x), I_B(x), F_B(x))$ is a single-valued neutrosophic BCK-subalgebra of $\overline{X'}$, and since $T_A(0) = T_A(b) = 1$, $I_A(0) = I_A(b) = F_A(0) = F_A(b) = 0$, we get that $|X'| = |\overline{X'}| + 1$. □

Corollary 5. *Each nonempty set can be constructed to an extendable single-valued neutrosophic BCK-subalgebra.*

4. Neutro Hyper BCK-Algebras

Smarandache in [17] introduced the concept of neutro hyper operation. An n-ary (for integer $n \geq 1$) hyperoperation $\vartheta: X^n \longrightarrow P(Y)$ is called a neutro hyper operation if it has n-plets in X^n for which the hyperoperation is well-defined $\vartheta(a_1, a_2, \ldots, a_n) \in P(Y)$ (degree of truth (T)), n-plets in X^n for which the hyperoperation is indeterminate (degree of indeterminacy (I)), and n-plets in X^n for which the hyperoperation is outer-defined $\vartheta(a_1, a_2, \ldots, a_n) \notin P(Y)$ (degree of falsehood (F)), where $T, I, F \in [0, 1]$, with

$(T, I, F) \neq (1,0,0)$ that represents the n-ary (total) hyper operation and $(T, I, F) \neq (0,0,1)$ that represents the n-ary anti hyper operation.

In this section, we introduce a novel concept of neutro hyper BCK-algebras as a generalization of neutro BCK-algebras and analyze their properties. The main motivation of the concept of neutro hyper BCK-algebra is a generalization of neutro BCK-algebra, which is defined as follows.

Definition 7. Let $X \neq \varnothing$ and $P^*(X) = \{Y \mid \varnothing \neq Y \subseteq X\}$. Then for a map $\vartheta\colon X^2 \longrightarrow P^*(X)$, a hyperalgebraic system $(X, \vartheta, 0)$ is called a neutro hyper BCK-algebra if it satisfies in the following neutro axioms:

(H1) ($\exists x, y, z \in X$ that $(x \vartheta z) \vartheta (y \vartheta z) \ll x \vartheta y$) and ($\exists x', y', z' \in X$ that $(x' \vartheta z') \vartheta (y' \vartheta z') \not\ll x' \vartheta y'$ or indeterminate)

(H2) ($\exists x, y, z \in X$ that $(x \vartheta y) \vartheta z = (x \vartheta z) \vartheta y$) and ($\exists x', y', z' \in X$ that $(x' \vartheta y') \vartheta z\prime \neq (x' \vartheta z') \vartheta y'$ or indeterminate)

(H3) ($\exists x \in X$ that $x \vartheta X \ll x$) and ($\exists x' \in X$ that $x' \vartheta X \not\ll x'$ or indeterminate)

(H4) ($\exists x, y \in X$ that if $x \ll y$ and $y \ll x$ imply $x = y$) and ($\exists x', y' \in X$ that if $x' \ll y'$ and $y' \ll x'$ imply $x' \neq y'$ or indeterminate),

where $a \ll b$ is defined by $0 \in a \vartheta b$, and $\forall A, B \subseteq H$, $A \ll B \Longleftrightarrow \forall a \in A\, \exists b \in B \text{ s.t } a \ll b$

If $(X, \vartheta, 0)$ is a neutro hyperalgebra and satisfies in condition (H1) to (H4), then we will call it is a neutro hyper BCK-algebra of type 4 (i.e., it satisfies 4 neutro axioms).

Investigation of partial order relation on neutro hyper BCK-algebra plays a main role in Hass diagram, so we have the following results.

Theorem 12. *Let $(X, \vartheta, 0)$ be a neutro hyper BCK-algebra, $x, y, z \in X$ and $A, B, C \subseteq X$. Then*

(i) $\exists x, y \in X$ such that $(x \vartheta y) \ll x$

(ii) $\exists x, y \in X$ such that $(x \vartheta y) \not\ll x$

(iii) $\exists x \in X$ such that $x \ll x$

(iv) $\exists x \in X$ such that $x \not\ll x$

(v) $\exists A, B \subseteq X$ such that $A \ll A$

(vi) $\exists A, B \subseteq X$ such that $A \not\ll A$

Proof. We prove only the item (ii), and other items are similar to it. Since $(X, \vartheta, 0)$ is a neutro hyper BCK-algebra, there exists $x \in X$such that $(x \vartheta X) \not\ll X$. It follows that there exist $a, y \in X$ such that $a \in x \vartheta y$ and $a \not\ll x$. Hence, $(x \vartheta y) \not\ll x$. □

Theorem 13. *Let $(X, \vartheta, 0)$ be a neutro hyper BCK-algebra, $x, y, z \in X$ and $A, B, C \subseteq X$. Then*

(i) if $A \ll B$, then $(A \cup C) \ll (B \cup C)$

(ii) if $A \not\ll B$, then $(A \cup C) \not\ll (B \cup C)$

Proof

(i) Let $a \in A$ be arbitrary. Since $A \ll B$, there exists $b \in B$ such that $a \ll b$. Hence, for $a \in (A \cup C)$, there exists $b \in (B \cup C)$ such that $a \ll b$ and so $(A \cup C) \ll (B \cup C)$.

(ii) Since $A \not\ll B$, there exists $a \in A$ such that for all, $b \in B$, we have $a \not\ll b$. Hence, there exists $a \in (A \cup C)$ such that for all, $b \in (B \cup C)$, we get that $a \not\ll b$ and so $(A \cup C) \not\ll (B \cup C)$. □

Example 2. (i) Every neutro BCK-algebra $(X, \vartheta, 0)$ is a neutro hyper BCK-algebra. Since, for all, $x, y \in X$, can define a hyperoperation ϑ on X by $x \vartheta y = \{x \varrho y\}$.

(ii) Consider $\mathbb{N}^* = \{0, 1, 2, 3, \ldots\}$. Define $x \vartheta y = \begin{cases} \{0, x\} & \text{if } x \leq y \\ 0 & (x, y) = (2,3) \text{ or } (x, y) = (3,2) \\ 2 & x = y = 1 \text{ or } (x, y) = (0,1) \\ x & \text{otherwise} \end{cases}$. Clearly, $(\mathbb{N}^*, \vartheta, 0)$ is a neutro hyper BCK-algebra.

The following theorem shows that neutro hyper BCK-algebras are the generalization of hyper BCK-algebras.

Theorem 14. *Every hyper BCK-algebra can be extended to a neutro hyper BCK-algebra.*

Proof. Let $(X, \vartheta, 0)$ be a hyper BCK-algebra and $\alpha \notin X$. For all, $x, y \in X \cup \{\alpha\}$, define ϑ_α on $X \cup \{\alpha\}$ by $x \vartheta_\alpha y = x \vartheta y$, where, $x, y \in X$ and whence $\alpha \in \{x, y\}$, define $x \vartheta_\alpha y$ is indeterminate or $x \vartheta_\alpha y \in X \cup \{\alpha\}$.

We show that how to construct neutro hyper BCK-algebras from BCK-algebras. □

Example 3. Let $X = \{0, 1, 2, 3, 4\}$ and consider Table 3.
Then

(i) If $a = 0$, then $(X, \vartheta_1, 0)$ is a neutro hyper BCK-algebra and if $a = 1$, then $(X \backslash \{3, 4, 5\}, \vartheta_1, 0)$ is a hyper BCK-algebra

(ii) $(X, \vartheta_2, 0)$ is a neutro hyper BCK-algebra and $(X \backslash \{4, 5\}, \vartheta_2, 0)$ is a hyper BCK-algebra

(iii) If $s = z = 0, w = 3$, then $(X, \vartheta_3, 0)$ is a neutro hyper BCK-algebra, and for $s = 1$, $z = 3$, $(X \backslash \{5\}, \vartheta_3, 0)$ is a hyper BCK-algebra. If $s = z = 0$, $w = \sqrt{2}$, then $(X, \vartheta_3, 0)$ is a neutro hyper BCK-algebra of type 4

The importance of the following theorem is to construct of neutro hyper BCK-algebra from any given nonempty set.

Theorem 15. *Let $0 \notin X \neq \varnothing$. Then there exists a hyperoperation "ϑ" on $X' = X \cup \{0\}$ such that $(X', \vartheta, 0)$ is a neutro hyper BCK-algebra.*

Proof. Let $0 \notin X \neq \varnothing$. Using Theorem 4, there exist a hyperoperation "ϑ" on $X' = X \cup \{0\}$ such that $(X', \vartheta, 0)$ is a hyper BCK-algebra. Now, apply Theorem 14; there exist a hyperoperation "ϑ'" on $X' = X \cup \{0\}$ such that $(X', \vartheta', 0)$ is a neutro hyper BCK-algebra.

Table 3: Neutro hyper BCK-algebras.

ϑ_1	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0	a	2	4	3
2	2	2	0, 2	0	2	0
3	3	0	1	2	4	5
4	1	4	2	1	4	3
5	0	4	0	1	4	0
ϑ_2	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0, 1	0	0, 1	4	5
2	2	2	0	2	5	0
3	3	3	3	0	0	0
4	2	1	2	4	1	2
5	5	0	4	0	0	x
ϑ_3	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0, 2	1	1	s	0
2	2	0, 2	0, 2	0, 2	0, 2	3
3	3	3	3	0, 2	z	0
4	4	4	4	4	0, 2	1
5	2	0	2	2	2	w

Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be two neutro hyper BCK-algebras. Define ϑ on $X_1 \times X_2$ by $(x, y)\vartheta(x', y') = (x\vartheta_1 x', y\vartheta_2 y')$, where $(x, y), (x', y') \in X_1 \times X_2$ and say that $(x, y) \ll (x', y') \iff (0_1, 0_2) \in (x, y)\vartheta(x', y')$. The following theorem investigates the properties of partial order relation on product of Neutro hyper BCK algebras. □

Theorem 16. *Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be two neutro hyper BCK-algebras. Then*

(i) $\forall (x, y), (x', y') \in X_1 \times X_2$, $(x, y) \ll (x', y') \iff (x \not\ll_1 x')$ and $(y \not\ll_2 y')$

(ii) $\forall (x, y), (x', y') \in X_1 \times X_2$, $(x, y) \not\ll (x', y') \iff (x \ll_1 x')$ or $(y \ll_2 y')$

(iii) $\exists (x, y), (x', y') \in X_1 \times X_2$, $(0_1, 0_2) \in ((x, y)\vartheta(x', y'))\vartheta(x, y)$

(iv) $\exists (x, y), (x', y') \in X_1 \times X_2$, $(0_1, 0_2) \notin ((x, y)\vartheta(x', y'))\vartheta(x, y)$

Proof

(i) Immediate

(ii) Let $(x, y), (x', y') \in X_1 \times X_2$. Then $(0_1, 0_2) \in (x, y)\vartheta(x', y')$, if and only if $(0_1, 0_2) \in (x\vartheta_1 x', y\vartheta_2 y')$, if and if only $0_1 \notin x\vartheta x'$ or $0_2 \notin y\vartheta y'$, and if and only if $(x \ll_1 x')$ or $(y \ll_2 y')$

(iii) Since $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be two neutro hyper BCK-algebras, there exist $x, y \in X_1, x', y' \in X_2$ such that $0_1 \in (x\vartheta y)\vartheta x$ and $0_2 \in (x'\vartheta y')\vartheta x'$. It follows that $\exists (x, y), (x', y') \in X_1 \times X_2$, $(0_1, 0_2) \in ((x, y)\vartheta(x', y'))\vartheta(x, y)$

(iv) Since $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be two neutro hyper BCK-algebras, there exist $x, y \in X_1, x', y' \in X_2$ such that $0_1 \notin (x\vartheta y)\vartheta x$ and $0_2 \notin (x'\vartheta y')\vartheta x'$. It follows that $\exists (x, y), (x', y') \in X_1 \times X_2$, $(0_1, 0_2) \notin ((x, y)\vartheta(x', y'))\vartheta(x, y)$

We need to extend neutro hyper BCK-algebras to a larger class of neutro hyper BCK-algebras, so we apply the notation of product on neutro hyper BCK-algebras as follows. □

Theorem 17. *Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be two neutro hyper BCK-algebras. Then $(X_1 \times X_2, \vartheta, (0_1, 0_2))$ is a neutro hyper BCK-algebra.*

Proof. We prove only the item (H4), and other items by Theorem 16 are valid. Since $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ are neutro hyper BCK-algebras, there exist $(x_1, x_2), (y_1, y_2), (x_1', x_2'), (y_1', y_2') \in X_1 \times X_2$ that if $(x_1 \ll_1 y_1, y_1 \ll_1 x_1)$, then $x_1 = y_1$, and if $(x_2 \ll_2 y_2, y_2 \ll_2 x_2)$, then $x_2 = y_2$. Also, if $(x_1' \ll_1 y_1', y_1' \ll_1 x_1')$, then $x_1 \neq y_1$, and if $(x_2' \ll_2 y_2', y_2' \ll_2 x_2')$, then $x_2 \neq y_2$. By (i), it follows that there exist $(x_1, x_2), (y_1, y_2), (x_1', x_2'), (y_1', y_2') \in X_1 \times X_2$ that if $(x_1, x_2) \ll (y_1, y_2)$, $(y_1, y_2) \ll (x_1, x_2)$, we have $(x_1, x_2) = (y_1, y_2)$, and if $(x_1', x_2') \ll (y_1', y_2')$, $(y_1', y_2') \ll (x_1', x_2')$, we have $(x_1', x_2') \neq (y_1', y_2')$.

Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be hyper BCK-algebras, where $X_1 \cap X_2 = \varnothing$. For some $x, y \in X$, define a hyperoperations ϑ_t, ϑ_s as follows:

$$x\vartheta_t y = \begin{cases} (x\vartheta_1 y)\backslash\{0_1\}, & \text{if } x, y \in X_1\backslash X_2, \\ x\vartheta_2 y, & \text{if } x, y \in X_2\backslash X_1, \\ t, & \text{if } x \in X_1,\ y \in X_2, \\ 0_2, & \text{if } x \in X_2,\ y \in X_1, \end{cases}$$
$$x\vartheta_s y = \begin{cases} x\vartheta_1 y, & \text{if } x, y \in X_1\backslash X_2, \\ (x\vartheta_2 y)\backslash\{0_2\}, & \text{if } x, y \in X_2\backslash X_1, \\ s, & \text{if } x \in X_1,\ y \in X_2, \\ 0_1, & \text{if } x \in X_2,\ y \in X_1, \end{cases} \tag{18}$$

and $0_1\vartheta_t 0_1 = 0_1$ $\vartheta_t 0_2 = 0_2\vartheta_t 0_1 = 0_1$, $0_1\vartheta_s 0_2 = 0_2\vartheta_s 0_1 = 0_2\vartheta_s 0_2 = 0_2$, where $0_2 \neq t \in X_2$, $0_1 \neq s \in X_1$. Thus, we have the following theorem.

We want to extend neutro hyper BCK-algebras to a larger class of neutro hyper BCK-algebras, so we apply the notation of union on neutro hyper BCK-algebras as follows. □

Theorem 18. *Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be hyper BCK-algebras, where $X_1 \cap X_2 = \varnothing$ and $X = X_1 \cup X_2$. Then*

(i) For all, $A \subseteq X_1, A \not\ll \{0_1, t\}$

(ii) For all, $A \subseteq X_1, A \not\ll 0_2$

(iii) For all, $A \subseteq X_1, A \not\ll A$, and for all, $B \subseteq X_2, B \not\ll B$

(iv) For all, $A \subseteq X_2, A \not\ll \{0_2, s\}$

(v) For all, $A \subseteq X_2, A \not\ll 0_1$

Proof

(i) Let $A \subseteq X_1$. Then $A\vartheta_t 0_1 = \cup_{a\in A}(a\vartheta_t 0_1) = \cup_{a\in A}((a\vartheta 0_1)\backslash\{0_1\})$. It follows that $0_1 \notin A\vartheta_t 0_1$, so $A \not\ll \{0_1\}$. In addition, $A\vartheta_t t = \cup_{a\in A}(a\vartheta_t t) = \{t\}$ and $0_1 \notin t\vartheta_t 0_1$. It follows that $0_1 \notin A\vartheta_t 0_1$, so $A \not\ll \{t\}$.

(ii) Let $A \subseteq X_1$. Then $A\vartheta_t 0_2 = \cup_{a\in A}(a\vartheta_t 0_t) = \{t\}$ and $0_1 \notin t\vartheta_t 0_2$. It follows that $0_1 \notin A\vartheta_t 0_1$, so $A \not\ll \{0_2\}$. In addition, $A\vartheta_t t = \cup_{a\in A}(a\vartheta_t t) = \{t\}$ and $0_1 \notin t\vartheta_t 0_1$. It follows that $0_1 \notin A\vartheta_t 0_1$, so $A \not\ll \{t\}$.

(iii) Let $A \subseteq X_1$ and $B \subseteq X_2$. Since $A\vartheta_t A = \cup_{a,a'\in A}(a\vartheta_t a') = \cup_{a,a_I\in A}((a\vartheta_t a')\backslash\{0_1\})$ and $B\vartheta_s S = \cup_{b,b'\in B}(b\vartheta_t b') = \cup_{b,b'\in B}((b\vartheta_s b')\backslash\{0_2\})$, we get that $0_1 \not\in A\vartheta_t A$ and $0_2 \not\in B\vartheta_s B$. Thus $A \not\ll A$ and $B \not\ll B$.

(iv) and (v) are similar to (i) and (ii), respectively. □

Theorem 19. *Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be hyper BCK-algebras, where $X_1 \cap X_2 = \varnothing$ and $X = X_1 \cup X_2$. Then*

(i) $(X, \vartheta_t, 0_1)$ is a neutro hyper BCK-algebra

(ii) $(X, \vartheta_s, 0_2)$ is a neutro hyper BCK-algebra

Proof

(i) $\overline{(H_1:)}$ For some, $x, y, z \in X_2\backslash X_1$, $(x\vartheta_t z)\vartheta_t \overline{(y\vartheta_t z) \ll (x\vartheta_t y)}$. Since, for $x \in X_1$, $(((x\vartheta 0_1)\backslash\{0_1\})\backslash\{0_1\})\vartheta_t 0_2 = t \neq 0_2$, we get that

$$\begin{aligned}(x\vartheta_t 0_1)\vartheta_t(0_2\vartheta_t 0_1) &= ((x\vartheta 0_1)\backslash\{0_1\})\vartheta_t 0_1\\ &= ((x\vartheta 0_1)\backslash\{0_1\})\backslash\{0_1\} \ll 0_2 = 0_1\vartheta_t 0_2.\end{aligned} \tag{19}$$

$\overline{(H_2:)}$ For some, $x, y, z \in X_2\backslash X_1$, $(x\vartheta_t y)\vartheta_t z = \overline{(x\vartheta_t z)\vartheta_t} y$. In addition, for $x \in X_1$,

$$\begin{aligned}(x\vartheta_t 0_2)\vartheta_t 0_1 &= t\vartheta_t 0_1 = 0_2 \neq t = ((x\vartheta 0_1)\backslash\{0_1\})\vartheta_t 0_2\\ &= (x\vartheta_t 0_1)\vartheta_t 0_2.\end{aligned} \tag{20}$$

$\overline{(H_3:)}$ For some, $x \in X_2 X_1, x\vartheta_t X = x\vartheta X_2 \ll X_2 = X$. Since $\overline{t\vartheta_t 0_1} = 0_2$ and $(\cup_{x\in X_1}((0_1\vartheta x)\ \{0_1\}))\vartheta_t 0_1 = (\cup_{x\in X_1}((0_1\vartheta x)\backslash\{0_1\}))\backslash\{0_1\}$, we get that

$$\begin{aligned}0_1\vartheta_t X &= (0_1\vartheta_t X_1)\cup(0_1\vartheta_t X_2) = \left(\bigcup_{x\in X_1}(0_1\vartheta_t x)\right)\cup\left(\bigcup_{y\in X_2}(0_1\vartheta_t y)\right)\\ &= \left(\bigcup_{x\in X_1}(0_1\vartheta x)\backslash\{0_1\}\right)\cup\{t\} \ll 0_1.\end{aligned} \tag{21}$$

$\overline{(H_3:)}$ Because $0_1 \ll 0_1$ and $0_1 \in 0_1\vartheta_t 0_2$ and $0_1 \in \overline{0_2\vartheta_t 0_1}$, while $0_1 \neq 0_2$, we get the item $\overline{(H_3:)}$ is valid. Therefore, $(X, \vartheta_t, 0_1)$ is a neutro hyper BC$\overline{\text{K-alge}}$bra.

(ii) It is similar to item (i). □

4.1. Application of Neutro Hyper BCK-Algebras and Single-Valued Neutrosophic Hyper BCK-Subalgebras. In this subsection, we describe some applications of neutro hyper BCK-algebra and single-valued neutrosophic hyper BCK-subalgebra in some complex (hyper) networks.

Table 4: Neutro hyper BCK-algebra of an economic network.

ϑ	a	b	c	d	e	f
a	a	a	a	a	a	f
b	b	a,c	b	b	a	a
c	c	a,c	a,c	a,c	a,c	d
d	d	d	d	a,c	a	a
e	e	e	e	e	a,c	b
f	c	a	c	c	c	???

Table 5: Single-valued neutrosophic hyper BCK-subalgebra of a data network.

ϑ	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{a,b\}$	$\{a,b\}$	$\{e,b\}$	$\{a,b\}$
c	$\{c\}$	$\{c\}$	$\{a,b\}$	$\{c\}$	$\{c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{a,b\}$	$\{d\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{a,b\}$
	a	b	c	d	e
T_B	1	1	0.2	0.4	0.6
I_B	0.1	0.1	0.3	0.7	0.9
F_B	0.05	0.05	0.25	0.45	0.65

Example 4 (economic network). Let $X = \{a = \text{China}, b = \text{Italy}, c = \text{Iran}, d = \text{Spain}, e = \text{Germany}, f = \text{USA}\}$ be a set of top countries, which are in an economic network. Suppose ϑ is the relations on X, which is described in Table 4, and for $x \neq y$, $x * y = D$ means that D is the set of countries that benefit from this economic partnership, whence the country x starts to country y, and for $x = y$, it means that the country x maintains its capital.

Clearly, $(X, *, \text{China})$ is a neutro hyper BCK-algebra in this model. We obtain that the USA is main source of this network; since if the USA starts to any other country, it does not benefit. In addition, if the USA starts to itself, this participation becomes indeterminate. Also, if any country starts to China, we conclude that China loss, else with USA, and if China starts to any other country, then China benefit else USA.

Example 5 (data network). Let $Y = \{a, b, c, d, e\}$ be a set of mobile sets, which are in a data network. Suppose ϑ is the relations on Y, which is described in Table 3, and for all, $x \neq , x * y = D$ means that D is a set of mobile sets that receive contents of messages that mobile set x starts to mobile set y, and for $x = y$, it means that the mobile set x retains its information. In addition, for any $y \in Y, T_B(y), I_B(y), F_B(y)$ are the cryptographic power, battery life, and RAM of mobile set y, respectively. Then $B = (T_B, I_B, F_B)$ is a single-valued neutrosophic hyper BCK-subalgebra of (Y, ϑ, a) in Table 5.

It is clear that if mobile set named "a" starts, then none of the devices receive the message, and if other devices start to name a mobile set "a", then this device (mobile set a) cannot receive their messages; hence, it is not suitable node in this network, since furthermore to its complex cryptography, its battery life, and RAM is weak. Also, one can see that the mobile set b is the best in this regard.

5. Conclusion

To conclude, the current paper has presented and analyzed the notion of single-valued neutrosophic hyper BCK-subalgebras and neutro hyper BCK-algebras and investigated some of their new useful properties. We defined the concept of the extended single-valued neutrosophic BCK-subalgebras and showed that for any $\alpha \in [0,1]$ and a single-valued neutrosophic subset hyper BCK-subalgebra, $A = (T_A, I_A, F_A)$, $A = (T_{A\alpha}, I_{A\alpha}, F_{A\alpha})$ is a hyper BCK-subalgebra. Through the concept of fundamental relation $C(\beta)$, we have generated the single-valued neutrosophic BCK-subalgebras from single-valued neutrosophic hyper BCK-subalgebras, so some categorical properties of single-valued neutrosophic BCK-subalgebras are investigated based on the categorical properties of single-valued neutrosophic hyper BCK-subalgebras. In addition, on any nonempty set, we have constructed at least one single-valued neutrosophic BCK-subalgebra and one extendable single-valued neutrosophic BCK-subalgebra. The concept of neutro hyper BCK-algebra as a generalization of neutro BCK-algebra is introduced in this study, and it is constructed the class of product of neutro hyper BCK-algebras and union of neutro hyper BCK-algebras via hyper BCK-algebras. In study of neutro hyper BCK-algebras, despite having key mathematical tools, there are some limitations. The union of two neutro hyper BCK-algebras is not necessarily; a neutro hyper BCK-algebras so the class of neutro hyper BCK-algebras is not closed under any given algebraic operation. In addition, neutro hyper BCK-algebras are different from single-valued neutrosophic hyper BCK-subalgebras so could not generalize the capabilities of single-valued neutrosophic hyper BCK-subalgebras to neutro hyper BCK-algebras and conversely. In final, we can apply these concepts in real world, especially in some complex (hyper) networks.

We hope that these results are helpful for further studies in single-valued neutrosophic logical algebras. In our future studies, we hope to obtain more results regarding single-valued neutrosophic (hyper) logical-subalgebras, neutro (hyper) logical-subalgebras, and their applications.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] Y. B. Jun, M. M. Zahedi, X. L. Xin, and R. A. Borzooei, "On hyper BCK-algebras," *Italian Journal of Pure and Applied Mathematics*, vol. 10, pp. 127–136, 2020.

[2] Y. Imai and K. Iseki, "On axiom systems of propositional calculi, XIV," *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, vol. 42, pp. 19–22, 1996.

[3] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[4] F. Smarandache, *Neutrosophy, Neutrosophic Probability, Set, and Logic*, Pro Quest Information & Learning, Ann Arbor, MI, USA, 1998.

[5] F. Smarandache, "Neutrosophic set, a generalisation of the intuitionistic fuzzy sets," *International Journal of Pure and Applied Mathematics*, vol. 24, pp. 287–297, 2005.

[6] A. Alsubie and A. Al-Masarwah, "MBJ-neutrosophic hyper BCK-ideals in hyper BCK-algebras," *AIMS Mathematics*, vol. 6, no. 6, pp. 6107–6121, 2021.

[7] A. Alsubie, A. Al-Masarwah, Y. B. Jun, and A. G. Ahmad, "An approach to BMBJ-neutrosophic hyper-BCK-ideals of hyper-BCK-algebras," *Journal of Mathematics*, vol. 2021, Article ID 5552060, 10 pages, 2021.

[8] A. Al-Masarwah and A. G. Ahmad, "Structures on doubt neutrosophic ideals of BCK/BCI-algebras under (S, T)-norms," *Neutrosophic Sets and Systems*, vol. 33, 2020.

[9] B. Bordbar, M. M. Takallo, R. A. Borzooei, and Y. B. Jun, "BMBJ-neutrosophic subalgebras in BCI/BCK-algebras," *Neutrosophic Sets System*, vol. 13, pp. 1–13, 2020.

[10] Y. B. Jun and E. H. Roh, "MBJ-neutrosophic ideals of BCK/BCI-algebras," *Open Mathematics*, vol. 17, no. 1, pp. 588–601, 2019.

[11] Y. Jun, S. Kim, and F. Smarandache, "Interval neutrosophic sets with applications in BCK/BCI-algebra," *Axioms*, vol. 7, no. 2, p. 23, 2018.

[12] S. Khademan, M. M. Zahedi, R. A. Borzooei, and Y. B. Jun, "Neutrosophic hyper BCK-ideals," *Neutrosophic Sets and Systems*, vol. 27, p. 201, 2019.

[13] G. Muhiuddin, A. Al-Kenani, E. Roh, and Y. Jun, "Implicative neutrosophic quadruple BCK-algebras and ideals," *Symmetry*, vol. 11, no. 2, p. 277, 2019.

[14] G. Muhiuddin, H. Harizavi, and Y. B. Jun, "Bipolar-valued fuzzy soft hyper BCK ideals in hyper BCK algebras," *Discrete Mathematics, Algorithms and Applications*, vol. 12, no. 2, Article ID 2050018, 2020.

[15] Y. Saber, F. Alsharari, and F. Smarandache, "On single-valued neutrosophic ideals in Sostak sense," *Symmetry*, vol. 12, no. 2, pp. 1–21, 2020.

[16] Y. J. Seo, H. S. Kim, Y. B. Jun, and S. S. Ahn, "Multipolar intuitionistic fuzzy hyper BCK-ideals in hyper BCK-algebras," *Mathematics*, vol. 8, no. 8, p. 1373, 2020.

[17] F. Smarandache, "Neutro algebra is a generalization of partial algebra," *International Journal of Neutrosophic Science*, vol. 2, no. 1, pp. 08–17, 2020.

[18] M. Hamidi and F. Smarandache, "Neutro-BCK-algebra," *International Journal of Neutrosophic Science*, vol. 8, pp. 110–117, 2018.

[19] A. A. A. Agboola, "Introduction to neutro groups," *International Journal of Neutrosophic Science*, vol. 6, pp. 41–47, 2020.

[20] F. Smarandache, "Extension of hypergraph to n-superhypergraph and to plithogenic n-superhypergraph, and extension of hyperalgebra to n-ary (classical-/neutro-/anti-) hyperalgebra," *Neutrosophic Sets and Systems*, vol. 33, pp. 290–296, 2020.

[21] R. A. Borzooei, R. Ameri, and M. Hamidi, "Fundamental relation on hyper BCK-algebras," *Analele Universității Oradea*, vol. 21, no. 1, pp. 123–136, 2014.

[22] J. Dongho, "Category of fuzzy hyper BCK-algebras," 2011, https://arxiv.org/abs/1101.2471.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 9938693, 7 pages
https://doi.org/10.1155/2021/9938693

Research Article

Neutrosophic Logic-Based Document Summarization

O. G. El Barbary [1] and Radwan Abu Gdairi [2]

[1] *Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt*
[2] *Department of Mathematics, Faculty of Science, Zarqa University, Zarqa, Jordan*

Correspondence should be addressed to O. G. El Barbary; omniaelbarbary@yahoo.com

Received 5 March 2021; Revised 3 July 2021; Accepted 13 July 2021; Published 26 July 2021

Academic Editor: Broumi Said

Nowadays, rich quantity of information is offered on the Net which makes it hard for the clients to detect necessary information. Programmed techniques are desirable to effectively filter and search useful data from the Net. The purpose of purported text summarization is to get satisfied content handling with information variety. The main factor of document summarization is to extract benefit feature. In this paper, we extract word feature in three group called important words. Also, we extract sentence feature depending on the extracted words. With increasing knowledge on the Internet, it turns out to be an extremely time-consuming, exhausting, and boring mission to read the whole content and papers and get the relevant information on precise topics

1. Introduction

By increasing the knowledge on the Internet, it turns out to be an extremely time-consuming and boring mission to read whole content and papers and get the relevant information on precise topics. Content summarization is recognized as a key for this matter as it generates programmed briefing of the data. Summarization of text can be defined as an abbreviated version of generated text from several documents without down core contents or impression of the original documents and expressive summary of a certain manuscript by covering greatest imperative part of the contents and with smallest redundancy from different contribution resources. There are various types of content summarization depending on rate of recurrence of input sources, the technique of generated summary, the goal of summary, and the input and output language of summarization process.

Recently, the theory of neutrosophic logic and sets has been introduced. Florentin [1, 2] presented the neutrosophic logic. It is a decision in which each proposition is valued to have three grades such as a grade of truth (T), a grade of indeterminacy (I), and a grade of falsity (F). A neutrosophic set is defined as a set where every component of the universe has a grade of truth, indeterminacy, and falsity, respectively, and lies between $[0, 1]^*$, which is the nonstandard unit interval [3–5]. There are various applications using neutrosophic logic as in [6, 7].

In this paper, we propose neutrosophic logic centered multidocument summarization procedure to debrief vital sentences to create nonredundant summary. The projected approach is associate degree extractive primarily built generic report system, and outline within the context of this projected work is matter outline created from one or many news connected documents.

The paper is well-structured as follows. In Section 2, we give some basic concepts on the text summarization system. Section 3 introduces the proposed summarization technique. The fundamentals of neutrosophic sets are introduced in Section 4. The basics of using neutrosophic sets based on information retrieval are introduced in Section 5. Section 6 is devoted to present our approach to document summarization using distance between neutrosophic sets. The conclusion of paper is given in Section 7.

2. Text Summarization

As previously said, text summary is a condensed version of a document that retains the major points and ideas of the original material(s). The goal of a summarizing system is to offer a concise and fluid overview of a given text by addressing the most important parts of the material while minimizing redundancy from various in-place sources.

There exists a range of taxonomies for text summarization [8–12] supported frequency of input sources, the means of outline generated, purpose of outline, and language of input sources.

There are 2 varieties of algorithms regarding which varied works are printed around text summarization. They are extraction-based summarization and abstraction-based summarization.

The extraction-based technique works by extracting sentences from a document. There is no compression in any format during this technique. It is just a matter of memorizing sentences in order to create a more compact outline.

Abstraction-based reports, on the other hand, are effective. Apart from memorizing the most important sentences, it alters the way a text is organized. The retrieved text is regenerated. It is categorized as a single document or multidocument report depending on the number of input sources considered for generating the outline. Once a document has been provided as an input for a text report, it is known as a single document report, whereas a multidocument report uses a collection of papers as input to create the outline.

The outline of a domain-specific report is generated using domain-specific data, whereas the outline of a domain freelance report (generic) is generated using generic alternatives. Domain-specific report approaches have become popular among academics.

In this research, we offer a document summarization system based on neutrosophic logic for extracting relevant sentences and generating a summary. The planned approach is an extraction-based generic report system, and the outline in this planned work is a matter outline created from one or more news-related papers.

3. The Proposed Document Summarization Technique

Summary is not sufficient to just generate words and phrases that apprehend the source document. Summary also must be accurate and read fluently as a new separate document. Summarization of text [3, 13–15] is the duty of creating a brief and fluent summary while retaining the overall meaning and information content. The process of summarization takes some steps: first is the preprocessing of data; second is the feature word extraction; third is the feature sentence extraction; and the last step is the organization of the set of documents to produce the summary. In the last step, we use the neutrosophic logic, and we illustrate it later.

3.1. Input Preprocessing. Some preprocessing activities are required for the set of raw documents before they can be entered into the planned technique.

(i) Words that should be avoided or removed: The most commonly used terms, such as “a,” “an,” and “the,” do not have any linguistics data related to the text area unit. All of the stop words have been preprogrammed and saved in a separate file.

(ii) *Stemming.* This is the process of converting all words to their root type by eliminating their prefix and suffix. For the stemming procedure, we employed a porter stemmer.

(iii) *Removal of Special Characters.* House character removes all special characters from a collection of input documents, including punctuation, interrogation, and exclamation.

(iv) *Segmentation Process.* This is a method of extracting each sentence from a document independently. All sentences from documents are retrieved and saved in this manner.

(v) When a sentence is segmented, the tokenization process is applied to all of the sentences. It is a technique for isolating words from sentences. It is used to define the character structure, such as the date, time, punctuation, and number.

3.2. Feature Extraction. To perform an efficient document summarization, we consider the feature extraction. Feature extraction is not only limited on words but also on sentence. In the following subsections, we illustrate our method to extract words with different levels of strength. Also, the sentence extraction depends on words feature.

The preprocessed knowledge in word is used to see sentence score in the feature extraction phase. The effectiveness of different sentence evaluation methods is determined by the type of text, genre of text, language, and structure of contribution text. The main belief is that completely distinct themes will enjoy different characteristics, which can be differentiated by a variety of possibilities.

All the text selections are divided into two categories: word level and sentence level alternatives. We have run tests on various combinations of shallow text options on various datasets to find the optimum mix of options that will deliver the greatest results in terms of coverage and relevancy for the news domain. The options that were used in the planned strategy are listed below.

3.2.1. Word Features. The previous methods of text summarization depend on word information in the whole documents. Another way, we can extract feature that recognizes topics by using words without reading the whole document. For example, word “algorithm” can indicate the document field “computer science”; the appearance of this word in any sentence means that this sentence is also important.

The term “document field” refers to basic and mutual information that is useful in human communication.

A field tree is a visual representation of document field relationships. The field tree’s leaf nodes are parallel to terminal fields, super-fields are nodes connected to the root, and other nodes are middle fields. Text field can be cleared efficiently if there are many important words and if the frequency rate is high. Therefore, we can define three levels of important words (IM-W) which will be more effective than using full documents as traditional methods. The three levels of IM-W are defined as follows:

(IM-W) 1. This appears with title of document and in one terminal field, and we can calculate it as follows: for the root of supper field F, the child field is F/c; the following formula is used to justify whether or not the word w is (IM-W) 1.

$$\text{concentration}(w, c) = \left[\frac{\text{appearance}(w, < c >)}{\text{appearance}(w, < F >)}\right] \geq \partial, \partial \leq 0.5,$$
$$\text{appearance}(w, < F >) = \frac{\text{the frequency number for appear the word } w \text{ in the field } F}{\text{total number of words in field } F}. \tag{1}$$

(IM-W) 2. This appears with more than one terminal field in one medium field.

(IM-W) 3. This appears only with one medium field.

3.2.2. Sentence Features. Sentence features are the most important to construct the summary. Two features of sentences are identified: the first is the sentence that contains IM-W and the second is sentence length, and the short sentences do not give any vital information, so short sentences are not recommended. Sentence length score is computed as follows:

$$\text{length of sentence}(S_i) = \frac{\text{number of of word occuring in sentence } S}{\text{number of words occuring in a long sentence}}. \tag{2}$$

3.3. Summarization Process. The summarization process [16–18] is done with three steps. First, all the sentences are arranged from the highest to lowest score achieved using the neutrosophic approach. Sentences are chosen based on their degree of resemblance to other sentences in the summary. We used the following formula to determine sentence similarity: Euclidian distance between two neutrosophic sets which is explained in Section 6. The second step is the optimization process; in this step, we delete the repeated sentence and delete the similar sentence which contains the largest number of similar words. The third step is sentences arrangement. Sentences are organized in the final summary in the order in which they appeared in the foundation documents. We have laid up certain guidelines for you, which are as follows:

(1) Sentences are arranged in declining order of their importance

(2) If two sentences in the same document have the same score and are at the same location, the sentence in the earlier document is given priority over the other sentence

4. Neutrosophic Sets

The neutrosophic set is an influential general frame that has been recently proposed by F. Smarandache in [1, 2]. He presented the grade of indeterminacy (I) as an independent component. At this point, the scale of truth, indeterminacy, and falsity corresponds to any element of a neutrosophic set in an ordinary unit interval [0, 1].

Neutrosophic set definition: Let D be a general set, and a single-valued neutrosophic set is an item $W = \{\langle d, T_w(d), I_w(d), F_w(d)\rangle: d \in D$ that is categorized by three membership functions. $T_w(d)\colon D \longrightarrow [0,1]$ is a truth-membership function, $I_w(d)\colon D \longrightarrow [0,1]$ is an indeterminacy-membership function, and $F_w(d)\colon D \longrightarrow [0,1]$ is a falsity-membership function. The total sum $T_w(d) + I_w(d) + F_w(d)$ of any element $d \in D$ deceptions in the closed interval [0, 3].

5. Information Retrieval Based on Neutrosophic Sets $\tilde{N}$

El in [19] discusses the fundamentals of information retrieval using neutrosophic sets as follows.

Let D be a limited set of documents, $D = \{d_1, d_2, \ldots, d_n\}$. W is a set of words, $W = \{w_1, w_1, \ldots, w_j\}, w_j \in d_i$; the neutrosophic set $\tilde{N}$ in D is considered by a truth-membership function $t_{\tilde{N}}$, an indeterminacy-membership function $i_{\tilde{N}}$, and a falsity-membership function $f_{\tilde{N}}$, wherever $t_{\tilde{N}}, i_{\tilde{N}}, f_{\tilde{N}}\colon D \longrightarrow [0,1]$ are functions and $\forall d \in D$, $d \equiv d(t_{\tilde{N}}d(w), i_{\tilde{N}}d(w), f_{\tilde{N}}d(w)) \in N$. Consider a neutrosophic single-valued element of $\tilde{N}$.

A neutrosophic single-valued [8–12, 20] set $\tilde{N}$ over a limited universe $D = \{d_1, d_2, \ldots, d_n\}$ is characterized as follows:

$$\begin{aligned} N = {} & (d_1, < t_{\tilde{\mathrm{N}}} d_1(w_i), i_{\tilde{\mathrm{N}}} d_1(w_i), f_{\tilde{\mathrm{N}}} d_1(w_i) >) \\ & + (d_2, < t_{\tilde{\mathrm{N}}} d_2(w_i), i_{\tilde{\mathrm{N}}} d_2(w_i), f_{\tilde{\mathrm{N}}} d_2(w_i) >) \\ & + \cdots + (d_n, < t_{\tilde{\mathrm{N}}} d_n(w_i), i_{\tilde{\mathrm{N}}} d_n(w_i), f_{\tilde{\mathrm{N}}} d_n(w_i) >), \end{aligned} \tag{3}$$

where $t_{\tilde{\mathrm{N}}} d_i(w_j) = S - r_{d_i}(w_j)/S$,

$$\begin{aligned} i_{\tilde{\mathrm{N}}} d_i(w_j) &= \frac{r_{d_i}(w_j)}{M}, \\ f_{\tilde{\mathrm{N}}} d_i(w_j) &= \frac{r_{d_i}(w_j)}{S}, \\ S &= \sum_{j=1}^{n} r d_i(w_j), \\ M &= \sum_{k=1}^{m} r d_k(w_j), \end{aligned} \tag{4}$$

where r is the number of appearance of the word w_j in the document d_i, S is the number of appearance of the word w_j in the set D, and M is the number of appearance of the word w_j in the subset $\tilde{\mathrm{N}}$.

6. Document Summarization Based on Neutrosophic Sets

We use the distance between two Neutrosophic sets [21, 22] to create a summary with related and closely-related sentences. Single-valued neutrosophic sets [18, 23] are a type of neutrosophic set that were motivated by a practical argument and can be employed in real-world applications like science and engineering. Distance and similarity are important concepts in a variety of fields, including psychology, linguistics, and computer intelligence.

6.1. Neutrosophic Summarization Technique Using Euclidian Distance between Two Neutrosophic Sets. We introduce the distance between two sentences as a single-valued neutrosophic.

Let the sets S_1 and S_2 be defined over the finite universe $D = \{S_1, S_2, \ldots, S_n\}$, and let S_1 *and* S_2 be two single-valued neutrosophic sets in $D = \{S_1, S_2, \ldots, S_n\}$. Then, the distance between S_1 and S_2 is as follows:

$$S_n(S_i, S_j) = \sum_{k=1}^{m} \left[\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left\{ \left| t_{S_i}(w_k) - t_{S_j}(w_k) \right| + \left| i_{S_i}(w_k) - i_{S_j}(w_k) \right| + \left| f_{S_i}(w_k) - f_{S_j}(w_k) \right| \right\} \right]. \tag{5}$$

The Euclidian distance between S_i and S_j is defined as follows:

$$e(S_i, S_j) = \sqrt{\sum_{k=1}^{m} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left\{ \left(t_{S_i}(w_k) - t_{S_j}(w_k) \right)^2 + \left(I_{S_i}(w_k) - I_{S_j}(w_k) \right)^2 + \left(f_{S_i}(w_k) - f_{S_j}(w_k) \right)^2 \right\}}. \tag{6}$$

The normalized Euclidian distance between S_1 and S_2 is defined as follows:

$$q_N(S_i, S_j) = \sqrt{\frac{1}{3n} \sum_{k=1}^{m} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left\{ \left(t_{S_i}(w_k) - t_{S_j}(w_k) \right)^2 + \left(I_{S_i}(w_k) - I_{S_j}(w_k) \right)^2 + \left(f_{S_i}(w_k) - f_{S_j}(w_k) \right)^2 \right\}}. \tag{7}$$

Example 1. In this example, we explain the whole method in one document, let us have a topic called "computer and math," and this topic considers a field and a part from the field tree as shown in Figure 1.

We take an article from the subfield "computer science," an article under title "Environmental impact of computation and the future of green computing." Assume that $\mathbf{S} = \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4, \mathbf{S}_5, \mathbf{S}_6\}$ is a set of extracted sentence from the document, the set of important words are $\mathbf{W}$ ={ Environmental, impact, computation, future, green, computing }, and $\tilde{\mathrm{N}}$ is a subset of sentence from $\aleph = \{\mathbf{S}_1, \mathbf{S}_3, \mathbf{S}_5\}$. They were selected according to the occurrence of the set of

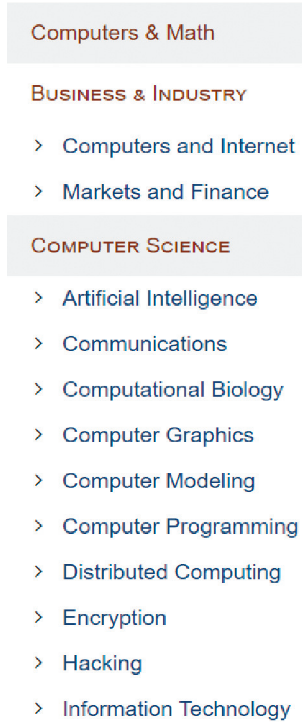


Figure 1: Part from field tree.

keywords W where $\mathbf{t}_{\tilde{N}}\, \mathbf{S_i}\,(\mathbf{w_j})$, $\mathbf{i}_{\tilde{N}}\, \mathbf{S_i}\,(\mathbf{w_j})$, and $\mathbf{f}_{\tilde{N}}\, \mathbf{S_i}\,(\mathbf{w_j})$ are a degree of 'strong occurrence of important words,' a degree of 'indeterminacy of important words,' and a degree of 'poor occurrence of important words,' respectively. The following step is to determine the Euclidean distance between two sentences like S_1 and S_2:

$$
\begin{aligned}
q_N(S_1, S_3) &= \sqrt{\frac{1}{3n}\sum_{k=1}^{m}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left\{\left(t_{S_i}(w_k) - t_{S_j}(w_k)\right)^2 + \left(I_{S_i}(w_k) - I_{S_j}(w_k)\right)^2 + \left(f_{S_i}(w_k) - f_{S_j}(w_k)\right)^2\right\}} \\
&= \sqrt{\frac{1}{3n}\sum_{k=1}^{m}\left\{\left(t_{S_1}(w_k) - t_{S_3}(w_k)\right)^2 + \left(I_{S_1}(w_k) - I_{S_3}(w_k)\right)^2 + \left(f_{S_1}(w_k) - f_{S_3}(w_k)\right)^2\right\}}.
\end{aligned}
\tag{8}
$$

Number of occurrence of keywords in the documents is as follows: Environmental "7," impact "6," computation "6," future "3," green "4," and computing "13." A single value for neutrosophic set N is given in Table 1.

Table 1: A single value for neutrosophic set N.

D	Environmental	Impact	Computation	Future	Green	Computing
$\mathbf{S}_1$	{0.67, 0.18, 0.023}	{0.66, 0.21, 0.13}	{0.56, 0.12, 0.19}	{0.78, 0.34, 0.18}	{0.49, 0.32, 0.21}	{0.71, 0.12, 0.16}
$\mathbf{S}_3$	{0.87, 0.22, 0.13}	{0.84, 0.12, 0.16}	{0.61, 0.11, 0.13}	{0.52, 0.41, 0.13}	{0.84, 0.15, 0.09}	{0.69, 0.21, 0.19}
$\mathbf{S}_5$	{0.75, 0.18, 0.25}	{0.73, 0.16, 0.14}	{0.67, 0.32, 0.04}	{0.71, 0.23, 0.14}	{0.71, 0.13, 0.16}	{0.74, 0.19, 0.21}

Example 2. From the data of Example 1 and Table 1, the normalized Euclidian distance between S_1 and S_3 is given as follows:

$$
\begin{aligned}
q_N(d_1, d_3) &= \sqrt{\frac{1}{3n}\sum_{k=1}^{m}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left\{\left(t_{s_i}(w_k)-t_{s_j}(w_k)\right)^2+\left(I_{s_i}(w_k)-I_{s_j}(w_k)\right)^2+\left(f_{s_i}(w_k)-f_{s_j}(w_k)\right)^2\right\}},\\
&= \sqrt{\frac{1}{3n}\sum_{k=1}^{m}\left\{\left(t_{s_1}(w_k)-t_{s_3}(w_k)\right)^2+\left(I_{s_1}(w_k)-I_{s_3}(w_k)\right)^2+\left(f_{s_1}(w_k)-f_{s_3}(w_k)\right)^2\right\}}\\
&= \sqrt{\frac{1}{3n}\sum_{k=1}^{m}\left\{\left(t_{s_1}(w_1)-t_{s_3}(w_k)\right)^2+\left(I_{s_1}(w_k)-I_{s_3}(w_k)\right)^2+\left(f_{s_1}(w_k)-f_{s_3}(w_k)\right)^2\right\}}\\
&= \sqrt{\frac{1}{3n}\left\{\begin{array}{c}
\left[(0.67-0.87)^2+(0.18-0.22)^2+(0.023-0.13)^2\right]+\\
\left[(0.66-0.84)^2+(0.21-0.12)^2+(0.13-0.16)^2\right]+\\
\left[(0.56-0.61)^2+(0.12-0.11)^2+(0.19-0.13)^2\right]+\\
\left[(0.78-0.52)^2+(0.34-0.41)^2+(0.18-0.13)^2\right]+\\
\left[(0.49-0.84)^2+(0.32-0.15)^2+(0.21-0.09)^2\right]+\\
\left[(0.71-0.69)^2+(0.12-0.21)^2+(0.16-0.19)^2\right]
\end{array}\right\}}\\
&= \sqrt{\frac{1}{6}(0.05304+0.0414+0.0062+0.075+0.1658+0.0094)}\\
&= 0.2418.
\end{aligned}
\tag{9}
$$

7. Conclusions and Future Works

The aim of our work is to study another method of text summarization based on neutrosophic sets. The benefit of using neutrosophic sets is that they are used as a good mathematical tool for document summarization via distance between two neutrosophic sets.

The expected future work for our paper is to compare this method of document summarization with other methods like fuzzy logic and fuzzy ontology.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] F. Smarandache, "(t, i, f)-Neutrosophic Structures & I-Neutrosophic Structures," *Neutrosophic Sets and Systems*, vol. 8, pp. 3–9, 2015.

[2] F. Smarandache, *A Unifying Field in Logics, Neutrosophy: NeutrosophicProbability, Set and Logic*, American Research Press, Champaign, IL, USA, 1999.

[3] M. E. Abd El-Monsef, A. El-Sayed, and O. G. El-Barbary, "Combining FA words with vector space models for Arabic text categorization," *International Information*

Institute (Tokyo) Information, vol. 16, no. 6, pp. 3517–3528, 2013.

[4] C. Antony Crispin Sweety and I. Arockiarani, "Topological structures of fuzzy neutrosophic rough sets," *Neutrosophic Sets and Systems*, vol. 9, pp. 50–57, 2015.

[5] A. Q. Ansari, R. Biswas, and S. Aggarwal, "Neutrosophic classifier: an extension of fuzzy classifier," *Applied Soft Computing*, vol. 13, no. 1, pp. 563–573, 2013.

[6] S. Broumi, K. Ullah, A. Bakali et al., "Novel system and method for telephone network planning based on neutrosophic graph," *Global Journal of Computer Science and Technology*, vol. 18, no. 2, pp. 1–10, 2018.

[7] S. Broumi, P. Kumar Singh, P. K. Singh et al., "Single-valued neutrosophic techniques for analysis of WIFI connection," *Advances in Intelligent Systems and Computing* in *Proceedings of the International Conference on Advanced Intelligent Systems for Sustainable Development AI2SD'2019*, vol. 915, pp. 405–412, Tangier, Morocco, March 2019.

[8] M. Moradi and N. Ghadiri, "Different approaches for identifying important concepts in probabilistic biomedical text summarization," *Artificial Intelligence in Medicine*, vol. 84, pp. 101–116, 2017.

[9] K. Nandhini and S. R. Balasundaram, "Improving readability through extractive summarization for learners with reading difficulties," *Egyptian Informatics Journal*, vol. 14, no. 3, pp. 195–204, 2013, http://texlexan.sourceforge.net/.

[10] H. Oufaida, O. Nouali, and P. Blache, "Minimum redundancy and maximum relevance for single and multi-document Arabic text summarization," *Journal of King Saud University-Computer and Information Sciences*, vol. 26, no. 4, pp. 450–461, 2014.

[11] J.-P. Qiang, P. Chen, W. Ding et al., "Multi-document summarization using closed patterns," *Knowledge-Based Systems*, vol. 99, pp. 28–38, 2016.

[12] U. Rivieccio, "Neutrosophic logics: prospects and problems," *Fuzzy Sets and Systems*, vol. 159, no. 14, pp. 1860–1868, 2008.

[13] R. Z. Al-Abdallah and A. T. Al-Taani, "Arabic single-document text summarization using particle swarm optimization algorithm," *Procedia Computer Science*, vol. 117, pp. 30–37, 2017.

[14] R. M. Alguliev, R. M. Aliguliyev, and C. A. Mehdiyev, "Sentence selection for generic document summarization using an adaptive differential evolution algorithm," *Swarm and Evolutionary Computation*, vol. 1, no. 4, pp. 213–222, 2011.

[15] R. M. Aliguliyev, "A new sentence similarity measure and sentence based extractive technique for automatic text summarization," *Expert Systems with Applications*, vol. 36, no. 4, pp. 7764–7772, 2009.

[16] Z.-P. Tian, H.-Y. Zhang, J. Wang, J.-Q. Wang, and X.-H. Chen, "Multi-criteria decision-making method based on a cross-entropy with interval neutrosophic sets," *International Journal of Systems Science*, vol. 47, no. 15, pp. 3598–3608, 2015.

[17] A. R. Pal and D. Saha, "An approach to automatic text summarization using WordNet," in *Proceedings of the Advance Computing Conference (IACC), 2014 IEEE International*, pp. 1169–1173, Gurgaon, India, February 2014.

[18] D. Patel and H. Chhinkaniwala, "Fuzzy logic-based single document summarisation with improved sentence scoring technique," *International Journal of Knowledge Engineering and Data Mining*, vol. 5, no. 1/2, pp. 125–138, 2018.

[19] B. El, "Document classification in information retrieval system based on neutrosophic sets," *Filomat*, vol. 34, no. 1, pp. 89–97, 2020.

[20] J. M. Sanchez-Gomez, M. A. Vega-Rodríguez, and C. J. Pérez, "Extractive multi- document text summarization using a multi-objective artificial bee colony op- timization approach," *Knowledge-Based Systems*, vol. 159, pp. 1–8, 2017.

[21] J. Ye, "Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses," *Artificial Intelligence in Medicine*, vol. 63, no. 3, pp. 135–220, 2015.

[22] R. Rautray and R. C. Balabantaray, "Cat swarm optimization based evolutionary framework for multi document summarization," *Physica A: Statistical Mechanics and its Applications*, vol. 477, pp. 174–186, 2017.

[23] W. Yang, J. Shi, Y. Pang et al., "Linear assignment method for interval neutrosophic sets," *Neural Computing & Applications*, vol. 27, pp. 554–564, 2016.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 5594623, 13 pages
https://doi.org/10.1155/2021/5594623

Research Article

Optimal Solutions for Constrained Bimatrix Games with Payoffs Represented by Single-Valued Trapezoidal Neutrosophic Numbers

Mohamed Gaber,[1] **Majed G. Alharbi**,[2] **Abd Alwahed Dagestani**,[3] **and El-Saeed Ammar**[4]

[1]*School of Mathematics and Statistics, Central South University, Changsha 410083, Hunan, China*
[2]*Department of Mathematics, College of Arts and Sciences, Methnab, Qassim University, Buridah, Saudi Arabia*
[3]*School of Business, Central South University, Changsha 410083, China*
[4]*Department of Mathematics Faculty of Science, Tanta University, Tanta, Egypt*

Correspondence should be addressed to Mohamed Gaber; mohamedgaber@csu.edu.cn

Received 25 January 2021; Revised 25 April 2021; Accepted 2 June 2021; Published 18 June 2021

Academic Editor: Parimala Mani

Single-valued neutrosophic set (SVNS) is considered as generalization and extension of fuzzy set, intuitionistic fuzzy set (IFS), and crisp set for expressing the imprecise, incomplete, and indeterminate information about real-life decision-oriented models. The theme of this research is to develop a solution approach to solve constrained bimatrix games with payoffs of single-valued trapezoidal neutrosophic numbers (SVTNNs). In this approach, the concepts and suitable ranking function of SVTNNs are defined. Hereby, the equilibrium optimal strategies and equilibrium values for both players can be determined by solving the parameterized mathematical programming problems, which are obtained from two novel auxiliary SVTNNs programming problems based on the proposed ranking approach of SVTNNs. Moreover, an application example is examined to verify the effectiveness and superiority of the developed algorithm. Finally, a comparison analysis between the proposed and the existing approaches is conducted to expose the advantages of our work.

1. Introduction

Constrained bimatrix games are nonzero-sum two-player noncooperative games which play a dominant role in many real-life applications such as in military, finance, economy, strategic welfares, cartel behaviour, management models, social problems or auctions, political voting systems, races, and development research [1, 2]. Usually, the constrained bimatrix game makes the assumption that the payoff values are described with crisp elements and exactly known by each player. However, players are not able to evaluate the games outcomes exactly due to the unavailability and ambiguity of information. To handle that, Zadeh [3] introduced the fuzzy set concept and since then various researchers have extended it to the different sets such as interval intuitionistic fuzzy set, IFS, linguistic interval IFS, and cubic IFS. Many scholars have studied various kinds of noncooperative games under uncertainty. For instance, Li et al. [4] proposed a bilinear programming algorithm for solving bimatrix games with intuitionistic fuzzy (IF) payoffs. Figueroa et al. [5] studied group matrix games with interval-valued fuzzy numbers payoffs. Jana et al. [6] introduced novel similarity measure to solve matrix games with dual hesitant fuzzy payoffs. Singh et al. [7] established 2-tuple linguistic matrix games. Zhou et al. [8] constructed novel matrix game with generalized Dempster-Shafer payoffs. Seikh et al. [9] solved matrix games with payoffs of hesitant fuzzy numbers. Han et al. [10] described new matrix game with Maxitive Belief information. Roy et al. [11] discussed Stackelberg game with payoffs of type-2 fuzzy numbers. Bhaumik et al. [12] solved

Prisoners' dilemma matrix game with hesitant interval-valued intuitionistic fuzzy-linguistic payoffs elements. Ammar et al. [13] studied bimatrix games with rough interval payoffs. Brikaa et al. [14] developed fuzzy multi-objective programming technique to solve fuzzy rough constrained matrix games. Bhaumik et al. [15] introduced multiobjective linguistic-neutrosophic matrix game with applications to tourism management. Brikaa et al. [16] applied resolving indeterminacy technique to find optimal solutions of multicriteria matrix games with IF goals. So far, as the authors are aware, there are only four articles that studied constraint bimatrix games. Jing-Jing et al. [17] proposed linear programming method for solving constrained bimatrix games with IF payoffs. Koorosh et al. [18] presented constrained bimatrix games and their application in wireless communications. Fanyong et al. [19] applied two approaches to solve the classical constrained bimatrix games. Bigdeli et al. [20] discussed constrained bimatrix games with fuzzy goals.

However, the IFS and fuzzy set theories are unable to deal with inconsistent and indeterminate data correctly. To consider that, Smarandache [21] introduced the theory of neutrosophic set (NS), defining the three components of indeterminacy, falsity, and truth; all lie in $]0^-, 1^+[$ and are independent. As NS is difficult to implement on realistic applications, Wang et al. [22] developed the single-valued neutrosophic set (SVNS) concept, which is an extension of the NS. Due to its importance, many scholars have applied the SVNS theory in various disciplines. For example, Garg [23] studied the analysis of decision-making based on sine trigonometric operational laws for SVNSs. Murugappan [24] presented neutrosophic inventory problem with immediate return for deficient items. Garg [25] proposed new neutrality aggregation operators with multiattribute decision-making (MADM) approach for single-valued neutrosophic numbers (SVNNs). Abdel-Basset et al. [26] investigated resource levelling model in construction projects with neutrosophic information. Garai et al. [27] discussed variance, standard deviation, and possibility mean of SVNNs with applications to MADM models. Broumi et al. [28] solved neutrosophic shortest path model by applying Bellman technique. Garg [29] proposed TOPSIS and clustering approaches to solve SVNNs decision-making model. Mullai et al. [30] presented inventory backorder model with neutrosophic environment. Garg et al. [31] studied MADM based on Frank Choquet Heronian mean operator for SVNSs. Leyva et al. [32] introduced a new problem of information technology project with neutrosophic information. Garg [33] presented nonlinear programming approach for solving MADM model with interval neutrosophic parameters. Sun et al. [34] developed new SVNN decision-making algorithms based on the theory of prospect. Garg [35] introduced biparametric distance measures on SVNSs and their applications in medical diagnosis and pattern recognition.

In the imprecise data game, players may encounter some assessment data that cannot be represented as real numbers when estimating the utility functions or uncertain subjects. Since SVNS has great superiority and flexibility in describing many uncertainties with complex environments, it is effective and convenient to represent the constrained bimatrix games with neutrosophic data. Due to decision-making growing requirements of expressing their judgments in a human friendly and neatly manner, it is important to extend the IF or fuzzy constrained bimatrix games into neutrosophic environment. The SVNS is an effective tool to satisfy the increasing requirement of higher uncertain and complicated constrained bimatrix game models. Probably, this is the first attempt of solving constrained bimatrix game with SVTNNs payoffs. The fundamental targets of this article are listed as follows:

(1) To propose a novel constrained bimatrix games model with SVTNNs payoffs
(2) To develop an effective algorithm for SVTNN constrained bimatrix games to obtain the optimal strategies for such games
(3) To formulate crisp linear optimization problems from the neutrosophic models based on the defined ambiguity and value indexes of SVTNN
(4) To present an application example to demonstrate the effectiveness and applicability of the proposed method
(5) To compare our results with other existing approaches

The remainder of the manuscript is summarized as follows. Section 2 introduces the concept, cut sets, and arithmetic operations of SVTNNs. Section 3 gives the concept of ambiguity and value indexes of SVTNNs and the ranking technique of SVTNNs. Section 4 formulates constrained bimatrix games with SVTNNs payoffs and the solution approach to solve such games. The illustrative example with comparative analysis is discussed in Section 5. Lastly, a short conclusion is given in Section 6.

2. Preliminaries

In the following, we introduce the basic concepts of fuzzy sets, IFSs, NSs, SVNSs, and SVNNs.

Definition 1 (see [36]). A fuzzy number $\widetilde{B} = (b_1, b_2, b_3, b_4)$ is said to be a trapezoidal fuzzy number (TFN), if its membership function $\delta_{\widetilde{B}}(y)$ is given by

$$\delta_{\widetilde{B}}(y) = \begin{cases} \dfrac{y - b_1}{b_2 - b_1}, & \text{if } b_1 \leq y \leq b_2, \\ 1, & \text{if } b_2 \leq y \leq b_3, \\ \dfrac{b_4 - y}{b_4 - b_3}, & \text{if } b_3 \leq y \leq b_4, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Definition 2 (see [37]). Suppose that Y is a universal set. An IFS $\widetilde{C}$ is defined as follows:

$$\widetilde{C} = \left\{ \langle y, \delta_{\widetilde{C}}(y), \gamma_{\widetilde{C}}(y) \rangle: \quad y \in Y \right\}, \tag{2}$$

where $\gamma_{\widetilde{C}}: Y \longrightarrow [0,1]$ and $\delta_{\widetilde{C}}: Y \longrightarrow [0,1]$ are the non-membership degree and the membership degree of $y \in Y$ to the set $\widetilde{C} \subseteq Y$, such that $0 \leq \delta_{\widetilde{C}}(y) + \gamma_{\widetilde{C}}(y) \leq 1$, $\forall y \in Y$.

Definition 3 (see [22]). An SVNS $\widetilde{B}$ in a universe Y is defined by

$$\widetilde{B} = \left\{ \langle y, T_{\widetilde{B}}(y), I_{\widetilde{B}}(y), F_{\widetilde{B}}(y) \rangle: \quad y \in Y \right\}, \tag{3}$$

where $T_{\widetilde{B}}(y): Y \longrightarrow [0,1]$, $I_{\widetilde{B}}(y): Y \longrightarrow [0,1]$, and $F_{\widetilde{B}}(y): Y \longrightarrow [0,1]$ such that $0 \leq T_{\widetilde{B}}(y) + I_{\widetilde{B}}(y), + F_{\widetilde{B}}(y) \leq 3$, $\forall y \in Y$. The values$F_{\widetilde{B}}(y), I_{\widetilde{B}}(y)$ and $T_{\widetilde{B}}(y)$, respectively, express the falsity membership, indeterminacy membership, and truth membership degree of y to $\widetilde{B}$.

Definition 4 (see [22]). An (α, β, γ)-cut set of SVNS $\widetilde{B}$, a crisp subset of $\mathbb{R}$, is given by

$$\widetilde{B}_{(\alpha,\beta,\gamma)} = \left\{ y: T_{\widetilde{B}}(y) \geq \alpha, I_{\widetilde{B}}(y) \leq \beta, F_{\widetilde{B}}(y) \leq \gamma \right\}, \tag{4}$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 1$, and $0 \leq \alpha + \beta + \gamma \leq 3$.

Definition 5 (see [22]). An SVNS $\widetilde{B} = \left\{ \langle y, T_{\widetilde{B}}(y), I_{\widetilde{B}}(y), F_{\widetilde{B}}(y) \rangle: y \in Y \right\}$ is called neutrosophic normal, if there exist at least three points $y_1, y_2, y_3 \in Y$ such that $T_{\widetilde{B}}(y_1) = I_{\widetilde{B}}(y_2) = F_{\widetilde{B}}(y_3) = 1$.

Definition 6 (see [22]). An SVNS $\widetilde{B} = \{ \langle y, T_{\widetilde{B}}(y), I_{\widetilde{B}}(y), F_{\widetilde{B}}(y) \rangle: y \in Y \}$ is said to be neutrosophic convex, if, $\forall y_1, y_2 \in Y$ and $\xi \in [0,1]$, the following conditions are satisfied:

(i) $T_{\widetilde{B}}(\xi y_1 + (1-\xi) y_2) \geq \min(T_{\widetilde{B}}(y_1), T_{\widetilde{B}}(y_2))$
(ii) $I_{\widetilde{B}}(\xi y_1 + (1-\xi) y_2) \leq \max(I_{\widetilde{B}}(y_1), I_{\widetilde{B}}(y_2))$
(iii) $F_{\widetilde{B}}(\xi y_1 + (1-\xi) y_2) \leq \max(F_{\widetilde{B}}(y_1), F_{\widetilde{B}}(y_2))$

Definition 7 (see [22]). An SVNS $\widetilde{B} = \left\{ \langle y, T_{\widetilde{B}}(y), I_{\widetilde{B}}(y), F_{\widetilde{B}}(y) \rangle: y \in Y \right\}$, is said to be single-valued neutrosophic number when

(1) $\widetilde{B}$ is neutrosophic normal
(2) $\widetilde{B}$ is neutrosophic convex
(3) $T_{\widetilde{B}}(y)$ is upper semicontinuous, $I_{\widetilde{B}}(y)$ is lower semicontinuous, and $F_{\widetilde{B}}(y)$ is lower semicontinuous
(4) The support of $\widetilde{B}$, that is, $S(\widetilde{B}) = \left\{ \langle T_{\widetilde{B}}(y) > 0, I_{\widetilde{B}}(y) < 1, F_{\widetilde{B}}(y) < 1, \forall y \in Y \rangle \right\}$, is bounded

Definition 8 (see [38]). An SVTNN $\widetilde{b} = \langle (k, l, m, n); u_{\widetilde{b}}, v_{\widetilde{b}}, w_{\widetilde{b}} \rangle$ is a special neutrosophic set on the set of real numbers $\mathbb{R}$, whose truth membership, indeterminacy membership, and falsity membership are represented as

$$\mu_{\widetilde{b}}(y) = \begin{cases} \dfrac{(y-k)u_{\widetilde{b}}}{l-k}, & \text{if } k \leq y < l, \\ u_{\widetilde{b}}, & \text{if } l \leq y \leq m, \\ \dfrac{(n-y)u_{\widetilde{b}}}{n-m}, & \text{if } m < y \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

$$\delta_{\widetilde{b}}(y) = \begin{cases} \dfrac{\left(l - y + (y-k)v_{\widetilde{b}}\right)}{l-k}, & \text{if } k \leq y < l, \\ v_{\widetilde{b}}, & \text{if } l \leq y \leq m, \\ \dfrac{\left(y - m + (n-y)v_{\widetilde{b}}\right)}{n-m}, & \text{if } m < y \leq n, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

$$\eta_{\widetilde{b}}(y) = \begin{cases} \dfrac{\left(l - y + (y-k)w_{\widetilde{b}}\right)}{l-k}, & \text{if } k \leq y < l, \\ w_{\widetilde{b}}, & \text{if } l \leq y \leq m, \\ \dfrac{\left(y - m + (n-y)w_{\widetilde{b}}\right)}{n-m}, & \text{if } m < y \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

respectively.

Definition 9 (see [38]). Let $\widetilde{c} = \langle (k_1, l_1, m_1, n_1); u_{\widetilde{c}}, v_{\widetilde{c}}, w_{\widetilde{c}} \rangle$ and $\widetilde{d} = \langle (k_2, l_2, m_2, n_2); u_{\widetilde{d}}, v_{\widetilde{d}}, w_{\widetilde{d}} \rangle$ be two SVTNNs and let $\lambda \neq 0$ be any real number. Then,

(1) $\widetilde{c} + \widetilde{d} = \langle (k_1 + k_2, l_1 + l_2, m_1 + m_2, n_1 + n_2); u_{\widetilde{c}} \wedge u_{\widetilde{d}}, v_{\widetilde{c}} \vee v_{\widetilde{d}}, w_{\widetilde{c}} \vee w_{\widetilde{d}} \rangle$
(2) $\widetilde{c}\widetilde{d} = \left\{ \langle (k_1k_2, l_1l_2, m_1m_2, n_1n_2); u_{\widetilde{c}} \wedge u_{\widetilde{d}}, v_{\widetilde{c}} \vee v_{\widetilde{d}}, w_{\widetilde{c}} \vee w_{\widetilde{d}} \rangle (n_1 > 0, \ n_2 > 0) \langle (k_1n_2, l_1m_2, m_1l_2, n_1k_2); u_{\widetilde{c}} \wedge u_{\widetilde{d}}, v_{\widetilde{c}} \vee v_{\widetilde{d}}, w_{\widetilde{c}} \vee w_{\widetilde{d}} \rangle (n_1 < 0, n_2 > 0) \langle (n_1n_2, m_1m_2, l_1l_2, k_1k_2); u_{\widetilde{c}} \wedge u_{\widetilde{d}}, v_{\widetilde{c}} \vee v_{\widetilde{d}}, w_{\widetilde{c}} \vee w_{\widetilde{d}} \rangle (n_1 < 0, n_2 < 0) \right.$
(3) $\lambda \widetilde{c} = \begin{cases} \langle (\lambda k_1, \lambda l_1, \lambda m_1, \lambda n_1); u_{\widetilde{c}}, v_{\widetilde{c}}, w_{\widetilde{c}} \rangle & (\lambda > 0) \\ \langle (\lambda n_1, \lambda m_1, \lambda l_1, \lambda k_1); u_{\widetilde{c}}, v_{\widetilde{c}}, w_{\widetilde{c}} \rangle & (\lambda < 0) \end{cases}$

Definition 10 (see [38]). Let $\widetilde{b} = \langle ((k_1, l_1, m_1, \ n_1), u_{\widetilde{b}}), ((k_2, l_2, m_2, n_2), v_{\widetilde{b}}), ((k_3, l_3, \ m_3, n_3), w_{\widetilde{b}}) \rangle$ be an SVTNN. Then, $\langle \alpha, \beta, \gamma \rangle$-cut set of the SVTNN $\widetilde{b}$, represented by $\widetilde{b}_{\langle \alpha,\beta,\gamma \rangle}$, is given as

$$\tilde{b}_{\langle\alpha,\beta,\gamma\rangle} = \left\{ y\colon \mu_{\tilde{b}}(y) \geq \alpha, \delta_{\tilde{b}}(y) \leq \beta, \eta_{\tilde{b}}(y) \leq \gamma, \quad y \in \mathbb{R} \right\}, \tag{6}$$

which satisfies the following conditions:

$$\begin{aligned} &0 \leq \alpha \leq u_{\tilde{b}},\\ &v_{\tilde{b}} \leq \beta \leq 1,\\ &w_{\tilde{b}} \leq \gamma \leq 1,\\ &0 \leq \alpha + \beta + \gamma \leq 3. \end{aligned} \tag{7}$$

Obviously, any $\langle\alpha,\beta,\gamma\rangle$-cut set $\tilde{b}_{\langle\alpha,\beta,\gamma\rangle}$ of an SVTNN $\tilde{b}$ is a crisp subset over the set of real numbers $\mathbb{R}$.

Definition 11 (see [38]). Let $\tilde{b} = \langle((k_1, l_1, m_1, n_1), u_{\tilde{b}}), ((k_2, l_2, m_2, n_2), v_{\tilde{b}}), ((k_3, l_3, m_3, n_3), w_{\tilde{b}})\rangle$ be an SVTNN. Then, α-cut set of the SVTNN $\tilde{b}$, represented by $\tilde{b}_\alpha$, is given as

$$\tilde{b}_\alpha = \left\{ y\colon \mu_{\tilde{b}}(y) \geq \alpha, \quad y \in \mathbb{R} \right\}, \tag{8}$$

where $\alpha \in [0, u_{\tilde{b}}]$.

Obviously, any α-cut set $\tilde{b}_\alpha$ of an SVTNN $\tilde{b}$ is a crisp subset over the set of real numbers $\mathbb{R}$.

Here, any α-cut set of an SVTNN $\tilde{b}$ for the truth membership function is a closed interval, represented by $\tilde{b}_\alpha = [L^\alpha(\tilde{b}), R^\alpha(\tilde{b})]$.

Definition 12 (see [38]). Let $\tilde{b} = \langle((k_1, l_1, m_1, n_1), u_{\tilde{b}}), ((k_2, l_2, m_2, n_2), v_{\tilde{b}}), ((k_3, l_3, m_3, n_3), w_{\tilde{b}})\rangle$ be an SVTNN. Then, β-cut set of the SVTNN $\tilde{b}$, represented by $\tilde{b}_\beta$, is given as

$$\tilde{b}_\beta = \left\{ y\colon \delta_{\tilde{b}}(y) \leq \beta, \quad y \in \mathbb{R} \right\}, \tag{9}$$

where $\beta \in [v_{\tilde{b}}, 1]$.

Obviously, any β-cut set $\tilde{b}_\beta$ of an SVTNN $\tilde{b}$ is a crisp subset over the set of real numbers $\mathbb{R}$.

Here, any β-cut set of an SVTNN $\tilde{b}$ for the indeterminacy membership function is a closed interval, represented by $\tilde{b}_\beta = [L^\beta(\tilde{b}), R^\beta(\tilde{b})]$.

Definition 13 (see [38]). Let $\tilde{b} = \langle((k_1, l_1, m_1, n_1), u_{\tilde{b}}), ((k_2, l_2, m_2, n_2), v_{\tilde{b}}), ((k_3, l_3, m_3, n_3), w_{\tilde{b}})\rangle$ be an SVTNN. Then, γ-cut set of the SVTNN $\tilde{b}$, represented by $\tilde{b}_\gamma$, is given as

$$\tilde{b}_\gamma = \left\{ y\colon \eta_{\tilde{b}}(y) \leq \gamma, \quad y \in \mathbb{R} \right\}, \tag{10}$$

where $\gamma \in [w_{\tilde{b}}, 1]$.

Obviously, any γ-cut set $\tilde{b}_\gamma$ of an SVTNN $\tilde{b}$ is a crisp subset over the set of real numbers $\mathbb{R}$.

Here, any γ-cut set of an SVTNN $\tilde{b}$ for the falsity membership function is a closed interval, represented by $\tilde{b}_\gamma = [L^\gamma(\tilde{b}), R^\gamma(\tilde{b})]$.

3. Characteristics and the Ranking Approach for SVTNNs

3.1. Value and Ambiguity of SVTNNs. Here, we introduce the basic definitions of value and ambiguity indices of SVTNN.

Definition 14 (see [38]). Let $\tilde{b} = \langle((k_1, l_1, m_1, n_1), u_{\tilde{b}}), ((k_2, l_2, m_2, n_2), v_{\tilde{b}}), ((k_3, l_3, m_3, n_3), w_{\tilde{b}})\rangle$ be an SVTNN and let $\tilde{b}_\alpha = [L^\alpha(\tilde{b}), R^\alpha(\tilde{b})]$, $\tilde{b}_\beta = [L^\beta(\tilde{b}), R^\beta(\tilde{b})]$, and $\tilde{b}_\gamma = [L^\gamma(\tilde{b}), R^\gamma(\tilde{b})]$ be any α-cut set, β-cut set, and γ-cut set of the SVTNN $\tilde{b}$, respectively. Then, we have the following.

(1) The value of the SVTNN $\tilde{b}$ for α-cut set, represented by $V_\mu(\tilde{b})$, is given as

$$V_\mu(\tilde{b}) = \int_0^{u_{\tilde{b}}} \left(L^\alpha(\tilde{b}) + R^\alpha(\tilde{b})\right) h(\alpha) \mathrm{d}\alpha, \tag{11}$$

where $h(\alpha) \in [0,1]$ $(\alpha \in [0, u_{\tilde{b}}])$, $h(0) = 0$, and $h(\alpha)$is nondecreasing and monotonic of $\alpha \in [0, u_{\tilde{b}}]$.

(2) The value of the SVTNN $\tilde{b}$ for β-cut set, represented by $V_\delta(\tilde{b})$, is given as

$$V_\delta(\tilde{b}) = \int_{v_{\tilde{b}}}^1 \left(L^\beta(\tilde{b}) + R^\beta(\tilde{b})\right) f(\beta) \mathrm{d}\beta, \tag{12}$$

where $f(\beta) \in [0,1]$ $(\beta \in [v_{\tilde{b}}, 1])$, $f(1) = 0$, and $f(\beta)$ is nondecreasing and monotonic of $\beta \in [v_{\tilde{b}}, 1]$.

(3) The value of the SVTNN $\tilde{b}$ for γ-cut set, represented by $V_\eta(\tilde{b})$, is given as

$$V_\eta(\tilde{b}) = \int_{w_{\tilde{b}}}^1 \left(L^\gamma(\tilde{b}) + R^\gamma(\tilde{b})\right) g(\gamma) \mathrm{d}\gamma, \tag{13}$$

where $g(\gamma) \in [0,1]$ $(\gamma \in [w_{\tilde{b}}, 1])$, $g(1) = 0$, and $g(\gamma)$ is nondecreasing and monotonic of $\gamma \in [w_{\tilde{b}}, 1]$.

Definition 15 (see [38]). Let $\tilde{b} = \langle((k_1, l_1, m_1, n_1), u_{\tilde{b}}), ((k_2, l_2, m_2, n_2), v_{\tilde{b}}), ((k_3, l_3, m_3, n_3), w_{\tilde{b}})\rangle$ be an SVTNN and let $\tilde{b}_\alpha = [L^\alpha(\tilde{b}), R^\alpha(\tilde{b})]$, $\tilde{b}_\beta = [L^\beta(\tilde{b}), R^\beta(\tilde{b})]$, and $\tilde{b}_\gamma = [L^\gamma(\tilde{b}), R^\gamma(\tilde{b})]$ be any α-cut set, β-cut set, and γ-cut set of the SVNN $\tilde{b}$, respectively. Then, we have the following.

(1) The ambiguities of the SVTNN $\tilde{b}$ for α-cut set, represented by $A_\mu(\tilde{b})$, are given as

$$A_\mu(\tilde{b}) = \int_0^{u_{\tilde{b}}} \left(R^\alpha(\tilde{b}) - L^\alpha(\tilde{b})\right) h(\alpha) \mathrm{d}\alpha, \tag{14}$$

where $h(\alpha) \in [0,1]$ $(\alpha \in [0, u_{\tilde{b}}])$, $h(0) = 0$, and $h(\alpha)$is nondecreasing and monotonic of $\alpha \in [0, u_{\tilde{b}}]$.

(2) The ambiguities of the SVTNN $\tilde{b}$ for β-cut set, represented by $A_\delta(\tilde{b})$, are given as

$$A_\delta(\tilde{b}) = \int_{v_{\tilde{b}}}^1 \left(R^\beta(\tilde{b}) - L^\beta(\tilde{b})\right) f(\beta) \mathrm{d}\beta, \tag{15}$$

where $f(\beta) \in [0,1]$ $(\beta \in [v_{\tilde{b}}, 1])$, $f(1) = 0$, and $f(\beta)$ is nondecreasing and monotonic of $\beta \in [v_{\tilde{b}}, 1]$.

(3) The ambiguities of the SVTNN $\widetilde{b}$ for γ-cut set, represented by $A_\eta(\widetilde{b})$, are given as

$$A_\eta(\widetilde{b}) = \int_{w_{\widetilde{b}}}^{1} \left(R^\gamma(\widetilde{b}) - L^\gamma(\widetilde{b})\right) h(\gamma)\mathrm{d}\gamma, \tag{16}$$

where $g(\gamma) \in [0,1]$ $(\gamma \in [w_{\widetilde{b}}, 1])$, $g(1) = 0$a, and $g(\gamma)$ is nondecreasing and monotonic of $\gamma \in [w_{\widetilde{b}}, 1]$.

Here, the weighting functions $h(\alpha)$, $f(\beta)$, and $g(\gamma)$ can be supposed according to the decision-making model nature. Suppose that $h(\alpha) = \alpha$, $f(\beta) = 1-\beta$, and $g(\gamma) = 1-\gamma$.

Let $\widetilde{b} = \langle (k,l,m,n); u_{\widetilde{b}}, v_{\widetilde{b}}, w_{\widetilde{b}} \rangle$ be an SVTNN. Then the value and ambiguity indices, using the above descriptions, are constructed as

$$\begin{aligned}
V_\mu(\widetilde{b}) &= \frac{(k+2l+2m+n)u_{\widetilde{b}}^2}{6}, \quad A_\mu(\widetilde{b}) = \frac{(n-k+2m-2l)u_{\widetilde{b}}^2}{6},\\
V_\delta(\widetilde{b}) &= \frac{(k+2l+2m+n)\left(1-v_{\widetilde{b}}\right)^2}{6}, \quad A_\delta(\widetilde{b}) = \frac{(n-k+2m-2l)\left(1-v_{\widetilde{b}}\right)^2}{6},\\
V_\eta(\widetilde{b}) &= \frac{(k+2l+2m+n)\left(1-w_{\widetilde{b}}\right)^2}{6}, \quad A_\eta(\widetilde{b}) = \frac{(n-k+2m-2l)\left(1-w_{\widetilde{b}}\right)^2}{6}.
\end{aligned} \tag{17}$$

3.2. A Ranking Approach of an SVTNN Based on Value and Ambiguity Indices. This section provides a ranking approach of SVTNNs based on the ambiguity and value indices of SVTNNs in a similar way to those of SVNNs introduced by A. Bhaumik et al. [39].

Definition 16. Let $\widetilde{b} = \langle (k,l,m,n); u_{\widetilde{b}}, v_{\widetilde{b}}, w_{\widetilde{b}} \rangle$ be an SVTNN. The weighted value ambiguity index for an SVTNN $\widetilde{b}$ is given as

$$R_{\lambda_1,\lambda_2,\lambda_3}(\widetilde{b}) = \left[\lambda_1 V_\mu(\widetilde{b}) + (1-\lambda_1)A_\mu(\widetilde{b})\right] + \left[\lambda_2 V_\delta(\widetilde{b}) + (1-\lambda_2)A_\delta(\widetilde{b})\right] + \left[\lambda_3 V_\eta(\widetilde{b}) + (1-\lambda_3)A_\eta(\widetilde{b})\right], \tag{18}$$

with $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$.

Definition 17. Let $\widetilde{c}$ and $\widetilde{d}$ be two SVTNNs and let $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$. For the weighted value ambiguity index of the SVTNNs $\widetilde{c}$ and $\widetilde{d}$, the ranking order of $\widetilde{c}$ and $\widetilde{d}$ is given as follows:

(1) if $R_{\lambda_1,\lambda_2,\lambda_3}(\widetilde{c}) >_N R_{\lambda_1,\lambda_2,\lambda_3}(\widetilde{d})$, then $\widetilde{c} >_N \widetilde{d}$
(2) if $R_{\lambda_1,\lambda_2,\lambda_3}(\widetilde{c}) <_N R_{\lambda_1,\lambda_2,\lambda_3}(\widetilde{d})$, then $\widetilde{c} <_N \widetilde{d}$
(3) if $R_{\lambda_1,\lambda_2,\lambda_3}(\widetilde{c}) =_N R_{\lambda_1,\lambda_2,\lambda_3}(\widetilde{d})$, then $\widetilde{c} =_N \widetilde{d}$

where "$>_N$" and "$<_N$" are neutrosophic versions of the order relations "$>$" and "$<$" in the real line, respectively.

4. Constrained Bimatrix Games with SVTNNs Payoffs and Solution Method

Let us consider the constrained bimatrix game with SVTNNs payoffs. Suppose that $T_1 = \{\xi_1, \xi_2, \ldots, \xi_\kappa\}$ and $T_2 = \{\eta_1, \eta_2, \ldots, \eta_\ell\}$ are pure strategies sets for two players I and II, respectively. When player II selects pure strategy $\eta_j \in T_2$ and player I selects pure strategy $\xi_i \in T_1$, at the situation (ξ_i, η_j), player II gains payoff and player I gains payoff, which are expressed with SVTNNs as $\widetilde{\mathbf{C}} = (\widetilde{c}_{ij})_{\kappa\times\ell}$ and $\widetilde{\mathbf{D}} = (\widetilde{d}_{ij})_{\kappa\times\ell}$, where each $\widetilde{c}_{ij} = \langle (a_{ij}, b_{ij}, f_{ij}, h_{ij}); u_{\widetilde{c}_{ij}}, v_{\widetilde{c}_{ij}}, w_{\widetilde{c}_{ij}} \rangle$ and $\widetilde{d}_{ij} = \langle (k_{ij}, l_{ij}, m_{ij}, n_{ij}); u_{\widetilde{d}_{ij}}, v_{\widetilde{d}_{ij}}, w_{\widetilde{d}_{ij}} \rangle$ $(i = 1, 2, \ldots, \kappa; j = 1, 2, \ldots, \ell)$ are SVTNNs defined as above. The mixed strategies vectors are represented as $\mathbf{r} = (r_1, r_2, \ldots, r_\kappa)^T$ and $\mathbf{s} = (s_1, s_2, \ldots, s_\ell)^T$, where r_i $(i = 1, 2, \ldots, \kappa)$ and s_j $(j = 1, 2, \ldots, \ell)$ are probabilities for both players selecting their pure strategies $\xi_i \in T_1$ and $\eta_j \in T_2$, respectively. The mixed strategies r_i and s_j are affiliated with the strategies sets (convex polyhedron) which are described by some inequalities and equations. Let $R = \{\mathbf{r}\colon \mathbf{G}^T\mathbf{r} \geq \mathbf{p},\ \mathbf{r} \geq 0\}$ represent the strategy constraint set of player I, where $\mathbf{p} = (p_1, p_2, \ldots, p_e)^T$, $\mathbf{G} = (g_{in})_{\kappa\times e}$, and e is a positive integer. Let $S = \{\mathbf{s}\colon \mathbf{H}\mathbf{s} \geq \mathbf{q},\ \mathbf{s} \geq 0\}$ express the strategy constraint set of player II, where $\mathbf{q} = (q_1, q_2, \ldots, q_b)$, $\mathbf{H} = (h_{mj})_{b\times\ell}$, and b is a positive integer. Note that $\mathbf{G}^T\mathbf{r} \geq \mathbf{p}$ contains $\sum_{i=1}^{\kappa} r_i = 1$, since $\sum_{i=1}^{\kappa} r_i = 1$ is equivalent to $\sum_{i=1}^{\kappa} r_i \geq 1$ and $-\sum_{i=1}^{\kappa} r_i \geq -1$. Similarly, $\mathbf{H}\mathbf{s} \geq \mathbf{q}$ contains $\sum_{j=1}^{\ell} s_j = 1$. In the sequel, the above SVTNN constrained bimatrix game is simply denoted by $(\widetilde{\mathbf{C}}, \widetilde{\mathbf{D}})$ for short.

Without loss of generality, suppose that both players I and II, respectively, select mixed strategies $r \in R$ and $s \in S$ in order to maximize their own payoffs; then their expected payoffs can be obtained as follows:

$$E_1(\mathbf{r},\mathbf{s},\widetilde{\mathbf{C}}) = \mathbf{r}^T\widetilde{\mathbf{C}}\mathbf{s} = \sum_{i=1}^{\kappa}\sum_{j=1}^{\ell} r_i\widetilde{c}_{ij}s_j,$$
$$E_2(\mathbf{r},\mathbf{s},\widetilde{\mathbf{D}}) = \mathbf{r}^T\widetilde{\mathbf{D}}\mathbf{s} = \sum_{i=1}^{\kappa}\sum_{j=1}^{\ell} r_i\widetilde{d}_{ij}s_j. \quad (19)$$

Definition 18 (see [40]). If $(\mathbf{r}^*,\mathbf{s}^*)\in R\times S$ satisfies the following conditions:

$$\mathbf{r}^{*T}\widetilde{\mathbf{C}}\mathbf{s}^* = \min_{s\in S}\mathbf{r}^{*T}\widetilde{\mathbf{C}}\mathbf{s} = \max_{r\in R}\min_{s\in S}\mathbf{r}^T\widetilde{\mathbf{C}}\mathbf{s},$$
$$\mathbf{r}^{*T}\widetilde{\mathbf{D}}\mathbf{s}^* = \min_{r\in R}\mathbf{r}^T\widetilde{\mathbf{D}}\mathbf{s}^* = \max_{s\in S}\min_{r\in R}\mathbf{r}^T\widetilde{\mathbf{D}}\mathbf{s}, \quad (20)$$

for any mixed strategies $\mathbf{r}\in R$ and $\mathbf{s}\in S$, then $\mathbf{r}^*$ and $\mathbf{s}^*$ are called equilibrium strategies, and $U^* = \mathbf{r}^{*T}\widetilde{\mathbf{C}}\mathbf{s}^*$ and $W^* = \mathbf{r}^{*T}\widetilde{\mathbf{D}}\mathbf{s}^*$ are called equilibrium values of players I and II, respectively.

Theorem 1. *If* $(\mathbf{r}^*,\mathbf{y}^*)$ *and* $(\mathbf{s}^*,\mathbf{z}^*)$ *are the optimal solutions of the following linear programming problems:*

$$\max\{\mathbf{q}^{\mathbf{T}}\mathbf{y}\}$$
$$\text{s.t.}\begin{cases}\mathbf{H}^{\mathbf{T}}\mathbf{y}\le_N\widetilde{\mathbf{C}}^{\mathbf{T}}\mathbf{r},\\ \mathbf{G}^{\mathbf{T}}\mathbf{r}\ge\mathbf{p},\\ \mathbf{r}\ge\mathbf{0},\\ \mathbf{y}\ge\mathbf{0},\end{cases} \quad (21)$$

$$\max\{\mathbf{p}^{\mathbf{T}}\mathbf{z}\},$$
$$\text{s.t.}\begin{cases}\mathbf{G}\mathbf{z}\le_N\widetilde{\mathbf{D}}\mathbf{s},\\ \mathbf{H}\mathbf{s}\ge\mathbf{q},\\ \mathbf{s}\ge\mathbf{0},\\ \mathbf{z}\ge\mathbf{0},\end{cases} \quad (22)$$

respectively, then $\mathbf{r}^*$ *and* $\mathbf{s}^*$ *are equilibrium strategies of the SVTNN constrained bimatrix game* $(\widetilde{\mathbf{C}},\widetilde{\mathbf{D}})$*, and* $U^* = \mathbf{q}^{\mathbf{T}}\mathbf{y}^* = \mathbf{r}^{*T}\widetilde{\mathbf{C}}\mathbf{s}^*$ *and* $W^* = \mathbf{p}^{\mathbf{T}}\mathbf{z}^* = \mathbf{r}^{*T}\widetilde{\mathbf{D}}\mathbf{s}^*$ *are equilibrium values of players I and II, respectively.*

Proof:. The proof of this theorem is similar to the proof given by Jing-Jing et al. [17].

It is obvious that the two players often cannot calculate the payoffs accurately in each situation, and the game values of the SVTNN constrained bimatrix games are not equal to $\mathbf{q}^{\mathbf{T}}\mathbf{y}$ in (21) and $\mathbf{p}^{\mathbf{T}}\mathbf{z}$ in (22). The two players may allow some violations on the set of constraints $\mathbf{H}^{\mathbf{T}}\mathbf{y}\le_N\widetilde{\mathbf{C}}^{\mathbf{T}}\mathbf{r}$ and $\mathbf{G}\mathbf{z}\le_N\widetilde{\mathbf{D}}\mathbf{s}$.

Therefore, the equilibrium strategies $\mathbf{r}^*$ and $\mathbf{s}^*$ and equilibrium values U^* and W^* of the SVTNN constrained bimatrix games are equal to the optimal values and optimal solutions of (23 and 24) as follows:

$$\max\{\mathbf{q}^{\mathbf{T}}\mathbf{y}\},$$
$$\text{s.t.}\begin{cases}\mathbf{H}^{\mathbf{T}}\mathbf{y}-\widetilde{\mathbf{C}}^{\mathbf{T}}\mathbf{r}\le_N(1-\rho)\widetilde{\mathbf{m}},\\ \mathbf{G}^{\mathbf{T}}\mathbf{r}\ge\mathbf{p},\\ \mathbf{r}\ge\mathbf{0},\\ \mathbf{y}\ge\mathbf{0},\end{cases} \quad (23)$$

$$\max\{\mathbf{p}^{\mathbf{T}}\mathbf{z}\},$$
$$\text{s.t.}\begin{cases}\mathbf{G}\mathbf{z}-\widetilde{\mathbf{D}}\mathbf{s}\le_N(1-\rho)\widetilde{\mathbf{n}},\\ \mathbf{H}\mathbf{s}\ge\mathbf{q},\\ \mathbf{s}\ge\mathbf{0},\\ \mathbf{z}\ge\mathbf{0},\end{cases} \quad (24)$$

respectively, where $\widetilde{\mathbf{m}} = (\widetilde{m}_1,\widetilde{m}_2,\ldots,\widetilde{m}_\ell)^T$, $\widetilde{\mathbf{n}} = (\widetilde{n}_1,\widetilde{n}_2,\ldots,\widetilde{n}_k)^T$, and all the vectors elements of $\widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{n}}$ are SVTNNs that are approximately equal to zero, which represent the maximum violations that the two players may permit on the set of constraints. The parameter ρ $(0\le\rho\le 1)$ is a real number.

Applying the ranking approach of SVTNNs, as proposed in Subsection 3.2, the SVTNN mathematical programming problems (equations (23) and (24)) can be transformed into the following parameterized programming problems:

$$\max\{\mathbf{q}^{\mathbf{T}}\mathbf{y}\},$$
$$\text{s.t.}\begin{cases}\mathbf{H}^{\mathbf{T}}\mathbf{y}-R_{\lambda_1,\lambda_2,\lambda_3}\left(\widetilde{\mathbf{C}}^{\mathbf{T}}\right)\mathbf{r}\le_N(1-\rho)R_{\lambda_1,\lambda_2,\lambda_3}(\widetilde{\mathbf{m}}),\\ \mathbf{G}^{\mathbf{T}}\mathbf{r}\ge\mathbf{p},\\ \mathbf{r}\ge\mathbf{0},\\ \mathbf{y}\ge\mathbf{0},\end{cases} \quad (25)$$

$$\max\{\mathbf{p}^{\mathbf{T}}\mathbf{z}\},$$
$$\text{s.t.}\begin{cases}\mathbf{G}\mathbf{z}-R_{\lambda_1,\lambda_2,\lambda_3}(\widetilde{\mathbf{D}})\mathbf{s}\le_N(1-\rho)R_{\lambda_1,\lambda_2,\lambda_3}(\widetilde{\mathbf{n}}),\\ \mathbf{H}\mathbf{s}\ge\mathbf{q},\\ \mathbf{s}\ge\mathbf{0},\\ \mathbf{z}\ge\mathbf{0},\end{cases} \quad (26)$$

respectively.

For given $\lambda_1,\lambda_2,\lambda_3\in[0,1]$, solving equations (25) and (26), we can obtain the optimal game values $\mathbf{q}^{\mathbf{T}}\mathbf{y}^*(\rho)$ and $\mathbf{p}^{\mathbf{T}}\mathbf{z}^*(\rho)$ and the optimal solutions $(\mathbf{r}^*(\rho),\mathbf{y}^*(\rho))$ and $(\mathbf{s}^*(\rho),\mathbf{z}^*(\rho))$, respectively. □

Theorem 2. *If* $(\mathbf{r}^*(\rho),\mathbf{y}^*(\rho))$ *and* $(\mathbf{s}^*(\rho),\mathbf{z}^*(\rho))(\lambda_1,\lambda_2,\lambda_3\in[0,1])$ *are optimal solutions of equations (25) and (26), respectively, then* $\mathbf{r}^*(\rho)$ *and* $\mathbf{s}^*(\rho)$ *are equilibrium strategies, and* $U^* = \mathbf{q}^{\mathbf{T}}\mathbf{y}^*(\rho)$ *and* $W^* = \mathbf{p}^{\mathbf{T}}\mathbf{z}^*(\rho)$ *are equilibrium values of both players for SVTNN constrained bimatrix games, respectively.*

Table 1: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (0.2, 0.3, 0.5)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	4.868	(0.4, 0.6)	4.855
0.1	(0.75, 0.25)	4.854	(0.4, 0.6)	4.849
0.2	(0.75, 0.25)	4.839	(0.4, 0.6)	4.844
0.3	(0.75, 0.25)	4.825	(0.4, 0.6)	4.838
0.4	(0.75, 0.25)	4.811	(0.4, 0.6)	4.832
0.5	(0.75, 0.25)	4.797	(0.4, 0.6)	4.827
0.6	(0.75, 0.25)	4.782	(0.4, 0.6)	4.821
0.7	(0.75, 0.25)	4.768	(0.4, 0.6)	4.815
0.8	(0.75, 0.25)	4.754	(0.4, 0.6)	4.809
0.9	(0.75, 0.25)	4.739	(0.4, 0.6)	4.804
1.0	(0.75, 0.25)	4.725	(0.4, 0.6)	4.798

Table 2: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (0.4, 0.5, 0.6)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	6.223	(0.4, 0.6)	6.666
0.1	(0.75, 0.25)	6.204	(0.4, 0.6)	6.658
0.2	(0.75, 0.25)	6.186	(0.4, 0.6)	6.649
0.3	(0.75, 0.25)	6.168	(0.4, 0.6)	6.642
0.4	(0.75, 0.25)	6.149	(0.4, 0.6)	6.634
0.5	(0.75, 0.25)	6.132	(0.4, 0.6)	6.626
0.6	(0.75, 0.25)	6.113	(0.4, 0.6)	6.617
0.7	(0.75, 0.25)	6.095	(0.4, 0.6)	6.609
0.8	(0.75, 0.25)	6.076	(0.4, 0.6)	6.601
0.9	(0.75, 0.25)	6.059	(0.4, 0.6)	6.593
1.0	(0.75, 0.25)	6.04	(0.4, 0.6)	6.585

Table 3: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (0.5, 0.5, 0.5)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	6.232	(0.4, 0.6)	6.801
0.1	(0.75, 0.25)	6.214	(0.4, 0.6)	6.792
0.2	(0.75, 0.25)	6.196	(0.4, 0.6)	6.784
0.3	(0.75, 0.25)	6.178	(0.4, 0.6)	6.776
0.4	(0.75, 0.25)	6.16	(0.4, 0.6)	6.767
0.5	(0.75, 0.25)	6.143	(0.4, 0.6)	6.759
0.6	(0.75, 0.25)	6.125	(0.4, 0.6)	6.751
0.7	(0.75, 0.25)	6.107	(0.4, 0.6)	6.743
0.8	(0.75, 0.25)	6089	(0.4, 0.6)	6.735
0.9	(0.75, 0.25)	6.071	(0.4, 0.6)	6.726
1.0	(0.75, 0.25)	6.054	(0.4, 0.6)	6.718

5. Application Example

In this section, an example of the company development strategy choice model adapted from Jing-Jing et al. [17] is used to illustrate the solution procedure of a constrained bimatrix game with payoffs of SVTNNs.

Table 4: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (0.6, 0.4, 0.7)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	6.785	(0.4, 0.6)	7.309
0.1	(0.75, 0.25)	6.765	(0.4, 0.6)	7.301
0.2	(0.75, 0.25)	6.746	(0.4, 0.6)	7.292
0.3	(0.75, 0.25)	6.726	(0.4, 0.6)	7.283
0.4	(0.75, 0.25)	6.706	(0.4, 0.6)	7.274
0.5	(0.75, 0.25)	6.687	(0.4, 0.6)	7.265
0.6	(0.75, 0.25)	6.667	(0.4, 0.6)	7.256
0.7	(0.75, 0.25)	6.647	(0.4, 0.6)	7.247
0.8	(0.75, 0.25)	6.628	(0.4, 0.6)	7.238
0.9	(0.75, 0.25)	6.608	(0.4, 0.6)	7.229
1.0	(0.75, 0.25)	6.588	(0.4, 0.6)	7.219

Table 5: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (0.7, 0.6, 0.8)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	7.861	(0.4, 0.6)	8.732
0.1	(0.75, 0.25)	7.838	(0.4, 0.6)	8.721
0.2	(0.75, 0.25)	7.815	(0.4, 0.6)	8.709
0.3	(0.75, 0.25)	7.792	(0.4, 0.6)	8.699
0.4	(0.75, 0.25)	7.769	(0.4, 0.6)	8.688
0.5	(0.75, 0.25)	7.746	(0.4, 0.6)	8.677
0.6	(0.75, 0.25)	7.723	(0.4, 0.6)	8.666
0.7	(0.75, 0.25)	7.7	(0.4, 0.6)	8.655
0.8	(0.75, 0.25)	7.678	(0.4, 0.6)	8.644
0.9	(0.75, 0.25)	7.655	(0.4, 0.6)	8.633
1.0	(0.75, 0.25)	7.632	(0.4, 0.6)	8.622

5.1. The Company Development Strategy Choice Model. "We consider two companies E_1 and E_2 (i.e., players I and II). In order to improve the two companies competitiveness, both players have two strategies: introducing the advanced equipment ξ_1 or η_1 and introducing the senior talent ξ_2 or η_2. When player I chooses pure strategies ξ_1 and ξ_2, he wants to invest 7 million and 5 million dollars, respectively. Due to a lack of fund, player I can invest up to 6.5 million dollars, which means that player I has a constraint, $7r_1 + 5r_2 \le 6.5$, when selecting strategy. Likewise, player II wants to invest 4 million and 6.5 million dollars when he chooses pure strategies η_1 and η_2, respectively. However, due to a lack of fund, player II can invest up to 5.5 million dollars. Namely, player II has a constraint, $4s_1 + 6.5s_2 \le 5.5$, when choosing strategies." This is a typical SVTN constrained bimatrix game. According to the previous description of the matrix game model, the two players' constrained strategy sets are given as follows:

$$\begin{aligned} R &= \{r \mid 7r_1 + 5r_2 \le 6.5, r_1 + r_2 = 1, \quad r_1 \ge 0, r_2 \ge 0\}, \\ S &= \{s \mid 4s_1 + 6.5s_2 \le 5.5, s_1 + s_2 = 1, \quad s_1 \ge 0, s_2 \ge 0\}, \end{aligned} \tag{27}$$

respectively. The SVTNNs payoff matrices of the two players are given by

$$\tilde{\mathbf{C}} = \begin{pmatrix} \langle(6,7,9,1);0.9,0.2,0.4\rangle & \langle(3.5,5,7,9);0.5,0.4,0.2\rangle \\ \langle(3,5,6,8);0.6,0.5,0.1\rangle & \langle(5,6.5,8,10);0.7,0.3,0.5\rangle \end{pmatrix},$$
$$\tilde{\mathbf{D}} = \begin{pmatrix} \langle(5,6.5,8,9);0.8,0.2,0.3\rangle & \langle(4,5,7,8.5);0.8,0.3,0.1\rangle \\ \langle(3.5,4.5,6,7.5);0.6,0.4,0.2\rangle & \langle(6,7,8,9);0.9,0.1,0.4\rangle \end{pmatrix}. \quad (28)$$

The vectors of the constraints and the coefficient matrices are given by

$$\begin{aligned} \mathbf{G} &= \begin{pmatrix} -7 & 1 & -1 \\ -5 & 1 & -1 \end{pmatrix}, \\ \mathbf{H}^T &= \begin{pmatrix} -4 & 1 & -1 \\ -6.5 & 1 & -1 \end{pmatrix}, \\ \mathbf{p} &= \begin{pmatrix} -6.5 & 1 & -1 \end{pmatrix}^T, \\ \mathbf{q} &= \begin{pmatrix} -5.5 & 1 & -1 \end{pmatrix}^T. \end{aligned} \quad (29)$$

Let the two players select $\tilde{m}_1 = \tilde{m}_2 = \langle(0.18, 0.1, 0.21, 0.13); 0.7, 0.2, 0.1\rangle$ and $\tilde{n}_1 = \tilde{n}_2 = \langle(0.04, 0.1, 0.13, 0.02); 0.8, 0.2, 0.3\rangle$, respectively.

5.2. The Solution Procedure. Applying the ranking approach presented in Section 3 to the SVTN constrained bimatrix game, we have

$$\begin{aligned} R_{\lambda_1,\lambda_2,\lambda_3}(\tilde{C}) &= \begin{pmatrix} 5.4\lambda_1 + 4.267\lambda_2 + 2.4\lambda_3 + 2.715 & 1.125\lambda_1 + 1.62\lambda_2 + 2.88\lambda_3 + 1.979 \\ 1.56\lambda_1 + 1.083\lambda_2 + 3.51\lambda_3 + 1.657 & 2.94\lambda_1 + 2.94\lambda_2 + 1.5\lambda_3 + 1.64 \end{pmatrix}, \\ R_{\lambda_1,\lambda_2,\lambda_3}(\tilde{D}) &= \begin{pmatrix} 3.84\lambda_1 + 3.84\lambda_2 + 2.94\lambda_3 + 2.065 & 2.987\lambda_1 + 2.287\lambda_2 + 3.78\lambda_3 + 2.748 \\ 1.5\lambda_1 + 1.5\lambda_2 + 2.667\lambda_3 + 1.587 & 5.4\lambda_1 + 5.4\lambda_2 + 2.4\lambda_3 + 1.65 \end{pmatrix}. \end{aligned} \quad (30)$$

According to equations (25) and (26), we can formulate the optimization problems with four parameters $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$, and $\rho \in [0,1]$ as follows:

$$\begin{aligned} &\text{maximize}\{-5.5y_1 + y_2 - y_3\}, \\ &\text{subject to} \begin{cases} -4y_1 + y_2 - y_3 - (5.4\lambda_1 + 4.267\lambda_2 + 2.4\lambda_3 + 2.715)r_1 - (1.56\lambda_1 + 1.083\lambda_2 + 3.51\lambda_3 + 1.657)r_2 \\ \le (0.062\lambda_1 + 0.081\lambda_2 + 0.103\lambda_3 + 0.055)(1-\rho), \\ -6.5y_1 + y_2 - y_3 - (1.125\lambda_1 + 1.62\lambda_2 + 2.88\lambda_3 + 1.979)r_1 - (2.94\lambda_1 + 2.94\lambda_2 + 1.5\lambda_3 + 1.64)r_2 \\ \le (0.062\lambda_1 + 0.081\lambda_2 + 0.103\lambda_3 + 0.055)(1-\rho), \\ 7r_1 + 5r_2 \le 6.5, \\ r_1 + r_2 = 1, \\ y_1, y_2, y_3, r_1, r_2 \ge 0, \end{cases} \end{aligned} \quad (31)$$

$$\begin{aligned} &\text{maximize}\{-6.5z_1 + z_2 - z_3\}, \\ &\text{subject to} \begin{cases} -7z_1 + z_2 - z_3 - (3.84\lambda_1 + 3.84\lambda_2 + 2.94\lambda_3 + 2.065)s_1 - (2.987\lambda_1 + 2.287\lambda_2 + 3.78\lambda_3 + 2.748)s_2 \\ \le (0.051\lambda_1 + 0.051\lambda_2 + 0.039\lambda_3 + 0.012)(1-\rho), \\ -5z_1 + z_2 - z_3 - (1.5\lambda_1 + 1.5\lambda_2 + 2.667\lambda_3 + 1.587)s_1 - (5.4\lambda_1 + 5.4\lambda_2 + 2.4\lambda_3 + 1.65)s_2 \\ \le (0.051\lambda_1 + 0.051\lambda_2 + 0.039\lambda_3 + 0.012)(1-\rho), \\ 4s_1 + 6.5s_2 \le 5.5, \\ s_1 + s_2 = 1, \\ z_1, z_2, z_3 s_1, s_2 \ge 0. \end{cases} \end{aligned} \quad (32)$$

Table 6: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (0.7, 0.7, 0.7)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	7.856	(0.4, 0.6)	8.866
0.1	(0.75, 0.25)	7.833	(0.4, 0.6)	8.855
0.2	(0.75, 0.25)	7.81	(0.4, 0.6)	8.844
0.3	(0.75, 0.25)	7.787	(0.4, 0.6)	8.833
0.4	(0.75, 0.25)	7.765	(0.4, 0.6)	8.822
0.5	(0.75, 0.25)	7.742	(0.4, 0.6)	8.811
0.6	(0.75, 0.25)	7.719	(0.4, 0.6)	8.799
0.7	(0.75, 0.25)	7.696	(0.4, 0.6)	8.789
0.8	(0.75, 0.25)	7.674	(0.4, 0.6)	8.778
0.9	(0.75, 0.25)	7.651	(0.4, 0.6)	8.767
1.0	(0.75, 0.25)	7.628	(0.4, 0.6)	8.756

Table 7: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (0.8, 0.7, 0.9)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	8.673	(0.4, 0.6)	9.765
0.1	(0.75, 0.25)	8.648	(0.4, 0.6)	9.752
0.2	(0.75, 0.25)	8.622	(0.4, 0.6)	9.739
0.3	(0.75, 0.25)	8.597	(0.4, 0.6)	9.727
0.4	(0.75, 0.25)	8.571	(0.4, 0.6)	9.715
0.5	(0.75, 0.25)	8.546	(0.4, 0.6)	9.703
0.6	(0.75, 0.25)	8.521	(0.4, 0.6)	9.690
0.7	(0.75, 0.25)	8.495	(0.4, 0.6)	9.678
0.8	(0.75, 0.25)	8.469	(0.4, 0.6)	9.665
0.9	(0.75, 0.25)	8.444	(0.4, 0.6)	9.653
1.0	(0.75, 0.25)	8.419	(0.4, 0.6)	9.641

Table 8: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (0.8, 0.8, 0.8)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	8.667	(0.4, 0.6)	9.899
0.1	(0.75, 0.25)	8.642	(0.4, 0.6)	9.887
0.2	(0.75, 0.25)	8.617	(0.4, 0.6)	9.874
0.3	(0.75, 0.25)	8.591	(0.4, 0.6)	9.862
0.4	(0.75, 0.25)	8.567	(0.4, 0.6)	9.849
0.5	(0.75, 0.25)	8.542	(0.4, 0.6)	9.837
0.6	(0.75, 0.25)	8.516	(0.4, 0.6)	9.824
0.7	(0.75, 0.25)	8.491	(0.4, 0.6)	9.812
0.8	(0.75, 0.25)	8.466	(0.4, 0.6)	9.799
0.9	(0.75, 0.25)	8.441	(0.4, 0.6)	9.787
1.0	(0.75, 0.25)	8.416	(0.4, 0.6)	9.774

Table 9: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (0.9, 0.8, 0.8)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	8.946	(0.4, 0.6)	10.288
0.1	(0.75, 0.25)	8.92	(0.4, 0.6)	10.275
0.2	(0.75, 0.25)	8.894	(0.4, 0.6)	10.262
0.3	(0.75, 0.25)	8.869	(0.4, 0.6)	10.249
0.4	(0.75, 0.25)	8.843	(0.4, 0.6)	10.236
0.5	(0.75, 0.25)	8.817	(0.4, 0.6)	10.223
0.6	(0.75, 0.25)	8.791	(0.4, 0.6)	10.21
0.7	(0.75, 0.25)	8.765	(0.4, 0.6)	10.197
0.8	(0.75, 0.25)	8.739	(0.4, 0.6)	10.184
0.9	(0.75, 0.25)	8.714	(0.4, 0.6)	10.171
1.0	(0.75, 0.25)	8.688	(0.4, 0.6)	10.158

Table 10: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (0.9, 0.9, 0.9)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	9.479	(0.4, 0.6)	10.932
0.1	(0.75, 0.25)	9.452	(0.4, 0.6)	10.918
0.2	(0.75, 0.25)	9.424	(0.4, 0.6)	10.904
0.3	(0.75, 0.25)	9.397	(0.4, 0.6)	10.89
0.4	(0.75, 0.25)	9.369	(0.4, 0.6)	10.876
0.5	(0.75, 0.25)	9.341	(0.4, 0.6)	10.862
0.6	(0.75, 0.25)	9.314	(0.4, 0.6)	10.848
0.7	(0.75, 0.25)	9.286	(0.4, 0.6)	10.835
0.8	(0.75, 0.25)	9.258	(0.4, 0.6)	10.821
0.9	(0.75, 0.25)	9.231	(0.4, 0.6)	10.807
1.0	(0.75, 0.25)	9.203	(0.4, 0.6)	10.793

Table 11: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (1.0, 0.9, 0.8)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	9.488	(0.4, 0.6)	11.066
0.1	(0.75, 0.25)	9.461	(0.4, 0.6)	11.052
0.2	(0.75, 0.25)	9.434	(0.4, 0.6)	11.038
0.3	(0.75, 0.25)	9.407	(0.4, 0.6)	11.024
0.4	(0.75, 0.25)	9.379	(0.4, 0.6)	11.01
0.5	(0.75, 0.25)	9.352	(0.4, 0.6)	10.996
0.6	(0.75, 0.25)	9.325	(0.4, 0.6)	10.982
0.7	(0.75, 0.25)	9.298	(0.4, 0.6)	10.968
0.8	(0.75, 0.25)	9.27	(0.4, 0.6)	10.954
0.9	(0.75, 0.25)	9.243	(0.4, 0.6)	10.94
1.0	(0.75, 0.25)	9.216	(0.4, 0.6)	10.926

For different values $\lambda_1, \lambda_2, \lambda_3$, and ρ, the equilibrium strategies and the equilibrium values of both players can be obtained by solving equations (31) and (32), as depicted in Tables 1–12.

It can be easily seen from Table 1 that when $\lambda_1 = 0.2, \lambda_2 = 0.3, \lambda_3 = 0.5$, and $\rho = 0$, the equilibrium value and the equilibrium strategy for player I are $U^* = \mathbf{q}^{\mathrm{T}}\mathbf{y}^* = 4.868$ and $\mathbf{r}^* = (0.75, 0.25)^T$, respectively; and the equilibrium value and the equilibrium strategy for player II are $W^* = \mathbf{p}^{\mathrm{T}}\mathbf{z}^* = 4.855$ and $\mathbf{s}^* = (0.4, 0.6)^T$, respectively. The results indicate that different optimal solutions can be obtained for different values of $\lambda_1, \lambda_2, \lambda_3$, and ρ. Thus, it is essential to take all the parameters into consideration.

Table 12: The equilibrium strategies and the equilibrium values of player I and player II when $(\lambda_1, \lambda_2, \lambda_3) = (1.0, 1.0, 1.0)$.

ρ	Player I		Player II	
	$\mathbf{r}^*$	U^*	$\mathbf{s}^*$	W^*
0	(0.75, 0.25)	10.291	(0.4, 0.6)	11.965
0.1	(0.75, 0.25)	10.261	(0.4, 0.6)	11.949
0.2	(0.75, 0.25)	10.231	(0.4, 0.6)	11.934
0.3	(0.75, 0.25)	10.201	(0.4, 0.6)	11.919
0.4	(0.75, 0.25)	10.171	(0.4, 0.6)	11.903
0.5	(0.75, 0.25)	10.141	(0.4, 0.6)	11.888
0.6	(0.75, 0.25)	10.111	(0.4, 0.6)	11.873
0.7	(0.75, 0.25)	10.081	(0.4, 0.6)	11.858
0.8	(0.75, 0.25)	10.051	(0.4, 0.6)	11.842
0.9	(0.75, 0.25)	10.021	(0.4, 0.6)	11.827
1.0	(0.75, 0.25)	9.99	(0.4, 0.6)	11.812

Table 13: The equilibrium strategies and the equilibrium values of player I and player II when $\theta = 0$.

ρ	Player I		Player II	
	r^*	U^*	s^*	W^*
0	(0.75, 0.25)	6.809	(0.4, 0.6)	7.496
0.1	(0.75, 0.25)	6.787	(0.4, 0.6)	7.486
0.2	(0.75, 0.25)	6.764	(0.4, 0.6)	7.476
0.3	(0.75, 0.25)	6.742	(0.4, 0.6)	7.467
0.4	(0.75, 0.25)	6.719	(0.4, 0.6)	7.457
0.5	(0.75, 0.25)	6.697	(0.4, 0.6)	7.447
0.6	(0.75, 0.25)	6.674	(0.4, 0.6)	7.438
0.7	(0.75, 0.25)	6.652	(0.4, 0.6)	7.427
0.8	(0.75, 0.25)	6.629	(0.4, 0.6)	7.418
0.9	(0.75, 0.25)	6.607	(0.4, 0.6)	7.408
1.0	(0.75, 0.25)	6.585	(0.4, 0.6)	7.398

Table 14: The equilibrium strategies and the equilibrium values of player I and player II when $\theta = 0.2$.

ρ	Player I		Player II	
	r^*	U^*	s^*	W^*
0	(0.75, 0.25)	6.144	(0.4, 0.6)	6.891
0.1	(0.75, 0.25)	6.124	(0.4, 0.6)	6.882
0.2	(0.75, 0.25)	6.105	(0.4, 0.6)	6.873
0.3	(0.75, 0.25)	6.085	(0.4, 0.6)	6.864
0.4	(0.75, 0.25)	6.066	(0.4, 0.6)	6.854
0.5	(0.75, 0.25)	6.046	(0.4, 0.6)	6.846
0.6	(0.75, 0.25)	6.027	(0.4, 0.6)	6.837
0.7	(0.75, 0.25)	6.007	(0.4, 0.6)	6.828
0.8	(0.75, 0.25)	5.988	(0.4, 0.6)	6.819
0.9	(0.75, 0.25)	5.968	(0.4, 0.6)	6.810
1.0	(0.75, 0.25)	5.949	(0.4, 0.6)	6.801

5.3. Comparison Analysis. In this subsection, the proposed ranking approach is compared with three other approaches that were introduced by Khalifa [41], Ye [42], and Garai et al. [43].

We compare our results with those of Khalifa [41], where a score function is described by

Table 15: The equilibrium strategies and the equilibrium values of player I and player II when $\theta = 0.4$.

ρ	Player I		Player II	
	r^*	U^*	s^*	W^*
0	(0.75, 0.25)	5.478	(0.4, 0.6)	6.285
0.1	(0.75, 0.25)	5.462	(0.4, 0.6)	6.277
0.2	(0.75, 0.25)	5.445	(0.4, 0.6)	6.269
0.3	(0.75, 0.25)	5.429	(0.4, 0.6)	6.261
0.4	(0.75, 0.25)	5.412	(0.4, 0.6)	6.253
0.5	(0.75, 0.25)	5.396	(0.4, 0.6)	6.245
0.6	(0.75, 0.25)	5.379	(0.4, 0.6)	6.237
0.7	(0.75, 0.25)	5.363	(0.4, 0.6)	6.228
0.8	(0.75, 0.25)	5.346	(0.4, 0.6)	6.220
0.9	(0.75, 0.25)	5.329	(0.4, 0.6)	6.212
1.0	(0.75, 0.25)	5.313	(0.4, 0.6)	6.204

$$S(\widetilde{b}) = \frac{1}{16}(k+l+m+n)\left(u_{\widetilde{b}} + \left(1 - v_{\widetilde{b}}\right) + \left(1 - w_{\widetilde{b}}\right)\right). \tag{33}$$

Here, $\widetilde{b} = \langle (k,l,m,n); u_{\widetilde{b}}, v_{\widetilde{b}}, w_{\widetilde{b}}\rangle$ expresses an SVTNN. Based on this score function, we obtain a set of linear optimization models as follows:

$$\begin{aligned}
&\max\{-5.5y_1 + y_2 - y_3\},\\
&\text{s.t.}\begin{cases} -4y_1 + y_2 - y_3 - 4.744r_1 - 2.75r_2 \le 0,\\ -6.5y_1 + y_2 - y_3 - 2.909r_1 - 3.503r_2 \le 0,\\ 7r_1 + 5r_2 \le 6.5,\\ r_1 + r_2 = 1,\\ y_1, y_2, y_3, r_1, r_2 \ge 0,\end{cases}\\
&\max\{-6.5z_1 + z_2 - z_3\},\\
&\text{s.t.}\begin{cases} -7z_1 + z_2 - z_3 - 4.097s_1 - 3.675s_2 \le 0,\\ -5z_1 + z_2 - z_3 - 2.688s_1 - 4.5s_2 \le 0,\\ 4s_1 + 6.5s_2 \le 5.5,\\ s_1 + s_2 = 1,\\ z_1, z_2, z_3 s_1, s_2 \ge 0.\end{cases}
\end{aligned} \tag{34}$$

Using the Simplex technique, we can obtain that the equilibrium value and the equilibrium strategy for player I are $U^* = \mathbf{q}^{\mathrm{T}}\mathbf{y}^* = 3.533$ and $\mathbf{r}^* = (0.75, 0.25)^T$, respectively; and the equilibrium value and the equilibrium strategy for player II are $W^* = \mathbf{p}^{\mathrm{T}}\mathbf{z}^* = 3.775$ and $\mathbf{s}^* = (0.4, 0.6)$, respectively, although this approach provides the same optimal solutions as our results.

We compare our results with those of Jun Ye [42], where the score function is given by

$$S(\widetilde{b}) = \frac{1}{12}(k+l+m+n)\left(2 + u_{\widetilde{b}} - v_{\widetilde{b}} - w_{\widetilde{b}}\right). \tag{35}$$

Here, $\widetilde{b} = \langle (k,l,m,n); u_{\widetilde{b}}, v_{\widetilde{b}}, w_{\widetilde{b}}\rangle$ expresses an SVTNN. Based on this score function, we obtain the following mathematical programming models:

Table 16: The equilibrium strategies and the equilibrium values of player I and player II when $\theta = 0.6$.

ρ	Player I		Player II	
	r^*	U^*	s^*	W^*
0	(0.75, 0.25)	4.813	(0.4, 0.6)	5.679
0.1	(0.75, 0.25)	4.799	(0.4, 0.6)	5.672
0.2	(0.75, 0.25)	4.786	(0.4, 0.6)	5.665
0.3	(0.75, 0.25)	4.772	(0.4, 0.6)	5.658
0.4	(0.75, 0.25)	4.759	(0.4, 0.6)	5.651
0.5	(0.75, 0.25)	4.745	(0.4, 0.6)	5.643
0.6	(0.75, 0.25)	4.732	(0.4, 0.6)	5.636
0.7	(0.75, 0.25)	4.718	(0.4, 0.6)	5.629
0.8	(0.75, 0.25)	4.704	(0.4, 0.6)	5.622
0.9	(0.75, 0.25)	4.691	(0.4, 0.6)	5.614
1.0	(0.75, 0.25)	4.677	(0.4, 0.6)	5.607

$$
\begin{aligned}
&\max \quad \{-5.5y_1 + y_2 - y_3\} \\
&\text{s.t.} \quad \begin{cases} -4y_1 + y_2 - y_3 - 6.325r_1 - 3.667r_2 \le 0, \\ -6.5y_1 + y_2 - y_3 - 3.879r_1 - 4.671r_2 \le 0, \\ 7r_1 + 5r_2 \le 6.5, \\ r_1 + r_2 = 1, \\ y_1, y_2, y_3, r_1, r_2 \ge 0, \end{cases} \\
&\max \quad \{-6.5z_1 + z_2 - z_3\}, \\
&\text{s.t.} \quad \begin{cases} -7z_1 + z_2 - z_3 - 5.463s_1 - 4.9s_2 \le 0, \\ -5z_1 + z_2 - z_3 - 3.583s_1 - 6s_2 \le 0, \\ 4s_1 + 6.5s_2 \le 5.5, \\ s_1 + s_2 = 1, \\ z_1, z_2, z_3 s_1, s_2 \ge 0. \end{cases}
\end{aligned} \tag{36}
$$

Using the Simplex technique, we can obtain the equilibrium value and the equilibrium strategy for player I as $U^* = \mathbf{q}^{\mathrm{T}}\mathbf{y}^* = 4.71$ and $\mathbf{r}^* = (0.75, 0.25)^T$, respectively; and the equilibrium value and the equilibrium strategy for player II are $W^* = \mathbf{p}^{\mathrm{T}}\mathbf{z}^* = 5.033$ and $\mathbf{s}^* = (0.4, 0.6)$, respectively, although this approach provides the same optimal solutions as our results.

Finally, we compare our results with those of Garai et al. [43], where the ranking function is described by

$$
M(\tilde{b}) = \frac{1}{6}(k + 2l + 2m + n)\left(\theta u_{\tilde{b}}^2 + (1-\theta)\left(1 - v_{\tilde{b}}\right)^2 + (1-\theta)\left(1 - w_{\tilde{b}}\right)^2\right). \tag{37}
$$

Here, $\tilde{b} = \langle (k, l, m, n); u_{\tilde{b}}, v_{\tilde{b}}, w_{\tilde{b}} \rangle$ represents an SVTNN. Based on this ranking function, we can get a set of optimization models as follows:

$$
\begin{aligned}
&\text{maximize} \quad \{-5.5y_1 + y_2 - y_3\}, \\
&\qquad -4y_1 + y_2 - y_3 - (8.16667 - 1.551667\theta)r_1 - (5.83 - 3.85\theta)r_2 \le (0.22475 - 0.1488\theta)(1-\rho), \\
&\qquad -6.5y_1 + y_2 - y_3 - (6.08333 - 4.5625\theta)r_1 - (5.42667 - 1.833333\theta)r_2 \le (0.22475 - 0.1488\theta)(1-\rho), \\
&\text{subject to} \quad 7r_1 + 5r_2 \le 6.5, \\
&\qquad r_1 + r_2 = 1, \\
&\qquad y_1, y_2, y_3, r_1, r_2 \ge 0. \\
&\text{maximize} \quad \{-6.5z_1 + z_2 - z_3\}, \\
&\qquad -7z_1 + z_2 - z_3 - (8.09833 - 3.511667\theta)s_1 - (7.90833 - 4.015\theta)s_2 \le (0.0979333 - 0.04246667\theta)(1-\rho), \\
&\qquad -5z_1 + z_2 - z_3 - (5.33333 - 3.413333\theta)s_1 - (8.775 - 2.7\theta)s_2 \le (0.0979333 - 0.04246667\theta)(1-\rho), \\
&\text{subject to} \quad 4s_1 + 6.5s_2 \le 5.5, \\
&\qquad s_1 + s_2 = 1, \\
&\qquad z_1, z_2, z_3 s_1, s_2 \ge 0.
\end{aligned} \tag{38}
$$

By solving the above mathematical programming models, we obtain the following tabulated optimal solutions, given in Tables 13–18. From the results shown in Tables 1–18, the optimal strategies obtained by different

Table 17: The equilibrium strategies and the equilibrium values of player I and player II when $\theta = 0.8$.

ρ	Player I		Player II	
	r^*	U^*	s^*	W^*
0	(0.75, 0.25)	4.147	(0.4, 0.6)	5.017
0.1	(0.75, 0.25)	4.137	(0.4, 0.6)	5.010
0.2	(0.75, 0.25)	4.126	(0.4, 0.6)	5.003
0.3	(0.75, 0.25)	4.116	(0.4, 0.6)	4.997
0.4	(0.75, 0.25)	4.105	(0.4, 0.6)	4.991
0.5	(0.75, 0.25)	4.094	(0.4, 0.6)	4.985
0.6	(0.75, 0.25)	4.084	(0.4, 0.6)	4.978
0.7	(0.75, 0.25)	4.073	(0.4, 0.6)	4.972
0.8	(0.75, 0.25)	4.063	(0.4, 0.6)	4.965
0.9	(0.75, 0.25)	4.052	(0.4, 0.6)	4.959
1.0	(0.75, 0.25)	4.042	(0.4, 0.6)	4.953

Table 18: The equilibrium strategies and the equilibrium values of player I and player II when $\theta = 1.0$.

ρ	Player I		Player II	
	r^*	U^*	s^*	W^*
0	(0.75, 0.25)	3.482	(0.4, 0.6)	4.287
0.1	(0.75, 0.25)	3.474	(0.4, 0.6)	4.281
0.2	(0.75, 0.25)	3.467	(0.4, 0.6)	4.276
0.3	(0.75, 0.25)	3.459	(0.4, 0.6)	4.270
0.4	(0.75, 0.25)	3.451	(0.4, 0.6)	4.265
0.5	(0.75, 0.25)	3.443	(0.4, 0.6)	4.259
0.6	(0.75, 0.25)	3.436	(0.4, 0.6)	4.253
0.7	(0.75, 0.25)	3.429	(0.4, 0.6)	4.248
0.8	(0.75, 0.25)	3.421	(0.4, 0.6)	4.242
0.9	(0.75, 0.25)	3.413	(0.4, 0.6)	4.237
1.0	(0.75, 0.25)	3.406	(0.4, 0.6)	4.231

ranking approaches are the same as those of the proposed approach. So, the proposed approach is feasible and effective.

6. Conclusion

The constrained bimatrix games with payoffs of SVTNNs are studied and constructed in this article. The ranking order relation, important theorems, and arithmetic operations of SVTNNs are outlined. Novel neutrosophic optimization problems for both players are established from the arithmetic operations of SVTNNs and solution method for SVTNNs constrained bimatrix games. Based on the ranking approach of SVTNNs presented in this article, the neutrosophic optimization problems for both players are converted into crisp parameterized problems, which are solved to obtain the equilibrium optimal strategies and equilibrium values for two players. Moreover, the ranking approach proposed in this article is demonstrated with a numerical simulation. Finally, our article is the first to study the constrained bimatrix games under neutrosophic environment and provide algorithm and practicable application for SVTNNs constrained bimatrix games.

In the future, we will study game theory under other types of uncertain environment such as linguistic neutrosophic, interval neutrosophic, linguistic interval neutrosophic, and linguistic interval intuitionistic neutrosophic. Furthermore, we will apply the proposed ranking approach to other areas such as pattern recognition, supply chain, risk evaluation, teacher selection, and optimization models.

Data Availability

The data used to support the findings of this research are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References

[1] J. Von Neumann and D. Morgenstern, *The Theory of Games in Economic Bahavior*, John Wiley & Sons, Inc., New York, NY, USA, 1944.

[2] I. C. Hung, K. H. Hsia, and L. W. Chen, "Fuzzy differential game of guarding a movable territory," *Information Sciences*, vol. 91, pp. 113–131, 1996.

[3] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[4] D.-F. Li, "Bi-matrix games with payoffs of intuitionistic fuzzy sets and bilinear programming method," in *Decision and Game Theory in Management with Intuitionistic Fuzzy Sets*, pp. 421–441, Springer, Berlin, Germany, 2014.

[5] J. C. Figueroa-García and A. Mehra, S. Chandra, "Optimal solutions for group matrix games involving interval-valued fuzzy numbers," *Fuzzy Sets and Systems*, vol. 362, pp. 55–70, 2019.

[6] J. Jana and S. K. Roy, "Dual hesitant fuzzy matrix games: based on new similarity measure," *Soft Computing*, vol. 23, no. 18, pp. 8873–8886, 2018.

[7] A. Singh, A. Gupta, and A. Mehra, "Matrix games with 2-tuple linguistic information," *Annals of Operations Research*, vol. 287, no. 2, pp. 895–910, 2020.

[8] L. Zhou and F. Xiao, "A new matrix game with payoffs of generalized Dempster-Shafer structures," *International Journal of Intelligent Systems*, vol. 34, no. 9, pp. 2253–2268, 2019.

[9] M. R. Seikh, S. Karmakar, and M. Xia, "Solving matrix games with hesitant fuzzy pay-offs," *Iranian Journal of Fuzzy Systems*, vol. 17, no. 4, pp. 25–40, 2020.

[10] Y. Han and Y. Deng, "A novel matrix game with payoffs of Maxitive Belief Structure," *International Journal of Intelligent Systems*, vol. 34, no. 4, pp. 690–706, 2019.

[11] S. K. Roy and S. K. Maiti, "Reduction methods of type-2 fuzzy variables and their applications to Stackelberg game," *Applied Intelligence*, vol. 50, pp. 1398–1415, 2020.

[12] A. Bhaumik, S. K. Roy, and G. W. Weber, "Hesitant interval-valued intuitionistic fuzzy-linguistic term set approach in Prisoners' dilemma game theory using TOPSIS: a case study on Human-trafficking," *Central European Journal of Operations Research*, vol. 28, no. 2, pp. 797–816, 2020.

[13] E. Ammar and M. G. Brikaa, "Solving bi-matrix games in tourism planning management under rough interval approach," *International Journal of Mathematical Sciences and Computing(IJMSC)*, vol. 4, no. 44–62, 2019.

[14] M. G. Brikaa, Z. Zheng, and E.-S. Ammar, "Fuzzy multi-objective programming approach for constrained matrix games with payoffs of fuzzy rough numbers," *Symmetry*, vol. 11, no. 5, p. 702, 2019.

[15] A. Bhaumik, S. K. Roy, and G. W. Weber, "Multi-objective linguistic-neutrosophic matrix game and its applications to tourism management," *Journal of Dynamics & Games*, vol. 8, no. 2, pp. 101–118, 2021.

[16] M. G. Brikaa, Z. Zheng, and E.-S. Ammar, "Resolving indeterminacy approach to solve multi-criteria zero-sum matrix games with intuitionistic fuzzy goals," *Mathematics*, vol. 8, no. 3, p. 305, 2020.

[17] J.-J. An and D.-F. Li, "A linear programming approach to solve constrained bi-matrix games with intuitionistic fuzzy payoffs," *International Journal of Fuzzy Systems*, vol. 21, no. 3, pp. 908–915, 2019.

[18] K. Firouzbakht, G. Noubir, and M. Salehi, "Linearly constrained bimatrix games in wireless communications," *IEEE Transactions on Communications*, vol. 64, no. 1, pp. 429–440, 2015.

[19] F. Meng and J. Zhan, "Two methods for solving constrained bi-matrix games," *The Open Cybernetics and Systemics Journal*, vol. 8, no. 1, 2014.

[20] H. Bigdeli, H. Hassanpour, and J. Tayyebi, "Constrained bimatrix games with fuzzy goals and its application in nuclear negotiations," *Iranian Journal of Numerical Analysis and Optimization*, vol. 8, no. 1, pp. 81–110, 2018.

[21] F. Smarandache, "Neutrosophic set a generalization of the intuitionistic fuzzy set," *International Journal of Pure and Applied Mathematics*, vol. 24, pp. 287–297, 2005.

[22] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman, "Single valued neutrosophic sets," *Multispace & Multistructure*, vol. 4, pp. 410–413, 2010.

[23] H. Garg, "Decision making analysis based on sine trigonometric operational laws for single-valued neutrosophic sets and their applications," *Applied and Computational Mathematics*, vol. 19, no. 2, pp. 255–276, 2020.

[24] M. Murugappan, "Neutrosophic inventory model under immediate return for deficient items," *Annals of Optimization Theory and Practice*, vol. 3, no. 4, pp. 1–9, 2020.

[25] G. Harish, "Novel neutrality aggregation operators-based multiattribute group decision making method for single-valued neutrosophic numbers," *Soft Computing*, vol. 24, pp. 10327–10349, 2020.

[26] M. Abdel-Basset, M. Ali, and A. Atef, "Resource levelling problem in construction projects under neutrosophic environment," *The Journal of Supercomputing*, vol. 76, no. 2, pp. 964–988, 2020.

[27] T. Garai, S. Dalapati, H. Garg, and T. K. Roy, "Possibility mean, variance and standard deviation of single-valued neutrosophic numbers and its applications to multi-attribute decision-making problems," *Soft Computing*, vol. 24, no. 24, pp. 18795–18809, 2020.

[28] S. Broumi, A. Dey, M. Talea et al., "Shortest path problem using Bellman algorithm under neutrosophic environment," *Complex & Intelligent Systems*, vol. 5, no. 4, pp. 409–416, 2019.

[29] H. Garg, "Algorithms for single-valued neutrosophic decision making based on TOPSIS and clustering methods with new distance measure," *AIMS Mathematics*, vol. 5, no. 3, pp. 2671–2693, 2020.

[30] M. Mullai and R. Surya, "Neutrosophic inventory backorder problem using triangular neutrosophic numbers," *Neutrosophic Sets and Systems*, vol. 31, pp. 148–155, 2020.

[31] H. Garg and Nancy, "Multiple criteria decision making based on frank choquet heronian mean operator for single-valued neutrosophic sets," *Applied and Computational Mathematics*, vol. 18, no. 2, pp. 163–188, 2019.

[32] M. Leyva-Vázquez, M. A. Quiroz-Martínez, E. Portilla-Castell, Y. Hechavarría-Hernández, and J. R. González-Caballero, "A new model for the selection of information technology project in a neutrosophic environment," *Neutrosophic Sets and Systems*, vol. 32, no. 1, pp. 344–360, 2020.

[33] H. Garg and Nancy, "Non-linear programming method for multi-criteria decision making problems under interval neutrosophic set environment," *Applied Intelligence*, vol. 48, no. 8, pp. 2199–2213, 2018.

[34] R. Sun, J. Hu, and X. Chen, "Novel single-valued neutrosophic decision-making approaches based on prospect theory and their applications in physician selection," *Soft Computing*, vol. 23, no. 1, pp. 211–225, 2019.

[35] H. Garg and Nancy, "Some new biparametric distance measures on single-valued neutrosophic sets with applications to pattern recognition and medical diagnosis," *Information*, vol. 8, no. 4, p. 162, 2017.

[36] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, NY, USA, 1980.

[37] K. Atanassov, "Intuitionistic fuzzy sets," *Theory and Applications, Physica*, Academic Press, New York, NY, USA, 1999.

[38] I. Deli and Y. Şubaş, "A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems," *International Journal of Machine Learning and Cybernetics*, vol. 8, no. 4, pp. 1309–1322, 2017.

[39] A. Bhaumik, S. Kumar, and D. Li, "(α, β, γ)—cut set based ranking approach to solving bi-matrix games in neutrosophic environment," *Soft Computing*, vol. 25, no. 8, pp. 1–11, 2020.

[40] W. Fei and D.-F. Li, "Bilinear programming approach to solve interval bimatrix games in tourism planning management," *International Journal of Fuzzy Systems*, vol. 18, no. 3, pp. 504–510, 2016.

[41] H. A. Khalifa, "An approach for solving two-person zero-sum matrix games in neutrosophic environment," *Journal of Industrial and Systems Engineering*, vol. 12, no. 2, pp. 186–198, 2019.

[42] J. Ye, "Some weighted aggregation operators of trapezoidal neutrosophic numbers and their multiple attribute decision making method," *Informatica*, vol. 28, no. 2, pp. 387–402, 2017.

[43] T. Garai, H. Garg, and T. K. Roy, "A ranking method based on possibility mean for multi-attribute decision making with single valued neutrosophic numbers," *Journal of Ambient Intelligence and Humanized Computing*, vol. 11, pp. 5245–5258, 2020.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 5583218, 16 pages
https://doi.org/10.1155/2021/5583218

Research Article

Neutrosophic Semiopen Hypersoft Sets with an Application to MAGDM under the COVID-19 Scenario

D. Ajay,[1] J. Joseline Charisma,[1] N. Boonsatit ,[2] P. Hammachukiattikul ,[3] and G. Rajchakit [4]

[1]*Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Vellore 635 601, Tamilnadu, India*
[2]*Department of Mathematics, Faculty of Science and Technology, Rajamangala University of Technology Suvarnabhumi, Nonthaburi 11000, Thailand*
[3]*Department of Mathematics, Faculty of Science and Technology, Phuket Rajabhat University, Phuket 83000, Thailand*
[4]*Department of Mathematics, Faculty of Science, Maejo University, Chiang Mai 50290, Thailand*

Correspondence should be addressed to N. Boonsatit; nattakan.b@rmutsb.ac.th

Received 24 February 2021; Revised 19 March 2021; Accepted 5 April 2021; Published 30 April 2021

Academic Editor: Broumi Said

Hypersoft set is a generalization of soft sets, which takes into account a multiargument function. The main objective of this work is to introduce fuzzy semiopen and closed hypersoft sets and study some of their characterizations and also to present neutrosophic semiopen and closed hypersoft sets, an extension of fuzzy hypersoft sets, along with few basic properties. We propose two algorithms based on neutrosophic hypersoft open sets and topology to obtain optimal decisions in MAGDM. The efficiency of the algorithms proposed is demonstrated by applying them to the current COVID-19 scenario.

1. Introduction

Fuzzy set theory [1] is an important tool for dealing with vagueness and incomplete data and is much more evolving and applied in different fields. Fuzzy set, which is an extension of general sets, has elements with membership function within the interval [0, 1]. In view of other options of human thinking, fuzzy set along with some conditions is extended to the intuitionistic fuzzy set [2]. The intuitionistic fuzzy set assigns membership and nonmembership functions to each object which satisfies the constraint that the sum of both membership functions is between 0 and 1.

Fuzziness was improved and extended from intuitionistic sets to neutrosophic sets. Smarandache [3] proposed neutrosophic sets, an essential mathematical tool which deals with incomplete, indeterminant, and inconsistent information. Neutrosophic set is characterized by the elements with truth, indeterminacy, and false membership functions which assume values within the range of 0 and 1. Wang et al. [4] proposed the concept of single-valued neutrosophic sets, a generalization of intuitionistic sets and a subclass of neutrosophic sets, which comprise elements with three membership functions which they belong to interval [0, 1]. Under this neutrosophic environment, many researchers have worked on their extensions and developed many applications and results. A ranking approach based on the outranking relations of simplified neutrosophic numbers is developed in order to solve MCDM problems. Practical examples are provided to illustrate the proposed approach with a comparison analysis [5]. A comparison analysis is performed for this method with two examples [6], and the developed single-valued neutrosophic TOPSIS extension is demonstrated on a numerical illustration of the evaluation and selection of e-commerce development strategies [7].

Molodtsov [8] introduced the idea of soft theory as a new approach to dealing with uncertainty, and now, there is a rapid growth of soft theory along with applications in various fields. Maji et al. [9] defined various basic concepts of soft theory, and the study of soft semirings by using the soft set theory has been initiated, and the notions of soft semirings, soft sub-semirings, soft ideals, idealistic soft semirings, and soft semiring homomorphisms with several related properties are investigated [10, 11]. Maji et al. [12] developed the fuzzy soft set theory, which is a combination of soft and fuzzy sets.

The idea of soft sets was generalized into hypersoft sets by Smarandache [13] by transforming the argument function F into a multiargument function. He also introduced many results on hypersoft sets. Saqlain et al. [14] utilized this notion and proposed a generalized TOPSIS method for decision-making. Neutrosophic sets [15], from their very introduction, have seen many such extensions and have been very successful in applications. A new hybrid methodology for the selection of offshore wind power station location combining the Analytical Hierarchy Process and Preference Ranking Organization Method for Enrichment Evaluations methods in the neutrosophic environment has been proposed [16], a neutrosophic preference ranking organization method for enrichment evaluation technique for multicriteria decision-making problems to describe fuzzy information efficiently was proposed and applied to a real case study to select proper security service for FMEC in the presence of fuzzy information [17], and a model is proposed based on a plithogenic set and is applied to differentiate between COVID-19 and other four viral chest diseases under the uncertainty environment [18].

In 2019, Rana et al. [19] introduced the plithogenic fuzzy hypersoft set (PFHS) in the matrix form and defined some operations on the PFHS. Single- and multivalued neutrosophic hypersoft sets were proposed by Saqlain et al. [20], who also defined tangent similarity measure for single-valued sets and an application of the same in a decision-making scenario. In another effort, Saqlain et al. [21] also introduced aggregation operators for neutrosophic hypersoft sets. A recent development in this area of research is the introduction of basic operations on hypersoft sets in which hypersoft points in different fuzzy environments are also introduced [22].

Fuzzy topology, a collection of fuzzy sets fulfilling the axioms, was defined by Chang [23]. A new definition of fuzzy space compactness and observed to have α–compactness along with a Tychonoff theorem for an arbitrary product of α–compact fuzzy spaces and a 1-point compactification [24], filters in the lattice I^X, where I is the unit interval and X an arbitrary set, have all been studied and using this study the convergence is defined in fuzzy topological space which leads to characterise fuzzy continuity and compactness [25]. Then, the basic concepts of intuitionistic fuzzy topological spaces were constructed, and the definitions of fuzzy continuity, fuzzy compactness, fuzzy connectedness, and fuzzy Hausdorff space and some characterizations concerning fuzzy compactness and fuzzy connectedness were defined [26]. Neutrosophic topological spaces were introduced by Salama and Alblowi [27], and further concepts such as connectedness, semiclosed sets, and generalized closed sets [28] were developed.

The concept of fuzzy soft topology and some of its structural properties such as neighborhood of a fuzzy soft set, interior fuzzy soft set, fuzzy soft basis, and fuzzy soft subspace topology were studied [29]. The soft topological spaces, soft continuity of soft mappings, soft product topology, and soft compactness, as well as properties of soft projection mappings, have all been defined [30], and a relationship between a fuzzy soft set's closure and its fuzzy soft limit points has been constructed on fuzzy soft topological spaces [31]. Subspace, separation axioms, compactness, and connectedness on intuitionistic fuzzy soft topological spaces were defined along with some base theorems [32], some important properties of intuitionistic fuzzy soft topological spaces and intuitionistic fuzzy soft closure and interior of an intuitionistic fuzzy soft set were introduced, and an intuitionistic fuzzy soft continuous mapping with structural characteristics was studied [33]. A topology on a neutrosophic soft set was constructed, neutrosophic soft interior, neutrosophic soft closure, neutrosophic soft neighbourhood, and neutrosophic soft boundary were introduced, some of their basic properties were studied, and the concept of separation axioms on the neutrosophic soft topological space was introduced [34]. The concept of fuzzy hypersoft sets was applied to fuzzy topological spaces, and fuzzy hypersoft topological spaces were presented by Ajay and Charisma in [35]. In the same work, fuzzy hypersoft topology has been extended to intuitionistic and neutrosophic hypersoft topological spaces along with their properties. In this paper, we define the idea of semiopen sets in fuzzy hypersoft topological spaces with their characterization and extend to semiopen sets in intuitionistic and neutrosophic hypersoft topological spaces.

The paper is structured as follows: Section 2 recalls few basic terminologies and definitions of fuzzy hypersoft topological spaces. In Section 3, we define semiopen sets in fuzzy hypersoft topological spaces along with some of their properties. Sections 4 and 5 elaborate the logical extension of fuzzy hypersoft semiopen sets to intuitionistic and neutrosophic hypersoft semiopen sets. In Section 6, we present an application of the neutrosophic hypersoft open set and topology in an MAGDM and conclude in Section 7.

2. Preliminaries

Definition 1 (see [35]). Let $(\varpi, \mathfrak{X})$ be an element of $\mathfrak{P}(V, \mathbb{E})$ (where $\mathfrak{X} = X_1 x X_2 x X_3 \ldots x X_n$ with each X_i a subset of $E_i\,(i = 1, 2, \ldots, n)$), and let the set of all fuzzy hypersoft (FH) subsets of $(\varpi, \mathfrak{X})$ be $P(\varpi, \mathfrak{X})$ and τ, a subcollection of $P(\varpi, \mathfrak{X})$.

(i) $\varnothing_{\mathfrak{X}}, (\varpi, \mathfrak{X}) \in \tau$

(ii) $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in \tau \Longrightarrow (\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B}) \in \tau$

(iii) $\{(\Theta, \mathfrak{J})_l | l \in L\} \in \tau \Longrightarrow \cup_{l \in L} (\Theta, \mathfrak{J})_l \in \tau$

If the above axioms are satisfied, then τ is a fuzzy hypersoft topology (FHT) on $(\varpi, \mathfrak{X}) \cdot (\mathfrak{X}_{\varpi}, \tau)$ which is called a fuzzy hypersoft topological space (FHTS). Every member of τ is called an open fuzzy hypersoft set (OFHS). A fuzzy hypersoft set is said to be a closed fuzzy hypersoft set (CFHS) if its complement is OFHS.

Example 1. Let $V = \{x_1, x_2, x_3, x_4\}$ and the attributes be $E_1 = \{a_1, a_2\}$, $E_2 = \{a_3, a_4\}$, and $E_3 = \{a_5, a_6\}$. Then, the fuzzy hypersoft set is

$$\begin{aligned}
&\left((a_1, a_3, a_5), \left\{\frac{x_2}{0.4}, \frac{x_4}{0.6}\right\}\right),\\
&\left((a_1, a_3, a_6), \left\{\frac{x_1}{0.7}\right\}\right),\\
&\left((a_1, a_4, a_5), \left\{\frac{x_1}{0.4}, \frac{x_2}{0.3}\right\}\right),\\
&\left((a_1, a_4, a_6), \left\{\frac{x_1}{0.5}, \frac{x_3}{0.7}\right\}\right),\\
&\left((a_2, a_3, a_5), \left\{\frac{x_2}{0.3}, \frac{x_3}{0.5}\right\}\right),\\
&\left((a_2, a_3, a_6), \left\{\frac{x_3}{0.8}\right\}\right),\\
&\left((a_2, a_4, a_5), \left\{\frac{x_4}{0.9}\right\}\right),\\
&\left((a_2, a_4, a_6), \left\{\frac{x_2}{0.6}\right\}\right).
\end{aligned} \tag{1}$$

Let us consider this fuzzy hypersoft as $(\varpi, \mathfrak{X})$. Then, the subfamily

$$\tau = \varnothing_{\mathfrak{X}}, (\varpi, \mathfrak{X}), \cdot \left\{ \begin{array}{c} \left((a_1, a_3, a_5), \left\{\frac{x_1}{0.3}, \frac{x_2}{0.6}\right\}\right), \left((a_2, a_3, a_5), \left\{\frac{x_2}{0.4}, \frac{x_3}{0.5}\right\}\right), \left((a_1, a_3, a_5), \left\{\frac{x_2}{0.4}\right\}\right), \left((a_2, a_3, a_5), \left\{\frac{x_2}{0.3}, \frac{x_3}{0.5}\right\}\right) \\ \left((a_1, a_3, a_5), \left\{\frac{x_1}{0.3}, \frac{x_2}{0.6}, \frac{x_3}{0.6}\right\}\right), \left((a_1, a_4, a_6), \left\{\frac{x_1}{0.5}, \frac{x_3}{0.7}\right\}\right), \left((a_1, a_3, a_6), \left\{\frac{x_1}{0.7}\right\}\right), \left((a_1, a_4, a_5), \left\{\frac{x_1}{0.4}, \frac{x_2}{0.3}\right\}\right), \\ \left((a_2, a_3, a_5), \left\{\frac{x_2}{0.4}, \frac{x_3}{0.5}\right\}\right), \left((a_2, a_3, a_6), \left\{\frac{x_3}{0.8}\right\}\right)\left((a_2, a_4, a_5), \left\{\frac{x_4}{0.9}\right\}\right)(a_2, a_4, a_6), \left\{\frac{x_2}{0.6}\right\}, \end{array} \right\} \tag{2}$$

of $P(\varpi, \mathfrak{X})$ is a FHT on $(\varpi, \mathfrak{X})$.

Definition 2 (see [35]). Let τ be a FHT on $(\varpi, \mathfrak{X}) \in \mathfrak{P}(V, \mathbb{E})$ and $(\chi, \mathfrak{B})$ be a FH set in $P(\varpi, \mathfrak{X})$. A FH set $(\Theta, \mathfrak{J})$ in $P(\varpi, \mathfrak{X})$ is a neighbourhood of the FH set of $(\chi, \mathfrak{B})$ if and only if there exists an OFHS $(\xi, \mathfrak{C})$ such that $(\chi, \mathfrak{B}) \subset (\xi, \mathfrak{C}) \subset (\Theta, \mathfrak{J})$.

Definition 3 (see [35]). Let $(\mathfrak{X}_{\varpi}, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B})$ be FH sets in $P(\varpi, \mathfrak{X})$ such that $(\chi, \mathfrak{B}) \subset (\Theta, \mathfrak{J})$. Then, $(\chi, \mathfrak{B})$ is said to be an interior fuzzy hypersoft set (IFHS) of $(\Theta, \mathfrak{J})$ if and only if $(\Theta, \mathfrak{J})$ is a neighbourhood of $(\chi, \mathfrak{B})$. The union of the whole IFHS of $(\Theta, \mathfrak{J})$ is named the interior of $(\Theta, \mathfrak{J})$ and denoted as $(\Theta, \mathfrak{J})^{\circ}$ or $\mathrm{FHint}(\Theta, \mathfrak{J})$.

Definition 4 (see [35]). Let $(\mathfrak{X}_{\varpi}, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}) \in \mathfrak{P}(V, \mathbb{E})$. The fuzzy hypersoft closure (FHC) of $(\Theta, \mathfrak{J})$ is the intersection of all CFH sets that contain $(\Theta, \mathfrak{J})$ which is denoted by $\overline{(\Theta, \mathfrak{J})}$ or $\mathrm{FHcl}(\Theta, \mathfrak{J})$.

Definition 5 (see [35]). Let $(\varpi, \mathfrak{X})$ be an element of $\mathfrak{P}(V, \mathbb{E})$ (where $\mathfrak{X} = X_1 x X_2 x X_3 \ldots x X_n$ with each X_i a subset of $E_i (i = 1, 2, \ldots, n)$. Let the set of all neutrosophic hypersoft

(NH) subsets of $(\varpi, \mathfrak{X})$ be $P(\varpi, \mathfrak{X})$ and τ, a subcollection of $P(\varpi, \mathfrak{X})$.

(i) $\varnothing_{\mathfrak{X}}, (\varpi, \mathfrak{X}) \in \tau$

(ii) $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in \tau \Longrightarrow (\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B}) \in \tau$

(iii) $\{(\Theta, \mathfrak{J})_l | l \in L\} \in \tau \Longrightarrow \cup_{l\in L} (\Theta, \mathfrak{J})_l \in \tau$

If the above axioms are satisfied, then τ is a neutrosophic hypersoft topology (NHT) on $(\varpi, \mathfrak{X}) \cdot (\mathfrak{X}_{\varpi}, \tau)$ which is called a neutrosophic hypersoft topological space (NHTS). Every member of τ is called an open neutrosophic hypersoft set (ONHS). A neutrosophic hypersoft set is called a closed neutrosophic hypersoft set (CNHS) if its complement is an ONHS.

For example, $\{\varnothing_{\mathfrak{X}}, (\varpi, \mathfrak{X})\}$ and $P(\varpi, \mathfrak{X})$ are neutrosophic hypersoft topologies on $(\varpi, \mathfrak{X})$ and are called indiscrete NHT and discrete NHT, respectively.

Definition 6 (see [35]). Let τ be a NHT on $(\varpi, \mathfrak{X}) \in \mathfrak{P}(V, \mathbb{E})$ and $(\chi, \mathfrak{B})$ be a NH set in $P(\varpi, \mathfrak{X})$. A NH set $(\Theta, \mathfrak{J})$ in $P(\varpi, \mathfrak{X})$ is a neighbourhood of the NH set of $(\chi, \mathfrak{B})$ iff there exists an ONHS $(\xi, \mathfrak{C})$ such that $(\chi, \mathfrak{B}) \subset (\xi, \mathfrak{C}) \subset (\Theta, \mathfrak{J})$.

Definition 7 (see [35]). Let $(\mathfrak{X}_{\varpi}, \tau)$ be a NHTS and $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B})$ be NH sets in $P(\varpi, \mathfrak{X})$ such that $(\chi, \mathfrak{B}) \subset (\Theta, \mathfrak{J})$. Then, $(\chi, \mathfrak{B})$ is said to be an interior neutrosophic hypersoft set (INHS) of $(\Theta, \mathfrak{J})$ if and only if $(\Theta, \mathfrak{J})$ is a neighbourhood of $(\chi, \mathfrak{B})$. The union of the whole INHS of $(\Theta, \mathfrak{J})$ is named the interior of $(\Theta, \mathfrak{J})$ and denoted as $(\Theta, \mathfrak{J})^{\circ}$or $\mathrm{FHint}(\Theta, \mathfrak{J})$.

Definition 8 (see [35]). Let $(\mathfrak{X}_{\varpi}, \tau)$ be a NHTS and $(\Theta, \mathfrak{J}) \in \mathfrak{P}(V, \mathbb{E})$. The neutrosophic hypersoft closure (NHC) of $(\Theta, \mathfrak{J})$ is the intersection of all CNH sets that contain $(\Theta, \mathfrak{J})$ which is denoted by $\overline{(\Theta, \mathfrak{J})}$ or $\mathrm{FHcl}(\Theta, \mathfrak{J})$. Thus, $\overline{(\Theta, \mathfrak{J})}$ is the smallest CNHS which has $(\Theta, \mathfrak{J})$, and $\overline{(\Theta, \mathfrak{J})}$ is the CNHS.

3. Fuzzy Semiopen and Closed Hypersoft Sets

Definition 9. Let $(\mathfrak{X}_{\varpi}, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}) \in P(\varpi, \mathfrak{X})$. If $(\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}))$, then $(\Theta, \mathfrak{J})$ is called the fuzzy semiopen hypersoft set (FSOHS). We denote the set of all fuzzy semiopen hypersoft sets by $\mathrm{FSOHS}(\mathfrak{X})$.

Definition 10. A fuzzy hypersoft set $(\Theta, \mathfrak{J})$ in the FHST space is a fuzzy semiclosed hypersoft set (FSCHS) if and only if its complement $(\Theta, \mathfrak{J})^C$ is FSOHS. The class of FSCHS is denoted by FSCHS $(\mathfrak{X})$.

Example 2. Let $X = \{y_1, y_2, y_3\}$ and the attributes be $E_1 = \{a_1, a_2, a_3\}$, $E_2 = \{b_1, b_2\}$, and $E_3 = \{c_1, c_2\}$.

The fuzzy hypersoft topological space is τ:

$$\begin{aligned}\tau = \Bigg\{&\left\{\left\langle (a_1, b_1, c_2), \left\{\frac{y_1}{0.9}, \frac{y_2}{0.2}, \frac{y_3}{0.2}\right\}\right\rangle, \left\langle (a_1, b_2, c_2)\left\{\frac{y_1}{0.8}, \frac{y_2}{0.7}, \frac{y_3}{0.4}\right\}\right\rangle\right\}\Bigg\},\\ &\left\{\left\langle (a_1, b_1, c_1), \left\{\frac{y_1}{0.8}, \frac{y_2}{0.6}, \frac{y_3}{0.3}\right\}\right\rangle \left\langle (a_3, b_1, c_1), \left\{\frac{y_1}{0.6}, \frac{y_2}{0.5}, \frac{y_3}{0.7}\right\}\right\rangle\right\},\\ &\left\{\left\langle (a_1, b_1, c_2)\left\{\frac{y_1}{0.9}, \frac{y_2}{0.2}, \frac{y_3}{0.2}\right\}\right\rangle, \left\langle (a_1, b_2, c_2), \left\{\frac{y_1}{0.8}, \frac{y_2}{0.7}, \frac{y_3}{0.4}\right\}\right\rangle \left\langle (a_1, b_1, c_1), \left\{\frac{y_1}{0.8}, \frac{y_2}{0.6}, \frac{y_3}{0.3}\right\}\right\rangle, \left\langle (a_3, b_1, c_1)\left\{\frac{y_1}{0.6}, \frac{y_2}{0.5}, \frac{y_3}{0.7}\right\}\right\rangle\right\}.\end{aligned} \tag{3}$$

The fuzzy hypersoft set

$$\left\{\begin{array}{l}\left\langle (a_1, b_1, c_2), \left\{\frac{y_1}{0.9}, \frac{y_2}{0.3}, \frac{y_3}{0.5}\right\}\right\rangle, \left\langle (a_1, b_2, c_2), \left\{\frac{y_1}{0.9}, \frac{y_2}{0.8}, \frac{y_3}{0.5}\right\}\right\rangle,\\ \left\langle (a_1, b_1, c_1), \left\{\frac{y_1}{0.9}, \frac{y_2}{0.7}, \frac{y_3}{0.5}\right\}\right\rangle, \left\langle (a_3, b_1, c_1), \left\{\frac{y_1}{0.7}, \frac{y_2}{0.6}, \frac{y_3}{0.8}\right\}\right\rangle\end{array}\right\} \tag{4}$$

is a FSOHS.

Theorem 1. *Let $(\mathfrak{X}_{\varpi}, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}) \in FSOHS(\mathfrak{X})$; then,*

(i) Arbitrary fuzzy hypersoft union of FSOHS is FSOHS

(ii) Arbitrary fuzzy hypersoft intersection of FSCHS is FSCHS

Proof

(i) Let $\{(\Theta, \mathfrak{J})_j \colon j \in J\} \subseteq \mathrm{FSOHS}(\mathfrak{X})$.

Then, $\forall j \in J$, $(\Theta, \mathfrak{J})_j \subseteq \mathrm{FHcl}(\Theta, \mathfrak{J})_j$.

Hence, $\cup_j (\Theta, \mathfrak{J})_j \subseteq \bigcup\{\mathrm{FHclFHint}\}(\Theta, \mathfrak{J})_j \subseteq \mathrm{FHcl}(\mathrm{FHint}(\cup_j (\Theta, \mathfrak{J})_j))$.

Therefore, $\cup_j (\Theta, \mathfrak{J})_j \in \mathrm{FSOHS}(\mathfrak{X})$.

Similarly, (ii) is proved. □

Theorem 2. *Let $(\mathfrak{X}_{\varpi}, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}) \in P(\varpi, \mathfrak{X})$. Then,*

(i) $(\Theta, \mathfrak{J}) \in FSOHS(\mathfrak{X})$ if and only if there exists $(\chi, \mathfrak{B}) \in \tau$ such that $(\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) \subseteq FHcl(\chi, \mathfrak{B})$

(ii) If $(\Theta, \mathfrak{J}) \in FSOHS(\mathfrak{X})$ and $(\Theta, \mathfrak{J}) \subseteq (\xi, \mathfrak{C}) \subseteq FHcl(\Theta, \mathfrak{J})$, then $(\xi, \mathfrak{C}) \in FSOHS(\mathfrak{X})$

Proof

(i) Let $(\Theta, \mathfrak{J}) \in \mathrm{FSOHS}(\mathfrak{X})$.

Then, $(\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}))$.

We know that $\mathrm{FHint}(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})$; thus, $\mathrm{FHint}(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}))$.

Let $(\chi, \mathfrak{B}) = \mathrm{FHint}(\Theta, \mathfrak{J})$; thus, we get $(\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\chi, \mathfrak{B})$.

Conversely, let $(\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\chi, \mathfrak{B})$ for some $(\chi, \mathfrak{B}) \in \tau$. Then, $(\chi, \mathfrak{B}) \subseteq \mathrm{FHint}(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})$.

$\Longrightarrow \mathrm{FHcl}(\chi, \mathfrak{B}) \subseteq \mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}))$.

Thus, $(\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}))$.

Therefore, $(\Theta, \mathfrak{J}) \in \mathrm{FSOHS}(\mathfrak{X})$.

(ii) Let $(\Theta, \mathfrak{J}) \in \mathrm{FSOHS}(\mathfrak{X})$. Then, for some $(\chi, \mathfrak{B}) \in \tau$, $(\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\chi, \mathfrak{B})$. If $(\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) \subseteq (\xi, \mathfrak{C})$, then $(\chi, \mathfrak{B}) \subseteq (\xi, \mathfrak{C}) \subseteq \mathrm{FHcl}(\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\chi, \mathfrak{B})$.

Hence, $(\chi, \mathfrak{B}) \subseteq (\xi, \mathfrak{C}) \subseteq \mathrm{FHcl}(\chi, \mathfrak{B})$. Thus, by (i), $(\xi, \mathfrak{C}) \in \mathrm{FSOHS}(\mathfrak{X})$. □

Definition 11. Let $(\mathfrak{X}_{\varpi}, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}) \in P(\varpi, \mathfrak{X})$.

Then, the largest fuzzy semiopen hypersoft set contained in $(\Theta, \mathfrak{J})$ is called the fuzzy semi-hypersoft interior of $(\Theta, \mathfrak{J})$ and denoted by $\mathrm{FSHSint}(\Theta, \mathfrak{J})$, i.e., $\mathrm{FSHSint}(\Theta,$
$\mathfrak{J}) = \cup\{(\chi, \mathfrak{B})\colon (\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) (\chi, \mathfrak{B}) \in \mathrm{FSOHS}(\mathfrak{X})\}$.

And the smallest fuzzy semiclosed hypersoft set containing $(\Theta, \mathfrak{J})$ is called the fuzzy semi-hypersoft closure of $(\Theta, \mathfrak{J})$ and denoted by $\mathrm{FSHScl}(\Theta, \mathfrak{J})$.

$\mathrm{FSHScl}(\Theta, \mathfrak{J}) = \cap\{(\xi, \mathfrak{C})\colon (\xi, \mathfrak{C}) \supseteq (\Theta, \mathfrak{J})$, and $(\xi, \mathfrak{C}) \in \mathrm{FSCHS}(\mathfrak{X})$.

Theorem 3. *Let $(\mathfrak{X}_{\varpi}, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in P(\varpi, \mathfrak{X})$. Then, the following properties hold:*

(i) $FSHS\mathrm{int}(\varnothing_{\mathfrak{X}}) = \varnothing_{\mathfrak{X}}$ *and* $FSHS\mathrm{int}(\varpi, \mathfrak{X}) = (\varpi, \mathfrak{X})$

(ii) $FSHS\mathrm{int}(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})$

(iii) $FSHS\,int\,(\Theta, \mathfrak{J})$ *is the largest fuzzy semiopen hypersoft set contained in* $(\Theta, \mathfrak{J})$

(iv) If $(\Theta, \mathfrak{J}) \subseteq (\chi, \mathfrak{B})$, *then* $FSHS\mathrm{int}(\Theta, \mathfrak{J}) \subseteq FSHS\mathrm{int}(\chi, \mathfrak{B})$

(v) $FHSH\mathrm{int}(FHSH\mathrm{int}(\Theta, \mathfrak{J})) = FHSH\mathrm{int}(\Theta, \mathfrak{J})$

(vi) $FHSH\mathrm{int}(\Theta, \mathfrak{J}) \cup FHSH\mathrm{int}(\chi, \mathfrak{B}) \subseteq FHSH\mathrm{int}[(\Theta, \mathfrak{J}) \cup (\chi, \mathfrak{B})]$

(vii) $FHSH\mathrm{int}[(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})] \subseteq FHSH\mathrm{int}(\Theta, \mathfrak{J}) \cap FHSH\mathrm{int}(\chi, \mathfrak{B})$

Theorem 4. *Let $(\mathfrak{X}_{\varpi}, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in P(\varpi, \mathfrak{X})$. Then, the following properties hold:*

(i) $FSHScl(\varnothing_{\mathfrak{X}}) = \varnothing_{\mathfrak{X}}$, *and* $FSHScl(\varpi, \mathfrak{X}) = (\varpi, \mathfrak{X})$

(ii) $FSHScl(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})$

(iii) $FSHScl(\Theta, \mathfrak{J})$ *is the smallest fuzzy semiclosed hypersoft set that contains* $(\Theta, \mathfrak{J})$

(iv) If $(\Theta, \mathfrak{J}) \subseteq (g, B)$, *then* $FSHScl(\Theta, \mathfrak{J}) \subseteq FSHScl(\chi, \mathfrak{B})$

(v) $FHSHcl(FHSHcl(\Theta, \mathfrak{J})) = FHSHcl(\Theta, \mathfrak{J})$

(vi) $FHSHcl(\Theta, \mathfrak{J}) \cup FHSHcl\ (\chi, \mathfrak{B}) \subseteq FHSHcl\ (\Theta, \mathfrak{J}) \cup [(\chi, \mathfrak{B})]$

(vii) $FHSHcl[(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})] \subseteq FHSHcl(\Theta, \mathfrak{J}) \cap FHSHcl(\chi, \mathfrak{B})$

Theorem 5. *Every fuzzy open (closed) hypersoft set in a FHTS $(\mathfrak{X}_{\varpi}, \tau)$ is a fuzzy semiopen (closed) hypersoft set.*

Proof. Let $(\Theta, \mathfrak{J})$ be a fuzzy open hypersoft set. Then, $\mathrm{FHint}(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})$. Since $(\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\Theta, \mathfrak{J})$, $(\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}))$. Thus, $(\Theta, \mathfrak{J}) \in \mathrm{FSOHS}(\mathfrak{X})$. □

Theorem 6. *Let $(\mathfrak{X}_{\varpi}, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in P(\varpi, \mathfrak{X})$. If either $(\Theta, \mathfrak{J}) \in FSOHS(X)$ or $(\chi, \mathfrak{B}) \in FSOHS(\mathfrak{X})$, then $FHcl(FH\mathrm{int}(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})) = FHcl(FH\mathrm{int}(\Theta, \mathfrak{J}) \cap FHclFH\mathrm{int}(\chi, \mathfrak{B}))$.*

Proof. Let $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in P(\varpi, \mathfrak{X})$.

Then, we have

$$\begin{aligned}
&\mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})) \subseteq \\
&\mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}) \cap \mathrm{FHclFHint}(\chi, \mathfrak{B})) \\
&\mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}) \cap \mathrm{FHclFHint}(\chi, \mathfrak{B})) \subseteq \\
&\mathrm{FHcl}[\mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J})) \cap \mathrm{FHcl}(\mathrm{FHint}(\chi, \mathfrak{B}))] \\
&= \mathrm{FHcl}[\mathrm{FHcl}[\mathrm{FHint}(\Theta, \mathfrak{J}) \cap \mathrm{FHint}(\chi, \mathfrak{B})]] \\
&= \mathrm{FHcl}[\mathrm{FHcl}[\mathrm{FHint}[(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})]]] \\
&\subseteq \mathrm{FHcl}[\mathrm{FHint}(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})] \\
&\Longrightarrow \mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J})) \cap \mathrm{FHcl}(\mathrm{FHint}(\chi, \mathfrak{B})) \\
&\subseteq \mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})).
\end{aligned} \tag{5}$$

Thus, $\mathrm{FHcl}(\mathrm{FHint}(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})) = \mathrm{FHclFHint}(\Theta, \mathfrak{J}) \cap \mathrm{FHcl}(\mathrm{FHint}(\chi, \mathfrak{B}))$. □

Theorem 7. *Let $(\mathfrak{X}_{\varpi}, \tau)$ be a FHTS, $(\Theta, \mathfrak{J})$ be a fuzzy hypersoft open set, and $(\chi, \mathfrak{B}) \in FSOHS(\mathfrak{X})$. Then, $(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B}) \in FSOHS(\mathfrak{X})$.*

Proof. Let $(\Theta, \mathfrak{J})$ be a FOHS and $(\chi, \mathfrak{B})$ be a FSOHS.

Then, $(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B}) \supseteq \mathrm{FHint}((\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})) \Longrightarrow \mathrm{FHint}((\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})) \subseteq (\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})$.

Then, $\mathrm{FHint}((\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})) \subseteq (\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B}) \backslash \subseteq \mathrm{FHclFHint}((\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B}))$ $\mathrm{FHint}((\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})) \subseteq (\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B}) \backslash \subseteq \mathrm{FHclFHint}((\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B}))$ $(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B}) \subseteq \mathrm{FHclFHint}((\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B}))$.

Therefore, $(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})$ is a FSOHS. □

Proposition 1. *Let $(\Theta, \mathfrak{J})$ be a fuzzy hypersoft set in the FHTS $(\mathfrak{X}_{\varpi}, \tau)$. Then, $(\Theta, \mathfrak{J})$ is the FSCHS if and only if there exists an FCHS set $(\xi, \mathfrak{J})$ such that $FH\mathrm{int}(\xi, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J}) \subseteq (\xi, \mathfrak{J})$.*

Proposition 2. *Every fuzzy hypersoft closed set is a FSCHS in a FHTS $(\mathfrak{X}_{\varpi}, \tau)$, but the converse need not be true.*

Theorem 8. *Let $(\Theta, \mathfrak{J})$ be a FHS in a FHTS $(\mathfrak{X}_{\varpi}, \tau)$. Then, $(\Theta, \mathfrak{J})$ is a FSCHS if and only if $FH\mathrm{int}(FHcl(\Theta, \mathfrak{J})) \subseteq (\Theta, \mathfrak{J})$.*

Proof. Suppose $(\Theta, \mathfrak{J})$ is a FSCHS. Then, there exists a FHCS $(\Theta, \mathfrak{J})$ such that $\mathrm{FHint}(\xi, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J}) \subseteq (\xi, \mathfrak{J})$. $\mathrm{FHcl}(\Theta, \mathfrak{J}) \subseteq \mathrm{FHcl}(\xi, \mathfrak{J}) = (\xi, \mathfrak{J})$.

Thus, $\text{FHint}(\text{FHcl}(\Theta,\mathfrak{J}))\subseteq\text{FHint}(\xi,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\Longrightarrow \text{FHint}(\text{FHcl}(\xi,\mathfrak{J}))\subseteq(\Theta,\mathfrak{J})$.

Conversely, let $(\Theta,\mathfrak{J})$ be a fuzzy hypersoft set in $(\mathfrak{X}_{\varpi},\tau)$ such that $\text{FHint}(\text{FHcl}(\Theta,\mathfrak{J}))\subseteq(\Theta,\mathfrak{J})$. Let $\text{FHcl}(\Theta,\mathfrak{J})=(\xi,\mathfrak{J})$. Then, $\text{FHint}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. Thus, $(\Theta,\mathfrak{J})$ is a FSCHS. □

Theorem 9. *Let $\{(\Theta,\mathfrak{J})_\beta\colon \beta\in I\}$ be a family of FSCHS in a FHTS $(\mathfrak{X}_{\varpi},\tau)$. Then, the intersection $\cap_{\beta\in I}(\Theta,\mathfrak{J})_\beta$ is a FSCHS in $(\mathfrak{X}_{\varpi},\tau)$.*

Proof. Since each $\beta\in$ I, $(\Theta,\mathfrak{J})_\beta$ is a FSCHS. Then, there exists a FCHS $(\xi,\mathfrak{J})_\beta$ such that $\text{FHint}((h,A)_\beta)\subseteq(\Theta,\mathfrak{J})_\beta\subseteq(\xi,\mathfrak{J})_\beta$. Thus, $\cap_{\beta\in I}(\text{FHint}((\xi,\mathfrak{J})_\beta))\subseteq\cap_{\beta\in I}(\Theta,\mathfrak{J})_\beta\subseteq\cap_{\beta\in I}(\xi,\mathfrak{J})_\beta$. Consider $\cap_{\beta\in I}(\xi,\mathfrak{J})_\beta=(\xi,\mathfrak{J})$. Then, $(\xi,\mathfrak{J})$ is a FCHS, and hence, $\cap_{\beta\in I}(\Theta,\mathfrak{J})_\beta$ is a FSCHS. □

Theorem 10. *Let $(\Theta,\mathfrak{J})$ be a FSCHS and $(\vartheta,\mathfrak{J})$ be a FCHS in $(\mathfrak{X}_{\varpi},\tau)$. If $FH\text{int}(\Theta,\mathfrak{J})\subseteq(\vartheta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})$, then $(\Theta,\mathfrak{J})$ is a FSCHS.*

Proof. Since $(\Theta,\mathfrak{J})$ is a FSCHS, there exists a FCHS $(\xi,\mathfrak{J})$ such that $\text{FHint}(\xi,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. Then, $(\vartheta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. Also, $\text{FHintFHint}(\xi,\mathfrak{J})\subseteq\text{FHint}(\xi,\mathfrak{J})\subseteq\text{FHint}(\Theta,\mathfrak{J})\Longrightarrow\text{FHint}(\xi,\mathfrak{J})\subseteq(\vartheta,\mathfrak{J})$. Therefore, $\text{FHint}(\xi,\mathfrak{J})\subseteq(\vartheta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. Hence, $(\Theta,\mathfrak{J})$ is a FSCHS. □

Remark 1. For any FCHS $(\Theta,\mathfrak{J})$, $\text{FSHScl}(\Theta,\mathfrak{J})=(\Theta,\mathfrak{J})$. And for any FOHS $(\zeta,\mathfrak{J})$, $\text{FSHSint}(\zeta,\mathfrak{J})=(\zeta,\mathfrak{J})$.

Remark 2. If $(\Theta,\mathfrak{J})$ is a fuzzy hypersoft set in $(\mathfrak{X}_{\varpi},\tau)$, then $\text{FHint}(\Theta,\mathfrak{J})\subseteq\text{FSHSint}\ (\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq\text{FSHScl}(\Theta,\mathfrak{J})\subseteq\text{FHcl}(\Theta,\mathfrak{J})$.

Theorem 11. *Let $(\Theta,\mathfrak{J})$ be a FHS in $(\mathfrak{X}_{\varpi},\tau)$. Then,*

(i) $(FSH\text{Sint}(\Theta,\mathfrak{J}))^C=FSHScl((\Theta,\mathfrak{J})^C)$

(ii) $(FSHScl(\Theta,\mathfrak{J}))^C=FSH\text{Sint}((\Theta,\mathfrak{J})^C)$

(iii) $FSH\text{Sint}(FH\text{int}(\Theta,\mathfrak{J}))=FH\text{int}(FSH\text{Sint}(\Theta,\mathfrak{J}))=FH\text{int}(\Theta,\mathfrak{J})$

(iv) $FSHScl(FHcl(\Theta,\mathfrak{J}))=FHcl(FSHScl(\Theta,\mathfrak{J}))\backslash\backslash=FHcl(\Theta,\mathfrak{J})$

Proof

(i) $\text{FSHSint}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\Longrightarrow(\Theta,\mathfrak{J})^C\subseteq(\text{FSHSint}(\Theta,\mathfrak{J}))^C$.

Since $(\text{FSHSint}(\Theta,\mathfrak{J}))^C$ is a FSCHS,$\text{FSHScl}(\Theta,\mathfrak{J})^C\subseteq\text{FSHScl}((\text{FSHSint}(\Theta,\mathfrak{J}))^C)=(\text{FSHSint}(\Theta,\mathfrak{J}))^C$.

Conversely, $(\Theta,\mathfrak{J})^C\subseteq\text{FSHScl}((\Theta,\mathfrak{J})^C)\Longrightarrow\text{FSHScl}((\Theta,\mathfrak{J})^C)^C\subseteq((\Theta,\mathfrak{J})^C)^C=(\Theta,\mathfrak{J})$.

$\text{FSHScl}((\Theta,\mathfrak{J})^C)$ being FSCHS implies that $\text{FSHScl}((\Theta,\mathfrak{J})^C)^C$ is a FSOHS set.

Thus, $\text{FSHScl}((\Theta,\mathfrak{J})^C)^C\subseteq\text{FSHSint}(\Theta,\mathfrak{J})$.

And hence, $(\text{FSHSint}(\Theta,\mathfrak{J}))^C\subseteq(\text{FSHScl}((\Theta,\mathfrak{J})^C)^C)^C=(\text{FSHScl}((\Theta,\mathfrak{J})^C))$.

(ii) The proof is the same as that of (i).

(iii) $\text{FHint}(\Theta,\mathfrak{J})$ is FOHS implying that it is FSOHS.

Therefore, $\text{FSHSint}(\text{FHint}(\Theta,\mathfrak{J}))=\text{FHint}(\Theta,\mathfrak{J})$.

Now, $\text{FHint}(\Theta,\mathfrak{J})\subseteq\text{FSHSint}(\Theta,\mathfrak{J})=(\Theta,\mathfrak{J})$.

Thus, $\text{FSHSint}(\text{FHint}(\Theta,\mathfrak{J}))=\text{FHint}(\Theta,\mathfrak{J})$.

(iv) $\text{FHcl}(\Theta,\mathfrak{J})$ is fuzzy closed hypersoft implying that is FSCHS. Therefore, $\text{FSHScl}(\text{FHcl}(\Theta,\mathfrak{J}))=\text{FHcl}(\Theta,\mathfrak{J})$. Now, $(\Theta,\mathfrak{J})\subseteq\text{FSHScl}(\Theta,\mathfrak{J})\subseteq\text{FHcl}(\Theta,\mathfrak{J})$.

Hence, $\text{FSHScl}(\Theta,\mathfrak{J})\subseteq\text{FHclFSHScl}((\Theta,\mathfrak{J}))\subseteq\text{FSHScl}(\Theta,\mathfrak{J})$.

This implies $\text{IHcl}(\text{ISHScl}(\Theta,\mathfrak{J}))\subseteq\text{IHcl}(\Theta,\mathfrak{J})$. □

4. Intuitionistic Semiopen and Closed Hypersoft Sets

Definition 12. Let $(\mathfrak{X}_{\varpi},\tau)$ be an IHTS and $(\Theta,\mathfrak{J})\in P(\varpi,\mathfrak{X})$. If $(\Theta,\mathfrak{J})\subseteq\text{IHcl}(\text{IHint}(\Theta,\mathfrak{J}))$, then $(\Theta,\mathfrak{J})$ is called an intuitionistic semiopen hypersoft set (ISOHS). We denote the set of all intuitionistic semiopen hypersoft sets by $\text{ISOHS}(\mathfrak{X})$.

Definition. 13An intuitionistic hypersoft set $(\Theta,\mathfrak{J})$ in the IHST space is an intuitionistic semiclosed hypersoft set (ISCHS) if and only if its complement $(\Theta,\mathfrak{J})^C$ is ISOHS. The class of ISCHS is denoted by $\text{ISCHS}(\mathfrak{X})$.

Example 3. Let $X=\{y_1,y_2,y_3\}$ and the attributes be $E_1=\{a_1,a_2,a_3\}$, $E_2=\{b_1,b_2\}$, and $E_3=\{c_1,c_2\}$.

The intuitionistic hypersoft topological space is τ:

$$\tau=\left\{\left\langle(a_1,b_1,c_2),\left\{\frac{y_1}{0.9,0.1},\frac{y_2}{0.2,0.4},\frac{y_3}{0.2,0.7}\right\}\right\rangle,\left\langle(a_1,b_2,c_2),\left\{\frac{y_1}{0.8,0.3},\frac{y_2}{0.7,0.2},\frac{y_3}{0.4,0.8}\right\}\right\rangle\right\},$$

$$\cdot\left\{\left\langle(a_1,b_1,c_1)\left\{\frac{y_1}{0.8,0.2},\frac{y_2}{0.6,0.4},\frac{y_3}{0.3,0.2}\right\}\right\rangle,\left\langle(a_3,b_1,c_1),\left\{\frac{y_1}{0.6,0.5},\frac{y_2}{0.5,0.4}\ \frac{y_3}{0.7,0.2}\right\}\right\rangle\right\},$$

$$\cdot\left\{\begin{array}{c}\left\langle (a_1,b_1,c_2),\left\{\frac{y_1}{0.9,0.1},\frac{y_2}{0.2,0.4},\frac{y_3}{0.2,0.7}\right\}\right\rangle,\\ \left\langle (a_1,b_2,c_2),\left\{\frac{y_1}{0.8,0.3},\frac{y_2}{0.7,0.2},\frac{y_3}{0.4,0.8}\right\}\right\rangle,\left\langle (a_1,b_1,c_1)\left\{\frac{y_1}{0.8,0.2},\frac{y_2}{0.6,0.4},\frac{y_3}{0.3,0.2}\right\}\right\rangle,\\ \left\langle (a_3,b_1,c_1),\left\{\frac{y_1}{0.6,0.5}\,\frac{y_2}{0.5,0.4},\frac{y_3}{0.7,0.2}\right\}\right\rangle\end{array}\right\}. \tag{6}$$

The intuitionistic hypersoft set

$$\left\{\begin{array}{c}\left\langle (a_1,b_1,c_2)\left\{\frac{y_1}{0.9,0.1},\frac{y_2}{0.3,0.3},\frac{y_3}{0.5,0.5}\right\}\right\rangle,\left\langle (a_1,b_2,c_2)\left\{\frac{y_1}{0.9,0.2},\frac{y_2}{0.8,0.1},\frac{y_3}{0.5,0.7,0.5}\right\}\right\rangle,\\ \left\langle (a_1,b_1,c_1),\left\{\frac{y_1}{0.9,0.1},\frac{y_2}{0.7,0.3},\frac{y_3}{0.5,0.1}\right\}\right\rangle,\left\langle (a_3,b_1,c_1),\left\{\frac{y_1}{0.7,0.3},\frac{y_2}{0.6,0.3},\frac{y_3}{0.8,0.1}\right\}\right\rangle\end{array}\right\} \tag{7}$$

is ISOHS.

Theorem 12. *Let $(\mathfrak{X}_{\varpi},\tau)$ be an IHTS and $(\Theta,\mathfrak{J})\in ISOHS(\mathfrak{X})$; then,*

(i) *Arbitrary intuitionistic hypersoft union of ISOHS is an ISOHS*

(ii) *Arbitrary intuitionistic hypersoft intersection of ISCHS is an ISCHS*

Proof

(i) Let $\{(\Theta,\mathfrak{J})_j\colon j\in J\}\subseteq \mathrm{ISOHS}(\mathfrak{X})$.
Then, $\forall j\in J$, $(\Theta,\mathfrak{J})_j\subseteq \mathrm{FHcl}(\Theta,\mathfrak{J})_j$.
Hence, $\cup_j(\Theta,\mathfrak{J})_j\subseteq \cup\,\mathrm{IHc}(\mathrm{IHint}(\Theta,\mathfrak{J})_j)\subseteq \mathrm{IHcl}(\mathrm{IHint}(\cup_j(\Theta,\mathfrak{J})_j))$.
Therefore, $\cup_j(\Theta,\mathfrak{J})_j\in \mathrm{ISOHS}(\mathfrak{X})$.
Similarly, (ii) is proved. □

Theorem 13. *Let $(\mathfrak{X}_{\varpi},\tau)$ be an IHTS and $(\Theta,\mathfrak{J})\in P(\varpi,\mathfrak{X})$. Then,*

(i) *$(\Theta,\mathfrak{J})\in ISOHS(\mathfrak{X})$ if and only if there exists $(\chi,\mathfrak{B})\in\tau$ such that $(\chi,\mathfrak{B})\subseteq(\Theta,\mathfrak{J})\subseteq IHcl(\chi,\mathfrak{B})$*

(ii) *If $(\Theta,\mathfrak{J})\in ISOHS(\mathfrak{X})$ and $(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{C})\subseteq Icl(\Theta,\mathfrak{J})$, then $(\xi,\mathfrak{C})\in ISOHS(\mathfrak{X})$*

Proof

(i) Let $(\Theta,\mathfrak{J})\in \mathrm{ISOHS}(\mathfrak{X})$. Then, $(\Theta,\mathfrak{J})\subseteq \mathrm{IHcl}(\mathrm{IHint}(\Theta,\mathfrak{J}))$. We know that $\mathrm{IHint}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})$; thus, $\mathrm{IHint}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq \mathrm{IHcl}(\mathrm{IHint}(\Theta,\mathfrak{J}))$.
Let $(\chi,\mathfrak{B})=\mathrm{IHint}(\Theta,\mathfrak{J})$; thus, we get $(\chi,\mathfrak{B})\subseteq(\Theta,\mathfrak{J})\subseteq \mathrm{IHcl}(\chi,\mathfrak{B})$.
Conversely, let $(\chi,\mathfrak{B})\subseteq(\Theta,\mathfrak{J})\subseteq \mathrm{IHcl}(\chi,\mathfrak{B})$ for some $(\chi,\mathfrak{B})\in\tau$. Then, $(\chi,\mathfrak{B})\subseteq \mathrm{IHint}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})$. $\Longrightarrow \mathrm{IHcl}(\chi,\mathfrak{B})\subseteq \mathrm{IHcl}(\mathrm{IHint}(\Theta,\mathfrak{J}))$.
Thus, $(\Theta,\mathfrak{J})\subseteq \mathrm{IHcl}(\mathrm{IHint}(\Theta,\mathfrak{J}))$.
Therefore, $(\Theta,\mathfrak{J})\in \mathrm{ISOHS}(\mathfrak{X})$.

(ii) Let $(\Theta,\mathfrak{J})\in \mathrm{ISOHS}(\mathfrak{X})$. Then, for some $(\chi,\mathfrak{B})\in\tau$, $(\chi,\mathfrak{B})\subseteq(\Theta,\mathfrak{J})\subseteq \mathrm{IHcl}(\chi,\mathfrak{B})$. If $(\chi,\mathfrak{B})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{C})$, then $(\chi,\mathfrak{B})\subseteq(\xi,\mathfrak{C})\subseteq \mathrm{IHcl}(\Theta,\mathfrak{J})\subseteq \mathrm{IHcl}(\chi,\mathfrak{B})$. Hence, $(\chi,\mathfrak{B})\subseteq(\xi,\mathfrak{C})\subseteq \mathrm{IHcl}(\chi,\mathfrak{B})$. Thus, by (i), $(\xi,\mathfrak{C})\in \mathrm{ISOHS}(\mathfrak{X})$. □

Definition 14. Let $(\mathfrak{X}_{\varpi},\tau)$ be an IHTS and $(\Theta,\mathfrak{J})\in P(\varpi,\mathfrak{X})$.

Then, the largest intuitionistic semiopen hypersoft set contained in $(\Theta,\mathfrak{J})$ is called the intuitionistic semi-hypersoft interior of $(\Theta,\mathfrak{J})$ and denoted by $\mathrm{ISHSint}(\Theta,\mathfrak{J})$, i.e., $\mathrm{ISHSint}(\Theta,\mathfrak{J})=\cup\{(\chi,\mathfrak{B})\colon(\chi,\mathfrak{B})\subseteq(\Theta,\mathfrak{J}),(\chi,\mathfrak{B})\in \mathrm{ISOHS}(\mathfrak{X})\}$.

And the smallest intuitionistic semiclosed hypersoft set containing $(\Theta,\mathfrak{J})$ is called the intuitionistic semi-hypersoft closure of $(\Theta,\mathfrak{J})$ and denoted by $\mathrm{ISHScl}(\Theta,\mathfrak{J})$. $\mathrm{ISHScl}(\Theta,\mathfrak{J})=\cap\{(\xi,\mathfrak{C})\colon(\xi,\mathfrak{C})\supseteq(\Theta,\mathfrak{J})$ and $(\xi,\mathfrak{C})\in \mathrm{ISCHS}(\mathfrak{X})\}$.

Theorem 14. *Let $(\mathfrak{X}_{\varpi},\tau)$ be an IHTS and $(\Theta,\mathfrak{J}),(\chi,\mathfrak{B})\in P(\varpi,\mathfrak{X})$. Then, the following properties hold:*

(i) *$ISHS\mathrm{int}(\varnothing_{\mathfrak{X}})=\varnothing_{\mathfrak{X}}$ and $ISHS\mathrm{int}(\varpi,\mathfrak{X})=(\varpi,\mathfrak{X})$*

(ii) *$ISHS\mathrm{int}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})$*

(iii) *$ISHS\mathrm{int}(\Theta,\mathfrak{J})$ is the largest intuitionistic semiopen hypersoft set contained in $(\Theta,\mathfrak{J})$*

(iv) *If $(\Theta,\mathfrak{J})\subseteq(\chi,\mathfrak{B})$, then $ISHS\mathrm{int}(\Theta,\mathfrak{J})\subseteq ISHS\mathrm{int}(\chi,\mathfrak{B})$*

(v) *$IHSH\mathrm{int}(IHSH\mathrm{int}(\Theta,\mathfrak{J}))=IHSH\mathrm{int}(\Theta,\mathfrak{J})$*

(vi) $IHSH\mathrm{int}(\Theta,\mathfrak{J})\cup IHSH\mathrm{int}(\chi,\mathfrak{B})\subseteq IHSH\mathrm{int}[(\Theta,\mathfrak{J})\cup(\chi,\mathfrak{B})]$

(vii) $IHSH\mathrm{int}[(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})]\subseteq IHSH\mathrm{int}(\Theta,\mathfrak{J})\cap IHSH\mathrm{int}(\chi,\mathfrak{B})$

Theorem 15. *Let* $(\mathfrak{X}_{\varpi},\tau)$ *be an IHTS and* $(\Theta,\mathfrak{J}),(\chi,\mathfrak{B})\in P(\varpi,\mathfrak{X})$. *Then, the following properties hold:*

(i) $ISHScl(\varnothing_{\mathfrak{X}})=\varnothing_{\mathfrak{X}}$ *and* $ISHScl(\varpi,\mathfrak{X})=(\varpi,\mathfrak{X})$

(ii) $ISHScl(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})$

(iii) $ISHScl(\Theta,\mathfrak{J})$ *is the smallest intuitionistic semiclosed hypersoft set that contains* $(\Theta,\mathfrak{J})$

(iv) *If* $(\Theta,\mathfrak{J})\subseteq(g,B)$, *then* $ISHScl(\Theta,\mathfrak{J})\subseteq ISHScl(\chi,\mathfrak{B})$

(v) $IHSHcl(IHSHcl(\Theta,\mathfrak{J}))=IHSHcl(\Theta,\mathfrak{J})$

(vi) $IHSHcl(\Theta,\mathfrak{J})\cup IHSHcl(\chi,\mathfrak{B})\subseteq IHSHcl[(\Theta,\mathfrak{J})\cup(\chi,\mathfrak{B})]$

(vii) $IHSHcl[(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})]\subseteq IHSHcl(\Theta,\mathfrak{J})\cap IHSHcl(\chi,\mathfrak{B})$

Theorem 16. *Every intuitionistic open (closed) hypersoft set in an IHTS* $(\mathfrak{X}_{\varpi},\tau)$ *is an intuitionistic semiopen (closed) hypersoft set.*

Proof. Let $(\Theta,\mathfrak{J})$ be an intuitionistic open hypersoft set. Then, $\mathrm{IHint}(\Theta,\mathfrak{J})=(\Theta,\mathfrak{J})$. Since $(\Theta,\mathfrak{J})\subseteq\mathrm{IHcl}(\Theta,\mathfrak{J})$, $(\Theta,\mathfrak{J})\subseteq\mathrm{IHcl}(\mathrm{IHint}(\Theta,J))$. Thus, $(\Theta,\mathfrak{J})\in\mathrm{ISOHS}(\mathfrak{X})$. □

Theorem 17. *Let* $(\mathfrak{X}_{\varpi},\tau)$ *be an IHTS and* $(\Theta,\mathfrak{J}),(\chi,\mathfrak{B})\in P(\varpi,\mathfrak{X})$. *If either* $(\Theta,\mathfrak{J})\in ISOHS(X)$ *or* $(\chi,\mathfrak{B})\in ISOHS(\mathfrak{X})$, *then* $IHcl(IH\mathrm{int}(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))=IHcl(IH\mathrm{int}(\Theta,\mathfrak{J})\cap FHclIH\mathrm{int}(\chi,\mathfrak{B}))$.

Proof. Let then, we have

$$\begin{aligned}
&\mathrm{IHcl}(\mathrm{IHint}(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))\subseteq\\
&\mathrm{IHcl}(\mathrm{IHint}(\Theta,\mathfrak{J})\cap\mathrm{IclIHint}(\chi,\mathfrak{B}))\\
&\mathrm{FHcl}(\mathrm{IHint}(\Theta,\mathfrak{J})\cap\mathrm{IHclIHint}(\chi,\mathfrak{B}))\subseteq\\
&\mathrm{IHcl}[\mathrm{IHcl}(\mathrm{IHint}(\Theta,\mathfrak{J}))\cap\mathrm{IHcl}(\mathrm{IHint}(\chi,\mathfrak{B}))]\\
&=\mathrm{IHcl}[\mathrm{IHcl}[\mathrm{IHint}(\Theta,\mathfrak{J})\cap\mathrm{IHint}(\chi,\mathfrak{B})]]\\
&=\mathrm{IHcl}[\mathrm{IHcl}[\mathrm{IHint}[(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})]]]\\
&\subseteq\mathrm{IHcl}[\mathrm{IHint}(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})]\\
&\Longrightarrow\mathrm{IHcl}(\mathrm{IHint}(\Theta,\mathfrak{J}))\cap\mathrm{IHcl}(\mathrm{IHint}(\chi,\mathfrak{B}))\\
&\subseteq\mathrm{IHcl}(\mathrm{IHint}(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})).\\
&\cdot(\Theta,\mathfrak{J}),(\chi,\mathfrak{B})\in P(\varpi,\mathfrak{X})
\end{aligned}\tag{8}$$

Thus, $\mathrm{IHcl}(\mathrm{IHint}(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))=\mathrm{IHclIHint}(\Theta,\mathfrak{J})\cap\mathrm{IHcl}(\mathrm{IHint}(\chi,\mathfrak{B}))$. □

Theorem 18. *Let* $(\mathfrak{X}_{\varpi},\tau)$ *be an IHTS,* $(\Theta,\mathfrak{J})$ *be an intuitionistic hypersoft open set, and* $(\chi,\mathfrak{B})\in ISOHS(\mathfrak{X})$. *Then,* $(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})\in ISOHS(\mathfrak{X})$.

Proof. Let $(\Theta,\mathfrak{J})$ be an IOHS and $(\chi,\mathfrak{B})$ be an ISOHS.

Then, $(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})\supseteq\mathrm{IHint}((\Theta,\mathfrak{J})\cap(?,\mathfrak{B}))\Longrightarrow\mathrm{IHint}((\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))\subseteq(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})$.

Then, $\mathrm{IHint}((\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))\subseteq(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})\subseteq\mathrm{IHclIHint}((\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))$. $\Longrightarrow(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})\subseteq\mathrm{IHclIHint}((\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))$.

Therefore, $(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})$ is an ISOHS. □

Proposition 3. *Let* $(\Theta,\mathfrak{J})$ *be an intuitionistic hypersoft set in the IHTS* $(\mathfrak{X}_{\varpi},\tau)$. *Then,* $(\Theta,\mathfrak{J})$ *is ISCHS if and only if there exists an ICHS set* $(\xi,\mathfrak{J})$ *such that* $IH\mathrm{int}(\xi,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$.

Proposition 4. *Every intuitionistic hypersoft closed set is an ISCHS in an IHTS* $(\mathfrak{X}_{\varpi},\tau)$, *but the converse need not be true.*

Theorem 19. *Let* $(\Theta,\mathfrak{J})$ *be an IHS in an IHTS* $(\mathfrak{X}_{\varpi},\tau)$. *Then,* $(\Theta,\mathfrak{J})$ *is an ISCHS if and only if* $IH\mathrm{int}(IHcl(\Theta,\mathfrak{J}))\subseteq(\Theta,\mathfrak{J})$.

Proof. Suppose $(\Theta,\mathfrak{J})$ is an ISCHS; then, there exists an $\mathrm{IHCS}(\Theta,\mathfrak{J})$ such that $\mathrm{IHint}(\xi,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. $\mathrm{IHcl}(\Theta,\mathfrak{J})\subseteq\mathrm{IHcl}(\xi,\mathfrak{J})=(\xi,\mathfrak{J})$.

Thus, $\mathrm{IHint}(\mathrm{IHcl}(\Theta,\mathfrak{J}))\subseteq\mathrm{IHint}(\xi,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\Longrightarrow\mathrm{IHint}(\mathrm{IHcl}(\xi,\mathfrak{J}))\subseteq(\Theta,\mathfrak{J})$.

Conversely, let $(\Theta,\mathfrak{J})$ be an intuitionistic hypersoft set in $(\mathfrak{X}_{?},\tau)$ such that $\mathrm{IHint}(\mathrm{IHcl}(\Theta,\mathfrak{J}))\subseteq(\Theta,\mathfrak{J})$. Let $\mathrm{IHcl}(\Theta,\mathfrak{J})=(\xi,\mathfrak{J})$. Then, $\mathrm{IHint}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. Thus, $(\Theta,\mathfrak{J})$ is an ISCHS. □

Theorem 20. *Let* $\{(\Theta,\mathfrak{J})_{\beta}\colon\beta\in I\}$ *be a family of ISCHSs in an IHTS* $(\mathfrak{X}_{\varpi},\tau)$. *Then, the intersection* $\cup_{\beta\in I}(\Theta,\mathfrak{J})_{\beta}$ *is an ISCHS in* $(\mathfrak{X}_{\varpi},\tau)$.

Proof. Since each $\beta\in I$, $(\Theta,\mathfrak{J})_{\beta}$ is an ISCHS. Then, there exists an ICHS $(\xi,\mathfrak{J})_{\beta}$ such that $\mathrm{IHint}((h,A)_{\beta})\subseteq(\Theta,\mathfrak{J})_{\beta}\subseteq(\xi,\mathfrak{J})_{\beta}$.

Thus, $\cap_{\beta\in I}(\mathrm{IHint}((\xi,\mathfrak{J})_{\beta}))\subseteq\cap_{\beta\in I}(\Theta,\mathfrak{J})_{\beta}\subseteq\cap_{\beta\in I}(\xi,\mathfrak{J})_{\beta}$. Consider $\cap_{\beta\in I}(\xi,\mathfrak{J})_{\beta}=(\xi,\mathfrak{J})$. Then, $(\xi,\mathfrak{J})$ is an ICHS, and hence, $\cap_{\beta\in I}(\Theta,\mathfrak{J})_{\beta}$ is an ISCHS. □

Theorem 21. *Let* $(\Theta,\mathfrak{J})$ *be an ISCHS and* $(\vartheta,\mathfrak{J})$ *be an ICHS in* $(\mathfrak{X}_{\varpi},\tau)$. *If* $IH\mathrm{int}(\Theta,\mathfrak{J})\subseteq(\vartheta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})$, *then* $(\Theta,\mathfrak{J})$ *is an ISCHS.*

Proof. Since $(\Theta,\mathfrak{J})$ is an ISCHS, there exists an ICHS $(\xi,\mathfrak{J})$ such that $\mathrm{IHint}(\xi,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. Then, $(\vartheta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. Also, $\mathrm{IHintIHint}(\xi,\mathfrak{J})\subseteq\mathrm{IHint}(\xi,\mathfrak{J})\subseteq\mathrm{IHint}(\Theta,\mathfrak{J})\Longrightarrow\mathrm{IHint}(\xi,\mathfrak{J})\subseteq(\vartheta,\mathfrak{J})$. Therefore, $\mathrm{IHint}(\xi,\mathfrak{J})\subseteq(\vartheta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. Hence, $(\Theta,\mathfrak{J})$ is an ISCHS. □

Remark 3. For any ICHS $(\Theta,\mathfrak{J})$, $\mathrm{ISHScl}(\Theta,\mathfrak{J})=(\Theta,\mathfrak{J})$. And for any IOHS $(\zeta,\mathfrak{J})$, $\mathrm{ISHSint}(\zeta,\mathfrak{J})=(\zeta,\mathfrak{J})$.

Remark 4. If $(\Theta,\mathfrak{J})$ is an intuitionistic hypersoft set in $(\mathfrak{X}_{\varpi},\tau)$, then $\mathrm{IHint}(\Theta,\mathfrak{J})\subseteq\mathrm{ISHSint}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq\mathrm{ISHScl}(\Theta,\mathfrak{J})\subseteq\mathrm{IHcl}(\Theta,\mathfrak{J})$.

Theorem 22. *Let* $(\Theta,\mathfrak{J})$ *be an IHS in* $(\mathfrak{X}_{\varpi},\tau)$. *Then,* *(i)* $(ISHS\mathrm{int}(\Theta, \mathfrak{J}))^C = ISHScl((\Theta, \mathfrak{J})^C)$

(ii) $(ISHScl(\Theta, \mathfrak{J}))^C = ISHS\mathrm{int}((\Theta, \mathfrak{J})^C)$

(iii) $ISHS\mathrm{int}(IH\mathrm{int}(\Theta, \mathfrak{J})) = IH\mathrm{int}(ISHS\mathrm{int}(\Theta, \mathfrak{J}))\backslash\backslash = IH\mathrm{int}(\Theta, \mathfrak{J})$

(iv) $ISHScl(IHcl(\Theta, \mathfrak{J})) = IHcl(ISHScl(\Theta, \mathfrak{J}))\backslash\backslash = IHcl(\Theta, \mathfrak{J})$

Proof

(i) $\mathrm{ISHSint}(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J}) \Longrightarrow (\Theta, \mathfrak{J})^C \subseteq (\mathrm{ISHSint}(\Theta, \mathfrak{J}))^C$.

Since $(\mathrm{ISHSint}(\Theta, \mathfrak{J}))^C$ is an ISCHS, $\mathrm{ISHScl}(\Theta, \mathfrak{J})^C \subseteq \mathrm{ISHScl}((\mathrm{FSHSint}(\Theta, \mathfrak{J}))^C) = (\mathrm{ISHSint}(\Theta, \mathfrak{J}))^C$.

Conversely, $(\Theta, \mathfrak{J})^C \subseteq \mathrm{ISHScl}((\Theta, \mathfrak{J})^C) \Longrightarrow (\mathrm{ISHScl}(\Theta, \mathfrak{J})^C)^C \subseteq ((\Theta, \mathfrak{J})^C)^C = (\Theta, \mathfrak{J})$.

$\mathrm{ISHScl}((\Theta, \mathfrak{J})^C)$ being ISCHS implies that $\mathrm{ISHScl}((\Theta, \mathfrak{J})^C)^C$ is an ISOHS set. Thus, $\mathrm{ISHScl}((\Theta, \mathfrak{J})^C)^C \subseteq \mathrm{ISHSint}(\Theta, \mathfrak{J})$. And hence, $(\mathrm{ISHSint}(\Theta, \mathfrak{J}))^C \subseteq (\mathrm{ISHScl}((\Theta, \mathfrak{J})^C)^C)^C = (\mathrm{ISHScl}((\Theta, \mathfrak{J})^C))$.

(ii) The proof is the same as that of (i).

(iii) $\mathrm{IHint}(\Theta, \mathfrak{J})$ is IOHS which implies that it is ISOHS.

Therefore, $\mathrm{ISHSint}(\mathrm{IHint}(\Theta, \mathfrak{J})) = \mathrm{IHint}(\Theta, \mathfrak{J})$.

Now, $\mathrm{IHint}(\Theta, \mathfrak{J}) \subseteq \mathrm{ISHSint}(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})$.

Thus, $\mathrm{ISHSint}(\mathrm{IHint}(\Theta, \mathfrak{J})) = \mathrm{IHint}(\Theta, \mathfrak{J})$.

(iv) $\mathrm{IHcl}(\Theta, \mathfrak{J})$ is an intuitionistic closed hypersoft set, and this implies that it is an ISCHS. Therefore, $\mathrm{ISHScl}(\mathrm{IHcl}(\Theta, \mathfrak{J})) = \mathrm{IHcl}(\Theta, \mathfrak{J})$. Now, $(\Theta, \mathfrak{J}) \subseteq \mathrm{ISHScl}(\Theta, \mathfrak{J}) \subseteq \mathrm{IHcl}(\Theta, \mathfrak{J})$.

Hence, $\mathrm{ISHScl}(\Theta, \mathfrak{J}) \subseteq \mathrm{IHclISHScl}((\Theta, \mathfrak{J})) \subseteq \mathrm{ISHScl}(\Theta, \mathfrak{J})$.

This implies $\mathrm{IHcl}(\mathrm{ISHScl}(\Theta, \mathfrak{J})) \subseteq \mathrm{IHcl}(\Theta, \mathfrak{J})$. □

5. Neutrosophic Semiopen and Closed Hypersoft Sets

Definition 15. Let $(\mathfrak{X}_{\varpi}, \tau)$ be a NHTS and $(\Theta, \mathfrak{J}) \in P(\varpi, \mathfrak{X})$. If $(\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\mathrm{NHint}(\Theta, \mathfrak{J}))$, then $(\Theta, \mathfrak{J})$ is called a neutrosophic semiopen hypersoft set (NSOHS). We represent the collection of all neutrosophic semiopen hypersoft sets by $\mathrm{NSOHS}(\mathfrak{X})$.

Definition 16. A neutrosophic hypersoft set $(\Theta, \mathfrak{J})$ in the NHST space is a neutrosophic semiclosed hypersoft set (NFSCHS) iff its complement $(\Theta, \mathfrak{J})^C$ is NSOHS. The class of NSCHS is denoted by $\mathrm{NSCHS}(\mathfrak{X})$.

Example 4. Let $X = \{y_1, y_2, y_3\}$ and the attributes be $E_1 = \{a_1, a_2, a_3\}$, $E_2 = \{b_1, b_2\}$, and $E_3 = \{c_1, c_2\}$.

The neutrosophic hypersoft topological space is

$$
\begin{aligned}
\tau = \Bigg\{ & \left\{ \left\langle (a_1, b_1, c_2), \left\{ \frac{y_1}{0.9, 0.2, 0.1}, \frac{y_2}{0.2, 0.2, 0.4}, \frac{y_3}{0.2, 0.1, 0.7} \right\} \right\rangle, \left\langle (a_1, b_2, c_2), \left\{ \frac{y_1}{0.8, 0.4, 0.3}, \frac{y_2}{0.7, 0.4, 0.2}, \frac{y_3}{0.4, 0.6, 0.8} \right\} \right\rangle \right\} \Bigg\} \\
& \cdot \left\{ \left\langle (a_1, b_1, c_1), \left\{ \frac{y_1}{0.8, 0.5, 0.2}, \frac{y_2}{0.6, 0.5, 0.4}, \frac{y_3}{0.3, 0.6, 0.2} \right\} \right\rangle, \left\langle (a_3, b_1, c_1), \left\{ \frac{y_1}{0.6, 0.4, 0.5}, \frac{y_2}{0.5, 0.7, 0.4}, \frac{y_3}{0.7, 0.3, 0.2} \right\} \right\rangle \right\} \\
& \cdot \left\{ \begin{array}{l} \left\langle (a_1, b_1, c_2), \left\{ \frac{y_1}{0.9, 0.2, 0.1}, \frac{y_2}{0.2, 0.2, 0.4}, \frac{y_3}{0.2, 0.1, 0.7} \right\} \right\rangle, \left\langle (a_1, b_2, c_2), \left\{ \frac{y_1}{0.8, 0.4, 0.3}, \frac{y_2}{0.7, 0.4, 0.2}, \frac{y_3}{0.4, 0.6, 0.8} \right\} \right\rangle, \\[2ex] \left\langle (a_1, b_1, c_1), \left\{ \frac{y_1}{0.8, 0.5, 0.2}, \frac{y_2}{0.6, 0.5, 0.4}, \frac{y_3}{0.3, 0.6, 0.2} \right\} \right\rangle, \left\langle (a_3, b_1, c_1), \left\{ \frac{y_1}{0.6, 0.4, 0.5}, \frac{y_2}{0.5, 0.7, 0.4}, \frac{y_3}{0.7, 0.3, 0.2} \right\} \right\rangle \end{array} \right\}.
\end{aligned} \tag{9}
$$

The neutrosophic hypersoft set

$$
\left\{ \begin{array}{l} \left\langle (a_1, b_1, c_2) \left\{ \frac{y_1}{0.9, 0.3, 0.1}, \frac{y_2}{0.3, 0.4, 0.3}, \frac{y_3}{0.5, 0.2, 0.5} \right\} \right\rangle, \left\langle (a_1, b_2, c_2) \left\{ \frac{y_1}{0.9, 0.5, 0.2}, \frac{y_2}{0.8, 0.5, 0.1}, \frac{y_3}{0.5, 0.7, 0.5} \right\} \right\rangle, \\[2ex] \left\langle (a_1, b_1, c_1) \left\{ \frac{y_1}{0.9, 0.6, 0.1}, \frac{y_2}{0.7, 0.7, 0.3}, \frac{y_3}{0.5, 0.7, 0.1} \right\} \right\rangle, \left\langle (a_3, b_1, c_1) \left\{ \frac{y_1}{0.7, 0.5, 0.3}, \frac{y_2}{0.6, 0.8, 0.3}, \frac{y_3}{0.8, 0.4, 0.1} \right\} \right\rangle \end{array} \right\} \tag{10}
$$

is NSOHS.

Theorem 23. *Let* $(\mathfrak{X}_{\varpi}, \tau)$ *be a NHTS and* $(\Theta, \mathfrak{J}) \in NSOHS(\mathfrak{X})$*; then,*

(i) Arbitrary neutrosophic hypersoft union of NSOHS is a NSOHS

(ii) Arbitrary neutrosophic hypersoft intersection of NSCHS is a NFSCHS

Proof

(i) Let $\{(\Theta, \mathfrak{J})_j \colon j \in J\} \subseteq \mathrm{NSOHS}(\mathfrak{X})$.

Then, $\forall j \in J$, $(\Theta, \mathfrak{J})_j \subseteq \mathrm{NHcl}(\Theta, \mathfrak{J})_j$.

Hence, $\cup_j (\Theta, \mathfrak{J})_j \subseteq \cup\, \mathrm{NHcl}(\mathrm{NHint}(\Theta, \mathfrak{J})_j) \subseteq \backslash\backslash \mathrm{NHcl}(\mathrm{NHint}(\cup_j (\Theta, \mathfrak{J})_j))$.

Therefore, $\cup_j (\Theta, \mathfrak{J})_j \in \mathrm{NSOHS}(\mathfrak{X})$.

Similarly, (ii) is proved. □

Theorem 24. *Let* $(\mathfrak{X}_{\varpi}, \tau)$ *be a NHTS and* $(\Theta, \mathfrak{J}) \in P(\varpi, \mathfrak{X})$*. Then,*

(i) $(\Theta, \mathfrak{J}) \in NSOHS(\mathfrak{X})$ *if and only if there exists* $(\chi, \mathfrak{B}) \in \tau$ *such that* $(\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) \subseteq NHcl(\chi, \mathfrak{B})$

(ii) If $(\Theta, \mathfrak{J}) \in NSOHS(\mathfrak{X})$ *and* $(\Theta, \mathfrak{J}) \subseteq (\xi, \mathfrak{C}) \subseteq Ncl(\Theta, \mathfrak{J})$*, then* $(\xi, \mathfrak{C}) \in NSOHS(\mathfrak{X})$

Proof

(i) Let $(\Theta, \mathfrak{J}) \in \mathrm{NSOHS}(\mathfrak{X})$. Then, $(\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\mathrm{NHint}(\Theta, \mathfrak{J}))$. We know that $\mathrm{NHint}(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})$; thus, $\mathrm{NHint}(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\mathrm{NHint}(\Theta, \mathfrak{J}))$. Let $(\chi, \mathfrak{B}) = \mathrm{NHint}(\Theta, \mathfrak{J})$; thus, we get $(\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\chi, \mathfrak{B})$.

Conversely, let $(\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\chi, \mathfrak{B})$ for some $(\chi, \mathfrak{B}) \in \tau$. Then, $(\chi, \mathfrak{B}) \subseteq \mathrm{NHint}(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})$. $\Longrightarrow \mathrm{NHcl}(\chi, \mathfrak{B}) \subseteq \mathrm{NHcl}(\mathrm{NHint}(\Theta, \mathfrak{J}))$. Thus, $(\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\mathrm{NHint}(\Theta, \mathfrak{J}))$.

Therefore, $(\Theta, \mathfrak{J}) \in \mathrm{NSOHS}(\mathfrak{X})$.

(ii) Let $(\Theta, \mathfrak{J}) \in \mathrm{NSOHS}(\mathfrak{X})$. Then, for some $(\chi, \mathfrak{B}) \in \tau$, $(\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\chi, \mathfrak{B})$. If $(\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}) \subseteq (\xi, \mathfrak{C})$, then $(\chi, \mathfrak{B}) \subseteq (\xi, \mathfrak{C}) \subseteq \mathrm{NHcl}(\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\chi, \mathfrak{B})$. Hence, $(\chi, \mathfrak{B}) \subseteq (\xi, \mathfrak{C}) \subseteq \mathrm{NHcl}(\chi, \mathfrak{B})$. Thus, by (i), $(\xi, \mathfrak{C}) \in \mathrm{NSOHS}(\mathfrak{X})$. □

Definition 17. Let $(\mathfrak{X}_{\varpi}, \tau)$ be a NHTS and $(\Theta, \mathfrak{J}) \in P(\varpi, \mathfrak{X})$. Then, the largest neutrosophic semiopen hypersoft set contained in $(\Theta, \mathfrak{J})$ is known as the neutrosophic semi-hypersoft interior of $(\Theta, \mathfrak{J})$ and is denoted by $\mathrm{FSHSint}(\Theta, \mathfrak{J})$, i.e., $\mathrm{FSHSint}(\Theta, \mathfrak{J}) = \cup \{(\chi, \mathfrak{B}) \colon (\chi, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in \mathrm{FSOHS}(\mathfrak{X})\}$.

And the smallest neutrosophic semiclosed hypersoft set containing $(\Theta, \mathfrak{J})$ is called the neutrosophic semi-hypersoft closure of $(\Theta, \mathfrak{J})$ and is denoted by $\mathrm{NSHScl}(\Theta, \mathfrak{J})$.

$$\mathrm{NSHScl}(\Theta, \mathfrak{J}) = \cap \{(\xi, \mathfrak{C}) \colon (\xi, \mathfrak{C}) \supseteq (\Theta, \mathfrak{J}) \text{ and } (\xi, \mathfrak{C}) \in \mathrm{NSCHS}(\mathfrak{X})\}. \tag{11}$$

Theorem 25. *Let* $(\mathfrak{X}_{\varpi}, \tau)$ *be a NHTS and* $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in P(\varpi, \mathfrak{X})$*. Then, the following properties hold:*

(i) $NSHS\mathrm{int}(\varnothing_{\mathfrak{X}}) = \varnothing_{\mathfrak{X}}$ *and* $NSHS\mathrm{int}(\varpi, \mathfrak{X}) = (\varpi, \mathfrak{X})$

(ii) $NSHS\mathrm{int}(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})$

(iii) $NSHS\mathrm{int}(\Theta, \mathfrak{J})$ *is the largest neutrosophic semiopen hypersoft set contained in* $(\Theta, \mathfrak{J})$

(iv) If $(\Theta, \mathfrak{J}) \subseteq (\chi, \mathfrak{B})$*, then* $NSHS\mathrm{int}(\Theta, \mathfrak{J}) \subseteq NSHS\mathrm{int}(\chi, \mathfrak{B})$

(v) $NHSH\mathrm{int}(NHSH\mathrm{int}(\Theta, \mathfrak{J})) = NHSH\mathrm{int}(\Theta, \mathfrak{J})$

(vi) $NHSH\mathrm{int}(\Theta, \mathfrak{J}) \cup NHSH\mathrm{int}(\chi, \mathfrak{B}) \subseteq NHSH\mathrm{int}[(\Theta, \mathfrak{J}) \cup (\chi, \mathfrak{B})]$

(vii) $NHSH\mathrm{int}[(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})] \subseteq NHSH\mathrm{int}(\Theta, \mathfrak{J}) \cap NHSH\mathrm{int}(\chi, \mathfrak{B})$

Theorem 26. *Let* $(\mathfrak{X}_{\varpi}, \tau)$ *be a NHTS and* $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in P(\varpi, \mathfrak{X})$*. Then, the following properties hold:*

(i) $NSHScl(\varnothing_{\mathfrak{X}}) = \varnothing_{\mathfrak{X}}$ *and* $NSHScl(\varpi, \mathfrak{X}) = (\varpi, \mathfrak{X})$

(ii) $NSHScl(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})$

(iii) $NSHScl(\Theta, \mathfrak{J})$ *is the minutest neutrosophic semi-closed hypersoft set that holds* $(\Theta, \mathfrak{J})$

(iv) If $(\Theta, \mathfrak{J}) \subseteq (\chi, B)$*, then* $NSHScl(\Theta, \mathfrak{J}) \subseteq NSHScl(\chi, \mathfrak{B})$

(v) $NHSHcl(NHSHcl(\Theta, \mathfrak{J})) = NHSHcl(\Theta, \mathfrak{J})$

(vi) $NHSHcl(\Theta, \mathfrak{J}) \cup NHSHcl(\chi, \mathfrak{B}) \subseteq NHSHcl[(\Theta, \mathfrak{J}) \cup (\chi, \mathfrak{B})]$

(vii) $NHSHcl[(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})] \subseteq NHSHcl(\Theta, \mathfrak{J}) \cap NHSHcl(\chi, \mathfrak{B})$

Theorem 27. *Every neutrosophic open (closed) hypersoft set in a NHTS* $(\mathfrak{X}_{\varpi}, \tau)$ *is a neutrosophic semiopen (closed) hypersoft set.*

Proof. Let $(\Theta, \mathfrak{J})$ be a neutrosophic open hypersoft set. Then, $\mathrm{NHint}(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})$. Since $(\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\Theta, \mathfrak{J})$, $(\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\mathrm{NHint}(\Theta, \mathfrak{J}))$. Thus, $(\Theta, \mathfrak{J}) \in \mathrm{NSOHS}(\mathfrak{X})$. □

Theorem 28. *Let* $(\mathfrak{X}_{\varpi}, \tau)$ *be a NHTS and* $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in P(\varpi, \mathfrak{X})$*. If either* $(\Theta, \mathfrak{J}) \in NSOHS(X)$ *or* $(\chi, \mathfrak{B}) \in NSOHS(\mathfrak{X})$*, then* $NHcl(NH\mathrm{int}(\Theta, \mathfrak{J}) \cap (\chi, \mathfrak{B})) = NHcl(NH\mathrm{int}\ (\Theta, \mathfrak{J}) \cap NHclNH\mathrm{int}(\chi, \mathfrak{B}))$.

Proof. Let $(\Theta, \mathfrak{J}), (\chi, \mathfrak{B}) \in P(\varpi, \mathfrak{X})$. Then, we have

$$\begin{aligned}
&\mathrm{NHcl}(\mathrm{NHint}(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))\subseteq\\
&\mathrm{NHcl}(\mathrm{NHint}(\Theta,\mathfrak{J})\cap\mathrm{NHclNHint}(\chi,\mathfrak{B}))\\
&\mathrm{NHcl}(\mathrm{NHint}(\Theta,\mathfrak{J})\cap\mathrm{NHclNHint}(\chi,\mathfrak{B}))\subseteq\\
&\mathrm{NHcl}[\mathrm{NHcl}(\mathrm{NHint}(\Theta,\mathfrak{J}))\cap\mathrm{NHcl}(\mathrm{NHint}(\chi,\mathfrak{B}))]\\
&=\mathrm{NHcl}[\mathrm{NHcl}[\mathrm{NHint}(\Theta,\mathfrak{J})\cap NH\mathrm{int}(\chi,\mathfrak{B})]]\\
&=\mathrm{NHcl}[\mathrm{NHcl}[\mathrm{NHint}[(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})]]]\\
&\subseteq\mathrm{NHcl}[\mathrm{NHint}(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})]\\
&\Longrightarrow\mathrm{NHcl}(\mathrm{NHint}(\Theta,\mathfrak{J}))\cap NHcl(\mathrm{NHint}(\chi,\mathfrak{B}))\subseteq\mathrm{NHcl}(\mathrm{NHint}(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})).
\end{aligned}\tag{12}$$

Thus, $\mathrm{NHcl}(\mathrm{NHint}(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))=\mathrm{NHclNHint}(\Theta,\mathfrak{J})\cap NHcl(\mathrm{NHint}(\chi,\mathfrak{B}))$. □

Theorem 29. *Let $(\mathfrak{X}_{\varpi},\tau)$ be a NHTS, $(\Theta,\mathfrak{J})$ be a neutrosophic hypersoft open set, and $(\chi,\mathfrak{B})\in NSOHS(\mathfrak{X})$. Then, $(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})\in NSOHS(\mathfrak{X})$.*

Proof. Let $(\Theta,\mathfrak{J})$ be a NOHS set and $(\chi,\mathfrak{B})$ be a NSOHS. Then, $(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})\supseteq\mathrm{NHint}((\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))\Longrightarrow\mathrm{NHint}((\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))\subseteq(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})$. Then, $\mathrm{NHint}((\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))\subseteq(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})\subseteq\mathrm{NHclNHint}((\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))$. $\Longrightarrow(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})\subseteq\mathrm{NHclNHint}((\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B}))$.

Therefore, $(\Theta,\mathfrak{J})\cap(\chi,\mathfrak{B})$ is a NSOHS. □

Proposition 5. *Let $(\Theta,\mathfrak{J})$ be a neutrosophic hypersoft set in the NHTS $(\mathfrak{X}_{\varpi},\tau)$. Then, $(\Theta,\mathfrak{J})$ is NSCHS if and only if there exists a NCHS set $(\xi,\mathfrak{J})$ such that $NH\mathrm{int}(\xi,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$.*

Proposition 6. *Every neutrosophic hypersoft closed set is a NSCHS set in a NHTS $(\mathfrak{X}_{\varpi},\tau)$, but the converse need not be true.*

Theorem 30. *Let $(\Theta,\mathfrak{J})$ be a NHS in a NHTS $(\mathfrak{X}_{\varpi},\tau)$. Then, $(\Theta,\mathfrak{J})$ is NSCHS if and only if $NH\mathrm{int}(NHcl(\Theta,\mathfrak{J}))\subseteq(\Theta,\mathfrak{J})$.*

Proof. Suppose $(\Theta,\mathfrak{J})$ is a NSCHS set; then, there exists a $\mathrm{NHCS}(\Theta,\mathfrak{J})$ such that $\mathrm{NHint}(\xi,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. $\mathrm{NHcl}(\Theta,\mathfrak{J})\subseteq\mathrm{NHcl}(\xi,\mathfrak{J})=(\xi,\mathfrak{J})$.

Thus, $\mathrm{NHint}(\mathrm{NHcl}(\Theta,\mathfrak{J}))\subseteq\mathrm{NHint}(\xi,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\Longrightarrow\mathrm{NHint}(\mathrm{NHcl}(\xi,\mathfrak{J}))\subseteq(\Theta,\mathfrak{J})$.

Conversely, let $(\Theta,\mathfrak{J})$ be a neutrosophic hypersoft set in $(\mathfrak{X}_{\varpi},\tau)$ such that $\mathrm{NHint}(\mathrm{NHcl}(\Theta,\mathfrak{J}))\subseteq(\Theta,\mathfrak{J})$.

Let $\mathrm{NHcl}(\Theta,\mathfrak{J})=(\xi,\mathfrak{J})$.

Then, $\mathrm{NHint}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$.

Thus, $(\Theta,\mathfrak{J})$ is a NSCHS. □

Theorem 31. *Let $\left\{(\Theta,\mathfrak{J})_{\beta}\colon\beta\in I\right\}$ be a family of NSCHSs in a NHTS $(\mathfrak{X}_{\varpi},\tau)$. Then, the intersection $\cap_{\beta\in I}(\Theta,\mathfrak{J})_{\beta}$ is a NSCHS in $(\mathfrak{X}_{\varpi},\tau)$.*

Proof. Since each $\beta\in I$, $(\Theta,\mathfrak{J})_{\beta}$ is a NSCHS.

Then, there exists a NCHS $(\xi,\mathfrak{J})_{\beta}$ such that $\mathrm{NHint}((h,A)_{\beta})\subseteq(\Theta,\mathfrak{J})_{\beta}\subseteq(\xi,\mathfrak{J})_{\beta}$.

Thus, $\cap_{\beta\in I}(\mathrm{NHint}((\xi,\mathfrak{J})_{\beta}))\subseteq\cap_{\beta\in I}(\Theta,\mathfrak{J})_{\beta}$ $\cap_{\beta\in I}(\mathrm{NHint}((\xi,\mathfrak{J})_{\beta}))\subseteq\cap_{\beta\in I}(\Theta,\mathfrak{J})_{\beta}\subseteq\cap_{\beta\in I}(\xi,\mathfrak{J})_{\beta}$. Consider $\cap_{\beta\in I}(\xi,\mathfrak{J})_{\beta}=(\xi,\mathfrak{J})$.

Then, $(\xi,\mathfrak{J})$ is a NCHS, and hence, $\cap_{\beta\in I}(\Theta,\mathfrak{J})_{\beta}$ is a NSCHS. □

Theorem 32. *Let $(\Theta,\mathfrak{J})$ be a NSCHS and $(\vartheta,\mathfrak{J})$ be a NCHS in $(\mathfrak{X}_{\varpi},\tau)$. If $NH\mathrm{int}(\Theta,\mathfrak{J})\subseteq(\vartheta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})$, then $(\vartheta,\mathfrak{J})$ is a NSCHS.*

Proof. Since $(\Theta,\mathfrak{J})$ is a NSCHS, there exists a NCHS $(\xi,\mathfrak{J})$ such that $\mathrm{NHint}(\xi,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. Then, $(\vartheta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$. Also, $\mathrm{NHintFHint}(\xi,\mathfrak{J})\subseteq\mathrm{NHint}(\xi,\mathfrak{J})\subseteq\mathrm{NHint}(\Theta,\mathfrak{J})$. $\Longrightarrow\mathrm{NHint}(\xi,\mathfrak{J})\subseteq(\vartheta,\mathfrak{J})$. Therefore, $\mathrm{NHint}(\xi,\mathfrak{J})\subseteq(\vartheta,\mathfrak{J})\subseteq(\xi,\mathfrak{J})$.

Hence, $(\Theta,\mathfrak{J})$ is a NSCHS. □

Remark 5. For any NCHS $(\Theta,\mathfrak{J})$, $\mathrm{NSHScl}(\Theta,\mathfrak{J})=(\Theta,\mathfrak{J})$. And for any NOHS $(\zeta,\mathfrak{J})$, $\mathrm{NSHSint}(\zeta,\mathfrak{J})=(\zeta,\mathfrak{J})$.

Remark 6. If $(\Theta,\mathfrak{J})$ is a neutrosophic hypersoft set in $(\mathfrak{X}_{\varpi},\tau)$, then $\mathrm{NHint}(\Theta,\mathfrak{J})\subseteq\mathrm{NSHSint}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\subseteq\mathrm{NSHScl}(\Theta,\mathfrak{J})\subseteq\mathrm{NHcl}(\Theta,\mathfrak{J})$.

Theorem 33. *Let $(\Theta,\mathfrak{J})$ be a NHS in $(\mathfrak{X}_{\varpi},\tau)$. Then,*

(i) $(NSHS\mathrm{int}(\Theta,\mathfrak{J}))^{C}=NSHScl((\Theta,\mathfrak{J})^{C})$

(ii) $(NSHScl(\Theta,\mathfrak{J}))^{C}=NSHS\mathrm{int}((\Theta,\mathfrak{J})^{C})$

(iii) $NSHS\mathrm{int}(FH\mathrm{int}(\Theta,\mathfrak{J}))\backslash\backslash=NH\mathrm{int}(FSHS\mathrm{int}(\Theta,\mathfrak{J}))=NH\mathrm{int}(\Theta,\mathfrak{J})$

(iv) $NSHScl(FHcl(\Theta,\mathfrak{J}))=NHcl(FSHScl(\Theta,\mathfrak{J}))\backslash\backslash=NHcl(\Theta,\mathfrak{J})$

Proof

(i) $\mathrm{NSHSint}(\Theta,\mathfrak{J})\subseteq(\Theta,\mathfrak{J})\Longrightarrow(\Theta,\mathfrak{J})^{C}\subseteq(\mathrm{NSHSint}(\Theta,\mathfrak{J}))^{C}$.

Since $(\mathrm{NSHSint}(\Theta,\mathfrak{J}))^{C}$ is a NSCHS, $\mathrm{NSHScl}(\Theta,\mathfrak{J})^{C}\subseteq\mathrm{NSHScl}((\mathrm{NSHSint}(\Theta,\mathfrak{J}))^{C})=(\mathrm{NSHS\ int}(\Theta,\mathfrak{J}))^{C}$.

Conversely, $(\Theta, \mathfrak{J})^C \subseteq \mathrm{NSHScl}((\Theta, \mathfrak{J})^C) \Longrightarrow \mathrm{NSHScl}((\Theta, \mathfrak{J})^C)^C \subseteq ((\Theta, \mathfrak{J})^C)^C = (\Theta, \mathfrak{J})$.

$\mathrm{NSHScl}((\Theta, \mathfrak{J})^C)$ being FSCHS implies that $\mathrm{NSHScl}((\Theta, \mathfrak{J})^C)^C$ is a FSOHS.

Thus, $\mathrm{NSHScl}((\Theta, \mathfrak{J})^C)^C \subseteq \mathrm{NSHSint}(\Theta, \mathfrak{J})$.

And hence, $(\mathrm{NSHSint}(\Theta, \mathfrak{J}))^C \subseteq (\mathrm{NSHScl}((\Theta, \mathfrak{J})^C)^C)^C = (\mathrm{NSHScl}((\Theta, \mathfrak{J})^C))$.

(ii) The proof is the same as that of (i).

(iii) $\mathrm{NHint}(\Theta, \mathfrak{J})$ is FOHS implying that it is FSOHS.

Therefore,
$\mathrm{NSHSint}(\mathrm{NHint}(\Theta, \mathfrak{J})) = \mathrm{NHint}(\Theta, \mathfrak{J})$.

Now, $\mathrm{NHint}(\Theta, \mathfrak{J}) \subseteq \mathrm{NSHSint}(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})$.

Thus, $\mathrm{NSHSint}(\mathrm{NHint}(\Theta, \mathfrak{J})) = \mathrm{NHint}(\Theta, \mathfrak{J})$.

(iv) $\mathrm{NHcl}(\Theta, \mathfrak{J})$ is neutrosophic closed hypersoft implying that it is NSCHS.

Therefore, $\mathrm{NSHScl}(\mathrm{NHcl}(\Theta, \mathfrak{J})) = \mathrm{NHcl}(\Theta, \mathfrak{J})$.

Now, $(\Theta, \mathfrak{J}) \subseteq \mathrm{NSHScl}(\Theta, \mathfrak{J}) \subseteq \mathrm{NHcl}(\Theta, \mathfrak{J})$.

Hence, $\mathrm{NSHScl}(\Theta, \mathfrak{J}) \subseteq \mathrm{NHclNSHScl}((\Theta, \mathfrak{J}))\backslash\backslash\subseteq \mathrm{NSHScl}(\Theta, \mathfrak{J})$.

This implies $\mathrm{NHcl}(\mathrm{NSHScl}(\Theta, \mathfrak{J})) \subseteq \mathrm{NHcl}(\Theta, \mathfrak{J})$. □

6. Application

In this section, we present a multiattribute group decision-making (MAGDM) application of the NHS and NHS topology using two different algorithms, and the results of both algorithms are compared at the end. The algorithms proposed in [36] are considered, and some of their techniques are followed. Hypersoft sets are more feasible than soft sets and are more advantageous to use for applications since they can be dealt with more uncertainties. There are many methods proposed for multiattribute group decision-making applications, but the proposed method is feasible than the methods which were proposed beforehand and done by using the more advanced recent work.

6.1. Numerical Example. We propose to analyse the risk of COVID-19 by two MAGDM methods described by Algorithms 1 and 2 based on neutrosophic hypersoft sets and topology. We have all been affected by the current COVID-19 pandemic. However, the impact and consequences of the pandemic vary depending on our status as individuals and members of the society. We all find it difficult to be treated in hospitals because COVID affects everyone regardless of age. As a result, determining who should be treated first and assisting the most affected in becoming cured are difficult. The following method proposes methods for reducing the risk and treating patients based on their high risk of virus infection. Suppose that a committee of doctors have to give a report on patients having risk of COVID-19 in a particular area or hospital.

Let $X = \{p_1, p_2, p_3, p_4, p_5\}$ be the patients reported to the hospital. Suppose that the doctors consider the following set of attributes: $E = \{e_1, e_2, e_3, e_4, e_5\}$, where the attributes are e_1 = age, e_2 = illness, and e_3 = symptoms of the patients.

The attributes are subclassified as $E_1 = \{e_{11}, e_{12}, e_{13}, e_{14}, e_{15}\}$ = age, where $e11$ is people of age 0 to 17, $e12$ is people of age 18 to 44, $e13$ is people of age 45 to 64, $e14$ is people of age 65 to 74, and $e15$ is people of age 75+.

$E_2 = \{e_{21}, e_{22}, e_{23}, e_{24}\}$ = illness, where e_{21}, $e22$, e_{23}, and e_{24} represent the patients with diabetes and hypertension, cardiovascular disease, chronic respiratory disease, and cancer, respectively.

$E_3 = \{e_{31}, e_{32}, e_{33}\}$ = immune level, where $e31$, $e32$, and $e33$ represent people with low, medium, and high level of immune count.

$E_4 = \{e_{41}, e_{42}, e_{43}\}$ = symptoms, where e_{41} is the person having most common symptoms (fever, dry cough, and tiredness), e_{42} is the person having less common symptoms (aches and pain, sore throat, diarrhoea, headache, and loss of taste or smell), and e_{43} is the person having serious symptoms (shortness of breath, chest pain, and loss of speech or movement).

Doctors divide the criteria into two subsets, $\mathfrak{A}$ (category 1, for higher risk) and $\mathfrak{B}$ (category 2, for medium risk).

Category 1: $\mathfrak{A}$ represents attributes e_3 and e_4

Category 2: $\mathfrak{B}$ represents attributes e_2, e_3, and e_4

First, we solve the problem by using the NHS-MAGDM method as described in Algorithm 1.

Step 1: two NHSs, namely, $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$ over X, are constructed after receiving all the required data from the committee.

$(f, \mathfrak{A}) = \{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = f(e_{31}, e_{43}), \alpha_2 = f(e_{32}, e_{43})$, and $\alpha_3 = f(e_{31}, e_{42})$, and $(g, \mathfrak{B}) = \{\beta_1, \beta_2, \beta_3\}$, where $\beta_1 = g(e_{21}, e_{31}, e_{43}), \beta_2 = g(e_{21}, e_{32}, e_{42})$, and $\beta_3 = g(e_{22}, e_{32}, e_{41})$. The values for the NHS $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$ are given in Tables 1 and 2.

Step 2: we are now constructing the NHS topology given by $\tau = \{\varnothing, X, (f, \mathfrak{A}), (g, \mathfrak{B})\}$, where $\varnothing$, X are NHS empty and full sets. The neutrosophic hypersoft open set $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$ are formed in Tables 3 and 4, respectively, by taking the average for each element from Tables 1 and 2.

Step 3: the score matrix of NHS sets $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$ is calculated in Tables 5 and 6, respectively.

Step 4: we are now calculating the decision table of $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$ by averaging the score values correspondingly. Table 7 gives the decision values of $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$.

Step 5: now, by adding the decision values of $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$, we find the final decision value. Table 8 is the required final decision table.

Step 6: using Table 8, the final ranking of the patients is given by

$$p_2 > p_3 > p_5 > p_1 > p_4. \tag{13}$$

We see that patient 2 has the maximum value. So, patient 2 is selected for the immediate treatment.

Step 1: input NHS $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$.
Step 2: construct NHS topology τ such that $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$ are ONHS in τ. Construct the hypersoft open set $(f, \mathfrak{A})$ such that, for each element, the average is taken to form the table.
Step 3: calculate the score matrix corresponding to each ONHS. $(f, \mathfrak{A})$ denotes the neutrosophic hypersoft set; then, the neutrosophic set $(f_T, \mathfrak{A})$ in which each entry in the set $f_T(e)$ is the score function of the respective entries in the hypersoft set $f(e)$ is called the score matrix. For each hypersoft element $f(x)$, $s(f) = (1/n(f))\sum_{\varphi\in f(x)}\varphi$ is the score function of $f(x)$, where $n(f)$ is the number of values in $f(x)$.
Step 4: calculate the average of each ONHS for each p_i, and let it be denoted by d_i and e^i. This is the decision table for each ONHS.
Step 5: add the decision table of ONHS $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$. This is the final decision table.
Step 6: select the optimal alternative p_i using $\max\{d_i + e_i\}$.

Algorithm 1:

Step 1: input NHS $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$.
Step 2: construct NHS topology τ such that $(f, \mathfrak{A})$ and $(g, \mathfrak{B})$ are ONHS in τ. Construct the hypersoft open set $(f, \mathfrak{A})$ such that, for each element, the average is taken to form the table.
Step 3: calculate the score matrix corresponding to each ONHS. $(f, \mathfrak{A})$ denotes the neutrosophic hypersoft set; then, the neutrosophic set $(f_T, \mathfrak{A})$ in which each entry in the set $f_T(e)$ is the score function of the respective entries in the hypersoft set $f(e)$ is called the score matrix. For each hypersoft element $f(x)$, $s(f) = (1/n(f))\sum_{\varphi\in f(x)}\varphi$ is the score function of $f(x)$, where $n(f)$ is the number of values in $f(x)$.
Step 4: find the cardinality of all ONHSs by using $C(f, \mathfrak{A}) = \{Cf(a)/a\colon a \in \mathfrak{A}\}$, where $Cf(a) = \sum_{p\in X} f(p)/\langle X\rangle$.
Step 5: find the aggregate fuzzy set of the score matrix by using $|E|^* M_{(f,\mathfrak{A})^*} = M_{(f,\mathfrak{A})} * M^t_{C(f,\mathfrak{A})}$, where $M_{(f,\mathfrak{A})^*}$, $M_{(f,\mathfrak{A})}$, and $M^t_{C(f,\mathfrak{A})}$ represent the aggregate fuzzy matrix, score matrix, and transpose of the cardinal set, respectively.
Step 6: add $(f, \mathfrak{A})^*$ and $(g, \mathfrak{B})^*$ to find decision NHS.
Step 7: determine the optimal choice given by $\max\{(f, \mathfrak{A})^* + (g, \mathfrak{B})^*(p)\}$.

Algorithm 2:

Table 1: Values for $(f, \mathfrak{A})$.

(f, A)	α_1		α_2		α_3	
X	e_{31}	e_{43}	e_{32}	e_{43}	e_{31}	e_{42}
p_1	(0.5, 0.4, 0.1)	(0.1, 0.1, 0.1)	(0.5, 0.3, 0.2)	(0.1, 0.1, 0.1)	(0.5, 0.4, 0.1)	(0.8, 0.2, 0.1)
p_2	(0.7, 0.2, 0.2)	(0.7, 0.2, 0.1)	(0.6, 0.4, 0.2)	(0.7, 0.2, 0.1)	(0.7, 0.2, 0.2)	(0.6, 0.1, 0.2)
p_3	(0.1, 0.4, 0.4)	(0.8, 0.4, 0.2)	(0.8, 0.1, 0.1)	(0.8, 0.4, 0.2)	(0.1, 0.4, 0.4)	(0.5, 0.2, 0.1)
p_4	(0.1, 0.4, 0.1)	(0.5, 0.2, 0.1)	(0.1, 0.2, 0.1)	(0.5, 0.2, 0.1)	(0.1, 0.4, 0.1)	(0.3, 0.2, 0.1)
p_5	(0.8, 0.3, 0.1)	(0.6, 0.4, 0.2)	(0.6, 0.2, 0.1)	(0.6, 0.4, 0.2)	(0.8, 0.3, 0.1)	(0.4, 0.1, 0.1)

Table 2: Values for $(g, \mathfrak{B})$.

X	β_1			β_2			β_3		
	e_{21}	e_{31}	e_{43}	e_{21}	e_{32}	e_{42}	e_{22}	e_{32}	e_{41}
p_1	(0.8, 0.2, 0.1)	(0.5, 0.4, 0.1)	(0.1, 0.1, 0.1)	(0.8, 0.2, 0.1)	(0.5, 0.3, 0.2)	(0.8, 0.2, 0.1)	(0.4, 0.2, 0.2)	(0.5, 0.3, 0.2)	(0.9, 0.1, 0.1)
p_2	(0.9, 0.2, 0.1)	(0.7, 0.2, 0.2)	(0.7, 0.2, 0.1)	(0.9, 0.2, 0.1)	(0.6, 0.4, 0.2)	(0.6, 0.1, 0.2)	(0.6, 0.3, 0.2)	(0.6, 0.4, 0.2)	(0.8, 0.2, 0.2)
p_3	(0.5, 0.3, 0.4)	(0.1, 0.4, 0.4)	(0.8, 0.4, 0.2)	(0.5, 0.3, 0.4)	(0.8, 0.1, 0.1)	(0.5, 0.2, 0.1)	(0.8, 0.2, 0.2)	(0.8, 0.1, 0.1)	(0.6, 0.4, 0.2)
p_4	(0.6, 0.4, 0.2)	(0.1, 0.4, 0.1)	(0.5, 0.2, 0.1)	(0.6, 0.4, 0.2)	(0.1, 0.2, 0.1)	(0.3, 0.2, 0.1)	(0.7, 0.3, 0.2)	(0.1, 0.2, 0.1)	(0.5, 0.3, 0.2)
p_5	(0.3, 0.2, 0.1)	(0.8, 0.3, 0.1)	(0.6, 0.4, 0.2)	(0.3, 0.2, 0.1)	(0.6, 0.2, 0.1)	(0.4, 0.1, 0.1)	(0.4, 0.2, 0.1)	(0.6, 0.2, 0.1)	(0.6, 0.3, 0.2)

Table 3: The tabular representation of $(f, \mathfrak{A})$.

X	α_1	α_2	α_3
p_1	(0.3, 0.25, 0.1)	(0.3, 0.2, 0.15)	(0.65, 0.30, 0.1)
p_2	(0.7, 0.2, 0.15)	(0.65, 0.3, 0.15)	(0.65, 0.15, 0.2)
p_3	(0.45, 0.4, 0.3)	(0.8, 0.2, 0,1)	(0.3, 0.3, 0.25)
p_4	(0.3, 0.3, 0.1)	(0.3, 0.2, 0.1)	(0.2, 0.3, 0.1)
p_5	(0.7, 0.35, 0.15)	(0.6, 0.3, 0.15)	(0.6, 0.2, 0.1)

Table 4: The tabular representation of $(g, \mathfrak{B})$.

X	β_1	β_2	β_3
p_1	(0.47, 0.23, 0.1)	(0.7, 0.23, 0.13)	(0.6, 0.2, 0.17)
p_2	(0.77, 0.2, 0.13)	(0.7, 0.23, 0.17)	(0.67, 0.3, 0.2)
p_3	(0.47, 0.37, 0.33)	(0.6, 0.2, 0.2)	(0.73, 0.23, 0.17)
p_4	(0.4, 0.33, 0.13)	(0.33, 0.27, 0.13)	(0.43, 0.27, 0.17)
p_5	(0.57, 0.3, 0.13)	(0.43, 0.17, 0.1)	(0.53, 0.23, 0.13)

Table 5: Score matrix of $(f, \mathfrak{A})$.

X	α_1	α_2	α_3
p_1	0.217	0.217	0.35
p_2	0.35	0.37	0.33
p_3	0.38	0.37	0.28
p_4	0.23	0.2	0.2
p_5	0.4	0.35	0.3

Table 6: Score matrix of $(g, \mathfrak{B})$.

X	β_1	β_2	β_3
p_1	0.27	0.35	0.32
p_2	0.37	0.37	0.39
p_3	0.39	0.33	0.38
p_4	0.29	0.24	0.29
p_5	0.33	0.23	0.29

Table 7: Decision table.

$(f, \mathfrak{A})$		$(g, \mathfrak{B})$	
d_i	Values	e_i	Values
d_1	0.261	e_1	0.313
d_2	0.35	e_2	0.376
d_3	0.34	e_3	0.367
d_4	0.21	e_4	0.273
d_5	0.35	e_5	0.283

Table 8: Final decision table.

$d_i + e_i$	Values
$d_1 + e_1$	0.574
$d_2 + e_2$	0.726
$d_3 + e_3$	0.707
$d_4 + e_4$	0.483
$d_5 + e_5$	0.633

Using Algorithm 2, we are now solving the same problem.

Steps 1, 2, and 3 are identical to those in Algorithm 1.
Step 4: the cardinal is computed by the formula given above in the algorithm. The cardinal for $(f, \mathfrak{A})$ is

$$C(f, \mathfrak{A}) = \{0.315, 0.301, 0.292\}. \tag{14}$$

Similarly, the cardinal for $(g, \mathfrak{B})$ is

$$C(g, \mathfrak{B}) = \{0.33, 0.304, 0.334\}, \tag{15}$$

and the cardinal for empty and full sets is completely 0 and 1, respectively.

Step 5: we are now finding the fuzzy matrix aggregate $M_{(f,\mathfrak{A})^*}$:

$$\begin{aligned} M_{(f,\mathfrak{A})^*} &= \frac{1}{4}\begin{bmatrix} 0.217 & 0.217 & 0.35 \\ 0.35 & 0.37 & 0.33 \\ 0.38 & 0.37 & 0.28 \\ 0.23 & 0.2 & 0.2 \\ 0.4 & 0.35 & 0.3 \end{bmatrix}\begin{bmatrix} 0.315 \\ 0.301 \\ 0.292 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 0.236 \\ 0.318 \\ 0.313 \\ 0.191 \\ 0.319 \end{bmatrix} = \begin{bmatrix} 0.059 \\ 0.0795 \\ 0.07825 \\ 0.04775 \\ 0.07975 \end{bmatrix}. \end{aligned} \tag{16}$$

Thus, we obtain aggregate fuzzy set $(f, \mathfrak{A})^*$ given by $(f, \mathfrak{A})^* = \{(p_1, 0.059), (p_2, 0.0795), (p_3, 0.07825), (p_4, 0.04775), (p_5, 0.07975)\}$.

We can also find an aggregate fuzzy matrix, $M_{(g,\mathfrak{B})^*}$:

$$\begin{aligned} M_{(g,\mathfrak{B})^*} &= \frac{1}{4}\begin{bmatrix} 0.27 & 0.35 & 0.32 \\ 0.37 & 0.37 & 0.39 \\ 0.39 & 0.33 & 0.38 \\ 0.29 & 0.24 & 0.29 \\ 0.33 & 0.23 & 0.29 \end{bmatrix}\begin{bmatrix} 0.33 \\ 0.304 \\ 0.334 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 0.3525 \\ 0.3649 \\ 0.356 \\ 0.2657 \\ 0.2758 \end{bmatrix} = \begin{bmatrix} 0.076 \\ 0.091 \\ 0.089 \\ 0.066 \\ 0.0689 \end{bmatrix}, \end{aligned} \tag{17}$$

$$(g, \mathfrak{B})^* = \{(p_1, 0.076), (p_2, 0.091), (p_3, 0.089), (p_4, 0.066), (p_5, 0.0689)\}.$$

Step 6: now, by adding the aggregate fuzzy sets, we find the final decision set $(f, \mathfrak{A})^*$ and $(g, \mathfrak{B})^*$:

Table 9: Comparison of the final ranking obtained by both algorithms.

Method	Ranking	Optimal decision
Algorithm 1	$p_2 > p_3 > p_5 > p_1 > p_4$	p_2
Algorithm 2	$p_2 > p_3 > p_5 > p_1 > p_4$	p_2

$$\begin{aligned}(f,\mathfrak{A})^*(p) &= \{(p_1, 0.059), (p_2, 0.0795), (p_3, 0.07825), (p_4, 0.04775), (p_5, 0.07975)\},\\ (g,\mathfrak{B})^*(p) &= \{(p_1, 0.076), (p_2, 0.091), (p_3, 0.089), (p_4, 0.066), (p_5, 0.0689)\}.\end{aligned} \tag{18}$$

Thus, $\{(f,\mathfrak{A})^*(p) + (g,\mathfrak{B})^*(p)\} = \{(p_1, 0.135)$ $(p_2, 0.170), (p_3, 0.167), (p_4, 0.114), (p_5, 0.149)\}$.

Step 7: by using the optimal decision function $\max\{(f,\mathfrak{A})^* + (g,\mathfrak{B})^*(p)\}$, we have the ranking of the patients who are of high risk of COVID-19. The final ranking according to Algorithm 2 is given by

$$p_2 > p_3 > p_5 > p_1 > p_4. \tag{19}$$

6.2. Comparison Analysis. Using NHS, cardinal sets, score matrices, and aggregate fuzzy sets, we produced two MAGDM techniques. Table 9 provides a comparison of both algorithms, showing the optimal alternative and results. Both algorithms provide the same optimum decision, as can be seen in the comparison table.

7. Conclusion

The idea of hypersoft sets is a newly emerging technique in dealing with problems in the real world. Herein, we have defined the new concept of semi-hypersoft sets of the fuzzy hypersoft topological space. Then, it has been extended to intuitionistic and neutrosophic semisets of intuitionistic and neutrosophic hypersoft topological spaces along with basic characterizations. Also, a real-life example in the current scenario of COVID-19 to make decision on the critical stage of medical diagnosis has been projected in MAGDM. This hypersoft topological space will also be extended to Pythagorean hypersoft topological spaces, as well as various forms of open sets, and more fuzzy topological space properties will be investigated. The concept of open sets introduced in this work may be extended to pre-, alpha-open neutrosophic hypersoft sets and strong open neutrosophic hypersoft sets based on which more such applications to real-world problems can be explored.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

N.B. contributed to funding acquisition. G.R. and N.B. conceptualized the study, contributed to software, performed formal analysis, developed the methodology, wrote the original draft, and validated the study. G.R., P.H., J.J.C., and D.A. supervised the study. G.R., D.A., and N.B. reviewed and edited the article. All authors read and agreed to the published version of the manuscript.

Acknowledgments

This research was supported by the Rajamangala University of Technology Suvarnabhumi.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.
[3] F. Smarandache, *A unifying Field of Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*, American Research Press, Rehoboth, MA, USA, 1998.
[4] H. Wang, F. Smarandache, R. Sunderraman, and Y. Q. Zhang, "Single valued neutrosophic sets," *Review of the Air Force Academy*, vol. 4, pp. 410–413, 2010.
[5] J.-J. Peng, J.-Q. Wang, H.-Y. Zhang, and X.-H. Chen, "An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets," *Applied Soft Computing*, vol. 25, pp. 336–346, 2014.
[6] H. Zhang, J. Wang, and X. Chen, "An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets," *Neural Computing and Applications*, vol. 27, no. 3, pp. 615–627, 2016.
[7] D. Karabasevi, D. Tanujkic, E. K. Zavadskas et al., "A novel extension of the TOPSIS method adapted for the use of single-valued neutrosophic sets and hamming distance for E-commerce development strategies selection," *Symmetry*, vol. 12, p. 1263, 2020.
[8] D. Molodtsov, "Soft set theory-first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.
[9] P. K. Maji, R. Biswas, and A. R. Roy, "Soft set theory," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 555–562, 2003.
[10] F. Feng, Y. B. Jun, and X. Zhao, "Soft semirings," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2621–2628, 2008.
[11] H. Aktap and N. Çağman, "Soft sets and soft groups," *Information Sciences*, vol. 177, no. 13, pp. 2726–2735, 2007.
[12] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, pp. 589–602, 2001.
[13] F. Smarandache, "Extension of soft set to hypersoft set, and then to plithogenic hypersoft set," *Neutrosophic Sets and Systems*, vol. 22, pp. 168–170, 2018.

[14] M. Saqlain, M. Saeed, M. R. Ahmad, and F. Smarandache, “Generalization of TOPSIS for neutrosophic hypersoft set using accuracy function and its application,” *Neutrosophic Sets and Systems*, vol. 27, pp. 131–137, 2019.

[15] F. Smarandache, “*A Unifying Field in Logics, Neutrosophic Logic, Neutrosophy, Neutrosophic Set and Neutrosophic Probabilty*, ” Ed., American Research Press, Rehoboth, DE, USA, 4th edition, 1999.

[16] M. Abdel-Basset, A. Gamal, R. K. Chakrabortty, and M. Ryan, “A new hybrid multi-criteria decision-making approach for location selection of sustainable offshore wind energy stations: a case study,” *Journal of Cleaner Production*, vol. 280, pp. 124–462, 2021.

[17] M. Abdel-Basset, G. Manogaran, and M. Mohamed, “A neutrosophic theory based security approach for fog and mobile-edge computing,” *Computer Networks*, vol. 157, pp. 122–132, 2019.

[18] M. Abdel-Basst, R. Mohamed, and M. Elhoseny, “A model for the effective COVID-19 identification in uncertainty environment using primary symptoms and CT scans,” *Health Informatics Journal*, vol. 26, no. 4, pp. 3088–3105, 2020.

[19] S. Rana, M. Qayyum, M. Saeed, F. Smarandache, and B. A. Khan, “Plithogenic fuzzy whole hypersoft set, construction of operators and their application in frequency matrix multi attribute decision making technique,” *Neutrosophic Sets and Systems*, vol. 28, pp. 34–50, 2019.

[20] M. Saqlain, N. Jafar, S. Moin, M. Saeed, and S. Broumi, “Single and multi-valued neutrosophic hypersoft set and tangent similarity measure of single valued neutrosophic hypersoft sets,” *Neutrosophic Sets and Systems*, vol. 32, pp. 317–329, 2020.

[21] M. Saqlain, S. Moin, M. N. Jafar, M. Saeed, and F. Smarandache, “Aggregate operators of neutrosophic hypersoft set,” *Neutrosophic Sets and Systems*, vol. 32, pp. 294–306, 2020.

[22] M. Abbas, G. Murtaza, and F. Smarandache, “Basic operations on hypersoft sets and hypersoft point,” *Neutrosophic Sets and Systems*, vol. 35, pp. 407–421, 2020.

[23] C. L. Chang, “Fuzzy topological spaces,” *Journal of Mathematical Analysis and Applications*, vol. 24, no. 1, pp. 182–190, 1968.

[24] T. E. Gantner, R. C. Steinlage, and R. H. Warren, “Compactness in fuzzy topological spaces,” *Journal of Mathematical Analysis and Applications*, vol. 62, no. 3, pp. 547–562, 1978.

[25] R. Lowen, “Convergence in fuzzy topological spaces,” *General Topology and Its Applications*, vol. 10, no. 2, pp. 147–160, 1979.

[26] D. Çoker, “An introduction to intuitionistic fuzzy topological spaces,” *Fuzzy Sets and Systems*, vol. 88, no. 1, pp. 81–89, 1997.

[27] A. A. Salama and S. A. Alblowi, “Neutrosophic set and neutrosophic topological spaces,” *IOSR Journal of Mathematics*, vol. 3, no. 4, pp. 31–35, 2012.

[28] P. Iswarya and K. Bageerathi, “On neutrosophic semi-open sets in neutrosophic topological spaces,” *International Journal of Mathematics Trends and Technology*, vol. 37, no. 3, pp. 214–223, 2016.

[29] B. Tanay and M. B. Kandemir, “Topological structure of fuzzy soft sets,” *Computers & Mathematics with Applications*, vol. 61, no. 10, pp. 2952–2957, 2011.

[30] A. Aygünoğlu and H. Aygün, “Some notes on soft topological spaces,” *Neural Computing and Applications*, vol. 21, no. 1, pp. 113–119, 2012.

[31] S. Roy and T. K. Samanta, “An introduction to open and closed sets on fuzzy soft topological spaces,” *Annals of Fuzzy Mathematics and Informatics*, vol. 6, no. 2, pp. 425–431, 2013.

[32] I. Osmanoglu and D. Tokat, “On intuitionistic Fuzzy soft topology,” *General Mathematics Notes*, vol. 19, no. 2, pp. 59–70, 2013.

[33] S. Bayramov and C. Gunduz, “On intuitionistic fuzzy soft topological spaces,” *Turkic World Mathematical Society (TWMS) Journal of Pure and Applied Mathematics*, vol. 5, no. 1, pp. 66–79, 2014.

[34] T. Bera and N. K. Mahapatra, “Introduction to neutrosophic soft topological space,” *Opsearch*, vol. 54, no. 4, pp. 841–867, 2017.

[35] D. Ajay and J. J. Charisma, “Neutrosophic hypersoft topological spaces,” *Neutrosophic Sets and Systems*, vol. 40, pp. 179–194, 2021.

[36] M. Riaz, B. Davvaz, A. Fakhar, and A. Firdous, “Hesitant fuzzy soft topology and its applications to multi-attribute group decision-making,” *Soft Computing*, vol. 24, no. 21, pp. 16269–16289, 2020.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6668711, 8 pages
https://doi.org/10.1155/2021/6668711

Research Article

Neutrosophic Number Optimization Models and Their Application in the Practical Production Process

Angyan Tu [1,2], Jun Ye [2], and Bing Wang [1]

[1]School of Mechatronic Engineering and Automation, Shanghai University, 99 Shangda Road, Shanghai 200444, China
[2]Department of Electrical and Information Engineering, Shaoxing University, 508 Huancheng West Road, Shaoxing, Zhejiang 312000, China

Correspondence should be addressed to Bing Wang; susanbwang@shu.edu.cn

Received 15 December 2020; Revised 30 March 2021; Accepted 3 April 2021; Published 19 April 2021

Academic Editor: Broumi Said

In order to simplify the complex calculation and solve the difficult solution problems of neutrosophic number optimization models (NNOMs) in the practical production process, this paper presents two methods to solve NNOMs, where Matlab built-in function "fmincon()" and neutrosophic number operations (NNOs) are used in indeterminate environments. Next, the two methods are applied to linear and nonlinear programming problems with neutrosophic number information to obtain the optimal solution of the maximum/minimum objective function under the constrained conditions of practical productions by neutrosophic number optimization programming (NNOP) examples. Finally, under indeterminate environments, the fit optimal solutions of the examples can also be achieved by using some specified indeterminate scales to fulfill some specified actual requirements. The NNOP methods can obtain the feasible and flexible optimal solutions and indicate the advantage of simple calculations in practical applications.

1. Introduction

Traditional inventory models [1–4] and production planning models [5–7] involve deterministic constrained functions and/or objective functions in deterministic environments. Nevertheless, uncertainty is nearly universal in real world. Therefore, many uncertain optimization methods were proposed for optimization problems with uncertain variables, interval numbers, stochastic, and fuzzy logics [8–15]. In many applied fields, such as management, engineering, and design problems, uncertain programming has been broadly carried out so far. In order to obtain the optimal crisp values of the objective function and the optimal feasible crisp solutions of the decision variables, the constrained functions and/or objective functions are usually changed into some crisp or deterministic programming problems in existing uncertain programming approaches. So, the aforementioned transformed methods are not really meaningful indeterminate approaches because the real indeterminate optimization problems can only indicate indeterminate solutions rather than optimal crisp solutions in indeterminate environments. Nevertheless, indeterminate programming problems imply the corresponding indeterminate optimal values of the objective function and indeterminate optimal solutions for the decision variables under indeterminate environments. So, it is necessary to find some fit optimization approaches for dealing with indeterminate programming problems with indeterminate solutions.

Smarandache [16–18] is a pioneer of indeterminacy theories which provide the new minds to solve indeterminacy problems. He adopted the imaginary value denoted by I and then introduced a neutrosophic number (NN) $z = x + yI$ for $x, y \in R$ (R: the set of all real numbers) composed of the determinate part x and indeterminate part yI. As for describing indeterminate and incomplete information, obviously, NNs in the indeterminacy theories are a useful mathematical tool. With the development of indeterminacy theories, NNs were also applied to fault diagnosis [19, 20] and decision making [21, 22] under indeterminate environments.

Further, thick function or interval function named neutrosophic function, neutrosophic precalculus, and neutrosophic calculus were provided by Smarandache [23] in 2015, where thick function $e: S \longrightarrow E(S)$ ($E(S)$ is the set of all interval functions) as the form of an interval function $e(x) = [e_1(x),\ e_2(x)]$. The indeterminate function was applied in engineering problems successfully. For example, Ye et al. [24, 25] and Chen et al. [26, 27] proposed expressions of neutrosophic function and applied NNs in analyzing the joint roughness coefficient. Later, Ye [28] used neutrosophic linear equations of NNs to solve traffic flow problems.

At present, neutrosophic linguistic numbers, hesitant neutrosophic linguistic numbers, and their aggregation operators were applied to multiattribute decision making [29–31].

But in real situations, affected by each kind subjective and objective reasons, such as absences of precise information judged by decision makers or experts, loss of data, and measurement errors, there exist some indeterminate problems. As for the concepts of NNs, NN functions containing indeterminacy I can represent the indeterminate problems with partial certainty and partial uncertainty under indeterminate environments. Ye [32] and Jiang and Ye [33] introduced NN nonlinear and linear programming models and their preliminary solution methods. However, existing methods for solving complex NN optimization problems imply some difficulty and calculational complexity in their solution process. Inspired by the previous solution methods, this paper first selects the models of practical applications in production process, such as inventory models and production planning models. Then, NN nonlinear and linear mathematical models and their solution methods (Matlab built-in function "fmincon()" and operations of NNs) are built with indeterminacy I as our preliminary application study. Finally, real examples of NN linear programming (NN-LP) and NN nonlinear programming (NN-NP) problems illustrate the feasibility of the proposed methods. The advantage of the proposed methods is that the optimization calculations are simple and effective in practical applications.

The remainder of this paper is organized as follows. Section 2 depicts some concepts and their operations of NNs. Section 3 first introduces NN-NP problems with an inventory mathematical model and model formation and then uses two methods (Matlab built-in function "fmincon()" and operations of NNs) to solve the NN-NP problems in indeterminate setting. Section 4 presents NN-LP problems with the production planning mathematical model and model formation and then applies two methods regarding the Matlab built-in function "fmincon()" and operations of NNs to solve the solutions in the NN-NP problems and to show the simplicity and effectiveness of the proposed NN-LP methods. Conclusions and future research are provided in Section 5.

2. Mathematical Preliminaries

2.1. Some Concepts and Their Operations of Neutrosophic Numbers (NNs). The concept of NN was first proposed by Smarandache [34, 35], which consists of two parts (a determinate part and an indeterminate part). He defined the mathematical expression form $z = x + yI$ for $x, y \in R$, where R represents all real numbers and I is indeterminacy. So, it is conveniently used in indeterminate environments.

For example, consider that a NN is $z = 13 + 5I$. Then, its determinate part value is 13 and its indeterminate part value is $5I$. When $I \in [0, 0.5]$, it is equivalent to $z \in [13, 15.5]$ for sure $z \geq 13$.

Let $z_1 = x_1 + y_1 I$ and $z_2 = x_2 + y_2 I$ be two NNs. Then, Smarandache [34, 35] gave their operations of NNs in the following:

(1) $z_1 + z_2 = x_1 + x_2 + (y_1 + y_2)I$.

(2) $z_1 - z_2 = x_1 - x_2 + (y_1 - y_2)I$.

(3) $z_1 \times z_2 = x_1x_2 + (x_1y_2 + x_2y_1 + y_1y_2)I$, in particular, when $z_1 = 0$ and $z_2 = I$, we get the equation with $0 \times I = 0$.

(4) $z_1^2 = (x_1 + y_1 I)^2 = x_1^2 + (2x_1y_1 + y_1^2)\mathrm{I}$, in particular, when $z_1 = I$, we get the equation with $I2 = I$.

(5) $z_1/z_2 = x_1 + y_1 I/x_2 + y_2 I = x_1/x_2 + x_2y_1 - x_1y_2/x_2(x_2 + y_2)I$ for $x_2 \neq 0$ and $x_2 \neq -y_2$.

(6) $$\sqrt{z_1} = \sqrt{x_1 + y_1 I} = \begin{cases} \sqrt{x_1} - \left(\sqrt{x_1} + \sqrt{x_1 + y_1}\right)I \\ \sqrt{x_1} - \left(\sqrt{x_1} - \sqrt{x_1 + y_1}\right)I \\ -\sqrt{x_1} + \left(\sqrt{x_1} + \sqrt{x_1 + y_1}\right)I \\ -\sqrt{x_1} + \left(\sqrt{x_1} - \sqrt{x_1 + y_1}\right)I \end{cases}.$$

2.2. Example

There are two NNs $z_1 = 5 + 3I$ and $z_2 = 2 + 5I$. Then, we can obtain the following results according to the above operations:

(1) $z_1 + z_2 = x_1 + x_2 + (y_1 + y_2)I = 5 + 2 + (3 + 5)I = 7 + 8I$.

(2) $z_1 - z_2 = x_1 - x_2 + (y_1 - y_2)I = 5 - 2 + (3 - 5)I = 3 - 2I$.

(3) $z_1 \times z_2 = x_1x_2 + (x_1y_2 + x_2y_1 + y_1y_2)I = 5 \times 2 + (5 \times 5 + 3 \times 2 + 3 \times 5)I = 10 + 46I$.

(4) $z_1^2 = (x_1 + y_1 I)^2 = x_1^2 + (2x_1y_1 + y_1^2)I = 5^2 + (2 \times 5 \times 3 + 3^2)I = 25 + 39I$, $z_2^2 = (x_2 + y_2 I)^2 = x_2^2 + (2x_2y_2 + y_2^2)I = 2^2 + (2 \times 2 \times 5 + 5^2)\mathrm{I} = 4 + 45I$.

(5) $z_1/z_2 = x_1 + y_1 I/x_2 + y_2 I = x_1/x_2 + x_2y_1 - \mathrm{x_1 y_2/x_2}(x_2 + y_2)I = 5/2 + 2 \times 3 - 5 \times 5/2(2 + 5)I = 2.5 - 1.3571I$.

(6) $$\sqrt{z_1} = \sqrt{x_1 + y_1 I} = \begin{cases} \sqrt{x_1} - \left(\sqrt{x_1} + \sqrt{x_1 + y_1}\right)I \\ \sqrt{x_1} - \left(\sqrt{x_1} - \sqrt{x_1 + y_1}\right)I \\ -\sqrt{x_1} + \left(\sqrt{x_1} + \sqrt{x_1 + y_1}\right)I \\ -\sqrt{x_1} + \left(\sqrt{x_1} - \sqrt{x_1 + y_1}\right)I \end{cases} = \begin{cases} \sqrt{5} - \left(\sqrt{5} + \sqrt{5 + 3}\right)I = 2.2361 - 5.0645I \\ \sqrt{5} - \left(\sqrt{5} - \sqrt{5 + 3}\right)I = 2.2361 + 0.5924I \\ -\sqrt{5} + \left(\sqrt{5} + \sqrt{5 + 3}\right)I = -2.2361 + 5.0645I \\ -\sqrt{5} + \left(\sqrt{5} - \sqrt{5 + 3}\right)I = -2.2361 - 0.5924I \end{cases}$$

$$\sqrt{z_2} = \sqrt{x_2 + y_2 I} = \begin{cases} \sqrt{x_2} - \left(\sqrt{x_2} + \sqrt{x_2 + y_2}\right)I \\ \sqrt{x_2} - \left(\sqrt{x_2} - \sqrt{x_2 + y_2}\right)I \\ -\sqrt{x_2} + \left(\sqrt{x_2} + \sqrt{x_2 + y_2}\right)I \\ -\sqrt{x_2} + \left(\sqrt{x_2} - \sqrt{x_2 + y_2}\right)I \end{cases} = \begin{cases} \sqrt{2} - \left(\sqrt{2} + \sqrt{2 + 5}\right)I = 1.4142 - 4.0600I \\ \sqrt{2} - \left(\sqrt{2} - \sqrt{2 + 5}\right)I = 1.4142 + 1.2315I \\ -\sqrt{2} + \left(\sqrt{2} + \sqrt{2 + 5}\right)I = -1.4142 + 4.0600I \\ -\sqrt{2} + \left(\sqrt{2} - \sqrt{2 + 5}\right)I = -1.4142 - 1.2315I \end{cases}.$$

3. Neutrosophic Number Nonlinear Programming (NN-NP)

3.1. NN-NP Mathematical Model. The usual mathematical model of NN-NP is represented in the following form [36, 37]:

$$\begin{aligned} &\operatorname{Min} F(x, I) \\ &\text{s.t.} g_i(x, I) \leq c_i, \quad i = 1, 2, \ldots, p, \\ &h_j(x, I) = 0, \quad j = 1, 2, \ldots, q, \\ &x \in Z^n, \end{aligned} \tag{1}$$

where $g_1(x, I), g_2(x, I), \ldots, g_p(x, I), h_1(x, I), h_2(x, I)$: $\boldsymbol{Z^n} \longrightarrow \boldsymbol{Z}$($\boldsymbol{Z}$ is the set of all NNs), and $I \in [I^L, I^U]$(the interval range of I).

3.2. Inventory Mathematical Model [38]

3.2.1. Notations. The following notations are used in the inventory model.

Three decision variables:

(i) D: demand/unit/time
(ii) Q_p: production quantity/batch
(iii) C_s: setup cost/unit/time

Except the above cost variable CS, three other cost variables are

(i) C_{ta}: total average cost/unit/time
(ii) C_{tp}: total production cost/cycle
(iii) C_h: time depending on holding cost/unit/item

Other time and space variables:

(i) T: every cycle of length
(ii) $Q(t)$: inventory level at time t $(t \geq 0)$
(iii) S: total storage space area
(iv) s_0: space area/unit/quantity.

3.2.2. Assumptions. The inventory model is developed by considering the following assumptions:

(i) Only one item is involved in the inventory system.
(ii) The replenishment occurs with the near instantaneous response.
(iii) The startup time can be ignored.
(iv) The demand rate at any time is constant.
(v) The total production cost C_{tp} is related to the setup cost CS and production quantity Q_P.
(vi) Holding cost is the time depended function.

3.3. Model Formation. As shown in Figure 1, in every time period T, the value of the production quantity $Q_{(t)}$ decreases from Q_p to zero. The slope of the line is constant negative D and denoted by $(dQ(t)/dt) = -D (0 \leq t \leq T)$.

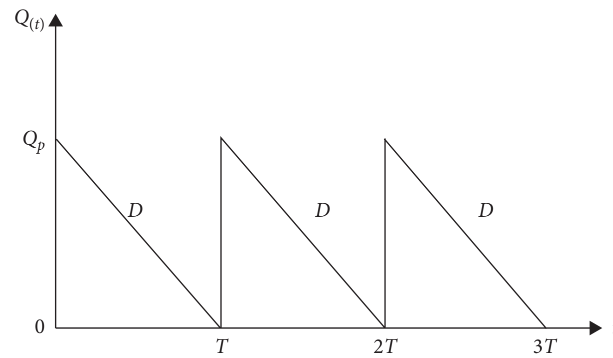


Figure 1: Crisp inventory model.

The total average cost of the cycle T (denoted by C_{ta}) consists of three sections: setup cost (denoted by C_1), holding cost (denoted by C_2), and production cost (denoted by C_3).

$$C_{ta} = C_1 + C_2 + C_3. \tag{2}$$

Because we have the equation $Q(t) = Q_p - Dt$, we obtain the cycle T, $T = (Q_p/D)$.

$$\begin{aligned} C_1 &= \frac{C_s}{T} = \frac{C_s D}{Q_p}, \\ C_2 &= \frac{\int_0^T C_h Q(t) dt}{T} = \frac{\int_0^T etQ(t) dt}{T} = \frac{\left(eQ_p^3/6D^2\right)}{T} \\ C_3 &= \frac{C_{tp}}{T} = \frac{f C_s^{-x} Q_p^{-y}}{T} = \frac{f\ D}{C_s^x Q_p^{1+y}} \frac{eQ_p^2}{6D}. \end{aligned} \tag{3}$$

Based on equations (2) and (3), we obtain the following equation:

$$C_{ta} = \frac{C_s D}{Q_p} + \frac{eQ_p^2}{6D} + \frac{f\ D}{C_s^x Q_p^{1+y}}. \tag{4}$$

So, the inventory model is constructed as follows:

$$\begin{aligned} &\operatorname{Min} C_{ta}\left(D, C_s, Q_p\right) = \frac{C_s D}{Q_p} + \frac{eQ_p^2}{6D} + \frac{f\ D}{C_s^x Q_p^{1+y}}, \\ &\text{s.t. } s_0 Q_p \leq S, \\ &D, C_s, Q_p > 0. \end{aligned} \tag{5}$$

3.4. Solution Corresponding to Matlab Built-In Function "fmincon()". In order to conveniently calculate the solutions, we simplify some parameters and set some constants with history records, where $e = 18$, $f = 5$, $x = 1$, $y = 3$, $s_0 = 200$, and $S = 1100$. When we assume $D = x_1$, $C_s = x_2$, and $Q_p = x_3$, we can obtain the following mathematical model:

$$\begin{aligned}
&\operatorname{Min} C_{ta}\left(x_1, x_2, x_3\right) = \frac{x_1 x_2}{x_3} + \frac{18 x_3^2}{6 x_1} + \frac{5 x_1}{x_2 x_3^4},\\
&s.\text{t. } 200 x_3 \leq 1100 \text{ with maximum allowable tolerance}\\
&\quad 400\ x_1,\ x_2,\ x_3 \geq 0.
\end{aligned} \tag{6}$$

Assume $a_1 = x_1 - 40.496I$, $a_2 = x_2 + 0.058I$, and $a_3 = x_3 - 2I$; then, equation (6) can be expressed in the following form:

$$\begin{aligned}
&s.t.\ 200\left(x3 - 2I\right) \geq 1100 \text{ with maximum allowable tolerance } 400,\\
&x_1 - 40.496I > 0,\\
&x_2 + 0.058I > 0,\\
&x_3 - 2I > 0.
\end{aligned} \tag{7}$$

According to the de-neutrosophication technique proposed by Ye [39] and considering $I = 0$ or 0.5 or 1 as the minimum or moderate or maximum indeterminacy, we can obtain three optimal solutions as follows:

(1) $x_1^* = 80.615$, $x_2^* = 0.097$, $x_3^* = 7.500$, and $f\left(x^*, I\right) = 4.187$ for $I = 0$.

(2) $x_1^* = 60.367$, $x_2^* = 0.126$, $x_3^* = 6.5$, and $f\left(x^*, I\right) = 4.343$ for $I = 0.5$.

(3) $x_1^* = 40.119$, $x_2^* = 0.155$, $x_3^* = 5.5$, and $f\left(x^*, I\right) = 4.525$ for $I = 1$.

Clearly, using the indeterminacy $I \in [0, 1]$, different optimal results are revealed. The optimal solutions of the optimization problem are $x_1^* = [40.119, 80.615]$, $x_2^* = [0.097, 0.155]$, and $x_3^* = [5.5, 7.5]$ for $f\left(x^*, I\right) = [4.187, 4.525]$, which show the interval optimal ranges.

3.5. Solution Corresponding to Operations of NNs. According to the front optimal solutions, we assume $a_1 = x_1 + y_1 I = 80.615 - 40.496\text{I}$, $a_2 = x_2 + y_2 I = 0.097 + 0.058I$, and $a_3 = x_3 - y_3 I = 7.500 - 2I$, and then we give the results by equation (9):

$$\begin{aligned}
\operatorname{Min} C_{ta}\left(a_1, a_2, a_3\right) &= \frac{a_1 a_2}{a_3} + \frac{18 a_3^2}{6 a_1} + \frac{5 a_1}{a_2 a_3^4} = \frac{\left(x_1 + y_1 I\right)\left(x_2 + y_2 I\right)}{x_3 + y_3 I} + \frac{18\left(x_3 + y_3 I\right)^2}{6\left(x_1 + y_1 I\right)} + \frac{5\left(x_1 + y_1 I\right)}{\left(x_2 + y_2 I\right)\left(x_3 + y_3 I\right)^4}\\
&= \frac{x_1 x_2 + \left(x_1 y_2 + x_2 y_1 + y_1 y_2\right) I}{x_3 + y_3 I} + \frac{3\left[x_3^2 + \left(2 x_3 y_3 + y_3^2\right) I\right]}{\left(x_1 + y_1 I\right)} + \frac{5\left(x_1 + y_1 I\right)}{\left(x_2 + y_2 I\right)\left[x_3^2 + \left(2 x_3 y_3 + y_3^2\right) I\right]^2}\\
&= \frac{x_1 x_2 + \left(x_1 y_2 + x_2 y_1 + y_1 y_2\right) I}{x_3 + y_3 I} + \frac{3\left[x_3^2 + \left(2 x_3 y_3 + y_3^2\right) I\right]}{\left(x_1 + y_1 I\right)}\\
&\quad + \frac{5\left(x_1 + y_1 I\right)}{\left(x_2 + y_2 I\right)\left\{x_3^4 + \left[2\left(2 x_3 y_3 + y_3^2\right) x_3^2 + \left(2 x_3 y_3 + y_3^2\right)^2\right] I\right\}}\\
&\quad \frac{5 x_1}{x_2 x_3^4} + 5 \frac{x_2 x_3^4 y_1 - x_1 \left\{\begin{array}{c} x_2\left[2\left(2 x_3 y_3 + y_3^2\right) x_3^2 + \left(2 x_3 y_3 + y_3^2\right)^2\right] \\ + y_2 x_3^4 \\ + y_2\left[2\left(2 x_3 y_3 + y_3^2\right) x_3^2 + \left(2 x_3 y_3 + y_3^2\right)^2\right] \end{array}\right\}}{x_2 x_3^4 \left(x_2 + y_2\right)\left[\left(x_3^4 + y_3\left(2 x_3 + y_3\right)\right)\left(2 x_3^2 + 2 x_3 y_3 + y_3^2\right)\right]} I.
\end{aligned} \tag{8}$$

Because $a_1 = 80.615 - 40.496I$, $a_2 = 0.097 + 0.058\ I$, and $a_3 = 7.500 - 2I$, we can get $x_1 = 80.615$, $y_1 = -40.496$, $x_2 = 0.097$, $y_2 = 0.058$, $x_3 = 7.5$, and $y_3 = -2$. Then, we calculate the three costs, respectively, as follows:

Setup cost:

$$C_1 = \frac{\left(x_1 + y_1 I\right)\left(x_2 + y_2 I\right)}{x_3 + y_3 I} = 1.407 + 0.087I. \tag{9}$$

Holding cost:

$$C_2 = \frac{18\left(x_3 + y_3 I\right)^2}{6\left(x_1 + y_1 I\right)} = 2.093 + 0.169I \tag{10}$$

Production cost:

$$C_3 = \frac{5\left(x_1 + y_1 I\right)}{\left(x_2 + y_2 I\right)\left(x_3 + y_3 I\right)^4} = 1.047 + 0.082I \tag{11}$$

Then, we add the three costs and obtain the total cost C_{ta} with equation (2) as follows:

$$\begin{aligned}C_{ta} = C_1 + C_2 + C_3 &= \frac{(x_1 + y_1 I)(x_2 + y_2 I)}{x_3 + y_3 I} + \frac{18(x_3 + y_3 I)^2}{6(x_1 + y_1 I)} \\ &+ \frac{5(x_1 + y_1 I)}{(x_2 + y_2 I)(x_3 + y_3 I)^4} \\ &= (1.047 + 0.087I) + (2.093 + 0.169I) + (1.047 + 0.082I) \\ &= 4.187 + 0.338I.\end{aligned} \tag{12}$$

So, the calculational results validate that the same solution is obtained by using the two methods of both the Matlab built-in function "fmincon" and the operations of NNs, which are $x_1^* = [40.119,\ 80.615]$, $x_2^* = [0.097,\ 0.155]$, and $x_3^* = [5.5,\ 7.5]$ for $f(x^*, I) = [4.187,\ 4.525]$. We also obtain every cost $C_1 = [1.047, 1.134]$, $C_2 = [\ 2.093, 2.262]$, and $C_3 = [1.047,\ 1.129]$, which are the interval optimal ranges.

4. Production Planning Mathematical Model

4.1. NN-LP Mathematical Model. The usual mathematical model of NN-LP is similar to mathematical model (1), so we omit it.

4.2. Production Planning Mathematical Model

4.2.1. Notations. The following notations are used in the production planning model.

Nine decision variables:

(i) a_1 to a_6: product quantities of six plans of type I

(ii) a_7 to a_8: product quantities of two plans of type II

(iii) a_9: product quantities of two plans of type III

Objective function:

(i) b: maximum profit

4.2.2. Assumptions. The production planning model is developed by considering the following assumptions:

(i) Every product must pass two working procedures: A and B.

(ii) The startup time of two working procedures can be ignored.

(iii) Product quantities are only affected by validity time of machines.

(iv) The demand rate at any time is constant.

4.3. Model Formation. As shown in Table 1, we consider an application in production planning studied by Hu [40]. A company manufactures three types of products: Types I, II, and III. All types must pass two working procedures: A and B. We consider that procedure A can be operated on machine A_1 or A_2 and procedure B can be operated on the machines B_1, B_2, and B_3. Type I can be operated on all machines of procedure A and procedure B; Type II can be operated on all machines of procedure A and only machine B_1 of procedure B; Type III can be operated on only machine A_2 of procedure A and machine B_2 of procedure B. Our aim is to schedule the optimal production planning, which can pursue for the maximum profits. All used data are listed in Table 1, including required procedure time of every working procedure, processing fees, material cost, and selling price per unit. So, Type I has six plans to produce products, along with (A_1, B_1) or (A_1, B_2) or (A_1, B_3) or (A_2, B_1) or (A_2, B_2) or (A_2, B_3), respectively. Similarly, we consider the product quantities of the six plans a_1, a_2, a_3, a_4, a_5, and a_6, respectively. Type II has two plans to produce products, along with (A_1, B_1) or (A_2, B_1), and Type III has one plan to produce products, along with (A_2, B_2). We consider the product quantities of the remaining three plans a_7, a_8, and a_9, respectively. So, we can get the following objective function.

$$\begin{aligned}b &= [(1.20 + 0.03I) - (0.23 + 0.03I)] \times (a_1 + a_2 + a_3 + a_4 + a_5 + a_6) + [(1.60 + 0.5I) - (0.30 + 0.07I)] \\ &\times (a_7 + a_8) + [(2.30 + 0.3I) - (0.30 + 0.05I)] \times a9 - (0.04 + 0.02I) \times [(4.5 + 1.7I) \times (a_1 + a_2 + a_3) + (8 + I) \times a_7] \\ &- (0.02 + 0.01I) \times [(0.67 + 0.8I) \times (a_4 + a_5 + a_6) + (8.6 + 1.4I) \times a_8 + (11 - I) \times a_9] - (0.05 + 0.02I) \\ &\times [(5.6 - 0.1I) \times (a1 + a4) + (7.8 + 1.2I) \times (a_7 + a_8)] - (0.10 + 0.02I) \times [(3.5 + 2.5I) \times (a_2 + a_5) + (10 + 2I) \times a_9] \\ &- (0.04 + 0.02I) \times [(6.7 + 1.3I) \times (a_3 + a_6)].\end{aligned} \tag{13}$$

So, we get the followed production planning mathematical model:

Table 1: All data of three types of products.

Machine	Product I	Product II	Product III	Validity time (hours/machine)	Processing fees ($/hour/machine)
A_1	$4.5+1.7I$	$8+I$	—	$5600+700I$	$0.04+0.02I$
A_2	$6.7+1.8I$	$8.6+1.4I$	$11\text{-}I$	$9000+700I$	$0.02+0.01I$
B_1	$5.6–0.1I$	$7.8+1.2I$	—	$3700+1100I$	$0.05+0.02I$
B_2	$3.5+2.5I$	—	$10+2I$	$6000+1000I$	$0.10+0.02I$
B_3	$6.7+1.3I$	—	—	$3500+1500I$	$0.04+0.02I$
Material cost ($/piece)	$0.23+0.03I$	$0.30+0.07I$	$0.30+0.05I$	—	—
Selling price ($/piece)	$1.20+0.03I$	$1.60+0.5I$	$2.30+0.3I$		

$$
\begin{aligned}
\max\quad & b\\
s.t.\quad & (4.5+1.71)\times(x_1+x_2+x_3)+(8+I)\times x7\le 5600+700I,\\
& (6.7+1.8I)\times(x_4+x_5+x_6)+(8.6+1.4I)\times x_8+(11-I)\times x_9\le 0.02+0.01I,\\
& (5.6-0.1I)\times(x_1+x_4)+(7.8+1.2I)\times(x_7+x_8)\le 0.05+0.02I,\\
& (3.5+2.5I)\times(x_2+x_5)+(10+2I)\times x_9\le 0.10+0.02I,\\
& (6.7+1.3I)\times(x_3+x_6)\le 0.04+0.02I.
\end{aligned}
\tag{14}
$$

4.4. Solution regarding Matlab Built-In Function "fmincon()". According to the de-neutrosophication technique proposed by Ye [37] and considering $I=0$ or 0.5 or 1 as the minimum or moderate or maximum indeterminacy, we can obtain three optimal solutions as follows:

(1) $x_1^*=0$, $x_2^*=778.508$, $x_3^*=465.936$, $x_4^*=0$, $x_5^*=677.953$, $x_6^*=56.452$, $x_7^*=0$, $x_8^*=474.359$, $x_9^*=0$, and $f(x^*,I)=1297.389$ for $I=0$.

(2) $x_1^*=0$, $x_2^*=0$, $x_3^*=578.231$, $x_4^*=0$, $x_5^*=0$, $x_6^*=0$, $x_7^*=167.732$, $x_8^*=338.221$, $x_9^*=590.909$, and $f(x^*,I)=940.871$ for $I=0.5$.

(3) $x_1^*=0$, $x_2^*=0$, $x_3^*=625$, $x_4^*=0$, $x_5^*=0$, $x_6^*=0$, $x_7^*=146.667$, $x_8^*=386.667$, $x_9^*=583.333$, and $f(x^*,I)=762.717$ for $I=1$.

Clearly, using the indeterminacy $I\in[0,1]$, different optimal results are revealed. The optimal solutions of the optimization problem are $x_1^*=[0,0]$, $x_2^*=[0,778.508]$, $x_3^*=[465.936,625]$, $x_4^*=[0,0]$, $x_5^*=[0,677.953]$, $x_6^*=[0,56.452]$, $x_7^*=[0,146.667]$, $x_8^*=[386.667,474.359]$, and$x_9^*=[0,583.333]$ for $f(x^*,I)=[762.717,1297.389]$, which shows the interval optimal ranges.

4.5. Solution regarding Operations of NNs. According the front optimal solutions, we next calculate the nine relation formulas of the indeterminacy I and variables a_1, a_2, a_3, a_4, a_5, a_6 , a_7, a_8, and a_9. For example, let us calculate $a_3=465.936+159.064I$. Firstly, according to three points (0, 465.936), (0.5, 578.231), and (1, 625), we obtain the linear equation ($a_3=159.06I+476.86$). Next we amend the intercept of trend curve on the vertical coordinate. The other linear equations are obtained in the same way. So, $a_1=x_1+y_1I=0+0I=0$, $a_2=x_2+y_2I=778.508–778.508I$, $a_3=x_3+y_3I=465.936+159.064I$, $a_4=x_4+y_4I=0+0I=0$, $a_5=x_5+y_5I=677.953–677.953I$, $a_6=x_6+y_6I=56.452-56.452I$, $a_7=x_7–y_7I=0+146.667I$, $a_8=x_8+y_8I=474.359-87.692I$, and $a_9=x_9+y_9I=0+583.333I$; then, we calculate the results of equation (13) as follows:

$b=[(1.2+0.03I)-(0.23+0.03I)]\times(a_1+a_2+a_3+a_4+a_5+a_6)+[(1.60+0.5I)-(0.30+0.07I)]\times(a_7+a_8)+[(2.30+0.3I)-(0.30+0.05I)]\times a_9-(0.04+0.02I)\times[(4.5+1.7I)\times(a_1+a_2+a_3)+(8+I)\times a_7]-(0.02+0.01I)\times[(6.7+1.8I)\times(a_4+a_5+a_6)+(8.6+1.4I)\times a_8+(11-I)\times a_9]-(0.05+0.02I)\times[(5.6-0.1I)\times(a_1+a_4)+(7.8+1.2I)\times(a_7+a_8)]-(0.10+0.02I)\times[(3.5+2.5I)\times(a_2+a_5)+(10+2I)\times a_9]-(0.04+0.02I)\times[(6.7+1.3I)\times(a_3+a_6)]=0.97\times(a_1+a_2+a_3+a_4+a_5+a_6)+(1.30+0.43I)\times(a_7+a_8)+(2.0+0.25I)\times a_9-(0.04+0.02I)\times[(4.5+1.7I)\times(a_1+a_2+a_3)+(8+\mathrm{I})\times a_7]-(0.02+0.01I)\times[(6.7+1.8I)\times(a_4+a_5+a_6)+(8.6+1.4I)\times a_8+(11-I)\times a_9]-(0.05+0.02I)\times[(5.6-0.1\mathrm{I})\times(a_1+a_4)+(7.8+1.2I)\times(a_7+a_8)]-(0.10+0.02I)\times[(3.5+2.5I)\times(a_2+a_5)+(10+2I)\times a_9]-(0.04+0.02I)\times[(6.7+1.3I)\times(a_3+a_6)]=0.97\times(a_2+a_3+a_5+a_6)+(1.30+0.43I)\times(a_7+a_8)+(2.0+0.25I)\times a_9-(0.04+0.02I)\times[(4.5+1.7I)\times(a_2+a_3)+(8+I)\times a_7]-(0.02+0.01I)\times[(6.7+1.8I)\times(a_5+a_6)+(8.6+1.4I)\times a_8+(11-I)\times a_9]-(0.05+0.02I)\times[(7.8+1.2I)\times(a_7+a_8)]-(0.10+0.02\mathrm{I})\times[(3.5+2.5I)\times(a_2+a_5)+(10+2I)\times a_9]-(0.04+0.02I)\times[(6.7+1.3I)\times(a_3+a_6)]=0.97\times(778.508-778.508I+465.936+159.064I+677.953-677.953I+56.452-56.452I)+(1.30+0.43I)\times(0+146.667I+474.359-87.692I)+(2.0+0.25I)\times(0+583.333I)-(0.04+0.02I)\times[(4.5+1.7I)\times(778.508-778.508I+465.936+159.064I)+(8+\mathrm{I})\times(0+146.667I)]-(0.02+0.01I)\times[(6.7+1.8I)\times(0+677.953-677.953I+56.452-56.452I)+(8.6+1.4I)\times(474.359-87.692I)+(11-I)\times(0+583.333I)]-(0.05+0.02\mathrm{I})\times[(7.8+1.2I)\times(0+146.667I+474.359-87.692I)]-(0.10+0.02\mathrm{I})\times[(3.5+2.5\mathrm{I})\times(778.508-778.508I+677.953-677.953I)+(10+2\mathrm{I})\times(0+583.333I)]-(0.04+0.02I)\times[(6.7+1.3I)\times(465.936+159.064I+$

$56.452-56.452I)] = 0.97 \times (1978.849 - 1353.849I) + (1.3 + 0.43I) \times (474.359 + 58.975I) + (2 + 0.25I) \times (0 + 583.333I) - (0.04 + 0.02I) \times (5599.998 - 404.995I) - (0.02 + 0.01I) \times (9000.001 + 699.9991I) - (0.05 + 0.02I) \times (3700 + 1100.006I) - (0.10 + 0.02I) \times (5097.614 + 1902.383I) - (0.04 + 0.02I) \times (3500 + 1500I) = 1919.484 - 1313.234I + 616.6667 + 306.001I + 1312.499I - 224 - 87.7I - 180 - 111.000I - 185 - 509.761 - 330.238I - 140 - 160.000I = 1297.389 - 534.672I.$

So, these calculational results validate that the same solution is obtained by using the two methods of both the Matlab built-in function "fmincon()" and the operations of NNs, which are $x_1^* = [0, 0]$, $x_2^* = [0, 778.508]$, $x_3^* = [465.936, 625]$, $x_4^* = [0, 0]$, $x_5^* = [0, 677.953]$, $x_6^* = [0, 56.452]$, $x_7^* = [0, 146.667]$, $x_8^* = [386.667, 474.359]$, and$x_9^* = [0, 583.333]$ for $f(x^*, I) = [762.717, 1297.389]$ and show the interval optimal ranges.

5. Conclusion

This paper first introduced some concepts and their operations of NNs with indeterminacy I. Next, we built a mathematical model with constrained conditions and then constructed the corresponding inventory model and production planning model. Finally, we obtained the optimal solutions by using the two methods of the Matlab built-in function "fmincon()" and the operations of NNs to solve the NN-NP and NN-LP problems with constrained conditions as preliminary application study in indeterminate setting. The final results show that the two methods obtained the same effective solutions, but the former needs the Matlab built-in function along with the simple calculational process, while the latter needs a lot of operations of NNs along with the complex calculational process. Some contributions in this study are that (1) different methods can obtain the same optimal results, (2) the NN-NP and NN-LP methods provided the new application ways for engineering management, (3) the NN-NP and NN-LP methods are more suitable than other ones under uncertain environments as the generalization of traditional programming methods, and (4) the two approaches can obtain the interval solutions for avoiding determinate solutions of traditional programming methods.

Obviously, the proposed NN-LP and NN-NP methods can handle indeterminate and/or determinate mathematical programming problems, which are the generalization of existing uncertain or certain linear and nonlinear programming methods. As the preliminary application study in this paper, however, there exist a lot of mathematical solution methods and proof problems along with some complexity/difficulty in the nonlinear programming problems which need to be studied further. Hence, as our future works, one is to further analyze the two presented methods of this paper from the mathematical problems, such as the convexity problem in the nonlinear programming, the stability and solution range problems regarding the changeability of NNs, and the sensitivities of NNs on the solution results, and then NN-LP and NN-NP approaches will be extended to other fields, such as engineering design and management science.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. Kumar, N. Kumar, and S. Liu, "An inventory model for deteriorating items under inflation and permissible delay in payments by genetic algorithm," *Cogent Business & Management*, vol. 3, no. 1, 2016.

[2] A. Oliveira, T. Oliveira, and A. Seijasmacías, "The uniform distribution product: an approach to the (Q, r) inventory model using R," *Journal of Applied Statistics*, vol. 45, no. 2, pp. 284–297, 2018.

[3] P. Mahata, G. C. Mahata, and S. Kumar De, "Optimal replenishment and credit policy in supply chain inventory model under two levels of trade credit with time- and credit-sensitive demand involving default risk," *Journal of Industrial Engineering International*, vol. 14, no. 1, pp. 31–42, 2018.

[4] I. D. Wangsa and M. W. Hui, "An integrated vendor–buyer inventory model with transportation cost and stochastic demand," *International Journal of Systems Science*, vol. 5, no. 4, pp. 295–309, 2018.

[5] C. P. T. Hedenstierna and S. M. Disney, "Avoiding the capacity cost trap: three means of smoothing under cyclical production planning," *International Journal of Production Economics*, vol. 201, pp. 149–162, 2018.

[6] H.-M. Cho and I.-J. Jeong, "A two-level method of production planning and scheduling for bi-objective reentrant hybrid flow shops," *Computers & Industrial Engineering*, vol. 106, pp. 174–181, 2017.

[7] C. Jiang, Z. G. Zhang, Q. F. Zhang, X. Han, H. C. Xie, and J. Liu, "A new nonlinear interval programming method for uncertain problems with dependent interval variables," *European Journal of Operational Research*, vol. 238, no. 1, pp. 245–253, 2014.

[8] I. Pergher and A. T. De Almeida, "A multi-attribute decision model for setting production planning parameters," *Journal of Manufacturing Systems*, vol. 42, pp. 224–232, 2017.

[9] G. S. Veresnikov, L. A. Pankova, and V. A. Pronina, "Uncertain programming in preliminary design of technical systems with uncertain parameters," *Procedia Computer Science*, vol. 103, pp. 36–43, 2017.

[10] C. Jiang, X. Y. Long, X. Han, Y. R. Tao, and J. Liu, "Probability-interval hybrid reliability analysis for cracked structures existing epistemic uncertainty," *Engineering Fracture Mechanics*, vol. 112-113, pp. 148–164, 2013.

[11] B. Zhang and J. Peng, "Uncertain programming model for uncertain optimal assignment problem," *Applied Mathematical Modelling*, vol. 37, no. 9, pp. 6458–6468, 2013.

[12] B. D. Liu and X. W. Chen, "Uncertain multiobjective programming and uncertain goal programming," *Journal of Uncertainty Analysis and Applications*, vol. 3, p. 10, 2015.

[13] L. Chen, J. Peng, and B. Zhang, "Uncertain goal programming models for bicriteria solid transportation problem," *Applied Soft Computing*, vol. 51, pp. 49–59, 2017.

[14] L. Wang, Z. Chen, G. Yang, Q. Sun, and J. Ge, "An interval uncertain optimization method using back-propagation

neural network differentiation," *Computer Methods in Applied Mechanics and Engineering*, vol. 366, p. 113065, 2020.
[15] L. Wang, Z. Chen, and G. Yang, "An interval uncertainty analysis method for structural response bounds using feed-forward neural network differentiation," *Applied Mathematical Modelling*, vol. 82, pp. 449–468, 2020.
[16] F. Smarandache, *Neutrosophy: Neutrosophic Probability, Set, and Logic*, American Research Press, Rehoboth, MA, USA, 1998.
[17] F. Smarandache, *Introduction to Neutrosophic Statistics*, Sitech & Education Publishing, Columbus, OH, USA, 2014.
[18] F. Smarandache, *Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability*, Sitech & Education Publisher, Columbus, OH, USA, 2013.
[19] L. Kong, Y. Wu, and J. Ye, "Misfire fault diagnosis method of gasoline engines using the cosine similarity measure of neutrosophic numbers," *Neutrosophic Sets and Systems*, vol. 8, pp. 43–46, 2015.
[20] J. Ye, "Fault diagnoses of steam turbine using the exponential similarity measure of neutrosophic numbers," *Journal of Intelligent & Fuzzy Systems*, vol. 30, no. 4, pp. 1927–1934, 2016.
[21] J. Ye, "Multiple-attribute group decision-making method under a neutrosophic number environment," *Journal of Intelligent Systems*, vol. 25, no. 3, pp. 377–386, 2016.
[22] J. Ye, "Bidirectional projection method for multiple attribute group decision making with neutrosophic numbers," *Neural Computing and Applications*, vol. 28, no. 5, pp. 1021–1029, 2017.
[23] F. Smarandache, *Neutrosophic Precalculus and Neutrosophic Calculus*, EuropaNova, Brussels, Belgium, 2015.
[24] J. Ye, R. Yong, Q. F. Liang, M. Huang, and S. G. Du, "Neutrosophic functions of the joint roughness coefficient (JRC) and the shear strength: a case study from the pyroclastic rock mass in Shaoxing City, China," *Mathematical Problems in Engineering*, vol. 2016, Article ID 4825709, , 2016.
[25] J. Ye, J. Chen, R. Yong, and S. Du, "Expression and analysis of joint roughness coefficient using neutrosophic number functions," *Information*, vol. 8, no. 2, p. 69, 2017.
[26] J. Chen, J. Ye, S. Du, and R. Yong, "Scale effect and anisotropy analyzed for neutrosophic numbers of rock joint roughness coefficient based on neutrosophic statistics," *Symmetry*, vol. 9, no. 10, p. 208, 2017.
[27] J. Chen, J. Ye, S. Du, and R. Yong, "Expressions of rock joint roughness coefficient using neutrosophic interval statistical numbers," *Symmetry*, vol. 9, no. 7, p. 123, 2017.
[28] J. Ye, "Neutrosophic linear equations and application in traffic flow problems," *Algorithms*, vol. 10, no. 4, p. 133, 2017.
[29] J. Ye, "Aggregation operators of neutrosophic linguistic numbers for multiple attribute group decision making," vol. 5, no. 1, , p. 1691, Springer Plus, 2016.
[30] Z. Fang and J. Ye, "Multiple attribute group decision-making method based on linguistic neutrosophic numbers," *Symmetry*, vol. 9, no. 7, p. 111, 2017.
[31] J. Ye, "Multiple attribute decision-making methods based on the expected value and the similarity measure of hesitant neutrosophic linguistic numbers," *Cognitive Computation*, vol. 10, no. 2, pp. 454–463, 2018.
[32] J. Ye, "Neutrosophic number linear programming method and its application under neutrosophic number environments," *Soft Computing*, vol. 22, no. 14, pp. 4639–4646, 2018.
[33] W. Jiang and J. Ye, "Optimal design of truss structures using a neutrosophic number optimization model under an indeterminate environment," *Neutrosophic Sets and Systems*, vol. 14, pp. 93–97, 2016.
[34] F. Smarandache, *Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability*, Sitech & Education Publishing, Craiova, Romania, 2013.
[35] F. Smrandache, *Introduction to Neutrosophic Statistics*, Sitech & Education Publishing, Craiova, Romania, 2014.
[36] J. Ye, "Neutrosophic number linear programming method and its application under neutrosophic number environments," *Soft Computing*, vol. 22, no. 14, pp. 4639–4646, 2018.
[37] J. Ye, W. Cui, and Z. Lu, "Neutrosophic number nonlinear programming problems and their general solution methods under neutrosophic number environments," *Axioms*, vol. 7, p. 13, 2018.
[38] C. Kar, B. Mondal, and T. K. Roy, "An inventory model under space constraint in neutrosophic environment: a neutrosophic geometric programming approach," *Neutrosophic Sets and Systems*, vol. 21, pp. 93–109, 2018.
[39] J. Ye, "An improved neutrosophic number optimization method for optimal design of truss structures," *New Mathematics and Natural Computation*, vol. 14, no. 3, pp. 295–305, 2018.
[40] Y. Hu and Y. Guo, *Operations Research Tutorial*, Tsinghua University Press, Beijing, China, Fourth edition, 2012.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6636029, 12 pages
https://doi.org/10.1155/2021/6636029

Research Article

Single-Valued Neutrosophic DEMATEL for Segregating Types of Criteria: A Case of Subcontractors' Selection

Lazim Abdullah,[1] **Zheeching Ong**,[2] **and Shalela Mohd Mahali**[1]

[1]*Management Science Research Group, Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, Kuala Nerus 21030, Malaysia*
[2]*Public Bank Berhad, Jalan Sultan Yahya Petra, Kota Bharu 15200, Malaysia*

Correspondence should be addressed to Lazim Abdullah; lazim_m@umt.edu.my

Received 8 December 2020; Revised 8 March 2021; Accepted 29 March 2021; Published 16 April 2021

Academic Editor: Parimala Mani

The decision-making trial and evaluation laboratory (DEMATEL) has been used to solve numerous multicriteria decision-making (MCDM) problems, where real numbers are utilised in defining linguistic variables. Although the DEMATEL has shown its success in solving many decision-making problems, researchers have not fully understood how the DEMATEL works on non-real-number linguistic variables. Recent discovery of single-valued neutrosophic sets (SVNSs) can offer a new method to solve decision-making problems, where three memberships of SVNSs are used to define experts' linguistic judgment. This paper aims to propose a novel MCDM method, where SVNSs and the DEMATEL are fully utilised. Different from the DEMATEL, which directly utilises real numbers, this proposed method introduces SVNSs to better deal with truth, indeterminacy, and falsity in solving MCDM problem. As an application of the proposed method, subcontractors' selection problem is investigated using the proposed method, where four types of criteria are developed. A group of experts were invited to provide opinions and linguistic judgment regarding the degree of influence between criteria of subcontractors' selection. The linguistic evaluations defined in SVNSs were computed using the eight-step procedures of the proposed method. Based on the degree of influence, the computational results successfully segregated all ten criteria into four types, in which two to three criteria are grouped in each type. The results also suggest that "Experience" and "Quality" are the most influential criteria in subcontractors' selection. The segregation based on degree of influence would be greatly significant for the practical implementation of the subcontractors' selection.

1. Introduction

The decision-making trial and evaluation laboratory (DEMATEL) method is one of the many multicriteria decision-making (MCDM) methods available in literature. The DEMATEL was initially developed by the Science and Human Affairs Program of the Battelle Memorial Institute of Geneva between 1972 and 1976 to resolve the complicated and intertwined problems. Compared with other MCDM techniques such as the analytic hierarchy process (AHP), where evaluation criteria are independent, this method is one of the structural modelling techniques that can identify the interdependencies of criteria through causality diagram and unidirectional analysis. The causal diagram uses digraphs rather than directionless graphs to portray the basic concept of contextual relationships and the strengths of influence among the elements or criteria [1]. This method has been applied in analysing and developing the cause-and-effect relationship among evaluation criteria [2]. In other words, the DEMATEL is used to derive interrelationship among evaluation criteria or factors [3]. In other words, the DEMATEL is a comprehensive method for developing a basic model that contains causal connections between a number of complex criteria of decision problems. Using the DEMATEL, all evaluation criteria are partitioned into two groups, in which the first group is known as cause group and the second group is called as effect group. Owing to these positive features, the DEMATEL has been successfully applied in many recent decision-making problems (see [4–7]). It is good to note that pairwise comparisons between criteria in DEMATEL are measured using a scale of real numbers accompanied by five linguistic terms.

Despite all these advantages, the linguistic terms used in DEMATEL suffer from several limitations. The linguistic scales based on real numbers are insufficient to provide a good evaluation or judgment because information is regularly exorbitant and, more importantly, many are vague and incomplete. In addition, elicitation of decision-makers' opinions using these linguistic scales could be misconstrued due to the restricted or incomplete information. In fact, the fuzziness in decision-makers' opinions or insufficient knowledge about an issue could make the decision-making process complicated [8]. In response to the limitation in dealing with incomplete information, neutrosophic sets were introduced [9]. A year later, neutrosophic sets were extended to single-valued neutrosophic sets (SVNSs) as to ease their applications to real scientific and engineering areas [10]. With the simplicity of SVNSs, these sets have been assimilated with other scientific knowledge such as aggregation operators, correlation studies, score functions, distance, and similarity measures. Ye [11], for example, presented the correlation coefficient between SVNSs and applied the proposed method to an illustrative example. Peng et al. [12] pointed out that some SVNS operations defined by Ye [11] may also be invalid and they defined novel operations and aggregation operators and applied them to similarity-measures problems. Peng et al. [13] also defined the multivalued neutrosophic sets and proposed two aggregation operators for the sets. Liu and Wang [14] defined a normalised weighted Bonferroni mean aggregation operator of SVNS. Şahin and Küçük [15] proposed the concept of neutrosophic subsethood based on distance measures for SVNSs. Majumdar and Samanta [16] studied the notions of distance and several similarity measures between two SVNSs as well as entropy of a SVNS. A hybrid model of score accuracy functions and SVNS was developed by Mondal and Pramanik [17], where this hybrid model was applied in teacher recruitment. Ye and Fu [18] proposed similarity measures between SVNSs based on tangent function and applied them to medical diagnosis problems. Very recently, Zhao et al. [19] and Tian et al. [20] proposed some new power Heronian aggregation operators for SVNNs and introduced a novel decision-making method using the proposed operators. Garai et al. [21] presented a new ranking method of SVN numbers based on possibility theory for solving a decision-making problem. The concept of possibility mean of SVN numbers was defined and the properties of single-valued trapezoidal neutrosophic (SVTN) numbers were studied. Finally, they developed a new ranking approach using the concept of weighted possibility mean, and Qin and Wang [22] studied the similarity and entropy measures of SVNS by proposing the axiomatic definitions of similarity and entropy for single-valued neutrosophic values (SVNVs) with respect to a new kind of inclusion relation between SVNVs. On the basis of Hamming distance, cosine function, and cotangent function, three similarity measures and three entropies for SVNVs were constructed. Other related researches about SVNS and its application in multicriteria decision-making, matrices operations, and similarity measures can be retrieved from [23, 24] and [25], respectively. It can be seen that all these related researches have discussed the theoretical decision analyses or pattern recognition methods such as similarity measures, entropy, accuracy functions, aggregation operators, and distance measures without really applying to a real case data or experiment.

Turning now to related research of neutrosophic sets integrated with a specific MCDM method, Nabeeh et al. [26], for example, developed an integration of AHP-triangular neutrosophic numbers and applied it to estimate influential factors for a successful IoT enterprise. Abdel-Basset et al. [27] proposed a novel type-2 neutrosophic number-TOPSIS strategy by combining type-2 neutrosophic numbers and TOPSIS for supplier selection. Abdel Basset et al. [28] proposed an integration of bipolar neutrosophic numbers with Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and applied it to medical device selection. However, the extent of integration between specific type of neutrosophic sets and DEMATEL is yet to be fully understood. The following section provides a brief of latest research that elucidates the merging of neutrosophic sets and DEMATEL. These reviews would provide an insight into the research gap between the recently published works and the proposed work.

2. Related Research and Identification of Research Gap

This section summarises the latest research that elucidates the integration of neutrosophic sets and DEMATEL. These reviews highlighted the type of neutrosophic numbers used, the integration of DEMATEL with other methods, and the fields of applications. Table 1 provides the recent researches that were carried out as well as research gaps.

It seems that little information is available on direct integration of SVNS linguistic variables to DEMATEL method. In addition, there were no researches that applied to subcontractors' selection and, more importantly, the absence of quadrant analysis in the analysis of their respective applications. To bridge these research gaps, this paper aims to propose an integration of SVNS and the DEMATEL (SVN-DEMATEL), where linguistic variables defined in SVNS are merged into the DEMATEL procedures. The integration of DEMATEL and SVNS ensued when linguistic variables used are now defined in three independent memberships of SVNS. In the SVN-DEMATEL framework, the eight-step computational procedures are characterised by truth-membership function, indeterminacy-membership function, and falsity-membership function. To illustrate the proposed method, a case of subcontractors' selection is investigated, where a quadrant analysis supplemented the other typical analysis in DEMATEL. Detailed descriptions of the subcontractors' selection problem, related definitions of SVNS, and the proposed SVN-DEMATEL method are presented in the subsequent sections.

Table 1: Summary of related researches and research gaps.

Authors	Type of NS	Method DEMATEL	Application	Research gap
Nabeeh [29]	Neutrosophic sets	DEMATEL method and data envelopment analysis	Technology selection process	Did not consider SVNS -No quadrant analysis
Abdel-Basset et al. [30]	Trapezoidal neutrosophic number	Integration of DEMATEL and TOPSIS	Project selection	An integration method No quadrant analysis
Awang et al. [31]	Left–right neutrosophic numbers	Multiplicative inverse of decision matrix in DEMATEL	Coastal erosion	-Focused on improving inverse matrix No quadrant analysis
Tan and Zhang [32]	Trapezoidal fuzzy neutrosophic	DEMATEL, fuzzy distance, and linear assignment method	Typhoon disaster evaluation	Many other methods were integrated to DEMATEL Did not consider SVNS No quadrant analysis
Tian et al [33]	Single-valued neutrosophic sets	DEMATEL with quality function deployment TODIM	Market segment evaluation and selection	-An integration of DEMATEL with two other methods No quadrant analysis
Feng et al. [34]	Neutrosophic sets	DEMATEL with VIKOR, TOPSIS, and ELECTRE III	Photovoltaic plan selection	An integration of DEMATEL with three other methods No quadrant analysis

3. Preliminaries

In light of the idea of big data as branch of information theory, it is essential to have a tool that can be used for managing vulnerability and irregularity of information. Therefore, Wang et al. [10] coined the concept of SVNS because SVNS is a subclass of the neutrosophic set and is very valuable in engineering application models. To ease the computation of SVNS in real-life applications, theoretical operations between two SVNSs are defined and some fundamental properties of these tasks are studied. This section provides the related definitions of SVNS and its operations.

Definition 1 (see [9]). Let X be a space of points (objects) with generic elements in S denoted by x. A neutrosophic set S in X is characterised by truth-membership function $T_S(x)$, indeterminacy-membership function $I_S(x)$, and falsity-membership function $F_S(x)$. The functions $T_S(x), I_S(x)$ and $F_S(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$. That is, $T_S(x) \longrightarrow]0^-, 1^+[$, $I_S(x) \longrightarrow]0^-, 1^+[$, and $F_S(x) \longrightarrow]0^-, 1^+[$. Thus, there is no restriction on the sum of $T_S(x), I_S(x)$ and $F_S(x)$, so $0^- \le \sup T_S(x) + \sup I_S(x) + \sup F_S(x) \le 3^+$.

Obviously, it is difficult to apply in real scientific and engineering areas because of the nonstandard subsets of neutrosophic set. Hence, Wang et al. [10] introduced the definition of SVNS as follows.

Definition 2 (see [10]). Let X be a space of points (objects) with generic elements in X denoted by x. An SVNS S in X is characterised by truth-membership function $T_S(x)$, indeterminacy-membership function $I_S(x)$, and falsity-membership function $F_S(x)$. Then, an SVNS S can be denoted by $S = \{\langle x, T_S(x), I_S(x), F_S(x)\rangle x \in X\}$, where $T_S(x), I_S(x)$, $F_S(x) \in [0,1]$ for each point x in X. Therefore, the sum of $T_S(x), I_S(x)$ and $F_S(x)$ satisfies the condition $0 \le T_S(x) + I_S(x) + F_S(x) \le 3$.

In decision-making, human language, commonly referred to as linguistic variables, is normally used. Ratings of criteria of decision problems can be expressed using linguistic variables that can be transformed into SVNNs. These SVNNs are a subset or a special case of SVNSs and defined as follows.

Definition 3 (see [10]). If an SVNS S can be denoted by $S = \{\langle x, T_S(x), I_S(x), F_S(x)\rangle x \in X\}$, where $T_S(x), I_S(x)$, $F_S(x) \in [0,1]$ for each point x in X and the sum of $T_S(x), I_S(x)$ and $F_S(x)$ satisfies the condition $0 \le T_S(x) + I_S(x) + F_S(x) \le 3$, for convenience, $\alpha = \langle T_S, I_S F_S\rangle$ to represent a SVNN.

These three membership functions work under specific arithmetic operations. The basic arithmetic operations of SVNNs are defined as follows.

Definition 4 (see [14]). Arithmetic operations between two SVNNs are defined as follows.

Let $x = (T_1, I_1, F_1)$ and $y = (T_2, I_2, F_2)$ be two SVNNs; then the arithmetic operations are defined as follows:

(i) $x \oplus y = (T_1 + T_2 - T_1, T_2, I_1, I_2, F_1, F_2)$
(ii) $x \otimes y = (T_1 T_2, I_1 + I_2 - I_1 I_2, F_1 + F_2 - F_1 F_2)$
(iii) $\lambda x = ((1 - (1 - T_1)^\lambda, I_1^\lambda, F_1^\lambda)$
(iv) $x^\lambda = (T_1^\lambda, 1 - (1 - I_1^\lambda), 1 - (1 - F_1^\lambda))$

Definition 5 (see [10]). If $x = (T_1, I_1, F_1)$ and $y = (T_2, I_2, F_2)$ are two SVNNs, then some properties of set theoretic operators are defined as follows:

(i) Commutative:
$x \cup y = (T_1, I_1, F_1) \cup (T_2, I_2, F_2) = (T_2, I_2, F_2) \cup (T_1, I_1, F_1) = y \cup x$
(ii) Idempotent:
$x \cup x = (T_1, I_1, F_1) \cup (T_1, I_1, F_1) = (T_1, I_1, F_1) = x$,
$y \cap y = (T_1, I_1, F_1) \cap (T_1, I_1, F_1) = (T_1, I_1, F_1) = y$

(iii) Absorption: $x \cup x \cap y = (T_1, I_1, F_1) \cup (T_1, I_1, F_1) \cap (T_2, I_2, F_2) = (T_1, I_1, F_1) = x$

(iv) De Morgan's laws: $k\ (x \cup y) = k\ ((T_1, I_1, F_1) \cap k\ (T_2, I_2, F_2))$; $k\ (x \cap y) = k\ (T_1, I_1, F_1) \cup k\ ((T_2, I_2, F_2))$, where k is a constant

(v) Involution: $k\ (k\ (x)) = k\ (k\ (T_1, I_1, F_1)) = (T_1, I_1, F_1) = x$, where k is a constant

The definitions of complement, union, and intersection of SVNS satisfy most properties of sets. In this paper, the SVNS is integrated with the DEMATEL with some of the above definitions and properties being prevalently used in the computational procedures. Detailed description of this integration is presented in the following section.

4. Proposed SVN-DEMATEL

The algorithm of DEMATEL, proposed by Fontela and Gabus [35] and Gabus and Fontela [36], is used as a basis in proposing the SVN-DEMATEL. Different from DEMATEL where real numbers are used in defining linguistic scales, the proposed method used SVNSs instead. Several new innovations are made in this proposed method compared to the DEMATEL and the existing SVN-DEMATEL. Apart from substitution of real numbers with SVNS, the proposed method also includes relative importance of decision-makers' weight. The importance of each decision-maker is measured using the proportion equation proposed by Boran et al. [37]. Instead of taking equal weights for decision-makers, this proposed method introduced relative weights, where each decision-maker has different weight. Another innovation is the way of transforming SVNN into real numbers. In this proposed method, the concept of average using the equation proposed by Radwan and Fouda [38] is used. The three memberships of SVNS are averaged to obtain a real number. This step would avoid the invalidity of finding multiplicative inverse of matrix in DEMATEL. Detailed discussion of validity of multiplicative inverse matrix can be retrieved from Awang et al. [31]. Different from most of the DEMATEL-based methods where the last computational step is drawing a causal-effect diagram, this proposed method extends with another step to establish four types of criteria. In summary, the framework of the proposed method is illustrated in Figure 1.

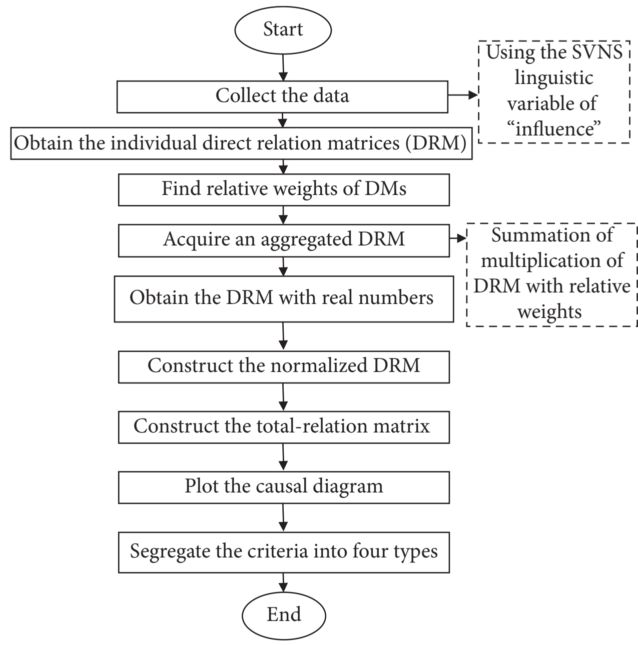


Figure 1: Framework of the proposed method.

This flowchart is translated into stepwise algorithm. Our proposed algorithm of SVN-DEMATEL is presented as follows.

Step 1. Construct direct-relation matrix (DRM).

Each DM judgment is collected and pooled into a direct relation matrix $X_{n\times n}$ (total number of criteria is n) which is an assessment of interrelationship between elements utilising a 5-linguistic rating scale. The table indicates the interrelationship of selection of subcontractors and performances on each other.

Step 2. Find relative weights of decision-makers.

Each decision-maker's judgment has a particular weight that must be considered to determine total averaged crisp matrix. As the work experience and knowledge of decision-makers' fluctuate, we assume distinctive overall weights for decision-makers' opinions in deciding the total averaged crisp matrix. Table 2 shows the linguistic variable used for relative importance weights of decision-makers and its respective SVNN.

Assume that $\lambda_k = (T_k, I_k, F_k)$ is the SVNN for relative importance weights of kth expert. The value of kth expert can be obtained using the following equation:

$$\lambda_k = \frac{T_k(x) + I_k(x)\left(\left(T_k(x)/T_k(x) + F_k(x)\right)\right)}{\sum_{k=1}^{l} T_k(x) + I_k(x)\left(\left(T_k(x)/T_k(x) + F_k(x)\right)\right)}, \qquad (1)$$

where $\lambda_k \geq 0, \sum_{k=1}^{l} \lambda_k = 1$.

Step 3. Construct aggregated DRM.

Each decision-maker's opinions need to be aggregated to assemble a collective neutrosophic set decision matrix. Let $z_{ij}^k = (T_{ij}^k, I_{ij}^k, F_{ij}^k)$ be the SVN given by kth expert on the assessment of criterion i on j. The single-valued neutrosophic set weighted aggregation (SVNSWA) operator is used to aggregate single-valued neutrosophic number rating, and x_{ij} represents the influence level of criterion i on j.

$$\begin{aligned} a_{ij} &= \mathrm{SVNSWA}\left(z_{ij}^1, z_{ij}^2, \ldots, z_{ij}^k\right) \\ &= \sum_{k=1}^{l} \lambda_k z_{ij}^k = \left\langle 1 - \prod_{k=1}^{l}\left(1 - T_j\right)^{w_j}, \prod_{k=1}^{l}\left(I_j\right)^{w_j}, \prod_{k=1}^{l}\left(F_j\right)^{w_j} \right\rangle \\ i &= 1, 2, \ldots, m;\ j = 1, 2, \ldots, n, \end{aligned} \qquad (2)$$

where λ_k is the importance weight of k th expert; a_{ij}^k is corresponding to SVN of k h expert's opinion when comparing i to j.

Table 2: Linguistic variable for relative importance weight of DM [39].

Linguistic variable	SVNN $\langle T, I, F\rangle$
Very important	$\langle 0.90, 0.10, 0.10\rangle$
Important	$\langle 0.80, 0.20, 0.15\rangle$
Medium	$\langle 0.50, 0.40, 0.45\rangle$
Unimportant	$\langle 0.35, 0.60, 0.70\rangle$
Very unimportant	$\langle 0.10, 0.80, 0.90\rangle$

Step 4. Construct DRM with real numbers

Transform the aggregated single neutrosophic relation matrix into real number matrix using the following equation:

$$E(z) = \frac{(3 + T - 2I - F)}{4}. \tag{3}$$

Step 5. Construct normalised DRM.

Calculate the normalised DRM (matrix X) using the following equation:

$$X = k \times A, \tag{4}$$

where

$$k = \min\left(\frac{1}{\max\limits_{1\le i\le n}\sum_{j=1}^{n}\left|a_{ij}\right|}, \frac{1}{\max\limits_{1\le j\le n}\sum_{i=1}^{n}\left|a_{ij}\right|}\right) i, j \in \{1, 2, 3, \ldots, n\}, \tag{5}$$

and A is the normalised DRM.

Step 6. Obtain total-relation matrix (TRM).

The TRM, T, is then calculated using the following equation:

$$T = X(I - X)^{-1}, \tag{6}$$

where I is an identity matrix.

Step 7. Plot causal diagram.

Compute R and D from TRM, T, using equation (7) and equation (8).

Given T,

$$T = \left[t_y\right]_{n\times n}, \quad i, j = 1, 2, \ldots, n, \tag{7}$$

$$\begin{aligned} R &= \left[\sum_{i=1}^{n} t_y\right]_{1\times n} = \left[t_j\right]_{1\times n}, \\ D &= \left[\sum_{j=1}^{n} t_y\right]_{n\times 1} = \left[t_i\right]_{n\times 1}, \end{aligned} \tag{8}$$

where R denotes the total of rows for the matrix and D denotes the total of columns for the matrix.A criterion is considered as a cause-and-effect criterion if (R − D) is positive and (R − D) is negative, respectively.

Step 8. Identify types of criteria.

Coordinates of $(R + D$, R − D) in Cartesian plane are used to segregate criteria into four types.

The proposed eight-step computational procedures are used to establish four types of criteria based on the degree of influence. Detailed implementation of the case of subcontractors' selection is presented in the following section.

4.1. A Case of Subcontractors' Selection. Subcontractors' selection is a critical part of construction or industrial management, where a major challenge is the existence of multiple criteria that the project management team needs to evaluate in the selection process [40, 41]. Subcontractors usually help main contractor to overcome problems related to the need for special expertise, limitation in finances, and shortage in resources. Specialist subcontractor can be utilised, when the main contractor acquires products or administrations, which the main contractor does not deliver or cannot deliver by his own company. Therefore, selecting the deliverable subcontractors is critical in making sure the implementation of the project is successful and completed within the stipulated times.

In solving subcontractors' selection problem, information about criteria, linguistic terms are required other than the algorithm of SVNS-DEMATEL. It is presented in the following sections.

4.2. Criteria, Linguistic Scale, and Decision-Makers. Criteria that influence subcontractors' selection are retrieved from literature (see [42–44]). In this experiment, ten criteria are Price (C_1), Completing on Time (C_2), Experience (C_3), Financial Stability (C_4), Compliance with Regulations (C_5), Quality (C_6), Performance History (C_7), Safety Management (C_8), Timely Payment to Labour (C_9), and Length of Time in Industry (C_{10}). These evaluation criteria are judged by a group of decision-makers using a five-point linguistic scale. The judgments are made in pairwise comparison manner, in which one criterion is compared to the other criteria in terms of degree of influence. Table 3 presents linguistic variable of "influence," five linguistic terms and their respective SVNS.

In this study, five decision-makers denoted as DM1, DM2, DM3, DM4, and DM5, respectively, are assigned to provide pairwise comparative linguistic judgments of criteria using the defined linguistic scale. All decision-makers are experts in selecting subcontractors and currently hold key positions in a construction company. A formal letter was sent to the decision-makers and they were requested to rate a criterion with respect to other criteria in terms of degree of influence of selecting subcontractors using the linguistic scale. Linguistic data obtained from decision-makers are implemented to the proposed SVNS-DEMATEL.

4.3. Implementation. In accordance with the proposed algorithm (see Section 3), the following computations are implemented.

Table 3: Five-point linguistic scale [39].

Linguistic terms	SVNS $\langle T, I, F\rangle$
No influence (NI)	$\langle 0.00, 1.00, 1.00\rangle$
Extremely low influence (ELI)	$\langle 0.20, 0.85, 0.80\rangle$
Low influence (LI)	$\langle 0.40, 0.65, 0.60\rangle$
High influence (HI)	$\langle 0.60, 0.35, 0.40\rangle$
Extremely high influence (EHI)	$\langle 0.80, 0.15, 0.20\rangle$

Step 1: construct DRM

All individual decision-makers' DRM are constructed. Table 4 summarises the judgments of DM1 regarding the influences of the criteria on subcontractors' selection.

Similar DRM matrices are constructed for DM2, DM3, DM4, and DM5. It is good to recall that the linguistic terms in the matrices indicate the interrelationship between criteria in subcontractors' selection.

Step 2: find relative weight of decision-makers

Relative weights of the decisionmakers λ_k are computed using equation (1). They are presented in Table 5.

Step 3: construct aggregated DRM

The aggregated DRM is constructed usingequation (2). For example,

$$\begin{aligned}
a_{11} 1-\prod_{k=1}^{l}\left(1-T_j\right)^{\lambda_k} &= 1-((1-0.00)\wedge 0.2913 * (1-0.00)\wedge 0.2849 * (1-0.00)\wedge 0.2090 * (1-0.00)\wedge 0.1618 * (1-0.00)\wedge 0.0530)\\
&= 0.000,\\
\prod_{k=1}^{l}\left(I_j\right)^{\lambda_k} &= 1\wedge(0.2913) * 1\wedge(0.2849) * 1\wedge(0.2090) * 1\wedge(0.1618) * 1\wedge(0.0530) = 1.000,\\
\prod_{k=1}^{l}\left(F_j\right)^{\lambda_k} &= 1\wedge(0.2913) * 1\wedge(0.2849) * 1\wedge(0.2090) * 1\wedge(0.1618) * 1\wedge(0.0530) = 1.000,\\
a_{21} 1-\prod_{k=1}^{l}\left(1-T_j\right)^{\lambda_k} &= 1-((1-0.60)\wedge 0.2913 * (1-0.80)\wedge 0.2849 * (1-0.60)\wedge 0.2090 * (1-0.20)\wedge 0.1618 * (1-0.60)\wedge 0.0530)\\
&= 0.6327,\\
\prod_{k=1}^{l}\left(I_j\right)^{\lambda_k} &= 0.35\wedge(0.2913) * 0.15\wedge(0.2849) * 0.35\wedge(0.2090) * 0.85\wedge(0.1618) * 0.35\wedge(0.0530)\\
&= 0.3174,\\
\prod_{k=1}^{l}\left(F_j\right)^{\lambda_k} &= 0.40\wedge(0.2913) * 0.20\wedge(0.2929) * 0.40\wedge(0.2090) * 0.80\wedge(0.1618) * 0.40\wedge(0.0530)\\
&= 0.3673,
\end{aligned} \tag{9}$$

Part of the aggregated DRM is shown in Table 6.

Step 4: construct DRM with real numbers

Transform the aggregated SVNS matrix into aggregated real number DRM using equation (3).

Table 4: Judgments of criteria (DM1).

Criteria	C1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C1		HI	NI	ELI	ELI	HI	ELI	NI	NI	NI
C2	LI		ELI	NI	NI	ELI	LI	ELI	ELI	LI
C3	HI	HI		ELI	ELI	HI	LI	NI	ELI	HI
...	...	...	...	...	...	...	...	...	...	...
C_{10}	ELI	LI	HI	NI	NI	ELI	LI	ELI	NI	LI

Table 5: Relative weights of decision-makers.

Decision-makers	DM1	DM2	DM3	DM4	DM5
Lambda, λ_k	0.2913	0.2849	0.2090	0.1618	0.0530

Table 6: Aggregated DRM.

Criteria	C_1	C_2	C_3	...	C_8	C_9	C_{10}
C_1	$\langle 0.0000, 1.0000, 1.0000 \rangle$	$\langle 0.5742, 0.3947, 0.4258 \rangle$	$\langle 0.2566, 0.7754, 0.7434 \rangle$	...	$\langle 0.2630, 0.7875, 0.7370 \rangle$	$\langle 0.6289, 0.3296, 0.3711 \rangle$	$\langle 0.2566, 0.7754, 0.7434 \rangle$
C_2	$\langle 0.6327, 0.3174, 0.3673 \rangle$	$\langle 0.0000, 1.0000, 1.0000 \rangle$	$\langle 0.3475, 0.6867, 0.6525 \rangle$	...	$\langle 0.2643, 0.7861, 0.7357 \rangle$	$\langle 0.5269, 0.4592, 0.4731 \rangle$	$\langle 0.5777, 0.3796, 0.4223 \rangle$
C_3	$\langle 0.7611, 0.1902, 0.2389 \rangle$	$\langle 0.7925, 0.1569, 0.2075 \rangle$	$\langle 0.0000, 1.0000, 1.0000 \rangle$	...	$\langle 0.5759, 0.3861, 0.4241 \rangle$	$\langle 0.4444, 0.5577, 0.5556 \rangle$	$\langle 0.6622, 0.2870, 0.3378 \rangle$
...	...	...	...		...	...	...
C_{10}	$\langle 0.6424, 0.3184, 0.3576 \rangle$	$\langle 0.6553, 0.2918, 0.3447 \rangle$	$\langle 0.7522, 0.1989, 0.2478 \rangle$	...	$\langle 0.3908, 0.6593, 0.6092 \rangle$	$\langle 0.4130, 0.5718, 0.5870 \rangle$	$\langle 0.0000, 1.0000, 1.0000 \rangle$

For C_1, the computations are

$$\begin{aligned}
a_{11} &= \frac{(3+0.0000-2*1.0000-1.0000)}{4} = 0.0000,\\
a_{21} &= \frac{(3+0.6327-2*0.3174-0.3673)}{4} = 0.6577,\\
a_{31} &= \frac{(3+0.7611-2*0.1902-0.2389)}{4} = 0.7855,\\
a_{41} &= \frac{(3+0.4444-2*0.5577-0.5556)}{4} = 0.4433,\\
a_{51} &= \frac{(3+0.4610-2*0.5185-0.5390)}{4} = 0.4713,\\
a_{61} &= \frac{(3+0.7688-2*0.1791-0.2312)}{4} = 0.7949\\
a_{71} &= \frac{(3+0.6731-2*0.2735-0.3269)}{4} = 0.6998,\\
a_{81} &= \frac{(3+0.6150-2*0.3455-0.3850)}{4} = 0.6347,\\
a_{91} &= \frac{(3+0.4405-2*0.5406-0.5595)}{4} = 0.4499,\\
a_{101} &= \frac{(3+0.6424-2*0.3184-0.3576)}{4} = 0.6620,
\end{aligned} \tag{10}$$

This matrix is presented in Table 7.

Step 5: construct normalised DRM

In order to construct normalised DRM, summation of rows and summation of columns of DRM are computed first. The summation of rows and summation of columns are shown in Table 8.

The maximum numbers from summation of rows and summation of columns are identified (bold). With these maximum numbers, k is calculated using equation (4).

$$k = 0.1586. \tag{11}$$

The DRM in Table 6 is normalised by multiplying with k.

The maximum numbers from summation of rows and summation of columns have been chosen, respectively, as

$$\begin{aligned}
k &= \min\left(\frac{1}{5.8834}, \frac{1}{6.3056}\right)\\
&= \min(0.1700, 0.1586)\\
&= 0.1586.
\end{aligned} \tag{12}$$

Multiply the Direct-Relation Matrix with k to normalise it.

Table 7: Aggregated DRM with real numbers.

Criteria	C_1	C_2	C_3	...	C_8	C_9	C_{10}
C_1	0.0000	0.5898	0.2406	...	0.2378	0.6496	0.2406
C_2	0.6577	0.0000	0.3304	...	0.2391	0.5339	0.5991
C_3	0.7855	0.8178	0.0000	...	0.5949	0.4433	0.6876
C_4	0.4433	0.7484	0.1870	...	0.5345	0.6818	0.2378
C_5	0.4713	0.7333	0.3295	...	0.7738	0.6152	0.5373
C_6	0.7949	0.7727	0.5391	...	0.5019	0.3798	0.4959
C_7	0.6998	0.6797	0.7018	...	0.4959	0.5165	0.5535
C_8	0.6347	0.6922	0.4657	...	0.0000	0.2489	0.4250
C_9	0.4499	0.5901	0.4235	...	0.2300	0.0000	0.3919
C_{10}	0.6620	0.6818	0.7766	...	0.3657	0.4206	0.0000

Table 8: Summation of rows and columns.

Criteria	Summation of rows	Summation of columns
C_1	4.1834	5.5991
C_2	4.1228	6.3056
C_3	5.8834	3.9943
C_4	3.8961	4.0049
C_5	4.8499	3.9644
C_6	5.4122	4.3552
C_7	5.0220	5.4200
C_8	4.6333	3.9735
C_9	3.4352	4.4896
C_{10}	4.8369	4.1686

C_1:

$$\begin{aligned}
a_{11} &= 0.0000 * 0.1586 = 0.0000,\\
a_{21} &= 0.5898 * 0.1586 = 0.0935,\\
a_{31} &= 0.2406 * 0.1586 = 0.0382,\\
a_{41} &= 0.6508 * 0.1586 = 0.1032,\\
a_{51} &= 0.4206 * 0.1586 = 0.0667,\\
a_{61} &= 0.7855 * 0.1586 = 0.1246,\\
a_{71} &= 0.3682 * 0.1586 = 0.0584,\\
a_{81} &= 0.2378 * 0.1586 = 0.0377,\\
a_{91} &= 0.6496 * 0.1586 = 0.1030,\\
a_{101} &= 0.2406 * 0.1586 = 0.0382.
\end{aligned} \tag{13}$$

The normalised DRM is shown in Table 9.

The ten-by-ten matrix represents the normalised DRM with real numbers.

Step 6: obtain total-relation matrix (TRM)

The TRM is obtained using equation (6). This is the matrix obtained as a result of multiplicative inverse of DRM with differences of identity matrix and DRM.

Table 9: Normalised DRM.

Criteria	C_1	C_2	C_3	...	C_8	C_9	C_{10}
C_1	0.0000	0.0935	0.0382	...	0.0377	0.1030	0.0382
C_2	0.1043	0.0000	0.0524	...	0.0379	0.0847	0.0950
C_3	0.1246	0.1297	0.0000	...	0.0943	0.0703	0.1090
C_4	0.0703	0.1187	0.0297	...	0.0848	0.1081	0.0377
C_5	0.0747	0.1163	0.0523	...	0.1227	0.0976	0.0852
C_6	0.1261	0.1225	0.0855	...	0.0796	0.0602	0.0786
C_7	0.1110	0.1078	0.1113	...	0.0786	0.0819	0.0878
C_8	0.1007	0.1098	0.0739	...	0.0000	0.0395	0.0674
C_9	0.0714	0.0936	0.0672	...	0.0365	0.0000	0.0621
C_{10}	0.1050	0.1081	0.1232	...	0.0580	0.0667	0.0000

For example, total-relation matrix can be found by multiplying X with $(I - X)^{-1}$.

$$\begin{aligned}
a_{11} = {} & 0.0000(1.2174) + 0.0935(0.3262)\\
& + 0.0382(0.1934) + 0.1032(0.2559)\\
& + 0.0667(0.2172) + 0.1246(0.2854)\\
& + 0.0584(0.2620) + 0.0377(0.1899)\\
& + 0.1030(0.2751) + 0.0382(0.2020)\\
= {} & 0.2174.
\end{aligned} \tag{14}$$

Table 10 shows the TRM.

Step 7: plot causal diagram

Cause-and-effect diagram is obtained by calculating the sum of rows, R, and the sum of columns, D. These two sums are used to compute $R + D$ and R − D values.

We have the following example.

Summation of rows:

$$\begin{aligned}
C_1 = {} & 0.2174 + 0.3262 + 0.1934 + 0.2559 + 0.2172\\
& + 0.2854 + 0.2620 + 0.1899 + 0.2751 + 0.2020\\
= {} & 2.4247.
\end{aligned} \tag{15}$$

Summation of columns:

$$\begin{aligned}
C_1 = {} & 0.2174 + 0.3147 + 0.4201 + 0.2660 + 0.3248\\
& + 0.3954 + 0.3646 + 0.3402 + 0.2512 + 0.3548\\
= {} & 3.2492.
\end{aligned} \tag{16}$$

Table 11 presents these values according to criteria.

The results can be obtained by mapping the data set of $(R + D, R - D)$ into Cartesian plane, in which the

Table 10: TRM.

Criteria	C_1	C_2	C_3	...	C_8	C_9	C_{10}
C_1	0.2174	0.3262	0.1934	...	0.1899	0.2751	0.2020
C_2	0.3147	0.2411	0.2108	...	0.1897	0.2588	0.2531
C_3	0.4201	0.4540	0.2236	...	0.3014	0.3180	0.3324
C_4	0.2660	0.3276	0.1751	...	0.2164	0.2645	0.1896
C_5	0.3248	0.3843	0.2372	...	0.2886	0.2963	0.2733
C_6	0.3954	0.4205	0.2819	...	0.2712	0.2905	0.2862
C_7	0.3646	0.3885	0.2918	...	0.2582	0.2934	0.2820
C_8	0.3402	0.3718	0.2483	...	0.1746	0.2427	0.2522
C_9	0.2512	0.2886	0.1964	...	0.1646	0.1526	0.1995
C_{10}	0.3548	0.3823	0.2988	...	0.2365	0.2751	0.1979

Table 11: R, D, $R+D$, and R − D for criteria.

Criteria	R	D	$R+D$	$R-D$
C_1	2.4247	3.2492	5.6739	−0.8245
C_2	2.4327	3.5848	6.0176	−1.1521
C_3	3.4056	2.3573	5.7628	1.0483
C_4	2.2383	2.3834	4.6218	−0.1451
C_5	2.8249	2.3416	5.1665	0.4833
C_6	3.1288	2.5993	5.7281	0.5295
C_7	2.9301	3.0943	6.0244	−0.1642
C_8	2.7429	2.2910	5.0339	0.4519
C_9	2.0477	2.6672	4.7149	−0.6195
C_{10}	2.8608	2.4683	5.3291	0.3925

performance of each criterion of the entire subcontractors' selection system can be measured or interpreted.

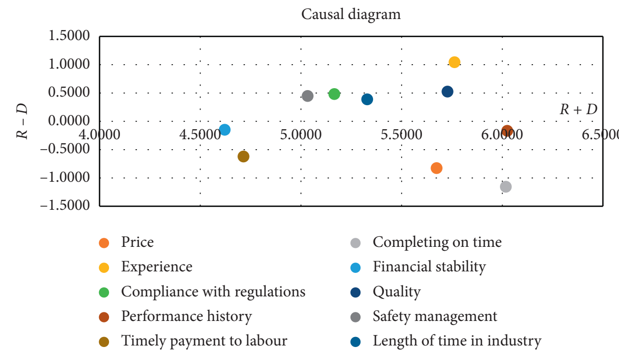


Figure 2: Causal diagram of criteria.

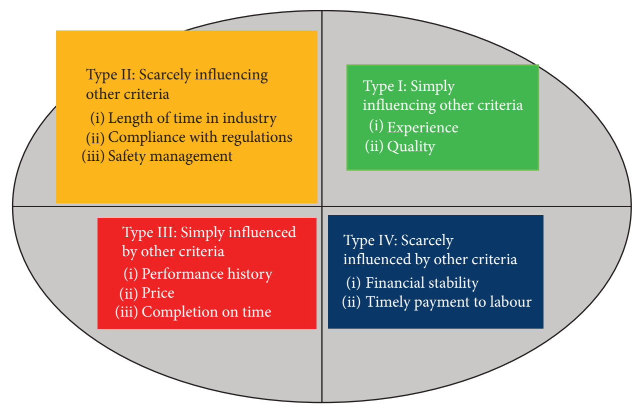


Figure 3: Quadrant analysis.

5. Results

The $(R+D)$ and (R − D) values are translated into a causal diagram. Figure 2 shows the causal diagram where cause group and effect group of criteria are separated by $R+D$ axis.

The above causal diagram visualises the cause criteria and the effect criteria. The cause criteria are Experience, Quality, Length of Time in Industry, Compliance with Regulations, and Safety Management as their values of (R − D) are positives. On the other hand, the effect criteria are Financial Stability, Performance History, Timely Payment to Labour, Price, and Completing on Time as their values of (R − D) are negatives. It is suggested that the criteria in cause group ought to be given priority as these criteria influence other criteria in suggesting the best subcontractors. This result also indicates that "Experience" is the most influential factor in subcontractors' selection owing to the largest value of (R − D).

Step 8: identify types of criteria

The interpretation of this diagram can be further made based on the coordinates of $(R+D$, R − D). Tsai et al. [45] suggest that criteria can be divided into four types. In this analysis, all criteria are mapped into four quadrants based on the coordinates of $(R+D$, R − D). The first type is ensued when (R − D) is positive and $(R+D)$ is large. This indicates that the criteria are the cause criteria which are also driving cause for solving problems. Therefore, the criterion "Experience" is the most important cause in influencing subcontractors' selection. The second type happens when (R − D) is positive and $(R+D)$ is small. This indicates that the criteria are independent and can influence only a few other criteria. In this subcontractors' selection, the criterion "Safety Management" is an independent criterion and does not influence other criteria much. The third type is ensued when (R − D) is negative and $(R+D)$ is large. This indicates that the criteria are effect-type in which can be directly improved. The criterion "Completing on Time" is an effect criterion, where it depends heavily on other criteria. Finally, the interpretation can be made when (R − D) is negative and $(R+D)$ is small. This indicates that the criteria are independent and hardly influenced by other criteria. In the case of subcontractors' selection, the criterion "Financial Stability" is seen as an independent criterion. Summarily, these types of criteria and their respective criteria of subcontractors' selectors are divided into four quadrants.

Figure 3 depicts the quadrant analysis in which four types of criteria are identified.

Looking at the results from the two figures, it is shown that "Experience" and "Quality" are the driving factors of influencing the selection of subcontractors. Therefore, subcontractors who had vast experience and produced quality works would have an advantage to be chosen as subcontractors. This result is different from that of [46] which suggested that "on-time delivery of materials," "failure to complete contract," and "reputation" are the most influencing criteria. Perhaps the different research frameworks used in these studies contributed to the different results.

6. Conclusions

Multicriteria decision-making methods under neutrosophic environment are an active research area and many relevant integration methods have been investigated over the years. However, real applicability of the decision-making methods can be achieved when the detailed integration of the decision-making method and neutrosophic sets is well understood. In this paper, an extended neutrosophic set is integrated with a decision-making method to gain better understanding about the use of neutrosophic sets in decision-making. The SVNS was proposed to substitute the neutrosophic sets due to its complexity in computations, particularly in real scientific and engineering case applications. The SVNS also has no direct integration with causal analysis decision-making methods such as DEMATEL despite the advantages of its three memberships in dealing with indeterminacy information. This paper proposed the SVNS-DEMATEL method, where the real numbers in DEMATEL are substituted with SVNN. This proposed method is applied to subcontractors' selection, where ten criteria are evaluated. The aim of the proposed method is a plot of causal diagram. In this paper, we identified the cause criteria and the effect criteria that could be used in subcontractors' selection. Truth membership, indeterminacy membership, and falsity membership of SVNS provide a comprehensive evaluation of criteria, in which all criteria are successfully separated into two groups. The proposed SVN-DEMATEL method is a valuable instrument to decide the key criteria that could become cause criteria and effect criteria. The experimental results show that the proposed method can successfully capture the important result of decision-making, where the criteria "Experience" and "Quality" are the main causes that need to be highly considered in subcontractors' selection, while "Completing on Time" is a criterion that has no effect in subcontractors' selection. Differentiating the important criteria while choosing subcontractors would really help the main contractor in ensuring the success of construction projects.

The contributions of this paper are fivefold:

(1) We propose using relative weights of decision-makers based on three memberships of SVNS instead of considering equal weights among the five decision-makers. The proposed method uses a proportion equation that makes the weights of decision-makers more suitable for real-life application.

(2) We propose using a weighted averaging operator to find aggregated direct relation matrix, where a series of multiplications of assessment scales and relative weights of decision-makers are accounted.

(3) We propose introducing a transformation equation instead of typical averaged defuzzification method to transform three memberships of SVNS to single real numbers.

(4) We propose an extension to the computational procedures of DEMATEL, where all criteria under investigation are segregated into four types based on degree of influence.

(5) We extend the analysis in the application part with quadrant analysis, where all criteria are mapped onto one of the four quadrants. This analysis is in addition to the causal diagram, which is typically used in the analysis of DEMATEL. These five contributions are embedded in the proposed SVNS-DEMATEL, in which ten criteria of subcontractors' selection are segregated into four types. In future studies, we would like to extend the SVN-DEMATEL beyond the scope of causal diagram. As the SVN-DEMATEL can effectively identify the criteria, the two obtained groups of criteria contain useful information about which criteria specifically influenced other criteria. These unidirectional relationships can be explored as part of future research direction.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this manuscript and read and approved the final manuscript.

Acknowledgments

The authors would like to acknowledge the Centre of Research and Innovation Management of University Malaysia Terengganu for providing excellent research and academic environments. This study was funded by the Fundamental Research Grant Scheme, FRGS/1/2018/STG06/UMT/01/1, vote no. 59522, Ministry of Higher Education Malaysia, and University Malaysia Terengganu.

References

[1] M. L. Tseng and Y. H. Lin, "Application of fuzzy DEMATEL to develop a cause and effect model of municipal solid waste management in Metro Manila," *Environmental Monitoring and Assessment*, vol. 158, no. 1-4, p. 519, 2009.

[2] Z. Yang, Y. Shi, and H. Yan, "Scale, congestion, efficiency and effectiveness in e-commerce firms," *Electronic Commerce Research and Applications*, vol. 20, pp. 171–182, 2016.

[3] K.-P. Lin, M.-L. Tseng, and P.-F. Pai, "Sustainable supply chain management using approximate fuzzy DEMATEL method," *Resources, Conservation and Recycling*, vol. 128, pp. 134–142, 2018.

[4] R. Gholamnia, M. Ebrahimian, S. B. Gendeshmin, R. Saeedi, and S. Firooznia, "Effective factors on the occurrence of falling from height accidents in construction projects by using DEMATEL method," *Advances in Intelligent Systems and Computing*, vol. 819, pp. 293–305, 2019.

[5] K. Sivakumar, R. Jeyapaul, K. Vimal, and P. Ravi, "A DEMATEL approach for evaluating barriers for sustainable end-of-life practices," *Journal of Manufacturing Technology Management*, vol. 29, no. 6, pp. 1065–1091, 2018.
[6] F. Altuntas and M. S. Gok, "The effect of COVID-19 pandemic on domestic tourism: a DEMATEL method analysis on quarantine decisions," *International Journal of Hospitality Management*, vol. 92, Article ID 102719, 2021.
[7] H. Asefi, Y. Zhang, S. Lim, M. Maghrebi, and S. Shahparvari, "A multi-criteria decision support framework for municipal solid waste landfill siting: a case study of New South Wales (Australia)," *Environmental Monitoring and Assessment*, vol. 192, no. 11, Article ID 682, 2020.
[8] G. Büyüközkan, S. Güleryüz, and B. Karpak, "A new combined IF-DEMATEL and IF-ANP approach for CRM partner evaluation," *International Journal of Production Economics*, vol. 191, pp. 194–206, 2017.
[9] F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic*, pp. 1–141, American Research Press, Champaign, IL, USA, 1999.
[10] H. Wang, F. Smarandache, Y. Zhang et al., "Single valued neutrosophic sets," *Multispace and Multistructure*, vol. 2010, no. 4, pp. 410–413, 2010.
[11] J. Ye, "Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment," *International Journal of General Systems*, vol. 42, no. 4, pp. 386–394, 2013.
[12] J.-j. Peng, J.-q. Wang, X.-h. Wu, J. Wang, and X.-h. Chen, "Multi-valued neutrosophic sets and power aggregation operators with their applications in multi-criteria group decision-making problems," *International Journal of Computational Intelligence Systems*, vol. 8, no. 2, pp. 345–363, 2014.
[13] J.-j. Peng, J.-q. Wang, J. Wang, H.-y. Zhang, and X.-h. Chen, "Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems," *International Journal of Systems Science*, vol. 47, no. 10, pp. 2342–2358, 2015.
[14] P. Liu and Y. Wang, "Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean," *Neural Computing and Applications*, vol. 25, no. 7-8, pp. 2001–2010, 2014.
[15] R. Şahin and A. Küçük, "Subsethood measure for single valued neutrosophic sets," *Journal of Intelligent & Fuzzy Systems*, vol. 29, no. 2, pp. 525–530, 2015.
[16] P. Majumdar and S. K. Samanta, "On similarity and entropy of neutrosophic sets," *Journal of Intelligent & Fuzzy Systems*, vol. 26, no. 3, pp. 1245–1252, 2014.
[17] K. Mondal and S. Pramanik, "Multi-criteria group decision making approach for teacher recruitment in higher education under simplified neutrosophic environment," *Global Journal of Engineering Science and Research Management*, vol. 6, pp. 28–34, 2014.
[18] J. Ye and J. Fu, "Multi-period medical diagnosis method using a single valued neutrosophic similarity measure based on tangent function," *Computer Methods and Programs in Biomedicine*, vol. 123, pp. 142–149, 2016.
[19] S. Zhao, D. Wang, C. Liang, Y. Leng, and J. Xu, "Some single-valued neutrosophic power Heronian aggregation operators and their application to multiple-attribute group decision-making," *Symmetry*, vol. 11, no. 5, Article ID 653, 2019.
[20] C. Tian, J. J. Peng, Z. Q. Zhang, M. Goh, and J. Q. Wang, "A multi-criteria decision-making method based on single-valued neutrosophic partitioned heronian mean operator," *Mathematics*, vol. 8, no. 7, Article ID 1189, 2020.
[21] T. Garai, H. Garg, and T. K. Roy, "A ranking method based on possibility mean for multi-attribute decision making with single valued neutrosophic numbers," *Journal of Ambient Intelligence and Humanized Computing*, vol. 11, no. 11, pp. 5245–5258, 2020.
[22] K. Qin and L. Wang, "New similarity and entropy measures of single-valued neutrosophic sets with applications in multi-attribute decision making," *Soft Computing*, vol. 24, no. 21, pp. 16165–16176, 2020.
[23] F. Karaaslan and F. Hunu, "Type-2 single-valued neutrosophic sets and their applications in multi-criteria group decision making based on TOPSIS method," *Journal of Ambient Intelligent Human Computing*, vol. 11, pp. 4113–4132, 2020.
[24] F. Karaaslan and K. Hayat, "Some new operations on single-valued neutrosophic matrices and their applications in multi-criteria group decision making," *Applied Intelligence*, vol. 48, no. 12, pp. 4594–4614, 2018.
[25] F. Karaaslan, "Multicriteria decision-making method based on similarity measures under single-valued neutrosophic refined and interval neutrosophic refined environments," *International Journal of Intelligent Systems*, vol. 33, no. 5, pp. 928–952, 2018.
[26] N. A. Nabeeh, M. Abdel-Basset, H. A. El-Ghareeb, M. Abdel-Basset, H. A. El-Ghareeb, and A. Aboelfetouh, "Neutrosophic multi-criteria decision making approach for IoT-based enterprises," *IEEE Access*, vol. 7, pp. 59559–59574, 2019.
[27] M. Abdel-Basset, M. Saleh, A. Gamal, and F. Smarandache, "An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number," *Applied Soft Computing*, vol. 77, pp. 438–452, 2019.
[28] M. Abdel-Basset, G. Manogaran, and A. Gamal, "A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection," *Journal of Medical Systems*, vol. 43, p. 38, 2019.
[29] N. A. Nabeeh, "A hybrid neutrosophic approach of DEMATEL with AR-DEA in technology selection," *Neutrosophic Sets and Systems*, vol. 31, pp. 17–30, 2020.
[30] M. Abdel-Basset, A. Atef, and F. Smarandache, "A hybrid neutrosophic multiple criteria group decision making approach for project selection," *Cognitive Systems Research*, vol. 57, pp. 216–227, 2019.
[31] A. Awang, N. Aizam, and L. Abdullah, "An integrated decision-making method based on neutrosophic numbers for investigating factors of coastal erosion," *Symmetry*, vol. 11, no. 3, p. 328, 2019.
[32] R. Tan and W. Zhang, "Multiple attribute decision making method based on DEMATEL and fuzzy distance of trapezoidal fuzzy neutrosophic numbers and its application in typhoon disaster evaluation," *Journal of Intelligent & Fuzzy Systems*, vol. 39, no. 3, pp. 3413–3439, 2020.
[33] Z.-P. Tian, J.-Q. Wang, and H.-Y. Zhang, "Hybrid single-valued neutrosophic MCGDM with QFD for market segment evaluation and selection," *Journal of Intelligent & Fuzzy Systems*, vol. 34, no. 1, pp. 177–187, 2018.
[34] J. Feng, M. Li, and Y. Li, "Study of decision framework of shopping mall photovoltaic plan selection based on DEMATEL and ELECTRE III with symmetry under neutrosophic set environment," *Symmetry*, vol. 10, no. 5, p. 150, 2018.

[35] E. Fontela and A. Gabus, *The DEMATEL Observer, DEMATEL 1976 Report*, Battelle Geneva Research Center, Switzerland, Geneva, 1976.

[36] A. Gabus and E. Fontela, *World Problems, an Invitation to Further Thought within the Framework of DEMATEL*, Battelle Geneva Research Center, Geneva, Switzerland, 1972.

[37] F. E. Boran, S. Genç, M. Kurt, and D. Akay, "A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method," *Expert Systems with Applications*, vol. 36, no. 8, pp. 11363–11368, 2009.

[38] A. G. Radwan and M. E. Fouda, "On the mathematical modeling of memristor, memcapacitor, and meminductor," *Studies in Systems, Decision and control*, Vol. 26, Springer, Berlin, Germany, 2016.

[39] P. Biswas, S. Pramanik, and B. C. Giri, "TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment," *Neural Computing and Applications*, vol. 27, no. 3, pp. 727–737, 2016.

[40] H. Abbasianjahromi, M. Sepehri, and O. Abbasi, "A decision-making framework for subcontractor selection in construction projects," *Engineering Management Journal*, vol. 30, no. 2, pp. 141–152, 2018.

[41] M. Keshavarz-Ghorabaee, M. Amiri, E. K. Zavadskas, Z. Turskis, and J. Antucheviciene, "A dynamic fuzzy approach based on the EDAS method for multi-criteria subcontractor evaluation," *Information (Switzerland)*, vol. 9, no. 3, Article ID 68, 2018.

[42] H. Doloi, "Analysis of pre-qualification criteria in contractor selection and their impacts on project success," *Construction Management and Economics*, vol. 27, no. 12, pp. 1245–1263, 2009.

[43] M. M. Marzouk, A. A. El Kherbawy, and M. Khalifa, "Factors influencing subcontractors selection in construction projects," *HBRC Journal*, vol. 9, no. 2, pp. 150–158, 2013.

[44] T. S. Ng, M. Skitmore, and W. F. Chung, "Ten basic factors to identify suitable subcontractors for construction projects," *CIB-International Council for Research and Innovation in Building and Construction*, pp. 1–8, 2003.

[45] S. B. Tsai, M. F. Chien, Y. Xue et al., "Using the Fuzzy DEMATEL to Determine Environmental Performance: A Case of Printed Circuit Board Industry in Taiwan," *PLoS One*, vol. 10, no. 6, Article ID e0129153, 2015.

[46] H. A. El-khalek, R. F. Aziz, and E. S. Morgan, "Identification of construction subcontractor prequalification evaluation criteria and their impact on project success," *Alexandria Engineering Journal*, vol. 58, no. 1, pp. 217–223, 2019.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6649349, 7 pages
https://doi.org/10.1155/2021/6649349

Research Article

Applications of (Neutro/Anti)sophications to Semihypergroups

A. Rezaei,[1] **F. Smarandache**,[2] **and S. Mirvakili**[1]

[1]*Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran*
[2]*Department of Mathematics and Science, University of New Mexico, Gallup, New Mexico 87301, USA*

Correspondence should be addressed to A. Rezaei; rezaei@pnu.ac.ir

Received 17 December 2020; Revised 18 February 2021; Accepted 25 February 2021; Published 30 March 2021

Academic Editor: Parimala Mani

In this paper, we extend the notion of semi-hypergroups (resp. hypergroups) to neutro-semihypergroups (resp. neutro-hypergroups). We investigate the property of anti-semihypergroups (resp. anti-hypergroups). We also give a new alternative of neutro-hyperoperations (resp. anti-hyperoperations), neutro-hyperoperation-sophications (resp. anti-hypersophications). Moreover, we show that these new concepts are different from classical concepts by several examples.

1. Introduction

A hypergroup, as a generalization of the notion of a group, was introduced by F. Marty [1] in 1934. The first book in hypergroup theory was published by Corsini [2]. Nowadays, hypergroups have found applications to many subjects of pure and applied mathematics, for example, in geometry, topology, cryptography and coding theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets and automata theory, physics, and also in biological inheritance [3–7]. The first book in semi-hypergroup theory was published by Davvaz in 2016 (see [8]). In recent years, several other valuable books in hyperstructures have been written by Davvaz et al. [6, 9, 10].

M. Al-Tahan et al. introduced the Corsini hypergroup and studied its properties as a special hypergroup that was defined by Corsini. They investigated a necessary and sufficient condition for the productional hypergroup to be a Corsini hypergroup, and they characterized all Corsini hypergroups of orders 2 and 3 up to isomorphism [3]. Semi-hypergroup, hypergroup, and fuzzy hypergroup of order 2 are enumerated in [7, 11, 12]. S. Hoskova-Mayerova et al. used the fuzzy multisets to introduce the concept of fuzzy multi-hypergroups as a generalization of fuzzy hypergroups, defined the different operations on fuzzy multi-hypergroups, and extended the fuzzy hypergroups to fuzzy multi-hypergroups [13].

In 2019 and 2020, within the field of neutrosophy, Smarandache [14–16] generalized the classical algebraic structures to neutroalgebraic structures (or neutroalgebras) (whose operations and axioms are partially true, partially indeterminate, and partially false) as extensions of partial algebra and to antialgebraic structures (or antialgebras) (whose operations and axioms are totally false). Furthermore, he extended any classical structure, no matter what field of knowledge, to a neutrostructure and an antistructure. These are new fields of research within neutrosophy. Smarandache in [16] revisited the notions of neutroalgebras and anti-algebras, where he studied partial algebras, universal algebras, effect algebras, and Boole's partial algebras and showed that neutroalgebras are the generalization of partial algebras. Also, with respect to the classical hypergraph (that contains hyperedges), Smarandache added the supervertices (a group of vertices put together to form a supervertex), in order to form a super-hypergraph. Then, he extended the super-hypergraph to n-super-hypergraph, by extending the power set $P(V)$ to $P^n(V)$ that is the n-power set of the set V (the n-super-hypergraph, through its n -super-hypergraph-vertices and n -superhypergraph-edges that belong to $P^n(V)$, can be the best (so far) to model our complex and sophisticated reality). Furthermore, he extended the classical hyperalgebra to n-ary hyperalgebra and its alternatives n -ary neutro-hyperalgebra and n -ary anti-hyperalgebra [17]. The notion of neutrogroup was defined and studied by Agboola in [18].

Recently, M. Al-Tahan et al. studied neutro-ordered algebra and some related terms such as neutro-ordered subalgebra and neutro-ordered homomorphism in [19].

In this paper, the concept of neutro-semihypergroup and anti-semihypergroup is formally presented. And, new alternatives are introduced, such as neutro-hyperoperations (resp. anti-hyperoperations), neutro-hyperaxioms, and anti-hyperaxioms. We show that these definitions are different from classical definitions by presenting several examples. Also, we enumerate neutro-hypergroup and anti-hypergroup of order 2 (see Table 1) and obtain some known results (see Table 2).

2. Preliminaries

In this section, we recall some basic notions and results regarding hyperstructures.

Definition 1 (see [2, 8]). A hypergroupoid $(H, \circ)$ is a nonempty set H together with a map $\circ\colon H\times H \longrightarrow P^*(H)$ called (binary) hyperoperation, where $P^*(H)$ denotes the set of all nonempty subsets of H. The hyperstructure $(H, \circ)$ is called a hypergroupoid, and the image of the pair (x, y) is denoted by $x\circ y$.

If A and B are nonempty subsets of H and $x\in H$, then by $A\circ B$, $A\circ x$, and $x\circ B$ we mean $A\circ B = \cup_{a\in A, b\in B} a\circ b$, $A\circ x = A\circ\{x\}$, and $x\circ B = \{x\}\circ B$.

Definition 2 (see [2, 8]). A hypergroupoid $(H, \circ)$ is called a semi-hypergroup if it satisfies the following:

(A) $(\forall a,b,c\in H)\ (a\circ(b\circ c) = (a\circ b)\circ c)$ (associativity).

Definition 3 (see [2, 8]). A hypergroupoid $(H, \circ)$ is called a quasi-hypergroup if reproduction axiom is valid. This means that, for all a of H, we have

(R) $(\forall a\in H)\ (H\circ a = a\circ H = H)$ (i.e. $(\forall a,b\in H)$ $(\exists c,d\in H)$ s.t. $b\in c\circ a, b\in a\circ d$).

Definition 4 (see [2, 8]). A hypergroupoid $(H, \circ)$ which is both a semi-hypergroup and a quasi-hypergroup is called a hypergroup.

Example 1 (see [2, 8])

(i) Let H be a nonempty set, and for all $x, y\in H$, we define $x\circ y = H$. Then, $(H, \circ)$ is a hypergroup, called the total hypergroup.

(ii) Let G be a group and H a normal subgroup of G, and for all $x, y\in G$, we define $x\circ y = xyH$. Then, $(G, \circ)$ is a hypergroup.

Definition 5 (see [2, 12]). Let $(H, \circ)$ be a hypergroupoid. The commutative law on $(H, \circ)$ is defined as follows:

(C) $(\forall a, b\in H)\ (a\circ b = b\circ a)$.

$(H, \circ)$ is called a commutative hypergroupoid.

Table 1: Classification of the hypergroupoids of order 2.

		A	NA	AA
C	R	6	4	—
	NR	—	—	—
	AR	—	—	—
	Etc.	3	2	—
NC	R	—	—	—
	NR	—	—	—
	AR	—	—	—
	Etc.	—	—	—
AC	R	2	8	—
	NR	—	—	—
	AR	—	—	—
	Etc.	6	10	4

Table 2: Classification of the semi-hypergroups of order 2.

	Com	Noncom	N
Semigroup	3	2	5
Group	1	—	1
Semi-hypergroup	9	8	17
Hypergroup	6	2	8

Example 2 (see [13]). Let $\mathbb{Z}$ be the set of integers, and define $\circ_1$ on $\mathbb{Z}$ as follows. For all $x, y\in\mathbb{Z}$,

$$x\circ_1 y = \left\{\begin{array}{ll} 2\mathbb{Z}, & \text{if } x, y \text{ have same partiy,} \\ 2\mathbb{Z}+1, & \text{otherwise.}\end{array}\right\}. \tag{1}$$

Then, $(\mathbb{Z}, \circ_1)$ is a commutative hypergroup.

3. On Neutro-hypergroups and Anti-hypergroups

F. Smarandache generalized the classical algebraic structures to the neutroalgebraic structures and antialgebraic structures. Neutro-sophication of an item C (that may be a concept, a space, an idea, a hyperoperation, an axiom, a theorem, a theory, an algebra, etc.) means to split C into three parts (two parts opposite to each other, and another part which is the neutral/indeterminacy between the opposites), as pertinent to neutrosophy $((\langle A\rangle, \langle \text{neut}A\rangle, \langle \text{anti}A\rangle)$, or with other notation $(T, I, F))$, meaning cases where C is partially true (T), partially indeterminate (I), and partially false (F), while antisophication of C means to totally deny C (meaning that C is made false on its whole domain) (see [14, 15, 17, 20]).

Neutrosophication of an axiom on a given set X means to split the set X into three regions such that, on one region, the axiom is true (we say the degree of truth T of the axiom), on another region, the axiom is indeterminate (we say the degree of indeterminacy I of the axiom), and on the third region, the axiom is false (we say the degree of falsehood F of the axiom), such that the union of the regions covers the whole set, while the regions may or may not be disjoint, where (T, I, F) is different from $(1, 0, 0)$ and from $(0, 0, 1)$.

Antisophication of an axiom on a given set X means to have the axiom false on the whole set X (we say total degree of falsehood F of the axiom) or $(0, 0, 1)$.

Neutrosophication of a hyperoperation defined on a given set X means to split the set X into three regions such that, on one region, the hyperoperation is well-defined (or inner-defined) (we say the degree of truth T of the hyperoperation), on another region, the hyperoperation is indeterminate (we say the degree of indeterminacy I of the hyperoperation), and on the third region, the hyperoperation is outer-defined (we say the degree of falsehood F of the hyperoperation), such that the union of the regions covers the whole set, while the regions may or may not be disjoint, where (T, I, F) is different from $(1, 0, 0)$ and from $(0, 0, 1)$.

Antisophication of a hyperoperation on a given set X means to have the hyperoperation outer-defined on the whole set X (we say total degree of falsehood F of the axiom) or $(0, 0, 1)$.

In this section, we will define the neutro-hypergroups and anti-hypergroups.

Definition 6. A neutro-hyperoperation is a map $\circ\colon H \times H \longrightarrow P(U)$, where U is a universe of discourse that contains H that satisfies the below neutrosophication process.

The neutrosophication (degree of well-defined, degree of indeterminacy, and degree of outer-defined) of the hyperoperation is the following neutrohyperoperation (NH):

(NR) $(\exists x, y \in H)\ (x \circ y \in P^*(H))$ and $(\exists x, y \in H)(x \circ y$ is an indeterminate subset, or $x \circ y \notin P^*(H))$.

The neutrosophication (degree of truth, degree of indeterminacy, and degree of falsehood) of the hypergroup axiom of associativity is the following neutroassociativity (NA):

(NA) $(\exists a, b, c \in H)\ (a \circ (b \circ c) = (a \circ b) \circ c)$ and $(\exists d, e, f \in H)(d \circ (e \circ f) \neq (d \circ e) \circ f$ or $d \circ (e \circ f) =$ indeterminate, or $(d \circ e) \circ f =$ indeterminate).

Neutroreproduction axiom (NR):

(NR) $(\exists a \in H)(H \circ a = a \circ H = H)$ and $(\exists b \in H)$ $(H \circ b, b \circ H$, and H are not all three equal, or some of them are indeterminate).

Also, we define the neutrocommutativity (NC) on $(H, \circ)$ as follows:

(NC) $(\exists a, b \in H)\,(a \circ b = b \circ a)$ and $(\exists c, d \in H)$ $(c \circ d \neq d \circ c$, or $c \circ d =$ indeterminate, or $d \circ c =$ indeterminate).

Now, we define a neutro-hyperalgebraic system $S = \langle H, F, A \rangle$, where H is a set or neutrosophic set, F is a set of the hyperoperations, and A is the set of hyperaxioms, such that there exists at least one neutro-hyperoperation or at least one neutro-hyperaxiom and no anti-hyperoperation and no anti-hyperaxiom.

Definition 7. The anti-hypersophication (totally outer-defined) of the hyperoperation defines anti-hyperoperation (AH): (AH) $(\forall x, y \in H)\ (x \circ y \notin P^*(H))$.

The anti-hypersophication (totally false) of the hypergroup is as follows:

(AA) $(\forall x, y, z \in H)\ (x \circ (y \circ z) \neq (x \circ y) \circ z)$ (antiassociativity)

(AR) $(\forall a \in H)(H \circ a,\ a \circ H$, and H are not equal) (antireproduction axiom)

Also, we define the anticommutativity (AC) on $(H, \circ)$ as follows:

(AC) $(\forall a, b \in H$ with $a \neq b)$ $(a \circ b \neq b \circ a)$.

Definition 8. A neutro-semihypergroup is an alternative of semi-hypergroup that has at least (NH) or (NA), which does not have (AA).

Example 3

(i) Let $H = \{a, b, c\}$ and $U = \{a, b, c, d\}$ be a universe of discourse that contains H. Define the neutro-hyperoperation $\circ_2$ on H with Cayley's table.

$\circ_2$	a	b	c
a	a	a	a
b	b	$\{a, b\}$	$\{a, b, d\}$
c	c	?	H

Then, $(H, \circ_2)$ is a neutro-semihypergroup. Since $a \circ_2 b \in P^*(H)$, $b \circ_2 c = \{a, b, d\} \notin P^*(H)$, and $c \circ_2 b =$ indeterminate, so (NH) holds.

(ii) Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_3$ on H with Cayley's table.

$\circ_3$	a	b	c
a	a	a	a
b	b	$\{a, b\}$	$\{a, b\}$
c	c	$\{b, c\}$	H

Then, $(H, \circ_3)$ is a neutro-semihypergroup. (NA) is valid, since $(b \circ_3 c) \circ_3 a = \{a, b\} \circ_3 a = (a \circ_3 a) \cup (b \circ_3 a) = \{a\} \cup \{b\} = \{a, b\}$ and $b \circ_3 (c \circ_3 a) = b \circ_3 \{c\} = b \circ_3 c = \{a, b\}$.

Hence, $(b \circ_3 c) \circ_3 a = b \circ_3 (c \circ_3 a)$. Also, $\{b \circ_3 a\} \circ_3 c = \{b\} \circ_3 c = b \circ_3 c = \{a, b\}$ and $b \circ_3 (a \circ_3 c) = b \circ_3 \{a\} = b \circ_3 a = \{b\}$, so $(b \circ_3 a) \circ_3 c \neq b \circ_3 (a \circ_3 c)$.

Definition 9. A neutrocommutative semi-hypergroup is a semi-hypergroup that satisfies (NC).

Example 4. Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_4$ on H with Cayley's table.

$\circ_4$	a	b	c
a	$\{a, c\}$	a	a
b	a	b	c
c	a	$\{b, c\}$	$\{b, c\}$

Then, $(H, \circ_4)$ is a semi-hypergroup, but not a hypergroup, since $a \circ_4 H = H \circ_4 a = \{a, c\} \neq H$. (NC) is valid, since $a \circ_4 b = \{a\} = b \circ_4 a$ and $c \circ_4 b = \{b, c\} \neq b \circ_4 c = \{c\}$.

Definition 10. A neutrocommutative hypergroup is a hypergroup that satisfies (NC).

Example 5. Let $H = \{a, b, c, d, e, f\}$. Define the operation $\circ_5$ on H with Cayley's table.

$\circ_5$	a	b	c	d	e	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	f	d	e	b	a
d	d	c	f	a	e	b
f	f	d	c	b	a	e

Then, $(H, \circ_5, e)$ is a group and so is a natural hypergroup. Also, it is a neutrocommutative hypergroup, since $a \circ_5 b = e = b \circ_5 a$ and $a \circ_5 c = d \neq c \circ_5 a = f$.

Definition 11. A neutrohypergroup is an alternative of hypergroup that has at least (NH) or (NA) or (NR), which does not have (AA) and (AR).

Example 6. Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_6$ on H with Cayley's table.

$\circ_6$	a	b	c
a	a	b	c
b	b	b	b
c	c	c	a

Then, $(H, \circ_6)$ is a neutrohypergroup. The hyperoperation $\circ_6$ is associative. (NR) is valid, since $a \circ_6 H = (a \circ_6 a) \cup (a \circ_6 b) \cup (a \circ_6 c) = H = (a \circ_6 a) \cup (b \circ_6 a) \cup (c \circ_6 a) = H \circ_6 a$, $b \circ_6 H = (b \circ_6 a) \cup (b \circ_6 b) \cup (b \circ_6 c) = \{b\} \neq H \neq \{c, b\} = (a \circ_6 b) \cup (b \circ_6 b) \cup (c \circ_6 b) = H \circ_6 b$, and $c \circ_6 H = (c \circ_6 a) \cup (c \circ_6 b) \cup (c \circ_6 c) = \{a, c\} \neq H$, but $H \circ_6 c = (a \circ_6 c) \cup (b \circ_6 c) \cup (c \circ_6 c) = \{a, b, c\} = H$.

Note that every neutro-semihypergroup, neutrohypergroup, neutrocommutative semi-hypergroup, and neutrocommutative hypergroup are neutro-hyperalgebraic systems.

Definition 12. An anti-semihypergroup is an alternative of semi-hypergroup that has at least (AH) or (AA).

Example 7

(i) Let $\mathbb{N}$ be the set of natural numbers except 0. Define hyperoperation $\circ_7$ on $\mathbb{N}$ by $x \circ_7 y = \{(x^2/x^2 + 1), y\}$. Then, $(\mathbb{N}, \circ_7)$ is an anti-semihypergroup. (AH) is valid, since, for all $x, y \in \mathbb{N}$, $x \circ_7 y \notin P^*(\mathbb{N})$. Thus, (AH) holds.

(ii) Let $H = \{a, b\}$. Define the hyperoperation $\circ_8$ on H with Cayley's table.

$\circ_8$	a	b
a	b	a
b	b	a

Then, $(H, \circ_8)$ is an anti-semihypergroup. (AA) is valid, since, for all $x, y, z \in H$, $x \circ_8 (y \circ_8 z) \neq (x \circ_8 y) \circ_8 z$.

(iii) Let $H = \{a, b\}$. Define the hyperoperation $\circ_9$ on H with Cayley's table.

$\circ_9$	a	b
a	b	H
b	a	a

Then, $(H, \circ_9)$ is an anticommutative semi-hypergroup. (AC) is valid, since $a \circ_9 b = H \neq b \circ_9 a = \{a\}$.

Definition 13. An anti-hypergroup is an anti-semihypergroup, or it satisfies (AR).

Example 8

(i) Let $\mathbb{R}$ be the set of real numbers. Define hyperoperation $\circ_{10}$ on $\mathbb{R}$ by $x \circ_{10} y = \{x^2 + 1, x^2 - 1\}$. Then, $(\mathbb{R}, \circ_{10})$ is an anti-semihypergroup, since, for all $x, y, z \in \mathbb{R}$, $x \circ_{10} (y \circ_{10} z) \neq (x \circ_{10} y) \circ_{10} z$. Because $x \circ_{10} (y \circ_{10} z) = x \circ_{10} \{y^2 + 1, y^2 - 1\} = \{x \circ_{10} (y^2 + 1), x \circ_{10} (y^2 - 1)\} = \{x^2 + 1, x^2 - 1\}$, but $(x \circ_{10} y) \circ_{10} z = \{x^2 + 1, x^2 - 1\} \circ_{10} z = ((x^2 + 1) \circ_{10} z) \cup ((x^2 - 1) \circ_{10} z) = \{(x^2 + 1)^2 + 1, (x^2 - 1)^2 + 1\}$. Hence, (AA) is valid.

(ii) Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_{11}$ on H with Cayley's table.

$\circ_{11}$	a	b	c
a	a	a	b
b	a	a	a
c	c	c	c

Then, $(H, \circ_{11})$ is an anti-semihypergroup. The hyperoperation $\circ_{11}$ is associative. Also, (AR) holds, since $a \circ_{11} H = (a \circ_{11} a) \cup (a \circ_{11} b) \cup (a \circ_{11} c) = \{c\} \neq H \neq \{b, c\} = (a \circ_{11} a) \cup (b \circ_{11} a) \cup (c \circ_{11} a) = H \circ_{11} a$, $b \circ_{11} H = (b \circ_{11} a) \cup (b \circ_{11} b) \cup (b \circ_{11} c) = \{b\} \neq H \neq \{b, c\} = (a \circ_{11} b) \cup (b \circ_{11} b) \cup (c \circ_{11} b) = H \circ_{11} b$, and $c \circ_{11} H = (c \circ_{11} a) \cup (c \circ_{11} b) \cup (c \circ_{11} c) = \{c\} \neq H \neq \{b, c\} = (a \circ_{11} c) \cup (b \circ_{11} c) \cup (c \circ_{11} c) = H \circ_{11} c$.

(iii) Let $\mathbb{R}$ be the set of real numbers. Define hyperoperation $\circ_{12}$ on $\mathbb{R}$ by $x \circ_{12} y = \{x, 1\}$. Then, $(\mathbb{R}, \circ_{12})$ is an anti-semihypergroup. The hyperoperation $\circ_{12}$ is associative, since, for all $x, y, z \in \mathbb{R}$, we have $x \circ_{12} (y \circ_{12} z) = x \circ_{12} \{y, 1\} = (x \circ_{12} y) \cup (x \circ_{12} 1) = \{x, 1\} \cup \{x, 1\} = \{x, 1\}$ and $(x \circ_{12} y) \circ_{12} z = \{x, 1\} \circ_{12} z = (x \circ_{12} z) \cup (1 \circ_{12} z) = \{x, 1\} \cup \{1, 1\} = \{x, 1\}$, so $x \circ_{12} (y \circ_{12} z) = (x \circ_{12} y) \circ_{12} z$. However, for $a \in \mathbb{R}$, we have $a \circ_{12} \mathbb{R} = \cup_{x \in \mathbb{R}} a \circ_{12} x = \cup_{x \in \mathbb{R}}$

$\{a, 1\} = \{a, 1\} \neq \mathbb{R}$ and $R \circ_{12} a = \cup_{x \in \mathbb{R}} x \circ_{12} a = \cup_{x \in R} \{x, 1\} = \mathbb{R}$. Thus, $a \circ_{12} \mathbb{R} \neq \mathbb{R} \circ_{12} a$.

Definition 14. An anticommutative semi-hypergroup is a semi-hypergroup that satisfies (AC).

Example 9

(i) Let $H = \{a, b\}$. Define the hyperoperation $\circ_{13}$ on H with Cayley's table.

$\circ_{13}$	a	b
a	a	a
b	H	b

Then, $(H, \circ_{13})$ is a semi-hypergroup and (AC) is valid, since $a \circ_{13} b = \{a\} \neq b \circ_{13} a = H$. Thus, $(H, \circ_{13})$ is an anticommutative semi-hypergroup.

(ii) Let $H = \{a, b\}$. Define the hyperoperation $\circ_{14}$ on H with Cayley's table.

$\circ_{14}$	a	b
a	b	a
b	b	a

Then, $(H, \circ_{14})$ is an anticommutative semi-hypergroup, and the hyperoperation $\circ_{14}$ is not associative, since $(a \circ_{14} a) \circ_{14} a = \{b\} \circ_{14} a = \{b\} \neq a \circ_{14} (a \circ_{14} a) = a \circ_{14} \{b\} = \{a\}$. (AC) is valid, since $a \circ_{14} b = \{a\} \neq b \circ_{14} a = \{b\}$.

Definition 15. An anticommutative hypergroup is a hypergroup that satisfies (AR).

Example 10

(i) Let $H = \{a, b\}$. Define the hyperoperation $\circ_{15}$ on H with Cayley's table.

$\circ_{15}$	a	b
a	H	a
b	H	H

Then, $(H, \circ_{15})$ is an anticommutative hypergroup. (AC) is valid, since $a \circ_{15} b = \{a\} \neq b \circ_{15} a = H$.

(ii) Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_{16}$ on H with Cayley's table.

$\circ_{16}$	a	b	c
a	a	a	H
b	b	b	H
c	c	c	H

Then, $(H, \circ_{16})$ is an anticommutative hypergroup. The hyperoperation $\circ_{16}$ is associative. Also, (AC) holds, since $a \circ_{16} b = \{a\} \neq b \circ_{16} a = \{b\}$, $a \circ_{16} c = H \neq c \circ_{16} a = \{c\}$, and $b \circ_{16} c = H \neq c \circ_{16} b = \{c\}$.

(iii) Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_{17}$ on H with Cayley's table.

$\circ_{17}$	a	b	c
a	a	b	c
b	a	b	c
c	H	H	H

Then, $(H, \circ_{17})$ is an anticommutative hypergroup, (AC) holds, since $a \circ_{17} b = \{b\} \neq b \circ_{17} a = \{a\}$, $a \circ_{17} c = \{c\} \neq c \circ_{17} a = H$, and $b \circ_{17} c = \{c\} \neq c \circ_{17} b = H$.

Note that every anti-semihypergroup, antihypergroup, anticommutative semi-hypergroup, and anticommutative hypergroup are anti-hyperalgebraic systems.

In the following results, we use hyperoperation instead of neutro-hyperoperation.

Note that if $(H, \circ)$ is a neutro-semihypergroup and $(G, \circ)$ is an anti-semihypergroup, then $(H \cap G, \circ)$ is not a neutro-semihypergroup, but it is an anti-semihypergroup. Also, let $(H, \circ_H)$ be a neutro-semihypergroup, $(G, \circ_G)$ be an anti-semihypergroup, and $H \cap G = \varnothing$. Define hyperoperation $\circ$ on $H \uplus G$ by

$$x \circ y = \left\{ \begin{array}{ll} x \circ_H y, & \text{if } x, y \in H, \\ x \circ_G y, & \text{if } x, y \in G, \\ \{x, y\}, & \text{otherwise.} \end{array} \right\}. \tag{2}$$

Then, $(H \uplus G, \circ)$ is a neutro-semihypergroup, but it is not an anti-semihypergroup.

Proposition 1. *Let $(H, \circ)$ be an antisemihypergroup and $e \in H$. Then, $(H \cup \{e\}, *)$ is a neutrosemihypergroup, where $*$ is defined on $H \cup \{e\}$ by*

$$x * y = \left\{ \begin{array}{ll} x \circ_H y, & \text{if } x, y \in H, \\ \{e, x, y\}, & \text{otherwise.} \end{array} \right\}. \tag{3}$$

Proof. It is straightforward.

Proposition 2. *Let $(H, \circ)$ be a commutative hypergroupoid. Then, $(H, \circ)$ cannot be an anti-semihypergroup.*

Proof. Let $a \in H$. Then, $a \circ (a \circ a) = (a \circ a) \circ a$, so $(H, \circ)$ cannot be an anti-semihypergroup.

Corollary 1. *Let $(H, \circ)$ be a hypergroupoid, and there exists $a \in H$ such that $a^{\circ}a$ commuted with a. Then, $(H, \circ)$ cannot be an anti-semihypergroup.*

Corollary 2. *Let $(H, \circ)$ be a hypergroupoid with a scalar idempotent, i.e., there exists $a \in H$ such that $a^{\circ}a = a$. Then, $(H, \circ)$ cannot be an anti-semihypergroup.*

Proposition 3. *Let $(H, \circ_H)$ and $(G, \circ_G)$ be two neutro-semihypergroups (resp. anti-semihypergroups). Then, $(H \times$

$G, *$) is a neutro-semihypergroup (resp. anti-semi-hypergroups), where $*$ is defined on $H \times G$. For any $(x_1, y_1), (x_2, y_2) \in H \times G$,

$$(x_1, y_1) * (x_2, y_2) = (x_1 \circ_H x_2, y_1 \circ_G y_2). \tag{4}$$

Note that if $(H, \circ)$ is a neutro-semihypergroup, then if there is a nonempty set $H_1 \subseteq H$, such that $(H_1, \circ)$ is a semi-hypergroup, we call it Smarandache semi-hypergroup.

Suppose $(H, \circ_H)$ and $(G, \circ_G)$ are two hypergroupoids. A function $f: H \longrightarrow G$ is called a homomorphism if, for all $a, b \in H$, $f(a \circ_H b) = f(a) \circ_G f(b)$ (see [21, 22], for details).

Proposition 4. *Let $(H, \circ_H)$ be a semi-hypergroup, $(G, \circ_G)$ be a neutro-hypergroup, and $f: H \longrightarrow G$ be a homomorphism. Then, $(f(H), \circ_G)$ is a semi-hypergroup, where $f(H) = \{f(h): h \in H\}$.*

Proof. Assume that $(H, \circ_H)$ is a semi-hypergroup and $x, y, z \in f(H)$. Then, there exist $h_1, h_2, h_3 \in f(H)$ such that $f(h_1) = x$, $f(h_2) = y$, and $f(h_3) = z$, so we have

$$\begin{aligned} x \circ_G (y \circ_G z) &= f(h_1) \circ_G (f(h_2) \circ_G (h_3)) \\ &= f(h_1) \circ_G f(h_2 \circ_H h_3) = f(h_1 \circ_H (h_2 \circ_H h_3)) \\ &= f((h_1 \circ_H h_2) \circ_H h_3) = f(h_1 \circ_H h_2) \circ_G f(h_3) \\ &= (f(h_1) \circ_G f(h_2)) \circ_G f(h_3) = (x \circ_G y) \circ_G z. \end{aligned} \tag{5}$$

Then, $(f(H), \circ_G)$ is a semi-hypergroup. □

Definition 16. Let $(H, \circ_H)$ and $(G, \circ_G)$ be two hypergroupoids. A bijection $f: H \longrightarrow G$ is an isomorphism if it conserves the multiplication (i.e., $f(a \circ_H b) = f(a) \circ_G f(b)$) and write $H \cong G$. A bijection $f: H \longrightarrow G$ is an antiisomorphism if for all $a, b \in H$, $f(a \circ_H b) \neq f(b) \circ_G f(a)$. A bijection $f: H \longrightarrow G$ is a neutroisomorphism if there exist $a, b \in H$, $f(a \circ_H b) = f(b) \circ_G f(a)$, i.e., degree of truth (T), there exist $c, d \in H$ and $f(c \circ_H d)$ or $f(c) \circ_G f(d)$ are indeterminate, i.e., degree of indeterminacy (I), and there exist $e, h \in H$, $f(e \circ_H h) \neq f(e) \circ_G f(h)$, i.e., degree of falsehood (F), where (T, I, F) are different from $(1, 0, 0)$ and $(0, 0, 1)$, and $T, I, F \in [0, 1]$.

Let $\circ$ be a hyperoperation on $H = \{a, b\}$ and $(A_{11}, A_{12}, A_{21}, A_{22})$ inside of Cayley's table.

$\circ$	a	b
a	A_{11}	A_{12}
b	A_{21}	A_{22}

Lemma 1 (see [5]). *Let $(H = \{a, b\}, \circ_H)$ and $(G = \{a', b'\}, \circ_G)$ be hypergroupoids with Cayley's tables (A, B, C, D) and (A', B', C', D'), respectively. Then, $H \cong G$ if and only if, for all $i, j \in \{1, 2\}$, $A_{ij} = A'_{ij}$ or*

$$A'_{ij} = \left\{ \begin{array}{ll} A^d_{ij}, & \text{if } A_{ij} = H, \\ G \setminus A'_{ij}, & \text{if } A_{ij} \neq H, \end{array} \right\}, \tag{6}$$

where $A^d_{11} = A_{22}$, $A^d_{12} = A_{12}$, $A^d_{21} = A_{21}$, and $A^d_{22} = A_{11}$.

Lemma 2 (see [6]). *If $(H, \circ)$ is a hypergroupoid, then $(H, *)$ is a hypergroupoid when $x * y = y \circ x$ for all $x, y \in H$.*

$(H, *)$ in Lemma 2 is called dual hypergroupoid of $(H, \circ)$.

Theorem 1. *Let $(H = \{a, b\}, \circ)$. Then, $(H, \circ) \cong (H, *)$ if and only if $(H, \circ)$ is anticommutative.*

Lemma 3. *There exist 4 anticommutative anti-semi-hypergroup of order 2 (up to isomorphism).*

Proof. Let $(H, \circ)$ be an anticommutative anti-semihypergroup. By Corollary 2, we have $a \circ a \neq a$ and $b \circ b \neq b$. Also, $a \circ b \neq b \circ a$. Consider the following.

If $a \circ a = H$, then $a \circ (a \circ a) = a \circ H = H = H \circ a = (a \circ a) \circ a$, a contradiction. Then, we get $a \circ a = b$ and $b \circ b = a$.

Now, we have

Case 1. If $a \circ b = a$, then $b \circ a = H$ or $b \circ a = b$, so we get (b, a, b, a) and (b, a, H, a) are two anti-semihypergroups
Case 2. If $a \circ b = b$, then $b \circ a = H$ or $b \circ a = a$, so we get (b, b, a, a) and (b, b, H, a) are two anti-semihypergroups
Case 3. If $a \circ b = H$, then $b \circ a = a$ or $b \circ a = b$, so we get (b, H, a, a) and (b, H, b, a) are two anti-semihypergroups

It can be see that $(b, a, H, a) \cong (b, H, b, a)$ and $(b, H, a, a) \cong (b, b, H, a)$. Therefore, (b, b, a, a), (b, a, b, a), (b, a, H, a), and (b, H, a, a) are 4 nonisomorphic anti-semihypergroups of order 2. □

Corollary 3. *There exists two nonisomorphic anti-semigroups of order 2: (b, b, a, a) and (b, a, b, a). Anti-semigroup (b, b, a, a) is the dual form of the anti-semigroup (b, a, b, a).*

Corollary 4. *There exists two nonisomorphic anti-semihypergroups of order 2: (b, a, H, a) and (b, H, a, a). Anti-semihypergroup (b, a, H, a) is the dual form of the anti-semihypergroup (b, H, a, a).*

Theorem 2. *Let $(H, \circ)$ be a hypergroupoid of order 2. Then, $(H, \circ)$ does not have (NR) or (AR).*

Proof. Let $H = \{a, b\}$. Suppose $Ha \neq H$, $aH \neq H$, and $Ha \neq aH$. Hence, $Ha = \{a\}$ or $Ha = \{b\}$. First, give $Ha = \{a\}$, then $aH \neq H$ and $Ha \neq aH$ implies that $aH = \{b\}$. Then, $a \circ a \subseteq Ha = \{b\}$ and $a \circ a \subseteq Ha = \{a\}$. Therefore, $\{b\} = a \circ a \circ a = \{a\}$, and this is a contradiction. In the similar way, we obtain $Hb \neq H$, $bH \neq H$, and $Hb \neq bH$, a contradiction.

Using Lemmas 1 and 2 and Theorem 1, we can find 45 nonisomorphic classes hypergroupoids of the order 2. We characterize these 45 classes in Table 1.

Note that semi-hypergroups, hypergroups, and fuzzy hypergroups of order 2 are enumerated in [7, 11, 12].

We obtain anti-semihypergroups and neutro-semihypergroups of order 2 and the classification of the hypergroupoids of order 2 (classes up to isomorphism).

R, NR, AR, A, NA, AA, C, NC, and AC in Table 1 are denoted in Sections 2 and 3.

A result from Table 1 confirms the enumeration of the hyperstructure of order 2 [11, 23, 24], which is summarized as follows. □

4. Conclusion and Future Work

In this paper, we have studied several special types of hypergroups, neutro-semihypergroups, anti-semihypergroups, neutro-hypergroups, and anti-hypergroups. New results and examples on these new algebraic structures have been investigated. Also, we characterize all neutro-hypergroups and anti-hypergroups of order two up to isomorphism. These concepts can further be generalized.

Future research to be done related to this topic are

(a) Define neutro-quasihypergroup, anti-quasihypergroup, neutrocommutative quasi-hypergroup, and anticommutative quasi-hypergroup

(b) Define neutro-hypergroups, anti-hypergroups, neutrocommutative hypergroups, and anti-commutative hypergroups

(c) Define and investigate neutroHv-groups, antiHv-groups, neutroHv-rings, and antiHv-rings

(d) It will be interesting to characterize infinite neutro-hypergroups and anti-hypergroups up to isomorphism

(e) These results can be applied to other hyper-algebraic structures, such as hyper-rings, hyper-spaces, hyper-BCK-algebra, hyper-BE-algebras, and hyper-K-algebras.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] F. Marty, *Sur une Generalization de la Notion de Groupe*, pp. 45–49, Huitieme Congress de Mathematiciens, Scandinaves, Stockholm, 1934.

[2] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani, Udine, Italy, 1993.

[3] M. Al-Tahan and B. Davvaz, "On Corsini hypergroups and their productional hypergroups," *The Korean Journal of Mathematics*, vol. 27, no. 1, pp. 63–80, 2019.

[4] P. Corsini and V. Leoreanu-Fotea, *Applications of Hyperstruture Theory*, Springer Science+Business Media, Berlin, Germany, 2003.

[5] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, Cambridge, MA, USA, 2007.

[6] B. Davvaz and T. Vougiouklis, *A Walk through Weak Hyperstructures; H_v-Structure*, World Scientific Publishing, Hackensack, NJ, USA, 2019.

[7] T. Vougiouklis, *Cyclic Hypergroups*, Ph. D Thesis, Democritous University of Thrace, Komotini, Greece, 1980.

[8] B. Davvaz, *Semihypergroup Theory*, Elsevier, Amsterdam, Netherlands, 2016.

[9] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific, Singapore, 2013.

[10] B. Davvaz and I. Cristea, *Fuzzy Algebraic Hyperstructures; An Introduction*, Springer, Berlin, Germany, 2015.

[11] C. G. Massouros and G. G. Massouros, "On 2-element fuzzy and mimic fuzzy hypergroups," *AIP Conference Proceedings*, vol. 1479, pp. 2213–2216, 2012.

[12] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Palm Harbor, FL, USA, 1994.

[13] S. Hoskova-Mayerova, M. Al Tahan, and B. Davvaz, "Fuzzy multi-hypergroups," *Mathematics*, vol. 8, no. 2, p. 244, 2020.

[14] F. Smarandache, "Neutroalgebra is a generalization of partial algebra," *International Journal of Neutrosophic Science*, vol. 2, no. 1, pp. 8–17, 2020.

[15] F. Smarandache, "Introduction to neutroalgebraic structures and antialgebraic structures," in *Advances of Standard and Nonstandard Neutrosophic Theories*, Pons Publishing House, Brussels, Belgium, 2019.

[16] F. Smarandache, "Introduction to NeutroAlgebraic structures and AntiAlgebraic structures (revisited)," *Neutrosophic Sets and Systems*, vol. 31, pp. 1–16, 2020.

[17] F. Smarandache, "Extension of hypergraph to n-superhypergraph and to plithogenic n-superhypergraph, and extension of hyperalgebra to n-ary (classical-/neutro-/anti-) hyperalgebra," *Neutrosophic Sets and Systems*, vol. 33, pp. 290–296, 2020.

[18] A. A. A. Agboola, "Introduction to NeutroGroups," *International Journal of Neutrosophic Science*, vol. 6, no. 1, pp. 41–47, 2020.

[19] M. Al-Tahan, F. Smarandache, and B. Davvaz, "Neutro ordered algebra: applications to semigroups," *Neutrosophic Sets and Systems*, vol. 39, pp. 133–147, 2021.

[20] F. Smarandache, *Neutrosophy/Neutrosophic Probability, Set, and Logic*, American Research Press, Santa Fe, NM, USA, 1998.

[21] R. Ameri and M. M. Zahedi, "Hyperalgebraic system," *Italian Journal of Pure and Applied Mathematics*, vol. 6, pp. 21–32, 1999.

[22] P. Corsini and V. Leoreanu-Fotea, "Hypergroups and binary relations," *Algebra Universalis*, vol. 43, pp. 321–330, 2000.

[23] R. Bayon and N. Lygeros, "Catégories spécifiques dhypergroupes dordre 3, in: structure elements of hyper-structures," pp. 17–33, Spanidis, Xanthi, Greece, 2005.

[24] S. Worawiset, J. Koppitz, and S. Chotchaisthit, "The class of all semigroups related to semihypergroups of order 2," *Mathematica Slovaca*, vol. 69, no. 2, pp. 371–380, 2019.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6649751, 10 pages
https://doi.org/10.1155/2021/6649751

Research Article

Study of Two Kinds of Quasi AG-Neutrosophic Extended Triplet Loops

Xiaogang An [1] and Mingming Chen [1,2]

[1] *School of Arts and Sciences, Shaanxi University of Science & Technology, Xi'an 710021, China*
[2] *Shaanxi Joint Laboratory of Artificial Intelligence, Shaanxi University of Science & Technology, Xi'an 710021, China*

Correspondence should be addressed to Xiaogang An; anxiaogang@sust.edu.cn

Received 27 December 2020; Revised 27 January 2021; Accepted 18 February 2021; Published 8 March 2021

Academic Editor: Parimala Mani

Abel-Grassmann's groupoid and neutrosophic extended triplet loop are two important algebraic structures that describe two kinds of generalized symmetries. In this paper, we investigate quasi AG-neutrosophic extended triplet loop, which is a fusion structure of the two kinds of algebraic structures mentioned above. We propose new notions of AG-(l,r)-Loop and AG-(r,l)-Loop, deeply study their basic properties and structural characteristics, and prove strictly the following statements: (1) each strong AG-(l,r)-Loop can be represented as the union of its disjoint sub-AG-groups, (2) the concepts of strong AG-(l,r)-Loop, strong AG-(l,l)-Loop, and AG-(l,lr)-Loop are equivalent, and (3) the concepts of strong AG-(r,l)-Loop and strong AG-(r,r)-Loop are equivalent.

1. Introduction

The so-called left almost semigroup (LA-semigroup) was actually the concept of an Abel-Grassmann's groupoid (AG-groupoid), which was put forward by Kazim and Naseeruddin [1] at the first time in 1972. Different classes of AG-groupoids and their concerned characteristics have been studied in [2–5].

Neutrosophic set (NS) was first put forward by Smarandache in [6]. Then, it has been growing promptly over the previous 15 years. Nowadays, NS theory is widely used in a couple of sectors such as professional selection [7], integrated speech and text sentiment analysis [8], finite automata [9], clustering methods [10], and deep learning [11]. Besides, more new theoretical studies on NS in [12–17] have been conducted and a few significant results have been gained.

The concept of Abel-Grassmann's neutrosophic extended triplet loop (AG-NET-Loop), which plays a significant role in neutrosophic triplet algebraic structures, was proposed in [18], that is, an AG-NET-Loop is both an AG-groupoid and a neutrosophic extended triplet loop (NET-Loop). In [19], the concept of neutrosophic triplet elements (NT-elements) and quasi neutrosophic triplet loops were introduced. In [20], two kinds of quasi AG-NET-Loops (AG-(l,l)-Loop and AG-(r,r)-Loop) were proposed and their basic properties were investigated. As a continuation of [20], we propose two other kinds of quasi AG-NET-Loops, which are the AG-(l,r)-Loop and the AG-(r,l)-Loop. We study their properties and analyze their relationship.

The rest of this paper is arranged as follows. In Section 2, some definitions and properties on quasi AG-NET-Loop are given. Some properties and structures about the AG-(l,r)-Loop are discussed in Section 3. The relations among four kinds of quasi AG-NET-Loops are analyzed in Section 4. Some properties about the alternative quasi AG-NET-Loops are discussed in Section 5. Lastly, Section 6 presents the summary and the direction of future efforts.

2. Preliminaries

A groupoid $(G, *)$ is called an AG-groupoid if it holds the left invertive law, that is, for all $x, y, z \in G$, $(x * y) * z = (z * y) * x$. In an AG-groupoid $(G, *)$ the medial law holds, for all $x_1, x_2, x_3, x_4 \in G$, $(x_1 * x_2) * (x_3 * x_4) = (x_1 * x_3) * (x_2 * x_4)$. An AG-groupoid $(G, *)$ is called locally associative if for all

$x \in G$, $(x * x) * x = x * (x * x)$. In an AG-groupoid $(G, *)$, for all $x \in G$, $k \in Z^+$, x^k is defined as follows: $x^1 = x, x^2 = x * x, x^3 = x^2 * x, x^4 = x^3 * x, \ldots, x^k = x^{k-1} * x$.

Definition 1 (see [21]). Let G be a nonempty set together with a binary operation *. Then, G is called a neutrosophic extended triplet set if, for all $x \in G$, there exist a neutral of "x" and an opposite of "x" (denoted by $\mathrm{neut}(x)$ and $\mathrm{anti}(x)$, respectively), such that $\mathrm{neut}(x), \mathrm{anti}(x) \in G$, and $\mathrm{neut}(x) * x = x * \mathrm{neut}(x) = x, \mathrm{anti}\ (x) * x = x * \mathrm{anti}(x) = \mathrm{neut}(x)$. The triplet $(x, \mathrm{neut}(x), \mathrm{anti}(x))$ is called a neutrosophic extended triplet (NET).

Definition 2 (see [18]). An NET set $(G, *)$ is called an NET-Loop, if, for all $x, y \in G$, one has $x * y \in G$.

Definition 3 (see [18]). An AG-groupoid $(G, *)$ is called an AG-NET-Loop if it is an NET-Loop.

An AG-NET-Loop G is called a commutative AG-NET-Loop if for all $x, y \in G, x * y = y * x$.

Theorem 1 (see [18]). *Let $(G, *)$ be an AG-NET-Loop. Then,*

(1) For all $x \in G$, $neut(x)$ is unique

(2) For all $x \in G$, $(neut(x))^2 = neut(x)$

Definition 4 (see [2]). AG-groupoid $(G, *)$ is called regular if, for all $a \in G$, there exists $m \in G$, $a = (a * m) * a$.

Definition 5 (see [20]). Let $(G, *)$ be an AG-groupoid. Then, G is called an AG-*(l,l)*-Loop if, for all $a \in G$, there exist a local *(l,l)*-neutral element of"a" and a local *(l,l)*-opposite element of"a" (denoted by $nll(a)$ and $oll(a)$, respectively), such that $nll(a) \in G, oll(a) \in G$, and $nll(a) * a = a$ and $oll(a) * a = nll(a)$.

Definition 6 (see [20]). Let $(G, *)$ be an AG-groupoid. Then, G is called an AG-(r,r)-Loop if, for all $a \in G$, there exist a local (r,r)-neutral element of "a" and a local (r,r)-opposite element of "a" (denoted by $nrr(a)$ and $orr(a)$, respectively), such that $nrr(a) \in G, orr(a) \in G$, and $a * nrr(a) = a$ and $a * orr(a) = nrr(a)$.

Definition 7. Let $(G, *)$ be an AG-groupoid. Then, G is called an AG-(l,r)-Loop if, for all $a \in G$, there exist a local (l,r)-neutral element of "a" and a local (l,r)-opposite element of "a" (denoted by $nlr(a)$ and $olr(a)$, respectively), such that $nlr(a) \in G, olr(a) \in G$, and $nlr(a) * a = a$ and $a * olr(a) = nlr(a)$.

Remark 1. For quasi AG-NET-Loop, we will use the notations such as AG-NET-Loop. If $nlr(a)$ and $olr(a)$ are not unique, then the set of all local (l,r)-neutral elements of"a" and the set of all local (l,r)-opposite elements of "a" are denoted by $\{nlr(a)\}$ and $\{olr(a)\}$, respectively.

Definition 8. Let $(G, *)$ be an AG-groupoid. Then, G is called an AG-(r,l)-Loop if, for all $a \in G$, there exist a local (r,l)-neutral element of "a" and a local (r,l)-opposite element of"a" (denoted by $nrl(a)$ and $orl(a)$, respectively), such that $nrl(a) \in G, orl(a) \in G$, and $a * nrl(a) = a$ and $orl(a) * a = nrl(a)$.

Definition 9. Let $(G, *)$ be an AG-(l,r)-Loop. Then, G is called an AG-(l,lr)-Loop if, for all $a \in G$, $olr(a) * a = a * olr(a) = nlr(a)$.

Definition 10 (see [22]). An AG-groupoid G with a left identity is called an AG-group if each $a \in G$ has an inverse element a'.

3. AG-(l,r)-Loop and Strong AG-(l,r)-Loop

Theorem 2. *Let $(G, *)$ be a groupoid. Then, G is an AG-(l,r)-Loop iff it is a regular AG-groupoid.*

Proof. Necessity: if G is an AG-(l,r)-Loop, from Definition 7, for all $a \in G$, there exist $nlr(a), olr(a) \in G, nlr(a) * a = a$, and $a * olr(a) = nlr(a)$. We have $(a * olr(a)) * a = a$. By Definition 4, G is a regular AG-groupoid.

Sufficiency: if G is a regular AG-groupoid, from Definition 4, for all $a \in G$, there exists $m \in G$ and $a = (a * m) * a$. Set $nlr(a) = a * m$, by Definition 7, G is an AG-(l,r)-Loop.

Example 1 illustrates that an AG-groupoid may be neither an AG-(l,l)-Loop nor an AG-(l,r)-Loop nor an AG-*(r,r)*-Loop nor an AG-(r,l)-Loop. □

Example 1. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the definition of operation $*$ on G is shown in Table 1. There is no $oll(2), olr(2), orr(2)$, and $orl(2)$ in G. That is, the element "2" in G has no local *(l,l)*-opposite element, no local (l,r)-opposite element, no local *(r,r)*-opposite element, and no local *(r,l)*-opposite element. From Definitions 5–8, G is neither an AG-*(l,l)*-Loop nor an AG-(l,r)-Loop nor an AG-*(r,r)*-Loop nor an AG-(r,l)-Loop.

Example 2 illustrates that an AG-*(l,r)*-Loop may be neither an AG-*(l,l)*-Loop nor an AG-*(r,r)*-Loop nor an AG-*(r,l)*-Loop.

Example 2. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 2. From Definition 7, G is an AG-*(l,r)*-Loop. However, there is no $oll(2), nrr(2)$, and $nrl(2)$ in G. From Definitions 5, 6, and 8, G is neither an AG-*(l,l)*-Loop nor an AG-*(r,r)*-Loop nor an AG-*(r,l)*-Loop.

Definition 11. An AG-*(l,r)*-Loop $(G, *)$ is called a strong AG-*(l,r)*-Loop if, for all $a \in G, nlr(a)^2 = nlr(a)$.

Example 3 illustrates that an AG-*(l,r)*-Loop is not always a strong AG-*(l,r)*-Loop.

Example 3. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 3. From Definition 7, G is an AG-*(l,r)*-Loop. However, $nlr(2) = 3, 3 * 3 = 1$; thus, G is not a strong AG-*(l,r)*-Loop.

Table 1: Table of Example 1.

*	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	2	1	1	1	1
3	1	1	3	1	1	1	1	1
4	1	2	1	4	1	1	1	1
5	1	1	1	1	5	1	1	1
6	1	1	1	1	1	6	8	8
7	1	1	1	1	1	8	7	8
8	1	1	1	1	1	8	8	8

Table 2: Table of Example 2.

*	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	1	1	4	4	1	1
3	1	1	3	1	3	3	7
4	1	2	1	1	2	1	1
5	1	2	3	4	5	3	7
6	1	1	3	1	3	6	7
7	1	1	7	1	7	7	7

Table 3: Table of Example 3.

*	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	1	3	3	1	3	3
3	1	2	1	2	1	2	2
4	1	2	3	4	5	6	7
5	1	1	1	5	5	1	1
6	1	2	3	6	1	6	6
7	1	2	3	7	1	6	7

Example 4 illustrates that a strong AG-*(l,r)*-Loop is not always an AG-NET-Loop.

Example 4. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 4. By Definition 11, G is a strong AG-*(l,r)*-Loop. However, since $1 * 4 \neq 4 * 1$, G is not an AG-NET-Loop.

Table 4: Table of Example 4.

*	1	2	3	4	5	6	7
1	1	1	3	4	1	1	1
2	1	2	3	4	1	1	1
3	4	4	1	3	4	4	4
4	3	3	4	1	3	3	3
5	1	1	3	4	5	1	1
6	1	1	3	4	1	6	6
7	1	1	3	4	1	6	7

Theorem 3. *Let* $(G, *)$ *be a strong AG-(l,r)-Loop. Then,*

(1) *For all* $a \in G$, $nlr(a)$ *is unique*

(2) *For all* $a \in G$, $nlr(nlr(a)) = nlr(a)$

(3) *For all* $a \in G$ *and for any* $r \in \{olr(a)\}$, $nlr(a) * r \in \{olr(a)\}$

(4) *For all* $a, b \in G$, $nlr(a * b) = nlr(a) * nlr(b)$

Proof

(1) If $(G, *)$ is a strong AG-*(l,r)*-Loop, suppose $a \in G$, there exist $nlr_1, nlr_2 \in \{nlr(a)\}$. By Definition 11, $nlr_1 * a = a, nlr_2 * a = a$, $nlr_1 * nlr_1 = nlr_1$, and $nlr_2 * nlr_2 = nlr_2$, and there exist $olr_1, olr_2 \in G$ which satisfy $a * olr_1 = nlr_1$ and $a * olr_2 = nlr_2$. We have

$$\begin{aligned}
nlr_1 * nlr_2 &= (nlr_1 * nlr_1) * nlr_2 \\
&= (nlr_2 * nlr_1) * nlr_1 \\
&= (nlr_2 * nlr_1) * (a * olr_1) \\
&= (nlr_2 * a) * (nlr_1 * olr_1) \\
&\quad \text{(by the medial law)} \\
&= (nlr_1 * a) * (nlr_1 * olr_1) \\
&= (nlr_1 * nlr_1) * (a * olr_1) \\
&\quad \text{(by the medial law)} \\
&= nlr_1 * nlr_1 = nlr_1, \\
nlr_2 * nlr_1 &= (nlr_2 * nlr_2) * nlr_1 \\
&= (nlr_1 * nlr_2) * nlr_2 \\
&= (nlr_1 * nlr_2) * (a * olr_2) \\
&= (nlr_1 * a) * (nlr_2 * olr_2) \\
&\quad \text{(by the medial law)} \\
&= (nlr_2 * a) * (nlr_2 * olr_2) \\
&= (nlr_2 * nlr_2) * (a * olr_2) \\
&\quad \text{(by the medial law)} \\
&= nlr_2 * nlr_2 = nlr_2, \\
nlr_2 &= nlr_2 * nlr_1 \\
&= (nlr_2 * nlr_2) * nlr_1 \\
&= (nlr_1 * nlr_2) * nlr_2 \\
&= nlr_1 * nlr_2 = nlr_1.
\end{aligned} \tag{1}$$

We know that $nlr_2 = nlr_1$, and $nlr(a)$ is unique.

(2) If $(G, *)$ is a strong AG-*(l,r)*-Loop, from Definition 11, we have, for all $a \in G$, $nlr(a)^2 = nlr(a)$. Thus, $nlr(nlr(a)) = nlr(a)$.

(3) Suppose $r \in \{olr(a)\}$; then,

$$\begin{aligned}
a * (nlr(a) * r) &= (nlr(a) * a) * (nlr(a) * r) \\
&= (nlr(a) * nlr(a)) * (a * r) \quad \text{(by the medial law)} \\
&= nlr(a) * nlr(a) \\
&= nlr(a).
\end{aligned} \tag{2}$$

So, we get $nlr(a) * r \in \{olr(a)\}$.

(4) From Definition 11, we have, for all $a, b \in G$,

$$
\begin{aligned}
a * b &= (nlr(a) * a) * (nlr(b) * b) \\
&= (nlr(a) * nlr(b)) * (a * b), \\
nlr(a) * nlr(b) &= (a * olr(a)) * (b * olr(b)) \\
&= (a * b) * (olr(a) * olr(b)).
\end{aligned} \tag{3}
$$

Therefore, $nlr(a * b) = nlr(a) * nlr(b)$. □

Example 5. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 5. It is a strong AG-*(l,r)*-Loop. We have (corresponding to the results of Theorem 3)

(1) For all $a \in G$, we can verify that $nlr(a)$ is unique.

(2) Being $nlr(nlr(1)) = nlr(1)$, $nlr(nlr(2)) = nlr(2)$, $nlr(nlr(3)) = nlr(3)$, $nlr(nlr(4)) = nlr(4)$, $nlr(nlr(5)) = nlr(5)$, $nlr(nlr(6)) = nlr(6)$, and $nlr(nlr(7)) = nlr(7)$, that is, for all $a \in G$, $nlr(nlr(a)) = nlr(a)$.

(3) For any $a \in G$, let $a = 1$, and we can get $nlr(1) = 1$ and $\{olr(1)\} = \{1, 2, 5, 6, 7\}$. Being $1 * 1 = 1 * 2 = 1 * 5 = 1 * 6 = 1 * 7 = 1 \in \{olr(1)\}$, that is, $nlr(1) * olr(1) \in \{olr(1)\}$, let $a = 3$, and we can get $nlr(3) = 1$, $olr(3) = 3$. Being $1 * 3 = 3 = olr(3)$, that is, $nlr(3) * olr(3) \in \{olr(3)\}$, we can verify other cases; thus, $nlr(a) * r \in \{olr(a)\}$.

(4) For any $a, b \in G$, without loss of generality, let $a = 1$ and $b = 3$; we can get $nlr(1 * 3) = nlr(1) * nlr(3)$. We can verify other cases; thus, $nlr(a * b) = nlr(a) * nlr(b)$.

Table 5: Table of Example 5.

*	1	2	3	4	5	6	7
1	1	1	3	4	1	1	1
2	1	2	3	4	1	1	2
3	4	4	1	3	4	4	4
4	3	3	4	1	3	3	3
5	1	1	3	4	5	1	5
6	1	1	3	4	1	6	1
7	1	2	3	4	5	1	7

Theorem 4. *Let $(G, *)$ be a strong AG-(l,r)-Loop. A binary $\approx$ on G is introduced as follows:*

$$\text{for all } a, b \in G, a \approx b \Leftrightarrow nlr(a) = nlr(b). \tag{4}$$

Then,

(1) The binary $\approx$ on G is an equivalence relation, and the equivalent class contained x is denoted by $[x]_\approx$

(2) For all $x \in G$, $[x]_\approx$ is a sub-AG-group

(3) $G = \cup_{x\in G}[x]_\approx$, that is, each strong AG-(l,r)-Loop can be represented as the union of its disjoint sub-AG-groups

Proof

(1) From the binary $\approx$ definition, it is easy to verify that $\approx$ has the properties of reflexive, symmetric, and transitive. Thus, it is an equivalence relation.

(2) For all $a \in [x]_\approx$, let $nlr(x) = e_x$, and we have $nlr(a) = nlr(x) = e_x$. From Theorem 3 (2), $nlr(e_x) = e_x$, and we have $e_x \in [x]_\approx$:

(i) By Definition 11, we have $e_x * a = nlr(a) * a = a$; thus, e_x is a left identity of $[x]_\approx$.

(ii) For all $a, b, c \in [x]_\approx$, the left invertive law holds directly.

(iii) For all $a, b \in [x]_\approx$, $nlr(a) = nlr(b) = e_x$; from Theorem 3 (4), $nlr(a * b) = nlr(a) * nlr(b) = e_x$; thus, $a * b \in [x]_\approx$.

(iv) For all $a \in [x]_\approx$, let $nlr(a) = e_x$, and suppose $p \in \{olr(a)\}$, $q = nlr(a) * p$; by Theorem 3 (3), we have $q \in \{olr(a)\}$, $a * q = nlr(a) = e_x$, and

$$
\begin{aligned}
nlr(q) &= nlr(nlr(a) * p) \\
&= nlr(nlr(a)) * nlr(p) \quad (\text{by Theorem } 3(4)) \\
&= nlr(a) * nlr(p) \quad (\text{by Theorem } 3(2)) \\
&= nlr(a * p) \quad (\text{by Theorem } 3(4)) \\
&= nlr(nlr(a)) \\
&= nlr(a) \quad (\text{by Theorem } 3(2)) \\
&= e_x.
\end{aligned} \tag{5}
$$

(v) $q * a = (nlr(q) * q) * a = (e_x * q) * a = (a * q) * e_x = e_x$. Thus, $q \in [x]_\approx$ and q is an inverse element of a. From Definition 10, $[x]_\approx$ is a sub-AG-group of G.

(3) By Theorem 3 (1), for all $a \in [x]_\approx$, $nlr(a)$ is unique. Then, $G = \cup_{x\in G}[x]_\approx$. □

Example 6. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the definition of operation $*$ on G is shown in Table 6. $[1]_\approx = \{1, 2, 3, 4\}$ and $[5]_\approx = \{5, 6, 7, 8\}$. $G = [1]_\approx \cup [5]_\approx$, and $[1]_\approx$ and $[5]_\approx$ are sub-AG-groups of G.

Let G be an AG-groupoid; then, a is an idempotent in G if $a \in G$, $a^2 = a$. The set of all idempotents in G is denoted by $E(G)$. An AG-groupoid G is called an AG-band if $G = E(G)$.

From now on, we assume that G is a strong AG-*(l,r)*-Loop, which is the same as Theorem 4. Let Y be an AG-band, $Y \subset G$, and for any $\alpha \in Y$, the equivalent class $[\alpha]_\approx$, which is defined in Theorem 4, will be denoted by S_α, and the elements of S_α will be denoted by $a_\alpha, b_\alpha, \ldots,$.

Theorem 5. *Let $(G, *)$ be a groupoid, Y be an AG-band, $Y \subset G$. $G = \cup_{\alpha\in Y}S_\alpha$, $(S_\alpha, *)$ is a strong AG-(l,r)-Loop with a left identity e_α for each $\alpha \in Y$, and $S_\alpha \cap S_\beta = \varnothing$, $\alpha, \beta \in Y$ and $\alpha \neq \beta$. If, for all $a_\alpha \in S_\alpha$, for all $b_\beta \in S_\beta$, $a_\alpha * b_\beta = a_\alpha * e_\alpha$, and $b_\beta * a_\alpha = a_\alpha$, then G is a strong AG-(l,r)-Loop.*

Table 6: Table of Example 6.

*	1	2	3	4	5	6	7	8
1	1	2	3	4	1	1	1	1
2	2	1	4	3	2	2	2	2
3	4	3	2	1	4	4	4	4
4	3	4	1	2	3	3	3	3
5	1	2	3	4	5	6	7	8
6	1	2	3	4	6	5	8	7
7	1	2	3	4	8	7	6	5
8	1	2	3	4	7	8	5	6

Proof. Suppose $G=\cup_{\alpha\in Y}S_\alpha$ is the groupoid, Y is an AG-band, for each $\alpha\in Y$, and S_α is a strong AG-*(l,r)*-Loop with a left identity e_α and $S_\alpha\cap S_\beta=\varnothing$ if $\alpha\neq\beta$ in Y.

We first prove that G is an AG-groupoid. Let $a_\alpha\in S_\alpha$, $b_\beta\in S_\beta$, and $c_\gamma\in S_\gamma$ be arbitrary elements. Since S_α, S_β, and S_γ are strong AG-*(l,r)*-Loops, we have

$$\begin{aligned}\left(a_\alpha * b_\beta\right) * c_\gamma &= \left(a_\alpha * e_\alpha\right) * c_\gamma\\ &= \left(a_\alpha * e_\alpha\right) * e_\alpha\\ &= \left(e_\alpha * e_\alpha\right) * a_\alpha \quad \text{(by the left invertive law)}\\ &= e_\alpha * a_\alpha = a_\alpha,\end{aligned} \tag{6}$$

where $(c_\gamma * b_\beta) * a_\alpha = b_\beta * a_\alpha = a_\alpha = (a_\alpha * b_\beta) * c_\gamma$. Since S_α is a strong AG-*(l,r)*-Loop, the left invertive law holds directly for elements $a_\alpha, b_\alpha, c_\alpha\in S_\alpha$. Thus, G is an AG-groupoid.

For any $b_\beta\in S_\beta$, we have $nlr(b_\beta)=e_\beta$ and $olr(b_\beta) * b_\beta = b_\beta * olr(b_\beta)=e_\beta$. Let $x\in G-S_\beta$, we denote e_x is the left identity in $[x]_\approx$, $LS_\beta=\left\{x|x*b_\beta=x*e_x, b_\beta*x=x, x\in G-S_\beta\right\}$, and $RS_\beta=\left\{x|x*b_\beta=b_\beta, b_\beta*x=b_\beta*e_\beta, x\in G-S_\beta\right\}$. Being $S_\alpha\cap S_\beta=\varnothing$ if $\alpha\neq\beta$ in Y, we can get $LS_\beta\cap S_\beta\cap RS_\beta=\varnothing$ and $LS_\beta\cup S_\beta\cup RS_\beta=G$.

Depending on S_β, we have three cases to discuss. □

case 1. $LS_\beta=G-S_\beta, RS_\beta=\varnothing, x\in LS_\beta, x*b_\beta=x*e_x$, and $b_\beta*x=x$. Being $S_\alpha\cap S_\beta=\varnothing$ if $\alpha\neq\beta$ in Y, we can get $x*e_x\in[x]_\approx, x*b_\beta\notin S_\beta$. That is, there is no element $x\notin S_\beta$ such that $x*b_\beta=b_\beta$.

case 2. $LS_\beta=\varnothing, RS_\beta=G-S_\beta, x\in RS_\beta,\ x*b_\beta=b_\beta$, and $b_\beta * x=b_\beta*e_\beta$. Being $S_\alpha\cap S_\beta=\varnothing$ if $\alpha\neq\beta$ in Y, we can get $b_\beta*x=b_\beta*e_\beta\in S_\beta$. That is, there is no element $x\notin S_\beta$ such that $x*b_\beta=b_\beta$ and $b_\beta*y=x$, and there exists $y\in G-S_\beta$.

case 3. $LS_\beta\neq\varnothing$ and $RS_\beta\neq\varnothing$, when $x\in LS_\beta, x*b_\beta=x*e_x\notin S_\beta$, and $b_\beta*x=x\notin RS_\beta$; when $x\in RS_\beta, x*b_\beta=b_\beta$, $b_\beta*x=b_\beta*e_\beta\notin RS_\beta$. That is, there is no element $x\notin S_\beta$ such that $x*b_\beta=b_\beta$ and $b_\beta*y=x$, and there exists $y\in G-S_\beta$.

From all the above cases, b_β has a unique $nlr(b_\beta)=e_\beta$ and $\left\{olr(b_\beta)\right\}\subseteq S_\beta$. Consequently, G is a strong AG-*(l,r)*-Loop.

Example 7. Let $G=\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$, and the definition of operation $*$ on G is shown in Table 7. An AG-band $Y=\{1,5,9\}$ and $S_1=\{1,2,3,4\}, e_1=1$, $S_5=\{5,6,7,8\}, e_5=5$, and $S_9=\{9,10,11,12,13\}, e_9=9$. For any $a_1\in S_1, b_5\in S_5$, and $c_9\in S_9$, without losing generality, let $a_1=3, b_5=7$, and $c_9=10$, and we have $3*7=3*1$ and $7*3=3$, $3*10=3*1$ and $10*3=3$, $7*10=7*5$ and $10*7=7$, and $(3*7)*10=(10*7)*3$. The other cases can be verified; thus, G is an AG-groupoid.

Table 7: Table of Example 7.

*	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	2	3	4	1	1	1	1	1	1	1	1	1
2	2	1	4	3	2	2	2	2	2	2	2	2	2
3	4	3	2	1	4	4	4	4	4	4	4	4	4
4	3	4	1	2	3	3	3	3	3	3	3	3	3
5	1	2	3	4	5	6	7	8	5	5	5	5	5
6	1	2	3	4	6	5	8	7	6	6	6	6	6
7	1	2	3	4	8	7	5	6	8	8	8	8	8
8	1	2	3	4	7	8	6	5	7	7	7	7	7
9	1	2	3	4	5	6	7	8	9	10	11	12	13
10	1	2	3	4	5	6	7	8	10	11	12	13	9
11	1	2	3	4	5	6	7	8	11	12	13	9	10
12	1	2	3	4	5	6	7	8	12	13	9	10	11
13	1	2	3	4	5	6	7	8	13	9	10	11	12

Let $c_9=10$, $LS_9=G-S_9=\{1,2,3,4,5,6,7,8\}$ and $RS_9=\varnothing$; for all $x\in LS_9$, there is no element x such that $x*10=10$. That is, the element "10" has a unique $nlr(10)=9$ and $\{olr(10)\}=\{13\}\subseteq S_9$.

Let $a_1=3$, $LS_1=\varnothing, RS_1=G-S_1=\{5,6,7,8,9,10,11,12,13\}$; for all $x\in RS_1$, $3*x=3*e_1=3*1=4\notin RS_1$; thus, there is no element x such that there exists $y\in RS_1, x*3=3, 3*y=x$. That is, the element "3" has a unique $nlr(3)=1$ and $\{olr(3)\}=\{4\}\subseteq S_1$.

Let $b_5=7$, $LS_5=\{1,2,3,4\}$, and $RS_5=\{9,10,11,12,13\}$, when $x\in LS_5, x*7=x*e_x\notin S_5, 7*x=x\notin RS_5$; when $x\in RS_5, x*7=7, 7*x=7*e_5=7*5=8\notin RS_5$. That is, there is no element $x\notin S_5$ such that $x*7=7, 7*y=x$, and there exists $y\in G-S_5$. The element "7" has a unique $nlr(7)=5$ and $\{olr(7)\}=\{7\}\subseteq S_5$.

The other cases can be verified; thus, G is a strong AG-*(l,r)*-Loop.

Theorem 6. *Let* $(G, *)$ *be a groupoid,* Y *be an AG-band,* $Y\subset G$. $G=\cup_{\alpha\in Y}S_\alpha$, $(S_\alpha, *)$ *be a strong AG-(l,r)-Loop with a left identity* e_α *for each* $\alpha\in Y$, *and* $S_\alpha\cap S_\beta=\varnothing$, $\alpha,\beta\in Y, \alpha\neq\beta$. *If, for all* $a_\alpha\in S_\alpha$, *for all* $b_\beta\in S_\beta$, $a_\alpha*b_\beta=b_\beta, b_\beta*a_\alpha=b_\beta*e_\beta$, *then* G *is a strong AG-(l,r)-Loop.*

Proof. Theorem 6 is proved similarly to Theorem 5.

The strong AG-(l,r)-Loop constructed by Theorem 5 is not isomorphic to the strong AG-(l,r)-Loop constructed by Theorem 6. □

Definition 12 (see [20]). An AG-*(l,l)*-Loop $(G, *)$ is called a strong AG-*(l,l)*-Loop if for all $a\in G, nll(a)^2=nll(a)$.

Example 8 illustrates that an AG-(l,l)-Loop is not always a strong AG-(l,l)-Loop.

Example 8. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the definition of operation $*$ on G is shown in Table 8. From Definitions 5 and 7, G is both an AG-*(l,l)*-Loop and an AG-*(l,r)*-Loop. However, $nll(1) = nlr(1) = 3, 3 * 3 = 4 \neq 3$; thus, it is neither a strong AG-*(l,l)*-Loop nor a strong AG-*(l,r)*-Loop.

Table 8: Table of Example 8.

*	1	2	3	4	5	6	7	8
1	2	4	3	1	7	5	6	8
2	3	1	2	4	6	8	7	5
3	1	3	4	2	8	6	5	7
4	4	2	1	3	5	7	8	6
5	8	6	5	7	6	8	7	5
6	5	7	8	6	7	5	6	8
7	7	5	6	8	5	7	8	6
8	6	8	7	5	8	6	5	7

Theorem 7. *Let $(G, *)$ be an AG-groupoid. Then, the following three statements are equivalent:*

(1) G is a strong AG-(l,r)-Loop

(2) G is a strong AG-(l,l)-Loop

(3) G is an AG-(l,lr)-Loop

Proof

(1) $\Longrightarrow$ (2). Suppose G is a strong AG-*(l,r)*-Loop; from Definition 11, for all $a \in G$, there exist $nlr(a), olr(a) \in G$, $nlr(a) * a = a$, $a * olr(a) = nlr(a)$, and $nlr(a)^2 = nlr(a)$. Let $d = nlr(a) * olr(a)$, and we have $d * a = (nlr(a) * olr(a)) * a = (a * olr(a)) * nlr(a) = nlr(a)^2 = nlr(a)$. From Definition 12, G is a strong AG-*(l,l)*-Loop.

(2) $\Longrightarrow$ (3). Suppose G is a strong AG-*(l,l)*-Loop; from Definition 12, for all $a \in G$, there exist $nll(a), oll(a) \in G$, $nll(a) * a = a$, $oll(a) * a = nll(a)$, and $nll(a)^2 = nll(a)$. So, $a * oll(a) = (nll(a) * a) * oll(a) = (oll(a) * a) * nll(a) = nll(a)^2 = nll(a)$. By Definition 9, G is an AG-*(l,lr)*-Loop.

(3) $\Longrightarrow$ (1). If G is an AG-*(l,lr)*-Loop, from Definition 9, for all $a \in G$, there exist $nlr(a), olr(a) \in G$, $nlr(a) * a = a$, and $olr(a) * a = a * olr(a) = nlr(a)$. So, $nlr(a) * nlr(a) = (olr(a) * a) * nlr(a) = (nlr(a) * a) * olr(a) = a * olr(a) = nlr(a)$. By Definition 11, G is a strong AG-*(l,r)*-Loop.

Figure 1 shows the relationships among AG-(l,l)-Loop and AG-(l,r)-Loop. Here, A stands for AG-NET-Loop, B stands for strong AG-(l,r)-Loop shown in Example 4 rather than AG-NET-Loop, C stands for AG-(l,r)-Loop and AG-(l,l)-Loop shown in Example 8, which is, however, not strong AG-(l,r)-Loop, D stands for AG-(l,l)-Loop rather than AG-(l,r)-Loop, E stands for AG-(l,r)-Loop shown in Example 2 rather than AG-(l,l)-Loop, and F stands for AG-groupoid shown in Example 1, which is, however, not either AG-(l,l)-Loop or AG-(l,r)-Loop. A + B stands for strong AG-(l,r)-Loop, A + B + C + D stands for AG-(l,l)-Loop, A + B + C + E stands for AG-(l,r)-Loop, and A + B + C + D + E + F stands for AG-groupoid. □

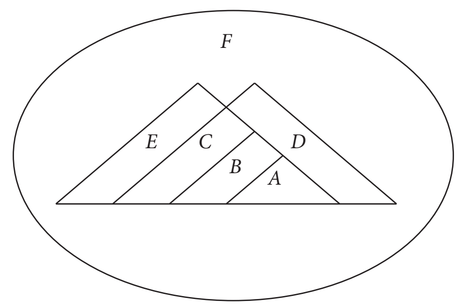


Figure 1: The relationships among AG-*(l,l)*-Loop and AG-*(l,r)*-Loop.

4. AG-*(r,r)*-Loop and AG-*(r,l)*-Loop

Theorem 8. *Let $(G, *)$ be an AG-(r,r)-Loop. Then,*

(1) G is an AG-(r,l)-Loop

(2) G is an AG-(l,l)-Loop

Proof

(1) Suppose G is an AG-*(r,r)*-Loop; from Definition 6, for all $a \in G$, there exist $nrr(a), orr(a) \in G$, $a * nrr(a) = a$, and $a * orr(a) = nrr(a)$. Let $q = orr(a) * nrr(a)$, and we have $q * a = (orr(a) * nrr(a)) * a = (a * nrr(a)) * orr(a) = a * orr(a) = nrr(a)$. By Definition 8, G is an AG-*(r,l)*-Loop.

(2) Suppose G is an AG-*(r,r)*-Loop; from Definition 6, for all $a \in G$, there exist $nrr(a), orr(a) \in G$, $a * nrr(a) = a$, and $a * orr(a) = nrr(a)$. Let $d = nrr(a)^2$ and $q = nrr(a) * orr(a)$, and we have $d * a = (nrr(a) * nrr(a)) * a = (a * nrr(a)) * nrr(a) = a * nrr(a) = a$ and $q * a = (nrr(a) * orr(a)) * a = (a * orr(a)) * nrr(a) = nrr(a) * nrr(a) = d$.

By Definition 5, G is an AG-*(l,l)*-Loop. □

Definition 13. An AG-(r,r)-Loop $(G, *)$ is called a strong AG-(r,r)-Loop if for all $a \in G, nrr(a)^2 = nrr(a)$.

Definition 14. An AG-*(r,l)*-Loop $(G, *)$ is called a strong AG-*(r,l)*-Loop if for all $a \in G, nrl(a)^2 = nrl(a)$.

Example 9 illustrates that an AG-*(r,r)*-Loop is not always a strong AG-*(r,r)*-Loop and an AG-*(r,l)*-Loop is not always a strong AG-(r,l)-Loop.

Example 9. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the definition of operation $*$ on G is shown in Table 9. From Definitions 6, 8, 5, and 7, G is both an AG-(r,r)-Loop and an AG-(r,l)-Loop and an AG-(l,l)-Loop and AG-(l,r)-Loop. However, $nrr(1) = 4, nrl(1) = 4, 4 * 4 = 3 \neq 4$; $nll(1) = 3, nlr(1) = 3, 3 * 3 = 4 \neq 3$. Thus, G is neither a strong AG-(r,r)-Loop nor a

Table 9: Table of Example 9.

$*$	1	2	3	4	5	6	7	8
1	2	4	3	1	3	1	2	4
2	3	1	2	4	2	4	3	1
3	1	3	4	2	4	2	1	3
4	4	2	1	3	1	3	4	2
5	1	3	4	2	6	8	7	5
6	4	2	1	3	7	5	6	8
7	2	4	3	1	5	7	8	6
8	3	1	2	4	8	6	5	7

strong AG-(r,l)-Loop nor a strong AG-(l,l)-Loop nor a strong AG-(l,r)-Loop.

Theorem 9. *Let $(G, *)$ be an AG-groupoid. Then, the following three statements are equivalent:*

(1) G is a strong AG-(r,r)-Loop

(2) G is a strong AG-(r,l)-Loop

(3) G is an AG-NET-Loop

Proof

(1) $\Longrightarrow$(2). Suppose G is a strong AG-*(r,r)*-Loop; from Definition 13, for all $a \in G$, there exist $nrr(a)$, $orr(a) \in G$, $a * nrr(a) = a$, $a * orr(a) = nrr(a)$, and $nrr(a)^2 = nrr(a)$. Let $q = orr(a) * nrr(a)$, and we have $q * a = (orr(a) * nrr(a)) * a = (a * nrr(a)) * orr(a) = a * orr(a) = nrr(a)$. By Definition 14, G is a strong AG-*(r,l)*-Loop.

(2) $\Longrightarrow$(3). Suppose G is a strong AG-*(r,l)*-Loop; from Definition 14, for all $a \in G$, there exist $nrl(a), orl(a) \in G$, $a * nrl(a) = a$, $orl(a) * a = nrl(a)$, and $nrl(a)^2 = nrl(a)$. So, $nrl(a) * a = (nrl(a) * nrl(a)) * a = (a * nrl(a)) * nrl(a) = a * nrl(a) = a$ and $a * orl(a) = (nrl(a) * a) * orl(a) = (orl(a) * a) * nrl(a) = nrl(a)^2 = nrl(a)$. By Definition 3, G is an AG-NET-Loop.

(3) $\Longrightarrow$(1). It is obvious that an AG-NET-Loop is a strong AG-*(r,r)*-Loop.

Figure 2 shows the relationships among AG-(r,l)-Loop and AG-(l,r)-Loop. Here, A stands for AG-NET-Loop, B stands for AG-(r,l)-Loop and strong AG-(l,r)-Loop shown in Example 4, which is, however, not AG-NET-Loop, C stands for AG-(r,l)-Loop and AG-(l,r)-Loop shown in Example 9, which is, however, not strong AG-(l,r)-Loop, D stands for AG-(r,l)-Loop rather than AG-(l,r)-Loop, E stands for strong AG-(l,r)-Loop rather than AG-(r,l)-Loop, F stands for AG-(l,r)-Loop shown in Example 2, which is, however, not either AG-(r,l)-Loop or strong AG-(l,r)-Loop, and G stands for AG-groupoid shown in Example 1, which is, however, not either AG-(l,r)-Loop or AG-(r,l)-Loop. A + B + E stands for strong AG-(l,r)-Loop, A + B + C + D stands for AG-(r,l)-Loop, A + B + C + E + F stands for AG-(l,r)-Loop, and A + B + C + D + E + F + G stands for AG-groupoid.

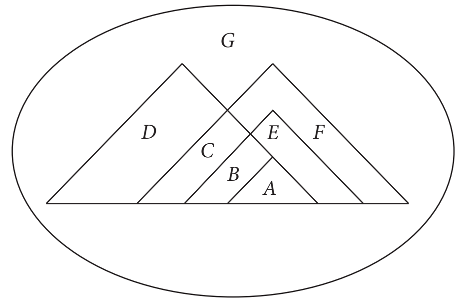


Figure 2: The relationships among AG-*(r,l)*-Loop and AG-*(l,r)*-Loop.

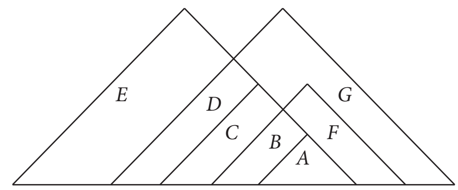


Figure 3: The relationships among AG-(r,l)-Loop and AG-(l,l)-Loop.

Figure 3 shows the relationships among AG-(r,l)-Loop and AG-(l,l)-Loop. Here, A stands for AG-NET-Loop, B stands for AG-(r,r)-Loop and strong AG-(l,l)-Loop shown in Example 4, which is, however, not AG-NET-Loop, C stands for AG-(r,r)-Loop shown in Example 9 rather than strong AG-(l,l)-Loop, D stands for AG-(r,l)-Loop and AG-(l,l)-Loop rather than AG-(r,r)-Loop, E stands for AG-(r,l)-Loop rather than AG-(l,l)-Loop, F stands for strong AG-(l,l)-Loop rather than AG-(r,l)-Loop, and G stands for AG-(l,l)-Loop, which is, however, not either AG-(r,l)-Loop or a strong AG-(l,l)-Loop. A + B + C stands for AG-(r,r)-Loop, A + B + F stands for strong AG-(l,l)-Loop, A + B + C + D + E stands for AG-(r,l)-Loop, and A + B + C + D + F + G stands for AG-(l,l)-Loop. □

5. Alternative Quasi AG-NET-Loop

Definition 15. Let $(G, *)$ be an AG-NET-Loop (AG-(l,l)-Loop, AG-(l,r)-Loop, AG-(r,r)-Loop, and AG-(r,l)-Loop). Then, G is called a right alternative AG-NET-Loop (AG-(l,l)-Loop, AG-(l,r)-Loop, AG-(r,r)-Loop, and AG-(r,l)-Loop) if $b * (a * a) = (b * a) * a$, for all $a, b \in G$.

Definition 16. Let $(G, *)$ be an AG-NET-Loop (AG-(l,l)-Loop, AG-(l,r)-Loop, AG-(r,r)-Loop, and AG-(r,l)-Loop). Then, G is called an alternative AG-NET-Loop (AG-(l,l)-Loop, AG-(l,r)-Loop, AG-(r,r)-Loop, and AG-(r,l)-Loop), if for all $a, b \in G$, $(a * a) * b = a * (a * b)$, $a * (b * b) = (a * b) * b$.

Example 10 illustrates that an AG-NET-Loop is not always an alternative AG-NET-Loop.

Example 10. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 10. By Definition 3, G is

Table 10: Table of Example 10.

$*$	1	2	3	4	5	6	7
1	1	4	2	3	3	1	2
2	3	2	4	1	1	3	4
3	4	1	3	2	2	4	3
4	2	3	1	4	4	2	1
5	2	3	1	4	5	2	1
6	1	4	2	3	3	6	2
7	4	1	3	2	2	4	7

an AG-NET-Loop. However, G is not an alternative AG-NET-Loop because $(3*4)*4 \neq 3*(4*4)$.

Theorem 10. *Let $(G,*)$ be an AG-NET-Loop. Then, the following three statements are equivalent:*

(1) G is a right alternative AG-NET-Loop
(2) G is a commutative AG-NET-Loop
(3) G is an alternative AG-NET-Loop

Proof

(1) $\Rightarrow$(2). Suppose G is a right alternative AG-NET-Loop; from Definition 15, for all $a,b\in G$,

$$\begin{aligned} a*\text{neut}(b) &= a*(\text{neut}(b)*\text{neut}(b)) && \text{(by Theorem 1 (2))}\\ &= (a*\text{neut}(b))*\text{neut}(b) && \text{(by the right alternative law)}\\ &= (\text{neut}(b)*\text{neut}(b))*a && \text{(by the left invertive law)}\\ &= \text{neut}(b)*a, \end{aligned} \tag{7}$$

so

$$\begin{aligned} a*b &= (\text{neut}(a)*a)*(b*\text{neut}(b))\\ &= (\text{neut}(a)*b)*(a*\text{neut}(b)) \quad \text{(by the medial law)}\\ &= (b*\text{neut}(a))*(\text{neut}(b)*a)\\ &= (b*\text{neut}(b))*(\text{neut}(a)*a)\\ &= b*a. \end{aligned} \tag{8}$$

Consequently, G is a commutative AG-NET-Loop.

(2) $\Rightarrow$(3). If G is a commutative AG-NET-Loop, for all $m,n\in G$, $m*(n*n)=(n*n)*m=(m*n)*n$ and $(m*m)*n=(n*m)*m=m*(n*m)=m*(m*n)$. By Definition 16, G is an alternative AG-NET-Loop.

(3) $\Rightarrow$(1). It is obvious that an alternative AG-NET-Loop is a right alternative AG-NET-Loop. □

Theorem 11 (see [23]). *Let $(G,*)$ be a locally associative AG-groupoid. If G is finite, then there exists $a\in G, a^2=a$.*

Theorem 12. *Let $(G,*)$ be a right alternative AG-(r,l)-Loop. If G is finite, then, for all $a\in G$, there exist $s,p\in G, a*s=a, p*a=s$, and $s^2=s$.*

Proof. If G is a finite right alternative AG-*(r,l)*-Loop. Then, for all $a\in G$, there exist $s,p\in G, a*s=a$, and $p*a=s$, and we have $a*s^2=a*(s*s)=(a*s)*s=a*s=a$.

When $k\in Z^+$, $k>2$,

$$\begin{aligned} a*s^k &= (a*s)*\left(s^2*s^{k-2}\right)\\ &=\left(a*s^2\right)*\left(s*s^{k-2}\right) \quad \text{(by the medial law)}\\ &= a*s^{k-1}\\ &= \cdots\cdots\\ &= a*s^2=a. \end{aligned} \tag{9}$$

Thus, $s,s^2,s^3,\ldots,s^k,\ldots$ are all right neutral element.

By Theorem 11, we get that there is an idempotent right neutral element in G. □

Theorem 13 (see [23]). *Let $(G,*)$ be a finite alternative AG-(l,l)-Loop. Then, G is a strong AG-(l,l)-Loop.*

Theorem 14. *Let $(G,*)$ be an AG-groupoid. Then, the following three statements are equivalent:*

(1) G is a finite right alternative AG-(r,l)-Loop
(2) G is a finite alternative AG-NET-Loop
(3) G is a finite alternative AG-(l,l)-Loop

Proof

(1) $\Rightarrow$(2). If G is a finite right alternative AG-*(r,l)*-Loop, applying Theorem 12, we get that G is a strong AG-*(r,l)*-Loop. From Theorem 9, we get that G is a right alternative AG-NET-Loop. Applying Theorem 10, G is a finite alternative AG-NET-Loop.

(2) $\Rightarrow$(3). It is obvious that a finite alternative AG-NET-Loop is a finite alternative AG-*(l,l)*-Loop.

(3) $\Rightarrow$(1). If G is a finite alternative AG-*(l,l)*-Loop, applying Theorem 13, we get that G is a strong AG-(l,l)-Loop. From Definition 12, for all $a\in G$, there exist $nll(a), oll(a)\in G$, $nll(a)*a=a$, $oll(a)*a=nll(a)$, and $nll(a)^2=nll(a)$. We have

Table 11: Table of Example 11.

$*$	1	2	3	4	5	6	7
1	2	5	4	1	3	1	1
2	5	3	1	2	4	2	2
3	4	1	5	3	2	3	3
4	1	2	3	4	5	4	4
5	3	4	2	5	1	5	5
6	1	2	3	4	5	6	4
7	1	2	3	4	5	4	7

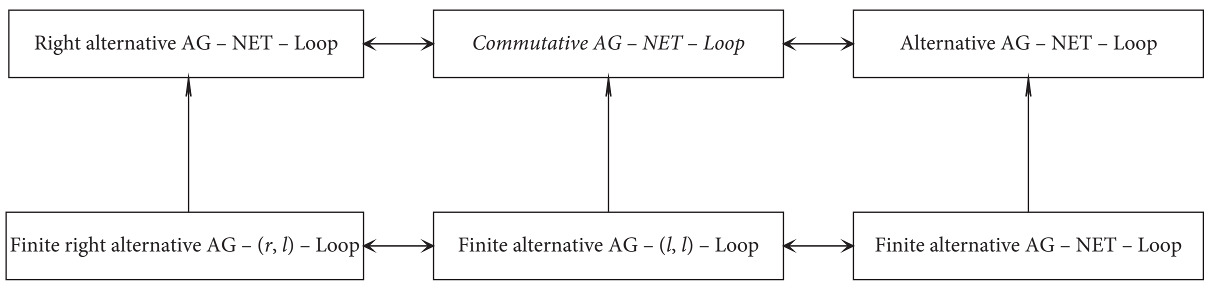


Figure 4: The relationships among alternative AG-NET-Loop and other alternative quasi AG-NET-Loops.

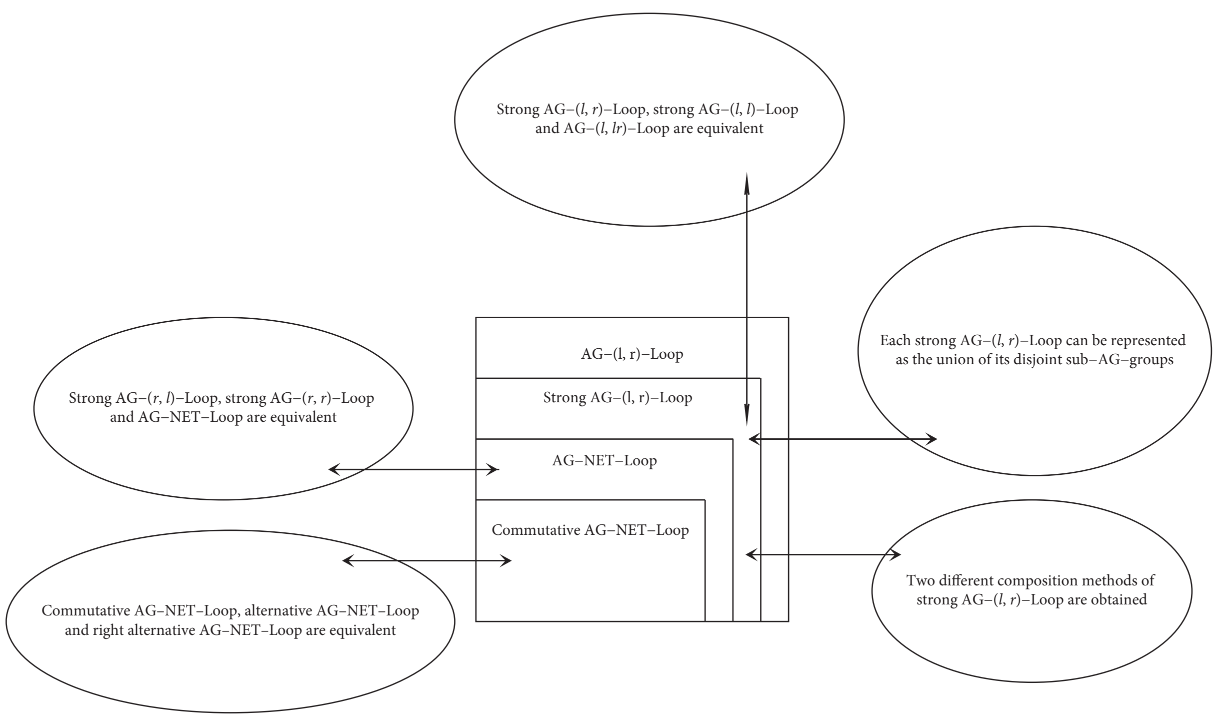


Figure 5: The main results of this paper.

$$\begin{aligned} a * nll(a) &= a * (nll(a) * nll(a)) \\ &= (a * nll(a)) * nll(a) \quad \text{(by the right alternative law)} \\ &= (nll(a) * nll(a)) * a \quad \text{(by the left invertive law)} \\ &= nll(a) * a = a. \end{aligned} \tag{10}$$

By Definition 15, G is a finite right alternative AG-*(r,l)*-Loop. □

Example 11. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 11. We can easily verify that G satisfies the alternative law. Being each element in G has a neutral element and an opposite element; by Definition 16, G is a finite alternative AG-NET-Loop. Obviously, a finite alternative AG-NET-Loop is both a finite right alternative AG-(r,l)-Loop and a finite alternative AG-(l,l)-Loop. Since for all $a, b \in G$ and $a * b = b * a$, we have G as a commutative AG-NET-Loop.

Figure 4 shows the relationships among alternative AG-NET-Loop and other alternative quasi AG-NET-Loops. In

Figure 4, we prove that the right alternative AG-NET-Loop is equivalent to the commutative AG-NET-Loop, and the commutative AG-NET-Loop is equivalent to the alternative AG-NET-Loop. As the finite right alternative AG-*(r,l)*-Loop is equivalent to the finite alternative AG-*(l,l)*-Loop, the finite alternative AG-*(l,l)*-Loop is equivalent to the finite alternative AG-NET-Loop; therefore, they are equivalent to each other.

6. Conclusion

In this paper, the AG-*(l,r)*-Loop and AG-*(r,l)*-Loop have been introduced, the structure of the quasi AG-NET-Loops have been studied further, and some important results have been obtained. We prove that the strong AG-*(l,r)*-Loop, the strong AG-*(l,l)*-Loop, and the AG-*(l,lr)*-Loop are equivalent (see Theorem 7); the strong AG-(*r,l)*-Loop, the strong AG-*(r,r)*-Loop, and the AG-NET-Loop are equivalent (see Theorem 9); the commutative AG-NET-Loop, the alternative AG-NET-Loop, and the right alternative AG-NET-Loop are equivalent (see Theorem 10). Furthermore, the decomposition theorem of strong AG-*(l,r)*-Loop (see Theorem 4) and two different ways how to make a strong AG-*(l,r)*-Loop are obtained (see Theorem 5 and Theorem 6), thus illuminating the structure of strong AG-*(l,r)*-Loop. Figure 5 shows the main results of this paper. Future efforts will be directed towards discussing the relationship between strong AG-*(l,r)*-Loop and other related AG-groupoid bands, such as root of band, AG-4-band, and AG-3-band (see [24]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research was supported by Scientific Research Program funded by Shaanxi Provincial Education Department (Program No. 20JK0549).

References

[1] M. Kazim and M. Naseeruddin, "On almost semigroups," *The Aligarh Bulletin of Mathematics*, vol. 2, pp. 1–7, 1972.

[2] M. Khan, Faisal, and V. Amjid, "On some classes of Abel-Grassmann's groupoids," *Journal of Advanced Research in Pure Mathematics*, vol. 3, pp. 109–119, 2011.

[3] Faisal, A. Khan, and B. Davvaz, "On fully regular AG-groupoids," *Afrika Matematika*, vol. 25, no. 2, pp. 449–459, 2014.

[4] M. Rashad, I. Ahmad, M. Shah, and Z. U. A. Khuhro, "Left transitive AG-groupoids," *Sindh University Research Journal (Science Series)*, vol. 46, pp. 547–552, 2014.

[5] M. Iqbal and I. Ahmad, "Ideals in CA-AG-groupoids," *Indian Journal of Pure and Applied Mathematics*, vol. 49, no. 2, pp. 265–284, 2018.

[6] F. Smarandache, "Neutrosophic set–a generalization of the intuitionistic fuzzy set," *International Journal of Pure and Applied Mathematics*, vol. 24, pp. 287–297, 2005.

[7] M. Abdel-Basset, A. Gamal, L. H. Son, and F. Smarandache, "A bipolar neutrosophic multi criteria decision making framework for professional selection," *Applied Sciences*, vol. 10, no. 4, p. 1202, 2020.

[8] K. Mishra, I. Kandasamy, V. Kandasamy, and F. Smarandache, "A novel framework using neutrosophy for integrated speech and text sentiment analysis," *Symmetry*, vol. 12, no. 10, p. 1715, 2020.

[9] J. Kavikumar, D. Nagarajan, S. P. Tiwari, S. Broumi, and F. Smarandache, "Composite neutrosophic finite automata," *Neutrosophic Sets and Systems*, vol. 36, pp. 282–291, 2020.

[10] E. Rashno, B. Minaei-Bidgoli, and Y. Guo, "An effective clustering method based on data indeterminacy in neutrosophic set domain," *Engineering Applications of Artificial Intelligence*, vol. 89, Article ID 103411, 2020.

[11] G. Cai, Y. Guo, W. Chen, H. Zeng, Y. Zhou, and Y. Lu, "Computer-aided detection and diagnosis of microcalcification clusters on full field digital mammograms based on deep learning method using neutrosophic boosting," *Multimedia Tools and Applications*, vol. 79, no. 23-24, pp. 17147–17167, 2020.

[12] X. Zhang, X. Mao, F. Smarandache, and C. Park, "On homomorphism theorem for perfect neutrosophic extended triplet groups," *Information*, vol. 9, no. 9, p. 237, 2018.

[13] X. Zhang, Y. Ma, and P. Yu, "On two conjectures of Abel Grassmann's groupoids," *Symmetry*, vol. 11, no. 6, p. 816, 2019.

[14] X. Zhang and X. Wu, "Involution Abel-Grassmann's groups and filter theory of Abel-Grassmann's groups," *Symmetry*, vol. 11, no. 4, p. 553, 2019.

[15] M. Abdel-Basset, V. Chang, M. Mohamed, and F. Smarandche, "A refined approach for forecasting based on neutrosophic time series," *Symmetry*, vol. 11, no. 4, p. 457, 2019.

[16] W. Yuan and X. Zhang, "Regular CA-groupoids and cyclic associative neutrosophic extended triplet groupoids (CA-NET-groupoids) with green relations," *Mathematics*, vol. 8, no. 2, p. 204, 2020.

[17] M. Hu, F. Smarandache, and X. Zhang, "On neutrosophic extended triplet LA-hypergroups and strong pure LA-semihypergroups," *Symmetry*, vol. 12, no. 1, p. 163, 2020.

[18] X. Zhang, X. Wu, X. Mao, F. Smarandache, and C. Park, "On neutrosophic extended triplet groups (loops) and Abel-Grassmann's groupoids (AG-groupoids)," *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 4, pp. 5743–5753, 2019.

[19] X. Zhang, X. Wu, F. Smarandache, and M. Hu, "Left (right)-quasi neutrosophic triplet loops (groups) and generalized BE-algebras," *Symmetry*, vol. 10, no. 7, p. 241, 2018.

[20] X. Wu and X. Zhang, "The decomposition theorems of AG-neutrosophic extended triplet loops and strong AG-(l,l)-loops," *Mathematics*, vol. 7, p. 268, 2019.

[21] F. Smarandache, *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras and Applications*, Pons Publishing House, Brussels, Belgium, 2017.

[22] P. V. Protić, "Some remarks on Abel-Grassmann's groups," *Quasigroups and Related Systems*, vol. 20, pp. 267–274, 2012.

[23] X. An, X. Zhang, and Y. Ma, "Generalized Abel-Grassmann's neutrosophic extended triplet loop," *Mathematics*, vol. 7, no. 12, p. 1206, 2019.

[24] N. Stevanović and P. V. Protić, "Composition of Abel-Grassmann's 3-bands," *Novi Sad Journal of Mathematics*, vol. 34, pp. 175–182, 2004.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6664330, 13 pages
https://doi.org/10.1155/2021/6664330

Research Article

Multicriteria Decision-Making Method and Application in the Setting of Trapezoidal Neutrosophic Z-Numbers

Rui Yong,[1] Jun Ye ,[1,2] and Shigui Du[1]

[1]*School of Civil and Environmental Engineering, Ningbo University, Ningbo 315211, China*
[2]*School of Mechanical and Electrical Engineering, Shaoxing University, Shaoxing 312000, China*

Correspondence should be addressed to Jun Ye; yejun1@nbu.edu.cn

Received 10 November 2020; Revised 30 November 2020; Accepted 27 January 2021; Published 5 March 2021

Academic Editor: Basil K.Papadopoulos

The information expression and modeling of decision-making are critical problems in the fuzzy decision theory and method. However, existing trapezoidal neutrosophic numbers (TrNNs) and neutrosophic Z-numbers (NZNs) and their multicriteria decision-making (MDM) methods reveal their insufficiencies, such as without considering the reliability measures in TrNN and continuous Z-numbers in NZN. To overcome the insufficiencies, it is necessary that one needs to propose trapezoidal neutrosophic Z-numbers (TrNZNs), their aggregation operations, and an MDM method for solving MDM problems with TrNZN information. Hence, this study first proposes a TrNZN set, some basic operations of TrNZNs, and the score and accuracy functions of TrNZN and their ranking laws. Then, the TrNZN weighted arithmetic averaging (TrNZNWAA) and TrNZN weighted geometric averaging (TrNZNWGA) operators are presented based on the operations of TrNZNs. Next, an MDM approach using the proposed aggregation operators and score and accuracy functions is established to carry out MDM problems under the environment of TrNZNs. In the end, the established MDM approach is applied to an MDM example of software selection for revealing its rationality and efficiency in the setting of TrNZNs. The main advantage of this study is that the established approach not only makes assessment information continuous and reliable but also strengthens the decision rationality and efficiency in the setting of TrNZNs.

1. Introduction

In fuzzy decision-making problems, various new fuzzy decision-making methods [1–3] have received many applications under neutrosophic, simplified neutrosophic hesitant fuzzy, and bipolar neutrosophic environments. Then, triangular and trapezoidal fuzzy numbers are usually used for real decision-making problems because they can be depicted by the continuous fuzzy numbers of membership functions rather than exact/discrete fuzzy values. Hence, some researchers extended triangular fuzzy numbers to intuitionistic fuzzy sets (IFSs) and presented triangular intuitionistic fuzzy sets (TIFSs), where the values of the membership and nonmembership functions are triangular fuzzy numbers, and some triangular intuitionistic fuzzy aggregation operators for multicriteria decision-making (MDM) problems with triangular intuitionistic fuzzy information [4–7]. As the extension of TIFSs, Ye [8] introduced a trapezoidal intuitionistic fuzzy set (TrIFS), in which the values of its membership and nonmembership functions are trapezoidal fuzzy numbers rather than triangular fuzzy numbers, and some prioritized weighted aggregation operators of trapezoidal intuitionistic fuzzy numbers (TrIFNs) for MDM problems with TrIFNs. However, TIFSs and TrIFSs cannot depict inconsistence and indeterminacy information. Hence, Ye [9] generalized TrIFS and proposed a trapezoidal neutrosophic set (TrNS), in which the values of its truth, falsity, and indeterminacy membership functions are trapezoidal fuzzy numbers, to express incomplete, indeterminate, and inconsistent information, and then he presented some basic operations of trapezoidal neutrosophic numbers (TrNNs), score and accuracy functions of TrNNs, and TrNN weighted arithmetic averaging (TrNNWAA) and TrNN weighted geometric averaging (TrNNWGA) operators for MDM problems in the setting of TrNNs. Then, some researchers utilized the integrated approach [10] and defuzzification method [11] for the evaluation and MDM problems with interval-valued TrNNs. Further, Giri et al. [12] applied TOPSIS method in MDM problems with interval-valued TrNNs. Also, Jana et al. [13] and Khatter [14] presented some basic operations of interval-valued TrNNs, score and accuracy functions of an interval-valued TrNN, and the interval-valued TrNNWAA and TrNNWGA operators for MDM problems in the setting of interval-valued TrNNs.

The notion of a Z-number introduced by Zadeh [15] is described by a fuzzy number and its reliability measure to strengthen the reliability of the fuzzy information. After that, Z-numbers have been used for many areas [16–22]. Based on the truth, falsity, and indeterminacy Z-numbers, Du et al. [23] extended the Z-number concept and proposed neutrosophic Z-numbers (NZNs) to enhance the reliability of the neutrosophic information, and then they presented basic operations of NZNs, score and accuracy functions of NZN, and the NZN weighted geometric averaging (NZNWGA) and NZN weighted arithmetic averaging (NZNWAA) operators and further established their MDM method under the environment of NZNs.

However, TrNN is described only by the trapezoidal fuzzy numbers of its truth, falsity, and indeterminacy membership functions without considering their reliability measures, while NZN is depicted only by exact/discrete truth, falsity, and indeterminacy Z-numbers rather than continuous Z-numbers. Hence, TrNN and NZN and their MDM methods reveal their insufficiencies in their information expressions and applications. To express both the continuous Z-numbers of truth, falsity, and indeterminacy membership functions and the reliability measures in MDM problems, it is necessary that this study needs to propose an MDM method based on trapezoidal neutrosophic Z-numbers (TrNZNs) to make up such insufficiencies of existing information expressions and MDM methods in the environments of TrNNs and NZNs. To do so, the main aims of this article are (1) to propose a TrNZN set and some basic operations of TrNZNs, (2) to introduce score and accuracy functions of TrNZN for ranking TrNZNs, (3) to put forward the TrNZNWAA and TrNZNWGA operators for aggregating TrNZNs, (4) to develop a MDM approach using the proposed aggregation operators and score and accuracy functions for solving MDM problems under the environment of TrNZNs, and (5) to apply the established MDM approach to an MDM example of software selection for revealing its efficiency in the setting of TrNZNs.

The rest of the article is composed of the following sections. Section 2 introduces some basic notions of TrNNs as preliminaries of this study. Section 3 proposes a TrNZN set, basic operations of TrNZNs, the score and accuracy functions of TrNZN, and their ranking laws of TrNZNs. Then, the TrNZNWAA and TrNZNWGA operators and their relative properties are presented in section 4. Section 5 develops an MDM approach using the TrNZNWAA and TrNZNWGA operators and score and accuracy functions of TrNZNs. In Section 6, the developed MDM approach is applied to an MDM example of software selection to indicate its efficiency in the setting of TrNZNs. In the end, conclusions and further study are contained in Section 7.

2. Preliminaries of TrNSs

In this section, we introduce preliminaries of TrNSs, including TrNNs, operations of TrNNs, two TrNN weighted aggregation operators, and score and accuracy functions of TrNNs for ranking TrNNs.

Ye [9] first proposed TrNS in a universe set U, which is denoted as

$$\tilde{Y} = \left\{ \left\langle u, \mathrm{TN}_{\tilde{Y}}(u), \mathrm{IN}_{\tilde{Y}}(u), \mathrm{FN}_{\tilde{Y}}(u) \right\rangle, \quad u \in U \right\}, \tag{1}$$

where $\mathrm{TN}_{\tilde{Y}}(u) \subseteq [0,1]$, $\mathrm{IN}_{\tilde{Y}}(u) \subseteq [0,1]$, and $\mathrm{FN}_{\tilde{Y}}(u) \subseteq [0,1]$ are the truth, indeterminacy, and falsity membership functions; then their values are three trapezoidal fuzzy numbers $\mathrm{TN}_{\tilde{Y}}(u) = (\mathrm{TN}_1(u), \mathrm{TN}_2(u), \mathrm{TN}_3(u), \mathrm{TN}_4(u))$: $U \longrightarrow [0, 1]$, $\mathrm{IN}_{\tilde{Y}}(u) = (\mathrm{IN}_1(u), \mathrm{IN}_2(u), \mathrm{IN}_3(u), \mathrm{IN}_4(u))$: $U \longrightarrow [0, 1]$, and $\mathrm{FN}_{\tilde{Y}}(u) = (\mathrm{FN}_1(u), \mathrm{FN}_2(u), \mathrm{FN}_3(u), \mathrm{FN}_4(u))$: $U \longrightarrow [0, 1]$ with the condition $0 \le \mathrm{TN}_4(u) + \mathrm{IN}_4(u) + \mathrm{FN}_4(u) \le 3$ for $u \in U$. For convenience, a TrNN in $\tilde{Y}$ is simply denoted by $\tilde{y}$ = <(TN1, TN2, TN3, TN4), (IN1, IN2, IN3, IN4), (FN1, FN2, FN3, FN4)>.

Regarding two TrNNs $\tilde{y}_1 = \langle (TN_{11}, TN_{12}, TN_{13}, TN_{14}), (IN_{11}, IN_{12}, IN_{13}, IN_{14}), (FN_{11}, FN_{12}, FN_{13}, FN_{14})\rangle$ and $\tilde{y}_2 = \langle (TN_{21}, TN_{22}, TN_{23}, TN_{24}), (IN_{21}, IN_{22}, IN_{23}, IN_{24}), (FN_{21}, FN_{22}, FN_{23}, FN_{24})\rangle$, Ye [14] defined the following basic operations:

(1) $\tilde{y}_1 \oplus \tilde{y}_2 = \langle (TN_{11} + TN_{21} - TN_{11}TN_{21}, TN_{12} + TN_{22} - TN_{12}TN_{22}, TN_{13} + TN_{23} - TN_{13}TN_{23}, TN_{14} + TN_{24} - TN_{14}TN_{24}), (IN_{11}IN_{21}, IN_{12}IN_{22}, IN_{13}IN_{23}, IN_{14}IN_{24}), (FN_{11}FN_{21}, FN_{12}FN_{22}, FN_{13}FN_{23}, FN_{14}FN_{24}) \rangle$

(2) $\tilde{y}_1 \otimes \tilde{y}_2 = \langle (TN_{11}TN_{21}, TN_{12}TN_{22}, TN_{13}TN_{23}, TN_{14}TN_{24}), (IN_{11} + IN_{21} - IN_{11}IN_{21}, IN_{12} + IN_{22} - IN_{12}IN_{22}, IN_{13} + IN_{23} - IN_{13}IN_{23}, IN_{14} + IN_{24} - IN_{14} \quad IN_{24}), (FN_{11} + FN_{21} - FN_{11}FN_{21}, FN_{12} + FN_{22} - FN_{12}FN_{22}, FN_{13} + FN_{23} - FN_{13} \; FN_{23}, FN_{14} + FN_{24} - FN_{14}FN_{24}) \rangle$

(3) $\lambda\tilde{y}_1 = \langle (1 - (1 - TN_{11})^{\lambda}, 1 - (1 - TN_{12})^{\lambda}, 1 - (1 - TN_{13})^{\lambda}, 1 - (1 - TN_{14})^{\lambda}), (IN_{11}^{\lambda}, IN_{12}^{\lambda}, IN_{13}^{\lambda}, IN_{14}^{\lambda}), (FN_{11}^{\lambda}, FN_{12}^{\lambda}, FN_{13}^{\lambda}, FN_{14}^{\lambda}) \rangle$, $\lambda > 0$

(4) $\tilde{y}_1^{\lambda} = \langle (TN_{11}^{\lambda}, TN_{12}^{\lambda}, TN_{13}^{\lambda}, TN_{14}^{\lambda}), (1 - (1 - IN_{11})^{\lambda}, 1 - (1 - IN_{12})^{\lambda}, 1 - (1 - IN_{13})^{\lambda}, 1 - (1 - IN_{14})^{\lambda}), (1 - (1 - FN_{11})^{\lambda}, 1 - (1 - FN_{12})^{\lambda}, 1 - (1 - FN_{13})^{\lambda}, 1 - (1 - FN_{14})^{\lambda}) \rangle$, $\lambda \ge 0$

Regarding a group of TrNNs $\tilde{y}_j = \langle (TN_{j1}, TN_{j2}, TN_{j3}, TN_{j4}), (IN_{j1}, IN_{j2}, IN_{j3}, IN_{j4}), (FN_{j1}, FN_{j2}, FN_{j3}, FN_{j4})\rangle$ $(j = 1, 2, \ldots, n)$ with their weights λ_j $(j = 1, 2, \ldots, n)$ for $\lambda_j \in [0, 1]$ and $\sum_{j=1}^{n} \lambda_j = 1$, Ye [9] proposed the TrNNWAA and TrNNWGA operators:

$$\begin{aligned}
\mathrm{TrNNWAA}(\tilde{y}_1,\tilde{y}_2,\ldots,\tilde{y}_n) &= \mathop{\oplus}_{j=1}^{n} \lambda_j \tilde{y}_j \\
&= \Bigg\langle \Bigg(1-\prod_{j=1}^{n}\left(1-TN_{j1}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-TN_{j2}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-TN_{j3}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-TN_{j4}\right)^{\lambda_j} \Bigg), \\
&\quad \left(\prod_{j=1}^{n} IN_{j1}^{\lambda_j}, \prod_{j=1}^{n} IN_{j2}^{\lambda_j}, \prod_{j=1}^{n} IN_{j3}^{\lambda_j}, \prod_{j=1}^{n} IN_{j4}^{\lambda_j} \right), \left(\prod_{j=1}^{n} FN_{j1}^{\lambda_j}, \prod_{j=1}^{n} FN_{j2}^{\lambda_j}, \prod_{j=1}^{n} FN_{j3}^{\lambda_j}, \prod_{j=1}^{n} FN_{j4}^{\lambda_j} \right) \Bigg\rangle,
\end{aligned} \tag{2}$$

$$\begin{aligned}
\mathrm{TrNNWGA}\left(\tilde{y}_1,\tilde{y}_2,\ldots,\tilde{y}_n\right) &= \mathop{\oplus}_{j=1}^{n} \tilde{y}_j^{\lambda_j} \\
&= \Bigg\langle \left(\prod_{j=1}^{n} TN_{j1}^{\lambda_j}, \prod_{j=1}^{n} TN_{j2}^{\lambda_j}, \prod_{j=1}^{n} TN_{j3}^{\lambda_j}, \prod_{j=1}^{n} TN_{j4}^{\lambda_j} \right), \\
&\quad \left(1-\prod_{j=1}^{n}\left(1-IN_{j1}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-IN_{j2}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-IN_{j3}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-IN_{j4}\right)^{\lambda_j} \right), \\
&\quad \left(1-\prod_{j=1}^{n}\left(1-FN_{j1}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-FN_{j2}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-FN_{j3}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-FN_{j4}\right)^{\lambda_j} \right) \Bigg\rangle.
\end{aligned} \tag{3}$$

Then, the score and accuracy functions of the TrNN $\tilde{y} = <(TN_1, TN_2, TN_3, TN_4), (IN_1, IN_2, IN_3, IN_4), (FN_1, FN_2, FN_3, FN_4)>$ were defined as follows [9]:

$$S(\tilde{y}) = \frac{1}{3}\left(2 + \frac{TN_1+TN_2+TN_3+TN_4}{4} - \frac{IN_1+IN_2+IN_3+IN_4}{4} - \frac{FN_1+FN_2+FN_3+FN_4}{4}\right), \quad S(\tilde{y}) \in [0,1], \tag{4}$$

$$H(\tilde{y}) = \frac{TN_1+TN_2+TN_3+TN_4}{4} - \frac{FN_1+FN_2+FN_3+FN_4}{4}, \quad H(\tilde{y}) \in [-1,1]. \tag{5}$$

Based on the score and accuracy functions of TrNNs, the ranking relations between two TrNNs $\tilde{y}_1 = <(TN_{11}, TN_{12}, TN_{13}, TN_{14}), (IN_{11}, IN_{12}, IN_{13}, IN_{14}), (FN_{11}, FN_{12}, FN_{13}, FN_{14})>$ and $\tilde{y}_2 = <(TN_{21}, TN_{22}, TN_{23}, TN_{24}), (IN_{21}, IN_{22}, IN_{23}, IN_{24}), (FN_{21}, FN_{22}, FN_{23}, FN_{24})>$ were defined as follows [9]:

(1) $\tilde{y}_1 \succ \tilde{y}_2$ for $S(\tilde{y}_1) = S(\tilde{y}_2)$

(2) $\tilde{y}_1 \succ \tilde{y}_2$ for $S(\tilde{y}_1) = S(\tilde{y}_2)$ and $H(\tilde{y}_1) > H(\tilde{y}_2)$

(3) $\tilde{y}_1 \cong \tilde{y}_2$ for $S(\tilde{y}_1) = S(\tilde{y}_2)$ and $H(\tilde{y}_1) = H(\tilde{y}_2)$

3. Trapezoidal Neutrosophic Z-Number (TrNZN) Sets

To make trapezoidal neutrosophic information reliable, this section gives the following definitions of a TrNZN set, operations of TrNZNs, score and accuracy functions of TrNZN, and ranking laws of TrNZNs.

Definition 1. Set U as a universe set; then, a TrNZN set in U is defined as the following mathematical representation:

$$\tilde{Z} = \left\{u, \left\langle \left(TZ_{\tilde{V}}(u), TZ_{\tilde{R}}(u)\right), \left(IZ_{\tilde{V}}(u), IZ_{\tilde{R}}(u)\right), \left(FZ_{\tilde{V}}(u), FZ_{\tilde{R}}(u)\right) \right\rangle | u \in U\right\}, \tag{6}$$

where $(TZ_{\tilde{V}}(u), TZ_{\tilde{R}}(u))$, $(IZ_{\tilde{V}}(u), IZ_{\tilde{R}}(u))$, and $(FZ_{\tilde{V}}(u), FZ_{\tilde{R}}(u))$ are the truth, indeterminacy, and falsity trapezoidal Z-numbers that are composed of the truth, indeterminacy, and falsity trapezoidal fuzzy numbers and their reliability measures, denoted as $(TZ_{\tilde{V}}(u), TZ_{\tilde{R}}(u)) = ((T_{V1}(u), T_{V2}(u), T_{V3}(u), T_{V4}(u)), (T_{R1}(u), T_{R2}(u), T_{R3}(u), T_{R4}(u)))$: $U \longrightarrow [0,1] \times [0,1]$, $((IZ_{\tilde{V}}(u), IZ_{\tilde{R}}(u) = ((I_{V1}(u), I_{V2}(u), I_{V3}(u), I_{V4}(u)), (I_{R1}(u), I_{R2}(u), I_{R3}(u), I_{R4}(u)))$: U $\longrightarrow [0,1]\times[0,1]$, and $(FZ_{\tilde{V}}(u), FZ_{\tilde{R}}(u)) = ((F_{V1}(u), F_{V2}(u), F_{V3}(u), F_{V4}(u)), (F_{R1}(u), F_{R2}(u), F_{R3}(u), F_{R4}(u)))$: U $\longrightarrow [0,1]\times[0,1]$ with the conditions $0 \le T_{V4}(u) + I_{V4}(u) + F_{V4}(u) \le 3$ and $0 \le T_{R4}(u) + I_{R4}(u) + F_{R4}(u) \le 3$ for $u \in U$.

For convenience, the three trapezoidal Z-numbers in $\tilde{Z}$ are simply denoted as $(TZ_{\tilde{V}}(u), TZ_{\tilde{R}}(u)) = ((T_{V1},$

$T_{V2}, T_{V3}, T_{V4}), (T_{R1}, T_{R2}, T_{R3}, T_{R4})), \quad (IZ_{\widetilde{V}}(u), IZ_{\widetilde{R}}(u)) = ((I_{V1}, I_{V2}, I_{V3}, I_{V4}), (I_{R1}, I_{R2}, I_{R3}, I_{R4}))$, and $(FZ_{\widetilde{V}}(u), FZ_{\widetilde{R}}(u)) = ((F_{V1}, F_{V2}, F_{V3}, F_{V4}), (F_{R1}, F_{R2}, F_{R3}, F_{R4}))$. Thus, a TrNZN in $\widetilde{Z}$ is simply denoted as $\widetilde{z} = <((T_{V1}, T_{V2}, T_{V3}, T_{V4}), (T_{R1}, T_{R2}, T_{R3}, T_{R4})), ((I_{V1}, I_{V2}, I_{V3}, I_{V4}), (I_{R1}, I_{R2}, I_{R3}, T_{R4})), ((F_{V1}, F_{V2}, F_{V3}, F_{V4}), (F_{R1}, F_{R2}, F_{R3}, F_{R4}))>$.

If $T_{V2} = T_{V3}$, $T_{R2} = T_{R3}$, $I_{V2} = I_{V3}$, $I_{R2} = I_{R3}$, and $F_{V2} = F_{V3}$, $F_{R2} = F_{R3}$ hold in the TrNZN $\widetilde{z}$; it is reduced to the triangular neutrosophic Z-number, which is a special case of TrNZN.

Definition 2. Set $\widetilde{z}_1 = <((T_{V11}, T_{V12}, T_{V13}, T_{V14}), (T_{R11}, T_{R12}, T_{R13}, T_{R14})), ((I_{V11}, I_{V12}, I_{V13}, I_{V14}), (I_{R11}, I_{R12}, I_{R13}, T_{R14})), ((F_{V11}, F_{V12}, F_{V13}, F_{V14}), (F_{R11}, F_{R12}, F_{R13}, F_{R14}))>$ and $\widetilde{z}_2 = <((T_{V21}, T_{V22}, T_{V23}, T_{V24}), (T_{R21}, T_{R22}, T_{R23}, T_{R24})), ((I_{V21}, I_{V22}, I_{V23}, I_{V24}), (I_{R21}, I_{R22}, I_{R23}, T_{R24})), ((F_{V21}, F_{V22}, F_{V23}, F_{V24}), (F_{R21}, F_{R22}, F_{R23}, F_{R24}))>$ as two TrNZNs. Then they are defined as the following basic operations:

(1) $\widetilde{z}_1 \oplus \widetilde{z}_2 = \langle ((T_{V11} + T_{V21} - T_{V11}T_{V21}, T_{V12} + T_{V22} - T_{V12}T_{V22}, T_{V13} + T_{V23} - T_{V13}T_{V23}, T_{V14} + T_{V24} - T_{V14}T_{V24}), (T_{R11} + T_{R21} - T_{R11}T_{R21}, T_{R12} + T_{R22} - T_{R12}T_{R22}, T_{R13} + T_{R23} - T_{R13}T_{R23}, T_{R14} + T_{R24} - T_{R14}T_{R24})), ((I_{V11}I_{V21}, I_{V12}I_{V22}, I_{V13}I_{V23}, I_{V14}I_{V24}), (I_{R11}I_{R21}, I_{R12}I_{R22}, I_{R13}I_{R23}, I_{R14}I_{R24})), ((F_{V11}F_{V21}, F_{V12}F_{V22}, F_{V13}F_{V23}, F_{V14}F_{V24}), (F_{R11}F_{R21}, F_{R12}F_{R22}, F_{R13}F_{R23}, F_{R14}F_{R24}))\rangle$

(2) $\widetilde{z}_1 \otimes \widetilde{z}_2 = \langle ((T_{V11}T_{V21}, T_{V12}T_{V22}, T_{V13}T_{V23}, T_{V14}T_{V24})(T_{R11}T_{R21}, T_{R12}T_{R22}, T_{R13}T_{R23}, T_{R14}T_{R24})), ((I_{V11} + I_{V21} - I_{V11}I_{V21}, I_{V12} + I_{V22} - I_{V12}I_{V22}, I_{V13} + I_{V23} - I_{V13}I_{V23}, I_{V14} + I_{V24} - I_{V14}I_{V24}), (I_{R11} + I_{R21} - I_{R11}I_{R21}, I_{R12} + I_{R22} - I_{R12}I_{R22}, I_{R13} + I_{R23} - I_{R13}I_{R23}, I_{R14} + I_{R24} - I_{R14}I_{R24})), ((F_{V11} + F_{V21} - F_{V11}F_{V21}, F_{V12} + F_{V22} - F_{V12}F_{V22}, F_{V13} + F_{V23} - F_{V13}F_{V23}, F_{V14} + F_{V24} - F_{V14}F_{V24}), (F_{R11} + F_{R21} - F_{R11}F_{R21}, F_{R12} + F_{R22} - F_{R12}F_{R22}, F_{R13} + F_{R23} - F_{R13}F_{R23}, F_{R14} + F_{R24} - F_{R14}F_{R24}))\rangle$

(3) $\lambda\widetilde{z}_1 = \langle ((1 - (1 - T_{V11})^\lambda, 1 - (1 - T_{V12})^\lambda, 1 - (1 - T_{V13})^\lambda, 1 - (1 - T_{V14})^\lambda), (1 - (1 - T_{R11})^\lambda, 1 - (1 - T_{R12})^\lambda, 1 - (1 - T_{R13})^\lambda, 1 - (1 - T_{R14})^\lambda)), ((I_{V11}^\lambda, I_{V12}^\lambda, I_{V13}^\lambda, I_{V14}^\lambda), (I_{R11}^\lambda, I_{R12}^\lambda, I_{R13}^\lambda, I_{R14}^\lambda)), ((F_{V11}^\lambda, F_{V12}^\lambda, F_{V13}^\lambda, F_{V14}^\lambda), (F_{R11}^\lambda, F_{R12}^\lambda, F_{R13}^\lambda, F_{R14}^\lambda))\rangle, \lambda > 0$

(4) $\widetilde{z}_1^\lambda = \langle ((T_{V11}^\lambda, T_{V12}^\lambda, T_{V13}^\lambda, T_{V14}^\lambda), (T_{R11}^\lambda, T_{R12}^\lambda, T_{R13}^\lambda, T_{R14}^\lambda)), ((1 - (1 - I_{V11})^\lambda, 1 - (1 - I_{V12})^\lambda, 1 - (1 - I_{V13})^\lambda, 1 - (1 - I_{V14})^\lambda), (1 - (1 - I_{R11})^\lambda, 1 - (1 - I_{R12})^\lambda, 1 - (1 - I_{R13})^\lambda, 1 - (1 - I_{R14})^\lambda)), ((1 - (1 - F_{V11})^\lambda, 1 - (1 - F_{V12})^\lambda, 1 - (1 - F_{V13})^\lambda, 1 - (1 - F_{V14})^\lambda), (1 - (1 - F_{R11})^\lambda, 1 - (1 - F_{R12})^\lambda, 1 - (1 - F_{R13})^\lambda, 1 - (1 - F_{R14})^\lambda))\rangle, \lambda > 0$

For ranking TrNZNs, the score and accuracy functions of TrNZN are defined according to the expected value of a trapezoidal fuzzy number and score and accuracy functions of TrNN [9].

Definition 3. Set $\widetilde{z}_1 = <((T_{V11}, T_{V12}, T_{V13}, T_{V14}), (T_{R11}, T_{R12}, T_{R13}, T_{R14})), ((I_{V11}, I_{V12}, I_{V13}, I_{V14}), (I_{R11}, I_{R12}, I_{R13}, T_{R14})), ((F_{V11}, F_{V12}, F_{V13}, F_{V14}), (F_{R11}, F_{R12}, F_{R13}, F_{R14}))>$ as TrNZN. Then the score and accuracy functions of the TrNZN $\widetilde{z}_1$ can be defined as follows:

$$S(\widetilde{z}_1) = \frac{1}{3}\begin{pmatrix} 2 + \dfrac{T_{V11} + T_{V12} + T_{V13} + T_{V14}}{4} \times \dfrac{T_{R11} + T_{R12} + T_{R13} + T_{R14}}{4} \\ -\dfrac{I_{V11} + I_{V12} + I_{V13} + I_{V14}}{4} \times \dfrac{I_{R11} + I_{R12} + I_{R13} + I_{R14}}{4} \\ -\dfrac{F_{V11} + F_{V12} + F_{V13} + F_{V14}}{4} \times \dfrac{F_{R11} + F_{R12} + F_{R13} + F_{R14}}{4} \end{pmatrix}, \quad S(\widetilde{z}_1) \in [0, 1], \tag{7}$$

$$\begin{aligned} H(\widetilde{z}_1) = {} & \frac{T_{V11} + T_{V12} + T_{V13} + T_{V14}}{4} \times \frac{T_{R11} + T_{R12} + T_{R13} + T_{R14}}{4} - \frac{F_{V11} + F_{V12} + F_{V13} + F_{V14}}{4} \\ & \times \frac{F_{R11} + F_{R12} + F_{R13} + F_{R14}}{4}, \quad H(\widetilde{z}) \in [-1, 1]. \end{aligned} \tag{8}$$

Based on equations (7) and (8), ranking laws between two TrNZNs are given by the following definition.

Definition 4. Set $\widetilde{z}_1 = <((T_{V11}, T_{V12}, T_{V13}, T_{V14}), (T_{R11}, T_{R12}, T_{R13}, T_{R14})), ((I_{V11}, I_{V12}, I_{V13}, I_{V14}), (I_{R11}, I_{R12}, I_{R13}, T_{R14})), ((F_{V11}, F_{V12}, F_{V13}, F_{V14}), (F_{R11}, F_{R12}, F_{R13}, F_{R14}))>$ and $\widetilde{z}_2 = <((T_{V21}, T_{V22}, T_{V23}, T_{V24}), (T_{R21}, T_{R22}, T_{R23}, T_{R24})), ((I_{V21}, I_{V22}, I_{V23}, I_{V24}), (I_{R21}, I_{R22}, I_{R23}, T_{R24})), ((F_{V21}, F_{V22}, F_{V23}, F_{V24}), (F_{R21}, F_{R22}, F_{R23}, F_{R24}))>$ as two TrNZNs. Then, the ranking laws between two TrNZNs are defined as follows:

(1) If $S(\widetilde{z}_1) > S(\widetilde{z}_2)$, then $\widetilde{z}_1 \succ \widetilde{z}_2$

(2) If $S(\widetilde{z}_1) = S(\widetilde{z}_2)$ and $H(\widetilde{z}_1) > H(\widetilde{z}_2)$, then $\widetilde{z}_1 \succ \widetilde{z}_2$

(3) If $S(\widetilde{z}_1) = S(\widetilde{z}_2)$ and $H(\widetilde{z}_1) = H(\widetilde{z}_2)$, then $\widetilde{z}_1 \cong \widetilde{z}_2$

4. Weighted Aggregation Operators of TrNZNs

Regarding information aggregation in MDM problems, one usually utilizes the weighted arithmetic and geometric averaging operators as the most basic information aggregation approaches. To aggregate TrNZNs, therefore, this section proposes the two following weighted aggregation operators of TrNZNs based on the basic operations of TrNZNs in Definition 2.

4.1. Weighted Arithmetic Averaging Operator of TrNZNs

Definition 5. Set $\tilde{z}_j = <((T_{Vj1}, T_{Vj2}, T_{Vj3}, T_{Vj4}), (T_{Rj1}, T_{Rj2}, T_{Rj3}, T_{Rj4})), ((I_{Vj1}, I_{Vj2}, I_{Vj3}, I_{Vj4}), (I_{Rj1}, I_{Rj2}, I_{Rj3}, I_{Rj4})), ((F_{Vj1}, F_{Vj2}, F_{Vj3}, F_{Vj4}), (F_{Rj1}, F_{Rj2}, F_{Rj3}, F_{Rj4}))> (j = 1, 2, \ldots, n)$ as a series of TrNZNs. Then, the TrNZNWAA operator is defined as

$$\mathrm{TrNZNWAA}(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) = \mathop{\oplus}_{j=1}^{n} \lambda_j \tilde{z}_j, \tag{9}$$

where λ_j $(j = 1, 2, \ldots, n)$ is the weight of the jth TrNZN $\tilde{z}_j$ $(j = 1, 2, \ldots, n)$ for $\lambda_j \in [0, 1]$ and $\sum_{j=1}^{n} \lambda_j = 1$.

Based on the basic operations of TrNZNs in Definition 2 and equation (9), we have the following theorem.

Theorem 1. *Set $\tilde{z}_j = <((T_{Vj1}, T_{Vj2}, T_{Vj3}, T_{Vj4}), (T_{Rj1}, T_{Rj2}, T_{Rj3}, T_{Rj4})), ((I_{Vj1}, I_{Vj2}, I_{Vj3}, I_{Vj4}), (I_{Rj1}, I_{Rj2}, I_{Rj3}, I_{Rj4})), ((F_{Vj1}, F_{Vj2}, F_{Vj3}, F_{Vj4}), (F_{Rj1}, F_{Rj2}, F_{Rj3}, F_{Rj4}))> (j = 1, 2, \ldots, n)$ as a series of TrNZNs. Then, the aggregated value of equation (9) is also TrNZN, which is yielded by the following equation:*

$$\begin{aligned}
&\mathrm{TrNZNWAA}(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \\
&= \left\langle \begin{array}{c} \left(\left(1-\prod_{j=1}^{n}(1-T_{Vj1})^{\lambda_j}, 1-\prod_{j=1}^{n}(1-T_{Vj2})^{\lambda_j}, 1-\prod_{j=1}^{n}(1-T_{Vj3})^{\lambda_j}, 1-\prod_{j=1}^{n}(1-T_{Vj4})^{\lambda_j}\right), \left(1-\prod_{j=1}^{n}(1-T_{Rj1})^{\lambda_j}, 1-\prod_{j=1}^{n}(1-T_{Rj2})^{\lambda_j}, 1-\prod_{j=1}^{n}(1-T_{Rj3})^{\lambda_j}, 1-\prod_{j=1}^{n}(1-T_{Rj4})^{\lambda_j}\right)\right), \\ \left(\left(\prod_{j=1}^{n} I_{Vj1}^{\lambda_j}, \prod_{j=1}^{n} I_{Vj2}^{\lambda_j}, \prod_{j=1}^{n} I_{Vj3}^{\lambda_j}, \prod_{j=1}^{n} I_{Vj4}^{\lambda_j}\right), \left(\prod_{j=1}^{n} I_{Rj1}^{\lambda_j}, \prod_{j=1}^{n} I_{Rj2}^{\lambda_j}, \prod_{j=1}^{n} I_{Rj3}^{\lambda_j}, \prod_{j=1}^{n} I_{Rj4}^{\lambda_j}\right)\right), \left(\left(\prod_{j=1}^{n} F_{Vj1}^{\lambda_j}, \prod_{j=1}^{n} F_{Vj2}^{\lambda_j}, \prod_{j=1}^{n} F_{Vj3}^{\lambda_j}, \prod_{j=1}^{n} F_{Vj4}^{\lambda_j}\right), \left(\prod_{j=1}^{n} F_{Rj1}^{\lambda_j}, \prod_{j=1}^{n} F_{Rj2}^{\lambda_j}, \prod_{j=1}^{n} F_{Rj3}^{\lambda_j}, \prod_{j=1}^{n} F_{Rj4}^{\lambda_j}\right)\right) \end{array} \right\rangle,
\end{aligned} \tag{10}$$

where λ_j $(j = 1, 2, \ldots, n)$ is the weight of the jth TrNZN $\tilde{z}_j$ $(j = 1, 2, \ldots, n)$ for $\lambda_j \in [0, 1]$ and $\sum_{j=1}^{n} \lambda_j = 1$.

Proof. The proof of equation (10) can be given by mathematical induction.

(1) Set $n = 2$. Then there is the following result:

$$\begin{aligned}
\mathrm{TrNZNWAA}(\tilde{z}_1, \tilde{z}_2) &= \lambda_1 \tilde{z}_1 \oplus \lambda_2 \tilde{z}_2 \\
&= \Big\langle\Big(\Big(1-(1-T_{V11})^{\lambda_1} + 1-(1-T_{V21})^{\lambda_2} - \big(1-(1-T_{V11})^{\lambda_1}\big)\big(1-(1-T_{V21})^{\lambda_2}\big), \\
&\quad 1-(1-T_{V12})^{\lambda_1} + 1-(1-T_{V22})^{\lambda_2} - \big(1-(1-T_{V12})^{\lambda_1}\big)\big(1-(1-T_{V22})^{\lambda_2}\big), \\
&\quad 1-(1-T_{V13})^{\lambda_1} + 1-(1-T_{V23})^{\lambda_2} - \big(1-(1-T_{V13})^{\lambda_1}\big)\big(1-(1-T_{V23})^{\lambda_2}\big), \\
&\quad 1-(1-T_{V14})^{\lambda_1} + 1-(1-T_{V24})^{\lambda_2} - \big(1-(1-T_{V14})^{\lambda_1}\big)\big(1-(1-T_{V24})^{\lambda_2}\big)\Big), \\
&\quad \Big(1-(1-T_{R11})^{\lambda_1} + 1-(1-T_{R21})^{\lambda_2} - \big(1-(1-T_{R11})^{\lambda_1}\big)\big(1-(1-T_{R21})^{\lambda_2}\big), \\
&\quad 1-(1-T_{R12})^{\lambda_1} + 1-(1-T_{R22})^{\lambda_2} - \big(1-(1-T_{R12})^{\lambda_1}\big)\big(1-(1-T_{R22})^{\lambda_2}\big), \\
&\quad 1-(1-T_{R13})^{\lambda_1} + 1-(1-T_{R23})^{\lambda_2} - \big(1-(1-T_{R13})^{\lambda_1}\big)\big(1-(1-T_{R23})^{\lambda_2}\big), \\
&\quad 1-(1-T_{V14})^{\lambda_1} + 1-(1-T_{R24})^{\lambda_2} - \big(1-(1-T_{R14})^{\lambda_1}\big)\big(1-(1-T_{R24})^{\lambda_2}\big)\Big)\Big), \\
&\quad \Big(\big(I_{V11}^{\lambda_1} I_{V21}^{\lambda_2}, I_{V12}^{\lambda_1} I_{V22}^{\lambda_2}, I_{V13}^{\lambda_1} I_{V23}^{\lambda_2}, I_{V14}^{\lambda_1} I_{V24}^{\lambda_2}\big), \big(I_{R11}^{\lambda_1} I_{R21}^{\lambda_2}, I_{R12}^{\lambda_1} I_{R22}^{\lambda_2}, I_{R13}^{\lambda_1} I_{R23}^{\lambda_2}, I_{R14}^{\lambda_1} I_{R24}^{\lambda_2}\big)\Big), \\
&\quad \Big(\big(F_{V11}^{\lambda_1} F_{V21}^{\lambda_2}, F_{V12}^{\lambda_1} F_{V22}^{\lambda_2}, F_{V13}^{\lambda_1} F_{V23}^{\lambda_2}, F_{V14}^{\lambda_1} F_{V24}^{\lambda_2}\big), \big(F_{R11}^{\lambda_1} F_{R21}^{\lambda_2}, F_{R12}^{\lambda_1} F_{R22}^{\lambda_2}, F_{R13}^{\lambda_1} F_{R23}^{\lambda_2}, F_{R14}^{\lambda_1} F_{R24}^{\lambda_2}\big)\Big)\Big\rangle \\
&= \Big\langle\Big(\Big(1-(1-T_{V11})^{\lambda_1}(1-T_{V21})^{\lambda_2}, 1-(1-T_{V12})^{\lambda_1}(1-T_{V22})^{\lambda_2}, \\
&\quad 1-(1-T_{V13})^{\lambda_1}(1-T_{V23})^{\lambda_2}, 1-(1-T_{V14})^{\lambda_1}(1-T_{V24})^{\lambda_2}\Big) \\
&\quad \Big(1-(1-T_{R11})^{\lambda_1}(1-T_{R21})^{\lambda_2}, 1-(1-T_{R12})^{\lambda_1}(1-T_{R22})^{\lambda_2}, \\
&\quad 1-(1-T_{R13})^{\lambda_1}(1-T_{R23})^{\lambda_2}, 1-(1-T_{R14})^{\lambda_1}(1-T_{R24})^{\lambda_2}\Big)\Big), \\
&\quad \left(\left(\prod_{j=1}^{2} I_{Vj1}^{\lambda_j}, \prod_{j=1}^{2} I_{Vj2}^{\lambda_j}, \prod_{j=1}^{2} I_{Vj3}^{\lambda_j}, \prod_{j=1}^{2} I_{Vj4}^{\lambda_j}\right), \left(\prod_{j=1}^{2} I_{Rj1}^{\lambda_j}, \prod_{j=1}^{2} I_{Rj2}^{\lambda_j}, \prod_{j=1}^{2} I_{Rj3}^{\lambda_j}, \prod_{j=1}^{2} I_{Rj4}^{\lambda_j}\right)\right), \\
&\quad \left(\left(\prod_{j=1}^{2} F_{Vj1}^{\lambda_j}, \prod_{j=1}^{2} F_{Vj2}^{\lambda_j}, \prod_{j=1}^{2} F_{Vj3}^{\lambda_j}, \prod_{j=1}^{2} F_{Vj4}^{\lambda_j}\right), \left(\prod_{j=1}^{2} F_{Rj1}^{\lambda_j}, \prod_{j=1}^{2} F_{Rj2}^{\lambda_j}, \prod_{j=1}^{2} F_{Rj3}^{\lambda_j}, \prod_{j=1}^{2} F_{Rj4}^{\lambda_j}\right)\right)\Big\rangle.
\end{aligned} \tag{11}$$

(2) Set $n = k$. Then, equation (10) can hold in the following equation:

$$
\begin{aligned}
\operatorname{TrNZNWAA}\left(\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_k\right) &= \mathop{\oplus}_{j=1}^{k} \lambda_j \widetilde{z}_j \\
&= \Bigg\langle \Bigg(\Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Vj1}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k} \left(1 - T_{Vj2}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k} \left(1 - T_{Vj3}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k} \left(1 - T_{Vj4}\right)^{\lambda_j} \Bigg), \\
&\quad \Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Rj1}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k} \left(1 - T_{Rj2}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k} \left(1 - T_{Rj3}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k} \left(1 - T_{Rj4}\right)^{\lambda_j} \Bigg) \Bigg), \\
&\quad \Bigg(\Bigg(\prod_{j=1}^{k} I_{Vj1}^{\lambda_j}, \prod_{j=1}^{k} I_{Vj2}^{\lambda_j}, \prod_{j=1}^{k} I_{Vj3}^{\lambda_j}, \prod_{j=1}^{k} I_{Vj4}^{\lambda_j} \Bigg), \Bigg(\prod_{j=1}^{k} I_{Rj1}^{\lambda_j}, \prod_{j=1}^{k} I_{Rj2}^{\lambda_j}, \prod_{j=1}^{k} I_{Rj3}^{\lambda_j}, \prod_{j=1}^{k} I_{Rj4}^{\lambda_j} \Bigg) \Bigg), \\
&\quad \Bigg(\Bigg(\prod_{j=1}^{k} F_{Vj1}^{\lambda_j}, \prod_{j=1}^{k} F_{Vj2}^{\lambda_j}, \prod_{j=1}^{k} F_{Vj3}^{\lambda_j}, \prod_{j=1}^{k} F_{Vj4}^{\lambda_j} \Bigg), \Bigg(\prod_{j=1}^{k} F_{Rj1}^{\lambda_j}, \prod_{j=1}^{k} F_{Rj2}^{\lambda_j}, \prod_{j=1}^{k} F_{Rj3}^{\lambda_j}, \prod_{j=1}^{k} F_{Rj4}^{\lambda_j} \Bigg) \Bigg) \Bigg\rangle.
\end{aligned}
\tag{12}
$$

(3) Set $n = k + 1$. By equations (11) and (12), we can obtain

$$
\begin{aligned}
&\operatorname{TrNZNWAA}\left(\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_k, \widetilde{z}_{k+1}\right) = \mathop{\oplus}_{j=1}^{k} \lambda_j \widetilde{z}_j \oplus \lambda_{k+1} \widetilde{z}_{k+1} \\
&= \Bigg\langle \Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Vj1}\right)^{\lambda_j} + 1 - \left(1 - T_{V(k+1)1}\right)\Big)^{\lambda_{k+1}} - \Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Vj1}\right)^{\lambda_j} \Bigg) \left(1 - \left(1 - T_{V(k+1)1}\right)\right)^{\lambda_{k+1}} \Bigg), 1 - \prod_{j=1}^{k} \left(1 - T_{Vj2}\right)^{\lambda_j} + 1 - \left(1 - T_{V(k+1)2}\right)^{\lambda_{k+1}} - \Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Vj2}\right)^{\lambda_j} \Bigg) \left(1 - \left(1 - T_{V(k+1)2}\right)\right)^{\lambda_{k+1}} \Bigg), \\
&\qquad 1 - \prod_{j=1}^{k} \left(1 - T_{Vj3}\right)^{\lambda_j} + 1 - \left(1 - T_{V(k+1)3}\right)^{\lambda_{k+1}} - \Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Vj3}\right)^{\lambda_j} \Bigg) \left(1 - \left(1 - T_{V(k+1)3}\right)\right)^{\lambda_{k+1}} \Bigg), 1 - \prod_{j=1}^{k} \left(1 - T_{Vj4}\right)^{\lambda_j} + 1 - \left(1 - T_{V(k+1)4}\right)^{\lambda_{k+1}} - \Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Vj4}\right)^{\lambda_j} \Bigg) \left(1 - \left(1 - T_{V(k+1)4}\right)\right)^{\lambda_{k+1}} \Bigg) \\
&\qquad \Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Rj1}\right)^{\lambda_j} + 1 - \left(1 - T_{R(k+1)1}\right)^{\lambda_{k+1}} - \Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Rj1}\right)^{\lambda_j} \Bigg) \left(1 - \left(1 - T_{R(k+1)1}\right)\right)^{\lambda_{k+1}} \Bigg), 1 - \prod_{j=1}^{k} \left(1 - T_{Rj2}\right)^{\lambda_j} + 1 - \left(1 - T_{R(k+1)2}\right)^{\lambda_{k+1}} - \Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Rj2}\right)^{\lambda_j} \Bigg) \left(1 - \left(1 - T_{R(k+1)2}\right)\right)^{\lambda_{k+1}} \Bigg), \\
&\qquad 1 - \prod_{j=1}^{k} \left(1 - T_{Rj3}\right)^{\lambda_j} + 1 - \left(1 - T_{R(k+1)3}\right)^{\lambda_{k+1}} - \Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Rj3}\right)^{\lambda_j} \Bigg) \left(1 - \left(1 - T_{R(k+1)3}\right)\right)^{\lambda_{k+1}} \Bigg), 1 - \prod_{j=1}^{k} \left(1 - T_{Rj4}\right)^{\lambda_j} + 1 - \left(1 - T_{R(k+1)4}\right)^{\lambda_{k+1}} - \Bigg(\Bigg(1 - \prod_{j=1}^{k} \left(1 - T_{Rj4}\right)^{\lambda_j} \Bigg) \left(1 - \left(1 - T_{R(k+1)4}\right)\right)^{\lambda_{k+1}} \Bigg), \Bigg) \Bigg) \\
&\qquad \Bigg(\Bigg(\prod_{j=1}^{k+1} I_{Vj1}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Vj2}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Vj3}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Vj4}^{\lambda_j} \Bigg), \Bigg(\prod_{j=1}^{k+1} I_{Rj1}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Rj2}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Rj3}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Rj4}^{\lambda_j} \Bigg) \Bigg), \Bigg(\Bigg(\prod_{j=1}^{k+1} F_{Vj1}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Vj2}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Vj3}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Vj4}^{\lambda_j} \Bigg), \Bigg(\prod_{j=1}^{k+1} F_{Rj1}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Rj2}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Rj3}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Rj4}^{\lambda_j} \Bigg) \Bigg) \Bigg\rangle \\
&= \Bigg\langle \Bigg(\Bigg(1 - \prod_{j=1}^{k+1} \left(1 - T_{Vj1}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k+1} \left(1 - T_{Vj2}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k+1} \left(1 - T_{Vj3}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k+1} \left(1 - T_{Vj4}\right)^{\lambda_j}, \Bigg) \Bigg(1 - \prod_{j=1}^{k+1} \left(1 - T_{Rj1}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k+1} \left(1 - T_{Rj2}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k+1} \left(1 - T_{Rj3}\right)^{\lambda_j}, 1 - \prod_{j=1}^{k+1} \left(1 - T_{Rj4}\right)^{\lambda_j} \Bigg) \Bigg), \\
&\qquad \Bigg(\Bigg(\prod_{j=1}^{k+1} I_{Vj1}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Vj2}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Vj3}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Vj4}^{\lambda_j} \Bigg), \Bigg(\prod_{j=1}^{k+1} I_{Rj1}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Rj2}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Rj3}^{\lambda_j}, \prod_{j=1}^{k+1} I_{Rj4}^{\lambda_j} \Bigg) \Bigg), \Bigg(\Bigg(\prod_{j=1}^{k+1} F_{Vj1}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Vj2}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Vj3}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Vj4}^{\lambda_j} \Bigg), \Bigg(\prod_{j=1}^{k+1} F_{Rj1}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Rj2}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Rj3}^{\lambda_j}, \prod_{j=1}^{k+1} F_{Rj4}^{\lambda_j} \Bigg) \Bigg) \Bigg\rangle.
\end{aligned}
\tag{13}
$$

Regarding the above results, equation (10) can hold for any n. Thus, the proof is completed.

Especially when $\lambda_j = 1/n$ $(j = 1, 2, \ldots, n)$, the TrNZNWAA operator is reduced to the TrNZN arithmetic averaging operator. □

Theorem 2. *The TrNZNWAA operator contains the three following properties:*

(P1) Idempotency: set $\widetilde{z}_j = <((T_{Vj1}, T_{Vj2}, T_{Vj3}, T_{Vj4}), (T_{Rj1}, T_{Rj2}, T_{Rj3}, T_{Rj4})), ((I_{Vj1}, I_{Vj2}, I_{Vj3}, I_{Vj4}), (I_{Rj1}, I_{Rj2}, I_{Rj3}, I_{Rj4})), ((F_{Vj1}, F_{Vj2}, F_{Vj3}, F_{Vj4}), (F_{Rj1}, F_{Rj2}, F_{Rj3}, F_{Rj4})) > (j = 1, 2, \ldots, n)$ *as a series of TrNZNs. If* $\widetilde{z}_j = \widetilde{z}$ *for* $j = 1, 2, \ldots, n$*, then there exists* $\operatorname{TrNZNWAA}(\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_n) = \widetilde{z}$.

(P2) Set $\widetilde{z}_j = <((T_{Vj1}, T_{Vj2}, T_{Vj3}, T_{Vj4}), (T_{Rj1}, T_{Rj2}, T_{Rj3}, T_{Rj4})), ((I_{Vj1}, I_{Vj2}, I_{Vj3}, I_{Vj4}), (I_{Rj1}, I_{Rj2}, I_{Rj3}, I_{Rj4})), ((F_{Vj1},$

F_{Vj2}, F_{Vj3}, F_{Vj4}), (F_{Rj1}, F_{Rj2}, F_{Rj3}, F_{Rj4})) > (j = 1, 2, . . ., n) as a series of TrNZNs; then, set the minimum and maximum TrNZNs as

$$\tilde{z}^- = \left\langle \begin{array}{c} \left(\left(\min_j T_{Vj1}, \min_j T_{Vj2}, \min_j T_{Vj3}, \min_j T_{Vj4}\right), \left(\min_j T_{Rj1}, \min_j T_{Rj2}, \min_j T_{Rj3}, \min_j T_{Rj4}\right)\right), \\ \left(\left(\max_j I_{Vj1}, \max_j I_{Vj2}, \max_j I_{Vj3}, \max_j I_{Vj4}\right), \left(\max_j I_{Rj1}, \max_j I_{Rj2}, \max_j I_{Rj3}, \max_j I_{Rj4}\right)\right), \\ \left(\left(\max_j F_{Vj1}, \max_j F_{Vj2}, \max_j F_{Vj3}, \max_j F_{Vj4}\right), \left(\max_j F_{Rj1}, \max_j F_{Rj2}, \max_j F_{Rj3}, \max_j F_{Rj4}\right)\right) \end{array} \right\rangle,$$
$$\tilde{z}^+ = \left\langle \begin{array}{c} \left(\left(\max_j T_{Vj1}, \max_j T_{Vj2}, \max_j T_{Vj3}, \max_j T_{Vj4}\right), \left(\max_j T_{Rj1}, \max_j T_{Rj2}, \max_j T_{Rj3}, \max_j T_{Rj4}\right)\right), \\ \left(\left(\min_j I_{Vj1}, \min_j I_{Vj2}, \min_j I_{Vj3}, \min_j I_{Vj4}\right), \left(\min_j I_{Rj1}, \min_j I_{Rj2}, \min_j I_{Rj3}, \min_j I_{Rj4}\right)\right), \\ \left(\left(\min_j F_{Vj1}, \min_j F_{Vj2}, \min_j F_{Vj3}, \min_j F_{Vj4}\right), \left(\min_j F_{Rj1}, \min_j F_{Rj2}, \min_j F_{Rj3}, \min_j F_{Rj4}\right)\right) \end{array} \right\rangle. \tag{14}$$

Then, there is $\tilde{z}^- \le \mathrm{TrNZNWAA}(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \le \tilde{z}^+$.

(P3) Monotony: set $\tilde{z}_j$ = <((TVj1, TVj2, TVj3, TVj4), (TRj1, TRj2, TRj3, TRj4)), ((IVj1, IVj2, IVj3, IVj4), (IRj1, IRj2, IRj3, IRj4)), ((FVj1, FVj2, FVj3, FVj4), (FRj1, FRj2, FRj3, FRj4))> (j = 1, 2, . . ., n) as a series of TrNZNs. If $\tilde{z}_j \le \tilde{z}_j^*$ for j = 1, 2, . . ., n, then there is $\mathrm{TrNZNWAA}(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \le \mathrm{TrNZNWAA}(\tilde{z}_1^*, \tilde{z}_2^*, \ldots, \tilde{z}_n^*)$.

Proof.

(P1) Owing to $\tilde{z}_j = \tilde{z}$ for $j = 1, 2, \ldots, n$, there is the following result:

$$\begin{aligned}
\mathrm{TrNZNWAA}(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) &= \bigoplus_{j=1}^{n} \lambda_j \tilde{z}_j \\
&= \left\langle \left(\left(1 - \prod_{j=1}^{n} \left(1 - T_{Vj1}\right)^{\lambda_j}, 1 - \prod_{j=1}^{n} \left(1 - T_{Vj2}\right)^{\lambda_j}, 1 - \prod_{j=1}^{n} \left(1 - T_{Vj3}\right)^{\lambda_j}, 1 - \prod_{j=1}^{n} \left(1 - T_{Vj4}\right)^{\lambda_j} \right), \right.\right. \\
&\quad \left. \left(1 - \prod_{j=1}^{n} \left(1 - T_{Rj1}\right)^{\lambda_j}, 1 - \prod_{j=1}^{n} \left(1 - T_{Rj2}\right)^{\lambda_j}, 1 - \prod_{j=1}^{n} \left(1 - T_{Rj3}\right)^{\lambda_j}, 1 - \prod_{j=1}^{n} \left(1 - T_{Rj4}\right)^{\lambda_j} \right) \right), \\
&\quad \left(\left(\prod_{j=1}^{n} I_{Vj1}^{\lambda_j}, \prod_{j=1}^{n} I_{Vj2}^{\lambda_j}, \prod_{j=1}^{n} I_{Vj3}^{\lambda_j}, \prod_{j=1}^{n} I_{Vj4}^{\lambda_j} \right), \left(\prod_{j=1}^{n} I_{Rj1}^{\lambda_j}, \prod_{j=1}^{n} I_{Rj2}^{\lambda_j}, \prod_{j=1}^{n} I_{Rj3}^{\lambda_j}, \prod_{j=1}^{n} I_{Rj4}^{\lambda_j} \right) \right), \\
&\quad \left.\left(\left(\prod_{j=1}^{n} F_{Vj1}^{\lambda_j}, \prod_{j=1}^{n} F_{Vj2}^{\lambda_j}, \prod_{j=1}^{n} F_{Vj3}^{\lambda_j}, \prod_{j=1}^{n} F_{Vj4}^{\lambda_j} \right), \left(\prod_{j=1}^{n} F_{Rj1}^{\lambda_j}, \prod_{j=1}^{n} F_{Rj2}^{\lambda_j}, \prod_{j=1}^{n} F_{Rj3}^{\lambda_j}, \prod_{j=1}^{n} F_{Rj4}^{\lambda_j} \right) \right) \right\rangle \\
&= \left\langle \left(\left(1 - \left(1 - T_{V1}\right) \sum_{j=1} n\lambda_j, 1 - \left(1 - T_{V2}\right) \sum_{j=1} n\lambda_j, 1 - \left(1 - T_{V3}\right) \sum_{j=1} n\lambda_j, 1 - \left(1 - T_{V4}\right) \sum_{j=1} n\lambda_j \right), \right.\right. \\
&\quad \left. \left(1 - \left(1 - T_{R1}\right) \sum_{j=1} n\lambda_j, 1 - \left(1 - T_{R2}\right) \sum_{j=1} n\lambda_j, 1 - \left(1 - T_{R3}\right) \sum_{j=1} n\lambda_j, 1 - \left(1 - T_{R4}\right) \sum_{j=1} n\lambda_j \right) \right),
\end{aligned}$$

$$\left(\left(I\sum_{V1} j=1^n w_j, I\sum_{V2} j=1^n w_j, I\sum_{V3} j=1^n w_j, I\sum_{V4} j=1^n w_j\right),\right.$$
$$\left.\left(I\sum_{R1} j=1^n w_j, I\sum_{R2} j=1^n w_j, I\sum_{R3} j=1^n w_j, I\sum_{R4} j=1^n w_j\right)\right),$$
$$\left(\left(F\sum_{V1} j=1^n w_j, F\sum_{V2} j=1^n w_j, F\sum_{V3} j=1^n w_j, F\sum_{V4} j=1^n w_j\right),\right.$$
$$\left.\left.\left(F\sum_{R1} j=1^n w_j, F\sum_{R2} j=1^n w_j, F\sum_{R3} j=1^n w_j, F\sum_{R4} j=1^n w_j\right)\right)\right\rangle$$
$$=\langle(((T_{V1},T_{V2},T_{V3},T_{V4}),(T_{R1},T_{R2},T_{R3},T_{R4})),((I_{V1},I_{V2},I_{V3},I_{V4}),(I_{R1},I_{R2},I_{R3},I_{R4})),$$
$$((I_{V1},I_{V2},I_{V3},I_{V4}),(I_{R1},I_{R2},I_{R3},I_{R4}))\rangle=\tilde{z}. \tag{15}$$

(P2) Due to $\tilde{z}^- \leq \tilde{z}_j \leq \tilde{z}^+$ for $j = 1, 2, \ldots, n$, there exists $\oplus_{j=1}^n \lambda_j \tilde{z}^- \leq \oplus_{j=1}^n \lambda_j \tilde{z}_j \leq \oplus_{j=1}^n \lambda_j \tilde{z}^+$. So, the inequality $\tilde{z}^- \leq \oplus_{j=1}^n \lambda_j \tilde{z}_j \leq \tilde{z}^+$ can hold according to (P1); that is, $\tilde{z}^- \leq \mathrm{TrNZNWAA}(\tilde{z}_1, \tilde{\tilde{n}}_2, \ldots, \tilde{\tilde{n}}_n) \leq \tilde{z}^+$.

(P3) Due to $\tilde{z}_j \leq \tilde{z}_j^*$ for $j = 1, 2, \ldots, n$, there is $\oplus_{j=1}^n \lambda_j \tilde{z}_j \leq \oplus_{j=1}^n \lambda_j \tilde{z}_j^*$; that is, $\mathrm{TrNZNWAA}(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \leq \mathrm{TrNZNWAA}(\tilde{z}_1^*, \tilde{z}_2^*, \ldots, \tilde{z}_n^*)$.

Thus, the proof of these properties is completed. □

4.2. Weighted Geometric Averaging Operator of TrNZNs

Definition 6. Set $\tilde{z}_j$= <((T_{Vj1}, T_{Vj2}, T_{Vj3}, T_{Vj4}), (T_{Rj1}, T_{Rj2}, T_{Rj3}, T_{Rj4})), ((I_{Vj1}, I_{Vj2}, I_{Vj3}, I_{Vj4}), (I_{Rj1}, I_{Rj2}, I_{Rj3}, I_{Rj4})), ((F_{Vj1}, F_{Vj2}, F_{Vj3}, F_{Vj4}), (F_{Rj1}, F_{Rj2}, F_{Rj3}, F_{Rj4}))> ($j = 1, 2, \ldots, n$) as a series of TrNZNs. Then, the TrNZNWGA operator is defined as

$$\mathrm{TrNZNWGA}(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) = \otimes_{j=1}^{n} \tilde{z}_j^{\lambda_j}, \tag{16}$$

where λ_j ($j = 1, 2, \ldots, n$) is the weight of the jth TrNZN $\tilde{z}_j$ for $\lambda_j \in [0, 1]$ and $\sum_{j=1}^n \lambda_j = 1$.

Regarding the basic operations of TrNZNs in Definition 2 and equation (16), we can give the theorem below.

Theorem 3. *Set $\tilde{z}_j$= <((T_{Vj1}, T_{Vj2}, T_{Vj3}, T_{Vj4}), (T_{Rj1}, T_{Rj2}, T_{Rj3}, T_{Rj4})), ((I_{Vj1}, I_{Vj2}, I_{Vj3}, I_{Vj4}), (I_{Rj1}, I_{Rj2}, I_{Rj3}, I_{Rj4})), ((F_{Vj1}, F_{Vj2}, F_{Vj3}, F_{Vj4}), (F_{Rj1}, F_{Rj2}, F_{Rj3}, F_{Rj4}))> ($j = 1, 2, \ldots, n$) as a series of TrNZNs. Then, the aggregated value of the TrNZNWGA operator is also TrNZN, which is obtained by*

$$\begin{aligned}
\mathrm{TrNZNWGA}(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) &= \otimes_{j=1}^{n} \tilde{z}_j^{\lambda_j} \\
&= \left\langle\left(\left(\prod_{j=1}^n T_{Vj1}^{\lambda_j}, \prod_{j=1}^n T_{Vj2}^{\lambda_j}, \prod_{j=1}^n T_{Vj3}^{\lambda_j}, \prod_{j=1}^n T_{Vj4}^{\lambda_j}\right), \left(\prod_{j=1}^n T_{Rj1}^{\lambda_j}, \prod_{j=1}^n T_{Rj2}^{\lambda_j}, \prod_{j=1}^n T_{Rj3}^{\lambda_j}, \prod_{j=1}^n T_{Rj4}^{\lambda_j}\right)\right),\right. \\
&\left(\left(1-\prod_{j=1}^n (1-I_{Vj1})^{\lambda_j}, 1-\prod_{j=1}^n (1-I_{Vj2})^{\lambda_j}, 1-\prod_{j=1}^n (1-I_{Vj3})^{\lambda_j}, 1-\prod_{j=1}^n (1-I_{Vj4})^{\lambda_j}\right),\right. \\
&\left.\left(1-\prod_{j=1}^n (1-I_{Rj1})^{\lambda_j}, 1-\prod_{j=1}^n (1-I_{Rj2})^{\lambda_j}, 1-\prod_{j=1}^n (1-I_{Rj3})^{\lambda_j}, 1-\prod_{j=1}^n (1-I_{Rj4})^{\lambda_j}\right)\right), \\
&\left(\left(1-\prod_{j=1}^n (1-F_{Vj1})^{\lambda_j}, 1-\prod_{j=1}^n (1-F_{Vj2})^{\lambda_j}, 1-\prod_{j=1}^n (1-F_{Vj3})^{\lambda_j}, 1-\prod_{j=1}^n (1-F_{Vj4})^{\lambda_j}\right),\right. \\
&\left.\left.\left(1-\prod_{j=1}^n (1-F_{Rj1})^{\lambda_j}, 1-\prod_{j=1}^n (1-F_{Rj2})^{\lambda_j}, 1-\prod_{j=1}^n (1-F_{Rj3})^{\lambda_j}, 1-\prod_{j=1}^n (1-F_{Rj4})^{\lambda_j}\right)\right)\right\rangle.
\end{aligned} \tag{17}$$

where λ_j (j = 1, 2, . . ., n) is the weight of the jth TrNZN $\tilde{z}_j$ for $\lambda_j \in [0, 1]$ and $\sum_{j=1}^{n} \lambda_j = 1$.

Based on the similar proof process of Theorem 1, we can verify Theorem 3, which is omitted.

In particular, the TrNZNWGA operator is reduced to the TrNZN geometric averaging operator when $\lambda_j = 1/n$ (j = 1, 2, . . ., n).

Theorem 4. *The TrNZNWGA operator also contains the three following properties:*

(P1) Idempotency: set $\tilde{z}_j$ = <((T_{Vj1}, T_{Vj2}, T_{Vj3}, T_{Vj4}), (T_{Rj1}, T_{Rj2}, T_{Rj3}, T_{Rj4})), ((I_{Vj1}, I_{Vj2}, I_{Vj3}, I_{Vj4}), (I_{Rj1}, I_{Rj2}, I_{Rj3}, I_{Rj4})), ((F_{Vj1}, F_{Vj2}, F_{Vj3}, F_{Vj4}), (F_{Rj1}, F_{Rj2}, F_{Rj3}, F_{Rj4})) > (j = 1, 2,. . .,n) as a series of TrNZNs. If $\tilde{z}_j = \tilde{z}$ for j = 1, 2, . . ., n, then there exists TrNZNWGA $(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) = \tilde{z}$.

(P2) Boundedness: set $\tilde{z}_j$ = <((T_{Vj1}, T_{Vj2}, T_{Vj3}, T_{Vj4}), (T_{Rj1}, T_{Rj2}, T_{Rj3}, T_{Rj4})), ((I_{Vj1}, I_{Vj2}, I_{Vj3}, I_{Vj4}), (I_{Rj1}, I_{Rj2}, I_{Rj3}, I_{Rj4})), ((F_{Vj1}, F_{Vj2}, F_{Vj3}, F_{Vj4}), (F_{Rj1}, F_{Rj2}, F_{Rj3}, F_{Rj4})) > (j = 1, 2, . . ., n) as a series of TrNZNs; then set the minimum and maximum TrNZNs as

$$\begin{aligned}
\tilde{z}^- &= \left\langle \begin{array}{c} \left(\left(\min_j T_{Vj1}, \min_j T_{Vj2}, \min_j T_{Vj3}, \min_j T_{Vj4}\right), \left(\min_j T_{Rj1}, \min_j T_{Rj2}, \min_j T_{Rj3}, \min_j T_{Rj4}\right)\right), \\ \left(\left(\max_j I_{Vj1}, \max_j I_{Vj2}, \max_j I_{Vj3}, \max_j I_{Vj4}\right), \left(\max_j I_{Rj1}, \max_j I_{Rj2}, \max_j I_{Rj3}, \max_j I_{Rj4}\right)\right), \\ \left(\left(\max_j F_{Vj1}, \max_j F_{Vj2}, \max_j F_{Vj3}, \max_j F_{Vj4}\right), \left(\max_j F_{Rj1}, \max_j F_{Rj2}, \max_j F_{Rj3}, \max_j F_{Rj4}\right)\right) \end{array} \right\rangle, \\
\tilde{z}^+ &= \left\langle \begin{array}{c} \left(\left(\max_j T_{Vj1}, \max_j T_{Vj2}, \max_j T_{Vj3}, \max_j T_{Vj4}\right), \left(\max_j T_{Rj1}, \max_j T_{Rj2}, \max_j T_{Rj3}, \max_j T_{Rj4}\right)\right), \\ \left(\left(\min_j I_{Vj1}, \min_j I_{Vj2}, \min_j I_{Vj3}, \min_j I_{Vj4}\right), \left(\min_j I_{Rj1}, \min_j I_{Rj2}, \min_j I_{Rj3}, \min_j I_{Rj4}\right)\right), \\ \left(\left(\min_j F_{Vj1}, \min_j F_{Vj2}, \min_j F_{Vj3}, \min_j F_{Vj4}\right), \left(\min_j F_{Rj1}, \min_j F_{Rj2}, \min_j F_{Rj3}, \min_j F_{Rj4}\right)\right) \end{array} \right\rangle
\end{aligned} \tag{18}$$

Then, there is $\tilde{z}^- \le$ TrNZNWGA $(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \le \tilde{z}^+$.

(P3) Monotony: set $\tilde{z}_j$ = <((T_{Vj1}, T_{Vj2}, T_{Vj3}, T_{Vj4}), (T_{Rj1}, T_{Rj2}, T_{Rj3}, T_{Rj4})), ((I_{Vj1}, I_{Vj2}, I_{Vj3}, I_{Vj4}), (I_{Rj1}, I_{Rj2}, I_{Rj3}, I_{Rj4})), ((F_{Vj1}, F_{Vj2}, F_{Vj3}, F_{Vj4}), (F_{Rj1}, F_{Rj2}, F_{Rj3}, F_{Rj4}))> (j = 1, 2, . . ., n) as a series of TrNZNs. If $\tilde{z}_j \le \tilde{z}_j^*$ for j = 1, 2, . . ., n, then there exists TrNZNWGA $(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \le$ TrNZNWGA $(\tilde{z}_1^*, \tilde{z}_2^*, \ldots, \tilde{z}_n^*)$.

By the same proof process of Theorem 2, the properties of the TrNZNWGA operator can be also verified, which are not repeated here.

5. MDM Approach Using the TrNZNWAA and TrNZNWGA Operators and Score and Accuracy Functions

This section establishes an MDM approach by using the TrNZNWAA and TrNZNWGA operators and score and accuracy functions to handle MDM problems with TrNZN information.

Regarding an MDM problem with TrNZN information, a set of alternatives $Q = \{Q_1, Q_2, \ldots, Q_m\}$ are commonly presented and satisfactorily assessed by a set of criteria $S = \{s_1, s_2, \ldots, s_n\}$. Each alternative over criteria is assessed by decision makers and then their given assessment values are expressed in the form of TrNZNs $\tilde{z}_{ij}$ = <((T_{Vij1}, T_{Vij2}, T_{Vij3}, T_{Vij4}), (T_{Rij1}, T_{Rij2}, T_{Rij3}, T_{Rij4})), ((I_{Vij1}, I_{Vij2}, I_{Vij3}, I_{Vij4}), (I_{Rij1}, I_{Rij2}, I_{Rij3}, I_{Rij4})), ((F_{Vij1}, F_{Vij2}, F_{Vij3}, F_{Vij4}), (F_{Rij1}, F_{Rij2}, F_{Rij3}, F_{Rij4}))> (j = 1, 2,. . .,n; i = 1, 2,. . .,m), where (T_{Vij1}, T_{Vij2}, T_{Vij3}, T_{Vij4}) ⊆ [0, 1] and (T_{Rij1}, T_{Rij2}, T_{Rij3}, T_{Rij4}) ⊆ [0, 1] indicate the truth degrees and reliability measures of the alternative Q_i over the criteria s_j, (I_{Vij1}, I_{Vij2}, I_{Vij3}, I_{Vij4}) ⊆ [0, 1] and (I_{Rij1}, I_{Rij2}, I_{Rij3}, I_{Rij4}) ⊆ [0, 1] indicate the indeterminate degrees and reliability measures of the alternative Q_i over the criteria s_j, and (F_{Vij1}, F_{Vij2}, F_{Vij3}, F_{Vij4}) ⊆ [0, 1] and (F_{Rij1}, F_{Rij2}, F_{Rij3}, F_{Rij4}) ⊆ [0, 1] indicate the falsity degrees and reliability measures of the alternative Q_i over the criteria s_j, along with $0 \le T_{Vij4} + I_{Vij4} + F_{Vij4} \le 3$ and $0 \le T_{Rij4} + I_{Rij4} + F_{Rij4} \le 3$ for j = 1, 2, . . ., n and i = 1, 2, . . ., m. Then, all the specified TrNZNs are constructed as their decision matrix $\tilde{Z} = (\tilde{z}_{ij})_{m\times n}$.

Thus, the TrNZNWAA and TrNZNWGA operators and the score and accuracy functions can be applied to MDM problems with TrNZN information, and then their MDM approach can be indicated by the following procedures:

Step 1: the aggregated TrNZN $\tilde{z}_i$ for Q_i (i = 1, 2, . . ., m) is obtained by applying the TrNZNWAA or TrNZNWGA operator:

$$\begin{aligned}
\tilde{z}_i &= \mathrm{TrNZNWAA}\left(\tilde{z}_{i1}, \tilde{z}_{i2}, \ldots, \tilde{z}_{in}\right) = \mathop{\oplus}_{j=1}^{n} \lambda_j \tilde{z}_{ij} \\
&= \Bigg\langle \Bigg(\Bigg(\Bigg(1-\prod_{j=1}^{n}\left(1-T_{Vij1}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-T_{Vij2}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-T_{Vij3}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-T_{Vij4}\right)^{\lambda_j} \Bigg), \\
&\quad \Bigg(1-\prod_{j=1}^{n}\left(1-T_{Rij1}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-T_{Rij2}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-T_{Rij3}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-T_{Rij4}\right)^{\lambda_j} \Bigg) \Bigg), \\
&\quad \Bigg(\Bigg(\prod_{j=1}^{n} I_{Vij1}^{\lambda_j}, \prod_{j=1}^{n} I_{Vij2}^{\lambda_j}, \prod_{j=1}^{n} I_{Vij3}^{\lambda_j}, \prod_{j=1}^{n} I_{Vij4}^{\lambda_j} \Bigg), \Bigg(\prod_{j=1}^{n} I_{Rij1}^{\lambda_j}, \prod_{j=1}^{n} I_{Rij2}^{\lambda_j}, \prod_{j=1}^{n} I_{Rij3}^{\lambda_j}, \prod_{j=1}^{n} I_{Rij4}^{\lambda_j} \Bigg) \Bigg), \\
&\quad \Bigg(\Bigg(\prod_{j=1}^{n} F_{Vij1}^{\lambda_j}, \prod_{j=1}^{n} F_{Vij2}^{\lambda_j}, \prod_{j=1}^{n} F_{Vij3}^{\lambda_j}, \prod_{j=1}^{n} F_{Vij4}^{\lambda_j} \Bigg), \Bigg(\prod_{j=1}^{n} F_{Rij1}^{\lambda_j}, \prod_{j=1}^{n} F_{Rij2}^{\lambda_j}, \prod_{j=1}^{n} F_{Rij3}^{\lambda_j}, \prod_{j=1}^{n} F_{Rij4}^{\lambda_j} \Bigg) \Bigg) \Bigg\rangle,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\tilde{z}_i &= \mathrm{TrNZNWGA}\left(\tilde{z}_{i1}, \tilde{z}_{i2}, \ldots, \tilde{z}_{\mathrm{in}}\right) = \mathop{\otimes}_{j=1}^{n} \tilde{z}_{ij}^{\lambda_j} \\
&= \Bigg\langle \Bigg(\Bigg(\prod_{j=1}^{n} T_{Vij1}^{\lambda_j}, \prod_{j=1}^{n} T_{Vij2}^{\lambda_j}, \prod_{j=1}^{n} T_{Vij3}^{\lambda_j}, \prod_{j=1}^{n} T_{Vij4}^{\lambda_j} \Bigg), \Bigg(\prod_{j=1}^{n} T_{Rij1}^{\lambda_j}, \prod_{j=1}^{n} T_{Rij2}^{\lambda_j}, \prod_{j=1}^{n} T_{Rij3}^{\lambda_j}, \prod_{j=1}^{n} T_{Rij4}^{\lambda_j} \Bigg) \Bigg), \\
&\quad \Bigg(\Bigg(1-\prod_{j=1}^{n}\left(1-I_{Vij1}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-I_{Vij2}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-I_{Vij3}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-I_{Vij4}\right)^{\lambda_j} \Bigg), \\
&\quad \Bigg(1-\prod_{j=1}^{n}\left(1-I_{Rij1}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-I_{Rij2}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-I_{Rij3}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-I_{Rij4}\right)^{\lambda_j} \Bigg) \Bigg), \\
&\quad \Bigg(\Bigg(1-\prod_{j=1}^{n}\left(1-F_{Vij1}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-F_{Vij2}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-F_{Vij3}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-F_{Vij4}\right)^{\lambda_j} \Bigg), \\
&\quad \Bigg(1-\prod_{j=1}^{n}\left(1-F_{Rij1}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-F_{Rij2}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-F_{Rij3}\right)^{\lambda_j}, 1-\prod_{j=1}^{n}\left(1-F_{Rij4}\right)^{\lambda_j} \Bigg) \Bigg) \Bigg\rangle.
\end{aligned} \tag{20}$$

Step 2: by equation (7), we calculate the score values of $S(\tilde{z}_i)$. If necessary, we calculate the accuracy values of $H(\tilde{z}_i)$ ($i = 1, 2, \ldots, m$) by equation (8).

Step 3: all the alternatives Q_i ($i = 1, 2, \ldots, m$) are ranked corresponding to the score values (the accuracy values) and the best one(s) is chosen in the set of alternatives.

Step 4: end.

6. MDM Example and Comparison with Existing MDM Approaches

6.1. MDM Example of Software Selection. This section indicates an MDM example of software selection adapted from [9] to reveal the usability and efficiency of the established MDM approach under the environment of TrNZNs.

In an MDM example, an investment company needs to select a suitable software system from potential software systems, where five candidate software systems are provided preliminarily and denoted as a set of five alternatives $Q = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$. Then, these alternatives must satisfy the requirements of the four criteria: s_1 (the contribution to organization performance), s_2 (the effort to transform from current system), s_3 (the costs of hardware/software investment), and s_4 (the outsourcing software developer reliability). Regarding the importance of the four criteria, the weight values of the four criteria are specified as the weight vector $\lambda = (0.25, 0.25, 0.3, 0.2)$. Thus, decision makers/experts assess the satisfiability of the five alternatives over the four criteria by TrNZNs $\tilde{z}_{ij} = <((T_{Vij1}, T_{Vij2}, T_{Vij3}, T_{Vij4}), (T_{Rij1}, T_{Rij2}, T_{Rij3}, T_{Rij4})), ((I_{Vij1}, I_{Vij2}, I_{Vij3}, I_{Vij4}), (I_{Rij1}, I_{Rij2}, I_{Rij3}, I_{Rij4})), ((F_{Vij1}, F_{Vij2}, F_{Vij3}, F_{Vij4}), (F_{Rij1}, F_{Rij2}, F_{Rij3}, F_{Rij4}))>$ ($j = 1, 2, 3, 4$; $i = 1, 2, 3, 4, 5$), where $(T_{Vij1}, T_{Vij2}, T_{Vij3}, T_{Vij4}) \subseteq [0, 1]$ and $(T_{Rij1}, T_{Rij2}, T_{Rij3}, T_{Rij4}) \subseteq [0, 1]$ indicate that the alternative Q_i satisfies the degrees and reliability measures of the criteria s_j, $(I_{Vij1}, I_{Vij2}, I_{Vij3}, I_{Vij4}) \subseteq [0, 1]$ and $(I_{Rij1}, I_{Rij2}, I_{Rij3}, I_{Rij4}) \subseteq [0, 1]$ indicate the indeterminate degrees and reliability measures of the alternative Q_i over the criteria s_j, and $(F_{Vij1}, F_{Vij2}, F_{Vij3}, F_{Vij4}) \subseteq [0, 1]$ and $(F_{Rij1}, F_{Rij2}, F_{Rij3}, F_{Rij4}) \subseteq [0, 1]$ indicate that the alternative A_i does not satisfy the degrees and reliability measures of the criteria s_j, along with $0 \leq T_{Vij4} + I_{Vij4} + F_{Vij4} \leq 3$ and $0 \leq T_{Rij4} + I_{Rij4} + F_{Rij4} \leq 3$. Hence, all the specified TrNZNs can be constructed as the following decision matrix $\tilde{Z} = (\tilde{z}_{ij})_{5\times 4}$:

$$\widetilde{Z}=\begin{bmatrix}\langle((0.4,0.5,0.6,0.7),(0.4,0.5,0.6,0.7)),((0.0,0.1,0.2,0.3),(0.3,0.4,0.5,0.6)),((0.1,0.1,0.1,0.1),(0.3,0.4,0.5,0.6)\rangle\\ \langle((0.3,0.4,0.5,0.5),(0.5,0.6,0.7,0.8)),((0.1,0.2,0.3,0.4),(0.4,0.5,0.6,0.7)),((0.0,0.1,0.1,0.1),(0.5,0.6,0.7,0.8))\rangle\\ \langle((0.1,0.1,0.1,0.1),(0.5,0.6,0.7,0.8)),((0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.6,0.7,0.8,0.9),(0.5,0.6,0.7,0.8))\rangle\\ \langle((0.7,0.7,0.7,0.7),(0.4,0.5,0.6,0.7)),((0.0,0.1,0.2,0.3),(0.4,0.5,0.6,0.7)),((0.1,0.1,0.1,0.1),(0.3,0.4,0.5,0.6))\rangle\\ \langle((0.0,0.1,0.2,0.2),(0.4,0.5,0.6,0.7)),((0.1,0.1,0.1,0.1),(0.3,0.4,0.5,0.6)),((0.5,0.6,0.7,0.8),(0.5,0.6,0.7,0.8))\rangle\\ \langle((0.0,0.1,0.2,0.3),(0.3,0.4,0.5,0.6)),((0.0,0.1,0.2,0.3),(0.4,0.5,0.6,0.7)),(0.2,0.3,0.4,0.5),(0.3,0.4,0.5,0.6)\rangle\\ \langle((0.2,0.3,0.4,0.5),(0.6,0.7,0.8,0.9)),((0.0,0.1,0.2,0.3),(0.5,0.6,0.7,0.8)),((0.0,0.1,0.2,0.3),(0.4,0.5,0.6,0.7))\rangle\\ \langle((0.0,0.1,0.1,0.2),(0.5,0.6,0.7,0.8)),((0.0,0.1,0.2,0.3),(0.5,0.6,0.6,0.7)),((0.3,0.4,0.5,0.6),(0.3,0.4,0.5,0.6))\rangle\\ \langle((0.4,0.5,0.6,0.7),(0.5,0.6,0.7,0.8)),((0.1,0.1,0.1,0.1),(0.6,0.7,0.7,0.8)),((0.0,0.1,0.2,0.2),(0.5,0.6,0.7,0.8))\rangle\\ \langle((0.4,0.4,0.4,0.4),(0.3,0.4,0.5,0.6)),((0.0,0.1,0.2,0.3),(0.4,0.5,0.6,0.7)),((0.0,0.1,0.2,0.3),(0.5,0.6,0.7,0.8))\rangle\\ \langle((0.3,0.4,0.5,0.6),(0.4,0.5,0.6,0.7)),((0.0,0.1,0.2,0.3),(0.4,0.5,0.5,0.6)),((0.1,0.1,0.1,0.1),(0.4,0.5,0.5,0.6))\rangle\\ \langle((0.0,0.1,0.1,0.2),(0.5,0.6,0.7,0.8)),((0.1,0.1,0.1,0.1),(0.4,0.5,0.6,0.7)),((0.5,0.6,0.7,0.8),(0.3,0.4,0.5,0.6))\rangle\\ \langle((0.2,0.3,0.4,0.5),(0.3,0.4,0.5,0.6)),((0.0,0.1,0.2,0.3),(0.5,0.6,0.7,0.8)),((0.1,0.2,0.2,0.3),(0.4,0.5,0.6,0.7))\rangle\\ \langle((0.2,0.3,0.4,0.5),(0.5,0.6,0.6,0.7)),((0.0,0.1,0.2,0.3),(0.5,0.6,0.7,0.8)),((0.1,0.2,0.3,0.3),(0.3,0.4,0.5,0.6))\rangle\\ \langle((0.6,0.7,0.7,0.8),(0.5,0.5,0.5,0.5)),((0.1,0.1,0.1,0.1),(0.5,0.6,0.7,0.8)),((0.0,0.1,0.1,0.2),(0.4,0.5,0.5,0.6))\rangle\\ \langle((0.3,0.4,0.5,0.6),(0.3,0.4,0.5,0.6)),((0.1,0.1,0.1,0.1),(0.4,0.5,0.6,0.7)),((0.1,0.2,0.3,0.4),(0.5,0.6,0.6,0.7))\rangle\\ \langle((0.3,0.4,0.5,0.5),(0.5,0.6,0.6,0.7)),((0.0,0.1,0.2,0.3),(0.3,0.4,0.5,0.6)),((0.0,0.1,0.1,0.2),(0.4,0.5,0.6,0.7))\rangle\\ \langle((0.1,0.2,0.3,0.4),(0.6,0.7,0.8,0.9)),((0.1,0.1,0.1,0.1),(0.5,0.6,0.7,0.8)),((0.3,0.4,0.5,0.6),(0.4,0.5,0.6,0.7))\rangle\\ \langle((0.1,0.2,0.3,0.4),(0.5,0.6,0.7,0.8)),((0.1,0.1,0.1,0.1),(0.4,0.5,0.5,0.6)),((0.4,0.5,0.6,0.6),(0.3,0.4,0.5,0.6))\rangle\\ \langle((0.1,0.2,0.3,0.3),(0.4,0.5,0.6,0.7)),((0.1,0.2,0.3,0.4),(0.4,0.5,0.5,0.6)),((0.2,0.3,0.4,0.5),(0.4,0.5,0.6,0.7))\rangle\end{bmatrix}. \tag{21}$$

Thus, we utilize the established MDM approach to obtain the most suitable software system(s), which can be depicted by the following decision process.

First, by equation (19) or equation (20), we obtain the following aggregated TrNZNs $\widetilde{z}_i$ $(i = 1, 2, 3, 4, 5)$:

$\widetilde{z}_1$= <((0.2636, 0.3656, 0.4682, 0.5719), (0.3569, 0.4572, 0.5577, 0.6585)), ((0, 0.1000, 0.1741, 0.2408), (0.3722, 0.4729, 0.5428, 0.6431)), ((0.1189, 0.1512, 0.1762, 0.1973), (0.3622, 0.4638, 0.5186, 0.6188))>

$\widetilde{z}_2$= <((0.1945, 0.2958, 0.3758, 0.4243), (0.5271, 0.6278, 0.7129, 0.8176)), ((0, 0.1189, 0.1798, 0.2319), (0.3993, 0.5005, 0.6012, 0.7018)), ((0, 0.1712, 0.2132, 0.2821), (0.3880, 0.4894, 0.5904, 0.6911))>

$\widetilde{z}_3$= <((0.1081, 0.1848, 0.2421, 0.3245), (0.4710, 0.5735, 0.6776, 0.7856)), ((0, 0.1000, 0.1464, 0.1830), (0.5233, 0.6236, 0.6964, 0.7969)), ((0.2566, 0.3737, 0.4272, 0.5393), (0.3936, 0.4949, 0.5958, 0.6964))>

$\widetilde{z}_4$= <((0.4035, 0.4652, 0.5298, 0.5983), (0.4767, 0.5771, 0.6486, 0.7500)), ((0, 0.1000, 0.1464, 0.1830), (0.4733, 0.5745, 0.6297, 0.7305)), ((0, 0.1699, 0.2366, 0.2366), (0.3409, 0.4427, 0.5439, 0.6447))>

$\widetilde{z}_5$= <((0.3454, 0.4287, 0.4599, 0.5218), (0.4096, 0.4767, 0.5478, 0.6242)), ((0, 0.1149, 0.1481, 0.1737), (0.3980, 0.4995, 0.5789, 0.6798)), ((0, 0.1950, 0.2552, 0.3760), (0.4472, 0.5477, 0.6136, 0.7145))>

Or we obtain the following aggregated TrNZNs $\widetilde{z}_i$ $(i = 1, 2, 3, 4, 5)$:

$\widetilde{z}_1$= <((0, 0.2991, 0.4162, 0.5244), (0.3514, 0.4522, 0.5527, 0.6531)), ((0.0209, 0.1000, 0.1809, 0.2639), (0.3764, 0.4767, 0.5478, 0.6486)), ((0.1261, 0.1745, 0.2266, 0.2835), (0.3751, 0.4762, 0.5218, 0.6224))>

$\widetilde{z}_2$= <((0, 0.2456, 0.2918, 0.3798), (0.5233, 0.6236, 0.7018, 0.8022)), ((0.0563, 0.1261, 0.1984, 0.2737), (0.4088, 0.5096, 0.6108, 0.7129)), ((0.1877, 0.2944, 0.3715, 0.4743), (0.3996, 0.5005, 0.6020, 0.7045))>

$\widetilde{z}_3$= <((0, 0.1597, 0.1888, 0.2543), (0.4449, 0.5479, 0.6499, 0.7513)), ((0.0463, 0.1000, 0.1565, 0.2162), (0.5271, 0.6278, 0.7087, 0.8139)), ((0.3437, 0.4500, 0.5422, 0.6655), (0.4042, 0.5051, 0.6064, 0.7087))>

$\widetilde{z}_4$= <((0.2832, 0.3885, 0.4807, 0.5658), (0.4729, 0.5733, 0.6431, 0.7434)), ((0.0463, 0.1000, 0.1565, 0.2162), (0.4867, 0.5884, 0.6430, 0.7458)), ((0.1480, 0.2276, 0.3109, 0.3109), (0.3565, 0.4578, 0.5599, 0.6636))>

$\widetilde{z}_5$= <((0, 0.2912, 0.3756, 0.3910), (0.3980, 0.4729, 0.5428, 0.6089)), ((0.0760, 0.1210, 0.1690, 0.2206), (0.4096, 0.5106, 0.5943, 0.6976)), ((0.1958, 0.3012, 0.3877, 0.5020), (0.4523, 0.5528, 0.6296, 0.7330))>

Then, the results of the MDM approach based on the TrNZNWAA and TrNNWGA operators and the score function are shown in Table 1.

From the results of Table 1, the ranking orders based on the TrNZNWAA and TrNZNWGA operators are identical and the best one indicates the same selection as the software system Q_4.

6.2. Comparison with Existing MDM Approaches. For convenient comparison with existing MDM approach in the setting of TrNNs [9], we may ignore the reliability measures in TrNZNs and only contain the decision matrix of TrNNs in the MDM example as its special case. Thus, existing MDM approach in the setting of TrNNs [9] can be used for the special case of the MDM example. In this case, the decision results based on the TrNNWAA and TrNNWGA operators

Table 1: Results of the MDM approach based on the TrNZNWAA and TrNZNWGA operators and the score function.

Aggregation operator	Score value	Ranking
TrNZNWAA	0.6892, 0.6845, 0.6154, 0.7207, 0.6824	Q4 > Q1 > Q2 > Q5 > Q3
TrNZNWGA	0.6607, 0.6257, 0.5750, 0.6848, 0.6158	Q4 > Q1 > Q2 > Q5 > Q3

Table 2: Results of the MDM approach based on the TrNNWAA and TrNNWGA operators and the score function [9].

Aggregation operator	Score value	Ranking
TrNNWAA	0.7092, 0.6744, 0.5694, 0.7437, 0.7077	Q4 > Q1 > Q5 > Q2 > Q3
TrNNWGA	0.6553, 0.5779, 0.5069, 0.6835, 0.5904	Q4 > Q1 > Q5 > Q2 > Q3

(equations (2) and (3)) and the score function of TrNNs (equation (4)) are introduced from [9], which are shown in Table 2.

Based on the decision results in Tables 1 and 2, we can see that the ranking orders based on the established MDM approach and the existing MDM approach [9] reveal their difference, but the best alternative Q_4 (the best software system) is identical. Then, the reason for their ranking difference is that decision information in the existing MDM approach [9] only contains TrNNs without considering the reliability measures of TrNNs in this MDM example, while decision information in the established MDM approach contains both TrNNs and their reliability measures. Hence, different decision information can result in different ranking results. It is obvious that the reliability measures in this example can affect the ranking order of alternatives, which shows the efficiency and rationality of the established MDM approach under the environment of TrNZNs.

However, the different decision information and decision methods can have an impact on the ranking of alternatives in the MDM problem, which reveals their importance in MDM applications. Thus, existing MDM methods [11–14, 23] only contain the TrNN or NZN information without considering the reliability measures in TrNNs or continuous Z-numbers in NZNs; they may lose some useful decision information so as to result in decision distortion/unreasonable decision results, which reveal some insufficiencies, while the new established approach can contain much more information than existing MDM methods and overcome the insufficiencies. Furthermore, existing methods [11–14, 23] also cannot deal with such MDM problems with TrNZNs.

Based on the above comparative analysis, the new established approach in setting of TrNZNs not only makes assessment information of TrNNs more reliable but also strengthens the effectiveness and continuity of decision information by comparison with existing MDM methods with TrNN and NZN information [9, 11–14, 23], which reveals the highlighting advantages of the new established approach in the information representation and MDM applications. Therefore, the new established approach not only extends existing methods but also demonstrates its superiority over them.

7. Conclusion

To make TrNN reliable, this paper presented a TrNZN set based on the truth, falsity, and indeterminacy trapezoidal Z-numbers as the generalization of the Z-number concept and then defined basic operations of TrNZNs, score and accuracy functions of TrNZNs, and ranking laws of TrNZNs. Next, the TrNZNWAA and TrNZNWGA operators were proposed to aggregate the TrNZN information. Furthermore, an MDM approach based on the two aggregation operators and score and accuracy functions was established in the setting of TrNZNs, in which the assessment values of alternatives over the criteria take the form of TrNZNs containing TrNNs and their reliability measures. Finally, an MDM example of software selection was provided to reveal the suitability and efficiency of the established MDM approach in the setting of TrNZNs.

The main advantage of this study is that the established method not only makes assessment information of TrNNs more reliable but also strengthens the decision rationality and efficiency in solving MDM problems with TrNZN information. However, the established method only uses the basic aggregation algorithms of TrNZNWAA and TrNZNWGA for MDM problems without considering the interactions of some evaluation criteria with each other, which implies the limitation of the proposed method in MDM applications. For capturing these relationships, the future study is to develop other aggregation algorithms and to use them for some other MDM problems including slope design schemes, energy and environmental managements, and medicine options.

Data Availability

There are no underlying data supporting the results of your study.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] M. Akram, S. Naz, and F. Smarandache, “Generalization of maximizing deviation and TOPSIS method for MADM in simplified neutrosophic hesitant fuzzy environment,” *Symmetry*, vol. 11, no. 8, p. 1058, 2019.

[2] J. Zhan, M. Akram, and M. Sitara, “Novel decision-making method based on bipolar neutrosophic information,” *Soft Computing*, vol. 23, no. 20, pp. 9955–9977, 2019.

[3] S. Habib, W. us Salam, M. A. Butt, M. Akram, and F. Smarandache, “A neutrosophic clinical decision-making system for cardiovascular diseases risk analysis,” *Journal of Intelligent & Fuzzy Systems*, vol. 39, no. 5, pp. 7807–7829, 2020.

[4] F. Liu and X. H. Yuan, “Fuzzy number intuitionistic fuzzy set,” *Fuzzy Systems and Mathematics*, vol. 21, no. 1, pp. 88–91, 2007.

[5] X. F. Wang, “Fuzzy number intuitionistic fuzzy geometric aggregation operators and their application to decision making,” *Control and Decision*, vol. 23, no. 6, pp. 607–612, 2008.
[6] X. F. Wang, “Fuzzy number intuitionistic fuzzy arithmetic aggregation operators,” *International Journal of Fuzzy Systems*, vol. 10, no. 2, pp. 104–111, 2008.
[7] G. Wei, X. Zhao, and R. Lin, “Some induced aggregating operators with fuzzy intuitionistic number information and their applications for group decision making,” *International Journal of Computational Intelligence Systems*, vol. 3, no. 1, pp. 84–95, 2010.
[8] J. Ye, “Prioritized aggregation operators of trapezoidal intuitionistic fuzzy sets and their application to multicriteria decision-making,” *Neural Computing and Applications*, vol. 25, no. 6, pp. 1447–1454, 2014.
[9] J. Ye, “Trapezoidal neutrosophic set and its application to multiple attribute decision-making,” *Neural Computing and Applications*, vol. 26, no. 5, pp. 1157–1166, 2015.
[10] R. Liang, J. Wang, and H. Zhang, “Evaluation of e-commerce websites: an integrated approach under a single-valued trapezoidal neutrosophic environment,” *Knowledge-Based Systems*, vol. 135, pp. 44–59, 2017.
[11] İ. Deli, “A novel defuzzification method of SV-trapezoidal neutrosophic numbers and multi-attribute decision making: a comparative analysis,” *Soft Computing*, vol. 23, no. 23, pp. 12529–12545, 2019.
[12] B. C. Giri, M. U. Molla, and P. Biswas, “TOPSIS method for MADM based on interval trapezoidal neutrosophic number,” *Neutrosophic Sets and Systems*, vol. 22, pp. 151–167, 2018.
[13] C. Jana, M. Pal, F. Karaaslan, and J. Wang, “Trapezoidal neutrosophic aggregation operators and their application to the multi-attribute decision-making process,” *Scientia Iranica*, vol. 27, no. 3, pp. 1655–1673, 2020.
[14] K. Khatter, “Interval valued trapezoidal neutrosophic set: multi-attribute decision making for prioritization of non-functional requirements,” *Journal of Ambient Intelligence and Humanized Computing*, vol. 12, pp. 1039–1055, 2020.
[15] L. A. Zadeh, “A note on z-numbers,” *Information Sciences*, vol. 181, no. 14, pp. 2923–2932, 2011.
[16] P. Patel, E. S. Khorasani, and S. Rahimi, “Modeling and implementation of z-number,” *Soft Computing*, vol. 20, no. 4, pp. 1341–1364, 2016.
[17] B. Kang, Y. Hu, Y. Deng, and D. Zhou, “A new methodology of multicriteria decision-making in supplier selection based onZ-numbers,” *Mathematical Problems in Engineering*, vol. 2016, Article ID 8475987, 17 pages, 2016.
[18] W. Jiang, C. Xie, M. Zhuang, Y. Shou, and Y. Tang, “Sensor data fusion with z-numbers and its application in fault diagnosis,” *Sensors*, vol. 16, no. 9, pp. 1–22, 2016.
[19] R. A. Aliev, W. Pedrycz, O. H. Huseynov, and S. Z. Eyupoglu, “Approximate reasoning on a basis of z-number-valued if-then rules,” *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 6, pp. 1589–1600, 2017.
[20] D. Qiu, R. Dong, S. Chen, and A. Li, “On an optimization method based on Z-numbers and the multi-objective evolutionary algorithm,” *Intelligent Automation and Soft Computing*, vol. 24, no. 1, pp. 147–150, 2018.
[21] B. Kang, P. Zhang, Z. Gao, G. Chhipi-Shrestha, K. Hewage, and R. Sadiq, “Environmental assessment under uncertainty using Dempster-Shafer theory and Z-numbers,” *Journal of Ambient Intelligence and Humanized Computing*, vol. 11, no. 5, pp. 2041–2060, 2020.
[22] Z. Ren, H. Liao, and Y. Liu, “Generalized Z-numbers with hesitant fuzzy linguistic information and its application to medicine selection for the patients with mild symptoms of the COVID-19,” *Computers & Industrial Engineering*, vol. 145, 2020 UNSP 106517, Article ID 106517.
[23] S. G. Du, J. Ye, R. Yong, and F. W. Zhang, “Some aggregation operators of neutrosophic Z-numbers and their multicriteria decision making method,” *Complex & Intelligent Systems*, vol. 7, pp. 429–438, 2021.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 5591576, 5 pages
https://doi.org/10.1155/2021/5591576

Research Article

On the Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations

Mohammad Abobala

Faculty of Science, Department of Mathematics, Tishreen University, Latakia, Syria

Correspondence should be addressed to Mohammad Abobala; mohammadabobala777@gmail.com

Received 8 January 2021; Revised 25 January 2021; Accepted 9 February 2021; Published 25 February 2021

Academic Editor: Parimala Mani

The objective of this paper is to study the representation of neutrosophic matrices defined over a neutrosophic field by neutrosophic linear transformations between neutrosophic vector spaces, where it proves that every neutrosophic matrix can be represented uniquely by a neutrosophic linear transformation. Also, this work proves that every neutrosophic linear transformation must be an AH-linear transformation; i.e., it can be represented by classical linear transformations.

1. Introduction

Neutrosophy is a new branch of philosophy founded by Smarandache [1, 2] to deal with uncertainty in real-life problems.

Neutrosophic concepts found their way in many other fields, such as classification [3, 4], number theory [5, 6], algebraic equations [7, 8], Boolean algebra [9] and optimization [10].

Neutrosophic algebra began with Smarandache and Kandasamy in [11], where they defined neutrosophic rings and fields for the first time. Lately, neutrosophic fields [12] were used in the study of neutrosophic vector spaces [13–16].

Neutrosophic matrices were defined to deal with indeterminacy problems, and many applications and theorems can be found in [17–19].

If V is a vector space over the field F, then $V(I) = \{x + yI; x, y \in V\}$ is the corresponding strong neutrosophic vector space over the neutrosophic field $F(I)$.

In [4, 20–24], Abobala et al. proposed the concept of AH substructures in groups, rings, spaces, and modules as a neutrosophic structures with two classical parts; for example, in the strong neutrosophic vector space $V(I)$, an AH subspace is the set $W(I) = T + SI$, where T and S are two classical subspaces of V. In a similar way, an AH linear transformation is a function f between two neutrosophic vector spaces $V(I)$ and $W(I)$ with two classical parts $f = g + hI$, where g and h are classical linear transformations between V and W.

It is known that classical matrices can be represented by linear transformations; from this point of view, we will study this problem in single valued neutrosophic systems.

In this work, we study neutrosophic matrices as linear neutrosophic functions. In particular, we prove that every linear transformation between two neutrosophic vector spaces must have an AH structure.

2. Preliminaries

Definition 1 (see [16]). Let (V, +, .) be a vector space over the field K, then ($V(I)$, +, .) is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field $K(I)$.

A neutrosophic field $K(I)$ is a triple ($K(I)$, +, .), where K is a classical field. A neutrosophic field is not a field by classical meaning, but it is a ring.

Elements of $V(I)$ have the following form: $\mathbf{x} + \mathbf{yI}; \mathbf{x}, \mathbf{y} \in \mathbf{V}$; i.e., $V(I)$ can be written as $\mathbf{V(I)} = \mathbf{V} + \mathbf{VI}$.

Definition 2 (see [16]). Let $V(I)$ be a strong neutrosophic vector space over the neutrosophic field $K(I)$ and $W(I)$ be a nonempty set of $V(I)$, then $W(I)$ is called a strong neutrosophic subspace if $W(I)$ itself is a strong neutrosophic vector space.

Definition 3 (see [16]). Let $v_1, v_2.v_s \in V(I)$, and $x \in V(I)$ we say that x is a linear combination of $\{v_i; i = 1,..,s\}$ if

$$x = a_1 v_1 + \ldots + a_s v_s \text{ such } a_i \in K(I). \tag{1}$$

The set $\{v_i; i = 1,..,s\}$ is called linearly independent if $a_1 v_1 + \cdots + a_s v_s = 0$ implies $a_i = 0$ for all i.

Definition 4 (see [18]). Let $M_{m\times n} = \{(a_{ij}): a_{ij} \in K(I)\}$ where $K(I)$ is a neutrosophic field. We call it the neutrosophic matrix.

3. Main Discussion

Theorem 1. *Let V, W be two vector spaces over the field F with $dim(V) = n$, $dim(W) = m$ and $V(I), W(I)$ be the corresponding neutrosophic vector spaces over the corresponding neutrosophic field $F(I)$. Let $g, h: V \longrightarrow W$ be two linear transformations, then there exists a neutrosophic linear transformation $f = g + hI: V(I) \longrightarrow W(I)$, where f is defined as follows:*

$$f(x + yI) = g(x) + [(g + h)(x + y) - g(x)]I. \tag{2}$$

Proof. We define $f = g + hI: V(I) \longrightarrow W(I)$, where

$$f(x + yI) = g(x) + [(g + h)(x + y) - g(x)]I, \tag{3}$$

in which f is a linear transformation, that is, because for every $m = x + yI, n = z + tI \in V(I)$, we have

$$\begin{aligned} f(m + n) &= f([x + z] + I[y + t]) = g(x + z) \\ &\quad + I[(g + h)(x + y + z + t) - g(x + z)] \\ &= (g(x) + [(g + h)(x + y) - g(x)]I) \\ &\quad + (g(z) + [(g + h)(z + t) - g(z)]I) \\ &= f(m) + f(n). \end{aligned} \tag{4}$$

On the contrary, consider an arbitrary neutrosophic number $a + bI \in F(I)$, then

$$\begin{aligned} f([a + bI]m) &= f([a + bI][x + yI]) = f(ax + I[ay + bx + by]) = f(ax + I[(a + b)(x + y) - ax]) \\ &= g(ax) + I[(g + h)[(a + b)(x + y)] - g(ax)] \\ &= ag(x) + I[(a + b)(g + h)[x + y] - ag(x)] \\ &= (a + bI)(g(x) + I[(g + h)(x + y) - g(x)] = (a + bI)f(m). \end{aligned} \tag{5}$$

Thus, f is a neutrosophic linear transformation. □

Definition 5. The neutrosophic linear transformation f defined in Theorem 1 is called a full AH-linear transformation.

Definition 6. Let $f = g + hI: V(I) \longrightarrow W(I)$ be a full AH-linear transformation and $M = A + BI$ be an $n \times m$ neutrosophic matrix over $F(I)$, and we call M the neutrosophic matrix of f if and only if $f(x + yI) = M(x + yI)$ for every $x + yI \in V(I)$.

Theorem 2. *Let $f = g + hI: V(I) \longrightarrow W(I)$ be any full AH-linear transformation, then $M = A + BI$ is the corresponding neutrosophic matrix if and only if A is the matrix of g and B is the matrix of h.*

Proof. We assume that A is the matrix of g and B is the matrix of h; hence, $Ax = g(x)$, $By = h(y)$, $(A + B)(x + y) = (g + h)(x + y)$. We have

$$\begin{aligned} M.(x + yI) &= (A + BI)(x + yI) = (Ax + I[Ay + Bx + By]) = (Ax + I[(A + B)(x + y) - Ax]) \\ &= g(x) + I[(g + h)(x + y) - g(x)] = f(x + yI). \end{aligned} \tag{6}$$

Thus, M is the neutrosophic matrix of f.

Conversely, suppose that M is the neutrosophic matrix of f, and we shall prove that A is the matrix of g and B is the matrix of h.

According to the assumption, we have $M(x + yI) = f(x + yI)$; hence,

$$\begin{aligned} &(Ax + I[(A + B)(x + y) - Ax]) \\ &\quad = g(x) + I[(g + h)(x + y) - g(x)]. \end{aligned} \tag{7}$$

This implies that $Ax = g(x)$, $(A + B)(x + y) = (g + h)(x + y)$ so that $B(x + y) = h(x + y)$. By considering the arbitrariness of x and y, we get that A is the matrix of g and B is the matrix of h. □

Example 1

(a) Let $V(I) = R^2(I) = \{(a,b) + (c,d)I = (a+cI, b+dI);\ a,b,\ c,d \in R\}$, consider the following neutrosophic matrix $M = \begin{pmatrix} 1+I & I \\ -I & 2-I \end{pmatrix}$. The corresponding neutrosophic linear transformation is defined as follows:

$$\begin{aligned} f(x+yI) &= M.\begin{pmatrix} a+cI \\ b+dI \end{pmatrix} = (a + I[c+a+c+b+d], -aI - cI + 2b + 2\ dI - bI - dI) \\ &= (a + I[a+2c+b+d], 2b + I[-a-c-b+d]) = (a, 2b) + I(a+2c+b+d, -a-c-b+d). \end{aligned} \tag{8}$$

(b) $f = g + hI$; $g(x,y) = (x, 2y)$, $h(x,y) = (x+y, -x-y)$, where $g, h\colon V \longrightarrow V$.

Theorem 3. *Let V, W be two vector spaces over the field F, with $dim(V) = n$, $dim(W) = m$, and let $M = A + BI$ be any $n \times m$ neutrosophic matrix over $F(I)$. Then, M can be represented by a unique full AH-linear transformation $f = g + hI$, where A is the matrix of g and B is the matrix of h.*

Proof. According to Theorem 2, the neutrosophic matrix $M = A + BI$ can be represented by a neutrosophic full AH-linear transformation $f = g + hI$, where A is the matrix of g and B is the matrix of h. For the uniqueness condition, we suppose that $F = G + HI$ is another linear AH-transformation with the property.

$M(x+yI) = F(x+yI)$. We have

$$M.(x+yI) = F(x+yI) = f(x+yI), \quad \text{for all } x + yI \in V(I). \tag{9}$$

Thus, $F = f$ and f is unique.

The following theorem shows an algorithm to find a basis for the neutrosophic vector space $V(I)$ from any basis of the corresponding classical vector space V. □

Theorem 4. *Let V(I) be any neutrosophic vector space over the neutrosophic field F(I) and V be its corresponding classical vector space over the field F. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a basis of V over F, then $L = \{l_{ij} = v_i + (v_j - v_i)I;\ 1 \le i, j \le n\}$ is a basis of V(I) over F(I).*

Proof. First of all, we must prove that L generates $V(I)$ over $F(I)$. Let $x + yI$ be any element of $V(I)$, where $x,\ y \in V$, and we have

$$\begin{aligned} x &= \sum_{i=1}^{n} a_i v_i, x + y = \sum_{j=1}^{n} b_j v_j, \text{We put } r_{ij} \\ &= a_i + (b_j - a_i)I \in F(I). \end{aligned} \tag{10}$$

Now, we compute $\sum_{i,j=1}^{n} r_{ij} l_{ij}$.

$$\begin{aligned} \sum_{i,j=1}^{n} r_{ij} l_{ij} &= \sum_{i,j=1}^{n} \left(a_i + (b_j - a_i)I\right)\left(v_i + (v_j - v_i)I\right) \\ &= \sum_{i,j=1}^{n} a_i v_i + I\left[b_j v_j - a_i v_i\right] \\ &= \sum_{i=1}^{n} a_i v_i + I\left[\sum_{j=1}^{n} b_j v_j - \sum_{i=1}^{n} a_i v_i\right] \\ &= x + I[(x+y) - x] = x + yI. \end{aligned} \tag{11}$$

Thus, L generates $V(I)$ over $F(I)$.

Now, we prove that L is linearly independent. For this purpose, we assume that $\sum_{i,j=1}^{n} (a_i + b_j I) l_{ij} = 0$; thus, we get

$$\begin{aligned} &\sum_{i,j=1}^{n} \left(a_i v_i + I\left[(a_i + b_j)v_j - a_i v_i\right)\right) = 0, \text{ hence, } \sum_{i=1}^{n} a_i v_i \\ &= \sum_{i,j}^{n} (a_i + b_j) v_j = 0, \text{ thus, } a_i = a_i + b_j = 0, \text{so that, } b_j = 0. \end{aligned} \tag{12}$$

This implies that L is linearly independent, and then it is a basis. □

Example 2. It is well known that $\{x = (1, 0),\ y = (0, 1)\}$ is a basis of $V = R^2$. The corresponding basis of $V(I) = R^2(I)$ is

$$\begin{aligned} &\{x,\ y,\ x + (y-x)I,\ y + (x-y)I\} \\ &= \{(1,0),\ (0,1),\ (1,0) + (-1,1)I,\ (0,1) + (1,-1)I\}. \end{aligned} \tag{13}$$

The following theorem shows that every linear transformation between $V(I)$ and $W(I)$ must be a full AH-linear transformation.

Theorem 5. *Let V, W be two vector spaces over the field F, with $dim(V) = n$, $dim(W) = m$ and let $V(I), W(I)$ be the corresponding neutrosophic vector spaces over $F(I)$. Let $f\colon V(I) \longrightarrow W(I)$ be any linear transformation, then f is a full AH-linear transformation.*

Proof. Let $f\colon V(I) \longrightarrow W(I)$ be any linear transformation, and we must prove that there exists two classical linear transformations $g, q\colon V \longrightarrow W$, where $f = g + qI$.

Suppose that $S = \{v_1, v_2, \ldots, v_n\}$ is a basis of V, then $L = \{l_{ij} = v_i + (v_j - v_i)I;\ 1 \le i, j \le n\}$ is a basis of $V\ (I)$. It is known that $f(L) = \{f(v_i + (v_j - v_i)I) = w_i + (w_j - w_i)I;\ w_i, w_j \in W\}$ is a basis of $W\ (I)$ and that is because the direct image of a basis by any linear transformation is a gain a basis.

Define $g: V \longrightarrow W; g(v_i) = w_i, h: V \longrightarrow W; h(v_j) = w_j$. It is clear that $f(v_i + (v_j - v_i)I) = g(v_i) + I[h(v_j) - g(v_i)]$. This means that $f = g + qI = g + (h - g)I$. Now, we must prove that $g, q = h - g$ are classical linear transformations.

Let x, y be any two elements of V, we have $x = x + 0I, y = y + 0I \in V(I)$. We have

$$f(x + y) = f([x + 0I] + [y + 0I]) = g(x + y) = g(x) + g(y). \tag{14}$$

For any $m \in F$, we have $m = m + 0I \in F(I)$, and $f([m + 0I][x + 0I]) = f(mx + 0I) = g(mx) = mg(x)$, and thus, g is a linear transformation.

On the other hand, we have

$$\begin{aligned} xI, yI \in V(I), \text{and } f(xI + yI) &= f([x + y]I) = f(0 + [x + y]I) = g(0) + I[(g + q)(x + y) - g(0)] = I[h(x + y)] \\ &= h(x)I + h(y)I, \text{thus } h(x + y) = h(x) + h(y), \\ f[m + 0I][0 + xI] &= f(0 + mxI) = g(0) + I[(g + q)(mx) - g(0)] = I[h(mx)] \\ &= mh(x)I, \text{so that } h(mx) = mh(x). \end{aligned} \tag{15}$$

This implies that g, h are two classical linear transformations; thus, g, q are linear transformations, which implies that $f = g + qI$ is a full AH-linear transformation. □

Remark 1. From Theorem 5 and Theorem 3, we get the following interesting result: every neutrosophic linear transformation $f: V(I) \longrightarrow W(I)$ can be represented by a unique neutrosophic matrix $M = A + BI$.

3.1. Further Applications. According to this work, we can use linear functions to study any problem that needs neutrosophic matrices. From this point of view, single-valued neutrosophic matrices used in [19] can be turned into algebraic linear functions.

4. Conclusion

In this paper, we have proved that every neutrosophic matrix can be represented uniquely by a neutrosophic linear vector space transformation. Also, we have showed that the linear property of any neutrosophic vector space function implies the AH-structure of this function.

This work opens a wide door to use neutrosophic vector spaces and matrices in classical representation theory of groups since it is well known that classical groups are represented by linear transformations from a vector space to itself. According to our results, we can find an important application of neutrosophic algebraic theory in the classical representation theory of groups. This application can be summarized by the following open question.

5. Open Problem

Determine the algebraic structure of all groups which can be represented by neutrosophic linear transformations from a neutrosophic vector space $V(I)$ to itself.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

This research has no external funding.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

[1] F. Smarandache, *Introduction to Neutrosophic Statistics*, Sitech & Educ89, Columbus. OH, USA.

[2] F. Smarandache, "Neutrosophic set a generalization of the intuitionistic fuzzy sets," *International Journal of Pure and Applied Mathematics*, vol. 24, pp. 287–297, 2005.

[3] M. Abobala, "Classical homomorphisms between n-refined neutrosophic rings," *International Journal of Neutrosophic Science*, vol. 7, pp. 74–78, 2020.

[4] M. Abobala, "Semi homomorphisms and algebraic relations between strong refined neutrosophic modules and strong neutrosophic modules," *Neutrosophic Sets and Systems*, vol. 39, 2021.

[5] H. Sankari and M. Abobala, "Neutrosophic linear diophantine equations with two variables," *Neutrosophic Sets and Systems*, vol. 38, 2020.

[6] M. Abobala, "Foundations of neutrosophic number theory," *Neutrosophic Sets and Systems*, vol. 39, 2021.

[7] M. Abobala, "On some neutrosophic algebraic equations," *Journal of New Theory*, vol. 33, 2020.

[8] S. A. Edalatpanah, "Systems of neutrosophic linear equations," *Neutrosophic Sets and Systems*, vol. 33, pp. 92–104, 2020.

[9] T. Chalapathi and L. Madhavi, "Neutrosophic boolean rings," *Neutrosophic Sets and Systems*, vol. 33, pp. 57–66, 2020.

[10] M. Abdel-Basset, R. Mohamed, A. E. N. H. Zaied, A. Gamal, and F. Smarandache, "Solving the supply chain problem using

the best-worst method based on a novel Plithogenic model," in *Optimization Theory Based on Neutrosophic and Plithogenic Sets*, pp. 1–19, Academic Press, Cambridge, CA, USA, 2020.

[11] W. B. V. Kandasamy and F. Smarandache, *Neutrosophic Rings*, Hexis, Phoenix, Arizona, 2006.

[12] M. Ali, F. Smarandache, M. Shabir, and L. Vladareanu, "Generalization of neutrosophic rings and neutrosophic fields," *Neutrosophic Sets and Systems*, vol. 5, pp. 9–14, 2014.

[13] F. Smarandache and M. Abobala, "n-Refined neutrosophic vector spaces," *International Journal of Neutrosophic Science*, vol. 7, pp. 47–54, 2020.

[14] H. Sankari and M. Abobala, "Solving three conjectures about neutrosophic quadruple vector spaces," *Neutrosophic Sets and Systems*, vol. 38, 2020.

[15] M. A. Ibrahim, A. A. A. Agboola, B. S. Badmus, and S. A. Akinleye, "On refined neutrosophic vector spaces I," *International Journal of Neutrosophic Science*, vol. 7, pp. 97–109, 2020.

[16] A. A. A. Agboola and S. A. Akinleye, "Neutrosophic vector spaces," *Neutrosophic Sets and Systems*, vol. 4, pp. 9–17, 2014.

[17] M. Dhar, S. Broumi, and F. Smarandache, "A note on square neutrosophic fuzzy matrices," *Neutrosophic Sets and Systems*, vol. 3, pp. 37–41, 2014.

[18] H. Khaled, A. Younus, and A. Mohammad, "The rectangle neutrosophic fuzzy matrices," *Faculty of Education Journal*, vol. 15, 2019 Arabic version.

[19] P. K. Singh, "Single-valued neutrosophic context analysis at distinct multi-granulation," *Computational and Applied Mathematics, June*, vol. 38, p. 80, 2019, https://link.springer.com/article/10.1007/s40314-019-0842-4.

[20] M. Abobala, "A study of AH-substructures in n-refined neutrosophic vector spaces," *International Journal of Neutrosophic Science*, vol. 9, pp. 74–85, 2020.

[21] M. Abobala, "On some special elements in neutrosophic rings and refined neutrosophic rings," *Journal of New Theory*, vol. 33, 2020.

[22] F. Smarandache and M. Abobala, "n-Refined neutrosophic rings," *International Journal of Neutrosophic Science*, vol. 6, pp. 83–90, 2020.

[23] H. Sankari and M. Abobala, "AH-homomorphisms in neutrosophic rings and refined neutrosophic rings," *Neutrosophic Sets and Systems*, vol. 38, 2020.

[24] H. Sankari and M. Abobala, "n-Refined neutrosophic modules," *Neutrosophic Sets and Systems*, vol. 36, 2020.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 5582102, 5 pages
https://doi.org/10.1155/2021/5582102

Research Article

Neutrosophic D'Agostino Test of Normality: An Application to Water Data

Mohammed Albassam [1] **Nasrullah Khan,** [2] **and Muhammad Aslam** [1]

[1]*Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21551, Saudi Arabia*
[2]*Department of Statistics, University of Veterinary and Animal Sciences, Jhang Campus, Lahore, Pakistan*

Correspondence should be addressed to Muhammad Aslam; aslam_ravian@hotmail.com

Received 25 January 2021; Revised 8 February 2021; Accepted 16 February 2021; Published 23 February 2021

Academic Editor: Parimala Mani

The D'Agostino test has been widely applied for testing the normality of the data. The existing D'Agostino test cannot be applied when the data have some indeterminate observations or observations which are obtained from the complex systems. In this paper, we present a D'Agostino test under neutrosophic statistics. We propose the D'Agostino test to test the normality of the data having indeterminate observations. The design of the proposed test is given and implemented with the help of real data. From the comparison, it is concluded that the proposed test is effective, adequate, and suitable to be applied in the presence of indeterminacy.

1. Introduction

The data obtained from various fields such as medical, physiological, education, and chemical process are assumed to follow the approximately normal distribution. Therefore, before some estimation and forecasting, the normality of the data in hand is checked first. If the data follow the normal distribution, the statistical techniques based on normal distribution are used; otherwise, the nonparametric methods are applied for the analysis of the data. Among many statistical tests, the D'Agostino test has been widely applied for testing the normality of the data. This test is used to test the null hypothesis that the data do not significantly differ from the normal distribution versus the alternative hypothesis that the data significantly differ from the normality. D'Agostino and Stephens [1] introduced statistical tests when the data follow the normal distribution. Öztuna et al. [2] studied the power of the test and type-I error rate for various tests under normality assumptions. Yap and Sim [3] discussed various statistical tests and showed that the D'Agostino test has better power. Chen and Xia [4] presented tests when data are nonnormal. Mishra et al. [5] presented the descriptive statistic for the test. More details on the statistical test for normality can be seen in [6–9].

The traditional statistical tests are applied to test the hypothesis that the data follow approximately normal distribution with exact mean and variance. In some situations, such as the measure of the water level, a lifetime of a product and melting of a material cannot be expressed in the exact form and have approximate mean and variances. In this case, the statistical test using the fuzzy logic is preferable to apply for the analysis of the data [10]. Hesamian and Akbari [11] presented the tests using fuzzy logic. Chachi and Taheri [12] worked on the optimal test using the fuzzy approach. Haktanır and Kahraman [13] discussed the role of tests in decision-making issues. For details, the reader may refer to [14–24].

The neutrosophic logic which is more efficient than the fuzzy logic and interval-based analysis was proposed by Smarandache [25]. This logic estimates the measures of truth, falsehood, and indeterminacy, while the fuzzy logic is unable to estimate the measure of indeterminacy. More applications of neutrosophic logic can be read in [26–36]. Based on the idea of neutrosophic logic, Smarandache [37] introduced the descriptive neutrosophic statistics which are applied for the analysis of the data having indeterminate observations. Kandasamy and Smarandache [38] introduced the neutrosophic numbers for the first time. Chen et al. [39] applied the neutrosophic numbers in rock measuring. Aslam [40] introduced a new branch of statistical quality control under neutrosophic statistics. Kolmogorov–Smirnov tests and Bartlett and Hartley tests using neutrosophic statistics were developed by Aslam [41, 42], respectively. More details on the application of neutrosophic statistics can be seen in [43, 44].

Although the D'Agostino test under classical statistics is available in the literature, the existing D'Agostino test cannot be applied if observations are imprecise, vague, and indeterminate. By exploring the literature and according to the best of our knowledge, there is work on the D'Agostino test. In this paper, we will propose and design the D'Agostino test under indeterminacy. The operational process of the proposed test is explained. The application of the proposed test will be given with the help of water data. We expect that the proposed test will be informative and adequate than the existing D'Agostino test under classical statistics in the indeterminate environment.

2. Preliminary

Suppose that a_i and $b_i I_N; I_N \epsilon [I_L, I_U]$ are determinate and indeterminate parts of neutrosophic random variable $z_N = a_i + b_i I_N; I_N \epsilon [I_L, I_U]$, $i = 1, 2, \ldots, n_N$, where n_N denotes the neutrosophic sample size. The values of z_N reduce to a_i when $I_N = 0$. Based on this information, compute the neutrosophic average for variable $z_N \epsilon [z_L, z_U]$ as follows:

$$\bar{z}_N = \bar{a} + \bar{b} I_N,\ I_N \epsilon [I_L, I_U], \tag{1}$$

where $\bar{a} = (1/n_N) \sum_{i=1}^{n_N} a_i$ and $\bar{b} = (1/n_N) \sum_{i=1}^{n_N} b_i$.

The neutrosophic sum of squares (NSS) by following [39] is computed as follows:

$$\sum_{i=1}^{n_N} (z_i - \bar{z}_{iN})^2 = \sum_{i=1}^{n_N} \begin{bmatrix} \min \begin{pmatrix} (a_i + b_i I_L)(\bar{a} + \bar{b} I_L), (a_i + b_i I_L)(\bar{a} + \bar{b} I_U) \\ (a_i + b_i I_U)(\bar{a} + \bar{b} I_L), (a_i + b_i I_U)(\bar{a} + \bar{b} I_U) \end{pmatrix} \\ \max \begin{pmatrix} (a_i + b_i I_L)(\bar{a} + \bar{b} I_L), (a_i + b_i I_L)(\bar{a} + \bar{b} I_U) \\ (a_i + b_i I_U)(\bar{a} + \bar{b} I_L), (a_i + b_i I_U)(\bar{a} + \bar{b} I_U) \end{pmatrix} \end{bmatrix}, \quad I_N \epsilon [I_L, I_U]. \tag{2}$$

3. Design of the Proposed D'Agostino Test under Neutrosophic Statistics

The main objective is to design D'Agostino test under neutrosophic statistics for testing the null hypothesis H_{0N} that the neutrosophic data follow the neutrosophic normal distribution versus the alternative hypothesis H_{1N} that the data do not belong to the neutrosophic normal distribution. The acceptance of the null hypothesis means that the data are not significantly away from the normal distribution. The operational procedure of the proposed test is stated as follows.

Step 1: Compute the neutrosophic averages of lower values $a_i (i = 1, 2, \ldots, n_L)$ and upper values $b_i (i = 1, 2, \ldots, n_U)$ as follows: $\bar{a} = (1/n_N) \sum_{i=1}^{n_N} a_i$ and. $\bar{b} = (1/n_N) \sum_{i=1}^{n_N} b_i$.

Step 2: Find neutrosophic average as follows:

$$\bar{z}_N = \bar{a} + \bar{b} I_N, I_N \epsilon [I_L, I_U]. \tag{3}$$

Step 3: The neutrosophic sum of squares (NSS) by following [39] is calculated using the following expression:

$$\sum_{i=1}^{n_N} (z_i - \bar{z}_{iN})^2 = \sum_{i=1}^{n_N} \begin{bmatrix} \min\left((a_i - \bar{a})^2, \left((a_i - \bar{a})\left((a_i - \bar{a}) + 1 \times (b_i - \bar{b})\right), (a_i - \bar{a}) + 1 \times (b_i - \bar{b})^2\right)\right) \\ \max\left((a_i - \bar{a})^2, \left((a_i - \bar{a})\left((a_i - \bar{a}) + 1 \times (b_i - \bar{b})\right), (a_i - \bar{a}) + 1 \times (b_i - \bar{b})^2\right)\right) \end{bmatrix}. \tag{4}$$

Step 4: Compute the neutrosophic numerator $T_N \epsilon [T_L, T_U]$ of the proposed test as follows:

$$T_N = \sum \left(i_N - \left(\frac{n_N + 1}{2} \right) \right) X_{iN} T_N \epsilon [T_L, T_U], \tag{5}$$

where i_N denotes the rank of neutrosophic observations X_{iN} for $a_i (i = 1, 2, \ldots, n_L)$ and $b_i (i = 1, 2, \ldots, n_U)$.

Step 5: Compute the neutrosophic test statistic $D_N \epsilon [D_L, D_U]$ of the proposed test as follows:

$$D_N = \frac{T_N}{\sqrt{n_N^3 \left(\sum_{i=1}^{n_N} (z_i - \bar{z}_{iN})^2 \right)}}, T_N \epsilon [T_L, T_U], D_N \epsilon [D_L, D_U]. \tag{6}$$

Step 6: Decide the level of significance α and select the critical values from the D'Agostino table. The null hypothesis will be accepted if $D_N \epsilon [D_L, D_U]$ lies within the range of the tabulated values.

Table 1: The PMW data.

Portuguese mineral	$n.$ 1		$n.$ 2		$n.$ 3		$n.$ 4		$n.$ 5	
	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i
HCO_3^-	21	41	113	119	2.2	4.2	8	11.6	4.6	5
CI^-	7	9	16.5	17.5	3.6	4	4.1	4.7	6.6	7.4
N_a^+	10	16	10.3	10.7	2.8	3.8	2.8	3.6	5.4	5.6
C_a^{2+}	3	4	15	21	0.01	1.01	1.9	2.9	0.72	0.84
SiO_2	23	29	13.7	14.9	1.01	7.8	5.8	6.8	16.7	18.3
pH	6.1	6.5	6.7	7.1	5.71	5.81	5.9	6	5.4	5.8

Table 2: Neutrosophic means of five different types of water.

Water	$\overline{a}_N$	$\overline{b}_N$	$\overline{z}_N$
$n.$ 1	11.68	17.58	[11.68, 29.26]
$n.$ 2	29.2	31.7	[29.2, 60.9]
$n.$ 3	2.55	4.43	[2.55, 6.98]
$n.$ 4	4.75	5.93	[4.75, 10.68]
$n.$ 5	5.57	7.15	[5.57, 12.72]

4. Application for Portuguese Mineral Water

In this section, we will give the application of the proposed test using the Portuguese mineral water (PMW) data. D'Urso and Giordani [45] used the same data and analyzed them using classical statistics. D'Urso and Giordani [45] conidered six mineral concentrations such as six mineral concentrations of HCO_3^-, CI^-, N_a^+, C_a^{2+}, SiO_2, and pH. The PMW data are reported in Table 1. Table 1 clearly indicates that the data are reported in intervals. Before any prediction or estimation is given for the data, it is necessary to see that the data do not significantly differ from the normal distribution. Therefore, we will apply the proposed test on these data to test whether the six variables are from the neutrosophic normal distribution or not.

The necessary computations for PMW data are given in the following steps.

Step 1: The neutrosophic averages of lower values $a_i\,(i = 1, 2, \ldots, n_L)$ and upper values $b_i\,(i = 1, 2, \ldots, n_U)$ of PMW data of five different types of water are given in Table 2.

Step 2: The neutrosophic averages $\overline{z}_N$; $I_N \epsilon [0, 1]$for the water data are also shown in Table 2.

Step 3: The values of NSS are given in Table 3 by following [39]:

$$\sum_{i=1}^{n_N} \left(z_i - \overline{z}_{iN}\right)^2 = \sum_{i=1}^{n_N} \begin{bmatrix} \min\left(\left(a_i - \overline{a}\right)^2, \left(\left(a_i - \overline{a}\right)\left(\left(a_i - \overline{a}\right) + 1 \times \left(b_i - \overline{b}\right)\right), \left(a_i - \overline{a}\right) + 1 \times \left(b_i - \overline{b}\right)^2\right)\right) \\ \max\left(\left(a_i - \overline{a}\right)^2, \left(\left(a_i - \overline{a}\right)\left(\left(a_i - \overline{a}\right) + 1 \times \left(b_i - \overline{b}\right)\right), \left(a_i - \overline{a}\right) + 1 \times \left(b_i - \overline{b}\right)^2\right)\right) \end{bmatrix}. \tag{7}$$

Step 4: The values $T_N \epsilon [T_L, T_U]$ and $D_N \epsilon [D_L, D_U]$ are also shown in Table 3.

Step 5: Let $\alpha = 0.05$; the range of the tabulated values is 0.2513, 0.2849. The null hypothesis that the data follow the normal distribution is accepted if $D_N \epsilon [D_L, D_U]$ is within the range of the tabulated values. The acceptance or rejection of H_{0N} is shown in Table 3. From Table 3, it is clear that the PMW data for all waters do not follow the neutrosophic normal distribution.

5. Comparative Study and Discussion

The proposed D'Agostino test under neutrosophic statistics is the extension of the D'Agostino test under classical statistics. The proposed test reduces to D'Agostino test under classical statistics when $D_N = D_L = 0$. We compare the proposed test with the existing D'Agostino test using the PMW data of five types of water with the same values of α. The values of statistic D for the existing test and the proposed test along with the measure of indeterminacy are shown in Table 4. From Table 4, it can be seen that the proposed test statistic $D_N \epsilon [D_L, D_U]$ has the results in the neutrosophic form with the probability of the indeterminacy. On the contrary, the existing test provides only the determined values of statistic D. For example, when $\alpha = 0.05$ and $n.1$, the null hypothesis H_{0N} will accepted the probability of 0.95, the chance to do not accept H_{0N} is 0.05, and the probability of indeterminacy is 0.0621. From the proposed test, it can be seen that $0.95 + 0.05 + 0.062 > 1$ which shows the case of paraconsistent neutrosophic probability, see [37].

Table 3: The values of NSS of five waters.

Water	NSS	$T_N \epsilon [T_L, T_U]$	$D_N \epsilon [D_L, D_U]$	Decision
n. 1	[811.01, 4915.23]	[73.85, 117.75]	[0.1764, 0.1142]	Do not accept H_{0N}
n. 2	[7858.36, 33176.61]	[274.2, 296.5]	[0.2104, 0.1107]	Do not accept H_{0N}
n. 3	[54.55, 268.79]	[18.43, 20.09]	[0.7847, 0.6489]	Do not accept H_{0N}
n. 4	[84.73, 521.31]	[20.75, 27.2]	[0.1533, 0.0810]	Do not accept H_{0N}
n. 5	[385.16, 1686.90]	[42.95, 47.35]	[0.1489, 0.0784]	Do not accept H_{0N}

Table 4: The comparison of two tests.

Water	The proposed test		The existing test
	$D_N \epsilon [D_L, D_U]$	Measure of indeterminacy $I_N \epsilon [I_L, L_U]$	D
n. 1	[0.1764, 0.1142]	0.1764–0.1142 I_N; [0, 0.0621]	0.1764
n. 2	[0.2104, 0.1107]	0.2104–0.1107 I_N; [0, 0.90]	0.2104
n. 3	[0.7847, 0.6489]	0.7847–0.6489 I_N; [0, 0.21]	0.7847
n. 4	[0.1533, 0.0810]	0.1533–0.0810 I_N; [0, 0.892]	0.1533
n. 5	[0.1489, 0.0784]	0.1489–0.0784 I_N; [0, 0.0.899]	0.1489

On the contrary, the existing test provides only the determined value which is not adequate when the data have interval, uncertain, and indeterminate values or the data are obtained from the complex system. From this comparison, it is concluded that the proposed test provides the values of statistic in the indeterminate interval, and this theory is the same as in [39]. Therefore, the use of the proposed test is adequate under an indeterminate environment.

6. Concluding Remarks

In this paper, we presented a D'Agostino test under neutrosophic statistics. We proposed the D'Agostino test to test the normality of the data having indeterminate observations. The design of the proposed test was given and implemented with the help of real data. The proposed test was the extension of an existing D'Agostino test under classical statistics. From the comparison, it was concluded that the proposed test is effective, adequate, and suitable to be applied in the presence of indeterminacy. The development of software for the proposed test will be a fruitful area of research. The application of the proposed test for big datasets such as testing the normality of ocean data, Facebook user data, and rail data can be considered as future research.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah (Grant no. D-103-130–1441). The authors, therefore, gratefully acknowledge DSR for the technical and financial support.

References

[1] R. D'Agostino and M. Stephens, *Tests for Normal Distribution in Goodness-Of-Fit Techniques, Statistics: Textbooks and Monographs*, R. B. D'Agostino and Stephens, Eds., Dekker, New York, NY, USA, 1986.

[2] D. Öztuna, A. H. Elhan, and E. Tüccar, "Investigation of four different normality tests in terms of type 1 error rate and power under different distributions," *Turkish Journal of Medical Sciences*, vol. 36, no. 3, pp. 171–176, 2006.

[3] B. W. Yap and C. H. Sim, "Comparisons of various types of normality tests," *Journal of Statistical Computation and Simulation*, vol. 81, no. 12, pp. 2141–2155, 2011.

[4] H. Chen and Y. Xia, "A nonparametric normality test for high-dimensional data," 2019, https://arxiv.org/pdf/1904.05289.pdf.

[5] P. Mishra, C. M. Pandey, U. Singh, A. Gupta, C. Sahu, and A. Keshri, "Descriptive statistics and normality tests for statistical data," *Annals of Cardiac Anaesthesia*, vol. 22, no. 1, p. 67, 2019.

[6] E. S. Pearson, R. B. D'"Agostino, and K. O. Bowman, "Tests for departure from normality: comparison of powers," *Biometrika*, vol. 64, no. 2, pp. 231–246, 1977.

[7] R. B. D'agostino, A. Belanger, and R. B. D'Agostino Jr, "A suggestion for using powerful and informative tests of normality," *The American Statistician*, vol. 44, no. 4, pp. 316–321, 1990.

[8] L. Baringhaus and N. Henze, "A test for uniformity with unknown limits based on d'agostino's D," *Statistics & Probability Letters*, vol. 9, no. 4, pp. 299–304, 1990.

[9] S. Kallithraka, I. S. Arvanitoyannis, P. Kefalas, A. El-Zajouli, E. Soufleros, and E. Psarra, "Instrumental and sensory analysis of Greek wines; implementation of principal component analysis (PCA) for classification according to geographical origin," *Food Chemistry*, vol. 73, no. 4, pp. 501–514, 2001.

[10] N. Watanabe and T. Imaizumi, "A fuzzy statistical test of fuzzy hypotheses," *Fuzzy Sets and Systems*, vol. 53, no. 2, pp. 167–178, 1993.

[11] G. Hesamian and M. G. Akbari, “Statistical test based on intuitionistic fuzzy hypotheses,” *Communications in Statistics-Theory and Methods*, vol. 46, no. 18, pp. 9324–9334, 2017.

[12] J. Chachi and S. M. Taheri, “Optimal statistical tests based on fuzzy random variables,” *Iranian Journal of Fuzzy Systems*, vol. 15, no. 5, pp. 27–45, 2018.

[13] E. Haktanır and C. Kahraman, “Fuzzy hypothesis testing in statistical decision making,” *Journal of Intelligent & Fuzzy Systems (Preprint)*, vol. 37, no. 5, pp. 6545–6555, 2019.

[14] B. Van Cutsem and I. Gath, “Detection of outliers and robust estimation using fuzzy clustering,” *Computational Statistics & Data Analysis*, vol. 15, no. 1, pp. 47–61, 1993.

[15] M. Montenegro, M. A. R. Casals, M. A. A. Lubiano, and M. A. A. Gil, “Two-sample hypothesis tests of means of a fuzzy random variable,” *Information Sciences*, vol. 133, no. 1-2, pp. 89–100, 2001.

[16] V. Mohanty and P. AnnanNaidu, “Fraud detection using outlier analysis: a survey,” *International Journal of Engineering Sciences and Research Technology*, vol. 2, no. 6, 2013.

[17] Y. M. Moradnezhadi, “Determination of a some simple methods for outlier detection in maximum daily rainfall (case study: baliglichay Watershed Basin–Ardebil Province–Iran),” *Bulletin of Environment, Pharmacology and Life Sciences*, vol. 3, no. 3, pp. 110–117, 2014.

[18] T.-Y. Ling, C.-L. Soo, J.-J. Liew, L. Nyanti, S.-F. Sim, and J. Grinang, “Application of multivariate statistical analysis in evaluation of surface river water quality of a tropical river,” *Journal of Chemistry*, vol. 2017, 2017.

[19] G. Shahzadi, M. Akram, and A. B. Saeid, “An application of single-valued neutrosophic sets in medical diagnosis,” *Neutrosophic Sets and Systems*, vol. 18, pp. 80–88, 2017.

[20] A. K. Shrestha and N. Basnet, “The Correlation and regression analysis of physicochemical parameters of River water for the evaluation of percentage contribution to electrical conductivity,” *Journal of Chemistry*, vol. 2018, 2018.

[21] Y. Choi, H. Lee, and Z. Irani, “Big data-driven fuzzy cognitive map for prioritising IT service procurement in the public sector,” *Annals of Operations Research*, vol. 270, no. 1-2, pp. 75–104, 2018.

[22] S. Habib and M. Akram, “Diagnostic methods and risk analysis based on fuzzy soft information,” *International Journal of Biomathematics*, vol. 11, no. 08, Article ID 1850096, 2018.

[23] S. Habib and M. Akram, “Medical decision support systems based on Fuzzy Cognitive Maps,” *International Journal of Biomathematics*, vol. 12, no. 06, Article ID 1950069, 2019.

[24] S. Habib, M. A. Butt, M. Akram, and F. Smarandache, “A neutrosophic clinical decision-making system for cardiovascular diseases risk analysis,” *Journal of Intelligent & Fuzzy Systems (Preprint)*, vol. 39, no. 5, pp. 7807–7829, 2020.

[25] F. Smarandache, *Neutrosophy. Neutrosophic Probability, Set, and Logic*, vol. 105, pp. 118–123, ProQuest Information & Learning, Ann Arbor, MI, USA, 1998.

[26] H. Wang, F. Smarandache, R. Sunderraman, and Y.-Q. Zhang, “Interval neutrosophic sets and logic: theory and applications in computing: theory and applications in computing,” *Infinite Study*, vol. 5, 2005.

[27] I. Hanafy, A. Salama, and K. Mahfouz, “Neutrosophic classical events and its probability,” *International Journal of Mathematics and Computer Applications Research (IJMCAR)*, vol. 3, no. 1, pp. 171–178, 2013.

[28] Y. Guo and A. Sengur, “NECM: neutrosophic evidential c-means clustering algorithm,” *Neural Computing and Applications*, vol. 26, no. 3, pp. 561–571, 2015.

[29] M. Abdel-Basset, G. Manogaran, A. Gamal, and F. Smarandache, “A hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria,” *Design Automation for Embedded Systems*, vol. 22, no. 3, pp. 257–278, 2018.

[30] R. Alhabib, M. M. Ranna, H. Farah, and A. Salama, “Some neutrosophic probability distributions,” *Neutrosophic Sets and Systems*, vol. 30, 2018.

[31] S. Broumi, A. Bakali, M. Talea, and F. Smarandache, “Bipolar neutrosophic minimum spanning tree,” *Infinite Study*, vol. 127, 2018.

[32] X. Peng and J. Dai, “Approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and new similarity measure with score function,” *Neural Computing and Applications*, vol. 29, no. 10, pp. 939–954, 2018.

[33] A. I. Shahin, K. M. Amin, A. A. Sharawi, and Y. Guo, “A novel enhancement technique for pathological microscopic image using neutrosophic similarity score scaling,” *Optik*, vol. 161, pp. 84–97, 2018.

[34] M. Abdel-Basset, M. Mohamed, M. Elhoseny, L. H. Son, F. Chiclana, and A. E.-N. H. Zaied, “Cosine similarity measures of bipolar neutrosophic set for diagnosis of bipolar disorder diseases,” *Artificial Intelligence in Medicine*, vol. 101, Article ID 101735, 2019.

[35] C. Jana and M. Pal, “A robust single-valued neutrosophic soft aggregation operators in multi-criteria decision making,” *Symmetry*, vol. 11, no. 1, Article ID 110, 2019.

[36] N. A. Nabeeh, M. Abdel-Basset, H. A. El-Ghareeb, and A. Aboelfetouh, “Neutrosophic multi-criteria decision making approach for iot-based enterprises,” *Institute of Electrical and Electronics Engineers Access*, vol. 7, pp. 59559–59574, 2019.

[37] F. Smarandache, “Introduction to neutrosophic statistics,” *Infinite Study*, https://arxiv.org/abs/1406.2000, 2014.

[38] W. V. Kandasamy and F. Smarandache, “Fuzzy cognitive maps and neutrosophic cognitive maps,” *Infinite Study*, https://arxiv.org/abs/math/0311063, 2003.

[39] J. Chen, J. Ye, and S. Du, “Scale effect and anisotropy analyzed for neutrosophic numbers of rock joint roughness coefficient based on neutrosophic statistics,” *Symmetry*, vol. 9, no. 10, p. 208, 2017.

[40] M. Aslam, “A new sampling plan using neutrosophic process loss consideration,” *Symmetry*, vol. 10, no. 5, p. 132, 2018.

[41] M. Aslam, “Introducing Kolmogorov–smirnov tests under uncertainty: an application to radioactive data,” *ACS Omega*, vol. 5, no. 1, pp. 914–917, 2019.

[42] M. Aslam, “Design of the Bartlett and Hartley tests for homogeneity of variances under indeterminacy environment,” *Journal of Taibah University for Science*, vol. 14, no. 1, pp. 6–10, 2020.

[43] M. Aslam, “Neutrosophic analysis of variance: application to university students,” *Complex & Intelligent Systems*, vol. 5, no. 4, pp. 403–407, 2019.

[44] M. Aslam and M. Albassam, “Application of neutrosophic logic to evaluate correlation between prostate cancer mortality and dietary fat assumption,” *Symmetry*, vol. 11, no. 3, p. 330, 2019.

[45] P. D'Urso and P. Giordani, “A least squares approach to principal component analysis for interval valued data,” *Chemometrics and Intelligent Laboratory Systems*, vol. 70, no. 2, pp. 179–192, 2004.

[46] J. Chen, J. Ye, S. Du, and R. Yong, “Expressions of rock joint roughness coefficient using neutrosophic interval statistical numbers,” *Symmetry*, vol. 9, no. 7, p. 123, 2017.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 5531093, 5 pages
https://doi.org/10.1155/2021/5531093

Research Article

On Refined Neutrosophic Matrices and Their Application in Refined Neutrosophic Algebraic Equations

Mohammad Abobala

Faculty of Science, Department of Mathematics, Tishreen University, Latakia, Syria

Correspondence should be addressed to Mohammad Abobala; mohammadabobala777@gmail.com

Received 7 January 2021; Accepted 1 February 2021; Published 13 February 2021

Academic Editor: Parimala Mani

The objective of this paper is to introduce the concept of refined neutrosophic matrices as matrices such as multiplication, addition, and ring property. Also, it determines the necessary and sufficient condition for the invertibility of these matrices with respect to multiplication. On the contrary, nilpotency and idempotency properties will be discussed.

1. Introduction

Neutrosophy is a new branch of generalized logic found by Smarandache to deal with indeterminacy in all fields of human knowledge. Neutrosophic sets were applicable in decision-making [1], number theory [2, 3], and space theory [4, 5].

The concept of refined neutrosophic structure was supposed firstly in [6] by splitting indeterminacy I into two levels of subindeterminacies I_1 and I_2. This idea was used in the study of refined neutrosophic rings [7–9], modules [10, 11], and groups [6]. Recently, the concept of n-refined neutrosophic structures was defined and used in [12–14].

Neutrosophic matrices were a useful tool to deal with indeterminacy in many studies; we find their basic definition and properties such as ring structure, multiplication, and other applications in [15, 16].

Through this work, we define, for the first time, the concept of refined neutrosophic matrices as a direct application of the refined neutrosophic set. Also, we determine the necessary and sufficient condition for the invertibility of these matrices with many related examples. On the contrary, we build an example to show how refined matrices can be used in refined neutrosophic equations defined in [17].

All refined neutrosophic matrices through this paper are defined over a neutrosophic field $F(I_1, I_2)$.

The structure of refined neutrosophic numbers is taken as $a + bI_1 + cI_2$ instead of (a, bI_1, cI_2). This representation is based on the theory of n-refined neutrosophic rings proposed in [12], where refined neutrosophic numbers can be represented by this form without any loss of generality or algebraic properties.

2. Preliminaries

Definition 1 (see [7]). Let K be a field, the neutrosophic field generated by $K \cup I$, which is denoted by $K(I) = K \cup I$.

Definition 2 (see [7]). Classical neutrosophic number has the form $a + bI$, where a and b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results in $I^n = I$ for all positive integers n.

Definition 3 (neutrosophic matrix; see [15]). Let $M_{m\times n} = \{(a_{ij}): a_{ij} \in K(I)\}$, where $K(I)$ is a neutrosophic field. We refer this to be the neutrosophic matrix.

Remark 1 (see [6]). The element I can be split into two indeterminacies I_1 and I_2 with conditions

$$\begin{aligned} I_1^2 &= I_1, \\ I_2^2 &= I_2, \\ I_1 I_2 &= I_2 I_1 = I_1. \end{aligned} \tag{1}$$

Definition 4 (see [1]). If X is a set, then $X(I_1, I_2) = \{(a, bI_1, cI_2)\colon a, b, c \in X\}$ is called the refined neutrosophic set generated by X, I_1, and I_2.

Definition 5 (see [7]). Let $(R, +, \times)$ be a ring; $(R(I_1, I_2), +, \times)$ is called a refined neutrosophic ring generated by R, I_1, and I_2.

Theorem 1 (see [7]). *Let $(R(I_1, I_2), +, \times)$ be a refined neutrosophic ring; then, it is a ring.*

It is called a neutrosophic field if R is a classical field.

Theorem 2 (see [17]). *Let $A_1X_1 + \cdots + A_nX_n = C$, $C = (c_0, c_1I_1, c_2I_2)$, $X_i = (x_0^{(i)}, x_1^{(i)}I_1, x_2^{(i)}I_2)$, and $A_i = (a_0^{(i)}, a_1^{(i)}I_1, a_2^{(i)}I_2)$, be a linear equation with n-variables over a refined neutrosophic field $F(I_1, I_2)$. Then, it is equivalent to the following system of classical linear equations over the classical field F:*

(a) $\sum_{i=1}^{n} a_0^{(i)} x_0^{(i)} = c_0$
(b) $\sum_{i=1}^{n} (a_0^{(i)} + a_2^{(i)})(x_0^{(i)} + x_2^{(i)}) = c_0 + c_2$
(c) $\sum_{i=1}^{n} (a_0^{(i)} + a_1^{(i)} + a_2^{(i)})(x_0^{(i)} + x_1^{(i)} + x_2^{(i)}) = c_0 + c_1 + c_2$

3. Main Concepts

Definition 6 (refined neutrosophic matrix)

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$ be an $n \times m$ matrix; if $a_{ij} = x + yI_1 + zI_2 \in R_2(I)$, then it is called an refined neutrosophic matrix, where $R_2(I)$ is an refined neutrosophic field.

Example 1. $X = \begin{pmatrix} I_1 & I_1 + I_2 \\ 3 - I_1 & 2I_2 \end{pmatrix}$ is a 2×2 refined neutrosophic matrix.

Remark 2 (addition and multiplication, ring structure)

(a) If A is an $m \times n$ matrix, then it can be represented as an element of the refined neutrosophic ring of matrices such as the following: $A = B + CI_1 + DI_2$, where D, B, and C are classical matrices with elements from ring R and from size $m \times n$.

For example, $A = \begin{pmatrix} 2 + I_1 + 3I_2 & 1 - I_1 - I_2 \\ 3 + 4I_2 & 1 + I_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} I_2.$

(b) The addition operation can be defined by using the representation in Remark 2 as follows:

$$(A + BI_1 + CI_2) + (X + YI_1 + ZI_2) = (A + X) + (B + Y)I_1 + (C + Z)I_2. \tag{2}$$

(c) Multiplication can be defined by using the same representation as a special case of multiplication on refined neutrosophic rings as follows:

$$(A + BI_1 + CI_2)(X + YI_1 + ZI_2) = (AX) + (AY + BX + BY + BZ + CY)I_1 + (AZ + CZ + CX)I_2. \tag{3}$$

This method of multiplication is exactly equivalent to the normal multiplication between matrices, but it is easier to deal with in this way.

Example 2. Let $X = \begin{pmatrix} I_1 & I_1 + I_2 \\ 3 - I_1 & 2I_2 \end{pmatrix}$ and $Y = \begin{pmatrix} -1 & I_1 \\ 1 + I_2 & 3I_1 \end{pmatrix}$ be two refined neutrosophic matrices over the refined neutrosophic field of reals. We have

(a) $X = A + BI_1 + CI_2$; $A = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$.

(b) $Y = M + NI_1 + SI_2$; $M = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$, and $S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

(c) $X + Y = \begin{pmatrix} -1 + I_1 & 2I_1 + I_2 \\ 4 - I_1 + I_2 & 3I_1 + 2I_2 \end{pmatrix}$.

(d) $XY = \begin{pmatrix} -I_1 + (I_1 + I_2)(1 + I_2) & I_1I_1 + (I_1 + I_2)(3I_1) \\ -3 + I_1 + (2I_2)(1 + I_2) & (3 - I_1)(I_1) + (2I_2)(3I_1) \end{pmatrix} = \begin{pmatrix} I_1 + 2I_2 & 7I_1 \\ -3 + I_1 + 4I_2 & 8I_1 \end{pmatrix}$.

(e) If we compute the multiplication using the representation of Remark 2, we get

$$AM = \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix},$$

$$AN = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix},$$

$$BM = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$BN = \begin{pmatrix} 0 & 4 \\ 0 & -1 \end{pmatrix},$$

$$BS = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (4)$$

$$CN = \begin{pmatrix} 0 & 3 \\ 0 & 6 \end{pmatrix},$$

$$AS = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$CS = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix},$$

$$CM = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}.$$

Hence, $XY = AM + I_1(AN + BN + BM + BS + CN) + I_2(AS + CS + CM) = \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix} + I_1\begin{pmatrix} 1 & 7 \\ 1 & 8 \end{pmatrix} + I_2\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} I_1 + 2I_2 & 7I_1 \\ -3 + I_1 + 4I_2 & 8I_1 \end{pmatrix}$.

Theorem 3. *The set of all square $n \times n$ refined neutrosophic matrices together makes a ring.*

Proof. The proof holds directly from the definition of n-refined neutrosophic rings by taking $n = 2$. □

Remark 3. The identity with respect to multiplication is the normal unitary matrix.

Definition 7. Let A be a square $n \times n$ refined neutrosophic matrix; then, it is called invertible if there exists a refined square $n \times n$ neutrosophic matrix B such that $AB = U_{n\times n}$, where $U_{n\times n}$ is the unitary classical matrix.

Theorem 5. *Let $X = A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix; then, it is invertible if and only if A, $A + C$, and $A + B + C$ are invertible. The inverse of X is $X^{-1} = A^{-1} + ((A + B + C)^{-1} - (A + C))^{-1}I_1 + ((A + C)^{-1} - A^{-1})I_2$.*

Proof. The proof holds as a special case of invertible elements in refined neutrosophic rings [8]. □

Definition 8. We define the determinant of a square $n \times n$ refined neutrosophic matrix as $\det X = \det A + [\det(A + B + C) - \det(A + C)]I_1 + [\det(A + C) - \det A]I_2$.

This definition is supported by the condition of invertibility.

Theorem 6. *Let $X = A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix; we have the following:*

(a) X is invertible if and only if $\det X \neq 0$

(b) If $Y = M + NI_1 + SI_2$ is a square $n \times n$ refined neutrosophic matrix, then $\det XY = \det X \det Y$

(c) $\det X^{-1} = (\det X)^{-1}$

Proof

(a) If $\det X \neq 0$, this will be equivalent to $\det A \neq 0$, $\det(A + C) \neq 0$, and $\det(A + B + C) \neq 0$, i.e., A, $A + C$, and $A + B + C$ are invertible; thus, X is invertible according to Theorem 5.

(b) $XY = AM + I_1[(A + B + C)(M + N + S) - (A + C)(M + S)] + I\ \ 2[(A + C)(M + S) - AM]$. Hence, $\det XY = \det(AM) + I_1[\det((A + B + C)\ \ (M + N + S))]\ \ + I_2[\det((A + C)(M + S))]\ \ = \det A \det M + I_1[\det(A + B + C)\det(M + N + S)] + I\ \ 2[\det(A + C)\det(M + S)] = (\det A + I_1[\det(A + B + C) - \det(A + C)] + I_2[\det(A + C) - \det A])\ \ (\det M + I_1[\det(M + N + S) - \det(M + S)] + I_2[\det(M + S) - \det M]) = \det X \det Y$.

(c) It holds directly from (b). □

Theorem 7. *Let $X = A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix; we have the following:*

(a) X is nilpotent if and only if A, $A + C$, and $A + B + C$ are nilpotent

(b) X is idempotent if and only if A, $A + C$, and $A + B + C$ are idempotent

Proof

(a) First of all, we will prove that $X^r = A^r + I_1[(A + B + C)^r - (A + C)^r] + I_2[(A + C)^r - A^r]$.

We use the induction, for $r = 1$, it is clear. Suppose that it is true for $r = k$, we prove it for $k + 1$.

$$\begin{aligned} X^{k+1} &= X^k X = \left(A^k + I_1\left[(A+B+C)^k - (A+C)^k\right] + I_2\left[(A+C)^k - A^k\right]\right)(A + BI_1 + CI_2) \\ &= A^{k+1} + I_1\left[A^k B + (A+B+C)^k A + (A+B+C)^k B + (A+B+C)^k C - (A+C)^k A - (A+C)^k B \right.\\ &\quad \left. - (A+C)^k C + (A+C)^k B - A^k B\right] + I_2\left[A^k C + (A+C)^k C - A^k C + (A+C)^k A - A^k A\right] \\ &= A^{k+1} + I_1\left[(A+B+C)^{k+1} - (A+C)^{k+1}\right] + I_2\left[(A+C)^{k+1} - A^{k+1}\right]. \end{aligned} \tag{5}$$

X is nilpotent if there is a positive integer r such that $X^r = O_{n\times n}$. This is equivalent to

$$A^r = (A+C)^r = (A+B+C)^r = O_{n\times n}, \tag{6}$$

which implies the proof.

(b) The proof is similar to (a). □

Example 4. Consider the following refined neutrosophic matrix $A = \begin{pmatrix} 2+I_1+3I_2 & 1-I_1-I_2 \\ 3+4I_2 & 1+I_1 \end{pmatrix}$; we have the following:

(a) $A = \begin{pmatrix} 2+I_1+3I_2 & 1-I_1-I_2 \\ 3+4I_2 & 1+I_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} I_2$.

$B = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$.

$B + D = \begin{pmatrix} 5 & 0 \\ 7 & 1 \end{pmatrix}$, and $B + C + D = \begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix}$.

(b) $B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$, $(B+D)^{-1} = \begin{pmatrix} 1/5 & 0 \\ -7/5 & 1 \end{pmatrix}$, and $(B+C+D)^{-1} = \begin{pmatrix} 2/19 & 1/19 \\ -7/19 & 6/19 \end{pmatrix}$.

(c) $A^{-1} = B^{-1} + I_1[(B+C+D)^{-1} - (B+D)^{-1}] + I_2[(B+D)^{-1} - B^{-1}] = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} + I_1\begin{pmatrix} -9/95 & 1/19 \\ 98/95 & -13/19 \end{pmatrix} + I_2\begin{pmatrix} 6/5 & -1 \\ -22/5 & 3 \end{pmatrix} = \begin{pmatrix} -1-(9/95)I_1+(6/5)I_2 & 1+(1/19)I_1 - I_2 \\ 3+(98/95)I_1-(22/5)I_2 & -2-(13/19)I_1+3I_2 \end{pmatrix}$.

It is easy to find that $A^{-1}A = AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) $\det B = -1$, $\det(B+D) = 5$, $\det(B+C+D) = 19$, $\det A = -1 + I_1[19-5] + I_2[5-(-1)] = -1 + 14I_1 + 6I_2$.

If we compute the determinant of A by using the classical way, we will get the same result.

Now, we illustrate an example to clarify the application of refined neutrosophic matrices in solving refined neutrosophic algebraic equations defined in [17].

Example 5. Consider the following system of refined neutrosophic linear equations:

$$\begin{aligned} (2+I_1+3I_2)X + (1-I_1-I_2)Y &= -I_1(*), \\ (3+4I_2)X + (1+I_1)Y &= I_2(**). \end{aligned} \tag{7}$$

The corresponding refined neutrosophic matrix is $A = \begin{pmatrix} 2+I_1+3I_2 & 1-I_1-I_2 \\ 3+4I_2 & 1+I_1 \end{pmatrix}$.

Since A is invertible, we get the solution of the previous system by computing the product:

$$\begin{aligned} A^{-1}\begin{pmatrix} -I_1 \\ I_2 \end{pmatrix} &= \begin{pmatrix} -1-\frac{9}{95}I_1+\frac{6}{5}I_2 & 1+\frac{1}{19}I_1 - I_2 \\ 3+\frac{98}{95}I_1-\frac{22}{5}I_2 & -2-\frac{13}{19}I_1+3I_2 \end{pmatrix}\begin{pmatrix} -I_1 \\ I_2 \end{pmatrix} \\ &= \begin{pmatrix} I_1\left[1+\frac{9}{95}-\frac{6}{5}+\frac{1}{19}\right] \\ I_1\left[-3-\frac{98}{95}+\frac{22}{5}-\frac{13}{19}\right] + I_2[-2+3] \end{pmatrix} \\ &= \begin{pmatrix} -I_1(1/19) \\ -(6/19)I_1 + I_2 \end{pmatrix}. \text{ Thus, } X = -\frac{1}{19}I_1, \\ Y &= -\frac{6}{19}I_1 + I_2. \end{aligned} \tag{8}$$

4. Conclusion

In this paper, we have used the concept of refined neutrosophic set to build the corresponding refined neutrosophic matrix. On the contrary, many interesting properties have been studied and proved such as idempotency, nilpotency, determinants, and invertibility of these matrices.

Also, a direct application of these matrices was proposed in solving refined neutrosophic equations.

As a future research direction, we aim to study the diagonalization properties with eigenvectors of these matrices.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

[1] M. Abdel-Basset, A. Gamal, L. H. Son, and F. Smarandache, "A bipolar neutrosophic multi criteria decision making framework for professional selection," *Applied Sciences*, vol. 10, no. 4, p. 1202, 2020.

[2] H. Sankari and M. Abobala, "Neutrosophic linear diophantine equations with two variables," *Neutrosophic Sets and Systems*, vol. 38, 2020.

[3] M. Abobala, "Foundations of neutrosophic number theory," *Neutrosophic Sets and Systems*, vol. 39, 2021.

[4] M. Abobala, "A study of AH-substructures in n-refined neutrosophic vector spaces," *International Journal of Neutrosophic Science*, vol. 9, pp. 74–85, 2020.

[5] M. Abobala and R. Alhamido, "AH-substructures in neutrosophic modules," *International Journal of Neutrosophic Science*, vol. 7, pp. 79–86, 2020.

[6] A. A. A. Agboola, "On refined neutrosophic algebraic structures," *Neutrosophic Sets and Systems*, vol. 10, pp. 99–101, 2015.

[7] E. O. Adeleke, A. A. A. Agboola, and F. Smarandache, "Refined neutrosophic rings II," *International Journal of Neutrosophic Science*, vol. 2, pp. 89–94, 2020.

[8] M. Abobala, "On some special elements in neutrosophic rings and refined neutrosophic rings," *Journal of New Theory*, vol. 33, 2020.

[9] H. Sankari and M. Abobala, "AH-homomorphisms in neutrosophic rings and refined neutrosophic rings," *Neutrosophic Sets and Systems*, vol. 38, 2020.

[10] M. Abobala and A. Hatip, "AH-substructures in strong refined neutrosophic modules," *International Journal of Neutrosophic Science*, vol. 9, pp. 110–116, 2020.

[11] M. Abobala, "Semi homomorphisms and algebraic relations between strong refined neutrosophic modules and strong neutrosophic modules," *Neutrosophic Sets and Systems*, vol. 39, 2021.

[12] F. Smarandache and M. Abobala, "n-Refined neutrosophic rings," *International Journal of Neutrosophic Science*, vol. 6, pp. 83–90, 2020.

[13] F. Smarandache and M. Abobala, "n-Refined neutrosophic vector spaces," *International Journal of Neutrosophic Science*, vol. 7, pp. 47–54, 2020.

[14] H. Sankari and M. Abobala, "n-Refined neutrosophic modules," *Neutrosophic Sets and Systems*, vol. 36, 2020.

[15] M. Dhar, S. Broumi, and F. Smarandache, "A note on square neutrosophic fuzzy matrices," *Neutrosophic Sets and Systems*, vol. 3, pp. 37–41, 2014.

[16] H. Khaled, A. Younus, and A. Mohammad, "The rectangle neutrosophic fuzzy matrices," *Faculty of Education Journal*, vol. 15, 2019 (Arabic version).

[17] M. Abobala, "On some neutrosophic algebraic equations," *Journal of New Theory*, vol. 33, 2020.