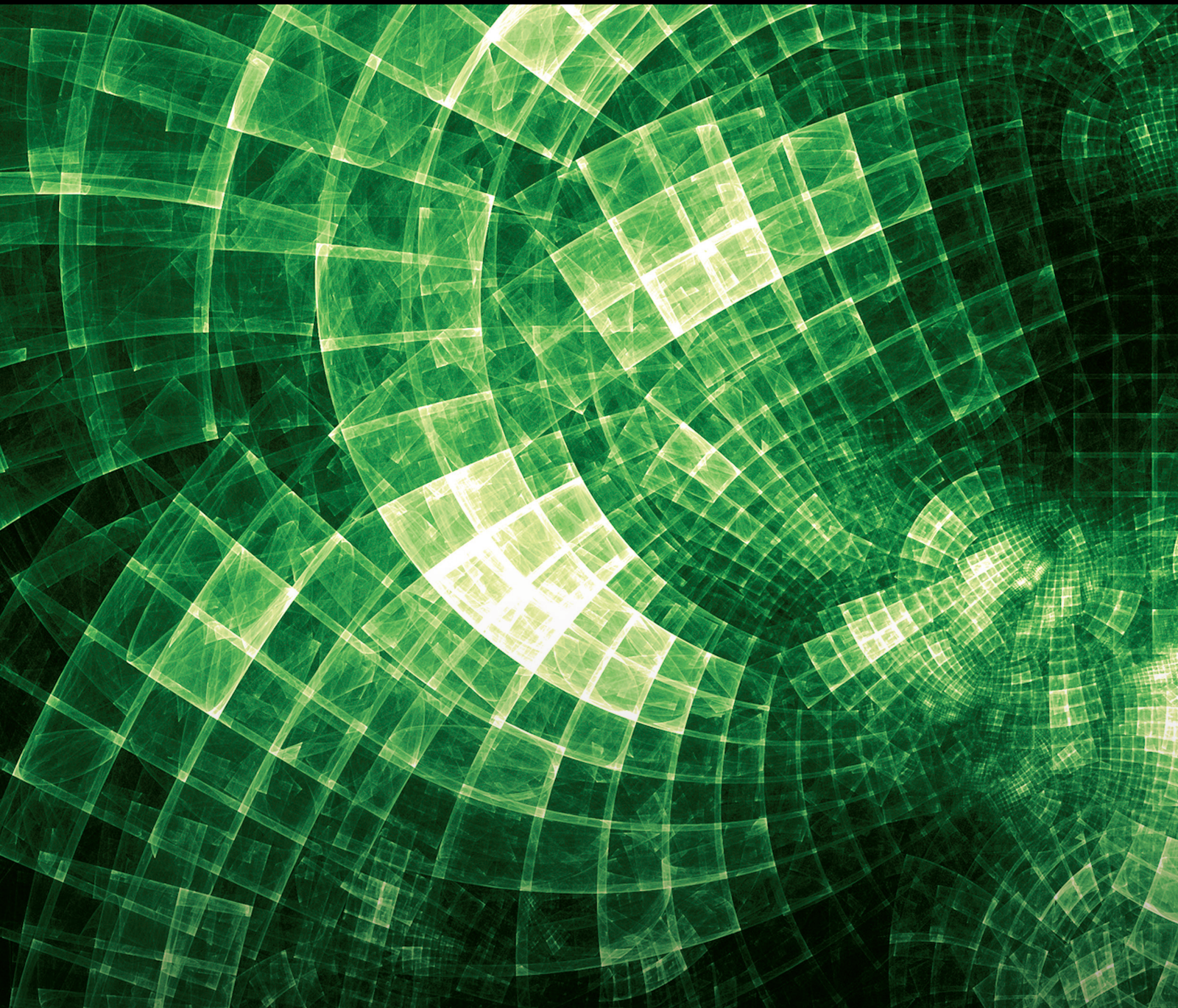


Advances in Geometric Function Theory

Lead Guest Editor: V. Ravichandran

Guest Editors: Om P. Ahuja and Adam Lecko





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Journal of Mathematics

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Research Article

New Class of Close-to-Convex Harmonic Functions Defined by a Fourth-Order Differential Inequality

Mohammad Faisal Khan ¹, Khaled Matarneh,² Shahid Khan ³, Saqib Hussain ⁴,
and Maslina Darus ⁵

¹Department of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia

²Faculty of Computer Science, Arab Open University, Riyadh, Saudi Arabia

³Department of Mathematics, Riphah International University, Islamabad 44000, Pakistan

⁴Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad 22060, Pakistan

⁵Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Malaysia

Correspondence should be addressed to Shahid Khan; shahidmath761@gmail.com

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In the recent past, various new subclasses of normalized harmonic functions have been defined in open unit disk U which satisfy second-order and third-order differential inequalities. Here, in this study, we define a new class of normalized harmonic functions in open unit disk U which is satisfying a fourth-order differential inequality. We investigate some useful results such as close-to-convexity, coefficient bounds, growth estimates, sufficient coefficient condition, and convolution for the functions belonging to this new class of harmonic functions. In addition, under convex combination and convolution of its members, we prove that this new class is closed, and we also give some lemmas to prove our main results.

1. Introduction and Definitions

Let \mathcal{H} represent the class of all harmonic functions $f = s + \bar{v}$ in open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, and all these harmonic functions are normalized by

$$f(0) = f_z(0) - 1 = 0, \quad (1)$$

and f can be expressed as $f = s + \bar{v}$, where s is the analytic part and v is the coanalytic part of f in U , and also, these functions have a series of the form

$$\begin{aligned} s(z) &= z + \sum_{j=2}^{\infty} a_j z^j, \\ v(z) &= \sum_{j=2}^{\infty} b_j z^j. \end{aligned} \quad (2)$$

Harmonic functions f is locally univalent and sense-preserving in U , if it satisfies a necessary sufficient condition $|s'(z)| > |v'(z)|$ [1, 2]. If coanalytic part of f is zero, then class \mathcal{H} of complex valued harmonic functions reduces to class \mathcal{A} of normalized analytic functions.

Let \mathcal{S} denote the family of analytic univalent and normalized functions in U and also $\mathcal{S} \subset \mathcal{S}_{\mathcal{H}}$ which are defined as

$$\mathcal{S} = \{f \in \mathcal{S}_{\mathcal{H}} : v = 0 \text{ in } U\}. \quad (3)$$

Also, let class \mathcal{H}^0 define as

$$\mathcal{H}^0 = \{f \in \mathcal{H} : f_{\bar{z}}(0) = 0\}, \quad (4)$$

$$\mathcal{S}_{\mathcal{H}}^0 = \{f \in \mathcal{S}_{\mathcal{H}} : f_{\bar{z}}(0) = 0\},$$

where class $\mathcal{S}_{\mathcal{H}}$ represent the class of functions f which are harmonic, univalent, and sense-preserving in open unit disk U .

We can see that class $\mathcal{S}_{\mathcal{H}}^0$ is compact and normal, but class $\mathcal{S}_{\mathcal{H}}$ is only normal. Let $\mathcal{S}_{\mathcal{H}}^{0,*}$, $\mathcal{K}_{\mathcal{H}}^0$, and $\mathcal{C}_{\mathcal{H}}^0$ are the subclasses of $\mathcal{S}_{\mathcal{H}}^0$ which map open unit disk U onto starlike, convex, and close-to-convex domains for harmonic functions, respectively. We can observe that

$$\begin{aligned}\mathcal{S}^* &\subset \mathcal{S}_{\mathcal{H}}^{0,*}, \\ \mathcal{K} &\subset \mathcal{K}_{\mathcal{H}}^0, \\ \mathcal{C} &\subset \mathcal{C}_{\mathcal{H}}^0.\end{aligned}\quad (5)$$

These subclasses \mathcal{S}^* , \mathcal{K} , and \mathcal{C} map open unit disk U onto their respective domains.

For [2], Ponnusamy et al. defined the class of harmonic functions $f \in \mathcal{H}^0$ which satisfy the condition

$$\operatorname{Re}(f_z(z)) > |\bar{f}_{\bar{z}}(z)|, \quad \text{for } z \in U. \quad (6)$$

In this class, they studied about close-to-convexity of harmonic functions. After that, Li and Ponnusamy [3, 4] discussed univalence and convexity of the abovementioned class. A class $\mathcal{W}_{\mathcal{H}}^0$ of harmonic functions $f = s + \bar{v} \in \mathcal{H}^0$ defined by Nagpal and Ravichandran in [5] and the functions in this class satisfy the condition

$$\operatorname{Re}(s'(z) + zs''(z)) > |\bar{v}'(z) + z\bar{v}''(z)|, \quad \text{for } z \in U. \quad (7)$$

Note that

$$\mathcal{W}_{\mathcal{H}}^0 \subset \mathcal{S}_{\mathcal{H}}^{0,*}, \quad (8)$$

and members of $\mathcal{W}_{\mathcal{H}}^0$ are fully starlike in U .

Recently, Ghosh and Vasudevarao [6] for $\alpha \geq 0$ defined a new class $\mathcal{W}_{\mathcal{H}}^0(\alpha)$ for harmonic functions $f = s + \bar{v} \in \mathcal{H}^0$ satisfying the condition

$$\operatorname{Re}(s'(z) + \alpha zs''(z)) > |\bar{v}'(z) + \alpha z\bar{v}''(z)|, \quad \text{for } z \in U. \quad (9)$$

Rajbala and Prajapat [7] for $\delta \geq 0$, $0 \leq \lambda < 1$, defined a new class $\mathcal{W}_{\mathcal{H}}^0(\delta, \lambda)$ of harmonic functions which satisfy the following inequality:

$$\operatorname{Re}(s'(z) + \delta zs''(z) - \lambda) > |s'(z) + \delta zv''(z)|. \quad (10)$$

For this class, authors used Gaussian hypergeometric function and created harmonic polynomials for the class $\mathcal{W}_{\mathcal{H}}^0(\delta, \lambda)$.

Very recently, for the functions $f = s + \bar{v} \in \mathcal{H}^0$, Yaşar and Yalçın [8] introduced the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ which satisfy the following inequality:

$$\begin{aligned}\operatorname{Re}(s'(z) + \lambda zs''(z) + \delta z^2 s'''(z)) \\ > |\bar{v}'(z) + \lambda z\bar{v}''(z) + \delta z^2 \bar{v}'''(z)|,\end{aligned}\quad (11)$$

for $\lambda \geq \delta \geq 0$. For further study about harmonic functions, refer [2, 5, 9–11].

By taking the inspiration from the abovementioned work, we define new class of harmonic functions in U which satisfy the fourth-order differential inequality.

Definition 1. For $\lambda \geq \delta \geq \gamma \geq 0$, let $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$ denote the class of functions $f = s + \bar{v} \in \mathcal{H}^0$ and satisfy the condition

$$\begin{aligned}\operatorname{Re}(s'(z) + \lambda zs''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z)) \\ > |\bar{v}'(z) + \lambda z\bar{v}''(z) + \delta z^2 \bar{v}'''(z) + \gamma z^3 \bar{v}''''(z)|.\end{aligned}\quad (12)$$

Definition 2. For $\lambda \geq \delta \geq \gamma \geq 0$, let $\mathcal{R}(\lambda, \delta, \gamma)$ denote a class of functions $f \in \mathcal{A}$ if it satisfies the inequality

$$\operatorname{Re}(f'(z) + \lambda z f''(z) + \delta z^2 f'''(z) + \gamma z^3 f''''(z)) > 0. \quad (13)$$

Note that

$$\mathcal{R}(\lambda, \delta, \gamma) \subset \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma). \quad (14)$$

Special cases are

- (1) $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma) = \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$, defined by Yaşar and Yalçın [8].
- (2) $\mathcal{R}_{\mathcal{H}}^0(1, 0, 0) = \mathcal{W}_{\mathcal{H}}^0$, discussed by Nagpal and Ravichandran [5].
- (3) $\mathcal{R}_{\mathcal{H}}^0(\alpha, 0, 0) = \mathcal{W}_{\mathcal{H}}^0(\alpha)$, defined by Ghosh and Vasudevarao [6].
- (4) $\mathcal{R}_{\mathcal{H}}^0(\delta, 1 - 2\lambda, 0) = \mathcal{W}_{\mathcal{H}}^0(\delta, \lambda)$, defined by Rajbala and Prajapat [7].

In this section, we prove that all the members of the class are close-to-convex. We will derive coefficient bounds, growth estimates, and sufficient coefficient condition for the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$. Furthermore, we investigate example of harmonic polynomial belonging to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

2. Main Results

Lemma 1 (see [1]). *Let s and v are analytic functions in open unit disk U along with $|s'(0)| < |v'(0)|$ and $\mathcal{L}_\mu = s + \mu v$ is close-to-convex for each $\mu \in \mathbb{C}$ ($|\mu| = 1$). Then,*

$$f = s + \bar{v} \text{ is close to convex in } U. \quad (15)$$

Theorem 1. *Let $f = s + \bar{v} \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$ if and only if $\mathcal{L}_\mu = s + \mu v \in \mathcal{R}(\lambda, \delta, \gamma)$ for each $\mu \in \mathbb{C}$ ($|\mu| = 1$).*

Proof. Suppose $f = s + \bar{v} \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$. For each $|\mu| = 1$,

$$\begin{aligned} & \operatorname{Re} \left\{ \mathcal{L}'_\mu(z) + \lambda z \mathcal{L}''_\mu(z) + \delta z^2 \mathcal{L}'''_\mu(z) + \gamma z^3 \mathcal{L}''''_\mu(z) \right\} \\ &= \operatorname{Re} \left\{ s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z) \right. \\ & \quad \left. + \mu \left(v'(z) + \lambda z v''(z) + \delta z^2 v'''(z) + \gamma z^3 v''''(z) \right) \right\} \\ &> \operatorname{Re} \left\{ s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z) \right\} \\ & \quad - |v'(z) + \lambda z v''(z) + \delta z^2 v'''(z) + \gamma z^3 v''''(z)| > 0, \quad z \in U. \end{aligned} \quad (16)$$

Thus, $\mathcal{L}_\mu \in \mathcal{R}(\lambda, \delta, \gamma)$ for each μ ($|\mu| = 1$). Conversely, let $\mathcal{L}_\mu = s + \mu v \in \mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$, then

$$\begin{aligned} & \operatorname{Re} \left\{ s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z) \right\} \\ &> \operatorname{Re} \left(-\mu \left(v'(z) + \lambda z v''(z) + \delta z^2 v'''(z) + \gamma z^3 v''''(z) \right) \right), \quad z \in U. \end{aligned} \quad (17)$$

By choosing μ ($|\mu| = 1$), we get

$$\begin{aligned} & \operatorname{Re} \left\{ s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z) \right\} \\ &> |v'(z) + \lambda z v''(z) + \delta z^2 v'''(z) + \gamma z^3 v''''(z)|, \quad z \in U, \end{aligned} \quad (18)$$

Hence, $f \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

Lemma 2 (see [12]). *Let $z_0 \neq 0$, $z_0 \in U$, with $r_0 = |z_0|$. Let $\varphi(z)$ defined by*

$$\varphi(z) = C_j z^j + C_{j+1} z^{j+1} + \dots, \quad (19)$$

be analytic in U , such that

$$|\varphi(z_0)| = \max_{|z| \leq |z_0|} |\varphi(z)|. \quad (20)$$

Then, for $n \in \mathbb{R}$, $n \geq j \geq 1$, such that

$$\begin{aligned} & \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = n, \\ & \operatorname{Re} \left(\frac{z_0''(z_0)}{\varphi'(z_0)} \right) \geq n - 1, \end{aligned} \quad (21)$$

$$\operatorname{Re} \left(1 + 3 \frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} + \frac{z_0^2 \varphi'''(z_0)}{\varphi'(z_0)} \right) \geq n^2,$$

or

$$\operatorname{Re} \left(\frac{z_0^2 \varphi'''(z_0)}{\varphi'(z_0)} \right) \geq n^2 - 3n + 2. \quad (22)$$

Then,

$$\operatorname{Re} \left(1 + 7 \frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} + 6 \left(\frac{z_0^2 \varphi'''(z_0)}{\varphi'(z_0)} \right) + \left(\frac{z_0^3 \varphi''''(z_0)}{\varphi'(z_0)} \right) \right) \geq n^3. \quad (23)$$

Lemma 3. *If $\mathcal{L} \in \mathcal{R}(\lambda, \delta, \gamma)$ with $\lambda \geq \delta \geq \gamma \geq 0$, then*

$$\operatorname{Re}(\mathcal{L}'(z)) > 0, \quad (24)$$

and hence,

$$\mathcal{L}(z) \text{ is close - to - convex in } U. \quad (25)$$

Proof. Let $\mathcal{L} \in \mathcal{R}(\lambda, \delta, \gamma)$, and we have

$$\mathcal{L}'(z) + \lambda z \mathcal{L}''(z) + \delta z^2 \mathcal{L}'''(z) + \gamma z^3 \mathcal{L}''''(z) = \vartheta(z). \quad (26)$$

Then,

$$\operatorname{Re}(\vartheta(z)) > 0, \quad \text{for } z \in U. \quad (27)$$

Let φ be an analytic function in U with

$$\varphi(0) = 0,$$

$$\mathcal{L}'(z) = \frac{1 + \varphi(z)}{1 - \varphi(z)}, \quad (28)$$

$$\varphi(z) \neq 1.$$

We have to show that $|\varphi(z)| < 1$, $\forall z \in U$. Then,

$$\vartheta(z) = \mathcal{L}'(z) + \lambda z \mathcal{L}''(z) + \delta z^2 \mathcal{L}'''(z) + \gamma z^3 \mathcal{L}''''(z),$$

$$\begin{aligned} \vartheta(z) = & \left\{ \left(\frac{1 + \varphi(z)}{1 - \varphi(z)} \right) + 2\lambda \left(\frac{z\varphi'(z)}{(1 - \varphi(z))^2} \right) \right. \\ & + 2\delta \frac{z\varphi'(z)}{(1 - \varphi(z))^2} \left(\frac{z\varphi''(z)}{\varphi'(z)} \right) \\ & + 4\delta \left(\frac{(z\varphi'(z))^2}{(1 - \varphi(z))^3} \right) + \frac{2\gamma z\varphi'(z)}{(1 - \varphi(z))^2} \left(\frac{z^2\varphi'''(z)}{\varphi'(z)} \right) \\ & \left. + \frac{12\gamma z\varphi'(z)}{(1 - \varphi(z))^3} \left(\frac{z\varphi''(z)}{\varphi'(z)} \right) + \frac{12\gamma(z\varphi'(z))^3}{(1 - \varphi(z))^4} \right\}. \end{aligned} \quad (29)$$

Since φ is analytic in U ,

$$\varphi(0) = 0, \quad \text{if } \exists z_0 \in U, \quad (30)$$

such that

$$\max_{|z| \leq |z_0|} |\varphi(z)| = |\varphi(z_0)| = 1. \quad (31)$$

Then, by using Lemma 2, we may write

$$\varphi(z_0) = e^{i\theta},$$

$$z_0 \varphi'(z_0) = n\varphi'(z_0) = ne^{i\theta}, \quad n \geq 1, \quad 0 < \theta < 2\pi,$$

$$\operatorname{Re} \left(\frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} \right) \geq n - 1, \quad (32)$$

$$\operatorname{Re} \left(\frac{z_0^2 \varphi'''(z_0)}{\varphi'(z_0)} \right) \geq n^2 - 3n + 2.$$

For all $z_0 \in U$, we get

$$\begin{aligned} \operatorname{Re} \vartheta(z_0) &= \operatorname{Re} \left(\left(\frac{1 + \varphi(z)}{1 - \varphi(z)} \right) + 2\lambda \left(\frac{z\varphi'(z)}{(1 - \varphi(z))^2} \right) + 2\delta \frac{z\varphi'(z)}{(1 - \varphi(z))^2} \left(\frac{z\varphi''(z)}{\varphi'(z)} \right) + 4\delta \frac{(z\varphi'(z))^2}{(1 - \varphi(z))^3} + \frac{2\gamma z\varphi'(z)}{(1 - \varphi(z))^2} \left(\frac{z^2\varphi'''(z)}{\varphi'(z)} \right) \right. \\ &\quad \left. + \frac{12\gamma z\varphi'(z)}{(1 - \varphi(z))^3} \left(\frac{z\varphi''(z)}{\varphi'(z)} \right) + \frac{12\gamma(z\varphi'(z))^3}{(1 - \varphi(z))^4} \right) \\ &= \operatorname{Re} \left\{ \frac{1 + e^{i\theta}}{1 - e^{i\theta}} + \frac{2\lambda ne^{i\theta}}{(1 - e^{i\theta})^2} + \frac{2\delta ne^{i\theta}}{(1 - e^{i\theta})^2} \left(\frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} \right) + \frac{4\delta (ne^{i\theta})^2}{(1 - e^{i\theta})^3} + \frac{2\gamma ne^{i\theta}}{(1 - e^{i\theta})^2} \left(\frac{z_0^2 \varphi'''(z_0)}{\varphi'(z_0)} \right) \right. \\ &\quad \left. + \frac{12\gamma ne^{i\theta}}{(1 - e^{i\theta})^3} \left(\frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} \right) + \frac{12\gamma (ne^{i\theta})^3}{(1 - e^{i\theta})^4} \right\}, \quad (33) \\ &= \left(\frac{-\lambda n}{1 - \cos \theta} - \frac{\delta n}{(1 - \cos \theta)} \operatorname{Re} \left(\frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} \right) + \frac{\delta n^2}{(1 - \cos \theta)} - \frac{\gamma n}{(1 - \cos \theta)} \operatorname{Re} \left(\frac{z_0 \varphi'''(z_0)}{\varphi'(z_0)} \right) \right. \\ &\quad \left. - \frac{3\gamma n}{(1 - \cos \theta)} \operatorname{Re} \left(\frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} \right) - \frac{3\gamma n^3}{2(1 - \cos \theta)} \right) \\ &\leq \frac{-\lambda n}{1 - \cos \theta} + \frac{\delta n}{(1 - \cos \theta)} (n - 1) + \frac{\delta n^2}{(1 - \cos \theta)} - \frac{\gamma n}{(1 - \cos \theta)} (n^2 - 3n + 2) - \frac{3\gamma n}{(1 - \cos \theta)} (n - 1) - \frac{3\gamma n^3}{2(1 - \cos \theta)}, \\ \operatorname{Re}(\vartheta(z_0)) &\leq \frac{-\lambda n}{(1 - \cos \theta)} + \frac{\delta n}{(1 - \cos \theta)} - \frac{\gamma n(5n^2 - 1)}{2(1 - \cos \theta)} \leq \frac{(\delta - \lambda)n}{(1 - \cos \theta)} - \frac{\gamma n(5n^2 - 1)}{2(1 - \cos \theta)} \leq 0, \end{aligned}$$

which opposes the hypothesis. Hence, there is no $z_0 \in U$, such that

$$|\varphi(z_0)| = 1. \quad (34)$$

Hence, $|\varphi(z)| < 1$ for all $z \in U$. So, we get

$$\operatorname{Re}(\mathcal{L}'(z)) > 0. \quad (35)$$

□

Theorem 2. A function $f = s + \bar{v} \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$ is close-to-convex in U .

Proof. According to Lemma 3, we derive that $\mathcal{L}_\mu = s + \mu v \in \mathcal{R}(\lambda, \delta, \gamma)$ are close-to-convex in U , $\forall \mu (|\mu| = 1)$. Therefore, in the light of Theorem 1 and Lemma 1, we get $\mathcal{L}_\mu \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$ are close-to-convex in U . □

Theorem 3. Let $f = s + \bar{v} \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$. Then,

$$|b_j| \leq \frac{1}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]}, \quad \text{for } j \geq 2. \quad (36)$$

The equality is satisfied for

$$f(z) = z + \frac{1}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]} \bar{z}^j. \quad (37)$$

Proof. Let $f = s + \bar{v} \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$. Applying the series of $v(z)$, we get

$$\begin{aligned} & r^{j-1} [j + j\lambda(j-1) + \delta j(j-1)(j-2) + j\gamma(j-1)(j-2)(j-3)] |b_j| \\ & \leq \frac{1}{2\pi} \int_0^{2\pi} |v'(re^{i\theta}) + \lambda re^{i\theta} v''(re^{i\theta}) + \delta r^2 e^{2i\theta} v'''(re^{i\theta}) + \gamma r^3 e^{3i\theta} v''''(re^{i\theta})| d\theta \\ & < \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\{s'(re^{i\theta}) + \lambda re^{i\theta} s''(re^{i\theta}) + \delta r^2 e^{2i\theta} s'''(re^{i\theta}) + \gamma r^3 e^{3i\theta} s''''(re^{i\theta})\} d\theta \\ & = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left\{1 + \sum_{j=2}^{\infty} j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}] \times a_j r^{j-1} e^{i(j-1)\theta}\right\} d\theta = 1. \end{aligned} \quad (38)$$

Taking $r \rightarrow 1-$, we get required result.

Theorem 4. Let for $j \geq 2$, $f = s + \bar{v} \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$. Then,

$$\begin{aligned} (1) \quad & |a_j| + |b_j| \leq \frac{2}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]}, \\ (2) \quad & ||a_j| - |b_j|| \leq \frac{2}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]}, \\ (3) \quad & |a_j| \leq \frac{2}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]}. \end{aligned} \quad (39)$$

The equality holds in each case for the function

$$f(z) = z + \sum_{j=2}^{\infty} \frac{2}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]} z^j. \quad (40)$$

Proof. Let

$$f = s + \bar{v} \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma). \quad (41)$$

Then, from Theorem 1, we have

$$\mathcal{L}_\mu = s + \mu v \in \mathcal{R}(\lambda, \delta, \gamma), \quad (42)$$

for each $\mu (|\mu| = 1)$. Thus, for each $|\mu| = 1$, we have

$$\operatorname{Re}\{\Phi(s, v)\} > 0, \quad (43)$$

for $z \in U$, where

$$\Phi(s, v) = (s + \mu v)' + \lambda z(s + \mu v)'' + \delta z^2(s + \mu v)''' + \gamma z^3(s + \mu v)'''. \quad (44)$$

Then, there exists an analytic function p of the form

$$p(z) = 1 + \sum_{j=1}^{\infty} p_j z^j, \quad (45)$$

with

$$\operatorname{Re} p(z) > 0, \quad (46)$$

such that

$$\{\Phi_1(s, v) + \mu \Phi_2(s, v)\} = p(z), \quad (47)$$

where

$$\begin{aligned} \Phi_1(s, v) &= s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z), \\ \Phi_2(s, v) &= \left(v'(z) + \lambda z v''(z) + \delta z^2 v'''(z) + \gamma z^3 v''''(z) \right). \end{aligned} \quad (48)$$

Comparing coefficients on both sides of (47), we yield

$$j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}] \\ (a_j + \mu b_j) = p_{j-1}, \quad j \geq 2. \quad (49)$$

Since $|p_j| \leq 2$ for $j \geq 1$ and $\mu(|\mu| = 1)$ is arbitrary, first part of Theorem 4 is complete. Similarly, we can prove (2) and (3).

Now, we investigate sufficient condition for $f \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

Theorem 5. Let $f = s + \bar{v} \in \mathcal{H}^0$ with

$$\sum_{j=2}^{\infty} j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}](|a_j| + |b_j|) \leq 1. \quad (50)$$

Then, $f \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

Proof. Let $f = s + \bar{v} \in \mathcal{H}^0$. Then by using (50), we have

$$\begin{aligned} & \operatorname{Re}\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z)\} \\ &= \operatorname{Re}\left\{1 + \sum_{j=2}^{\infty} j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]a_j z^{j-1}\right\} \\ &> 1 - \sum_{j=2}^{\infty} j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]\left|a_j\right| \\ &\geq \sum_{j=2}^{\infty} j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]\left|b_j\right| \\ &> \left|\sum_{j=2}^{\infty} j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]b_j z^{j-1}\right| \\ &= |v'(z) + \lambda z v''(z) + \delta z^2 v'''(z) + \gamma z^3 v''''(z)|. \end{aligned} \quad (51)$$

Hence, $f \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

Corollary 1. Let $f = s + \bar{v} \in \mathcal{H}^0$. If

$$\sum_{j=2}^{\infty} j^2 [3 - j + \lambda(j-1)](|a_j| + |b_j|) \leq 2, \quad (52)$$

then $f \in \mathcal{R}_{\mathcal{H}}^0(\lambda, ((\lambda-1)/2), 0)$ with $\lambda \geq 1$.

Example 1. By taking $\lambda = 0.5$ and $\delta = 0.05$, $\gamma = 0$ in the light of Theorem 5, then harmonic polynomials

$$\begin{aligned} f_1(z) &= z + 0.15\bar{z}^3, \\ f_1(z) &= z - 0.079z^3 + 0.079\bar{z}^3, \end{aligned} \quad (53)$$

belong to $\mathcal{R}_{\mathcal{H}}^0(0.5, 0.05, 0)$.

Example 2. By taking $\lambda = 3$ and $\delta = 1$, $\gamma = 0$ in the light of Theorem 5, then harmonic polynomials

$$f_3(z) = z - \frac{1}{16}\bar{z}^2 + \frac{1}{54}\bar{z}^3, \quad (54)$$

$$f_1(z) = z - 0.079z^3 + 0.079\bar{z}^3,$$

belong to $\mathcal{R}_{\mathcal{H}}^0(3, 1, 0)$.

Remark 1. The results which we obtained above lead to the results of the classes $\mathcal{R}_{\mathcal{H}}^0(0, 0, 0)$ and $\mathcal{R}_{\mathcal{H}}^0(\lambda, 0, 0)$ which are defined and studied in [2–4, 6], respectively.

Remark 2. The class $\mathcal{R}(\lambda, ((\lambda-1)/2), 0)$ with $\lambda \geq 1$ is a special case of the class defined in [13] and also our results lead the results of [13].

Now, we investigate convex combinations and convolutions for the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

Theorem 6. The class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$ of harmonic functions is closed under convex combinations.

Proof. Let, for $k = 1, 2, \dots, j$ and $f_k = s_k + \bar{v}_k \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$,

$$\sum_{k=1}^j t_k = 1, \quad 0 \leq t_k \leq 1, \quad (55)$$

and convex combination of functions f_k ($k = 1, 2, \dots, j$) can be defined as

$$f(z) = \sum_{k=1}^j t_k f_k(z) = s(z) + \overline{v(z)}, \quad (56)$$

where

$$\begin{aligned} s(z) &= \sum_{k=1}^j t_k s_k(z), \\ v(z) &= \sum_{k=1}^j t_k v_k(z), \end{aligned} \quad (57)$$

and s and v are the analytic in U with

$$s(0) = v(0) = s'(0) - 1 = v'(0) = 0. \quad (58)$$

Now,

$$\begin{aligned} & \operatorname{Re}\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z)\} \\ &= \operatorname{Re}\left\{\sum_{k=1}^j t_k \left(s'_k(z) + \lambda z s''_k(z) + \delta z^2 s'''_k(z) + \gamma z^3 s''''_k(z)\right)\right\} \\ &> \sum_{k=1}^j t_k \left|v'_k(z) + \lambda z v''_k(z) + \delta z^2 v'''_k(z) + \gamma z^3 v''''_k(z)\right| \\ &\geq |v'(z) + \lambda z v''(z) + \delta z^2 v'''(z) + \gamma z^3 v''''(z)|, \end{aligned} \quad (59)$$

showing that $f \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

A sequence $\{B_j\}_{j=0}^{\infty}$ of nonnegative real numbers is said to be a convex null sequence, if $B_j \rightarrow 0$ as $j \rightarrow \infty$ and $B_0 - B_1 \geq B_1 - B_2 \geq B_2 - B_3 \geq \dots \geq B_{j-1} - B_j \geq \dots \geq 0$. (60)

We need Lemma 4 and Lemma [14] to prove results for convolution.

Lemma 4 (see [15]). *If $\{B_j\}_{j=0}^{\infty}$ is a convex null sequence, then*

$$B(z) = \frac{B_0}{2} + \sum_{j=1}^{\infty} B_j z^j, \quad (61)$$

is analytic and

$$\operatorname{Re}(B(z)) > 0, \quad \text{in } U. \quad (62)$$

Lemma 5. *Let $\mathcal{L} \in \mathcal{R}(\lambda, \delta, \gamma)$. Then, $\operatorname{Re}(\mathcal{L}(z)/z) > (1/2)$.*

Proof. Suppose $\mathcal{L} \in \mathcal{R}(\lambda, \delta, \gamma)$ be given by $\mathcal{L}(z) = z + \sum_{j=2}^{\infty} A_j z^j$. Then,

$$\operatorname{Re} \left\{ 1 + \sum_{j=2}^{\infty} j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}] A_j z^{j-1} \right\} > 0, \quad (63)$$

which is equivalent to

$$\operatorname{Re}(p(z)) > \frac{1}{2}, \quad \text{in } U, \quad (64)$$

where

$$p(z) = 1 + \frac{1}{2} \sum_{j=2}^{\infty} j[1 + (j-1)(\lambda + (j-2)(\delta + \gamma(j-3)))] A_j z^{j-1}. \quad (65)$$

Now, for $j \geq 2$, we consider a sequence $\{B_j\}_{j=0}^{\infty}$ defined by

$$B_0 = 1, \quad (66)$$

$$B_{j-1} = \frac{2}{j[1 + (j-1)(\lambda + (j-2)(\delta + \gamma(j-3)))]}.$$

Since $\{B_j\}_{j=0}^{\infty}$ is a convex null sequence and using Lemma 4, the function

$$B(z) = 1 + \sum_{j=2}^{\infty} \frac{2}{j[1 + (j-1)(\lambda + (j-2)(\delta + \gamma(j-3)))]} z^{j-1}, \quad (67)$$

is analytic and $\operatorname{Re}(B(z)) > (1/2)$ in U . Writing

$$\frac{\mathcal{L}(z)}{z} = p(z) * \left(1 + \sum_{j=2}^{\infty} \frac{2}{j[1 + (j-1)(\lambda + (j-2)(\delta + \gamma(j-3)))]} z^{j-1} \right), \quad (68)$$

and using Lemma [14], we get $\operatorname{Re}(\mathcal{L}(z)/z) > (1/2)$.

Lemma 6. *Let $\mathcal{L}_i \in \mathcal{R}(\lambda, \delta, \gamma)$, for $k = 1, 2$. Then, $\mathcal{L}_1 * \mathcal{L}_2 \in \mathcal{R}(\lambda, \delta, \gamma)$.*

Proof. Suppose $\mathcal{L}_1(z) = z + \sum_{j=2}^{\infty} A_j z^j$ and $\mathcal{L}_2(z) = z + \sum_{j=2}^{\infty} B_j z^j$. Then, the convolution of $\mathcal{L}_1(z)$ and $\mathcal{L}_2(z)$ is defined by

$$\mathcal{L}(z) = (\mathcal{L}_1 * \mathcal{L}_2)(z) = z + \sum_{j=2}^{\infty} A_j B_j z^j. \quad (69)$$

Since

$$\begin{aligned} \mathcal{L}'(z) &= \mathcal{L}'_1(z) * \frac{\mathcal{L}_2(z)}{z}, \\ z\mathcal{L}''(z) &= z\mathcal{L}''_1(z) * \frac{\mathcal{L}_2(z)}{z}, \\ z^2\mathcal{L}'''(z) &= z^2\mathcal{L}'''_1(z) * \frac{\mathcal{L}_2(z)}{z}, \\ z^3\mathcal{L}''''(z) &= z^3\mathcal{L}''''_1(z) * \frac{\mathcal{L}_2(z)}{z}. \end{aligned} \quad (70)$$

Then, we have

$$\begin{aligned} \mathcal{L}'(z) + \lambda z\mathcal{L}''(z) + \delta z^2\mathcal{L}'''(z) + \gamma z^3\mathcal{L}''''(z) \\ = \left(\mathcal{L}'_1(z) + \lambda z\mathcal{L}''_1(z) + \delta z^2\mathcal{L}'''_1(z) + \gamma z^3\mathcal{L}''''_1(z) \right) * \frac{\mathcal{L}_2(z)}{z}. \end{aligned} \quad (71)$$

Since $\mathcal{L}_1 \in \mathcal{R}(\lambda, \delta, \gamma)$,

$$\operatorname{Re} \left\{ \mathcal{L}'_1(z) + \lambda z\mathcal{L}''_1(z) + \delta z^2\mathcal{L}'''_1(z) + \gamma z^3\mathcal{L}''''_1(z) \right\} > 0, \quad (72)$$

and using Lemma 5, $\operatorname{Re}\{\mathcal{L}(z)/z\} > (1/2)$ in U .

Now applying Lemma [14] on (71) yields

$$\operatorname{Re} \left\{ \mathcal{L}'(z) + \lambda z\mathcal{L}''(z) + \delta z^2\mathcal{L}'''(z) + \gamma z^3\mathcal{L}''''(z) \right\} > 0, \quad (73)$$

in U . Thus, $\mathcal{L} = \mathcal{L}_1 * \mathcal{L}_2 \in \mathcal{R}(\lambda, \delta, \gamma)$.

Now, by using Lemma 6, let us prove that the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$ is closed under convolutions of its members.

Theorem 7. *Let $f_k \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$, for $k = 1, 2$. Then,*

$$f_1 * f_2 \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma). \quad (74)$$

Proof. Let

$$f_k = s_k + \overline{v_k} \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma), \quad k = 1, 2, \quad (75)$$

Then, the convolution of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2 = s_1 * s_2 + \overline{v_1 * v_2}, \quad (76)$$

in order to show that

$$f_1 * f_2 \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma). \quad (77)$$

We have to prove that

$$\mathcal{L}_\mu = s_1 * s_2 + \mu(v_1 * v_2) \in \mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma), \quad (78)$$

for each $\mu (|\mu| = 1)$. By Lemma 6, the class $\mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$ is closed under convolutions for each $\mu (|\mu| = 1)$, $s_k + \mu v_k \in \mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$ for $(k = 1, 2)$. Then, both \mathcal{L}_1 and \mathcal{L}_1 given by

$$\begin{aligned} \mathcal{L}_1 &= (s_1 - v_1) * (s_2 - \mu v_2), \\ \mathcal{L}_2 &= (s_1 + v_1) * (s_2 + \mu v_2), \end{aligned} \quad (79)$$

belong to $\mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$. Since $\mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$ is closed under convex combinations, then

$$\mathcal{L} = \frac{1}{2}(\mathcal{L}_1 + \mathcal{L}_2) = s_1 * s_2 + \mu(v_1 * v_2), \quad (80)$$

belongs to $\mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$. Hence, $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$ is closed under convolution.

Goodloe [16] defined the Hadamard product of a harmonic function as follows:

$$f \tilde{*} \varphi = s * \varphi + \overline{v * \varphi}, \quad (81)$$

where $f = s + \bar{v} \in \mathcal{H}$ and $\varphi \in \mathcal{A}$. By considering this Hadamard product of a harmonic function, we investigate following result.

Theorem 8. Let $f \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$ and $\varphi \in \mathcal{A}$ be such that $\operatorname{Re}(\varphi(z)/z) > (1/2)$, for $z \in U$. Then, $f \tilde{*} \varphi \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

Proof. Suppose

$$f = s + \bar{v} \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma). \quad (82)$$

Then,

$$\mathcal{L}_\mu = s + \mu v \in \mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma), \quad (83)$$

for each $\mu (|\mu| = 1)$. Using Theorem 1 and in order to show that $f \tilde{*} \varphi \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$, we need to show that

$$G = s * \varphi + \mu(v * \varphi) \in \mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma), \quad (84)$$

for each $\mu (|\mu| = 1)$. Write $G = \mathcal{L}_\mu * \varphi$, and

$$\begin{aligned} & \left(G'(z) + \lambda z G''(z) + \delta z^2 G'''(z) + \gamma z^3 G''''(z) \right) \\ &= \left(\mathcal{L}'_\mu(z) + \lambda z \mathcal{L}''_\mu(z) + \delta z^2 \mathcal{L}'''_\mu(z) + \gamma z^3 \mathcal{L}''''_\mu(z) \right) * \frac{\varphi(z)}{z}. \end{aligned} \quad (85)$$

Since $\operatorname{Re}(\varphi(z)/z) > (1/2)$ and

$$\operatorname{Re} \left(\mathcal{L}'_\mu(z) + \lambda z \mathcal{L}''_\mu(z) + \delta z^2 \mathcal{L}'''_\mu(z) + \gamma z^3 \mathcal{L}''''_\mu(z) \right) > 0, \quad (86)$$

in U . Lemma [14] proves that $G \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

Corollary 2. Let $f \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$ and $\varphi \in \mathcal{H}$. Then, $f \tilde{*} \varphi \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

Proof. Suppose $\varphi \in \mathcal{H}$. Then, $\operatorname{Re}(\varphi(z)/z) > (1/2)$ for $z \in U$. Theorem 8 concludes that $f \tilde{*} \varphi \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta, \gamma)$.

3. Conclusion

Various new subclasses of normalized harmonic functions have been defined in open unit disk U , satisfying second-order and third-order differential inequalities. In this study, we defined a new class of normalized harmonic functions in open unit disk U , satisfying a fourth-order differential inequality. We gave some useful results such that close-to-convexity, coefficient bounds, growth estimates, sufficient coefficient condition, and convolution for the functions belong to this new class of harmonic functions. Further using the concepts of fourth-order differential inequality, all these problems can be studied for classes of meromorphic harmonic functions, Bazilevic harmonic functions, and for p valent harmonic functions as well.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors equally contributed to this study.

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Research Article

The Third Logarithmic Coefficient for Certain Close-to-Convex Functions

Najla M. Alarifi 

Department of Mathematics, Imam Abdulrahman Bin Faisal University, Dammam 31113, Saudi Arabia

Correspondence should be addressed to Najla M. Alarifi; nalareefi@iau.edu.sa

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The logarithmic coefficients γ_n of a normalized analytic functions f are defined by $(\log f(z)/z) = 2 \sum_{n=1}^{\infty} c_n z^n$. For certain close-to-convex functions $f(z) = z + a_2 z^2 + \dots$, Cho et al. (on the third logarithmic coefficient in some subclasses of close-to-convex functions) has obtained the upper bound of the third logarithmic coefficient γ_3 when the second coefficient a_2 is real. In the present paper, the upper bound of the third logarithmic coefficient γ_3 is computed with no restriction on the second coefficient a_2 .

1. Introduction and Preliminaries

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane \mathbb{C} and let \mathcal{A} be the set of all analytic normalized functions $f: \mathbb{D} \rightarrow \mathbb{C}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \quad (1)$$

Let \mathcal{S} be its subclass consisting of functions that are univalent in \mathbb{D} . Given a function $f \in \mathcal{S}$, the coefficients γ_n are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D} \setminus \{0\}, \quad \log 1 := 0. \quad (2)$$

For example (see Figure 1), for the Koebe function k given by $k(z) = (z/(1-z^2))^2$, the logarithmic coefficients $\gamma_n = (1/n)$ are as follows

$$\log \frac{k(z)}{z} = 2 \sum_{n=1}^{\infty} \frac{1}{n} z^n. \quad (3)$$

The Milin conjecture ([1] and ([2] p. 155)) gives an inequality satisfied by the logarithmic coefficients. For $f \in \mathcal{S}$, the logarithmic coefficients satisfy

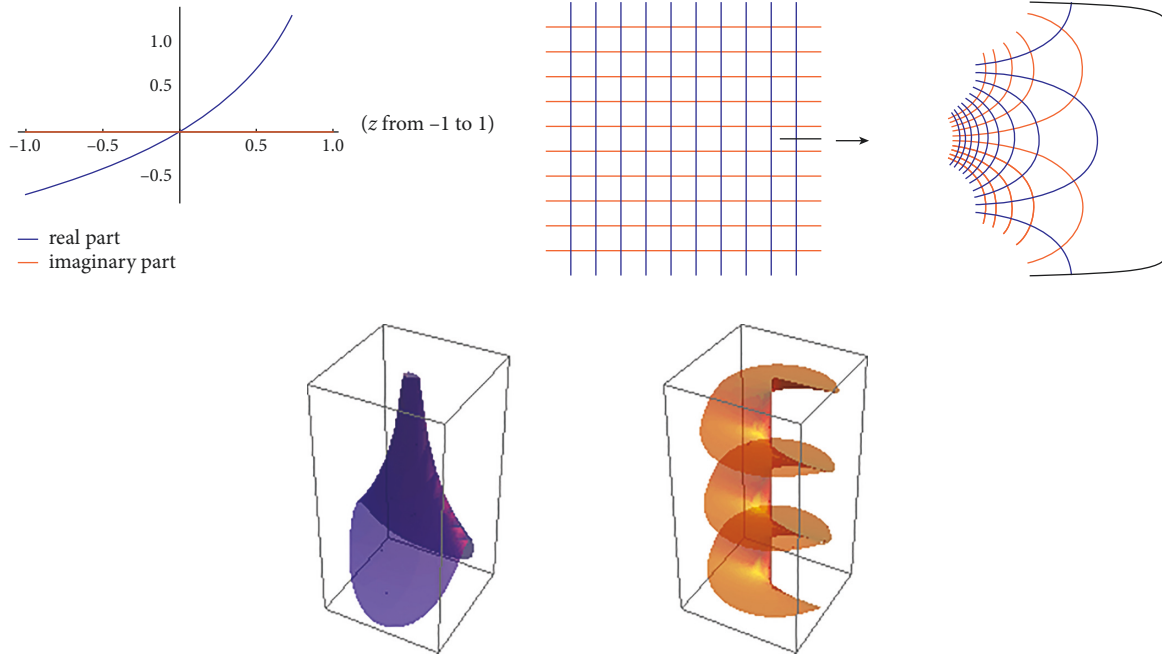
$$\sum_{m=1}^n \sum_{k=1}^m \left(k |\gamma_k|^2 - \frac{1}{k} \right) \leq 0. \quad (4)$$

The Milin conjecture was confirmed (e.g., ([2] p. 37), by Branges [3] and implies the famous Bieberbach conjecture that $|a_n| \leq n$ for $f \in \mathcal{S}$. Sharp estimates for the class \mathcal{S} are known only for the first two coefficients:

$$\begin{aligned} |\gamma_1| &\leq 1, \\ |\gamma_2| &\leq \frac{1}{2} + \frac{1}{e} = 0.635 \dots \end{aligned} \quad (5)$$

Note that Obradović and Tuneski [4] obtained an upper bound of $|\gamma_3|$ for the class \mathcal{S} . The problem of estimating the modulus of the first three logarithmic coefficients is significantly studied for the subclasses of \mathcal{S} , and in some cases, sharp bounds are obtained. For instance, sharp estimates for the class of starlike functions \mathcal{S}^* are given by the inequality $|\gamma_n| \leq (1/n)$ holds for $n \in \mathbb{N}$ ([5], p. 42).

Furthermore, for $f \in \mathcal{S}^*$, the class of strongly starlike function of the order β , $(0 \leq \beta \leq 1)$, it holds that $|\gamma_n| \leq (\beta/n)$ ($n \in \mathbb{N}$) [6]. The bounds of γ_n for functions in subclasses of \mathcal{S} have been widely studied in recent years. Sharp estimates for different subclasses are given in [6, 7] and ([5], p. 116) and [8], respectively, while nonsharp

FIGURE 1: Plot of $\log(f(z)/z)$, $f(z) = (z/(1-z))^2$.

estimates for the class of Bazilevic and close-to-convex are given in [9–11], respectively.

Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be the subclasses of \mathcal{S} satisfying, respectively, the next conditions:

$$\begin{aligned} \Re\{(1-z)f'(z)\} &> 0, \quad z \in \mathbb{D}, \\ \Re\{(1-z^2)f'(z)\} &> 0, \quad z \in \mathbb{D}, \\ \Re\{(1-z+z^2)f'(z)\} &> 0, \quad z \in \mathbb{D}. \end{aligned} \quad (6)$$

Note that each class defined above is the subclass of the well-known class of close-to-convex functions; consequently, families \mathcal{F}_i , $i = 1, 2, 3$, contain only univalent functions ([2], Vol. II, p. 2). The sharp bounds of γ_1 , γ_2 and partial results for γ_3 of the subclasses $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of \mathcal{S} were determined by Pranav Kumar and Vasudevarao [12].

Moreover, Cho et al. [13] computed the sharp upper bounds for the third logarithmic coefficient γ_3 of f when a_2 is a real number. Differentiating (1) and comparing the coefficients with (2), we get $\gamma_1 = (1/2)a_1$, $\gamma_2 = (1/2)(a_3 - (1/2)a^2)$, and

$$\gamma_3 = \frac{1}{2} \left(a_4 - a_2 a_3 + \frac{1}{3} a_2^3 \right). \quad (7)$$

The main aim of this paper is to determine the upper bound of the third logarithmic coefficient in the general case of a_2 . The following lemma is needed to prove our main results.

Lemma 1 (see [14]). Let $w(z) = c_1 z + c_2 z^2 + \dots$ be a Schwarz function. Then

$$\begin{aligned} |c_1| &\leq 1, \\ |c_2| &\leq 1 - |c_1|^2, \\ |c_3| &\leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}. \end{aligned} \quad (8)$$

2. Main Results

Our main result is as follows:

Theorem 1. Let $f \in \mathcal{F}_1$. Then

$$|\gamma_3| \leq \frac{15.75}{48} = 0.328125. \quad (9)$$

Proof. Since $f \in \mathcal{F}_1$, and for analytic function w in \mathbb{D} with $w(0) = 0$ satisfying the formula

$$(1-z)f'(z) = \frac{1+w(z)}{1-w(z)} = 1 + 2w(z) + 2w^2(z) + \dots \quad (10)$$

We obtain

$$w(z) = c_1 z + c_2 z^2 + \dots \quad (11)$$

Then, by using (10) along with (11) leads to

$$\begin{aligned}
 a_2 &= \frac{1}{2}(1 + 2c_1), \\
 a_3 &= \frac{1}{3}(1 + 2c_1 + 2c_1^2 + 2c_2), \\
 a_4 &= \frac{1}{4}(1 + 2c_1 + 2c_2 + 2c_3 + 2c_1^2 + 4c_1c_2 + 2c_1^3).
 \end{aligned} \tag{12}$$

From (7) and (12), we obtain

$$\gamma_3 = \frac{1}{48}(3 + 2c_1 + 4c_2 + 12c_3 + 8c_1c_2 + 4c_1^3). \tag{13}$$

In view of Lemma 1, we attain

$$\begin{aligned}
 48|\gamma_3| &\leq 3 + 2|c_1| + 4|c_2| + 12|c_3| + 8|c_1||c_2| + 4|c_1|^3 \\
 &\leq 3 + 2|c_1| + 4|c_2| + 12\left(1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}\right) \\
 &\quad + 8|c_1||c_2| + 4|c_1|^3 =: f_1(|c_1|, |c_2|),
 \end{aligned} \tag{14}$$

where

$$\begin{aligned}
 f_1(x, y) &= 3 + 2x + 4y + 12\left(1 - x^2 - \frac{y^2}{1 + x}\right) \\
 &\quad + 8xy + 4x^3,
 \end{aligned} \tag{15}$$

$$(x, y) \in E: 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2.$$

The system

$$\frac{\partial f_1(x, y)}{\partial x} = 2 - 24x + 12\left(\frac{y}{1 + x}\right)^2 + 8y + 12x^2 = 0, \tag{16}$$

$$\frac{\partial f_1(x, y)}{\partial y} = 4 - \frac{24y}{1 + x} + 8x = 0,$$

has a unique solution $(x_1, y_1) = ((1/4), (5/16)) \in E \setminus \partial E$ with

$$f_1(x_1, y_1) = 15.75. \tag{17}$$

The maximum value of f_1 is obtained when (x, y) is a point on the boundary of E . In view of this, we have

$$\begin{aligned}
 f_1(x, 0) &= 15 + 2x - 12x^2 + 4x^3 \\
 &\leq 9 + \frac{10\sqrt{30}}{9} = 15.08580\dots,
 \end{aligned} \tag{18}$$

$$\begin{aligned}
 f_1(0, y) &= 3 + 4y + 12(1 - y^2) \\
 &= 15 + 4y - 12y^2 \leq \frac{46}{3} = 15.33\dots,
 \end{aligned}$$

and

$$f_1(x, 1 - x^2) = 7 + 22x - 4x^2 - 16x^3 \leq 15.304035\dots \tag{19}$$

Using (14) and (17)–(19), we conclude the following outcome:

$$48|\gamma_3| \leq 15.75, \quad \text{i.e., } |\gamma_3| \leq 0.328125. \tag{20}$$

This completes the proof. \square

Remark 1. If $f \in \mathcal{F}_1$, where $f''(0)$ is a real number, then we get the result in [13]

$$|\gamma_3| \leq \frac{1}{288}(11 + 15\sqrt{30}) = 0.323466\dots \tag{21}$$

Theorem 2. Let $f \in \mathcal{F}_2$. Then

$$|\gamma_3| \leq 0.258765\dots \tag{22}$$

Proof. Since $f \in \mathcal{F}_2$, then there exists an analytic function w in \mathbb{D} with $w(0) = 0$ and

$$(1 - z^2)f'(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + 2w(z) + 2w^2(z) + \dots \tag{23}$$

The coefficients can be determined by comparing the information in (11) and (23)

$$a_2 = c_1,$$

$$a_3 = \frac{1}{3}(1 + 2c_2 + 2c_1^2), \tag{24}$$

$$a_4 = \frac{1}{2}(c_1 + c_3 + 2c_1c_2 + c_1^3).$$

From (7) and (24), we have the following conclusion:

$$\gamma_3 = \frac{1}{12}(c_1 + 3c_3 + 2c_1c_2 + c_1^3). \tag{25}$$

Moreover, according to Lemma 1, we get the following inequality:

$$\begin{aligned}
 12|\gamma_3| &\leq |c_1| + 3|c_3| + 2|c_1||c_2| + |c_1|^3 \\
 &\leq |c_1| + 3\left(1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}\right) \\
 &\quad + 2|c_1||c_2| + |c_1|^3 =: f_2(|c_1|, |c_2|),
 \end{aligned} \tag{26}$$

where

$$f_2(x, y) = 3\left(1 - x^2 - \frac{y^2}{1 + x}\right) + 2xy + x + x^3, \tag{27}$$

$$(x, y) \in E: 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2.$$

From the system,

$$\frac{\partial f_2(x, y)}{\partial x} = 6x + 3\left(\frac{y}{1+x}\right)^2 + 2y + 1 + 3x^2 = 0, \quad (28)$$

$$\frac{\partial f_2(x, y)}{\partial y} = -\frac{6y}{1+x} + 2x = 0,$$

only one solution (x_2, y_2) lies in the interior of E , where

$$x_2 = \frac{4 - \sqrt{7}}{6} = 0.22570 \dots, \quad (29)$$

$$y_2 = \frac{47 - 14\sqrt{7}}{108} = 0.092217 \dots,$$

and

$$f_2(x_2, y_2) = 3.10518 \dots \quad (30)$$

On the boundary of E , we have the next property

$$f_2(x, 0) = 3(1 - x^2) + x + x^3 \leq 2 + \frac{4}{9}\sqrt{6} = 3.08866,$$

$$\text{for } 0 \leq x \leq 1,$$

$$f_2(0, y) = 3(1 - y^2) \leq 3, \quad \text{for } 0 \leq y \leq 1,$$

$$f_2(x, 1 - x^2) = 6x - 4x^3 \leq 2\sqrt{2} = 2.82842 \dots \quad (31)$$

Consequently, (26), (30), and (31) yield

$$12|\gamma_3| \leq 3.10518 \dots, \quad \text{i.e., } |\gamma_3| \leq 0.258765 \dots \quad (32) \quad \square$$

Remark 2. If $f \in \mathcal{F}_2$, where $f''(0)$ is a real number, then [13]

$$|\gamma_3| \leq \frac{1}{972} (95 + 23\sqrt{46}) = 0.258223 \dots \quad (33)$$

Theorem 3. Let $f \in \mathcal{F}_3$. Then

$$|\gamma_3| \leq \frac{17.75}{48} = 0.36979 \dots \quad (34)$$

Proof. Let $f \in \mathcal{F}_3$ and an analytic function w in \mathbb{D} with $w(0) = 0$ such that

$$(1 - z + z^2)f'(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + 2w(z) + 2w^2(z) + \dots \quad (35)$$

Substituting (11) into (35), we have

$$a_2 = \frac{1}{2}(1 + 2c_1),$$

$$a_3 = \frac{2}{3}(c_1 + c_2 + c_1^2), \quad (36)$$

$$a_4 = \frac{1}{4}(2c_2 + 2c_3 + 2c_1^2 + 2c_1^3 + 4c_1c_2 - 1).$$

By using (7) and (36), we obtain

$$\gamma_3 = \frac{1}{48}(-5 - 2c_1 + 4c_2 + 12c_3 + 8c_1c_2 + 4c_1^3). \quad (37)$$

According to Lemma 1, we conclude that

$$\begin{aligned} 48|\gamma_3| &\leq 5 + 2|c_1| + 4|c_2| + 12|c_3| + 8|c_1||c_2| + 4|c_1|^3 \\ &\leq 5 + 2|c_1| + 4|c_2| + 12\left(1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}\right) \end{aligned} \quad (38)$$

$$+ 8|c_1||c_2| + 4|c_1|^3 =: f_3(|c_1|, |c_2|),$$

where

$$\begin{aligned} f_3(x, y) &= 5 + 2x + 4y + 12\left(1 - x^2 - \frac{y^2}{1+x}\right) + 8xy + 4x^3, \\ (x, y) \in E: 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2. \end{aligned} \quad (39)$$

The system

$$\frac{\partial f_3(x, y)}{\partial x} = 2 - 24x + 12\left(\frac{y}{1+x}\right)^2 + 8y + 12x^2 = 0, \quad (40)$$

$$\frac{\partial f_3(x, y)}{\partial y} = 4 - \frac{24y}{1+x} + 8x = 0,$$

admits a unique solution $(x_3, y_3) = ((1/4), (5/16))$ in the interior of E such that

$$f_1(x_3, y_3) = 17.75. \quad (41)$$

On the boundary of E , the following cases are observed:

$$\begin{aligned} f_3(x, 0) &= 17 + 2x - 12x^2 + 4x^3 \\ &\leq 11 + \frac{10\sqrt{30}}{9} = 17.08580 \dots, \end{aligned} \quad (42)$$

$$\begin{aligned} f_3(0, y) &= 3 + 4y + 12(1 - y^2) \\ &= 17 + 4y - 12y^2 \leq \frac{46}{3} = 17.33 \dots, \end{aligned}$$

and

$$f_3(x, 1 - x^2) = 9 + 22x - 4x^2 - 20x^3 \leq 16.56455 \dots \quad (43)$$

Equations (38), (41)–(43) show that

$$|\gamma_3| \leq \frac{17.75}{48} = 0.36979 \dots \quad (44)$$

□

Remark 3. Let $f \in \mathcal{F}_3$, where $f''(0)$ is a real number. Then [13]

$$|\gamma_3| \leq \frac{1}{7776} (743 + 131\sqrt{262}) = 0.368238 \dots \quad (45)$$

Data Availability

No data were used in this study.

Disclosure

The author would like to declare that a preprint of this article has previously been published in [15].

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Convolution Results and Fekete–Szegő Inequalities for Certain Classes of Symmetric q -Starlike and Symmetric q -Convex Functions

Tamer M. Seoudy ^{1,2}

¹Department of Mathematics, Jamoum University College, Umm Al-Qura University, Makkah, Saudi Arabia

²Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt

Correspondence should be addressed to Tamer M. Seoudy; tmsaman@uqu.edu.sa

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In this paper, by using the concept of the symmetric q -difference operator, we introduce certain classes of symmetric q -starlike and symmetric q -convex functions. Convolution results, coefficient estimates, and Fekete–Szegő inequalities for the analytic functions belonging to these classes are obtained.

1. Introduction

Let us represent the class of analytic (or holomorphic or regular) functions in $\Delta = \{\xi \in \mathbb{C} : |\xi| < 1\}$ by $\mathcal{H}(\Delta)$ and suppose that \mathcal{A} is the subclass of $\mathcal{H}(\Delta)$ defined as follows:

$$\mathcal{A} = \left\{ g \in \mathcal{H}(\Delta) : g(\xi) = \xi + \sum_{j=2}^{\infty} b_j \xi^j \right\}. \quad (1)$$

Further, let $\mathcal{S}(\alpha)$ and $\mathcal{C}(\alpha)$ denote the subclasses of \mathcal{A} that consist, respectively, of starlike of order α and convex of order α ($0 \leq \alpha < 1$) in Δ (see [1]). For two given regular functions $h_1, h_2 \in \mathcal{H}(\Delta)$, we say that $h_1(\xi)$ is subordinate to $h_2(\xi)$ or $h_2(\xi)$ is superordinate to $h_1(\xi)$, written symbolically as $h_1(\xi) \prec h_2(\xi)$ if there exists a Schwarz function ω , which (by definition) is regular in Δ with $\omega(0) = 0$ and $|\omega(\xi)| < 1$ for all $\xi \in \Delta$, such that

$$h_1(\xi) = h_2(\omega(\xi)), \quad (\xi \in \Delta). \quad (2)$$

Moreover, if $h_1(\xi)$ is univalent function in Δ , then the following equivalence holds true (see [2, 3]):

$$\begin{aligned} h_1(\xi) \prec h_2(\xi) &\Leftrightarrow h_1(0) = h_2(0), \\ h_1(\Delta) &\subset h_2(\Delta). \end{aligned} \quad (3)$$

For functions g given by (1) and h given by

$$h(\xi) = \xi + \sum_{j=2}^{\infty} c_j \xi^j, \quad (4)$$

the convolution or Hadamard product of g and h is defined by

$$(g * h)(\xi) = \xi + \sum_{j=2}^{\infty} b_j c_j \xi^j = (h * g)(\xi). \quad (5)$$

Let $\mathcal{S}[L, M]$ and $\mathcal{C}[L, M]$ denote the subclasses of \mathcal{A} for $-1 \leq M < L \leq 1$ which are defined by (see [4–9])

$$\begin{aligned} \mathcal{S}[L, M] &= \left\{ g(\xi) \in \mathcal{A} : \frac{\xi g'(\xi)}{g(\xi)} \prec \frac{1 + L\xi}{1 + M\xi} \right\}, \\ \mathcal{C}[L, M] &= \left\{ g(\xi) \in \mathcal{A} : \frac{(\xi g'(\xi))'}{g'(\xi)} \prec \frac{1 + L\xi}{1 + M\xi} \right\}. \end{aligned} \quad (6)$$

We note that $\mathcal{S}[1-2\alpha, -1] = \mathcal{S}(\alpha)$ and $\mathcal{C}[1-2\alpha, -1] = \mathcal{C}(\alpha)$ ($0 \leq \alpha < 1$).

For $0 < q < 1$, the symmetric q -difference of a function g is defined as follows (see [10]):

$$\tilde{D}_q g(\xi) = \frac{g(q\xi) - g(q^{-1}\xi)}{(q - q^{-1})\xi}, \quad (\xi \neq 0), \quad (7)$$

and $\tilde{D}_q g(0) = g'(0)$ provided that $g(\xi)$ is differentiable at 0. From (1) and (7), we deduce that

$$\tilde{D}_q g(\xi) = 1 + \sum_{j=2}^{\infty} [\widetilde{j}]_q b_j \xi^{j-1}, \quad (8)$$

where $[\widetilde{j}]_q$ is the q -number given by

$$[\widetilde{j}]_q = \frac{q^j - q^{-j}}{q - q^{-1}}. \quad (9)$$

Furthermore, if $g(\xi)$ and $h(\xi)$ are the two functions, then

$$\tilde{D}_q [\gamma_1 g(\xi) \pm \gamma_2 h(\xi)] = \gamma_1 \tilde{D}_q g(\xi) \pm \gamma_2 \tilde{D}_q h(\xi), \quad (10)$$

where γ_1 and γ_2 are constants.

$$\tilde{D}_q [g(\xi)h(\xi)] = h(q\xi)\tilde{D}_q g(\xi) + g(q^{-1}\xi)\tilde{D}_q h(\xi),$$

$$\tilde{D}_q \left[\frac{g(\xi)}{h(\xi)} \right] = \frac{h(q\xi)\tilde{D}_q g(\xi) - g(q\xi)\tilde{D}_q h(\xi)}{h(q\xi)h(q^{-1}\xi)}. \quad (11)$$

Making use of the symmetric q -difference $\tilde{D}_q g(\xi)$ given by (7), we introduce the subclasses $\tilde{\mathcal{S}}_q[L, M]$ and $\tilde{\mathcal{C}}_q[L, M]$ of \mathcal{A} for $0 < q < 1$ and $-1 \leq M < L \leq 1$ as follows:

$$\tilde{\mathcal{S}}_q[L, M] = \left\{ g(\xi) \in \mathcal{A} : \frac{\xi \tilde{D}_q g(\xi)}{g(\xi)} < \frac{1+L\xi}{1+M\xi} \right\}, \quad (12)$$

$$\tilde{\mathcal{C}}_q[L, M] = \left\{ g(\xi) \in \mathcal{A} : \frac{\tilde{D}_q(\xi \tilde{D}_q g(\xi))}{\tilde{D}_q g(\xi)} < \frac{1+L\xi}{1+M\xi} \right\}. \quad (13)$$

From (12) and (13), we have

$$g(\xi) \in \tilde{\mathcal{C}}_q[L, M] \Leftrightarrow \xi \tilde{D}_q g(\xi) \in \tilde{\mathcal{S}}_q[L, M]. \quad (14)$$

We also note that

$$(i) \lim_{q \rightarrow 1^-} \tilde{\mathcal{S}}_q[L, M] = \mathcal{S}[L, M] \quad \text{and} \quad \lim_{q \rightarrow 1^-} \tilde{\mathcal{C}}_q[L, M] = \mathcal{C}[L, M] \quad (\text{see [6, 7, 9, 11, 12]});$$

$$(ii) \tilde{\mathcal{S}}_q[1-2\alpha, -1] = \tilde{\mathcal{S}}_q(\alpha) \quad (0 \leq \alpha < 1),$$

$$\tilde{\mathcal{S}}_q(\alpha) = \left\{ g(\xi) \in \mathcal{A} : \Re \left\{ \frac{\xi \tilde{D}_q g(\xi)}{g(\xi)} \right\} > \alpha, \xi \in \Delta \right\}, \quad (15)$$

$$\text{and } \lim_{q \rightarrow 1^-} \tilde{\mathcal{S}}_q(\alpha) = \mathcal{S}(\alpha) \quad (\text{see [1, 13, 14]}).$$

$$(iii) \tilde{\mathcal{C}}_q[1-2\alpha, -1] = \tilde{\mathcal{C}}_q(\alpha) \quad (0 \leq \alpha < 1),$$

$$\tilde{\mathcal{C}}_q(\alpha) = \left\{ g(\xi) \in \mathcal{A} : \Re \left\{ \frac{\tilde{D}_q(\xi \tilde{D}_q g(\xi))}{\tilde{D}_q g(\xi)} \right\} > \alpha, \xi \in \Delta \right\}, \quad (16)$$

$$\text{and } \lim_{q \rightarrow 1^-} \tilde{\mathcal{C}}_q(\alpha) = \mathcal{C}(\alpha) \quad (\text{see [1, 8, 15]}).$$

$$(iv) \tilde{\mathcal{S}}_q[(1-2\alpha)\beta, -\beta] = \tilde{\mathcal{S}}_q(\alpha, \beta) \quad (0 \leq \alpha < 1, 0 < \beta \leq 1),$$

$$\tilde{\mathcal{S}}_q(\alpha, \beta) = \left\{ g(\xi) \in \mathcal{A} : \left| \frac{(\xi \tilde{D}_q g(\xi)/g(\xi)) - 1}{(\xi \tilde{D}_q g(\xi)/g(\xi)) + 1 - 2\alpha} \right| < \beta, \xi \in \Delta \right\}, \quad (17)$$

$$\text{and } \lim_{q \rightarrow 1^-} \tilde{\mathcal{S}}_q(\alpha, \beta) = \mathcal{S}(\alpha, \beta) \quad (\text{see [16]}).$$

$$(v) \tilde{\mathcal{C}}_q[(1-2\alpha)\beta, -\beta] = \tilde{\mathcal{C}}_q(\alpha, \beta) \quad (0 \leq \alpha < 1, 0 < \beta \leq 1),$$

$$\tilde{\mathcal{C}}_q(\alpha, \beta) = \left\{ g(\xi) \in \mathcal{A} : \left| \frac{(\tilde{D}_q(\xi \tilde{D}_q g(\xi))/\tilde{D}_q g(\xi)) - 1}{(\tilde{D}_q(\xi \tilde{D}_q g(\xi))/\tilde{D}_q g(\xi)) + 1 - 2\alpha} \right| < \beta, \xi \in \Delta \right\}, \quad (18)$$

and $\lim_{q \rightarrow 1^-} \tilde{\mathcal{C}}_q(\alpha, \beta) = \mathcal{C}(\alpha, \beta)$ (see [16]).

In the present article, our aim is to investigate convolution properties, coefficient estimates, and Fekete–Szegő inequalities for the classes $\tilde{\mathcal{S}}_q[L, M]$ and $\tilde{\mathcal{C}}_q[L, M]$. The motivation of this article is to generalize and improve previously known results.

2. Convolution Results and Coefficient Estimates

Unless otherwise mentioned, we assume throughout this investigation that

$$\begin{aligned}
0 &\leq \theta < 2\pi, \\
-1 &\leq M < L \leq 1, \\
0 &< q < 1, \\
\xi &\in \Delta.
\end{aligned} \tag{19}$$

Theorem 1. If $g(\xi) \in \mathcal{A}$ has the series form (1), then $g(\xi) \in \tilde{\mathcal{S}}_q[L, M]$ if and only if

$$\frac{1}{\xi} \left[g(\xi) * \frac{\xi - [1 + R(q - 1 + q^{-1})]\xi^2 + R\xi^3}{(1 - q\xi)(1 - \xi)(1 - q^{-1}\xi)} \right] \neq 0, \tag{20}$$

where R is given by

$$R = R(\theta, L, M) = \frac{e^{-i\theta} + L}{L - M}. \tag{21}$$

Proof. It is easy to check that

$$\begin{aligned}
g(\xi) * \frac{\xi}{1 - \xi} &= g(\xi), \\
g(\xi) * \frac{\xi}{(1 - q\xi)(1 - q^{-1}\xi)} &= \xi \tilde{D}_q g(\xi).
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\frac{1}{\xi} \left[(1 + Me^{i\theta}) \left(g(\xi) * \frac{\xi}{(1 - q\xi)(1 - q^{-1}\xi)} \right) - (1 + Le^{i\theta}) \left(g(\xi) * \frac{\xi}{1 - \xi} \right) \right] \\
&= \frac{(M - L)e^{i\theta}}{\xi} \left\{ g(\xi) * \frac{\xi - [1 + ((e^{-i\theta} + L)/(L - M))(q - 1 + q^{-1})]\xi^2 + ((e^{-i\theta} + L)/(L - M))\xi^3}{(1 - q\xi)(1 - \xi)(1 - q^{-1}\xi)} \right\} \neq 0,
\end{aligned} \tag{26}$$

which proves the necessary condition (20) for Theorem 1.

Reversely, suppose that $g(\xi) \in \mathcal{A}$ satisfies condition (20). Since it was shown in the first part of the proof that assumption (20) is equivalent to (25), we obtain that

$$\frac{\xi \tilde{D}_q g(\xi)}{g(\xi)} \neq \frac{1 + Le^{i\theta}}{1 + Me^{i\theta}}, \quad (0 \leq \theta < 2\pi). \tag{27}$$

If we denote

$$\varphi(\xi) = \frac{\xi \tilde{D}_q g(\xi)}{g(\xi)}, \tag{28}$$

$$\psi(\xi) = \frac{1 + L\xi}{1 + M\xi},$$

relation (27) means that

$$\varphi(\Delta) \cap \psi(\partial\Delta) = \emptyset. \tag{29}$$

Thus, the simply connected domain $\varphi(\Delta)$ is included in a connected component of $\mathbb{C} \setminus \psi(\partial\Delta)$. Therefore, using the fact that $\varphi(0) = \psi(0)$ and the univalence of the function $\psi(\xi)$, it

In order to prove that equation (20) holds, we will write (12) by using the definition of the subordination as follows:

$$\frac{\xi \tilde{D}_q g(\xi)}{g(\xi)} = \frac{1 + Lw(\xi)}{1 + Mw(\xi)}, \tag{23}$$

where $w(\xi)$ is Schwarz function; hence, we have

$$\frac{\xi \tilde{D}_q g(\xi)}{g(\xi)} \neq \frac{1 + Le^{i\theta}}{1 + Me^{i\theta}}, \tag{24}$$

which is equivalent to

$$\frac{1}{\xi} \left[(1 + Me^{i\theta}) \xi \tilde{D}_q g(\xi) - (1 + Le^{i\theta}) g(\xi) \right] \neq 0. \tag{25}$$

By using (22) in (25), we obtain

follows that $\varphi(\xi) \prec \psi(\xi)$, which implies that $g(\xi) \in \tilde{\mathcal{S}}_q[L, M]$. This completes the proof of Theorem 1.

Putting $L = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $M = -1$ in Theorem 1, we get the following convolution result for $\tilde{\mathcal{S}}_q(\alpha)$. \square

Corollary 1. If $g(\xi) \in \mathcal{A}$ has the series form (1), then $g(\xi) \in \tilde{\mathcal{S}}_q(\alpha)$ if and only if

$$\frac{1}{\xi} \left[g(\xi) * \frac{\xi - [1 + T(q - 1 + q^{-1})]\xi^2 + T\xi^3}{(1 - q\xi)(1 - \xi)(1 - q^{-1}\xi)} \right] \neq 0, \tag{30}$$

where T is given by

$$T = T(\theta, \alpha) = \frac{e^{-i\theta} + 1 - 2\alpha}{2(1 - \alpha)}, \quad (0 \leq \alpha < 1). \tag{31}$$

Theorem 2. If $g(\xi) \in \mathcal{A}$ has the series form (1), then $g(\xi) \in \tilde{\mathcal{C}}_q[L, M]$ if and only if

$$\frac{1}{\xi} \left[g(\xi) * \frac{\xi - [R(q + q^{-1}) + 1](q - 1 + q^{-1})\xi^2 + [R(q + 2 + q^{-1}) - 1](q - 1 + q^{-1})\xi^3 - (2R - 1)\xi^4}{(1 - q^2\xi)(1 - q\xi)(1 - \xi)(1 - q^{-1}\xi)(1 - q^{-2}\xi)} \right] \neq 0, \quad (32)$$

where R is given by (21).

Proof. From relation (14), we have

$$g(\xi) \in \tilde{\mathcal{C}}_q[L, M] \Leftrightarrow \xi \tilde{D}_q g(\xi) \in \tilde{\mathcal{S}}_q[L, M]. \quad (33)$$

Then, according to Theorem 1, we obtain

$$g(\xi) \in \tilde{\mathcal{C}}_q[L, M] \Leftrightarrow \frac{1}{\xi} [\xi \tilde{D}_q g(\xi) * \phi(\xi)] \neq 0, \quad (34)$$

where $\phi(\xi)$ is given by

$$\phi(\xi) = \frac{\xi - [1 + R(q - 1 + q^{-1})]\xi^2 + R\xi^3}{(1 - q\xi)(1 - \xi)(1 - q^{-1}\xi)}, \quad (35)$$

and we note that

$$\xi \tilde{D}_q \phi(\xi) = \frac{\xi - [R(q + q^{-1}) + 1](q - 1 + q^{-1})\xi^2 + [R(q + 2 + q^{-1}) - 1](q - 1 + q^{-1})\xi^3 - (2R - 1)\xi^4}{(1 - q^2\xi)(1 - q\xi)(1 - \xi)(1 - q^{-1}\xi)(1 - q^{-2}\xi)}. \quad (36)$$

Using relation (36) and the following identity:

$$\xi \tilde{D}_q g(\xi) * \phi(\xi) = g(\xi) * \xi \tilde{D}_q \phi(\xi), \quad (g, \phi \in \mathcal{A}), \quad (37)$$

it is easy to check that (34) is equivalent to (32). Thus, the proof of Theorem 2 is completed.

Putting $L = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $M = -1$ in Theorem 2, we obtain the following result for $\tilde{\mathcal{C}}_q(\alpha)$. \square

Corollary 2. If $g(\xi) \in \mathcal{A}$ has the series form (1), then $g(\xi) \in \tilde{\mathcal{C}}_q(\alpha)$ ($0 \leq \alpha < 1$) if and only if

$$\frac{1}{\xi} \left[g(\xi) * \frac{\xi - [T(q + q^{-1}) + 1](q - 1 + q^{-1})\xi^2 + [T(q + 2 + q^{-1}) - 1](q - 1 + q^{-1})\xi^3 - (2T - 1)\xi^4}{(1 - q^2\xi)(1 - q\xi)(1 - \xi)(1 - q^{-1}\xi)(1 - q^{-2}\xi)} \right] \neq 0, \quad (38)$$

where T is given by (31).

Theorem 3. If $g(\xi) \in \mathcal{A}$ has the series form (1), then $g(\xi) \in \tilde{\mathcal{S}}_q[L, M]$ if and only if

$$1 - \sum_{j=2}^{\infty} \frac{([\widetilde{j}]_q - 1)(e^{-i\theta} + M) - L + M}{L - M} b_j \xi^{j-1} \neq 0. \quad (39)$$

Proof. From Theorem 1, we find that $g(\xi) \in \tilde{\mathcal{S}}_q[L, M]$ if and only if

$$\frac{1}{\xi} \left[g(\xi) * \frac{\xi - [1 + R(q - 1 + q^{-1})]\xi^2 + R\xi^3}{(1 - q\xi)(1 - \xi)(1 - q^{-1}\xi)} \right] \neq 0, \quad (40)$$

for all R given by (21). The left hand side of (38) can be written as

$$\begin{aligned} & \frac{1}{\xi} \left[g(\xi) * \left(\frac{R\xi}{(1 - \xi)} - \frac{(R - 1)\xi}{(1 - q\xi)(1 - q^{-1}\xi)} \right) \right] \\ &= \frac{1}{\xi} [Rg(\xi) - (R - 1)\xi \tilde{D}_q g(\xi)] \\ &= 1 - \sum_{j=2}^{\infty} ([\widetilde{j}]_q (R - 1) - R) b_j \xi^{j-1} \\ &= 1 - \sum_{j=2}^{\infty} \frac{([\widetilde{j}]_q - 1)(e^{-i\theta} + M) - L + M}{L - M} b_j \xi^{j-1}. \end{aligned} \quad (41)$$

Thus, the proof of Theorem 3 is completed.

Putting $L = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $M = -1$ in Theorem 3, we obtain the following result for $\tilde{\mathcal{S}}_q(\alpha)$. \square

Corollary 3. If $g(\xi) \in \mathcal{A}$ has the series form (1), then $g(\xi) \in \tilde{\mathcal{S}}_q(\alpha)$ if and only if

$$1 - \sum_{j=2}^{\infty} \frac{(\widetilde{[j]}_q - 1)(e^{-i\theta} - 1) - 2(1 - \alpha)}{2(1 - \alpha)} b_j \xi^{j-1} \neq 0. \quad (42)$$

Theorem 4. If $g(\xi) \in \mathcal{A}$ has the series form (1), then $g(\xi) \in \tilde{\mathcal{C}}_q[L, M]$ if and only if

$$1 - \sum_{j=2}^{\infty} \frac{(\widetilde{[j]}_q - 1)(e^{-i\theta} + M) - L + M}{L - M} b_j \xi^{j-1} \neq 0 \quad (\xi \in \Delta). \quad (43)$$

Proof. From Theorem 1, we find that $g(\xi) \in \tilde{\mathcal{C}}_q[L, M]$ if and only if

$$\frac{1}{\xi} \left[g(\xi) * \frac{\xi - [R(q + q^{-1}) + 1](q - 1 + q^{-1})\xi^2 + [R(q + 2 + q^{-1}) - 1](q - 1 + q^{-1})\xi^3 - (2R - 1)\xi^4}{(1 - q^2\xi)(1 - q\xi)(1 - \xi)(1 - q^{-1}\xi)(1 - q^{-2}\xi)} \right] \neq 0, \quad (44)$$

for all R given by (21). The left hand side of (44) can be written as

$$\begin{aligned} & \frac{1}{\xi} \left[g(\xi) * \left(\frac{R\xi}{(1 - q\xi)(1 - q^{-1}\xi)} - \frac{(R - 1)(\xi + \xi^2)}{(1 - q^2\xi)(1 - \xi)(1 - q^{-2}\xi)(1 - \xi)} \right) \right] \\ &= \frac{1}{\xi} [R\xi \tilde{D}_q g(\xi) - (R - 1)\xi \tilde{D}_q (\xi \tilde{D}_q g(\xi))] \\ &= 1 - \sum_{j=2}^{\infty} \widetilde{[j]}_q (\widetilde{[j]}_q (R - 1) - R) b_j \xi^{j-1} \\ &= 1 - \sum_{j=2}^{\infty} \widetilde{[j]}_q \frac{(\widetilde{[j]}_q - 1)(e^{-i\theta} + M) - L + M}{L - M} b_j \xi^{j-1}, \end{aligned} \quad (45)$$

and this proves Theorem 4.

Putting $L = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $M = -1$ in Theorem 4, we obtain the following convolution result for $\tilde{\mathcal{C}}_q(\alpha)$. \square

Corollary 4. If $g(\xi) \in \mathcal{A}$ has the series form (1), then $g(\xi) \in \tilde{\mathcal{S}}_q(\alpha)$ if and only if

$$1 - \sum_{j=2}^{\infty} \widetilde{[j]}_q \frac{(\widetilde{[j]}_q - 1)(e^{-i\theta} - 1) - 2(1 - \alpha)}{2(1 - \alpha)} b_j \xi^{j-1} \neq 0. \quad (46)$$

Theorem 5. If the function $g(\xi) \in \mathcal{A}$ has the series form (1) satisfying the inequality

$$\sum_{j=2}^{\infty} \{(\widetilde{[j]}_q - 1)(1 - M) + L - M\} |b_j| \leq L - M, \quad (47)$$

then $g(\xi) \in \tilde{\mathcal{S}}_q[L, M]$.

Proof. Hence,

$$\begin{aligned} & \left| 1 - \sum_{j=2}^{\infty} \frac{(\widetilde{[j]}_q - 1)(e^{-i\theta} + M) + M - L}{L - M} b_j \xi^{j-1} \right| \\ & \geq 1 - \left| \sum_{j=2}^{\infty} \frac{(\widetilde{[j]}_q - 1)(e^{-i\theta} + M) + M - L}{L - M} b_j \xi^{j-1} \right| \\ & \geq 1 - \sum_{j=2}^{\infty} \frac{(\widetilde{[j]}_q - 1)(1 - M) + L - M}{L - M} |b_j| > 0. \end{aligned} \quad (48)$$

Thus, inequality (47) holds, and our result follows from Theorem 3.

Putting $L = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $M = -1$ in Theorem 5, we obtain the following result for $\tilde{\mathcal{S}}_q(\alpha)$. \square

Corollary 5. If the function $g(\xi) \in \mathcal{A}$ has the series form (1) satisfying the inequality

$$\sum_{j=2}^{\infty} (\widetilde{[j]}_q - \alpha) |b_j| \leq 1 - \alpha, \quad (49)$$

then $g(\xi) \in \tilde{\mathcal{S}}_q(\alpha)$.

By using arguments and analysis to those in the proof of Theorem 5, we can derive the following theorem.

Theorem 6. If the function $g(\xi) \in \mathcal{A}$ has the series form (1) satisfying the inequality

$$\sum_{j=2}^{\infty} \widetilde{[j]}_q \{(\widetilde{[j]}_q - 1)(1 - M) + L - M\} |b_j| \leq L - M, \quad (50)$$

then $g(\xi) \in \tilde{\mathcal{C}}_q[L, M]$.

Putting $L = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $M = -1$ in Theorem 6, we obtain the following result for $\tilde{\mathcal{C}}_q(\alpha)$.

Corollary 6. If the function $g(\xi) \in \mathcal{A}$ has the series form (1) satisfying the inequality

$$\sum_{j=2}^{\infty} \widetilde{[j]}_q (\widetilde{[j]}_q - \alpha) |b_j| \leq 1 - \alpha, \quad (51)$$

then $g(\xi) \in \tilde{\mathcal{C}}_q(\alpha)$.

3. Fekete–Szegő Inequalities

In this section, we obtain the Fekete–Szegő inequalities for the classes $\tilde{\mathcal{S}}_q[L, M]$ and $\tilde{\mathcal{C}}_q[L, M]$. In order to establish our results, we need the following lemmas.

Lemma 1 (see [17]). If $p(\xi) = 1 + \delta_1 \xi + \delta_2 \xi^2 + \dots$ is a function with positive real part in Δ and ν is a complex number, then

$$|\delta_2 - \nu \delta_1^2| \leq 2 \max\{1; |2\nu - 1|\}. \quad (52)$$

The result is sharp for the functions given by

$$p(\xi) = \frac{1 + \xi^2}{1 - \xi^2}, \quad (53)$$

$$p(\xi) = \frac{1 + \xi}{1 - \xi}.$$

Lemma 2 (see [17]). If $p(\xi) = 1 + \delta_1 \xi + \delta_2 \xi^2 + \dots$ is an analytic function with a positive real part in Δ , then

$$|\delta_2 - \nu \delta_1^2| \leq \begin{cases} -4\nu + 2, & \text{if } \nu \leq 0, \\ 2, & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2, & \text{if } \nu \geq 1. \end{cases} \quad (54)$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(\xi)$ is $((1 + \xi)/(1 - \xi))$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(\xi)$ is $((1 + \xi^2)/(1 - \xi^2))$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p(\xi) = \left(\frac{1 + \lambda}{2} \right) \frac{1 + \xi}{1 - \xi} + \left(\frac{1 - \lambda}{2} \right) \frac{1 - \xi}{1 + \xi}, \quad (0 \leq \lambda \leq 1), \quad (55)$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if $p(\xi)$ is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$.

Also, the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$\begin{aligned} |\delta_2 - \nu \delta_1^2| + \nu |\delta_1|^2 &\leq 2, \quad \left(0 \leq \nu \leq \frac{1}{2}\right), \\ |\delta_2 - \nu \delta_1^2| + (1 - \nu) |\delta_1|^2 &\leq 2, \quad \left(\frac{1}{2} \leq \nu \leq 1\right). \end{aligned} \quad (56)$$

Theorem 7. If $g(\xi)$ defined by (1) belongs to the class $\tilde{\mathcal{S}}_q[L, M]$, then

$$|b_3 - \mu b_2^2| \leq \frac{L - M}{\widetilde{[3]}_q - 1} \max \left\{ 1; \left| M - \frac{L - M}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q - 1}{\widetilde{[2]}_q - 1} \mu \right) \right| \right\}. \quad (57)$$

Proof. If $g(\xi) \in \tilde{\mathcal{S}}_q[L, M]$, then there is a Schwarz function w in Δ such that

$$\frac{\xi \tilde{D}_q g(\xi)}{g(\xi)} = \frac{1 + Lw(\xi)}{1 + Mw(\xi)}. \quad (58)$$

Define the function $p(\xi)$ by

$$p(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} = 1 + \delta_1 \xi + \delta_2 \xi^2 + \dots \quad (59)$$

Since $w(\xi)$ is a Schwarz function, we see that $\Re\{p(\xi)\} > 0$ and $p(0) = 1$. Now, by substituting (59) in (58), we have

$$\frac{\xi \tilde{D}_q g(\xi)}{g(\xi)} = 1 + \frac{(L - M)}{2} \delta_1 \xi + \frac{(L - M)}{2} \left[\delta_2 - \frac{(1 + M)}{2} \delta_1^2 \right] \xi^2 + \dots \quad (60)$$

From the above equation, we obtain

$$\begin{aligned} (\widetilde{[2]}_q - 1) b_2 &= \frac{(L - M)}{2} \delta_1, \\ (\widetilde{[3]}_q - 1) b_3 - (\widetilde{[2]}_q - 1) b_2^2 &= \frac{(L - M)}{2} \left[\delta_2 - \frac{(1 + M)}{2} \delta_1^2 \right], \end{aligned} \quad (61)$$

or, equivalently,

$$\begin{aligned} b_2 &= \frac{(L - M)}{2(\widetilde{[2]}_q - 1)} \delta_1, \\ b_3 &= \frac{(L - M)}{2(\widetilde{[3]}_q - 1)} \left[\delta_2 - \frac{1}{2} \left(1 + M - \frac{(L - M)}{\widetilde{[2]}_q - 1} \right) \delta_1^2 \right]. \end{aligned} \quad (62)$$

Therefore, we have

$$b_3 - \mu b_2^2 = \frac{L-M}{2(\widetilde{[3]}_q - 1)} \{\delta_2 - \nu \delta_1^2\}, \quad (63)$$

where

$$\nu = \frac{1}{2} \left[1 + M - \frac{L-M}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q - 1}{\widetilde{[2]}_q - 1} \mu \right) \right]. \quad (64)$$

Our result now follows from Lemma 1. This completes the proof of Theorem 1.

Similarly, we can prove the following theorem for the class $\widetilde{\mathcal{C}}_q[L, M]$. \square

Theorem 8. *If $g(\xi)$ given by (1) belongs to the class $\widetilde{\mathcal{C}}_q[L, M]$, then*

$$|b_3 - \mu b_2^2| \leq \frac{L-M}{\widetilde{[3]}_q(\widetilde{[3]}_q - 1)} \max \left\{ 1; \left| M - \frac{L-M}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q(\widetilde{[3]}_q - 1)}{\widetilde{[2]}_q^2(\widetilde{[2]}_q - 1)} \mu \right) \right| \right\}. \quad (65)$$

The result is sharp.

Putting $L = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $M = -1$ in Theorems 7 and 8, we obtain the following corollaries.

Corollary 7. *If $g(\xi)$ given by (1) belongs to the class $\widetilde{\mathcal{S}}_q(\alpha)$ ($0 \leq \alpha < 1$), then*

$$|b_3 - \mu b_2^2| \leq \frac{2(1-\alpha)}{\widetilde{[3]}_q - 1} \max \left\{ 1; \left| 1 + \frac{2(1-\alpha)}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q - 1}{\widetilde{[2]}_q - 1} \mu \right) \right| \right\}. \quad (66)$$

Corollary 8. *If $g(\xi)$ given by (1) belongs to the class $\widetilde{\mathcal{C}}_q(\alpha)$ ($0 \leq \alpha < 1$), then*

$$|b_3 - \mu b_2^2| \leq \frac{2(1-\alpha)}{\widetilde{[3]}_q(\widetilde{[3]}_q - 1)} \max \left\{ 1; \left| 1 + \frac{2(1-\alpha)}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q(\widetilde{[3]}_q - 1)}{\widetilde{[2]}_q^2(\widetilde{[2]}_q - 1)} \mu \right) \right| \right\}. \quad (67)$$

The results are sharp.

Theorem 9. *Let*

$$\begin{aligned} \sigma_1 &= \frac{(\widetilde{[2]}_q - 1)(L-M) - (\widetilde{[2]}_q - 1)^2(1+M)}{(\widetilde{[3]}_q - 1)(L-M)}, \\ \sigma_2 &= \frac{(\widetilde{[2]}_q - 1)(L-M) + (\widetilde{[2]}_q - 1)^2(1-M)}{(\widetilde{[3]}_q - 1)(L-M)}, \\ \sigma_3 &= \frac{(\widetilde{[2]}_q - 1)(L-M) - (\widetilde{[2]}_q - 1)^2 M}{(\widetilde{[3]}_q - 1)(L-M)}. \end{aligned} \quad (68)$$

If g given by (1) belongs to the class $\widetilde{\mathcal{S}}_q[L, M]$, then

$$|b_3 - \mu b_2^2| \leq \begin{cases} -\frac{L-M}{[\overline{3}]_q-1} \left[M - \frac{L-M}{[\overline{2}]_q-1} \left(1 - \frac{[\overline{3}]_q-1}{[\overline{2}]_q-1} \mu \right) \right], & (\mu \leq \sigma_1), \\ \frac{L-M}{[\overline{3}]_q-1}, & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{L-M}{[\overline{3}]_q-1} \left[M - \frac{L-M}{[\overline{2}]_q-1} \left(1 - \frac{[\overline{3}]_q-1}{[\overline{2}]_q-1} \mu \right) \right], & (\mu \geq \sigma_2). \end{cases} \quad (69)$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|b_3 - \mu b_2^2| + \frac{([\overline{2}]_q-1)^2}{([\overline{3}]_q-1)} \left[\frac{1+M}{L-M} - \frac{1}{[\overline{2}]_q-1} \left(1 - \frac{[\overline{3}]_q-1}{[\overline{2}]_q-1} \mu \right) \right] |b_2|^2 \leq \frac{L-M}{[\overline{3}]_q-1}. \quad (70)$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|b_3 - \mu b_2^2| + \frac{([\overline{2}]_q-1)^2}{([\overline{3}]_q-1)} \left[\frac{1-M}{L-M} + \frac{1}{[\overline{2}]_q-1} \left(1 - \frac{[\overline{3}]_q-1}{[\overline{2}]_q-1} \mu \right) \right] |b_2|^2 \leq \frac{L-M}{[\overline{3}]_q-1}. \quad (71)$$

Proof. Applying Lemma 2 to (63) and (64), we can obtain our results asserted by Theorem 9.

Putting $L = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $M = -1$ in Theorem 9, we obtain the following result. \square

Corollary 9. Let

$$\begin{aligned} \sigma_4 &= \frac{[\overline{2}]_q-1}{[\overline{3}]_q-1}, \\ \sigma_5 &= \frac{([\overline{2}]_q-1)(1-\alpha) + ([\overline{2}]_q-1)^2}{([\overline{3}]_q-1)(1-\alpha)}, \\ \sigma_6 &= \frac{2([\overline{2}]_q-1)(1-\alpha) + ([\overline{2}]_q-1)^2}{2([\overline{3}]_q-1)(1-\alpha)}. \end{aligned} \quad (72)$$

If g given by (1) belongs to the class $\tilde{\mathcal{F}}_q(\alpha)$ ($0 \leq \alpha < 1$), then

$$|b_3 - \mu b_2^2| \leq \begin{cases} \frac{2(1-\alpha)}{[\widetilde{3}]_q - 1} \left[1 + \frac{2(1-\alpha)}{[\widetilde{2}]_q - 1} \left(1 - \frac{[\widetilde{3}]_q - 1}{[\widetilde{2}]_q - 1} \mu \right) \right], & (\mu \leq \sigma_4), \\ \frac{2(1-\alpha)}{[\widetilde{3}]_q - 1}, & (\sigma_4 \leq \mu \leq \sigma_5), \\ -\frac{2(1-\alpha)}{[\widetilde{3}]_q - 1} \left[1 + \frac{2(1-\alpha)}{[\widetilde{2}]_q - 1} \left(1 - \frac{[\widetilde{3}]_q - 1}{[\widetilde{2}]_q - 1} \mu \right) \right], & (\mu \geq \sigma_5). \end{cases} \quad (73)$$

Further, if $\sigma_4 \leq \mu \leq \sigma_6$, then

$$|b_3 - \mu b_2^2| - \frac{[\widetilde{2}]_q - 1}{[\widetilde{3}]_q - 1} \left(1 - \frac{[\widetilde{3}]_q - 1}{[\widetilde{2}]_q - 1} \mu \right) |b_2|^2 \leq \frac{2(1-\alpha)}{[\widetilde{3}]_q - 1}. \quad (74)$$

If $\sigma_6 \leq \mu \leq \sigma_5$, then

$$|b_3 - \mu b_2^2| + \frac{([\widetilde{2}]_q - 1)^2}{([\widetilde{3}]_q - 1)} \left[\frac{1}{(1-\alpha)} + \frac{1}{[\widetilde{2}]_q - 1} \left(1 - \frac{[\widetilde{3}]_q - 1}{[\widetilde{2}]_q - 1} \mu \right) \right] |b_2|^2 \leq \frac{2(1-\alpha)}{[\widetilde{3}]_q - 1}. \quad (75)$$

Similarly, we can obtain the following theorem.

If g given by (1) belongs to the class $\tilde{\mathcal{E}}_q[L, M]$, then

Theorem 10. Let

$$\begin{aligned} \chi_1 &= \frac{[\widetilde{2}]_q^2 ([\widetilde{2}]_q - 1) [L - M - ([\widetilde{2}]_q - 1)(1 + M)]}{([\widetilde{3}]_q - 1)(L - M)}, \\ \chi_2 &= \frac{[\widetilde{2}]_q^2 ([\widetilde{2}]_q - 1) [L - M + ([\widetilde{2}]_q - 1)(1 - M)]}{([\widetilde{3}]_q - 1)(L - M)}, \\ \chi_3 &= \frac{[\widetilde{2}]_q^2 ([\widetilde{2}]_q - 1) [L - M - ([\widetilde{2}]_q - 1)M]}{([\widetilde{3}]_q - 1)(L - M)}. \end{aligned} \quad (76)$$

$$|b_3 - \mu b_2^2| = \begin{cases} -\frac{L - M}{[\widetilde{3}]_q ([\widetilde{3}]_q - 1)} \left[M - \frac{L - M}{[\widetilde{2}]_q - 1} \left(1 - \frac{[\widetilde{3}]_q ([\widetilde{3}]_q - 1)}{[\widetilde{2}]_q^2 ([\widetilde{2}]_q - 1)} \mu \right) \right], & (\mu \leq \chi_1), \\ \frac{L - M}{[\widetilde{3}]_q ([\widetilde{3}]_q - 1)}, & (\chi_1 \leq \mu \leq \chi_2), \\ \frac{L - M}{[\widetilde{3}]_q ([\widetilde{3}]_q - 1)} \left[M - \frac{L - M}{[\widetilde{2}]_q - 1} \left(1 - \frac{[\widetilde{3}]_q ([\widetilde{3}]_q - 1)}{[\widetilde{2}]_q^2 ([\widetilde{2}]_q - 1)} \mu \right) \right], & (\mu \geq \chi_2). \end{cases} \quad (77)$$

Further, if $\chi_1 \leq \mu \leq \chi_3$, then

$$|b_3 - \mu b_2^2| + \frac{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)^2}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)} \left[\frac{1+M}{L-M} - \frac{1}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)} \mu \right) \right] |b_2|^2 \leq \frac{L-M}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}. \quad (78)$$

If $\chi_3 \leq \mu \leq \chi_2$, then

$$|b_3 - \mu b_2^2| + \frac{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)^2}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)} \left[\frac{1-M}{L-M} + \frac{1}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)} \mu \right) \right] |b_2|^2 \leq \frac{L-M}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}. \quad (79)$$

Putting $L = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $M = -1$ in Theorem 10, we obtain the following result.

If g given by (1) belongs to the class $\widetilde{\mathcal{G}}_q(\alpha)$ ($0 \leq \alpha < 1$), then

Corollary 10. *Let*

$$\begin{aligned} \chi_4 &= \frac{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)}{\widetilde{[3]}_q - 1}, \\ \chi_5 &= \frac{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1) [1 - \alpha + (\widetilde{[2]}_q - 1)]}{(\widetilde{[3]}_q - 1)(1 - \alpha)}, \\ \chi_6 &= \frac{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1) [2(1 - \alpha) + (\widetilde{[2]}_q - 1)]}{2(\widetilde{[3]}_q - 1)(1 - \alpha)}. \end{aligned} \quad (80)$$

$$|b_3 - \mu b_2^2| \leq \begin{cases} \frac{2(1-\alpha)}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)} \left[1 + \frac{2(1-\alpha)}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)} \mu \right) \right], & (\mu \leq \chi_4), \\ \frac{2(1-\alpha)}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}, & (\chi_4 \leq \mu \leq \chi_5), \\ -\frac{2(1-\alpha)}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)} \left[1 + \frac{2(1-\alpha)}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)} \mu \right) \right], & (\mu \geq \chi_5). \end{cases} \quad (81)$$

Further, if $\chi_4 \leq \mu \leq \chi_6$, then

$$|b_3 - \mu b_2^2| - \frac{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)} \left(1 - \frac{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)} \mu \right) |b_2|^2 \leq \frac{2(1-\alpha)}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}. \quad (82)$$

If $\chi_6 \leq \mu \leq \chi_5$, then

$$|b_3 - \mu b_2^2| + \frac{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)^2}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)} \left[\frac{1}{1-\alpha} + \frac{1}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}{\widetilde{[2]}_q^2 (\widetilde{[2]}_q - 1)} \mu \right) \right] |b_2|^2 \leq \frac{2(1-\alpha)}{\widetilde{[3]}_q (\widetilde{[3]}_q - 1)}. \quad (83)$$

4. Conclusion

In this present investigation, we have introduced two classes $\mathcal{S}_q[L, M]$ and $\mathcal{C}_q[L, M]$ of analytic functions by using the symmetric q -difference operator linked to an open unit disc $\Delta = \{\xi \in \mathbb{C}: |\xi| < 1\}$. We also studied convolution results, coefficient estimates, and Fekete–Szegő inequalities for the newly defined classes. We note that our results naturally include several results that are known for those subclasses, which are listed in the introduction section.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Essential Norms of Stević–Sharma Operators from General Banach Spaces into Zygmund-Type Spaces

M. A. Bakhit 

Department of Mathematics, Faculty of Science, Jazan University, Jazan 45142, Saudi Arabia

Correspondence should be addressed to M. A. Bakhit; mabakhit@jazanu.edu.sa

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A Stević–Sharma operator denoted by $T_{\psi_1, \psi_2, \varphi}$ is a generalization product of multiplication, differentiation, and composition operators. Using several restrictive terms, we characterize an approximation of the essential norm of the Stević–Sharma operator $T_{\psi_1, \psi_2, \varphi}$ from a general class X of holomorphic function spaces into Zygmund-type spaces with some of the most convenient test functions on the open unit disk. As an application, we show that our results hold up for several other domain spaces of $T_{\psi_1, \psi_2, \varphi}$, such as the Hardy space and the weighted Bergman space.

1. Introduction

Let $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the open unit disk of the complex plane \mathbb{C} . We denote by $\mathcal{HO}(\mathbb{D})$ the family of analytic functions in \mathbb{D} ; also, we denote by $\mathcal{AS}(\mathbb{D})$ the family of analytic self-maps of \mathbb{D} , and $\text{Aut}(\mathbb{D})$ the set of conformal automorphisms of the disk \mathbb{D} . As usual, the Banach space H^∞ is the family of bounded functions $h \in \mathcal{HO}(\mathbb{D})$, defined by the norm $\|h\|^\infty = \sup_{\zeta \in \mathbb{D}} |h(\zeta)| < \infty$. For any weighted

function μ , the weighted Banach space $H_{\mu,0}^\infty$ is the space of functions $h \in \mathcal{HO}(\mathbb{D})$ such that

$$\|h\|_\mu^\infty = \sup_{\zeta \in \mathbb{D}} |h(\zeta)|\mu(\zeta) < \infty. \quad (1)$$

Moreover, the little weighted Banach space $H_{\mu,0}^\infty$ is the space of functions $h \in \mathcal{HO}(\mathbb{D})$ such that

$$\lim_{|\zeta| \rightarrow 1^-} |h(\zeta)|\mu(\zeta) = 0. \quad (2)$$

The Bloch-type space \mathcal{B}_μ is the Banach space of functions $h \in \mathcal{HO}(\mathbb{D})$ such that $h' \in H_{\mu,0}^\infty$, with the norm $\|h\|_{\mathcal{B}_\mu} = |h(0)| + \|h'\|_\mu^\infty$. The little Bloch-type space $\mathcal{B}_{\mu,0}$ is the closed subspace of \mathcal{B}_μ consisting of functions $h \in \mathcal{HO}(\mathbb{D})$ such that $h' \in H_{\mu,0}^\infty$.

The space of all functions $h \in \mathcal{HO}(\mathbb{D})$ is said to be a weighted Zygmund-type space \mathcal{Z}_μ if all functions h are such

that $h'' \in H_{\mu,0}^\infty$. The space \mathcal{Z}_μ becomes a Banach space with the norm

$$\|h\|_{\mathcal{Z}_\mu} = |h(0)| + |h'(0)| + \sup_{\zeta \in \mathbb{D}} |h''(\zeta)|\mu(\zeta). \quad (3)$$

The little weighted Zygmund-type space $\mathcal{Z}_{\mu,0}$ is the closed subspace of \mathcal{Z}_μ and consists of functions $h \in \mathcal{HO}(\mathbb{D})$, such that $\lim_{|\zeta| \rightarrow 1^-} |h''(\zeta)|\mu(\zeta) = 0$.

Exactly as in the aforementioned spaces, when $\mu(\zeta) = (1 - |\zeta|^2)^d$, we get the Zygmund-type space \mathcal{Z}^d . In addition, the \mathcal{Z}^d space is a Banach space with the norm $|h(0)| + |h'(0)| + \|h\|_{\mathcal{Z}^d}$, where $\|h\|_{\mathcal{Z}^d} = \sup_{\zeta \in \mathbb{D}} |h''(\zeta)|(1 - |\zeta|^2)^d < \infty$. The Zygmund spaces also satisfy $\mathcal{Z}^d \subsetneq \mathcal{Z}^l$ whenever $0 < d < l$. When $d = 1$, we get the classical Zygmund space denoted by \mathcal{Z} . In fact, the motivation for the name and for studying the Zygmund-type spaces comes from the Zygmund class; see, for example, Chapter 5 of Duren's book [1]. The Zygmund space and its subspace \mathcal{Z}_0 play an important role in connection to the theory of the Hardy spaces H^q , ($0 < q < 1$). Indeed, in [2], it was shown that \mathcal{Z} can be viewed as the dual of the Hardy space H^q , $q = 1/2$. More generally, the spaces defined by replacing the function h in the definition of \mathcal{Z} with its n th derivative, under the assumption $h, h', \dots, h^{(n)} \in \mathcal{HO}(\mathbb{D}) \cap \mathcal{C}(\mathbb{D})$ for $n \geq 1$, can be viewed as duals of the Hardy spaces H^q , $q = 1/n + 2$. A recent study of several operators on

Zygmund-type spaces has attracted significant research attention; see, for example, [3–10].

For any $\Psi \in \mathcal{HO}(\mathbb{D})$ and $\varphi \in \mathcal{AS}(\mathbb{D})$, we have the following linear operators:

$Dh(\zeta) = h'(\zeta)$, the differentiation operator

$M_\Psi h(\zeta) = \Psi(\zeta)h(\zeta)$, the multiplication operator

$C_\varphi h(\zeta) = (h \circ \varphi)(\zeta) = h(\varphi(\zeta))$, the composition operator

$W_{\Psi, \varphi} h(\zeta) = \Psi(\zeta) \cdot h(\varphi(\zeta))$, the weighted composition operator

For a unified manner of treatment of these operators, many researchers seek to present various products of multiplication, composition, and differentiation operators; see, for example, [8, 11–18]. Stević et al. was the first to introduce the operator $T_{\psi_1, \psi_2, \varphi}$ in [19]. So, for any $\psi_1, \psi_2 \in \mathcal{HO}(\mathbb{D})$ and $\varphi \in \mathcal{AS}(\mathbb{D})$, the operator $T_{\psi_1, \psi_2, \varphi}$ is called the Stević–Sharma operator, which is defined as

$$T_{\psi_1, \psi_2, \varphi} h(\zeta) = \psi_1(\zeta)h(\varphi(\zeta)) + \psi_2(\zeta)h'(\varphi(\zeta)), \quad h \in \mathcal{HO}(\mathbb{D}). \quad (4)$$

Over the past 10 years, the boundedness and compactness of this operator have been studied extensively in the most well-known spaces of holomorphic functions; for example, see [3, 19–27].

This paper proceeds as follows. In Section 2, we present basic facts and prerequisites required for the general class X of holomorphic function spaces. In Section 3, we characterize upper and lower bounds for the essential norm of the $T_{\psi_1, \psi_2, \varphi}$ operators from X into the Zygmund-type space \mathcal{Z}_μ (or $\mathcal{Z}_{\mu, 0}$). Finally, in Section 4, we show that our estimations withstand any choice space, provided that it is reflexive, such as the Hardy spaces H^q , ($1 \leq q \leq \infty$), and the weighted Bergman space A_β^q ($\beta > -1$ and $1 \leq q < \infty$). This work is a continuation of our not yet published article regarding the boundedness and compactness of $T_{\psi_1, \psi_2, \varphi}$ between the currently considered spaces; see [20].

In this paper, for any two quantities $Q_1(h)$ and $Q_2(h)$ that are jointly dependent on $h \in \mathcal{HO}(\mathbb{D})$, we stipulate that $Q_1(h) \lesssim Q_2(h)$, meaning that there is a positive constant C that fulfills $Q_1(h) \leq CQ_2(h)$. Thus, when $Q_2(h) \lesssim Q_1(h) \lesssim Q_2(h)$, we hold that $Q_1(h) \approx Q_2(h)$ and the quantities $Q_1(h)$ and $Q_2(h)$ are said to be equivalent. If $Q_1(h) \approx Q_2(h)$, then $Q_2(h) < \infty$ if and only if $Q_1(h) < \infty$.

2. Preliminaries

Following popular terminology in functional analysis, we stipulate a Banach space X (or Y) whose elements are functions $h \in \mathcal{HO}(\mathbb{D})$ with the norm $\|h\|_X$ and whose the functionals of point-evaluation are bounded.

Let E_ζ indicate the functional of point-evaluation at ζ . So, for any $\zeta \in \mathbb{D}$, we can define the norm

$$\Lambda_X(\zeta) = \sup\{|h(\zeta)| : h \in X, \|h\|_X \leq 1\} = \|E\|_{\zeta, X}. \quad (5)$$

For each $h \in X$ and $\zeta \in \mathbb{D}$, we obtain

$$|h(\zeta)| \leq \|h\|_X \Lambda_X(\zeta). \quad (6)$$

The following sets of conditions on Banach space X , given in [28], include the conditions used to formulate the results of the present work. For all $s \in (0, 1)$ and $\zeta \in \mathbb{D}$, if $h \in X$, we obtain

(A₁) $\Lambda_X(s\zeta) \leq \Lambda_X(\zeta)$, for all $s \in (0, 1)$.

(A₂) By a positive constant δ , the norm $\Lambda_X(\zeta)$ is bounded below in compact subsets of \mathbb{D} and $\lim_{|\zeta| \rightarrow 1} \Lambda_X(\zeta) = \infty$.

(A₃) With respect to the uniform convergence topology on compact subsets of \mathbb{D} , the unit ball \mathbb{B}_X of a Banach space X is comparatively compact.

(A₄) For all $h \in X$ and $\mathcal{S} \in \text{Aut}(\mathbb{D})$, we have $\mathcal{S}^k h \in X$ and $\|\mathcal{S}^k h\|_X \leq \|h\|_X$, where $k = 1, 2, 3$.

(A₅) For $s \in (0, 1)$, let $h_s(\zeta) = h(s\zeta)$, $\zeta \in \mathbb{D}$, where $h \in X$. Then, the linear map $\mathcal{E}_s : h \rightarrow h_s$ is a compact map on X .

(A₆) For $s \in (0, 1)$, $\sup_{0 < s < 1} \|\mathcal{E}_s\|_{s, X} < \infty$.

(A₇) For all $\zeta \in \mathbb{D}$, let $h \in X$, then $(1 - |\zeta|^2)^k |h^{(k)}(\zeta)| \leq \|h\|_X \Lambda_X(\zeta)$, where $k = 1, 2, 3$.

If a Banach space X contains polynomials and satisfies the conditions (A₃), (A₄), and (A₇), then X is said to be an admissible space (see [28–30]). If the set of polynomials in an admissible space X is dense, it is said to be polynomial dense. Examples of a polynomial dense space include the Hardy spaces H^q , ($1 \leq q \leq \infty$) and the weighted Bergman space A_β^q for all $\beta > -1$ and $1 \leq q < \infty$ (see [31]).

The following proposition is Proposition 1 in [28].

Proposition 1. For $\zeta \in \mathbb{D}$, the map $\zeta \rightarrow \Lambda_X(\zeta)$ is bounded on compact subsets of \mathbb{D} in a Banach space X . Further, if a Banach space X is reflexive, then \mathbb{B}_X is compact and shows relatively of uniform convergence with respect to the topology on a compact subset of \mathbb{D} .

The following lemma helps to distinguish the properties of the operator $T_{\psi_1, \psi_2, \varphi}$.

Lemma 1. Let there be a Banach space X satisfying the above conditions (A₄) and (A₇). Suppose that $\varphi \in \mathcal{AS}(\mathbb{D})$, then for any $w \in \mathbb{D}$ and $h \in X$, such that $\|h\|_X \leq 1$, there is a set of functions $F_{w,k}$ defined in X , for $k = 1, 2, 3$, such that $\sup_{w \in \mathbb{D}} \|F_{w,k}\|_X < \infty$,

$$F_{w,k}^{(j)}(\varphi(w)) = 0, \quad \text{if } k > j, \quad (7)$$

and

$$F_{w,k}^{(j)}(\varphi(w)) = \frac{k!(-1)^k h(\varphi(w))}{(1 - |\varphi(w)|^2)^j}, \quad \text{if } k = j. \quad (8)$$

Moreover, the sequences $\{F_{w,k}\}$ converge to 0 uniformly on compact subsets of \mathbb{D} , when $|w| \rightarrow 1$.

Proof. Let $\rho_b(\zeta) = b - \zeta/1 - \bar{b}\zeta$, which is the automorphism of \mathbb{D} that changes zero and b , that is, $\rho_b \in \text{Aut}(\mathbb{D})$, for any $\zeta \in \mathbb{D}$. Note that

$$\rho'_b(b) = \frac{-1}{1-|b|^2}, \quad \rho''_b(b) = \frac{-2\bar{b}}{(1-|b|)^{22}}, \quad (9)$$

and

$$\rho_b^{(3)}(b) = \frac{4(\bar{b})^2}{(1-|b|^2)^3}. \quad (10)$$

Now, fix $h \in X$, such that $\|h\|_X \leq 1$ and consider the functions $F_{w,k}(w) = (\rho_{\varphi(w)}(w))^k h(w)$, for all $k \in \{1, 2, 3\}$ and $w \in \mathbb{D}$. Let $\mathcal{S} = \rho_{\varphi(w)}$ in condition (A_4) , then for all $k = 1, 2, 3$, we have $F_{w,k} \in X$ and

$$\|F_{w,k}\|_X = \|\mathcal{S}^k h\|_X \leq \|h\|_X \leq 1. \quad (11)$$

Also, it is not difficult to prove that

$$\begin{aligned} F_{w,1}(\varphi(w)) &= 0; \\ F'_{w,1}(\varphi(w)) &= \frac{-h(\varphi(w))}{1-|\varphi(w)|^2}; \\ F_{w,2}(\varphi(w)) &= F'_{w,2}(\varphi(w)) = 0; \\ F''_{w,2}(\varphi(w)) &= \frac{2h(\varphi(w))}{(1-|\varphi(w)|^2)^2}; \\ F_{w,3}(\varphi(w)) &= F'_{w,3}(\varphi(w)) = F''_{w,3}(\varphi(w)) = 0; \\ F^{(3)}_{w,3}(\varphi(w)) &= \frac{-6h(\varphi(w))}{(1-|\varphi(w)|^2)^3}. \end{aligned} \quad (12)$$

Finally, the uniform convergence to 0 of the sequences $\{F_{w,k}\}$ is self-evident when $|w| \rightarrow 1$. \square

This work is a continuation of [20], where we describe the boundedness and the compactness of $T_{\psi_1, \psi_2, \varphi}: X \rightarrow \mathcal{X}_\mu$. The following boundedness result has been proven.

$$\begin{aligned} \widetilde{K}_0(\mu, \psi, \varphi) &= \sup_{w \in \mathbb{D}} \mu(w) |\psi'_1(w) \Lambda_X(\varphi(w))|, \\ \widetilde{K}_1(\mu, \psi, \varphi) &= \sup_{w \in \mathbb{D}} \mu(w) \frac{|2\psi'_1(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{(1-|\varphi(w)|^2)} \Lambda_X(\varphi(w)), \\ \widetilde{K}_2(\mu, \psi, \varphi) &= \sup_{w \in \mathbb{D}} \mu(w) \frac{|\psi_1(w)\varphi'^2(w) + 2\psi'_2(w)\varphi'(w) + \psi_2(w)\varphi''(w)|}{(1-|\varphi(w)|^2)^2} \Lambda_X(\varphi(w)), \\ \widetilde{K}_3(\mu, \psi, \varphi) &= \sup_{w \in \mathbb{D}} \mu(w) \frac{|\psi_2(w)\varphi'^2(w)|}{(1-|\varphi(w)|^2)^3} \Lambda_X(\varphi(w)). \end{aligned} \quad (14)$$

Theorem 1. Let there be a Banach space X satisfying the above conditions (A_4) and (A_7) . Then, $T_{\psi_1, \psi_2, \varphi}: X \rightarrow \mathcal{X}_\mu$ is a bounded operator if and only if all the quantities $\widetilde{K}_j(\mu, \psi, \varphi)$ are finite, for $j = 0, 1, 2, 3$. Moreover, in which case

$$\|T_{\psi_1, \psi_2, \varphi}\|_{X \rightarrow \mathcal{X}_\mu} \approx \sum_{j=0}^3 \widetilde{K}_j(\mu, \psi, \varphi), \quad (13)$$

where the quantities $\widetilde{K}_j(\mu, \psi, \varphi)$, ($j = 0, 1, 2, 3$) are defined as follows:

The following lemma is proved in a standard way; see, for example [11, 28].

Lemma 2. Suppose that X, Y are two Banach spaces satisfying the following conditions:

- (B₁) The point evaluation functionals E_ζ are continuous on X
- (B₂) With respect to the uniform convergence topology on compact subsets of \mathbb{D} , the closed unit ball \overline{B}_X of a Banach space X is comparatively compact
- (B₃) The operator T is continuous from X into Y when X and Y are given the topology of uniform convergence on compact sets

Then, the bounded operator $T: X \rightarrow Y$ is compact if and only if for any bounded sequence $\{h_m\}$ in X converges uniformly to 0 as $m \rightarrow \infty$ on compact subsets of \mathbb{D} , and we have $\lim_{m \rightarrow \infty} \|Th_m\|_Y = 0$.

We know that the essential norm $\|T\|_e$ of an operator T is its distance from the compact operators in the operator norm. Specifically, let X and Y be two Banach spaces and let $T: X \rightarrow Y$ be a bounded linear operator, then the essential norm of T between X and Y is referred to as $\|T\|_{e, X \rightarrow Y}$, and its definition is

$$\|T\|_{e, X \rightarrow Y} = \inf \{ \|T - \mathcal{T}\|_{X \rightarrow Y} : \mathcal{T}: X \rightarrow Y \text{ is compact} \}. \quad (15)$$

Lemma 3 (see [28]). Let X be a Banach space of functions $h \in \mathcal{HO}(\mathbb{D})$, such that $\|h\|_X \leq 1$. Fix any $\varepsilon > 0, r \in (0, 1)$ and choose $s \in (0, 1)$.

- (a) If the Banach space X includes some functions non-vanishing at zero and fulfill either (A₂) or (A₆), then

$$\sup_{\|h\|_X \leq 1} \sup_{\zeta \in \mathbb{D}} \frac{|(I - \mathcal{E}_s)h(\zeta)|}{\Lambda_X(\zeta)} < \infty. \quad (16)$$

- (b) If the Banach space X satisfies one of the following sets of conditions:

- (i) Both (A₁) and (A₅)
- (ii) (A₅) and X is reflexive
- (iii) (A₇)

then

$$\sup_{\|h\|_X \leq 1} \sup_{|\zeta| \leq r} |(I - \mathcal{E}_s)h(\zeta)| < \varepsilon. \quad (17)$$

- (c) If the Banach space X satisfies (A₂) and one of the conditions in part (b), then

$$\sup_{\|h\|_X \leq 1} \sup_{\zeta \in \mathbb{D}} \frac{|((I - \mathcal{E}_s)h)^{(k)}(\zeta)|}{\Lambda_X(\zeta)} < \varepsilon, \quad \forall k = 1, 2. \quad (18)$$

Remark 1. Fix any $\varepsilon > 0, r \in (0, 1)$ and choose $s \in (0, 1)$. By part (c) of Lemma 3, we have $((I - \mathcal{E}_s)h)^{(3)}$ converging uniformly to 0 on compact subsets as $s \rightarrow 1^-$. Since X satisfies (A₂), under the same conditions, we obtain

$$\sup_{\|h\|_X \leq 1} \sup_{\zeta \in \mathbb{D}} \frac{|((I - \mathcal{E}_s)h)^{(3)}(\zeta)|}{\Lambda_X(\zeta)} < \varepsilon, \quad (19)$$

Remark 2. For any $\varphi \in \mathcal{AS}(\mathbb{D})$, if we suppose that

$$\inf_{w \in \mathbb{D}} \Lambda_X(\varphi(w)) > 0, \quad (20)$$

we observe that assumption (20) is sufficient to apply Lemma 3, while part (c) of Lemma 3 requires condition (A₂).

3. Essential Norm of $T_{\psi_1, \psi_2, \varphi}: X \rightarrow \mathcal{X}_\mu$

Here, we provide an approximation of the essential norm of $T_{\psi_1, \psi_2, \varphi}$ from a large class X of Banach spaces into the Zygmund-type space \mathcal{X}_μ , under specific conditions on class X on the open unit disk. With quantities $K_j(\mu, \psi, \varphi)$ ($j = 0, 1, 2, 3$) defined below, the following result complements Theorem 3 in [32]. To simplify the formulation of the main results in this section, for $\Lambda_X(w)$ as in (5) and $\delta > 0$, we set

$$\begin{aligned} K_0(\mu, \psi, \varphi) &= \lim_{\delta \rightarrow 1} \sup_{|\varphi(w)| > \delta} \mu(w) |\psi_1''(w)| \Lambda_X(\varphi(w)), \\ K_1(\mu, \psi, \varphi) &= \lim_{\delta \rightarrow 1} \sup_{|\varphi(w)| > \delta} \mu(w) \frac{|2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{(1 - |\varphi(w)|^2)} \Lambda_X(\varphi(w)), \\ K_2(\mu, \psi, \varphi) &= \lim_{\delta \rightarrow 1} \sup_{|\varphi(w)| > \delta} \mu(w) \frac{|\psi_1(w)\varphi'^2(w) + 2\psi_2'(w)\varphi'(w) + \psi_2(w)\varphi''(w)|}{(1 - |\varphi(w)|^2)^2} \Lambda_X(\varphi(w)), \\ K_3(\mu, \psi, \varphi) &= \lim_{\delta \rightarrow 1} \sup_{|\varphi(w)| > \delta} \mu(w) \frac{|\psi_2(w)\varphi'^2(w)|}{(1 - |\varphi(w)|^2)^3} \Lambda_X(\varphi(w)). \end{aligned} \quad (21)$$

Theorem 2. Let a Banach space X of functions in $\mathcal{HO}(\mathbb{D})$ be reflexive and satisfy the above conditions (A_4) , (A_5) , and (A_7) , together with either (A_1) or (A_6) . Suppose that $\inf_{w \in \mathbb{D}} \Lambda_X(\varphi(w)) > 0$, if $T_{\psi_1, \psi_2, \varphi}: X \rightarrow \mathcal{F}_\mu$ is a bounded operator. Then,

$$\|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{F}_\mu} \approx \sum_{j=0}^3 K_j(\mu, \psi, \varphi). \quad (22)$$

Proof. First, we prove that,

$$\|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{F}_\mu} \geq r \sin \sum_{j=0}^3 K_j(\mu, \psi, \varphi). \quad (23)$$

If we suppose that $\|\varphi\|^\infty < 1$, then it follows immediately that $K_j(\mu, \psi, \varphi) = 0$ for all $j = 0, 1, 2, 3$. This is why, hereafter, we suppose $\|\varphi\|^\infty = 1$, then we prove that $K_j(\mu, \psi, \varphi) \leq \|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{F}_\mu}$ for all $j = 0, 1, 2, 3$.

For each $m \in \mathbb{N}$ and fix $M > 0$, we choose $h_m \in X$ such that $\|h_m\|_X \leq 1$,

$$|h_m(\varphi(b_m))| > \Lambda_X(\varphi(b_m)) - M. \quad (24)$$

By assuming that X is reflexive, the unit ball \mathbb{B}_X of a Banach space X is compact under the uniform convergence topology on compact subsets of \mathbb{D} . Using the fact that $|h_m(w)| \leq \|h_m\|_X \Lambda_X(w)$, then $\{h_m\}$ is a uniformly bounded sequence on compact sets of \mathbb{D} . For all $w \in \mathbb{D}$ and $m \in \mathbb{N}$, set the functions

$$f_m(w) = \frac{(1 - |\varphi(b_m)|^2)w}{1 - \overline{\varphi(b_m)}w} h_m(w). \quad (25)$$

Then, observe that

$$|f_m(w)| = \frac{1 - |\varphi(b_m)|^2}{1 - |w|} \Lambda_X(w), \quad (26)$$

and

$$f_m(\varphi(b_m)) = \varphi(b_m) h_m(\varphi(b_m)). \quad (27)$$

Clearly $\{f_m\}$ converges uniformly to 0 on compact subsets of \mathbb{D} . By the condition (A_4) , we obtain $f_m \in X$ and $\|f_m\|_X \leq \|h_m\|_X \leq 1$.

Since $f_m \in X$, by the functions defined in Lemma 1, we consider the functions

$$F_{m,k}(b_m) = \left(\rho_\varphi(b_m)(b_m)\right)^k f_m(b_m), \quad \text{for } k = 1, 2, 3. \quad (28)$$

By Lemma 1, observe that

$$F_{m,k}^{(j)}(\varphi(b_m)) = 0, \text{ if } k > j \text{ and}$$

$$F_{m,k}^{(j)}(\varphi(b_m)) = \frac{k!(-1)^k \Lambda_X(\varphi(b_m))}{(1 - |\varphi(b_m)|^2)^j}, \text{ if } k = j. \quad (29)$$

Further, $F_{m,k} \in X$ and $\|F_{m,k}\|_X \leq \|f_m\|_X \leq 1$, for all $k = 1, 2, 3$.

Using the hypothesis that conditions (A_4) and (A_7) hold, then all sequences $\{F_{w,k}\}$ are bounded in a Banach space X . Moreover, the sequences $\{F_{w,k}\}$ converge to 0 uniformly on compact subsets of \mathbb{D} , when $|w| \rightarrow 1$. \square

Step 1. In the case of $j = 3$, let $\{b_m\}$ be a sequence in \mathbb{D} , such that $\lim_{m \rightarrow \infty} |\varphi(b_m)| = 1$ and

$$K_3(\mu, \psi, \varphi) := \lim_{m \rightarrow \infty} \mu(b_m) \frac{|\psi_2(b_m) \varphi'^2(b_m)|}{(1 - |\varphi(b_m)|^2)^3} \Lambda_X(\varphi(b_m)). \quad (30)$$

Let $k = 3$ in (29), then we have

$$\begin{aligned} F_{m,3}(\varphi(b_m)) &= F_{b_m,3}'(\varphi(b_m)) \\ &= F_{m,3}''(\varphi(b_m)) = 0, \end{aligned} \quad (31)$$

$$F_{m,3}^{(3)}(\varphi(b_m)) = \frac{-6f_m(\varphi(b_m))}{(1 - |\varphi(b_m)|^2)^3}.$$

Thus,

$$\begin{aligned} &|(T_{\psi_1, \psi_2, \varphi} F_{m,3})''(b_m)| \\ &= |\psi_2(b_m) \varphi'^2(b_m) F_{m,3}^{(3)}(\varphi(b_m))| \\ &= \frac{|6\psi_2(b_m) \varphi'^2(b_m) f_m(\varphi(b_m))|}{(1 - |\varphi(b_m)|^2)^3}. \end{aligned} \quad (32)$$

Now, supposing that $\mathcal{T}: X \rightarrow \mathcal{F}_\mu$ is a compact operator, then the operator $T_{\psi_1, \psi_2, \varphi} - \mathcal{T}$ is bounded. Therefore, by Proposition 1 and Lemma 2, we obtain

$$\lim_{m \rightarrow \infty} \|\mathcal{T} F_{m,3}\|_{\mathcal{F}_\mu} = 0. \quad (33)$$

By (32) and (33), we have

$$\begin{aligned} &\|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{F}_\mu} \\ &\geq \limsup_{m \rightarrow \infty} \|F_{m,3}\|_X \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_X \\ &\geq \limsup_{m \rightarrow \infty} \|(T_{\psi_1, \psi_2, \varphi} - \mathcal{T}) F_{m,3}\|_{\mathcal{F}_\mu} \\ &= \limsup_{m \rightarrow \infty} \|T_{\psi_1, \psi_2, \varphi} F_{m,3}\|_{\mathcal{F}_\mu} \\ &\geq \limsup_{m \rightarrow \infty} \mu(b_m) |(T_{\psi_1, \psi_2, \varphi} F_{m,3})''(b_m)| \\ &= \limsup_{m \rightarrow \infty} \mu(b_m) \frac{|6\psi_2(b_m) \varphi'^2(b_m) f_m(\varphi(b_m))|}{(1 - |\varphi(b_m)|^2)^3}. \end{aligned} \quad (34)$$

Using (24) and (27), we obtain

$$\begin{aligned}
& \mu(b_m) \frac{|6\psi_2(b_m)\varphi'^2(b_m)|}{(1-|\varphi(b_m)|^2)^3} (\Lambda_X(\varphi(b_m)) - M) \\
& \leq \mu(b_m) \frac{|6\psi_2(b_m)\varphi'^2(b_m)|}{(1-|\varphi(b_m)|^2)^3} |h_m(\varphi(b_m))| \\
& \leq \mu(b_m) \frac{|6\psi_2(b_m)\varphi'^2(b_m)|}{(1-|\varphi(b_m)|^2)^3} |f_m(\varphi(b_m))|.
\end{aligned} \tag{35}$$

Thus, by (30), (34), and (35), we have

$$\begin{aligned}
K_3(\mu, \psi, \varphi) &:= \lim_{m \rightarrow \infty} \mu(b_m) \frac{|\psi_2(b_m)\varphi'^2(b_m)|}{(1-|\varphi(b_m)|^2)^3} \Lambda_X(\varphi(b_m)) \\
&\leq \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} \\
&\leq \inf_{\mathcal{T}} \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} \\
&\leq \|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{X}_\mu}.
\end{aligned} \tag{36}$$

Step 2. In the case of $j = 2$, we let a sequence $\{b_m\}$ in \mathbb{D} be such that $\lim_{m \rightarrow \infty} |\varphi(b_m)| = 1$ and

$$K_2(\mu, \psi, \varphi) = \lim_{m \rightarrow \infty} \mu(b_m) \frac{|\psi_1(b_m)\varphi'^2(b_m) + 2\psi_2'(b_m)\varphi'(b_m) + \psi_2(b_m)\varphi''(b_m)|}{(1-|\varphi(b_m)|^2)^2} \Lambda_X(\varphi(b_m)). \tag{37}$$

Letting $k = 2$ in (29), we observe that

$$\begin{aligned}
F_{m,2}(\varphi(b_m)) &= F'_{m,2}(\varphi(b_m)) \\
F_{m,2}''(\varphi(b_m)) &= \frac{2f_m(\varphi(b_m))}{(1-|\varphi(b_m)|^2)^2}.
\end{aligned} \tag{38}$$

So, we have

$$\begin{aligned}
&= \left(\psi_1(b_m)\varphi'^2(b_m) + 2\psi_2'(b_m)\varphi'(b_m) + \psi_2(b_m)\varphi''(b_m) \right) F_{m,2}''(\varphi(b_m)) \\
&\quad + \psi_2(b_m)\varphi'^2(b_m) F_{m,2}^{(3)}(\varphi(b_m)) \\
&\geq \left| \left(\psi_1(b_m)\varphi'^2(b_m) + 2\psi_2'(b_m)\varphi'(b_m) + \psi_2(b_m)\varphi''(b_m) \right) F_{m,2}''(\varphi(b_m)) \right| \\
&\quad - \left| \psi_2(b_m)\varphi'^2(b_m) F_{m,2}^{(3)}(\varphi(b_m)) \right|.
\end{aligned} \tag{39}$$

By Proposition 1 and Lemma 2, if $\mathcal{T}: X \rightarrow \mathcal{X}_\mu$ is a compact operator, then we obtain

$$\lim_{m \rightarrow \infty} \|\mathcal{T}F_{m,2}\|_{\mathcal{X}_\mu} = 0. \tag{40}$$

By (39) and (40), we have

$$\begin{aligned}
&\|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} \\
&\geq \limsup_{m \rightarrow \infty} \|F_{m,2}\|_X \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_X \\
&\geq \limsup_{m \rightarrow \infty} \|(T_{\psi_1, \psi_2, \varphi} - \mathcal{T})F_{m,2}\|_{\mathcal{X}_\mu} \\
&\geq \limsup_{m \rightarrow \infty} \mu(b_m) \left| \left(\psi_1(b_m)\varphi'^2(b_m) + 2\psi_2'(b_m)\varphi'(b_m) \right. \right. \\
&\quad \left. \left. + \psi_2(b_m)\varphi''(b_m) \right) \frac{2f_m(\varphi(b_m))}{(1-|\varphi(b_m)|^2)^2} \right| \\
&\quad - \limsup_{m \rightarrow \infty} \mu(b_m) \frac{|6\psi_2(b_m)\varphi'^2(b_m)f_m(\varphi(b_m))|}{(1-|\varphi(b_m)|^2)^3}.
\end{aligned} \tag{41}$$

Hence,

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \mu(b_m) \left| \left(\psi_1(b_m) \varphi'^2(b_m) + 2\psi_2'(b_m) \varphi'(b_m) + \psi_2(b_m) \varphi''(b_m) \right) \frac{2f_m(\varphi(b_m))}{(1 - |\varphi(b_m)|^2)^2} \right| \\ & \leq \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{F}_\mu} + \limsup_{m \rightarrow \infty} \mu(b_m) \frac{|6\psi_2(b_m) \varphi'^2(b_m) f_m(\varphi(b_m))|}{(1 - |\varphi(b_m)|^2)^3}. \end{aligned} \quad (42)$$

Using (24) and (36), then (42) gives

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \mu(b_m) \frac{|\psi_1(b_m) \varphi'^2(b_m) + 2\psi_2'(b_m) \varphi'(b_m) + \psi_2(b_m) \varphi''(b_m)|}{(1 - |\varphi(b_m)|^2)^2} \Lambda_X(\varphi(b_m)) \\ & \leq \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{F}_\mu} + M + K_3(\mu, \psi, \varphi). \end{aligned} \quad (43)$$

Since M is an arbitrary constant, combining (43) with (36) and (37), we conclude that

$$\begin{aligned} & K_2(\mu, \psi, \varphi) \\ & \leq \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{F}_\mu} + K_3(\mu, \psi, \varphi) \\ & \leq \inf_{\mathcal{T}} \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{F}_\mu} + K_3(\mu, \psi, \varphi) \\ & \leq \|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{F}_\mu}. \end{aligned} \quad (44)$$

Step 3. In the case of $j = 1$, we let a sequence $\{b_m\}$ in \mathbb{D} be such that $\lim_{m \rightarrow \infty} |\varphi(b_m)| = 1$ and

$$K_1(\mu, \psi, \varphi) := \lim_{m \rightarrow \infty} \mu(b_m) \frac{|2\psi_1'(b_m) \varphi'(b_m) + \psi_1(b_m) \varphi''(b_m) + \psi_2''(b_m)|}{1 - |\varphi(b_m)|^2} \Lambda_X(\varphi(b_m)). \quad (45)$$

Letting $k = 1$ in (29), we observe that

$$\begin{aligned} & F_{m,1}(\varphi(b_m)) = 0, \\ & F_{m,1}'(\varphi(b_m)) = \frac{-f_m(\varphi(b_m))}{1 - |\varphi(b_m)|^2}. \end{aligned} \quad (46)$$

Thus, we have

$$\begin{aligned} & |((T_{\psi_1, \psi_2, \varphi} F_{m,1})''(b_m))| \\ & = |(2\psi_1'(b_m) \varphi'(b_m) + \psi_1(b_m) \varphi''(b_m) + \psi_2''(b_m)) F_{m,1}'(\varphi(b_m)) \\ & \quad + (\psi_1(b_m) \varphi'^2(b_m) + 2\psi_2'(b_m) \varphi'(b_m) + \psi_2(b_m) \varphi''(b_m)) F_{m,1}''(\varphi(b_m)) \\ & \quad + \psi_2(b_m) \varphi'^2(b_m) F_{m,1}^{(3)}(\varphi(b_m))|. \end{aligned} \quad (47)$$

By Proposition 1 and Lemma 2, if $\mathcal{T}: X \longrightarrow \mathcal{X}_\mu$ is a compact operator, then we obtain

$$\lim_{m \rightarrow \infty} \|\mathcal{T}F_{m,1}\|_{\mathcal{X}_\mu} = 0. \quad (48)$$

By (47) and (50), we have

$$\begin{aligned} & \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} \\ & \geq \limsup_{m \rightarrow \infty} \|F_{m,1}\|_X \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_X \\ & \geq \limsup_{m \rightarrow \infty} \|(T_{\psi_1, \psi_2, \varphi} - \mathcal{T})F_{m,1}\|_{\mathcal{X}_\mu} \\ & \geq \limsup_{m \rightarrow \infty} \mu(b_m) \left| (2\psi_1'(b_m)\varphi'(b_m) + \psi_1(b_m)\varphi''(b_m) + \psi_2''(b_m)) \frac{-f_m(\varphi(b_m))}{1 - |\varphi(b_m)|^2} \right| \\ & \quad - \limsup_{m \rightarrow \infty} \mu(b_m) \left| \left(\psi_1(b_m)\varphi'^2(b_m) + 2\psi_2'(b_m)\varphi'(b_m) + \psi_2(b_m)\varphi''(b_m) \right) \frac{2f_m(\varphi(b_m))}{(1 - |\varphi(b_m)|^2)^2} \right| \\ & \quad - \limsup_{m \rightarrow \infty} \mu(b_m) \frac{|6\psi_2(b_m)\varphi'^2(b_m)f_m(\varphi(b_m))|}{(1 - |\varphi(b_m)|^2)^3}. \end{aligned} \quad (49)$$

Hence,

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \mu(b_m) \left| (2\psi_1'(b_m)\varphi'(b_m) + \psi_1(b_m)\varphi''(b_m) + \psi_2''(b_m)) \frac{-f_m(\varphi(b_m))}{1 - |\varphi(b_m)|^2} \right| \\ & \leq \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} + \limsup_{m \rightarrow \infty} \mu(b_m) \frac{|6\psi_2(b_m)\varphi'^2(b_m)f_m(\varphi(b_m))|}{(1 - |\varphi(b_m)|^2)^3} \\ & \quad + \limsup_{m \rightarrow \infty} \mu(b_m) \left| \left(\psi_1(b_m)\varphi'^2(b_m) + 2\psi_2'(b_m)\varphi'(b_m) + \psi_2(b_m)\varphi''(b_m) \right) \frac{2f_m(\varphi(b_m))}{(1 - |\varphi(b_m)|^2)^2} \right|. \end{aligned} \quad (50)$$

Using (24), (36), (44), and then (50) gives

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \mu(b_m) \frac{|2\psi_1'(b_m)\varphi'(b_m) + \psi_1(b_m)\varphi''(b_m) + \psi_2''(b_m)|}{1 - |\varphi(b_m)|^2} \Lambda_X(\varphi(b_m)) \\ & \leq \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} + M + K_3(\mu, \psi, \varphi) + K_2(\mu, \psi, \varphi). \end{aligned} \quad (51)$$

Since M is an arbitrary constant, combining (51) with (44) and (45), we conclude that

$$\begin{aligned}
K_1(\mu, \psi, \varphi) &\leq \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} + K_2(\mu, \psi, \varphi) + K_3(\mu, \psi, \varphi) \\
&\leq \inf_{\mathcal{T}} \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} + K_2(\mu, \psi, \varphi) + K_3(\mu, \psi, \varphi) \\
&\leq \|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{X}_\mu}.
\end{aligned} \tag{52}$$

Step 4. In the case of $j = 0$, for each $m \in \mathbb{N}$ and fix $M > 0$, let h_m and f_m be such that $f_m(\varphi(b_m)) = \varphi(b_m)h_m(\varphi(b_m))$ in terms of the sequence $\{b_m\}$ in \mathbb{D} such that $\lim_{m \rightarrow \infty} |\varphi(b_m)| = 1$ and

$$K_0(\mu, \psi, \varphi) = \lim_{m \rightarrow \infty} \mu(b_m) |\psi_1''(b_m)| \Lambda_X(\varphi(b_m)). \tag{53}$$

Then,

$$\begin{aligned}
&|((T_{\psi_1, \psi_2, \varphi} f_m)''(b_m))| \\
&= |\psi_1''(b_m)| f_m(b_m) \\
&\quad + |(2\psi_1'(b_m)\varphi'(b_m) + \psi_1(b_m)\varphi''(b_m) + \psi_2''(b_m))f_m'(\varphi(b_m))| \\
&\quad + \left| \left(\psi_1(b_m)\varphi'^2(b_m) + 2\psi_2'(b_m)\varphi'(b_m) + \psi_2(b_m)\varphi''(b_m) \right) f_m''(\varphi(b_m)) \right| \\
&\quad + |\psi_2(b_m)\varphi'^2(b_m)f_m^{(3)}(\varphi(b_m))|.
\end{aligned} \tag{54}$$

If $\mathcal{T}: X \rightarrow \mathcal{X}_\mu$ is a compact operator, then by Lemma 2, we obtain

$$\lim_{m \rightarrow \infty} \|\mathcal{T} f_m\|_{\mathcal{X}_\mu} = 0. \tag{55}$$

By (54) and (55), we have

$$\begin{aligned}
&\|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} \\
&\geq \limsup_{m \rightarrow \infty} \|f_m\|_X \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_X \\
&\geq \limsup_{m \rightarrow \infty} \|(T_{\psi_1, \psi_2, \varphi} - \mathcal{T})f_m\|_{\mathcal{X}_\mu} \\
&\geq \limsup_{m \rightarrow \infty} \mu(b_m) |\psi_1''(b_m)| f_m(b_m) \\
&\quad - \limsup_{m \rightarrow \infty} \mu(b_m) \left| (2\psi_1'(b_m)\varphi'(b_m) + \psi_1(b_m)\varphi''(b_m) + \psi_2''(b_m)) \frac{-f_m(\varphi(b_m))}{1 - |\varphi(b_m)|^2} \right| \\
&\quad - \limsup_{m \rightarrow \infty} \mu(b_m) \left| \left(\psi_1(b_m)\varphi'^2(b_m) + 2\psi_2'(b_m)\varphi'(b_m) + \psi_2(b_m)\varphi''(b_m) \right) \frac{2f_m(\varphi(b_m))}{(1 - |\varphi(b_m)|^2)^2} \right| \\
&\quad - \limsup_{m \rightarrow \infty} \mu(b_m) \frac{|6\psi_2(b_m)\varphi'^2(b_m)f_m(\varphi(b_m))|}{(1 - |\varphi(b_m)|^2)^3}.
\end{aligned} \tag{56}$$

Hence, by condition (A_7) and (56), we have

$$\begin{aligned}
& \limsup_{m \rightarrow \infty} \mu(b_m) |\psi_1''(b_m) f_m(\varphi(b_m))| \\
& \leq \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} + \limsup_{m \rightarrow \infty} \mu(b_m) \frac{|6\psi_2(b_m)\varphi'^2(b_m)f_m(\varphi(b_m))|}{(1 - |\varphi(b_m)|^2)^3} \\
& \quad + \limsup_{m \rightarrow \infty} \mu(b_m) \left| \left(\psi_1(b_m)\varphi'^2(b_m) + 2\psi_2'(b_m)\varphi'(b_m) + \psi_2(b_m)\varphi''(b_m) \right) \frac{2f_m(\varphi(b_m))}{(1 - |\varphi(b_m)|^2)^2} \right| \\
& \quad + \limsup_{m \rightarrow \infty} \mu(b_m) \left| \left(2\psi_1'(b_m)\varphi'(b_m) + \psi_1(b_m)\varphi''(b_m) + \psi_2''(b_m) \right) \frac{-f_m(\varphi(b_m))}{1 - |\varphi(b_m)|^2} \right|.
\end{aligned} \tag{57}$$

Using (6), (14), (21), and (50), then (57) gives

$$\begin{aligned}
& \limsup_{m \rightarrow \infty} \mu(b_m) |\psi_1''(b_m) \Lambda_X(\varphi(b_m))| \\
& \leq \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} + M + K_3(\mu, \psi, \varphi) + K_2(\mu, \psi, \varphi) + K_1(\mu, \psi, \varphi).
\end{aligned} \tag{58}$$

Since M is an arbitrary constant, combining (58) with (52) and (53), we conclude that

$$\begin{aligned}
K_0(\mu, \psi, \varphi) & \leq \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} + K_1(\mu, \psi, \varphi) + K_2(\mu, \psi, \varphi) + K_3(\mu, \psi, \varphi) \\
& \leq \inf_{\mathcal{T}} \|T_{\psi_1, \psi_2, \varphi} - \mathcal{T}\|_{X \rightarrow \mathcal{X}_\mu} + K_1(\mu, \psi, \varphi) + K_2(\mu, \psi, \varphi) + K_3(\mu, \psi, \varphi) \\
& \leq \|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{X}_\mu}.
\end{aligned} \tag{59}$$

We thus obtain the lower estimate on the essential norm. Now, let us prove the upper estimate. By the following,

$$\|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{X}_\mu} \leq \sum_{j=0}^3 K_j(\mu, \psi, \varphi). \tag{60}$$

In the statement of the theorem, we assume that $\inf_{w \in \mathbb{D}} \Lambda_X(\varphi(w)) > 0$, which is the same as assumption (20) in Remark 2. By reading carefully the detailed proof of Lemma

3 in [28], while part (c) of the lemma requires condition (A_2) , we can conclude that the assumption $\inf_{w \in \mathbb{D}} \Lambda_X(\varphi(w)) > 0$ is sufficient to apply Lemma 3. Thus, we fix $\varepsilon > 0$, $r \in (0, 1)$ and choose $s \in (0, 1)$ as in Lemma 3 and Remark 2.

By condition (A_5) , since $T_{\psi_1, \psi_2, \varphi}: X \rightarrow \mathcal{X}_\mu$ is a bounded operator, the linear map \mathcal{C}_s is compact on X , and the product $T_{\psi_1, \psi_2, \varphi} \mathcal{C}_s: X \rightarrow \mathcal{X}_\mu$ is compact. Therefore,

$$\begin{aligned}
\|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{X}_\mu} & \leq \|T_{\psi_1, \psi_2, \varphi} - T_{\psi_1, \psi_2, \varphi} \mathcal{C}_s\|_{e, X \rightarrow \mathcal{X}_\mu} \\
& = \sup_{\|h\|_X \leq 1} \|T_{\psi_1, \psi_2, \varphi} (I - \mathcal{C}_s)h\|_{\mathcal{X}_\mu} \\
& \leq \sup_{\|h\|_X \leq 1} \left\{ \left| (T_{\psi_1, \psi_2, \varphi} (I - \mathcal{C}_s)h)(0) \right| \right. \\
& \quad \left. + \left| (T_{\psi_1, \psi_2, \varphi} (I - \mathcal{C}_s)h)'(0) \right| \right\} + \sup_{\|h\|_X \leq 1} \sum_{j=0}^3 L_j(\mu, \psi, \varphi),
\end{aligned} \tag{61}$$

where

$$\begin{aligned}
L_0(\mu, \psi, \varphi) &= \sup_{w \in \mathbb{D}} \mu(w) |\psi_1''(w)(I - \mathcal{E}_s)h(\varphi(w))|, \\
L_1(\mu, \psi, \varphi) &= \sup_{w \in \mathbb{D}} \mu(w) | (2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w))((I - \mathcal{E}_s)h)'(\varphi(w))|, \\
L_2(\mu, \psi, \varphi) &= \sup_{w \in \mathbb{D}} \mu(w) \left| \left(\psi_1(w)\varphi'^2(w) + 2\psi_2'(w)\varphi'(w) + \psi_2(w)\varphi''(w) \right) ((I - \mathcal{E}_s)h)''(\varphi(w)) \right|, \\
L_3(\mu, \psi, \varphi) &= \sup_{w \in \mathbb{D}} \mu(w) \left| \left(\psi_2(w)\varphi'^2(w) \right) ((I - \mathcal{E}_s)h)^{(3)}(\varphi(w)) \right|.
\end{aligned} \tag{62}$$

By Lemma 3, we obtain

$$\begin{aligned}
& \sup_{\|h\|_X \leq 1} \left\{ \left| (T_{\psi_1, \psi_2, \varphi}(I - \mathcal{E}_s)h)(0) \right| + \left| (T_{\psi_1, \psi_2, \varphi}(I - \mathcal{E}_s)h)'(0) \right| \right\} \\
& \leq \sup_{\|h\|_X \leq 1} \left\{ \left| \psi_1(0)(I - \mathcal{E}_s)h(\varphi(0)) + \psi_2(0)((I - \mathcal{E}_s)h)'(\varphi(0)) \right| \right\} \\
& \quad + \sup_{\|h\|_X \leq 1} \left\{ \left| \psi_1'(0)(I - \mathcal{E}_s)h(\varphi(0)) + \psi_1(0)\varphi'(0)((I - \mathcal{E}_s)h)'(\varphi(0)) \right| \right\} \\
& \quad + \sup_{\|h\|_X \leq 1} \left\{ \left| \psi_2'((I - \mathcal{E}_s)h)'(\varphi(0)) + \psi_2(0)\varphi'(0)((I - \mathcal{E}_s)h)''(\varphi(0)) \right| \right\} \\
& \leq \varepsilon \left\{ |\psi_1(0)| + |\psi_1'(0)| \right\} + \varepsilon \left\{ |\psi_2(0)| + |\psi_1(0)\varphi'(0)| + |\psi_2'(0)| + |\psi_2(0)\varphi'(0)| \right\} \Lambda_X(\varphi(0)).
\end{aligned} \tag{63}$$

Now, combining condition (A_7) with Lemma 3, Remark 1, Theorem 1, and (20), we obtain

$$\begin{aligned}
\sup_{\|h\|_X \leq 1} L_0(\mu, \psi, \varphi) & \leq \sup_{\|h\|_X \leq 1} \sup_{|\varphi(w)| \leq r} \mu(w) |\psi_1''(w)| \Lambda_X(\varphi(w)) \frac{|(I - \mathcal{E}_s)h(\varphi(w))|}{\Lambda_X(\varphi(w))} \\
& \quad + \sup_{\|h\|_X \leq 1} \sup_{|\varphi(w)| > r} \mu(w) |\psi_1''(w)| \Lambda_X(\varphi(w)) \frac{|(I - \mathcal{E}_s)h(\varphi(w))|}{\Lambda_X(\varphi(w))} \\
& \leq \varepsilon \widetilde{K}_0(\mu, \psi, \varphi) + \sup_{|\varphi(w)| > r} \mu(w) |\psi_1''(w)| \Lambda_X(\varphi(w)), \\
\sup_{\|h\|_X \leq 1} L_1(\mu, \psi, \varphi) & \leq \sup_{\|h\|_X \leq 1} \sup_{|\varphi(w)| \leq r} \mu(w) \frac{|2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{(1 - |\varphi(w)|^2)} \\
& \quad \cdot \Lambda_X(\varphi(w)) \times (1 - |\varphi(w)|^2) \frac{|((I - \mathcal{E}_s)h)'(\varphi(w))|}{\Lambda_X(\varphi(w))} \\
& \quad + \sup_{\|h\|_X \leq 1} \sup_{|\varphi(w)| > r} \mu(w) \frac{|2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{(1 - |\varphi(w)|^2)} \\
& \quad \cdot \Lambda_X(\varphi(w)) \times (1 - |\varphi(w)|^2) \frac{|((I - \mathcal{E}_s)h)'(\varphi(w))|}{\Lambda_X(\varphi(w))} \\
& \leq \varepsilon \widetilde{K}_1(\mu, \psi, \varphi) + \sup_{|\varphi(w)| > r} \mu(w) \frac{|2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{(1 - |\varphi(w)|^2)} \Lambda_X(\varphi(w)),
\end{aligned} \tag{64}$$

$$\begin{aligned}
& \sup_{\|h\|_X \leq 1} L_2(\mu, \psi, \varphi) \leq \sup_{\|h\|_X \leq 1} \sup_{|\varphi(w)| \leq r} \mu(w) \frac{|2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{(1 - |\varphi(w)|^2)} \\
& \quad \cdot \Lambda_X(\varphi(w)) \times (1 - |\varphi(w)|^2) \frac{|((I - \mathcal{E}_s)h)'(\varphi(w))|}{\Lambda_X(\varphi(w))} \\
& \quad + \sup_{\|h\|_X \leq 1} \sup_{|\varphi(w)| > r} \mu(w) \frac{|2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{(1 - |\varphi(w)|^2)} \\
& \quad \cdot \Lambda_X(\varphi(w)) \times (1 - |\varphi(w)|^2) \frac{|((I - \mathcal{E}_s)h)'(\varphi(w))|}{\Lambda_X(\varphi(w))} \\
& \leq \varepsilon \widetilde{K}_1(\mu, \psi, \varphi) + \sup_{|\varphi(w)| > r} \mu(w) \frac{|2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{(1 - |\varphi(w)|^2)} \Lambda_X(\varphi(w)),
\end{aligned} \tag{65}$$

$$\begin{aligned}
\sup_{\|h\|_X \leq 1} L_2(\mu, \psi, \varphi) &\leq \sup_{\|h\|_X \leq 1} \sup_{|\varphi(w)| \leq r} \mu(w) \frac{|\psi_1(w)\varphi'^2(w) + 2\psi_2'(w)\varphi'(w) + \psi_2(w)\varphi''(w)|}{(1 - |\varphi(w)|^2)^2} \\
&\quad \cdot \Lambda_X(\varphi(w)) \times (1 - |\varphi(w)|^2)^2 \frac{|((I - \mathcal{E}_s)h)''(\varphi(w))|}{\Lambda_X(\varphi(w))} \\
&\quad + \sup_{\|h\|_X \leq 1} \sup_{|\varphi(w)| > r} \mu(w) \frac{|\psi_1(w)\varphi'^2(w) + 2\psi_2'(w)\varphi'(w) + \psi_2(w)\varphi''(w)|}{(1 - |\varphi(w)|^2)^2} \\
&\quad \cdot \Lambda_X(\varphi(w)) \times (1 - |\varphi(w)|^2)^2 \frac{|((I - \mathcal{E}_s)h)''(\varphi(w))|}{\Lambda_X(\varphi(w))} \\
&\leq \varepsilon \widetilde{K}_2(\mu, \psi, \varphi) + \sup_{|\varphi(w)| > r} \mu(w) \frac{|\psi_1(w)\varphi'^2(w) + 2\psi_2'(w)\varphi'(w) + \psi_2(w)\varphi''(w)|}{(1 - |\varphi(w)|^2)^2} \Lambda_X(\varphi(w)),
\end{aligned} \tag{66}$$

$$\begin{aligned}
\sup_{\|h\|_X \leq 1} L_3(\mu, \psi, \varphi) &\leq \sup_{\|h\|_X \leq 1} \sup_{|\varphi(w)| \leq r} \mu(w) \frac{|\psi_2(w)\varphi'^2(w)|}{(1 - |\varphi(w)|^2)^3} \Lambda_X(\varphi(w)) \\
&\quad \times (1 - |\varphi(w)|^2)^3 \frac{|((I - \mathcal{E}_s)h)^{(3)}(\varphi(w))|}{\Lambda_X(\varphi(w))} \\
&\quad + \sup_{\|h\|_X \leq 1} \sup_{|\varphi(w)| > r} \mu(w) \frac{|\psi_2(w)\varphi'^2(w)|}{(1 - |\varphi(w)|^2)^3} \Lambda_X(\varphi(w)) \\
&\quad \times (1 - |\varphi(w)|^2)^3 \frac{|((I - \mathcal{E}_s)h)^{(3)}(\varphi(w))|}{\Lambda_X(\varphi(w))} \\
&\leq \varepsilon \widetilde{K}_3(\mu, \psi, \varphi) + \sup_{|\varphi(w)| > r} \mu(w) \frac{|\psi_2(w)\varphi'^2(w)|}{(1 - |\varphi(w)|^2)^3} \Lambda_X(\varphi(w)).
\end{aligned} \tag{67}$$

Using (38), (39), (40), and (67), we obtain

$$\begin{aligned}
&\|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{F}_\mu} \\
&\leq \varepsilon + \sup_{|\varphi(w)| > r} \mu(w) |\psi_1''(w)| \Lambda_X(\varphi(w)) \\
&\quad + \sup_{|\varphi(w)| > r} \mu(w) \frac{|2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{(1 - |\varphi(w)|^2)} \Lambda_X(\varphi(w)) \\
&\quad + \sup_{|\varphi(w)| > r} \mu(w) \frac{|\psi_1(w)\varphi'^2(w) + 2\psi_2'(w)\varphi'(w) + \psi_2(w)\varphi''(w)|}{(1 - |\varphi(w)|^2)^2} \Lambda_X(\varphi(w)) \\
&\quad + \sup_{|\varphi(w)| > r} \mu(w) \frac{|\psi_2(w)\varphi'^2(w)|}{(1 - |\varphi(w)|^2)^3} \Lambda_X(\varphi(w)).
\end{aligned} \tag{68}$$

Since ε is arbitrary, if $\|\varphi\|^\infty < 1$, then we have $\|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{F}_\mu} = 0$, and $T_{\psi_1, \psi_2, \varphi}: X \rightarrow \mathcal{F}_\mu$ is compact. If $\|\varphi\|^\infty = 1$, letting $r \rightarrow 1$, we have

$$\|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{F}_\mu} \leq \sum_{j=0}^3 K_j(\mu, \psi, \varphi). \tag{69}$$

Remark 3. Note that the assumption that the Banach space X is reflexive is used only in the case of a lower estimate of Theorem 2. This brings about an interesting question. What is the possibility of approximating the essential norm in Theorem 2 without assuming the reflectivity on the Banach space X ?

Actually, by condition (B_3) in Lemma 2, if $T_{\psi_1, \psi_2, \varphi}: X \rightarrow \mathcal{X}_\mu$ is a compact operator and the sequence $\{h_m\}$ is bounded in the Banach space X converging to zero uniformly on compact subsets of \mathbb{D} , then

$\lim_{m \rightarrow \infty} \|Th_m\|_{\mathcal{X}_\mu} = 0$. Thus, our argument in the proof of Theorem 2 for h_m taken to be $F_{m,k}$, $k = 1, 2, 3$ leads to a lower estimate on $\|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{X}_\mu}$. An example of a space satisfying condition B_3 in Lemma 2 is the Hardy spaces H^∞ .

Thus, the approximation of $\|T_{\psi_1, \psi_2, \varphi}\|_{e, X \rightarrow \mathcal{X}_\mu}$ also holds if the conditions (B_1) – (B_3) in Lemma 2 hold.

In the cases $\psi_1(w) = \psi(w)$ and $\psi_2(w) = 0$ in Theorem 2, we deduce the following result as a corollary; see Theorem 3.3 in [4].

$$\begin{aligned} R_0 &= \lim_{\delta \rightarrow 1} \sup_{|\varphi(w)| > \delta} \mu(w) |\psi''(w)| \Lambda_X(\varphi(w)), \\ R_1 &= \lim_{\delta \rightarrow 1} \sup_{|\varphi(w)| > \delta} \mu(w) \frac{|2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)|}{(1 - |\varphi(w)|^2)} \Lambda_X(\varphi(w)), \\ R_2 &= \lim_{\delta \rightarrow 1} \sup_{|\varphi(w)| > \delta} \mu(w) \frac{|\psi(w)\varphi'^2(w)|}{(1 - |\varphi(w)|^2)^2} \Lambda_X(\varphi(w)). \end{aligned} \quad (71)$$

Corollary 1. Let a Banach space X of functions in $\mathcal{H}\mathcal{O}(\mathbb{D})$ be reflexive and satisfy the above conditions (A_4) , (A_5) , and (A_7) , together with either (A_1) or (A_6) . Suppose that $\inf_{w \in \mathbb{D}} \Lambda_X(\varphi(w)) > 0$, if $T_{\psi, \varphi}: X \rightarrow \mathcal{X}_\mu$ is a bounded operator. Then,

$$\|T_{\psi, \varphi}\|_{e, X \rightarrow \mathcal{X}_\mu} \approx \sum_{j=0}^2 R_j(\mu, \psi, \varphi), \quad (70)$$

where

4. Applications of Essential Norm on Other Domains

We now show that our estimations hold up for any domain space choice, provided that the domain is reflexive. Note that, all spaces handled below contain the constant functions. Thus, condition (20) applies to all cases. First, set the following quantities:

$$\begin{aligned} Q_0(\mu, \psi, \varphi; \alpha) &= \mu(w) \frac{|\psi''(w)|}{(1 - |\varphi(w)|^2)^\alpha}, \\ Q_1(\mu, \psi, \varphi; \alpha) &= \mu(w) \frac{|2\psi'_1(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{(1 - |\varphi(w)|^2)^{\alpha+1}}, \\ Q_2(\mu, \psi, \varphi; \alpha) &= \mu(w) \frac{|\psi_1(w)\varphi'^2(w) + 2\psi_2'(w)\varphi'(w) + \psi_2(w)\varphi''(w)|}{(1 - |\varphi(w)|^2)^{\alpha+2}}, \\ Q_3(\mu, \psi, \varphi; \alpha) &= \mu(w) \frac{|\psi_2(w)\varphi'^2(w)|}{(1 - |\varphi(w)|^2)^{\alpha+3}}. \end{aligned} \quad (72)$$

Second, we discuss the approximation of $\|T_{\psi_1, \psi_2, \varphi}\|_e$ on other domain spaces.

4.1. The Hardy Spaces H^q , $(1 \leq q \leq \infty)$. Observe that the map $\zeta \rightarrow \Lambda(\zeta)$ is a constant 1 mapping. Clearly for the H^∞ space, all conditions (A_2) – (A_5) hold. By Theorem 5.4 in

[33], for any integer k , we know that the space H^∞ and the Bloch space \mathcal{B} are connected by the condition as follows:

$$(1 - |\zeta|)^k |h^{(k)}(\zeta)| \leq \|h\|_{\mathcal{B}}, \quad \text{for } \zeta \in \mathbb{D}, h \in \mathcal{B}. \quad (73)$$

Thus, the H^∞ space also satisfies the condition (A_7) . So, we deduce that the approximation of the essential norm

$\|T_{\psi_1, \psi_2, \varphi}\|_{e, H^\infty, X} \longrightarrow \mathcal{X}_\mu$ is applicable. The theory below summarizes this approach, and the results are applicable for the domain space H^∞ .

Theorem 3. *If the operator $T_{\psi_1, \psi_2, \varphi}: H^\infty \longrightarrow \mathcal{X}_\mu$ is a bounded operator, then*

$$\|T_{\psi_1, \psi_2, \varphi}\|_{e, H^\infty} \longrightarrow \mathcal{X}_\mu \approx \sum_{j=0}^3 Q_j(\mu, \psi, \varphi; 0). \quad (74)$$

For $q \geq 1$ and $h \in \mathcal{HO}(\mathbb{D})$, the Hardy space $H^q(\mathbb{D}) = H^q$ is the Banach space with the norm as follows:

$$\|h\|_{H^q}^q = \sup_{0 < s < 1} \int_0^{2\pi} (|hse^{i\theta}|)^q \frac{d\theta}{2\pi}. \quad (75)$$

We know that the explicit formula for Hardy space H^q is given by

$$\Lambda_{H^q}(\zeta) = \frac{1}{(1 - |\zeta|^2)^{1/q}}. \quad (76)$$

Moreover, for a positive integer k , $\zeta \in \mathbb{D}$ and $1 \leq q < \infty$ if $h \in H^q$, then

$$|h^{(k)}(\zeta)| \leq \frac{\|h\|_{H^q}}{(1 - |\zeta|^2)^{k+1/q}}. \quad (77)$$

From (76) and (77), we have that the conditions (A_4) and (A_7) hold for all $k = 1, 2, 3$ on Hardy space H^q . Thus, the norm estimate of the operator is an outcome instant from Theorem 1.

The results for $T_{\psi_1, \psi_2, \varphi}: H^q \longrightarrow \mathcal{X}_{\mu, 0}$ are an instant outcome from Theorem 2. We can then summarize the results as follows.

Theorem 4. *If $T_{\psi_1, \psi_2, \varphi}: H^q \longrightarrow \mathcal{X}_\mu$ is a bounded operator, then*

$$\|T_{\psi_1, \psi_2, \varphi}\|_{e, H^q} \longrightarrow \mathcal{X}_\mu \approx \sum_{j=0}^3 Q_j(\mu, \psi, \varphi; \alpha), \quad (78)$$

where $\alpha = 1/q$.

4.2. The Weighted Bergman Space A_β^q . For all $\beta > -1$ and $1 < q < \infty$, the weighted Bergman space A_β^q is the set of all functions $h \in \mathcal{HO}(\mathbb{D})$, such that

$$\|h\|_{A_\beta^q}^q = \int_{\mathbb{D}} |h(\zeta)|^q dA_\beta(\zeta) < \infty, \quad (79)$$

where $dA_\beta(\zeta) = (\beta + 1)(1 - |\zeta|^2)^\beta dA(\zeta)$ is the weighted Lebesgue measure. When $\beta > -1$, the positive measure dA_β is normalized to become a probability measure. The weighted Bergman-type space A_β^q is a Banach space when $1 \leq q < \infty$, and also it is a complete metric space when $0 < q < 1$. Several properties of A_β^q spaces are discussed in [31, 34].

The following lemma is proved in [31] and Lemma 5.3 in [4].

Lemma 4. *For a positive integer k and $\zeta \in \mathbb{D}$, let $\beta > -1$ and $1 \leq q < \infty$ and let $h \in A_\beta^q$. Then,*

(i)

$$\Lambda_{A_\beta^q}(\zeta) = \sup \left\{ |h(\zeta)| : h \in A_\beta^q, \|h\|_{A_\beta^q} \leq 1 \right\} = \frac{1}{(1 - |\zeta|^2)^{\beta+2/q}}. \quad (80)$$

$$(ii) |h^{(k)}(\zeta)| \leq \|h\|_{A_\beta^q} (1 - |\zeta|^2)^{k+\beta+2/q}.$$

Let us summarize $\|T_{\psi_1, \psi_2, \varphi}\|_e$ from the domain space A_β^q .

Theorem 5. *If $T_{\psi_1, \psi_2, \varphi}: A_\beta^q \longrightarrow \mathcal{X}_\mu$ is a bounded operator, then*

$$\|T_{\psi_1, \psi_2, \varphi}\|_{e, A_\beta^q} \longrightarrow \mathcal{X}_\mu \approx \sum_{j=0}^3 Q_j(\mu, \psi, \varphi; \alpha), \quad (81)$$

where $\alpha = \beta + 2/q$.

Proof. From Lemma 4, we have that conditions (A_4) and (A_7) hold. Thus, the norm estimate of the operator is an outcome instant from Theorem 1.

For the essential norm estimates of $\|T_{\psi_1, \psi_2, \varphi}\|_{A_\beta^q} \longrightarrow \mathcal{X}_\mu$, we observe that conditions (A_1) – (A_3) and (A_5) hold for A_β^q , $1 < q < \infty$ (see [28]). Since A_β^q is a reflexive Banach space if $1 < q < \infty$, in the case of an arbitrary weight μ , the hypotheses of Theorem 2 hold. \square

Data Availability

The research conducted in this paper does not make use of separate data.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Majorization for Certain Classes of Analytic Functions Defined by Fournier–Ruscheweyh Integral Operator

Murli Manohar Gour ¹, Pranay Goswami ², Basem Aref Frasin ³,
and Saad Althobaiti ⁴

¹Department of Mathematics and Statistics, Manipal University, Jaipur, India

²School of Liberal Studies, Ambedkar University, Delhi, India

³Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan

⁴Department of Sciences and Technology, Ranyah University Collage, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

Correspondence should be addressed to Basem Aref Frasin; bafrasin@yahoo.com

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In this paper, we introduce three new classes $S_k^a[M, N; \mu]$, $R_k^a(\mu)$, and $T_k^a(\theta)$ of analytic functions defined by Fournier–Ruscheweyh integral operator. For these classes, we investigate the majorization problem. Furthermore, a number of new results are shown to follow upon specializing the parameters involved in our main results.

1. Introduction and Definitions

For the two functions u and v which are analytic in the open unit disk $D = \{z \in \mathbb{C}: |z| < 1\}$, we can define the majorization for these two functions as follows (see [1]):

$$u(z) \ll v(z) \quad (z \in D). \quad (1)$$

If there exists a function $\psi(z)$ that is analytic in D , then

$$|\psi(z)| \leq 1 \text{ and } u(z) = \psi(z)v(z) \quad (z \in D). \quad (2)$$

For the two functions u and v , we say that the function u is subordinate to the function v defined as $u(z) \prec v(z)$, if there is a Schwarz function w , that is analytic in D with $w(0) = 0$ and $|w(z)| < 1$, $z \in D$, such that $u(z) = v(w(z))$, $z \in D$.

Now, on combining subordination and majorization, we define quasi-subordination as follows. For two functions u and v , we say that u is quasi-subordinate to v (see [2]) and it is defined as

$$u(z) \prec_q v(z) \quad (z \in D), \quad (3)$$

If there are two functions $\psi(z)$ and $w(z)$ that are analytic in D , then $(u(z)/\psi(z))$ is analytic in D and

$$|\psi(z)| \leq 1 \text{ and } w(0) = 0, |w(z)| \leq |z| \leq 1 \quad (z \in D), \quad (4)$$

satisfying

$$u(z) = \psi(z)v(w(z)) \quad (z \in D). \quad (5)$$

Remark 1

- (i) If we put $\psi(z) = 1$ in (5), we have the usual definition of subordination
- (ii) If we put $w(z) = z$ in (6), we have the usual definition of majorization

Let A denote the class of all functions of the form

$$f(z) = z + \sum_{m=0}^{\infty} a_m z^m, \quad (z \in D), \quad (6)$$

which are analytic in open unit disk D .

The function class Φ has been introduced and studied by Li and Srivastava [3] and is defined as

$$\Phi = \left\{ k(t): k(t) \geq 0, (0 \leq t \leq 1), \int_0^1 k(t) dt = 1 \right\}. \quad (7)$$

Fournier and Ruscheweyh [3, 4] considered an integral operator with a nonnegative function:

$$k_a: [0, 1] \longrightarrow \mathbb{R} \text{ such that } \int_0^1 k_a(t) dt = 1. \quad (8)$$

By substituting suitable values of parameter a , there are lots of special cases of function $k_a(t)$. We therefore consider the Fournier–Ruscheweyh integral operator to be in the following modified form [3] (see [5]):

$$\mathcal{J}_k^a f(z) = \int_0^1 k_a(t) \frac{f(tz)}{t} dt, \quad (f \in A). \quad (9)$$

where the real-valued functions k_a and k_{a-1} fulfill the requirements:

- (1) For an acceptable parameter a ,

$$k_{a-1}(t) \in \Phi, k_a(t) \in \Phi \text{ and } k_a(1) = 0. \quad (10)$$

- (2) There exists a constant λ ($-1 < \lambda \leq 2$) such that

$$\lambda k_a(t) - t k_a'(t) = (\lambda + 1) k_{a-1}(t), \quad (11)$$

where $t \in (0, 1)$ and $-1 < \lambda \leq 2$.

For \mathcal{J}_k^a operator, we have

$$z(\mathcal{J}_k^a u(z))' = -\lambda(\mathcal{J}_k^a u(z)) + (\lambda + 1)\mathcal{J}_k^{a-1} u(z). \quad (12)$$

Remark 2

- (i) If we take

$$k_a(t) = \frac{2^a}{\mu(a)} t \left(\log \frac{1}{t} \right)^{a-1} = k_1 \quad (a > 0), \quad (13)$$

in (9), we get the integral operator \mathcal{P}^a as

$$\mathcal{P}^a(t) = \frac{2^a}{z\mu(a)} \int_0^z \left(\log \frac{z}{t} \right)^{a-1} f(t) dt \quad (f \in A, a > 0). \quad (14)$$

The integral operator \mathcal{P}^a is exactly the same as the transformation I^k given by Flett [6] and studied subsequently by Li [7], Li and Srivastava [8], and many others. In the case when $a > 1$, then we have $\lambda = 1$.

- (ii) If we take

$$k_a(t) = \binom{a+b}{a} a(1-t)^{a-1} t^b = k_2, \quad (a > 0, b > -1), \quad (15)$$

in (9), we get the Jung–Kim–Srivastava integral operator Q_b^a [9] (see [10–12]) as

$$Q_b^a f(z) = \binom{a+b}{a} \frac{a}{z^b} \int_0^z \left(1 - \frac{t}{z} \right)^{a-1} t^{b-1} f(t) dt \quad (16)$$

$$(f \in A, a > 0, b > -1),$$

where

$$\binom{a}{b} = \frac{\mu(a+1)}{\mu(b+1)\mu(a-b+1)} = \binom{a}{b}, \quad (a, b \in \mathbb{C}). \quad (17)$$

In terms of known Gamma functions, the integral operator $Q_b^a u(z)$ is analogous to the convolution operator $L(a, d)$ by Carlson and Shaffer [13]. In the case when $a > 1, b > -1$, and $0 < a + b \leq 3$, we have $\lambda = a + b - 1$.

Now, we describe the following classes of analytical functions using integral operator (9).

Definition 1. The function $f \in A$ is said to be in the class $S_k^a[M, N; \mu]$ if and only if

$$1 + \frac{1}{\mu} \left(\frac{z(\mathcal{J}_k^a f(z))'}{\mathcal{J}_k^a f(z)} - 1 \right) \prec \frac{1 + Nz}{1 + Nz}, \quad (18)$$

with $-1 \leq N < M \leq 1, k \in \Phi, \mu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

If we take the value of k as defined in (13) and (15), this class becomes $S_{k_1}^a[M, N; \mu]$ and $S_{k_2}^a[M, N; \mu]$, respectively.

Definition 2. The function $f \in A$ is said to be in the class $R_k^a(\mu)$ if and only if

$$\left[\frac{z(\mathcal{J}_k^a f(z))'}{\mathcal{J}_k^a f(z)} - \mu \left| \frac{z(\mathcal{J}_k^a f(z))'}{\mathcal{J}_k^a f(z)} - 1 \right| \right] \prec e^z, \quad (z \in D), \quad (19)$$

where $a \geq 0, k \in \Phi$, and $\mu \geq 0$.

If we take the value of k as defined in (13) and (15), this class becomes $R_{k_1}^a(\mu)$ and $R_{k_2}^a(\mu)$, respectively.

Definition 3. The function $f \in A$ is said to be in the class $T_k^a(\theta)$ if and only if

$$e^{i\theta} \left(\frac{z(\mathcal{J}_k^a f(z))'}{\mathcal{J}_k^a f(z)} \right) \prec e^z \cos \theta + i \sin \theta, \quad (z \in D), \quad (20)$$

where $a \geq 0, k \in \Phi$, and $-(\Pi/2) < \theta < (\Pi/2)$.

If we take the value of k as defined in (13) and (15), this class becomes $T_{k_1}^a(\theta)$ and $T_{k_2}^a(\theta)$, respectively.

A majorization problem for the normalized class of starlike functions has been investigated by MacGregor [1] and further studied by Altintas et al. [14]. Recently, a number of researchers have studied several majorization problems for univalent and multivalent functions or meromorphic and multivalent meromorphic functions involving different operators and different classes [14–20, 22–24]. By motivating the above work, the majorization problems of the classes $S_k^a[M, N, \mu]$, $R_k^a(\mu)$, and $T_k^a(\theta)$ are investigated as follows.

2. Problem of Majorization for the Classes

$S_k^a[M, N, \mu]$, $R_k^a(\mu)$, and $T_k^a(\theta)$

Theorem 1. Let the function $f \in A$, and assume that $g \in S_k^a[M, N, \mu]$. If $\mathcal{J}_k^a f(z)$ is majorized by $\mathcal{J}_k^a g(z)$ in D , then

$$|\mathcal{J}_k^{a-1} f(z)| \leq |\mathcal{J}_k^{a-1} g(z)| \text{ for } |z| \leq \rho_0, \quad (21)$$

where ρ_0 is the smallest positive root of the equation

$$\begin{aligned} & |\mu(M-N) + (1+\lambda)N|\rho^3 - (2|N| + \lambda + 1)\rho^2 \\ & - (2 + |\mu(M-N) + (\lambda+1)N|)\rho + (\lambda+1) = 0, \end{aligned} \quad (22)$$

where $-1 \leq N < M \leq 1, k \in \Phi, \mu \in \mathbb{C}^*, -1 < \lambda \leq 2$, and $(\lambda+1) \geq |\mu(M-N) + (\lambda+1)N|$.

Proof. Since $g \in S_k^a[M, N, \mu]$, then, from (18),

$$1 + \frac{1}{\mu} \left(\frac{z(\mathcal{J}_k^a g(z))'}{\mathcal{J}_k^a g(z)} - 1 \right) = \frac{1 + Mw(z)}{1 + Nw(z)}, \quad (23)$$

where w is the analytic function in D , with $w(0) = 0$ and $|w(z)| \leq |z| < 1 \forall z \in D$.

Now, from the previous equality,

$$\frac{z(\mathcal{J}_k^a g(z))'}{\mathcal{J}_k^a g(z)} = \frac{1 + (\mu(M-N) + N)w(z)}{1 + Nw(z)}. \quad (24)$$

Now, we make use of relation (12), that is,

$$z(\mathcal{J}_k^a g(z))' = -\lambda(\mathcal{J}_k^a g(z)) + (\lambda+1)\mathcal{J}_k^{a-1}g(z), \quad (25)$$

For $-1 < \lambda \leq 2$, then, from (24), we have

$$\frac{\mathcal{J}_k^{a-1}g(z)}{\mathcal{J}_k^a g(z)} = \frac{\lambda + 1 + (\mu(M-N) + (\lambda+1)N)w(z)}{(\lambda+1)(1 + Nw(z))}, \quad (26)$$

which implies that

$$|\mathcal{J}_k^{a-1}f(z)| \leq \left[\frac{|z|(1 - |\psi(z)|^2)(1 + |N||z|)}{(1 - |z|^2)((\lambda+1) - |\mu(M-N) + (\lambda+1)N||z|)} + |\psi(z)| \right] |\mathcal{J}_k^{a-1}g(z)|. \quad (33)$$

Setting $|z| = \rho$ and $|\psi(z)| = c$, then inequality (33) leads to

$$|\mathcal{J}_k^{a-1}f(z)| \leq \frac{\zeta(\rho, c) \|\mathcal{J}_k^{a-1}g(z)\|}{(1 - \rho^2)((\lambda+1) - |\mu(M-N) + (\lambda+1)N|\rho)}, \quad (34)$$

where

$$\begin{aligned} \zeta(\rho, c) &= \rho(1 - c^2)(1 + |N|\rho) + c(1 - \rho^2) \\ &\cdot [(\lambda+1) - |\mu(M-N) + (\lambda+1)N|\rho]. \end{aligned} \quad (35)$$

Then, from (34),

$$|\mathcal{J}_k^{a-1}f(z)| \leq \eta(\rho, c) |\mathcal{J}_k^{a-1}g(z)|, \quad (36)$$

where

$$\eta(\rho, c) = \frac{\zeta(\rho, c)}{(1 - \rho^2)((\lambda+1) - |\mu(M-N) + (\lambda+1)N|\rho)}. \quad (37)$$

From relation (36), in order to prove our result, we need to determine

$$|\mathcal{J}_k^a g(z)| \leq \frac{(\lambda+1)(1 + |N||z|) |\mathcal{J}_k^{a-1}g(z)|}{(\lambda+1) - |\mu(M-N) + (\lambda+1)N||z|}. \quad (27)$$

Now, since $\mathcal{J}_k^a f(z)$ is majorized by $\mathcal{J}_k^a g(z)$ in open unit disk D , then

$$\mathcal{J}_k^a f(z) = \psi(z) \mathcal{J}_k^a g(z). \quad (28)$$

Differentiating the previous equality with respect to z and then multiplying by z , we get

$$z(\mathcal{J}_k^a f(z))' = z\psi(z)(\mathcal{J}_k^a g(z))' + z\psi'(z)(\mathcal{J}_k^a g(z)). \quad (29)$$

On using relation (12), we have

$$(\lambda+1)\mathcal{J}_k^{a-1}f(z) = z\psi'(z)\mathcal{J}_k^a g(z) + (\lambda+1)\psi(z)\mathcal{J}_k^{a-1}g(z). \quad (30)$$

This implies

$$(\lambda+1)|\mathcal{J}_k^{a-1}f(z)| \leq |z|\|\psi'(z)\| \|\mathcal{J}_k^a g(z)\| + (\lambda+1)|\psi(z)| \|\mathcal{J}_k^{a-1}g(z)\|. \quad (31)$$

Note, therefore, that the Schwarz function ψ satisfies the inequality (see [21])

$$|\psi'(z)| \leq \frac{1 - |\psi(z)|^2}{1 - |z|^2}; \quad (z \in D). \quad (32)$$

On using (27) and (32) in (31), we have

$$\begin{aligned} \rho_0 &= \max\{\rho \in [0, 1]; \eta(\rho, c) \leq 1 \forall c \in [0, 1]\} \\ &= \max\{\rho \in [0, 1]; G(\rho, c) \geq 0 \forall c \in [0, 1]\}, \end{aligned} \quad (38)$$

where

$$\begin{aligned} G(\rho, c) &= (1 - \rho^2)(1 - c)[(\lambda+1) - |\mu(M-N) \\ &+ (\lambda+1)N|\rho] - \rho(1 - c^2)(1 + |N|\rho). \end{aligned} \quad (39)$$

A simple calculation shows that the $G(\rho, c) \geq 0$ inequality is equivalent to

$$\begin{aligned} u(\rho, c) &= [(\lambda+1) - |\mu(M-N) + (\lambda+1)N|\rho](1 - \rho^2) \\ &- \rho(1 + c)(1 + |N|\rho) \geq 0. \end{aligned} \quad (40)$$

However, the function $u(\rho, c)$ has a minimum value at $c = 1$, that is,

$$\min\{u(\rho, c): c \in [0, 1]\} = u(\rho, 1) = v(\rho), \quad (41)$$

where

$$\begin{aligned} u(\rho) &= |\mu(M-N) + (\lambda+1)N|\rho^3 - (2|N| + \lambda + 1)\rho^2 \\ &- [2 + |\mu(M-N) + (\lambda+1)N|]\rho + (\lambda+1) = 0. \end{aligned} \quad (42)$$

It follows that $v(\rho) \geq 0 \forall \rho \in [0, \rho_0]$, where $\rho_0 = \rho_0(\mu, \lambda, M, N)$ is the smallest positive root of equation (22), which proves conclusion (21). \square

Theorem 2. Let the function $f \in A$, and assume that $g \in R_k^a(\mu)$. If $\mathcal{J}_k^a f(z)$ is majorized by $\mathcal{J}_k^a g(z)$ in D , then

$$|\mathcal{J}_k^{a-1} f(z)| \leq |\mathcal{J}_k^{a-1} g(z)| \text{ for } |z| \leq \rho_1, \quad (43)$$

where ρ_1 is the smallest positive root of the equation

$$(e^\rho + \mu(\lambda + 1) - |\lambda|)\rho^2 - 2(1 + \mu)\rho + (|\lambda| - \mu(\lambda + 1) - e^\rho) = 0, \quad (44)$$

where $a \geq 0, k \in \Phi, \mu \geq 0, -1 < \lambda \leq 2$, and $|\lambda| > \mu(\lambda + 1) + e$.

Proof. Since $g \in R_k^a(\mu)$, then, from (19) and the subordination relation,

$$\left[\frac{z(\mathcal{J}_k^a g(z))'}{\mathcal{J}_k^a g(z)} - \mu \left| \frac{z(\mathcal{J}_k^a g(z))'}{\mathcal{J}_k^a g(z)} - 1 \right| \right] = e^{w(z)} (z \in D), \quad (45)$$

where w is the analytic function in D , with $w(0) = 0$ and $|w(z)| \leq |z| \leq 1, \forall z \in D$. Now, let

$$W = \frac{z(\mathcal{J}_k^a g(z))'}{\mathcal{J}_k^a g(z)}. \quad (46)$$

In (45), we have

$$W - \mu|W - 1| = e^{w(z)}, \quad (47)$$

which implies that

$$W - \mu(W - 1)e^{i\phi} = e^{w(z)}. \quad (48)$$

Then, we have

$$W = \frac{e^{w(z) - \mu e^{i\phi}}}{1 - \mu e^{i\phi}}. \quad (49)$$

From (46 and 49), we have

$$\frac{z(\mathcal{J}_k^a g(z))'}{\mathcal{J}_k^a g(z)} = \frac{e^{w(z) - \mu e^{i\phi}}}{1 - \mu e^{i\phi}}. \quad (50)$$

Now, on using (12) in (50), for $-1 < \lambda \leq 2$, we have the following:

$$\frac{\mathcal{J}_k^a g(z)}{\mathcal{J}_k^{a-1} g(z)} = \frac{e^{w(z)} + \lambda - (\lambda + 1)\mu e^{i\phi}}{(\lambda + 1)(1 - \mu e^{i\phi})}, \quad (51)$$

which implies that

$$|\mathcal{J}_k^a g(z)| \leq \frac{(\lambda + 1)(1 + \mu)}{|\lambda| - \mu(\lambda + 1) - e^{|z|}} |\mathcal{J}_k^{a-1} g(z)|. \quad (52)$$

Now, since $\mathcal{J}_k^a f(z)$ is majorized by $\mathcal{J}_k^a g(z)$ in D , then we have

$$\mathcal{J}_k^a f(z) = \psi(z) \mathcal{J}_k^a g(z). \quad (53)$$

Differentiating the previous equality with respect to z and then multiplying by z , we get

$$z(\mathcal{J}_k^a f(z))' = z\psi(z)(\mathcal{J}_k^a g(z))' + z\psi'(z)(\mathcal{J}_k^a g(z)). \quad (54)$$

On using relation (12), we have

$$(\lambda + 1)\mathcal{J}_k^{a-1} f(z) = z\psi'(z)\mathcal{J}_k^a g(z) + (\lambda + 1)\psi(z)\mathcal{J}_k^{a-1} g(z). \quad (55)$$

This implies

$$(\lambda + 1)|\mathcal{J}_k^{a-1} f(z)| \leq |z|\|\psi'(z)\|\mathcal{J}_k^a g(z) + (\lambda + 1)|\psi(z)|\mathcal{J}_k^{a-1} g(z). \quad (56)$$

Thus, note that the Schwarz function ψ satisfies the inequality (see [21])

$$|\psi'(z)| \leq \frac{1 - |\psi(z)|^2}{1 - |z|^2}; (z \in D). \quad (57)$$

On using (52) and (57) in (56), we have

$$|\mathcal{J}_k^{a-1} f(z)| \leq \left(\frac{|z|(1 + \mu)(1 - |\psi(z)|^2)}{(1 - |z|^2)(|\lambda| - \mu(\lambda + 1) - e^{|z|})} + |\psi(z)| \right) |\mathcal{J}_k^{a-1} g(z)|. \quad (58)$$

Setting $|z| = \rho$ and $|\psi(z)| = c$ ($0 \leq c \leq 1$), then inequality (58) leads to

$$|\mathcal{J}_k^{a-1} f(z)| \leq \frac{\zeta_1(\rho, c)}{(1 - \rho^2)(|\lambda| - \mu(\lambda + 1) - e^\rho)} |\mathcal{J}_k^{a-1} g(z)|, \quad (59)$$

where

$$\zeta_1(\rho, c) = \rho(1 + \mu)(1 - c^2) + c(1 - \rho^2)(|\lambda| - \mu(\lambda + 1) - e^\rho). \quad (60)$$

Then, from (59),

$$|\mathcal{J}_k^{a-1} f(z)| \leq \eta_1(\rho, c) |\mathcal{J}_k^{a-1} g(z)|. \quad (61)$$

Here,

$$\eta_1(\rho, c) = \frac{\zeta_1(\rho, c)}{(1 - \rho^2)(|\lambda| - \mu(\lambda + 1) - e^\rho)}. \quad (62)$$

From relation (61), in order to prove our result, we need to determine

$$\rho_1 = \max\{\rho \in [0, 1]; \eta_1(\rho, c) \leq 1 \forall c \in [0, 1]\}, \quad (63)$$

$$= [\max\{\rho \in [0, 1]; G_1(\rho, c) \geq 0 \forall c \in [0, 1]\}],$$

where

$$G_1(\rho, c) = (1 - \rho^2)(1 - c)(|\lambda| - \mu(\lambda + 1) - e^\rho) - \rho(1 + \mu)(1 - c^2). \quad (64)$$

A simple calculus shows that the inequality $G_1(\rho, c) \geq 0$ is equivalent to

$$u_1(\rho, c) = (1 - \rho^2)(|\lambda| - \mu(\lambda + 1) - e^\rho) - \rho(1 + \mu)(1 + c) \geq 0. \quad (65)$$

However, the function $u_1(\rho, c)$ takes its minimum value at $c = 1$, that is,

$$\min\{u_1(\rho, c): c \in [0, 1]\} = u_1(\rho, 1) = v_1(\rho), \quad (66)$$

where

$$v_1(\rho) = (1 - \rho^2)(|\lambda| - \mu(\lambda + 1) - e^\rho) - 2\rho(1 + \mu) = 0. \quad (67)$$

It follows that $v_1(\rho) \geq 0 \forall \rho \in [0, \rho_1]$, where $\rho_1 = \rho_1(\mu, \lambda)$ is the smallest positive root of (44), which prove conclusion (43). \square

Theorem 3. Let the function $f \in A$, and assume that $g \in T_k^a(\theta)$. If $\mathcal{F}_k^a f(z)$ is majorized by $\mathcal{F}_k^a g(z)$ in D , then

$$|\mathcal{F}_k^{a-1} f(z)| \leq |\mathcal{F}_k^{a-1} g(z)| \text{ for } |z| \leq \rho^*, \quad (68)$$

where ρ^* is the smallest positive root of the equation

$$(e^\rho + (\lambda + 1)|\tan \theta| - |\lambda|)\rho^2 - 2|\sec \theta|\rho + (|\lambda| - (\lambda + 1)|\tan \theta| - e^\rho) = 0, \quad (69)$$

where $a \geq 0, k \in \Phi, -(\Pi/2) < \theta < (\Pi/2), -1 < \lambda \leq 2$, and $|\lambda| > (\lambda + 1)|\tan \theta| + e$.

Proof. Since $g \in T_k^a(\theta)$, then, from (20) and the subordination relation,

$$e^{i\theta} \left(\frac{z(\mathcal{F}_k^a g(z))'}{\mathcal{F}_k^a g(z)} \right) = e^{w(z)} \cos \theta + i \sin \theta, \quad (70)$$

where w is the analytic function in D , with $w(0) = 0$ and $|w(z)| \leq |z| \leq 1, \forall z \in D$. From (70), we have

$$\frac{z(\mathcal{F}_k^a g(z))'}{\mathcal{F}_k^a g(z)} = \frac{e^{w(z)+i \tan \theta}}{1 + i \tan \theta}. \quad (71)$$

Now, on using (12) in (71), for $-1 < \lambda \leq 2$, we have the following:

$$\frac{\mathcal{F}_k^{a-1} g(z)}{\mathcal{F}_k^a g(z)} = \frac{e^{w(z)} + \lambda + (\lambda + 1)i \tan \theta}{(\lambda + 1)(1 + i \tan \theta)}, \quad (72)$$

which implies that

$$|\mathcal{F}_k^a g(z)| \leq \frac{(\lambda + 1)|\sec \theta|}{(|\lambda| - (\lambda + 1)|\tan \theta| - e^{|z|})} |\mathcal{F}_k^{a-1} g(z)|. \quad (73)$$

Now, since $\mathcal{F}_k^a f(z)$ is majorized by $\mathcal{F}_k^a g(z)$ in D , then we have

$$\mathcal{F}_k^a f(z) = \psi(z) \mathcal{F}_k^a g(z). \quad (74)$$

Differentiating the previous equality with respect to z and then multiplying by z , we get

$$z(\mathcal{F}_k^a f(z))' = z\psi(z)(\mathcal{F}_k^a g(z))' + z\psi'(z)(\mathcal{F}_k^a g(z)). \quad (75)$$

On using relation (12), we have

$$(\lambda + 1)\mathcal{F}_k^{a-1} f(z) = z\psi'(z)\mathcal{F}_k^a g(z) + (\lambda + 1)\psi(z)\mathcal{F}_k^{a-1} g(z). \quad (76)$$

This implies

$$(\lambda + 1)|\mathcal{F}_k^{a-1} f(z)| \leq |z|\|\psi'(z)\|\mathcal{F}_k^a g(z) + (\lambda + 1)|\psi(z)|\mathcal{F}_k^{a-1} g(z). \quad (77)$$

Thus, note that the Schwarz function ϕ satisfies the inequality (see [21])

$$|\psi'(z)| \leq \frac{1 - |\psi(z)|^2}{1 - |z|^2}; \quad (z \in D). \quad (78)$$

On using (71) and (78) in (77), we have

$$|\mathcal{F}_k^{a-1} f(z)| \leq \left(\frac{|z|(1 - |\psi(z)|^2)|\sec \theta|}{(1 - |z|^2)(|\lambda| - (\lambda + 1)|\tan \theta| - e^{|z|})} + |\psi(z)| \right) \cdot |\mathcal{F}_k^{a-1} g(z)|. \quad (79)$$

Setting $|z| = \rho$ and $|\psi(z)| = c$ ($0 \leq c \leq 1$), then inequality (79) leads to

$$|\mathcal{F}_k^{a-1} f(z)| \leq \frac{\zeta_2(\rho, c)}{(1 - \rho^2)(|\lambda| - (\lambda + 1)|\tan \theta| - e^\rho)} |\mathcal{F}_k^{a-1} g(z)|, \quad (80)$$

where

$$\zeta_2(\rho, c) = \rho(1 - c^2)|\sec \theta| + c(1 - \rho^2)(|\lambda| - (\lambda + 1)|\tan \theta| - e^\rho). \quad (81)$$

Then, from (80),

$$|\mathcal{F}_k^{a-1} f(z)| \leq \eta_2(\rho, c) |\mathcal{F}_k^{a-1} g(z)|, \quad (82)$$

where

$$\eta_2(\rho, c) = \frac{\zeta_2(\rho, c)}{(1 - \rho^2)(|\lambda| - (\lambda + 1)|\tan \theta| - e^\rho)}. \quad (83)$$

From relation (82), in order to prove our result, we need to determine

$$\begin{aligned} \rho^* &= \max\{\rho \in [0, 1]; \eta_2(\rho, c) \leq 1 \forall c \in [0, 1]\} \\ &= \max\{\rho \in [0, 1]; G_2(\rho, c) \geq 0 \forall c \in [0, 1]\}, \end{aligned} \quad (84)$$

where

$$\begin{aligned} G_2(\rho, c) &= (1 - \rho^2)(1 - c)(|\lambda| - (\lambda + 1)|\tan \theta| - e^\rho) \\ &\quad - \rho(1 - c^2)|\sec \theta|. \end{aligned} \quad (85)$$

A simple calculus shows that the inequality $G_2(r, \rho) \geq 0$ is equivalent to

$$\begin{aligned} u_2(\rho, c) &= (1 - \rho^2)(|\lambda| - (\lambda + 1)|\tan \theta| - e^\rho) \\ &\quad - \rho(1 + c)|\sec \theta| \geq 0. \end{aligned} \quad (86)$$

However, the function $u_2(\rho, c)$ takes its minimum value at $c = 1$, that is,

$$\min\{u_2(\rho, c): c \in [0, 1]\} = u_2(\rho, 1) = v_2(\rho), \quad (87)$$

where

$$v_2(\rho) = (1 - \rho^2)(|\lambda| - (\lambda + 1)|\tan \theta| - e^\rho) - 2\rho|\sec \theta| = 0. \quad (88)$$

It follows that $v_2(\rho) \geq 0 \forall \rho \in [0, \rho^*]$, where $\rho^* = \rho^*(\theta, \lambda)$ is the smallest positive root of (69), which proves conclusion (68). \square

3. Corollaries and Consequences

If we take the values of k defined in (13) and (15), then the above theorems give the following corollaries.

Corollary 1. Let the function $f \in A$, and assume that $g \in S_{k_1}^a[M, N, \mu]$. If $\mathcal{P}^a f(z)$ is majorized by $\mathcal{P}^a g(z)$ in D , then

$$|\mathcal{P}^{a-1} f(z)| \leq |\mathcal{P}^{a-1} g(z)| \text{ for } |z| \leq \rho_2, \quad (89)$$

where ρ_2 is the smallest positive root of the equation

$$\begin{aligned} & \mu(M - N) + 2N|\rho^3 - 2(|N| + 1)\rho^2 \\ & - (2 + |\mu(M - N) + 2N|)\rho + 2 = 0, \end{aligned} \quad (90)$$

where $-1 \leq N < M \leq 1, \mu \in \mathbb{C}^*$, and $2 \geq |\mu(M - N) + 2N|$.

Corollary 2. Let the function $f \in A$, and assume that $g \in R_{k_1}^a(\mu)$. If $\mathcal{P}^a f(z)$ is majorized by $\mathcal{P}^a g(z)$ in D , then

$$|\mathcal{P}^{a-1} f(z)| \leq |\mathcal{P}^{a-1} g(z)| \text{ for } |z| \leq \rho_3, \quad (91)$$

where ρ_3 is the smallest positive root of the equation

$$(e^\rho + 2\mu - 1)\rho^2 - 2(1 + \mu)\rho + (1 - 2\mu - e^\rho) = 0, \quad (92)$$

where $a \geq 0, \mu \geq 0$, and $1 > 2\mu + e$.

Corollary 3. Let the function $f \in A$, and assume that $g \in T_{k_1}^a(\theta)$. If $\mathcal{P}^a f(z)$ is majorized by $\mathcal{P}^a g(z)$ in D , then

$$|\mathcal{P}^{a-1} f(z)| \leq |\mathcal{P}^{a-1} g(z)| \text{ for } |z| \leq \rho_1^*, \quad (93)$$

where ρ_1^* is the smallest positive root of the equation

$$(e^\rho + 2|\tan \theta| - 1)\rho^2 - 2|\sec \theta|\rho + (1 - 2|\tan \theta| - e^\rho) = 0, \quad (94)$$

where $a \geq 0, -(\pi/2) < \theta < (\pi/2)$, and $1 > 2|\tan \theta| + e$.

Corollary 4. Let the function $f \in A$, and assume that $g \in S_{k_2}^a[M, N, \mu]$. If $Q_b^a f(z)$ is majorized by $Q_b^a g(z)$ in D , then

$$|Q_b^{a-1} f(z)| \leq |Q_b^{a-1} g(z)| \text{ for } |z| \leq \rho_4, \quad (95)$$

where ρ_4 is the smallest positive root of the equation

$$\begin{aligned} & \mu(M - N) + (a + b)N|\rho^3 - (2|N| + a + b)\rho^2 \\ & - (2 + |\mu(M - N) + (a + b)N|)\rho + (a + b) = 0, \end{aligned} \quad (96)$$

where $-1 \leq N < M \leq 1, \mu \in \mathbb{C}^*, a > 1, b > -1$, and $(a + b) \geq |\mu(M - N) + (a + b)N|$.

Corollary 5. Let the function $f \in A$, and assume that $g \in R_{k_2}^a(\mu)$. If $Q_b^a f(z)$ is majorized by $Q_b^a g(z)$ in D , then

$$|Q_b^{a-1} f(z)| \leq |Q_b^{a-1} g(z)| \text{ for } |z| \leq \rho_5, \quad (97)$$

where ρ_5 is the smallest positive root of the equation

$$\begin{aligned} & (e^\rho + \mu(a + b) - |a + b - 1|)\rho^2 - 2(1 + \mu)\rho \\ & + (|a + b - 1| - \mu(a + b) - e^\rho) = 0, \end{aligned} \quad (98)$$

where $a \geq 0, b > -1, \mu \geq 0$, and $|a + b - 1| > \mu(a + b) + e$.

Corollary 6. Let the function $f \in A$, and assume that $g \in T_{k_2}^a(\theta)$. If $Q_b^a f(z)$ is majorized by $Q_b^a g(z)$ in D , then

$$|Q_b^{a-1} f(z)| \leq |Q_b^{a-1} g(z)| \text{ for } |z| \leq \rho_2^*, \quad (99)$$

where ρ_2^* is the smallest positive root of the equation

$$\begin{aligned} & (e^\rho + (a + b)|\tan \theta| - |a + b - 1|)\rho^2 - 2|\sec \theta|\rho \\ & + (|a + b - 1| - (a + b)|\tan \theta| - e^\rho) = 0, \end{aligned} \quad (100)$$

where $a \geq 0, b > -1, -(\pi/2) < \theta < (\pi/2)$, and $|a + b - 1| > (a + b)|\tan \theta| + e$.

If we take $M = 1$ and $N = -1$, then Theorem 1, Corollary 1, and Corollary 4 give the following results.

Corollary 7. Let the function $f \in A$, and assume that $g \in S_k^a[1, -1, \mu]$. If $\mathcal{F}_k^a f(z)$ is majorized by $\mathcal{F}_k^a g(z)$ in D , then

$$|\mathcal{F}_k^{a-1} f(z)| \leq |\mathcal{F}_k^{a-1} g(z)| \text{ for } |z| \leq \rho_0, \quad (101)$$

where ρ_0 is the smallest positive root of the equation

$$|2\mu - (1 + \lambda)|\rho^3 - (\lambda + 3)\rho^2 - (2 + |2\mu - (\lambda + 1)|)\rho + (\lambda + 1) = 0, \quad (102)$$

where $k \in \Phi, \mu \in \mathbb{C}^*, -1 < \lambda \leq 2$, and $(\lambda + 1) \geq |2\mu - (\lambda + 1)|$.

Corollary 8. Let the function $f \in A$, and assume that $g \in S_{k_1}^a[1, -1, \mu]$. If $\mathcal{P}^a f(z)$ is majorized by $\mathcal{P}^a g(z)$ in D , then

$$|\mathcal{P}^{a-1} f(z)| \leq |\mathcal{P}^{a-1} g(z)| \text{ for } |z| \leq \rho_2, \quad (103)$$

where ρ_2 is the smallest positive root of the equation

$$|\mu - 1|\rho^3 - 2\rho^2 - (1 + |\mu - 1|)\rho + 1 = 0, \quad (104)$$

where $\mu \in \mathbb{C}^*$ and $1 \geq |\mu - 1|$.

Corollary 9. Let the function $f \in A$, and assume that $g \in S_{k_2}^a[1, -1, \mu]$. If $Q_b^a f(z)$ is majorized by $Q_b^a g(z)$ in D , then

$$|Q_b^{a-1} f(z)| \leq |Q_b^{a-1} g(z)| \text{ for } |z| \leq \rho_4, \quad (105)$$

where ρ_4 is the smallest positive root of the equation

$$|2\mu - (a + b)|\rho^3 - (2 + a + b)\rho^2 - (2 + |2\mu - (a + b)|)\rho + (a + b) = 0, \quad (106)$$

where $\mu \in \mathbb{C}^*, a > 1, b > -1$, and $(a + b) \geq |2\mu - (a + b)|$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have contributed equally to the paper.

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Research Article

Applications of q -Derivative Operator to the Subclass of Bi-Univalent Functions Involving q -Chebyshev Polynomials

Bilal Khan ¹, Zhi-Guo Liu,¹ Timilehin Gideon Shaba,² Serkan Araci ³,
Nazar Khan,⁴ and Muhammad Ghaffar Khan ⁵

¹School of Mathematical Sciences and Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China

²Department of Mathematics, University of Ilorin, P. M. B., Ilorin 1515, Nigeria

³Department of Economics, Faculty of Economics Administrative and Social Sciences, Hasan Kalyoncu University, TR-27410, Gaziantep, Turkey

⁴Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad, Pakistan

⁵Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat, Pakistan

Correspondence should be addressed to Serkan Araci; mtrskn@gmail.com

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In recent years, the usage of the q -derivative and symmetric q -derivative operators is significant. In this study, firstly, many known concepts of the q -derivative operator are highlighted and given. We then use the symmetric q -derivative operator and certain q -Chebyshev polynomials to define a new subclass of analytic and bi-univalent functions. For this newly defined functions' classes, a number of coefficient bounds, along with the Fekete-Szegő inequalities, are also given. To validate our results, we give some known consequences in form of remarks.

1. Introduction and Definitions

Let $\mathcal{H}(\mathbb{D})$ denote the class of functions which are analytic in the open unit disk:

$$\mathbb{D} = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}. \quad (1)$$

Let \mathcal{A} be the subclass of functions $f \in \mathcal{H}(\mathbb{D})$, which satisfy the normalization condition given by

$$f(0) = f'(0) - 1 = 0, \quad (2)$$

that is, which are represented by the following Taylor-Maclaurin series expansion:

$$f(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad z \in \mathbb{D}. \quad (3)$$

Also, let \mathcal{S} be the class of functions in \mathcal{A} , which are univalent in \mathbb{D} .

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$\begin{aligned} f^{-1}(f(z)) &= z, \quad z \in \mathbb{D}, \\ f^{-1}(f(w)) &= w, \quad |w| < r_0(f); r_0(f) \geq \frac{1}{4}, \end{aligned} \quad (4)$$

where

$$\begin{aligned} f^{-1}(w) &= g(w) = w - b_2 w^2 + (2b_2^2 - b_3) w^3 \\ &\quad - (5b_2^3 - 5b_2 b_3 + b_4) w^4 + \dots \end{aligned} \quad (5)$$

A function is said to be bi-univalent in \mathbb{D} if both f and f^{-1} are univalent in \mathbb{D} .

Let Σ denote the class of bi-univalent function in \mathbb{D} given by (3). Example of functions in the class Σ is

$$\frac{z}{1-z}, \log \frac{1}{1-z} \text{ and } \log \sqrt{\frac{1+z}{1-z}}. \quad (6)$$

However, the familiar Koebe function is an example of the class Σ . Other common examples of functions in \mathcal{S} , such as

$$\frac{2z - z^2}{2} \text{ and } \frac{z}{1 - z^2}, \quad (7)$$

are also not members of Σ .

Lewin [1] investigated a bi-univalent functions class Σ and showed that $|b_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|b_2| < \sqrt{2}$. Netanyahu [3], on the contrary, showed that

$$\max_{f \in \Sigma} |b_2| = \frac{4}{3}. \quad (8)$$

The coefficient for each of the Taylor–Maclaurin coefficients $|a_n|$ ($n \geq 3$, $n \in \mathbb{N}$) is presumably still an open problem.

Similar to the familiar subclasses $\mathcal{S}^*(\zeta)$ and $\mathcal{K}(\zeta)$ of star-like and convex functions of order ζ ($0 \leq \zeta < 1$), respectively, Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ , namely, $\mathcal{S}_\Sigma^*(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$ of bi-star-like functions and bi-convex functions of order ζ ($0 \leq \zeta < 1$), respectively. For each of the function classes $\mathcal{S}_\Sigma^*(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$, they found nonsharp bounds on the first two Taylor–Maclaurin coefficients $|b_2|$ and $|b_3|$.

Furthermore, let s_1 and s_2 be analytic functions in open unit disc \mathbb{D} ; then, the function s_1 is subordinated to s_2 and symbolically denoted as

$$s_1(z) \prec s_2(z), \quad z \in \mathbb{D}, \quad (9)$$

if there occurs an analytic function w with properties that

$$w(0) = 0 \text{ and } |w(z)| < 1. \quad (10)$$

Suppose w holomorphic in \mathbb{D} , such that

$$s_1(z) = s_2(w(z)). \quad (11)$$

If the function s_2 is univalent in \mathbb{D} , then the above condition is equivalent to

$$s_1(z) \prec s_2(z) \iff s_1(0) = s_2(0) \text{ and } s_1(\mathbb{D}) \subset s_2(\mathbb{D}). \quad (12)$$

Jackson [5] introduced and studied the q -derivative operator \mathfrak{D}_q of a function as follows:

$$\mathfrak{D}_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} = \frac{1}{z} \left\{ z + \sum_{k=2}^{\infty} \left(\frac{1-q^k}{1-q} \right) a_k z^k \right\} \quad (13)$$

and $\mathfrak{D}_q f(0) = f'(0)$. In case $f(z) = z^k$, for k is a positive integer, the q -derivative of f is given by

$$\mathfrak{D}_q z^k = \frac{(zq)^k - z^k}{z(q-1)} = \left(\frac{1-q^k}{1-q} \right) z^{k-1}, \quad (14)$$

$$\lim_{q \rightarrow 1^-} [k]_q = \lim_{q \rightarrow 1^-} \frac{1-q^k}{1-q} = k, \quad (15)$$

where ($z \neq 0$, $q \neq 0$). For more details on the concepts of q -derivative, see [6, 7].

The quantum (or q -) calculus is an essential tool for studying diverse families of analytic functions, and its applications in mathematics and related fields have inspired researchers. Srivastava [8] was the first person to apply it in the context of univalent functions. Numerous scholars conducted substantial work on q -calculus and examined its various applications due to the applicability of q -analysis in mathematics and other domains. For example, with the help of certain higher-order q -derivative operators, Khan et al. [7] defined and studied a number of subclasses of q -star-like functions. Also, Shi et al. [9] (see also [10]) used the q -differential operator and defined a new subclass of Janowski-type multivalent q -star-like functions. In [7, 9], a number of sufficient conditions and some other interesting properties have been examined. More importantly, the convolution theory enables us to investigate various properties of analytic functions. Due to the large range of applications of q -calculus and the importance of q -operators instead of regular operators, many researchers have explored q -calculus in depth. In addition, Srivastava [11] recently published survey-cum-expository review paper which is useful for researchers and scholars (see, for example, [12, 13]) working on these subjects. Also, Srivastava's recent survey-cum-expository review article [11] further motivates the use of the q -analysis in geometric function theory, as well as commenting on the triviality of the so-called (\mathfrak{p}, q) -analysis involving an insignificant and redundant parameter (\mathfrak{p}, q) (see p. 340 of [11]).

Utilizing the idea of q -derivative operator, in 2013, Brahim et al. introduced and studied the symmetric q -derivative operator $(\mathfrak{D}_q f)$ for a function f as follows:

$$(\mathfrak{D}_q f)(z) = \begin{cases} \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}, & (z \neq 0), \\ f'(0), & (z = 0). \end{cases} \quad (16)$$

It is easy to see that

$$\begin{aligned} \mathfrak{D}_q z^k &= [\widetilde{k}]_q z^{k-1}, \\ \mathfrak{D}_q f(z) &= 1 + \sum_{k=2}^{\infty} [\widetilde{k}]_q b_k z^{k-1}, \end{aligned} \quad (17)$$

where

$$[\widetilde{k}]_q = \frac{q^k - q^{-k}}{q - q^{-1}}. \quad (18)$$

The relation between q -derivative operator and symmetric q -derivative operator is given by

$$(\mathfrak{D}_q f)(z) = \mathfrak{D}_{q^2} f(q^{-1}z). \quad (19)$$

Suppose f^{-1} is the inverse of f ; then,

$$(\mathfrak{D}_q f^{-1})(w) = 1 - [2]_q b_2 w + [3]_q (2b_2^2 - b_3) w^2 - [4]_q (5b_2^3 - 5b_2 b_3 + b_4) w^3 + \dots \quad (20)$$

Al Salam and Ismail [14] discovered a family of polynomials that can be understood as q -analogues of the second-order bivariate Chebyshev polynomials. In 2012, Johann Cigler introduced and studied the q -Chebyshev polynomials as follows.

Definition 1 (see [15]). The polynomials

$$\begin{aligned} U_m(t, y, q) &= P_{m+1}(t, -1, y, q)(-q; q)_m \\ &= \sum_{k=0}^{(n/2)} q^{k^2} \begin{bmatrix} m-k \\ k \end{bmatrix} (1+q^{k+1}) \cdots (1+q^{m-k}) y^k t^{m-2k} \end{aligned} \quad (21)$$

are called q -Chebyshev polynomial of the second kind.

Theorem 1 (see [15]). The q -Chebyshev polynomials of the second kind satisfy

$$U_m(t, y, q) = (1+q^m)tU_{m-1}(t, y, q) + q^{m-1}yU_{m-2}(t, y, q), \quad (22)$$

with initial values

$$U_0(t, y, q) = 1 \text{ and } U_1(t, y, q) = (1+q)t. \quad (23)$$

Remark 1. It is clear that

$$U_m(t, -1, 1) = U_m(t), \quad (24)$$

where $U_m(t)$ is the classical Chebyshev polynomial of the second kind.

Now, making use q -Chebyshev polynomials, we define the following.

Definition 2. Let $\mathfrak{M}(z, t, y, q)$ be defined as follows:

$$\mathfrak{M}(z, t, y, q) = 1 + \sum_{j=1}^{\infty} U_j(t, y, q) z^j. \quad (25)$$

By using the principal of subordination and the symmetric q -derivative operator \mathfrak{D}_q , we define the following subclasses of analytic and bi-univalent functions.

Definition 3. A function $f \in \Sigma$ given by (3) is said to be in the class $\tilde{M}_{\Sigma}^{q,y}(t)$ if the following conditions are satisfied:

$$(\mathfrak{D}_q f(z)) < \mathfrak{M}(z, t, y, q), \quad \frac{1}{2} < t < 1, 0 < q < 1, z \in \mathbb{D}, \quad (26)$$

$$(\mathfrak{D}_q f^{-1}(w)) < \mathfrak{M}(w, t, y, q), \quad \frac{1}{2} < t < 1, 0 < q < 1, w \in \mathbb{D}. \quad (27)$$

We note from (25) that

$$\begin{aligned} \mathfrak{M}(z, t, y, q) &= 1 + U_1(t, y, q)z + U_2(t, y, q)z^2 \\ &\quad + U_3(t, y, q)z^3 + \dots, \end{aligned} \quad (28)$$

where $z \in \mathbb{D}$ and $t \in (-1, 1)$.

Also, from (22), we have the following:

$$\left[\begin{array}{l} U_1(t, y, q) = (1+q)t \\ U_2(t, y, q) = t^2(1+q)(1+q^2) + qy \\ U_3(t, y, q) = (1+q)(1+q^2)(1+q^3)t^3 + q(1+q)(1+q^2)yt \\ U_4(t, y, q) = (1+q)(1+q^2)(1+q^3)(1+q^4)t^4 + q(1+q)(1+q^2)(1+q^4+q^2)y^2t + q^4y \end{array} \right]. \quad (29)$$

The goal of this research is to investigate q -Chebyshev polynomial expansions in order to derive initial coefficient estimates for some subclasses of analytic and bi-univalent functions defined by the symmetric q -derivative operator. In addition, Fekete–Szegő inequalities for the class $\tilde{M}_{\Sigma}^{q,y}(t)$ are established.

Lemma 1 (see [16]). Let the function p be given by

$$p(z) = 1 + p_1z + p_2z^2 + \dots, \quad (30)$$

be in the class \mathcal{D} of functions with positive real part. Then,

$$|p_n| \leq 2, \quad n \in \mathbb{N}. \quad (31)$$

This last inequality is sharp.

2. Coefficients Bounds for $f \in \tilde{M}_{\Sigma}^{q,y}(t)$

Theorem 2. Let $f \in \tilde{M}_{\Sigma}^{q,y}(t)$. Then,

$$|b_2| \leq \frac{(1+q)t\sqrt{(1+q)t}}{\sqrt{(1+q)t^2 \left[\widetilde{[3]_q}(1+q) - (1+q^2)\widetilde{[2]_q^2} - qy\widetilde{[2]_q^2} + (1+q) + \widetilde{[2]_q^2} \right]}}, \quad (32)$$

$$|b_3| \leq \frac{(1+q)^2 t^2}{\widetilde{[2]_q^2}} + \frac{(1+q)t}{\widetilde{[3]_q}}, \quad (33)$$

$$|b_4| \leq \frac{5(1+q)^2 t^2}{2\widetilde{[2]_q}\widetilde{[3]_q}} + \frac{(1+q)t}{\widetilde{[4]_q}} + \frac{2t(1+q)[t(1+q^2) - 1] + 2qy}{\widetilde{[4]_q}} + \frac{(1+q)t[1 - 2t(1+q^2) + (1+q^2)(1+q^3)t^2 + q(1+q^2)y] - 2qy}{\widetilde{[4]_q}}. \quad (34)$$

Proof. Let $f \in \sigma$ given by (3) be in the class $\tilde{M}_{\Sigma}^{q,y}(t)$. Then,

$$(\tilde{\mathfrak{D}}_q f(z)) = \mathfrak{M}(\omega(z), t, y, q), \quad (35)$$

$$(\tilde{\mathfrak{D}}_q f^{-1}(w)) = \mathfrak{M}(\omega(w), t, y, q). \quad (36)$$

Let $p, y \in \mathbb{D}$ be defined as

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \quad (37)$$

$$\implies \omega(z) = \frac{p(z) - 1}{p(z) + 1}, \quad z \in \mathbb{D}.$$

$$y(w) = \frac{1 + \omega(w)}{1 - \omega(w)} = 1 + y_1 w + y_2 w^2 + y_3 w^3 + \dots \quad (38)$$

$$\implies \omega(w) = \frac{y(w) - 1}{y(w) + 1}, \quad w \in \mathbb{D}.$$

It follows that, from (37) and (38),

$$\omega(z) = \frac{1}{2} \left[p_1 z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) z^3 + \dots \right], \quad (39)$$

$$\omega(w) = \frac{1}{2} \left[y_1 w + \left(y_2 - \frac{y_1^2}{2} \right) w^2 + \left(y_3 - y_1 y_2 + \frac{y_1^3}{4} \right) w^3 + \dots \right]. \quad (40)$$

From (39) and (40), applying $\mathfrak{M}(z, t, y, q)$ as given in (25), we see that

$$\begin{aligned} \mathfrak{M}(\omega(z), t, y, q) &= 1 + \frac{U_1(t, y, q)}{2} p_1 z \\ &+ \left[\frac{U_1(t, y, q)}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{U_2(t, y, q)}{4} p_1^2 \right] z^2 \end{aligned} \quad (41)$$

$$+ \left[\frac{U_1(t, y, q)}{2} \cdot \left(p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) + \frac{U_2(t, y, q)}{2} p_1 \left(p_2 - \frac{p_1^2}{2} \right) + \frac{U_3(t, y, q)}{8} p_1^3 \right] z^3 + \dots,$$

$$\begin{aligned} \mathfrak{M}(\omega(w), t, y, q) &= 1 + \frac{U_1(t, y, q)}{2} y_1 w + \left[\frac{U_1(t, y, q)}{2} \left(y_2 - \frac{y_1^2}{2} \right) + \frac{U_2(t, y, q)}{4} y_1^2 \right] w^2 \\ &+ \left[\frac{U_1(t, y, q)}{2} \left(y_3 - y_1 y_2 + \frac{y_1^3}{4} \right) + \frac{U_2(t, y, q)}{2} y_1 \left(y_2 - \frac{y_1^2}{2} \right) + \frac{U_3(t, y, q)}{8} y_1^3 \right] w^3 + \dots. \end{aligned} \quad (42)$$

From (35), (41) and (36), (42), we have

$$\widetilde{[2]}_q b_2 = \frac{U_1(t, y, q)}{2} p_1, \quad (43)$$

$$\widetilde{[3]}_q b_3 = \frac{U_1(t, y, q)}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{U_2(t, y, q)}{4} p_1^2, \quad (44)$$

$$\begin{aligned} \widetilde{[4]}_q b_4 &= \frac{U_1(t, y, q)}{2} \left(p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) \\ &+ \frac{U_2(t, y, q)}{2} p_1 \left(p_2 - \frac{p_1^2}{2} \right) + \frac{U_3(t, y, q)}{8} p_1^3, \end{aligned} \quad (45)$$

$$-\widetilde{[2]}_q b_2 = \frac{U_1(t, y, q)}{2} y_1, \quad (46)$$

$$\widetilde{[3]}_q (2b_2^2 - b_3) = \frac{U_1(t, y, q)}{2} \left(y_2 - \frac{y_1^2}{2} \right) + \frac{U_2(t, y, q)}{4} y_1^2, \quad (47)$$

$$\begin{aligned} -\widetilde{[4]}_q (5b_2^3 - 5b_2 b_3 + b_4) &= \frac{U_1(t, y, q)}{2} \left(y_3 - y_1 y_2 + \frac{y_1^3}{4} \right) \\ &+ \frac{U_2(t, y, q)}{2} y_1 \left(y_2 - \frac{y_1^2}{2} \right) \\ &+ \frac{U_3(t, y, q)}{8} y_1^3. \end{aligned} \quad (48)$$

Adding (43) and (46), we have

$$p_1 = -y_1, \quad p_1^2 = y_1^2 \text{ and } p_1^3 = -y_1^3, \quad (49)$$

$$b_2^2 = \frac{U_1^2(t, y, q)(p_1^2 + y_1^2)}{8\widetilde{[2]}_q^2}. \quad (50)$$

Also, adding (44) and (47) and applying (49) yields

$$2\widetilde{[3]}_q b_2^2 = \frac{U_1(t, y, q)}{2} (p_2 + y_2) - y_1^2 (U_1(t, y, q) - U_2(t, y, q)). \quad (51)$$

Applying (49) in (50) gives

$$y_1^2 = \frac{4\widetilde{[2]}_q^2 b_2^2}{U_1^2(t, y, q)}. \quad (52)$$

Putting (52) into (51) with some calculations, we have

$$|b_2|^2 = \left| \frac{U_1^3(t, y, q)(p_2 + y_2)}{4\left[\widetilde{[3]}_q U_1^2(t, y, q) - (U_2(t, y, q) - U_1(t, y, q))\widetilde{[2]}_q^2\right]} \right|. \quad (53)$$

Applying triangular inequality and Lemma 1, we have

$$|b_2| \leq \frac{(1+q)t\sqrt{(1+q)t}}{\sqrt{\left| (1+q)t^2\left[\widetilde{[3]}_q(1+q) - (1+q^2)\widetilde{[2]}_q^2\right] - qy\widetilde{[2]}_q^2 + (1+q)\widetilde{[2]}_q^2 \right|}}. \quad (54)$$

Subtracting (47) from (44) with some calculations, we have

$$b_3 = b_2^2 + \frac{U_1(t, y, q)[p_2 - y_2]}{4\widetilde{[3]}_q}, \quad (55)$$

$$b_3 = \frac{U_1^2(t, y, q)p_1^2}{4\widetilde{[2]}_q^2} + \frac{U_1(t, y, q)[p_2 - y_2]}{4\widetilde{[3]}_q}. \quad (56)$$

Applying triangular inequality and Lemma 1, we have

$$|b_3| \leq \frac{(1+q)^2 t^2}{\widetilde{[2]}_q^2} + \frac{(1+q)t}{\widetilde{[3]}_q}. \quad (57)$$

Subtracting (48) from (45), we have

$$\begin{aligned} 2\widetilde{[4]}_q b_4 &= \frac{5\widetilde{[4]}_q U_1^2(t, y, q)p_1(p_2 - y_2)}{8\widetilde{[2]}_q \widetilde{[3]}_q} + \frac{U_1(t, y, q)(p_3 - y_3)}{2} \\ &+ \frac{[U_2(t, y, q) - U_1(t, y, q)]p_1(p_2 + y_2)}{2} \\ &+ \frac{(U_1(t, y, q) - 2U_2(t, y, q) + U_3(t, y, q))p_1^3}{4}. \end{aligned} \quad (58)$$

Applying triangular inequality and Lemma 1, we have

$$\begin{aligned} |b_4| &\leq \frac{5(1+q)^2 t^2}{2\widetilde{[2]}_q \widetilde{[3]}_q} + \frac{(1+q)t}{\widetilde{[4]}_q} + \frac{2t(1+q)[t(1+q^2) - 1] + 2qy}{\widetilde{[4]}_q} \\ &+ \frac{(1+q)t[1 - 2t(1+q^2) + (1+q^2)(1+q^3)t^2 + q(1+q^2)y] - 2qy}{\widetilde{[4]}_q}. \end{aligned} \quad (59)$$

□

3. Fekete–Szegő Inequalities for the Function Class $\tilde{M}_{\Sigma}^{q,y}(t)$

The n th coefficient of a function class \mathcal{S} is well known to be restricted by n , and the coefficient limits give information about the functions geometric characteristics. The famous problem solved by Fekete–Szegő [17] is to determine the greatest value of the coefficient functional $\Omega_{\sigma}(f)/\text{coloneq}|b_3 - \sigma b_2^2|$ over the class \mathcal{S} for each $\sigma \in [0, 1]$, which was demonstrated using the Loewner technique. In this section, we aim to determine the upper bounds of the coefficient functional $|b_3 - \delta b_2^2|$ for the function class $\tilde{M}_{\Sigma}^{q,y}(t)$.

Theorem 3. Let $f \in \tilde{M}_{\Sigma}^{q,y}(t)$. Then, for some $\delta \in \mathbb{R}$,

$$|b_3 - \delta b_2^2| \leq \begin{cases} \frac{(1+q)t}{[3]_q}, & |\delta - 1| \leq \frac{\Lambda_q(q^{-1}, y, t)}{[3]_q(1+q)^2 t^2}, \\ \frac{(1+q)^3 t^3 |\delta - 1|}{\Lambda_q(q^{-1}, y, t)}, & |\delta - 1| \geq \frac{\Lambda_q(q^{-1}, y, t)}{[3]_q(1+q)^2 t^2}, \end{cases} \quad (60)$$

where

$$\Lambda_q(q^{-1}, y, t) = (1+q)t^2 \left[\widetilde{[3]_q^2}(1+q) - (1+q^2)\widetilde{[2]_q^2} \right] - qy\widetilde{[2]_q^2} + (1+q)t\widetilde{[2]_q^2}. \quad (61)$$

Proof. From (51) and (55), we have

$$\begin{aligned} b_3 - \delta b_2^2 &= \frac{(1-\delta)U_1^3(t, y, q)(p_2 + y_2)}{4\left[\widetilde{[3]_q^2}U_1^2(t, y, q) - (U_2(t, y, q) - U_1(t, y, q))\widetilde{[2]_q^2}\right]} \\ &\quad + \frac{U_1(t, y, q)[p_2 - y_2]}{4[3]_q} \\ &= U_1(t, y, q) \left[\left(J(\delta) + \frac{1}{4[3]_q} \right) p_2 + \left(J(\delta) - \frac{1}{4[3]_q} \right) y_2 \right], \end{aligned} \quad (62)$$

where

$$J(\delta) = \frac{(1-\delta)U_1^2(t, y, q)}{4\left[\widetilde{[3]_q^2}U_1^2(t, y, q) - (U_2(t, y, q) - U_1(t, y, q))\widetilde{[2]_q^2}\right]}. \quad (63)$$

Applying Lemma 1, we have

$$H_2(2) = |b_2 b_4 - b_3^2| \leq \begin{cases} \frac{(1+q)t}{[3]_q}, & 0 \leq |J(\delta)| \leq \frac{1}{4[3]_q}, \\ 4(1+q)t|J(\delta)|, & |J(\delta)| \geq \frac{1}{4[3]_q}. \end{cases} \quad (64)$$

□

Remark 2. Taking $q = 1$ and $y = -1$ in Theorem 2 and Theorem 3, we have the results obtained by Altinkaya and Yalcin [18].

4. Conclusion

Recently, the q -derivative and symmetric q -derivative operators are particularly applicable in many diverse areas of mathematics and physics. In this study, firstly, many known concepts of the q -derivative operator have been highlighted and given. We have then used the symmetric q -derivative operator and certain q -Chebyshev Polynomials and have defined a new subclass of analytic and bi-univalent functions. For these newly defined functions' classes, a number of coefficients bounds, along with the Fekete–Szegő inequalities, have also been given. To validate our results, we have given some known consequence in the form of Remarks.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors jointly worked on the results, and they have read and approved the final manuscript.


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Research Article

Bicomplex Landau and Ikehara Theorems for the Dirichlet Series

Ritu Agarwal ¹, Urvashi Purohit Sharma ^{1,2}, Ravi P. Agarwal ³, Daya Lal Suthar ⁴,
and Sunil Dutt Purohit ⁵

¹Department of Mathematics, Malaviya National Institute of Technology, Jaipur-302017, India

²Department of Science and Humanities, Kalaniketan Polytechnic College, Jabalpur-482001, India

³Department of Mathematics, Texas A&M University-Kingsville, 700 University Blvd., Kingsville, USA

⁴Department of Mathematics, Wollo University, P.O. Box: 1145, Dessie, Ethiopia

⁵Department of HEAS(Mathematics), Rajasthan Technical University, Kota-324010, India

Correspondence should be addressed to Daya Lal Suthar; dlsuthar@gmail.com

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The aim of this paper is to generalize the Landau-type Tauberian theorem for the bicomplex variables. Our findings extend and improve on previous versions of the Ikehara theorem. Also boundedness result for the bicomplex version of Ikehara–Korevaar theorem is derived. The purpose of this article is to substantially extend the various complex Tauberian theorems for the Dirichlet series to the bicomplex domain.

1. Introduction

For a long time, bicomplex numbers have been investigated, and a lot of work has been carried out in this area. Bicomplex numbers are introduced by Segre [1] in 1882. Different algebraic and geometric features of bicomplex numbers, as well as their applications, have been the focus of recent research. Many properties and applications of bicomplex numbers have been discovered (see, [2–8]). In recent developments, efforts have been made to extend the integral transforms [9–14], and a number of special functions like [5, 15–19] to the bicomplex variable from their complex counterparts.

The aim of this paper is to extend the various complex Tauberian theorems for the Dirichlet series to the bicomplex domain. Generalization of Landau-type theorem and Ikehara theorem is introduced. Boundedness condition for the bicomplex Tauberian theorem has been included. In the proof of these results, the decomposition theorem of Ringleb plays a vital role.

1.1. Bicomplex Numbers. The set of bicomplex numbers was defined by Segre [1] in the following way:

Definition 1 (Bicomplex number). The set of bicomplex numbers is defined in terms of real components as

$$\mathbb{T} = \{\xi: \xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3, x_0, x_1, x_2, x_3 \in \mathbb{R}\}, \quad (1)$$

and it can be represented as in terms of complex numbers as

$$\mathbb{T} = \{\xi: \xi = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}\}, \quad (2)$$

where $i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1 = j, j^2 = 1$.

The notations we will use are as follows:

$$x_0 = \operatorname{Re}(\xi), x_1 = \operatorname{Im}_{i_1}(\xi), x_2 = \operatorname{Im}_{i_2}(\xi), x_3 = \operatorname{Im}_j(\xi).$$

The set of all zero divisor elements of \mathbb{T} is called null cone, and it is denoted by \mathbb{NC} and is defined as follows:

$$\mathbb{NC} = \{z_1 + z_2 i_2 | z_1^2 + z_2^2 = 0\}. \quad (3)$$

Segre [1] noticed that the two zero divisor elements $(1 + i_1 i_2)/2$ and $(1 - i_1 i_2)/2$ are idempotent elements and play a vital role in the theory of the bicomplex numbers. e_1 and e_2 , the two nontrivial idempotent elements of \mathbb{T} , are defined as follows:

$$\begin{aligned} e_1 &= \frac{1 + i_1 i_2}{2} = \frac{1 + j}{2}, \\ e_2 &= \frac{1 - i_1 i_2}{2} = \frac{1 - j}{2}. \end{aligned} \quad (4)$$

Also,

$$\begin{aligned} e_1 + e_2 &= 1, \\ e_1 \cdot e_2 &= 0, \\ e_1^2 &= e_1, \\ e_2^2 &= e_2. \end{aligned} \quad (5)$$

and comparing them, we get $\xi_1 = (x_0 + x_3) + i_1(x_1 - x_2)$ and $\xi_2 = (x_0 - x_3) + i_1(x_1 + x_2)$.

The set of hyperbolic numbers $\mathbb{D} = \{x_1 + x_3 j | x_1, x_3 \in \mathbb{R}, j^2 = 1 \text{ and } j \notin \mathbb{R}\}$ and the set of complex numbers \mathbb{C} are two important proper subsets which are unified by the set of bicomplex numbers \mathbb{T} (see, [[6], p.19]). The sets \mathbb{T}, \mathbb{D} are connected to the theory of Clifford algebras. The set of bicomplex number is a two-dimensional complex Clifford algebra which has a set of hyperbolic numbers as its real (Clifford) subalgebra (see [[6], p.24]), or $\mathbb{T} \cong CI_{\mathbb{C}}(1, 0) \cong CI_{\mathbb{C}}(0, 1)$ and $\mathbb{D} \cong CI_{\mathbb{R}}(0, 1)$ (see [[7], p.1]).

Definition 3 (bicomplex moduli). Let $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2 = x_0 + x_1 i_1 + x_2 i_2 + x_3 j \in \mathbb{T}$ (see [6, 22, 23]).

The norm of ξ is defined as

Definition 2 (idempotent representation). \mathbb{T} has a unique idempotent representation for each element [4, 20–22] defined by

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2. \quad (6)$$

So, if $\xi_1 = (z_1 - i_1 z_2)$ and $\xi_2 = (z_1 + i_1 z_2)$, then $\xi = \xi_1 e_1 + \xi_2 e_2$.

Writing ξ in real components and idempotent components as

$$\xi = x_0 + i_1 x_1 + i_2 x_2 + i_1 i_2 x_3 = (x_0 + i_1 x_1) + i_2(x_2 + i_1 x_3) = \xi_1 e_1 + \xi_2 e_2, \quad (7)$$

$$\|\xi\| = \sqrt{|z_1|^2 + |z_2|^2} = \frac{1}{\sqrt{2}} \sqrt{|\xi_1|^2 + |\xi_2|^2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}. \quad (8)$$

The i_1 modulus of ξ is given by

$$|\xi|_{i_1} = \sqrt{z_1^2 + z_2^2}. \quad (9)$$

The i_2 modulus of ξ is given by

$$|\xi|_{i_2} = \sqrt{(|z_1|^2 - |z_2|^2) + 2\operatorname{Re}(z_1 \bar{z}_2)i_2}. \quad (10)$$

The j modulus of ξ is given by

$$|\xi|_j = |z_1 - i_1 z_2|e_1 + |z_1 + i_1 z_2|e_2. \quad (11)$$

The absolute value of ξ is given by

$$|\xi|_{abs} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{|(z_1 - i_1 z_2)(z_1 + i_1 z_2)|} = \sqrt{|\xi_1 \xi_2|} = \sqrt{|\xi_1| |\xi_2|}. \quad (12)$$

Ringleb [24] (see also [22]), investigated the analyticity of a bicomplex function with respect to its idempotent complex component functions in the following theorem. When studying the convergence of bicomplex functions, this theorem is crucial.

Theorem 1 (decomposition theorem of Ringleb [24]). Let $f(\xi)$ be analytic in a region $U \subseteq \mathbb{T}$, and let $T_1 \subseteq \mathbb{C}$ and $T_2 \subseteq \mathbb{C}$ be the component regions of \mathbb{T} , in the ξ_1 and ξ_2 planes, respectively. Then, there exists a unique pair of complex-valued

analytic functions, $f_1(\xi_1)$ and $f_2(\xi_2)$, defined in $U_1 \subseteq T_1$ and $U_2 \subseteq T_2$, respectively, such that

$$f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2, \quad \xi \in U. \quad (13)$$

Conversely, if $f_1(\xi_1)$ is any complex-valued analytic function in a region T_1 and $f_2(\xi_2)$ any complex-valued analytic function in a region T_2 , then the bicomplex-valued function $f(\xi)$ defined by equation (13) is an analytic function of the bicomplex variable ξ in the product region $U = U_1 \times_e U_2$.

In 1826, Abel proved the following result for the real power series (see [25–27]).

Theorem 2 (Abel’s theorem). *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad (14)$$

be a power series with coefficients $a_n \in \mathbb{R}$ that converges on $(-1, 1)$. We assume that $\sum_{n=0}^{\infty} a_n$ converges. Then,

$$\lim_{x \rightarrow 1} f(x) = \sum_{n=0}^{\infty} a_n. \quad (15)$$

In general, the converse is not true, i.e., if $\lim_{x \rightarrow 1} f(x)$ exists, one cannot conclude that $\sum_{n=0}^{\infty} a_n$ converges. In 1897, Tauber [28] proved the converse to Abel’s theorem but under an additional hypothesis.

Theorem 3 (Tauberian theorem). *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad (16)$$

be a power series with coefficients $a_n \in \mathbb{R}$ that converges on the real interval $(-1, 1)$. We assume that

$$\lim_{x \rightarrow 1} f(x) = A, \quad (17)$$

exists, and moreover,

$$\lim_{n \rightarrow \infty} n a_n = 0. \quad (18)$$

Then, $\sum_{n=0}^{\infty} a_n$ converges and is equal to A .

Detailed proof of the above theorem may be found in [[27], p.435].

Tauber’s result directed to many other Tauberian theorems. Later, various other converse theorems have been proved by Hardy and Littlewood and they named them the “Tauberian theorems” (see [26, 29]).

Tauberian theory provides many techniques for resolving difficult problems in analysis. Tauberian type theorems have numerous applications in mathematics, including rapidly decaying distributions and their applications to stable laws [30], generalized functions [31], Dirichlet series [32], and the solution of the prime number theorem [26]. In the bicomplex variable [10], the Tauberian theorem for the Laplace–Stieltjes transform is proved. Tauberian theory provides novel answers to complex situations. It has a variety of applications in number theory [26, 33]. In the area of mathematical physics, applications are studied in the quantum field theory [31, 34].

Landau [35] (see also [[32], p.4]) studied the following Tauberian result for complex power series.

Theorem 4 (Landau’s theorem). *Let G be given for $\operatorname{Re}(w) > 1, w \in \mathbb{C}$ by a convergent Dirichlet series*

$$G(w) = \sum_{n=1}^{\infty} \frac{a_n}{n^w}, \quad (19)$$

with $a_n \geq 0, \forall n \in \mathbb{N}$. We suppose that for some constant α , the analytic function

$$H(w) = G(w) - \frac{\alpha}{w-1}, \quad \operatorname{Re}(w) > 1, \quad (20)$$

has an analytic or just continuous extension (also called H) to the closed half-plane $\operatorname{Re}(w) \geq 1$. Finally, we suppose that there is a constant K such that

$$H(w) = O(|w|^K), \quad \operatorname{Re}(w) \geq 1, K > 0. \quad (21)$$

Then,

$$\frac{1}{n} \sum_{k=1}^n a_k \rightarrow \alpha, \quad \text{as } n \rightarrow \infty. \quad (22)$$

Ikehara’s theorem [25] extends the result of Landau (see [29]).

Theorem 5 (Ikehara’s theorem). *Let G be given by the Dirichlet series*

$$G(w) = \sum_{n=1}^{\infty} \frac{a_n}{n^w}, \quad (23)$$

convergent for $\operatorname{Re}(w) > 1$, where the coefficients satisfy the Tauberian condition $a_n \geq 0, \forall n \in \mathbb{N}$. If there exists a constant α such that

$$G(w) - \frac{\alpha}{w-1}, \quad (24)$$

admits a continuous extension to the line $\operatorname{Re}(w) = 1$, then

$$\sum_{k=1}^n a_k \sim \alpha n, \quad \text{as } n \rightarrow \infty. \quad (25)$$

In [36, 37], the authors defined the bicomplex Dirichlet series as $f(\xi) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n \xi}, \xi \in \mathbb{T}$, where $\{a_n\}, a_n = \alpha_1 e_1 + \alpha_2 e_2$ is a bicomplex number sequence. Substituting $\lambda_n = \log n$, the following form of the bicomplex Dirichlet series is obtained:

$$f(\xi) = \sum_{n=1}^{\infty} a_n n^{-\xi}. \quad (26)$$

In terms of idempotent components, $f(\xi)$ can be written as

$$\begin{aligned} f(\xi) &= \sum_{n=1}^{\infty} a_n n^{-\xi} = \sum_{n=1}^{\infty} \alpha_1 n n^{-\xi_1} e_1 + \sum_{n=1}^{\infty} \alpha_2 n n^{-\xi_2} e_2 \\ &= f_1(\xi_1) e_1 + f_2(\xi_2) e_2. \end{aligned} \quad (27)$$

The idempotent components of $f(\xi)$, $f_1(\xi_1) = \sum_{n=1}^{\infty} \alpha_1 n n^{-\xi_1}$ and $f_2(\xi_2) = \sum_{n=1}^{\infty} \alpha_2 n n^{-\xi_2}$ are the complex Dirichlet Series.

If the abscissae of convergence of the series $f_1(\xi_1) = \sum_{n=1}^{\infty} \alpha_{1n} n^{-\xi_1}$ and $f_2(\xi_2) = \sum_{n=1}^{\infty} \alpha_{2n} n^{-\xi_2}$ are denoted by σ_1 and σ_2 , respectively, then the region

$$\mathbb{E} = \{\xi \in \mathbb{T} : \operatorname{Re}(\xi_1) > \sigma_1 \text{ and } \operatorname{Re}(\xi_2) > \sigma_2\}, \quad (28)$$

or equivalently

$$\mathbb{E} = \{\xi \in \mathbb{T} : -\operatorname{Re}(\xi) + \sigma_1 < \operatorname{Im}_j(\xi) < \operatorname{Re}(\xi) - \sigma_2\}, \quad (29)$$

is the region of convergence of the bicomplex Dirichlet series $f(\xi)$ defined in equation (26).

Inspired by the work of Agarwal et al. [10] and Srivastava and Kumar [37], here, the bicomplex Landau-type Tauberian theorem is investigated. Also, the bicomplex version of the Ikehara's Tauberian theorem, which is generalization of the Landau-type Tauberian theorem, has been studied.

2. Bicomplex Versions of the Landau and Ikehara Theorems

Motivated by the work of Landau, we have derived the bicomplex version of Theorem 4 as follows:

Theorem 6 (bicomplex Landau theorem). *Let f be given for $\xi = \xi_1 e_1 + \xi_2 e_2$, $|\operatorname{Im}_j(\xi)| < \operatorname{Re}(\xi) - 1$ by a convergent Dirichlet series*

$$f(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n^{\xi}}, \quad \xi, a_n \in \mathbb{T}, \quad (30)$$

where $a_n = a_{n_1} + ja_{n_2} \in \mathbb{D}$ with $a_{n_1} \geq |a_{n_2}|$, $\forall n \in \mathbb{N}$. We suppose that for some hyperbolic constant $A = A_1 e_1 + A_2 e_2$, the analytic function

$$g(\xi) = f(\xi) - \frac{A}{\xi - 1}, \quad |\operatorname{Im}_j(\xi)| < \operatorname{Re}(\xi) - 1, \quad (31)$$

has an analytic or just continuous extension (also called g) to the closed half-plane $|\operatorname{Im}_j(\xi)| \leq \operatorname{Re}(\xi) - 1$.

Finally, we suppose that there is a constant M such that

$$g(\xi) = O(|\xi|_j^M), \quad (32)$$

for $|\operatorname{Im}_j(\xi)| \leq \operatorname{Re}(\xi) - 1$. Then,

$$\frac{1}{n} S_n = \frac{1}{n} \sum_{k=1}^n a_k \longrightarrow A, \quad \text{as } n \longrightarrow \infty. \quad (33)$$

Proof. We consider the Dirichlet series

$$f(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n^{\xi}} = f(\xi_1) e_1 + f(\xi_2) e_2, \quad a_n, \xi \in \mathbb{T}, \quad (34)$$

and let $A_1 e_1 + A_2 e_2 = A \in \mathbb{T}$. Here,

$$f(\xi_1) = \sum_{n=1}^{\infty} \frac{\alpha_{1n}}{n^{\xi_1}}, \quad \operatorname{Re}(\alpha_{1n}) > 0, \quad (35)$$

$$f(\xi_2) = \sum_{n=1}^{\infty} \frac{\alpha_{2n}}{n^{\xi_2}}, \quad \operatorname{Re}(\alpha_{2n}) > 0,$$

are convergent for $\operatorname{Re}(\xi_1) > 1$ and $\operatorname{Re}(\xi_2) > 1$, respectively. For some constants A_i , $(i = 1, 2)$,

$$g_i(\xi_i) = f(\xi_i) - \frac{A_i}{\xi_i - 1}, \quad \operatorname{Re}(\xi_i) > 1, \quad i = 1, 2, \quad (36)$$

are analytic functions in the complex domain. By Theorem 4, function $g_i(\xi_i)$, $(i = 1, 2)$ has an analytic or just continuous extension (also called g_i) to the closed half plane $\operatorname{Re}(\xi_i) \geq 1$, $(i = 1, 2)$.

Since $g_1(\xi_1)$ and $g_2(\xi_2)$ are analytic functions, thereby taking the idempotent linear combination of (36) for $i = 1, 2$,

$$\begin{aligned} g(\xi) &= g_1(\xi_1) e_1 + g_2(\xi_2) e_2 \\ &= \left(f(\xi_1) - \frac{A_1}{\xi_1 - 1} \right) e_1 + \left(f(\xi_2) - \frac{A_2}{\xi_2 - 1} \right) e_2 \\ &= f(\xi_1) e_1 + f(\xi_2) e_2 - \left(\frac{A_1 e_1 + A_2 e_2}{\xi_1 e_1 + \xi_2 e_2 - 1} \right) \\ &= f(\xi) - \frac{A}{\xi - 1}. \end{aligned} \quad (37)$$

With the help of equation (7), the conditions $\operatorname{Re}(\xi_1) > 1, \operatorname{Re}(\xi_2) > 1$ can be rewritten as

$$\begin{aligned} x_0 + x_3 &> 1, \\ x_0 - x_3 &> 1, \\ \Rightarrow |x_3| &< x_0 - 1 \\ \Rightarrow |\operatorname{Im}_j(\xi)| &< \operatorname{Re}(\xi) - 1. \end{aligned} \quad (38)$$

By assumption of the theorem, the j -modulus of $g(\xi)$, in (37), $\xi \in \mathbb{T}$, we have

$$\begin{aligned} g(\xi) &= O(|\xi|_j^M), \\ \text{Since, } |\xi|_j^M &= |\xi_1|^M e_1 + |\xi_2|^M e_2, \\ \Rightarrow g(\xi_1) &= O(|\xi_1|^M) \\ g(\xi_2) &= O(|\xi_2|^M). \end{aligned} \quad (39)$$

Thus, by Theorem 4 for complex domain,

$$\begin{aligned} \frac{1}{n} S_{1n} &= \frac{1}{n} \sum_{k=1}^n \alpha_{1k} \longrightarrow A_1, \\ \frac{1}{n} S_{2n} &= \frac{1}{n} \sum_{k=1}^n \alpha_{2k} \longrightarrow A_2, \end{aligned} \quad (40)$$

as $n \longrightarrow \infty$.

By idempotent combination of the above series,

$$\begin{aligned} \frac{1}{n}S_{1n}e_1 + \frac{1}{n}S_{2n}e_2 &= \frac{1}{n}S_n \longrightarrow A_1e_1 + A_2e_2 = A, \\ \frac{1}{n}S_n &\longrightarrow A, \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (41)$$

Furthermore, the relation $a_n = \alpha_{1n}e_1 + \alpha_{2n}e_2 = a_{n_1} + i_1a_{n_2} + i_2a_{n_3} + ja_{n_4}$ gives

$$\begin{aligned} \alpha_{1n} &= (a_{n_1} + a_{n_4}) + i_1(a_{n_2} - a_{n_3}), \\ \alpha_{2n} &= (a_{n_1} - a_{n_4}) + i_1(a_{n_2} + a_{n_3}). \end{aligned} \quad (42)$$

The conditions $\alpha_{1n} \geq 0$, $\alpha_{2n} \geq 0$ imply

$$\begin{aligned} a_{n_1} + a_{n_4} &\geq 0, \\ a_{n_2} - a_{n_3} &= 0; \\ a_{n_1} - a_{n_4} &\geq 0, \\ a_{n_2} + a_{n_3} &= 0. \\ \Rightarrow a_{n_1} &\geq |a_{n_4}|; \\ a_{n_2} &= a_{n_3} = 0. \\ \Rightarrow a_n &= (a_{n_1} + a_{n_4})e_1 + (a_{n_1} - a_{n_4})e_2 = a_{n_1} + ja_{n_4}. \end{aligned} \quad (43)$$

Hence, a_n is a hyperbolic number with $a_{n_1} \geq |a_{n_4}|$. \square

Remark 1. In the proof of the above theorem, it is observed that the results and conditions focus on the hyperbolic coefficients and not on coefficients of imaginary units i_1 and i_2 ; hence, it can be called the hyperbolic version of the Landau theorem.

Theorem 7 (bicomplex Ikehara theorem). *Let $\xi, a_n \in \mathbb{T}$ where $\xi = \xi_1e_1 + \xi_2e_2$ and $a_n = a_{n_1} + ja_{n_4}$, $n \in \mathbb{N}$ is a sequence of hyperbolic numbers [6]. Let f be given by the Dirichlet series*

$$f(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n^\xi}, \quad a_{n_1} \geq |a_{n_4}|, \quad (44)$$

convergent for $|\text{Im}_j(\xi)| < \text{Re}(\xi) - 1$.

If there exists a hyperbolic constant $\beta \in \mathbb{D}$ such that

$$f(\xi) - \frac{\beta}{\xi - 1}, \quad (45)$$

admits a continuous extension to the plane $\text{Re}(\xi) = 1$, $\text{Im}_j(\xi) = 0$, then

$$S_n = \sum_{k=1}^n a_k \sim \beta n, \quad \text{as } n \longrightarrow \infty. \quad (46)$$

Proof. We consider the Dirichlet Series

$$f(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n^\xi} = f(\xi_1)e_1 + f(\xi_2)e_2, \quad (47)$$

where

$$f(\xi_i) = \sum_{n=1}^{\infty} \frac{\alpha_{in}}{n^{\xi_i}}, \quad \alpha_{in} \geq 0, \quad \text{Re}(\xi_i) > 1, \quad i = 1, 2, \quad (48)$$

where $f(\xi)$ is convergent for $\alpha_{1n} \geq 0$, $\alpha_{2n} \geq 0$ i.e. $a_{n_1} \geq |a_{n_4}|$ (from equation (43)) and $|\text{Im}_j(\xi)| < \text{Re}(\xi) - 1$.

By Theorem 5, for some constants β_i , ($i = 1, 2$), the analytic functions

$$f_i(\xi_i) - \frac{\beta_i}{\xi_i - 1}, \quad i = 1, 2, \quad (49)$$

admit a continuous extension to the lines $\text{Re}(\xi_i) = 1$, ($i = 1, 2$). Taking idempotent linear combination of the functions defined in equation (49), we get for $\text{Re}(\xi_i) > 1$, ($i = 1, 2$) or equivalently $|\text{Im}_j(\xi)| < \text{Re}(\xi) - 1$,

$$\left(f(\xi_1) - \frac{\beta_1}{\xi_1 - 1}\right)e_1 + \left(f(\xi_2) - \frac{\beta_2}{\xi_2 - 1}\right)e_2 = f(\xi) - \frac{\beta}{\xi - 1}, \quad (50)$$

where $\beta = \beta_1e_1 + \beta_2e_2 \in \mathbb{D}$. Hence, $f(\xi) - (\beta/(\xi - 1))$ admits a continuous extension to the plane $\text{Re}(\xi_1) = 1$, $\text{Re}(\xi_2) = 1$, i.e., $\text{Re}(\xi) = 1$, $\text{Im}_j(\xi) = 0$ which means $\xi = 1 + x_1i_1 + x_2i_2$, $x_1, x_2 \in \mathbb{R}$.

Furthermore,

$$\sum_{k=1}^n a_k = \sum_{k=1}^n \alpha_{1k}e_1 + \sum_{k=1}^n \alpha_{2k}e_2 \sim \beta n, \quad \text{as } n \longrightarrow \infty. \quad (51)$$

\square

3. Ikehara's Theorem Involving Boundedness

In this section, we discuss some results about Schwartz functions, tempered distributions, and the Fourier transform (see [38–40]). Schwartz [41] (see also [38]) chooses the class of test function ϕ that is infinitely continuously differentiable and that vanishes outside some bounded set. All functionals defined on this class that are linear and continuous are named distributions by Schwartz.

Space $S(\mathbb{R})$ is the Schwartz space of rapidly decreasing smooth test functions ϕ (see [29]), i.e., those C^∞ functions over the real field such that

$$\sup_{u \in \mathbb{R}} |u^p \phi^{(q)}(u)| < \infty, \quad p, q \in \mathbb{N}. \quad (52)$$

The space of tempered distributions is represented by $S'(\mathbb{R})$, which is the dual of $S(\mathbb{R})$ (see [29]). The evaluation of $g \in S'(\mathbb{R})$ at $\psi \in S(\mathbb{R})$ is denoted by $\langle g, \psi \rangle = \int_{-\infty}^{\infty} g(u)\psi(u)du$. Thus, $g \in S'(\mathbb{R})$ if and only if

$$\begin{aligned} \langle g, a\psi + \phi \rangle &= a\langle g, \psi \rangle + \langle g, \phi \rangle, \\ \lim_{n \rightarrow \infty} \langle g, \psi_n \rangle &= \langle g, \lim_{n \rightarrow \infty} \psi_n \rangle, \end{aligned} \quad (53)$$

whenever $\{\psi_n\}_{n=0}^{\infty}$ is convergent in $S(\mathbb{R})$.

If a tempered distribution is the Fourier transform of a bounded (measurable) function, then it is called a pseudomeasure.

Let $\sum_{n=1}^{\infty} a_n/n^w$ be a complex Dirichlet series with coefficients $a_n \geq 0$ that converges to a function $f(w)$ for

$\operatorname{Re}(w) > 1$. In 2008, Korevaar [42] proved following theorem for boundedness of S_N/N in complex space as follows:

Theorem 8 (Ikehara–Korevaar theorem). *Let $\sum_{n=1}^{\infty} a_n/n^w$ be a Dirichlet series with coefficients $a_n \geq 0$ converging to $g(w)$ for $\operatorname{Re}(w) > 1$. Let $S_N = \sum_{n \leq N} a_n$; the sequence $\{S_N/N\}$ will remain bounded if the quotient*

$$f(w) = \frac{g(w)}{w}, \quad w = u + i_1 v \in \mathbb{C}, \operatorname{Re}(w) = u > 1, \quad (54)$$

converges in the sense of tempered distribution to a pseudomeasure $f(1 + i_1 v)$, as $u \rightarrow 1$.

Remark 2. The distributional convergence in the above theorem is convergence in the Schwartz space S' . In other words,

$$\langle f(u + i_1 v), \phi(v) \rangle \rightarrow \langle f(1 + i_1 v), \phi(v) \rangle, \quad \text{as } u \rightarrow 1, \quad (55)$$

for all testing functions $\phi(v) \in S$, that is, all rapidly decreasing C^∞ functions.

We hereby provide the bicomplex version of Theorem 8.

Theorem 9 (bicomplex Ikehara–Korevaar theorem). *Let $\sum_{n=1}^{\infty} a_n/n^\xi$ be a bicomplex Dirichlet series where $\xi = \xi_1 e_1 + \xi_2 e_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 \in \mathbb{T}$, $z_1 = x_1 + i_1 y_1$, $z_2 = x_2 + i_1 y_2 \in \mathbb{C}$, and $a_n = a_{n_1} + j a_{n_2} \in \mathbb{D}$ with $a_{n_1} \geq |a_{n_2}|$ that converges to $f(\xi)$ for $|\operatorname{Im}_j(\xi)| < \operatorname{Re}(\xi) - 1$.*

Let $S_N = \sum_{n \leq N} a_n$; then, a necessary and sufficient condition for the boundedness of S_N/N is that the quotient $q(\xi) = (f(\xi)/\xi)$, $\xi \notin \mathbb{NC}$ converges in the sense of tempered distribution to a pseudomeasure $q(1 + i_1 y_1 + i_2 x_2)$, as $x_1 \rightarrow 1, y_2 \rightarrow 0$.

Proof. Let the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^\xi$ converges to $f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$ where

$$\sum_{n=1}^{\infty} \frac{a_n}{n^\xi} = \sum_{n=1}^{\infty} \frac{\alpha_{1n}}{n^{\xi_1}} e_1 + \sum_{n=1}^{\infty} \frac{\alpha_{2n}}{n^{\xi_2}} e_2, \quad a_n = \alpha_{1n} e_1 + \alpha_{2n} e_2. \quad (56)$$

For $\operatorname{Re}(\xi_1) > 1$ and $\operatorname{Re}(\xi_2) > 1$, equivalently, $|\operatorname{Im}_j(\xi)| < \operatorname{Re}(\xi) - 1$, the Dirichlet series $\sum_{n=1}^{\infty} \alpha_{1n}/n^{\xi_1}$, $\alpha_{1n} \geq 0$ converges to the function $f_1(\xi_1)$ and the Dirichlet series $\sum_{n=1}^{\infty} \alpha_{2n}/n^{\xi_2}$, $\alpha_{2n} \geq 0$ converges to the function $f_2(\xi_2)$.

Let us denote $\sum_{n \leq N} \alpha_{1n} = S_{1N}$ and $\sum_{n \leq N} \alpha_{2n} = S_{2N}$; then,

$$\begin{aligned} S_N &= \sum_{n \leq N} a_n \\ &= \sum_{n \leq N} \alpha_{1n} e_1 + \sum_{n \leq N} \alpha_{2n} e_2 \\ &= S_{1N} e_1 + S_{2N} e_2. \end{aligned} \quad (57)$$

From Theorem 8, the necessary and sufficient condition for the boundedness of S_{1N}/N is that the quotient

$$q_1(z_1 - i_1 z_2) = q_1((x_1 + y_2) + i_1(y_1 - x_2)) = \frac{f_1(z_1 - i_1 z_2)}{z_1 - i_1 z_2}, \quad (58)$$

converges in the sense of tempered distribution to a pseudomeasure $q_1(1 + i_1(y_1 - x_2))$, as $x_1 + y_2 \rightarrow 1$.

Similarly, the necessary and sufficient condition for boundedness of S_{2N}/N is that the quotient

$$q_2(z_1 + i_1 z_2) = q_2((x_1 - y_2) + i_1(y_1 + x_2)) = \frac{f_2(z_1 + i_1 z_2)}{z_1 + i_1 z_2}, \quad (59)$$

converges in the sense of tempered distribution to a pseudomeasure $q_2(1 + i_1(y_1 + x_2))$, as $x_1 - y_2 \rightarrow 1$.

Again, by the application of the Ringleb theorem, the necessary and sufficient condition for the boundedness of $S_N/N = (S_{1N}/N)e_1 + (S_{2N}/N)e_2$ is that the quotient

$$\begin{aligned} q(z_1 + i_2 z_2) &= q_1 e_1 + q_2 e_2 \\ &= \left(\frac{f_1(z_1 - i_1 z_2)}{z_1 - i_1 z_2} \right) e_1 + \left(\frac{f_2(z_1 + i_1 z_2)}{z_1 + i_1 z_2} \right) e_2 \\ &= (f_1(z_1 - i_1 z_2)e_1 + f_2(z_1 + i_1 z_2)e_2) \left(\frac{1}{z_1 + i_2 z_2} \right) \\ &= \frac{f(\xi)}{\xi}, \quad \xi \notin \mathbb{NC}, \end{aligned} \quad (60)$$

converges to $q_1(1 + i_1(y_1 - x_2))e_1 + q_2(1 + i_1(y_1 + x_2))e_2 = q(1 + i_1 y_1 + i_2 x_2)$ in the sense of tempered distribution to a pseudomeasure as $x_1 + y_2 \rightarrow 1$ and $x_1 - y_2 \rightarrow 1$, i.e., $x_1 \rightarrow 1, y_2 \rightarrow 0$. \square

4. Conclusion

In this paper, Landau-type Tauberian theorem in bicomplex space which is the generalization of Landau-type Tauberian theorem has been derived. The necessary and sufficient condition for the boundedness of the partial sum $S_N = \sum_{n \leq N} a_n$ for bicomplex Dirichlet series with hyperbolic coefficients is obtained. The conditions of convergence are affected by the j coefficient of bicomplex numbers, and hence the theorems can be seen as the hyperbolic versions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Research Article

On the Partial Sums of the q -Generalized Dini Function

Alaa H. El-Qadeem ¹, Mohamed A. Mamon ², and Ibrahim S. Elshazly ³

¹Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt

²Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt

³Department of Basic Sciences, Common First Year, King Saud University, Alriyad 11451, Saudi Arabia

Correspondence should be addressed to Alaa H. El-Qadeem; ahhassan@science.zu.edu.eg

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Our objective in this paper is to introduce a q -analog of the generalized Dini function. Also, we investigate the lower bound for the ratio of the q -generalized Dini function to its sequences of partial sums.

1. Introduction and Basic Concepts

Let \mathcal{A} denote the class of functions f that have the following Maclaurin's form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which is analytic and univalent in the open unit disc $\mathcal{U} = \{z: |z| < 1\}$ and satisfies the normalization conditions $f(0) = f'(0) - 1 = 0$.

Special functions play an inspired role in applied mathematics and physics. The widespread use of these functions has attracted many researchers to work in many directions. Lately, many authors studied the geometric properties of some special functions such as starlikeness,

univalence, and convexity, see [1–6]. There are several results related to partial sums of analytic univalent functions that were developed by the authors in [7–9]. Specifically, the authors in [10] investigated the partial sums of the generalized Bessel function, and then, a lot of authors followed them in studying the same problem for different special functions such as Bessel [11, 12], Struve [13], Lommel [14], Wright [15], and Mittag-Leffler [16], see also [17].

Our aim in this study is to develop a q -analog of the generalized Dini function, which is inspired by early studies on analytic and special functions. We also provide lower bounds for the ratio of q -generalized Dini function to its sequences of partial sums, for $m \in \mathbb{N} = \{1, 2, 3, \dots\}$, $(\psi_{\gamma, b, c}^a(z; q))_m = z + \sum_{n=1}^m \zeta_n z^{n+1}$. We will investigate the following:

$$\Re \left\{ \frac{\psi_{\gamma, b, c}^a(z; q)}{(\psi_{\gamma, b, c}^a(z; q))_m} \right\}, \Re \left\{ \frac{(\psi_{\gamma, b, c}^a(z; q))_m}{\psi_{\gamma, b, c}^a(z; q)} \right\}, \Re \left\{ \frac{(\psi_{\gamma, b, c}^a(z; q))'_m}{(\psi_{\gamma, b, c}^a(z; q))'_m} \right\}, \text{ and } \Re \left\{ \frac{(\psi_{\gamma, b, c}^a(z; q))'_m}{(\psi_{\gamma, b, c}^a(z; q))'_m} \right\}. \quad (2)$$

To introduce the main results, we would like to recall some fundamentals and concepts related to geometric function theory and the definition of q -generalized Bessel

function. At first, let us consider the following second-order linear homogenous differential equation (for more details, see [18–20]):

$$z^2 w''(z) + bz w'(z) + [cz^2 - \nu^2 + (1-b)\nu] w(z) = 0, \quad (3)$$

$$(b, c, \nu \in \mathbb{C}).$$

The function $w_{\nu, b, c}(z)$ is known as the generalized Bessel function of the first kind of order ν , which is a particular solution of equation (3). The function $w_{\nu, b, c}(z)$ has the following series representation:

$$w_{\nu, b, c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \Gamma(\nu + n + ((b+1)/2))} \left(\frac{z}{2}\right)^{2n+\nu}, \quad (4)$$

where Γ stands for the Gamma function.

Now, let $q \in (0, 1)$ and $a \in \mathbb{C}$; the q -shifted factorial is defined by

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1} (1 - aq^k), & n \in \mathbb{N}. \end{cases} \quad (5)$$

The limit of $(a; q)_n$ as n tends to infinity exists and is denoted by $(a; q)_{\infty}$:

$$(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k). \quad (6)$$

The multiple q -shifted factorial for complex numbers a_1, a_2, \dots, a_r is defined by

$$(a_1, a_2, \dots, a_r; q)_n = \prod_{j=1}^r (a_j; q)_n. \quad (7)$$

If $aq^{\alpha} \neq q^{-n}$, for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define $(a; q)_{\alpha}$ to be

$$(a; q)_{\alpha} = \frac{(a; q)_{\infty}}{(aq^{\alpha}; q)_{\infty}}. \quad (8)$$

Definition 1. Let $b, \nu \in \mathbb{R}$ and $c \in \mathbb{C}$; the q -generalized Bessel function is defined by

$$\begin{aligned} \omega_{\nu, b, c}(z; q) &= \frac{(q^k; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-c)^n (1-q)^{((b-1)/2)}}{(q; q)_n (q^k; q)_n} \left(\frac{z}{2}\right)^{2n+\nu} \\ &= \frac{1}{(q; q)_{k-1}} \sum_{n=0}^{\infty} \frac{(-c)^n (1-q)^{((b-1)/2)}}{(q, q^k; q)_n} \left(\frac{z}{2}\right)^{2n+\nu}, \end{aligned} \quad (9)$$

where $k = \nu + b + 1/2 \neq 0, -1, -2, \dots$, $z \in \mathcal{U}$ and $q \in (0, 1)$.

Now, we introduce the q -generalized Dini function $\phi_{\nu, b, c}^a(z; q)$ in terms of $\omega_{\nu, b, c}(z; q)$.

Definition 2. Let $a \in \mathbb{R}^+$; the q -generalized Dini function $\phi_{\nu, b, c}^a(z; q)$ is defined by

$$\begin{aligned} \phi_{\nu, b, c}^a(z; q) &= (a - \nu) \omega_{\nu, b, c}(\sqrt{z}; q) + \sqrt{z} \omega'_{\nu, b, c}(z; q), \\ &= \frac{1}{(q; q)_{k-1}} \sum_{n=0}^{\infty} \frac{(-c)^n (1-q)^{((b-1)/2)} (a + 2n)}{(q, q^k; q)_n} \left(\frac{z}{4}\right)^{(n+(v/2))}. \end{aligned} \quad (10)$$

Remark 1. By specializing the value of a, b, c , and q , we see that

- (1) $\lim_{q \rightarrow 1^-} \omega_{\nu, b, c}((1-q)z; q) = \omega_{\nu, b, c}(z)$ is the generalized Bessel function of first kind introduced by Orhan and Yagmur [10]
- (2) By putting $b = c = 1$, then $\omega_{\nu, 1, 1}(z; q) = J_{\nu}^{(1)}(z; q)$ is the first kind of q -Bessel function given by Annaby and Mansour [21]. Also, $\lim_{q \rightarrow 1^-} \omega_{\nu, 1, 1}((1-q)z; q) = J_{\nu}(z)$ is the familiar Bessel function defined by Baricz [18].
- (3) By putting $b = -c = 1$, then $\omega_{\nu, 1, -1}(z; q) = I_{\nu}^{(1)}(z; q)$ is the first kind of modified q -Bessel function given by Annaby and Mansour [21]. Also, $\lim_{q \rightarrow 1^-} \omega_{\nu, 1, -1}((1-q)z; q) = I_{\nu}(z)$ is the modified Bessel function defined by Baricz [18].
- (4) $\lim_{q \rightarrow 1^-} \phi_{\nu, b, c}^a((1-q)^2 z; q) = D_{\nu, a, b, c}(z)$ is the generalized Dini function investigated by Deniz and Gören [3]. Also, by putting $b = c = 1$ in the last expression, we get $d_{\nu, a}(z)$ introduced by Aktaş and Orhan [22]. In addition, by putting $a = 1$, we obtain the Dini function $d_{\nu}(z)$ which is introduced by Baricz et al. [2].

Because the function $\phi_{\nu, b, c}^a(z; q)$ defined by (10) do not belong to the class \mathcal{A} , we consider the following normalized form of the q -generalized Dini function, $\psi_{\nu, b, c}^a(z; q): \mathcal{U} \rightarrow \mathbb{C}$, as

$$\begin{aligned} \psi_{\nu, b, c}^a(z; q) &= \frac{2^{\nu} (q; q)_{k-1}}{a(1-q)^{b-1/2}} z^{1-(v/2)} \phi_{\nu, b, c}^a(z; q) \\ &= z + \sum_{n=1}^{\infty} \zeta_n z^{n+1}, \end{aligned} \quad (11)$$

where

$$\zeta_n = \frac{(-c)^n (a + 2n)}{4^n a (q, q^k; q)_n}. \quad (12)$$

Definition 3 (subordination principle, see [23–25]). An analytic function f is said to be subordinate to another analytic function g , written as $f(z) \prec g(z)$ ($z \in \mathcal{U}$), if there exists a Schwarz function ω , which is analytic in \mathcal{U} with $\omega(0) = 0$ and $|\omega(z)| < 1$ ($z \in \mathcal{U}$), such that $f(z) = g(\omega(z))$. In particular, if the function g is univalent in \mathcal{U} , then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}). \quad (13)$$

Remark 2. It is observed that the function $\Omega(z) = 1 + z/1 - z$ maps \mathcal{U} conformally into the right half plane such that $\Re\{\Omega(z)\} > 0$. That function plays a great roll in proving our main results.

Here, we would like to mention the following inequalities,

$$(q; q)_n \geq (1-q)^n, \quad (14)$$

and

$$(q^\alpha; q)_n \geq (1 - q^\alpha)^n, \quad (15)$$

are valid for $q \in (0, 1)$, $\alpha \in \mathbb{R}$, and $n \in \mathbb{N}$.

Lemma 1. Let us consider $a \in \mathbb{R}^+$, $b, \nu \in \mathbb{R}$, $c \in \mathbb{C}$ ($|c| < 4$), and $k = \nu + (b + 1/2)$; the function $\psi_{\nu, b, c}^a(z; q)$, referred by (11), satisfies the following inequalities, for all $z \in \mathcal{U}$:

$$\begin{aligned} |\psi_{\nu, b, c}^a(z; q)| &\leq 1 + \frac{4|c|(a+2)(1-q)(1-q^k) - a|c|^2}{a[4(1-q)(1-q^k) - |c|]^2}, \\ |(\psi_{\nu, b, c}^a(z; q))'| &\leq 1 + \frac{32|c|(a+2)(1-q)^2(1-q^k)^2 - 4|c|^2(3a+2)(1-q)(1-q^k) + a|c|^3}{a[4(1-q)(1-q^k) - |c|]^3}. \end{aligned} \quad (16)$$

Proof. By taking into consideration inequalities (14) and (15), we obtain

$$\begin{aligned} |\psi_{\nu, b, c}^a(z; q)| &= \left| z + \sum_{n=1}^{\infty} \frac{(-c)^n(a+2n)}{4^n a(q, q^k; q)_n} z^{n+1} \right| \leq 1 + \sum_{n=1}^{\infty} \frac{|c|^n(a+2n)}{a4^n(q; q)_n(q^k; q)_n} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{|c|^n(a+2n)}{a4^n(1-q)^n(1-q^k)^n} \\ &= 1 + \frac{|c|}{4(1-q)(1-q^k)} \sum_{n=1}^{\infty} \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^{n-1} + \frac{|c|}{2a(1-q)(1-q^k)} \sum_{n=1}^{\infty} n \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^{n-1} \\ &\quad + \frac{4|c|(a+2)(1-q)(1-q^k) - a|c|^2}{a[4(1-q)(1-q^k) - |c|]^2} \end{aligned} \quad (17)$$

and

$$\begin{aligned} |(\psi_{\nu, b, c}^a(z; q))'| &= \left| 1 + \sum_{n=1}^{\infty} \frac{(-c)^n(n+1)(a+2n)}{a4^n(q, q^k; q)_n} z^n \right| \leq 1 + \sum_{n=1}^{\infty} \frac{|c|^n(2n^2 + (a+2)n + a)}{a4^n(q; q)_n(q^k; q)_n} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{(2n^2 + (a+2)n + a)}{a} \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^n \\ &= 1 + \frac{|c|}{4(1-q)(1-q^k)} \sum_{n=1}^{\infty} \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^{n-1} + \frac{|c|(a+2)}{4a(1-q)(1-q^k)} \sum_{n=1}^{\infty} n \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^{n-1} \\ &\quad + \frac{|c|}{2a(1-q)(1-q^k)} \sum_{n=1}^{\infty} n^2 \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^{n-1} \\ &= 1 + \frac{32|c|(a+2)(1-q)^2(1-q^k)^2 - 4|c|^2(3a+2)(1-q)(1-q^k)}{a[4(1-q)(1-q^k) - |c|]^3}. \end{aligned} \quad (18)$$

Thus, we complete the proof. \square

is valid, then

2. Main Results

Theorem 1. Let us consider $a \in \mathbb{R}^+$, $b, \nu \in \mathbb{R}$, and $c \in \mathbb{C}$ ($|c| < 4$), $k = \nu + b + 1/2$, and the function $\psi_{\nu,b,c}^a(z; q)$ be defined by (11) and its partial sum $(\psi_{\nu,b,c}^a(z; q))_m = z + \sum_{n=1}^m \zeta_n z^{n+1}$. If the inequality,

$$16a(1-q)^2(1-q^k)^2 + 2a|c|^2 > 4(3a+2)|c|(1-q)(1-q^k), \quad (19)$$

$$\Re \left\{ \frac{\psi_{\nu,b,c}^a(z; q)}{(\psi_{\nu,b,c}^a(z; q))_m} \right\} > \beta_1 \quad (20)$$

and

$$\Re \left\{ \frac{(\psi_{\nu,b,c}^a(z; q))_m}{\psi_{\nu,b,c}^a(z; q)} \right\} > \beta_2 \quad (21)$$

holds true, for all $z \in \mathcal{U}$, where

$$\beta_1 = \frac{16a(1-q)^2(1-q^k)^2 + 4|c|(2-a)(1-q)(1-q^k) + 2a|c|^2}{a[4(1-q)(1-q^k) - |c|]^2},$$

$$\beta_2 = \frac{a[4(1-q)(1-q^k) - |c|]^2}{16a(1-q)^2(1-q^k)^2 + 4|c|(2-a)(1-q)(1-q^k)}.$$

Proof. From the steps of proving Lemma 1, we observe

$$1 + \sum_{n=1}^{\infty} |\zeta_n| \leq 1 + \frac{4|c|(a+2)(1-q)(1-q^k) - a|c|^2}{a[4(1-q)(1-q^k) - |c|]^2}, \quad (23)$$

which is equivalent to

$$\mu \sum_{n=1}^{\infty} |\zeta_n| \leq 1, \quad (24)$$

where

$$\mu = a[4(1-q)(1-q^k) - |c|]^2 / 4|c|(a+2)(1-q)(1-q^k) - a|c|^2.$$

Now, let us consider

$$\begin{aligned} \frac{1 + \omega(z)}{1 - \omega(z)} &= \mu \left[\frac{\psi_{\nu,b,c}^a(z; q)}{(\psi_{\nu,b,c}^a(z; q))_m} - \beta_1 \right] \\ &= \frac{1 + \sum_{n=1}^m \zeta_n z^n + \mu \sum_{n=m+1}^{\infty} \zeta_n z^n}{1 + \sum_{n=1}^m \zeta_n z^n}. \end{aligned} \quad (25)$$

Therefore,

$$\omega(z) = \frac{\mu \sum_{n=m+1}^{\infty} \zeta_n z^n}{2 + 2 \sum_{n=1}^m \zeta_n z^n + \mu \sum_{n=m+1}^{\infty} \zeta_n z^n} \quad (26)$$

and

$$|\omega(z)| \leq \frac{\mu \sum_{n=m+1}^{\infty} |\zeta_n|}{2 - 2 \sum_{n=1}^m |\zeta_n| - \mu \sum_{n=m+1}^{\infty} |\zeta_n|}. \quad (27)$$

Inequality (20) holds true if $|\omega(z)| \leq 1$ according to the definition of subordination. Then, the upcoming inequality,

$$\sum_{n=1}^m |\zeta_n| + \mu \sum_{n=m+1}^{\infty} |\zeta_n| \leq 1, \quad (28)$$

implies that $|\omega(z)| \leq 1$. It suffices to show that the left-hand side of (28) is bounded above by

$$\mu \sum_{n=1}^{\infty} |\zeta_n|, \quad (29)$$

which is equivalent to

$$(\mu - 1) \sum_{n=1}^m |\zeta_n| \geq 0. \quad (30)$$

On the contrary, to prove inequality (21), we consider

$$\begin{aligned} \frac{1 + p(z)}{1 - p(z)} &= (\mu + 1) \left[\frac{(\psi_{\nu,b,c}^a(z; q))_m}{\psi_{\nu,b,c}^a(z; q)} - \beta_2 \right], \\ &= \frac{1 + \sum_{n=1}^m \zeta_n z^n - \mu \sum_{n=m+1}^{\infty} \zeta_n z^n}{1 + \sum_{n=1}^m \zeta_n z^n}. \end{aligned} \quad (31)$$

Therefore,

$$p(z) = \frac{-(1 + \mu) \sum_{n=m+1}^{\infty} \zeta_n z^n}{2 + 2 \sum_{n=1}^m \zeta_n z^n + (1 - \mu) \sum_{n=m+1}^{\infty} \zeta_n z^n} \quad (32)$$

and

$$|p(z)| \leq \frac{(1 + \mu) \sum_{n=m+1}^{\infty} |\zeta_n|}{2 - 2 \sum_{n=1}^m |\zeta_n| - (\mu - 1) \sum_{n=m+1}^{\infty} |\zeta_n|}. \quad (33)$$

Inequality (21) holds true if $|p(z)| \leq 1$ according to the definition of subordination. Then, the upcoming inequality,

$$\sum_{n=1}^m |\zeta_n| + \mu \sum_{n=m+1}^{\infty} |\zeta_n| \leq 1, \quad (34)$$

implies that $|p(z)| \leq 1$. Since the left-hand side of (34) is bounded above $\mu \sum_{n=1}^{\infty} |\zeta_n|$, thus, we complete the proof. \square

Theorem 2. Let us consider $a \in \mathbb{R}^+$, $b, v \in \mathbb{R}$, $c \in \mathbb{C} (|c| < 4)$, defined by (11), and its partial sum be $(\psi_{v,b,c}^a(z; q))_m$ and $k = v + b + 1/2$, the function $\psi_{v,b,c}^a(z; q): \mathcal{U} \rightarrow \mathbb{C}$, be $= z + \sum_{n=1}^m \zeta_n z^{n+1}$. If the inequalities,

$$\left. \begin{aligned} 32(a+2)|c|(1-q)^2(1-q^k)^2 + a|c|^3 &> 4(3a+2)|c|^2(1-q)(1-q^k), \\ 32a(1-q)^3(1-q^k)^3 + 4(3a+1)|c|^2(1-q)(1-q^k) &> 8(5a+4)|c|(1-q)^2(1-q^k)^2 + a|c|^3, \end{aligned} \right\} \quad (35)$$

are valid, then

$$\Re \left\{ \frac{(\psi_{v,b,c}^a(z; q))'_m}{((\psi_{v,b,c}^a(z; q))'_m)'} \right\} > \gamma_1 \quad (36) \quad \Re \left\{ \frac{((\psi_{v,b,c}^a(z; q))'_m)'}{(\psi_{v,b,c}^a(z; q))'_m} \right\} > \gamma_2, \quad (37)$$

holds true, for all $z \in \mathcal{U}$, where

and

$$\gamma_1 = \frac{64a(1-q)^3(1-q^k)^3 - 16|c|(5a+4)(1-q)^2(1-q^k)^2 + 5(3a+1)|c|^2(1-q)(1-q^k) - 2a|c|^3}{a[4(1-q)(1-q^k) - |c|]^3} \quad (38)$$

and

$$\gamma_2 = \frac{a[4(1-q)(1-q^k) - |c|]^3}{64a(1-q)^3(1-q^k)^3 + 16|c|(4-a)(1-q)^2(1-q^k)^2 - 8|c|(1-q)(1-q^k)}. \quad (39)$$

Proof. From Lemma 1, we observe that

$$1 + \sum_{n=1}^{\infty} |\eta_n| \leq 1 + \frac{32|c|(a+2)(1-q)^2(1-q^k)^2 - 4|c|^2(3a+2)(1-q)(1-q^k) + a|c|^3}{a[4(1-q)(1-q^k) - |c|]^3}, \quad (40)$$

which is equivalent to

where $\eta_n = (-c)^n(n+1)(a+2n)/4^n a(q, q^k; q)_n$ and

$$\lambda \sum_{n=1}^{\infty} |\eta_n| \leq 1, \quad (41)$$

$$\lambda = \frac{a[4(1-q)(1-q^k) - |c|]^3}{32|c|(a+2)(1-q)^2(1-q^k)^2 - 4|c|^2(3a+2)(1-q)(1-q^k) + a|c|^3}. \quad (42)$$

Now, let us consider

$$\frac{1+r(z)}{1-r(z)} = \lambda \left[\frac{(\psi_{\gamma,b,c}^a(z;q))'}{((\psi_{\gamma,b,c}^a(z;q))_m)'} - \gamma_1 \right] \quad (43)$$

$$= \frac{1 + \sum_{n=1}^m \eta_n z^n + \lambda \sum_{n=m+1}^{\infty} \eta_n z^n}{1 + \sum_{n=1}^m \eta_n z^n}.$$

Therefore,

$$r(z) = \frac{\lambda \sum_{n=m+1}^{\infty} \eta_n z^n}{2 + 2 \sum_{n=1}^m \eta_n z^n + \lambda \sum_{n=m+1}^{\infty} \eta_n z^n} \quad (44)$$

and

$$|r(z)| \leq \frac{\lambda \sum_{n=m+1}^{\infty} |\eta_n|}{2 - 2 \sum_{n=1}^m |\eta_n| - \lambda \sum_{n=m+1}^{\infty} |\eta_n|}. \quad (45)$$

Inequality (36) holds true if $|r(z)| \leq 1$ according to subordination principle. Then, the upcoming inequality,

$$\sum_{n=1}^m |\eta_n| + \lambda \sum_{n=m+1}^{\infty} |\eta_n| \leq 1, \quad (46)$$

implies that $|r(z)| \leq 1$. It suffices to show that the left-hand side of (47) is bounded above by

$$\lambda \sum_{n=1}^{\infty} |\eta_n|, \quad (47)$$

which is equivalent to

$$(\lambda - 1) \sum_{n=1}^m |\eta_n| \geq 0. \quad (48)$$

On the contrary, to prove inequality (37), we consider

$$\frac{1+s(z)}{1-s(z)} = (\lambda + 1) \left[\frac{((\psi_{\gamma,b,c}^a(z;q))_m)'}{(\psi_{\gamma,b,c}^a(z;q))'} - \gamma_2 \right], \quad (49)$$

$$= \frac{1 + \sum_{n=1}^m \eta_n z^n - \lambda \sum_{n=m+1}^{\infty} \eta_n z^n}{1 + \sum_{n=1}^m \eta_n z^n}.$$

Therefore,

$$s(z) = \frac{-(1+\lambda) \sum_{n=m+1}^{\infty} \eta_n z^n}{2 + 2 \sum_{n=1}^m \eta_n z^n - (\lambda - 1) \sum_{n=m+1}^{\infty} \eta_n z^n} \quad (50)$$

and

$$|s(z)| \leq \frac{(1+\lambda) \sum_{n=m+1}^{\infty} |\eta_n|}{2 - 2 \sum_{n=1}^m |\eta_n| - (\lambda - 1) \sum_{n=m+1}^{\infty} |\eta_n|}. \quad (51)$$

Inequality (37) holds true if $|s(z)| \leq 1$ according to subordination principle. Then, the upcoming inequality,

$$\sum_{n=1}^m |\eta_n| + \lambda \sum_{n=m+1}^{\infty} |\eta_n| \leq 1, \quad (52)$$

implies that $|s(z)| \leq 1$. Since the left-hand side of (52) is bounded above by $\lambda \sum_{n=1}^{\infty} |\eta_n|$; thus, the proof is now completed. \square

Data Availability

No data were used in this paper.

Ethical Approval

This study does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors confirm no conflicts of interest.

Authors' Contributions

The authors contributed to the draft of the manuscript; they have read and approved the final manuscript.



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Research Article

Certain Families of Analytic Functions Characterized by (p, q) -Difference Operator

Rashid Murtaza ¹, Adnan ¹, Amna Aziz,¹ Abdulaziz H. Alghtani,² Ilyas Khan ³,
and Mulugeta Andualem ⁴

¹Department of Mathematics, Mohi-Ud-Din Islamic University, Nerian Sharif AJK 12080, Pakistan

²Mechanical Engineering Department Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

³Department of Mathematics, College of Science Al-zulfi, Majmaah University, Al Majma'ah 11952, Saudi Arabia

⁴Department of Mathematics, Bonga University, Bonga, Saudi Arabia

Correspondence should be addressed to Rashid Murtaza; rashid.murtaza765@gmail.com, Adnan; adnan_abbasi89@yahoo.com, and Mulugeta Andualem; mulugetaandualem4@gmail.com

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The major motivation behind this work is to utilize (p, q) -calculus and inspected the generalized (p, q) -Ruscheweyh differential operator. At the present time, we utilized (p, q) -Ruscheweyh differential operator and examined some new subfamilies of \mathcal{S} and study some essential properties, for example, inclusion and subordination properties.

1. Introduction

The q -calculus is the speculation of ordinary calculus where the limits has been supplanted by q . The chronicled background of q -calculus may be portrayed by its wide variety of use in quantum mechanics, logical number hypothesis, theta functions, Bernoulli and Euler polynomials, Sobolev spaces, and more in the analytic and univalent function theory. Jackson ought to be praised for the specific start of quantum calculus in his work in [1, 2], by presenting the q -derivative and integral, respectively. In q -calculus, we essentially center around q -analogue that arise normally, rather than in arbitrary q -analogues of known results. While zeroing in on advancements and improvement of q -calculus and its applications in explicit fields of numerical and actual science, we will moreover look at q -analogues of a part of the new results in Geometric Function Theory in unit disk. The early work of q -calculus in the field of Geometric Function Theory was finished by Ismail et al. [3], by summing up the class of star-like functions into the class of q -star-like functions.

Assume that $f \in \mathbb{C}$. Then,

$$f \text{ is normalized analytic function} \iff f'(0) - 1 = 0 = f(0) \quad (1)$$

and characterized as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}. \quad (2)$$

The symbol \mathcal{A} stands for the family of all these type of functions. Also, by $\mathcal{S} \subset \mathcal{A}$, comprising of functions, which are univalent in \mathcal{E} . By $\mathcal{S}^*(\alpha)$ ($0 \leq \alpha < 1$), we meant the subfamily of \mathcal{S} consisting of star-like functions of order α . If $\alpha = 0$, then $\mathcal{S}^*(\alpha) = \mathcal{S}^*$.

Let $\wp \in \mathbb{C}$ be analytic in \mathcal{E} . Furthermore, we say that

$$\wp \in \mathcal{P} \iff \wp(0) = 1, \quad \operatorname{Re}(\wp(z)) > 0, z \in \mathcal{E}, \quad (3)$$

and presented as

$$\wp(z) = 1 + \sum_{k=1}^{\infty} c_k z^k. \quad (4)$$

For f_1 and f_2 given by

$$\begin{aligned}
 f_1(z) &= z + \sum_{k=2}^{\infty} a_k z^k, \\
 f_2(z) &= z + \sum_{k=2}^{\infty} b_k z^k, \\
 z &\in \mathcal{E},
 \end{aligned} \tag{5}$$

the convolution is characterized as

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (f_2 * f_1)(z), \quad z \in \mathcal{E}. \tag{6}$$

Assume f and g be analytic in \mathcal{E} . Then, $f < g$ if there exists ω analytic in \mathcal{E} , along $\omega(0) = 0$ and $|\omega(z)| < 1$, in \mathcal{E} such as

$$f(z) = g(\omega(z)), \quad z \in \mathcal{E}. \tag{7}$$

Also, it is observed by [4] that

$$f < g \iff f(0) = g(0) \text{ and } f(\mathcal{E}) \subset g(\mathcal{E}). \tag{8}$$

Kanas and Wisniowska [5] extended the parabolic region and characterized the region $Y_k, 0 \leq k < \infty$ as

$$Y_k = \left\{ w = s + it \in \mathbb{C} : s > k\sqrt{(s-1)^2 + t^2} \right\}, \tag{9}$$

with $1 \in Y_k$. Assume

$$\partial Y_k = \left\{ w = s + it : s^2 = k^2((s-1)^2 + t^2) \right\}. \tag{10}$$

Therefore,

$$\begin{aligned}
 k = 0 &\iff Y_0 = s > 0, \\
 k = 1 &\iff Y_1 = t^2 < 2s - 1, \\
 k \in (0, 1) &\iff Y_k = s > k\sqrt{(s-1)^2 + t^2}.
 \end{aligned} \tag{11}$$

Identified with the region Y_k , the accompanying functions are extremal and $\wp_k(\mathcal{E}) \subset Y_k$

$$\wp_k(z) = \begin{cases} \frac{1+z}{1-z}, & k=0, \\ 1 + \frac{2}{1-k^2} \sinh^2 \left[\left(\frac{2}{\pi} \arccos k \right) \arctan h\sqrt{z} \right], & 0 < k < 1, \\ 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & k=1. \end{cases} \tag{12}$$

Two fascinating subfamilies of \mathcal{S} are $k-\mathcal{UCV}$ and $k-\mathcal{ST}$ which consist of k -uniformly convex and k -uniformly star-like functions, respectively, and characterized by [5, 6] for $k \geq 0$ as

$$\begin{aligned}
 k-\mathcal{UCV} &= \left\{ g \in \mathcal{S} : \operatorname{Re} \left(1 + \frac{zg''(z)}{g'(z)} \right) > k \left| \frac{zg''(z)}{g'(z)} \right|, z \in \mathcal{E} \right\}, \\
 k-\mathcal{ST} &= \left\{ g \in \mathcal{S} : \operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) > k \left| \frac{zg'(z)}{g(z)} - 1 \right|, z \in \mathcal{E} \right\}.
 \end{aligned} \tag{13}$$

Note that

$$g \in k-\mathcal{UCV} \iff zg' \in k-\mathcal{ST}. \tag{14}$$

Geometrically, $g \in k-\mathcal{UCV}$ means that the image of \mathcal{E} under $[(zg'(z)) / g'(z)]$ is contained in Y_k , where Y_k is defined by (9).

In recent years, researchers are using these conic domain defined by (9) very efficiently and obtained very excellent results some of them are [7, 8] and reference therein.

Remark 1. All through this study, we mean by \mathcal{H} the family of functions h that are convex univalent in \mathcal{E} along $h(0) = 1$, except if in any case referenced.

2. (p, q)-Calculus

Expanding the concept of q -calculus, (p, q) -number was freely considered by [9]. Let $0 < q < p \leq 1$; then, (p, q) -number is characterized for $j \in \mathbb{N}$ by

$$[j]_{(p,q)} = \frac{p^j - q^j}{p - q} = p^{j-1} + p^{j-2}q + \dots + pq^{j-2} + q^{j-1}, \tag{15}$$

which is the common generalization of q -number. Note that $[j]_{(1,1)} = j$. Also, for $q \in \mathbb{N}$, we obtain

$$\begin{aligned}
 [q+1]_{(p,q)} &= q^q + p[q]_{(p,q)}, \\
 [q+j]_{(p,q)} &= q^{q+j-1} + p^j [q]_{(p,q)}.
 \end{aligned} \tag{16}$$

Furthermore,

$$\begin{aligned}
 [q+j]_{(p,q)} - p^j [q]_{(p,q)} &= \frac{p^{q+j} - q^{q+j}}{p - q} - \frac{p^j (p^q - q^q)}{p - q} \\
 &= \frac{q^q (p^j - q^j)}{p - q} \\
 &= q^q [j]_{(p,q)}.
 \end{aligned} \tag{17}$$

In [9], for $f \in \mathbb{C}$, (p, q) -derivative of is characterized as

$$\mathfrak{d}_{(p,q)} f(z) = \begin{cases} \frac{f(qz) - f(pz)}{(q-p)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad (18)$$

Moreover,

$$f(z) = z^k \iff \mathfrak{d}_{(p,q)} f(z) = [k]_{(p,q)} z^{k-1}. \quad (19)$$

Also, by utilizing (2), we attain

$$\mathfrak{d}_{(p,q)} f(z) = 1 + \sum_{k=2}^{\infty} [k]_{(p,q)} a_k z^{k-1}. \quad (20)$$

Note, if $p = 1$, at that point

$$\mathfrak{d}_{(1,q)} f(z) = \mathfrak{d}_q f(z). \quad (21)$$

Also, $\mathfrak{d}_{(1,1)} f(z) = f'(z)$. It is notable [10] that

$$\begin{aligned} \mathfrak{d}_{(p,q)} (v(z)u(z)) &= v(pz)\mathfrak{d}_{(p,q)} u(z) \\ &\quad + u(qz)\mathfrak{d}_{(p,q)} v(z), \end{aligned} \quad (22)$$

$$\begin{aligned} \text{or } \mathfrak{d}_{(p,q)} (v(z)u(z)) &= u(pz)\mathfrak{d}_{(p,q)} v(z) \\ &\quad + v(qz)\mathfrak{d}_{(p,q)} u(z). \end{aligned} \quad (23)$$

The (p, q) -integral of f presented in [10] is given by

$$\frac{1}{z(p-q)} \int_0^z \mathfrak{d}_{(p,q)} t = \sum_{j=0}^{\infty} \left(\frac{q^j}{p^{j+1}} \right) f \left(\frac{q^j}{p^{j+1}} z \right), \quad \text{if } \left| \frac{p}{q} \right| > 1. \quad (24)$$

Note that if $p = 1$, definition (24) reduces to the well-known Jackson integral defined in [2]. We refer to a notion of (p, q) -operators, that is, (p, q) -operator, that play imperative job in the hypothesis of hypergeometric series, operator theory, and quantum physics. The utilization of q -calculus was started by Jackson.

As of late, Akbar et al. [11] started the possibility of symmetric (p, q) -calculus, presented symmetric (p, q) -derivative, and examined another class of star-like functions along intriguing results.

2.1. (p, q) -Generalized Pochhammer Symbol. The investigation of star-like functions via (p, q) -calculus and various analytic functions classes were reported in [11, 12], respectively. The (p, q) -Gamma function is characterized by the accompanying relation:

$$\begin{aligned} \Gamma_{(p,q)}(r) &= [r]_{(p,q)} \Gamma_{(p,q)}(r-1), \\ \Gamma_{(p,q)}(1) &= 1. \end{aligned} \quad (25)$$

Expanding the concept of q -generalized Pochhammer symbol, the (p, q) -shifted factorial ((p, q) -generalized symbol) is characterized as

$$[r]_{(p,q)} = \begin{cases} 1, & k = 0, \\ [r]_{(p,q)} [r+1]_{(p,q)} \cdots [r+k-1]_{(p,q)}, & k \in \mathbb{N}. \end{cases} \quad (26)$$

Note

$$\left([r]_{(1,q)} \right)_k = \left([r]_q \right)_k \text{ and } (r)_{(1,1)k} = (r)_k. \quad (27)$$

Now, by generalizing the concept of [13], we introduced the (p, q) -Ruscheweyh differential operator as

$$\mathfrak{R}_{(p,q)}^q f(z) = f(z) * \mathfrak{F}_{(p,q)}^{q+1}(z), \quad q > -1, 0 < q < p \leq 1, z \in \mathcal{E}, \quad (28)$$

where

$$\mathfrak{F}_{(p,q)}^q(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma_{(p,q)}(k+q)}{[k-1]_{(p,q)}! \Gamma_{(p,q)}(1+q)} z^k, \quad (29)$$

$$= z + \sum_{k=2}^{\infty} \frac{\left([q+1]_{(p,q)} \right)_{k-1}}{[k-1]_{(p,q)}!} z^k. \quad (30)$$

2.2. Special Cases. If $p = 1, q \iff 1^-$, then

$$\mathfrak{R}_{(1,1)}^q f(z) = f(z) * \mathfrak{F}_{(1,1)}^{q+1}(z) = \mathfrak{R}^q f(z), \quad (31)$$

see [14].

Also, if $p = 1$, then

$$\mathfrak{R}_{(1,q)}^q f(z) = \mathfrak{R}_q^q(z) = f(z) * \mathfrak{F}_q^{q+1}(z), \quad (32)$$

see [13].

Therefore,

$$\mathfrak{R}_{(p,q)}^q f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma_{(p,q)}(k+q)}{\Gamma_{(p,q)}(1+q)[k-1]_{(p,q)}!} a_k z^k, \quad z \in \mathcal{E}. \quad (33)$$

Using (p, q) -derivative defined by (19), we observed that

$$\mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^q f(z) \right) = 1 + \sum_{k=2}^{\infty} \frac{\Gamma_{(p,q)}(k+q)}{\Gamma_{(p,q)}(1+q)[k-1]_{(p,q)}!} [k]_{(p,q)} a_k z^{k-1}. \quad (34)$$

If $p = 1, q \iff 1^-$ and $q = 0$, then

$$\mathfrak{d}_{(1,1)} \left(\mathfrak{R}_{(p,q)}^q f(z) \right) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} = f'(z), \quad z \in \mathcal{E}. \quad (35)$$

Utilizing (35) and $q \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, it can, without much of a stretch, be confirmed that

$$\begin{aligned} [q+1]_{(p,q)} \mathfrak{R}_{(p,q)}^q f(z) &= q^q z \left(\mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^q f(z) \right) \right) \\ &\quad + p[q]_{(p,q)} \mathfrak{R}_{(p,q)}^q f(z). \end{aligned} \quad (36)$$

By taking $p = 1$, we obtain

$$[q+1]_q \mathfrak{R}_q^q f(z) = q^q z \left(d_q \left(\mathfrak{R}_q^q f(z) \right) \right) + [q]_q \mathfrak{R}_q^q f(z), \quad (37)$$

see [15].

Also, if $p = 1, q \iff 1^-$, we observe

$$(\varrho + 1)\mathfrak{R}^{\varrho} f(z) = z(\mathfrak{R}^{\varrho} f(z))' + \varrho \mathfrak{R}^{\varrho} f(z), \quad (38)$$

see [14].

Using $(\mathfrak{p}, \mathfrak{q})$ -derivative defined by (19), we define the class $k - \mathcal{ST}_{(\mathfrak{p}, \mathfrak{q})}(\varrho)$ by the following.

Definition 1. Assume that $f \in \mathcal{A}$ is defined by (2) and $h \in \mathcal{H}$ is given as in Remark 1. Then, for $0 < \mathfrak{q} < \mathfrak{p} \leq 1$,

$$f \in k - \mathcal{ST}_{(\mathfrak{p}, \mathfrak{q})}(\varrho) \iff \left(\frac{z \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(\mathfrak{R}^{\varrho}_{(\mathfrak{p}, \mathfrak{q})} f(z))}{\mathfrak{R}^{\varrho}_{(\mathfrak{p}, \mathfrak{q})} f(z)} \right) \prec \wp_k(z), \quad (39)$$

$$k \geq 0, z \in \mathcal{E},$$

where \wp_k are defined by (12).

We can define the corresponding class $k - \mathcal{UCV}_{(\mathfrak{p}, \mathfrak{q})}(\varrho)$ as

$$f \in k - \mathcal{UCV}_{(\mathfrak{p}, \mathfrak{q})}(\varrho) \iff z(\mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})} f) \in k - \mathcal{ST}_{(\mathfrak{p}, \mathfrak{q})}(\varrho). \quad (40)$$

Equivalently,

$$(\mathfrak{R}^{\varrho}_{(\mathfrak{p}, \mathfrak{q})} f) \in k - \mathcal{ST} \iff f \in k - \mathcal{ST}_{(\mathfrak{p}, \mathfrak{q})}(\varrho). \quad (41)$$

Also,

$$f \in \mathcal{S}^*_{(\mathfrak{p}, \mathfrak{q})}(\varrho) \iff (\mathfrak{R}^{\varrho}_{(\mathfrak{p}, \mathfrak{q})} f) \in \mathcal{S}^*. \quad (42)$$

3. Main Results

To get our main results, the accompanying lemmas are helpful.

Lemma 1. Assume that $s = s_1 + is_2, t = t_1 + it_2$ and $\chi: D \subset \mathbb{C} \times \mathbb{C} \iff \mathbb{C}$ along

- (i) $\chi(s, t)$ is continuous in D
- (ii) $\chi(1, 0) > 0$ and $(1, 0) \in D$
- (iii) $\operatorname{Re}(\chi(is_2, t_1)) \leq 0$, for $(is_2, t_1) \in D$ and $t_1 \leq -(1/2)(1 + s_2^2)$

Suppose that $\wp(z) = 1 + c_1 z + c_2 z^2 + \dots$ belongs to \mathcal{P} along $(\wp, z\wp') \in D \forall z \in \mathcal{E}$. If $\operatorname{Re}(\chi(\wp(z), z\wp'(z))) > 0$, then $\operatorname{Re}(\wp(z)) > 0$; for details, see [16].

The attendant lemma is $(\mathfrak{p}, \mathfrak{q})$ -analogue of the lemma presented by [17].

Lemma 2. Assume that $h \in \mathcal{H}$ be defined as in Remark 1. Furthermore, let $\beta_0, \gamma_0 \in \mathbb{C}$ along $\beta_0 \neq 0$. If $\wp \in \mathcal{P}$, fulfill $\operatorname{Re}(\beta_0 \wp(z) + \gamma_0) > 0$ and

$$\wp(z) + \frac{z \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})} f(z)}{\beta_0 \wp(z) + \gamma_0} \prec h(z) \iff \wp(z) \prec h(z), \quad z \in \mathcal{E}. \quad (43)$$

The accompanying lemma is $(\mathfrak{p}, \mathfrak{q})$ -analogue of the lemma presented by [15] and will be required for later use.

Lemma 3. Assume that $h \in \mathcal{H}$ and $s(z) = 1 + s_1 z + s_2 z^2 + \dots$ be analytic in \mathcal{E} . Furthermore, if

$$s(qz) + \frac{\mathfrak{p}^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} z \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(s(z))}{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} \prec h(z), \quad z \in \mathcal{E}. \quad (44)$$

Then, for $[\gamma]_{(\mathfrak{p}, \mathfrak{q})} \neq 0$, we have

$$s(qz) \prec t(qz) \prec h(z), \quad (45)$$

where

$$t(z) = \frac{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}{z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}} \int_0^z t^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}-1} h(t) d_{(\mathfrak{p}, \mathfrak{q})} t \quad (46)$$

$$= (\mathfrak{p} - \mathfrak{q}) [\gamma]_{(\mathfrak{p}, \mathfrak{q})} \sum_{j=0}^{\infty} \frac{\mathfrak{q}^{j[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}}{\mathfrak{p}^{(j+1)[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}} h\left(\frac{\mathfrak{q}^j}{\mathfrak{p}^{j+1}} z\right),$$

is the best super-ordinate of (46).

Proof. Assume that $h \in \mathcal{H}$ and

$$s(qz) + \frac{\mathfrak{p}^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} z \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(s(z))}{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} \prec h(z), \quad z \in \mathcal{E}. \quad (47)$$

To find the best dominant of (47), we have to find the solution of corresponding differential equation:

$$t(qz) + \frac{\mathfrak{p}^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} z \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(t(z))}{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} = h(z), \quad z \in \mathcal{E}. \quad (48)$$

This means that, making use of product rules (22) or (23), we obtain

$$[\gamma]_{(\mathfrak{p}, \mathfrak{q})} z^{\gamma-1} h(z) = (\mathfrak{p}z)^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(t(z))$$

$$+ [\gamma]_{(\mathfrak{p}, \mathfrak{q})} z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}-1} t(qz) = (\mathfrak{p}z)^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(t(z))$$

$$+ t(qz) d_{(\mathfrak{p}, \mathfrak{q})}\left(z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}\right) = \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}\left(z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} t(z)\right). \quad (49)$$

Consequently,

$$t(z) = \frac{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}{z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}} \int_0^z t^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}-1} h(t) d_{(\mathfrak{p}, \mathfrak{q})} t, \quad (50)$$

which is best dominant of (46). By using (24), we can right

$$t(z) = \frac{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}{z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}} z (\mathfrak{p} - \mathfrak{q}) \sum_{j=0}^{\infty} \left(\frac{\mathfrak{q}^j}{\mathfrak{p}^{j+1}} \right) \left(\frac{\mathfrak{q}^j}{\mathfrak{p}^{j+1}} z \right)^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}-1} h$$

$$\left(\frac{\mathfrak{q}^j}{\mathfrak{p}^{j+1}} z \right) [\gamma]_{(\mathfrak{p}, \mathfrak{q})} (\mathfrak{p} - \mathfrak{q}) \sum_{j=0}^{\infty} \left(\frac{\mathfrak{q}^{j[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}}{\mathfrak{p}^{(j+1)[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}} \right) h\left(\frac{\mathfrak{q}^j}{\mathfrak{p}^{j+1}} z\right), \quad (51)$$

which is required.

If we take $\mathfrak{p} = 1$ in Lemma 3, we obtain the well-known result from [15]. \square

Theorem 1. Assume that $k \geq 0, 0 < \mathfrak{q} < \mathfrak{p} \leq 1$,

$$k - \mathcal{UCV}_{(p,q)}(\varrho) \subset k - \mathcal{ST}_{(p,q)}(\varrho), \quad z \in \mathcal{E}. \quad (52)$$

$$\frac{z \mathfrak{d}_{(p,q)}(\mathfrak{R}_{(p,q)}^{\varrho} f(z))}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} = h(z), \quad h \in \mathcal{P}. \quad (53)$$

Proof. Assume $f \in k - \mathcal{UCV}_{(p,q)}(\varrho)$ and consider

Presently utilizing (36), we obtain

$$h(z) = \frac{\left([\varrho + 1]_{(p,q)} / \mathfrak{q}^{\varrho}\right) \mathfrak{R}_{(p,q)}^{\varrho+1} f(z) - \left(\mathfrak{p} [\varrho]_{(p,q)} / \mathfrak{q}^{\varrho}\right) \mathfrak{R}_{(p,q)}^{\varrho} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} = \frac{\left([\varrho + 1]_{(p,q)} / \mathfrak{q}^{\varrho}\right) \mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} - \frac{\mathfrak{p} [\varrho]_{(p,q)}}{\mathfrak{q}^{\varrho}}. \quad (54)$$

Differentiating (53), (p, q) -logarithmically, we have

$$h(z) + \frac{z \mathfrak{d}_{(p,q)} h(z)}{h(z)} = \frac{\mathfrak{d}_{(p,q)} \left(z \mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^{\varrho} f(z) \right) \right)}{\mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^{\varrho} f(z) \right)} < \wp_k(z). \quad (55)$$

Therefore,

$$h(z) + \frac{z \mathfrak{d}_{(p,q)} h(z)}{h(z)} < \wp_k(z), \quad z \in \mathcal{E}. \quad (56)$$

Along these lines by utilizing Lemma 2, we have $h(z) < \wp_k(z)$, and thus,

$$\frac{z \mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^{\varrho} f(z) \right)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} < \wp_k \iff f \in k - \mathcal{ST}_{(p,q)}(\varrho) \quad (57)$$

which is required.

Note that, by taking specific values to the parameters in Theorem 1, we can get some notable results.

For $\varrho = 0$, $\mathfrak{p} = 1$, and $q \iff 1^-$, we obtain accompanying crucial result, see [6]. \square

Corollary 1. Assume that $k \geq 0$ and $\forall z \in \mathcal{E}$,

$$k - \mathcal{UCV} \subseteq k - \mathcal{ST}. \quad (58)$$

Corollary 2. Assuming that $k = 0$ and $\forall z \in \mathcal{E}$, we obtain $\mathcal{C} \subseteq \mathcal{S}^*$.

Theorem 2. Suppose that, for $k \geq 0$, and $\varrho \in \mathbb{N}_0$,

$$k - \mathcal{ST}_{(p,q)}(\varrho + 1) \subseteq k - \mathcal{ST}_{(p,q)}(\varrho) \subseteq \dots \subseteq k - \mathcal{ST} \subseteq \mathcal{S}^*(\eta_0) \subseteq \mathcal{S}^* \subseteq \mathcal{S}. \quad (59)$$

Equivalently, for $\eta_0 = k/k + 1$,

$$\mathcal{S}_{(p,q)}^*(\varrho + 1, \eta_0) \subseteq \mathcal{S}_{(p,q)}^*(\varrho, \eta_1) \subseteq \dots \subseteq \mathcal{S}_{(p,q)}^* \quad (60)$$

$$(\varrho - (m - 1), \eta_m) \subseteq \mathcal{S}_{(p,q)}^*(\varrho - m, \eta_{m+1}),$$

where $\varrho > m \in \mathbb{N}_0$ and

$$\eta_{m+1} = \frac{\left[2\mathfrak{q}^{\varrho} \eta_0 - 2\mathfrak{p} \left([\varrho]_{(p,q)} - [m]_{(p,q)} \right) - \mathfrak{q}^{\varrho} \right] - \sqrt{\left[2\mathfrak{q}^{\varrho} \eta_0 - 2\mathfrak{p} \left([\varrho]_{(p,q)} - [m]_{(p,q)} \right) - \mathfrak{q}^{\varrho} \right]^2 + 8 \left[2\eta_0 \mathfrak{p} \left([\varrho]_{(p,q)} - [m]_{(p,q)} \right) + \mathfrak{q}^{\varrho} \right]}}{4}. \quad (61)$$

Proof. Firstly, consider

$$k - \mathcal{ST}_{(p,q)}(\varrho + 1) \subseteq k - \mathcal{ST}_{(p,q)}(\varrho), \quad (62)$$

where

or

$$\mathcal{S}_{(p,q)}^*(\varrho + 1, \eta_0) \subseteq \mathcal{S}_{(p,q)}^*(\varrho, \eta_1), \quad (63)$$

$$\eta_1 = \frac{\left(2\mathfrak{q}^{\varrho} \eta_0 - 2\mathfrak{p} [\varrho]_{(p,q)} - \mathfrak{q}^{\varrho} \right) - \sqrt{\left(2\mathfrak{q}^{\varrho} \eta_0 - 2\mathfrak{p} [\varrho]_{(p,q)} - \mathfrak{q}^{\varrho} \right)^2 + 8 \left(2\eta_0 \mathfrak{p} [\varrho]_{(p,q)} + \mathfrak{q}^{\varrho} \right)}}{4}. \quad (64)$$

Let $f \in \mathcal{S}_{(p,q)}^*(\varrho + 1, \eta_0)$. Furthermore, let

$$\frac{z \mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^{\varrho} f(z) \right)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} = (1 - \eta_1) p_1(z) + \eta_1, \quad p_1 \in \mathcal{P}. \quad (65)$$

Using (37), we obtain

$$\begin{aligned} & \frac{\left(\left([\varrho + 1]_{(p,q)} \right) / \mathfrak{q}^{\varrho} \right) \mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} - \frac{\mathfrak{p}[\varrho]_{(p,q)}}{\mathfrak{q}^{\varrho}} \\ &= (1 - \eta_1) p_1(z) + \eta_1. \end{aligned} \quad (66)$$

This implies that

$$\frac{[\varrho + 1]_{(p,q)} \mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} = \mathfrak{q}^{\varrho} (1 - \eta_1) p_1(z) + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)}. \quad (67)$$

Differentiating (p, q) -logarithmically, we attain

$$\begin{aligned} & \frac{z \mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^{\varrho+1} f(z) \right)}{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z)} = (1 - \eta_1) p_1(z) + \eta_1 \\ & + \frac{(1 - \eta_1) z \mathfrak{d}_{(p,q)} (p_1(z))}{\mathfrak{q}^{\varrho} (1 - \eta_1) p_1(z) + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)}}. \end{aligned} \quad (68)$$

Since

$$f \in k - \mathcal{ST}_{(p,q)}(\varrho + 1) \subseteq \mathcal{S}_{(p,q)}^*(\varrho + 1, \eta_0), \quad \eta_0 = \frac{k}{k+1}. \quad (69)$$

This means that

$$\begin{aligned} & \operatorname{Re} \left((1 - \eta_1) p_1(z) + \eta_1 + \frac{\mathfrak{q}^{\varrho} (1 - \eta_1) z \mathfrak{d}_{(p,q)} (p_1(z))}{\mathfrak{q}^{\varrho} (1 - \eta_1) p_1(z) + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)}} \right) \\ & > \eta_0, z \in \mathcal{E}. \end{aligned} \quad (70)$$

Now, we construct a functional $\Psi(s, \nu)$ with $s = s_1 + is_2 = p_1(z)$ and $\nu = \nu_1 + i\nu_2 = z \mathfrak{d}_{(p,q)} (p_1(z))$ as

$$\Psi(s, \nu) = (1 - \eta_1)s + (\eta_1 - \eta_0) + \frac{\mathfrak{q}^{\varrho} (1 - \eta_1)\nu}{\mathfrak{q}^{\varrho} (1 - \eta_1)s + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)}}. \quad (71)$$

It very well may be seen that, for the functional $\Psi(s, \nu)$,

- (i) $\Psi(s, \nu)$ is continuous in $\mathbb{D} = \mathbb{C} - \left\{ \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)} / \mathfrak{q}^{\varrho} (\eta_1 - 1) \right\} \times \mathbb{C}$
- (ii) $(1, 0) \in \mathbb{D} = \mathbb{C} - \left\{ \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)} / \mathfrak{q}^{\varrho} (\eta_1 - 1) \right\} \times \mathbb{C}$ and $\operatorname{Re}(\Psi(1, 0)) = 1 - \eta_0 > 0$

Now, consider

$$\begin{aligned} & \operatorname{Re}(\Psi(is, \nu_1)) - \operatorname{Re} \left((1 - \eta)is_2 + (\eta_1 - \eta_0) + \frac{\mathfrak{q}^{\varrho} (1 - \eta_1)\nu_1}{\mathfrak{q}^{\varrho} (1 - \eta_1)is_2 + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)}} \right) \\ &= (\eta_1 - \eta_0) + \frac{\mathfrak{q}^{\varrho} (1 - \eta_1)\nu_1}{\mathfrak{q}^{2\varrho} (1 - \eta_1)^2 s_2^2 + (\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)})^2} \leq (\eta_1 - \eta_0) - \frac{(\mathfrak{q}^{\varrho} (1 - \eta_1)^2 (1 + s_2^2) + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)})}{2(\mathfrak{q}^{2\varrho} (1 - \eta_1)^2 s_2^2 + (\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)})^2)}, \end{aligned} \quad (72)$$

because $\nu_1 \leq -(1 + s_2^2/2)$. Therefore, after some simplifications, we obtain

$$\operatorname{Re}(\Psi(is_2, \nu_1)) = \frac{A + Bs_2^2}{2C}, \quad (73)$$

where

$$\begin{aligned} A &= \left(2(\eta_1 - \eta_0) \left(\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)} \right)^2 - \mathfrak{q}^{\varrho} (1 - \eta_1) \left(\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)} \right) \right), \\ B &= \left(2\mathfrak{q}^{2\varrho} (\eta_1 - \eta_0) (1 - \eta_1)^2 - \mathfrak{q}^{\varrho} (1 - \eta_1) \left(\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)} \right) \right), \\ C &= 2 \left(\mathfrak{q}^{2\varrho} (1 - \eta_1)^2 s_2^2 + (\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)})^2 \right). \end{aligned} \quad (74)$$

The R.H.S of (76) is negative if $A \leq 0, B \leq 0$. By applying Lemma 1, we finally attain

$$\mathcal{S}_{(p,q)}^*(\varrho, \eta_0) \subseteq \mathcal{S}_{(p,q)}^*(\varrho - 1, \eta_1), \quad (75)$$

and $A \leq 0$ yields

$$\eta_1 = \frac{\left(2q^e \eta_0 - 2p[\varrho]_{(p,q)} - q^e\right) - \sqrt{\left(2q^e \eta_0 - 2p[\varrho]_{(p,q)} - q^e\right)^2 + 8\left(2\eta_0 p[\varrho]_{(p,q)} + q^e\right)}}{4}, \quad (76)$$

where $\eta_0 = k/k + 1$.

Similarly, proceeding as before, one can prove

where

$$\mathcal{S}_{(p,q)}^*(\varrho, \eta_1) \subseteq \mathcal{S}_{(p,q)}^*(\varrho - 1, \eta_2), \quad (77)$$

$$\eta_2 = \frac{\left[2q^e \eta_0 - 2p\left([\varrho]_{(p,q)} - [1]_{(p,q)}\right) - q^e\right] - \sqrt{\left[2q^e \eta_0 - 2p\left([\varrho]_{(p,q)} - [1]_{(p,q)}\right) - q^e\right]^2 + 8\left[2\eta_0 p\left([\varrho]_{(p,q)} - [1]_{(p,q)}\right) + q^e\right]}}{4}. \quad (78)$$

Continuing this process, we obtain

where

$$\mathcal{S}_{(p,q)}^*(\varrho - (m-1), \eta_m) \subseteq \mathcal{S}_{(p,q)}^*(\varrho - m, \eta_{m+1}), \quad (79)$$

$$\eta_{m+1} = \frac{\left[2q^e \eta_0 - 2p\left([\varrho]_{(p,q)} - [m]_{(p,q)}\right) - q^e\right] - \sqrt{\left[2q^e \eta_0 - 2p\left([\varrho]_{(p,q)} - [m]_{(p,q)}\right) - q^e\right]^2 + 8\left[2\eta_0 p\left([\varrho]_{(p,q)} - [m]_{(p,q)}\right) + q^e\right]}}{4}. \quad (80)$$

By using the principal of mathematical induction for post-quantum calculus, one can prove that formula (83) is true for all $m \in \mathbb{N}_0$ and the proof is complete.

Note by allocating special values for parameters involved in Theorem 2, we gain some notable results. Assuming $p = 1$ and $q \iff 1^-$, in Theorem 2, we acquire notable results from [14]. \square

Corollary 3. Assume that $p = 1$ and $q \iff 1^-$, which implies

$$k - \mathcal{ST}(\varrho + 1) \subseteq k - \mathcal{ST}(\varrho) \quad (81)$$

Equivalently,

$$\mathcal{S}^*(\varrho + 1, \eta_0) \subseteq \mathcal{S}^*(\varrho, \eta_1) \quad (82)$$

where

$$\eta_1 = \frac{(2\eta_0 - 2\varrho - 1) - \sqrt{(2\eta_0 - 2\varrho - 1)^2 + 8(2\eta_0\varrho + 1)}}{4},$$

$$\eta_0 = \frac{k}{k+1}. \quad (83)$$

In this portion, we generalize some subordination results presented in [15].

Theorem 3. Let $\alpha > 0, \varrho > 0, 0 < q < p \leq 1, -1 \leq \mathcal{D} < \mathcal{C} \leq 1$. If $f \in \mathcal{A}$ presented by (2) fulfills

$$(1 - \alpha) \frac{\Re_{(p,q)}^e f(z)}{z} + \alpha \frac{\Re_{(p,q)}^{e+1} f(z)}{z} < \frac{1 + \mathcal{C}z}{1 + \mathcal{D}z}, \quad (84)$$

then, for $\eta \geq 1$, we obtain

$$\operatorname{Re}(t(z))^{1/\eta} > \left(\frac{[\varrho + 1]_{(p,q)}}{\alpha q^e p^{[\gamma]_{(p,q)}}} \int_0^1 s \left(([\varrho + 1]_{(p,q)}) / \left(\alpha_q^{ep^{[\gamma]_{(p,q)}}} \right) \right)^{-1} \frac{1 - \mathcal{C}s}{1 - \mathcal{D}s} d_{(p,q)} s \right)^{1/\eta}. \quad (85)$$

Proof. Let

$$g(z) = \frac{\mathcal{R}_{(p,q)}^e f(z)}{z}, \quad g \in \mathcal{P}. \quad (86)$$

Differentiating this equation (p, q) -logarithmically, we obtain

$$\frac{z \mathfrak{d}_{(p,q)}(g(z))}{g(z)} = \frac{z \mathfrak{d}_{(p,q)}\left(\frac{\mathcal{R}_{(p,q)}^e f(z)}{z}\right)}{\mathcal{R}_{(p,q)}^e f(z)} - 1. \quad (87)$$

Making use of (37), we can obtain

$$\frac{z \mathfrak{d}_{(p,q)}(g(z))}{g(z)} = \frac{[\varrho + 1]_{(p,q)}}{q^{\varrho}} \frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} - \frac{p[\varrho]_{(p,q)}}{q^{\varrho}} - 1. \quad (88)$$

This implies from (16) that

$$\frac{z \mathfrak{d}_{(p,q)}(g(z))}{g(z)} = \frac{[\varrho + 1]_{(p,q)}}{q^{\varrho}} \left(\frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z) - \mathfrak{R}_{(p,q)}^{\varrho} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} \right) \frac{q^{\varrho} z \mathfrak{d}_{(p,q)}(g(z))}{[\varrho + 1]_{(p,q)}} = \left(\frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z) - \mathfrak{R}_{(p,q)}^{\varrho} f(z)}{z} \right). \quad (89)$$

Equivalently,

$$g(z) + \frac{\alpha q^{\varrho} z \mathfrak{d}_{(p,q)}(g(z))}{[\varrho + 1]_{(p,q)}} = (1 - \alpha) \frac{\mathfrak{R}_{(p,q)}^{\varrho} f(z)}{z} + \alpha \frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{z} < \frac{1 + \mathcal{C}z}{1 + \mathcal{D}z}, \quad z \in \mathcal{E}. \quad (90)$$

Therefore,

$$g(z) + \frac{\alpha q^{\varrho} z \mathfrak{d}_{(p,q)}(g(z))}{[\varrho + 1]_{(p,q)}} < \frac{1 + \mathcal{C}z}{1 + \mathcal{D}z}, \quad z \in \mathcal{E} \quad (91)$$

or

$$g(z) + \frac{p^{[y]}_{(p,q)} z \mathfrak{d}_{(p,q)}(g(z))}{\left([\varrho + 1]_{(p,q)}\right) \left(\alpha q^{\varrho} p^{[y]}_{(p,q)}\right)} < \frac{1 + \mathcal{C}z}{1 + \mathcal{D}z}, \quad z \in \mathcal{E}. \quad (92)$$

Using Lemma 3, we obtain

$$g(z) < t(z) = \frac{[\varrho + 1]_{(p,q)}}{\alpha q^{\varrho} p^{[y]}_{(p,q)}} z^{-\left(\left([\varrho + 1]_{(p,q)}\right) / \left(\alpha q^{\varrho} p^{[y]}_{(p,q)}\right)\right)} \int_0^z s^{\left(\left([\varrho + 1]_{(p,q)}\right) / \left(\alpha q^{\varrho} p^{[y]}_{(p,q)}\right)\right) - 1} \frac{1 + \mathcal{C}w(z)}{1 + \mathcal{D}w(z)} \mathfrak{d}_{(p,q)} s. \quad (93)$$

Therefore,

$$Re(t(z)) > \frac{[\varrho + 1]_{(p,q)}}{\alpha q^{\varrho} p^{[y]}_{(p,q)}} \int_0^1 s^{\left(\left([\varrho + 1]_{(p,q)}\right) / \left(\alpha q^{\varrho} p^{[y]}_{(p,q)}\right)\right) - 1} \frac{1 - \mathcal{C}s}{1 - \mathcal{D}s} \mathfrak{d}_{(p,q)} s. \quad (94)$$

Presently, utilizing the reality $(Rew)^{1/\eta} < Re(w^{1/\eta})$ for $0 < Re(w)$ and $1 \leq \eta$, we get (85), which is required.

Note that, by offering specific values to parameters in Theorem 3, we attain certain fascinating results from [15]. \square

Corollary 4. Assumes that $\alpha > 0, \varrho > 0, -1 \leq \mathcal{D} < \mathcal{C} \leq 1$, and $p = 1$. Furthermore, if $f \in \mathcal{A}$ is presented as in (2), which satisfies

$$(1 - \alpha) \frac{\mathfrak{R}_{(p,q)}^{\varrho} f(z)}{z} + \alpha \frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{z} < \frac{1 + \mathcal{C}z}{1 + \mathcal{D}z}, \quad (95)$$

Then, for $\eta \geq 1$, one can have

$$\begin{aligned} & Re\left(\frac{\mathfrak{R}_{(p,q)}^{\varrho} f(z)}{z}\right)^{1/\eta} \\ & > \left(\frac{[\varrho + 1]_q}{\alpha q^{\varrho}} \int_0^1 s^{([\varrho + 1]_q / \alpha q^{\varrho}) - 1} \frac{1 - \mathcal{C}s}{1 - \mathcal{D}s} d_q s\right)^{1/\eta}. \end{aligned} \quad (96)$$

Corollary 5. Assume that $\alpha > 1, \varrho > 1, \mathcal{C} = 1 - 2\beta$, $0 \leq \beta < 1, \mathcal{D} = 1$ and $p = 1$. Furthermore, if $f \in \mathcal{A}$ is presented as in (2), which satisfies

$$\begin{aligned} & (1 - \alpha) \frac{\mathfrak{R}_{(p,q)}^{\varrho} f(z)}{z} + \alpha \frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{z} \\ & < \frac{1 + (1 - 2\beta)z}{1 - z}, \end{aligned} \quad (97)$$

then, for $\eta \geq 1$, we obtain

$$\operatorname{Re}\left(\frac{\mathfrak{R}_q^{\varrho} f(z)}{z}\right)^{1/\eta} > \left((2\beta - 1)s^{([q+1]_q)/\alpha q^{\varrho}} + \frac{2(1 - \beta)[q + 1]_q}{\alpha q^{\varrho}} \int_0^1 \frac{s^{([q+1]_q)/\alpha q^{\varrho} - 1}}{1 + s} d_q s\right)^{1/\eta}. \quad (98)$$

4. Conclusion

The essential explanation behind this investigation work is to use (p, q) -calculus and inspected the generalized (p, q) -Ruscheweyh differential operator. At the present time, we utilized (p, q) -Ruscheweyh differential operator and examined some new subfamilies of \mathcal{S} and study some essential properties, for example, inclusion and subordination properties. Our results sum up some charming known consequences. The way is open for researchers to investigate all the more right currently related areas.

Data Availability

No data were used to support this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Characterizations of Integral Type for Weighted Classes of Analytic Banach Function Spaces

Amnah E. Shammaky¹ and A. El-Sayed Ahmed ²

¹Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia

²Mathematics Department, Faculty of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

Correspondence should be addressed to A. El-Sayed Ahmed; ahsayed80@hotmail.com

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The aim of this study is to give some new definitions of Banach spaces of holomorphic functions. Some holomorphic characterizations of integral type for some classes of Banach spaces of holomorphic functions are established in the unit disc U .

1. Introduction and Preliminaries

Numerous studies of analytic function spaces by several classes of functions are introduced and intensively studied. The theory of function spaces provides interesting tools in many active branches of mathematics, especially in mathematical analysis such as in operator theory, measure theory, and differential equations. In the present study, we aim to give the definition of g -Bloch space of holomorphic functions. Using the new class of functions, some essential relations between it and some other known classes are investigated.

Next, we report the recent advancements of the concepts of specific-weighted classes of holomorphic function spaces. The choice of the appropriate functions gives the specific essential properties of the underlying weighted classes of functions that can have an important impact for the study.

Specific weighted classes of holomorphic function spaces and concepts are presented. Let $U = \{w: w \in \mathbb{C}, |w| < 1\}$, and $H(U)$ denote the class of all holomorphic functions that belonging to U .

The known analytic Bloch-type space [1–5] is defined by

$$\mathcal{B} = \left\{ f: f \in H(U) \text{ and } \sup_{w \in U} (1 - |w|^2) |f'(w)| < \infty \right\}. \quad (1)$$

The analytic little Bloch-type space \mathcal{B}_0 is symbolized by \mathcal{B}_0 , for which

$$\mathcal{B}_0 = \left\{ f: f \in H(U) \text{ and } \lim_{|w| \rightarrow 1^-} (1 - |w|^2) |f'(w)| = 0 \right\}. \quad (2)$$

For numerous global studies on Bloch-type spaces, we refer to [6–13] and others.

The analytic Dirichlet-type space [1, 14] is given by

$$\mathcal{D} = \left\{ f: f \in H(U) \text{ and } \int_U |f'(w)|^2 d\sigma_w < \infty \right\}, \quad (3)$$

where $d\sigma_w = dx dy$.

Using Green's function $g(w, w_0) = \ln|(1 - \bar{w}_0 w)/(w_0 - w)| = \ln(1/|\phi_{w_0}(w)|)$ as a weight function, the analytic \mathcal{Q}_p -spaces is defined in [1] by

$$\mathcal{Q}_p = \left\{ f: f \in H(U) \text{ and } \sup_{w_0 \in U} \int_U |f'(w)|^2 g^p(w, w_0) d\sigma_w < \infty \right\}. \quad (4)$$

By analytic \mathcal{Q}_p -spaces, some interesting relationships between analytic Dirichlet-type space and the analytic Bloch-type space are obtained [1].

For intensive research on analytic \mathcal{Q}_p spaces, we may refer to [1, 15–18] and others. Also, there are certain specific generalizations of these weighted classes of analytic functions in C^n [12, 19, 20]. On the other hand, there are some interesting extensions using quaternion-valued functions setting ([21–24]).

Hereafter, we set

$$\varphi_{w_0}(w) = \frac{w_0 - w}{1 - \bar{w}_0 w}, \quad w \neq w_0, \quad (5)$$

and set

$$\varphi_{w_0}(w) = C < 1, \quad \text{when } w = w_0. \quad (6)$$

The modified Green's function is introduced by

$$\mathbf{g}(w, w_0) = \ln \left| \frac{1 - \bar{w}_0 w}{w_0 - w} \right| = \ln \frac{1}{|\varphi_{w_0}(w)|}. \quad (7)$$

Motivated by the modified Green's function, the following definitions can be presented.

Definition 1. Let $0 < m < \infty$ and $0 < n < \infty$. For the function $f \in H(U)$, we define the analytic \mathbf{g} -Bloch space $\mathcal{B}_g^{n,m}$ as follows:

$$\mathcal{B}_g^{n,m} = \left\{ f: \sup_{w, w_0 \in U} \frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |f'(w)| < \infty \right\}. \quad (8)$$

Furthermore, assume that

$$\mathcal{B}_g^{n,m}(f) = \sup_{w, w_0 \in U} \frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |f'(w)|. \quad (9)$$

Example 1. Let $m \in (0, \infty)$. Suppose that

$$f(w) = \int_{|w| < 1} \left(\log \frac{1}{|\varphi_a(w)|} \right)^m dw. \quad (10)$$

It is very obvious that the function f is a \mathbf{g} -Bloch function.

Remark 1. Using Definition 1, relationships between analytic Dirichlet-type functions and analytic Bloch functions can be characterized.

When $m = 0$ and $0 < n = \alpha < \infty$, then we will obtain α -Bloch space. The case $m \in (-\infty, 0)$ induces some other types of analytic function spaces with different behaviors and can be studied separately.

The little analytic \mathbf{g} -Bloch space $\mathcal{B}_g^{n,m,0}$ can be considered as a subspace of the analytic \mathbf{g} -Bloch space that includes all $f \in \mathcal{B}_g^{n,m}$, with

$$\lim_{|w| \rightarrow 1^-} \lim_{|w_0| \rightarrow 1^-} \frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |f'(w)| = 0. \quad (11)$$

Remark 2. The symbol $\mathcal{B}_g^{n,m}(f)$ stands for a seminorm, while the usual norm can be defined as

$$\|f\|_{\mathcal{B}_g^{n,m}} = |f(0)| + \mathcal{B}_g^{n,m}(f). \quad (12)$$

Applying the above norm, the space $\mathcal{B}_g^{n,m}$ is a complete normed space (Banach space).

The next lemma can be applied for some results in this study.

Lemma 1 (See [14]). Let $0 < s < \infty$ and assume that $|w_0| < 1$. Then,

$$\int_{\Gamma_w} \frac{1}{|1 - \bar{w}_0 w|^{2s}} d\Gamma_w \leq \frac{k}{(1 - |w_0|)^s}, \quad (13)$$

where $k > 0$, and Γ_w defines the boundary of U .

2. \mathbf{g} -Bloch Space and Dirichlet Space

Some essential characterizations between the analytic Dirichlet-type space and the analytic \mathbf{g} -Bloch space are given in this section.

Proposition 1. Let $f \in H(U)$ and assume that $|w_0| < 1$. Then,

$$\frac{(1 - |w|^2)^2}{\mathbf{g}^m(w, w_0)} |f'(w_0)|^2 \leq \frac{4}{\pi R^2} \int_U |f'(w_0)|^2 d\sigma_w, \quad (14)$$

where $0 < R < 1$.

Proof. Because the pseudohyperbolic disk can be symbolized by

$$D(w_0, R) = \left\{ w: \rho(w, w_0) = \left| \frac{w - w_0}{1 - \overline{w_0}w} \right| < R \right\}, \quad (15)$$

where w_0 is its center and R is its radius, then we infer that

$$\begin{aligned} \int_U |f'(w)|^2 d\sigma_w &\geq \int_{D(w_0, R)} |f'(w)|^2 d\sigma_w \\ &= \pi R^2 (1 - |w_0|^2)^2 |f'(w_0)|^2 \geq \pi R^2 (1 - |w_0|^2)^2 |f'(w_0)|^2. \end{aligned} \quad (16)$$

Using the definition of the modified Green's function as well as the inequalities,

$$\begin{aligned} 0 < \varphi_{w_0}(w_0) &= C < 1, \\ \left(1 - |\varphi_{w_0}(w)|^2\right) &\leq 2\mathbf{g}(w, w_0), \text{ (see [25])}. \end{aligned} \quad (17)$$

We can obtain

$$\begin{aligned} \int_U |f'(w)|^2 d\sigma_w &\geq \pi R^2 \frac{(1 - |w_0|^2)^2}{\left(1 - |\varphi_{w_0}(w_0)|^2\right)^2} |f'(w_0)|^2 \\ &> 2\pi R^2 \frac{(1 - |w_0|^2)^2}{C = \mathbf{g}^2(w_0, w_0)} |f'(w_0)|^2. \end{aligned} \quad (18)$$

The proof of Proposition 1 is therefore finished. \square

Corollary 1. In view of Proposition 1, for $|w_0| < 1$, we have the following interesting inclusion:

$$\mathcal{D} \subset \mathcal{B}_{\mathbf{g}}^{2,2}. \quad (19)$$

Proposition 2. Let $f \in H(U)$ and assume that $|w_0| < 1$. Also, we suppose that $n \in [1, \infty)$ and $m \in (0, \infty)$, then

$$\frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^m(w_0, w_0)} |f'(w_0)|^2 \leq \frac{2^m}{\pi R^2} \int_U |f'(w)|^2 d\sigma_w, \quad (20)$$

where $0 < R < 1$.

Proof. The proof of Proposition 2 can be obtained as in the proof of Proposition 1, with the following changes:

$$\begin{aligned} \int_U |f'(w)|^2 d\sigma_w &\geq \pi R^2 \frac{(1 - |w_0|^2)^2}{\left(1 - |\varphi_{w_0}(w_0)|^2\right)^m} |f'(w_0)|^2 \\ &\geq \frac{\pi}{2^m} R^2 \frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^m(w_0, w_0)} |f'(w_0)|^2. \end{aligned} \quad (21)$$

\square

Corollary 2. Proposition 2 results that

$$\mathcal{D} \subset \mathcal{B}_{\mathbf{g}}^{2n,m}, \quad (22)$$

with $|w_0| < 1$, $n \in [1, \infty)$, and $m \in (0, \infty)$.

Remark 3. Corollaries 1 and 2 interpret that the analytic Dirichlet-type space can be considered as a subspace of the analytic \mathbf{g} -Bloch space when $m = n = 2$ as well as for $n \in [1, \infty)$ and $m \in (0, \infty)$, respectively.

Proposition 3. Let $f \in H(U)$ and $f \in \mathcal{B}_{\mathbf{g}}^{n,m}$. Suppose that

$$I(m, n) = \int_0^1 \frac{(\ln(1/r))^m}{(1 - r^2)^{1-n}} r dr < \infty. \quad (23)$$

Then, for $0 < m < \infty$ and $0 < n < \infty$ with $m + n > 0$, we obtain that

$$\sup_{w \in U} \int_U |f'(w)|^2 d\sigma_w \leq (2)^{4-m} \pi I(m, n) (\mathcal{B}_{\mathbf{g}}^{n,m}(f))^2. \quad (24)$$

Proof. From the definition of \mathbf{g} -Bloch space, we have

$$\frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^m(w_0, w_0)} |f'(w)| \leq \mathcal{B}_{\mathbf{g}}^{n,m}(f). \quad (25)$$

By a change of the variables technique, we infer that

$$\begin{aligned} \int_U |f'(w)|^2 d\sigma_w &\leq (\mathcal{B}_{\mathbf{g}}^{n,m}(f))^2 \int_U \frac{(\ln(1/|w_0|))^m}{\left(1 - |\varphi_{w_0}(w_0)|^2\right)^n} \\ &\quad \frac{(1 - |\varphi_{w_0}(w_0)|^2)^2}{(1 - |w_0|^2)^2} d\sigma_w. \end{aligned} \quad (26)$$

Since,

$$\left(1 - |\varphi_{w_0}(w_0)|^2\right) = \frac{(1 - |w_0|^2)(1 - |w|^2)}{|1 - \overline{w_0}w|^2}. \quad (27)$$

Therefore,

$$\begin{aligned} \int_U |f'(w)|^2 d\sigma_w &\leq (\mathcal{B}_{\mathbf{g}}^{n,m}(f))^2 \int_U \frac{(\ln(1/|w_0|))^m}{\left(1 - |\varphi_{w_0}(w_0)|^2\right)^n} d\sigma_w \\ &\leq \pi 2^{4-n} (\mathcal{B}_{\mathbf{g}}^{n,m}(f))^2 \int_0^1 \frac{(\ln(1/r))^m}{(1 - r^2)^{1-n}} r dr, \end{aligned} \quad (28)$$

which implies that

$$\int_U |f'(w)|^2 d\sigma_w \leq (\mathcal{B}_{\mathbf{g}}^{n,m}(f))^2 k_1, \quad (29)$$

where $0 < k_1 < \infty$. The proof is therefore established.

Propositions 2 and 3 result in the following fundamental theorem. \square

Theorem 1. Let $f \in H(U)$; then, we have equivalence between the following statements:

- (i) $f \in \mathcal{B}_{\mathbf{g}}^{n,m}$ with $0 \leq m < \infty$ and $0 < n < \infty$ with $m + n > 0$
- (ii) $f \in \mathcal{D}$.

Remark 4. The obtained results in Theorem 1 reflexed the major role of the newly definition of the analytic \mathbf{g} -Bloch space which has used to get relations between analytic Dirichlet-type space \mathcal{D} and $\mathcal{B}_{\mathbf{g}}^{n,m}$ space.

3. \mathbf{g} -Bloch Functions and \mathcal{Q}_p Functions

Proposition 4. For $0 \leq m < \infty$, $1 \leq n < \infty$, let $f \in H(U)$ and $|w_0| < 1$; hence,

$$\frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^m(w_0, w_0)} |f'(w_0)|^2 \leq \frac{2^{2m+4}}{\rho^2(1 - \rho^2)^p} \int_U |f'(w)|^2 (1 - |\varphi_{w_0}(w)|^2)^p d\sigma_w, \quad (30)$$

where $0 < \rho < 1$ and $p \in (0, \infty)$.

Proof. From subharmonicity principle, the following inequality can be easily obtained:

$$|h(0)|^2 \leq \frac{1}{\rho^2} \int_{D(0, \rho)} |h(z)|^2 d\sigma_w, \quad (31)$$

Since $f \in H(U)$ is holomorphic on $D(w_0, \rho)$, then use (31) to the function

$$h = f' \circ \varphi_{w_0}. \quad (32)$$

Applying the technique of change of variables, the next inequality can be inferred:

$$\begin{aligned} |f'(w_0)|^2 &\leq \frac{1}{\rho^2} \int_{D(0, \rho)} |f'(\varphi_{w_0}(w))|^2 d\sigma_w \\ &= \frac{1}{\rho^2} \int_{D(w_0, \rho)} |f'(w)|^2 \left(\frac{1 - |\varphi_{w_0}(w)|^2}{1 - |w|^2} \right)^2 d\sigma_w. \end{aligned} \quad (33)$$

From [5], we have

$$\left(\frac{1 - |\varphi_{w_0}(w)|^2}{1 - |w|^2} \right)^2 \leq \left(\frac{4}{(1 - |w_0|^2)} \right)^2. \quad (34)$$

This yields that

$$|f'(w_0)|^2 \leq \frac{(4/\rho)^2}{(1 - |w_0|^2)^2} \int_{D(w_0, \rho)} |f'(w)|^2 d\sigma_w. \quad (35)$$

Because,

$$\begin{aligned} (1 - |\varphi_{w_0}(w)|^2)^p &\geq (1 - \rho^2)^p, \\ \varphi_{w_0}(w_0) &= 0. \end{aligned} \quad (36)$$

Then,

$$\begin{aligned} \int_{D(w_0, \rho)} |f'(w)|^2 (1 - |\varphi_{w_0}(w)|^2)^p d\sigma_w &\geq \frac{\rho^2(1 - \rho^2)^p}{16} (1 - |w_0|^2)^2 |f'(w_0)|^2 \\ &\geq \frac{\rho^2(1 - \rho^2)^p}{16} \frac{(1 - |w_0|^2)^2}{(1 - |\varphi_{w_0}(w_0)|^2)^2} |f'(w_0)|^2 \\ &\geq \frac{\rho^2(1 - \rho^2)^p}{2^{2m+4}} \frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^m(w, w_0)} |f'(w_0)|^2. \end{aligned} \quad (37)$$

Using the inequality

$$\int_U |f'(w)|^2 \left(1 - |\varphi_{w_0}(w)|^2\right)^p d\sigma_w \geq \int_{D(w_0, \rho)} |f'(w)|^2 \left(1 - |\varphi_{w_0}(w_0)|^2\right)^p d\sigma_w, \quad (38)$$

we can deduce that

$$\frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^{2m}(w_0, w)} |f'(w_0)|^2 \leq \frac{2^{2m+4}}{\rho^2 (1 - \rho^2)^p} \int_U |f'(w)|^2 \left(1 - |\varphi_{w_0}(w)|^2\right)^p d\sigma_w. \quad (39)$$

The proof is therefore completely finished. \square

Corollary 3. By Proposition 4, for $p > 0$, $0 \leq m < \infty$, $1 \leq n < \infty$, and $|w_0| < 1$, the following inclusions can be easily proved:

where

$$\mathcal{B}_{\mathbf{g}}^{2n, 2m}(f) \leq \frac{2^{2m+4}}{\rho^2 (1 - \rho^2)^p} \mathcal{Q}_p^2(f), \quad (40)$$

$$\mathcal{Q}_p(f) = \sup_{w_0 \in U} \int_U |f'(w)|^2 \left(1 - |\varphi_{w_0}(w)|^2\right)^p d\sigma_w \approx \sup_{w_0 \in U} \int_U |f'(w)|^2 (\mathbf{g}(w, w_0))^p d\sigma_w < \infty. \quad (41)$$

Proposition 5. Let $f \in H(U)$. Assume that $0 \leq m < \infty$ and $0 < n < \infty$ with $m + n + p > 0$. Then, we have that

$$I(m, n, p) = \lambda \int_0^1 \frac{(\ln(1/r))^{m+p}}{(1-r)^{1-n}} r dr, \quad (43)$$

and λ is an absolute positive constant.

$$\int_U |f'(w)|^2 (\mathbf{g}(w, w_0))^p d\sigma_w \leq I(m, n, p) (\mathcal{B}_{\mathbf{g}}^{m, n})^2(f), \quad (42)$$

Proof. It is not hard to see that

where

$$\int_U |f'(w)|^2 \mathbf{g}^p(w_0, w) d\sigma_w \leq 4 (\mathcal{B}_{\mathbf{g}}^{m, n})^2(f) \int_U \frac{(\ln(1/|w|))^{m+p}}{(1 - |w|^2)^{2-2n}} d\sigma_w. \quad (44)$$

Applying Lemma 1, we obtain

$$\begin{aligned} \int_U |f'(w)|^2 \mathbf{g}^p(w_0, w) d\sigma_w &\leq 8\pi (\mathcal{B}_{\mathbf{g}}^{m, n})^2(f) \int_0^1 \frac{(\ln(1/r))^{m+p}}{(1-r)^{1-n}} r dr \\ &= I(m, n, p) (\mathcal{B}_{\mathbf{g}}^{m, n})^2(f). \end{aligned} \quad (45)$$

Combining Corollary 3 and Proposition 5, we have the following theorem: \square

Theorem 2. Let $h \in H(U)$. Assume that $0 \leq m < \infty$ and $0 < n < \infty$ with $m + n + p > 0$ as well as $0 < p < \infty$. Therefore, the next assertions can be equivalent:

$$(a) \ h \in \mathcal{B}_{\mathbf{g}}^{m, n}$$

$$(b) \ h \in \mathcal{Q}_p$$

$$(c) \ h \in \mathcal{Q}_p \text{ for some } 0 < p < \infty.$$

Proof. (a) \Rightarrow (b) can be established using Proposition 5. (b) \Rightarrow (c) is quite obvious. (c) \Rightarrow (a) can be proved by Corollary 3. \square

Remark 5. Theorem 2 investigates relations between the new type of holomorphic g -Bloch functions and the weighted holomorphic Dirichlet-type functions.

4. A Specific Criteria

The following symbol stands for a non-Euclidean distance of hyperbolic-type between the points w_0 and w in U .

$$d(w_0, w) = \ln \sqrt{\frac{|1 - \overline{w_0}w| + |w_0 - w|}{|1 - \overline{w_0}w| - |w_0 - w|}} \quad (\text{see [26]}). \quad (46)$$

Now, for $0 < R < \infty$, set

$$\begin{aligned} T(w, R) &= \{w_0 \in U : d(w_0, w) < R\}, \\ T_1(w, R) &= \{w_0 \in U : d(w_0, w) = R\}. \end{aligned} \quad (47)$$

Let $h \in H(U)$ be the nonconstant analytic function, and let $S_h(w, R)$ define the concerned area of the Riemannian image $P(w, R)$ of $T(w, R)$ by the function h , and assume that $\mathcal{S}_h^*(w, R)$ defines the area of the image $P(w, R)$ of $T(w, R)$ using the function h . It should be noted that $P(w, R)$ defines the projection to C . The length of the Riemannian image of

$T_1(w, R)$ by h is denoted by $M_h(z, \rho)$, and the symbol $\mathcal{M}_h(z, \rho)$ defines the length of the outer boundary of $P(w, R)$. If Γ defines a specific bounded domain in C , the concept of outer boundary means that we are working in the boundary of $C \setminus B$, where B may be defined as the unbounded component for the complement $C \setminus \Gamma$.

Now, we clearly have the following inequalities:

$$\begin{aligned} S_h(w, R) &\geq \mathcal{S}_h^*(w, R), \\ M_h(w, R) &\geq \mathcal{M}_h(w, R), \end{aligned} \quad (48)$$

where $0 < R < \infty$ and each $w \in U$.

Proposition 6. Let $h \in H(U)$ be a nonconstant analytic function. Suppose that $1 \leq n < \infty$ and $0 \leq m < \infty$, then the following inequalities can be deduced:

$$\frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |h'(w)| \leq \frac{1}{\theta(R)} \sqrt{\frac{2^m \mathcal{S}_h^*(w, R)}{\pi(1 - \rho^2)^m}}. \quad (49)$$

Also,

$$\frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |h'(w)| \leq \frac{2^m (1 - |w|^2)^n}{(1 - |\varphi_{w_0}(w)|^2)^m} |h'(w)| \leq \frac{2^{m-1} \mathcal{M}_h(w, R)}{(1 - \rho^2)^m \pi \theta(R)}, \quad (50)$$

where $\theta(R) = ((e^{2R} - 1)/(e^{2R} + 1))$.

Proof. First, we suppose that $h'(w) \neq 0$. Now, we suppose that

$$f(w_0) = h\left(\frac{w_0 + w}{1 + \overline{w_0}w}\right) = b_0 + b_1 w_0 + b_2 w_0^2 + b_3 w_0^3 + \dots, \quad (51)$$

with $|w_0| < 1$, where $b_0, b_1, b_2, b_3, \dots$ are the positive constants. Now,

$$(1 - |w|^2)^n |h'(w)| \leq b_1 = (1 - |w|^2) |h'(w)| \neq 0, \quad (52)$$

which implies that

$$\frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |h'(w)| \leq \frac{(1 - |w|^2)}{\mathbf{g}^m(w, w_0)} |h'(w)| \neq 0 < \infty. \quad (53)$$

Using Theorems 1 and 2 in [27], the following inequalities can be deduced:

$$\begin{aligned} \pi \theta^2(R) |b_1|^2 &\leq \mathcal{S}_h^*(w, R), \\ 2\pi \theta(R) |b_1| &\leq \mathcal{M}_h(w, R). \end{aligned} \quad (54)$$

Since, $0 < |\varphi_{w_0}(w)| < \rho < 1$, then

$$\frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |h'(w)| \leq \frac{1}{\theta(R)} \sqrt{\frac{2^m \mathcal{S}_h(w, R)}{\pi(1 - \rho^2)^m}}. \quad (55)$$

Also,

$$\frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |h'(w)| \leq \frac{2^m (1 - |w|^2)^n}{(1 - |\varphi_{w_0}(w)|^2)^m} |h'(w)| \leq \frac{2^{m-1} \mathcal{M}_h(w, R)}{(1 - \rho^2)^m \pi \theta(R)}. \quad (56)$$

Remark 6. In the proof of Proposition 6, positive coefficients are considered to keep the convergence of Taylor or Fourier power series in its region.

Theorem 3. Let $h \in H(U)$ be the nonconstant analytic function. Suppose that $1 \leq n < \infty$ and $0 \leq m < \infty$; therefore, the equivalence between next assertions can be mutually obtained:

□

- (1) The function $h \in \mathcal{B}_g^{n,m}$
 (2) The constant $0 < R < \infty$ can be obtained, for which

$$\sup_{w_0 \in U} S_h(w, R) < \infty. \quad (57)$$

- (3) The constant $0 < R < \infty$ can be obtained, for which

$$\sup_{w_0 \in U} \mathcal{S}_h^*(w, R) < \infty. \quad (58)$$

- (4) The constant $0 < R < \infty$ can be obtained, for which

$$\sup_{w_0 \in U} M_h(w, R) < \infty. \quad (59)$$

- (5) The constant $0 < R < \infty$ can be obtained, for which

$$\sup_{w_0 \in U} \mathcal{M}_h(w, R) < \infty. \quad (60)$$

Proof. The assertions (2) \Rightarrow (3) and (4) \Rightarrow (5) can be deduced clearly. Additionally, (1) \Rightarrow (2) and (1) \Rightarrow (4) are obvious. Next, suppose that (1) with

$$\sup_{w_0 \in U} \frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} h'(w) = \mathcal{B}_g^{n,m}(h) < \infty. \quad (61)$$

Hence,

$$\begin{aligned} S_h(w, R) &= \int_{T(w, R)} |h'(w_0)|^2 d\sigma_{w_0} \\ &\leq (\mathcal{B}_g^{n,m}(h))^2 \int_{|w_0| < \theta(R)} \frac{\mathbf{g}^m(w, w_0)}{(1 - |w|^2)^n} d\sigma_{w_0} \\ &= \eta_1(R) (\mathcal{B}_g^{n,m}(h)), \end{aligned} \quad (62)$$

where $\eta_1(R)$ is a positive constant depending on R . In addition, using the same way, we deduce that

$$\begin{aligned} \frac{(\mathcal{B}_g^{n,m}(h))^2 (\theta(R))^{2n}}{(1 - (\theta(R))^2)^{2m}} &= M_h(w, R) = \eta(R) \int_{T_1(w, R)} |f'(w_0)| d\sigma_{w_0} \\ &\leq \mathcal{B}_g^{n,m}(h) \int_{T_1(w, R)} \frac{\mathbf{g}^m(w, w_0)}{(1 - |w|^2)^n} d\sigma_{w_0} \\ &= \eta_2(R) \mathcal{B}_g^{n,m}(h), \end{aligned} \quad (63)$$

where $\eta(R)$ and $\eta_2(R)$ are the positive constants depending on R .

Additionally, the assertion (3) \Rightarrow (1) as well as (5) \Rightarrow (1) can be proved in view of Proposition 6. \square

Remark 7. Theorem 3 gives an interesting and global criterion for analytic \mathbf{g} -Bloch-type function by the help of the concrete area as well as the concerned length of the images of both non-Euclidean unified discs and unified circles,

respectively. The obtained results in this section extended and improved some specific results in the study by Yamashita [26].

Remark 8. Quite recently, a new study of bicomplex functions was introduced in [28].

An interesting question can be formulated for the newly defined \mathbf{g} -Bloch functions as follows. Can we discuss and study the new defined class of analytic \mathbf{g} -Bloch type functions in the case of bicomplex functions?

5. Conclusions

Function spaces theory is developed, extended, and generalized to spaces of several complex variables ([8–13, 29]) also using quaternion-valued functions ([21–24, 30–33]). The intention of this study is to introduce a new type of analytic function spaces, which plays an interesting and global rule of studying complex function spaces. It should be emphasized that both the worked plane of the study (i.e., U) and the considered holomorphic functions of Bloch-type as well as specific properties of Green's function are extremely needed for the new classes.

The holomorphic classes of \mathbf{g} -Bloch functions are defined and deeply considered using a modified Green's function. By the new function classes, some specific relations and inclusions for the holomorphic Dirichlet-type space as well as holomorphic \mathcal{Q}_p spaces are obtained. On the other hand, an extension of Yamashita's result [26] is presented using holomorphic \mathbf{g} -Bloch functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Mittag-Leffler Operator Connected with Certain Subclasses of Bazilevič Functions

Om Ahuja ¹, Asena Çetinkaya ² and Naveen Kumar Jain ³

¹Department of Mathematical Sciences, Kent State University, Ohio 44021, USA

²Department of Mathematics and Computer Science, İstanbul Kültür University, İstanbul, Turkey

³Department of Mathematics, Aryabhata College, Delhi 110021, India

Correspondence should be addressed to Om Ahuja; oahuja@kent.edu

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In this paper, we introduce a new generalized class of analytic functions involving the Mittag-Leffler operator and Bazilevič functions. We examine inclusion properties, radius problems, and an application of the generalized Bernardi–Libera–Livingston integral operator for this function class.

1. Introduction

Let \mathcal{A} be the family of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disc $\mathbb{D} := \{z: |z| < 1\}$. Denote by \mathcal{S} the subfamily of \mathcal{A} consisting of functions that are univalent in \mathbb{D} . Let \mathcal{K} , \mathcal{S}^* , and \mathcal{C} be the well-known subclasses of \mathcal{S} consisting of functions that are, respectively, convex, starlike (with respect to the origin), and close-to-convex in \mathbb{D} . The class \mathcal{P} consists of analytic functions that are analytic in \mathbb{D} and satisfies the conditions $p(0) = 1$ and

$\operatorname{Re} p(z) > 0$ in \mathbb{D} and is also well-known in the theory of univalent functions. For definitions, properties, and history of these classes, one may refer to a survey article by the first author [1, 2]. Recently, Ali et al. [3] and Anand et al. [4] studied these classes to find various radius problems.

The Mittag-Leffler function E_α , defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbb{C}; \operatorname{Re}(\alpha) > 0; z \in \mathbb{C}), \quad (2)$$

was introduced in 1903 by Mittag-Leffler [5, 6] in connection with his method of summation of some divergent series. A general form of this special function (2) given by

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta \in \mathbb{C}; \operatorname{Re}(\alpha) > 0; \operatorname{Re}(\beta) > 0; z \in \mathbb{C}), \quad (3)$$

which was studied by Wiman [7] in 1905. During these last twenty-five years, interest in Mittag-Leffler type functions (2) and (3) has significantly increased among engineers and scientists due to their applications in numerous applied

problems, such as fluid flow, diffusive transport skin to diffusion, electric networks, probability, and statistical distribution theory. For detailed account of various properties and references related to applications, one may refer to [8, 9].

Motivated by Sivastava and Tomovski [10], Attiya [11] studied certain applications of the generalized Mittag-Leffler operator involving differential subordination.

Corresponding to the function $E_{\alpha,\beta}$, Elhaddad et al. [12] introduced the Mittag-Leffler linear operator $\mathcal{E}_{\lambda,\alpha,\beta}^m f: \mathcal{A} \longrightarrow \mathcal{A}$ given by

$$\mathcal{E}_{\lambda,\alpha,\beta}^m f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)(1+(n-1)\lambda)^m}{\Gamma(\alpha(n-1)+\beta)} a_n z^n, \quad (4)$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda \geq 0$, $\alpha, \beta \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$ and when $\operatorname{Re}(\alpha) = 0$ and $\beta \neq 0$. From (4), the following recurrence formula can be easily obtained:

$$\mathcal{E}_{\lambda,\alpha,\beta}^{m+1} f(z) = (1-\lambda)\mathcal{E}_{\lambda,\alpha,\beta}^m f(z) + \lambda z \left(\mathcal{E}_{\lambda,\alpha,\beta}^m f(z) \right)', \quad (5)$$

where $m \in \mathbb{N}_0$ and $\lambda \geq 0$. For suitable values of the parameters m, α, β , and λ , we may get several linear operators; for example,

(1) For $\alpha = 0$ and $\beta = 1$, we get Al-Oboudi operator [13].

(2) For $\alpha = 0$, $\beta = 1$, and $\lambda = 1$, we get Sălăgean operator [14].

(3) For $m = 0$ and $\lambda = 1$, we get the operator

$$\mathbb{E}_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n. \quad (6)$$

Let us denote by $\mathcal{B}(\vartheta, \tau, g, p)$ or briefly denote by \mathcal{B} , a class of functions $f \in \mathcal{A}$ for which $p \in \mathcal{P}$, $g \in \mathcal{S}^*$, and real numbers ϑ, τ with $\vartheta > 0$ such that

$$f(z) = \left[(\vartheta + i\tau) \int_0^z p(t)g(t)^{\vartheta} t^{i\tau-1} dt \right]^{1/(\vartheta+i\tau)}, \quad (7)$$

where powers are taken as principal values. Bazilevich [15] proved that $\mathcal{B} \subset \mathcal{S}$. In fact, it is known that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{B} \subset \mathcal{S}$. For $\vartheta > 0$ and $\rho < 1$, Ponnusamy and Karunakaran [16] showed that

$$\mathcal{B}_1(\vartheta, \rho) = \left\{ f \in \mathcal{A}: \operatorname{Re} \left(\frac{zf'(z)}{f^{1-\vartheta}(z)g^{\vartheta}(z)} \right) > \rho, z \in \mathbb{D} \right\}. \quad (8)$$

For $g(z) \equiv z$, these authors observed that

$$\begin{aligned} \mathcal{B}_2(\vartheta, \rho) &= \left\{ f \in \mathcal{A}: \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\vartheta} \right) > \rho, z \in \mathbb{D} \right\}, \\ \mathcal{B}_3(\rho) &:= \mathcal{B}_2(0, \rho) = \left\{ f \in \mathcal{A}: \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \rho, z \in \mathbb{D} \right\} = \mathcal{S}^*(\rho), \\ \mathcal{B}_4(\rho) &:= \mathcal{B}_2(1, \rho) = \left\{ f \in \mathcal{A}: \operatorname{Re}(f'(z)) > \rho, z \in \mathbb{D} \right\} = \mathcal{P}'(\rho). \end{aligned} \quad (9)$$

In view of (7), Singh [17] observed that $\mathcal{B}_1(\vartheta, 0)$, $\mathcal{B}_2(\vartheta, 0)$, $\mathcal{B}_3(0) = \mathcal{S}^*$, and $\mathcal{B}_4(0) = \mathcal{P}'$ are subclasses of \mathcal{B} . For further details, one may refer to [15, 18].

In 1976, Padmanabhan and Parvatham [19] introduced the class $\mathcal{P}_k(\rho)$ of analytic functions p defined in \mathbb{D} satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi, \quad (10)$$

where $z = re^{i\theta}$, $k \geq 2$, and $0 \leq \rho < 1$. In fact, he proved the following important result.

Lemma 1 (see [19]). *If $p \in \mathcal{P}_k(\rho)$, then*

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1-2\rho)ze^{-i\theta}}{1 - ze^{-i\theta}} d\mu(\theta), \quad (11)$$

where $\mu(\theta)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\begin{aligned} \int_0^{2\pi} d\mu(\theta) &= 2\pi, \\ \int_0^{2\pi} |d\mu(\theta)| &\leq k\pi. \end{aligned} \quad (12)$$

From (10), it is observed that $p \in \mathcal{P}_k(\rho)$ if and only if there exists $p_1, p_2 \in \mathcal{P}(\rho)$ such that (see [18])

$$p(z) = \left(\frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) p_2(z), \quad (z \in \mathbb{D}). \quad (13)$$

We note that, for $\rho = 0$, we obtain the class $\mathcal{P}_k(0) := \mathcal{P}_k$ defined by Pinchuk [20]. For $k = 2$, we get $\mathcal{P}_2(\rho) := \mathcal{P}(\rho)$ the class of analytic functions with positive real part greater than ρ , and for $k = 2$ and $\rho = 0$, we have the class $\mathcal{P}_2(0) := \mathcal{P}$ of functions with positive real part.

Motivated by many researchers in [5–7, 10–12, 15, 17, 18, 21], we introduce the following generalized class of Mittag-Leffler–Bazilevič operator involving the class $\mathcal{P}_k(\rho)$.

Definition 1. Let $k \geq 2$, $0 \leq \rho < 1$, $\vartheta > 0$, $m \in \mathbb{N}_0$, $\lambda \geq 0$, and $\gamma \in \mathbb{C}$ be such that $\operatorname{Re}(\gamma) > 0$. Then, a function $f \in \mathcal{A}$ given by (1) is in the class $\mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma}(k, \vartheta, \rho)$ if it satisfies the condition

$$\left\{ (1 - \gamma) \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^\vartheta + \gamma \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^{m+1} f(z)}{z} \right) \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^{\vartheta-1} \right\} \in \mathcal{P}_k(\rho), \quad (14)$$

where $z \in \mathbb{D}$.

This new class $\mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma}(k, \vartheta, \rho)$ involves several subclasses of the family \mathcal{B} of Bazilevič functions defined by (7); see [15]. For example, $\mathcal{M}_{1,0,1}^{0,1}(2, \vartheta, 0) = \mathcal{B}_2(\vartheta, 0)$, $\mathcal{M}_{1,0,1}^{0,1}(2, 0, 0) = \mathcal{B}_3(0)$, and $\mathcal{M}_{1,0,1}^{0,1}(2, 1, 0) = \mathcal{B}_4(0)$ are subclasses of \mathcal{B} ; see [15, 17]. In fact, for different values of

$m, \vartheta, \lambda, \alpha, \beta, \gamma, k$ and ρ , the class $\mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma}(k, \vartheta, \rho)$ reduces to many important subclasses studied by various researchers. For instance,

- (i) Setting $m = 0$, $\lambda = 1$, $\alpha = 0$, and $\beta = 1$, we get (see [18])

$$\mathcal{M}_{1,0,1}^{0,\gamma}(k, \vartheta, \rho) =: \mathcal{M}^\gamma(k, \vartheta, \rho) = \left\{ f \in \mathcal{A}: (1 - \gamma) \left(\frac{f(z)}{z} \right)^\vartheta + \gamma \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^\vartheta \in \mathcal{P}_k(\rho), 0 \leq \rho < 1 \right\}. \quad (15)$$

- (ii) Setting $m = 0$, $\lambda = 1$, $\alpha = 0$, $\gamma = 1$, $\beta = 1$, and $k = 2$, we get (see [15])

$$\mathcal{M}_{1,0,1}^{0,1}(2, \vartheta, \rho) =: \mathcal{B}_2(\vartheta, \rho) = \left\{ f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^\vartheta \in \mathcal{P}(\rho), 0 \leq \rho < 1 \right\}. \quad (16)$$

- (iii) Setting $m = 0$, $\lambda = 1$, $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\vartheta = 1$, and $k = 2$, we get (see [21])

$$\mathcal{M}_{1,0,1}^{0,0}(2, 1, \rho) =: \mathcal{M}(\rho) = \left\{ f \in \mathcal{A}: \frac{f(z)}{z} \in \mathcal{P}(\rho), 0 \leq \rho < 1 \right\}. \quad (17)$$

In view of the above examples, we conclude that the notion of generalized class $\mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma}(k, \vartheta, \rho)$ unifies several known subclasses of \mathcal{A} .

In this paper, we study various properties of the class $\mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma}(k, \vartheta, \rho)$. In particular, we investigate inclusion properties, radius problem, and an application of the generalized Bernardi–Libera–Livingston integral operator for this function class.

We list some preliminary lemmas required for proving our main results.

Lemma 2 (see [22]). Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$, and suppose $\Psi(u, v)$ is a complex function satisfying the following conditions:

- (i) $\Psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$

- (ii) $(1, 0) \in D$ and $\operatorname{Re} \Psi(1, 0) > 0$

- (iii) $\operatorname{Re} \Psi(iu_2, v_1) \leq 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \leq -((1 + u_2^2)/2)$

If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is an analytic function in \mathbb{D} such that $(p(z), zp'(z)) \in D$ and $\operatorname{Re} \Psi(p(z), zp'(z)) > 0$ for $z \in \mathbb{D}$, then $\operatorname{Re} p(z) > 0$ in \mathbb{D} .

Lemma 3 (see [23]). If p is in \mathcal{P} , and if ζ is a complex number satisfying $\operatorname{Re}(\zeta) \geq 0$, $\zeta \neq 0$, then $\operatorname{Re}\{p(z) + \zeta zp'(z)\} > \rho$ ($0 \leq \rho < 1$) implies that

$$\operatorname{Re} p(z) > \rho + (1 - \rho)(2\iota_1 - 1), \quad (18)$$

where

$$\iota_1 = \int_0^1 (1 + t^{\operatorname{Re}(\zeta)})^{-1} dt, \quad (19)$$

ι_1 is an increasing function of $\operatorname{Re}(\zeta)$ and $1/2 \leq \iota_1 < 1$. This estimate cannot be improved in general.

2. Inclusion Properties

In this section, we examine some inclusion properties for the class $\mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma}(k, \vartheta, \rho)$.

Theorem 1. Let $\gamma > 0$ and $f \in \mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma}(k, \vartheta, \rho)$. Then,

$$\left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^\vartheta \in \mathcal{P}_k(\rho_1), \quad (20)$$

where ρ_1 is given by

$$\rho_1 = \frac{2\vartheta\rho + \lambda\gamma}{2\vartheta + \lambda\gamma}. \quad (21)$$

$$\left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^\vartheta = (1 - \rho_1)p(z) + \rho_1, \quad (22)$$

where $p(0) = 1$ and

$$p(z) = \left(\frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) p_2(z). \quad (23)$$

Thus, by using (14) and (22), we obtain

Proof. In view of (20), let

$$\left\{ (1 - \gamma) \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^\vartheta + \gamma \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^{m+1} f(z)}{z} \right) \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^{\vartheta-1} \right\} = (1 - \gamma) ((1 - \rho_1)p(z) + \rho_1) + \gamma \left[\frac{\mathcal{E}_{\lambda, \alpha, \beta}^{m+1} f(z)}{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)} ((1 - \rho_1)p(z) + \rho_1) \right]. \quad (24)$$

Taking logarithmic differentiation of (22), we get

$$\vartheta \left[\frac{z \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)'}{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)} - 1 \right] = \frac{(1 - \rho_1)z p'(z)}{(1 - \rho_1)p(z) + \rho_1}. \quad (25)$$

By using the identity (5) in the last expression, we obtain

$$\frac{\mathcal{E}_{\lambda, \alpha, \beta}^{m+1} f(z)}{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)} = \frac{\lambda(1 - \rho_1)z p'(z)}{\vartheta((1 - \rho_1)p(z) + \rho_1)} + 1. \quad (26)$$

Substituting (26) into (24) and using (13), we arrive at

$$\begin{aligned} (1 - \rho_1)p(z) + \rho_1 + \frac{\lambda\gamma(1 - \rho_1)z p'(z)}{\vartheta} &= \left(\frac{k}{4} + \frac{1}{2} \right) \left\{ (1 - \rho_1)p_1(z) + \rho_1 + \frac{\lambda\gamma(1 - \rho_1)z p_1'(z)}{\vartheta} \right\} \\ &\quad - \left(\frac{k}{4} - \frac{1}{2} \right) \left\{ (1 - \rho_1)p_2(z) + \rho_1 + \frac{\lambda\gamma(1 - \rho_1)z p_2'(z)}{\vartheta} \right\}. \end{aligned} \quad (27)$$

Since $f \in \mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma}(k, \vartheta, \rho)$, it follows that

$$\left\{ (1 - \rho_1)p_i(z) + \rho_1 + \frac{\lambda\gamma(1 - \rho_1)z p_i'(z)}{\vartheta} \right\} \in \mathcal{P}(\rho), \quad (0 \leq \rho < 1, i = 1, 2). \quad (28)$$

That is,

$$\frac{1}{1 - \rho} \left\{ (1 - \rho_1)p_i(z) + \rho_1 + \frac{\lambda\gamma(1 - \rho_1)z p_i'(z)}{\vartheta} - \rho \right\} \in \mathcal{P}. \quad (29)$$

To prove the theorem, we will show that $p_i \in \mathcal{P}$ ($i = 1, 2$). We form the functional $\Psi(u, v)$ by taking $u = u_1 + iu_2$ and $v = v_1 + iv_2$ such that

$$\Psi(u, v) = (1 - \rho_1)u + \rho_1 - \rho + \frac{\lambda\gamma(1 - \rho_1)v}{\vartheta}. \quad (30)$$

Using (29), it is easy to show that the first two conditions of Lemma 2 are satisfied. To verify condition (iii), we obtain

$$\begin{aligned} \operatorname{Re} \Psi(iu_2, v_1) &= \rho_1 - \rho + \operatorname{Re} \left\{ \frac{\lambda\gamma(1 - \rho_1)v_1}{\vartheta} \right\} \\ &\leq \rho_1 - \rho - \frac{\lambda\gamma(1 - \rho_1)(1 + u_2^2)}{2\vartheta} \\ &= \frac{2\vartheta\rho_1 - 2\vartheta\rho - \lambda\gamma(1 - \rho_1)(1 + u_2^2)}{2\vartheta} \\ &= \frac{2\vartheta(\rho_1 - \rho) - \lambda\gamma(1 - \rho_1) - \lambda\gamma(1 - \rho_1)u_2^2}{2\vartheta} \\ &:= \frac{A + Bu_2^2}{2C}, \end{aligned} \quad (31)$$

where $v_1 \leq -((1 + u_2^2)/2)$. Now, $\operatorname{Re}\Psi(iu_2, v_1) \leq 0$ if

$$\begin{aligned} A &= 2\vartheta(\rho_1 - \rho) - \lambda\gamma(1 - \rho_1) \leq 0, \\ B &= -\lambda\gamma(1 - \rho_1) \leq 0, \\ C &= \vartheta > 0. \end{aligned} \quad (32)$$

It follows from $A \leq 0$, ρ_1 given by (21) and $B \leq 0$ that $0 \leq \rho_1 < 1$. In view of Lemma 2, for $p(z) = p_i(z)$, $p_i \in \mathcal{P}$ ($i = 1, 2$), we get $p \in \mathcal{P}_k(\rho_1)$. This proves the result.

For $m = 0$, $\lambda = 1$, $\alpha = 0$, and $\beta = 1$, Theorem 1 reduces to the following new result. \square

Corollary 1. Let $\gamma > 0$ and $f \in \mathcal{M}^\gamma(k, \vartheta, \rho)$. Then, $(f(z)/z)^\vartheta \in \mathcal{P}_k(\rho_1)$, where ρ_1 is given by

$$\rho_1 = \frac{2\vartheta\rho + \gamma}{2\vartheta + \gamma}. \quad (33)$$

By using the inclusion relation given in Theorem 1, we prove the following result.

Theorem 2. Let $\vartheta > 0$ and $0 \leq \gamma_1 < \gamma_2$. Then, $\mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma_2}(k, \vartheta, \rho) \subset \mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma_1}(k, \vartheta, \rho)$.

Proof. Let $f \in \mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma_2}(k, \vartheta, \rho)$. Then, we have

$$H_2(z) = \left\{ (1 - \gamma_2) \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^\vartheta + \gamma_2 \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^{m+1} f(z)}{z} \right) \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^{\vartheta-1} \right\} \in \mathcal{P}_k(\rho). \quad (34)$$

In view of Theorem 1, we conclude that

$$\left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^\vartheta := H_1(z) \in \mathcal{P}_k(\rho_1) \subset \mathcal{P}_k(\rho). \quad (35)$$

Thus, for $\gamma_1 \geq 0$, we have

$$\left\{ (1 - \gamma_1) \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^\vartheta + \gamma_1 \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^{m+1} f(z)}{z} \right) \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^{\vartheta-1} \right\} = \left(1 - \frac{\gamma_1}{\gamma_2} \right) H_1(z) + \frac{\gamma_1}{\gamma_2} H_2(z). \quad (36)$$

Because the class $\mathcal{P}_k(\rho)$ is a convex set (see [18]), it follows that the right side of (36) belongs to $\mathcal{P}_k(\rho)$, and therefore $f \in \mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma_1}(k, \vartheta, \rho)$.

For $m = 0$, $\lambda = 1$, $\alpha = 0$, and $\beta = 1$, Theorem 2 reduces to the following inclusion result. \square

Remark 1 (see [18]). Let $\vartheta > 0$ and $0 \leq \gamma_1 < \gamma_2$. Then, $\mathcal{M}^{\gamma_2}(k, \vartheta, \rho) \subset \mathcal{M}^{\gamma_1}(k, \vartheta, \rho)$.

3. Radius Problem

In this section, we examine certain radius problems.

Theorem 3. If a function $f \in \mathcal{A}$ satisfies

$$\left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^\vartheta \in \mathcal{P}_k(\rho), \quad (37)$$

then $f \in \mathcal{M}_{\lambda, \alpha, \beta}^{m, \gamma}(k, \vartheta, \rho)$ for $|z| < r_1$, where

$$r_1 = \frac{\lambda\gamma + \vartheta - \sqrt{\lambda^2\gamma^2 + 2\lambda\gamma\vartheta}}{\vartheta}. \quad (38)$$

Proof. In view of (37), we have

$$\left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^\vartheta = (1 - \rho)p(z) + \rho, \quad (39)$$

where $p \in \mathcal{P}_k$. Hence, by using (5) and (39), we easily get

$$\begin{aligned} & \frac{1}{1 - \rho} \left\{ (1 - \gamma) \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^\vartheta + \gamma \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^{m+1} f(z)}{z} \right) \left(\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right)^{\vartheta-1} - \rho \right\} = p(z) + \frac{\lambda\gamma z p'(z)}{\vartheta} \\ & = \left(\frac{k}{4} + \frac{1}{2} \right) \left\{ p_1(z) + \frac{\lambda\gamma z p_1'(z)}{\vartheta} \right\} - \left(\frac{k}{4} - \frac{1}{2} \right) \left\{ p_2(z) + \frac{\lambda\gamma z p_2'(z)}{\vartheta} \right\}, \end{aligned} \quad (40)$$

where $p_1, p_2 \in \mathcal{P}$ and $z \in \mathbb{D}$.

Now, by using well-known estimates (see [2]) for the class \mathcal{P} given by

$$|zp_i'(z)| \leq \frac{2r \operatorname{Re} p_i(z)}{(1-r)^2}, \quad (41)$$

$$\operatorname{Re} p_i(z) \geq \frac{1-r}{1+r}, \quad (|z| < r < 1; i = 1, 2; z \in \mathbb{D}),$$

we have

$$\begin{aligned} \operatorname{Re} \left\{ p_i(z) + \frac{\lambda \gamma z p_i'(z)}{9} \right\} &\geq \operatorname{Re} \left\{ p_i(z) - \frac{\lambda \gamma |z p_i'(z)|}{9} \right\} \\ &\geq \operatorname{Re} p_i(z) \left\{ 1 - \frac{2\lambda \gamma r}{9(1-r)^2} \right\} \\ &= \operatorname{Re} p_i(z) \left\{ \frac{9(1-r)^2 - 2\lambda \gamma r}{9(1-r)^2} \right\}. \end{aligned} \quad (42)$$

The right hand side of the last inequality is positive if $|z| = r < r_1$, where r_1 is given by (38).

Letting $m = 0$, $\lambda = 1$, $\alpha = 0$, and $\beta = 1$, Theorem 3 reduces to the following new result. \square

Corollary 2. If a function $f \in \mathcal{A}$ satisfies $(f(z)/z)^\vartheta \in \mathcal{P}_k(\rho)$, then $f \in \mathcal{M}^\gamma(k, \vartheta, \rho)$ for $|z| < r_2$, where

$$r_2 = \frac{\gamma + \vartheta - \sqrt{\gamma^2 + 2\gamma\vartheta}}{9}. \quad (43)$$

Remark 2. If $m = \alpha = 0$, $\gamma = \beta = \lambda = \vartheta = 1$, and $k = 2$ in Theorem 3, then f is in $\mathcal{M}_{1,0,1}^{0,1}(2, 1, \rho) := \mathcal{B}_2^1(1, 1, 1, \rho, 1, 0)$ for $|z| < 2 - \sqrt{3} \approx 0.2679$. This result was proved in Theorem 3.4 in [24].

4. Application of an Integral Operator

In this section, we consider an application of the generalized Mittag-Leffler operator given by (4) involving the generalized Bernardi-Libera-Livingston integral operator $\mathcal{L}_\sigma: \mathcal{A} \rightarrow \mathcal{A}$ given by

$$\mathcal{L}_\sigma f(z) = \frac{\sigma+1}{z^\sigma} \int_0^z t^{\sigma-1} f(t) dt, \quad (\sigma > -1). \quad (44)$$

From this operator, we easily get

$$z \left(\mathcal{E}_{\lambda, \alpha, \beta}^m \mathcal{L}_\sigma f(z) \right)' = (\sigma+1) \mathcal{E}_{\lambda, \alpha, \beta}^m f(z) - \sigma \mathcal{E}_{\lambda, \alpha, \beta}^m \mathcal{L}_\sigma f(z). \quad (45)$$

For several special cases of this operator and related operators, one may refer to a survey article by the first two authors [25] and related references therein.

Theorem 4. Let $f \in \mathcal{A}$ and $\mathcal{L}_\sigma f$ be given by (44). If

$$\left\{ (1-\gamma) \frac{\mathcal{E}_{\lambda, \alpha, \beta}^m \mathcal{L}_\sigma f(z)}{z} + \gamma \frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} \right\} \in \mathcal{P}_k(\rho), \quad (46)$$

then

$$\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m \mathcal{L}_\sigma f(z)}{z} \in \mathcal{P}_k(\iota), \quad (z \in \mathbb{D}), \quad (47)$$

where ι is given by

$$\begin{aligned} \iota &= \rho + (1-\rho)(2\iota_1 - 1), \\ \iota_1 &= \int_0^1 \left(1 + t\sigma + 1 \right)^{-1} dt. \end{aligned} \quad (48)$$

Proof. Consider the function

$$\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m \mathcal{L}_\sigma f(z)}{z} = p(z) = \left\{ \left(\frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) p_2(z) \right\}. \quad (49)$$

Differentiating both sides and using (45), we get

$$\frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} = p(z) + \frac{z p'(z)}{\sigma+1}. \quad (50)$$

If we use the identity given by (46), we obtain

$$(1-\gamma) \frac{\mathcal{E}_{\lambda, \alpha, \beta}^m \mathcal{L}_\sigma f(z)}{z} + \gamma \frac{\mathcal{E}_{\lambda, \alpha, \beta}^m f(z)}{z} = p(z) + \frac{\gamma z p'(z)}{\sigma+1} \in \mathcal{P}_k(\rho). \quad (51)$$

This implies that

$$\operatorname{Re} \left\{ p_i(z) + \frac{\gamma z p_i'(z)}{\sigma+1} \right\} > \rho, \quad (i = 1, 2). \quad (52)$$

By using Lemma 3, we see that $\operatorname{Re}\{p_i(z)\} > \iota$, where ι is given by (48). Thus, we arrive at $p \in \mathcal{P}_k(\iota)$. This completes the proof.

Setting $m = 0$, $\lambda = 1$, $\alpha = 0$, and $\beta = 1$ in operator $\mathcal{E}_{\lambda, \alpha, \beta}^m f$, Theorem 4 gives the following result. \square

Corollary 3. Let $f \in \mathcal{A}$ and $\mathcal{L}_\sigma f$ be given by (44). If

$$\left\{ (1-\gamma) \frac{\mathcal{L}_\sigma f(z)}{z} + \gamma \frac{f(z)}{z} \right\} \in \mathcal{P}_k(\rho), \quad (53)$$

then $(\mathcal{L}_\sigma f(z))/z \in \mathcal{P}_k(\iota)$, where ι is given by

$$\begin{aligned} \iota &= \rho + (1-\rho)(2\iota_1 - 1), \\ \iota_1 &= \int_0^1 \left(1 + t\sigma + 1 \right)^{-1} dt. \end{aligned} \quad (54)$$

5. Conclusion

We conclude our investigation by remarking that the defined new generalized class of analytic functions involving the Mittag-Leffler operator and Bazilevič functions gives various well-known subclasses of Bazilevič functions as particular cases which in turn yields many known results as corollaries.

Data Availability

No data were used to support this study.

Disclosure

A preprint of an earlier version is available at <https://arxiv.org/abs/2109.13509>.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Characteristics of Regular Functions Defined on a Semicommutative Subalgebra of 4-Dimensional Complex Matrix Algebra

Ji Eun Kim 

Department of Mathematics, Dongguk University, Gyeongju-Si 38066, Republic of Korea

Correspondence should be addressed to Ji Eun Kim; jeunkim@pusan.ac.kr

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In this paper, we give an extended quaternion as a matrix form involving complex components. We introduce a semicommutative subalgebra $\mathbb{C}(\mathbb{C}^2)$ of the complex matrix algebra $M(4, \mathbb{C})$. We exhibit regular functions defined on a domain in \mathbb{C}^4 but taking values in $\mathbb{C}(\mathbb{C}^2)$. By using the characteristics of these regular functions, we propose the corresponding Cauchy–Riemann equations. In addition, we demonstrate several properties of these regular functions using these novel Cauchy–Riemann equations. Mathematical Subject Classification is 32G35, 32W50, 32A99, and 11E88.

1. Introduction

Introduced by Hamilton in 1894, quaternions form an algebra generated as a noncommutative division (associative) algebra. Quaternions are used in physics and engineering fields such as electromechanics, quantum mechanics, and 3D animation (see [1–3]). To increase the utilization of quaternions and expand their application, several researchers have attempted to modify, supplement, or expand quaternions. In particular, considering that quaternions form a number system extending the complex numbers, there have been several studies already made since the past few decades regarding holomorphic functions of a quaternion variable. These researchers have investigated whether the properties of complex functions and the definition of holomorphic functions are applicable to functions of a quaternion variable. To do this, researchers have defined holomorphic (regular) functions of a quaternion variable and they have investigated their properties. Fueter [4] defined regular functions over the quaternion field identified with \mathbb{R}^4 . Based on this definition, Fueter investigated a generalization of the Cauchy–Riemann equations in the complex holomorphic function theory. Delanghe [5] established Stokes’ theorem and Cauchy’s and Green’s

formulas for functions with values in Clifford algebra over a quadratic n -dimensional real vector space V with an orthogonal base. Using a generalization of the Cauchy–Riemann equation, Ryan [6] introduced a regularity of quaternion-valued functions and developed a regular function theory of complex Clifford algebra. Sudbery [7] demonstrated basic algebraic properties of quaternions and quaternionic differential forms. Using exterior differential calculus, Sudbery proposed new and simple proofs of most of the main theorems and clarified the relationship between quaternionic analysis and complex analysis. Naser [8] and Nôno [9] introduced certain quaternionic differential operators and defined the hyperholomorphy of quaternionic functions. Nôno and Inenaga [10] developed hyperholomorphic functions of quaternionic variables as holomorphic function theory of \mathbb{C}^2 . Kim et al. [11, 12] researched the properties of regular functions with values in the three-dimensional real skew field (called ternary field \mathbb{T}) and reduced quaternions using Clifford analysis. They proposed the corresponding Cauchy–Riemann equations with applications defined on \mathbb{T} . Sommen [13] constructed kernels and monogenic and holomorphic functions, leading to connections between the theory of holomorphic functions of several variables and the theory of monogenic functions.

Based on these studies, in this paper, we propose a form distinct from the previously attempted expansion of quaternions. In previous studies, extensions of the quaternionic number system and various combinations of quaternions were suggested. Baez [14] applied octonions to spinors, Bott periodicity, projective and Lorentzian geometry, Jordan algebras, and exceptional Lie groups. Baez also dealt with their applications in quantum logic, special relativity, and supersymmetry. Imaeda K and Imaeda M [15] presented a 16-dimensional Cayley–Dickson algebra and found its algebraic properties, zero-divisors, and solutions to a general linear equation. Tian [16] provided a complete investigation of real matrix representations of octonions and considered their various applications to octonions, based on the fact that the octonion is an extension of the quaternion by the Cayley–Dickson construction. Gotô and Nôno [17] established a commutative algebra $\mathbb{C}(\mathbb{C}^2)$ identified with \mathbb{C} as the subalgebra of the four-dimensional real matrix algebra $M(4, \mathbb{R})$. They provided a regularity and properties of the functions of two complex variables with values in $\mathbb{C}(\mathbb{C}^2)$. Rudin [18] studied the integral formulas of the holomorphic functions of several complex variables and presented a function theory of including the boundary behavior of complex functions, complex-tangential phenomena, and quantitative theorems about zero-varieties in the unit ball of \mathbb{C}^n .

In this paper, we give algebra using the matrix as the basis by mapping the basis of the quaternion to matrix form. We define an algebraic system based on a matrix with a matrix as a component. This can be expressed as a matrix with complex numbers as components while retaining the characteristics of the basis for generating quaternions. Furthermore, a 4×4 matrix is constructed with a matrix that is isomorphic to the basis of quaternions. The algebra introduced in this paper has eight bases, some of which are commutative for products and some are noncommutative. Therefore, in this paper, a function is defined on an algebra generated by a semicommutative basis. Regularity of

functions and properties of these regular functions is investigated. In addition, the corresponding Cauchy–Riemann equations, which can be an alternative of the definition of holomorphic function in complex analysis, are derived for the algebra.

We develop the foundation of a function theory over a subalgebra of four-dimensional matrix algebra to replace the basis of the quaternion. In Section 2, we present preliminaries and notations required for the theory presented in this paper. Sections 3 and 4 present the definitions and propositions related to the semicommutative subalgebra $\mathbb{C}(\mathbb{C}^2)$ of the complex matrix algebra $M(4, \mathbb{C})$, and we provide a regularity of functions defined on a domain in \mathbb{C}^4 with values in $\mathbb{C}(\mathbb{C}^2)$. From the notion of regularity over $\mathbb{C}(\mathbb{C}^2)$, we propose corresponding Cauchy–Riemann equations and several properties of regular functions in $\mathbb{C}(\mathbb{C}^2)$. Finally, in Section 5, the conclusions of this paper are presented.

2. Preliminaries and Notations

In this section, we give notations that are needed to prove our main results. Let $M(n, \mathbb{F})$ be the ring of all $n \times n$ matrices over the field \mathbb{F} . For our purposes, we would be interested in the case where $n = 2$ and $\mathbb{F} = \mathbb{C}$. Put

$$\begin{aligned} e_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ e_1 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\ e_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ e_3 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \end{aligned} \tag{1}$$

These have the following rules:

$$\begin{aligned} e_0^2 &= e_0, \\ e_1^2 &= e_2^2 = e_3^2 = -e_0, \\ e_1 e_2 &= e_3 = -e_2 e_1, \\ e_2 e_3 &= e_1 = -e_3 e_2, \\ e_3 e_1 &= e_2 = -e_1 e_3. \end{aligned} \tag{2}$$

Let \mathcal{S} be a set of matrices defined as follows:

$$\mathcal{S} = \left\{ \sum_{r=0}^3 x_r e_r \mid x_r \in \mathbb{R} (r = 0, 1, 2, 3) \right\}. \tag{3}$$

From the rules of $e_r (r = 0, 1, 2, 3)$, we note that \mathcal{S} is a noncommutative subalgebra of $M(2, \mathbb{C})$. By using the matrices $e_r (r = 0, 1, 2, 3)$, we define four 4×4 matrices:

$$\begin{aligned}
\varepsilon_0 &= \begin{pmatrix} e_0 & \mathbf{0} \\ \mathbf{0} & e_0 \end{pmatrix}, \\
\varepsilon_1 &= \begin{pmatrix} e_1 & \mathbf{0} \\ \mathbf{0} & -e_1 \end{pmatrix}, \\
\varepsilon_2 &= \begin{pmatrix} e_0 & \mathbf{0} \\ \mathbf{0} & -e_0 \end{pmatrix}, \\
\varepsilon_3 &= \begin{pmatrix} e_1 & \mathbf{0} \\ \mathbf{0} & e_1 \end{pmatrix}, \\
\varepsilon_4 &= \begin{pmatrix} 0 & e_2 \\ -e_2 & 0 \end{pmatrix}, \\
\varepsilon_5 &= \begin{pmatrix} 0 & e_3 \\ e_3 & 0 \end{pmatrix}, \\
\varepsilon_6 &= \begin{pmatrix} 0 & e_2 \\ e_2 & 0 \end{pmatrix}, \\
\varepsilon_7 &= \begin{pmatrix} 0 & e_3 \\ -e_3 & 0 \end{pmatrix},
\end{aligned} \tag{4}$$

where $\mathbf{0}$ is the 2×2 null matrix. Then, we have the following properties:

$$\begin{aligned}
\varepsilon_0^2 &= \varepsilon_2^2 = \varepsilon_4^2 = \varepsilon_6^2 = \varepsilon_0, \\
\varepsilon_1^2 &= \varepsilon_3^2 = \varepsilon_5^2 = \varepsilon_7^2 = -\varepsilon_0,
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\varepsilon_1 \varepsilon_2 &= \varepsilon_3 = \varepsilon_2 \varepsilon_1, \\
\varepsilon_1 \varepsilon_6 &= \varepsilon_7 = \varepsilon_6 \varepsilon_1, \\
\varepsilon_5 \varepsilon_4 &= \varepsilon_1 = \varepsilon_4 \varepsilon_5, \\
\varepsilon_2 \varepsilon_5 &= \varepsilon_7 = -\varepsilon_5 \varepsilon_2, \\
\varepsilon_5 \varepsilon_3 &= \varepsilon_6 = -\varepsilon_3 \varepsilon_5, \\
\varepsilon_3 \varepsilon_4 &= \varepsilon_7 = -\varepsilon_4 \varepsilon_3, \\
\varepsilon_6 \varepsilon_4 &= \varepsilon_2 = -\varepsilon_4 \varepsilon_6.
\end{aligned} \tag{6}$$

Now, let the set $\mathcal{S}(4, \mathbb{C})$ be a subalgebra of $M(4, \mathbb{C})$, written as

$$\mathcal{S}(4, \mathbb{C}) = \left\{ z = \sum_{r=0}^7 x_r \varepsilon_r \mid x_r \in \mathbb{R}, r = 0, 1, \dots, 7 \right\}, \tag{7}$$

Here, we deal with ε_0 as 1. The element z of $\mathcal{S}(4, \mathbb{C})$ is also written as

$$\begin{aligned}
z &= (x_0 + x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3) + (x_4 + x_5 \varepsilon_1 + x_6 \varepsilon_2 + x_7 \varepsilon_3) \varepsilon_4 \\
&= \{(x_0 + x_1 \varepsilon_1) + (x_2 + x_3 \varepsilon_1) \varepsilon_2\} \\
&\quad + \{(x_4 + x_5 \varepsilon_1) + (x_6 + x_7 \varepsilon_1) \varepsilon_2\} \varepsilon_4.
\end{aligned} \tag{8}$$

From the above expressions, by simply making the following substitutions, we put

$$\begin{aligned}
\zeta_0 &= x_0 + x_1 \varepsilon_1, \\
\zeta_1 &= x_2 + x_3 \varepsilon_1, \\
\zeta_2 &= x_4 + x_5 \varepsilon_1, \\
\zeta_3 &= x_6 + x_7 \varepsilon_1,
\end{aligned} \tag{9}$$

and we obtain

$$\begin{aligned}
z_0 &= \zeta_0 + \zeta_1 \varepsilon_2, \\
z_1 &= \zeta_2 + \zeta_3 \varepsilon_2, \\
\bar{z}_0 &= \bar{\zeta}_0 + \bar{\zeta}_1 \varepsilon_2, \\
\bar{z}_1 &= \bar{\zeta}_2 + \bar{\zeta}_3 \varepsilon_2, \\
\tilde{z}_0 &= \zeta_0 - \zeta_1 \varepsilon_2, \\
\tilde{z}_1 &= \zeta_2 - \zeta_3 \varepsilon_2,
\end{aligned} \tag{10}$$

where \bar{z}_t ($t = 0, 1$) denotes the complex conjugate of the complex components of ε_1 and \tilde{z}_t is a changing symbol of a part of the basis of ε_2 . Then, the elements of $\mathcal{S}(4, \mathbb{C})$ can be represented in the form $z_0 + z_1 \varepsilon_4$. Let $\mathbb{C}(\mathbb{C}^2)$ be the set $\mathcal{S}(4, \mathbb{C})$, which is the set of elements of $\mathcal{S}(4, \mathbb{C})$ expressed in the form of $z_0 + z_1 \varepsilon_4$, denoted by

$$\mathbb{C}(\mathbb{C}^2) = \{z = z_0 + z_1 \varepsilon_4 \mid z_0, z_1 \in M(4, \mathbb{C})\}. \tag{11}$$

In addition, the conjugate of z in $\mathbb{C}(\mathbb{C}^2)$ is defined as $z = \bar{z}_0 + \bar{z}_1 \varepsilon_4$. Explicitly, we have

$$\begin{aligned}
\bar{z} &= x_0 - x_1 \varepsilon_1 + x_2 \varepsilon_2 - x_3 \varepsilon_3 + x_4 \varepsilon_4 - x_5 \varepsilon_5 + x_6 \varepsilon_6 - x_7 \varepsilon_7 \\
&= (x_0 - x_1 \varepsilon_1 + x_2 \varepsilon_2 - x_3 \varepsilon_3) + (x_4 - x_5 \varepsilon_1 + x_6 \varepsilon_2 - x_7 \varepsilon_3) \varepsilon_4 \\
&= \{(x_0 - x_1 \varepsilon_1) + (x_2 - x_3 \varepsilon_1) \varepsilon_2\} + \{(x_4 - x_5 \varepsilon_1) + (x_6 - x_7 \varepsilon_1) \varepsilon_2\} \varepsilon_4 \\
&= (\bar{\zeta}_0 + \bar{\zeta}_1 \varepsilon_2) + (\bar{\zeta}_2 + \bar{\zeta}_3 \varepsilon_2) \varepsilon_4 \\
&= \bar{z}_0 + \bar{z}_1 \varepsilon_4.
\end{aligned} \tag{12}$$

For $z, w \in \mathbb{C}(\mathbb{C}^2)$, we can write $z = z_0 + z_1 \varepsilon_4$ and $w = w_0 + w_1 \varepsilon_4$, then owing to the identities

$$\begin{aligned}
z_0 \varepsilon_4 &= \varepsilon_4 \tilde{z}_0, \\
z_1 \varepsilon_4 &= \varepsilon_4 \tilde{z}_1.
\end{aligned} \tag{13}$$

Thus, we obtain

$$zw = (z_0 w_0 + z_1 \tilde{w}_1) + (z_0 w_1 + z_1 \tilde{w}_0) \varepsilon_4. \tag{14}$$

Since $\varepsilon_1 \varepsilon_2 = \varepsilon_2 \varepsilon_1$, we have

$$z_r w_t = w_t z_r, \quad (r, t = 0, 1). \tag{15}$$

However,

$$\begin{aligned}
z_r \varepsilon_4 &= \varepsilon_4 \tilde{z}_r, \\
w_t \varepsilon_4 &= \varepsilon_4 \tilde{w}_t.
\end{aligned} \tag{16}$$

Although, as a result, $zw \neq wz$, it is equal to a part of zw and wz ; hence, $\mathbb{C}(\mathbb{C}^2)$ is a semicommutative subalgebra of $M(4, \mathbb{C})$. We first obtain an expression for z^* in order to define $\|z\|$. Let $z^* = \overline{(z^t)}$, where z^t is the transposed matrix of z . Explicitly, z^* (as an element of $\mathbb{C}(\mathbb{C}^2)$) is given by

$$\begin{aligned}
z^* &= \overline{(z^t)} \\
&= x_0 - x_1\varepsilon_1 + x_2\varepsilon_2 - x_3\varepsilon_3 + x_4\varepsilon_4 - x_5\varepsilon_5 - x_6\varepsilon_6 + x_7\varepsilon_7 \\
&= \{(x_0 - x_1\varepsilon_1) + (x_2 - x_3\varepsilon_1)\varepsilon_2\} \\
&\quad + \{(x_4 - x_5\varepsilon_1) - (x_6 - x_7\varepsilon_1)\varepsilon_2\}\varepsilon_4 \\
&= (\bar{\zeta}_0 + \bar{\zeta}_1\varepsilon_2) + (\bar{\zeta}_2 - \bar{\zeta}_3\varepsilon_2)\varepsilon_4 \\
&= \bar{z}_0 + \bar{z}_1\varepsilon_4.
\end{aligned} \tag{17}$$

From the above results, the norm $\|z\|$ of z is therefore given by

$$\|z\| = \frac{1}{2} \{tr(zz^*)\}^{1/2} = \left(\sum_{j=0}^7 x_j^2 \right)^{1/2}, \tag{18}$$

where $tr(z)$ is the trace of matrix (zz^*) .

Proposition 1. For $z, w \in \mathbb{C}(\mathbb{C}^2)$, the following properties for the norm $\|z\|$ are satisfied:

- (1) $\|z + w\| \leq \|z\| + \|w\|$
- (2) $\|zw\| \leq \|z\|\|w\|$, $z, w \in \mathbb{C}(\mathbb{C}^2)$

We now define differential operators as algebraic expressions for the elemental form of $\mathbb{C}(\mathbb{C}^2)$. The differential operator here includes elements in the form of a matrix. We consider the following differential operators over $\mathbb{C}(\mathbb{C}^2)$:

$$\begin{aligned}
\tilde{D}_1 &= \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}_0} - \varepsilon_4 \frac{\partial}{\partial z_1} \right), \\
D_1 &= \frac{1}{2} \left(\frac{\partial}{\partial z_0} + \varepsilon_4 \frac{\partial}{\partial \bar{z}_1} \right), \\
\tilde{D}_2 &= \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}_0} - \varepsilon_4 \frac{\partial}{\partial z_1} \right), \\
D_2 &= \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}_0} + \varepsilon_4 \frac{\partial}{\partial z_1} \right), \\
\tilde{D}_3 &= \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}_0} - \varepsilon_4 \frac{\partial}{\partial \bar{z}_1} \right), \\
D_3 &= \frac{1}{2} \left(\frac{\partial}{\partial z_0} + \varepsilon_4 \frac{\partial}{\partial \bar{z}_1} \right), \\
\tilde{D}_4 &= \frac{1}{2} \left(\frac{\partial}{\partial z_0} - \varepsilon_4 \frac{\partial}{\partial \bar{z}_1} \right), \\
D_4 &= \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}_0} + \varepsilon_4 \frac{\partial}{\partial \bar{z}_1} \right), \\
\frac{\partial}{\partial z_0} &= \frac{1}{2} \left(\frac{\partial}{\partial \zeta_0} + \varepsilon_2 \frac{\partial}{\partial \zeta_1} \right), \\
\frac{\partial}{\partial z_1} &= \frac{1}{2} \left(\frac{\partial}{\partial \zeta_2} + \varepsilon_2 \frac{\partial}{\partial \zeta_3} \right), \\
\frac{\partial}{\partial \bar{z}_0} &= \frac{1}{2} \left(\frac{\partial}{\partial \zeta_0} - \varepsilon_2 \frac{\partial}{\partial \zeta_1} \right), \\
\frac{\partial}{\partial \bar{z}_1} &= \frac{1}{2} \left(\frac{\partial}{\partial \zeta_2} - \varepsilon_2 \frac{\partial}{\partial \zeta_3} \right),
\end{aligned} \tag{19}$$

where $\partial/\partial\zeta_j$ ($j = 0, 1, 2, 3$) has the same pattern operator form as usual complex differential operators.

As a concrete example, we have

$$\begin{aligned} \frac{\partial}{\partial\zeta_0}\zeta_0 &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \varepsilon_1\frac{\partial}{\partial x_1}\right)(x_0 + x_1\varepsilon_1) \\ &= \frac{1}{2}\begin{pmatrix} \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} \end{pmatrix} \\ &\quad \begin{pmatrix} x_0 + ix_1 & 0 & 0 & 0 \\ 0 & x_0 - ix_1 & 0 & 0 \\ 0 & 0 & x_0 - ix_1 & 0 \\ 0 & 0 & 0 & x_0 + ix_1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (20)$$

In this case, let the 4×4 identity matrix be 1. If the calculation is performed by expressing it as a matrix, the following results are obtained:

$$\begin{aligned} \frac{\partial}{\partial z_0}z_0 &= 1, \\ \frac{\partial}{\partial \bar{z}_0}z_0 &= 0. \end{aligned} \quad (21)$$

Next, let Ω be a domain in \mathbb{C}^4 . We consider a function $f: \Omega \longrightarrow \mathbb{C}(\mathbb{C}^2)$ by

$$\begin{aligned} z &= (z_0, z_1) \mapsto f(z) \\ &= f_0(z_0, z_1) + f_1(z_0, z_1)\varepsilon_4, \end{aligned} \quad (22)$$

where f_0 and f_1 have the form $\varphi + \psi\varepsilon_2$ with φ and ψ being complex-valued functions. For example, for a function $f(z) = z^2$, this takes the form $f(z) = (z_0^2 + z_1\bar{z}_1) + (z_0z_1 + z_1\bar{z}_0)\varepsilon_4$, denoted by $f_0 = z_0^2 + z_1\bar{z}_1$ and $f_1 = z_0z_1 + z_1\bar{z}_0$.

The differential operator \tilde{D}_k ($k = 1, 2, 3, 4$) operates on f as follows: in the case of $n = 1$, applying the differential operator from the left, we have

$$\begin{aligned} \tilde{D}_1 f &= \frac{1}{2}\left(\frac{\partial}{\partial \bar{z}_0} - \varepsilon_4\frac{\partial}{\partial z_1}\right)(f_0 + f_1\varepsilon_4) \\ &= \frac{1}{2}\left\{\left(\frac{\partial f_0}{\partial \bar{z}_0} - \frac{\partial \tilde{f}_1}{\partial \bar{z}_1}\right) + \left(\frac{\partial f_1}{\partial \bar{z}_0} - \frac{\partial \tilde{f}_0}{\partial \bar{z}_1}\right)\varepsilon_4\right\}, \end{aligned} \quad (23)$$

and applying the differential operator from the right, we obtain

$$\begin{aligned} f\tilde{D}_1 &= \frac{1}{2}(f_0 + f_1\varepsilon_4)\left(\frac{\partial}{\partial \bar{z}_0} - \varepsilon_4\frac{\partial}{\partial z_1}\right) \\ &= \frac{1}{2}\left\{\left(\frac{\partial f_0}{\partial \bar{z}_0} - \frac{\partial f_1}{\partial z_1}\right) + \left(-\frac{\partial f_0}{\partial \bar{z}_1} + \frac{\partial f_1}{\partial z_0}\right)\varepsilon_4\right\}. \end{aligned} \quad (24)$$

Now, let us define the regular functions to be applied in $\mathbb{C}(\mathbb{C}^2)$ by using the differential operator $\tilde{D}_k(D_k)$ ($k = 1, 2, 3, 4$) for functions defined in $\mathbb{C}(\mathbb{C}^2)$.

3. Definitions and Properties of Regularity

Definition 1. Let Ω be a domain in \mathbb{C}^4 . Let $f = f_0 + f_1\varepsilon_4$ be a function defined on Ω . The function f is said to be $J_{L_k}(J_{R_k})$ -regular function ($k = 1, 2, 3, 4$) in Ω if

- (1) f_0 and f_1 are separately continuously holomorphic functions defined on Ω
- (2) $\tilde{D}_k f = 0$, ($f\tilde{D}_k = 0$) in Ω , where $k = 1, 2, 3, 4$

When a function f is $J_{L_k}(J_{R_k})$ -regular ($k = 1, 2, 3, 4$), it means that at least the above definition is satisfied for at least one of $k = 1, 2, 3, 4$.

Remark 1. Regarding f_0 and f_1 being separately continuous holomorphic functions, we mean that $f_0 = \varphi_1 + \varphi_2\varepsilon_2$ and $f_1 = \psi_1 + \psi_2\varepsilon_2$ such that the components φ_r and ψ_r ($r = 1, 2$) of each f_t ($t = 0, 1$) are holomorphic complex-valued functions.

Lemma 1 (Cauchy–Riemann equation in $\mathbb{C}(\mathbb{C}^2)$). *Condition 2 of Definition 1 for J_{L_k} -regular function is equivalent to*

$$\begin{aligned} \frac{\partial f_0}{\partial \bar{z}_0} &= \frac{\partial \tilde{f}_1}{\partial \bar{z}_1} \frac{\partial f_1}{\partial \bar{z}_0} = \frac{\partial \tilde{f}_0}{\partial \bar{z}_1}, \quad \text{if } k = 1, \\ \frac{\partial f_0}{\partial z_0} &= \frac{\partial \tilde{f}_1}{\partial \bar{z}_1} \frac{\partial f_1}{\partial z_0} = \frac{\partial \tilde{f}_0}{\partial \bar{z}_1}, \quad \text{if } k = 2, \\ \frac{\partial f_0}{\partial \bar{z}_0} &= \frac{\partial \tilde{f}_1}{\partial z_1} \frac{\partial f_1}{\partial \bar{z}_0} = \frac{\partial \tilde{f}_0}{\partial z_1}, \quad \text{if } k = 3, \\ \frac{\partial f_0}{\partial z_0} &= \frac{\partial \tilde{f}_1}{\partial z_1} \frac{\partial f_1}{\partial z_0} = \frac{\partial \tilde{f}_0}{\partial z_1}, \quad \text{if } k = 4. \end{aligned} \quad (25)$$

In addition, for J_{R_k} -regular function, Condition 2 is equivalent to

$$\begin{aligned} \frac{\partial f_0}{\partial \bar{z}_0} &= \frac{\partial f_1}{\partial \bar{z}_1}, \frac{\partial f_1}{\partial \bar{z}_0} = \frac{\partial f_0}{\partial \bar{z}_1}, & \text{if } k = 1, \\ \frac{\partial f_0}{\partial \bar{z}_0} &= \frac{\partial f_1}{\partial \bar{z}_1}, \frac{\partial f_1}{\partial \bar{z}_0} = \frac{\partial \tilde{f}_0}{\partial \bar{z}_1}, & \text{if } k = 2, \\ \frac{\partial f_0}{\partial \bar{z}_0} &= \frac{\partial f_1}{\partial \bar{z}_1}, \frac{\partial f_1}{\partial \bar{z}_0} = \frac{\partial f_0}{\partial \bar{z}_1}, & \text{if } k = 3, \\ \frac{\partial f_0}{\partial \bar{z}_0} &= \frac{\partial f_1}{\partial \bar{z}_1}, \frac{\partial f_1}{\partial \bar{z}_0} = \frac{\partial f_0}{\partial \bar{z}_1}, & \text{if } k = 4. \end{aligned} \quad (26)$$

Theorem 1. Let Ω be an open set in \mathbb{C}^4 , and let f be a regular function on Ω . Then, the derivative f' of f , also denoted as Df , is defined as

$$f' = \frac{\partial f}{\partial x_0} = -i \frac{\partial f}{\partial \bar{z}_0}. \quad (27)$$

on Ω .

Proof. From the definition of J_{L_k} -regular function defined on $\mathbb{C}(\mathbb{C}^2)$, in case $k = 1$, Condition 2 can be expressed as follows:

$$\begin{aligned} \tilde{D}_1 f &= \frac{1}{2} \left\{ \left(\frac{\partial f_0}{\partial \bar{z}_0} - \frac{\partial \tilde{f}_1}{\partial \bar{z}_1} \right) + \left(\frac{\partial f_1}{\partial \bar{z}_0} - \frac{\partial \tilde{f}_0}{\partial \bar{z}_1} \right) \varepsilon_4 \right\} \\ &= 0. \end{aligned} \quad (28)$$

Since f satisfies

$$\begin{aligned} \frac{\partial f_0}{\partial \bar{z}_0} - \frac{\partial \tilde{f}_1}{\partial \bar{z}_1} &= 0, \\ \frac{\partial f_1}{\partial \bar{z}_0} - \frac{\partial \tilde{f}_0}{\partial \bar{z}_1} &= 0, \end{aligned} \quad (29)$$

we obtain

$$\begin{aligned} \frac{\partial f_0}{\partial \bar{z}_0} &= \frac{\partial \tilde{f}_1}{\partial \bar{z}_1}, \\ \frac{\partial f_1}{\partial \bar{z}_0} &= \frac{\partial \tilde{f}_0}{\partial \bar{z}_1}. \end{aligned} \quad (30)$$

Similarly, we can derive the Cauchy–Riemann equations for the other cases.

For example, let f be an identity function such that $f(z) = z$. The function f is J_{L_k} -regular ($k = 1, 2, 3, 4$) defined on $\mathbb{C}(\mathbb{C}^2)$. Let f be a function such that $f(z) = z^2$ be J_{L_1} -regular defined on $\mathbb{C}(\mathbb{C}^2)$. Let f be $J_{L_k}(J_{R_k})$ -regular functions ($k = 1, 2, 3, 4$) defined in Ω . Then, we propose the $J_{L_k}(J_{R_k})$ -derivative $D_k f (f D_k)$ ($k = 1, 2, 3, 4$) of f and find the following theorem. \square

Theorem 2. Let Ω be a domain in \mathbb{C}^4 . Let f be a $J_{L_k}(J_{R_k})$ -regular function defined in Ω . Then, the derivative $D_k f (f D_k)$ ($k = 1, 2, 3, 4$) of f satisfies

$$D_k f = f D_k = \frac{\partial f}{\partial \bar{\zeta}_0}, \quad (k = 1, 2, 3, 4). \quad (31)$$

Proof. Since f is J_{L_1} -regular defined on Ω , the function f satisfies $\tilde{D}_1 f = 0$. Hence, we have

$$\begin{aligned} D_1 f &= \frac{1}{2} \left\{ \left(\frac{\partial f_0}{\partial \bar{z}_0} + \frac{\partial \tilde{f}_1}{\partial \bar{z}_1} \right) + \left(\frac{\partial f_1}{\partial \bar{z}_0} - \frac{\partial \tilde{f}_0}{\partial \bar{z}_1} \right) \varepsilon_4 \right\} \\ &= \frac{1}{2} \left\{ \left(\frac{\partial f_0}{\partial \bar{z}_0} + \frac{\partial f_0}{\partial \bar{z}_0} \right) + \left(\frac{\partial f_1}{\partial \bar{z}_0} + \frac{\partial f_1}{\partial \bar{z}_0} \right) \varepsilon_4 \right\} \\ &= \frac{1}{2} \left(\frac{\partial f}{\partial \bar{z}_0} + \frac{\partial f}{\partial \bar{z}_0} \right) \\ &= \frac{\partial f}{\partial \bar{\zeta}_0}. \end{aligned} \quad (32)$$

When $k = 2, 3, 4$, by using the equation $\tilde{D}_k f = 0$, we obtain the same results, that is,

$$D_2 f = D_3 f = D_4 f = \frac{1}{2} \left(\frac{\partial f}{\partial \bar{z}_0} + \frac{\partial f}{\partial \bar{z}_0} \right) = \frac{\partial f}{\partial \bar{\zeta}_0}. \quad (33)$$

Similarly, when f is J_{R_k} -regular ($k = 1, 2, 3, 4$) defined on Ω , then by the equation $f \tilde{D}_k = 0$ ($k = 1, 2, 3, 4$), we obtain

$$f D_k = \frac{1}{2} \left(\frac{\partial f}{\partial \bar{z}_0} + \frac{\partial f}{\partial \bar{z}_0} \right), \quad (k = 1, 2, 3, 4). \quad (34)$$

Thus, the $J_{L_k}(J_{R_k})$ -derivative of f satisfies

$$D_k f = f D_k = \frac{\partial f}{\partial \bar{\zeta}_0}, \quad (k = 1, 2, 3, 4). \quad (35)$$

From Theorem 1, for a separately continuous holomorphic $J_{L_k}(J_{R_k})$ -regular function f ($k = 1, 2, 3, 4$) defined in Ω , we denote

$$(D_k^n f)(z) = (f D_k^n)(z) = \frac{\partial^n}{\partial \bar{\zeta}_0^n} f(z) = f(z) \frac{\partial^n}{\partial \bar{\zeta}_0^n}. \quad (36) \quad \square$$

4. Properties of $J_{L_k}(J_{R_k})$ -Regular ($k = 1, 2, 3, 4$) Functions

In this section, let Ω be a domain in \mathbb{C}^4 . Based on the definition and properties of J_{L_k} -regularity ($k = 1, 2, 3, 4$), we determine whether the properties of holomorphic function of complex variables extend J_{L_k} -regular function ($k = 1, 2, 3, 4$) defined on Ω . Suppose that $\Omega_1, \Omega_2 \subset \mathbb{C}^4$ are two open neighborhoods; then, the variables in Ω_1 can be written as $z = (z_0, z_1)$ and Ω_2 can be written as $w = (w_0, w_1)$. Any mapping $g: \Omega_1 \rightarrow \Omega_2$ can be described by

$$\begin{aligned} w_0 &= g_0(z_0, z_1), \\ w_1 &= g_1(z_0, z_1). \end{aligned} \quad (37)$$

The mapping g is called a J_{L_k} -regular mapping ($k = 1, 2, 3, 4$) if the functions g_0 and g_1 are J_{L_k} -regular functions ($k = 1, 2, 3, 4$) in Ω_1 . If $f(w_0, w_1) = f(w)$ is any function defined in Ω_2 , then composition $f(g)$ is a well-defined function in Ω_1 . If $f(w)$ is a J_{L_k} -regular function ($k = 1, 2, 3, 4$) in Ω_2 and if $g: \Omega_1 \rightarrow \Omega_2$ is a J_{L_k} -regular mapping ($k = 1, 2, 3, 4$), then the composition $w = f(g)$ is a J_{L_k} -regular function ($k = 1, 2, 3, 4$) in Ω_1 .

Next, the following quadratic form is factors that play an important role in the integral calculus of J_{L_k} -regular functions ($k = 1, 2, 3, 4$):

$$\begin{aligned} w_1 &= dz_0 \wedge dz_1 \wedge d\bar{z}_1 - dz_0 \wedge d\bar{z}_0 \wedge dz_1 \varepsilon_4, \\ w_2 &= d\bar{z}_0 \wedge dz_1 \wedge d\bar{z}_1 + dz_0 \wedge d\bar{z}_0 \wedge dz_1 \varepsilon_4, \\ w_3 &= dz_0 \wedge dz_1 \wedge d\bar{z}_1 + dz_0 \wedge d\bar{z}_0 \wedge d\bar{z}_1 \varepsilon_4, \\ w_4 &= d\bar{z}_0 \wedge dz_1 \wedge d\bar{z}_1 - dz_0 \wedge d\bar{z}_0 \wedge d\bar{z}_1 \varepsilon_4. \end{aligned} \quad (38)$$

Furthermore, we note that

$$\begin{aligned} dz_i \wedge dz_j &= -dz_j \wedge dz_i, \\ dz_i \wedge dz_i &= 0, \quad (i, j = 0, 1), \\ d\Phi &= dz_0 \wedge d\bar{z}_0 \wedge dz_1 \wedge d\bar{z}_1. \end{aligned} \quad (39)$$

Definition 2 (Morera's theorem in $\mathbb{C}(\mathbb{C}^2)$). Let a function f be continuous in Ω . Suppose the integral of $\omega_k f$ ($k = 1, 2, 3, 4$) over the boundary ∂U of any sphere U in Ω be equal to zero. Then, f is a J_{L_k} -regular function ($k = 1, 2, 3, 4$) in Ω .

Proof. We assume that the function f is smooth in Ω . Let U be a sphere in Ω . Based on this hypothesis, we have

$$\int_{\partial U} \omega_k f = 0, \quad (k = 1, 2, 3, 4). \quad (40)$$

From Stokes' formula in complex analysis, we obtain

$$\int_U d_k(\omega_k f) = \int_{\partial U} \omega_k f = 0, \quad (k = 1, 2, 3, 4). \quad (41)$$

Hence, for $k = 1$, the integral can be calculated as follows:

$$\int_U \left(\frac{\partial f_0}{\partial \bar{z}_0} - \frac{\partial \tilde{f}_1}{\partial \bar{z}_1} \right) d\Phi = \int_U \left(\frac{\partial f_1}{\partial \bar{z}_0} + \frac{\partial \tilde{f}_0}{\partial \bar{z}_1} \right) d\Phi = 0. \quad (42)$$

Due to the smoothness of f , the expression under the integral sign is continuous. Since the fact U is an arbitrary sphere in Ω , it follows that

$$\begin{aligned} \frac{\partial f_0}{\partial \bar{z}_0} &= \frac{\partial \tilde{f}_1}{\partial \bar{z}_1}, \\ \frac{\partial f_1}{\partial \bar{z}_0} &= -\frac{\partial \tilde{f}_0}{\partial \bar{z}_1}, \end{aligned} \quad (43)$$

in Ω . For cases $k = 2, 3, 4$, we can obtain similar conclusions.

Next, we consider the relationship between the J_{L_k} -regular function ($k = 1, 2, 3, 4$) and series expansions in $\mathbb{C}(\mathbb{C}^2)$. \square

Lemma 2. Let f be a homogeneous polynomial of degree n with respect to z_0 and z_1 . If f is a J_{L_k} -regular function ($k = 1, 2, 3, 4$) in \mathbb{C}^4 , then f satisfies

$$f(z) = \frac{1}{n!} \frac{\partial^n f(z)}{\partial z_0^n} z^n. \quad (44)$$

Proof. According to the hypothesis, the function $f(z)$ is a homogeneous polynomial; thus, f satisfies

$$f(z) = \frac{1}{n} \frac{\partial f(z)}{\partial z_0} z. \quad (45)$$

Since $\partial f / \partial z_0$ is a homogeneous polynomial of degree $n - 1$, we have

$$\frac{\partial f(z)}{\partial z_0} = \frac{1}{n-1} \frac{\partial^2 f(z)}{\partial z_0^2} z. \quad (46)$$

Continuing this process, we can obtain

$$f(z) = \frac{1}{n!} \frac{\partial^n f(z)}{\partial z_0^n} z^n. \quad (47)$$

\square

Proposition 2. Let Ω be a domain in \mathbb{C}^4 and $f(z)$ be a function defined on a neighborhood V of $\alpha \in \mathbb{C}^4$ with values in $\mathbb{C}(\mathbb{C}^2)$. The function f is a J_{L_k} -regular function ($k = 1, 2, 3, 4$) in Ω and $\alpha \in \Omega$ if and only if f has a power series expansion in V such that for $z \in V$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n, \quad (48)$$

where $a_n = (1/n!) (\partial^n f(\alpha) / \partial z_0^n)$ ($n = 0, 1, 2, \dots$).

Proof. Without loss of generality, let $\alpha = 0$. Suppose that f is J_{L_k} -regular ($k = 1, 2, 3, 4$) in Ω with a value in $\mathbb{C}(\mathbb{C}^2)$; then, f is holomorphic in Ω . Thus, there exists a neighborhood V of zero such that for $z \in V$,

$$f(z) = \sum_{n=0}^{\infty} P_n(z), \quad (49)$$

where $P_n(z)$ are homogeneous polynomials of degree n with respect to z_0 and z_1 . Since the above series converges uniformly in V and $P_n(z)$ are homogeneous polynomials of degree n with respect to z_0 and z_1 , when $k = 1$, we have

$$\begin{aligned} D_1^* f(z) &= \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}_0} + \varepsilon_4 \frac{\partial}{\partial z_1} \right) f(z) \\ &= \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}_0} + \varepsilon_4 \frac{\partial}{\partial z_1} \right) P_n(z) = 0. \end{aligned} \quad (50)$$

Hence, $P_n(z)$ is a J_{L_k} -regular ($k = 1, 2, 3, 4$) in Ω . By Lemma 2, since

$$\frac{\partial^n f(0)}{\partial z_0^n} = \frac{\partial^n}{\partial z_0^n} \sum_{n=0}^{\infty} P_n(z), \quad (51)$$

we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} P_n(z) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n P_n(z)}{\partial z_0^n} z^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f(0)}{\partial z_0^n} z^n \\ &= \sum_{n=0}^{\infty} a_n z^n, \\ z &\in V. \end{aligned} \quad (52)$$

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (53)$$

where $a_n = (1/n!) (\partial^n f(0)/\partial z_0^n)$ ($n = 0, 1, 2, \dots$). Since $f(z)$ converges uniformly in V , we obtain

$$\begin{aligned} D_1^* f(z) &= \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}_0} + \varepsilon_4 \frac{\partial}{\partial z_1} \right) f(z) \\ &= \sum_{n=0}^{\infty} \frac{1}{2} a_n \left(\frac{\partial}{\partial \bar{z}_0} + \varepsilon_4 \frac{\partial}{\partial z_1} \right) z^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} a_n \left(\frac{\partial}{\partial \bar{z}_0} (z_0 + z_1 \varepsilon_4)^n + \varepsilon_4 \frac{\partial}{\partial z_1} (z_0 + z_1 \varepsilon_4)^n \right) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} a_n (n(z_0 + \varepsilon_2 z_1)^{n-1} + \varepsilon_2 n(z_0 + \varepsilon_2 z_1)^{n-1} \varepsilon_2) \\ &= 0. \end{aligned} \quad (54)$$

Thus, $f(z)$ is a J_{L_k} -regular function ($k = 1, 2, 3, 4$) in V .

From Proposition 2, equivalently, we can define J_{L_k} -regular functions ($k = 1, 2, 3, 4$) in Ω which is a domain in \mathbb{C}^4 as follows. \square

Definition 3. Let Ω be a domain in \mathbb{C}^4 . A function $f = f_0 + f_1 \varepsilon_4$ is said to a J_{L_k} -regular function ($k = 1, 2, 3, 4$) on Ω if every point $a \in \Omega$ corresponds to a neighborhood U admitting a power series

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n, \quad (55)$$

which converges to $f(z)$ for $z \in U$.

From the above results, the following statements are equivalent to each other:

- (1) f is a J_{L_k} -regular function ($k = 1, 2, 3, 4$) defined on Ω .
- (2) For every point $a \in \Omega$, there is a neighborhood U admitting a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n, \quad z \in U. \quad (56)$$

- (3) For a smooth boundary ∂U such that $\bar{U} \subset \Omega$, we have

$$\int_{\partial U} \omega_k f = 0, \quad (57)$$

where $\omega_k f$ ($k = 1, 2, 3, 4$) is the quaternion product of the form of the function $f = f_1 + f_2 \varepsilon_4$.

Corollary 1. Let Ω be a domain in \mathbb{C}^4 and f be a J_{L_k} -regular function ($k = 1, 2, 3, 4$). If f is infinitely differentiable in $\mathbb{C}(\mathbb{C}^2)$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n, \quad (58)$$

then we have

$$a_n = \frac{1}{n!} (D_k^n f)(\alpha) = \frac{1}{n!} \frac{\partial^n}{\partial z_0^n} f(\alpha), \quad n = 0, 1, 2, \dots \quad (59)$$

5. Conclusion

In this paper, we developed the foundation of a function theory over a subalgebra of four-dimensional matrix algebra as an alternative to the standard basis of quaternion algebra. Furthermore, we introduced a semicommutative subalgebra $\mathbb{C}(\mathbb{C}^2)$ of the complex matrix algebra $M(4, \mathbb{C})$, and we propose regular functions defined in a domain in \mathbb{C}^4 with values in $\mathbb{C}(\mathbb{C}^2)$. From the definition of regularity over $\mathbb{C}(\mathbb{C}^2)$, we derived corresponding Cauchy–Riemann equations and several properties of regular functions in $\mathbb{C}(\mathbb{C}^2)$. Throughout this paper, it has been possible to make various attempts to utilize quaternions by demonstrating a series expansion of quaternion variable. Regular functions defined on quaternions can be applied to this extension of the algebra of quaternions introduced in this paper. In addition, it is possible to present the extensibility of formulas derived in this paper based on proposed quaternionic theory for differential operators and series expansion. In the future, we plan to verify properties, such as $\bar{\partial}$ -closed form and regularity over a domain of holomorphy, by using certain regular functions and Cauchy–Riemann equations.

Data Availability

No datasets were generated or analyzed during the current study.

Conflicts of Interest

The author declares no conflicts of interest.

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Review Article

L^p Smoothness on Weighted Besov–Triebel–Lizorkin Spaces in terms of Sharp Maximal Functions

Ferit Gürbüz¹ and Ahmed Loulit²

¹Hakkari University, Faculty of Education, Department of Mathematics Education, Hakkari 30000, Turkey

²Département de Mathématique, Research Center E. Bernheim Solvay Business School, Université Libre de Bruxelles, Av F.D. Roosevelt 21, CP 135/01, B-1050, Brussels, Belgium

Correspondence should be addressed to Ferit Gürbüz; feritgurbuz@hakkari.edu.tr

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It is known, in harmonic analysis theory, that maximal operators measure local smoothness of L^p functions. These operators are used to study many important problems of function theory such as the embedding theorems of Sobolev type and description of Sobolev space in terms of the metric and measure. We study the Sobolev-type embedding results on weighted Besov–Triebel–Lizorkin spaces via the sharp maximal functions. The purpose of this paper is to study the extent of smoothness on weighted function spaces under the condition $M_\alpha^\#(f) \in L^{p,\mu}$, where μ is a lower doubling measure, $M_\alpha^\#(f)$ stands for the sharp maximal function of f , and $0 \leq \alpha \leq 1$ is the degree of smoothness.

1. Introduction and Main Result

In this paper, we consider the some continuous embeddings on weighted Besov–Triebel–Lizorkin spaces via a general sharp maximal function introduced by Calderón and Scott [6]. Furthermore, we investigate the spaces introduced by Hajlasz [13] that are defined via pointwise inequalities and their connection with the Triebel–Lizorkin spaces. For more details, see [11, 12].

Now, let us begin by recalling some definitions and classical results in harmonic analysis on the n -dimensional Euclidean space \mathbb{R}^n needed for later sections.

(1) A cube on \mathbb{R}^n will always mean a cube with sides parallel to the axes and has nonempty interior. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we denote by Q_{jk} the dyadic cube $2^{-j}([0, 1]^n + k)$, where $l(Q_{jk}) = 2^{-j}$ is its side length, $x_{Q_{jk}} = 2^{-j}k$ is its lower “left-corner,” and $c_{Q_{jk}}$ is its center. We set $Q = \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ and $j_Q = -\log_2 l(Q)$ for all $Q \in \mathcal{Q}$. When the dyadic cube Q appears as an index, such as $\sum_{Q \in \mathcal{Q}}$, it is understood that Q runs over all dyadic cubes in \mathbb{R}^n . For a function ν and dyadic cube $Q = Q_{jk}$, set

$$\nu_Q(x) = |Q|^{-(1/2)} \nu(2^j x - k) = |Q|^{(1/2)} \nu_j(x - x_Q), \quad (1)$$

for all $x \in \mathbb{R}^n$, where $\nu_j(x) = 2^{nj} \nu(2^j x)$.

(2) Throughout the paper, w denotes a weight function, i.e., w is an almost every (a.e.) positive locally integrable function on \mathbb{R}^n . A function $f \in L^p(w)$, $0 < p < \infty \iff$

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty, \quad (2)$$

and f belongs to the weak- L^p spaces, denoted by $L^{p,\infty}(w) \iff$

$$\|f\|_{L^{p,\infty}(w)} = \sup_{\lambda > 0} \lambda w(\{x \in \mathbb{R}^n : f(x) > \lambda\})^{1/p} < \infty. \quad (3)$$

If $w = 1$, we do not write the subscription w .

A weight function w is said to be in the Muckenhoupt classes A_p , where $1 \leq p < \infty$, if there exists a constant $C_p > 0$ such that for every cube Q ,

$$\frac{1}{|Q|} \int_Q w dy \left(\frac{1}{|Q|} \int_Q w^{1-p'} dy \right)^{p-1} \leq C_p. \quad (4)$$

When $1 < p < \infty$, $(1/p) + (1/p') = 1$; for $p = 1$,

$$\frac{1}{|Q|} \int_Q w(y) dy \leq C_1 w(x), \quad (5)$$

for a.e. $x \in Q$, or equivalently $Mw(x) \leq C_1 w(x)$ for a.e. $x \in \mathbb{R}^n$, where M is the Hardy–Littlewood maximal operator.

The class A_p was introduced by Muckenhoupt [16] in order to characterize the boundedness of the Hardy–Littlewood maximal operator M on the weighted Lebesgue spaces [8, 12]. The pioneering work of Muckenhoupt [16] showed that

$$M: L^p(w) \longrightarrow L^p(w), \quad (6)$$

$\iff w \in A_p$ when $1 < p < \infty$ and

$$M: L^1(w) \longrightarrow L^{1,\infty}(w), \quad \iff w \in A_1. \quad (7)$$

A weight function w is in Muckenhoupt's class $A_p(\mathbb{R}^n)$, $1 \leq p < \infty$, of weights if there exists a constant $C_p > 0$ such that for all cubes Q in \mathbb{R}^n ,

$$\left(\frac{1}{|Q|} \int_Q w(y) dy \right) \left(\frac{1}{|Q|} \int_Q w(y)^{1-p'} dy \right)^{p-1} \leq C_p. \quad (8)$$

When $1 < p < \infty$, $(1/p) + (1/p') = 1$; well, for $p = 1$,

$$\frac{1}{|Q|} \int_Q w(y) dy \leq C_1 w(x), \quad (9)$$

for a.e. $x \in B$, or equivalently $Mw(x) \leq C_1 w(x)$ for a.e. $x \in \mathbb{R}^n$, where M is the Hardy–Littlewood maximal operator.

(3) Note that if $w \in A_p$, then w is a doubling measure, i.e., there exists a constant $C \geq 1$ such that for all x and all $r > 0$,

$$w(B(x, 2r)) \leq Cw(B(x, r)). \quad (10)$$

Another class of functions that plays an important role in harmonic analysis and in partial differential equation theory is the class of functions with bounded mean oscillation denoted by $BMO(w)$, i.e., $\varphi \in BMO(w)$, if there is a constant C :

$$\sup_Q \frac{1}{w(Q)} \int_Q |\varphi(y) - \varphi_Q| w(y) dy < C, \quad (11)$$

where $\varphi_Q = (1/w(Q)) \int_Q \varphi(y) w(y) dy$ is the average of f on Q with respect to dw . The smallest constant C for which (11) is satisfied is taken to be the norm of φ in the space $BMO(w)$ and is denoted by $\|\varphi\|_{BMO(w)}$.

(4) The sharp maximal function $M^\# f(x)$ of f is defined by

$$M^\# f(x) = \sup_Q \inf_{c \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f(x) - c| dx, \quad (12)$$

where Q is taken over all cubes in \mathbb{R}^n . Let $\alpha \geq 0$. The sharp fractional maximal function $M_\alpha^\#(f)$ of f is defined by

$$M_\alpha^\#(f)(x) = \sup_{x \in Q} \inf_{c \in \mathbb{R}} \frac{1}{|Q|^{(\alpha/n)}} \int_Q |f(x) - c| dx. \quad (13)$$

(5) The space of Schwartz functions: let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on \mathbb{R}^n with the classical topology generated by the family of seminorms:

$$\|v\|_{k,N} = \sup_{x \in \mathbb{R}^n} \sup_{|\beta| \leq N} (1 + |x|)^k |\partial^\beta v(x)| \quad k, N \in \mathbb{N}_0, v \in \mathcal{S}(\mathbb{R}^n). \quad (14)$$

The topological dual space $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$ is the set of all continuous linear functional the space $\mathcal{S}(\mathbb{R}^n)$ is endowed with the weak $*$ -topology. We denote by $\mathcal{S}_\infty(\mathbb{R}^n)$ the topological subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ having all vanishing moments:

$$\mathcal{S}_\infty(\mathbb{R}^n) = \left\{ v \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\beta v(x) dx = 0, \text{ for every } \beta \in \mathbb{N}^n \right\}. \quad (15)$$

$\mathcal{S}'_\infty(\mathbb{R}^n)$ denotes the topological dual space of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functional on $\mathcal{S}'_\infty(\mathbb{R}^n)$. The space $\mathcal{S}'_\infty(\mathbb{R}^n)$ is also endowed with the weak $*$ -topology. It is well known that $\mathcal{S}'_\infty(\mathbb{R}^n) = (\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n))$ as topological spaces, where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials on \mathbb{R}^n ; see, for example, ([21], Proposition 8.1). Similarly, for any $R \in \mathbb{N}$, the space $\mathcal{S}_R(\mathbb{R}^n)$ is defined to be the set of all Schwartz functions having vanishing moments of order R and $\mathcal{S}'_R(\mathbb{R}^n)$ is its topological dual space. We write $\mathcal{S}'_{-1}(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)$.

The Fourier transform, $\mathcal{F}v = \widehat{v}$, of Schwartz function v is defined by

$$\widehat{v}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} v(x) dx. \quad (16)$$

The convolution of two functions $v, \mu \in \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$v * \mu(x) = \int_{\mathbb{R}^n} v(x - y) \mu(y) dy \quad (17)$$

and still belongs to $\mathcal{S}'(\mathbb{R}^n)$.

The convolution operator can be extended to $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ via $v * f(x) = \langle f, \mu(x - \cdot) \rangle$. It makes sense point-wise and is a C^∞ function on \mathbb{R}^n of at most polynomial growth.

To simplify notation, we write often $vf = v * f$. In some other situations, to avoid confusion, we keep the notation $v * f$. As usual, v_t denotes the function defined by $v_t(x) = t^{-n} v(x/t)$.

(6) In the rest of this paper, C expresses unspecified positive constant, possibly different at each occurrence; the symbol $A \leq B$ means that $A \leq CB$. If $A \leq B$ and $B \leq A$, then we write $A \approx B$. The Greek letter χ_S denotes the characteristics function of a sphere S , where S is a measurable subset of \mathbb{R}^n and $|S|$ represents its Lebesgue measure; p' and s' always denote the conjugate index of any $p > 1$ and $s > 1$, that is, $(1/p') = 1 - (1/p)$ and $(1/s') = 1 - (1/s)$.

Function spaces play a crucial role in the genesis of functional analysis and are widely used in the development

of the modern analysis of partial differential equations. For instance, the classical Besov–Triebel–Lizorkin spaces are a class of function spaces containing many well-known classical function spaces and are more suitable in the treatment of a large type of partial differential equations (see for instance [5, 10]). A comprehensive treatment of these function spaces and their history can be found in Triebel’s monographs [18, 19] and in the fundamental paper of Frazier and Jawerth [11].

In recent years, there has been increasing interest in a new family of function spaces, called new class of Besov–Triebel–Lizorkin spaces. These spaces unify and generalize many classical spaces including Besov spaces, Morrey spaces, and Triebel–Lizorkin spaces (see for instance [20]).

In this paper, we study the extent of smoothness on weighted function spaces under the condition $M_\alpha^\# f \in L^{p,\mu}$, where μ is a lower doubling measure, $M_\alpha^\# f$ stands for the sharp maximal function of f , and $0 \leq \alpha \leq 1$ is the degree of smoothness. When $\alpha = 0$, $M_0^\# f = M^\# f$ is the classical sharp maximal function. It is well known that the Hardy–Littlewood maximal function Mf is controlled by the sharp maximal function $M^\# f$ via the celebrated Stein–Fefferman inequality: $\|Mf\|_p \leq \|M^\# f\|_p$ and in the case of $\alpha = 1$, it is shown that $\|M_1^\# f\|_p \leq \|f\|_{H_p}$ for some range of p . As a result, we extend the above results to the some general weighted spaces. Embedding results on weighted Besov–Triebel–Lizorkin spaces are obtained. Namely, $\|f\|_{\dot{F}_{p,w}^{\gamma,q}} \leq \|M_\alpha^\#(f)\|_{p,w}$ (Theorem 1). As a consequence, we obtain $\|f\|_{\dot{F}_{p,w}^{\gamma,q}} \leq \|f\|_{\dot{W}^{\alpha,p}(w)}$, where $\dot{W}^{\alpha,p}(w)$ stands for the fractional Sobolev space.

Now, we are ready to present the main theorem of this section.

Theorem 1. *Let α and γ be real numbers satisfying $0 \leq \alpha \leq 1$ and $\gamma < \alpha$, and w is the lower d -regular doubling measure. Suppose that $w\{f(x) > \varepsilon\} < \infty$ for every $\varepsilon > 0$ and $M_\alpha^\#(f) \in L^p(w)$ for $(d/(n+\alpha)) < p < (d/(\alpha-\gamma))$. Then, for each $0 < q \leq \infty$,*

$$\|f\|_{\dot{F}_{p,w}^{\gamma,q}} \leq \|M_\alpha^\#(f)\|_{p,w}, \quad \frac{1}{p_*} = \frac{1}{p} - \frac{\alpha-\gamma}{d}. \quad (18)$$

Remark 1. The condition that $w\{f(x) > \varepsilon\} < \infty$ for every $\varepsilon > 0$ is necessary. On the other hand, under this condition, $M_\alpha^\#(f) \in L^p(w)$ only if $p > (d/(n+\alpha))$.

Proof. If f is not a constant function, then there exists a ball $B(x_0, R)$ such that

$$\inf_{c \in \mathbb{R}} \int_{B(x_0, R)} |f(y) - c| dy = c_0 > 0. \quad (19)$$

Therefore, for all $x \in \mathbb{R}^n$,

$$M_\alpha^\#(f)(x) \pm (R + |x - x_0|)^{-\alpha-n}. \quad (20)$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} |M_\alpha^\#(f)|^p w(x) dy &\pm \int_{|x-x_0| > R} (R + |x - x_0|)^{(-\alpha-n)p} w(x) dx \\ &\pm \sum_{k \geq 0} (R + 2^{k+1}R)^{(-\alpha-n)p} \int_{|x-x_0| < 2^{k+1}R} w(x) dx \\ &\pm R^{(-\alpha-n)p+d} \sum_{k \geq 0} 2^{k(-\alpha-n)p+kd} = \infty, \end{aligned} \quad (21)$$

if $p \leq (d/(n+\alpha))$. \square

Corollary 1. *Under the same conditions in Theorem 1, we have, for each $0 < q \leq \infty$,*

$$\|f\|_{\dot{B}_{p,w}^{\gamma,q}} \leq \|M_\alpha^\#(f)\|_{p,w}, \quad \frac{1}{p_*} = \frac{1}{p} - \frac{\alpha-\gamma}{d}, \quad (22)$$

for each $0 < p_* < q \leq \infty$.

Proof. By Minkowski’s inequality, we have

$$\|f\|_{\dot{B}_{p,w}^{\gamma,q}} \leq \|f\|_{\dot{F}_{p,w}^{\gamma,q}}, \quad 0 < p_* < q \leq \infty. \quad (23)$$

Then, applying Theorem 1, we obtain (22). This completes the proof. \square

2. Preliminaries

In this section, we introduce some necessary and important definitions, notations, lemmas, and results.

Definition 1. Let v be in the Schwartz space with $\text{supp } \hat{v}$ contained in an annulus about the origin and

$$\sum_{j \in \mathbb{Z}} \hat{v}(2^{-j}\xi) = 1 \quad \text{for all } \xi \neq 0. \quad (24)$$

Let μ be a doubling measure and $0 < p, q \leq \infty$ and $\gamma \in \mathbb{R}$; the homogeneous Triebel–Lizorkin space $\dot{F}_{p,v}^{\gamma,q}$ is the set of all distributions f (modulo polynomials) such that

$$\begin{aligned} \|f\|_{\dot{F}_{p,\mu}^{\gamma,q}} &= \left\| \left(\sum_{j \in \mathbb{Z}} 2^{j\gamma q} |v_{2^{-j}} f|^q \right)^{1/q} \right\|_{p,\mu} < \infty; \quad 0 < p, q < \infty, \\ \|f\|_{\dot{F}_{\infty,\mu}^{\gamma,q}} &= \sup_Q \left\{ \frac{1}{\mu(Q)} \int_Q \sum_{j=-\log_2 l(Q)} 2^{j\gamma q} |v_{2^{-j}} f|^q d\mu(x) \right\}^{1/q} < \infty; \quad 0 < q \leq \infty, \end{aligned} \quad (25)$$

with the interpretation that when $q = \infty$,

$$\|f\|_{\dot{F}_{\infty,w}^{\gamma,q}} = \sup_Q \sup_{j \geq -\log_2 l(Q)} \frac{1}{\mu(Q)} \int_Q 2^{j\gamma} |v_{2^{-j}} f| d\mu(x) < \infty. \quad (26)$$

The homogeneous Besov–Lipschitz space $\dot{B}_p^{\gamma,q}$ is the set of all distributions f (modulo polynomials) such that

$$\|f\|_{\dot{B}_p^{\gamma,q}} = \left(\sum_{j \in \mathbb{Z}} 2^{j\gamma q} \|v_{2^{-j}} f\|_{p,w}^q \right)^{1/q} < \infty; \quad 0 < p, q < \infty. \quad (27)$$

The supremum is taken over all dyadic cubes Q , and $l(Q)$ denotes the length of sides of the cube Q .

Moreover, it is well known that the Besov–Lipschitz spaces and the Triebel–Lizorkin spaces are independent of the choices of ν (see, for example [2–4, 11]). Throughout this paper, ν will be taken as in Definition 1. It is well known that many classical smoothness spaces are covered by the Besov and Triebel–Lizorkin spaces. We recall some examples in the case when $d\mu = wdx$ and $w \in A_{\infty}$:

- (1) $\dot{F}_{p,w}^{0,2} = H_{p,w}$, $0 < p < \infty$.
- (2) $\dot{F}_{p,w}^{0,2} = h_{p,w}$, $0 < p < \infty$, where $H_{p,w}$ denotes the weighted Hardy spaces of $f \in \mathcal{S}'$ for which

$$\|f\|_{H_{p,w}} = \left\| \sup_{t>0} \mu_t * f \right\|_{p,w} < \infty \quad (28)$$

and $h_{p,w}$ is the local weighted Hardy space of $f \in \mathcal{S}'$ for which

$$\|f\|_{h_{p,w}} = \left\| \sup_{0<t<1} \mu_t * f \right\|_{p,w} < \infty, \quad (29)$$

where μ is a fixed function in \mathcal{S} with $\int_{\mathbb{R}^n} \mu(x) dx \neq 0$. By the fundamental work of Fefferman and Stein [9] adapted to the weighted case, $H_{p,w}$ or $h_{p,w}$ does not depend on the choices of μ . In particular,

$$\dot{F}_{p,w}^{0,2} = L^p(w), \quad 1 < p < \infty. \quad (30)$$

- (3) $\dot{F}_{p,w}^{\gamma,2} = H_{p,w}^\gamma$, $1 < p < \infty$, where $H_{p,w}^\gamma$ denotes the weighted Bessel potential space defined by

$$\|f\|_{H_{p,w}^\gamma} = \left\| \mathcal{F}^{-1} (1 + |\xi|^2)^{\gamma/2} \mathcal{F} f \right\|_{p,w}. \quad (31)$$

In particular, when the exponent is a natural number, say $\gamma = N \in \mathbb{N}$, then the weighted Bessel potential space can be identified with the classical Sobolev space:

$$W_{p,w}^N = \left\{ f \in L^{p,w} : \left\| \sum_{|\sigma| \leq N} \partial^\sigma f \right\|_{L^{p,w}} < \infty \right\}, \quad 1 < p < \infty. \quad (32)$$

- (4) $\dot{F}_{\infty,w}^{0,2} = \text{BMO}(w)$.

All the above identities have to be understood in the sense of equivalent quasi-norms.

Definition 2. We say that a doubling measure μ is lower d -regular, where $d \geq n$, if there is some constant $C > 0$ such that

$$\mu(B(x, t)) \geq Ct^d \quad (33)$$

holds for all ball $B(x, t) \subset \mathbb{R}^n$.

Remark 2. An example of measure μ lower d -regular is $d\mu = wdx$, where

$$w(x) = |x|^\alpha \log^\beta(2 + |x|^{-1}). \quad (34)$$

In fact, $w \in A_p$ if $-n < \alpha < n(p-1)$ and $\beta \in \mathbb{R}$; hence, w is doubling. Moreover, if $0 \leq \alpha < n(p-1)$ and $\beta \geq 0$, then w satisfies $w(B(x, t)) \geq Ct^{n+\alpha}$ for all $0 < t < \infty$ and all x .

Lemma 1. Let $w \in A_p$ and d -regular. Then, we have

$$\|M_\alpha^\#(f)\|_{p,w} \leq \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + d}} w(x) w(y) dx dy \right)^{1/p}, \quad 1 \leq p < \infty. \quad (35)$$

Proof. Let Q be a cube and $x, y \in Q$. Then,

$$\begin{aligned} |f(y) - f_Q| &= \left| f(y) - \frac{1}{|Q|} \int_Q f(z) dz \right| \leq \frac{1}{|Q|} \int_Q |f(y) - f(z)| dz \\ &\leq |f(y) - f(x)| + \frac{1}{|Q|} \int_Q |f(z) - f(x)| dz. \end{aligned} \quad (36)$$

Integrating over the cube Q with respect to y , we get

$$\int_Q |f(y) - f_Q| dy \leq \int_Q |f(y) - f(x)| dy. \quad (37)$$

If $w \in A_1$, then we have for almost all $x \in Q$, $w(x) \geq (1/|Q|) \int_Q w(y) dy \geq |Q|^{-1+(d/n)}$. Hence,

$$\int_Q |f(y) - f_Q| dy \leq |Q|^{1-(d/n)} \int_Q |f(y) - f(x)| w(y) dy. \quad (38)$$

The last inequality implies that

$$\begin{aligned} \frac{1}{|Q|^{(\alpha/n)+1}} \int_Q |f(y) - f_Q| dy &\leq |Q|^{(-\alpha-d)/n} \int_Q |f(y) - f(x)| w(y) dy \\ &\leq \int_{\mathbb{R}^n} \frac{|f(y) - f(x)|}{|y - x|^{\alpha+d}} w(y) dy. \end{aligned} \quad (39)$$

On the other hand, if $w \in A_p$, $p > 1$; then using (37) and Hölder's inequality, we obtain

$$\begin{aligned}
\int_Q |f(y) - f_Q| dy &\leq \left(\int_Q |f(y) - f(x)|^p w(y) dy \right)^{1/p} \left(\int_Q w^{1-p'}(y) dy \right)^{(p-1)/p} \\
&\leq |Q|^{1-(d/np)} \left(\int_Q |f(y) - f(x)|^p w(y) dy \right)^{1/p} \\
&\leq |Q|^{1-(d/np)} \left(\int_Q |f(y) - f(x)|^p w(y) dy \right)^{1/p} \\
&\leq |Q|^{1+(a/n)} \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + d}} w(y) dy \right)^{1/p}.
\end{aligned} \tag{40}$$

Therefore, we conclude that

$$M_\alpha^\#(f)(x) \leq \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + d}} w(y) dy \right)^{1/p}, \quad 1 \leq p < \infty. \tag{41}$$

This completes the proof. \square

Lemma 2. We say that f is in the fractional Sobolev space $\dot{W}^{\alpha,p}(w)$, $0 < \alpha < 1$, $1 \leq p < \infty$, if

$$\|f\|_{\dot{W}^{\alpha,p}(w)} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + d}} w(x) w(y) dx dy \right)^{1/p} < \infty. \tag{42}$$

Corollary 2. Let $w \in A_{p,\alpha}$ and lower d -regular, $0 < \alpha < 1$, $1 \leq p < \infty$, and $f \in \dot{W}^{\alpha,p}(w)$. Then,

$$\|M_\alpha^\#(f)\|_{p,w} \leq \|f\|_{\dot{W}^{\alpha,p}(w)}. \tag{43}$$

One can immediately obtain the following corollary.

Corollary 3. Let α and γ be real numbers satisfying $0 < \alpha < 1$ and $\gamma < \alpha$. Assume $w \in A_p$ and $f \in \dot{W}^{\alpha,p}(w)$ with $(d/(n+\alpha)) < p < (d/(\alpha-\gamma))$. Then, for each $0 < q \leq \infty$,

$$\|f\|_{\dot{F}_{p,w}^{\gamma,q}} \leq \|f\|_{\dot{W}^{\alpha,p}(w)}, \tag{44}$$

where p_* is given by $(1/p_*) = (1/p) - ((\alpha - \gamma)/d)$.

Recall that for $0 < \alpha < 1$, $1 < p < \infty$ and $w \in A_{\infty}$; we have (see [17])

$$\|f\|_{\dot{F}_{p,w}^{\alpha,\infty}} \approx \|M_\alpha^\#(f)\|_{p,w}. \tag{45}$$

In particular,

$$\|f\|_{\dot{F}_{p,w}^{\gamma,q}} \leq \|f\|_{\dot{F}_{p,w}^{\alpha,\infty}}, \tag{46}$$

with $0 < \alpha < 1$, $1 < p < \infty$, and p_* is as before.

Lemma 3. Let $f \in W_{loc}^{1,1}(\mathbb{R}^n)$ and $0 \leq \alpha \leq 1$. Then, for every $x \in \mathbb{R}^n$, there is a constant $C(n)$ such that

$$M_\alpha^\#(f)(x) \leq C(n) M_{1-\alpha}(|\nabla f|)(x). \tag{47}$$

Proof. The proof is an immediate consequence of the well-known Poincaré inequality.

For all ball $B(x, R)$ and all $f \in W_{loc}^{1,1}(\mathbb{R}^n)$, there is a constant $C(n)$ such that

$$\int_{B(x,R)} |f(y) - f_{B(x,R)}| dy \leq C(n) R \int_{B(x,R)} |\nabla f(y)| dy \tag{48}$$

holds. \square

Corollary 4. Let f be a locally integrable function such that $|\nabla f| \in H_r(w)$ and $0 < p_* \leq \infty$ are determined by

$$\frac{1}{p_*} = \frac{1}{r} + \frac{\gamma - 1}{d}. \tag{49}$$

Then, f is in $\dot{F}_{p_*,w}^{\gamma,q}$. Moreover, we have

$$\|f\|_{\dot{F}_{p_*,w}^{\gamma,q}} \leq \|\nabla f\|_{H_r(w)}. \tag{50}$$

Proof. Let $P_t(x) = (c_n t / (t^2 + |x|^2))^{(n+1)/2}$ be the Poisson kernel with the constant $C(n)$ such that $\int_{\mathbb{R}^n} P_t(x) dx = 1$. Then, there exists a constant $C = C(n)$ such that $Mf(x) \leq C \sup_{t>0} P_t * |f|$. In fact, if Q is a cube with $\text{diam}(Q) = t$ and $x \in Q$, then we have

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |f(y)| dy &= \frac{1}{|Q|} \int_Q \frac{(t^2 + |x - y|^2)^{(n+1)/2}}{C(n)t} \\
&\quad \cdot P_t(x - y) |f(y)| dy \\
&\leq C(n) \frac{t^n}{|Q|} \int_Q P_t(x - y) |f(y)| dy \\
&\leq C(n) P_t * |f|.
\end{aligned} \tag{51}$$

Thus, we have

$$Mf(x) \leq C \sup_{t>0} P_t * |f(x)|. \tag{52}$$

Using Lemma 3 with $\alpha = 1$ and Proposition 2 (see below), we obtain

$$\|f\|_{\dot{F}_{p_*,w}^{\gamma,q}} \leq \|M_1^\#(f)\|_{r,w} \leq \left\| \sup_{t>0} P_t * |\nabla f| \right\|_{r,w} \leq \|\nabla f\|_{H_r(w)}. \quad (53)$$

Remark 3. If we take $\gamma = 0$, $q = 2$, $r > 1$, and $w = 1$, in Corollary 4, we obtain the classical Sobolev–Gagliardo–Nirenberg inequality:

$$\|f\|_{p_*} \leq \|\nabla f\|_r, \quad (54)$$

with

$$\frac{1}{p_*} = \frac{1}{r} - \frac{1}{n}. \quad (55)$$

3. Some Useful Lemmas

We start this section with some useful lemmas that will be helpful in proving our main result.

Lemma 4 (see [7]). *Provided $\gamma < 1$, $\lambda > 0$, and $0 < q \leq 1$, there exist Schwartz functions v and μ on \mathbb{R}^n such that*

- (1) $\text{supp } v \subset B(0, 1)$ and $\hat{v}(0) = 0$
- (2) $\text{supp } \hat{\mu} \subset \{(1/2) \leq |\xi| \leq 2\}$ and $\hat{\mu}(\xi) \geq c > 0$ on $\{(3/5) \leq |\xi| \leq (25/3)\}$
- (3) $\sum_{j \in \mathbb{Z}} 2^{j\gamma q} |\mu_{2^{-j}}^* f|^q \leq C \sum_{j \in \mathbb{Z}} 2^{j\gamma q} |v_{2^{-j}}^* f|^q$

Lemma 5. *Assume that $w(B(x, t)) \geq C_w t^d$ for each $x \in \mathbb{R}^n$ and each $t > 0$, and let $v \in S$ supported on $B(0, 1)$ such that*

$$\int_{\mathbb{R}^n} v(x) dx = 0. \quad (56)$$

Fix a large $\lambda > 0$, and define

$$v_t^* f(x) = \sup_{y \in \mathbb{R}^n} \frac{|v_t * f(y)|}{(1 + (|x - y|/t))^\lambda}. \quad (57)$$

Then,

$$v_t^* f(x) \leq \min \left(s^\alpha M_\alpha^\#(f)(x), s^{\alpha-(d/p)} \|M_\alpha^\#(f)\|_{p,w} \right). \quad (58)$$

Proof. We adapt here the proof given in [7] in the unweighted case. Use the well-known estimate

$$\left(1 + \frac{|x - z|}{s} \right)^{-\lambda} \leq \sum_{k=1}^{\infty} 2^{-k\lambda} \chi \left(\frac{|x - z|}{2^k s} \right), \quad \lambda > 0, \quad (59)$$

where χ denotes the characteristic function of the interval $[0, 1]$, to obtain, for any $\lambda > 0$,

$$v_s^* f(x) \leq \sum_{k=1}^{\infty} 2^{-k\lambda} \sup_{z \in \mathbb{R}^n} |v_s * f(z)| \chi \left(\frac{|x - z|}{2^k s} \right). \quad (60)$$

By taking any $z \in B(x, 2^k s)$ and using the fact that v is supported in the unit ball and has mean equal zero, we obtain

$$\begin{aligned} |v_s * f(z)| &\leq \int_{B(z, (1+2^k)s)} |v_s(z - y)| \left(f(y) - f_{B(z, (1+2^k)s)} \right) dy \\ &\leq s^\alpha (1 + 2^k)^{n+\alpha} M_\alpha^\#(f)(x), \end{aligned} \quad (61)$$

which holds. Hence,

$$v_s^* f(x) \leq s^\alpha \sum_{k=1}^{\infty} 2^{-k\lambda} (1 + 2^k)^{n+\alpha} M_\alpha^\#(f)(x). \quad (62)$$

If we choose λ large enough, we obtain

$$v_s^* f(x) \leq s^\alpha M_\alpha^\#(f)(x). \quad (63)$$

On the other hand, by (61), we have for any fixed $x \in B(z, s)$,

$$|v_s * f(z)| \leq s^\alpha M_\alpha^\#(f)(x). \quad (64)$$

Rising (65) to the p th power and integrating over the ball $B(z, s)$ with respect to $w(x)dx$, one has that

$$\begin{aligned} |v_s * f(z)| &\leq w \left(\left(B(z, s)^{-(1/p)} s^\alpha \|M_\alpha^\#(f)\|_{L_p(w)} \right)^p \right. \\ &\quad \left. \leq s^{\alpha-(d/p)} \|M_\alpha^\#(f)\|_{p,w}^p \right). \end{aligned} \quad (65)$$

By using (60), we obtain

$$v_s^* f(x) \leq s^{\alpha-d/p} \|M_\alpha^\#(f)\|_{p,w}. \quad (66)$$

Proof. Proof of Theorem 1. □

Proof. We consider only the case when $0 < q \leq 1$. In the case when $1 < q \leq \infty$, estimate (18) follows from the case $q = 1$ by the embedding

$$\dot{F}_{p,w}^{\gamma,q_0} \subset \dot{F}_{p,w}^{\gamma,q_1}, \quad 0 < q_0 \leq q_1 \leq \infty. \quad (67)$$

Let $k > 0$ be chosen later and let μ and v be as in Lemma 4. Assume $0 < ((\alpha - \gamma < d)/p)$ and $0 < q \leq 1$. Then, using (58), we get

$$\begin{aligned}
\sum_{j \in \mathbb{Z}} 2^{j\gamma q} (\mu_{2^{-j}}^* f(x))^q &\leq \sum_{j \in \mathbb{Z}} 2^{j\gamma q} (\nu_{2^{-j}}^* f(x))^q \\
&\leq M_\alpha^\#(f)(x) \sum_{j \leq k} 2^{-j(\alpha-\gamma)q} + \|M_\alpha^\#(f)\|_{p,w} \sum_{j > k} 2^{-j((\alpha-\gamma-d)/p)q} \\
&\leq 2^{-k(\alpha-\gamma)q} M_\alpha^\#(f)(x) + 2^{-k((\alpha-\gamma-d)/p)q} \|M_\alpha^\#(f)\|_{p,w}.
\end{aligned} \tag{68}$$

Choose $2^{-k} = (M_\alpha^\#(f)(x)/\|M_\alpha^\#(f)\|_{p,w})$ to deduce that

$$\begin{aligned}
\left(\sum_{j \in \mathbb{Z}} 2^{j\gamma q} (\mu_{2^{-j}}^* f(x))^q \right)^{1/q} &\leq \left(M_\alpha^\#(f)(x) \right)^{p/p_*} \\
&\cdot \left(\|M_\alpha^\#(f)\|_{p,w} \right)^{1-(p/p_*)},
\end{aligned} \tag{69}$$

where p_* is given by $(1/p_*) = (1/p) - ((\alpha-\gamma)/d)$. Thus, we have

$$\|f\|_{\dot{F}_{p,w}^{\gamma,q}} \leq \|M_\alpha^\#(f)\|_{p,w}. \tag{70}$$

4. Some Extensions

In this section, we will assume that μ is a nonnegative Borel doubling measure on \mathbb{R}^n ; there exists $\beta = \beta(\mu) > 0$ such that

$$\mu(B_{2r}) \leq 2^\beta \mu(B_r), \tag{71}$$

for all ball B_r . The smallest such β is called a doubling constant of μ .

For each $N \in \mathbb{N} \cup \{-1\}$, $m \in \mathbb{N}_0$, and $l \in \mathbb{N}$, we set

$$\mathcal{A} = \mathcal{A}_{N,m}^l = \{ \nu \in \mathcal{S}_N(\mathbb{R}^n) : \|\nu\|_{m,N+l+1} \leq 1 \}. \tag{72}$$

Definition 3. Let $\gamma \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$. The homogeneous grand Triebel–Lizorkin space is the set of all tempered functions f such that when $0 < q < \infty$,

$$\|f\|_{\mathcal{A}\dot{F}_{p,\mu}^{\gamma,q}} = \left\| \left(\sum_{j \in \mathbb{Z}} 2^{j\gamma q} \sup_{v \in \mathcal{A}} |\nu_{2^{-j}} f|^q \right)^{1/q} \right\|_{p,\mu} < \infty \tag{73}$$

and when $q = \infty$,

$$\|f\|_{\mathcal{A}\dot{F}_{p,\mu}^{\gamma,\infty}} = \left\| \sup_{j \in \mathbb{Z}} 2^{j\gamma} \sup_{v \in \mathcal{A}} |\nu_{2^{-j}} f| \right\|_{p,\mu} < \infty. \tag{74}$$

Proposition 1. Let $\gamma \in \mathbb{R}$, $0 < p \leq \infty$, and $0 < q \leq \infty$, and μ is the doubling measure with a constant equal to β . Set $J = n\beta \max(1, (1/p), (1/q))$. If $\mathcal{A} = \mathcal{A}_{N,m}^l$ with $l \in \mathbb{N}$, $N+1 > \max(\gamma, J-n-\gamma)$, and $m > \max(J, n+N+1)$, then

$$\|f\|_{\mathcal{A}\dot{F}_{p,\mu}^{\gamma,q}} = \|f\|_{\dot{F}_{p,\mu}^{\gamma,q}}. \tag{75}$$

Proof. Arguing as in the proof in ([15], Theorem 1.2) and using the almost-diagonality theorem (see [1], Theorem 4.2), we obtain the desired result. \square

Proposition 2. Let α and γ be real numbers satisfying $0 \leq \alpha \leq 1$ and $\gamma < \alpha$ and μ be a lower d -regular doubling measure. Assume f is a smooth function and $M_\alpha^\#(f) \in L^p(w)$ with $(d/(n+\alpha)) < p < (d/(\alpha-\gamma))$. Then, for each $0 < q \leq \infty$,

(1)

$$\|f\|_{\mathcal{A}\dot{F}_{p,\mu}^{\gamma,q}} \leq \|M_\alpha^\#(f)\|_{p,\mu}, \tag{76}$$

where p_* is given by $(1/p_*) = (1/p) - ((\alpha-\gamma)/d)$

(2) For all $(n/(n+\alpha)) < p \leq \infty$ and $0 \leq \alpha < \infty$,

$$\|f\|_{\mathcal{A}\dot{F}_{p,\mu}^{\alpha,\infty}} \leq \|M_\alpha^\#(f)\|_{p,\mu}. \tag{77}$$

(3) For all $(n/(n+\alpha)) < p \leq \infty$, $0 \leq \alpha < \infty$,

$$\|f\|_{\mathcal{A}\dot{F}_{\infty,\mu}^{\gamma,((\alpha-d)/p),\infty}} \leq \|M_\alpha^\#(f)\|_{p,\mu}. \tag{78}$$

Proof. We have from (58) that if μ is a lower d -regular measure, then

$$\sup_{v \in \mathcal{A}} |\nu_{2^{-j}} f| \leq \min \left(2^{j\alpha} M_\alpha^\#(f)(x), 2^{j((\alpha-d)/p)} \|M_\alpha^\#(f)\|_{p,\mu} \right). \tag{79}$$

Arguing as in the proof of Proposition 1, we obtain the desired result easily. \square

Definition 4. Let μ be a doubling measure $0 < p < \infty$ and $0 < \alpha \leq 1$. The homogeneous fractional Hajlasz–Sobolev space $\dot{M}_\mu^{\alpha,p}(\mathbb{R}^n)$ is the set of all measurable functions $L_{\mu,\text{loc}}^p$ for which there exists a nonnegative function $g \in L_\mu^p$ such that

$$|f(x) - f(y)| \leq |x - y|^\alpha [g(x) + g(y)], \tag{80}$$

for μ -a.e. $x, y \in \mathbb{R}^n$.

$\dot{M}_\mu^{\alpha,p}(\mathbb{R}^n)$ is equipped with the seminorm

$$\|f\|_{\dot{M}_\mu^{\alpha,p}} = \inf_{g \in D(f)} \|g\|_{p,\mu}, \tag{81}$$

where $D(f)$ denotes the class of all nonnegative Borel functions g satisfying (80). Thus, Lemma 4.1 in [15] implies the following Sobolev embedding.

Lemma 6. Let $0 < \alpha \leq 1$, $0 < \delta < (n/\alpha)$, and p_* be given by $(1/p_*) = (1/\delta) - (\alpha/n)$. Then, for all $x \in \mathbb{R}^n$, $0 < r < \infty$, $f \in \dot{M}_\mu^{\alpha,p}(\mathbb{R}^n)$, and $g \in D(f)$,

$$\begin{aligned} & \inf_{c \in \mathbb{R}} \left(\frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - c|^{p_*} d\mu \right)^{1/p_*} \\ & \leq r^\alpha \left(\frac{1}{\mu(B(x, 2r))} \int_{B(x, 2r)} g(y)^\delta d\mu \right)^{1/\delta}. \end{aligned} \quad (82)$$

Remark 4. Lemma 6 is due to Hajlasz ([13], Theorem 8.7) when $\alpha = 1$.

Corollary 5. Let α, γ , and δ be real numbers satisfying $0 \leq \alpha \leq 1$, $\gamma < \alpha$, and $(n/(n + \alpha)) < p < (n/(\alpha - \gamma))$. Assume $f \in \dot{M}_\mu^{\alpha,p}(\mathbb{R}^n)$. Then, for each $0 < q \leq \infty$,

$$\|f\|_{\dot{F}_{p,q}^{\gamma,\alpha}} \leq \|f\|_{\dot{M}_\mu^{\alpha,p}}, \quad (83)$$

where $(1/p_*) = (1/p) - ((\alpha - \gamma)/n)$.

Proof. Fix a ball $B(x, 2r)$. Then, using Lemma 6 and by taking $\delta = (n/(n + \alpha))$ and Hölder's inequality, we obtain

$$\begin{aligned} & \inf_{c \in \mathbb{R}} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - c| d\mu \\ & \leq r^\alpha \left(\frac{1}{\mu(B(x, 2r))} \int_{B(x, 2r)} |g(y)|^\delta d\mu \right)^{1/\delta}. \end{aligned} \quad (84)$$

Hence,

$$M_\alpha^\#(f)(x) \leq (M_\mu(g^\delta))^{1/\delta}(x) \quad (85)$$

holds, where $M_\mu(g)$ is the maximal function with respect to the measure μ . The $L^{p/\delta}$ -boundedness of M_μ when $\delta < p < \infty$ and Proposition 2 lead to estimate (83). \square

Also, recall that $\dot{M}_\mu^{\alpha,p}(\mathbb{R}^n) = \dot{F}_{p,\mu}^{\alpha,\infty}(\mathbb{R}^n)$ for $0 < \alpha \leq 1$ and $(n/(n + \alpha)) < p < \infty$ in [15] and $\dot{M}_\mu^{1,p}(\mathbb{R}^n) = \dot{F}_{p,\mu}^{1,2}(\mathbb{R}^n) \approx \dot{H}_\mu^p$ for $(n/(n + 1)) < p < \infty$ in [14]. Here, \dot{H}_μ^p denotes, for $p > 0$, the homogeneous Hardy-Sobolev space, i.e., the space of tempered distributions f on \mathbb{R}^n , such that $\partial_j f \in H_\mu^p$ for each $j = 1, \dots, n$ and

$$\|f\|_{\dot{H}_\mu^p} = \sum_{j=1}^n \|\partial_j f\|_{H_{p,\mu}}. \quad (86)$$

Consequently, if $f \in \dot{H}_\mu^p$ with $(n/(n + 1)) < p < (n/(1 - \gamma))$, then

$$\|f\|_{\dot{F}_{p_*}^{\gamma,q}} \leq \|f\|_{\dot{H}_\mu^p}, \quad \frac{1}{p_*} = \frac{1}{p} - \frac{1 - \gamma}{n}. \quad (87)$$

In particular, we have, for $(n/(n + 1)) < p < n$, the following well-known result:

$$\|f\|_{p_*} \leq \|f\|_{\dot{H}^p}, \quad \frac{1}{p_*} = \frac{1}{p} - \frac{1}{n}. \quad (88)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.







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Research Article

A Study of New Class of Star-Like Functions Associated by Symmetric (p, q) -Calculus

Khalid Akbar ¹, Rashid Murtaza ¹, Adnan ¹, Umar Khan ², Ilyas Khan ³,
and Md. Fayz-Al-Asad ⁴

¹Department of Mathematics, Mohi-Ud-Din Islamic University, Nerian Sharif AJ&K, Trarkhel, Pakistan

²Department of Mathematics and Statistics, Hazara University Mansehra, Dhodial, Pakistan

³Department of Mathematics, College of Science, Al-Zulfi Majmaah University, Al-Majmaah 11952, Saudi Arabia

⁴Department of Mathematics, Bangladesh University of Engineering and Technology (BUET), Dhaka 1000, Bangladesh

Correspondence should be addressed to Md. Fayz-Al-Asad; fayzmath.buet@gmail.com

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As of late quantum calculus is broadly utilized in different parts of mathematics. Uniquely, the hypothesis of univalent functions can be newly portrayed by utilizing q -calculus. In this paper, we utilize our recently presented symmetric (p, q) -number $[\tilde{m}]_{(p,q)}$ to characterize new symmetric (p, q) -derivative $\mathfrak{D}_{(p,q)}$ of analytic function f in the open unit disk \mathbb{U} . Utilizing $\mathfrak{D}_{(p,q)}$, we introduce new class of analytic star-like functions and examine some fascinating results.

1. Introduction

The mathematical study of q -calculus has been a topic of great interest for researchers due to its wide applications in different fields. Some of the earlier work on the applications of q -calculus was introduced by Jackson [1]. Later on, q -calculus attained much popularity among the researchers. Recently, q -calculus has gained the attention of researchers because of its huge applications in mathematics and physics. The in-depth analysis of q -calculus was firstly discussed by Jackson [1, 2], where he defined q -integral and q -derivative in a very systematic way. Recently, authors are using these q -integral and q -derivative to define new subclasses of the class of univalent functions and obtained variety of new results. Extending the idea of q -number, which contains only one variable q , the (p, q) -number which contains two independent parameters p and q was independently considered by Chakrabarti and Jagannathan [3]. As q -calculus or quantum calculus originated by using q -number, similarly by using (p, q) -number, the (p, q) -calculus or postquantum calculus has been studied and discussed by several researchers (see, for example, Duran et al. [4] and references

therein). Let $f \in \mathcal{C}$. Furthermore, f is normalized analytic, if f is single valued and differentiable for $z \in \mathbb{U}$ along $f(0) = 0$, $f'(0) = 1$ and is represented as

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m. \quad (1)$$

The symbol \mathfrak{A} is used for this type of functions. Let $f \in \mathfrak{A}$ be given by (1). Then,

$$f \text{ is univalent} \iff \lambda_1 \neq \lambda_2 \implies f(\lambda_1) \neq f(\lambda_2) \quad \forall \lambda_1, \lambda_2 \in \mathbb{U}. \quad (2)$$

Assume that $\tilde{p} \in \mathcal{C}$ is analytic. Furthermore,

$$\wp \in \mathfrak{P} \iff \operatorname{Re}(\wp(z)) > 0 \text{ along } \wp(0) = 1, \quad (3)$$

and is presented as

$$\wp(z) = 1 + \sum_{m=1}^{\infty} c_m z^m. \quad (4)$$

Let $D \subset \mathbb{C}$. If w_0 is a constant in D and for all z_0 in D , we have

$$w_0 \text{ and } \forall z_0 \in D \implies (1-t)w_0 + tz_0 \in D, \quad 0 \leq t \leq 1. \quad (5)$$

At that point, we state that D is a star-formed region. Geometrically, $f \in \mathfrak{A}$ is star-like if $f(\mathbb{U})$ is a star-shaped region. We meant by \mathfrak{S}^* the family of these functions. Analytically, we attain

$$f \in \mathfrak{S}^* \iff \frac{zf'(z)}{f(z)} \in \mathfrak{P}. \quad (6)$$

Assume that $D \subset \mathbb{C}$. If $\zeta_1 \neq \zeta_2 \in D$, then

$$D \text{ is convex} \iff (1-t)\zeta_1 + t\zeta_2 \in D, \quad 0 \leq t \leq 1. \quad (7)$$

Geometrically,

$$f \in \mathfrak{A} \text{ is convex} \iff f(\mathbb{U}) \text{ is convex region.} \quad (8)$$

The family containing all the convex function is represented as \mathfrak{C} . Analytically $f \in \mathfrak{C} \iff ((zf')'/f') \in \mathfrak{P}$,

Note

$$f \in \mathfrak{C} \iff zf' \in \mathfrak{S}^*. \quad (9)$$

Let f be presented as in (1), and the convolution $(f * g)$ is characterized as

$$(g * f)(z) = \sum_{m=1}^{\infty} a_m b_m z^m = (f * g)(z), \quad z \in \mathbb{U}, \quad (10)$$

where

$$g(z) = z + b_2 z^2 + \dots = \sum_{m=1}^{\infty} b_m z^m. \quad (11)$$

Let h_1, h_2 be two functions. Then, $h_1 < h_2 \iff \exists$, if there exist a Schwarz function ω analytic along $\omega(0) = 0, |\omega(z)| < 1$, such that $h_1(z) = (h_2 \circ \omega)(z)$. It can be found in [5] that if $h_2 \in \mathfrak{S}$, then

$$\begin{aligned} h_1(0) &= h_2(0), \\ h_1(\mathbb{U}) &\subset h_2(\mathbb{U}) \iff h_1 < h_2, \end{aligned} \quad (12)$$

and for details, see [6].

Kanas and Wisniowska [7] presented the conic regions $\Lambda_k, k \in (0, \infty)$ by

$$\Lambda_k = \left\{ \tilde{w} = \tilde{\mu} + i\tilde{\nu} \in \mathbb{C} : \tilde{\mu} > k\sqrt{(\tilde{\mu}-1)^2 + \tilde{\nu}^2} \right\}, \quad (13)$$

with $1 \in \Lambda_k$. Assume

$$\partial\Lambda_k = \left\{ \tilde{w} = \tilde{\mu} + i\tilde{\nu} \in \mathbb{C} : \tilde{\mu}^2 = k^2(\tilde{\mu}-1)^2 + \tilde{\nu}^2 \right\}. \quad (14)$$

Therefore,

$$k = 0 \implies \Lambda_0 = \tilde{\mu} > 0, \text{ right half plane,}$$

$$k = 1 \implies \Lambda_1 = \tilde{\nu}^2 < 2\tilde{\mu} - 1, \text{ parabolic region,}$$

$$k \in (0, 1) \implies \Lambda_k = \tilde{\mu} > k\sqrt{(\tilde{\mu}-1)^2 + \tilde{\nu}^2}, \text{ hyperbolic region.} \quad (15)$$

Identified with the region Λ_k , the accompanying functions are extremal and $p_k(\mathbb{U}) \subset \Lambda_k$:

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2} \left(\log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2, & k = 1, \\ 1 + \frac{2}{1-k^2} \sin^2 \left[\left(\frac{2}{\pi} \arccos k \right) \arctan h\sqrt{z} \right], & 0 < k < 1. \end{cases} \quad (16)$$

The families $k - \mathfrak{UCB}$ and $k - \mathfrak{CS}$ are fascinating subfamilies of \mathfrak{S} and are characterized in [8] for all $k \geq 0$, $z \in \mathbb{U}$, as

$$\begin{aligned} f \text{ belongs to } k - \mathfrak{UCB} &\iff \operatorname{Re} \left[1 + \frac{zf'(z)}{f'(z)} \right] > k \left| \frac{zf''(z)}{f'(z)} \right|, \\ f \text{ belongs to } k - \mathfrak{CS} &\iff \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] > k \left| \frac{zf'(z)}{f(z)} - 1 \right|. \end{aligned} \quad (17)$$

Note from [8] that

$$f \in k - \mathfrak{UCB} \iff zf' \in k - \mathfrak{CS}. \quad (18)$$

Geometrically,

$$f \in k - \mathfrak{UCB} \iff \left[\frac{(zf'(z))'}{f'(z)} \right](\mathbb{U}) \subset \Lambda_k. \quad (19)$$

2. Postquantum Calculus

Extending the idea of q -number, the (p, q) -number with two variables p and q was independently considered in [3].

For $0 < q < p \leq 1$, the twin basic number or (p, q) -number is defined for $m \in \mathbb{N}$ as

$$[m]_{(p,q)} = \frac{p^m - q^m}{p - q} = p^{m-1} + p^{m-2}q + \cdots + pq^{m-2} + q^{m-1}, \quad (20)$$

which is the normal speculation of q -number with the end goal that for $p \rightarrow 1^-$, $q \rightarrow 1^-$ we have

$$\begin{aligned} [m]_{(p,q)} &= [m]_{(1,q)} = [m]_q, \\ \text{and} \quad (21) \\ [m]_{(p,q)} &= [m]_{(1,1)} = m. \end{aligned}$$

Furthermore, note that $[m]_{(p,q)} = [m]_{(q,p)}$.

In 1991, Chakrabarti and Jagannathan [3] characterized (p, q) -derivative of $f \in \mathfrak{A}$ as

$$D_{(p,q)}f(z) = \begin{cases} \frac{f(pz) - f(qz)}{(p - q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad (22)$$

Furthermore, if $f(z) = z^m$, then $D_{(p,q)}f(z) = [m]_{(p,q)}z^{m-1}$, and for (1), we have

$$D_{(p,q)}f(z) = 1 + \sum_{m=2}^{\infty} [m]_{(p,q)} a_m z^{m-1}, \quad z \in \mathbb{U}. \quad (23)$$

We remark here as

$$p \rightarrow 1^- \implies D_{(p,q)}f(z) = D_qf(z), \quad (24)$$

$$\begin{aligned} p \rightarrow 1^-, q \rightarrow 1^- &\implies [m]_{(p,q)} = m, \\ D_{(p,q)}f(z) &= f'(z). \end{aligned} \quad (25)$$

We now extend the idea in [9] and define symmetric (p, q) -number as follows:

$$[\tilde{m}]_{(p,q)} = \frac{p^m - q^{-m}}{p - q^{-1}}, \quad 0 < q \leq p < 1. \quad (26)$$

Analogous to (p, q) -derivative defined by (22), the symmetric (p, q) -derivative is characterized as

$$\mathfrak{D}_{(p,q)}f(z) = \begin{cases} \frac{f(pz) - f(q^{-1}z)}{(p - q^{-1})z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad (27)$$

If

$$f(z) = z^m \implies \mathfrak{D}_{(p,q)}f(z) = [\tilde{m}]_{p,q} z^{m-1}, \quad \forall z \in \mathbb{U}. \quad (28)$$

In this manner for (1), we acquire

$$\mathfrak{D}_{(p,q)}f(z) = 1 + \sum_{m=2}^{\infty} [\tilde{m}]_{(p,q)} a_m z^{m-1}, \quad z \in \mathbb{U}. \quad (29)$$

Equivalently, by using the same technique as in [9], it can be seen that

$$\begin{aligned} \mathfrak{D}_{(p,q)}(f(z) + g(z)) &= \mathfrak{D}_{(p,q)}f(z) + \mathfrak{D}_{(p,q)}g(z), \\ \mathfrak{D}_{(p,q)}(f(z)g(z)) &= g(pz)\mathfrak{D}_{(p,q)}f(z) + f(q^{-1}z)\mathfrak{D}_{(p,q)}g(z) \\ &= g(q^{-1}z)\mathfrak{D}_{(p,q)}f(z) + f(pz)\mathfrak{D}_{(p,q)}g(z), \\ \mathfrak{D}_{(p,q)}\left(\frac{f(z)}{g(z)}\right) &= \frac{g(q^{-1}z)\mathfrak{D}_{(p,q)}f(z) - f(q^{-1}z)\mathfrak{D}_{(p,q)}g(z)}{g(pz)g(q^{-1}z)} \\ &= \frac{g(pz)\mathfrak{D}_{(p,q)}f(z) - f(pz)\mathfrak{D}_{(p,q)}g(z)}{g(pz)g(q^{-1}z)}. \end{aligned} \quad (30)$$

Assume that \mathfrak{T} denotes the subfamily of \mathfrak{A} with negative coefficients such as

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m, \quad z \in \mathbb{U}. \quad (31)$$

For further developments about class \mathfrak{T} and its subclasses, one can refer to a wide range of extraordinary articles

written by famous mathematicians (see [10–13] and references therein).

Presently, utilizing the symmetric (p, q) -derivative characterized by (27) of f , we present another subclass of \mathfrak{A} as follows.

Definition 1. Let $0 \leq k \leq \infty, 0 \leq \eta < 1$ and $0 < q \leq p < 1$. Assume that $f \in \mathfrak{A}$ characterized by (1). Furthermore,

$$f \in k - \widetilde{\mathfrak{ST}}_{(p,q)}(\eta) \iff k \left| \frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} - 1 \right| + \eta < \operatorname{Re} \left[\frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} \right], \quad z \in \mathbb{U}. \quad (32)$$

Geometrically,

$$\left[\frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} \right](\mathbb{U}) \subset \Lambda_{k,\eta} \iff f \in k - \widetilde{\mathfrak{ST}}_{(p,q)}(\eta), \quad (33)$$

where $\Lambda_{k,\eta}$ is given by

$$\Lambda_{k,\eta} = (1 - \eta) + \eta \Lambda_k, \quad (34)$$

and Λ_k is defined as in (13).

Utilizing the functions $p_{k,\eta}$ defined in [14], we have

$$\left(\frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} \right) < p_{k,\eta} \iff f \in k - \widetilde{\mathfrak{ST}}_{(p,q)}(\eta), \quad z \in \mathbb{U}. \quad (35)$$

Remark 1. Likewise, we set $k - \widetilde{\mathfrak{ST}}_{(p,q)}(\eta) = k - \mathfrak{ST}_{(p,q)}(\eta) \cap \mathfrak{T}$.

Note that on the off chance that $p \rightarrow 1^-, q \rightarrow 1^-$, at that point

$$k - \widetilde{\mathfrak{ST}}_{(p,q)}(\eta) = k - \mathfrak{ST}(\eta). \quad (36)$$

3. Main Results

Now in this segment, we will demonstrate our principle results. It is worthy to mention here that our main results are extension of results studied by Kanas et al. [9]. We utilize symmetric (p, q) -derivative operator to obtain these results.

The accompanying lemmas are useful to prove main result.

Lemma 1. Assume that $r \in (0, 1)$ and $\beta > 0$. Furthermore, f and g are analytic in \mathbb{U} along $f < g$. Then,

$$\int_0^{2\pi} |f(re^{i\theta})|^\beta d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\beta d\theta, \quad (37)$$

$$z = re^{i\theta},$$

and for details, see [15].

Lemma 2. For the sequence $\{b_m\}_{m=1}^\infty$ and $\forall z \in \mathbb{U}$,

$$\operatorname{Re} \left(1 + 2 \sum_{m=1}^\infty b_m z^m \right) > 0 \iff \{b_m\}_{m=1}^\infty, \quad (38)$$

is a subordinating factor sequence (see [15]).

Now we will extend the existing results of [13] with the help of symmetric (p, q) -derivative.

Theorem 1. Assume that $0 < q \leq p < 1$ and let $f \in \mathfrak{S}$ be defined by (1). If the inequality

$$\sum_{m=2}^\infty ([\tilde{m}]_{(p,q)}(k+1) - (k+\eta)) |a_m| \leq (1-\eta), \quad (39)$$

holds true, then $f \in k - \widetilde{\mathfrak{ST}}_{(p,q)}(\eta)$.

Proof. To prove (39), it is adequate to prove that

$$k \left| \frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} - 1 \right| - \operatorname{Re} \left(\frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} - 1 \right) < (1-\eta). \quad (40)$$

Consider

$$\begin{aligned} & k \left| \frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} - 1 \right| - \operatorname{Re} \left(\frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} - 1 \right) \\ & \leq k \left| \frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} - 1 \right| + \left| \frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} - 1 \right| \\ & = (k+1) \left| \frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} - 1 \right|. \end{aligned} \quad (41)$$

Now using (1) and (29), we obtain

$$\begin{aligned} & k \left| \frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} - 1 \right| - \operatorname{Re} \left(\frac{z \mathfrak{D}_{(p,q)} f(z)}{f(z)} - 1 \right) \leq \left| \frac{\sum_{m=2}^\infty [\tilde{m}]_{p,q} a_m z^m - \sum_{m=2}^\infty a_m z^m}{z + \sum_{m=2}^\infty a_m z^m} \right| \\ & \leq (k+1) \left| \frac{\sum_{m=2}^\infty ([\tilde{m}]_{p,q} - 1) a_m z^m}{1 + \sum_{m=2}^\infty a_m z^m} \right| \\ & \leq (k+1) \frac{\sum_{m=2}^\infty ([\tilde{m}]_{p,q} - 1) |a_m|}{1 - \sum_{m=2}^\infty |a_m|} \\ & \leq (1-\eta). \end{aligned} \quad (42)$$

This can be composed as

$$(k+1) \sum_{m=2}^{\infty} ([\tilde{m}]_{(p,q)} - 1) |a_m| \leq (1-\eta) \left(1 - \sum_{m=2}^{\infty} |a_m| \right). \quad (43)$$

From (43), we obtain (39) and the proof is completed.

Remark 2. In Theorem 1, by putting $p = q$, we have the result presented in [9].

Corollary 1. Assume that $0 < q < 1$ and let $f \in \mathfrak{E}$ be presented as in (1). If

$$\sum_{m=2}^{\infty} ([\tilde{m}]_q (k+1) - (k+\eta)) |a_m| \leq (1-\eta), \quad (44)$$

holds true for $0 \leq k < \infty$, $0 \leq \eta < 1$, then $f \in k - \mathfrak{E}\mathfrak{T}_q(\eta)$.

Remark 3. It can be seen that the quantity

$$[\tilde{m}]_{(p,q)} (k+1) - (k+\eta) = \frac{p^m - q^{-m}}{p - q^{-1}} (k+1) - (k+\eta) > 0, \quad (45)$$

for all $0 \leq k < \infty$, $0 \leq \eta < 1$ and $0 < q \leq p < 1$, unless otherwise mentioned.

Now using (39), we can find some special members of $f \in k - \mathfrak{E}\mathfrak{T}_{(p,q)}(\eta)$. One of them is the following.

Corollary 2. Let for $0 \leq k < \infty$, $0 \leq \eta < 1$ and $0 < q \leq p < 1$. If, for $f(z) = z + a_m z^m$, the following inequality:

$$|a_m| \leq \frac{(1-\eta)}{[\tilde{m}]_{(p,q)} (k+1) - (k+\eta)}, \quad m \geq 2, \quad (46)$$

holds, then $f \in k - \mathfrak{E}\mathfrak{T}_{(p,q)}(\eta)$. Especially

$$f(z) = z + \frac{(1-\eta)q}{(pq+1)(k+1) - (k+\eta)q} z^2 \in k - \mathfrak{E}\mathfrak{T}_{(p,q)}(\eta), \quad z \in \mathbb{U}. \quad (47)$$

For $p=q$, we obtain the following known results (see [9]).

Corollary 3. Let $0 \leq k < \infty$, $0 \leq \eta < 1$ and $0 < q < 1$. If, for $f(z) = z + a_m z^m$, the following inequality:

$$|a_m| \leq \frac{(1-\eta)}{[\tilde{m}]_q (k+1) - (k+\eta)}, \quad m \geq 2, \quad (48)$$

holds, then $f \in k - \mathfrak{E}\mathfrak{T}_q(\eta)$. Especially

$$f(z) = z + \frac{(1-\eta)q}{(q^2+1)(k+1) - (k+\eta)q} z^2 \in k - \mathfrak{E}\mathfrak{T}_q(\eta), \quad z \in \mathbb{U}. \quad (49)$$

Also, for $k - \mathfrak{E}\mathfrak{T}_{(p,q)}^-(\eta)$, we obtain the following.

Corollary 4. Let $0 \leq k < \infty$, $0 \leq \eta < 1$ and $0 < q \leq p < 1$. An essential and adequate condition of $f(z) = z - a_2 z^2 - \dots$, ($a_m \geq 0$) belongs to $k - \mathfrak{E}\mathfrak{T}_{(p,q)}^-(\eta)$, such as

$$\sum_{m=2}^{\infty} ([\tilde{m}]_{(p,q)} (k+1) - (k+\eta)) |a_m| \leq (1-\eta). \quad (50)$$

The accompanying function yields quality.

$$f(z) = z - \frac{1-\eta}{[\tilde{m}]_{(p,q)} (k+1) - (k+\eta)} z^m. \quad (51)$$

Proof. Proof immediately follows by using Theorem 1.

On the off chance that we take $p = q$ in Corollary 4, we have the accompanying corollary (see [9]).

Corollary 5. Let $0 \leq k < \infty$, $0 \leq \eta < 1$ and $0 < q < 1$. A necessary and sufficient condition for f of the form $f(z) = z - a_2 z^2 - \dots$, ($a_m \geq 0$), to be in the class $k - \mathfrak{E}\mathfrak{T}_q^-(\eta)$ is that

$$\sum_{m=2}^{\infty} ([\tilde{m}]_q (k+1) - (k+\eta)) |a_m| \leq (1-\eta). \quad (52)$$

The result is sharp and accompanying function yields quality.

$$f(z) = z - \frac{1-\eta}{[\tilde{m}]_q (k+1) - (k+\eta)} z^m. \quad (53)$$

Theorem 2. Let $0 \leq k < \infty$, $0 \leq \eta < 1$ and $0 < q \leq p < 1$. Assume $f(z) = z - a_2 z^2 - \dots$, ($a_m \geq 0$) belongs to $k - \mathfrak{E}\mathfrak{T}_{(p,q)}^-(\eta)$. Furthermore, $|z| = r < 1$ yields

$$\begin{aligned} r - \frac{q(1-\eta)}{(pq+1)(k+1) - q(k+\eta)} r^2 &\leq |f(z)| \\ &\leq r + \frac{q(1-\eta)}{(pq+1)(k+1) - q(k+\eta)} r^2. \end{aligned} \quad (54)$$

Accompanying function yields quality for (54).

$$f(z) = z + \frac{q(1-\eta)}{(pq+1)(k+1) - q(k+\eta)} z^2, \quad z \in \mathbb{U}. \quad (55)$$

Proof. Let $f \in k - \mathfrak{E}\mathfrak{T}_{(p,q)}^-(\eta)$. By using Theorem 1, we have

$$\begin{aligned} &([\tilde{2}]_{(p,q)} (k+1) - (k+\eta)) \sum_{m=2}^{\infty} a_m \\ &\leq \sum_{m=2}^{\infty} ([\tilde{m}]_{(p,q)} (k+1) - (k+\eta)) \\ &\leq (1-\eta). \end{aligned} \quad (56)$$

This means that

$$\sum_{m=2}^{\infty} a_m \leq \frac{(1-\eta)}{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}. \quad (57)$$

Furthermore if $|z| = r < 1$,

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{m=2}^{\infty} a_m |z|^m \\ &\leq r + \frac{q(1-\eta)}{(pq+1)(1-\eta)(k+1) - q(k+\eta)} r^2, \end{aligned} \quad (58)$$

and

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{m=2}^{\infty} a_m |z|^m \\ &\geq r - \frac{q(1-\eta)}{(pq+1)(1-\eta)(k+1) - q(k+\eta)} r^2. \end{aligned} \quad (59)$$

Combining (58) and (59), we obtain (54).

Theorem 3. Suppose that $f \in k - \widetilde{\mathfrak{TS}}_{(p,q)}(\eta)$. Moreover,

$$f_2(z) = z - \frac{(1-\eta)}{[\tilde{2}]_{(p,q)}(k+1) - (k+\eta)} z^2. \quad (60)$$

If

$$\begin{aligned} \forall z = re^{i\theta}, \\ 0 < r < 1 \implies \int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta. \end{aligned} \quad (61)$$

Proof. Note that to prove (61) for $f(z) = z - \sum_{m=2}^{\infty} a_m |z|^m$, it is equivalent to prove

$$\begin{aligned} \int_0^{2\pi} \left| 1 - \sum_{m=2}^{\infty} a_m |z|^{m-1} \right|^\eta d\theta \\ \leq \int_0^{2\pi} \left| 1 - \frac{(1-\eta)}{[\tilde{2}]_{(p,q)}(k+1) - (k+\eta)} \right|^\eta d\theta. \end{aligned} \quad (62)$$

By Lemma 1, it is sufficient to show that

$$1 - \sum_{m=2}^{\infty} a_m |z|^{m-1} < 1 - \frac{(1-\eta)}{[\tilde{2}]_{(p,q)}(k+1) - (k+\eta)}. \quad (63)$$

Set

$$1 - \sum_{m=2}^{\infty} a_m |z|^{m-1} = 1 - \frac{(1-\eta)}{[\tilde{2}]_{(p,q)}(k+1) - (k+\eta)} w(z). \quad (64)$$

This means that

$$|w(z)| = \left| \sum_{m=2}^{\infty} \frac{[\tilde{2}]_{(p,q)}(k+1) - (k+\eta)}{(1-\eta)} a_m |z|^{m-1} \right|. \quad (65)$$

Now using (39), we get $|w(z)| \leq |z| < 1$ and this finishes the proof.

Theorem 4. Let $f \in k - \widetilde{\mathfrak{TS}}_{(p,q)}(\eta)$ along $\in \mathfrak{C}$. Then,

$$\frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{2\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} (f * g)(z) < g(z), \quad (66)$$

that is,

$$\begin{aligned} \operatorname{Re}(f(z)) &> - \frac{\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}}{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))} > 0, \\ &z \in \mathbb{U}. \end{aligned} \quad (67)$$

The constant $(([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))/2\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\})$ cannot be increased.

Proof. Let $f \in k - \widetilde{\mathfrak{TS}}_{(p,q)}(\eta)$ and $g \in \mathfrak{C}$.

Then, consider

$$\begin{aligned} &\frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{2\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} (f * g)(z) \\ &= \frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{2\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} \left(z + \sum_{m=2}^{\infty} a_m b_m z^m \right). \end{aligned} \quad (68)$$

Therefore, by using the definition of subordinating factor sequence, (66) holds true, if

$$\left(\frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{2\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} a_m \right)_{n=1}^{\infty}, \quad (69)$$

is a subordinating factor sequence with $a_1 = 1$. Equivalently, by Lemma 2, we have

$$\begin{aligned} \operatorname{Re} \left(1 + \sum_{m=1}^{\infty} \frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} a_m z^m \right) &> 0, \\ &z \in \mathbb{U}. \end{aligned} \quad (70)$$

Now for $|z| = r < 1$, we can write

$$\begin{aligned} & \operatorname{Re} \left(1 + \frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} \sum_{m=1}^{\infty} a_m z^m \right) \\ &= \operatorname{Re} \left(1 + \frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} z + \frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta)) \sum_{m=2}^{\infty} a_m z^m}{\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} \right) \\ &> 1 - \frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} r - \frac{\sum_{m=2}^{\infty} ([\tilde{m}]_{(p,q)}(k+1) - (k+\eta)) a_m r^j}{\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}}. \end{aligned} \quad (71)$$

It can be seen that the function defined by

$$\psi_{(p,q)}(n, k, \eta) = [\tilde{m}]_{(p,q)}(k+1) - (k+\eta), \quad (72)$$

is increasing for $n \geq 2$; furthermore, by utilizing affirmation (39) of Theorem 1, we have

$$\begin{aligned} & \operatorname{Re} \left(1 + \frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} \sum_{m=1}^{\infty} a_m z^m \right) \\ &> 1 - \frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} r - \frac{1-\eta}{\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} r \\ &> 0. \end{aligned} \quad (73)$$

This equivalently proves the subordination condition given by (66). The inequality given by (67) can be obtained from (66) by taking $g(z) = (z/1-z) \in \mathfrak{G}$. Now from (47), we obtained the following function:

$$F(z) = z + \frac{(1-\eta)}{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))} \in k - \widetilde{\mathfrak{G}\mathfrak{S}}_{(p,q)}(\eta). \quad (74)$$

Function (66) becomes

$$\frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{2\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} F(z) < \frac{z}{1-z}. \quad (75)$$

It is easily verified that

$$\min \left\{ \operatorname{Re} \left(\frac{([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))}{2\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\}} F(z) \right) \right\} = -\frac{1}{2}, \quad z \in \mathbb{U}. \quad (76)$$

This implies that the constant $(([\tilde{2}]_{(p,q)}(k+1) - (k+\eta))/2\{(1-\eta) + [\tilde{2}]_{(p,q)}(k+1) - (k+\eta)\})$ is possible.

Theorem 5. Let $0 \leq k < \infty, 0 \leq \eta < 1$ and $0 < q \leq p < 1$. If we set

$$f_1(z) = z f_m(z) = z - \frac{1-\eta}{[\tilde{m}]_{(p,q)}(k+1) - (k+\eta)} z^m, \quad (77)$$

then

$$f \in k - \widetilde{\mathfrak{G}\mathfrak{S}}_{(p,q)}(\eta) \iff f(z) = \sum_{m=1}^{\infty} \delta_m f_m(z), \sum_{m=1}^{\infty} \delta_m = 1. \quad (78)$$

Proof. Suppose that

$$f(z) = \sum_{m=1}^{\infty} \delta_m f_m(z) = \delta_1 f_1(z) + \sum_{m=2}^{\infty} \delta_m f_m(z). \quad (79)$$

Utilizing (77), we get

$$\begin{aligned} f(z) &= \delta_1 f_1(z) + \sum_{m=2}^{\infty} \delta_m \left\{ z - \frac{(1-\eta)}{[\tilde{m}]_{(p,q)}(k+1) - (k+\eta)} z^m \right\} \\ &= \left(\sum_{m=1}^{\infty} \delta_m \right) z - \sum_{m=2}^{\infty} \delta_m \frac{(1-\eta)}{[\tilde{m}]_{(p,q)}(k+1) - (k+\eta)} z^m \\ &= z - \sum_{m=2}^{\infty} \delta_m \frac{(1-\eta)}{[\tilde{m}]_{(p,q)}(k+1) - (k+\eta)} z^m. \end{aligned} \quad (80)$$

Since $\sum_{m=1}^{\infty} \delta_m = 1$, this means that $\sum_{m=2}^{\infty} \delta_m = \sum_{m=1}^{\infty} \delta_m - \delta_1 = 1 - \delta_1 \leq 1$, and using the assertion (51) of Corollary 4, we obtain $f \in k - \widetilde{\mathfrak{TS}}_{(p,q)}(\eta)$.

Conversely, suppose that $f \in k - \widetilde{\mathfrak{TS}}_{(p,q)}(\eta)$. Making use of (39), we may set

$$\delta_m = \frac{[\tilde{m}]_{(p,q)}(k+1) - (k+\eta)}{(1-\eta)} |a_m|, \quad \sum_{m=1}^{\infty} \delta_m = 1. \quad (81)$$

Then,

$$\begin{aligned} f(z) &= z - \sum_{m=2}^{\infty} a_m z^m \\ &= z - \sum_{m=2}^{\infty} \delta_m \frac{(1-\eta)}{[\tilde{m}]_{(p,q)}(k+1) - (k+\eta)} z^m \\ &= z - \sum_{m=2}^{\infty} \delta_m (z - f_m(z)) \\ &= \left(1 - \sum_{m=2}^{\infty} \delta_m\right) z + \sum_{m=2}^{\infty} \delta_m f_m(z) \\ &= \delta_1 z + \sum_{m=2}^{\infty} \delta_m f_m(z), \\ \delta_1 &= 1 - \sum_{m=2}^{\infty} \delta_m \\ &= \sum_{m=1}^{\infty} \delta_m f_m(z), \quad z \in \mathbb{U}, \end{aligned} \quad (82)$$

which is required.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Research Article

Starlikeness of Analytic Functions with Subordinate Ratios

Rosihan M. Ali,¹ Vaithiyanathan Ravichandran², and Kanika Sharma³

¹School of Mathematical Sciences, Universiti Sains Malaysia, George Town 11800 USM, Penang, Malaysia

²Department of Mathematics, National Institute of Technology, Tiruchirappalli 620015, India

³Department of Mathematics, Atma Ram Sanatan Dharma College, University of Delhi, Delhi 110021, India

Correspondence should be addressed to Vaithiyanathan Ravichandran; vravi68@gmail.com

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Let h be a nonvanishing analytic function in the open unit disc with $h(0) = 1$. Consider the class consisting of normalized analytic functions f whose ratios $f(z)/g(z)$, $g(z)/zp(z)$, and $p(z)$ are each subordinate to h for some analytic functions g and p . The radius of starlikeness of order α is obtained for this class when h is chosen to be either $h(z) = \sqrt{1+z}$ or $h(z) = e^z$. Further, starlikeness radii are also obtained for each of these two classes, which include the radius of Janowski starlikeness, and the radius of parabolic starlikeness.

1. Two Subclasses of Normalized Analytic Functions

Let \mathcal{A} denote the class of normalized analytic functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the unit disc $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$. A prominent subclass of \mathcal{A} is the class \mathcal{S}^* consisting of functions $f \in \mathcal{A}$ such that $f(\mathbb{D})$ is a starlike domain with respect to the origin. Geometrically, this means the linear segment joining the origin to every other point $w \in f(\mathbb{D})$ lies entirely in $f(\mathbb{D})$. Every starlike function in \mathcal{A} is necessarily univalent.

Since $f'(0)$ does not vanish, every function $f \in \mathcal{A}$ is locally univalent at $z = 0$. Further, each function $f \in \mathcal{A}$ mirrors the identity mapping near the origin and thus, in particular, maps small circles $|z| = r$ onto curves which bound starlike domains. If $f \in \mathcal{A}$ is also required to be univalent in \mathbb{D} , then it is known that f maps the disc $|z| < r$ onto a domain starlike with respect to the origin for every $r \leq r_0 := \tan h(\pi/4)$ (see [1], Corollary, p. 98). The constant r_0 cannot be improved. Denoting by \mathcal{S} the class of univalent functions $f \in \mathcal{A}$, the number $r_0 = \tan h(\pi/4)$ is commonly referred to as the radius of starlikeness for the class \mathcal{S} .

Another informative description of the class \mathcal{S} is its radius of convexity. Here, it is known that every $f \in \mathcal{S}$ maps the disc $|z| < r$ onto a convex domain for every

$r \leq r_0 := 2 - \sqrt{3}$ ([1], Corollary, p. 44). Thus, the radius of convexity for \mathcal{S} is $r_0 = 2 - \sqrt{3}$.

To formulate a radius description for other entities besides starlikeness and convexity, consider in general two families \mathcal{G} and \mathcal{M} of \mathcal{A} . The \mathcal{G} -radius for the class \mathcal{M} , denoted by $R_{\mathcal{G}}(\mathcal{M})$, is the largest number R such that $r^{-1}f(rz) \in \mathcal{G}$ for every $0 < r \leq R$ and $f \in \mathcal{M}$. Thus, for example, an equivalent description of the radius of starlikeness for \mathcal{S} is that the \mathcal{S}^* -radius for the class \mathcal{S} is $R_{\mathcal{S}^*}(\mathcal{S}) = \tanh(\pi/4)$.

In this paper, we seek to determine the radius of starlikeness and certain other \mathcal{G} -radius, for particular subclasses \mathcal{G} of \mathcal{A} . Several widely studied subclasses of \mathcal{A} have simple geometric descriptions; these functions are often expressed as a ratio between two functions. Among the very early studies in this direction is the class of close-to-convex functions introduced by Kaplan [2] and Reade's class [3] of close-to-starlike functions. Close-to-convex functions are necessarily univalent, but not so for close-to-starlike functions.

In this paper, we examine two different subclasses of functions in \mathcal{A} satisfying a certain subordination of ratios. Interestingly, these classes contain nonunivalent functions. An analytic function f is subordinate to an analytic function g , written $f < g$, if

$$f(z) = g(w(z)), \quad z \in \mathbb{D}, \quad (1)$$

for some analytic self-map w in \mathbb{D} with $|w(z)| \leq |z|$. The function w is often referred to as a Schwarz function.

Now, let h be a nonvanishing analytic function in \mathbb{D} with $h(0) = 1$. The classes treated in this paper consist of functions $f \in \mathcal{A}$ whose ratios $f(z)/g(z)$, $g(z)/zp(z)$, and $p(z)$ are each subordinate to h for some analytic functions g and p :

$$\begin{aligned} \frac{f(z)}{g(z)} &< h(z), \\ \frac{g(z)}{zp(z)} &< h(z), \\ p(z) &< h(z). \end{aligned} \quad (2)$$

When p is the constant one function, then the class contains functions $f \in \mathcal{A}$ satisfying the subordination of ratios

$$\begin{aligned} \frac{f(z)}{g(z)} &< h(z), \\ \frac{g(z)}{z} &< h(z). \end{aligned} \quad (3)$$

When $f \in \mathcal{A}$ satisfies $\operatorname{RE}(f(z)/g(z)) > 0$ and $\operatorname{RE}(g(z)/z) > 0$, or their variants, these functions have earlier been studied, notably by MacGregor in [4–7] and Ratti in [8, 9]. For related investigations, see [10, 11] and several recent references therein. Under the present context, this amounts to choosing $h(z) = (1+z)/(1-z)$ or some other appropriate choices of h .

In this paper, two specific choices of the function h are made: $h(z) = \sqrt{1+z}$ and $h(z) = e^z$.

The class \mathcal{T}_1 : this is the class given by

$$\mathcal{T}_1 := \left\{ f \in \mathcal{A} : \frac{f(z)}{g(z)} < \sqrt{1+z}, \frac{g(z)}{zp(z)} < \sqrt{1+z}, \text{ for some } g \in \mathcal{A}, p(z) < \sqrt{1+z} \right\}. \quad (4)$$

This class is nonempty: let $f_1, g_1, p_1: \mathbb{D} \rightarrow \mathbb{C}$ be given by

$$\begin{aligned} f_1(z) &= z(1+z)^{3/2}, \\ g_1(z) &= z(1+z), \\ p_1(z) &= \sqrt{1+z}. \end{aligned} \quad (5)$$

Then, $f_1(z)/g_1(z) < \sqrt{1+z}$ and $g_1(z)/zp_1(z) < \sqrt{1+z}$, so that $f_1 \in \mathcal{T}_1$. The function f_1 will be shown to play the role of an extremal function for the class \mathcal{T}_1 . Since f_1' vanishes at $z = -2/5$, the function f_1 is nonunivalent, and thus, the class \mathcal{T}_1 contains nonunivalent functions. Incidentally, f_1 demonstrates the radius of univalence for \mathcal{T}_1 is at most $2/5$. In Theorem 1, the radius of starlikeness for \mathcal{T}_1 is shown to be $2/5$, whence \mathcal{T}_1 has radius of univalence $2/5$.

The following is a useful result in investigating the starlikeness of the class \mathcal{T}_1 .

Lemma 1. *Let $p(z) < \sqrt{1+z}$. Then, p satisfies the sharp inequalities*

$$\sqrt{1-r} \leq |p(z)| \leq \sqrt{1+r}, \quad |z| \leq r, \quad (6)$$

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{r}{2(1-r)}, \quad |z| \leq r. \quad (7)$$

Proof. If $p(z) < \sqrt{1+z}$, then $p^2(z) = 1 + w(z)$ for some Schwarz function w . The well-known Schwarz lemma shows that $|w(z)| \leq |z|$ and

$$|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2}. \quad (8)$$

Therefore,

$$|p(z)|^2 = |1 + w(z)| \leq 1 + |w(z)| \leq 1 + |z| \leq 1 + r, \quad (9)$$

for $|z| \leq r$, that is, $|p(z)| \leq \sqrt{1+r}$ for $|z| \leq r$. Similarly, $|p(z)| \geq \sqrt{1-r}$ for $|z| \leq r$.

Since $2zp'(z)/p(z) = zw'(z)/(1 + w(z))$, the inequality (8) readily shows

$$\begin{aligned} 2 \left| \frac{zp'(z)}{p(z)} \right| &\leq \frac{|z||w'(z)|}{1 - |w(z)|} \leq \frac{|z|(1 + |w(z)|)}{1 - |z|^2} \\ &\leq \frac{|z|(1 + |z|)}{1 - |z|^2} = \frac{|z|}{1 - |z|} \leq \frac{r}{1 - r}, \end{aligned} \quad (10)$$

for $|z| \leq r$. This proves (7). The inequalities are sharp for the function $p: \mathbb{D} \rightarrow \mathbb{C}$ defined by $p(z) = \sqrt{1+z}$. \square

For $f \in \mathcal{T}_1$, let $p_1(z) = f(z)/g(z)$ and $p_2(z) = g(z)/zp(z)$. Then, $f(z) = zp(z)p_1(z)p_2(z)$ and

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{zp'(z)}{p(z)} \right| + \left| \frac{zp_1'(z)}{p_1(z)} \right| + \left| \frac{zp_2'(z)}{p_2(z)} \right|. \quad (11)$$

Since $p, p_1, p_2 < \sqrt{1+z}$, we deduce from (7) and (11) that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{3r}{2(1-r)}, \quad |z| \leq r, \quad (12)$$

for each function $f \in \mathcal{T}_1$. Sharp growth inequalities also follow from (6):

$$r(1-r)^{3/2} \leq |f(z)| \leq r(1+r)^{3/2}, \quad (13)$$

for each $f \in \mathcal{T}_1$. Crude distortion inequalities can readily be obtained from (12) and the growth inequality; however, finding sharp estimates remain an open problem.

The class \mathcal{T}_2 : this class is defined by

$$\mathcal{T}_2 := \left\{ f \in \mathcal{A} : \frac{f(z)}{g(z)} \prec e^z, \frac{g(z)}{zp(z)} \prec e^z, \text{ for some } g \in \mathcal{A}, p(z) \prec e^z \right\}. \quad (14)$$

Let $f_2, g_2, p_2: \mathbb{D} \rightarrow \mathbb{C}$ be given by

$$\begin{aligned} f_2(z) &= ze^{3z}, \\ g_2(z) &= ze^{2z}, \\ p_2(z) &= e^z. \end{aligned} \quad (15)$$

Evidently, $f_2(z)/g_2(z) \prec e^z$, $g_2(z)/zp_2(z) \prec e^z$, so that $f_2 \in \mathcal{T}_2$, and the class \mathcal{T}_2 is nonempty. Similar to $f_1 \in \mathcal{T}_1$, the function f_2 plays the role of an extremal function for the class \mathcal{T}_2 . The Taylor series expansion for f_2 is

$$f_2(z) = z + 3z^2 + \frac{9z^3}{2} + \frac{9z^4}{2} + \frac{27z^5}{8} + \dots \quad (16)$$

Comparing the second coefficient, it is clear that f_2 is nonunivalent. Hence, the class \mathcal{T}_2 contains nonunivalent functions. The derivative f_2' vanishes at $z = -1/3$, which shows the radius of univalence for \mathcal{T}_2 is at most $1/3$. From Theorem 1, the radius of starlikeness is shown to be $1/3$, and so the radius of univalence for \mathcal{T}_2 is $1/3$.

Lemma 2. Every $p(z) \prec e^z$ satisfies the sharp inequalities

$$e^{-r} \leq |p(z)| \leq e^r, \quad |z| \leq r, \quad (17)$$

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \begin{cases} r, & |z| \leq r \leq \sqrt{2} - 1, \\ \frac{(1+r^2)^2}{4(1-r^2)}, & |z| = r \geq \sqrt{2} - 1. \end{cases} \quad (18)$$

Proof. Let $p(z) \prec e^z$. Since $p(z) = e^{w(z)}$ for some Schwarz self-map w satisfying $|w(z)| \leq |z|$, it follows that

$$e^{-|z|} \leq e^{-|w(z)|} \leq |p(z)| = e^{\operatorname{Re} w(z)} \leq e^{|w(z)|} \leq e^{|z|}. \quad (19)$$

The inequalities become equality for the function $p: \mathbb{D} \rightarrow \mathbb{C}$ defined by $p(z) = e^z$ respectively at $z = -r$ and $z = r$.

The function w also satisfies the sharp inequality (see [1], Corollary, p. 199)

$$|w'(z)| \leq \begin{cases} 1, & r = |z| \leq \sqrt{2} - 1, \\ \frac{(1+r^2)^2}{4r(1-r^2)}, & r \geq \sqrt{2} - 1. \end{cases} \quad (20)$$

From $zp'(z)/p(z) = zw'(z)$, we conclude that

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \begin{cases} r, & r = |z| \leq \sqrt{2} - 1, \\ \frac{(1+r^2)^2}{4(1-r^2)}, & r \geq \sqrt{2} - 1. \end{cases} \quad (21)$$

This inequality is sharp for the function $p: \mathbb{D} \rightarrow \mathbb{C}$ defined by $p(z) = e^z$ when $r = |z| \leq \sqrt{2} - 1$. It is also sharp in the remaining interval for the function $p(z) = e^{w(z)}$, where w is the extremal function for which equality holds in (20). \square

For $f \in \mathcal{T}_2$, let $p_1(z) = f(z)/g(z)$ and $p_2(z) = g(z)/zp(z)$. Then, $f(z) = zp(z)p_1(z)p_2(z)$ and

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{zp'(z)}{p(z)} \right| + \left| \frac{zp_1'(z)}{p_1(z)} \right| + \left| \frac{zp_2'(z)}{p_2(z)} \right|. \quad (22)$$

Since $p, p_1, p_2 \prec e^z$, estimates (18) and (22) show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \begin{cases} 3r, & r = |z| \leq \sqrt{2} - 1, \\ \frac{3(1+r^2)^2}{4(1-r^2)}, & r \geq \sqrt{2} - 1, \end{cases} \quad (23)$$

for each function $f \in \mathcal{T}_2$. It also follows from (17) that

$$re^{-3r} \leq |f(z)| \leq re^{3r} \quad (24)$$

holds for each function $f \in \mathcal{T}_2$ and that these estimates are sharp.

In this paper, we shall adopt the commonly used notations for subclasses of \mathcal{A} . First, for $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ denote the class of starlike functions of order α consisting of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1-2\alpha)z}{1-z}. \quad (25)$$

Thus,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}. \quad (26)$$

The case $\alpha = 0$ corresponds to the classical functions whose image domains are starlike with respect to the origin. Various other starlike subclasses of \mathcal{A} occurring in the literature can be expressed in terms of the subordination

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad (27)$$

for suitable choices of the superordinate function φ . When $\varphi: \mathbb{D} \rightarrow \mathbb{C}$ is chosen to be $\varphi(z) := (1 + Az)/(1 + Bz)$, $-1 \leq B < A \leq 1$, the subclass derived is denoted by $\mathcal{S}^*[A, B]$. Functions $f \in \mathcal{S}^*[A, B]$ are known as Janowski starlike functions. When $\varphi(z) := 1 + (2/\pi^2)((\log((1 + \sqrt{z})/(1 - \sqrt{z})))^2)$, the subclass is denoted by \mathcal{S}_p^* , and its functions are called parabolic starlike functions.

In Section 2 of this paper, the radius of starlikeness of order α , Janowski starlikeness, and parabolic starlikeness are found for the classes \mathcal{T}_i , with $i = 1, 2$. Section 3 deals with the determination of the \mathcal{G} -radius for the class \mathcal{T}_i with $i = 1, 2$, for certain other subclasses \mathcal{G} occurring in the literature. These classes are associated with particular choices of the superordinate function φ in (27). As mentioned earlier, the \mathcal{G} -radius for a given class \mathcal{M} , denoted by $R_{\mathcal{G}}(\mathcal{M})$, is the largest number R such that $r^{-1}f(rz) \in \mathcal{G}$ for every $0 < r \leq R$ and $f \in \mathcal{M}$. It will become apparent in the forthcoming proofs that there are common features in the methodology of finding the \mathcal{G} -radius for each of these subclasses.

2. Starlikeness of Order α , Janowski Starlikeness, and Parabolic Starlikeness

The first result deals with the $\mathcal{S}^*(\alpha)$ -radius (radius of starlikeness of order α) for the classes \mathcal{T}_1 and \mathcal{T}_2 . This radius is shown to equal the \mathcal{S}_α^* -radius, where \mathcal{S}_α^* is the subclass containing functions $f \in \mathcal{A}$ satisfying $|zf'(z)/f(z) - 1| < 1 - \alpha$. The latter condition also implies that $\mathcal{S}_\alpha^* \subset \mathcal{S}^*(\alpha)$.

Theorem 1. *Let $0 \leq \alpha < 1$. The radii of starlikeness of order α for \mathcal{T}_1 and \mathcal{T}_2 are*

- (i) $R_{\mathcal{S}^*(\alpha)}(\mathcal{T}_1) = R_{\mathcal{S}_\alpha^*}(\mathcal{T}_1) = 2(1 - \alpha)/(5 - 2\alpha)$,
- (ii) $R_{\mathcal{S}^*(\alpha)}(\mathcal{T}_2) = R_{\mathcal{S}_\alpha^*}(\mathcal{T}_2) = (1 - \alpha)/3$.

Proof

- (i) The function $\sigma(r) = (2 - 5r)/(2 - 2r)$ is a decreasing function on $[0, 1)$. Further, the number $R_1 := 2(1 - \alpha)/(5 - 2\alpha)$ is the root of the equation $\sigma(r) = \alpha$. For $f \in \mathcal{T}_1$ and $0 < r = |z| \leq R_1$, the inequality (12) readily yields

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq 1 - \frac{3r}{2(1-r)} = \frac{2-5r}{2-2r} = \sigma(r) \geq \sigma(R_1) = \alpha,$$

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{3r}{2(1-r)} = 1 - \sigma(r) \leq 1 - \sigma(R_1) = 1 - \alpha. \quad (28)$$

At $z = -R_1$, the function $f_1 \in \mathcal{T}_1$ given by $f_1(z) = z(1+z)^{3/2}$ yields

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2+5z}{2+2z} = \frac{2-5R_1}{2-2R_1} = \alpha. \quad (29)$$

Thus,

$$\begin{aligned} \operatorname{Re} \frac{zf_1'(z)}{f_1(z)} &= \alpha, \\ \left| \frac{zf_1'(z)}{f_1(z)} - 1 \right| &= 1 - \alpha. \end{aligned} \quad (30)$$

This proves that the $\mathcal{S}^*(\alpha)$ and \mathcal{S}_α^* radii for \mathcal{T}_1 are the same number R_1 .

- (ii) Consider $\omega(r) = 1 - 3r$, $0 \leq r < 1$. The number $R_2 = (1 - \alpha)/3 < 1/3$ is clearly the root of the equation $\omega(r) = \alpha$. Since ω is decreasing, then $\omega(r) \geq \omega(R_2) = \alpha$ for $0 < r \leq R_2$. It follows from (23) that for $0 < r = |z| \leq R_2$,

$$\begin{aligned} \operatorname{Re} \frac{zf'(z)}{f(z)} &\geq 1 - 3r = \omega(r) \geq \alpha, \\ \left| \frac{zf'(z)}{f(z)} - 1 \right| &\leq 3r = 1 - \omega(r) \leq 1 - \alpha. \end{aligned} \quad (31)$$

Evaluating the function $f_2(z) = ze^{3z}$ at $z = -R_2$ yields

$$\frac{zf_2'(z)}{f_2(z)} = 1 - 3R_2 = \alpha. \quad (32)$$

Hence,

$$\begin{aligned} \operatorname{Re} \frac{zf_2'(z)}{f_2(z)} &= \alpha, \\ \left| \frac{zf_2'(z)}{f_2(z)} - 1 \right| &= 1 - \alpha. \end{aligned} \quad (33)$$

This proves that the $\mathcal{S}^*(\alpha)$ and \mathcal{S}_α^* radii for the class \mathcal{T}_2 are the same number R_2 . \square

Next, we find the $\mathcal{S}^*[A, B]$ -radius (Janowski starlikeness) for \mathcal{T}_1 and \mathcal{T}_2 . Recall that $\mathcal{S}^*[A, B]$ consists of analytic functions $f \in \mathcal{A}$ satisfying the subordination $zf'(z)/f(z) \prec (1 + Az)/(1 + Bz)$, $-1 \leq B < A \leq 1$.

Theorem 2.

- (i) Every $f \in \mathcal{T}_1$ is Janowski starlike in the disc $\mathbb{D}_r = \{z: |z| < r\}$ for $r \leq 2(A - B)/(3(1 + |B|) + 2(A - B))$. If $B < 0$, then $R_{\mathcal{S}^*[A, B]}(\mathcal{T}_1) = 2(A - B)/(3 + 2A - 5B)$.
- (ii) The radius of Janowski starlikeness for \mathcal{T}_2 is $R_{\mathcal{S}^*[A, B]}(\mathcal{T}_2) = (A - B)/(3(1 + |B|))$.

Proof. Since $\mathcal{S}^*[A, -1] = \mathcal{S}^*((1 - A)/2)$, the results in the case $B = -1$ follow from Theorem 1. We now prove the results when $-1 < B < A \leq 1$.

- (i) Let $f \in \mathcal{T}_1$ and write $w = zf'(z)/f(z)$. Then, (12) shows that $|w - 1| \leq 3r/(2(1 - r))$ for $|z| \leq r$. For $0 \leq r \leq R_1 := 2(A - B)/(3(1 + |B|) + 2(A - B))$, then $3R_1/(2(1 - R_1)) = (A - B)/(1 + |B|)$.

For $0 \leq r \leq R_1$, we first show that the disc

$$\left\{ w: |w - 1| \leq \frac{3R_1}{2(1 - R_1)} = \frac{A - B}{1 + |B|} \right\} \quad (34)$$

is contained in the images of the unit disc under the mapping $(1 + Az)/(1 + Bz)$. As $B \neq -1$, the image is the disc given by

$$\left\{ w: \left| w - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \right\}. \quad (35)$$

Silverman ([12], p. 50–51) has shown that the disc

$$\{w: |w-c| < d\} \subset \{w: |w-a| < b\}, \quad (36)$$

if and only if $|a-c| \leq b-d$. With the choices $c = 1$, $d = (A-B)/(1+|B|)$, $a = (1-AB)/(1-B^2)$, and $b = (A-B)/(1-B^2)$, then $|a-c| = |B|(A-B)/(1-B^2) = b-d$. This proves that $\mathcal{S}^*[A, B]$ radius is at least R_1 .

To prove sharpness, consider the function $f_1 \in \mathcal{T}_1$ given by $f_1(z) = z(1+z)^{3/2}$. Evidently, $zf'_1(z)/f_1(z) = (2+5z)/(2+2z)$. For $B < 0$, evaluating at $z = -R_1$, then $zf'_1(z)/f_1(z) = 1 + 3z/(2+2z) = 1 - (A-B)/(1+|B|) = (1-A)/(1-B)$. This shows that

$$\left| \frac{zf'_1(z)}{f_1(z)} - \frac{1-AB}{1-B^2} \right| = \left| \frac{1-A}{1-B} - \frac{1-AB}{1-B^2} \right| = \frac{A-B}{1-B^2}, \quad (37)$$

proving sharpness in the case $B < 0$.

- (ii) Let $f \in \mathcal{T}_2$ and $w := zf'(z)/f(z)$. It follows from (23) that $|w-1| \leq 3r$ for $|z| \leq r$. For $0 \leq r \leq R_2 := (A-B)/(3(1+|B|))$, we see that the disc $\{w: |w-1| \leq 3R_2 = (A-B)/(1+|B|)\}$ is contained in the disc $\{w: |w - (1-AB)/(1-B^2)| < (A-B)/(1-B^2)\}$, as in the proof of (i). This proves that $\mathcal{S}^*[A, B]$ radius is at least R_2 . The result is sharp for the function $f_2 \in \mathcal{T}_2$ given by the function $f_2(z) = ze^{3z}$. \square

The function $\varphi_{\text{PAR}}: \mathbb{D} \rightarrow \mathbb{C}$ given by

$$\varphi_{\text{PAR}}(z) := 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, \quad \text{Im} \sqrt{z} \geq 0, \quad (38)$$

maps \mathbb{D} into the parabolic region

$$\varphi_{\text{PAR}}(\mathbb{D}) = \{w = u + iv: v^2 < 2u-1\} = \{w: \text{RE}w > |w-1|\}. \quad (39)$$

The class $\mathcal{C}(\varphi_{\text{PAR}}) = \{f \in \mathcal{A}: 1 + zf''(z)/f'(z) < \varphi_{\text{PAR}}(z)\}$ is the class of uniformly convex functions introduced by Goodman [13]. The corresponding class $\mathcal{S}_p^* := \mathcal{S}^*(\varphi_{\text{PAR}}) = \{f \in \mathcal{A}: zf'(z)/f(z) < \varphi_{\text{PAR}}(z)\}$ introduced by Rønning [14] is known as the class of parabolic starlike functions. The class \mathcal{S}_p^* consists of functions $f \in \mathcal{A}$ satisfying

$$\text{Re} \left(\frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \mathbb{D}. \quad (40)$$

Evidently, every parabolic starlike function is also starlike of order $1/2$. The radius of parabolic starlikeness for the classes \mathcal{T}_1 and \mathcal{T}_2 is given in the next result.

Corollary 1. *The radius of parabolic starlikeness for \mathcal{T}_1 and \mathcal{T}_2 is respectively equal to its radius of starlikeness of order $1/2$. Thus,*

$$(i) R_{\mathcal{S}_p^*}(\mathcal{T}_1) = 1/4,$$

$$(ii) R_{\mathcal{S}_p^*}(\mathcal{T}_2) = 1/6.$$

Proof. Shanmugam and Ravichandran ([15], p. 321) proved that

$$\left\{ w: |w-a| < a - \frac{1}{2} \right\} \subseteq \{w: \text{RE}w > |w-1|\}, \quad (41)$$

for $1/2 < a \leq 3/2$. Choosing $a = 1$, this implies that $\mathcal{S}_{1/2}^* \subset \mathcal{S}_p^*$. Every parabolic starlike function is also starlike of order $1/2$, whence the inclusion $\mathcal{S}_{1/2}^* \subset \mathcal{S}_p^* \subset \mathcal{S}^*(1/2)$. Therefore, for any class \mathcal{F} , readily $R_{\mathcal{S}_{1/2}^*}(\mathcal{F}) \leq R_{\mathcal{S}_p^*}(\mathcal{F}) \leq R_{\mathcal{S}^*(1/2)}(\mathcal{F})$.

When $\mathcal{F} = \mathcal{T}_i$, $i = 1, 2$, Theorem 1 gives $R_{\mathcal{S}^*(a)}(\mathcal{T}_i) = R_{\mathcal{S}_p^*}(\mathcal{T}_i)$. This shows that $R_{\mathcal{S}_{1/2}^*}(\mathcal{T}_i) = R_{\mathcal{S}_p^*}(\mathcal{T}_i) = R_{\mathcal{S}^*(1/2)}(\mathcal{T}_i)$. Since $R_{\mathcal{S}^*(1/2)}(\mathcal{T}_1) = 1/4$ and $R_{\mathcal{S}^*(1/2)}(\mathcal{T}_2) = 1/6$ from Theorem 1, it follows that $R_{\mathcal{S}_p^*}(\mathcal{T}_1) = 1/4$ and $R_{\mathcal{S}_p^*}(\mathcal{T}_2) = 1/6$. \square

3. Further Radius of Starlikeness

In this section, we find the \mathcal{G} -radius for the class \mathcal{T}_i with $i = 1, 2$, for certain other widely studied subclasses \mathcal{G} . These are associated with particular choices of the superordinate function φ in (27).

Denote by $\mathcal{S}_{\text{exp}}^* := \mathcal{S}^*(e^z)$ the class associated with $\varphi(z) = e^z$ in (27). This class was introduced by Mendiratta et al. [16], and it consists of functions $f \in \mathcal{A}$ satisfying the condition $|\log(zf'(z)/f(z))| < 1$. The following result gives the radius of exponential starlikeness for the classes \mathcal{T}_1 and \mathcal{T}_2 .

Corollary 2. *The $\mathcal{S}_{\text{exp}}^*$ -radius for the class \mathcal{T}_1 is*

$$R_{\mathcal{S}_{\text{exp}}^*}(\mathcal{T}_1) = \frac{(2-2e)}{(2-5e)} \approx 0.296475, \quad (42)$$

while that of \mathcal{T}_2 is

$$R_{\mathcal{S}_{\text{exp}}^*}(\mathcal{T}_2) = \frac{(e-1)}{3e}. \quad (43)$$

Proof. Mendiratta et al. ([16], Lemma 2.2) proved that

$$\left\{ w: |w-a| < a - \frac{1}{e} \right\} \subseteq \{w: |\log w| < 1\}, \quad (44)$$

for $e^{-1} \leq a \leq (e+e^{-1})/2$, and this inclusion with $a = 1$ gives $\mathcal{S}_{1/e}^* \subset \mathcal{S}_{\text{exp}}^*$. It was also shown in ([16], Theorem 2.1 (i)) that $\mathcal{S}_{\text{exp}}^* \subset \mathcal{S}^*(1/e)$. Therefore, $\mathcal{S}_{1/e}^* \subset \mathcal{S}_{\text{exp}}^* \subset \mathcal{S}^*(1/e)$, which, as a consequence of Theorem 1, established the result. \square

Corollary 3 investigates the radius of cardioid starlikeness for each class \mathcal{T}_1 and \mathcal{T}_2 . The class $\mathcal{S}_C^* := \mathcal{S}^*(\varphi_{\text{CAR}})$,

where $\varphi_{\text{CAR}}(z) = 1 + 4z/3 + 2z^2/3$ in (27), was introduced and studied in [17]. Descriptively, $f \in S_C^*$ provided $zf'(z)/f(z)$ lies in the region bounded by the cardioid $\Omega_C := \{w = u + iv: (9u^2 + 9v^2 - 18u + 5)^2 - 16(9u^2 + 9v^2 - 6u + 1) = 0\}$.

Corollary 3. *The following are the S_C^* -radius for the classes \mathcal{T}_1 and \mathcal{T}_2 :*

- (i) $R_{S_C^*}(\mathcal{T}_1) = 4/13$,
- (ii) $R_{S_C^*}(\mathcal{T}_2) = 2/9$.

Proof. Sharma et al. [17] proved that $\{w: |w - a| < a - 1/3\} \subseteq \Omega_C$ for $1/3 < a \leq 5/3$, and this inclusion with $a = 1$ gives $\mathcal{S}_{1/3}^* \subset S_C^*$. Thus, $R_{\mathcal{S}_{1/3}^*}(\mathcal{T}_i) \leq R_{S_C^*}(\mathcal{T}_i)$ for $i = 1, 2$. To complete the proof, we demonstrate $R_{\mathcal{S}_{1/3}^*}(\mathcal{T}_i) \leq R_{\mathcal{S}_{1/3}^*}(\mathcal{T}_i)$ for $i = 1, 2$.

- (i) Evaluating the function $f_1(z) = z(1+z)^{3/2}$ at $z = -R = -R_{\mathcal{S}_{1/3}^*}(\mathcal{T}_1) = -4/13$ gives

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2+5z}{2+2z} = \frac{2-5R}{2-2R} = \frac{1}{3} = \varphi_{\text{CAR}}(-1). \quad (45)$$

Thus, $R_{S_C^*}(\mathcal{T}_1) \leq 4/13$.

- (ii) Similarly, at $z = -R = -R_{\mathcal{S}_{1/3}^*}(\mathcal{T}_2) = -2/9$, the function $f_2(z) = ze^{3z}$ yields

$$\frac{zf_2'(z)}{f_2(z)} = 1 + 3z = 1 - 3R = \frac{1}{3} = \varphi_{\text{CAR}}(-1). \quad (46)$$

This proves that $R_{S_C^*}(\mathcal{T}_2) \leq 2/9$. \square

In 2019, Cho et al. [18] studied the class $\mathcal{S}_{\sin}^* := \mathcal{S}^*(1 + \sin z)$ consisting of functions $f \in \mathcal{A}$ satisfying the condition $zf'(z)/f(z) < 1 + \sin z$. We find the \mathcal{S}_{\sin}^* -radius for the classes \mathcal{T}_1 and \mathcal{T}_2 .

Corollary 4. *The following are the \mathcal{S}_{\sin}^* -radius for each class \mathcal{T}_1 and \mathcal{T}_2 :*

- (i) $R_{\mathcal{S}_{\sin}^*}(\mathcal{T}_1) = 2(\sin 1)/(3 + 2 \sin 1) \approx 0.35938$,
- (ii) $R_{\mathcal{S}_{\sin}^*}(\mathcal{T}_2) = (\sin 1)/3$.

Proof. It was proved in [18] that $\{w: |w - a| < \sin 1 - |a - 1|\} \subseteq q(\mathbb{D})$ for $|a - 1| \leq \sin 1$, where $q(z) := 1 + \sin z$. For $a = 1$, this implies that $\mathcal{S}_{1-\sin 1}^* \subset \mathcal{S}_{\sin}^*$. Thus, $R_{\mathcal{S}_{1-\sin 1}^*}(\mathcal{T}_i) \leq R_{\mathcal{S}_{\sin}^*}(\mathcal{T}_i)$ for $i = 1, 2$. The proof is completed by demonstrating $R_{\mathcal{S}_{1-\sin 1}^*}(\mathcal{T}_i) \leq R_{\mathcal{S}_{\sin}^*}(\mathcal{T}_i)$ for $i = 1, 2$.

- (i) Evaluating the function $f_1(z) = z(1+z)^{3/2}$ at $z = -R = -R_{\mathcal{S}_{1-\sin 1}^*}(\mathcal{T}_1) = -2 \sin 1/(3 + 2 \sin 1)$ gives

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2+5z}{2+2z} = \frac{2-5R}{2-2R} = 1 - \sin 1 = q(-1). \quad (47)$$

Thus, $R_{\mathcal{S}_{\sin}^*}(\mathcal{T}_1) \leq 2 \sin 1/(3 + 2 \sin 1)$.

- (ii) Similarly, at $z = \pm R = \pm R_{\mathcal{S}_{1-\sin 1}^*}(\mathcal{T}_2) = \pm (\sin 1)/3$, the function $f_2(z) = ze^{3z}$ yields

$$\frac{zf_2'(z)}{f_2(z)} = 1 + 3z = 1 \pm 3R = 1 \pm \sin 1 = q(\pm 1). \quad (48)$$

This proves that $R_{\mathcal{S}_{\sin}^*}(\mathcal{T}_2) \leq (\sin 1)/3$. \square

Consider next the class $\mathcal{S}_{\odot}^* := \mathcal{S}^*(z + \sqrt{1+z^2})$ introduced by Raina and Sokół in [19]. Functions $f \in \mathcal{S}_{\odot}^*$ provided $zf'(z)/f(z)$ lies in the region bounded by the lune $\Omega_l := \{w: |w^2 - 1| < 2|w|\}$. The result below gives the radius of lune starlikeness for each class \mathcal{T}_1 and \mathcal{T}_2 .

Corollary 5. *The following are the \mathcal{S}_{\odot}^* -radius for each class \mathcal{T}_1 and \mathcal{T}_2 :*

- (i) $R_{\mathcal{S}_{\odot}^*}(\mathcal{T}_1) = 2(\sqrt{2} - 2)/(2\sqrt{2} - 7) \approx 0.280847$,
- (ii) $R_{\mathcal{S}_{\odot}^*}(\mathcal{T}_2) = (2 - \sqrt{2})/3$.

Proof. It was shown by Gandhi and Ravichandran ([20], Lemma 2.1) that $\{w: |w - a| < 1 - |\sqrt{2} - a|\} \subseteq \Omega_l$ for $\sqrt{2} - 1 < a \leq \sqrt{2} + 1$. Choosing $a = 1$, the inclusion gives $\mathcal{S}_{\sqrt{2}-1}^* \subset \mathcal{S}_{\odot}^*$. Thus, $R_{\mathcal{S}_{\sqrt{2}-1}^*}(\mathcal{T}_i) \leq R_{\mathcal{S}_{\odot}^*}(\mathcal{T}_i)$ for $i = 1, 2$. We complete the proof by demonstrating $R_{\mathcal{S}_{\sqrt{2}-1}^*}(\mathcal{T}_i) \leq R_{\mathcal{S}_{\odot}^*}(\mathcal{T}_i)$ for $i = 1, 2$.

- (i) Evaluating the function $f_1(z) = z(1+z)^{3/2}$ at $z = -R = -R_{\mathcal{S}_{\sqrt{2}-1}^*}(\mathcal{T}_1) = -2(\sqrt{2} - 2)/(2\sqrt{2} - 7)$ gives

$$\left| \left(\frac{zf_1'(z)}{f_1(z)} \right)^2 - 1 \right| = \left| \left(\frac{2+5z}{2+2z} \right)^2 - 1 \right| = \left| \left(\frac{2-5R}{2-2R} \right)^2 - 1 \right| = 2(\sqrt{2} - 1) = 2 \left| \frac{zf_1'(z)}{f_1(z)} \right|. \quad (49)$$

Thus, $R_{\mathcal{S}_{\odot}^*}(\mathcal{T}_1) \leq 2(\sqrt{2} - 2)/(2\sqrt{2} - 7)$.

- (ii) Similarly, at $z = -R = -R_{\mathcal{S}_{\sqrt{2}-1}^*}(\mathcal{T}_2) = -(2 - \sqrt{2})/3$, the function $f_2(z) = ze^{3z}$ yields

$$\left| \left(\frac{zf_2'(z)}{f_2(z)} \right)^2 - 1 \right| = |(1+3z)^2 - 1| = |(1-3R)^2 - 1| = 2(\sqrt{2} - 1) = 2 \left| \frac{zf_2'(z)}{f_2(z)} \right|. \quad (50)$$

This proves that $R_{\mathcal{S}_{\odot}^*}(\mathcal{T}_2) \leq (2 - \sqrt{2})/3$. \square

As a further example, consider next the class $\mathcal{S}_R^* := \mathcal{S}^*(\eta(z))$, where $\eta(z) = 1 + ((zk + z^2)/(k^2 - kz))$, $k = \sqrt{2} + 1$. This class associated with a rational function was introduced and studied by Kumar and Ravichandran in [21].

Corollary 6. *The following are the \mathcal{S}_R^* -radius for the classes \mathcal{T}_1 and \mathcal{T}_2 :*

- (i) $R_{\mathcal{S}_R^*}(\mathcal{T}_1) = 2(-3 + 2\sqrt{2})/(4\sqrt{2} - 9) \approx 0.102642$,
(ii) $R_{\mathcal{S}_R^*}(\mathcal{T}_2) = (3 - 2\sqrt{2})/3$.

Proof. It was shown in [21] that $\{w: |w - a| < a - 2(\sqrt{2} - 1)\} \subseteq \eta(\mathbb{D})$ for $2(\sqrt{2} - 1) < a \leq \sqrt{2}$. This inclusion with $a = 1$ gives $\mathcal{S}_{2(\sqrt{2}-1)}^* \subset \mathcal{S}_R^*$. Thus, $R_{\mathcal{S}_{2(\sqrt{2}-1)}^*}(\mathcal{T}_i) \leq R_{\mathcal{S}_R^*}(\mathcal{T}_i)$ for $i = 1, 2$. We next show that $R_{\mathcal{S}_R^*}(\mathcal{T}_i) \leq R_{\mathcal{S}_{2(\sqrt{2}-1)}^*}(\mathcal{T}_i)$ for $i = 1, 2$.

- (i) At $z = -R = -R_{\mathcal{S}_{2(\sqrt{2}-1)}^*}(\mathcal{T}_1) = -2(-3 + 2\sqrt{2})/(4\sqrt{2} - 9)$, the function $f_1(z) = z(1+z)^{3/2}$ yields

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2-5R}{2-2R} = 2(\sqrt{2}-1) = \eta(-1). \quad (51)$$

Thus, $R_{\mathcal{S}_R^*}(\mathcal{T}_1) \leq 2(-3 + 2\sqrt{2})/(4\sqrt{2} - 9)$.

- (ii) Evaluating $f_2(z) = ze^{3z}$ at $z = -R = -R_{\mathcal{S}_{2(\sqrt{2}-1)}^*}(\mathcal{T}_2) = -(3 - 2\sqrt{2})/3$ gives

$$\frac{zf_2'(z)}{f_2(z)} = 1 - 3R = 2(\sqrt{2}-1) = \eta(-1). \quad (52)$$

Thus, $R_{\mathcal{S}_R^*}(\mathcal{T}_2) \leq (3 - 2\sqrt{2})/3$. \square

The class $\mathcal{S}_{N_e}^* := \mathcal{S}^*(\psi(z))$, where $\psi(z) = 1 + z - z^3/3$, was introduced and studied by Wani and Swaminathan in [22]. Geometrically, $f \in \mathcal{S}_{N_e}^*$ provided $zf'(z)/f(z)$ lies in the region bounded by the nephroid: a 2-cusped kidney-shaped curve $\Omega_{N_e} := \{w = u + iv: ((u-1)^2 + v^2 - 4/9)^3 - 4v^2/3 = 0\}$.

Corollary 7. *The following are the $\mathcal{S}_{N_e}^*$ -radius for the classes \mathcal{T}_1 and \mathcal{T}_2 :*

- (i) $R_{\mathcal{S}_{N_e}^*}(\mathcal{T}_1) = 4/13$,
(ii) $R_{\mathcal{S}_{N_e}^*}(\mathcal{T}_2) = 2/9$.

Proof. It was shown in [22] that $\{w: |w - a| < a - 1/3\} \subseteq \Omega_{N_e}$ for $1/3 < a \leq 1$. This inclusion with $a = 1$ gives $\mathcal{S}_{1/3}^* \subset \mathcal{S}_{N_e}^*$. This shows that $R_{\mathcal{S}_{1/3}^*}(\mathcal{T}_i) \leq R_{\mathcal{S}_{N_e}^*}(\mathcal{T}_i)$ for $i = 1, 2$. We next show that $R_{\mathcal{S}_{N_e}^*}(\mathcal{T}_i) \leq R_{\mathcal{S}_{1/3}^*}(\mathcal{T}_i)$ for $i = 1, 2$.

- (i) Evaluating the function $f_1(z) = z(1+z)^{3/2}$ at $z = -R = -R_{\mathcal{S}_{1/3}^*}(\mathcal{T}_1) = -4/13$ results in

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2-5R}{2-2R} = \frac{1}{3} = \psi(-1). \quad (53)$$

Thus, $R_{\mathcal{S}_{N_e}^*}(\mathcal{T}_1) \leq 4/13$.

- (ii) Similarly, evaluating $f_2(z) = ze^{3z}$ at $z = -R = -R_{\mathcal{S}_{1/3}^*}(\mathcal{T}_2) = -2/9$ yields

$$\frac{zf_2'(z)}{f_2(z)} = 1 - 3R = \frac{1}{3} = \psi(-1). \quad (54)$$

This proves that $R_{\mathcal{S}_{N_e}^*}(\mathcal{T}_2) \leq 2/9$. \square

Finally, we consider the class $\mathcal{S}_{SG}^* := \mathcal{S}^*(2/(1+e^{-z}))$ introduced by Goel and Kumar in [23]. Here, $2/(1+e^{-z})$ is the modified sigmoid function that maps \mathbb{D} onto the region $\Omega_{SG} := \{w = u + iv: |\log(w/(2-w))| < 1\}$. Thus, $f \in \mathcal{S}_{SG}^*$ provided the function $zf'(z)/f(z)$ maps \mathbb{D} onto the region lying inside the domain Ω_{SG} .

Corollary 8. *The \mathcal{S}_{SG}^* -radius for the class \mathcal{T}_1 is*

$$R_{\mathcal{S}_{SG}^*}(\mathcal{T}_1) = \frac{(2e-2)}{(1+5e)} \approx 0.23552, \quad (55)$$

while that of \mathcal{T}_2 is

$$R_{\mathcal{S}_{SG}^*}(\mathcal{T}_2) = \frac{(e-1)}{(3(1+e))}. \quad (56)$$

Proof. The inclusion $\{w: |w - a| < ((e-1)/(e+1)) - |a-1|\} \subseteq \Omega_{SG}$ holds for $2/(1+e) < a < 2e/(1+e)$ (see [23]). At $a = 1$, the set inclusion shows that $\mathcal{S}_{2/(e+1)}^* \subset \mathcal{S}_{SG}^*$. It was also shown in [23] that $\mathcal{S}_{SG}^* \subset \mathcal{S}^*(\alpha)$ for $0 \leq \alpha \leq 2/(e+1)$. The desired result is now an immediate consequence of Theorem 1. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Some Applications of New Complex Function Space Constructed by Different Weights and Exponents

Awad A. Bakery ^{1,2} and Elsayed A. E. Mohamed ^{1,3}

¹University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia

²Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Egypt

³Department of Mathematics, Faculty of Education, Alzaeim Alazhari University, Khartoum, Sudan

Correspondence should be addressed to Elsayed A. E. Mohamed; elsayed.bayen2020@yahoo.com

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In this article, we develop and study a new complex function space formed by varying the weights and exponents under a definite function. We investigate the geometric and topological characteristics of mapping ideals created using s -numbers and this complex function space. Also, the action of shift mappings on this complex function space has been discussed. Finally, we introduced an extension of Caristi's fixed point theorem on it.

1. Introduction

Numerous researchers are attempting to extend the Banach fixed point theorem [1] in a realistic manner. Kannan [2] recognized a subclass of mappings that execute the same fixed point operations as contractions but are not continuous. Ghoncheh [3] pioneered the study of Kannan mappings in modular vector spaces. Lebesgue spaces with variable exponents, $L_{(r)}$, include Nakano sequence spaces. Across the second half of the twentieth century, it was thought that these variable exponent spaces offered an adequate framework for the mathematical components of a variety of problems for which the traditional Lebesgue spaces were inadequate. Due to the importance of these areas and their consequences, they have developed a reputation as an effective instrument for resolving a wide variety of problems; presently, the study of $L_{(r)}(\Omega)$ spaces is a developing field of research, with implications reaching across a broad range of mathematical disciplines [4]. The investigation of variable exponent Lebesgue spaces was accelerated further by the mathematical description of non-Newtonian fluid hydrodynamics [5, 6]. Non-Newtonian fluids, also known as electrorheological fluids, have a wide range of applications

in a number of fields ranging from military science to civil engineering to orthopedics and beyond. Mapping ideal theory has a diverse range of applications in Banach space geometry, fixed point theory, spectral theory, and other areas of mathematics, as well as other fields of knowledge (for further information, see [7–13]). Bakery and Mohamed [14] studied the notion of a pre-quasi norm on Nakano sequence space with a variable exponent in the range $(0, 1]$. They explored the conditions under which it generates pre-quasi Banach and closed space when endowed with a particular pre-quasi norm as well as the Fatou property of various pre-quasi norms on it. Additionally, they showed the existence of a fixed point for Kannan pre-quasi norm contraction mappings on it as well as on the pre-quasi Banach operator ideal formed from this sequence space's s -numbers. In [15], they investigated some fixed points results of Kannan non-expansive mappings on generalized Cesàro backward difference sequence space of non-absolute type.

We will mark the complex and non-negative integers as \mathbb{C} and $\mathbb{N} = \{0, 1, 2, \dots\}$, respectively. By $\mathbb{C}^{\mathbb{C}}$, we denote the space of all complex functions with complex variable. Assuming that $r = (r_y)_{y \in \mathbb{N}} \in (0, \infty)^{\mathbb{N}}$, Bakery and El Dewaik [16] defined the following function space:

$$(\mathcal{H}_w(r_v))_\psi = \left\{ h \in \mathbb{C}^{\mathbb{C}} : h(x) = \sum_{y=0}^{\infty} \widehat{h}_y x^y \in \mathbb{C}; \text{ and } \psi(\omega h) < \infty, \text{ for some } \omega > 0 \right\}, \quad (1)$$

where

$$\psi(h) = \sum_{y=0}^{\infty} \left| \frac{\widehat{h}_y}{y+1} \right|^{r_y}. \quad (2)$$

They studied several of the topological and geometric properties for $(\mathcal{H}_w((r_v)))_\psi$ and even a pre-quasi ideal construction based on the $(\mathcal{H}_w((r_v)))_\psi$ and s -numbers. Upper bounds for s -numbers of infinite series of the weighted ν -th power forward shift operator on $(\mathcal{H}_w((r_v)))_\psi$ were also introduced for some entire functions. Further, they evaluated Caristi's fixed point theorem in $(\mathcal{H}_w((r_v)))_\psi$. For extra information on formal power series spaces and their behaviors, see [17–20]. We denote the space of every, finite rank, approximable, and compact bounded linear mappings

from a Banach space X into a Banach space Y by $L(X, Y)$, $F(X, Y)$, $\Lambda(X, Y)$, and $L_c(X, Y)$, and if $X = Y$, we mark $L(X)$, $F(X)$, $\Lambda(X)$, and $L_c(X)$, respectively. The ideal of all, finite rank, approximable, and compact mappings are denoted by L , F , Λ , and L_c . We will indicate the sequence of s -numbers, approximation numbers, and Kolmogorov numbers for any bounded linear mapping G by $(s_a(G))_{a \in \mathbb{N}}$, $(\alpha_a(G))_{a \in \mathbb{N}}$, and $(d_a(G))_{a \in \mathbb{N}}$. The mapping ideals constructed by the sequence of s -numbers, approximation numbers, and Kolmogorov numbers in sequence space \mathcal{V} are marked by $S_{\mathcal{V}}$, $S_{\mathcal{V}}^{\text{app}}$, and $S_{\mathcal{V}}^{\text{Kol}}$. For any Banach spaces X and Y , we will use the following notations.

Notations 1 (see [16])

$$\begin{aligned} S_{\mathcal{H}} &:= \{S_{\mathcal{H}}(X, Y)\}, \text{ where } S_{\mathcal{H}}(X, Y) := \left\{ P \in L(X, Y) : h_s \in \mathcal{H}, \text{ where, } h_s(x) = \sum_{v=0}^{\infty} s_v(P)x^v \in \mathbb{C} \right\}. \\ S_{\mathcal{H}}^{\text{app}} &:= \{S_{\mathcal{H}}^{\text{app}}(X, Y)\}, \text{ where } S_{\mathcal{H}}^{\text{app}}(X, Y) := \left\{ P \in L(X, Y) : h_{\text{app}} \in \mathcal{H}, \text{ where, } h_{\text{app}}(x) = \sum_{v=0}^{\infty} \alpha_v(P)x^v \in \mathbb{C} \right\}. \\ S_{\mathcal{H}}^{\text{Kol}} &:= \{S_{\mathcal{H}}^{\text{Kol}}(X, Y)\}, \text{ where } S_{\mathcal{H}}^{\text{Kol}}(X, Y) := \left\{ P \in L(X, Y) : h_{\text{Kol}} \in \mathcal{H}, \text{ where, } h_{\text{Kol}}(x) = \sum_{v=0}^{\infty} d_v(P)x^v \in \mathbb{C} \right\}. \quad (3) \\ (S_{\mathcal{H}_\rho})^\lambda &:= \left\{ (S_{\mathcal{H}_\rho})^\lambda(X, Y) \right\}, \text{ where} \\ (S_{\mathcal{H}_\rho})^\lambda(X, Y) &:= \left\{ T \in L(X, Y) : h_\lambda \in \mathcal{H}_\rho, \text{ where, } h_\lambda(x) = \sum_{v=0}^{\infty} \lambda_v(P)x^v \in \mathbb{C} \text{ and } \|P - \lambda_v(P)I\| = 0, \forall v \in \mathbb{N} \right\}. \end{aligned}$$

The purpose of this study is straightforward, as follows. In Section 3, we introduce and investigate the complex function space $(\mathbb{H}((b_v), (p_v)))_\rho$ under the definite function ρ . In Section 4, the mapping ideals constructed by s -numbers and $(\mathbb{H}((b_v), (p_v)))_\rho$ are presented. We have studied their geometric and topological properties. Specifically, we explore in Section 5 the upper limits of s -numbers for infinite series of the weighted ν -th power forward and backward shift mapping on $(\mathbb{H}((b_v), (p_v)))_\rho$ and their applications to various entire functions. Finally, in Section 6, we present an extension of Caristi's fixed point theorem in $(\mathbb{H}((b_v), (p_v)))_\rho$.

2. Definitions and Preliminaries

Let $\mathbb{R}^{\mathbb{N}}$, ℓ_∞ , ℓ^r , and c_0 denote the spaces of each, bounded, r -absolutely summable, and null sequences of real numbers, respectively.

Definition 1 (see [16]). The function space $\mathcal{H} = \{h \in \mathbb{C}^{\mathbb{C}} : h(y) = \sum_{v=0}^{\infty} \widehat{h}_v y^v\}$ is called a special space of formal power series (or in short ssfps), if it shows the following settings:

- (1) $e^{(b)} \in \mathcal{H}$, for all $b \in \mathbb{N}$, where $e^{(b)}(y) = \sum_{v=0}^{\infty} \widehat{e_v^{(b)}} y^v = y^b$.
- (2) If $g \in \mathcal{H}$ and $|\widehat{h}_v| t \leq n |q \widehat{g}_v|$, for all $v \in \mathbb{N}$, one has $h \in \mathcal{H}$.
- (3) Suppose $h \in \mathcal{H}$, then $h_{[\cdot]} \in \mathcal{H}$, where $h_{[\cdot]}(y) = \sum_{b=0}^{\infty} \widehat{h_{[b/2]}} y^b$ and $[b/2]$ marks the integral part of $[b/2]$.

Theorem 1 (see [16]). $S_{\mathcal{H}}$ is a mapping ideal, when \mathcal{H} is a ssfps.

We denote the space of finite formal power series by \mathfrak{F} , i.e., if $h \in \mathfrak{F}$, one has $k \in \mathbb{N}$ with $h(y) = \sum_{v=0}^k \hat{h}_v y^v$. Also, θ indicates the zero function of \mathcal{H} .

Definition 2 (see [16]). A subspace \mathcal{H}_ρ of the ssfps is said to be a pre-quasi normed ssfps, if there is a function $\rho: \mathcal{H} \rightarrow [0, \infty)$ which verifies the next conditions:

- (i) For $h \in \mathcal{H}$, we have $\rho(h) \geq 0$ and $h = \theta \iff \rho(h) = 0$.
- (ii) Suppose $h \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, then there are $l \geq 1$ with $\rho(\lambda h) \leq |\lambda| \rho(h)$.
- (iii) Let $f, g \in \mathcal{H}$; then, there are $K \geq 1$ such that $\rho(f + g) \leq K(\rho(f) + \rho(g))$.

Recall that if the space \mathcal{H}_ρ is complete, then \mathcal{H}_ρ is called a pre-quasi Banach ssfps.

Definition 3 (see [16]). A subspace \mathcal{H}_ρ of the ssfps is called a pre-modular ssfps, if there is a function $\rho: \mathcal{H} \rightarrow [0, \infty)$ which verifies the next conditions:

- (i) For $h \in \mathcal{H}$, we have $\rho(h) \geq 0$ and $h = \theta \iff \rho(h) = 0$.
- (ii) Suppose $h \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, then there are $l \geq 1$ with $\rho(\lambda h) \leq |\lambda| \rho(h)$.
- (iii) Let $f, g \in \mathcal{H}$; then, there are $K \geq 1$ such that $\rho(f + g) \leq K(\rho(f) + \rho(g))$.
- (iv) Suppose $|\hat{f}_b| \leq n|q\hat{g}_b|$, for every $b \in \mathbb{N}$; then, $\rho(f) \leq \rho(g)$.
- (v) There are $K_0 \geq 1$ so that $\rho(f) \leq \rho(f_{[.]}) \leq K_0 \rho(f)$.
- (vi) $\overline{\mathfrak{F}} = \mathcal{H}_\rho$.
- (vii) One has $\xi > 0$ with $\rho(\lambda e^{(0)}) \geq \xi |\lambda| \rho(e^{(0)})$, where $\lambda \in \mathbb{C}$.

Theorem 2 (see [16]). Every pre-modular ssfps \mathcal{H}_ρ is a pre-quasi normed ssfps.

Definition 4 (see [21]). A function $s: L(X, Y) \rightarrow [0, \infty)^\mathbb{N}$ is called an s -number, if the sequence $(s_b(B))_{a=0}^\infty$, for any $B \in L(X, Y)$, satisfies the following setup:

- (a) If $B \in L(X, Y)$, then $\|B\| = s_0(B) \geq s_1(B) \geq s_2(B) \geq \dots \geq 0$.
- (b) $s_{b+a-1}(B_1 + B_2) \leq s_b(B_1) + s_a(B_2)$, for every $B_1, B_2 \in L(X, Y)$, $b, a \in \mathbb{N}$.
- (c) The inequality $s_a(ABD) \leq \|A\| s_a(B) \leq \|B\|$ holds, if $D \in L(X_0, X)$, $B \in L(X, Y)$, and $A \in L(Y, Y_0)$; suppose that X_0 and Y_0 are any two Banach spaces.
- (d) For $A \in L(X, Y)$ and $\lambda \in \mathbb{R}$, then $s_a(\lambda A) = |\lambda| s_a(A)$.
- (e) Suppose $\text{rank}(A) \leq b$; then, $s_b(A) = 0$, whenever $A \in L(X, Y)$.
- (f) Assume that I_b represents the unit map on the b -dimensional Hilbert space ℓ_b^2 ; then, $s_{r \geq b}(I_b) = 0$ or $s_{r < b}(I_b) = 1$.

The following are some instances of s -numbers:

- (i) The k -th approximation number, $\alpha_k(A)$, is presented as

$$\alpha_k(A) = \inf\{\|A - B\|: B \in L(X, Y) \text{ and } \text{rank}(B) \leq k\}. \quad (4)$$

- (ii) The k -th Kolmogorov number, $d_k(A)$, is presented as

$$d_k(A) = \inf_{\dim(Y)} \leq k \sup_{\|u\| \leq 1} \inf_{v \in Y} \|Au - v\|. \quad (5)$$

Lemma 1 (see [7]). Assume that $B \in L(X, Y)$ and $B \notin \Lambda(X, Y)$, and we have maps $D \in L(X)$ and $M \in L(Y)$ with $MBDe_b = e_b$, for each $b \in \mathbb{N}$.

Definition 5 (see [7]). A Banach space Y is named simple if $L(Y)$ contains one and only one non-trivial closed ideal.

Theorem 3 (see [7]). Suppose Z is a Banach space with $\dim(Z) = \infty$, and we have

$$F(Z) \subsetneq \Lambda(Z) \subsetneq L_c(Z) \subsetneq L(Z). \quad (6)$$

Definition 6 (see [7]). A class $\mathcal{U} \subseteq L$ is said to be a mapping ideal if every component $\mathcal{U}(X, Y) = \mathcal{U} \cap L(X, Y)$ satisfies the next setups:

- (i) $F \subseteq \mathcal{U}$.
- (ii) $\mathcal{U}(X, Y)$ is linear space on \mathbb{R} .
- (iii) Assume $D \in L(X_0, X)$, $B \in \mathcal{U}(X, Y)$, and $A \in L(Y, Y_0)$; then, $ABD \in \mathcal{U}(X_0, Y_0)$.

Definition 7 (see [10]). A function $g: \mathcal{U} \rightarrow [0, \infty)$ is called a pre-quasi norm on the ideal \mathcal{U} if it satisfies the following setups:

- (1) Suppose $B \in L(X, Y)$, $g(B) \geq 0$, and $g(B) = 0 \iff B = 0$.
- (2) There is $M \geq 1$ with $g(vA) \leq M|v|g(A)$, for all $v \in \mathbb{C}$ and $A \in \mathcal{U}(X, Y)$.
- (3) One has $K \geq 1$ so that $g(A_1 + A_2) \leq K[g(A_1) + g(A_2)]$, for every $A_1, A_2 \in \mathcal{U}(X, Y)$.
- (4) We get $C \geq 1$ so that if $A \in L(X_0, X)$, $B \in \mathcal{U}(X, Y)$, and $D \in L(Y, Y_0)$, then $g(DBA) \leq C\|D\|g(B)\|A\|$, where X_0 and Y_0 are normed spaces.

Theorem 4 (see [10]). Every quasi norm is a pre-quasi norm on the same ideal.

With finite non-zero coordinates, we denote the space of every sequence by \mathcal{F} .

Theorem 5 (see [22]). Suppose s -type $\mathcal{V}_v := \{f = (s_r(T)) \in \mathbb{R}^\mathbb{N}: T \in L(X, Y) \text{ and } v(f) < \infty\}$. If $S_{\mathcal{V}_v}$ is a mapping ideal, we have

- (1) $\mathcal{F} \subset s\text{-type } \mathcal{V}_v$.
- (2) Assume $(s_r(T_1))_{r=0}^\infty \in s\text{-type } \mathcal{V}_v$ and $(s_r(T_2))_{r=0}^\infty \in s\text{-type } \mathcal{V}_v$; then, $(s_r(T_1 + T_2))_{r=0}^\infty \in s\text{-type } \mathcal{V}_v$.
- (3) If $\lambda \in \mathbb{C}$ and $(s_r(T))_{r=0}^\infty \in s\text{-type } \mathcal{V}_v$, then $|\lambda|(s_r(T))_{r=0}^\infty \in s\text{-type } \mathcal{V}_v$.
- (4) \mathcal{V}_v is solid, i.e., if $(s_x(J))_{x=0}^\infty \in s\text{-type } \mathcal{V}_v$ and $s_x(H) \leq s_x(J)$, for all $x \in \mathbb{N}$ and $H, J \in L(X, Y)$, then $(s_x(H))_{x=0}^\infty \in s\text{-type } \mathcal{V}_v$.

By card (\mathfrak{G}) , we denote the number of elements of \mathfrak{G} .

Lemma 2 (see [23]). Suppose $\{\xi_i\}_{i \in \Psi}$ is a bounded family of \mathbb{R} . Hence,

$$\inf_{\text{card}(\mathfrak{G})=b} \sup_{i \notin \mathfrak{G}} \xi_i = \sup_{\text{card}(\mathfrak{G})=b+1} \inf_{i \in \mathfrak{G}} \xi_i. \quad (7)$$

We will apply the next inequality [24]. For all $(r_a), (t_a) \in \mathbb{C}^{\mathbb{N}}$ and $(q_a) \in (0, \infty)^{\mathbb{N}}$, we have

$$|r_a + t_a|^{q_a} \leq K(|r_a|^{q_a} + |t_a|^{q_a}), \quad (8)$$

where $K = \max\{1, 2^{\mathfrak{Q}_q-1}\}$ and $\mathfrak{Q}_q = \max\{1, \sup_a q_a\}$.

Definition 8 (see [16]). Assume \mathcal{H}_ρ is a pre-quasi normed ssfps. A mapping $V_y: \mathcal{H}_\rho \longrightarrow \mathcal{H}_\rho$ is called forward shift, if

$V_y h = yh$, for all $h \in \mathcal{H}_\rho$, where $V_y h(y) = \sum_{v=0}^\infty \widehat{h}_v y^{v+1} \in \mathbb{C}$ and $\rho(V_y h) < \infty$.

Definition 9 (see [16]). Suppose \mathcal{H}_ρ is a pre-quasi normed ssfps. A mapping $B_y: \mathcal{H}_\rho \longrightarrow \mathcal{H}_\rho$ is called backward shift, if $B_y h(y) = h(y) - h(0)/y$, for all $h \in \mathcal{H}_\rho$, where $B_y h(y) = \sum_{v=0}^\infty \widehat{h}_{v+1} y^v \in \mathbb{C}$ and $\rho(B_y h) < \infty$.

Definition 10 (see [20]). If $g(y) = \sum_{m=0}^\infty a_m y^m$, then $V_{g(y)}(h(y)) := (\sum_{m=0}^\infty a_m V_y^m)(h(y))$.

Definition 11 (see [20]). If $g(y) = \sum_{m=0}^\infty a_m y^m$, then $B_{g(y)}(h(y)) := (\sum_{m=0}^\infty a_m B_y^m)(h(y))$.

3. Pre-Modular ssfps

This section contains the space's definition $(\mathbb{H}((b_n), (p_n)))_\rho$ under the function ρ , where $\rho(h) = \sum_{v=0}^\infty |b_v \widehat{h}_v|^{p_v}$, for all $h \in \mathbb{H}((b_n), (p_n))$. We offer enough setups on $(\mathbb{H}((b_n), (p_n)))_\rho$ to become pre-modular ssfps, which implies that $(\mathbb{H}((b_n), (p_n)))_\rho$ is a pre-quasi Banach ssfps.

Let $p = (p_v)_{v \in \mathbb{N}}, (b_v)_{v \in \mathbb{N}} \in (0, \infty)^{\mathbb{N}}$, and we define the following function space:

$$\mathbb{H}((b_v), (p_v)) = \{h \in \mathbb{C}^{\mathbb{C}}: h(y) = \sum_{v=0}^\infty \widehat{h}_v y^v \text{ and } \rho(\gamma h) < \infty, \text{ for some } \gamma > 0\}. \quad (9)$$

Theorem 6. If $(p_v) \in \ell_\infty$, then

$$\mathbb{H}((b_v), (p_v)) = \{h \in \mathbb{C}^{\mathbb{C}}: h(y) = \sum_{v=0}^\infty \widehat{h}_v y^v \text{ and } \rho(\gamma h) < \infty, \text{ for all } \gamma > 0\}. \quad (10)$$

Proof.

$$\begin{aligned} \mathbb{H}((b_v), (p_v)) &= \left\{ h \in \mathbb{C}^{\mathbb{C}}: h(y) = \sum_{v=0}^\infty \widehat{h}_v y^v \text{ and } \rho(\gamma h) < \infty, \text{ for some } \gamma > 0 \right\} \\ &= \left\{ h \in \mathbb{C}^{\mathbb{C}}: h(y) = \sum_{v=0}^\infty \widehat{h}_v y^v \text{ and } \sum_{v=0}^\infty |\gamma b_v \widehat{h}_v|^{p_v} < \infty, \text{ for some } \gamma > 0 \right\} \\ &= \left\{ h \in \mathbb{C}^{\mathbb{C}}: h(y) = \sum_{v=0}^\infty \widehat{h}_v y^v \text{ and } \inf_v |\gamma|^{p_v} \sum_{v=0}^\infty |b_v \widehat{h}_v|^{p_v} < \infty, \text{ for some } \gamma > 0 \right\} \\ &= \left\{ h \in \mathbb{C}^{\mathbb{C}}: h(y) = \sum_{v=0}^\infty \widehat{h}_v y^v \text{ and } \sum_{v=0}^\infty |b_v \widehat{h}_v|^{p_v} < \infty, \text{ for any } \gamma > 0 \right\} \\ &= \left\{ h \in \mathbb{C}^{\mathbb{C}}: h(y) = \sum_{v=0}^\infty \widehat{h}_v y^v \text{ and } \rho(\gamma h) < \infty, \text{ for any } \gamma > 0 \right\}. \end{aligned} \quad (11)$$

□

Hereafter, we will denote the space of all monotonic decreasing and monotonic increasing sequences of positive reals by md_{\searrow} and mi_{\nearrow} , respectively.

Theorem 7. $\mathbb{H}((b_n), (p_n))$ is a ssfps, if it verifies the next setups:

- (a1) $(p_n) \in mi_{\nearrow} \cap \ell_{\infty}$.
- (a2) $(b_n) \in md_{\searrow}$, or $(b_n) \in mi_{\nearrow}$ with $C \geq 1$ so that $b_{2n+1} \leq Cb_n$.

Proof

(1-i) Assume $f, g \in \mathbb{H}((b_n), (p_n))$; then, $f(z) = \sum_{n=0}^{\infty} \hat{f}_n z^n \in \mathbb{C}$ and $g(z) = \sum_{n=0}^{\infty} \hat{g}_n z^n \in \mathbb{C}$. We have $(f+g)(z) = \sum_{n=0}^{\infty} (\hat{f}_n + \hat{g}_n) z^n \in \mathbb{C}$. Since (p_n) is bounded, we get

$$\sum_{n=0}^{\infty} |b_n \hat{f}_n + b_n \hat{g}_n|^{p_n} \leq K \left(\sum_{n=0}^{\infty} |b_n \hat{f}_n|^{p_n} + \sum_{n=0}^{\infty} |b_n \hat{g}_n|^{p_n} \right) < \infty, \quad (12)$$

and then $f+g \in \mathbb{H}((b_n), (p_n))$.

(1-ii) Let $\lambda \in \mathbb{C}$ and $f \in \mathbb{H}((b_n), (p_n))$. We have $(\lambda f)(z) = \sum_{n=0}^{\infty} \lambda \hat{f}_n z^n \in \mathbb{C}$. Since (p_n) is bounded, we have

$$\sum_{n=0}^{\infty} |\lambda b_n \hat{f}_n|^{p_n} \leq \sup_n |\lambda|^{p_n} \sum_{n=0}^{\infty} |b_n \hat{f}_n|^{p_n} < \infty. \quad (13)$$

Then, $\lambda f \in \mathbb{H}((b_n), (p_n))$. Therefore, by using components (1-i) and (1-ii), $\mathbb{H}((b_n), (p_n))$ is linear. Clearly, $e^{(k)} \in \mathbb{H}((b_n), (p_n))$, for all $k \in \mathbb{N}$, where $e^{(k)}(z) = \sum_{n=0}^{\infty} \widehat{e^{(k)}}_n z^n = z^k$ and $\sum_{n=0}^{\infty} |b_n \widehat{e^{(k)}}_n|^{p_n} = b_k^{p_k}$.

- (2) Let $|\hat{f}_v| \leq n|q\hat{g}_v|$, for all $v \in \mathbb{N}$ and $g \in \mathbb{H}((b_n), (p_n))$. Then, $g(z) = \sum_{v=0}^{\infty} \hat{g}_v z^v \in \mathbb{C}$. Since $b_v > 0$, for all $v \in \mathbb{N}$, then

$$\sum_{v=0}^{\infty} |b_v \hat{f}_v|^{p_v} \leq \sum_{v=0}^{\infty} |b_v \hat{g}_v|^{p_v} < \infty. \quad (14)$$

Hence, $f(z) = \sum_{v=0}^{\infty} \hat{f}_v z^v \in \mathbb{C}$ and $\rho(f) < \infty$. Therefore, $f \in \mathbb{H}((b_n), (p_n))$.

- (3) Let $f \in \mathbb{H}((b_n), (p_n))$, (b_n) be an increasing sequence, and there exists $C > 0$ such that $b_{2v+1} \leq Cb_v$ and (p_n) is increasing. Therefore, $f(z) = \sum_{v=0}^{\infty} \hat{f}_v z^v \in \mathbb{C}$ and $\rho(f) < \infty$. One has

$$\begin{aligned} \rho(f_{[\cdot]}) &= \sum_{v=0}^{\infty} |b_v \widehat{f_{[\cdot]}}_v|^{p_v} = \sum_{v=0}^{\infty} |b_{2v} \hat{f}_v|^{p_{2v}} + \sum_{v=0}^{\infty} |b_{2v+1} \hat{f}_v|^{p_{2v+1}} \\ &\leq \sum_{v=0}^{\infty} |b_{2v} \hat{f}_v|^{p_v} + \sum_{v=0}^{\infty} |b_{2v+1} \hat{f}_v|^{p_v} \\ &\leq \max\{1, 2\sup_n C^{p_n}\} \rho(f). \end{aligned} \quad (15)$$

This implies that $f_{[\cdot]}(z) = \sum_{v=0}^{\infty} \widehat{f_{[\cdot]}}_v z^v \in \mathbb{C}$ and $\rho(f_{[\cdot]}) < \infty$. Hence, $f_{[\cdot]} \in \mathbb{H}((b_n), (p_n))$. \square

Theorem 8. Let conditions (a1) and (a2) be satisfied; then, the space $(\mathbb{H}((b_n), (p_n)))_{\rho}$ is a pre-modular Banach ssfps.

Proof

- (i) Evidently, for all $f \in \mathbb{H}((b_n), (p_n))$, then $\rho(f) \geq 0$ and $\rho(f) = 0 \iff f = \theta$.

- (ii) We have $l = \max\{1, \sup_n |\eta|^{p_n-1}\} \geq 1$, for all $\eta \in \mathbb{R} \setminus \{0\}$ and $l \geq 1$, for $\eta = 0$ such that

$$\rho(\eta f) = \sum_{n=0}^{\infty} |\eta b_n \hat{f}_n|^{p_n} \leq \sup_n |\eta|^{p_n} \sum_{n=0}^{\infty} |b_n \hat{f}_n|^{p_n} \leq l |\eta| \rho(f), \quad (16)$$

for every $f \in \mathbb{H}((b_n), (p_n))$.

- (iii) For some $K = \max\{1, 2^{\sup_n p_n-1}\}$, we obtain

$$\rho(f+g) = \sum_{n=0}^{\infty} |b_n (\hat{f}_n + \hat{g}_n)|^{p_n} \leq K(\rho(f) + \rho(g)), \quad (17)$$

for all $f, g \in \mathbb{H}((b_n), (p_n))$.

- (iv) It is clear from the proof part (2) of Theorem 7.

- (v) From the proof part (3) of Theorem 7, we have that $K_0 = \max\{1, 2\sup_n C^{p_n}\} \geq 1$.

- (vi) It is apparent that $\overline{\mathfrak{F}} = \mathbb{H}((b_n), (p_n))$.

- (vii) There is ζ with $0 < \zeta \leq \eta^{p_0-1}$ such that $\rho(\eta e^{(0)}) \geq \zeta |\eta| \rho(e^{(0)})$, for each $\eta \neq 0$ and $\zeta > 0$, if $\eta = 0$.

Therefore, the space $(\mathbb{H}((b_n), (p_n)))_{\rho}$ is pre-modular ssfps. To show that $(\mathbb{H}((b_n), (p_n)))_{\rho}$ is a pre-modular Banach ssfps, suppose $f^{(n)}$ is a Cauchy sequence in $(\mathbb{H}((b_n), (p_n)))_{\rho}$; then, for all $\varepsilon \in (0, 1)$, there is $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, we get

$$\rho(f^{(n)} - f^{(m)}) = \sum_{v=0}^{\infty} |b_v (\widehat{f^{(n)}}_v - \widehat{f^{(m)}}_v)|^{p_v} < \varepsilon^{\varpi_p}. \quad (18)$$

For $n, m \geq n_0$ and $v \in \mathbb{N}$, we obtain

$$|\widehat{f^{(n)}}_v - \widehat{f^{(m)}}_v| < \varepsilon. \quad (19)$$

Hence, $(\widehat{f^{(n)}}_v)_{n \geq n_0}$ is a Cauchy sequence in \mathbb{C} , for fixed $v \in \mathbb{N}$, so $\lim_{n \rightarrow \infty} \widehat{f^{(n)}}_v = \widehat{f^{(0)}}_v$, for fixed $v \in \mathbb{N}$. Therefore, $\rho(f^{(n)} - f^{(0)}) < \varepsilon^{\varpi_p}$, for each $n \geq n_0$. Finally, to explain that $f^{(0)} \in \mathbb{H}((b_n), (p_n))$, we have

$$\begin{aligned} \rho(f^{(0)}) &= \rho(f^{(0)} - f^{(n)} + f^{(n)}) \leq K(\rho(f^{(n)} - f^{(0)}) \\ &\quad + \rho(f^{(n)})) < \infty. \end{aligned} \quad (20)$$

So, $f^{(0)} \in \mathbb{H}((b_n), (p_n))$. This implies that $(\mathbb{H}((b_n), (p_n)))_{\rho}$ is a pre-modular Banach ssfps. \square

Taking into consideration Theorem 2, we put forward the following theorem.

Theorem 9. *Let conditions (a1) and (a2) be satisfied. Then, the space $(\mathbb{H}((b_n), (p_n)))_\rho$ is a pre-quasi Banach ssfps.*

Theorem 10. *Let conditions (a1) and (a2) be satisfied. Then, the space $(\mathbb{H}((b_n), (p_n)))_\rho$ is a pre-quasi closed ssfps.*

Proof. Assume that the setups are verified. From Theorem 9, the space $(\mathbb{H}((b_n), (p_n)))_\rho$ is a pre-quasi normed ssfps. To show that $(\mathbb{H}((b_n), (p_n)))_\rho$ is a pre-quasi closed ssfps, assume $\{h^{(m)}\}_{m=0}^\infty \in (\mathbb{H}((b_n), (p_n)))_\rho$ and $\lim_{m \rightarrow \infty} \rho(h^{(m)} - h^{(0)}) = 0$; then, for every $\varepsilon \in (0, 1)$, there is $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, one has

$$\varepsilon > \rho(h^{(m)} - h^{(0)}) = \left[\sum_{a=0}^\infty \left| b_a (\widehat{h_a^{(m)}} - \widehat{h_a^{(0)}}) \right|^{p_a} \right]^{1/\omega_p}. \quad (21)$$

Hence, for $m \geq m_0$ and $a \in \mathbb{N}$, we get

$$\left| \widehat{h_a^{(m)}} - \widehat{h_a^{(0)}} \right| < \varepsilon. \quad (22)$$

So, $(\widehat{h_a^{(m)}})$ is a convergent sequence in \mathbb{C} , for fixed $a \in \mathbb{N}$.

Therefore, $\lim_{m \rightarrow \infty} \widehat{h_a^{(m)}} = \widehat{h_a^{(0)}}$, for fixed $a \in \mathbb{N}$. Finally to prove that $h^{(0)} \in (\mathbb{H}((b_n), (p_n)))_\rho$, we consider

$$\begin{aligned} \rho(h^{(0)}) &= \rho(h^{(0)} - h^{(m)} + h^{(m)}) \leq K(\rho(h^{(m)} - h^{(0)})) \\ &\quad + \rho(h^{(m)}) < \infty, \end{aligned} \quad (23)$$

so $h^{(0)} \in (\mathbb{H}((b_n), (p_n)))_\rho$. This finishes the proof. \square

4. Pre-Quasi Ideal

In this section, the mapping ideals constructed by s -numbers and $(\mathbb{H}((b_n), (p_n)))_\rho$ are presented. We have studied their geometric and topological structures. We will use the notation for $B \in S_{\mathbb{H}((b_n), (p_n))_\rho}$, that is, $g(B) = \rho(f_s)$, $f_s(z) = \sum_{v=0}^\infty s_v(B)z^v \in \mathbb{C}$, and $\rho(f_s) = \sum_{v=0}^\infty (b_v s_v(B))^{p_v}$, for every $f_s \in \mathbb{H}((b_n), (p_n))_\rho$.

In view of Theorems 1 and 7, we conclude the next theorem.

Theorem 11. *Let conditions (a1) and (a2) be satisfied. Then, $S_{\mathbb{H}((b_n), (p_n))}$ is a mapping ideal.*

4.1. Ideal of Finite Rank Mappings. In this section, enough setups (not necessary) on $\mathbb{H}((b_n), (p_n))_\rho$ so that F is dense in $S_{\mathbb{H}((b_n), (p_n))_\rho}$ are investigated. This explains the non-linearity of the s -type $\mathbb{H}((b_n), (p_n))_\rho$ spaces (Rhoades open problem [25]).

Example 1. The sequence $(b_n) = (n + 1/n + 2)_{n \in \mathbb{N}}$ satisfies $(b_n) \in mi_\nearrow$ and $b_{2n+1} \leq Cb_n$, for some $C \geq 2$.

Theorem 12. $\overline{F(X, Y)} = S_{\mathbb{H}((b_n), (p_n))_\rho}(X, Y)$, whenever conditions (a1) and (a2) are satisfied.

Proof. It is clear that $\overline{F(X, Y)} \subset S_{\mathbb{H}((b_n), (p_n))_\rho}(X, Y)$, since the space $S_{\mathbb{H}((b_n), (p_n))_\rho}$ is a mapping ideal. Currently, we substantiate that $S_{\mathbb{H}((b_n), (p_n))_\rho}(X, Y) \subseteq \overline{F(X, Y)}$. On taking $T \in S_{\mathbb{H}((b_n), (p_n))_\rho}(X, Y)$, $f_s \in \mathbb{H}((b_n), (p_n))_\rho$, with $f_s(z) = \sum_{n=0}^\infty s_n(T)z^n \in \mathbb{C}$. Hence, $\rho(f_s) < \infty$, and assume $\varepsilon \in (0, 1)$, so there is $m \in \mathbb{N} - \{0\}$ such that $\rho(f_s - \sum_{n=0}^{m-1} e^{(n)}) < \varepsilon/4C^2$, for some $C \geq 1$. While $(s_n(T))_{n \in \mathbb{N}}$ is decreasing, we get

$$\sum_{n=m+1}^{2m} (b_n s_{2m}(T))^{p_n} \leq \sum_{n=m+1}^{2m} (b_n s_n(T))^{p_n} \quad (24)$$

$$\leq \sum_{n=m}^\infty (b_n s_n(T))^{p_n} < \frac{\varepsilon}{4C^2}.$$

Hence, there exist $A \in F_{2m}(X, Y)$, $\text{rank}(A) \leq 2m$, and

$$\sum_{n=2m+1}^{3m} (b_n \|T - A\|)^{p_n} \leq \sum_{n=m+1}^{2m} (b_n \|T - A\|)^{p_n} < \frac{\varepsilon}{4C^2}. \quad (25)$$

Since (p_n) is bounded,

$$\sum_{n=0}^m (b_n \|T - A\|)^{p_n} < \frac{\varepsilon}{4C^2}. \quad (26)$$

Let (b_n) be monotonically increasing such that there exists a constant $C \geq 1$ for which $b_{2n+1} \leq Cb_n$. Then, we have for $n \geq m$ that

$$b_{2m+n} \leq b_{2m+2n+1} \leq Cb_{m+n} \leq Cb_{2n} \leq Cb_{2n+1} \leq C^2 b_n. \quad (27)$$

Since $T - A \in S_{\mathbb{H}((b_n), (p_n))_\rho}(X, Y)$, then $h_s \in \mathbb{H}((b_n), (p_n))_\rho$, where $h_s(z) = \sum_{n=0}^\infty s_n(T - A)z^n \in \mathbb{C}$. Since (p_n) is increasing, inequalities (2)–(5) give

$$\begin{aligned} d(T, A) &= \rho(h_s) = \sum_{n=0}^{3m-1} (b_n s_n(T - A))^{p_n} + \sum_{n=3m}^\infty (b_n s_n(T - A))^{p_n} \\ &\leq \sum_{n=0}^{3m} (b_n \|T - A\|)^{p_n} + \sum_{n=m}^\infty (b_{n+2m} s_{n+2m}(T - A))^{p_{n+2m}} \\ &\leq 3 \sum_{n=0}^m (b_n \|T - A\|)^{p_n} + C^{2\sup_n p_n} \sum_{n=m}^\infty (b_n s_n(T))^{p_n} < \varepsilon. \end{aligned} \quad (28)$$

Since $I_8 \in S_{\mathbb{H}((n+1), (1/n+1))_\rho}(X, Y)$ which gives a counter example of the converse statement, this finishes the proof. \square

According to Theorem 12, if (a1) and (a2) are fulfilled, then every compact mapping is represented by finite rank mappings; however, the reverse is not necessarily true.

4.2. Closed and Banach. In this part, we have investigated the sufficient conditions on $\mathbb{H}((b_v), (p_v))_\rho$ such that the pre-quasi mapping ideal $S_{\mathbb{H}((b_v), (p_v))_\rho}$ is Banach and closed.

Theorem 13. *If X and Y are Banach spaces and conditions (a1) and (a2) are satisfied, then the function $g(B) = \rho(f_s)$ is a pre-quasi norm on $S_{\mathbb{H}((b_v), (p_v))_\rho}$.*

Proof. Suppose the conditions are verified, so g verifies the next setups:

(1) Let $B \in S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)$, then we have $g(B) = \rho(f_s) \geq 0$, and it is clear that $g(B) = \rho(f_s) = 0$, if and only if, $s_v(B) = 0$, for all $v \in \mathbb{N}$, if and only if, $B = 0$.

(2) We have $l \geq 1$ with $g(\lambda B) = \rho(\lambda f_s) \leq l|\lambda|\rho(f_s) = l|\lambda|g(B)$, for every $B \in S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)$ and $\lambda \in \mathbb{C}$.

(3) One has $KK_0 \geq 1$ for $B_1, B_2 \in S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)$. Therefore, $f_{1s}(z) = \sum_{v=0}^{\infty} s_v(B_1)z^v \in \mathbb{C}$ and $f_{2s}(z) = \sum_{v=0}^{\infty} s_v(B_2)z^v \in \mathbb{C}$. Therefore, for $h_s(z) = \sum_{v=0}^{\infty} s_v(B_1 + B_2)z^v$, one can see that

$$g(B_1 + B_2) = \rho(h_s) \leq \rho(f_{1s} + f_{2s}) \leq K(\rho(f_{1s}) + \rho(f_{2s})) \leq KK_0(g(B_1) + g(B_2)). \quad (29)$$

(4) We have $C \geq 1$; suppose $A \in L(X_0, X)$, $B \in S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)$, and $D \in L(Y, Y_0)$. Therefore, $f_s(z) = \sum_{v=0}^{\infty} s_v(B)z^v \in \mathbb{C}$. Then, for $h_s(z) = \sum_{v=0}^{\infty} s_v(DBA)z^v$, one can see that

$$g(DBA) = \rho(h_s) \leq \rho(\|A\|\|D\|f_s) \leq C\|A\|g(B)\|D\|. \quad (30)$$

\square

Theorem 14. *If X and Y are Banach spaces and conditions (a1) and (a2) are satisfied, then $(S_{\mathbb{H}((b_v), (p_v))_\rho}, g)$ is a pre-quasi Banach mapping ideal.*

Proof. Suppose the conditions are verified, then the function $g(B) = \rho(f_s)$ is a pre-quasi norm on $S_{\mathbb{H}((b_v), (p_v))_\rho}$. Let (B_m) be a Cauchy sequence in $S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)$. Therefore, $f_s^{(m)} \in \mathbb{H}((b_v), (p_v))_\rho \in \mathbb{C}$ and $f_s^{(m)}(z) = \sum_{v=0}^{\infty} s_v(B_m)z^v \in \mathbb{C}$. Assume $h_s(z) = \sum_{v=0}^{\infty} s_v(B_i - B_j)z^v$; then, by using conditions (iv) and (vii) of Definition 3 and since $L(X, Y) \supseteq S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)$, we get

$$\begin{aligned} g(B_i - B_j) &= \rho(h_s) \geq \rho(s_0(B_i - B_j)e^{(0)}) \\ &= \rho(\|B_i - B_j\|e^{(0)}) \geq \xi\|B_i - B_j\|\rho(e^{(0)}). \end{aligned} \quad (31)$$

Thus, $(B_m)_{m \in \mathbb{N}}$ is a Cauchy sequence in $L(X, Y)$. While the space $L(X, Y)$ is a Banach space, there exists $B \in L(X, Y)$ with $\lim_{m \rightarrow \infty} \|B_m - B\| = 0$ and since $f_s^{(m)} \in \mathbb{H}((b_v), (p_v))_\rho$ for each $m \in \mathbb{N}$, using Theorem 13 and the continuity of ρ at θ , we obtain

$$\begin{aligned} g(B) &= g(B - B_m + B_m) \leq KK_0(g(B_m - B) + g(B_m)) \\ &= KK_0\rho\left(\|B_m - B\| \sum_{m=0}^{\infty} e^{(m)}\right) + KK_0\rho(f_s^{(m)}) < \varepsilon. \end{aligned} \quad (32)$$

Thus, we have $f_s \in \mathbb{H}((b_v), (p_v))_\rho$; then, $B \in S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)$. \square

Theorem 15. *If X and Y are Banach spaces and conditions (a1) and (a2) are satisfied, then $(S_{\mathbb{H}((b_v), (p_v))_\rho}, g)$ is a pre-quasi closed mapping ideal.*

Proof. Suppose the conditions are verified; then, the function $g(B) = \rho(f_s)$ is a pre-quasi norm on $S_{\mathbb{H}((b_v), (p_v))_\rho}$. Assume $B_m \in S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)$, with $m \in \mathbb{N}$ and $\lim_{m \rightarrow \infty} g(B_m - B) = 0$. Therefore, $f_s^{(m)} \in \mathbb{H}((b_v), (p_v))_\rho \in \mathbb{C}$ and $f_s^{(m)}(z) = \sum_{v=0}^{\infty} s_v(B_m)z^v \in \mathbb{C}$. Suppose $h_s(z) = \sum_{v=0}^{\infty} s_v(B_i - B_j)z^v$; then, from conditions (iv) and (vii) of Definition 3 and since $L(X, Y) \supseteq S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)$, we get

$$\begin{aligned} g(B - B_j) &= \rho(h_s) \geq \rho(s_0(B - B_j)e^{(0)}) = \rho(\|B - B_j\|e^{(0)}) \\ &\geq \xi\|B - B_j\|\rho(e^{(0)}), \end{aligned} \quad (33)$$

and then $(B_m)_{m \in \mathbb{N}}$ is a convergent sequence in $L(X, Y)$. While the space $L(X, Y)$ is a Banach space, there exists $B \in L(X, Y)$ with $\lim_{m \rightarrow \infty} \|B_m - B\| = 0$ and since $f_s^{(m)} \in \mathbb{H}((b_v), (p_v))_\rho$ for each $m \in \mathbb{N}$, using Theorem 13 and the continuity of ρ at θ , one can see that

$$\begin{aligned} g(B) &= g(B - B_m + B_m) \leq KK_0(g(B_m - B) + g(B_m)) \\ &= KK_0\rho\left(\|B_m - B\| \sum_{m=0}^{\infty} e^{(m)}\right) + KK_0\rho(f_s^{(m)}) < \varepsilon, \end{aligned} \quad (34)$$

and we have $f_s \in \mathbb{H}((b_v), (p_v))_\rho$; then, $B \in S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)$. \square

We deduce the following characteristics of the s -type $\mathbb{H}((b_v), (p_v))_\rho$ using Theorem 5.

Theorem 16. *For s -type $\mathbb{H}((b_v), (p_v))_\rho := \{(s_n(T)) \in \mathbb{R}^{\mathbb{N}} : T \in S_{\mathbb{H}((b_v), (p_v))_\rho}(X, Y)\}$, the following holds:*

(1) *We have s -type $\mathbb{H}((b_v), (p_v))_\rho^{\mathcal{F}}$.*

- (2) If $(s_r(T_1))_{r=0}^\infty \in s\text{-type } \mathbb{H}((b_n), (p_n))_\rho$ and $(s_r(T_2))_{r=0}^\infty \in s\text{-type } \mathbb{H}((b_n), (p_n))_\rho$ then $(s_r(T_1 + T_2))_{r=0}^\infty \in s\text{-type } \mathbb{H}((b_n), (p_n))_\rho$.
- (3) For all $\lambda \in \mathbb{C}$ and $(s_r(T))_{r=0}^\infty \in s\text{-type } \mathbb{H}((b_n), (p_n))_\rho$ then $|\lambda|(s_r(T))_{r=0}^\infty \in s\text{-type } \mathbb{H}((b_n), (p_n))_\rho$.
- (4) The $s\text{-type } \mathbb{H}((b_n), (p_n))_\rho$ is solid.

4.3. Smallness. We give here some inclusion relations concerning the space $S_{\mathbb{H}((b_n), (p_n))_\rho}$ for different (b_n) and (p_n) .

Theorem 17. If X and Y are Banach spaces with $\dim(X) = \dim(Y) = \infty$, $0 < p_n < q_n$, $0 < a_n < b_n$, for all $n \in \mathbb{N}$, and setups (a1) and (a2) are satisfied, it is true that

$$S_{\mathbb{H}((b_n), (p_n))_\rho}(X, Y) \subsetneq S_{\mathbb{H}((a_n), (q_n))_\rho}(X, Y) \subsetneq L(X, Y). \quad (35)$$

Proof. Suppose $T \in S_{\mathbb{H}((b_n), (p_n))_\rho}(X, Y)$. Therefore, $f_s \in \mathbb{H}((b_n), (p_n))_\rho$ and $f_s(z) = \sum_{n=0}^\infty s_n(T)z^n \in \mathbb{C}$. One can see that

$$\sum_{n=0}^\infty (a_n s_n(T))^{q_n} < \sum_{n=0}^\infty (b_n s_n(T))^{p_n} < \infty, \quad (36)$$

hence $T \in S_{\mathbb{H}((a_n), (q_n))_\rho}(X, Y)$. Next, if we take T with $s_n(T) = ((n+1)^{-1/p_n}/b_n)$, then $T \notin S_{\mathbb{H}((b_n), (p_n))_\rho}(X, Y)$ and $T \in S_{\mathbb{H}((a_n), (q_n))_\rho}(X, Y)$. Clearly, $S_{\mathbb{H}((a_n), (q_n))_\rho}(X, Y) \subset L(X, Y)$. By choosing T with $s_n(T) = ((n+1)^{-1/q_n}/a_n)$, then $T \notin S_{\mathbb{H}((a_n), (q_n))_\rho}(X, Y)$ and $T \in L(X, Y)$. This finishes the proof. \square

In this part, we investigate the setups for which $S_{\mathbb{H}((b_n), (p_n))_\rho}^{\text{app}}$ is small.

Theorem 18. If X and Y are Banach spaces with $\dim(X) = \dim(Y) = \infty$, assume that the conditions (a1), (a2), and $(b_n) \notin \ell^{(p_n)}$ are satisfied, and hence $(S_{\mathbb{H}((b_n), (p_n))_\rho}^{\text{app}}, g)$ is small, where $g(U) = \sum_{j=0}^\infty (b_j \alpha_j(U))^{p_j}$.

Proof. Let $S_{\mathbb{H}((b_n), (p_n))_\rho}^{\text{app}}(X, Y) = L(X, Y)$. Therefore, one gets $V > 0$ so that $g(U) \leq V\|U\|$, for every $U \in L(X, Y)$. According to Dvoretzky's theorem [26] with $r \in \mathbb{N}$, there are quotient spaces X/λ_r and subspaces η_r of Y that mapped onto ℓ_2^r by isomorphisms D_r and B_r with $\|D_r\|\|D_r^{-1}\| \leq 2$ and $\|B_r\|\|B_r^{-1}\| \leq 2$. Let I_r be the identity mapping on ℓ_2^r , ζ_r be the quotient mapping from X onto X/λ_r , and J_r be the natural embedding mapping from η_r into Y . Let h_a , for all $a \in \mathbb{N}$, be the Bernstein numbers [27]; we have then

$$\begin{aligned} 1 &= h_a(I_r) = h_a(B_r B_r^{-1} I_r D_r D_r^{-1}) \leq \|B_r\| h_a(B_r^{-1} I_r D_r) \|D_r^{-1}\|, \\ &= \|B_r\| h_a(J_r B_r^{-1} I_r D_r) \|D_r^{-1}\| \leq \|B_r\|_r d_a(J_r B_r^{-1} I_r D_r) \|D_r^{-1}\| \\ &= \|B_r\| d_a(J_r B_r^{-1} I_r D_r \zeta_r) \|D_r^{-1}\| \leq \|B_r\|_r \alpha_a(J_r B_r^{-1} I_r D_r \zeta_r) \|D_r^{-1}\|, \end{aligned} \quad (37)$$

for $0 \leq j \leq r$. We have $l \geq 1$ so that

$$\begin{aligned} b_j^{p_j} &\leq (\|B_r\| \|D_r^{-1}\|)^{p_j} (b_j \alpha_j(J_r B_r^{-1} I_r D_r \zeta_r))^{p_j}, \\ b_j^{p_j} &\leq l \|B_r\| (b_j \alpha_j(J_r B_r^{-1} I_r D_r \zeta_r))^{p_j} \|D_r^{-1}\|, \\ \sum_{j=0}^r b_j^{p_j} &\leq l \|B_r\| \|D_r^{-1}\| \sum_{j=0}^r b_j (\alpha_j(J_r B_r^{-1} I_r D_r \zeta_r))^{p_j}, \\ \sum_{j=0}^r b_j^{p_j} &\leq l \|B_r\| \|D_r^{-1}\| g(J_r B_r^{-1} I_r D_r \zeta_r), \\ \sum_{j=0}^r b_j^{p_j} &\leq l V \|B_r\| \|D_r^{-1}\| \|J_r B_r^{-1} I_r D_r \zeta_r\|_r, \\ \sum_{j=0}^r b_j^{p_j} &\leq l V \|B_r\| \|D_r^{-1}\| \|J_r\| \|B_r^{-1}\| \|I_r\| \|D_r \zeta_r\| \\ &= l V \|B_r\| \|D_r^{-1}\| \|B_r^{-1}\| \|I_r\| \|D_r\|, \\ \sum_{j=0}^r b_j^{p_j} &\leq 4lV. \end{aligned} \quad (38)$$

As $r \rightarrow \infty$, then $\sum_{j=0}^\infty b_j^{p_j} < \infty$. This contradicts $(b_n) \notin \ell^{(p_n)}$. Therefore, $\dim(X) < \infty$ and $\dim(Y) < \infty$. Hence, the space $S_{\mathbb{H}((b_n), (p_n))_\rho}^{\text{app}}$ is small. \square

By the same manner, we can easily conclude the next theorem.

Theorem 19. If X and Y are Banach spaces with $\dim(X) = \dim(Y) = \infty$, assume that conditions (a1), (a2), and $(b_n) \notin \ell^{(p_n)}$ are satisfied, and hence $(S_{\mathbb{H}((b_n), (p_n))_\rho}^{\text{Kol}}, g)$ is small, where $g(U) = \sum_{j=0}^\infty (b_j d_j(U))^{p_j}$.

4.4. Simplesness. We introduce an answer of the next question; for which $\mathbb{H}((b_n), (p_n))_\rho$ is the space $S_{\mathbb{H}((b_n), (p_n))_\rho}$ simple?

Theorem 20. If (p_n) , (q_n) verify $1 \leq p_n < q_n$ and $0 < a_n < b_n$, for all $n \in \mathbb{N}$, and the setups (a1), (a2) are satisfied, then

$$\begin{aligned} &L\left(S_{\mathbb{H}((a_n), (q_n))_\rho}, S_{\mathbb{H}((b_n), (p_n))_\rho}\right) \\ &= \Lambda\left(S_{\mathbb{H}((a_n), (q_n))_\rho}, S_{\mathbb{H}((b_n), (p_n))_\rho}\right). \end{aligned} \quad (39)$$

Proof. Suppose there is $T \in L(S_{\mathbb{H}((a_n), (q_n))_\rho}, S_{\mathbb{H}((b_n), (p_n))_\rho})$, and $T \notin \Lambda(S_{\mathbb{H}((a_n), (q_n))_\rho}, S_{\mathbb{H}((b_n), (p_n))_\rho})$. According to Lemma 1, we

can find $G \in L(S_{\mathbb{H}((a_n), (q_n))_\rho})$ and $B \in L(S_{\mathbb{H}((b_n), (p_n))_\rho})$ with $BTGI_m = I_m$. For every $m \in \mathbb{N}$, one has

$$\begin{aligned} \|I_m\|_{S_{\mathbb{H}((b_n), (p_n))_\rho}} &= \left(\sum_{v=0}^{\infty} (b_v \alpha_v (I_m))^{p_v} \right)^{(1/\sup p_v)} = \left(\sum_{v=0}^{m-1} b_v \right)^{(1/\sup p_v)} \leq \|BTG\| \|I_m\|_{S_{\mathbb{H}((a_n), (q_n))_\rho}} \\ &\leq \left(\sum_{v=0}^{\infty} (a_v \alpha_v (I_m))^{q_v} \right)^{(1/\sup q_v)} = \left(\sum_{v=0}^{m-1} a_v \right)^{(1/\sup q_v)}. \end{aligned} \quad (40)$$

This contradicts Theorem 17. \square

Corollary 1. If $(p_n), (q_n)$ verify $1 \leq p_n < q_n$ and $0 < a_n < b_n$, for all $n \in \mathbb{N}$, and the setups (a1), (a2) are satisfied, then $L((S_{\mathbb{H}((a_n), (q_n))_\rho}, S_{\mathbb{H}((b_n), (p_n))_\rho})) = L_C((S_{\mathbb{H}((a_n), (q_n))_\rho}, S_{\mathbb{H}((b_n), (p_n))_\rho}))$.

Proof. It is clear from $\Lambda \subseteq L_C$. \square

Theorem 21. If setups (a1), (a2) are satisfied with $p_0 \geq 1$, then the space $S_{\mathbb{H}((b_n), (p_n))_\rho}$ is simple.

Proof. Assume $T \in L_C(S_{\mathbb{H}((b_n), (p_n))_\rho})$ and $T \notin \Lambda(S_{\mathbb{H}((b_n), (p_n))_\rho})$. From Lemma 1, one has $G, B \in L(S_{\mathbb{H}((b_n), (p_n))_\rho})$ so that $BTGI_k = I_k$. We have $I_{\mathbb{H}((b_n), (p_n))_\rho} \in L_C(S_{\mathbb{H}((b_n), (p_n))_\rho})$. Therefore, $L(S_{\mathbb{H}((b_n), (p_n))_\rho}) = L_C(S_{\mathbb{H}((b_n), (p_n))_\rho})$. This implies that there is one non-trivial closed ideal $\Lambda(S_{\mathbb{H}((b_n), (p_n))_\rho})$ in $L(S_{\mathbb{H}((b_n), (p_n))_\rho})$. \square

4.5. Spectrum. In this part, we expound the sufficient conditions on $\mathbb{H}((b_n), (p_n))_\rho$ such that $(S_{\mathbb{H}((b_n), (p_n))_\rho})^\lambda$ equals $S_{\mathbb{H}((b_n), (p_n))_\rho}$.

Theorem 22. If X and Y are Banach spaces with $\dim(X) = \dim(Y) = \infty$ and suppose setups (a1), (a2) are satisfied and $\inf b_n^{p_n} > 0$, then

$$\left(S_{\mathbb{H}((b_n), (p_n))_\rho} \right)^\lambda (X, Y) = S_{\mathbb{H}((b_n), (p_n))_\rho} (X, Y). \quad (41)$$

Proof. Let $T \in (S_{\mathbb{H}((b_n), (p_n))_\rho})^\lambda (X, Y)$, and hence $f_\lambda \in \mathbb{H}((b_n), (p_n))_\rho$, where $f_\lambda(z) = \sum_{n=0}^{\infty} \lambda_n(T) z^n \in \mathbb{C}$ with $\rho(f_\lambda) = \sum_{n=0}^{\infty} |b_n \lambda_n(T)|^{p_n} < \infty$, and $\|T - \lambda_l(T)I\| = 0$, for all $l \in \mathbb{N}$. We have $T = \lambda_l(T)I$, with $l \in \mathbb{N}$, so $s_l(T) = s_l(\lambda_l(T)I) = |\lambda_l(T)|$, with $l \in \mathbb{N}$. Therefore, $f_s \in \mathbb{H}((b_n), (p_n))_\rho$, so $T \in S_{\mathbb{H}((b_n), (p_n))_\rho}(X, Y)$.

Secondly, assume $T \in S_{\mathbb{H}((b_n), (p_n))_\rho}(X, Y)$. Therefore, $f_s \in \mathbb{H}((b_n), (p_n))_\rho$. Hence, we have

$$\infty > \sum_{r=0}^{\infty} |b_r s_r(T)|^{p_r} \geq \inf_r b_r^{p_r} \sum_{r=0}^{\infty} [s_r(T)]^{p_r}. \quad (42)$$

Since $\inf b_r^{p_r} > 0$, then $\lim_{r \rightarrow \infty} s_r(T) = 0$. Assume $\|T - s_r(T)I\|^{-1}$ exists, for every $r \in \mathbb{N}$. Therefore, $(\|T -$

$s_r(T)I\|^{-1})_{r \in \mathbb{N}} \in \ell_\infty$. So, $\lim_{r \rightarrow \infty} \|T - s_r(T)I\|^{-1} = \|T\|^{-1}$ exists and is bounded. From the pre-quasi mapping ideal of $(S_{\mathbb{H}((b_n), (p_n))_\rho}, g)$, we obtain

$$\begin{aligned} I = TT^{-1} &\in S_{(\mathbb{H}((b_r), (p_r))_\rho)}(X, Y) \Rightarrow \sum_{r=0}^{\infty} e^{(r)} \\ &\in \mathbb{H}((b_r), (p_r))_\rho \Rightarrow \sum_{r=0}^{\infty} b_r^{p_r} < \infty. \end{aligned} \quad (43)$$

This contradicts $\inf b_r^{p_r} > 0$. Therefore, $\|T - s_r(T)I\| = 0$, for every $r \in \mathbb{N}$. This gives $T \in (S_{\mathbb{H}((b_n), (p_n))_\rho})^\lambda (X, Y)$. This provides the proof. \square

5. Application of Shift

Mappings on $\mathbb{H}((b_r), (p_r))_\rho$

Specifically, we explore the upper limits of s -numbers for infinite series of the weighted r -th power forward and backward shift mapping on $\mathbb{H}((b_r), (p_r))_\rho$ and their applications to various entire functions in this section, where $\rho(f) = [\sum_{r=0}^{\infty} |b_r \hat{f}_r|^{p_r}]^{(1/\omega_p)}$, for all $f \in \mathbb{H}((b_r), (p_r))_\rho$.

Theorem 23. Let conditions (a1) and (a2) be satisfied, $\inf b_n \geq 1$, and $\sup_r (b_{r+1}/b_r)^{p_{r+1}/\omega_p} < \infty$; then, $V_z \in L^n(\mathbb{H}((b_r), (p_r))_\rho)$ with $\|V_z\| = \sup_r (b_{r+1}/b_r)^{(p_{r+1}/\omega_p)}$.

Proof. Assume that the conditions are satisfied. For $f \in \mathbb{H}((b_r), (p_r))_\rho$, since (p_r) is increasing and bounded from above with $p_r > 0$, for all $r \in \mathbb{N}$, then

$$\begin{aligned} \rho(V_z f) &= \rho(z f) = \left[\sum_{r=0}^{\infty} |b_{r+1} \hat{f}_r|^{p_{r+1}} \right]^{1/\omega_p} \leq \sup_r \left(\frac{b_{r+1}}{b_r} \right)^{p_{r+1}/\omega_p} \\ &\quad \left[\sum_{r=0}^{\infty} |b_r \hat{f}_r|^{p_{r+1}} \right]^{1/\omega_p} \\ &\leq \sup_r \left(\frac{b_{r+1}}{b_r} \right)^{p_{r+1}/\omega_p} \rho(f). \end{aligned} \quad (44)$$

This gives $V_z \in L(\mathbb{H}((b_r), (p_r))_\rho)$ with $\|V_z\| \leq \sup_r (b_{r+1}/b_r)^{(p_{r+1}/\omega_p)}$. Since $V_z \in L(\mathbb{H}((b_r), (p_r))_\rho)$, then there is $A > 0$ with $\rho(V_z f) \leq A\rho(f)$, for all $f \in \mathbb{H}((b_r), (p_r))_\rho$. Hence, $\rho(V_z e^{(r)}) \leq A\rho(e^{(r)})$, and one

gets $\sup_r (b_{r+1}/b_r)^{(p_{r+1}/\omega_p)} \leq \|V_z\|$. This completes the proof. \square

Theorem 24. Let conditions (a1) and (a2) be satisfied, $\sup_n b_n \geq 1$, and $\sup_r (b_r/b_{r+1})^{(p_r/\omega_p)} < \infty$; then, $B_z \in L(\mathbb{H}((b_r), (p_r))_\rho)$ with $\|B_z\| = \sup_r (b_r/b_{r+1})^{(p_r/\omega_p)}$.

Proof. Assume the conditions are satisfied. For $f \in \mathbb{H}((b_r), (p_r))_\rho$, since (p_r) is increasing and bounded from above with $p_r > 0$, for all $r \in \mathbb{N}$, then

$$\begin{aligned} \rho(B_z f) &= \left[\sum_{r=0}^{\infty} |b_r \widehat{f_{r+1}}|^{p_r} \right]^{1/\omega_p} \leq \sup_r \left(\frac{b_r}{b_{r+1}} \right)^{p_r/\omega_p} \\ &= \left[\sum_{r=0}^{\infty} |b_{r+1} \widehat{f_{r+1}}|^{p_r} \right]^{1/\omega_p} \leq \sup_r \left(\frac{b_r}{b_{r+1}} \right)^{p_r/\omega_p} \rho(f). \end{aligned} \quad (45)$$

This gives $B_z \in L(\mathbb{H}((b_r), (p_r))_\rho)$ with $\|B_z\| \leq \sup_r (b_r/b_{r+1})^{(p_r/\omega_p)}$. Since $B_z \in L(\mathbb{H}((b_r), (p_r))_\rho)$, then there is $A > 0$ with $\rho(B_z f) \leq A\rho(f)$, for all $f \in \mathbb{H}((b_r), (p_r))_\rho$. Hence, $\rho(B_z e^{(r)}) \leq A\rho(e^{(r)})$, and one gets $\sup_r (b_r/b_{r+1})^{p_r/\omega_p} \leq \|B_z\|$. This completes the proof. \square

By \mathbb{U} , we denote the open unit disc in \mathbb{C} .

Theorem 25. Let conditions (a1) and (a2) be satisfied with $p_0 \geq 1$. If $\limsup \sqrt[p_r]{b_r} = 1$, then every function in $\mathbb{H}((b_r), (p_r))_\rho$ is analytic on \mathbb{U} . Furthermore, the convergence in $\mathbb{H}((b_r), (p_r))_\rho$ implies the uniform convergence on $B \subseteq \mathbb{U}$, where B is compact.

Proof. Let $\limsup \sqrt[p_r]{b_r} = 1$, and $h \in \mathbb{H}((b_r), (p_r))_\rho$. Then, $h(y) = \sum_{r=0}^{\infty} \widehat{h}_r y^r \in \mathbb{C}$, with $y \in \mathbb{C}$ and $\rho(h) = [\sum_{r=0}^{\infty} |b_r \widehat{h}_r|^{p_r}]^{1/\omega_p} < \infty$. Therefore, $\limsup \sqrt[p_r]{|b_r \widehat{h}_r|^{p_r}} < 1$. This gives

$$\limsup \sqrt[p_r]{|\widehat{h}_r|^{p_r}} < \frac{1}{\limsup \sqrt[p_r]{|b_r|^{p_r}}} = 1. \quad (46)$$

As $(p_r) \in mi_\gamma \cap \ell_\infty$, one gets $\limsup \sqrt[p_r]{|\widehat{h}_r|} |y| < |y| < 1$, with $y \in \mathbb{U}$. Hence, $h(y) = \sum_{r=0}^{\infty} \widehat{h}_r y^r \in \mathbb{C}$, with $y \in \mathbb{U}$. Assume $h^k(y) \in B$, with $k \in \mathbb{N}$. Suppose $\lim_{k \rightarrow \infty} \rho(h^k - h) = 0$, where $h \in \mathbb{H}((b_r), (p_r))_\rho$, and we have

$$\begin{aligned} |h^k(y) - h(y)| &= \left| \sum_{r=0}^{\infty} (\widehat{h}_r^k - \widehat{h}_r) y^r \right| \leq \left| \sum_{r=0}^{\infty} \|\widehat{h}_r^k - \widehat{h}_r\| y^r \right| \\ &\leq \left[\sum_{r=0}^{\infty} |\widehat{h}_r^k - \widehat{h}_r|^{p_r} b_r^{p_r} \right]^{1/\omega_p} \\ &= \left[\sum_{r=0}^{\infty} \frac{|y|^{r q_r}}{b_r^{q_r}} \right]^{1/\omega_q} \rho(h^k - h), \left[\sum_{r=0}^{\infty} \frac{|y|^{r q_r}}{b_r^{q_r}} \right]^{1/\omega_q}, \end{aligned} \quad (47)$$

where (q_r) is increasing and bounded with $q_0 \geq 1$ and $(1/p_r) + (1/q_r) = 1$, for all $r \in \mathbb{N}$. Clearly, $\limsup_{r \rightarrow \infty}$

$(|y|^{q_r}/b_r^{(q_r/r)}) < 1$; then, $\sum_{r=0}^{\infty} |y|^{r q_r}/b_r^{q_r} < \infty$. So, $\lim_{k \rightarrow \infty} h^k(y) = h(y) \in B$. \square

Theorem 26. If V_z is the forward shift mapping on $\mathbb{H}((b_r), (p_r))_\rho$ we have

$$\begin{aligned} \sup_{\text{card } \xi=r+1} \inf_{k \in \xi} \left(\frac{b_{k+n}}{b_k} \right)^{(p_{k+n}/\omega_p)} \frac{1}{A_n} &\leq s_r(V_z^n) \\ &\leq \sup_{\text{card } \xi=r+1} \inf_{k \in \xi} \left(\frac{b_{k+n}}{b_k} \right)^{(p_{k+n}/\omega_p)}, \end{aligned} \quad (48)$$

where $A_n = [[\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_k}]^{(1/\omega_p)} / [\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_{k+n}}]^{(1/\omega_p)}]$.

Proof. Let $\text{card } \xi = r + 1$ and $V_z^n f \in \mathbb{H}((b_r), (p_r))_\rho$, for all $f \in \mathbb{H}((b_r), (p_r))_\rho$, for which $f(y) = \sum_{k=0}^{\infty} \widehat{f}_k y^k \in \mathbb{C}$ with $y \in \mathbb{C}$ and $\rho(f) = [\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_k}]^{(1/\omega_p)} < \infty$. Therefore, $V_z^n f(z) = \sum_{k=0}^{\infty} \widehat{f}_k z^{k+n}$ and $\rho(V_z^n f) = [\sum_{k=0}^{\infty} |b_{k+n} \widehat{f}_k|^{p_{k+n}}]^{(1/\omega_p)} < \infty$.

Let P_ξ be a mapping on $\mathbb{H}((b_r), (p_r))_\rho$ with rank $P_\xi = r + 1$ defined by

$$(P_\xi g)(z) = P_\xi \left(\sum_{k=0}^{\infty} \widehat{f}_k z^{k+n} \right) = \sum_{k \in \xi} \widehat{f}_k z^{k+n}. \quad (49)$$

Since $\rho(P_\xi g) = [\sum_{k \in \xi} |b_{k+n} \widehat{f}_k|^{p_{k+n}}]^{(1/\omega_p)} \leq [\sum_{k=0}^{\infty} |b_{k+n} \widehat{f}_k|^{p_{k+n}}]^{(1/\omega_p)} = \rho(g)$, this gives $\|P_\xi\| \leq 1$. Define a mapping S_z^n by $(S_z^n h)(z) = S_z^n (\sum_{k \in \xi} \widehat{f}_k z^{k+n}) = \sum_{k=0}^{\infty} \widehat{f}_k z^k$, and we have

$$\rho(S_z^n h) = \left[\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_k} \right]^{(1/\omega_p)} \leq U_n \left[\sum_{k \in \xi} |b_{k+n} \widehat{f}_k|^{p_{k+n}} \right]^{(1/\omega_p)} = U_n \rho(g). \quad (50)$$

This implies that $\|S_z^n\| \leq U_n$, where $1 \leq U_n = [\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_k}]^{(1/\omega_p)} / [\sum_{k \in \xi} |b_{k+n} \widehat{f}_k|^{p_{k+n}}]^{(1/\omega_p)} < \infty$. Then, the identity mapping will be $I_{r+1} = P_\xi V_z^n S_z^n$, and from the definition of s -numbers, we have

$$s_r(I_{r+1}) = 1 \leq \|P_\xi\| s_r(V_z^n) \|S_z^n\| \leq s_r(V_z^n) \|S_z^n\| \Rightarrow$$

$$s_r(V_z^n) \geq \frac{1}{\|S_z^n\|} \geq \frac{1}{U_n} = \frac{[\sum_{k \in \xi} |b_{k+n} \widehat{f}_k|^{p_{k+n}}]^{(1/\omega_p)}}{[\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_k}]^{(1/\omega_p)}} \quad (51)$$

$$\geq \inf_{k \in \xi} \left(\frac{b_{k+n}}{b_k} \right)^{(p_{k+n}/\omega_p)} \frac{1}{A_n}.$$

Since for all $\text{card } \xi = r + 1$, the last inequality is verified, so one can see that

$$s_r(V_z^n) \geq \sup_{\text{card } \xi=r+1} \inf_{k \in \xi} \left(\frac{b_{k+n}}{b_k} \right)^{(p_{k+n}/\omega_p)} \frac{1}{A_n}. \quad (52)$$

In contrary, let $\text{card } \xi = r$, where $\xi \subset \mathbb{N}$. Define the mapping R_z^n as $(R_z^n v)(z) = R_z^n (\sum_{k=0}^{\infty} \widehat{f}_k z^k) = \sum_{k \in \xi} \widehat{f}_k z^k$

$z^{k+n} \sum_{k \in \xi} \widehat{f}_k z^{k+n}$. From the definition of approximation numbers, we have

$$\begin{aligned} s_r(V_z^n) &\leq \alpha_r(V_z^n) \leq \|V_z^n - R_z^n\| \leq \sup_{|f(z)| \neq 0} \frac{|(V_z^n - R_z^n)f(z)|}{|f(z)|} = \sup_{|f(z)| \neq 0} \frac{\sum_{k \notin \xi} \widehat{f}_k z^{k+n}}{|f(z)|} \\ &\leq \sup_{|f(z)| \neq 0} \frac{\left[\sum_{k \notin \xi} |b_{k+n} \widehat{f}_k|^{p_{k+n}} \right]^{(1/\omega_p)}}{|f(z)|} \leq \sup_{k \notin \xi} \left(\frac{b_{k+n}}{b_k} \right)^{(p_{k+n}/\omega_p)} \end{aligned} \quad (53)$$

Since for all $\text{card } \xi = r$, the last inequality holds and by using Lemma 2, one has

$$\sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \left(\frac{b_{k+n}}{b_k} \right)^{(p_{k+n}/\omega_p)} \frac{1}{A_n} \leq s_r(V_z^n) \leq \inf_{\text{card } \xi = r} \sup_{k \notin \xi} \left(\frac{b_{k+n}}{b_k} \right)^{(p_{k+n}/\omega_p)} = \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \left(\frac{b_{k+n}}{b_k} \right)^{(p_{k+n}/\omega_p)}. \quad (54)$$

This completes the proof. \square

Theorem 27. If B_z is the backward shift mapping on $\mathbb{H}((b_r), (p_r))_\rho$ then

$$\begin{aligned} \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \left(\frac{b_k}{b_{k+n}} \right)^{(p_k/\omega_p)} \frac{1}{G_n} &\leq s_r(B_z^n) \\ &\leq \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \left(\frac{b_k}{b_{k+n}} \right)^{(p_k/\omega_p)}, \end{aligned} \quad (55)$$

where $G_n = [\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_k}]^{(1/\omega_p)} / [(\sum_{k \in \xi} |b_{k+n} \widehat{f}_{k+n}|^{p_{k+n}})^{(1/\omega_p)}]$.

Proof. Assume $\text{card } \xi = r+1$ and $B_z^n f \in \mathbb{H}((b_r), (p_r))_\rho$ for every $f \in \mathbb{H}((b_r), (p_r))_\rho$ where $f(y) = \sum_{k=0}^{\infty} \widehat{f}_k y^k \in \mathbb{C}$ with $y \in \mathbb{C}$ and $\rho(f) = [\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_k}]^{(1/\omega_p)} < \infty$. Therefore, $B_z^n f(z) = \sum_{k=0}^{\infty} \widehat{f}_{k+n} z^k$ and $\rho(B_z^n f) = [\sum_{k=0}^{\infty} |b_k \widehat{f}_{k+n}|^{p_k}]^{(1/\omega_p)} < \infty$.

Suppose P_ξ is a mapping on $\mathbb{H}((b_r), (p_r))_\rho$ with rank $P_\xi = r+1$ evident by

$$(P_\xi g)(z) = P_\xi \left(\sum_{k=0}^{\infty} \widehat{f}_{k+n} z^k \right) = \sum_{k \in \xi} \widehat{f}_{k+n} z^k. \quad (56)$$

As $\rho(P_\xi g) = [\sum_{k \in \xi} |b_k \widehat{f}_{k+n}|^{p_k}]^{(1/\omega_p)} \leq [\sum_{k=0}^{\infty} |b_k \widehat{f}_{k+n}|^{p_k}]^{(1/\omega_p)} = \rho(g)$. This implies that $\|P_\xi\| \leq 1$. Define a mapping S_z^n by $(S_z^n h)(z) = S_z^n (\sum_{k \in \xi} \widehat{f}_{k+n} z^k) = \sum_{k=0}^{\infty} \widehat{f}_k z^k$, and one gets

$$\rho(S_z^n h) = \left[\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_k} \right]^{(1/\omega_p)} \leq U_n \left[\sum_{k \in \xi} |b_k \widehat{f}_{k+n}|^{p_k} \right]^{(1/\omega_p)} = U_n \rho(h). \quad (57)$$

Therefore, $\|S_z^n\| \leq U_n$, where $1 \leq U_n = [\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_k}]^{(1/\omega_p)} / [\sum_{k \in \xi} |b_k \widehat{f}_{k+n}|^{p_k}]^{(1/\omega_p)} < \infty$. Hence, the identity mapping will be $I_{r+1} = P_\xi B_z^n S_z^n$, and in view of the definition of s -numbers, one has

$$s_r(I_{r+1}) = 1 \leq \|P_\xi\| s_r(B_z^n) \|S_z^n\| \leq s_r(B_z^n) \|S_z^n\| \Rightarrow$$

$$s_r(B_z^n) \geq \frac{1}{\|S_z^n\|} \geq \frac{1}{U_n} = \frac{\left[\sum_{k \in \xi} |b_k \widehat{f}_{k+n}|^{p_k} \right]^{(1/\omega_p)}}{\left[\sum_{k=0}^{\infty} |b_k \widehat{f}_k|^{p_k} \right]^{(1/\omega_p)}} \quad (58)$$

$$\geq \inf_{k \in \xi} \left(\frac{b_k}{b_{k+n}} \right)^{(p_k/\omega_p)} \frac{1}{G_n}.$$

Since for every $\text{card } \xi = r+1$, the last inequality is confirmed, and one obtains

$$s_r(B_z^n) \geq \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \left(\frac{b_k}{b_{k+n}} \right)^{(p_k/\omega_p)} \frac{1}{G_n}. \quad (59)$$

In contrary, let $\text{card } \xi = r$, where $\xi \subset \mathbb{N}$. Define the mapping R_z^n as $(R_z^n v)(z) = R_z^n (\sum_{k=0}^{\infty} \widehat{f}_k z^k) = \sum_{k \in \xi} \widehat{f}_{k+n} z^k$. From the definition of approximation numbers, one gets

$$\begin{aligned}
s_r(B_z^n) &\leq \alpha_r(B_z^n) \leq \|B_z^n - R_z^n\| \leq \sup_{|f(z)| \neq 0} \frac{|(B_z^n - R_z^n)f(z)|}{|f(z)|} = \sup_{|f(z)| \neq 0} \frac{|\sum_{k \notin \xi} \widehat{f_{k+n}} z^k|}{|f(z)|} \\
&\leq \sup_{|f(z)| \neq 0} \frac{\left[\sum_{k \notin \xi} |\widehat{b_k f_{k+n}}|^{p_k} \right]^{(1/\omega_p)}}{|f(z)|} \leq \sup_{k \notin \xi} \left(\frac{b_k}{b_{k+n}} \right)^{(p_k/\omega_p)}.
\end{aligned} \tag{60}$$

Since for all card $\xi = r$, the last inequality holds, and by using Lemma 2, one has

$$\begin{aligned}
&\sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \left(\frac{b_k}{b_{k+n}} \right)^{(p_k/\omega_p)} \frac{1}{G_n} \leq s_r(B_z^n) \\
&\leq \inf_{\text{card } \xi = r} \sup_{k \notin \xi} \left(\frac{b_k}{b_{k+n}} \right)^{(p_k/\omega_p)} = \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \left(\frac{b_k}{b_{k+n}} \right)^{(p_k/\omega_p)}.
\end{aligned} \tag{61}$$

This finishes the proof. \square

Theorem 28. If conditions (a1) and (a2) are satisfied with $p_0 \geq 1$, let $\sum_{m=0}^{\infty} c_m V_z^m$ be a shift mapping on the space $\mathbb{H}((b_r), (p_r))_\rho$ and $(c_m)_{m=0}^{\infty} \in \ell^{((p_m)/\omega_p)}$; then,

$$\begin{aligned}
&\sup_j \left[\sum_{m=0}^{\infty} |c_m|^{p_{m+j}} \frac{b_{m+j}^{p_{m+j}}}{b_j^{p_j}} \right]^{(1/\omega_p)} \leq \left\| \sum_{m=0}^{\infty} c_m V_z^m \right\| \\
&\leq \sup_{m,j} \left(\frac{b_{m+j}}{b_j} \right)^{(p_{m+j}/\omega_p)} \sum_{m=0}^{\infty} |c_m|^{(p_m/\omega_p)}.
\end{aligned} \tag{62}$$

$$\begin{aligned}
\left\| \sum_{m=0}^{\infty} c_m V_z^m \right\| &= \sup_{\rho(f) \neq 0} \frac{\rho(\sum_{m=0}^{\infty} c_m V_z^m f)}{\rho(f)} \leq \sup_{\rho(f) \neq 0} \frac{\sum_{m=0}^{\infty} \left[\sum_{j=0}^{\infty} (|c_m| |\widehat{f_j}| b_{m+j})^{p_{m+j}} \right]^{(1/\omega_p)}}{\left[\sum_{j=0}^{\infty} |\widehat{f_j} b_j|^{p_j} \right]^{(1/\omega_p)}} \\
&\leq \sup_{m,j} \left(\frac{b_{m+j}}{b_j} \right)^{(p_{m+j}/\omega_p)} \frac{\sum_{m=0}^{\infty} \left[\sum_{j=0}^{\infty} (|c_m| |\widehat{f_j}| b_j)^{p_{m+j}} \right]^{(1/\omega_p)}}{\left[\sum_{j=0}^{\infty} |\widehat{f_j} b_j|^{p_j} \right]^{(1/\omega_p)}} \leq \sup_{m,j} \left(\frac{b_{m+j}}{b_j} \right)^{(p_{m+j}/\omega_p)} \sum_{m=0}^{\infty} |c_m|^{(p_m/\omega_p)}.
\end{aligned} \tag{64}$$

Theorem 29. If conditions (a1) and (a2) are satisfied with $p_0 \geq 1$, let $\sum_{j=0}^{\infty} c_j B_z^j$ be a shift mapping on the space $\mathbb{H}((b_r), (p_r))_\rho$ and $(c_j)_{j=0}^{\infty} \in \ell^{((p_j)/\omega_p)}$; then,

$$\begin{aligned}
&\sup_k \left[\sum_{j=0}^{\infty} |c_j|^{p_k} \frac{b_k^{p_k}}{b_{k+j}^{p_{k+j}}} \right]^{1/\omega_p} \leq \left\| \sum_{j=0}^{\infty} c_j B_z^j \right\| \\
&\leq \sup_{j,k} \left(\frac{b_k}{b_{k+j}} \right)^{p_k/\omega_p} \sum_{j=0}^{\infty} |c_j|^{p_j/\omega_p}.
\end{aligned} \tag{65}$$

Proof. For $f \in \mathbb{H}((b_r), (p_r))_\rho$, we have $\sum_{m=0}^{\infty} c_m V_z^m f(z) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} c_m \widehat{f_{j+m}} z^{j+m}$. One has

$$\begin{aligned}
\left\| \sum_{m=0}^{\infty} c_m V_z^m \right\| &\geq \frac{\rho(\sum_{m=0}^{\infty} c_m V_z^m e^{(j)})}{\rho(e^{(j)})} = \left[\frac{\sum_{m=0}^{\infty} |c_m b_{m+j}|^{p_{m+j}}}{b_j^{p_j}} \right]^{(1/\omega_p)} \\
&\geq \sup_j \left[\sum_{m=0}^{\infty} |c_m|^{p_{m+j}} \frac{b_{m+j}^{p_{m+j}}}{b_j^{p_j}} \right]^{(1/\omega_p)}.
\end{aligned} \tag{63}$$

Since ρ satisfies the triangle inequality, we have

Proof. Suppose $f \in \mathbb{H}((b_r), (p_r))_\rho$ and one has $\sum_{j=0}^{\infty} c_j B_z^j f(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_j \widehat{f_{k+j}} z^k$. We have

$$\begin{aligned}
\left\| \sum_{j=0}^{\infty} c_j B_z^j \right\| &\geq \frac{\rho(\sum_{j=0}^{\infty} c_j B_z^j e^{(k)})}{\rho(e^{(k)})} = \left[\frac{\sum_{j=0}^{\infty} |b_{k-j} c_j|^{p_{k-j}}}{b_k^{p_k}} \right]^{(1/\omega_p)} \\
&\geq \sup_k \left[\sum_{j=0}^{\infty} |c_j|^{p_k} \frac{b_k^{p_k}}{b_{k+j}^{p_{k+j}}} \right]^{(1/\omega_p)}.
\end{aligned} \tag{66}$$

As ρ verifies the triangle inequality, one can see that

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} c_j B_z^j \right\| &= \sup_{\rho(f) \neq 0} \frac{\rho(\sum_{j=0}^{\infty} c_j B_z^j f)}{\rho(f)} \leq \sup_{\rho(f) \neq 0} \frac{\sum_{j=0}^{\infty} \left[\sum_{k=0}^{\infty} (b_k |c_j \widehat{f_{k+j}}|)^{p_k} \right]^{(1/\omega_p)}}{\left[\sum_{k=0}^{\infty} |b_k \widehat{f_k}|^{p_k} \right]^{(1/\omega_p)}} \\ &\leq \sup_{j,k} \left(\frac{b_k}{b_{k+j}} \right)^{(p_k/\omega_p)} \frac{\sum_{j=0}^{\infty} \left[\sum_{k=0}^{\infty} (b_{k+j} |c_j| |\widehat{f_{k+j}}|)^{p_k} \right]^{(1/\omega_p)}}{\left[\sum_{k=0}^{\infty} |b_k \widehat{f_k}|^{p_k} \right]^{(1/\omega_p)}} \leq \sup_{j,k} \left(\frac{b_k}{b_{k+j}} \right)^{(p_k/\omega_p)} \sum_{j=0}^{\infty} |c_j|^{p_j/\omega_p}. \end{aligned} \quad (67)$$

Theorem 30. If conditions (a1) and (a2) are satisfied with $p_0 \geq 1$, let $\sum_{r=0}^{\infty} c_r V_z^r$ be a shift mapping on $\mathbb{H}((b_r), (p_r))_\rho$; then, the s -numbers of this mapping are given by

$$\begin{aligned} s_r \left(\sum_{j=0}^{\infty} c_j V_z^j \right) &\leq \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \sup_j \left(\frac{b_{j+k}}{b_k} \right)^{p_{j+k}/\omega_p} \\ &\sum_{j=0}^{\infty} |c_j|^{p_j/\omega_p}, \quad \text{for all } (c_j)_{j=0}^{\infty} \in \ell^{(p_j)/\omega_p}. \end{aligned} \quad (68)$$

Proof. Let $\text{card } \xi = r$, where $\xi \subset \mathbb{N}$. Define the mapping R as $Rf(z) = R(\sum_{k=0}^{\infty} \widehat{f_k} z^k) = \sum_{k \in \xi} \sum_{j=0}^k c_j \widehat{f_{k-j}} z^k$. Since the triangle inequality holds by ρ , we have

$$\begin{aligned} s_r \left(\sum_{j=0}^{\infty} c_j V_z^j \right) &\leq \alpha_r \left(\sum_{j=0}^{\infty} c_j V_z^j \right) \leq \left\| \sum_{j=0}^{\infty} c_j V_z^j - R \right\| \leq \sup_{\rho(f) \neq 0} \frac{\rho(\sum_{j=0}^{\infty} c_j V_z^j f - Rf)}{\rho(f)} \\ &\leq \sup_{\rho(f) \neq 0} \frac{\sum_{j=0}^{\infty} \left[\sum_{k \notin \xi} |c_j \widehat{f_k} b_{k+j}|^{p_{k+j}} \right]}{\rho(f)} \leq \sup_{k \notin \xi, j} \left(\frac{b_{j+k}}{b_k} \right)^{(p_{j+k}/\omega_p)} \sum_{j=0}^{\infty} |c_j|^{(p_j/\omega_p)}. \end{aligned} \quad (69)$$

As for all $\text{card } \xi = r$, the last inequality is verified, and one has

$$s_r \left(\sum_{j=0}^{\infty} c_j V_z^j \right) \leq \inf_{\text{card } \xi = r} \sup_{k \notin \xi, j} \left(\frac{b_{j+k}}{b_k} \right)^{p_{j+k}/\omega_p} \sum_{j=0}^{\infty} |c_j|^{p_j/\omega_p} = \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \sup_j \left(\frac{b_{j+k}}{b_k} \right)^{p_{j+k}/\omega_p} \sum_{j=0}^{\infty} |c_j|^{p_j/\omega_p}. \quad (70)$$

This completes the proof. \square

Theorem 31. If conditions (a1) and (a2) are satisfied with $p_0 \geq 1$, let $\sum_{j=0}^{\infty} c_j B_z^j$ be a shift mapping on $\mathbb{H}((b_r), (p_r))_\rho$; then, the s -numbers of this mapping are given by

$$\begin{aligned} s_r \left(\sum_{j=0}^{\infty} c_j B_z^j \right) &\leq \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \sup_j \left(\frac{b_k}{b_{k+j}} \right)^{p_k/\omega_p} \\ &\sum_{j=0}^{\infty} c_j^{p_j/\omega_p}, \quad \text{for all } (c_j)_{j=0}^{\infty} \in \ell^{(p_j)/\omega_p}. \end{aligned} \quad (71)$$

Proof. Let $\text{card } \xi = r$, where $\xi \subset \mathbb{N}$. Define the mapping R as $Rf(z) = R(\sum_{k=0}^{\infty} \widehat{f}_k z^k) = \sum_{k \in \xi} \sum_{j=0}^k c_j \widehat{f_{k-j}} z^k$. Since the triangle inequality holds by ρ , one gets

$$\begin{aligned} s_r \left(\sum_{j=0}^{\infty} c_j B_z^j \right) &\leq \alpha_r \left(\sum_{j=0}^{\infty} c_j B_z^j \right) \leq \left\| \sum_{j=0}^{\infty} c_j B_z^j - R \right\| \leq \sup_{\rho(f) \neq 0} \frac{\rho(\sum_{j=0}^{\infty} c_j B_z^j f - Rf)}{\rho(f)} \\ &\leq \sup_{\rho(f) \neq 0} \frac{\sum_{j=0}^{\infty} \left[\sum_{k \notin \xi} \left(b_k |c_j| \left\| \widehat{f_{k+j}} \right\| \right)^{p_k} \right]^{(1/\omega_p)}}{\rho(f)} \\ &\leq \sup_{k \notin \xi, j} \left(\frac{b_k}{b_{k+j}} \right)^{(p_k/\omega_p)} \sum_{j=0}^{\infty} |c_j|^{(p_j/\omega_p)}. \end{aligned} \quad (72)$$

As for all $\text{card } \xi = r$, the last inequality is verified, and one has

$$\begin{aligned} s_r \left(\sum_{j=0}^{\infty} c_j B_z^j \right) &\leq \inf_{\text{card } \xi = r} \sup_{k \notin \xi, j} \left(\frac{b_k}{b_{k+j}} \right)^{(p_k/\omega_p)} \sum_{j=0}^{\infty} |c_j|^{(p_j/\omega_p)} \\ &= \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \sup_j \left(\frac{b_k}{b_{k+j}} \right)^{(p_k/\omega_p)} \sum_{j=0}^{\infty} |c_j|^{(p_j/\omega_p)}. \end{aligned} \quad (73)$$

This completes the proof. \square

The following theorems are direct actions of Theorem 30 and Definition 10.

Theorem 32. If conditions (a1) and (a2) are satisfied with $p_0 \geq 1$, let V_{e^z} be a shift mapping on $\mathbb{H}((b_r), (p_r))_\rho$ and $e^z = \sum_{r=0}^{\infty} z^r/r!$. The upper estimation of the s -numbers of V_{e^z} is given by

$$s_a(V_{e^z}) \leq \sup_{\text{card } \xi = a+1} \inf_{j \in \xi} \sup_r \left(\frac{b_{r+j}}{b_j} \right)^{p_{r+j}/\omega_p} \sum_{r=0}^{\infty} \left(\frac{1}{r!} \right)^{p_r/\omega_p}. \quad (74)$$

Theorem 33. If conditions (a1) and (a2) are satisfied with $p_0 \geq 1$, let $V_{\sin(z)}$ be a shift mapping on $\mathbb{H}((b_r), (p_r))_\rho$ and $\sin(z) = \sum_{m=0}^{\infty} (-1)^m (z^{2m+1}/(2m+1)!)$. The upper estimation of the s -numbers of $V_{\sin(z)}$ is given by

$$\begin{aligned} s_a(V_{\sin(z)}) &\leq \sup_{\text{card } \xi = a+1} \inf_{j \in \xi} \sup_r \left(\frac{b_{r+j}}{b_j} \right)^{p_{r+j}/\omega_p} \\ &\quad \sum_{r=0}^{\infty} \left(\frac{1}{(2r+1)!} \right)^{p_r/\omega_p}. \end{aligned} \quad (75)$$

The following theorems are direct actions of Theorem 31 and Definition 11.

Theorem 34. If conditions (a1) and (a2) are satisfied with $p_0 \geq 1$, then the mapping B_{e^z} on $\mathbb{H}((b_r), (p_r))_\rho$ holds the following inequality:

$$s_r(B_{e^z}) \leq \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \sup_m \left(\frac{b_k}{b_{k+m}} \right)^{p_k/\omega_p} \sum_{m=0}^{\infty} \left(\frac{1}{m!} \right)^{p_m/\omega_p}. \quad (76)$$

Theorem 35. If conditions (a1) and (a2) are satisfied with $p_0 \geq 1$ and the mapping $B_{\sin(z)}$ is defined on $\mathbb{H}((b_r), (p_r))_\rho$, then the upper estimation of the s -numbers of $B_{\sin(z)}$ is given by

$$s_r(B_{\sin(z)}) \leq \sup_{\text{card } \xi = r+1} \inf_{k \in \xi} \sup_m \left(\frac{b_k}{b_{k+m}} \right)^{p_k/\omega_p} \sum_{m=0}^{\infty} \left(\frac{1}{(2m+1)!} \right)^{p_m/\omega_p}. \quad (77)$$

6. Caristi's Generalization of Fixed Point Theorem

In modular spaces, the Ekeland variational principle [28] cannot be applied because the modular does not really prove the triangle inequality. In this part, we consider an extension of Caristi's fixed point theorem in $\mathbb{H}((b_r), (p_r))_\rho$ in light of Farkas [28].

Definition 12

- The pre-quasi normed ssfps ρ on $\mathbb{H}((b_r), (p_r))_\rho$ is called ρ -convex, if $\rho(\omega v + (1-\omega)t) \leq \omega \rho(v) + (1-\omega)\rho(t)$, for each $\omega \in [0, 1]$ and $v, t \in \mathbb{H}((b_n), (p_n))_\rho$.
- $\{v^{(a)}\}_{a \in \mathbb{N}} \subseteq \mathbb{H}((b_n), (p_n))_\rho$ is ρ -convergent to $v \in \mathbb{H}((b_n), (p_n))_\rho$ if and only if, $\lim_{a \rightarrow \infty} \rho(v^{(a)} - v) = 0$. If the ρ -limit exists, then it is unique.
- $\{v^{(a)}\}_{a \in \mathbb{N}} \subseteq \mathbb{H}((b_n), (p_n))_\rho$ is ρ -Cauchy, when $\lim_{a, b \rightarrow \infty} \rho(v^{(a)} - v^{(b)}) = 0$.
- $Y \subseteq \mathbb{H}((b_n), (p_n))_\rho$ is ρ -closed, if for all ρ -converging $\{u^{(a)}\}_{a \in \mathbb{N}} \subset Y$ to u , and hence $u \in Y$.

(e) $Y \subset \mathbb{H}((b_n), (p_n))_\rho$ is ρ -bounded, when $\delta_\rho(Y) = \sup\{\rho(v-t): v, t \in Y\} < \infty$.

(f) The ρ -ball of radius $d \geq 0$ and center v , for every $v \in \mathbb{H}((b_n), (p_n))_\rho$, is defined as

$$\mathcal{B}_\rho(v, d) = \{t \in \mathbb{H}((b_n), (p_n))_\rho: \rho(v-t) \leq d\}. \quad (78)$$

(g) A pre-quasi normed ssfps ρ on $\mathbb{H}((b_n), (p_n))_\rho$ satisfies the Fatou property, if for any sequence $\{t^{(u)}\} \subseteq \mathbb{H}((b_n), (p_n))_\rho$ with $\lim_{u \rightarrow \infty} \rho(t^{(u)} - t) = 0$ and any $v \in \mathbb{H}((b_n), (p_n))_\rho$,

$$\rho(v-t) \leq \sup_m \inf_{u \geq m} \rho(v-t^{(u)}). \quad (79)$$

Consider the fact that the ρ -closedness of the ρ -balls is determined by the Fatou property.

Theorem 36. Suppose setups (a1) and (a2) are satisfied; then, $\rho(f) = [\sum_{r=0}^{\infty} |b_r \widehat{f}_r|^{p_r}]^{(1/\omega_\rho)}$, for all $f \in \mathbb{H}((b_n), (p_n))_\rho$, holds the Fatou property.

Proof. Assume the setups are fulfilled and $\{f^{(i)}\} \subseteq \mathbb{H}((b_n), (p_n))_\rho$ with $\lim_{i \rightarrow \infty} \rho(f^{(i)} - f) = 0$. Since the space $\mathbb{H}((b_n), (p_n))_\rho$ is a pre-quasi closed space, then $f \in \mathbb{H}((b_n), (p_n))_\rho$. Then, for any $g \in \mathbb{H}((b_n), (p_n))_\rho$, one can see that

$$\rho(g-f) = \left[\sum_{a=0}^{\infty} |b_a(\widehat{g}_a - \widehat{f}_a)|^{p_a} \right]^{1/\omega_\rho} \leq \left[\sum_{a=0}^{\infty} |b_a(\widehat{g}_a - \widehat{f}_a^{(i)})|^{p_a} \right]^{1/\omega_\rho} + \left[\sum_{a=0}^{\infty} |b_a(\widehat{f}_a^{(i)} - \widehat{f}_a)|^{p_a} \right]^{1/\omega_\rho} \leq \sup_j \inf_{i \geq j} \rho(g-f^{(i)}). \quad (80)$$

Theorem 37. The function $\rho(f) = \sum_{r=0}^{\infty} |b_r \widehat{f}_r|^{p_r}$, for all $f \in \mathbb{H}((b_n), (p_n))_\rho$, does not satisfy the Fatou property, if setups (a1) and (a2) are satisfied with $p_0 > 1$.

Proof. Let the conditions be fulfilled and $\{f^{(i)}\} \subseteq \mathbb{H}((b_n), (p_n))_\rho$ with $\lim_{i \rightarrow \infty} \rho(f^{(i)} - f) = 0$. Since the space $\mathbb{H}((b_n), (p_n))_\rho$ is a pre-quasi closed space, then $f \in \mathbb{H}((b_n), (p_n))_\rho$. Then, for any $g \in \mathbb{H}((b_n), (p_n))_\rho$, we have

$$\begin{aligned} \rho(g-f) &= \sum_{a=0}^{\infty} |b_a(\widehat{g}_a - \widehat{f}_a)|^{p_a} \leq 2^{\sup_a p_a - 1} \left[\sum_{a=0}^{\infty} |b_a(\widehat{g}_a - \widehat{f}_a^{(i)})|^{p_a} + \sum_{a=0}^{\infty} |b_a(\widehat{f}_a^{(i)} - \widehat{f}_a)|^{p_a} \right] \\ &\leq 2^{\sup_a p_a - 1} \sup_j \inf_{i \geq j} \rho(g-f^{(i)}). \end{aligned} \quad (81)$$

Hence, ρ does not satisfy the Fatou property. \square

Example 2. The space of functions $\mathbb{H}((a_r), (q_r))_\rho$ is a pre-quasi normed ssfps, not quasi normed ssfps, and not a normed ssfps, where $\delta(h) = [\sum_{r=0}^{\infty} |a_r \widehat{h}_r|^{q_r}]^{(1/\omega_q)}$, for all $h \in \mathbb{H}((a_r), (q_r))_\rho$.

Example 3. The space of functions $\mathbb{H}((a_r), (q))$, with $0 < q < 1$, is a pre-quasi normed ssfps, quasi normed ssfps, and not a normed ssfps, where $\delta(h) = [\sum_{r=0}^{\infty} |a_r \widehat{h}_r|^q]^{(1/q)}$, for each $h \in \mathbb{H}((a_r), (q_r))_\delta$.

Example 4. The space of functions $\mathbb{H}((a_r), (q_r))$ is a pre-quasi normed ssfps, a quasi normed ssfps, and a normed ssfps, where $\delta(h) = \inf\{t > 0: \sum_{r=0}^{\infty} |a_r \widehat{h}_r|/t^{q_r} \leq 1\}$, for all $h \in \mathbb{H}((a_r), (q_r))_\delta$.

Definition 13. The function $J: \mathbb{H}((b_r), (p_r))_\delta \rightarrow (-\infty, \infty]$ is said to be lower semicontinuous at $h^{(0)} \in \mathbb{H}((b_r), (p_r))_\delta$ if $\sup_{V \in \mathcal{V}(h^{(0)})} \inf_{h \in V} J(h) = J(h^{(0)})$, for which $\mathcal{V}(h^{(0)})$ denotes $h^{(0)}$'s neighborhood system.

Definition 14. The function $J: \mathbb{H}((b_r), (p_r))_\delta \rightarrow (-\infty, \infty]$ is said to be proper, when

$$\mathcal{D}(J) = \{f \in \mathbb{H}((b_r), (p_r))_\delta: J(f) < \infty\} \neq \emptyset. \quad (82)$$

Theorem 38. If $\Xi \neq \emptyset$ and Ξ is a δ -closed subset of $\mathbb{H}((b_x), (p_x))_\delta$ with $\delta(h) = [\sum_{x=0}^{\infty} |b_x \widehat{h}_x|^{p_x}]^{(1/\omega_p)}$, for all $h \in \mathbb{H}((b_x), (p_x))_\delta$ and $J: \Xi \rightarrow (-\infty, \infty]$ is a proper, δ -lower semicontinuous function with $\inf_{h \in \Xi} J(h) > -\infty$, assume that $\lambda > 0$, $\{\eta_x\} \subset (0, \infty)$, and $h^{(0)} \in \Xi$ with $J(h^{(0)}) \leq \inf_{h \in \Xi} J(h) + \lambda$. So, we have $\{h^{(x)}\} \in \Xi$ which δ -converges to few $h^{(\lambda)}$, under the following conditions:

- (i) $\delta(h^{(\lambda)} - h^{(x)}) \leq (\lambda/2^x \eta_0)$, for every $x \in \mathbb{N}$.
- (ii) $J(h^{(\lambda)}) + \sum_{x=0}^{\infty} \eta_x \delta(h^{(\lambda)} - h^{(x)}) \leq J(h^{(0)})$.
- (iii) When $h \neq h^{(\lambda)}$, then $J(h^{(\lambda)}) + \sum_{x=0}^{\infty} \eta_x \delta(h^{(\lambda)} - h^{(x)}) < J(h) + \sum_{x=0}^{\infty} \eta_x \delta(h - h^{(x)})$.

Proof. Set $S(h^{(0)}) = \{h \in \Xi: J(h) + \eta_0 \delta(h - h^{(0)}) \leq J(h^{(0)})\}$. Since $h^{(0)} \in S(h^{(0)})$, then $S(h^{(0)}) \neq \emptyset$. As J is δ -lower

semicontinuous, δ satisfies the Fatou property, and Ξ is δ -closed, we have that $S(h^{(0)})$ is δ -closed. Select $h^{(1)} \in S(h^{(0)})$ with

$$J(h^{(1)}) + \eta_0 \delta(h^{(1)} - h^{(0)}) \leq \inf_{h \in S(h^{(0)})} \left\{ J(h) + \eta_0 \delta(h - h^{(0)}) \right\} + \frac{\lambda \eta_1}{2\eta_0}. \quad (83)$$

Next set

$$S(h^{(1)}) = \left\{ h \in S(h^{(0)}): J(h) + \sum_{i=0}^1 \eta_i \delta(h - h^{(i)}) \leq J(h^{(1)}) + \eta_0 \delta(h^{(1)} - h^{(0)}) \right\}. \quad (84)$$

Similar to $S(h^{(0)})$, one has $S(h^{(1)}) \neq \emptyset$ and δ -closed. Suppose that we have built $\{h^{(0)}, h^{(1)}, h^{(2)}, \dots, h^{(x)}\}$ and $\{S(h^{(0)}), S(h^{(1)}), S(h^{(2)}), \dots, S(h^{(x)})\}$. After that, select $h^{(x+1)} \in S(h^{(x)})$ with

Suppose

$$J(h^{(x+1)}) + \sum_{i=0}^x \eta_i \delta(h^{(x+1)} - h^{(i)}) \leq \inf_{h \in S(h^{(x)})} \left\{ J(h) + \sum_{i=0}^x \eta_i \delta(h - h^{(i)}) \right\} + \frac{\lambda \eta_x}{2^x \eta_0}. \quad (85)$$

$$S(h^{(x+1)}) := \left\{ h \in S(h^{(x)}): J(h) + \sum_{i=0}^{x+1} \eta_i \delta(h - h^{(i)}) \leq J(h^{(x+1)}) + \sum_{i=0}^x \eta_i \delta(h^{(x+1)} - h^{(i)}) \right\}. \quad (86)$$

Therefore, we construct the sequences $\{h^{(x)}\}$ and $\{S(h^{(x)})\}$ by induction. For constant $x \in \mathbb{N}$, assume $y \in S(h^{(x)})$. One can see that

which gives

$$J(y) + \sum_{i=0}^x \eta_i \delta(y - h^{(i)}) \leq J(h^{(x)}) + \sum_{i=0}^{x-1} \eta_i \delta(h^{(x)} - h^{(i)}), \quad (87)$$

$$\begin{aligned} \eta_x \delta(y - h^{(x)}) &\leq J(h^{(x)}) + \sum_{i=0}^{x-1} \eta_i \delta(h^{(x)} - h^{(i)}) - \left[J(y) + \sum_{i=0}^{x-1} \eta_i \delta(y - h^{(i)}) \right] \\ &\leq J(h^{(x)}) + \sum_{i=0}^{x-1} \eta_i \delta(h^{(x)} - h^{(i)}) - \inf_{h \in S(h^{(x-1)})} \left[J(h) + \sum_{i=0}^{x-1} \eta_i \delta(h - h^{(i)}) \right] \leq \frac{\lambda \eta_x}{2^x \eta_0}. \end{aligned} \quad (88)$$

Since $\{S(h^{(x)})\}$ is decreasing with $h^{(x)} \in S(h^{(x)})$, for each $x \in \mathbb{N}$, one has

$$\delta(h^{(x+q)} - h^{(x)}) \leq \frac{\lambda}{2^x \eta_0}, \quad (89)$$

for each $x, q \in \mathbb{N}$, which gives that $\{h^{(x)}\}$ is δ -Cauchy. Since $\mathbb{H}((b_x), (p_x))_\delta$ is δ -Banach space, $\{h^{(x)}\}$ has δ -limits $h^{(\lambda)}$ and $\bigcap_{x \in \mathbb{N}} S(h^{(x)}) = \{h^{(\lambda)}\}$ satisfies. As $h^{(x+1)} \in S(h^{(x)})$, one has

$$\begin{aligned} J(h^{(x+1)}) + \sum_{i=0}^x \eta_i \delta(h^{(x+1)} - h^{(i)}) &\leq J(h^{(x)}) \\ &+ \sum_{i=0}^{x-1} \eta_i \delta(h^{(x)} - h^{(i)}), \end{aligned} \quad (90)$$

which implies that $\{J(h^{(x)}) + \sum_{i=0}^{x-1} \eta_i \delta(h^{(x)} - h^{(i)})\}$ is decreasing. After that, assume $h \neq h^{(\lambda)}$. So, we get $r \in \mathbb{N}$ for which $h \notin S(h^{(x)})$, for each $x \geq r$, i.e.,

$$J(h^{(x)}) + \sum_{i=0}^{x-1} \eta_i \delta(h^{(x)} - h^{(i)}) < J(h) + \sum_{i=0}^x \eta_i \delta(h - h^{(i)}). \quad (91)$$

As $h^{(\lambda)} \in S(h^{(x)})$, with $x \geq r$, one can see that

$$\begin{aligned} J(h^{(\lambda)}) + \sum_{i=0}^x \eta_i \delta(h^{(\lambda)} - h^{(i)}) &\leq J(h^{(x)}) + \sum_{i=0}^{x-1} \eta_i \delta(h^{(x)} - h^{(i)}) \\ &\leq J(h^{(r)}) + \sum_{i=0}^{r-1} \eta_i \delta(h^{(r)} - h^{(i)}). \end{aligned} \quad (92)$$

As $x \rightarrow \infty$ in the previous inequality, one gets

$$J(h^{(\lambda)}) + \sum_{i=0}^{\infty} \eta_i \delta(h^{(\lambda)} - h^{(i)}) \leq J(h^{(r)}) + \sum_{i=0}^{r-1} \eta_i \delta(h^{(r)} - h^{(i)}) < J(h) + \sum_{i=0}^r \eta_i \delta(h - h^{(i)}) \leq J(h) + \sum_{i=0}^{\infty} \eta_i \delta(h - h^{(i)}). \quad (93)$$

This implies that

$$J(h^{(\lambda)}) + \sum_{x=0}^{\infty} \eta_x \delta(h^{(\lambda)} - h^{(x)}) < J(h) + \sum_{x=0}^{\infty} \eta_x \delta(h - h^{(x)}). \quad (94)$$

This finishes the proof. \square

We discuss the concept of Caristi's fixed point theorem in $\mathbb{H}((b_x), (p_x))_\delta$ using Theorem 38.

Theorem 39. If $\Xi \neq \emptyset$ and Ξ is a δ -closed subset of $\mathbb{H}((b_x), (p_x))_\delta$ under $\delta(h) = [\sum_{x=0}^{\infty} |b_x \hat{h}_x|^{p_x}]^{1/\omega_p}$, with $h \in \mathbb{H}((b_x), (p_x))_\delta$, let $\lambda > 0$ and $\{\eta_n\}$ with $0 < \nu = \sum_{x=0}^{\infty} \eta_x < \infty$. $U: \Xi \rightarrow \Xi$ is a mapping and there is a function $J: \Xi \rightarrow (-\infty, \infty]$ which is a proper and δ -lower semicontinuous under $\inf_{h \in \Xi} J(h) > -\infty$ and

- (1) $\delta(U(h) - g) - \delta(h - g) \leq \delta(U(h) - h)$, for any $h, g \in \Xi$.
- (2) $\delta(U(h) - h) \leq J(h) - J(U(h))$, for any $h \in \Xi$.

Hence, there is a fixed point of U in Ξ .

Proof. As $0 < \nu = \sum_{x=0}^{\infty} \eta_x < \infty$, we have that $J_1 := \nu J$ is proper, bounded from below, and δ -lower semicontinuous. If $h \in \Xi$, one has

$$\nu \delta(U(h) - h) \leq J_1(h) - J_1(U(h)). \quad (95)$$

As $\inf_{h \in \Xi} J_1(h) > -\infty$, there is $h^{(0)} \in \Xi$ with $J_1(h^{(0)}) < \inf_{h \in \Xi} J_1(h) + \lambda$. From Theorem 38, there is $\{h^{(x)}\}$

which δ -converges to few $h^{(\lambda)} \in \Xi$, with

$$J_1(h^{(\lambda)}) + \sum_{x=0}^{\infty} \eta_x \delta(h^{(\lambda)} - h^{(x)}) < J_1(h) + \sum_{x=0}^{\infty} \eta_x \delta(h - h^{(x)}), \quad (96)$$

for all $h \neq h^{(\lambda)}$. Suppose that $U(h^{(\lambda)}) \neq h^{(\lambda)}$, and one has

$$\begin{aligned} J_1(h^{(\lambda)}) + \sum_{x=0}^{\infty} \eta_x \delta(h^{(\lambda)} - h^{(x)}) &< J_1(U(h^{(\lambda)})) \\ &+ \sum_{x=0}^{\infty} \eta_x \delta(U(h^{(\lambda)}) - h^{(x)}), \end{aligned} \quad (97)$$

which gives

$$J_1(h^{(\lambda)}) - J_1(U(h^{(\lambda)})) < \sum_{x=0}^{\infty} \eta_x \delta(U(h^{(\lambda)}) - h^{(x)}) - \sum_{x=0}^{\infty} \eta_x \delta(h^{(\lambda)} - h^{(x)}) = \sum_{x=0}^{\infty} \eta_x (\delta(U(h^{(\lambda)}) - h^{(x)}) - \delta(h^{(\lambda)} - h^{(x)})). \quad (98)$$

From condition (6), one can see that

$$J_1(h^{(\lambda)}) - J_1(U(h^{(\lambda)})) < \sum_{x=0}^{\infty} \eta_x \delta(U(h^{(\lambda)}) - h^{(\lambda)}) \quad (99)$$

$$= \nu \delta(U(h^{(\lambda)}) - h^{(\lambda)}).$$

Inequality (6) gives

$$\nu \delta(U(h^{(\lambda)}) - h^{(\lambda)}) \leq J_1(h^{(\lambda)}) - J_1(U(h^{(\lambda)})) \quad (100)$$

$$< \nu \delta(U(h^{(\lambda)})) - h^{(\lambda)}.$$

We have a contradiction. Hence, $U(h^{(\lambda)}) = h^{(\lambda)}$. This completes the proof. \square

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Certain Class of Analytic Functions with respect to Symmetric Points Defined by Q-Calculus

K. R. Karthikeyan ¹, G. Murugusundaramoorthy ², S. D. Purohit ³ and D. L. Suthar ⁴

¹Department of Applied Mathematics and Science, National University of Science & Technology (By Merger of Caledonian College of Engineering and Oman Medical College), Muscat, Oman

²Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology Deemed to be University, Vellore, Tamilnadu, India

³Department of HEAS (Mathematics), Rajasthan Technical University, Kota, India

⁴Department of Mathematics, Wollo University, P. O. Box: 1145, Amhara Region, Ethiopia

Correspondence should be addressed to D. L. Suthar; dlsuthar@gmail.com

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In this study, we familiarise a novel class of Janowski-type star-like functions of complex order with regard to (j, k) -symmetric points based on quantum calculus by subordinating with pedal-shaped regions. We found integral representation theorem and conditions for starlikeness. Furthermore, with regard to (j, k) -symmetric points, we successfully obtained the coefficient bounds for functions in the newly specified class. We also quantified few applications as special cases which are new (or known).

1. Definitions and Preliminaries

The set of all analytic functions constructed on the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ is symbolised by $\mathcal{H}(U)$. Also, \mathcal{A} indicates the subclass of $\mathcal{H}(U)$ that has a Taylor series representation:

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \quad (\xi \in U = \{\xi : |\xi| < 1\}). \quad (1)$$

The family of functions $f \in \mathcal{A}$ that are univalent in U is represented by \mathcal{S} . This is well established that if $f(\xi)$, assume by (1), is in \mathcal{S} , then $[f(\xi^k)]^{1/k}$ (k is a positive integer) is consequently in \mathcal{S} .

Definition 1 (see [1], Definition 3). Assume k is a positive integer. A domain \mathbb{D} is known to be k -fold symmetric if a rotation of \mathbb{D} about the origin through an angle $2\pi/k$ carries \mathbb{D} onto itself. For U , a function f is said to be k -fold symmetric if and only if for each ξ in U

$$f(e^{2\pi i/k} \xi) = e^{2\pi i/k} f(\xi). \quad (2)$$

\mathcal{F}_k represents the family including all k -fold symmetric functions.

The concept of k -symmetrical function was protracted to so-called (j, k) -symmetrical function by Liczberski and Połubiński in [2]. To be specific, a function $f(\xi)$ is reported for being (j, k) -symmetrical if

$$f(\varepsilon \xi) = \varepsilon^j f(\xi), \quad (\xi \in U), \quad (3)$$

where $k \geq 2$ is a fixed integer, $j = 0, 1, 2, \dots, k-1$ and $\varepsilon = \exp(2\pi i/k)$. The family of (j, k) -symmetrical functions will indeed be indicated by \mathcal{F}_k^j . We believe that \mathcal{F}_2^1 , \mathcal{F}_2^0 , and \mathcal{F}_k^1 are quite well groups of odd functions, even functions, and k -symmetrical functions. Consider the subsequent equivalence demarcate $f_{j,k}(\xi)$ as well

$$f_{j,k}(\xi) = \frac{1}{k} \sum_{v=0}^{k-1} \frac{f(\varepsilon^v \xi)}{\varepsilon^{vj}}, \quad (f \in \mathcal{A}; k = 1, 2, \dots; j = 0, 1, 2, \dots, (k-1)). \quad (4)$$

It is evident that $f_{j,k}(\xi)$ is a linear operator from \mathbf{U} into \mathbf{U} . If ν is an integer, then the subsequent assumptions result directly from (4):

$$\begin{aligned} f_{j,k}(\varepsilon^\nu \xi) &= \varepsilon^{\nu j} f_{j,k}(\xi), \\ f'_{j,k}(\varepsilon^\nu \xi) &= \varepsilon^{\nu j - \nu} f'_{j,k}(\xi) = \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{f'(\varepsilon^\nu \xi)}{\varepsilon^{\nu j - \nu}}, \\ f''_{j,k}(\varepsilon^\nu \xi) &= \varepsilon^{\nu j - 2\nu} f''_{j,k}(\xi) = \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{f''(\varepsilon^\nu \xi)}{\varepsilon^{\nu j - 2\nu}}. \end{aligned} \quad (5)$$

Let the function $f \in \mathcal{A}$ provided by (1) and $g \in \mathcal{A}$ of the form $g(\xi) = \xi + \sum_{n=2}^{\infty} \gamma_n \xi^n$, the Hadamard product (or convolution) of these two functions is indicated by

$$\mathcal{H}(\xi) := (f * g)(\xi) := \xi + \sum_{n=2}^{\infty} a_n \gamma_n \xi^n, \quad \xi \in \mathbf{U}. \quad (6)$$

Using Hadamard product, various authors studied the univalent function theory in dual with the theory of special functions, see [3–5] and references provided therein. Throughout this whole article, we will assume that $k \in \mathbb{N}$, $\varepsilon = \exp(2\pi i/k)$, and

$$\mathcal{H}_{j,k}(\xi) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu j} [(f * g)(\varepsilon^\nu \xi)] = \xi + \dots, \quad (7)$$

where

$$f, g \in \mathcal{A}; k = 1, 2, \dots; j = 0, 1, 2, \dots, (k-1). \quad (8)$$

From (7), we, thus, have

$$\begin{aligned} \mathcal{H}_{j,k}(\xi) &= \sum_{n=1}^{\infty} a_n \gamma_n \Lambda_{n,j} \xi^n, \quad (a_1 = \gamma_1 = 1), \\ \Lambda_{n,j} &= \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(n-j)\nu}. \end{aligned} \quad (9)$$

The investigation of q -calculus (q stands for quantum) fascinated and inspired many scholars due its use in various areas of the quantitative sciences. Jackson [6, 7] was among the key contributors of all the scientists who introduced and developed the q -calculus theory. Just like q -calculus was used in other mathematical sciences, the formulations of this idea are commonly used to examine the existence of various structures of function theory. Though it is the first article in which a link was established between certain geometric nature of the analytic function and the q -derivative operator and the usage of q -calculus in function theory, a solid and comprehensive foundation is given in [8] by Srivastava. After this development, many researchers introduced and studied some useful operators in q -analog with the applications of convolution concepts. For example, Kanas and Raducanu [9] established the q -differential operator and then examined the behavior of this operator in function theory. For more applications of this operator, see [10, 11].

For $f \in \mathcal{A}$ assumed by (1) and $0 < q < 1$, the Jackson's q -derivative operator or q -difference operator for $f \in \mathcal{A}$ is specified under (see [12–14])

$$\mathfrak{D}_q f(\xi) := \begin{cases} f'(0), & \text{if } \xi = 0, \\ \frac{f(\xi) - f(q\xi)}{(1-q)\xi}, & \text{if } \xi \neq 0. \end{cases} \quad (10)$$

From (10), if f is assumed as in (1), we can effortlessly see that

$$\mathfrak{D}_q f(\xi) = 1 + \sum_{n=2}^{\infty} [n]_q a_n \xi^{n-1}, \quad (11)$$

for $\xi \neq 0$, provided the q -integer number $[n]_q$ is represented by

$$[n]_q := \frac{1 - q^n}{1 - q}, \quad (12)$$

and take into consideration $\lim_{q \rightarrow 1^-} \mathfrak{D}_q f(\xi) = f'(\xi)$. During our study, we let signify

$$([n]_q)_\kappa := [n]_q [n+1]_q [n+2]_q \dots [n+\kappa-1]_q. \quad (13)$$

The q -Jackson integral is defined by (see [6])

$$I_q[f(\xi)] := \int_0^\xi f(t) d_q t = \xi(1-q) \sum_{k=0}^{\infty} q^k f(\xi q^k). \quad (14)$$

If the q -series converges, further witness that

$$\begin{aligned} \mathfrak{D}_q I_q f(\xi) &= f(\xi), \\ I_q \mathfrak{D}_q f(\xi) &= f(\xi) - f(0), \end{aligned} \quad (15)$$

where the second equality grasps if f is continuous at $\xi = 0$.

Let the classes of star-like functions of order η ($0 \leq \eta < 1$) and convex functions of order η ($0 \leq \eta < 1$) are symbolised by $\mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta)$, respectively. In \mathcal{A} , we categorize the collection \mathbf{P} of functions $p(\xi) \in \mathcal{A}$ with $p(0) = 1$ and $\Re p(\xi) > 0$. The functions in the \mathbf{P} class are not univalent.

With \mathbf{U} , let f, g be analytic. The function f is said to be subordinate to g in \mathbf{U} if the Schwarz function $\omega(\xi)$ exists in \mathbf{U} such that $|\omega(\xi)| < |\xi|$ and $f(\xi) = g(\omega(\xi))$, as shown through $f \prec g$. Whenever g is univalent in \mathbf{U} , consequently the subordination is identical to $f(0) = g(0)$ and $f(\mathbf{U}) \subset g(\mathbf{U})$.

Using the concept of subordination for holomorphic functions, Ma and Minda [15] proposed the classes:

$$\begin{aligned} \mathcal{S}^*(\psi) &= \left\{ f \in \mathcal{A} : \frac{\xi f'(\xi)}{f(\xi)} \prec \psi \right\}, \\ \mathcal{C}(\psi) &= \left\{ f \in \mathcal{A} : \left(1 + \frac{\xi f''(\xi)}{f'(\xi)} \right) \prec \psi \right\}, \end{aligned} \quad (16)$$

where $\psi \in \mathbf{P}$ with $\psi'(0) > 0$ maps \mathbf{U} onto a region star-like with respect to 1 and symmetric with respect to the real axis. By making a choice ψ to map unit disc on to some specific regions such as cardioid, parabolas, lemniscate of Bernoulli, and booth lemniscate in the right-half of the complex plane,

various interesting subclasses of star-like and convex functions could be gained well.

Lots of fascinating subclasses of star-like and convex functions may be constructed by using ψ to map unit disc on to particular areas such as cardioid, parabolas, lemniscate of Bernoulli, and booth lemniscate on the right-half of the complex plane.

For arbitrary fixed numbers C, D , $-1 < C \leq 1$, and $-1 \leq D < C$, we express through $\mathbf{P}(C, D)$ the family of functions $p(\xi) = 1 + p_1\xi + p_2\xi^2 + \dots$ analytic in the unit disc and $p(\xi) \in \mathbf{P}(C, D)$ if and only if

$$p(\xi) = \frac{1 + Cw(\xi)}{1 + Dw(\xi)}, \quad (17)$$

where $w(\xi)$ is the Schwarz function. Geometrically, $p(\xi) \in \mathbf{P}(C, D)$ if and only if $p(0) = 1$ and $p(\mathbf{U})$ lies inside an open disc centred with center $1 - C D / (1 - D^2)$ on the real axis having radius $C - D / (1 - D^2)$ with diameter end points $p_1(-1) = 1 - C / (1 - D)$ and $p_1(1) = 1 + C / (1 + D)$. On observing that $w(\xi) = p(\xi) - 1 / (p(\xi) + 1)$ for $p(\xi) \in \mathbf{P}$, we have $S(\xi) \in \mathbf{P}(C, D)$ if and only if for some $p(\xi) \in \mathbf{P}$

$$S(\xi) = \frac{(1 + C)p(\xi) + 1 - C}{(1 + D)p(\xi) + 1 - D}. \quad (18)$$

For detailed study on the class of Janowski functions, we refer [16]. The class of Janowski star-like functions and Janowski convex functions is defined as follows:

$$\begin{aligned} \mathcal{S}^*(C, D) &:= \left\{ f \in \mathcal{A}: \frac{\xi f'(\xi)}{f(\xi)} \prec \frac{1 + C\xi}{1 + D\xi}, -1 \leq D < C \leq 1 \right\} \\ \mathcal{C}(C, D) &:= \left\{ f \in \mathcal{A}: \left(1 + \frac{\xi f''(\xi)}{f'(\xi)} \right) \prec \frac{1 + C\xi}{1 + D\xi}, -1 \leq D < C \leq 1 \right\}. \end{aligned} \quad (19)$$

Inspired by the theory familiarized by Sakaguchi [17], and the study on analytic functions with respect to (j, k) -symmetrical points by various authors (see [18–22]), under this article, we formulate new subclasses listed in Definition 2.

Definition 2. For $-(\pi/2) < \theta < (\pi/2)$, $b \in \mathbb{C} \setminus \{0\}$, and $\mathcal{H}_{j,k}(\xi)/\xi \neq 0$ be defined as in (7). We say that $f \in \mathcal{H}_s^b(\vartheta; \theta; \psi; g; C, D)$ if $\mathcal{H}(\xi) = (f * g)(\xi)$ satisfies the subordination condition:

$$\begin{aligned} 1 + \frac{(1 + i \tan \theta)}{b} \left[\frac{\vartheta \xi^2 \mathcal{H}''(\xi) + \xi \mathcal{H}'(\xi)}{(1 - \vartheta) \mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathcal{H}_{j,k}'(\xi)} - 1 \right] \\ \prec \frac{(C + 1)\psi(\xi) - (C - 1)}{(D + 1)\psi(\xi) - (D - 1)}, \end{aligned} \quad (20)$$

where $\psi \in \mathbf{P}$ and is given by

$$\psi(\xi) = 1 + L_1\xi + L_2\xi^2 + L_3\xi^3 + \dots, \quad \xi \in \mathbf{U}, L_1 \neq 0. \quad (21)$$

Remark 1. Here, we list few exceptional cases of the defined class $\mathcal{H}_s^b(\vartheta; \theta; \psi; g; C, D)$.

- (1) If we let $C = 1, D = -1, \theta = 0, b = 1$ and $g(\xi) = \xi + \sum_{n=2}^{\infty} \xi^n$, then $\mathcal{H}_s^1(0; 0; \psi; g; 1, -1) \equiv \mathcal{S}_s^{(j,k)}(\psi)$ [19] and $\mathcal{H}_s^1(1; 0; \psi; g; 1, -1) \equiv \mathcal{C}_s^{(j,k)}(\psi)$ [19]
- (2) Fixing $\theta = 0, b = 1, g(\xi) = \xi + \sum_{n=2}^{\infty} \xi^n$ and $\psi(\xi) = 1 + \xi / (1 - \xi)$, then $\mathcal{H}_s^b(\vartheta; 0; \psi; g; C, D)$ reduces to the class $\mathcal{S}^{(j,k)}(C, D)$ ([18], Definition 5)

For completeness, we will now define q -analogue of the as follows.

Definition 3. For $-(\pi/2) < \theta < (\pi/2)$, $b \in \mathbb{C} \setminus \{0\}$ and $\mathcal{H}_{j,k}(\xi)/\xi \neq 0$ be defined as in (7). We say that $f \in \mathcal{Q}\mathcal{H}_s^b(\vartheta; \theta; \psi; g; C, D)$ if $\mathcal{H}(\xi) = (f * g)(\xi)$ holds the subordination condition:

$$\begin{aligned} 1 + \frac{(1 + i \tan \theta)}{b} \left[\frac{\vartheta q \xi^2 \mathcal{D}_q^2(\mathcal{H}(\xi)) + \xi \mathcal{D}_q \mathcal{H}(\xi)}{(1 - \vartheta) \mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathcal{D}_q \mathcal{H}_{j,k}(\xi)} - 1 \right] \\ \prec \frac{(C + 1)\psi(\xi) - (C - 1)}{(D + 1)\psi(\xi) - (D - 1)}, \end{aligned} \quad (22)$$

where $\psi \in \mathbf{P}$ and ψ is defined as in (21).

By letting $\psi(\xi) = 1 + \xi / (1 - q\xi)$, $q \in (0, 1)$ in $\mathcal{Q}\mathcal{H}_s^b(\vartheta; \theta; \psi; g; C, D)$, we have

$$\begin{aligned} 1 + \frac{(1 + i \tan \theta)}{b} \left[\frac{\vartheta q \xi^2 \mathcal{D}_q^2(\mathcal{H}(\xi)) + \xi \mathcal{D}_q \mathcal{H}(\xi)}{(1 - \vartheta) \mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathcal{D}_q \mathcal{H}_{j,k}(\xi)} - 1 \right] \\ = \frac{(C + 1)w(\xi) + 2 + (C - 1)qw(\xi)}{(D + 1)w(\xi) + 2 + (D - 1)qw(\xi)}, \end{aligned} \quad (23)$$

where $q \in (0, 1)$, $w(\xi)$ is analytic in \mathbf{U} , and $w(0) = 0, |w(\xi)| < 1$.

Remark 2. The impact of Janowski functions on a particular conic region was initiated by Noor and Malik [23] and was subsequently studied by various authors (see [11, 24, 25] and references provided therein).

2. Inclusion Relationships and Integral Representations of the Classes

$\mathcal{H}_s^b(\vartheta; \theta; \psi; g; C, D)$ and $\mathcal{Q}\mathcal{H}_s^b(\vartheta; \theta; \psi; g; C, D)$

Let us begin with the following.

Theorem 1. Let $F_{j,k}(\vartheta; \xi) = (1 - \vartheta) \mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathcal{H}_{j,k}'(\xi)$. If $f \in \mathcal{H}_s^b(\vartheta; \theta; \psi; g; C, D)$, then

$$\Re \left[1 + \frac{k(1 + i \tan \theta)}{b} \left(\frac{\xi F_{j,k}'(\vartheta; \xi)}{F_{j,k}(\vartheta; \xi)} - \frac{1}{k} \right) \right] > 0. \quad (24)$$

Proof. From the definition of $\mathcal{H}_s^b(\vartheta; \theta; \psi; h; C, D)$ and (18), we have

$$\Re \left[1 + \frac{(1+i \tan \theta)}{b} \left(\frac{\vartheta \xi^2 \mathcal{H}''(\xi) + \xi \mathcal{H}'(\xi)}{(1-\vartheta) \mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathcal{H}'_{j,k}(\xi)} - 1 \right) \right] > 0. \quad \text{Replacing } \xi \text{ by } \varepsilon^v \xi \text{ in (25), then for all } v = 0, 1, 2, \dots, k-1, \text{ we have}$$

(25)

$$\Re \left[1 + \frac{(1+i \tan \theta)}{b} \left(\frac{\varepsilon^{2v} \vartheta \xi^2 \mathcal{H}''(\varepsilon^v \xi) + \varepsilon^v \xi \mathcal{H}'(\varepsilon^v \xi)}{(1-\vartheta) \mathcal{H}_{j,k}(\varepsilon^v \xi) + \vartheta \varepsilon^v \xi \mathcal{H}'_{j,k}(\varepsilon^v \xi)} - 1 \right) \right] > 0, \quad (\xi \in \mathbf{U}). \quad (26)$$

Using (5) in (26), we get

$$\Re \left[1 + \frac{(1+i \tan \theta)}{b} \left(\frac{\varepsilon^{2v} \vartheta \xi^2 \mathcal{H}''(\varepsilon^v \xi) + \varepsilon^v \xi \mathcal{H}'(\varepsilon^v \xi)}{(1-\vartheta) \varepsilon^{vj} \mathcal{H}_{j,k}(\xi) + \vartheta \varepsilon^{vj} \xi \mathcal{H}'_{j,k}(\xi)} - 1 \right) \right] > 0, \quad (\xi \in \mathbf{U}). \quad (27)$$

Suppose $v = 0, 1, 2, \dots, k-1$ in (27), respectively, and summing them, we arrive at

$$\Re \left[1 + \frac{(1+i \tan \theta)}{b} \left(\frac{\vartheta \xi^2 \sum_{v=0}^{k-1} \varepsilon^{2v-vj} \mathcal{H}''(\varepsilon^v \xi) + \xi \sum_{v=0}^{k-1} \varepsilon^{v-vj} \mathcal{H}'(\varepsilon^v \xi)}{(1-\vartheta) \mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathcal{H}'_{j,k}(\xi)} - 1 \right) \right] > 0. \quad (28)$$

or equivalently,

$$\Re \left[1 + \frac{k(1+i \tan \theta)}{b} \left(\frac{\xi F'_{j,k}(\vartheta; \xi)}{F_{j,k}(\vartheta; \xi)} - \frac{1}{k} \right) \right] > 0, \quad (\xi \in \mathbf{U}). \quad (29)$$

Hence the proof.

Now, by using the following two equivalent forms (see ([14], page 3)) of product rule of the q -difference operator,

$$\begin{aligned} \mathfrak{D}_q[f(\xi)g(\xi)] &= g(\xi)\mathfrak{D}_q[f(\xi)] + f(q\xi)\mathfrak{D}_q[g(\xi)] \\ &= g(q\xi)\mathfrak{D}_q[f(\xi)] + f(\xi)\mathfrak{D}_q[g(\xi)], \end{aligned} \quad (30)$$

we can establish the following result by retracing the steps as in Theorem 1.

Theorem 2. Let $F_{j,k}(\vartheta; \xi) = (1-\vartheta)\mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathfrak{D}_q[\mathcal{H}_{j,k}(\xi)]$, where $H_{j,k}(\xi)/\xi \neq 0$. If $f \in \mathcal{Q}\mathcal{H}_s^b(\vartheta; \theta; \psi; g; C, D)$, then

$$\Re \left[1 + \frac{k(1+i \tan \theta)}{b} \left(\frac{\xi \mathfrak{D}_q[F_{j,k}(\vartheta; \xi)]}{F_{j,k}(\vartheta; \xi)} - \frac{1}{k} \right) \right] > 0. \quad (31)$$

Theorem 3. Let $f \in \mathcal{H}_s^b(\vartheta; \theta; \psi; g; C, D)$, then

$$\begin{aligned} \mathcal{H}_{j,k}(\xi) &= \xi \exp \left\{ \frac{1}{k} \sum_{v=0}^{k-1} \int_0^{\varepsilon^v \xi} \frac{b(C-D)[\psi(w(t)) - 1]}{(1+i \tan \theta)t[(D+1)\psi(w(t)) - (D-1)]} dt \right\}, \quad \text{if } \vartheta = 0, \\ \mathcal{H}_{j,k}(\xi) &= \int_0^\xi \exp \left\{ \frac{1}{k} \sum_{v=0}^{k-1} \int_0^{\varepsilon^v \zeta} \frac{b(C-D)[\psi(w(t)) - 1]}{(1+i \tan \theta)t[(D+1)\psi(w(t)) - (D-1)]} dt \right\} d\zeta, \quad \text{if } \vartheta = 1, \end{aligned} \quad (32)$$

where $\mathcal{H}_{j,k}(\xi)$ is given by (7), $w(\xi)$ is analytic in \mathbf{U} , and $w(0) = 0$, $|w(\xi)| < 1$.

Proof. Let $f \in \mathcal{H}_s^b(\vartheta; \theta; \psi; g; C, D)$. In view of (20), we have

$$\begin{aligned} & \frac{\vartheta \xi^2 \mathcal{H}''(\xi) + \xi \mathcal{H}'(\xi)}{(1-\vartheta) \mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathcal{H}'_{j,k}(\xi)} - 1 \\ &= \frac{b(C-D)[\psi(w(\xi)) - 1]}{(1+i \tan \theta)[(D+1)\psi(w(\xi)) - (D-1)]} \end{aligned} \quad (33)$$

where $w(\xi)$ is analytic in \mathbf{U} and $w(0) = 0$, $|w(\xi)| < 1$. Substituting ξ by $\varepsilon^y \xi$ in equality (33) and ensuing the steps as in Theorem 1, we get

$$\frac{\xi F'_{j,k}(\vartheta; \xi)}{F_{j,k}(\vartheta; \xi)} - 1 = \frac{1}{k} \sum_{\gamma=0}^{k-1} \frac{b(C-D)[\psi(w(\varepsilon^\gamma \xi)) - 1]}{(1+i \tan \theta)[(D+1)\psi(w(\varepsilon^\gamma \xi)) - (D-1)]}. \quad (34)$$

From this equality, we get

$$\frac{\xi F'_{j,k}(\vartheta; \xi)}{F_{j,k}(\vartheta; \xi)} - \frac{1}{\xi} = \frac{1}{k} \sum_{\gamma=0}^{k-1} \frac{b(C-D)[\psi(w(\varepsilon^\gamma \xi)) - 1]}{(1+i \tan \theta)\xi[(D+1)\psi(w(\varepsilon^\gamma \xi)) - (D-1)]}. \quad (35)$$

Upon integration, we get

$$\log \left\{ \frac{F_{j,k}(\vartheta; \xi)}{\xi} \right\} = \frac{1}{k} \sum_{\gamma=0}^{k-1} \int_0^\xi \frac{b(C-D)[\psi(w(\varepsilon^\gamma \zeta)) - 1]}{(1+i \tan \theta)\zeta[(D+1)\psi(w(\varepsilon^\gamma \zeta)) - (D-1)]} d\zeta. \quad (36)$$

or equivalently,

$$(1-\vartheta)\mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathcal{H}'_{j,k}(\xi) = \xi \exp \left\{ \frac{1}{k} \sum_{\gamma=0}^{k-1} \int_0^{\varepsilon^\gamma \xi} \frac{b(C-D)[\psi(w(t)) - 1]}{(1+i \tan \theta)t[(D+1)\psi(w(t)) - (D-1)]} dt \right\}. \quad (37)$$

This concludes the proof of Theorem 3. \square

Theorem 4. Let $f \in \mathcal{Q}\mathcal{K}_s^b(\vartheta; \theta; \psi; g; C, D)$, then we have

$$\begin{aligned} \mathcal{H}_{j,k}(\xi) &= \xi \exp \left\{ \frac{\ln q}{(q-1)k} \sum_{\gamma=0}^{k-1} \int_0^{\varepsilon^\gamma \xi} \frac{b(C-D)[\psi(w(t)) - 1]}{(1+i \tan \theta)t[(D+1)\psi(w(t)) - (D-1)]} d_q t \right\}, \quad (\text{if } \vartheta = 0), \\ \mathcal{H}_{j,k}(\xi) &= \int_0^\xi \exp \left\{ \frac{\ln q}{(q-1)k} \times \sum_{\gamma=0}^{k-1} \int_0^{\varepsilon^\gamma \zeta} \frac{b(C-D)[\psi(w(t)) - 1]}{(1+i \tan \theta)t[(D+1)\psi(w(t)) - (D-1)]} d_q t \right\} d_q \zeta, \quad (\text{if } \vartheta = 1). \end{aligned} \quad (38)$$

where $\mathcal{H}_{j,k}(\xi)$ defined by equality (7), $w(\xi)$ is analytic in \mathbf{U} , and $w(0) = 0$, $|w(\xi)| < 1$.

Proof. Let $f \in \mathcal{Q}\mathcal{K}_s^b(\vartheta; \theta; \psi; g; C, D)$. In sight of Theorems 2 and 3, we have

$$\frac{\xi F'_{j,k}(\vartheta; \xi)}{F_{j,k}(\vartheta; \xi)} - \frac{1}{\xi} = \frac{1}{k} \sum_{\gamma=0}^{k-1} \frac{b(C-D)[\psi(w(\varepsilon^\gamma \xi)) - 1]}{(1+i \tan \theta)\xi[(D+1)\psi(w(\varepsilon^\gamma \xi)) - (D-1)]}, \quad (39)$$

where $w(\xi)$ is analytic in \mathbf{U} and $w(0) = 0$, $|w(\xi)| < 1$. For $f \in \mathcal{H}(\mathbf{U})$ and $0 < q < 1$, we obtain (see [10])

$$I_q \frac{\mathfrak{D}_q f(\xi)}{f(\xi)} = \frac{q-1}{\ln q} \log f(\xi), \quad (40)$$

where $I_q f$ is the Jackson q -integral, defined as in (14). Integrating the above equality, we get

$$\frac{q-1}{\ln q} \log \left\{ \frac{F_{j,k}(\vartheta; \xi)}{\xi} \right\} = \frac{1}{k} \sum_{\nu=0}^{k-1} \int_0^{\xi} \frac{b(C-D)[\psi(w(\varepsilon^\nu \zeta)) - 1]}{(1+i \tan \theta) \zeta [(D+1)\psi(w(\varepsilon^\nu \zeta)) - (D-1)]} d_q \zeta, \quad (41)$$

or equivalently,

$$(1-\vartheta) \mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathcal{H}'_{j,k}(\xi) = \xi \exp \left\{ \frac{\ln q}{(q-1)k} \sum_{\nu=0}^{k-1} \int_0^{\varepsilon^\nu \xi} \frac{b(C-D)[\psi(w(t)) - 1]}{(1+i \tan \theta) t [(D+1)\psi(w(t)) - (D-1)]} d_q t \right\}. \quad (42)$$

This concludes the proof of Theorem 3.

By fixing $C=1, D=-1, \theta=0, b=1$, and $g(\xi) = \xi + \sum_{n=2}^{\infty} \xi^n$ in Theorem 3, we state the subsequent result.

Corollary 1 (see ([19], Theorems 3 and 4)). *Let $f_{j,k}(\xi) \neq 0$ be assumed as in (4).*

(i) *If $f \in \mathcal{S}_s^{(j,k)}(\phi)$, then*

$$f_{j,k}(\xi) = \xi \exp \left\{ \frac{1}{k} \sum_{\nu=0}^{k-1} \int_0^{\varepsilon^\nu \xi} \frac{\phi(w(t)) - 1}{t} dt \right\}. \quad (43)$$

(ii) *If $f \in \mathcal{C}_s^{(j,k)}(\phi)$, then*

$$f_{j,k}(\xi) = \int_0^{\xi} \exp \left\{ \frac{1}{k} \sum_{\nu=0}^{k-1} \int_0^{\varepsilon^\nu \xi} \frac{\phi(w(t)) - 1}{t} dt \right\} d\eta, \quad (44)$$

where $w(\xi)$ is analytic in \mathbf{U} and $w(0) = 0, |w(\xi)| < 1$.

Corollary 2. *Let $f_{j,k}(\xi) \neq 0$ be assumed as in (4). If $f \in \mathcal{S}^{(j,k)}(C, D)$, then*

$$f_{j,k}(\xi) = \xi \exp \left\{ \frac{1}{k} \sum_{\nu=0}^{k-1} \int_0^{\varepsilon^\nu \xi} \frac{1}{t} \left[\frac{1+Cw(t)}{1+Dw(t)} \right] dt \right\}, \quad (45)$$

where $w(\xi)$ is analytic in \mathbf{U} and $w(0) = 0, |w(\xi)| < 1$.

3. Coefficient Inequalities for

$\mathcal{K}_s^b(\vartheta; \theta; \psi; g; C, D)$ and $\mathcal{QK}_s^b(\vartheta; \theta; \psi; g; C, D)$

The coefficient estimate $|a_n|$ of the defined function classes is determined in this section.

Theorem 5. *Let $|\psi(\xi)| < |D - 1/(D+1)|, \forall \xi \in \mathbf{U}$ and Y_n 's be real. If $f \in \mathcal{K}_s^b(\vartheta; \theta; \psi; g; C, D)$, then for $n \geq 2$,*

$$|a_n| \leq \frac{1}{Y_n} \prod_{m=1}^{n-1} \frac{|[(1-\vartheta) + \vartheta m] \Lambda_{m,j} (C-D) L_1 b - 2(1+i \tan \theta) [\vartheta m(m-1 - \Lambda_{m,j}) + m - (1-\vartheta) \Lambda_{m,j}] D|}{2 \sec \theta [\vartheta(m+1)(m - \Lambda_{m+1,j}) + (m+1) - (1-\vartheta) \Lambda_{m+1,j}]}. \quad (46)$$

Proof. By the definition of $\mathcal{K}_s^b(\vartheta; \theta; \psi; g; C, D)$, we have

$$1 + \frac{(1+i \tan \theta)}{b} \left[\frac{\vartheta \xi^2 \mathcal{H}''(\xi) + \xi \mathcal{H}'(\xi)}{(1-\vartheta) \mathcal{H}_{j,k}(\xi) + \vartheta \xi \mathcal{H}'_{j,k}(\xi)} - 1 \right] = p(\xi), \quad (47)$$

where $p(\xi) \in \mathbf{P}$ and satisfies the condition $p(\xi) \prec ((C+1)\psi(\xi) - (C-1))/((D+1)\psi(\xi) - (D-1))$.

Equivalently, (47) can be rewritten as

$$\begin{aligned} & (1+i \tan \theta) \left[\sum_{n=2}^{\infty} [\vartheta n(n-1 - \Lambda_{n,j}) + n - (1-\vartheta) \Lambda_{n,j}] Y_n a_n \xi^n \right] \\ &= b \left[\sum_{n=1}^{\infty} [(1-\vartheta) + \vartheta n] \Lambda_{n,j} a_n Y_n \xi^n \right] \left[\sum_{n=1}^{\infty} p_n \xi^n \right], \quad (a_1 = Y_1 = \Lambda_{1,j} = 1). \end{aligned} \quad (48)$$

On equating the coefficient of ξ^n , we get

$$\begin{aligned} & (1+i \tan \theta) [\vartheta n(n-1 - \Lambda_{n,j}) + n - (1-\vartheta) \Lambda_{n,j}] Y_n a_n \\ &= b [p_{n-1} \Lambda_{1,j} + p_{n-2} (1+\vartheta) \Lambda_{2,j} Y_2 a_2 + \cdots + p_1 \\ & \quad \cdot (1+(n-2)\vartheta) Y_{n-1} a_{n-1}]. \end{aligned} \quad (49)$$

From Lemma 6 of [26], we have $|p_n| \leq |L_1|(C-D)/2, n \geq 1$. On computation, we have

$$\begin{aligned} |a_n| &\leq \frac{|b|(C-D)|L_1|}{2 \sec \theta [\vartheta n(n-1 - \Lambda_{n,j}) + n - (1-\vartheta) \Lambda_{n,j}] Y_n} \\ &\quad \cdot \left[\sum_{m=1}^{n-1} [(1-\vartheta) + \vartheta m] \Lambda_{m,j} Y_m |a_m| \right]. \end{aligned} \quad (50)$$

Taking $n = 2$, in (50), we get

$$|a_2| \leq \frac{|b|(C-D)|L_1|}{2 \sec \theta [2\vartheta(1-\Lambda_{2,j}) + 2 - (1-\vartheta)\Lambda_{2,j}]Y_2}. \quad (51)$$

On substituting $n = 2$ in (46), we can see the hypothesis is true for $n = 2$. Now, taking $n = 3$ in (50), we get

$$\begin{aligned} |a_3| &\leq \frac{|b|(C-D)|L_1|}{2 \sec \theta [3\vartheta(2-\Lambda_{3,j}) + 3 - (1-\vartheta)\Lambda_{3,j}]Y_3} [1 + (1+\vartheta)\Lambda_{2,j}Y_2|a_2|] \\ &\leq \frac{|b|(C-D)|L_1|}{2 \sec \theta [3\vartheta(2-\Lambda_{3,j}) + 3 - (1-\vartheta)\Lambda_{3,j}]Y_3} \left[1 + \frac{(1+\vartheta)\Lambda_{2,j}|b|(C-D)|L_1|}{2 \sec \theta [2\vartheta(1-\Lambda_{2,j}) + 2 - (1-\vartheta)\Lambda_{2,j}]} \right]. \end{aligned} \quad (52)$$

If we let $n = 3$, in (46), we have

$$\begin{aligned} |a_3| &\leq \frac{1}{Y_3} \left[\frac{|b|(C-D)|L_1|\Lambda_{1,j}}{2 \sec \theta [2\vartheta(1-\Lambda_{2,j}) + 2 - (1-\vartheta)\Lambda_{2,j}]} \times \frac{|(C-D)L_1b(1+\vartheta)\Lambda_{2,j} - 2(1+i \tan \theta)[2\vartheta(1-\Lambda_{2,j}) + 2 - (1-\vartheta)\Lambda_{2,j}]D|}{2 \sec \theta [3\vartheta(2-\Lambda_{3,j}) + 3 - (1-\vartheta)\Lambda_{3,j}]} \right] \\ &\leq \frac{1}{Y_3} \left[\frac{|b|(C-D)|L_1|\Lambda_{1,j}}{2 \sec \theta [3\vartheta(2-\Lambda_{3,j}) + 3 - (1-\vartheta)\Lambda_{3,j}]} \times \frac{|b|(C-D)|L_1|(1+\vartheta)\Lambda_{2,j} + 2 \sec \theta [2\vartheta(1-\Lambda_{2,j}) + 2 - (1-\vartheta)\Lambda_{2,j}]}{2 \sec \theta [2\vartheta(1-\Lambda_{2,j}) + 2 - (1-\vartheta)\Lambda_{2,j}]} \right] \\ &\leq \frac{|b|(C-D)|L_1|}{2 \sec \theta [3\vartheta(2-\Lambda_{3,j}) + 3 - (1-\vartheta)\Lambda_{3,j}]Y_3} \left[1 + \frac{(1+\vartheta)\Lambda_{2,j}|b|(C-D)|L_1|}{2 \sec \theta [2\vartheta(1-\Lambda_{2,j}) + 2 - (1-\vartheta)\Lambda_{2,j}]} \right]. \end{aligned} \quad (53)$$

Hence, the hypothesis of the theorem is true for $n = 3$. Now, let us suppose (46) is valid for $n = 2, 3, \dots, r$. On using triangle inequality in (46), we get

$$|a_r| \leq \frac{1}{Y_r} \prod_{m=1}^{r-1} \frac{[(1-\vartheta) + \vartheta m]\Lambda_{m,j}|b|(C-D)|L_1| + 2 \sec \theta [\vartheta m(m-1-\Lambda_{m,j}) + m - (1-\vartheta)\Lambda_{m,j}]}{2 \sec \theta [\vartheta(m+1)(m-\Lambda_{m+1,j}) + (m+1) - (1-\vartheta)\Lambda_{m+1,j}]}. \quad (54)$$

By induction hypothesis, we have

$$\begin{aligned} &\frac{|b|(C-D)|L_1|}{2 \sec \theta [\vartheta r(r-1-\Lambda_{r,j}) + r - (1-\vartheta)\Lambda_{r,j}]Y_r} \left[\sum_{m=1}^{r-1} [(1-\vartheta) + \vartheta m]\Lambda_{m,j}Y_m|a_m| \right] \\ &\leq \frac{1}{Y_r} \prod_{m=1}^{r-1} \frac{[(1-\vartheta) + \vartheta m]\Lambda_{m,j}|b|(C-D)|L_1| + 2 \sec \theta [\vartheta m(m-1-\Lambda_{m,j}) + m - (1-\vartheta)\Lambda_{m,j}]}{2 \sec \theta [\vartheta(m+1)(m-\Lambda_{m+1,j}) + (m+1) - (1-\vartheta)\Lambda_{m+1,j}]}. \end{aligned} \quad (55)$$

From the above inequality, we have

$$\prod_{m=1}^r \frac{[(1-\vartheta) + \vartheta m] \Lambda_{m,j} |b| (C-D) |L_1| + 2 \sec \theta [\vartheta m(m-1-\Lambda_{m,j}) + m - (1-\vartheta) \Lambda_{m,j}]}{2 Y_{r+1} \sec \theta [\vartheta(m+1)(m-\Lambda_{m+1,j}) + (m+1) - (1-\vartheta) \Lambda_{m+1,j}]} \geq \frac{|b| (C-D) |L_1|}{2 \sec \theta [\vartheta(r+1)(r-\Lambda_{r+1,j}) + (r+1) - (1-\vartheta) \Lambda_{r+1,j}]} Y_{r+1} \left[\sum_{m=1}^r [(1-\vartheta) + \vartheta m] \Lambda_{m,j} Y_m |a_m| \right], \quad (56)$$

which implies that inequality (46) is true for $n = r + 1$. Hence the proof of the theorem. \square

Theorem 6. Let $|\psi(\xi)| < |D - 1/(D+1)|$ for all $\xi \in \mathbf{U}$ and Y_n be real. If $f \in \mathcal{Q}\mathcal{K}_s^b(\vartheta; \theta; \psi; g; C, D)$, then for $n \geq 2$,

$$|a_n| \leq \frac{1}{Y_n} \prod_{m=1}^{n-1} \frac{[(1-\vartheta) + \vartheta[m]_q] \Lambda_{m,j} (C-D) L_1 b - 2(1+i \tan \theta) [q\vartheta[m]_q([m-1]_q - \Lambda_{m,j}) + [m]_q - (1-\vartheta) \Lambda_{m,j}] D}{2 \sec \theta [\vartheta q[m+1]_q([m]_q - \Lambda_{m+1,j}) + [m+1]_q - (1-\vartheta) \Lambda_{m+1,j}]} \quad (57)$$

Proof. From the definition of $\mathcal{Q}\mathcal{K}_s^b(\vartheta; \theta; \psi; g; C, D)$, we have

$$(1+i \tan \theta) \left[\sum_{n=2}^{\infty} [\vartheta[n]_q(q[n-1]_q - \Lambda_{n,j}) + [n]_q - (1-\vartheta) \Lambda_{n,j}] Y_n a_n \xi^n \right] = b \left[\sum_{n=1}^{\infty} \{(1-\vartheta) + \vartheta[n]_q\} \Lambda_{n,j} a_n Y_n \xi^n \right] \left[\sum_{n=1}^{\infty} p_n \xi^n \right]. \quad (58)$$

Equating the coefficient of ξ^n and retracing the steps as in Theorem 5, we get

$$|a_n| \leq \frac{|b| (C-D) |L_1|}{2 \sec \theta [\vartheta[n]_q(q[n-1]_q - \Lambda_{n,j}) + [n]_q - (1-\vartheta) \Lambda_{n,j}] Y_n} \times \left[\sum_{m=1}^{n-1} [(1-\vartheta) + \vartheta[m]_q] \Lambda_{m,j} Y_m |a_m| \right]. \quad (59)$$

Now, by repeating the processes in Theorem 5, we acquire the required outcome. \square

If we let $\psi(\xi) = 1 + \xi/(1-q\xi)$ in Theorem 6, we have the following.

Corollary 3. Let $f \in \mathcal{Q}\mathcal{K}_s^b(\vartheta; \theta; \psi; g; C, D)$ and Y_n be real, then for $n \geq 2$,

$$|a_n| \leq \frac{1}{Y_n} \prod_{m=1}^{n-1} \frac{[(1-\vartheta) + \vartheta[m]_q] \Lambda_{m,j} (C-D) (1+q)b - 2(1+i \tan \theta) [q\vartheta[m]_q([m-1]_q - \Lambda_{m,j}) + [m]_q - (1-\vartheta) \Lambda_{m,j}] [D(1+q) + (1-q)]}{2 \sec \theta [\vartheta q[m+1]_q([m]_q - \Lambda_{m+1,j}) + [m+1]_q - (1-\vartheta) \Lambda_{m+1,j}]} \quad (60)$$

If we let $\theta = 0$, $b = 1$, $g(\xi) = \xi + \sum_{n=2}^{\infty} \xi^n$, and $\psi(\xi) = 1 + \xi/(1-\xi)$ in Theorem 5, we have the following.

Corollary 4. (see [18], Theorem 2) Let $f \in \mathcal{S}^{(j,k)}(C, D)$ and Y_n be real, then for $n \geq 2$,

$$|a_n| \leq \prod_{m=1}^{n-1} \frac{\Lambda_{m,j} [(C-D) - 1] + m}{(m+1) - \Lambda_{m+1,j}}. \quad (61)$$

4. Conclusion

Very few studies have been showed on analytic functions with regard to (j, k) -symmetric points. Since we have articulated the problem differently so as to deviate from the similar studies, only few special cases could be discussed. Furthermore, by swapping the ordinary differentiation with quantum differentiation, we have tried at the discretization of some of the familiar findings.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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Research Article

On Convolution and Convex Combination of Harmonic Mappings

Ahmad Sulaiman Ahmad El-Faqeer , Zhen Chuan Ng , and Shamani Supramaniam 

School of Mathematical Sciences, Universiti Sains Malaysia, Penang 11800, Malaysia

Correspondence should be addressed to Zhen Chuan Ng; zhenchuanng@usm.my

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In this paper, the subclass of harmonic univalent functions by shearing construction is studied and this subclass of harmonic mappings needs a necessary and adequate condition to be convex in the horizontal direction. Furthermore, convolutions of two special subclasses of univalent harmonic mappings are shown to be convex in the horizontal direction. Also, the family of univalent harmonic mappings of the unit disk onto a region convex in the direction of the imaginary axis is introduced. Sufficient conditions for convex combinations of harmonic mappings of this family to be univalently convex in the direction of the imaginary axis are obtained.

1. Motivation and Preliminaries

A complex-valued function $f = u + iv$ defined on the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ is called harmonic mapping if u and v are real-valued harmonic functions. In addition, since U is a simply connected domain, f has a unique representation $f = h + \bar{g}$, where h and g are analytic and co-analytic parts of f , respectively. It is known from Lewy [1] that the mapping $f = h + \bar{g}$ defined on U is locally univalent and sense-preserving if and only if

$$|g'(z)| < |h'(z)|, \quad \forall z \in U. \quad (1)$$

The class of all univalent, harmonic sense-preserving mappings $f = h + \bar{g}$ defined in U is denoted by \mathcal{H} . Moreover, let S_H be the class of all functions $f \in \mathcal{H}$ which is normalized by $f(0) = f_z(0) - 1 = 0$. Therefore, each member of $f \in S_H$ has the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n + \sum_{n=1}^{\infty} \bar{c}_n \bar{z}^n \quad (|c_1| < 1), \quad (2)$$

for all $z \in U$.

The class of such type of functions given in (2) with $c_1 = 0 = f_{\bar{z}}(0)$ is a subclass of S_H and is denoted by S_H^0 . The dilatation of f belonging to S_H^0 is the function $\omega: U \rightarrow \mathbb{C}$

given by $\omega = g'(z)/h'(z)$ and satisfies $|\omega(z)| < 1$ for all $z \in U$. Further, denote by K_H (or K_H^0) all $f \in S_H$ (or S_H^0) which are mapping U onto convex regions. For comprehensive and fundamental knowledge on planar harmonic mappings, see Duren [2]. The subclass of S_H^0 denoted by S_{CHD}^0 consists of all univalent harmonic functions which maps onto domain convex in the direction of the real axis.

Section 2 demonstrates that the convolution of two special subclasses of univalent harmonic mappings is convex in the horizontal direction. The convolution of any two arbitrary harmonic functions

$$\begin{aligned} f(z) &= h(z) + \bar{g}(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n, \\ F(z) &= H(z) + \bar{G}(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \bar{B}_n \bar{z}^n \end{aligned} \quad (3)$$

is defined by

$$\begin{aligned} (f \ast F)(z) &= (h \ast H)(z) + \overline{(g \ast G)}(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n \\ &\quad + \sum_{n=1}^{\infty} \bar{b}_n \bar{B}_n \bar{z}^n. \end{aligned} \quad (4)$$

The convolution of harmonic functions is different from the convolution of univalent functions (for more details, one can refer to [3]). Indeed, the convolution of two harmonic functions does not preserve the convexity and the convolution of any two univalent harmonic functions $f \in S_H^0$ need not be univalent. These facts generated significant interest in the analysis of harmonic convolutions of univalent harmonic functions, and several articles on this subject recently appeared in the literature [3–11]. Specifically, the collection of the mappings $F = H + \overline{G} \in S_H^0$ that map U onto the right half-plane $\Psi = \{w: \operatorname{Re}(w) > -(1/2)\}$ and have the form $(H + G)(z) = z/(1 - z)$ with $|G'| < |H'|$ for all $z \in U$. Kumar et al. [12] proposed a class of right half-plane harmonic mappings, $F_a = H_a + \overline{G}_a$, $(-1 < a < 1)$, satisfying

$$\begin{aligned} H_a(z) + G_a(z) &= \frac{z}{1 - z}, \\ G'_a(z) &= \frac{a - z}{1 - az} H'_a(z). \end{aligned} \quad (5)$$

Using the shear construction of Clunie and Sheil-Small [13], it follows that

$$\begin{aligned} H_a(z) &= \frac{z/(1 + a) - 1/2z^2}{(1 - z)^2}, \\ G_a(z) &= \frac{(a/1 + a)z - (1/2)z^2}{(1 - z)^2}. \end{aligned} \quad (6)$$

Letting $a = 0$ in (5), the mapping $F_0 = H_0 + G_0$ with $(F_0 + G_0)(z) = z/(1 - z)$ and $G'_0(z) = -zH'_0(z)$, which is called the standard right half-plane mapping, is obtained. In [9], the following result was obtained.

Theorem 1 (see [9]). *Let $f = h + \overline{g} \in K_H^0$ satisfy the condition $(h + g)(z) = z/(1 - z)$ and dilatation $\omega(z) = g'(z)/h'(z) = e^{i\theta}z^n$ ($n \in \mathbb{Z}^+$; $\theta \in \mathbb{R}$). If $n = 1, 2$, then $F_0 * f \in S_{\text{CHD}}^0$.*

Recently, Liu and Ponnusamy [8] generalized Theorem 1 in the case $n = 1$ as follows.

Theorem 2 (see [8]). *Let $f = h + \overline{g} \in K_H^0$ satisfy the condition $(h + g)(z) = (1 + a)z/(1 - z)$ with dilatation $\omega_1(z) = (z + a)/(1 + az)$, where $a \in (-1, 1)$, and $f_1 = h_1 + \overline{g}_1 \in K_H^0$ with dilatation $\omega_2(z) = e^{i\theta}z$ ($\theta \in \mathbb{R}$). Then, $f * f_1 \in S_{\text{CHD}}^0$.*

Liu and Ponnusamy [8] also proposed the following problem.

Problem 1. Let $f = h + \overline{g} \in K_H$ with $(h + g)(z) = (1 + a)z/(1 - z)$ and dilatation function $\omega_1(z) = (z + a)/(1 + az)$, where $a \in (-1, 1)$, and let $f_n = h_n + \overline{g}_n \in K_H^0$ with dilatation $\omega_2(z) = e^{i\theta}z^n$ ($\theta \in \mathbb{R}, n \in \mathbb{N}^+$). Then, $f * f_n \in S_{\text{CHD}}^0$.

Ali et al. [14] solved Problem 1 for $\omega_2(z) = -z$. In [8], the authors conjectured that $f * f_n$ is not locally univalent for $n \geq 2$. In Section 2, new subclass of harmonic mappings $P_a[\phi]$ is obtained. It is also shown that the necessary and sufficient condition for $P_a[\phi] \in S_{\text{CHD}}^0$, if and only if $\phi(z)$ is convex. The function $f_n = h + \overline{g}$ satisfies the condition $(h -$

$g)(z) = z/(1 - z)$ and dilatation $\omega(z) = z^n$; $n \in \mathbb{N}$ with the mapping $P_a[\ell]$ belonging to subclass S_{CHD}^0 for $a \in (-1, (2 - n)/(2 + n))$ and for all $n \in \mathbb{N}$ are also explored.

The results proposed by Hengartner and Schober [15] and Pommerenke [16] are very useful in checking the convexity of an analytic function in the direction of the imaginary axis as well as the convexity in the direction of the real axis, respectively.

Lemma 1 (see [15]). *Suppose f is a nonconstant analytic function of U . Then,*

$$\operatorname{Re}\{(1 - z^2)f'(z)\} \geq 0, \quad (7)$$

if and only if

- (1) f is univalent in U .
- (2) f is convex in the direction of imaginary axis.
- (3) There exist sequences $\{z_n'\}$ converging to $z = 1$ and $\{z_n''\}$ converging to $z = -1$, such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \operatorname{Re}\{f(z_n')\} &= \sup_{|z| < 1} \operatorname{Re}\{f(z)\}, \\ \lim_{n \rightarrow \infty} \operatorname{Re}\{f(z_n'')\} &= \inf_{|z| < 1} \operatorname{Re}\{f(z)\}. \end{aligned} \quad (8)$$

Lemma 2 (see [16]). *Suppose f is a nonconstant analytic function of U satisfying the condition $f(0) = 0$ and $f'(0) = 0$, and suppose*

$$\varphi(z) = \frac{z}{(1 + ze^{i\theta_1})(1 + ze^{i\theta_2})}, \quad (9)$$

where $\theta_i, i = 1, 2 \in \mathbb{R}$. If

$$\operatorname{Re}\left(\frac{zf'(z)}{\varphi(z)}\right) > 0, \quad (10)$$

then f is convex in the direction of the real axis.

Recall that the region Ω is said to be convex in the direction θ , $0 \leq \theta < \pi$, if every line parallel to the line through 0 and $e^{i\theta}$ has either connected or empty intersection with Ω , and if Ω is convex in every direction, then it is called convex. If $\theta = 0$ (or $\theta = \pi/2$), then Ω is convex in the direction of the real axis “CHD” or (is convex in the direction of the imaginary axis “CID”). Clunie and Sheil [13] proposed an integrated way to construct a univalent harmonic mapping convex in a given direction.

Theorem 3 (see [13]). *Let $0 \leq \theta < \pi$. A locally univalent harmonic function $f = h + \overline{g}$ in U is a univalent mapping of U onto a domain convex in the direction of θ if and only if $h - e^{2i\theta}g$ is a univalent mapping of U onto a domain convex in the direction of θ .*

The study of the geometric properties of the convex combination of the univalent mappings is another important topic in the geometric function theory. Dorff and Rolf [17] developed a necessary condition for convex combination to be

univalent maps onto a domain convex in the direction of the imaginary axis.

Theorem 4 (see [17]). Let $f_1 = h_1 + \bar{g}_1$ and $f_2 = h_2 + \bar{g}_2$ be two univalent harmonic mappings defined on U , with f_1 and f_2 having the same second complex dilatation and satisfying conditions in (4); then,

$$\begin{aligned} f_3 &= h_3 + \bar{g}_3 \\ &= [th_1(z) + (1-t)h_2] + [t\bar{g}_1(z) + (1-t)\bar{g}_2(z)] \quad (11) \\ &= tf_1(z) + (1-t)f_2 \end{aligned}$$

is univalent and is convex in the direction of the imaginary axis.

It is known that even if f and g are convex analytic functions, the convex combination of f and g may not be a univalent function (see [18], and for more recent studies of linear combinations of harmonic mappings, see [17, 19–23]). Moreover, Kumar et al. [20] studied the convexity of linear combination of harmonic mappings, which are shears of the analytic mapping $z/(1-z)$ and $z(1-\beta z)/(1-z^2)$, for $-1 \leq \beta \leq 1$. Beig et al. [23] studied and found necessary conditions for the convex combination of the right half-plane mappings, the vertical strip mapping, their rotations, and some other harmonic mappings to be univalent and convex in a particular direction. In Section 3, the convex combination of mappings of the family of sense-preserving and locally univalent harmonic mappings $f_\alpha = h_\alpha + g_\alpha$, by shearing the function $h_\alpha + g_\alpha = K_\alpha$ where

$$K_\alpha(z) = h_\alpha(z) + g_\alpha(z) = \frac{z(1-\alpha z^2)}{1-z^2}, \quad \alpha \in [0, 1], \quad (12)$$

$$\left| \frac{g'_\alpha}{h'_\alpha} \right| < 1,$$

are studied. Further, the appropriate conditions for the convex combination in the vertical direction between univalent harmonic mappings to be univalent and convex in the vertical direction are also established.

2. Convolutions of Subclasses of Univalent Harmonic Right Half-Plane Mappings

In this section, we consider the harmonic mapping $P_a = h_a + \bar{g}_a$ with $(h_a - g_a)(z) = (1-a)z/(1-z)$ with the second complex dilatation $\omega(z) = (a+z)/(1+az)$, $a \in (-1, 1)$. Hence, we can solve for h_a and g_a as follows:

$$\begin{aligned} h_a(z) &= \left(\frac{z - (1/2)(1-a)z^2}{(1-z)^2} \right) = \frac{1}{2} \left(\frac{(1+a)z}{(1-z)^2} + \frac{(1-a)z}{(1-z)} \right), \\ g_a(z) &= h_a(z) - \frac{(1-a)z}{1-z} = \frac{1}{2} \left(\frac{(1+a)z}{(1-z)^2} - \frac{(1-a)z}{(1-z)} \right), \end{aligned} \quad (13)$$

and thus

$$\begin{aligned} P_a(z) &= h_a(z) + \bar{g}_a(z) = \frac{1}{2} \left(\left(\frac{(1+a)z}{(1-z)^2} + \frac{(1-a)z}{(1-z)} \right) \right. \\ &\quad \left. + \overline{\left(\frac{(1+a)z}{(1-z)^2} - \frac{(1-a)z}{(1-z)} \right)} \right). \end{aligned} \quad (14)$$

Taking $\ell(z) = z/(1-z)$, P_a given in (14) can be rewritten as

$$\begin{aligned} P_a(z) &= \frac{1}{2} \left(((1+a)z\ell'(z) + (1-a)\ell(z)) \right. \\ &\quad \left. + \overline{((1+a)z\ell'(z) - (1-a)\ell(z))} \right). \end{aligned} \quad (15)$$

For an analytic univalent function $\phi: U \rightarrow \mathbb{C}$ normalized by $\phi(0) = \phi'(0) - 1 = 0$, define

$$\begin{aligned} P_a[\phi](z) &= \frac{1}{2} \left(((1+a)z\phi'(z) + (1-a)\phi(z)) \right. \\ &\quad \left. + \overline{((1+a)z\phi'(z) - (1-a)\phi(z))} \right) \\ &= H_a[\phi](z) + \bar{G}_a[\phi](z), \end{aligned} \quad (16)$$

for all $z \in U$. It is clear that $P_a[\ell] = P_a$. Therefore,

$$\begin{aligned} P_a[\ell](z) &= \frac{1}{2} \left(((1+a)z\ell'(z) + (1-a)\ell(z)) \right. \\ &\quad \left. + \overline{((1+a)z\ell'(z) - (1-a)\ell(z))} \right) \\ &= H_a(z) + \bar{G}_a(z). \end{aligned} \quad (17)$$

If F is an analytic function with $F(0) = 0$, then

$$(H_a * F)(z) = \frac{1}{2} \left(((1+a)zF'(z) + (1-a)F(z)) \right), \quad (18)$$

and

$$(G_a * F)(z) = \frac{1}{2} \left(((1+a)zF'(z) - (1-a)F(z)) \right). \quad (19)$$

Lemma 3. Let $f = h + \bar{g}$ satisfy the condition $(h - g)(z) = z/(1-z)$ with second complex dilatation $\omega = g'/h'$ such that $|\omega| < 1$ and $P_a[\ell]$ be family of harmonic mappings given in (17). Then, $\hat{\omega}$, the second complex dilatation of $f * P_a[\ell]$, is given by

$$\hat{\omega}(z) = \frac{2(a+z)\omega(1-\omega) + (1+a)z\omega'(1-z)}{2(1+az)(1-\omega) + (1+a)z\omega'(1-z)}. \quad (20)$$

Proof. Since

$$f * P_a[\ell] = (h + \bar{g}) * (H_a + \bar{G}_a) = (h * H_a) + \overline{(g * G_a)}, \quad (21)$$

then

$$\hat{\omega}(z) = \frac{(g * G_a)'(z)}{(h * H_a)'(z)}. \quad (22)$$

Now, from (18) and (19), it follows that

$$\begin{aligned} \hat{\omega}(z) &= \frac{((1+a)zg'(z) - (1-a)g(z))'}{((1+a)zh'(z) + (1-a)h(z))'} \\ &= \frac{2ag'(z) + (1+a)zg''(z)}{2h'(z) + (1+a)zh''(z)}. \end{aligned} \quad (23)$$

Since $g'' = \omega'h' + \omega h''$, then

$$\hat{\omega}(z) = \frac{(2a\omega + (1+a)z\omega')h'(z) + (1+a)z\omega h''(z)}{2h'(z) + (1+a)zh''(z)}. \quad (24)$$

From $(h-g)(z) = z/(1-z)$, it follows that

$$\begin{aligned} h'(z) &= \frac{1}{(1-\omega)(1-z)^2}, \\ h''(z) &= \frac{2(1-\omega) + \omega'(1-z)}{(1-\omega)^2(1-z)^3}, \end{aligned} \quad (25)$$

and thus

$$\begin{aligned} \hat{\omega}(z) &= \frac{(2a\omega + (1+a)z\omega')(1-\omega)(1-z) + (1+a)z\omega[2(1-\omega) + \omega'(1-z)]}{2(1-\omega)(1-z) + (1+a)z[2(1-\omega) + \omega'(1-z)]} \\ &= \frac{2(a+z)\omega(1-\omega) + (1+a)z\omega'(1-z)}{(2+2az)(1-\omega) + (1+a)z\omega'(1-z)}. \end{aligned} \quad (26)$$

□

Lemma 4. Let $f = h + \bar{g} \in S_H^0$ with $(h-g)(z) = z/(1-z)$. If $P_a[\ell] * f$ is locally univalent, then $P_a[\ell] * f$ is in S_{CHD}^0 .

Proof. Since

$$\begin{aligned} H_a - G_a &= (1+a)\ell(z), \\ h - g &= \ell(z), \end{aligned} \quad (27)$$

then

$$\begin{aligned} (1+a)(h+g) &= (H_a - G_a) * (h+g) \\ &= (H_a * h + H_a * g - G_a * h - G_a * g), \\ (H_a + G_a) &= (H_a + G_a) * (h-g) \\ &= (H_a * h) - (H_a * g) + (G_a * h) - (G_a * g). \end{aligned} \quad (28)$$

Thus,

$$\begin{aligned} (H_a * h) - (G_a * g) &= \frac{1}{2}[(1+a)(h+g) + (H_a + G_a)], \\ ((H_a * h) - (G_a * g))' &= \frac{1}{2}[(1+a)(h' + g') + (H_a' + G_a')]. \end{aligned} \quad (29)$$

Let $\varphi(z) = z/(1-z)^2$. Consequently, we have

$$\begin{aligned} &\operatorname{Re}\left(\frac{z(H_a * h - G_a * g)'}{\varphi}\right) \\ &= \operatorname{Re}\left(\frac{(1+a)z}{2\varphi}(h' + g') + \frac{z}{2\varphi}(H_a' + G_a')\right) \\ &= \frac{1}{2}\left[\operatorname{Re}\left((1+a)\frac{z}{\varphi}(h' - g')\frac{(h' + g')}{(h' - g')}\right) + \operatorname{Re}\left(\frac{z}{\varphi}(H_a' - G_a')\frac{(H_a' + G_a')}{(H_a' - G_a')}\right)\right] \\ &= \frac{(1+a)}{2}\left[\operatorname{Re}\left(\frac{h' + g'}{h' - g'}\right) + \operatorname{Re}\left(\frac{H_a' + G_a'}{H_a' - G_a'}\right)\right] \\ &= \frac{(1+a)}{2}[\operatorname{Re}p_1(z) + \operatorname{Re}p_2(z)] > 0, \end{aligned} \quad (30)$$

where $p_1(z) = (1 + \omega_f)/(1 - \omega_f)$ and $p_2(z) = (1 + \omega_{P_a[\ell]})/(1 - \omega_{P_a[\ell]})$. Thus, from Lemma 3, we deduce that $(H_a * h) - (G_a * g)$ is convex in the direction of the real axis. Theorem 3 implies that $P_a[\ell] * f \in S_{\text{CHD}}^0$. \square

Lemma 5. *The mapping $P_a[\phi]$ given in (16) is locally univalent if and only if the function ϕ is convex.*

Proof. The mapping $P_a[\phi] = H_a[\phi] + \overline{G_a}[\phi]$ is given in (16). Since $P_a[\phi]$ is locally univalent if and only if $|G'_a[\phi]/H'_a[\phi]| < 1$,

$$\left| \frac{2a\phi'(z) + (1+a)z\phi''(z)}{2\phi'(z) + (1+a)z\phi''(z)} \right| < 1, \quad (31)$$

and hence

$$\left| \frac{2a/(1+a) + z\phi''(z)/\phi'(z)}{2/(1+a) + z\phi''(z)/\phi'(z)} \right| < 1. \quad (32)$$

Thus,

$$\left| \frac{(a-1)}{(1+a)} + \left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) \right| < \left| \frac{(1-a)}{(1+a)} + \left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) \right|. \quad (33)$$

It can be easily shown that the above inequality is identical to

$$\operatorname{Re} \left(1 + \frac{z\phi''(z)}{\phi'(z)} \right) > 0. \quad (34)$$

Hence, $P_a[\phi]$ is locally univalent if $\phi(z)$ is convex function. \square

Theorem 5. *The mapping $P_a[\phi]$ is in the class S_{CHD}^0 , if and only if ϕ is convex.*

Proof. The mapping $P_a[\phi] = H_a[\phi] + \overline{G_a}[\phi]$ is given in (16), and $H_a[\phi] - G_a[\phi] = (1-a)\phi$ where ϕ is convex. This implies that $H_a[\phi] - G_a[\phi]$ is convex in the direction of the real axis. Consequently, from Lemma 5 and Theorem 3, we deduce that $P_a[\phi]$ in S_{CHD}^0 .

In order to prove the next result, we will employ the distribution theory of the polynomial function roots of the disc unit. Given a polynomial function of degree m :

$$p(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0, \quad (35)$$

with complex coefficients, the parallel algorithm for discovering polynomial zeros for (35) inside U is worth exploring. Let

$$p^*(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)} = \bar{a}_m + \bar{a}_{m-1} z + \cdots + \bar{a}_1 z^{m-1} + \bar{a}_0 z^m. \quad (36)$$

The inverse zeros for (35) and (36) can be easily checked with respect to U . We need the following lemma to prove the main theorem. \square

Lemma 6 (Cohn's rule) (see [24]). *Let p be a polynomial function as given in (35) of degree m and let p^* be as given in (36). The number of zeros of p in or on the unit circle is denoted by r and s , respectively. If $|a_0| < |a_m|$, then*

$$p_1(z) = \frac{\bar{a}_m p_1(z) - a_0 p^*(z)}{z} \quad (37)$$

is a polynomial function of degree $m-1$ with $r_1 = r-1$ and $s_1 = s$, where r_1 and s_1 are the number of zeros of the polynomial p_1 in or on the unit circle, respectively.

Theorem 6. *Let $P_a[\ell]$ be mapping given in (17) and let $f_n = h + \bar{g}$ with $(h-g)(z) = z/(1-z)$ and dilatation $\omega_1(z) = z^n$, $n \in \mathbb{N}$. Then, $P_a[\ell] * f_n \in S_{\text{CHD}}^0$ for $a \in (-1, (2-n)/(2+n))$ and for all n belonging to \mathbb{N} .*

Proof. According Lemma 4, it suffices to show that the mapping $P_a[\ell] * f_n$ has dilatation ω such that $|\omega| < 1$ for all $z \in U$. Setting $\omega_1(z) = z^n$ in (20), we obtain

$$\begin{aligned} \omega(z) &= -z^n \frac{(a(2+n)+n) + (2-n-an)z - 2az^n - 2z^{1+n}}{-2 - 2az + (2-n-an)z^n + (a(2+n)+n)z^{1+n}} \\ &= -z^n \frac{q(z)}{q^*(z)}, \end{aligned} \quad (38)$$

where

$$\begin{aligned} q(z) &= (a(2+n)+n) + (2-n-an)z - 2az^n - 2z^{1+n} \\ &= a_0 + a_1 z + a_n z^n + a_{n+1} z^{n+1}, \\ q^*(z) &= z^{n+1} \overline{q\left(\frac{1}{\bar{z}}\right)} = -2 - 2az + (2-n-an)z^n \\ &\quad + (a(2+n)+n)z^{1+n} \\ &= \bar{a}_{n+1} + \bar{a}_n + \bar{a}_1 z^n + \bar{a}_0 z^{n+1}. \end{aligned} \quad (39)$$

If $z_0 \neq 0$, is zero of the polynomial function $q(z)$, then $1/z_0$ will be zero of the polynomial function q^* . Hence, we can rewrite ω as

$$\omega(z) = -z^n \frac{(z+A_1)(z+A_2)(z+A_3), \dots, (z+A_{n+1})}{(1+\bar{A}_1 z)(1+\bar{A}_2 z)(1+\bar{A}_3 z), \dots, (1+\bar{A}_{n+1} z)}. \quad (40)$$

In order to show that $|\omega| < 1$ in U , it suffices to prove that $|A_i| < 1$ for $i = 1, 2, 3, \dots, n+1$, or equivalently, all the zeros A_i for $i = 1, 2, 3, \dots, n+1$ of the polynomial function $q(z)$ lie in or on the unit circle $|z| = 1$.

Note that

$$|a_0| = |a(2+n)+n| < |2| = |a_{n+1}|, \quad \forall a \in \left(-1, \frac{2-n}{2+n}\right), n \in \mathbb{N}, \quad (41)$$

so Cohn's rule can be applied on q . Consider the polynomial function $q_1(z)$ given by

$$\begin{aligned} q_1(z) &= \frac{\bar{a}_{n+1}q(z) - a_0q^*(z)}{z} \\ &= (1+a)(-2+n+a(2+n))(2+nz^{n-1} - (2+n)z^n). \end{aligned} \quad (42)$$

It is easy to verify that Cohn's rule is applicable to q_1 also. So, let a polynomial function q_2 given by

$$\begin{aligned} q_2(z) &= \frac{-(2+n)q_1(z) - 2q^*(z)}{z} \\ &= (1+a)(-2+n+a(2+n))(4n+n^2)(-1+z)z^{n-2}. \end{aligned} \quad (43)$$

Then, the zeros of polynomial function q_2 lie on ∂U or in U . We deduce that all zeros of q_1 lie inside U or on ∂U , and this implies that all the zeros of q lie inside U or on ∂U . \square

Corollary 1. Let $P_a[\ell]$ be mapping given in (17) and let $f_2 = h + \bar{g}$ with $(h-g)(z) = z/(1-z)$ and dilatation $\omega_1(z) =$

$-z^n$, $n \in \mathbb{N}$. Then, $P_a[\ell] * f_n \in S_{CHD}^0$ for $a \in (-1, (2-n)/(2+n))$.

The following examples illustrate Theorems 5 and 6.

Example 1. In Theorem 6, if $n = 2$, $(h-g)(z) = z/(1-z)$, and $\omega_{f_2}(z) = z^2$, then

$$\begin{aligned} f_2(z) &= h(z) + \bar{g}(z) \\ &= \frac{1}{8} \left(\frac{2z(3-2z)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right) \\ &\quad + \frac{1}{8} \overline{\left(\frac{2z(2z-1)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right)}. \end{aligned} \quad (44)$$

It follows from (17) that

$$\begin{aligned} P_a[\ell](z) &= h_a + \bar{g}_a = \frac{1}{2} \left(\left(\frac{(1+a)z}{(1-z)^2} + \frac{(1-a)z}{(1-z)} \right) \right. \\ &\quad \left. + \overline{\left(\frac{(1+a)z}{(1-z)^2} - \frac{(1-a)z}{(1-z)} \right)} \right), \end{aligned} \quad (45)$$

and $P_a[\ell](z) * f_2(z) = H(z) + \bar{G}(z)$. Further,

$$\begin{aligned} H(z) &= (h_a * h)(z) \\ &= \frac{1}{16} \left[\left(\frac{(1+a)z}{(1-z)^2} + \frac{(1-a)z}{(1-z)} \right) * \left(\frac{2z(3-2z)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right) \right] \\ &= \frac{1}{16} \left[\frac{8(1+a)z}{(1-z)^3(1+z)} + (1-a) \left(\frac{2z(3-2z)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right) \right], \\ G(z) &= (g_a * g)(z) \\ &= \frac{1}{16} \left[\left(\frac{(1+a)z}{(1-z)^2} - \frac{(1-a)z}{(1-z)} \right) * \left(\frac{2z(2z-1)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right) \right] \\ &= \frac{1}{16} \left[\frac{8(1+a)z^3}{(1-z)^3(1+z)} - (1-a) \left(\frac{2z(2z-1)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right) \right]. \end{aligned} \quad (46)$$

Figures 1–4, respectively, display representations of concentrated circles within U under the harmonic mapping $P_a[\ell](z)$ and concentrated circles under the convolution map $P_a[\ell](z) * f_2(z)$. We take various values of $a = -0.001, -0.5 \in (-1, 0)$.

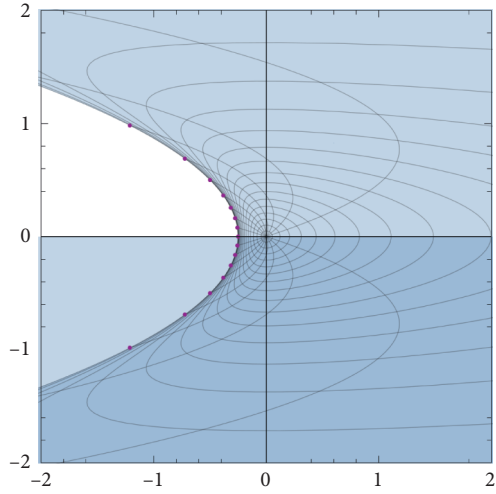
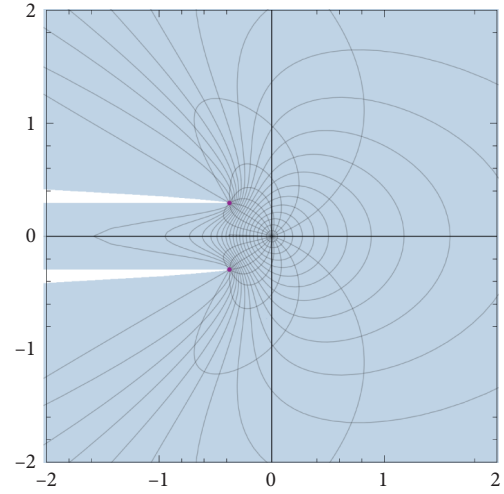
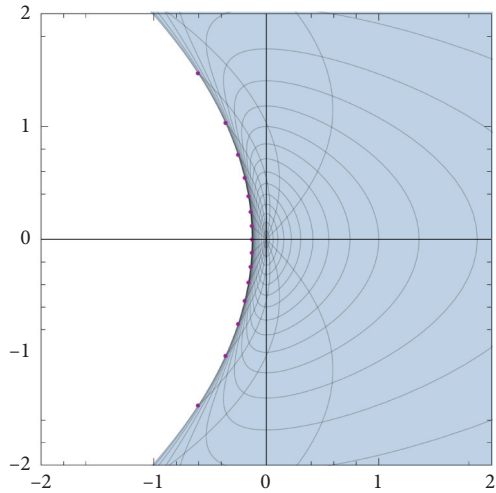
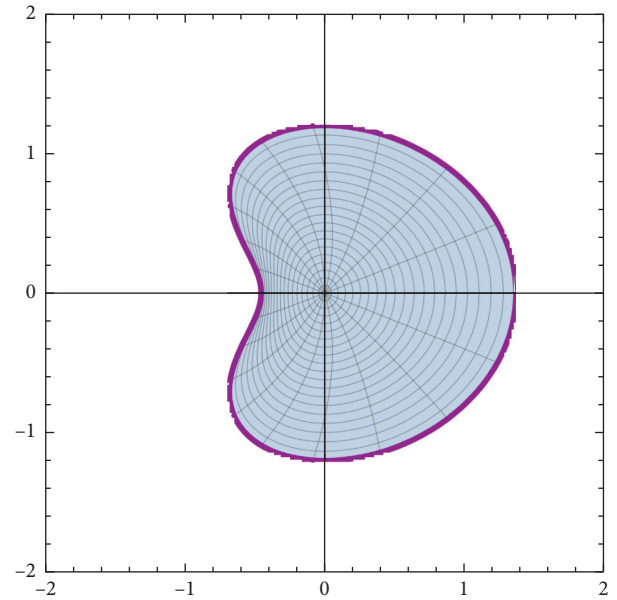
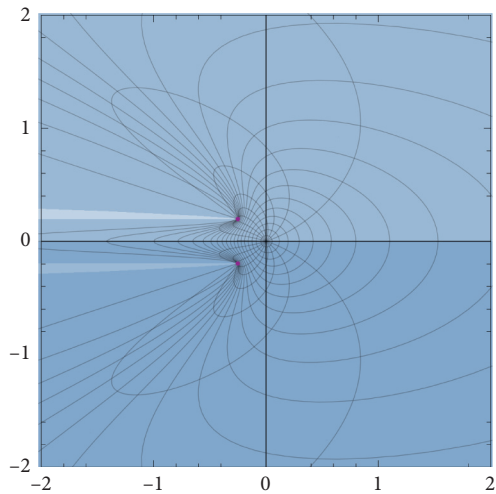
Example 2. Consider an analytic univalent function $\phi(z) = z + (z^2/4)$, where $\phi(0) = 0$ and $\phi'(0) = 1$. Since $\phi'(z) = 1 + (z/2)$ and $\phi''(z) = 1/2$,

$$\operatorname{Re} \left(1 + \frac{z\phi''(z)}{\phi'(z)} \right) = \operatorname{Re} \left(1 + \frac{(1/2)z}{(1+(1/2)z)} \right) \geq 0. \quad (47)$$

This implies that the function $\phi(z)$ maps U into convex region. In view of Theorem 1, the mapping $P_a[\phi](z)$ is in S_{CHD}^0 for $-1 < a < 1$. Now

$$\begin{aligned} P_a[\phi](z) &= \frac{1}{2} \left((1-a) \left(\frac{z^2}{4} + z \right) + (a+1) \left(\frac{z^2}{2} + z \right) \right) \\ &\quad + \frac{1}{2} \overline{\left((a+1) \left(\frac{z^2}{2} + z \right) - (1-a) \left(\frac{z^2}{4} + z \right) \right)}, \quad (z \in U). \end{aligned} \quad (48)$$

Figures 5–7 display images of U under $P_a[z + (1/4)z^2]$ for various values of $a \in (-1, 1)$, respectively. The image of

FIGURE 1: Image of $P_{-0.001}[\ell](z)$.FIGURE 4: Image of $P_{-0.5}[\ell](z) * f_2(z)$.FIGURE 2: Image of $P_{-0.5}[\ell](z)$.FIGURE 5: Image of $P_{-0.09}[z + (z^2/4)]$.FIGURE 3: Image of $P_{-0.001}[\ell](z) * f_2(z)$.

U under $P_a[z + (1/4)z^2]$ is convex in the direction of the real axis.

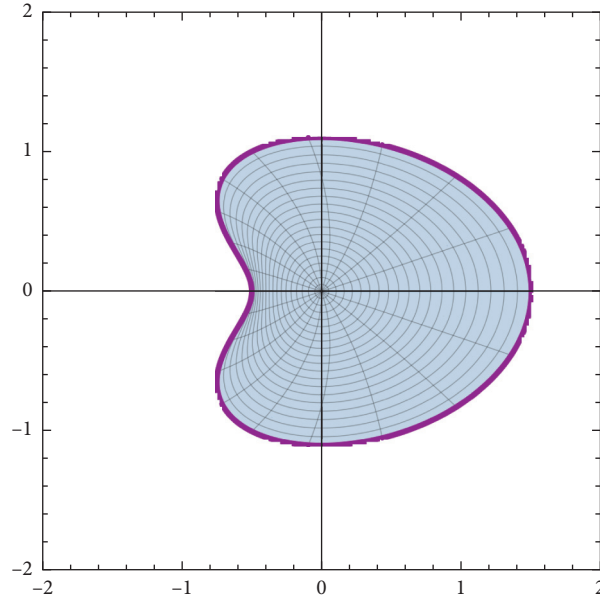
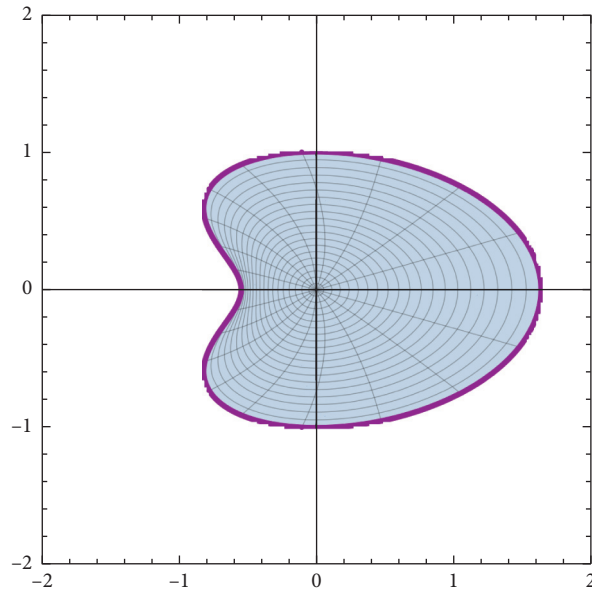
3. Convex Combination of a Family of Univalent Harmonic Mappings

Let $f_\alpha(z) = h_\alpha(z) + \overline{g}_\alpha(z)$, where

$$K_\alpha(z) = h_\alpha(z) + g_\alpha(z) = \frac{z(1 - \alpha z^2)}{1 - z^2}, \quad \alpha \in [0, 1], \quad (49)$$

$$\left| \frac{g'_\alpha}{h'_\alpha} \right| < 1,$$

be normalized, sense-preserving, and locally univalent harmonic mapping defined on U . First of all, we have to

FIGURE 6: Image of $P_0[z + (z^2/4)]$.FIGURE 7: Image of $P_{0.09}[z + (z^2/4)]$.

show that f_α belongs to class S_H and maps onto region convex in the direction of the imaginary axis. Let

$$\Psi(z) = (1 - z^2)K'_\alpha(z) = \frac{1 + \alpha z^4 + (1 - 3\alpha)z^2}{1 - z^2}, \quad (50)$$

for all $z \in U$, $0 \leq \alpha \leq 1$, and we will prove that

$$\operatorname{Re}(\Psi(z)) = \operatorname{Re} \frac{1 + \alpha z^4 + (1 - 3\alpha)z^2}{1 - z^2} > 0. \quad (51)$$

Since $|1 - z^2|^2 \neq 0$, for all z in U , $0 \leq \alpha \leq 1$ and

$$\begin{aligned} & \left((1 - \bar{z}^2)(1 + \alpha z^4 + (1 - 3\alpha)z^2) \right) \\ &= (1 - \bar{z}^2 + z^2 - |z|^4 + \alpha(z^4 - 3z^2 + 3|z|^4 - |z|^4 z^2)), \end{aligned} \quad (52)$$

it follows that

$$\operatorname{Re}(\Psi(z)) = \operatorname{Re} \frac{(1 - \bar{z}^2 + z^2 - |z|^4 + \alpha(z^4 - 3z^2 + 3|z|^4 - |z|^4 z^2))}{|1 - z^2|^2}. \quad (53)$$

Note that $\Psi(0) = 1$ and for each $\gamma \in \mathbb{R}$,

$$\begin{aligned}
\operatorname{Re}(\Psi(e^{i\gamma})) &= \operatorname{Re} \frac{\left(1 - (e^{-i\gamma})^2 + (e^{i\gamma})^2 - |e^{i\gamma}|^4 + \alpha \left((e^{i\gamma})^4 - 3(e^{i\gamma})^2 + 3|e^{i\gamma}|^4 - |e^{i\gamma}|^4 (e^{i\gamma})^2 \right)\right)}{|1 - (e^{i\gamma})^2|^2} \\
&= \operatorname{Re} \frac{\left(1 - (e^{-i\gamma})^2 + (e^{i\gamma})^2 - 1 + \alpha \left((e^{i\gamma})^4 - 3(e^{i\gamma})^2 + 3 - (e^{i\gamma})^2 \right)\right)}{|1 - (e^{i\gamma})^2|^2} \\
&= \operatorname{Re} \frac{\left(-4i \cos(\gamma) \sin(\gamma) + \alpha \left((e^{i\gamma})^4 - 3(e^{i\gamma})^2 + 3 - (e^{i\gamma})^2 \right)\right)}{|1 - (e^{i\gamma})^2|^2} \\
&= \operatorname{Re} \frac{\left(i(\sin(4\gamma) - 4 \sin(2\gamma)) - 4 \cos(\gamma) \sin(\gamma) + 2\alpha((\cos(2\gamma) - 1)^2)\right)}{|1 - (e^{i\gamma})^2|^2} \\
&= \frac{2\alpha}{|1 - (e^{i\gamma})^2|^2} ((\cos(2\gamma) - 1)^2) \geq 0, \quad 0 \leq \alpha \leq 1.
\end{aligned} \tag{54}$$

Hence, by the minimum principle for harmonic functions, we obtain

$$\operatorname{Re}(\Psi(z)) = \operatorname{Re}((1 - z^2)K'_\alpha(z)) > 0, \tag{55}$$

for all $z \in U$, and this yields the result.

In addition, by Lemma 1, we can deduce that the analytic function $K_\alpha = h_\alpha + g_\alpha$ is univalent and maps onto domain convex in the direction of the imaginary axis. Further, Theorem 3 yields that the harmonic mapping $h_\alpha + \bar{g}_\alpha$ belongs to the class S_H and is convex in the direction of

imaginary axis. Figure 8 gives an illustration of images of U under K_α for several values of α .

Let us begin by presenting our own dilatation ω .

Theorem 7. Let $f_{\alpha_j} = h_{\alpha_j} + \bar{g}_{\alpha_j}$, for $j = 1, 2$, be two normalized harmonic mapping satisfying $h_{\alpha_j}(z) + g_{\alpha_j}(z) = z(1 - \alpha_j z^2)/(1 - z^2)$, $0 \leq \alpha_j \leq 1$, and dilatation $\omega_j = g'_j/h'_j$ with $|\omega_j| < 1$ in U . Then, the second complex dilatation ω of the mapping $f = t f_{\alpha_1} + (1 - t) f_{\alpha_2}$, $0 \leq t \leq 1$, is defined to be

$$\omega(z) = \frac{(1 + z^2)(\omega_1 \omega_2 + t \omega_1 + (1 - t) \omega_2) + z^2(z^2 - 3)(\alpha_1 t \omega_1(1 + \omega_2) + \alpha_2(1 - t)(1 + \omega_1) \omega_2)}{(1 + z^2)(1 + t \omega_2 + (1 - t) \omega_1) + z^2(z^2 - 3)(\alpha_1 t(1 + \omega_2) + \alpha_2(1 - t)(1 + \omega_1))}. \tag{56}$$

Proof. Since

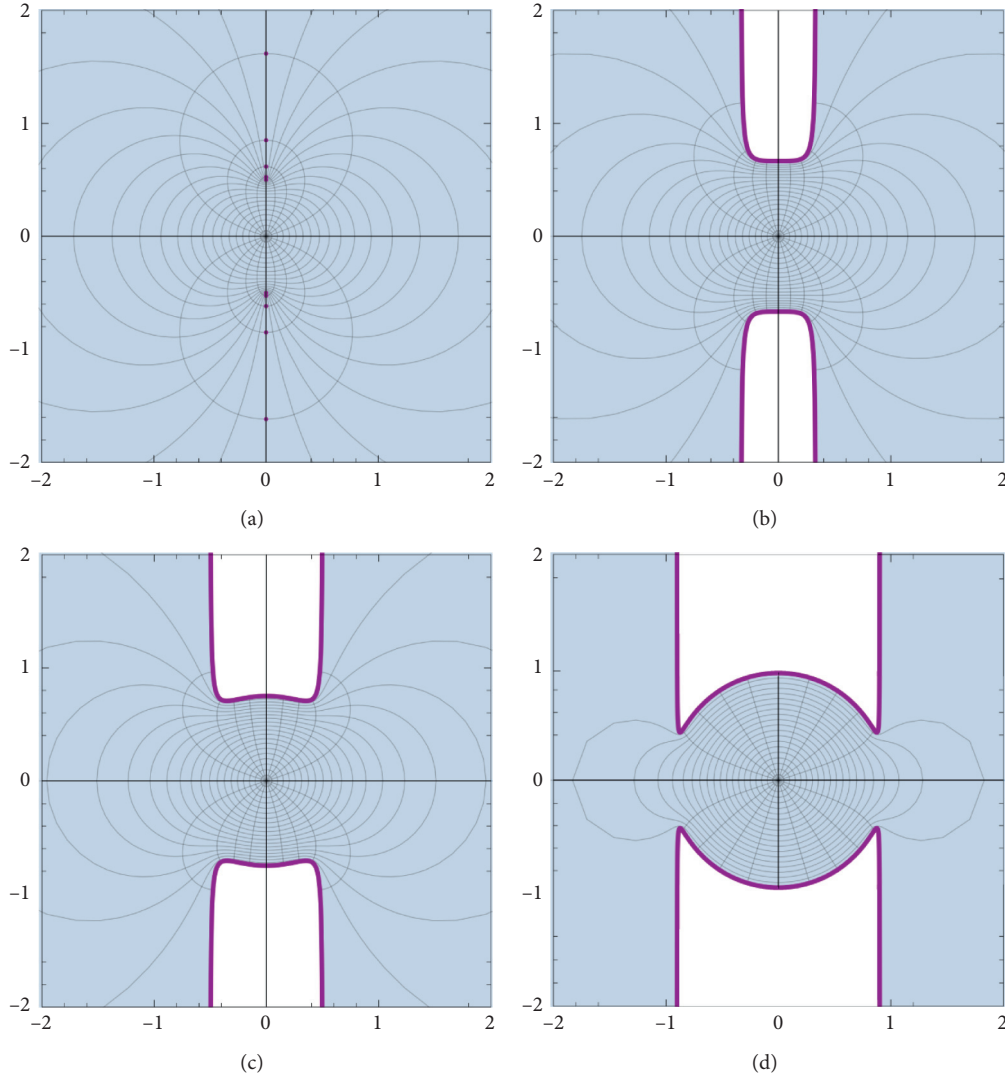
$f = t f_{\alpha_1} + (1 - t) f_{\alpha_2} = t h_{\alpha_1} + (1 - t) h_{\alpha_2} + t \bar{g}_{\alpha_1} + (1 - t) \bar{g}_{\alpha_2}$, it follows that the dilatation ω of the mapping f is given by

$$\omega = \frac{t g'_{\alpha_1} + (1 - t) g'_{\alpha_2}}{t h'_{\alpha_1} + (1 - t) h'_{\alpha_2}} = \frac{t \omega_1 h'_{\alpha_1} + (1 - t) \omega_2 h'_{\alpha_2}}{t h'_{\alpha_1} + (1 - t) h'_{\alpha_2}}. \tag{57}$$

Making use of the mapping $h_{\alpha_j}(z) + g_{\alpha_j}(z) = z(1 - \alpha_j z^2)/(1 - z^2)$ and $\omega_j = g'_{\alpha_j}/h'_{\alpha_j}$, for $j = 1, 2$, we obtain

$$\begin{aligned}
h'_{\alpha_1}(z) &= \frac{1 + z^2(1 + \alpha_1(z^2 - 3))}{(1 + \omega_1)(1 - z^2)^2}, \\
h'_{\alpha_2}(z) &= \frac{1 + z^2(1 + \alpha_2(z^2 - 3))}{(1 + \omega_2)(1 - z^2)^2}.
\end{aligned} \tag{58}$$

Replacing the above expressions with h'_{α_1} and h'_{α_2} in (57), it follows that

FIGURE 8: Images of (a) $K_0(U)$, (b) $K_{1/3}(U)$, (c) $K_{1/2}(U)$, and (d) $K_{0.9}(U)$.

$$\omega = \frac{t\omega_1 \left(1 + z^2(1 + \alpha_1(z^2 - 3)) / (1 + \omega_1)(1 - z^2)^2\right) + (1-t)\omega_2 \left(1 + z^2(1 + \alpha_2(z^2 - 3)) / (1 + \omega_2)(1 - z^2)^2\right)}{t \left(1 + z^2(1 + \alpha_1(z^2 - 3)) / (1 + \omega_1)(1 - z^2)^2\right) + (1-t) \left(1 + z^2(1 + \alpha_2(z^2 - 3)) / (1 + \omega_2)(1 - z^2)^2\right)} \quad (59)$$

$$= \frac{t\omega_1(1 + z^2 + \alpha_1 z^2(z^2 - 3))(1 + \omega_2) + (1-t)\omega_2(1 + z^2 + \alpha_2 z^2(z^2 - 3))(1 + \omega_1)}{t(1 + z^2 + \alpha_1 z^2(z^2 - 3))(1 + \omega_2) + (1-t)(1 + z^2 + \alpha_2 z^2(z^2 - 3))(1 + \omega_1)},$$

which yields the results in (56) through some computation.

The next result clarifies that the necessary and sufficient condition for the convex combination of f_{α_1} and f_{α_2} is in the class S_H and convex in the direction of the imaginary axis if it is locally univalent and sense-preserving. \square

Theorem 8. Let $f_{\alpha_j} = h_{\alpha_j} + \overline{g_{\alpha_j}}$, for $j = 1, 2$, be two normalized harmonic mappings satisfying $h_{\alpha_j}(z) + g_{\alpha_j}(z) = z(1 - \alpha_j z^2)/(1 - z^2)$, $0 \leq \alpha_j \leq 1$, and dilatation $\omega_j = g'_{\alpha_j}/h'_{\alpha_j}$

with $|\omega_j| < 1$ in U . Then, the mapping $f = tf_{\alpha_1} + (1-t)f_{\alpha_2}$, $0 \leq t \leq 1$ with dilatation ω as given in (56), is in S_H and maps into region convex in the direction of the imaginary axis, provided f is locally univalent and sense-preserving.

Proof. Define

$$f = h + \overline{g} = (th_{\alpha_1} + (1-t)h_{\alpha_2}) + \overline{tg_{\alpha_1} + (1-t)g_{\alpha_2}}. \quad (60)$$

Let

$$\begin{aligned}
h + g &= t(h_{\alpha_1} + g_{\alpha_1}) + (1-t)(h_{\alpha_2} + g_{\alpha_2}) \\
&= t(h_{\alpha_1} + g_{\alpha_1}) + (1-t)(h_{\alpha_2} + g_{\alpha_2}) \quad (61) \\
&= tK_{\alpha_1} + (1-t)K_{\alpha_2},
\end{aligned}$$

where $K_{\alpha_j} = h_{\alpha_j} + g_{\alpha_j}$, $j = 1, 2$. Since the mapping f_{α_j} satisfies (49), it follows from (7) that

$$\begin{aligned}
\operatorname{Re}\{(1-z^2)(h' + g')\} &= t\operatorname{Re}\{(1-z^2)K_{\alpha_1}'\} \\
&\quad + (1-t)\operatorname{Re}\{(1-z^2)K_{\alpha_2}'\} > 0, \quad (62)
\end{aligned}$$

on U for all $0 \leq t \leq 1$. Hence, according to Lemma 1, one can deduce that the mapping $h + g$ is univalently convex in the direction of the imaginary axis. Also, If $f = h + \bar{g}$ is sense-

preserving locally univalent mapping in U , then in view of Theorem 3, $f \in S_H$ is univalent analytic mapping of U onto a region convex in the direction of imaginary axis. \square

Theorem 9. For $j = 1, 2$, define $f_{\alpha_j} = h_{\alpha_j} + \bar{g}_{\alpha_j}$ to be normalized sense-preserving harmonic mapping such that $h_{\alpha_j}(z) + g_{\alpha_j}(z) = z(1 - \alpha_j z^2)/(1 - z^2)$, $0 \leq \alpha_j \leq 1$ with $\alpha_j = \alpha$. Let ω_j be the dilatation of f_{α_j} ($|\omega_j| < 1$ in U). If $\omega_1 \neq \omega_2$, then the mapping $f = tf_{\alpha_1} + (1-t)f_{\alpha_2}$, $0 \leq t \leq 1$, is in S_H and is convex in the direction of the imaginary axis.

Proof. From Theorem 8, it will be sufficient if we show that the dilatation of f given in (56) satisfies the condition $|\omega| < 1$ in U . Suppose that $\alpha_1 = \alpha_2 = \alpha$ in (56); then,

$$\begin{aligned}
\omega &= \frac{(1+z^2)[\omega_1\omega_2 + t\omega_1 + (1-t)\omega_2] + \alpha z^2(z^2-3)[\omega_1\omega_2 + t\omega_1 + (1-t)\omega_2]}{(1+z^2)[1+t\omega_2 + (1-t)\omega_1] + \alpha z^2(z^2-3)[1+t\omega_2 + (1-t)\omega_1]} \\
&= \frac{(\omega_1\omega_2 + t\omega_1 + (1-t)\omega_2)(1 + \alpha z^4 + (1-3\alpha)z^2)}{(1+t\omega_2 + (1-t)\omega_1)(1 + z^2 + \alpha z^2(z^2-3))} \quad (63) \\
&= \frac{\omega_1\omega_2 + t\omega_1 + (1-t)\omega_2}{1+t\omega_2 + (1-t)\omega_1}.
\end{aligned}$$

From the proof of Theorem 3 in [21], it is clear that $|\omega| < 1$. Hence, we deduce that the mapping f is locally univalent and sense-preserving.

Next, we will consider one of the harmonic mappings involved in the linear combination induced by shearing analytic mapping $z(1 - \beta z)/(1 - z^2)$ where $\beta \in [-1, 1]$. \square

Theorem 10. Define $f_\beta = h_\beta + \bar{g}_\beta$, satisfying $h_\beta(z) + g_\beta(z) = z(1 - \beta z)/(1 - z^2)$, $-1 \leq \beta \leq 1$, with dilatation $\omega_1 = g_\beta'/h_\beta'$ lies in U and let $f_\alpha = h_\alpha + \bar{g}_\alpha$, satisfying $h_\alpha(z) + g_\alpha(z) = z(1 - \alpha z^2)/(1 - z^2)$, $0 \leq \alpha \leq 1$, with dilatation $\omega_2 = g_\alpha'/h_\alpha'$ lies in U , be two normalized harmonic mapping. Then, the mapping $f_{\beta,\alpha} = tf_\beta + (1-t)f_\alpha$, $0 \leq t \leq 1$ is univalent and convex in the direction of the imaginary axis if $f_{\beta,\alpha}$ is locally univalent and sense-preserving.

Proof. Since $h_\alpha(z) + g_\alpha(z) = z(1 - \alpha z^2)/(1 - z^2)$ and $h_\beta(z) + g_\beta(z) = z(1 - \beta z)/(1 - z^2)$, it follows that

$$\begin{aligned}
h + g &= th_\alpha + (1-t)h_\beta + tg_\alpha + (1-t)g_\beta \\
&= t(h_\alpha + g_\alpha) + (1-t)(h_\beta + g_\beta), \quad (64)
\end{aligned}$$

which implies that

$$h' + g' = t(h_\alpha' + g_\alpha') + (1-t)(h_\beta' + g_\beta'). \quad (65)$$

Hence,

$$\begin{aligned}
\operatorname{Re}\{(1-z^2)(h' + g')\} &= t\operatorname{Re}\{(1-z^2)(h_\alpha' + g_\alpha')\} \\
&\quad + (1-t)\operatorname{Re}\{(1-z^2)(h_\beta' + g_\beta')\}, \quad (66)
\end{aligned}$$

from (55). Since

$$\operatorname{Re}\{(1-z^2)(h_\beta' + g_\beta')\} = \frac{(1-|z^2|)(1+|z|^2 - 2\beta\operatorname{Re}(z))}{|1-z^2|^2} > 0, \quad (67)$$

for all $0 \leq t \leq 1$, we can easily see that

$$\operatorname{Re}\{(1-z^2)(h' + g')\} > 0. \quad (68)$$

In view of Theorem 3, we deduce that the mapping $f = h + \bar{g}$ is univalent and maps onto domain convex in the direction of the imaginary axis provided f is locally univalent and sense-preserving on U .

It is known that for $0 \leq \theta < \pi$, the mapping $f_\theta = h_\theta + \bar{g}_\theta$ is called vertical strip mapping if it satisfies

$$h_\theta(z) + g_\theta(z) = \frac{1}{2i \sin \theta} \log \left(\frac{1 + e^{i\theta} z}{1 - e^{i\theta} z} \right). \quad (69)$$

Thus, Theorem 10 can be expressed in terms of vertical strip mappings. \square

Corollary 2. Let $f_\theta = h_\theta + \overline{g}_\theta$, where $h_\theta(z) + g_\theta(z) = 1/2i \sin \theta \log(1 + e^{i\theta}z/1 - e^{i\theta}z)$ with dilatation g'_θ/h'_θ lies in U and $f_\alpha = h_\alpha + \overline{g}_\alpha$, where $h_\alpha(z) + g_\alpha(z) = z(1 - \alpha z^2)/(1 - z^2)$, $0 \leq \alpha \leq 1$ with dilatation g'_α/h'_α lies in U , to be two normalized harmonic mapping. Then, the mapping $f_{\theta,\alpha} = tf_\theta + (1-t)f_\alpha$, $0 \leq t \leq 1$, is univalent and convex in the direction of the imaginary axis if $f_{\theta,\alpha}$ is locally univalent and sense-preserving.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Certain Classes of Analytic Functions Bound with Kober Operators in q -Calculus

S. D. Purohit ¹, M. M. Gour ², S. Joshi ² and D. L. Suthar ³

¹Department of HEAS (Mathematics), Rajasthan Technical University, Kota, India

²Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur, India

³Department of Mathematics, Wollo University, Dessie Campus, Wollo Dessie, Amhara, Ethiopia

Correspondence should be addressed to D. L. Suthar; dlsuthar@gmail.com

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Through applying the Kober fractional q -calculus apprehension, we preliminary implant and introduce new types of univalent analytical functions with a q -differintegral operator in the open disk $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$. The coefficient inequality and distortion theorems are among the results examined with these forms of functions. Specific cases are responded and addressed immediately. The findings include an expansion of the numerous established results in the q -theory of analytical functions.

1. Introduction and Preliminary

The q -analysis theory has been applied in recent times in several fields of science and engineering. The fractional q -calculus is indeed an analog of the conventional fractional calculus in q -theory. Very recently, Wang et al. [1] and Yan et al. [2] investigated the properties of subclasses of multivalent analytic or meromorphic functions expressed with q -difference operators. Furthermore, Srivastava [3] investigated the excellent work with q -calculus and fractional q -calculus operators, which is quite valuable for academics working on these issues. The applications of fractional q -calculus operators have been investigated by Purohit and Raina [4] to describe several new classes of analytic functions in open disk $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$. Moreover, Murugusundaramoorthy et al. [5], Purohit [6], and Purohit and Raina [4, 7] gave related work and added various classes of univalent and multivalently analytic functions in open unit disk \mathbb{U} . Several others have also released new classes of analytical functions with the resources of q -calculus operators. For any more inquiries on the analytic functions classes, we refer to [1, 2, 8–13] for functions described by applying q -calculus operators and subject related to this work.

In the current inquiry, we are planning to develop few additional families of analytic functions applying the Kober differential and integral operators in q -calculus. The results obtained must also provide the coefficient inequalities and distortion theorems for the subclasses established here below. First, we use the main notations and definitions in the q -calculus which are relevant to grasp the object of the study.

For each complex number \mathfrak{P} , the q -shifted factorials are delimited by

$$(\mathfrak{P}; q)_m = \prod_{j=0}^{m-1} (1 - q^j \mathfrak{P}); \quad m \in \mathbb{N}, \quad (1)$$

$$(\mathfrak{P}; q)_0 = 1,$$

and with regard to the basic analog of the gamma function,

$$(q^{\mathfrak{P}}; q)_m = \frac{\Gamma_q(\mathfrak{P} + m)(1 - q)^m}{\Gamma_q(\mathfrak{P})}; \quad m > 0, \quad (2)$$

in which the q -gamma function is set by (see [14])

$$\Gamma_q(\mathfrak{P}) = \frac{(q; q)_{\infty}(1 - q)^{1-\mathfrak{P}}}{(q^{\mathfrak{P}}; q)_{\infty}}; \quad 0 < q < 1. \quad (3)$$

The recurrence relationship specified by Gaspar and Rahman [15] for the q -gamma function is

$$\Gamma_q(1 + \mathfrak{P}) = \frac{(1 - q^{\mathfrak{P}})\Gamma_q(\mathfrak{P})}{1 - q}. \quad (4)$$

If $|q| < 1$, then equation (1) shall continue to play a role $m = \infty$ as an infinite product of convergence:

$$(\mathfrak{P}; q)_{\infty} = \prod_{j=0}^{\infty} (1 - \mathfrak{P}q^j), \quad (5)$$

and we have

$$(\mathfrak{P}; q)_m = \frac{(\mathfrak{P}; q)_{\infty}}{(\mathfrak{P}q^m; q)_{\infty}}, \quad m \in \mathbb{N} \cup \{\infty\}. \quad (6)$$

The q -binomial expansion is now as follows:

$$(a - b)_{\tau} = a^{\tau} \left(\frac{-b}{a}; q \right)_{\tau} = a^{\tau} {}_1\phi_0 \left[q^{-\tau}; -; q, \frac{bq^{\tau}}{a} \right]. \quad (7)$$

[15] accounts for Jackson's q -integral and q -derivative of a function f , which are described on a subset of \mathbb{C} , as

$$D_q f(\xi) = \frac{f(\xi) - f(q\xi)}{\xi(1 - q)}, \quad \xi \neq 0, q \neq 0, \quad (8)$$

with

$$D_{\xi}^{\mu} = \frac{(1 - q^{\mu})\xi^{\mu-1}}{1 - q}, \quad (9)$$

$$\int_0^{\xi} f(y) d_q y = \xi(1 - q) \sum_{k=0}^{\infty} q^k f(\xi q^k).$$

2. The Fractional q -Calculus Operators

Purohit and Raina [4] described the fractional q -integral operator of function $f(\xi)$ given by

$$I_{q,\xi}^{\mathfrak{P}} f(\xi) = \frac{1}{\Gamma_q(\mathfrak{P})} \int_0^{\xi} (\xi - yq)_{\mathfrak{P}-1} f(y) d_q y, \quad (10)$$

where $\mathfrak{P} > 0$ is the order of integral and $f(\xi)$ is an analytic function in \mathbb{U} , and (7) the $(\xi - yq)_{\mathfrak{P}-1}$ be expressed as

$$(\xi - yq)_{\mathfrak{P}-1} = \xi^{\mathfrak{P}-1} {}_1\phi_0 \left[q^{1-\mathfrak{P}}; -; q, \frac{yq^{\mathfrak{P}}}{\xi} \right], \quad (11)$$

where ${}_1\phi_0[\mathfrak{P}; -; q, \xi]$ is special case of basic hypergeometric series ${}_2\phi_1[\mathfrak{P}; \mathfrak{F}; \gamma; q, \xi]$ for $\gamma = \mathfrak{F}$ is single valued for $|\arg(\xi)| < \pi$ and $|\xi| < 1$ (see [15]).

Purohit and Raina [4] defined the $D_{q,\xi}^{\mathfrak{P}} f(\xi)$ fractional q -derivative operator of a function $f(\xi)$ by

$$D_{q,\xi}^{\mathfrak{P}} f(\xi) = \frac{1}{\Gamma_q(1 - \mathfrak{P})} D_{q,\xi} \int_0^{\xi} (\xi - yq)_{-\mathfrak{P}} f(y) d_q y, \quad (12)$$

where $0 \leq \mathfrak{P} < 1$ and $f(\xi)$ is suitably constrained with $D_{q,\xi}^{-\mathfrak{P}} f(\xi) = I_{q,\xi}^{\mathfrak{P}} f(\xi)$.

The Kober fractional q -integral operator for a real valued function $f(x)$ is determined by Garg and Chanchalani [16] as

$$I_q^{\gamma,\mathfrak{P}} f(x) = \frac{x^{-\gamma-\mathfrak{P}}}{\Gamma_q(\mathfrak{P})} \int_0^x (x - yq)_{\mathfrak{P}-1} y^{\gamma} f(y) d_q y, \quad (13)$$

where γ being real or complex and \mathfrak{P} is an absolute order of integration with $\Re(\mathfrak{P}) > 0$. For $q \rightarrow 1$, operator (13) is reduced to Kober operator $I^{\gamma,\mathfrak{P}} f(x)$ as defined in [17]. For $\gamma = 0$, this operator is converted to Riemann–Liouville fractional q -integral operator with a power weight function as $I_q^{0,\mathfrak{P}} f(x) = x^{-\mathfrak{P}} I_q^{\mathfrak{P}} f(x)$.

The Kober fractional q -derivative operator for a real valued function $f(x)$ is detailed by Garg and Chanchalani [16] as

$$D_q^{\gamma,\mathfrak{P}} f(x) = \prod_{j=1}^m ([\gamma + j]_q + xq^{\gamma+j} D_q) (I_q^{\gamma+\mathfrak{P},m-\mathfrak{P}} f(x)), \quad (14)$$

where \mathfrak{P} is order of derivative with $\Re(\mathfrak{P}) > 0$ and $m = [\Re(\mathfrak{P})] + 1, m \in \mathbb{N}$. For $q \rightarrow 1$, operator (14) is reduced to Kober operator $D^{\gamma,\mathfrak{P}} f(x)$ as defined in [17].

We are now defining q -calculus operators with a view to applying these operators to the geometric function theory of complex analysis.

Definition 1. Kober fractional q -integral operator:

For the function $f(\xi)$, the Kober fractional q -integral operator is demarcated by

$$I_q^{\gamma,\mathfrak{P}} f(\xi) = \frac{x^{-\gamma-\mathfrak{P}}}{\Gamma_q(\mathfrak{P})} \int_0^{\xi} (\xi - yq)_{\mathfrak{P}-1} y^{\gamma} f(y) d_q y, \quad (15)$$

where γ is the real or complex, \mathfrak{P} is an absolute order of integration with $\Re(\mathfrak{P}) > 0$, and the q -binomial $(\xi - yq)_{\mathfrak{P}-1}$ is expressed as in (11).

For $q \rightarrow 1$, operator (15) is reduced to Kober integral operator $I^{\gamma,\mathfrak{P}} f(\xi)$ as defined in [17].

Definition 2. Kober fractional q -derivative operator:

The Kober fractional q -derivative operator for the function $f(\xi)$ is demarcated by

$$D_q^{\gamma,\mathfrak{P}} f(\xi) = \prod_{j=1}^m ([\gamma + j]_q + \xi q^{\gamma+j} D_q) (I_q^{\gamma+\mathfrak{P},m-\mathfrak{P}} f(\xi)), \quad (16)$$

where \mathfrak{P} is the order of derivative with $\Re(\mathfrak{P}) > 0$ and $m = [\Re(\mathfrak{P})] + 1, m \in \mathbb{N}$. For $q \rightarrow 1$, operator (16) is reduced to Kober derivative operator $D^{\gamma,\mathfrak{P}} f(\xi)$ as defined in [17].

Under Kober q -integral and q -derivative operators fixed by (15) and (16), we offer the following image formulae for function ξ^{μ} .

Remark 1. If $\mathfrak{P}, \gamma, \mu \in \mathbb{C}$, $\Re(\gamma + \mathfrak{P} + \mu + 1) > 0$, and $\Re(\gamma + \mu + 1) > 0$, then

$$D_q^{\gamma, \mathfrak{P}} \xi^\mu = \frac{\Gamma_q(\gamma + \mathfrak{P} + \mu + 1)}{\Gamma_q(\gamma + \mu + 1)} \xi^\mu. \quad (17)$$

Remark 2. If $\mathfrak{P}, \gamma, \mu \in \mathbb{C}$, $\Re(\gamma + \mu + 1) > 0$, and $\Re(\gamma + \mathfrak{P} + \mu + 1) > 0$, then

$$I_q^{\gamma, \mathfrak{P}} \xi^\mu = \frac{\Gamma_q(\gamma + \mu + 1)}{\Gamma_q(\gamma + \mathfrak{P} + \mu + 1)} \xi^\mu. \quad (18)$$

3. New Classes of Functions

Let \mathcal{A}_m represent the function class of the form

$$f(\xi) = \xi + \sum_{k=m+1}^{\infty} a_k \xi^k; \quad m \in \mathbb{N}, \quad (19)$$

which are analytic and univalent in open unit disk \mathbb{U} . Above, let \mathcal{A}_m^- highlight the subclass of \mathcal{A}_m imposing of analytical and univalent functions articulated in the form

$$f(\xi) = \xi - \sum_{k=m+1}^{\infty} a_k \xi^k; \quad a_k \geq 0, m \in \mathbb{N}. \quad (20)$$

For the dedication of this work, we describe a fractional q -differintegral operator $\Omega_q^{\gamma, \mathfrak{P}}$ for a function $f(\xi)$ of the form (20) by

$$\begin{aligned} \Omega_q^{\gamma, \mathfrak{P}} f(\xi) &= \frac{\Gamma_q(\gamma + 2)}{\Gamma_q(\gamma + \mathfrak{P} + 2)} \xi^{-1} D_q^{\gamma, \mathfrak{P}} f(\xi) \\ &= 1 - \sum_{k=m+1}^{\infty} \frac{\Gamma_q(\gamma + 2) \Gamma_q(\gamma + \mathfrak{P} + k + 1)}{\Gamma_q(\gamma + \mathfrak{P} + 2) \Gamma_q(\gamma + k + 1)}, \quad (21) \\ \Omega_q^{\gamma, \mathfrak{P}} f(\xi) &= 1 - \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k \xi^{k-1}, \end{aligned}$$

where

$$A(\mathfrak{P}, \gamma, k, q) = \frac{\Gamma_q(\gamma + 2) \Gamma_q(\gamma + \mathfrak{P} + k + 1)}{\Gamma_q(\gamma + \mathfrak{P} + 2) \Gamma_q(\gamma + k + 1)}, \quad (22)$$

and $\Re(\gamma + 2) > 0$, $0 < q < 1$, $\xi \in \mathbb{U}$, $m \in \mathbb{N}$, $\Re(\mathfrak{P}) > 0$, and $D_q^{\gamma, \mathfrak{P}} f(\xi)$ represent a fractional q -derivative of $f(\xi)$ of order \mathfrak{P} . We announce here the alike classes of functions connecting operator (21):

$$\mathcal{S}_q^{\gamma, \mathfrak{P}}(\mathfrak{T}) = \left\{ f \in \mathcal{A}_m^-, \left| \frac{\Omega_q^{\gamma, \mathfrak{P}} f(\xi) - 1}{\Omega_q^{\gamma, \mathfrak{P}} f(\xi) - 2\mathfrak{T} + 1} \right| < \mathfrak{T} \right\}, \quad (23)$$

where

$$\Re(\mathfrak{P}) > 0, \Re(\gamma + 2) > 0, 0 \leq \mathfrak{T} < 1, 0 \leq \mathfrak{F} < 1, 0 < q < 1, \xi \in \mathbb{U}.$$

And

$$\mathcal{T}_q^{\gamma, \mathfrak{P}}(\tau) = \left\{ f \in \mathcal{A}_m^-, \Re \left((1 - \tau) \Omega_q^{\gamma, \mathfrak{P}} f(\xi) + \tau \left(\frac{1 - q^{1-\mathfrak{P}}}{1 - q} \right) \Omega_q^{\gamma, \mathfrak{P}+1} f(\xi) \right) > \mathfrak{F} \right\}. \quad (24)$$

The subsequent coefficient bounds for functions of the form (20) that belong to the classes $\mathcal{S}_q^{\gamma, \mathfrak{P}}(\mathfrak{T})$ and $\mathcal{T}_q^{\gamma, \mathfrak{P}}(\tau)$ are now obtained (interpreted above).

Theorem 1. A function f defined by (20) is connected to the class $\mathcal{S}_q^{\gamma, \mathfrak{P}}(\mathfrak{T})$ if and only if

$$\sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k (1 + \mathfrak{F}) \leq 2\mathfrak{F}(1 - \mathfrak{T}), \quad (25)$$

where

$$A(\mathfrak{P}, \gamma, k, q) = \frac{\Gamma_q(\gamma + 2) \Gamma_q(\gamma + \mathfrak{P} + k + 1)}{\Gamma_q(\gamma + \mathfrak{P} + 2) \Gamma_q(\gamma + k + 1)}. \quad (26)$$

The result is sharp.

Proof. Let us consider that inequality (25) holds, and for $|\xi| = 1$, we have

$$\begin{aligned} & \left| \Omega_q^{\gamma, \mathfrak{P}} f(\xi) - 1 \right| - \mathfrak{F} \left| \Omega_q^{\gamma, \mathfrak{P}} f(\xi) - 2\mathfrak{T} + 1 \right| \\ &= \left| - \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k \xi^{k-1} \right| - \mathfrak{F} \left| 2(1 - \mathfrak{T}) - \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k \xi^{k-1} \right| \\ &\leq \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k (1 + \mathfrak{F}) - 2\mathfrak{F}(1 - \mathfrak{T}) \leq 0, \end{aligned} \quad (27)$$

and by our assumption, this indicates that $f(\xi) \in \mathcal{S}_q^{\gamma, \mathfrak{P}}(\mathfrak{T})$.

For the proof of converse part, suppose that $f(\xi) \in \mathcal{S}_q^{\gamma, \mathfrak{P}}(\mathfrak{T})$, and then it follows that

$$\left| \frac{\Omega_q^{\gamma, \mathfrak{P}} f(\xi) - 1}{\Omega_q^{\gamma, \mathfrak{P}} f(\xi) - 2\mathfrak{T} + 1} \right| < \mathfrak{F}, \quad (28)$$

which implies that

$$\begin{aligned} & \left| \frac{\Omega_q^{\gamma, \mathfrak{P}} f(\xi) - 1}{\Omega_q^{\gamma, \mathfrak{P}} f(\xi) - 2\mathfrak{T} + 1} \right| \\ &= \left| - \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k \xi^{k-1} \right| \left| 2(1 - \mathfrak{T}) - \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k \xi^{k-1} \right|^{-1} < \mathfrak{F}. \end{aligned} \quad (29)$$

Since $|\mathfrak{R}(\xi)| \leq |\xi|$ for any ξ , therefore on choosing values of ξ on the real axis so that $\Omega_q^{\gamma, \mathfrak{P}} f(\xi)$ is real and allowing $\xi \rightarrow 1$ all through real values, we obtain from above inequality

$$\sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k \leq 2\mathfrak{F}(1 - \mathfrak{T}) - \mathfrak{F} \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k, \quad (30)$$

which implies that

$$\sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k (1 + \mathfrak{F}) \leq 2\mathfrak{F}(1 - \mathfrak{T}), \quad (31)$$

which is desired result. Here, we notice that assumption (25) of Theorem 1 is sharp and the external function is assumed by

$$f(\xi) = \xi - \frac{2\mathfrak{F}(1 - \mathfrak{T})}{(1 + \mathfrak{F})A(\mathfrak{P}, \gamma, m+1, q)} \xi^{m+1}; \quad m \in \mathbb{N}, \quad (32)$$

where $A(\mathfrak{P}, \gamma, k, q)$ is defined in (26). \square

Theorem 2. A function f defined by (20) is connected to the class $\mathcal{T}_q^{\gamma, \mathfrak{P}}(\tau)$ if and only if

$$\sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) \mathcal{A}_{k,q}(\mathfrak{P}, \gamma, \tau) a_k \leq (1 - \mathfrak{F} - \tau) + \tau(1 - q^{1-\mathfrak{P}}), \quad (33)$$

where

$$\mathcal{A}_{k,q}(\mathfrak{P}, \gamma, \tau) = \left[(1 - \tau)(1 - q) + \tau(1 - q^{1-\mathfrak{P}}) \left(\frac{1 - q^{\gamma+\mathfrak{P}+k+1}}{1 - q^{\gamma+\mathfrak{P}+2}} \right) \right]. \quad (34)$$

The accomplishment is sharp.

Proof. To prove above theorem, we address the elementary assertion that

$$\mathfrak{R}(g(\xi)) \geq \mathfrak{F} \Leftrightarrow \{1 - \mathfrak{F} + g(\xi)\} \geq \{1 + \mathfrak{F} - g(\xi)\}. \quad (35)$$

Now,

$$\begin{aligned} g(\xi) &= (1 - \tau) \Omega_q^{\gamma, \mathfrak{P}} f(\xi) + \tau \left(\frac{1 - q^{1-\mathfrak{P}}}{1 - q} \right) \Omega_q^{\gamma, \mathfrak{P}+1} f(\xi) \\ &= (1 - \tau) \left[1 - \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k \xi^{k-1} \right] + \tau \left(\frac{1 - q^{1-\mathfrak{P}}}{1 - q} \right) \left[1 - \sum_{k=m+1}^{\infty} A(\mathfrak{P} + 1, \gamma, k, q) a_k \xi^{k-1} \right] \\ &= (1 - \tau) + \tau \left(\frac{1 - q^{1-\mathfrak{P}}}{1 - q} \right) - \frac{1}{(1 - q)} \sum_{k=m+1}^{\infty} A_{k,q}(\mathfrak{P}, \gamma, \tau) a_k \xi^{k-1}, \end{aligned} \quad (36)$$

where

$$\mathcal{A}_{k,q}(\mathfrak{P}, \gamma, \tau) = \left[(1-\tau)(1-q) + \tau(1-q^{1-\mathfrak{P}}) \left(\frac{1-q^{\gamma+\mathfrak{P}+k+1}}{1-q^{\gamma+\mathfrak{P}+2}} \right) \right]. \quad (37)$$

In (35), it then suffices to show that

$$\begin{aligned} & \left| 2 - \mathfrak{F} - \tau + \tau \left(\frac{1-q^{1-\mathfrak{P}}}{1-q} \right) - \frac{1}{(1-q)} \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) \mathcal{A}_{k,q}(\mathfrak{P}, \gamma, \tau) a_k \xi^{k-1} \right| \\ & - \left| \mathfrak{F} + \tau - \tau \left(\frac{1-q^{1-\mathfrak{P}}}{1-q} \right) + \frac{1}{(1-q)} \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) \mathcal{A}_{k,q}(\mathfrak{P}, \gamma, \tau) a_k \xi^{k-1} \right| \geq 0. \end{aligned} \quad (38)$$

Now,

$$\begin{aligned} & \left| 2 - \mathfrak{F} - \tau + \tau \left(\frac{1-q^{1-\mathfrak{P}}}{1-q} \right) - \frac{1}{(1-q)} \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) \mathcal{A}_{k,q}(\mathfrak{P}, \gamma, \tau) a_k \xi^{k-1} \right| \\ & - \left| \mathfrak{F} + \tau - \tau \left(\frac{1-q^{1-\mathfrak{P}}}{1-q} \right) + \frac{1}{(1-q)} \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) \mathcal{A}_{k,q}(\mathfrak{P}, \gamma, \tau) a_k \xi^{k-1} \right| \\ & \geq \frac{2}{(1-q)} \left[(1-\mathfrak{F}-\tau)(1-q) + \tau(1-q^{1-\mathfrak{P}}) - \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) \mathcal{A}_{k,q}(\mathfrak{P}, \gamma, \tau) a_k |\xi|^{k-1} \right] \geq 0. \end{aligned} \quad (39)$$

This accomplishes the proof of theorem.

We accommodate that answer (33) is sharp. The external function is assumed by

$$f(\xi) = \xi - \frac{\{(1-\mathfrak{F}-\tau) + \tau(1-q^{1-\mathfrak{P}})\}}{A(\mathfrak{P}, \gamma, m+1, q) \mathcal{A}_{m+1,q}(\mathfrak{P}, \gamma, \tau)} \xi^{m+1}; \quad m \in \mathbb{N}, \quad (40)$$

where $A_{k,q}(\mathfrak{P}, \gamma, \tau)$ is given by (34). \square

4. Distortion Theorems

Theorem 3. Suppose that the function f is defined by (20) in the class $\mathcal{S}_q^{\gamma, \mathfrak{P}}(\mathfrak{T})$, then

$$|\xi| - |\xi|^{m+1} \left(\frac{2\mathfrak{F}(1-\mathfrak{T})}{1+\mathfrak{F}} \right) B(\mathfrak{P}, \gamma, m, q) \leq |f(\xi)| \leq |\xi| + |\xi|^{m+1} \left(\frac{2\mathfrak{F}(1-\mathfrak{T})}{1+\mathfrak{F}} \right) B(\mathfrak{P}, \gamma, m, q), \quad (41)$$

where

$$B(\mathfrak{P}, \gamma, m, q) = \frac{1}{A(\mathfrak{P}, \gamma, m+1, q)} = \frac{\Gamma_q(\gamma + \mathfrak{P} + 2) \Gamma_q(\gamma + m + 2)}{\Gamma_q(\gamma + 2) \Gamma_q(\gamma + \mathfrak{P} + m + 2)}. \quad (42)$$

Furthermore,

$$|\xi| - |\xi|^{m+1} \left(\frac{2\mathfrak{F}(1-\mathfrak{T})}{1+\mathfrak{F}} \right) \leq |\xi \Omega_q^{\gamma, \mathfrak{P}} f(\xi)| \leq |\xi| + |\xi|^{m+1} \left(\frac{2\mathfrak{F}(1-\mathfrak{T})}{1+\mathfrak{F}} \right), \quad (43)$$

where $\gamma > -2$, $\Re(\mathfrak{P}) > 0$, $0 < q < 1$.

Proof. Since $f(\xi) \in \mathcal{S}_q^{\gamma, \mathfrak{P}}(\mathfrak{T})$, then in interpretation of Theorem 1, we first show that the function

$$A(\mathfrak{P}, \gamma, k, q) = \frac{\Gamma_q(\gamma + 2)\Gamma_q(\gamma + \mathfrak{P} + k + 1)}{\Gamma_q(\gamma + \mathfrak{P} + 2)\Gamma_q(\gamma + k + 1)} = \phi(k), \text{ (let)} \quad (44)$$

is an increasing function of k for $\gamma > -2$ and $\Re(\mathfrak{P}) > 0$. It follows that

$$\frac{\phi(k+1)}{\phi(k)} = \frac{\Gamma_q(\gamma + \mathfrak{P} + k + 2)\Gamma_q(\gamma + k + 1)}{\Gamma_q(\gamma + \mathfrak{P} + k + 1)\Gamma_q(\gamma + k + 2)} = \frac{1 - q^{\gamma + \mathfrak{P} + k + 1}}{1 - q^{\gamma + k + 1}}, \quad 0 < q < 1. \quad (45)$$

Taking $k = m + 1$, then

$$\frac{\phi(m+2)}{\phi(m+1)} = \frac{1 - q^{\gamma + \mathfrak{P} + m + 2}}{1 - q^{\gamma + m + 2}}, \quad 0 < q < 1. \quad (46)$$

The function $\phi(k)$ is an increasing function of k if $(\phi(m+2)/\phi(m+1)) \geq 1$, and this gives

$$\frac{1 - q^{\gamma + \mathfrak{P} + m + 2}}{1 - q^{\gamma + m + 2}} \geq 1; \quad 0 < q < 1, \quad (47)$$

which implies

$$q^{\mathfrak{P}} \leq 1, \quad 0 < q < 1. \quad (48)$$

This inequality abides for $\Re(\mathfrak{P}) > 0$.

Thus, $\phi(k)$, ($k \geq m + 1$, $m \in \mathbb{N}$) is an increasing function of k for $\Re(\gamma + 2) > 0$, $\Re(\mathfrak{P}) > 0$ and $0 < q < 1$.

Now, (25) gives the alike inequality:

$$A(\mathfrak{P}, \gamma, m + 1, q) \sum_{k=m+1}^{\infty} a_k (1 + \mathfrak{T}) \leq \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k (1 + \mathfrak{T}) \leq 2\mathfrak{T}(1 - \mathfrak{T}), \quad (49)$$

which implies that

$$\sum_{k=m+1}^{\infty} a_k \leq \frac{2\mathfrak{T}(1 - \mathfrak{T})}{1 + \mathfrak{T}} B(\mathfrak{P}, \gamma, m, q), \quad (50)$$

where $B(\mathfrak{P}, \gamma, m, q)$ is defined in (42), and this last inequality is in the conjunction with the alike inequality (easily obtained from (20)):

$$|\xi| - |\xi|^{m+1} \sum_{k=m+1}^{\infty} a_k \leq |f(\xi)| \leq |\xi| + |\xi|^{m+1} \sum_{k=m+1}^{\infty} a_k, \quad (51)$$

and using (50), we have

$$|\xi| - |\xi|^{m+1} \left(\frac{2\mathfrak{T}(1 - \mathfrak{T})}{1 + \mathfrak{T}} \right) B(\mathfrak{P}, \gamma, m, q) \leq |f(\xi)| \leq |\xi| + |\xi|^{m+1} \left(\frac{2\mathfrak{T}(1 - \mathfrak{T})}{1 + \mathfrak{T}} \right) B(\mathfrak{P}, \gamma, m, q), \quad (52)$$

which is result (41) of Theorem 3.

Now, on using (21), we observe that for functions of form (20),

$$|\xi \Omega_q^{\gamma, \mathfrak{P}} f(\xi)| \geq |\xi| - \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k |\xi|^k \geq |\xi| - |\xi|^{m+1} \sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q) a_k, \quad (53)$$

which on using Theorem 1 gives

$$|\xi \Omega_q^{\gamma, \mathfrak{P}} f(\xi)| \geq |\xi| - 2\mathfrak{T} \left(\frac{1 - \mathfrak{T}}{1 + \mathfrak{T}} \right) |\xi|^{m+1}, \quad (54)$$

and similarly,

$$|\xi \Omega_q^{\gamma, \mathfrak{P}} f(\xi)| \leq |\xi| + 2\mathfrak{T} \left(\frac{1 - \mathfrak{T}}{1 + \mathfrak{T}} \right) |\xi|^{m+1}, \quad (55)$$

which implies that

$$|\xi| - 2\mathfrak{F}\left(\frac{1-\mathfrak{T}}{1+\mathfrak{F}}\right)|\xi|^{m+1} \leq |\xi\Omega_q^{\gamma,\mathfrak{P}} f(\xi)| \leq |\xi| + 2\mathfrak{P}\left(\frac{1-\mathfrak{T}}{1+\mathfrak{F}}\right)|\xi|^{m+1}. \quad (56)$$

Corollary 1. Let the function detailed in (20) be in the class $\mathcal{S}_q^{\gamma,\mathfrak{P}}(\mathfrak{T})$, then

$$\begin{aligned} \frac{\Gamma_q(\gamma + \mathfrak{P} + 2)}{\Gamma_q(\gamma + 2)} \left\{ |\xi| - 2\mathfrak{P}\left(\frac{1-\mathfrak{T}}{1+\mathfrak{F}}\right)|\xi|^{m+1} \right\} &\leq |D_q^{\gamma,\mathfrak{P}} f(\xi)| \\ &\leq \frac{\Gamma_q(\gamma + \mathfrak{P} + 2)}{\Gamma_q(\gamma + 2)} \left\{ |\xi| + 2\mathfrak{F}\left(\frac{1-\mathfrak{T}}{1+\mathfrak{F}}\right)|\xi|^{m+1} \right\}, \end{aligned} \quad (57)$$

where $\Re(\gamma + 2) > 0$, $\Re(\mathfrak{P}) > 0$, and $\xi \in \mathbb{U}$.

Corollary 2. Let the function detailed in (20) be in the class $\mathcal{S}_q^{\gamma,\mathfrak{P}}(\mathfrak{T})$, then

$$\begin{aligned} \frac{\Gamma_q(\gamma + 2)}{\Gamma_q(\gamma + \mathfrak{P} + 2)} \left\{ |\xi| - 2\mathfrak{F}\left(\frac{1-\mathfrak{T}}{1+\mathfrak{F}}\right) \left(\frac{\Gamma_q(\gamma + \mathfrak{P} + 2)}{\Gamma_q(\gamma + \mathfrak{P} + m + 2)} \right)^2 |\xi|^{m+1} \right\} &\leq |I_q^{\gamma,\mathfrak{P}} f(\xi)| \\ &\leq \frac{\Gamma_q(\gamma + 2)}{\Gamma_q(\gamma + \mathfrak{P} + 2)} \left\{ |\xi| + 2\mathfrak{F}\left(\frac{1-\mathfrak{T}}{1+\mathfrak{F}}\right) \left(\frac{\Gamma_q(\gamma + \mathfrak{P} + 2)}{\Gamma_q(\gamma + \mathfrak{P} + m + 2)} \right)^2 |\xi|^{m+1} \right\}, \end{aligned} \quad (58)$$

where $\Re(\gamma + 2) > 0$, $\Re(\mathfrak{P}) > 0$, and $\xi \in \mathbb{U}$.

Theorem 4. Suppose that the function $f(\xi)$ detailed in (20) be in the class $\mathcal{T}_q^{\gamma,\mathfrak{P}}(\tau)$, then for $\Re(\mathfrak{P}) > 0$, $\Re(\gamma + 2) > 0$, $\xi \in \mathbb{U}$, $0 < q < 1$,

$$|\xi| - B(\gamma, \mathfrak{P}, m, q)C|\xi|^{m+1} \leq |f(\xi)| \leq |\xi| + B(\gamma, \mathfrak{P}, m, q)C|\xi|^{m+1}, \quad (59)$$

also

$$|\xi| - C|\xi|^{m+1} \leq |\xi\Omega_q^{\gamma,\mathfrak{P}} f(\xi)| \leq |\xi| + C|\xi|^{m+1}, \quad (60)$$

where

$$C = \frac{(1 - \mathfrak{F} - \tau)(1 - q) + \tau(1 - q^{1-\mathfrak{P}})}{\mathcal{A}_{m+1,q}(\mathfrak{P}, \gamma, \tau)}, \quad (61)$$

and $\mathcal{A}_{k,q}(\mathfrak{P}, \gamma, \tau)$ and $B(\gamma, \mathfrak{P}, m, q)$ are detailed in (34) and (42), respectively.

Proof. Since $\mathcal{T}_q^{\gamma,\mathfrak{P}}(\tau)$, then under the hypothesis of Theorem 2, we have

$$\sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q)a_k \leq \frac{(1 - \mathfrak{F} - \tau)(1 - q) + \tau(1 - q^{1-\mathfrak{P}})}{\mathcal{A}_{m+1,q}(\mathfrak{P}, \gamma, \tau)}, \quad (62)$$

which implies that

$$\sum_{k=m+1}^{\infty} A(\mathfrak{P}, \gamma, k, q)a_k \leq C, \quad (63)$$

where $\mathcal{A}_{k,q}(\mathfrak{P}, \gamma, \tau)$ and C are given by (34) and (61), respectively, and this last inequality, when combined with the following inequality (which is conveniently obtained from (20)),

$$|\xi| - |\xi|^{m+1} \sum_{k=m+1}^{\infty} a_k \leq |f(\xi)| \leq |\xi| + |\xi|^{m+1} \sum_{k=m+1}^{\infty} a_k, \quad (64)$$

and using (63), we have

$$|\xi| - B(\gamma, \mathfrak{P}, m, q)C|\xi|^{m+1} \leq |f(\xi)| \leq |\xi| + B(\gamma, \mathfrak{P}, m, q)C|\xi|^{m+1}, \quad (65)$$

which is result (59) of Theorem 4.

Now, from (21), we obtain

$$|\xi| - |\xi|^{m+1} \sum_{k=m+1}^{\infty} A(\lambda, \gamma, k, q)a_k \leq |\xi\Omega_q^{\gamma,\lambda} f(\xi)|, \quad (66)$$

on using (63), this implies that

$$|\xi| - C|\xi|^{m+1} \leq |\xi\Omega_q^{\gamma,\lambda} f(\xi)|, \quad (67)$$

similarly, we have

$$\left| \xi \Omega_q^{\gamma, \lambda} f(\xi) \right| \leq |\xi| + C|\xi|^{m+1}, \quad (68)$$

and on combining above two results, we have

$$|\xi| - C|\xi|^{m+1} \leq \left| \xi \Omega_q^{\gamma, \lambda} f(\xi) \right| \leq |\xi| + C|\xi|^{m+1}. \quad (69)$$

□

Corollary 3. Let the function detailed in (20) be in the class $\mathcal{T}_q^{\gamma, \lambda}(\tau)$, then for all $\xi \in \mathbb{U}$, $\Re(\gamma + \lambda + 2) > 0$, $\Re(\gamma + 2) > 0$,

$$\frac{\Gamma_q(\gamma + \lambda + 2)}{\Gamma_q(\gamma + 2)} \{|\xi| - C|\xi|^{m+1}\} \leq \left| D_q^{\gamma, \lambda} f(\xi) \right| \leq \frac{\Gamma_q(\gamma + \lambda + 2)}{\Gamma_q(\gamma + 2)} \{|\xi| + C|\xi|^{m+1}\}. \quad (70)$$

The fractional q -calculus operators presented in Section 2 may be used to explore numerous different multivalent (or meromorphic) analytic function subclass and geometric characteristics which includes coefficient estimates, distortion bounds, radii of starlikeness, convexity, and so forth. The concept of fractional q -calculus can also be used to again with considerations.

Data Availability

No data were used to support this study.

Conflicts of Interest

There are no conflicts of interest regarding the publication of this article.

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