Mathematical Problems in Engineering

Applied Mathematics for Engineering Problems in Biomechanics and Robotics 2020

Lead Guest Editor: Carlos Llopis-Albert
Guest Editors: Francisco Rubio, Shouzhen Zeng, and Huchang Liao

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Editorial

Applied Mathematics for Engineering Problems in Biomechanics and Robotics 2020

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There is a disruptive impact of smart technologies in the 21st century, which is transforming the traditional industry and the healthcare sector into the Industry 5.0 and the Healthcare 5.0. In this sense, biomechanics and robotics are two closely interrelated disciplines that are boosting such transformation. On the one hand, biomechanics is the study of mechanical structures that exist in living beings and the subsequent application of mechanical laws to living structures, especially to the musculoskeletal system of the human body [1, 2]. On the other hand, robotics is the science or branch of technology that covers the analysis, design, construction, and application of robots to carry out human tasks, mainly laborious, repetitive, or dangerous [3–5].

In addition, both disciplines have a marked multidisciplinary character, including physics, mathematics, computing, and biology.

The special issue comprises works related to prostheses, exoskeletons, biomaterials, medical imaging, sensors, assistance, and remote monitoring of healthcare recipients, social robots, digital technologies, sustainable technologies, artificial intelligence, machine learning, nanotechnology, smart health technologies for the diagnosis and treatment of patients, injury prevention, and rehabilitation, sports, assembly line worker safety, product and service quality, ergonomics, production improvement, material handling systems, logistics, management, indoor transportation, vehicle dynamics, kinematics and dynamics of mechanisms, and manufacturing industries.

Therefore, the special issue considers mathematical works related to modeling, simulation, computational mechanics (e.g., multibody dynamics analysis and finite element analysis), optimization problems, statistics, etc.

Finally, the special issue presents 17 high-quality original research articles in the above-mentioned topics, which have been published after a peer review process. The accepted papers encompass a broad spectrum of topic,s and a brief description of them is given as follows.

In the paper entitled "An Intelligent and Robust Framework towards Anomaly Detection, Medical Diagnosis, and Shortest Path Problems Based on Interval-Valued T-Spherical Fuzzy Information," Jin et al. presented an application of the interval-valued T-spherical fuzzy set for anomaly detection, medical diagnosis, and shortest path problem.

Zhang et al. set out and discussed the ability to obtain accurate torque information and detect collision information for collaborative robots in the paper "Disturbance Elimination for the Modular Joint Torque Sensor of a Collaborative Robot."

The paper entitled "Formation Control Algorithm of Agents Based on Earth Mover's Distance" by Liu et al. introduced a fast formation control method that can be applied to a crow of unmanned aerial vehicles.

Ruan in the paper entitled "Hesitant Fuzzy Decision-Making Method Based on Correlation Coefficient under Confidence Levels with Application to Multisensor Electronic Reconnaissance" focused on the hesitant fuzzy

decision-making method based on correlation coefficient under confidence levels, which was applied to the multi-attribute decision-making problem in multisensor electronic reconnaissance.

Huang et al. developed a method for ship target detection while ensuring real-time performance, accuracy, and robustness in the paper entitled "Ship Target Detection Based on Improved YOLO Network."

Dealing with interconnected networks in the paper "Vertex-Disjoint Paths in a 3-Ary n-Cube with Faulty Vertices," Ma et al. dealt with the disjoint path cover problem, which can accelerate the transmission of large amounts of data by splitting data into disjoint communication paths of multiple vertices, thus avoiding communication congestion and enhancing the robustness of vertex failure and load balancing capability.

In the paper entitled "Prediction of Temperature Field and Thermal Damage of Multilayer Skin Tissues Subjected to Time-Varying Laser Heating and Fluid Cooling by a Semianalytical Method," Li et al. developed a semianalytical method, based on the separation of variables, to solve the problem of heat transfer in multilayer skin tissues.

Li et al. in the paper "Dynamic Path Planning for Bevel-Tip Flexible Needle Insertion into Soft Tissue Based on a Real-Time Finite Element Model" introduced a novel dynamic path planning methodology for needle steering into the soft tissue based on a real-time finite element model, which provided dynamic deformation information for path planning.

In the paper entitled "Study of Dynamical Behavior and Stability of Iterative Methods for Nonlinear Equation with Applications in Engineering," Rafiq et al. presented a numerical approach with applications in the fields of robotics in biomechanics.

Vu and Nguyen in the paper "Balancing Control of Two-Wheel Bicycle Problems" built, modeled, and designed a robust controller for self-balancing two-wheel bicycle, while introducing order reduction algorithms for stable and unstable systems.

In the paper entitled "Effect of Surface Roughness during Peristaltic Movement in a Nonuniform Channel," Shukla et al. addressed the problem of transport of physiological fluids that may be helpful when the inner surface of any biological organ is not fully smooth, for example, endothelium roughness in the coronary artery.

Quynh in the paper entitled "The Fuzzy PI Controller for PMSM's Speed to Track the Standard Model" developed a fuzzy PI controller to control the speed of a permanent magnet synchronous motor, while its correctness and stability were successfully tested and compared with other kinds of controllers.

In the paper entitled "Sample Entropy Analysis of Horses with and without Lameness at Trot before and after Sedation," Zhao et al. illustrated that the sample entropy can provide important information about the characteristics of locomotion pattern of the equine gait.

Zhang et al. dealt with the dynamic design and optimization of robot parallel manipulators in the paper "Elastostatic Stiffness Analysis for the US/UPS Parallel Manipulators," in which the modeling method was based on screw theory, Castigliano's theorem, and strain energy.

Dealing with mobile robots in the paper "Flocking Control of Mobile Robots with Obstacle Avoidance Based on Simulated Annealing Algorithm," Cheng and Wang proposed an algorithm and a motion control law to search the quasi-optimal position of robots in order to reduce the potential functions.

In the paper entitled "On Dynamics of Iterative Techniques for Nonlinear Equation with Applications in Engineering," Shams et al. constructed and showed the efficiency of an optimal family of iterative methods for finding the single root and then extended this family for determining all the distinct as well as multiple roots of single-variable nonlinear equations simultaneously.

Mao et al. in the paper "A Novel Collision-Free Navigation Approach for Multiple Nonholonomic Robots Based on ORCA and Linear MPC" presented a method for collision avoidance among multiple nonholonomic robots in planar environments, which combines the Optimal Reciprocal Collision Avoidance (ORCA) algorithm and Model Predictive Control (MPC) strategy.

Conflicts of Interest

The Guest Editors declare that they have no conflicts of interest regarding the publication of this special issue.

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Research Article

An Intelligent and Robust Framework towards Anomaly Detection, Medical Diagnosis, and Shortest Path Problems Based on Interval-Valued T-Spherical Fuzzy Information

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The recent emerging advancements in the domain of the fuzzy sets are the framework of the T-spherical fuzzy set (TSFS) and interval valued T-spherical fuzzy set (IVTSFS). Keeping in view the promising significance of the latest research trend in the fuzzy sets and the enabling impact of IVTSFS, we proposed a novel framework for decision assembly using interval valued TSFS based upon encompassing the four impressive dimensions of human judgement including favor, abstinence, disfavor, and refusal degree. Another remarkable contribution is the optimization of information modeling and prevention of information loss by redefining the concept of each membership in interval. Moreover, the proposed research made a worthy contribution work by demonstrating the effective utilization of the interval valued TSFS based framework in anomaly detection, medical diagnosis, and shortest path problem. The proposed work demonstrates the effective remedial measure for the anomaly detection problem based on several parameters using the aggregation operators of IVTSFS. Moreover, the interval valued T-spherical fuzzy relations and their composition are illustrated to investigate the medical diagnosis problem. Furthermore, the notion of interval valued T-spherical fuzzy graph is also presented and fundamental notions of graph theory are also demonstrated with the help of real world instances. In the context of interval valued T-spherical fuzzy graphs (IVTSFGs), a modified Dijkstra Algorithm (DA) is developed and applied to the shortest path problem. The in-depth quantitative assessment and comparative analysis revealed that the proposed notion outpaces contemporary progressive approaches.

1. Introduction

Zadeh [1] developed a novel way of describing an entity's membership to a set using the idea of a fuzzy set (FS) which was further enhanced by Atanassov [2] who proposed the perception of an intuitionistic fuzzy set (IFS). In an IFS, the relation of an object to a set is described by the membership as well as nonmembership grades (NMGs) denoted by ş and d on the unit interval in a way that $0 \leq \text{sum}(ş, d) \leq 1$. The condition $0 \leq \text{sum}(ş, d) \leq 1$ on IFS makes it less effective in some cases because it does not allow us to choose the values of ş and d independently. This issue was addressed by the framework of the Pythagorean fuzzy set (PyFS) developed with a relatively stronger condition, i.e., $0 \leq \text{sum}(ş^2, d^2) \leq 1$. The concept of PyFS also becomes limited in some cases and a new generalized ortho pair FS known as q-rung ortho pair FS (Q-ROPFS) introduced by Yager [3] with a condition that $0 \leq \text{sum}(ş^q, d^q) \leq 1$ such that $q \geq 1$. The comparative evaluation of the spaces of IFS, PyFS, and Q-ROPFS is described in Figure 1. In Figure 1, the turquoise color stripe models the aggregated perimeter of IFSs, the light blue semicircle represents PyFS, and the red curve illustrates Q-ROPFS for

Figure 1: Comparison of spaces of intuitionistic fuzzy set, Pythagorean fuzzy set, and q-rung ortho pair fuzzy set.

$q = 7$. IFS, PyFS, and Q-ROPFS are only effective in situations with discrete resolutions, i.e., yes or no.

Cuong [4, 5] proposed a new theory of picture fuzzy set (PFS) which described four types of grades ş, ı, d, and R denoting membership, nonmembership, abstinence, and refusal degree such that $0 \leq \text{sum}(ş, ı, d) \leq 1$ and $R = 1 - \text{sum}(ş, ı, d)$. Cuong's idea of PFS generalizes the theory of IFS but constrained by the condition $0 \leq \text{sum}(ş, ı, d) \leq 1$ due to which grades could not be assigned independently. Therefore, a novel idea of TSFS and SFS generalizing IFS, PyFS, Q-ROPFS, and PFS with a condition on membership grade (MG) $0 \leq \text{sum}(ş^n, ı^n, d^n) \leq 1$ and $R = \sqrt[n]{\text{sum}(ş^n, ı^n, d^n)}$ such that $n \in \mathbb{Z}^+$ and $n \geq 1$ was proposed by Mahmood [6]. The comparison of spaces of PFSs and SFSs is described in Figure 2 where the pink line represents the boundary space of PFSs, and the dark grey curves represent the boundary space of SFSs. One may also refer to [6, 7] for further knowledge of PFSs, SFSs, and TSFSs.

In real-life problems, the MG of an object could not be assigned due to some type of uncertainty, and hence, [8] proposed IIVFS with the definition of entity's membership with closed subinterval $(ş^L, ş^U)$ of the unit interval. An IVFS easily reduces to FS if we take $ş^L = ş^U$. Following this phenomenon, the same type of structures has been proposed for IFSs, PyFSs, and PFSs. The concept of interval valued IFS (IVIFS) is proposed in [9], interval valued PyFS (IVPyFS) is developed in [10], and interval valued PFS (IVPFS) is proposed in [5].

The concepts of IVIFS, IVPyFS, and IVPFS are reduced to IFS, PyFS, and PFS. Keeping the importance of these concepts, Ullah [11] proposed the framework of IVTSFS. Ullah [11] also described the importance of describing membership degrees. All these concepts have been applied to various practical problems, such as [12, 13] which described MADM problems using some normal interactive aggregation operators and VIKOR method based on IFSs, [14] which is based on multiple parameters based two-person zero sum game with IF information, and [15–22] which are dealing with MADM using different approaches.

Figure 2: Comparison of spaces of PFS and SFS.

The concept of fuzzy graph (FG) is developed by [23] as a generalized crisp graph and comprehensively examined by Rosenfeld [24]. The idea of FG is indeed very helpful in various problems including classification, clustering, anomaly/malware detection, shortest path problems, networking problems, slicing, and in the description of group structures, etc. Due to the uncertainties lying in these problems, FG was further generalized to the concept of an intuitionistic fuzzy graph (IFG) where nodes and edges are assumed to be intuitionistic fuzzy numbers (IFNs) in [25]. The graphical aspects of PyFS are discussed in [25], the notion of PFG is developed in [26], and its operations are studied. Reference [27] developed the theory of IVFG and [28] is based on IVIFG. The TSFG's notion is also proposed in [29] which provided a more flexible ground for dealing with imprecise information. Some quality work on these graphs can be found in [30–34].

Malware/Anomaly detection is a critical cyber world problem, and it is crucial with malware's prevalence because it functions as threat intelligence and detection system against anomaly based cyber-attacks. In anomaly detection, we establish a process using aggregation operators regarding the classification of a given sample of Windows based Portable Executable (PE) file. The aggregation operators are applied to the decision matrix containing the attributes of the given sample to calculate the score values. The score value of each sample file is used by the problem solver to determine the nature of the portable executable file, i.e., either anomaly (malicious) or benign (nonmalicious). Some novel research work in the subject domain has been so far addressed in [35–40]. Medical diagnosis is also another important practical problem that has been widely discussed in the context of FSs and its algebraic extensions. In medical diagnosis, we establish some relations between patients, symptoms and symptoms, diagnosis. Then by the composition of the previous two relations, we determine the diagnosis for patients. This type of phenomenon has been so far discussed in [6, 41–43]. Furthermore, the SP problem is also one of the interesting problems that have much impotence in computer sciences and fuzzy algebra. In this phenomenon, we find the SP in a network of nodes and edges from the source node to the destination node. Several algorithms have been developed for finding SP problems. Some notable work in this regard could be found in [44–48].

The problem of bounding the membership's aggregate, abstinence, and NMGs between one and zero is also

experienced by Cuong's PFS. The problem also revokes the privilege from the problem solvers to opt three aforementioned characteristic functions. Considering the problem, Mahmood [49] introduced a novel framework of the SFS. There is a dire need to develop innovative models keeping in view several characteristic functions with no shortcomings [53, 254]. The significance of representing the MGs in terms of intervals strongly motivated us for our proposed research work. The research work proposed the IVTSFS with WA and WG aggregation operators and demonstration of its effective application for MADM. The notable benefactions of this manuscript are as follows:

(1) To propose the novel concept of the IVTSFS that encompasses an event covering MGs in terms of intervals without any constraint

(2) To propose novel aggregation tools and its effective application in those specialized domains which remain a problem for the application of IVFS, IVIFS, and IVPFS

(3) To explore the problem of MADM and utilize the aggregation operators of IVTSFSs for decision making in real world scenarios

The extensive model evaluation followed by indepth comparative assessment has demonstrated that the proposed aggregation operators with appropriate constraints outperform the preexisting aggregation operators. In this manuscript, we developed a new concept of interval valued T-spherical fuzzy graphs (IVTSFGs). The concepts of IVTSFSs and IVTSFGs are applied to the problem of anomaly detection, medical diagnosis, and SP problem. Also, it is proved that none of the existing tools could handle the data provided in IVTSFS and on the other hand, it is also proved that the structure of IVTSFSs and IVTSFG process the problem handsomely in lieu of FS or FG, IFS or IFG, IVIFS or IVIFG, IVPFS, SFS, and TSFS.

This manuscript is arranged as follows: Section 1 described a brief history of existing work and motivation for new work, followed by a review of fundamental definitions in Section 2. Sections 3 and 4 covered the concept of IVTSFSs and its significance, respectively. Section 5 discussed the notion of IVTSFG and its some graphical notions. Section 6 discussed three applications of IVTSFSs and IVTSFGs including anomaly detection, medical diagnosis, and shortest path based on interval valued T-spherical fuzzy relations (IVTSFRs). Section 7 narrates the concluding remarks followed by the future direction.

2. Basic Concepts

In our study, we denote a universal set by X. Further, by $ş, ı, \mathrm{d}$, and R we denote membership, abstinence, nonmembership, and refusal functions maps on $[0,1]$ interval.

Definition 1 (see [2]). An IFS is expressed as $A = \{x, (ş(x), d(x))\}$ such that $0 \le \mathrm{sum}(ş, d) \le 1$ and $R = 1 - \mathrm{sum}(ş, d)$ are known as hesitancy index.

Definition 2 (see [4]). A PFS is expressed as $A = \{x, (ş(x), ı(x), \mathrm{d}(x))\}$ such that $0 \le \mathrm{sum}(ş, ı, \mathrm{d}) \le 1$ and $R = 1 - \mathrm{sum}(ş, ı, \mathrm{d})$ are termed as refusal degree.

Definition 3 (see [6]). A SFS is expressed as $A = \{x, (ş(x), ı(x), \mathrm{d}(x))\}$ such that $0 \le \mathrm{sum}(ş^2, ı^2, d^2) \le 1$ and $R = \sqrt{\mathrm{sum}(ş^2, ı^2, d^2)}$ are termed as refusal degree.

Definition 4 (see [6]). A TSFS is expressed as $A = \{x, (ş(x), ı(x), \mathrm{d}(x))\}$ such that $0 \le \mathrm{sum}(ş^n, ı^n, d^n) \le 1$ for $n \ge 1, n \in \mathbb{Z}^+$ and $R = \sqrt[n]{\mathrm{sum}(ş^n, ı^n, d^n)}$ are termed as refusal degree.

The triplet $(ş, ı, \mathrm{d})$ is a picture fuzzy and spherical fuzzy and T-spherical fuzzy numbers, respectively, for Definitions 2–4.

Definition 5 (see [9]). An IVIFS is expressed as $A = \{x, (ş(x), d(x))\}$ such that $ş = [ş^L. ş^U]$ and $d = [d^L, d^U]$ provided that $0 \le \mathrm{sum}(ş^U, d^U) \le 1$ and $R = [R^L, R^U] = [1 - \mathrm{sum}(ş^U, d^U), 1 - \mathrm{sum}(ş^L, d^L)]$ are known as hesitancy index. Further, the duplet $([ş^L. ş^U], [d^L, d^U])$ is an IVIFN.

Definition 6 (see [5]). An IVPFS is expressed as $A = \{x, (ş(x), ı(x), \mathrm{d}(x))\}$ such that $ş = [ş^L. ş^U], ı = [ı^L, ı^U]$ and $d = [d^L, d^U]$ provided that $0 \le \mathrm{sum}(ş^U, ı^U, d^U) \le 1$ and $R = [R^L, R^U] = [1 - \mathrm{sum}(ş^U, ı^U, d^U), 1 - \mathrm{sum}(ş^L, ı^L, d^L)]$ are known as refusal degree. Further, the triplet $([ş^L. ş^U], [ı^L, ı^U], [d^L, d^U])$ is an IVPFN.

Example 1. Figure 3 corresponds to FG, while Figure 4 to its complement.

Definition 7 (see [24]). An IFG is defined by $G = (V, E)$ V and E are the collections nodes and edges, respectively.

(1) For all $v \in V$ is categorized by $ş$ and d denoting the MG and NMG of $v \in V$ s.t $0 \le ş + d \le 1$. The term R denotes the hesitancy level of $v \in V$ such that $R = 1 - ş - d$.

(2) Every $e \in E$ is attributed $\hat{s}$ and $Đ$ denoting the MG and the NMG defined as follows:

$$\begin{aligned} \hat{S}(v_i, v_j) &\le \min(ş(v_i), ş(v_j)), \\ Đ(v_i, v_j) &\le \max(d(v_i), d(v_j)), \end{aligned} \tag{1}$$

s.t. $0 \le \hat{s} + Đ \le 1$. $\check{R}$ denotes the hesitancy degree of $e \in E$ such that $\check{R} = 1 - \hat{S} - Đ$.

Example 2. Figure 5 represents IFG.

3. Interval-Valued T-Spherical Fuzzy Sets

The recently introduced concept of IVTSFSs with properties is discussed and the notion of IVTSF relation is also presented with results and discussion.

Figure 3: Fuzzy graph w.r.t Definition 7.

Figure 4: Complement of fuzzy graph.

Figure 5: Intuitionistic fuzzy graph compatible with Definition 7.

Definition 8 (see [11]). An IVTSFS A= $\{x, (\text{ş}(x), \imath(x), d(x))\}$ where ş, ı and d are from X to subinterval of [0, 1] , i.e., $\text{ş}(x) = [\text{ş}^L(x), \text{ş}^U(x)], \imath(x) = [\imath^L(x), \imath^U(x)]$ and $d(x) = [d^L(x), d^U(x)]$ with a condition $0 \le (\text{ş}^U(x))^n + (\imath^U(x))^n + (d^U(x))^n \le 1$ and $n \in \mathbb{Z}^+$. Moreover, the interval $R(x) = [R^U(x), R^L(x)]$ represents the level of refusal $x \in X$, i.e., $R^U(x) = \sqrt[n]{1 - (\text{ş}^L(x))^n - (\imath^L(x))^n - (d^L(x))^n}$ and $R^L(x) = \sqrt[n]{1 - (\text{ş}^U(x))^n - (\imath^U(x))^n - (d^U(x))^n}$.

The following theorem proves the generalization of IVTSFSs over TSFSs, IVPFSs, PFSs, IVIFSs. IFSs, IVFSs, and FSs.

Theorem 1. *In Definition 8 of IVTSFSs, if we*

(1) Taking $\text{ş}^L = \text{ş}^U, \imath^L = \imath^U$ and $d^L = d^U$, then it reduces to TSFSs.

(2) Taking $n = 2$, then it reduces to the definition of IVSFS.

(3) Taking $n = 2$, $\text{ş}^L = \text{ş}^U, \imath^L = \imath^U$, and $d^L = d^U$, then it reduces to the definition of SFS.

(4) Taking $n = 1$, then it reduces to the definition of IVPFS.

(5) Taking $n = 1$, $\text{ş}^L = \text{ş}^U, \imath^L = \imath^U$ and $d^L = d^U$, then it reduces to the definition of PFS.

(6) Taking $n = 2$ and $\imath^L = \imath^U = 0$, then it reduces to the definition of IVPyFS.

(7) Taking $n = 2$, $\text{ş}^L = \text{ş}^U$, $\imath^L = \imath^U = 0$ and $d^L = d^U$, then it reduces to the definition of PyFS.

(8) Taking $n = 1$ and $\imath^L = \imath^U = 0$, then it reduces to the definition of IVIFS.

(9) Taking $n = 1$, $\text{ş}^L = \text{ş}^U, \imath^L = \imath^U = 0$ and $d^L = d^U$, then it reduces to the definition of IFS.

(10) Taking $n = 1$ and $\imath^L = \imath^U = d^L = d^U = 0$, then it reduces to the definition of IVFS.

(11) Taking $n = 1$ and $\text{ş}^L = \text{ş}^U, \imath^L = \imath^U = 0 = d^L = d^U$, then it reduces to the definition of FS.

Definition 9 For IVITSFSs $A = \{x, (\text{ş}_A(x), \imath_A(x), d_A(x))\}$ and $B = \{x, (\text{ş}_B(x), \imath_B(x), d_B(x))\}$:

(1) $A \subseteq B$ if $\text{ş}_A(x) \le \text{ş}_B(x)$, $\imath_A(x) \le \imath_B(x)$ and $d_A(x) \ge d_B(x)$, where

$$\begin{aligned} &\text{ş}_A(x) \le \text{ş}_B(x) \Longrightarrow \text{ş}_A^L(x) \le \text{ş}_B^L(x) \text{ and } \text{ş}_A^U(x) \le \text{ş}_B^U(x), \\ &\imath_A(x) \le \imath_B(x) \Longrightarrow \imath_A^L(x) \le \imath_B^L(x) \text{ and } \imath_A^U(x) \le \imath_B^U(x), \\ &d_A(x) \ge d_B(x) \Longrightarrow d_A^L(x) \ge d_B^L(x) \text{ and } d_A^U(x) \ge d_B^U(x). \end{aligned} \tag{2}$$

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(3) $A \cup B = \{x, (\text{ş}_A(x) \vee \text{ş}_B(x), \imath_A(x) \wedge \imath_B(x) \text{ and } d_A(x) \wedge d_B(x))\}$, where

$$\begin{aligned}
&\text{ş}_A(x) \vee \text{ş}_B(x) \Longrightarrow \text{ş}_A^L(x) \vee \text{ş}_B^L(x) \text{ and } \text{ş}_A^U(x) \vee \text{ş}_B^U(x),\\
&\text{ı}_A(x) \wedge \text{ı}_B(x) \Longrightarrow \text{ı}_A^L(x) \wedge \text{ı}_B^L(x) \text{ and } \text{ı}_A^U(x) \wedge \text{ş}_B^U(x),\\
&d_A(x) \wedge d_B(x) \Longrightarrow d_A^L(x) \wedge d_B^L(x) \text{ and } d_A^U(x) \wedge d_B^U(x).
\end{aligned} \tag{3}$$

(4) $A \cap B = \{x, (\text{ş}_A(x) \wedge \text{ş}_B(x), \text{ı}_A(x) \wedge \text{ı}_B(x) \text{ and } d_A(x) \vee d_B(x))\}$ where

$$\begin{aligned}
&\text{ş}_A(x) \wedge \text{ş}_B(x) \Longrightarrow \text{ş}_A^L(x) \wedge \text{ş}_B^L(x) \text{ and } \text{ş}_A^U(x) \wedge \text{ş}_B^U(x),\\
&\text{ı}_A(x) \wedge \text{ı}_B(x) \Longrightarrow \text{ı}_A^L(x) \wedge \text{ı}_B^L(x) \text{ and} \text{ı}_A^U(x) \wedge \text{ı}_B^U(x).
\end{aligned} \tag{4}$$

(5)
$d_A(x) \vee d_B(x) \Longrightarrow d_A^L(x) \vee d_B^L(x) \text{ and } d_A^U(x) \vee d_B^U(x).$

(6) $A^c = \{x, (d_A(x), \text{ı}_A(x), \text{ş}_A(x))\}$.

The following theorem shows that IVTSFSs preserve the basic set theoretic properties.

Theorem 2. *For three IVTSFSs A, B and C:*

(1) The transitive law w.r.t. inclusion holds true

(2) The union and intersection are commutative

(3) The union and intersection are associative

(4) Distributive laws of the union over intersection and intersection over union hold true

(5) DE Morgan's laws hold true

Proof. Let $A = \{x, (\text{ş}_A(x), \text{ı}_A(x), d_A(x))\}$, $B = \{x, (\text{ş}_B(x), \text{ı}_B(x), d_B(x))\}$ and $C = \{x, (\text{ş}_C(x), \text{ı}_C(x), d_C(x))\}$ be three IVTSFSs. The proof of the results is as follows:

(1) Transitivity: let us assume $A \le B$ and $B \le C$. To prove $A \le C$. As $A \le B$ so $\text{ş}_A(x) \le \text{ş}_B(x)$, $\text{ı}_A(x) \le B_B(x)$ and $d_A(x) \ge d_B(x)$. Also, as $B \le C$ so $\text{ş}_B(x) \le \text{ş}_C(x)$, $\text{ı}_B(x) \le \text{ı}_C(x)$ and $d_B(x) \ge d_C(x)$.
Now, $\text{ş}_A(x) \le \text{ş}_B(x)$ and $\text{ş}_B(x) \le \text{ş}_C(x) \Longrightarrow \text{ş}_A(x) \le \text{ş}_C(x)$.
Also, $\text{ı}_A(x) \le B_B(x)$ and $\text{ı}_B(x) \le \text{ı}_C(x) \Longrightarrow \text{ı}_A(x) \le \text{ı}_C(x)$.
Similarly, $d_A(x) \ge d_B(x)$ and $d_B(x) \ge d_C(x) \Longrightarrow d_A(x) \ge d_C(x)$.
Hence, $A \le C$.

(2) Commutativity of union: for IVTSFSs A and B,

$$\begin{aligned}
A \cup B &= \{x, (\text{ş}_A(x) \vee \text{ş}_B(x), \text{ı}_A(x) \wedge \text{ı}_B(x) \text{ and } \wedge_A(x) \wedge d_B(x))\}\\
&= \{x, (\text{ş}_B(x) \vee \text{ş}_A(x), \text{ı}_B(x) \wedge \text{ı}_A(x) \text{ and } \wedge_B(x) \wedge d_A(x))\}\\
&= B \cup A.
\end{aligned} \tag{5}$$

Commutativity of intersection is similar.

(3) Distributive law of union over intersection: to prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

$$\begin{aligned}
A \cup (B \cap C) &= \{x, (\text{ş}_A(x), \text{ı}_A(x), d_A(x))\} \cup (\{x, (\text{ş}_B(x) \wedge \text{ş}_C(x), \text{ı}_B(x) \wedge \text{ı}_C(x) \text{ and } d_B(x) \vee d_C(x))\})\\
&= \{x, (\text{ş}_A(x) \vee (\text{ş}_B(x) \wedge t\text{ş}_C n(x)), \text{ı}_A(x) \wedge (\text{ı}_B(x) \wedge \text{ı}_C(x)), d_A(x) \wedge (d_B(x) \vee d_C(x)))\}\\
&= \{x, ((\text{ş}_A(x) \vee t\text{ş}_B n(x)) \wedge (\text{ş}_A(x) \vee \text{ş}_C(x)), (\text{ı}_A(x) \vee \text{ı}_B(x)) \wedge (\text{ı}_A(x) \vee \text{ı}_C(x)), (d_A(x) \wedge d_B(x)) \vee (d_A(x) \wedge d_C(x)))\}\\
&= (A \cup B) \cap (A \cup C).
\end{aligned} \tag{6}$$

Distributive law of intersection over union is similar to prove.

(4) De Morgan's Laws: to prove $(A \cup B)^c = A^c \cap B^c$.

$$\begin{aligned}
(A \cup B)^c &= \{x, (\text{ş}_A(x) \vee \text{ş}_B(x), \text{ı}_A(x) \wedge B_B(x), d_A(x) \wedge d_B(x))\}^c\\
&= \{x, (d_A(x) \wedge d_B(x), \text{ı}_A(x) \wedge \text{ı}_B(x), \text{ş}_A(x) \vee \text{ş}_B(x))\}\\
&= \{x, (d_A(x), \text{ı}_A(x), \text{ş}_A(x))\} \cap \{x, (d_B(x), \text{ı}_B(x), \text{ş}_B(x))\}\\
&= A^c \cap B^c.
\end{aligned} \tag{7}$$

Similarly, $(A \cap B)^c = A^c \cup B^c$ can be proved. □

Definition 10. For IVITSFSs $A = \{x, (\text{ş}_A(x), \imath_A(x), d_A(x))\}$ and $B = \{x, (\text{ş}_B(x), \imath_B(x), d_B(x))\}$, we define

$$A \oplus B = \left\{ \left\{ x, \left(\begin{array}{c} \sqrt[n]{\overset{Ln}{\text{ş}}_A(x) + \overset{Ln}{\text{ş}}_B(x) - \overset{Ln}{\text{ş}}_A(x).\overset{Ln}{\text{ş}}_B(x)}, \left[\sqrt[n]{\overset{Un}{\text{ş}}_A(x) + \overset{Un}{\text{ş}}_B(x) - \overset{Un}{\text{ş}}_A(x).\overset{Un}{\text{ş}}_B(x)} \right], \\ \left[\overset{L}{\imath}_A(x).\overset{L}{\imath}_B(x), \overset{U}{\imath}_A(x).\overset{U}{\imath}_B(x) \right], \left[d_A^L(x).d_B^L(x), d_A^U(x).d_B^U(x) \right] \end{array} \right) \right\} \right\},$$
$$A \otimes B = \left\{ \left\{ x, \left(\begin{array}{c} \left[\text{ş}_A^L(x).\text{ş}_B^L(x), \text{ş}_A^U(x).\text{ş}_B^U(x) \right], \left[\overset{L}{\imath}_A(x).\overset{L}{\imath}_B(x), \overset{U}{\imath}_A(x).\overset{U}{\imath}_B(x) \right], \\ \sqrt[n]{d_A^{Ln}(x) + d_B^{Ln}(x) - d_A^{Ln}(x).d_B^{Ln}(x)} \left[, \sqrt[n]{d_A^{Un}(x) + d_B^{Un}(x) - d_A^{Un}(x).d_B^{Un}(x)} \right] \end{array} \right) \right\} \right\}. \tag{8}$$

In the following, the concept of Cartesian product and relations for IVTSFSs are proposed, and their properties are studied. These operations are the generalization of operations of IVIFSs and IVPFSs.

Definition 11. The Cartesian product of two IVTSFSs A and B over two universes X_1 and X_2 is of the following form:

$$\begin{aligned} A \times_1 B &= \{(x, y), (\text{ş}_A(x).\text{ş}_B(y), \imath_A(x).\imath_B(y), d_A(x).d_B(y))\} \\ &= \left\{(x, y), \left(\left[\text{ş}_A^L(x).t\text{ş}_B^L n(y)q, h\, \text{ş}_A^U x(x)7.C\text{ş}_B^U; (y) \right], \left[\overset{L}{\imath}_A(x).\overset{L}{\imath}_B(y), \overset{U}{\imath}_A(x).\overset{U}{\imath}_B(y) \right], \left[d_A^L(x).d_B^L(y), d_A^U(x).d_B^U(y) \right] \right) \right\}, \\ A \times_2 B &= \{(x, y), (\text{ş}_A(x) \wedge \text{ş}_B(y), \imath_A(x) \wedge \imath_B(y), d_A(x) \vee d_B(y))\} \\ &= \left\{(x, y), \left(\left[\text{ş}_A^L(x) \wedge t\text{ş}_B^L n(y)q, h\, \text{ş}_A^U x(x)7 \wedge C\text{ş}_B^U; (y) \right], \left[\overset{L}{\imath}_A(x) \wedge \overset{L}{\imath}_B(y), \overset{U}{\imath}_A(x) \wedge \overset{U}{\imath}_B(y) \right], \left[d_A^L(x) \vee d_B^L(y), d_A^U(x) \vee d_B^U(y) \right] \right) \right\}. \end{aligned} \tag{9}$$

Definition 12 An IVTSFR R is an IVTSF fuzzy subset of $X \times Y$ and is of the following form:

$$R = \{(x, y), (\text{ş}(x, y), \imath(x, y), \mathrm{d}(x, y))\}, \tag{10}$$

Provided that $0 \le \text{ş}^n(x, y) + \imath^n(x, y) + d^n(x, y) \le 1$ and $n \in \mathbb{Z}^+$.

Definition 13. For an IVTSFR R, R^{-1} is defined as follows:

$$R^{-1} = \left\{(y, x), \left(\text{ş}^{-1}(y, x), \imath^{-1}(y, x), d^{-1}(y, x)\right)\right\}, \tag{11}$$

where
$\text{ş}^{-1}(y, x) = \text{ş}(x, y), \imath^{-1}(y, x) = \imath(x, y), \mathrm{d}^{-1}(y, x) = \mathrm{d}(x, y)$.

Proposition 1. *The following results for three IVTSFRs R, R_1, and R_2 are Boolean positive.*

(1) $(R^{-1})^{-1} = R$

(2) If $R_1 \subseteq R_2$, *then* $R_1^{-1} \subseteq R_2^{-1}$

(3) $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$ *and* $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$

(4) $R_1 \cap R_2 \subseteq R_1$ *and* $R_1 \cap R_2 \subseteq R_2$

(5) If $R_1 \subseteq R$ *and* $R_2 \subseteq R$, *then* $R_1 \cup R_2 \subseteq R$

(6) If $R \subseteq R_1$ *and* $R \subseteq R_2$, *then* $R \subseteq R_1 \cap R_2$

Definition 14. For two IVTSFRs R_1 on $X \times Y$ and R_2 on $Y \times Z$, their composition is denoted by $R_2 \circ R_1$ and defined as follows:

$$R_2 \circ R_1 = \{(x, z), (\text{ş}(x, z), \imath(x, z), \mathrm{d}(x, z))\}, \tag{12}$$

where $\text{ş}(x, z) = \vee\{(\text{ş}(x, y) \wedge \text{ş}(y, z))\} = \vee_y \{([\text{ş}^L(x, y) \wedge \text{ş}^L(y, z), \text{ş}^U(x, y) \wedge \text{ş}^{Uy}(y, z)])\}$,

$$\begin{aligned} \imath(x, z) &= \wedge_y \{(\imath(x, y) \wedge \imath(y, z))\} \\ &= \wedge_y \left\{\left(\left[\imath^L(x, y) \wedge \imath^L(y, z), \imath^U(x, y) x \imath^U(y, z) \right] \right)\right\}, \\ d(x, z) &= \wedge_y \{(d(x, y) \vee d(y, z))\} \\ &= \wedge_y \left\{\left(\left[d^L(x, y) \vee d^L(y, z), d^U(x, y) \vee d^U(y, z) \right] \right)\right\}. \end{aligned} \tag{13}$$

Definition 15. The score function for an IVTSFN $A = ([\mathscr{Z}^L, \mathscr{Z}^U], [\overset{\vee}{\mathrm{r}}{}^L, \overset{\vee}{\imath}{}^U], [\eta^L, \eta^U])$ is defined as follows:

$$\mathrm{SC}(A) = \frac{\left(\mathscr{Z}^L\right)^n \left(1 - \left(\overset{\vee}{\mathrm{r}}{}^L\right)^n - \left(\eta^L\right)^n\right) + \left(\mathscr{Z}^U\right)^n \left(\left(1 - \left(\overset{\vee}{\imath}{}^U\right)^n - \left(\eta^U\right)^n\right)\right)}{3}, \ \mathrm{SC}(A) \in [0, 1] \tag{14}$$

Remark 1. Replacing $\imath^L = \imath^U = 0$ and $n = 1$ reduces the defined function for score calculation.

4. Importance of Structures: Interval-Valued Fuzzy

The worthy concept of modeling a MG of a FS using an interval was the inception of Gorzalczany [2], whereas the concepts of IFS, PyFS, Q-ROPFS, and PFS are established in [5, 6, 15, 50]. The IVFS's concept has shown great significance, especially in those situations where an occurrence is difficult to be attributed by a crisp number and therefore, the significance of interval valued frameworks is elaborated in more detail.

The main decision in the following instance is regarding the selection of the best candidate among five candidates ($\dot{g}_i$, $i = 1, 2, 3, 4$), with six attributes ($\dot{g}_j$, $j = 1, 2, 3, 4$) and weight vector $w = (0.22, 0.34, 0.27, 0.17)^T$. In this case, Table 1 illustrates the evaluation data.

The proposed aggregation tool by Xu and Cai [29] for the data provided in Table 1 is as follows:

$$\dot{g}_i = \text{IVIFWA}(\dot{g}_1, \dot{g}_2, \dot{g}_3, \ldots, \dot{g}_m) = \left(\left[1 - \prod_{j=1}^{m}\left(1 - \left(\mathcal{Z}_j^l\right)\right)^{w_j}, 1 - \prod_{j=1}^{m}\left(1 - \left(\mathcal{Z}_j^u\right)\right)^{w_j}\right], \left[\prod_{j=1}^{m}\left(\eta_j^l\right)^{w_j}, \prod_{j=1}^{m}\left(\eta_j^u\right)^{w_j}\right]\right). \tag{15}$$

The aggregated results are as follows:

$$\begin{aligned}
\dot{g}_1 &= \text{IVIFWA}(\dot{g}_{11}, \dot{g}_{12}, \dot{g}_{13}, \dot{g}_{14}) = ([0.6235, 0.7569], [0.0000, 0.3386]),\\
\dot{g}_2 &= \text{IVIFWA}(\dot{g}_{21}, \dot{g}_{22}, \dot{g}_{23}, \dot{g}_{24}) = ([0.6777, 0.8180], [0.0000, 0.2800]),\\
\dot{g}_3 &= \text{IVIFWA}(\dot{g}_{31}, \dot{g}_{32}, \dot{g}_{33}, \dot{g}_{34}) = ([0.5730, 0.7745], [0.1713, 0.3454]),\\
\dot{g}_4 &= \text{IVIFWA}(\dot{g}_{41}, \dot{g}_{42}, \dot{g}_{43}, \dot{g}_{44}) = ([0.6493, 0.8352], [0.1165, 0.2510]).
\end{aligned} \tag{16}$$

The score of IVFN $I = ([\eta^l, \eta^u], [\eta^l, \eta^u])$ is defined as follows [31]:

$$\begin{aligned}
\text{SC0}(I) &= \frac{1}{2}\left(\mathcal{Z}^l - \eta^l + \mathcal{Z}^u - \eta^u\right),\\
\text{SC}(\dot{g}_1) &= 0.5209, \text{SC}(\dot{g}_2) = 0.6079, \text{SC}(\dot{g}_3)\\
&= 0.4154, \text{SC}(\dot{g}_4) = 0.5585.
\end{aligned} \tag{17}$$

Clearly, $\text{SC0}(\dot{g}_3) > \text{SC0}(\dot{g}_4) > \text{SC0}(\dot{g}_1) > \text{SC} - (\dot{g}_2)$

Hence, according to the MADM method, $\dot{g}_2$ is considered to be the most appropriate selection candidate. The assignment of a crisp value to membership and NMG causes the information loss in Table 1 and can be aggregated using the aggregation operators Xu [22]. The decision matrix is illustrated in Table 2.

The weighted averaging aggregation operator of IFSs, proposed by Xu [22], is given by the following:

$$\begin{aligned}
\dot{g}_1 &= \text{IFWA}(\dot{g}_{11}, \dot{g}_{12}, \dot{g}_{13}, \ldots \dot{g}_m) = \left(1 - \prod_{j=1}^{m}\left(1 - \left(\mathcal{Z}_j^l\right)\right)^{w_j}, \prod_{j=1}^{m}\left(d_j^l\right)^{w_j}\right),\\
\dot{g}_1 &= \text{IFWA}(\dot{g}_{11}, \dot{g}_{12}, \dot{g}_{13}, \dot{g}_{14}) = (0.7569, 0.3386),\\
\dot{g}_2 &= \text{IFWA}(\dot{g}_{21}, \dot{g}_{22}, \dot{g}_{23}, \dot{g}_{24}) = (0.8180, 0.2800),\\
\dot{g}_3 &= \text{IFWA}(\dot{g}_{31}, \dot{g}_{32}, \dot{g}_{33}, \dot{g}_{34}) = (0.7745, 0.3454),\\
\dot{g}_4 &= \text{IFWA}(\dot{g}_{41}, \dot{g}_{42}, \dot{g}_{43}, \dot{g}_{44}) = (0.7489, 0.1165).
\end{aligned} \tag{18}$$

For ranking IFN I, the score function developed in [22] is given by $\text{SC0}(I) = \mathcal{Z} - \eta$.

Using this score functions, we have the following:

$$\begin{aligned}
\text{SC0}(\dot{g}_1) &= 0.0361, \text{SC0}(\dot{g}_2) = 0.2229, \text{SC0}(\dot{g}_3)\\
&= 0.1279, \text{SC0}(\dot{g}_4) = 0.3007.
\end{aligned} \tag{19}$$

Clearly, $\text{SC0}(\dot{g}_4) > \text{SC0}(\dot{g}_2) > \text{SC0}(\dot{g}_3) > \text{SC}(\dot{g}_1)$

Thus, according to the MADM approach, C_4 is the appropriate candidate with IVIFWA operators. The analysis of the results revealed the significance of the aggregation of IVIFS in contrast to the one obtained using IFS. The concept of TSFS [49] is a generalization of Q-ROPFS, SFS, PyFS, PFS, IFS, as well as FS, and therefore, the interval-valued structure for TSFSs with its diverse structure would have an outstanding impact. The concept of IVTSFS [6, 7, 49] gives further convincing strength to the notion of TSFSs due to its usefulness and flexibility towards the assignment of MGs of TSFS in terms of intervals.

Table 1: Decision matrix: evaluation data.

	$\dot{g}_1$	$\dot{g}_2$	$\dot{g}_3$	$\dot{g}_4$
$\dot{g}_1$	$\begin{pmatrix}[0.2, 0.3],\\ [0.0, 0.5]\end{pmatrix}$	$\begin{pmatrix}[0.5, 0.6],\\ [0.1, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.4, 0.5],\\ [0.2, 0.4]\end{pmatrix}$	$\begin{pmatrix}[0.7, 0.8],\\ [0.1, 0.2]\end{pmatrix}$
$\dot{g}_2$	$\begin{pmatrix}[0.6, 0.7],\\ [0.2, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.5, 0.6],\\ [0.0, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.6, 0.7],\\ [0.2, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.6, 0.7],\\ [0.1, 0.2]\end{pmatrix}$
$\dot{g}_3$	$\begin{pmatrix}[0.4, 0.5],\\ [0.3, 0.4]\end{pmatrix}$	$\begin{pmatrix}[0.3, 0.6],\\ [0.1, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.5, 0.6],\\ [0.3, 0.4]\end{pmatrix}$	$\begin{pmatrix}[0.6, 0.7],\\ [0.1, 0.3]\end{pmatrix}$
$\dot{g}_4$	$\begin{pmatrix}[0.6, 0.7],\\ [0.2, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.5, 0.7],\\ [0.1, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.7, 0.8],\\ [0.1, 0.2]\end{pmatrix}$	$\begin{pmatrix}[0.3, 0.4],\\ [0.1, 0.2]\end{pmatrix}$

Table 2: Decision matrix.

	$\dot{g}_2$	$\dot{g}_2$	$\dot{g}_2$	$\dot{g}_2$
$\dot{g}_1$	(0.3, 0.5)	(0.6, 0.3)	(0.5, 0.4)	(0.8, 0.2)
$\dot{g}_2$	(0.7, 0.3)	(0.6, 0.3)	(0.7, 0.3)	(0.7, 0.2)
$\dot{g}_3$	(05, 0.4)	(0.6, 0.3)	(0.6, 0.4)	(0.7, 0.3)
$\dot{g}_4$	(0.6, 0.2)	(0.5, 0.1)	(0.7, 0.1)	(0.3, 0.1)

5. Interval-Valued T-Spherical Fuzzy Graphs

The IVTSFS is a generalization of FS, IFSs, PFSs, IVIFSs, and IVPFSs, and therefore, it is of great importance. The notion of graph for IVTSFS, i.e., IVTSFG is proposed as a generalization of FG, IFG, PFG, IVIFG, and IVPFG along with fundamental terms and definitions with applications in Section 6.

Definition 16. An IVTSFG is $= (V, E)$, V is node set and E edges' collection and

(1) For all $v \in V$ is attributed with three functions $ş$, $ı$ and d degree of membership, abstinence, and nonmembership of $v \in V$. Basically, $ş = [ş^L, ş^L]$, $ı = [ı^L, ı^U]$ and $d = [d^L, d^L]$ are subintervals of the unit interval $[0, 1]$ with a condition that $0 \le (ş^U)^n + (ı^U)^n + (d^U)^n \le 1$ for $n \in \mathbb{Z}^+$. The $R = [R^U, R^L]$ denotes the refusal degree of $v \in V$ and $R^U = \sqrt[n]{1 - (ş^L)^n - (ı^L)^n - (d^L)^n}$ and $R^L = \sqrt[n]{1 - (ş^U)^n - (ı^U)^n - (d^U)^n}$.

(2) For all $e \in E$ is attributed with three functions $\widehat{S}$, $\hat{I}$ and $Đ$ degree of membership, abstinence, and nonmembership of $e = (\widehat{S}, \hat{I}, Đ) \in V \times V$. Basically $\widehat{S} = [\widehat{S}^L, \widehat{S}^L], \hat{I} = [\hat{I}^L, \hat{I}^U]$ and $Đ = [Đ^L, Đ^L]$ are as follows:
$\widehat{S}^L(v_i, v_j) \le \min(ş^L(v_i), ş^L(v_j))$ and $\widehat{S}^U(v_i, v_j) \le \min(ş^U(v_i), ş^U(v_j))$ s.t $\widehat{S}^U \ge \min(ş^L(v_i), ş^L(v_j))$.
$\hat{I}^L(v_i, v_j) \le \min(ı^L(v_i), ı^L(v_j))$ and $\hat{I}^U(v_i, v_j) \le \max(ı^U(v_i), ı^U(v_j))$ s.t $\hat{I}^U \ge \max(ı^L(v_i), ı^L(v_j))$.
$Đ^L(v_i, v_j) \le \max(d^L(v_i), d^L(v_j))$ and $Đ^U(v_i, v_j) \le \max(d^U(v_i), d^U(v_j))$ s.t $Đ^U \ge \max(d^L(v_i), d^L(v_j))$.
With a condition $0 \le (\widehat{S}^U)^n + (\hat{I}^U(\varkappa))^n + (Đ^U(\varkappa))^n \le 1$ for $n \in \mathbb{Z}^+$, the term $\check{R} = [\check{R}^U, \check{R}^L]$ denotes the refusal degree of $e \in E$ and $\check{R}^U = \sqrt[n]{1 - (\widehat{S}^L)^n - (\hat{I}^L)^n - (Đ^L)^n}$ and $\check{R}^L = \sqrt[n]{1 - (\widehat{S}^U)^n - (\hat{I}^U)^n - (Đ^U)^n}$.

Theorem 3. *IVTSFG is a generalization of IVPFG, IVIFG, and IVFG.*

(1) If we take $n = 1$, the IVTSFG diminishes to IVPFG

(2) If we take $n = 1$ and $ı^L = ı^U = 0$, the IVTSFG diminishes to IVIFG

(3) If we take $n = 1$, $ı^L = ı^U = 0$ and $d^L = d^U = 0$, the IVTSFG diminishes to IVFG

The outcome reveals the significance of the novel concept due to its generalization.

Example 3. The following Figure 6 is an IVTSFG with a set of nodes v_i $(i = 1, 2, 3, \ldots, 6)$ and a set of edges e_i i $(i = 1, 2, 3, 4 \text{ and } 5))$.

Definition 17. An IVTSFG $G' = (V', E')$ is IVTSFSG of graph $G = (V, E)$ iff $V' \subseteq V$ and $E' \subseteq E$

Example 4. The IVTSFSG of IVTSFG provided in Figure 6 is depicted in Figure 7

Figure 6: IV T-spherical fuzzy graph.

Figure 7: Interval-valued T-spherical fuzzy subgraph.

Definition 18. The complement, i.e., IVTSFG $G = (V, E)$ is represented by $G^c = (V^c, E^c)$ where $V^c = V$ and the MGs of E are defined by the following:

$$\begin{aligned}
\left(\widehat{S}^L\right)^c(v_i, v_j) &= \min\left(\text{ş}^L(v_i), \text{ş}^L(v_j)\right) - \widehat{S}^L\left((v_i, v_j)\right),\\
\left(\widehat{S}^U\right)^c(v_i, v_j) &= \min\left(\text{ş}^U(v_i), \text{ş}^U(v_j)\right) - \widehat{S}^U(v_i, v_j) + \min\left(\text{ş}^L(v_i), \text{ş}^L(v_j)\right),\\
\left(\hat{\text{I}}^L\right)^c(v_i, v_j) &= \min\left(\text{ı}^L(v_i), \text{ı}^L(v_j)\right) - \hat{\text{I}}^L\left((v_i, v_j)\right),\\
\left(\hat{\text{I}}^U\right)^c(v_i, v_j) &= \min\left(\text{ı}^U(v_i), \text{ı}^U(v_j)\right) - \hat{\text{I}}^U(v_i, v_j) + \max\left(\text{ı}^L(v_i), \text{ı}^L(v_j)\right),\\
\left(Đ^L\right)^c(v_i, v_j) &= \max\left(d^L(v_i), d^L(v_j)\right) - Đ^L(v_i, v_j),\\
\left(Đ^U\right)^c(v_i, v_j) &= \max\left(d^U(v_i), d^U(v_j)\right) - Đ^U(v_i, v_j) + \max\left(d^L(v_i), d^L(v_j)\right).
\end{aligned} \tag{20}$$

Theorem 4. *For IVTSFG $G = (V, E)$, $(G^c)^c = G$.*

Proof. Let $G = (V, E)$ be an IVTSFG. The proof is as follows:

$$\begin{aligned}
\left(\widehat{S}^L\right)^c(v_i, v_j) &= \min\left(ş^L(v_i), ş^L(v_j)\right) - \widehat{S}^L\left((v_i, v_j)\right),\\
\left(\left(\widehat{S}^L\right)^c\right)^c(v_i, v_j) &= \min\left(ş^L(v_i), ş^L(v_j)\right) - \left(\min\left(ş^L(v_i), ş^L(v_j)\right) - \widehat{S}^L\left((v_i, v_j)\right)\right) = \widehat{S}^L\left((v_i, v_j)\right),\\
\left(\widehat{S}^U\right)^c(v_i, v_j) &= \min\left(ş^U(v_i), ş^U(v_j)\right) - \widehat{S}^U(v_i, v_j) + \min\left(ş^L(v_i), ş^L(v_j)\right),\\
\left(\left(\widehat{S}^U\right)^c\right)^c(v_i, v_j) &= \min\left(ş^U(v_i), ş^U(v_j)\right) - \left(\min\left(ş^U(v_i), ş^U(v_j)\right) - \widehat{S}^U(v_i, v_j) + \min\left(ş^L(v_i), ş^L(v_j)\right)\right) + \min\left(ş^L(v_i), ş^L(v_j)\right).
\end{aligned} \tag{21}$$

$((\widehat{S}^U)^c)^c(v_i, v_j) = \widehat{S}^U(v_i, v_j)$. Similarly, we can prove that

$$\begin{aligned}
\left(\left(\hat{\mathrm{I}}^{\mathrm{L}}\right)^c\right)^c(v_i, v_j) &= \hat{\mathrm{I}}^{\mathrm{L}}(v_i, v_j),\\
\left(\left(\hat{\mathrm{I}}^{\mathrm{U}}\right)^c\right)^c(v_i, v_j) &= \hat{\mathrm{I}}^{\mathrm{U}}(v_i, v_j),\\
\left(\left(Đ^L\right)^c\right)^c(v_i, v_j) &= Đ^L(v_i, v_j),\\
\left(\left(Đ^U\right)^c\right)^c(v_i, v_j) &= Đ^U(v_i, v_j), \quad \text{Hence } (G^c)^c = G.
\end{aligned} \tag{22}$$

□

Example 5. Figure 8 represents the IVTSFGs graph, whereas Figure 9 represents its complement.

Definition 19. An IVTSFG is known as follows:

(1) Semi S strong: if $\widehat{S}^L(v_i, v_j) = \min(ş^L(v_i), ş^L(v_j))$ and $\widehat{S}^U(v_i, v_j) = \min(ş^U(v_i), ş^U(v_j))$

(2) Semi I strong: if $\hat{\mathrm{I}}^{\mathrm{L}}(v_i, v_j) = \min(\imath^{\mathrm{L}}(v_i), \imath^{\mathrm{L}}(v_j))$ and $\hat{\mathrm{I}}^{\mathrm{U}}(v_i, v_j) = \min(\imath^{\mathrm{U}}(v_i), \imath^{\mathrm{U}}(v_j))$

(3) Semi D strong: if $Đ^L(v_i, v_j) = \max(d^L(v_i), d^L(v_j))$ and $Đ^U(v_i, v_j) = \max(d^U(v_i), d^U(v_j))$

(4) Strong: if (1), (2), and (3) hold true.

Example 6. Figure 10 represents IVTSFG.

Definition 20. In an IVTSFG, a set of nonidentical nodes $v_i\,(i = 1, 2, , 3, \ldots, m)$ is regarded as a path provided an edge exists between every two vertices v_i and v_j for $i, j = 1, 2, , 3, \ldots, m$. The nonidentical nodes $v_i\,(i = 1, 2, , 3, \ldots, m)$ are equivalent to a path provided at least one of the following conditions holds true.

(1) $\widehat{S}(v_i, v_j)$ is a nonzero subinterval of [0, 1]

(2) $\hat{\mathrm{I}}(\mathrm{v_i}, \mathrm{v_j})$ is a nonzero subinterval of [0, 1]

(3) $Đ(v_i, v_j)$ is a nonzero subinterval of [0, 1]

Consequences of Definition 20:

(1) The length of a path is m−1 if it has m vertices

(2) If the first and last vertices of a path coincide, then it will be a cycle

(3) If two vertices are joined by a path, then they will be regarded as connected

Example 7. In Figure 11,

(1) $v_5v_1v_4v_2v_3$ is a path

(2) $v_1v_4v_2$ is a cycle

6. Applications

Several applications of IVTSFSs and IVTSFGs in anomaly detection, medical diagnosis, and SP problem are discussed in this section. The comparative assessment of the results is also presented.

6.1. Anomaly Detection. The Anomaly/Malware based threat has become an irresistible concern for businesses and organizations. Malware is an abbreviated form of "malicious software." It is a file or a set of instructions intended to bring controlled or catastrophic damage to organizations, facilities, processes, cloud infrastructure, industrial processes, and digital systems. In 2018, the number of users attacked with banking Trojans was 889,452, an increase of 15.9% in comparison with 767,072 in 2017, whereas in 2019, more than 100 million different hosts were attacked in H1 2019 as reported by Kaspersky Labs. Moreover, as per the statistics collected during research conducted by Juniper revealed that the cost of data breaches would rise from $3 trillion each year to over $5 trillion in 2024. Therefore, anomaly detection is one of the most critical elements for cyber security. As IVTSFS is a generalization of TSFSs and IVIFSs, therefore we used the approach of IVTSFSs using aggregation operators to handle anomaly detection problem that is proved to be of great significance. An anomaly is considered to be a file with malicious content or it can be Portable Executables (PE) file. In order to make a decision regarding anomaly detection problems, i.e., information is obtained about a given PE file and associated attributes/features are represented in an appropriate format for subsequent analysis. After the representation, aggregation operators are applied by the problem solvers. Subsequently, the score value is computed using the score function for IVTSFSs to make a decision regarding the classification of the anomaly, i.e., PE

Figure 8: Interval-valued T-spherical fuzzy graph.

Figure 9: Complement of Figure 8.

sample either as malicious or benign. The algorithm for malware detection is shown in Algorithm 1, and the algorithm for medical diagnosis is shown in Algorithm 2.

Consider a sample software file S being investigated by anomaly analyst under k attributes as A_k. The problem solvers investigate the a sample file under k attributes and represent the findings in the form of IVTSFNs. Further, $w = (w_1, w_2, w_3, \ldots, w_k)^T$ corresponds to the weight vector of attributes. The classification process is given as follows:

The proposed algorithm is illustrated in Figure 12.

Example 8. Consider a multinational cyber security consultancy firm who has to make a decision about the classification of a given zero-day anomaly. A zero-day malware/anomaly is an anomaly/malware one that is either unknown or not yet addressed by anyone. There is a team of domain experts in the research lab of that consultancy firm to investigate the problem and make a decision regarding the classification, i.e., malicious or benign (not malicious) of a given sample.

Initially, assume there are four samples to be examined. S_1: Windows-based PE Sample # 1, S_2: Windows-based PE Sample # 2, S_3: Windows-based PE Sample # 3, and S_4: Windows-based PE Sample # 4. In order to correctly analyze the samples, their attributes/features have to be extracted. So for this purpose, there are four features to be extracted and analyzed: A_1: entropy of the PE sample, A_2: resource size of the PE sample, A_3: virtual Size of the PE sample, and A_4: section mean entropy of the PE sample. All the four attributes are most significant ones for anomaly detection. Let $w = (0.23, 0.35, 0.26, 0.16)^T$ be the weight vector and represent their opinions in IVTSFNs format. The stepwise demonstration of the process is as follows:

Step 1: in the first step, the analyst examines the given samples S_i and extracts the four attributes/features including entropy, resource size, virtual size, and section mean entropy. After the extraction of the aforementioned four attributes (A_1, A_2, A_3, and A_4) from given PE samples (S_1, S_2, S_3, and S_4), a decision matrix $D_{4\times4}$ is constituted. The representation of attributes (A_1, A_2, A_3, and A_4) of PE samples (S_1, S_2, S_3, and S_4) in matrix $D_{4\times4}$ is given as in Table 3:

The assessment analysis of individual score against the threshold shows that Score of S_3 is greater than given criteria, i.e., 0.5; therefore, only S_3 among all samples is declared as malicious, i.e., anomaly. Moreover, as the score of S_1, S_2, and S_4 is below the defined threshold criteria, i.e., 0.5; therefore, they will be classified as benign in nature. Hence, it is shown that how can a critical and complex problem; i.e., Anomaly Detection can be solved using the proposed intelligent and robust decision making approach.

The classification decision of the given sample (s) either as malicious or benign is carried out by using the IVTSFWA operators. Only for $q = 5$ values are IVTSFNs.

Step 2: the IVTSFWA operators to the matrix gives the following output:

$$\begin{aligned}
S_1 &= \mathrm{IVTSFWA}(S_{11}, S_{12}, S_{13}, S_{14}) = ([0.4698, 0.8600], [0.2700, 0.4000], [0.2833, 0.3866]),\\
S_2 &= \mathrm{IVTSFWA}(S_{21}, S_{22}, S_{23}, S_{24}) = ([0.3836, 0.7655], [0.2703, 0.5060], [0.4257, 0.5579]),\\
S_3 &= \mathrm{IVTSFWA}(S_{31}, S_{32}, S_{33}, S_{34}) = ([0.7470, 0.9000], [0.0000, 0.2382], [0.2229, 0.3336]),
\end{aligned} \tag{23}$$

Figure 10: Strong interval-valued T-spherical fuzzy graph.

Figure 11: IV T-spherical fuzzy graph.

(1) Analyst will perform the extraction of features/attributes (A) from the given PE sample (S) and constitute a decision Matrix.
(2) Put on the following aggregation tools to the attribute matrix obtained in Step 1.
IVTSFWA$(I_1, I_2, I_3 \ldots I_m) = ([\sqrt[q]{1 - \prod_{j=1}^{m} (1 - (\mathring{s}_j^l)^q)^{w_j}}, \sqrt[q]{1 - \prod_{j=1}^{m} (1 - (\mathring{s}_j^u)^q)^{w_j}}], [\prod_{j=1}^{m} (i_j^l)^{w_j}, \prod_{j=1}^{m} (i_j^u)^{w_j}], [\prod_{j=1}^{m} (d_j^l)^{w_j}, \prod_{j=1}^{m} (d_j^u)^{w_j}])$,
where I1, I2, I3 . . .Im are interval-valued T-spherical fuzzy numbers.
(3) Calculate the rank/score of IVTSFNs acquired in the previous step.
(4) Analyze the score of the given PE sample, i.e., SC (S) takes the classification decision based upon the following threshold based criteria:
SC (Sample) > 0.5 then Classification = M i.e. Malicious else Classification = B i.e. Benign

Algorithm 1: Algorithm for malware detection.

(1) Formation of relation R_1: $\alpha \longrightarrow \beta$
(2) Formation of relation R_2: $\beta \longrightarrow \gamma$
(3) Computation of R_3: $\alpha \longrightarrow \gamma$ as a composition of R_1 and R_2
(4) Investigation of diagnosis for patients using the score function

Algorithm 2: Algorithm for medical diagnosis.

Figure 12: Flowchart describing the proposed anomaly detection process.

Step 3: score values are computed as follows:

$$\begin{aligned}
&\mathrm{SC}(S_1) = 0.2855, SC(S_2) = 0.1551, \mathrm{SC}(S_3) = 0.6429, \text{and}, \mathrm{SC}(S_4) = 0.1269,\\
&\mathrm{SC}(S_1) < 0.5;\ \mathrm{SC}(S_2) < 0.5,\\
&\quad S_4 = \mathrm{IVTSFWA}(S_{41}, S_{42}, S_{43}, S_{44}) = ([0.2090, 0.7814], [0.3818, 0.5595], [0.3426, 0.4288]),\\
&\mathrm{SC}(S_3) > 0.5.
\end{aligned} \tag{24}$$

Table 3: Decision matrix: anomaly detection.

$$= \begin{pmatrix} & A_1 & A_2 & A_3 & A_5 \\ S_1 & \begin{pmatrix} [0.3, 0.9], \\ [0.3, 0.4], \\ [0.2, 0.3] \end{pmatrix} & \begin{pmatrix} [0.4, 0.8], \\ [0.3, 0.4], \\ [0.4, 0.5] \end{pmatrix} & \begin{pmatrix} [0.5, 0.9], \\ [0.2, 0.4], \\ [0.3, 0.4] \end{pmatrix} & \begin{pmatrix} [0.6, 0.8], \\ [0.3, 0.4], \\ [0.2, 0.3] \end{pmatrix} \\ S_2 & \begin{pmatrix} [0.2, 0.5], \\ [0.4, 0.6], \\ [0.7, 0.8] \end{pmatrix} & \begin{pmatrix} [0.4, 0.8], \\ [0.3, 0.6], \\ [0.5, 0.8] \end{pmatrix} & \begin{pmatrix} [0.3, 0.8], \\ [0.2, 0.4], \\ [0.2, 0.2] \end{pmatrix} & \begin{pmatrix} [0.8, 0.9], \\ [0.0, 0.1], \\ [0.5, 0.6] \end{pmatrix} \\ S_3 & \begin{pmatrix} [0.8, 0.9], \\ [0.0, 0.2], \\ [0.2, 0.3] \end{pmatrix} & \begin{pmatrix} [0.7, 0.9], \\ [0.1, 0.2], \\ [0.3, 0.4] \end{pmatrix} & \begin{pmatrix} [0.7, 0.9], \\ [0.5, 0.6], \\ [0.1, 0.2] \end{pmatrix} & \begin{pmatrix} [0.8, 0.9], \\ [0.0, 0.1], \\ [0.5, 0.6] \end{pmatrix} \\ S_4 & \begin{pmatrix} [0.1, 0.2], \\ [0.6, 0.7], \\ [0.3, 0.4] \end{pmatrix} & \begin{pmatrix} [0.2, 0.8], \\ [0.3, 0.6], \\ [0.3, 0.3] \end{pmatrix} & \begin{pmatrix} [0.1, 0.9], \\ [0.3, 0.4], \\ [0.5, 0.6] \end{pmatrix} & \begin{pmatrix} [0.3, 0.4], \\ [0.5, 0.6], \\ [0.3, 0.6] \end{pmatrix} \end{pmatrix}$$

6.1.1. Comparative Analysis. The resolution of comparative assessment reiterates that none of the existing structures including FSs, IVFSs, IFSs, IVIFSs, PyFSs, IVPyFSs, q-OFS, IVq-OFS, PFSs, and IVPFSs processed the data except the proposed approach and its effectiveness is demonstrated by solving the similar problem using the existing IVq-OFNs method in Table 4.

In this section, previously discussed four similar Windows based PE samples are used. The decision is required regarding their anomaly classification, i.e., either as malicious or benign effectively addressed using the aggregation operators of IVq-OFNs by taking $\mathbf{q} = 2$ and $\mathbf{i}^{\mathbf{l}} = \mathbf{i}^{\mathbf{u}} = \mathbf{i} = 0$. The IVTSFWA operator for $\mathbf{q} = 2$ and $\mathbf{i}^{\mathbf{l}} = \mathbf{i}^{\mathbf{u}} = \mathbf{i} = 0$ becomes as follows:

$$= \left(\left[\sqrt[q]{1 - \prod_{j=1}^{m} \left(1 - \left(\mathfrak{s}_j^l\right)^q\right)^{w_j}}, \sqrt[q]{1 - \prod_{j=1}^{m} \left(1 - \left(\mathfrak{s}_j^u\right)^q\right)^{w_j}} \right], \left[\prod_{j=1}^{m} \left(d_j^l\right)^{w_j}, \prod_{j=1}^{m} \left(d_j^u\right)^{w_j} \right] \right) \tag{25}$$

The aggregation results are as follows:

$$\begin{aligned} S_1 &= \mathrm{IVTSFWA}\left(S_{11}, S_{12}, S_{13}, S_{14}\right) = ([0.4698, 0.8600], [0.2833, 0.3866]), \\ S_2 &= \mathrm{IVTSFWA}\left(S_{21}, S_{22}, S_{23}, S_{24}\right) = ([0.3836, 0.7655], [0.4257, 0.5579]), \\ S_3 &= \mathrm{IVTSFWA}\left(S_{31}, S_{32}, S_{33}, S_{34}\right) = ([0.7470, 0.9000], [0.2229, 0.3336]), \\ S_4 &= \mathrm{IVTSFWA}\left(S_{41}, S_{42}, S_{43}, S_{44}\right) = ([0.2090, 0.7918], [0.3426, 0.4288]). \end{aligned} \tag{26}$$

Using the score function, we get the following:

$$\begin{aligned} SC\left(S_1\right) = 0.5175, SC\left(S_2\right) = 0.3223, SC\left(S_3\right) \\ = 0.8902, \text{ and } SC\left(S_4\right) = 0.2944. \end{aligned} \tag{27}$$

The analysis of each given sample is summarized as follows:

$$\begin{aligned} \mathrm{SC}\left(S_1\right) > 0.5;\ \mathrm{SC}\left(S_3\right) > 0.5, \\ \mathrm{SC}\left(S_2\right) < 0.5, \mathrm{SC}\left(S_4\right) < 0.5. \end{aligned} \tag{28}$$

The assessment analysis of individual scores shows that Score of S_1 and S_3 is greater than given criteria, i.e., 0.5; therefore only S_1 and S_3 samples are declared as Malicious file, i.e., anomaly. Moreover, as the Score of S_2 and S_4 is below the defined threshold, i.e., 0.5; therefore, they will be classified as benign.

Comparing the respective score of all samples in both sections reveals an instance of wrong judgement about sample S_1. The classification of anomaly sample S_1 by using the IVq-OFNs method is Malicious which was previously classified as benign. The sample S_1 is actually benign. This occurrence of wrong judgement about sample S_1 is also known as False Positive; i.e., a benign PE file is classified as malicious. Therefore, the proposed claim using IVTSFWA operators is a worthy concept.

6.2. Medical Diagnosis. In this subsection, the process of medical diagnosis is carried out in the context of IVTSFSs. The medical diagnosis process based on TSFSs is established in [6] and proved to be of great importance as IFSs and PFSs failed to process such type of information. As IVTSFS is a generalization of TSFSs and IVIFSs so we used the approach of IVTSFRs to handle medical diagnosis problem.

In a medical diagnosis problem, some information is obtained about patients and symptoms and symptoms and diagnosis in terms of IVTSFRs showing the relation between them. Then, with the help of the composition of IVTSFRs, a relation between patient and diagnosis is obtained which is further analyzed using the score function for IVTSFSs. The proposed medical diagnosis problem is demonstrated through the following algorithm where α, β and γ represents a set of patients, symptoms, and diagnosis, respectively.

The proposed algorithm is depicted in Figure 13.

Example 9. The example that we are discussing here is adapted from [6], and instead of using TSFNs, we used

Table 4: Comparative analysis: decision matrix in the form of IVq-OFNs.

$$D4 \times 4 = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} \begin{pmatrix} [0.3, 0.9], \\ [0.2, 0.3] \end{pmatrix} & \begin{pmatrix} [0.4, 0.8], \\ [0.4, 0.5] \end{pmatrix} & \begin{pmatrix} [0.5, 0.9], \\ [0.3, 0.4] \end{pmatrix} & \begin{pmatrix} [0.6, 0.8], \\ [0.2, 0.3] \end{pmatrix} \\ \begin{pmatrix} [0.2, 0.5], \\ [0.7, 0.8] \end{pmatrix} & \begin{pmatrix} [0.4, 0.8], \\ [0.5, 0.8] \end{pmatrix} & \begin{pmatrix} [0.3, 0.8], \\ [0.2, 0.2] \end{pmatrix} & \begin{pmatrix} [0.5, 0.8], \\ [0.5, 0.8] \end{pmatrix} \\ \begin{pmatrix} [0.8, 0.9], \\ [0.2, 0.3] \end{pmatrix} & \begin{pmatrix} [0.7, 0.9], \\ [0.3, 0.4] \end{pmatrix} & \begin{pmatrix} [0.7, 0.9], \\ [0.1, 0.2] \end{pmatrix} & \begin{pmatrix} [0.8, 0.9], \\ [0.5, 0.6] \end{pmatrix} \\ \begin{pmatrix} [0.1, 0.2], \\ [0.3, 0.4] \end{pmatrix} & \begin{pmatrix} [0.2, 0.8], \\ [0.3, 0.3] \end{pmatrix} & \begin{pmatrix} [0.1, 0.9], \\ [0.5, 0.6] \end{pmatrix} & \begin{pmatrix} [0.3, 0.4], \\ [0.3, 0.6] \end{pmatrix} \end{pmatrix} \end{array}$$

Figure 13: Flowchart describing the proposed medical diagnosis process.

IVTSFNs. All the data is here being in the form of IVTSFNs, where $\alpha = \{\alpha_1 = \text{Amjad}, \alpha_2 = Ali, \alpha_3 = \text{Zeeshan}, \alpha_4 = \text{Naeem}\}$, $\beta = \{\beta_1 = \text{temperature}, \beta_2 = \text{headache}, \beta_3 = \text{cough}, \beta_4 = \text{chest pain}, \beta_5 = \text{flu}\}$, and $\gamma = \{\gamma_1 = \text{malaria}, \gamma_2 = \text{typhoid}, \gamma_3 = \text{allergy}, \gamma_4 = \text{heart problem}, \gamma_5 = \text{viral disease}\}$ denote the sets of patients, symptoms, and diagnosis, respectively. The detailed stepwise demonstration is as follows:

(1) Formation of relation R_1: $\alpha \longrightarrow \beta$ (see Table 5)

(2) Formation of relation R_2: $\beta \longrightarrow \gamma$ (see Table 6)

(3) Computation of R_3: $\alpha \longrightarrow \gamma$ as a composition of R_1 and R_2

(4) Investigation of diagnosis for patients using the score function

The scores of Table 8 represent that α_1 and α_4 , i.e., Amjad and Naeem have allergy problems. Zeeshan having symptoms of malaria and typhoid, while Ali is a patient of malaria too, as indicated by his symptoms.

6.2.1. Comparative Analysis. In this subsection, our aim is to show that if we consider the structures of FSs, IVFSs, IFSs, IVIFSs, PyFSs, IVPyFSs, PFSs, and IVPFS, they all will turn out be a fail case. Conversely, if we have the data in the existing structures, we are able to deal with it using the operations of IVTSFSs. For example, if we convert the data available in Tables 5 and 6 in the form of TSFNs by choosing MGs as a single number from $[0, 1]$ interval instead of closed subinterval of $[0, 1]$, then we will obtain the following relations shown in Table 9:

Now, the data provided in Tables 9 and 10 are purely in the form of TSFNs as described in [6]. This type of data can be shifted into equivalent IVTSFRs of closed subintervals without disturbing the grades.

The information in Tables 9 and 10 is similar to that in Tables 11 and 12. The process of medical diagnosis proposed in an intuitionistic and picture fuzzy environment [43, 51] can also be carried out using the proposed approach. Hence, we can use the tools of IVTSFSs to solve problems of TSFSs and consequently IVPFSs, PFSs, IVIFSs, IFSs, IVFSs, and FSs.

6.3. Shortest Path (SP) Problem. It is one of the important applications of graph theory and got great attention in past decades. A SP from a source to a destination node is required in most of the fields of engineering and other sciences. As briefly explained in the first section of this manuscript, the problem is highly valued in different fuzzy algebraic structures and several new approaches are developed to resolve it. We are going to follow the famous DA for finding an appropriate path problem in the context of IVTSFG. A directed network of finite nodes and edges is considered to find the SP form a source node (SN) to destination node (DN) based on alternative nodes and edges, respectively. The edges denoted the path from one vertex to the adjacent vertex. In real-life, the shortest paths mean a path having low cost to travel on or a path that required less time to reach a destination or a path having the least distance between SN and DN and the information about these parameters is provided in the context of IVTSFNs. The detailed algorithm for finding SP using DA is further described.

6.3.1. Dijkstra Algorithm (DA) for Finding SP in Interval-Valued T-Spherical Fuzzy Environment. DA is a baseline algorithm for the subject problem and for some quality work on fuzzy SP using DA, one may refer to [44–48]. Here, we

Table 5: Interval-valued T-spherical fuzzy relation showing the relation between patients and symptoms.

R_1	β_1	β_2	β_3	β_4	β_5
α_1	$\begin{pmatrix}[0.4,0.7],\\[0.1,0.4],\\[0.4,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.5],\\[0.1,0.4],\\[0.7,0.9]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.7],\\[0.6,0.8],\\[0.5,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.8],\\[0.0,0.4],\\[0.4,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.0,0.3],\\[0.7,0.9],\\[0.5,0.8]\end{pmatrix}$
α_2	$\begin{pmatrix}[0.6,0.9],\\[0.1,0.3],\\[0.0,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\[0.0,0.3],\\[0.1,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.0,0.2],\\[0.3,0.6],\\[0.5,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.4],\\[0.1,0.3],\\[0.2,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.3,0.6],\\[0.4,0.7],\\[0.5,0.9]\end{pmatrix}$
α_3	$\begin{pmatrix}[0.2,0.6],\\[0.3,0.6],\\[0.3,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.4],\\[0.0,0.3],\\[0.1,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.8],\\[0.3,0.6],\\[0.0,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.0,0.1],\\[0.6,0.9],\\[0.2,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.8],\\[0.1,0.3],\\[0.2,0.5]\end{pmatrix}$
α_4	$\begin{pmatrix}[0.0,0.3],\\[0.1,0.3],\\[0.4,0.9]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.7],\\[0.4,0.5],\\[0.3,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.9],\\[0.2,0.6],\\[0.1,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.6,1.0],\\[0.0,0.5],\\[0.0,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.5],\\[0.2,0.8],\\[0.5,1.0]\end{pmatrix}$

Table 6: Interval-valued T-spherical fuzzy relation showing the relation between symptoms and diagnosis.

R_2	γ_1	γ_2	γ_3	γ_4	γ_5
β_1	$\begin{pmatrix}[0.3,0.6],\\[0.1,0.5],\\[0.2,0.7]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\[0.3,0.7],\\[0.1,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.6],\\[0.0,0.1],\\[0.0,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.6],\\[0.2,0.5],\\[0.7,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.6],\\[0.1,0.4],\\[0.4,0.9]\end{pmatrix}$
β_2	$\begin{pmatrix}[0.6,0.7],\\[0.0,0.3],\\[0.0,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.0,0.6],\\[0.0,0.3],\\[0.2,0.7]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.8],\\[0.1,0.3],\\[0.2,0.7]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.7],\\[0.5,0.8],\\[0.3,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.8],\\[0.2,0.5],\\[0.6,0.9]\end{pmatrix}$
β_3	$\begin{pmatrix}[0.1,0.5],\\[0.1,0.4],\\[0.4,0.9]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.2],\\[0.0,0.4],\\[0.3,0.9]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.5],\\[0.2,0.7],\\[0.3,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.4,1.0],\\[0.0,0.6],\\[0.0,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.3],\\[0.2,0.5],\\[0.3,0.7]\end{pmatrix}$
β_4	$\begin{pmatrix}[0.0,0.4],\\[0.0,0.2],\\[0.4,1.0]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\[0.2,0.7],\\[0.0,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.8,0.9],\\[0.0,0.2],\\[0.1,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.0,0.3],\\[0.3,0.9],\\[0.5,0.9]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.8],\\[0.1,0.5],\\[0.0,0.4]\end{pmatrix}$
β_5	$\begin{pmatrix}[0.4,0.7],\\[0.1,0.5],\\[0.3,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.5],\\[0.1,0.7],\\[0.5,0.7]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.5],\\[0.3,0.7],\\[0.4,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.9],\\[0.2,0.4],\\[0.1,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.7],\\[0.0,0.4],\\[0.6,1.0]\end{pmatrix}$

discussed the DA for the information provided as IVTSFNs. The IVTSF modified DA [46] (IVTSFDA) is described as follows:

(1) Identify the source as permanent node (P) and assign it the label ([0, 0], [0, 0], [1, 1], −) as by default this node is included in low cost path and distance covered at this stage is 0.

(2) For every node j whose path is from node i, compute the label $[v_i \oplus d_{ij}, i]$ if j is not a permanent node. Further, if j is identified as $[v_j, k]$ via some other node, then replace $[v_j, k]$ by $[v_i \oplus d_{ij}, i]$ only if $\mathrm{SC}(v_i \oplus d_{ij})$ is less than $\mathrm{SC}(v_j)$.

(3) The algorithm terminates if all the nodes are permanently labeled. Else, select $[v_r, s]$ having low cost distance v_r and reiterate Step 2 by making $i = r$.

(4) Find the low cost path from SN to DN.

The flowchart is portrayed in Figure 14.

Remarks 2. The label $[v_i \oplus d_{ij}, i]$ means that we reached from node i and covered a distance $v_i \oplus d_{ij}$. Furthermore, it is

Table 7: Composition of two interval-valued T-spherical fuzzy relations in Tables 5 and 6 showing the diagnosis for a patient.

R_3	γ_1	γ_2	γ_3	γ_4	γ_5
α_1	$\begin{pmatrix}[0.4,0.6],\\[0.0,0.2],\\[0.4,0.7]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\[0.0,0.3],\\[0.4,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.8],\\[0.0,0.1],\\[0.4,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.7],\\[0.0,0.4],\\[0.5,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.8],\\[0.0,0.4],\\[0.4,0.6]\end{pmatrix}$
α_2	$\begin{pmatrix}[0.4,0.6],\\[0.0,0.2],\\[0.1,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\[0.0,0.3],\\[0.1,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\[0.0,0.1],\\[0.0,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.6],\\[0.0,0.3],\\[0.3,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.6],\\[0.0,0.3],\\[0.2,0.5]\end{pmatrix}$
α_3	$\begin{pmatrix}[0.4,0.7],\\[0.0,0.2],\\[0.1,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.6],\\[0.0,0.3],\\[0.2,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\[0.0,0.1],\\[0.2,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.8],\\[0.0,0.3],\\[0.0,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.7],\\[0.0,0.3],\\[0.2,0.5]\end{pmatrix}$
α_4	$\begin{pmatrix}[0.2,0.7],\\[0.0,0.2],\\[0.3,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\[0.0,0.3],\\[0.0,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.9],\\[0.0,0.1],\\[0.1,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.9],\\[0.0,0.3],\\[0.1,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.8],\\[0.0,0.3],\\[0.0,0.4]\end{pmatrix}$

Table 8: Score values of the data provided in Table 7.

R	γ_1	γ_2	γ_3	γ_4	γ_5
α_1	0.58	0.58	0.78	0.55	0.56
α_2	0.52	0.48	0.48	0.11	0.28
α_3	0.51	0.28	0.56	0.48	0.29
α_4	0.26	0.50	0.81	0.25	0.48

Table 9: T-spherical fuzzy relation showing the relation between patients and symptoms.

R_1	β_1	β_2	B_3	β_4	β_5
α_1	(0.7, 0.1, 0.4	(0.5, 0.1, 0.8)	(0.4, 0.6, 0.5)	(0.8, 0, 0.4)	(0, 0.8, 0.5)
α_2	(0.9, 0.1, 0)	(0.5, 0, 0.2)	(0, 0.6, 0.8)	(0.3, 0.3, 0.3)	(0.5, 0.5, 0.7)
α_3	(0.4, 0.4, 0.4)	(0.2, 0.1, 0.2)	(0.7, 0.4, 0.2)	(0, 0.8, 0.4)	(0.7, 0.3, 0.3)
α_4	(0, 0.1, 0.6)	(0.5, 0.5, 0.5)	(0.9, 0.4, 0.1)	(1, 0, 0)	(0.2, 0.5, 0.7)

Table 10: T-spherical fuzzy relation showing the relation between symptoms and diagnosis.

R_1	γ_1	γ_2	γ_3	γ_4	γ_5
β_1	(0.4, 0.3, 0.4)	(0.5, 0.5, 0.3)	(0.6, 0, 0.1)	(0.4, 0.3, 0.7)	(0.3, 0.3, 0.6)
β_2	(0.7, 0.1, 0.2)	(0, 0, 0.5)	(0.5, 0.1, 0.6)	(0.4, 0.6, 0.6)	(0.1, 0.3, 0.7)
β_3	(0.4, 0.2, 0.7)	(0.2, 0, 0.8)	(0.5, 0.2, 0.4)	(1, 0, 0)	(0.7, 0.4, 0.6)
β_4	(0, 0, 1)	(0.45, 0.4, 0)	(0.9, 0, 0.1)	(0.1, 0.6, 0.77)	(0.7, 0.2, 0.1)
β_5	(0.7, 0.1, 0.4)	(0.41, 0.3, 0.6)	(0.5, 0.5, 0.5)	(0.76, 0.3, 0.2)	(0.2, 0, 0.81)

important to note that we cannot proceed to a permeant node but can go reverse. For two directly connected adjacent nodes i and j respectively, node i is considered as the predecessor of node j if the path connecting them is directed from i to j.

Example 10. Consider Figure 15 where a network of 6 nodes with 8 edges is portrayed. Our aim is to apply modified DA to this network and find the low cost path from SN (N_1) to DN (N_2).

The edges involved in this network are listed in Table 13.

The step-by-step computations of modified DA is as follows:

Step 1. Identify node 1 as permanent due to its lowest cost.

Step 2. The ways directed from node 1 are also possible; i.e., we may either move to node 2 or to node 3. List of nodes is in Table 14.

Now we compute the scores of ([0.1, 0.3], [0.2, 0.6], [0.2, 0.8]) and ([0.3, 0.5], [0.2, 0.5], [0.5, 0.7]).

Table 11: Equivalent to Table 9.

R_1	β_1	β_2	β_3	β_4	β_5
α_1	$\begin{pmatrix}[0.7,0.7],\\[0.1,0.7],\\[0.4,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.5],\\[0.1,0.1],\\[0.8,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.4],\\[0.6,0.6],\\[0.5,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.8,0.8],\\[0.0,0.0],\\[0.4,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.0,0.0],\\[0.8,0.8],\\[0.5,0.5]\end{pmatrix}$
α_2	$\begin{pmatrix}[0.9,0.9],\\[0.1,0.1],\\[0.0,0.0]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.5],\\[0.0,0.0],\\[0.2,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.0,0.0],\\[0.6,0.6],\\[0.8,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.3,0.3],\\[0.3,0.3],\\[0.3,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.5],\\[0.5,0.5],\\[0.7,0.7]\end{pmatrix}$
α_3	$\begin{pmatrix}[0.4,0.4],\\[0.4,0.4],\\[0.4,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.2],\\[0.1,0.1],\\[0.2,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.7,0.7],\\[0.4,0.4],\\[0.2,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.0,0.0],\\[0.8,0.8],\\[0.4,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.7,0.7],\\[0.3,0.3],\\[0.3,0.3]\end{pmatrix}$
α_4	$\begin{pmatrix}[0.0,0.0],\\[0.1,0.1]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.5],\\[0.5,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.9,0.9],\\[0.4,0.4],\end{pmatrix}$	$\begin{pmatrix}[1.0,1.0],\\[0.0,0.0],\end{pmatrix}$	$\begin{pmatrix}[0.2,0.2],\\[0.5,0.5],\end{pmatrix}$

Table 12: Equivalent to Table 10.

R_2	γ_1	γ_2	γ_3	γ_4	γ_5
β_1	$\begin{pmatrix}[0.4,0.4],\\[0.3,0.3],\\[0.4,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.5],\\[0.5,0.5],\\[0.3,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.6],\\[0.0,0.0],\\[0.1,0.1]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.4],\\[0.3,0.3],\\[0.7,0.7]\end{pmatrix}$	$\begin{pmatrix}[0.3,0.3],\\[0.3,0.3],\\[0.6,0.6]\end{pmatrix}$
β_2	$\begin{pmatrix}[0.7,0.7],\\[0.1,0.1],\\[0.2,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.0,0.0],\\[0.0,0.0],\\[0.5,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.5],\\[0.1,0.1],\\[0.6,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.4],\\[0.6,0.6],\\[0.6,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.1],\\[0.3,0.3],\\[0.7,0.7]\end{pmatrix}$
β_3	$\begin{pmatrix}[0.4,0.4],\\[0.2,0.2],\\[0.7,0.7]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.2],\\[0.0,0.0],\\[0.8,0.8]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.5],\\[0.2,0.2],\\[0.4,0.4]\end{pmatrix}$	$\begin{pmatrix}[1.0,1.0],\\[0.0,0.0],\\[0.0,0.0]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.1],\\[0.4,0.4],\\[0.6,0.6]\end{pmatrix}$
β_4	$\begin{pmatrix}[0.0,0.0],\\[0.0,0.0],\\[1.0,1.0]\end{pmatrix}$	$\begin{pmatrix}[0.45,0.45],\\[0.4,0.4],\\[0.0,0.0]\end{pmatrix}$	$\begin{pmatrix}[0.9,0.9],\\[0.0,0.0],\\[0.1,0.1]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.1],\\[0.6,0.6],\\[0.7,0.7]\end{pmatrix}$	$\begin{pmatrix}[0.7,0.7],\\[0.2,0.2],\\[0.1,0.1]\end{pmatrix}$
β_5	$\begin{pmatrix}[0.7,0.7],\\[0.1,0.1],\\[0.4,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.41,0.41],\\[0.3,0.3],\\[0.6,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.5],\\[0.5,0.5],\\[0.5,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.76,0.76],\\[0.3,0.3],\\[0.2,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.2,0.2],\\[0.0,0.0],\\[0.81,0.81]\end{pmatrix}$

$$\begin{aligned}&\mathrm{SC}([0.1,\ 0.3],\ [0.2,\ 0.6],\ [0.2,0.8]) = 0.016667,\\&\mathrm{SC}([0.3,\ 0.5],\ [0.2,\ 0.5],\ [0.5,0.7]) = 0.042967.\end{aligned} \tag{29}$$

As the score of $([0.1, 0.3], [0.2, 0.6], [0.2, 0.8])$ is less than $([0.3, 0.5], [0.2, 0.5], [0.5, 0.7])$, we mark node 3 as $(([0.1, 0.3], [0.2, 0.6], [0.2, 0.8]), N_1)$ and labeled it permanent.

Step 3. Also, dual routes directed from node 3, i.e., we may either move to node 4 or to node 5. List of nodes is in Table 15.

Now, we compute the scores of $([0.009324, 0.120246], [0.02, 0.24], [0.0, 0.24])$ and $([0.000122, 0.019715], [0.08, 0.48], [0.14, 0.72])$ as follows:

$$\begin{aligned}&\mathrm{SC}([0.009324,\ 0.120246],\ [0.02,\ 0.24],\ [0.0,0.24]) = 0.063472,\\&\mathrm{SC}([0.000122,0.019715],\ [0.08,\ 0.48],\ [0.14,0.72]) = 0.035482.\end{aligned} \tag{30}$$

As the score of $([0.000122, 0.019715], [0.08, 0.48], [0.14, 0.72])$ is less than $([0.009324, 0.120246], [0.02, 0.24], [0.0, 0.24])$, therefore, we mark the node 5 as $(([0.000122, 0.019715], [0.08, 0.48], [0.14, 0.72]), N_3)$ and labeled it permanent.

Step 4. One way from node 5 exists only, i.e., we can only move to node 6 depicted in Table 16.

As there is only one way from nodes 5 to 6. Therefore, we mark node 6 as $(([0.002604, 0.007776], [0.016, 0.24], [0.028, 0.216]), N_5)$ and labeled it permanent.

Figure 14: Flowchart of modified DA for computing low cost path.

Figure 15: Interval valued T-spherical fuzzy network.

Step 5. Nodes 2 and 4 are the remaining temporary nodes, so their status is changed to permanent and depicted in Table 17.

Step 6. From Table 18, the lowest cost path pattern is from SN to DN, i.e., from nodes 1 to 6.

The SP computed as per modified DA is as follows:

$$N_1 \longrightarrow N_3 \longrightarrow N_5 \longrightarrow N_6. \tag{31}$$

6.3.2. Comparative Study. Consider a network based IVIF environment portrayed in Figure 16. All the path values are

Table 13: Weights of edges.

Edges	Interval-valued T-spherical distances
$(\mathbf{N}_1, \mathbf{N}_2)$	$([0.1, 0.3], [0.2, 0.6], [0.2, 0.8])$
$(\mathbf{N}_1, \mathbf{N}_3)$	$([0.3, 0.5], [0.2, 0.5], [0.5, 0.7])$
$(\mathbf{N}_2, \mathbf{N}_3)$	$([0.3, 0.8], [0.2, 0.4], [0.3, 0.8])$
$(\mathbf{N}_2, \mathbf{N}_5)$	$([0.5, 0.9], [0.1, 0.6], [0.3, 0.8])$
$(\mathbf{N}_3, \mathbf{N}_4)$	$([0.3, 0.7], [0.1, 0.4], [0.0, 0.3])$
$(\mathbf{N}_3, \mathbf{N}_5)$	$([0.3, 0.7], [0.4, 0.8], [0.7, 0.9])$
$(\mathbf{N}_4, \mathbf{N}_6)$	$([0.3, 0.5], [0.1, 0.4], [0.4, 0.8])$
$(\mathbf{N}_5, \mathbf{N}_6)$	$([0.5, 0.6], [0.2, 0.5], [0.2, 0.3])$

Table 14: List of nodes.

Nodes	Label	Status
$\mathbf{N}_1$	$(([0, 0], [0, 0], [1, 1]), -)$	Permanent
$\mathbf{N}_2$	$(([0.3, 0.5], [0.2, 0.5], [0.5, 0.7]), \mathbf{N}_1)$	Temporary
$\mathbf{N}_3$	$(([0.1, 0.3], [0.2, 0.6], [0.2, 0.8]), \mathbf{N}_1)$	Temporary

Table 15: List of nodes.

Nodes	Label	Status
$\mathbf{N}_1$	$(([0, 0], [0, 0], [1, 1]), -)$	Permanent
$\mathbf{N}_2$	$(([0.3, 0.5], [0.2, 0.5], [0.5, 0.7]), \mathbf{N}_1)$	Temporary
$\mathbf{N}_3$	$(([0.1, 0.3], [0.2, 0.6], [0.2, 0.8]), \mathbf{N}_1)$	Permanent
$\mathbf{N}_4$	$(([0.009324, 0.120246], [0.02, 0.24], [0.0, 0.24]), \mathbf{N}_3)$	Temporary
$\mathbf{N}_5$	$(([0.000122, 0.019715], [0.08, 0.48], [0.14, 0.72]), \mathbf{N}_3)$	Temporary

Table 16: List of nodes.

Nodes	Label	Status
$\mathbf{N}_1$	$(([0, 0], [0, 0], [1, 1]), -)$	Permanent
$\mathbf{N}_2$	$(([0.3, 0.5], [0.2, 0.5], [0.5, 0.7]), \mathbf{N}_1)$	Temporary
$\mathbf{N}_3$	$(([0.1, 0.3], [0.2, 0.6], [0.2, 0.8]), \mathbf{N}_1)$	Permanent
$\mathbf{N}_4$	$(([0.009324, 0.120246], [0.02, 0.24], [0.0, 0.24]), \mathbf{N}_3)$	Temporary
$\mathbf{N}_5$	$(([0.000122, 0.019715], [0.08, 0.48], [0.14, 0.72]), \mathbf{N}_3)$	Permanent
$\mathbf{N}_6$	$(([0.002604, 0.007776], [0.016, 0.24], [0.028, 0.216]), \mathbf{N}_5)$	Permanent

Table 17: List of nodes.

Nodes	Label	Status
$\mathbf{N}_1$	$(([0, 0], [0, 0], [1, 1]), -)$	Permanent
$\mathbf{N}_2$	$(([0.3, 0.5], [0.2, 0.5], [0.5, 0.7]), \mathbf{N}_1)$	Permanent
$\mathbf{N}_3$	$(([0.1, 0.3], [0.2, 0.6], [0.2, 0.8]), \mathbf{N}_1)$	Permanent
$\mathbf{N}_4$	$(([0.009324, 0.120246], [0.02, 0.24], [0.0, 0.24]), \mathbf{N}_3)$	Permanent
$\mathbf{N}_5$	$(([0.000122, 0.019715], [0.08, 0.48], [0.14, 0.72]), \mathbf{N}_3)$	Permanent
$\mathbf{N}_6$	$(([0.002604, 0.007776], [0.016, 0.24], [0.028, 0.216]), \mathbf{N}_5)$	Permanent

Table 18: List of nodes.

$\mathbf{N}_6$	$(([0.002604, 0.007776], [0.016, 0.24], [0.028, 0.216]), \mathbf{N}_5)$
$\mathbf{N}_5$	$(([0.000122, 0.019715], [0.08, 0.48], [0.14, 0.72]), \mathbf{N}_3)$
$\mathbf{N}_3$	$(([0.1, 0.3], [0.2, 0.6], [0.2, 0.8]), \mathbf{N}_1)$

Figure 16: Interval-valued intuitionistic fuzzy network.

Figure 17: Interval-valued fuzzy network.

in the form of IVIFNs that can be converted into IVTSFNs if we insert the abstinence grade [0.0, 0.0] in it and therefore by using the proposed approach of modified DA, we can find the SP.

Now, consider a network depicted in Figure 17, where all the path values are in the form of IVFNs. As an IVFN can be regarded as IVTSFN, if we insert [0.0, 0.0] in place of abstinence and nonmembership grade. Therefore, using the proposed approach of modified DA, we can solve a low cost path problem.

7. Conclusion

In this manuscript, a new concept of IVTSFSs which generalizes all the existing concepts like FSs, IVFSs, IFSs, IVIFSs, PyFSs, IVPyFS, PFSs, IVPFSs, SFSs, and TSFSs, is proposed. Such a concept is necessary when we have four types of opinion and information is not exact; i.e., information is provided in the form of intervals. Similarly, we developed the concept of IVTSFG which generalizes all the existing definitions of graph theory so far existed. The basic operations and related terms of IVTSFSs and IVTSFGs are defined with their properties. We utilized the aggregation operators of IVTSFSs in anomaly detection problems which is a challenging problem from computer science. We defined relations for IVTSFSs along with their compositions and applied them to solve anomaly detection and medical diagnosis problem. We also proposed a modified DA to solve a low cost path problem in a network based on IVTSFGs. The comparative evaluation demonstrated that the proposed structure is novel and addressed the shortcomings of the existing ideas. In the future, we shall try to develop some aggregation operators for IVTSFSs to deal with a scenario where the other tools fail to be applied.

Data Availability

All data, models, or codes generated or used during the study are available in a repository or online in accordance with funder data retention policies (full citations that include URLs or DOIs are provided).

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally to the manuscript.

Acknowledgments

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Research Article

Disturbance Elimination for the Modular Joint Torque Sensor of a Collaborative Robot

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The ability to obtain accurate torque information and detect collision information is the key demand for collaborative robots. Given the highly integrated modular joint, the joint torque-sensing system is disturbed by other elements during online measurement, affecting the accuracy of measurement. For example, the torque ripple caused by the periodic deformation of the flexspline of the harmonic driver and the temperature drift due to the heating of motor are the two main factors affecting the measurement of the torque sensor. In this study, a torque sensor suitable for a modular joint is developed. Moreover, a strain gauge configuration based on a specified phase difference for torque ripple cancelation and a temperature compensation method based on the combination of the Wheatstone bridge and a nonlinear model are proposed. Finally, the performance of the developed sensor is experimentally evaluated.

1. Introduction

Collaborative robots can be remarkably applied in various industries, for example, medical treatment, 3C electronics, and public service [1, 2]. To avoid injury from robots, the collision detection technology of robots has attracted extensive attention. This technology based on a visual sensor needs to collect and process a high volume of real-time data [3]. Collision detection based on a motor current should predict interference and assembly error model to accurately estimate torque [4]. To resolve these problems, many scholars have embedded a joint torque sensor into robot joints [5, 6]. In addition, the method based on strain gauges to measure force or moment is more mature and reliable, and this method is favored by researchers [7, 8].

At present, the harmonic drive is widely used in collaborative robots because of its high gear-reduction ratio and high torque ability in a compact geometry. During the operation of the harmonic driver, the elliptical wave generator facilitates the periodic deformation of the flexspline. This process couples the output of the torque sensor embedded in the robot joint with the periodic noise, which is also known as torque ripple [9]. Taghirad designed a Kalman filter to compensate for the torque ripple [10]. However, this method needs to estimate the model of the torque ripple in advance. Jung adopted the order tracking method based on resampling to eliminate the torque ripple [11]. However, the resampling method is only applicable when the position changes. Considering that the torque ripple mainly originates from the deformation of the flexspline, Hashimoto balanced the torque ripple by pasting strain gauges on the flexspline and forming the Wheatstone bridge [12]. This method avoids numerous calculations and improves response speed, but strain gauges should be pasted on the bottom of the flexspline in a certain configuration. In addition, the torsional stiffness of the flexspline should also be known. To reduce the placement error of the strain gauges and torque ripple, an elastomer with mutually perpendicular spokes and pasted strain gauges has been designed by adding two signals from two sets of stain gauges to eliminate the torque ripple [13, 14]. However, this method needs two sets of signal conditioning circuits.

During operation of the robot, the heat generated by the motor is easily transmitted to the torque sensor through the metal transmission shaft and metal flange. This process creates a temperature drift in the output signal of the torque

sensor. The Wheatstone bridge is commonly used in the joint torque measurement with strain gauge [12, 15]. However, when strain gauges are heated unevenly, this technique deteriorates the sensing performance. Francisco studied the temperature drift of a six-axis force torque sensor based on the strain gauge by assuming that the temperature drift of this sensor has a linear relationship with six-axis force. In this study, the calibration matrix and temperature drift coefficient of the sensor are estimated simultaneously using linear regression based on the least square method [16]. Sun modeled the temperature drift of a six-axis force torque sensor of a space manipulator by using the least square support vector machine and particle swarm optimization algorithm to optimize the parameters of the model [17]. However, the software compensation depends on the accuracy of the model. Moreover, the more accurate the model is, the more the calculation needed is, and this process introduces transmission delay. Therefore, to balance the accuracy and delay of the sensor, hardware and software are combined to compensate for the temperature drift.

In this study, a joint torque sensor embedded in a modular joint, which includes a motor, a harmonic drive, a torque sensor, and an encoder, is proposed to mainly address how a robot obtains accurate torque-sensing ability. The main research contents are as follows. A torque ripple cancelation method, which combines a specially designed elastic body and the Wheatstone bridge in a special form, is proposed. Thus, the periodic disturbance from the harmonic drive can be compensated. In addition, the temperature compensation method based on the combined hardware and software is proposed. Another Wheatstone bridge is distributed symmetrically with respect to the first Wheatstone bridge to cancel the torque ripple. A nonlinear model is also proposed to further reduce the error from the temperature drift.

This paper is organized as follows. The design of the torque sensor and the theoretical analysis of the torque ripple cancelation, temperature compensation, and collision detection are discussed in Section 2. In Section 3, the experimental setups for the developed torque sensor are presented. The evaluation results of the torque ripple cancelation, the temperature drift compensation, and collision detection are addressed in Section 4. Finally, the conclusion is provided in Section 5.

2. Proposed Approach

2.1. Problem Statement. A collaborative robot is a robot that can directly and physically interact with a human operator in a shared workspace. For example, a collaborative robot with surgery tool and several doctors perform an eye or orthopedic surgery together. For safety requirements, one of the most basic characteristics of a collaborative robot is the estimation of external force or collision detection. Industrial robots measure the external force by using a six-axis force torque sensor, which is only suitable for a force/torque acting on the robot end-effector. However, a torque sensor arranged on the robot joint can estimate the force acting on most areas of the robot. Therefore, a joint torque sensor is one of the necessary hardware devices for a collaborative robot. To facilitate repair and reconfiguration, collaborative robots need modular joints, which connect adjacent links and integrate the drive, sensing, and control.

A modular joint unit usually consists of a motor, a harmonic drive, a joint torque sensor, a joint encoder, and other modules. The shaft of the motor is directly connected to the wave generator of the harmonic driver, while the joint torque sensor is connected to the flexspline through the connector. As shown in Figure 1, the periodic deformation of the flexspline propagates to the joint torque sensor through the connector. This process mixes the output of the joint torque sensor with the torque ripple. The motor is the main heat source while the modular joint is used. The heat will be transferred to the joint torque sensor through the harmonic drive and connector, causing the output of the joint torque sensor to drift, even if the external load of the torque sensor is constant. Thus, for accurate collision detection, the design of the sensor should cancel the torque ripple and the temperature drift.

2.2. Design of Joint Torque Sensor. As shown in Figure 2, the elastic body of the sensor is designed as the spoke type; that is, the inner and outer rings are connected by spokes. The slit on the spoke makes the strain of the sensor equivalent to that of the spoke. Given the idle space outside the spoke, a signal processing circuit can be built in this space to make the sensor compact and thinner. In addition, four protective beams are added. When the sensor is overloaded, the protective beam will prevent irreversible damage to the sensor.

Based on the proposed torque sensor design, the finite element method (FEM) is conducted using ANSYS. The maximum sensing torque is 70 Nm, and the maximum load that the torque sensor can endure is set to 180 Nm. The maximum strain is 0.003 mm as calculated from the design of the conditioning circuit. The 6061 aluminum alloy is selected to fabricate the sensor because of its small elastic coefficient and large yield strength. The coordination of the torque sensor is shown in Figure 3. The applied load is $Mx = 180$ Nm. The structural parameters of the spoke as calculated by FEM are shown in Table 1. The deformation of the body is illustrated in Figure 4. The analytical results are used to specify the location of the strain gauge.

The mechanical strain of the torque sensor can be measured by using the strain gauge forming the Wheatstone bridge. However, the voltage signal of the Wheatstone bridge is too weak to collect. Hence, the weak electrical signal should be amplified and filtered. The analog signal is further converted into digital form to read the torque signal on the chip. In addition, a temperature-measuring circuit is installed to compensate for the temperature drift. A conditioning circuit board is designed and consists of an ADA4528 amplifier with a factor of 200, an RC low pass filter to cancel the noise signal, and an AD7790 with 16 bit buffered sigma-delta ADC to convert the analog signal to digital signal. The AD7790 transmits the digital signal to a microcontroller STM32F103T8 through the SPI

Figure 1: Expanded view of the developed modular joint.

Figure 2: Mechanical structure of the sensor.

Figure 3: Axis of the torque sensor for FEM analysis.

Table 1: Design parameters of the torque sensor by FEM.

Quantity	Length (mm)	Width (mm)	Height (mm)
Spoke	10	12	6
Slit	8	1.6	6

communication protocol. The microcontroller can directly collect the temperature signal by its ADC. The schematic diagram of the sensor circuit is shown in Figure 5, and the developed torque sensor is displayed in Figure 6.

2.3. Torque Ripple Cancelation. As shown in Figure 7(a), for each revolution of the elliptical wave generator of the harmonic drive, the flexspline produces two cycles of deformation, resulting in the torque ripple depicted by curve A in Figure 7(b). As shown in Figure 7(b), the periodic interference can be eliminated by adding the torque ripple curve A to the other curve $\overline{A}$ with the same amplitude and frequency but with phase difference of $\pi/2$.

To cancel the torque ripple, four strain gauges $R_1 - R_4$ are mounted on the optimum location at both sides of the spoke to collect the strain signals (Figure 8(a)) and form the Wheatstone bridge (Figure 8(b)). The theoretical analysis is as follows.

When a clockwise torque T is applied to the sensor, the odd-numbered strain gauges are under tension, while the even-numbered strain gauges are under compression. Strains ε_1 and ε_2 measured by gauges R_1 and R_2, respectively, can be expressed as follows:

$$\begin{cases} \varepsilon_1 = \varepsilon_t + \varepsilon_w, \\ \varepsilon_2 = -\varepsilon_t + \varepsilon_{w'}, \end{cases} \tag{1}$$

where ε_t is the tensile strain caused by the applied torque and ε_w is the tensile strain due to the deformation of the flexspline. To obtain the information of ε_t, the difference between strains ε_1 and ε_2 is written as follows:

Figure 4: Deformation behavior of the torque sensor.

Figure 5: Schematic circuit diagram of sensor.

Figure 6: Developed torque sensor.

$$\varepsilon_1 - \varepsilon_2 = 2\varepsilon_t + \Psi_0 \sin(2\beta), \tag{2}$$

where the difference $\varepsilon_\omega - \varepsilon_{\omega'}$ is assumed to be sinusoidal as explained by Hashimoto [12] and β is the rotational angle of the wave generator to the reference point.

To measure strain ε_t and eliminate $\Psi_0 \sin(2\beta)$, another signal $\overline{A}$ as mentioned above should be obtained. Gauges R_3 and R_4 are on the spoke rotating at 90° from the position of R_1 and R_2. Signal $\overline{A}$ can be obtained by the difference of strain ε_3 and ε_4, which are measured by gauges R_3 and R_4, respectively.

$$\varepsilon_3 - \varepsilon_4 = 2\varepsilon_t + \Psi_0 \sin(2\beta - \pi). \tag{3}$$

Equations (2) and (3) can be added to cancel the sinusoid component as follows:

$$\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 = 4\varepsilon_t. \tag{4}$$

Thus, the Wheatstone bridge composed of R_1, R_2, R_3, and R_4 can generate a voltage signal proportional to strain ε_t.

$$U_o = \left(\frac{K}{4}\right)(\varepsilon_1 + \varepsilon_3 - \varepsilon_2 - \varepsilon_4)U_i = K\varepsilon_t U_i, \tag{5}$$

where K is the strain coefficient of the gauges, and U_i and U_o represent the input and output voltages of the Wheatstone bridge.

If the bridge is subject to the same change in temperature, the strain due to temperature is canceled out. To minimize the sticking error and enhance the temperature

Figure 7: Relationship between the rotation of the wave generator and torque ripple: (a) deformation of the flexspline and (b) elimination of the torque ripple.

Figure 8: Sticking position of the strain gauge and the Wheatstone bridge circuit for joint torque sensing: (a) configuration of the strain gauges and the (b) Wheatstone bridge.

compensation, another four strain gauges, $R_5 - R_8$, are introduced to maintain the symmetry with $R_1 - R_4$, as shown in Figures 8(a) and 8(b). In this study, the sticking process of the strain gauges is carried out under a microscope with the scale display, minimizing the error caused by sticking the strain gauges.

2.4. Temperature Compensation Method. The Wheatstone bridge circuit, which is composed of eight strain gauges, can eliminate theoretically the effect of temperature. However, due to the distinct change in temperature of each strain gauge, accurate torque information is difficult to obtain if only the circuit is used. Therefore, on the basis of hardware compensation, this study adopts software to compensate for the temperature drift. The specific method is to measure the elastic body temperature with a built-in thermistor. Then, the experimental data are obtained to solve the temperature compensation model based on the least square method.

The change in the temperature of the sensor can be measured by the thermistor pasted on the body of the sensor. The temperature-measuring circuit and the pasting position of the thermistor are shown in Figure 9.

Figure 9: Temperature-measuring method: (a) thermistor measuring circuit and (b) pasting position of the thermistor.

According to Figure 9(a), the resistance of the thermistor can be calculated as follows:

$$R_t = \frac{V_{adc} R_a}{\mathrm{VCC} - V_{adc}}, \tag{6}$$

where R_a is the voltage divider; R_t is the thermistor; R_b and C_1 constitute the RC low pass filter circuit; and V_{adc} is the input voltage of the analog-to-digital converter.

In addition, the temperature of the thermistor related to the resistance of thermistor can be expressed as follows:

$$T_t = \frac{\mathrm{BT_{nom}}}{T_{nom}(\ln R_t - \ln R_{nom}) + B}, \tag{7}$$

where R_{nom} is the nominal resistance of the thermistor at 25°C; B is the known parameter of the thermistor; T_{nom} is the Kelvin temperature corresponding to 25°C; and T_t is the measurement temperature described by the Kelvin temperature. Substituting R_t as given in (6) into (7), the current temperature of the sensor can be easily calculated.

The applied torque of the sensor related to the output voltage of the Wheatstone bridge can be expressed as follows:

$$\tau = kV + b, \tag{8}$$

where k is the sensor sensitivity independent of temperature and b is the temperature drift that can be modeled as a second-degree polynomial related to temperature T as follows:

$$b = A_0 + A_1 T + A_2 T^2. \tag{9}$$

Substituting (9) into (8), the output of the sensor can be presented as follows:

$$\tau = kV + A_0 + A_1 T + A_2 T^2. \tag{10}$$

When the applied torque of the sensor is 0, the temperature drift model of the sensor can be obtained as follows:

$$V = -\left(\frac{A_0}{k}\right) - \left(\frac{A_1 T}{k}\right) - \left(\frac{A_2 T^2}{k}\right) = A_0' + A_1' T + A_2' T^2. \tag{11}$$

To acquire an accurate torque information, parameters A_0', A_1', and A_2' should be estimated. In this study, polynomial fitting is used to solve the parameters by using at least three sets of experimental data.

2.5. Collision Detection Method. To reflect the comprehensive performance of the joint torque sensor, the proposed torque sensor is applied to the collision detection, which is one of the most typical applications of a collaborative robot. The schematic diagram of collision detection is shown in Figure 10.

The dynamic of a robot system with n degrees of freedom can be modeled as follows:

$$\tau_m + \tau_c - \tau_f = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta), \tag{12}$$

where θ, $\dot{\theta}$, and $\ddot{\theta} \in R^n$ are the vectors of the generalized joint coordinates, velocities, and acceleration, respectively; $M(\theta) \in R^{n\times n}$ is the joint-space inertia matrix; $C(\theta, \dot{\theta}) \in R^{n\times n}$ is the Coriolis and centripetal coupling matrix; $G(\theta) \in R^n$ is the gravity loading; $\tau_f \in R^n$ is the friction force; $\tau_m \in R^n$ is the vector of generalized actuator force associated with the generalized coordinate θ; and $\tau_c \in R^n$ is the joint force due to the external torque.

Based on the robot dynamic characteristic that $\dot{M}(\theta) - 2C(\theta, \dot{\theta})$ is a skew symmetric matrix, the following equation is obtained:

$$\dot{M}(\theta) = C(\theta, \dot{\theta}) + C^T(\theta, \dot{\theta}). \tag{13}$$

To estimate the external torque, the generalized momentum of robot is modeled as follows:

$$p = M(\theta)\dot{\theta}. \tag{14}$$

Figure 10: Schematic diagram of collision detection.

Given the time derivative of p,

$$\dot{p} = M(\theta)\ddot{\theta} + C^T(\theta, \dot{\theta})\dot{\theta} + C(\theta, \dot{\theta})\dot{\theta}, \tag{15}$$

and substituting (15) into (12) yields

$$\tau_c = -\tau_m + \dot{p} - C^T(\theta, \dot{\theta})\dot{\theta} + g(\theta) + \tau_f. \tag{16}$$

The observer about joint force τ_c can be designed as follows:

$$\dot{r} = K_O(\tau_c - r), \tag{17}$$

where $K_O = \mathrm{diag}(k_{o,i}) > 0$ is a diagonal gain matrix. Substituting (16) into (17) and then integrating (17), the observer $r(t)$ with initial condition $r(0) = 0$ can be expressed as follows:

$$r(t) = K_O\left[p(t) - \int_0^t \left(\tau_m + C^T(\theta, \dot{\theta})\dot{\theta} - g(\theta) - \tau_f + r\right)\mathrm{du} - p(0)\right]. \tag{18}$$

The Laplace transform of (17) can be expressed as follows:

$$r(s) = \frac{Ko}{Ko + s}\tau_c(s). \tag{19}$$

Equation (19) is a first-order low pass filter. Hence,

$$K_O \longrightarrow \infty \Rightarrow r \approx \tau_c. \tag{20}$$

One of the basic characteristics of the first-order low pass filter is that high-frequency noise signal will not pass, and a greater value of K_O can obtain a faster step response. This good characteristic makes the generalized momentum observer a good virtual force sensor, which is used to measure the external force acting on the robot.

Given the errors of dynamic modeling and friction identification, $r(t)$ cannot be accurately calculated. Thus, the estimation $\hat{r}(t)$ is used in this experiment [18].

$$\hat{r}(t) = K_O\left[\hat{p}(t) - \int_0^t \left(\hat{\tau}_m + \hat{C}^T(\theta, \dot{\theta})\dot{\theta} - \hat{\tau}_f - \hat{g}(\theta) + \hat{r}\right)\mathrm{du} - \hat{p}(0)\right], \tag{21}$$

where $\hat{p}(t)$ is the estimated robot-generalized momentum at time $t \geq 0$; $\hat{\tau}_m$ is the measured torque; $\hat{\tau}_f$ is the friction torque; the matrix $\hat{C}^T(\theta, \dot{\theta})$ comprises the Coriolis and centrifugal terms; and $\hat{g}(\theta)$ is the gravity vector.

The collision of a robot is detected by the observer algorithm by setting the collision detection threshold as follows:

$$|\hat{r}(t)| > \Upsilon, \tag{22}$$

where $\Upsilon \in R^n$ is the vector of the collision detection threshold. In practical application, the threshold cannot be too small. Otherwise, misjudgment will occur. If the threshold is too high, the robot becomes less sensitive to collision. The threshold is minimized as error and noise are considered.

3. Experimental Setups

3.1. Calibration Setup. The dynamic calibration of the joint torque sensor is performed on the developed automatic calibration system, which includes a dynamic calibration platform and an application program based on LabVIEW. As shown in Figure 11, the dynamic calibration platform involves a magnetic powder brake, a slip-ring, a reference sensor (TS11-200-B, INTERFACE), the developed sensor, the developed modular joint, a potentiometer, a data acquisition device, an amplifier, a power supply, and a connecting part. The developed sensor is connected to the reference sensor by the connecting part. The application program collects the torque signal of the reference sensor through the data acquisition device and the output voltage signal and temperature signal of the developed sensor via RS485. Thus, the real-time drawing function of the signals is realized.

During calibration, the modular joint moves at a constant speed, and the brake force of the magnetic powder brake is controlled by adjusting the output voltage of the potentiometer to obtain the desired applied torque read from the reference torque. In this setup, a torque interval from 0 Nm to 70 Nm is applied, which is equal to the peak torque of the motor magnified by the harmonic drive.

3.2. Collision Detection Setup. An experiment system is used to examine the collision detection method (Figure 12). The system consists of the developed modular joint, the developed torque sensor, a motor driver (Golo Solo Whistle series, Elmo), a motor controller (Platinum Maestro, Elmo), and a one-link arm. The dynamics of the arm can be modeled as follows.

Figure 11: Developed dynamic calibration system.

Figure 12: Experiment setup for collision detection.

$$\tau = \left(\mathrm{ml}^2 + J\right)\ddot{q} + \mathrm{mgl}\sin(q), \tag{23}$$

where q and $\ddot{q}$ represent the arm angle and acceleration, respectively; m is the arm quality; l is the distance from the center of mass to the axis of the joint; J is the arm inertial; and τ is the measured torque from the torque sensor. The identified parameters of the dynamic model are listed in Table 2.

Table 2: Parameters of the one-link arm.

$\mathrm{ml}^2 + \mathrm{J}$ ($\mathrm{kg} \cdot \mathrm{m}^2$)	mgl (Nm)
0.0096	0.5022

4. Results and Discussion

4.1. Evaluation of the Torque Ripple Cancelation. Two of the developed sensors are needed to verify the effectiveness of the torque ripple cancelation method. One sensor adopts the strain gauge configuration as shown in Figure 13, while the other sensor adopts the strain gauge configuration as shown in Figure 8. The former is the type without torque ripple cancelation, and the latter has torque ripple cancelation. The modular joint is equipped with these two sensors. Then, the modular joint is rotated at a velocity of 35.16°/s without load. The sensor outputs of these two configurations are collected

Figure 13: Common strain gauge configuration.

separately, and then fast Fourier transform (FFT) is conducted.

Figure 14(a) shows that the voltage curve representing the torque ripple is fluctuating. Figure 14(b) demonstrates that the curve of the torque ripple becomes flatter, and its amplitude is almost equal to the temperature drift. In addition, Figure 15 shows the FFT plot of both torque signals. Figure 15(a) shows two peaks at 19.76 Hz and at 9.89 Hz, which correspond to the torque ripple. Figure 15(b) displays the FFT plot of the signal with the torque ripple cancelation. No peaks are observed at 19.76 Hz and 9.89 Hz. Therefore, the torque ripple is clearly reduced. This method does not need any additional software filtering algorithm and will not cause delay of the torque signal.

4.2. Evaluation of Temperature Compensation. The temperature drift of the sensor is caused by the increase in temperature rather than a drop in temperature [16]. Therefore, in this paper, the hysteresis of the temperature drift is ignored, and only the case of temperature rise is considered. To verify the temperature compensation method, the temperature of the sensor is between 20°C and 45°C with an increment of 5°C. Thus, the output of the sensor varies with the change in temperature. The output data are collected to fit the temperature compensation model by using the method described in Section 2.4. Figure 16 displays the relationship between the temperature and the output voltage of the sensor. The measured temperature and the voltage showed a nonlinear relationship. In this study, a second-degree polynomial curve is applied to fit the nonlinear curve. The fit equation can be expressed as follows:

$$b(T) = 0.0102T^2 - 0.8704T - 21.39. \tag{24}$$

To evaluate the effectiveness of the temperature compensation method, the sensor output before and after the compensation is collected and compared. As shown in Figure 17, the output torque of the sensor increases with temperature before the compensation, and the maximum value is 0.38 Nm. After the compensation, the output torque decreases apparently, and the maximum error is 0.08 Nm.

4.3. Calibration. During calibration, the modular joint moves at 2°/s in a counterclockwise direction, and the brake force of the magnetic powder brake is controlled to apply the desired torque that ranges between 0 Nm and 70 Nm with an increment of 5 Nm on the reference and developed sensors. The output data are collected from the reference and developed sensors, respectively. As shown in Figure 18, the x-axis corresponds to the voltage data from the developed sensor, and the y-axis corresponds to the applied torque measured by the reference torque sensor. The measured voltage and the applied torque show a linear relationship. In this study, a first-degree polynomial curve is applied to fit the linear curve. The fit equation can be expressed as follows:

$$\tau = 62.738x + 2.1835. \tag{25}$$

Therefore, the nonlinear error of the torque sensor is 0.878%, and the hysteresis is 0.903%. The sensitivity is $0.0627\ \mathrm{Nm} \cdot (\mathrm{mV})^{-1}$, and the temperature drift is −34.8 mV. The performance of the developed torque sensor is compared with that of the commercialized torque sensor M2210C1 (Sunrise Instruments). The comparative results are provided in Table 3. M2210C1 shows better performance than the developed sensor. However, because the output signal of M2210C1 is an analog signal, high-frequency noise can be mixed with this signal. In addition, the developed sensor is lighter than M2210C1 and can be calibrated online with the modular joint to improve sensing accuracy.

4.4. Evaluation of Collision Detection. To verify intuitively the effectiveness of the collision detection method described in Section 2.5, the experimental setup shown in Figure 19 is adopted. An egg is erected between the one-link arm and the

(a)

(b)

Figure 14: Experimental result of the modular joint at a constant velocity (a) before and (b) after torque ripple compensation.

(a)

Figure 15: Continued.

(b)

Figure 15: Fast Fourier transformation (FFT) results: FFT plots (a) without and (b) with torque ripple compensation.

Figure 16: Temperature drift fitting.

Figure 17: Evaluation of the temperature drift compensation.

Figure 18: Measured torque and fitting process.

obstacle. Then, the arm is controlled to collide horizontally with the egg. The obstacle limits the displacement of the egg after impact. When the impact torque is detected, the movement of the arm stops. In this experiment, the identified dynamic parameters (Table 2) of the arm are used for the observer of the external force, and the observer gain coefficient K_o is equal to 100. The speed signal is filtered, and the sampling frequency is set to 100 Hz. To set a reasonable collision detection threshold, the arm moves in different directions at distinct speeds without an external force acting on it. The estimated torque of the observer satisfies $|\hat{\tau}(0)| < 0.0255$ Nm. Thus, the collision detection threshold is set to 0.03 Nm.

The movement velocity of the arm is between 30°(s) and 60°(s). Figure 20 displays the estimated impact torque with no broken egg. These phenomena show that the designed joint torque sensor can effectively detect collision with short delay.

Table 3: Performance of the commercial and developed torque sensors.

Quantity	Height (mm)	Diameter (mm)	Weight (g)	Sensing range (Nm)	Nonlinearity (%)	Hysteresis (%)	Output type
M2210C1	7	80	180	100	0.5	1	Analog
Developed sensor	10	86	60.6	70	0.878	0.903	Digital

Figure 19: Experimental setup for the collision detection with egg.

Figure 20: Collision torque detection.

5. Conclusion and Future Work

In this paper, a torque sensor sensing technique is introduced for the developed modular joint. Based on theoretical and experimental research, the torque ripple is clearly reduced by using a spoke-type sensor body and a special strain gauge configuration. In addition, the temperature drift is compensated by using the combination of the Wheatstone bridge and the proposed temperature drift model.

The performance of the developed torque sensor is compared with that of a commercial sensor. The commercial torque sensor, namely, M2210C1, shows better resolution in its torque-sensing ability. However, the output voltage of this sensor is an analog signal, and, compared with the digital output, electromagnetic noise can be easily introduced to the analog signal. In addition, the developed sensor is lighter than M2210C1 and can be calibrated online with the modular joint to overcome the sensing error caused by assembly. Furthermore, the developed modular joint and the proposed torque sensor show good collision detection performance.

In this study, only a modular joint is employed to assess the torque ripple cancelation and the temperature drift compensation. A collaborative robot with multiple degrees of freedom will be evaluated in future studies.

Data Availability

As the project will not be finished until May 2022 and we have confidentiality agreement with the State Key Laboratory of Robotics and System, the data could not be released so far. For any information about the article, please contact us via zhanghe0451@hit.edu.cn.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Formation Control Algorithm of Agents Based on Earth Mover's Distance

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Massive sport, such as unmanned aerial vehicle performance, often needs fast and efficient calculation of formation morphing and individual path planning. This paper introduces a novel fast formation control method of a crowd. First, we get the agents' location in a 2D polygon with centroidal Voronoi tessellation and L-BFGS techniques. Then, we transform crowd formation shapes with a global shortest motion path pair assignment using earth mover's distance algorithm. Finally, the repulsing force between agents and obstacles is calculated based on the recursive velocity observer method control agents' motion. Extensive experimental results show the effectiveness and usefulness of our algorithm in 2D group formation transformation.

1. Introduction

Crowd animation simulation, which is a fundamental problem in computer graphics, augmented, virtual, and mixed reality, has received much attention in recent years [1, 2]. With the rapid development of computer hardware and software technology, especially the continuous improvement of GPU rendering technology, simulating animation of a massive number of virtual agents on a limited computing capability hardware is a problem that has recently become quite popular and crowd formation control has become one of the most important issues in the research field of Multirobot System (MRS). It plays an important role in a wide range of applications, including industry, military affairs, aviation, military drills, urban planning, art and sports, and video games. Especially, the manufacturing industry has an increasing demand for automatic processing of large objects or dangerous goods. In the military, the power and command efficiency can be improved by maintaining the military equipment or the army as reasonable and appropriate formation by the group simulation system. Agents usually interact with the user, and it not only improves operation efficiency but also ensures human safety. With the help of virtual agents, we can effectively complete the task and ensure the safety of humans in dangerous activities, such as exploring unknown environments, demining, patrolling, and reconnaissance. However, the crowd formation transform is still quite limited in terms of both agent number and individual intelligence. Designing and rehearsal of group formation are tedious and time-consuming; the crowd formation transformation simulation system is an effective solution for this problem. Through team formation control, formation design, and crowd simulation, the simulation system can combine with art and sports perfectly.

There are three major classes of algorithms for crowd formation control, namely, formation generation, formation maintenance, and formation transformation. The main difficulties are collision avoidance and path planning. If the motion matching between individuals is not accurate, the collision avoidance between individuals is difficult to be realized. It is complex and unrealistic to use artificial methods to realize the matching. Leader-follower dynamic technique [3] uses a linearization method to design a velocity

controller for transforming, but it depends on leader agents. When the leader follower fails, the formation will be out of control. Zheng et al. [4] proposed a group formation control method for formation based on geometric constraints, which can be applied to perform agent pairing, path planning, and collision avoidance for the crowd; the calculating process still needs a tedious iterative step and is time-consuming in real-time simulation.

To form a uniform group formation distribution, efficient calculation of individual motion matching, and smooth formation transformation, we propose a novel group formation transform approach to automatically generate a homogeneously distributed crowd and visually pleasing formation transformation. The group is uniformly sampled by the centroid Voronoi structure centroidal Voronoi tessellation (CVT). Based on the earth mover's distance (EMD) algorithm, we precompute a global shortest motion path assignment in the offline process. During the real-time transformation simulation, our contributions are summarized as follows:

(1) We introduce a novel group formation morphing approach that can automatically generate a global shortest motion path pair assignment based on geometric constraints

(2) We introduce a new method to enable a swarm of insects to follow the motion of a moving shape by applying EMD method

2. Related Work

2.1. Group Formation Morphing. Group formation control is a vital problem of many crowds and multiagent systems. Based on the Velocity Obstacle concept [5], Van den Berg et al. [6] proposed the concept of Reciprocal Velocity Obstacles for real-time multiagent navigation among autonomous decision-making entities, and it is widely used in the crowd simulation system. Anderson et al. [7] introduced a flocks animation simulation method with shape constraints to improve the motion concerning the underlying behavioral model. Ji et al. [8] proposed a method to simulate the movement modeling, path planning, and collision avoidance, which provided an auxiliary tool for the group calisthenics to improve the design of the formation and pattern. Zheng et al. [9] adopt geometric constraint idea for smooth formation animation of regulated crowds. Chen et al. [10] presented a robust multiagent model for large-scale controllable shape constrained simulation of flying insects. Wang et al. [11] proposed a linear formation control method under directed and switching topologies for distributed robot systems. Zheng et al. [4] proposed a geometric constraint group formation control method for group calisthenics, which is combined morphing, Centroidal Voronoi Tessellation, and Lloyd techniques and it is an efficient solution for individual distribution layout, individual matching, path planning, and collision avoidance. Furthermore, Zheng et al. [12] introduced a regulated heterogeneous flock formation control method based on a capacity-constrained centroidal power diagram to model the internal layouts of groups in the constraint shapes. Some researchers convert the formation problem into tracking problem, use the pseudorigid body method to calculate the path of the expected point of the agent, and use the reverse step method to design the formation control of the nonlinear control algorithm [13]. Based on backstepping control and bioheuristic neural network, Ding et al. [14] used a trajectory tracking method for three-dimensional formation control and obstacle avoidance. Henry et al. [15, 16] proposed a crowd control algorithm using deformed grids for single traversal search, and it can be used to control crowd interactive with the environment and collision avoidance. Xu et al. [1] proposed a natural group transformation and control method between source shape and target shape based on Kuhn–Munkres and mutual information method, which can be used for the interaction and collision avoidance of between individuals and environments.

2.2. Voronoi Diagram. In the centroidal Voronoi diagram, the distribution of seeds is very uniform. Due to these properties, centroidal Voronoi graphs can be used in a wide range of applications, including image compression, numerical integration, finite difference, location problem, and meshing [17]. The Voronoi diagram is suitable for not only uniform sampling but also sampling with constraints; it can be used to calculate a distribution as uniform as possible and compute region partition with capacity constraint. Based on the quasi-Newton method, Liu et al. [18] proposed a superlinear convergence algorithm for centroidal Voronoi diagram and successfully applied it to the important problem of remeshing. Yan et al. [19] proposed a method to compute the centroidal Voronoi diagram limited to the curved surface, which improved the efficiency of remeshing. After that, the centroidal Voronoi diagram has been widely studied in digital geometry. Lévy and Liu [20] introduced a centroidal Voronoi diagram method based on p-norm and demonstrated its application in quadrilateral meshing. Pan et al. [21] used the centroidal Voronoi diagram to construct grid surfaces with constant mean curvature. Wang et al. [22] proposed a fast algorithm for calculating the Riemannian center and the center of mass for any geodesic Voronoi diagram. Recently, Liu et al. [23] further improved the calculating speed of the geodesic centroid Voronoi diagram based on the evolutionary algorithm. To compute the Voronoi diagram with capacity constraint, Sorkine et al. [24] and Zheng et al. [25, 26] proposed some new optimization strategies, respectively, and their algorithms have obvious improvement in computing efficiency. The distribution problem with a capacity constraint can be interpreted by the optimal transmission theory, Desbrun et al. [27] provided a first-order convergent optimization algorithm and found that the blue noise generated by the capacity constraint has better spectral characteristics. Xin et al. [28] named this problem as the centroid power diagram problem and proposed an optimization algorithm with superlinear convergence.

3. Our Methods

Figure 1 presents an overview of the pipeline of our approach. Specifically, it can be divided into four phases: (1) sampling number calculation, (2) group distribution

Figure 1: The phases of our crowd formation transform approach.

sampling, (3) pair assignment precalculation, and (4) movement control phase. The sampling number calculation phase calculates the number of samples for each shape in a multishape formation. The group distribution sampling phase computes the sampling position based on the centroid Voronoi diagram. Pair assignment precalculation phase formulates the globally shortest path matching as a linear assignment optimization problem. The movement control phase combines target-oriented motion and interaction force between nearest agents to ensure collision-free movements for all the individual agents in real time.

3.1. Sampling Number Calculation. For the source and target formations, we first convert the formation shapes to Delaunay triangulation (DT) representations [29] in the interior of the shape. Then, we calculate the area s_k of each triangulated subshape k for a formation with multiple shapes. Finally, we use the followed formula to calculate the number n_k of sampled agents in each triangulated subshape:

$$n_k = \frac{s_k}{\sum_{i=1}^{m} s_i} \times N, \tag{1}$$

where N is the total number of sampled agents and m is the number of subshapes.

3.2. Group Distribution Sampling. The Voronoi diagram is a partitioning of a given domain $\Omega \subset E^d$ into regions based on distance to a set of points $\{p_i \subset E^d\}_{i=1}^{n}$ in a specific subset of the domain. That set of seeds is specified beforehand, and the region containing point p_i is called the Voronoi cells; for each seed, there is a Voronoi cell consisting of all points closer to that seed than to any other. $P = \{p_1, \ldots, p_n\}$ represents the discrete point set on the domain Ω; the Voronoi cell of the site p_i can be denoted as follows:

$$\Omega_i = \left\{ x \in \Omega \colon \|x - P\|_i \le \left\| x - P_j \right\|, \quad j = 1, \ldots, n, i \ne j \right\}. \tag{2}$$

Let the domain Ω be endowed with a density function $\rho(x) > 0$ which is bounded; then, the centroid c_i of each Voronoi cell Ω_i is given by

$$c_i = \frac{\int_{\Omega_i} \rho(x) x \mathrm{d}t}{\int_{\Omega_i} \rho(x) x}, \tag{3}$$

that is, the points c_i that serve as generators for the Voronoi regions Ω_i are themselves the mass centroids of those regions. We call such a region a centroidal Voronoi diagram. Then, the CVT energy function E of the Voronoi tessellation Ω_i is defined as [17]

$$E(x_i, \Omega_i) = \sum_{i=1}^{n} \int_{\Omega_i} \rho(x) \left\| x - p_i \right\|^2 \mathrm{d}x. \tag{4}$$

In this case, minimizing the energy function E, we can get all the centroids c_i of the tessellation which directly ensures that each position p_i is a representative of c_i. Liu et al. [18] and Xin et al. [28] use gradient descent method to calculate the center of CVT rapidly; the gradient of the energy with regard to x_i is defined as

$$\nabla E = \frac{\partial E}{\partial x_i} = 2(x_i - c_i) \int_{\Omega_i} \rho(x) \mathrm{d}x, \tag{5}$$

where c_i is the centroid and $\int_{\Omega_i} \rho(x) \mathrm{d}x$ is the mass of Voronoi regions Ω_i.

It is worth noting that all point sampling is located inside of shape using the above CVT method. However, groups who are strictly distributed according to user-designed shapes need to evenly distribute a part of the sampling on the shape boundary to achieve a more orderly crowd distribution in crowd formation transform simulation. Therefore, the sampling points of the group formation need to be calculated in two parts. In the first part, you need to sample m points on the boundary; namely, it needs to generate m points on the boundary $\partial\Omega$ of the shape and the Voronoi region of $\partial\Omega$ denoted as V_i. The remaining k of the sampling points need to be distributed in the internal region $\partial\overline{\Omega}$ of shape and W_j represents the Voronoi region of $\partial\overline{\Omega}$, as Figure 2 shows. The whole energy function of CVT is defined as

$$\begin{aligned} E(x_i, \Omega_i) = &\sum_{i=1}^{m} \int_{x \in V_i} \rho_1(x) \left\| x - p_i \right\|^2 \mathrm{d}x \\ &+ \sum_{j=1}^{k} \int_{x \in W_j} \rho_2(x) \left\| x - p_i \right\|^2 \mathrm{d}x, \end{aligned} \tag{6}$$

whose gradient with regard to x_i is

$$\begin{aligned} \nabla E = \frac{\partial E}{\partial x_i} = &2(x_i - c_i) \int_{x \in V_i} \rho_1(x) \mathrm{d}x + 2(x_i - c_j) \\ &\cdot \int_{x \in W_j} \rho_2(x) \mathrm{d}x, \end{aligned} \tag{7}$$

where $K = N - \mathrm{sign}(\lfloor s/l \rfloor) \times (\lfloor \sqrt{N} \rfloor - 1)$, $m = N - k$, s is the surface's area of shape, and l is the boundary's

Figure 2: CVT sampling for rectangle.

circumference of shape. We are able to adopt the L-BFGS solver [30] to minimize the objective function, leading to empirically superlinear convergence rate. The pseudocode of uniform sampling in shape with CVT is shown in Algorithm 1

3.3. Position Assignment. The goal of group formation transformation is to change all the agent from source group formation to target group formation. The formation transformation not only realizes collision avoidance during crowd movement but also achieves the global shortest moving distance. In group formation transformation simulation, agents need to observe the movement and position of other agents, maintain the coordination of the team, and avoid the free and scattered individual behavior. Therefore, it is necessary to realize the optimal movement control of the group and keep the number of the crowds unchanged. We apply the classical earth mover's distance (EMD) algorithm [31] to solve the global position assignment problem. The group formation transformation objective function is defined as

$$Z = \min \sum_{i=1}^{n} \sum_{j=1}^{m} D(s_i, t_j) f_{ij},$$
$$D(s_i, t_j) = \sqrt{\left\| s_i - t_j \right\|^2 + (\delta_i - \delta_j)^2}, \tag{8}$$

where s_i is the i-th position of from source group, t_j is the j-th position of target group, δ_i is the distance from s_i to its nearest neighboring point, and δ_j is the distance from t_j to its nearest neighboring point. The limiting condition of objective optimization function is subject to

$$\sum_{i=1}^{n} f_{ij} = 1,$$
$$\sum_{j=1}^{m} f_{ij} = 1, \tag{9}$$
$$f_{ij} \in \{0, 1\},$$

Input: collective crowd shape; an error tolerance $\varepsilon = 10^{-6}$.
Output: group sampling points $\{c_i\}_i^n$.
Generate the initial points $\{p_i\}_i^n$ as the seed;
Compute the Voronoi region Ω_i for each seed p_i
Compute the centroid c_i of each Voronoi region Ω_i;
While $\nabla E > \varepsilon$ **do**
  / ∗ L-BFGS∗/
  Compute the energy function $E(x_i, \Omega_i)$ in equation (6)
  Compute the gradient w.r.t. x; see equation (7)
  Update x by step search
end
return the final centroid points $\{c_i\}_i^n$

Algorithm 1: CVT-based method for group sampling.

where $\sum_{i=1}^{n} X_{i,j} = 1$ denotes that each agent in the source group can only move to one target position and $\sum_{j=1}^{m} X_{i,j} = 1$ means that a position of the target group can only be occupied by one agent, as shown in Figure 3.

3.4. Movement Control. In movement control phase, we can also mix our method with other methods for agent-based planning [6, 32]. Recursive Velocity Observer (RVO) method is used to calculate the actual speed of the agent. In addition to collision avoidance, the agent may change the target exit according to the updated environmental information or change the path through the area and then need to replan the path. For each time step, we first advance the individually controlled agents and then treat them as obstacles in the continuum model so that the crowd motion will flow around them.

4. Results and Performance

We implemented and experimented with our algorithm on a computer with a 64-bit version of Win7 system, a 3.07 GHz Intel(R) Core(TM) i7 CPU, and 6-GB memory. The coding language is C++. We shall use extensive experimental results to exhibit and demonstrate the algorithm's efficiency, as well as its insensitiveness to scene resolution. To demonstrate the effectiveness and practicability of our method, we carry out the evacuation simulation in different scenarios.

In our method, the most computationally time-consuming step is the position assignment, which has the time complexity of $O(N^3)$; we precalculate the result of the position assignment in the offline process. We recorded the time cost for the different simulation scenarios (Table 1); the numbers of agents are 200, 300, 400, 600, and 1000, respectively. As shown in Table 1, the cost time difference is very small on a different shape which contains the same number of agents. For example, the solving time of the objective optimization function is about 3 seconds for pair assignment of a group with 200 agents, while the cost time is 5 seconds and 45 seconds for groups which contain 300 and 1000 agents, respectively. It is suitable for offline formation preordering operations.

Figure 3: Pair assignment from source to target positions.

Table 1: Performance of our algorithm on the different shapes.

Sampling number	Shapes		Times (s)
	Source	Target	
200	"A" word	"H" word	2.83
200	"H" word	"Z" word	2.79
300	Rectangle	Pentagram	5.78
300	Pentagram	Circle	5.84
400	Pig	Rabbit	8.12
400	Rabbit	Skiing logo	8.45
200	"2018" word	Olympic rings logo	2.85
600	"Welcome" word	Chinese "welcome" word	14.83
400	Diamond	Four-triangle	6.91

4.1. Performance. Figure 4 illustrates the process of motion transformation of different groups in a single shape. Figure 4(a) shows that the "A" word shape formation transformed into "H" word formation and then into "H" word formation with 200 agents. Figure 4(b) shows a 300-person square formation transformed into a five-star formation and then into a circular formation. Figures 4(c) and 4(d) illustrate the group with 400-person formation from pig shape to rabbit shape and rabbit shape to skiing logo shape, respectively.

Figure 5 shows the process of motion transformation of crowds in multiple shapes. Figure 5(a) shows that the group has 200-agent formation from "2018" word with 4 subshapes to the Olympic rings logo which contains 5 subshapes. Figure 5(b) shows the transform process from the shapes of English word "welcome" to the shapes of Chinese word "welcome." From Figure 5, we can observe that our method has good adaptability in nonconvex and irregular formations with multiple shapes, and the transformation process is smooth and without any confusion.

Figure 6 shows the process of formation transformation with 400 agents in diamond shape under nonuniform density. We set the shape with an e density function which is defined as [4] $\rho(x, y) = |(x, y)(x + y)|$. Then, the diamond can also be decomposed into four symmetrical shapes. It can be seen that our algorithm can get different formation transformation effects.

4.2. Evaluations and Comparisons. To quantitatively analyze the effect of a crowd formation transformation, we introduce two different objective measures, as described as follows.

4.2.1. The Rangeability of Sampling Points. For agent s_i, the neighboring points set of s_i within its the nuclear radius r_i is denoted as $\{s_i\}_{np} = \{s_p \mid \|s_i - s_p\| < r_i$. Let min_dis$(s_i)$ be its the minimum distance from position s_i to its neighbors $\{s_i\}_{np}$, and t_i is the current position of s_i in the target group $G(T, P_T)$. The source position $s_p \in \{s_i\}_{np}$ will move to its target position t_p. The standard deviation σ_s and the average value μ_s can be used to quantify the stability rangeability of source group sampling during the transformation:

$$\mu_s = \frac{\sum_{i=1}^{N} \text{mean_dis}(s_i)}{N},$$
$$\text{mean_dis}(s_i) = \frac{1}{\left|\{s_i\}_{np}\right|} \sum_{t_p \in \{t_i\}_{np}} \left\|t_i - t_p\right\|, \tag{10}$$
$$\sigma_s = \sqrt{\frac{1}{N} \sum_{1}^{N} (\text{mean_dis}(s_i) - \mu_s)^2}.$$

Figure 4: Group formation between single formations. Group formation from (a) "A" word to "H" word and to "Z" word, (b) rectangle to five-star and to circular, (c) pig to rabbit, and (d) rabbit to skiing logo.

Clearly, the lower value of σ_s indicates that the agents have more similar distances from their neighbors.

4.2.2. Transform Distance from Source to Target Group. We also employ the standard deviation σ_{dt} and the average value μ_{dt} of transform distance to quantify the motion effect of agents. For an agent p_i, its transform distance is defined as follows:

$$\text{tra_dis}_i = \|s_i - t_i\|. \tag{11}$$

Here, s_i and t_i denote its source position and its current position, respectively. The average value μ_{dt} of all the agents' transform distance is defined as

$$\mu_{dt} = \frac{\sum_1^N \text{tra_dis}_i}{N}. \tag{12}$$

The standard deviation σ_{dt} of transform distance is calculated as follows:

$$\sigma_{\text{dt}} = \sqrt{\frac{1}{N}\sum_1^N \left(\text{tra_dis}_i - \mu_{\text{dt}}\right)^2}. \tag{13}$$

Clearly, the lower value of σ_{dt} and μ_{dt} indicates that the agents have less time consumption during the crowd transformation process.

We compare our method with the other three crowd formation transform simulation algorithms including RVO [6], geometry-constrained mechanism method [4], and mutual information-based method [1]. As shown in Figure 7, the four different methods are applied to the same source and target formations. The visual results show that our approach can achieve the best visual effect among the four approaches. Table 2 shows the comparison of results from our method on crowd simulation and other algorithms using simulation time, μ_s, σ_s, μ_{dt}, and σ_{dt}. From this table, we can observe that our simulation method on crowd formation transform outperforms other approaches no matter what kind of evaluation metric is used, except the Zheng method.

(a) (b)

Figure 5: Group formation between multiple formations. (a) Group formation from "2018" word with 4 subshapes to the Olympic rings; (b) group formation from English word "welcome" to Chinese word "welcome."

Figure 6: Group formation transformation under nonuniform density.

Figure 7: Comparisons of four different approaches given the same source and target formations. (a) Our method. (b) RVO [6]. (c) Zheng method [4]. (d) Xu method [1].

Table 2: Statistics comparison of representative algorithms transform from rectangle to star shapes.

Method	Times (s)	μ_s	σ_s	μ_{dt}	σ_{dt}
Our method	12.94	7.98	2.93	16.57	13.88
ROV	26.43	28.02	18.44	68.75	73.85
Zheng et al. [4]	58.43	9.56	3.13	23.45	19.23
Xu et al. [1]	15.32	16.11	8.06	36.53	26.23

When comparing with the Zheng method, our algorithm takes less simulation time.

5. Conclusions

In this paper, we introduce a novel crowd formation transform method based on Earth mover's distance. The main precompute step of our approach is to sample shapes with CVT technique and use EMD to calculate the global shortest position assignment. In the real-time simulation, the interactions between individuals and obstacles are calculated based on RVO which realize the collision avoidance effectively. Large experimental results show that our algorithm has an obvious speed advantage in the real-time simulation process, and our method can also be used to realize the formation transformation in three dimensions.

For large-scale group, formation transformation simulation, the precomputation process, and the collision avoidance calculation process are still time-consuming. In the future, we will keep improving the performance by advanced parallelization techniques. We will also explore more applications in visual computing.

Data Availability

The simulation data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work.

Acknowledgments

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Research Article

Hesitant Fuzzy Decision-Making Method Based on Correlation Coefficient under Confidence Levels with Application to Multisensor Electronic Reconnaissance

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In this paper, we focus on the hesitant fuzzy decision-making method based on correlation coefficient under confidence levels. Firstly, we propose several correlation coefficients based on confidence levels, and based on the correlation coefficient between attribute-evaluated values and the ideal values, several optimal attribute weights models are constructed. Secondly, we have defined the concepts of module, weight module, projection, and weighted projection in hesitant fuzzy sets on account of the projection theory. Finally, on the strength of our research results above, we construct a hesitant fuzzy multiattribute decision-making method and apply it to the multiattribute decision-making problem in multisensor electronic reconnaissance. Comparative analyses are made via simulation to test the rationality and validity of the proposed method.

1. Introduction

In the field of multiattribute decision-making, the decision-making process is complex and confuses researchers. Due to different knowledge backgrounds and thinking patterns of decision makers, the degree of membership may vary for the same elements and collections. For example, some people give 0.5, other people give 0.6 or 0.7. Especially, all decision makers insist on their own opinions, and they are deadlocked. To deal with this complex situation, the concept of hesitant fuzzy sets was proposed by Torra et al. [1, 2], which allowed the membership of an element in a given set to be different values. Xu and Xia [3, 4] proposed multiple distance measures for hesitant fuzzy sets. Farhadinia [5] studied the relationship between entropy, similarity measure and distance measure of hesitant fuzzy sets (HFSs), and interval-valued hesitant fuzzy sets (IVHFSs). Zhu et al. [6] and Yu et al. [7] proposed dual hesitant fuzzy sets (DHFSs) and researched the basic operation and characteristics of DHFs. The problem of fuzzy multiattribute group decision-making (MAGDM) with different priorities of attribute and expert was studied by Wei [8]. Yu [9] proposed a hesitant fuzzy multicriteria group decision-making method for performance and development review (PADR) evaluation. Farhadinia [10, 11] presents a series of scoring functions for hesitant fuzzy sets (HFSs) and extended the hesitant fuzzy set (HFS) to its higher order type. Rodriguez et al. [12] present a marvellous overview on hesitant fuzzy sets.

Many classic decision-making methods have been extended to uncertain environments, and some fuzzy multiattribute decision-making methods have been obtained; attribute weights play an important role in summarizing expert assessments in the decision-making process and attracts many researchers' attention. For example, Xu [13] studied some of the group decision-making problems concerning attribute weight information. Park [14] investigated the group decision-making problems, in which all the information provided by the decision makers is presented in the form of interval-valued intuitionistic fuzzy decision matrices. And fuzzy TOPSIS method under incomplete weight information [15–17], fuzzy ELECTRE II methods [18], fuzzy PROMETHEE method [19], fuzzy VIKOR method [20, 21], fuzzy QUALIFLEX method [22], fuzzy TWO-SIDED MATCHING model [23], etc. It can be said

that the research on fuzzy decision-making methods has formed a relatively complete system, which can basically cover decision-making problems in fuzzy environments. However, among these methods, the research on the hesitant fuzzy decision-making method under confidence levels has just started, and the content is less.

In many practical decision-making problems, the evaluation experts are required to provide two types of information such as the performance of the evaluation objects and the familiarity with the evaluation fields (called confidence levels). In this paper, we propose several correlation coefficients based on confidence levels. Then, we have defined the concepts of module, weight module, projection, and weighted projection in hesitant fuzzy sets under confidence levels on account of the projection theory. Finally, we construct a hesitant fuzzy multiattribute decision-making method and apply it to the multiattribute decision-making problem in multi-sensor electronic reconnaissance.

2. Preliminaries

Zadeh [24] first attempted to use membership function for uncertain information. However, in the decision-making process, because there is a lot of uncertain information, experts usually hesitate for one thing or another, and it is difficult to reach a final consensus, while hesitant fuzzy sets can effectively solve such problems.

Definition 1 (see [1, 2]). Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set; then, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a sunset of [0, 1].

For ease of understanding, Xia and Xu [25] represent the HFS with a mathematical symbol:

$$E = \{ < x, h_E(x) > \mid x \in X\}, \tag{1}$$

where $h_E(x)$ is a set of some values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, Xia and Xu [25] call $h = h_E(x)$ a hesitant fuzzy element (HFE) and H the set of all the HFEs.

In many practical decision-making problems, most existing hesitant fuzzy aggregation operators do not take into account the confidence level of aggregation parameters provided by information providers. Xia and Xu [26] propose a hesitant fuzzy aggregation operator considering absolute confidence levels.

Definition 2 (see [26]). Let $h_1, h_2, \ldots, h_n$ be a collection of HFSs, l_i be the confidence levels of HFS h_i, and $l_i \in [0, 1]$, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of them, such that $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$. If

$$\begin{aligned} \mathrm{CIHFWA}(h_1, h_2, \ldots, h_n) &= \mathop{\oplus}_{i=1}^{n} w_i (l_i h_i) \\ &= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left(1 - \prod_{i=1}^{n} (1 - l_i \gamma_i)^{w_i} \right), \end{aligned} \tag{2}$$

then CIHFWA is referred to as confidence-induced hesitant fuzzy weighted averaging (CIHFWA) operator. Peculiarly, if $l_1 = l_2 = \cdots = l_n = 1$, then the CIHFWA operator reduces to the hesitant fuzzy weighted averaging (HFWA) operator:

$$\begin{aligned} \mathrm{HFWA}(h_1, h_2, \ldots, h_n) &= \mathop{\oplus}_{j=1}^{n} w_j h_j \\ &= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left(1 - \prod_{j=1}^{n} (1 - \gamma_j)^{w_j} \right). \end{aligned} \tag{3}$$

Definition 3 (see [25]). Suppose $h_1, h_2, \ldots, h_n$ be a collection of hesitant fuzzy sets (HFSs), and let HFOWA be $\Omega_n \longrightarrow \Omega$. The hesitant fuzzy ordered weighted averaging (HFOWA) operator:

$$\begin{aligned} \mathrm{HFOWA}(h_1, h_2, \ldots, h_n) &= \mathop{\oplus}_{j=1}^{n} w_j h_{\sigma(j)} \\ &= \cup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \ldots, \gamma_{\sigma(n)} \in h_{\sigma(n)}} \left(1 - \prod_{j=1}^{n} (1 - \gamma_{\sigma(j)})^{w_j} \right), \end{aligned} \tag{4}$$

where $h_{\sigma(j)}$ is the jth largest element of h_j $(j = 1, 2, \ldots, n)$ and $w = (w_1, w_2, \ldots, w_n)^T$ is the aggregation-associated vector such that $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$.

3. Models and Method for Determining the Attribute Weights

In a multiattribute decision-making, the expert group is sometimes asked to provide the weights for selected attributes. At all events, the attribute weights are always incomplete because of the short of knowledge and data and the limitation of decision makers' expertise and time. Therefore, how to carry out the alternatives sorting by the known information is an interesting and important question. In this section, we propose a method based on the correlation coefficients of HFEs.

3.1. Correlation Coefficients of HFEs. We first introduce the concept of correlation coefficient, then give several correlation coefficient formulas, and discuss their properties.

Definition 4 (see [25]). For two HFEs h_1 and h_2, the correlation coefficient of h_1 and h_2, denoted as $C(h_1, h_2)$, should satisfy the following properties:

$$\begin{aligned} &(1)\ |C(h_1, h_2)| \le 1, \\ &(2)\ C(h_1, h_2) = 1, \\ &(3)\ C(h_1, h_2) = C(h_2, h_1). \end{aligned} \tag{5}$$

In most cases, $k_{h1} \neq k_{h_2}$, and for convenience, let $k = \max\{k_{h1}, k_{h_2}\}$. For proper operation, when comparing them, extend the shorter one to the point where both have the same length. The best way to extend the shorter one is to add a value to it. In fact, we can extend shorter values by adding shorter ones, depending on the DM's risk appetite. The optimists expect the desired outcome and may increase

the maximum, while the pessimists expect the adverse outcome and may increase the minimum.

Based on Definition 4, in [4], the following correlation coefficients for HFEs are given:

$$C_1(h_1,h_2)=\frac{\sum_{j=1}^{k}h_1^{\sigma(j)}\cdot h_2^{\sigma(j)}}{\max\left\{\sum_{j=1}^{k}\left(h_1^{\sigma(j)}\right)^2,\sum_{j=1}^{k}\left(h_2^{\sigma(j)}\right)^2\right\}},$$
$$C_2(h_1,h_2)=\frac{\sum_{j=1}^{k}h_1^{\sigma(j)}\cdot h_2^{\sigma(j)}}{\left(\sum_{j=1}^{k}\left(h_1^{\sigma(j)}\right)^2\cdot\sum_{j=1}^{k}\left(h_2^{\sigma(j)}\right)^2\right)^{1/2}}, \tag{6}$$

where $h_1^{\sigma(j)}$ and $h_2^{\sigma(j)}$ are the jth largest values in h_1 and h_2.

We give another correlation coefficient for HFEs:

$$C_3(h_1,h_2)=\frac{\sum_{j=1}^{k}h_1^{\sigma(j)}\cdot h_2^{\sigma(j)}}{\left(\sum_{j=1}^{k}\left(h_1^{\sigma(j)}\right)^2+\sum_{j=1}^{k}\left(h_2^{\sigma(j)}\right)^2\right)/2}. \tag{7}$$

Although all three formulas satisfy the properties of Definition 4, they each have their own characteristics. We will discuss this issue below.

Theorem 1. *Let h_1 and h_2 be two HFEs; then,*

$$C_2(h_1,h_2)\le C_3(h_1,h_2)\le C_1(h_1,h_2). \tag{8}$$

Proof. Because $\sqrt{ab}\le(a+b/2)\le\max\{a,b\}, a,b\ge 0$, we can obtain

$$\frac{\left(\sum_{j=1}^{k}\left(h_1^{\sigma(j)}\right)^2\cdot\sum_{j=1}^{k}\left(h_2^{\sigma(j)}\right)^2\right)^{1/2}\le\left(\sum_{j=1}^{k}\left(h_1^{\sigma(j)}\right)^2+\sum_{j=1}^{k}\left(h_2^{\sigma(j)}\right)^2\right)}{2\le\max\left\{\sum_{j=1}^{k}\left(h_1^{\sigma(j)}\right)^2,\sum_{j=1}^{k}\left(h_2^{\sigma(j)}\right)^2\right\}}, \tag{9}$$

then

$$\begin{aligned}&\frac{\sum_{j=1}^{k}h_1^{\sigma(j)}\cdot h_2^{\sigma(j)}}{\left(\sum_{j=1}^{k}\left(h_1^{\sigma(j)}\right)^2\cdot\sum_{j=1}^{k}\left(h_2^{\sigma(j)}\right)^2\right)^{1/2}}\\&\le\frac{\sum_{j=1}^{k}h_1^{\sigma(j)}\cdot h_2^{\sigma(j)}}{\left(\sum_{j=1}^{k}\left(h_1^{\sigma(j)}\right)^2+\sum_{j=1}^{k}\left(h_2^{\sigma(j)}\right)^2\right)/2}\\&\le\frac{\sum_{j=1}^{k}h_1^{\sigma(j)}\cdot h_2^{\sigma(j)}}{\max\left\{\sum_{j=1}^{k}\left(h_1^{\sigma(j)}\right)^2,\sum_{j=1}^{k}\left(h_2^{\sigma(j)}\right)^2\right\}}.\end{aligned} \tag{10}$$

This completes the proof of Theorem 1.

Considering the importance of the confidence levels, we propose several hesitant fuzzy correlation coefficients based on relative confidence levels and absolute confidence levels and discuss their relationship.

Definition 5. Let h_1 and h_2 be two HFEs, $k=\max(k_{h_1},k_{h_2})$, and k_{h_1} and k_{h_2} be the length in h_1 and h_2. Then, we can construct several correlation coefficients under confidence levels for HFEs:

$$\begin{aligned}C_l^1&=\frac{\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)}{\max\left\{\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)^2,\sum_{j=1}^{k}\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)^2\right\}},\\C_l^2&=\frac{\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)}{\left(\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)^2\cdot\sum_{j=1}^{k}\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)^2\right)^{1/2}},\\C_l^3&=\frac{\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)}{\left(\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)^2+\sum_{j=1}^{k}\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)^2\right)/2},\\C_l^4&=\frac{\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)}{\max\left\{\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)^2,\sum_{j=1}^{k}\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)^2\right\}},\\C_l^5&=\frac{\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)}{\left(\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)^2\cdot\sum_{j=1}^{k}\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)^2\right)^{1/2}},\\C_l^6&=\frac{\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)}{\left(\sum_{j=1}^{k}\left(l_1^{\tau(j)}h_1^{\tau(j)}\right)^2+\sum_{j=1}^{k}\left(l_2^{\tau(j)}h_2^{\tau(j)}\right)^2\right)/2},\end{aligned} \tag{11}$$

where $h_1^{\sigma(j)}$ and $h_2^{\sigma(j)}$ are the jth largest values in h_1 and h_2. $l_i^{\tau(j)}$ be the confidence levels of $h_i^{\tau(j)}$ and $l_i^{\tau(j)}\in[0,1]$, and $l_i^{(j)}=n\times(l_i^{\tau(j)}/\sum_{i=1}^{k}l_i^{\tau(j)})$ be the relative confidence levels of $h_i^{\tau(j)}$ and $l_i^{(j)}\in[0,1]$.

Theorem 2. *Let h_1 and h_2 be two HFEs, then*

$$\begin{aligned}&C_l^2(h_1,h_2)\le C_l^3(h_1,h_2)\le C_l^1(h_1,h_2),\\&C_l^5(h_1,h_2)\le C_l^6(h_1,h_2)\le C_l^4(h_1,h_2).\end{aligned} \tag{12}$$

3.2. Models for Determining the Attribute Weights. In real life, there are always some differences between the ideal point of attribute values and the attribute values' vector corresponding to the alternative $A_i\,(i = 1, 2, \ldots, m)$. We call $A^* = (a_1^*, a_2^*, \ldots, a_n^*)$ is an ideal selection for attribute values, where $a_j^* = \max_i\{a_{ij}\}$, $j = 1, 2, \ldots, n$. We define the correlation $C_1(A_i, A^*)$. Obviously, the bigger $C_1(A_i, A^*)$ is, the better the alternative A_i will be. Therefore, a reasonable weight vector $w^* = (w_1^*, w_2^*, \ldots, w_n^*)$ should be determined so that all the correlations are as bigger as possible, which means maximizing the following correlation vector [27–29].

Model 1:

$$\max\left(\sum_{j=1}^{n} w_j C_1\left(a_{1j}, a_j^*\right), \sum_{j=1}^{n} w_j C_1\left(a_{2j}, a_j^*\right), \ldots, \sum_{j=1}^{n} w_j C_1\left(a_{nj}, a_j^*\right)\right), \tag{13}$$

under the condition $w \in H$, where H is the set of the known weight information.

To this end, we set up the following multiobjective optimization model.

Model 2:

$$\begin{aligned} \max \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} w_j C_1\left(a_{ij}, a_j^*\right) \\ \text{s.t.} \quad & w \in H. \end{aligned} \tag{14}$$

Model 2 can be easily solved by the single-objective programming model; then, we can get the optimal solution $w^* = (w_1^*, w_2^*, \ldots, w_n^*)$, which can be used as the weight vector of attributes.

If the information about attribute weights is completely unknown, the following multiobjective programming model can be established.

Model 3:

$$\begin{aligned} \max \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} w_j C_1\left(a_{ij}, a_j^*\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_j \frac{\sum_{j=1}^{l} \gamma_1^{\sigma(j)} \cdot \gamma_2^{\sigma(j)}}{\max\left\{\sum_{j=1}^{l}\left(\gamma_1^{\sigma(j)}\right)^2, \sum_{j=1}^{l}\left(\gamma_2^{\sigma(j)}\right)^2\right\}} \\ \text{s.t.} \quad & \sum_{j=1}^{n} w_j^2 = 1, w_j \geq 0, j = 1, 2, \ldots, n. \end{aligned} \tag{15}$$

To solve this model, we construct the Lagrange function:

$$L(w, \lambda) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_j C_1\left(a_{ij}, a_j^*\right) + \frac{\lambda}{2}\left(\sum_{j=1}^{n} w_j^2 - 1\right), \tag{16}$$

where λ is the Lagrange multiplier.

Differentiating equation (16) with respect to w_j $(j = 1, 2, \ldots, n)$ and λ, set these partial derivatives to zero, which gives the following equation:

$$\begin{cases} \dfrac{\delta L\left(w_j, \lambda\right)}{\delta w_j} = \displaystyle\sum_{i=1}^{m} C_1\left(a_{ij}, a_j^*\right) + \lambda w_j = 0, \\[2ex] \dfrac{\delta L\left(w_j, \lambda\right)}{\delta \lambda} = \displaystyle\sum_{j=1}^{n} w_j^2 - 1 = 0. \end{cases} \tag{17}$$

By solving equation (17), we obtain a simple and precise formula to determine the attribute weights as shown below:

$$w_j^* = \frac{\sum_{i=1}^{m} C_1\left(a_{ij}, a_j^*\right)}{\sum_{j=1}^{n}\left(\sum_{i=1}^{m} C_1\left(a_{ij}, a_j^*\right)\right)}, \tag{18}$$

which can be used as the weight vector of attributes. Apparently, $w_j^* \geq 0$, for all j.

If $C_1(a_{ij}, a_j^*)$ is replaced with formula (4), then we have

$$w_j^* = \frac{\sum_{i=1}^{m}\left(\sum_{j=1}^{k} a_{ij} \cdot a_j^* / \left(\max\left\{\sum_{j=1}^{k}(a_{ij})^2, \sum_{j=1}^{k}(a_j^*)^2\right\}\right)\right)}{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left(\left(\sum_{j=1}^{k} a_{ij} \cdot a_j^*\right) / \left(\max\left\{\sum_{j=1}^{k}(a_{ij})^2, \sum_{j=1}^{k}(a_j^*)^2\right\}\right)\right)\right)}, \quad j = 1, 2, \ldots, n. \tag{19}$$

Through the above analysis, we know that, under the hesitant fuzzy information, both Model 2 and Model 3 can be used to determine attribute weights with incomplete weight information. Model 2 can be used when the weight information is incomplete, and the model can be constructed in various forms. Model 3 can be used when the information about attribute weights is completely unknown, which can be solved easily and obtained as a simple and accurate formula for determining the weight of an attribute [29].

4. Projection Method under Hesitant Fuzzy Environment

Wang [30] proposed a projection method for multiattribute group decision-making, the projection values of evaluation alternatives, and ideal alternatives are used to get the ideal sort. As an excellent measuring tool, the projection method not only reflects the closeness degree between the two objects but also reflects the intrinsic nature of the objects. Now, it is being applied to the decision-making method [31–33]. In this section, we focus on the projection method under hesitant fuzzy environment.

Definition 6 (see [30–33]). Suppose $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are two vectors, then cosine of included angle between vectors x and y can be defined as follows:

$$\cos(x, y) = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}}. \tag{20}$$

Evidently, cosine of included angle is in this range: $0 < \cos(x, y) \leq 1$, and the larger the value of $\cos(x, y)$ is, the more the direction of vector x and y is consistent.

Definition 7 (see [30–33]). Suppose $x = (x_1, x_2, \ldots, x_n)$, then $|x| = \sqrt{\sum_{i=1}^{n} x_i^2}$ is a norm of vector x.

A vector consists of two parts, the direction and the norm. The cosine of the angle $\cos(x, y)$ can only measure whether the direction is consistent, but it cannot reflect the magnitude of norm. In order to consider the norm magnitude and cosine of included angle together, the proximity of two vectors can be measured by the projection value, and the definition of projection is given as follows:

Definition 8 (see [30–33]). Suppose $x = (x_2, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots y_n)$ are two vectors, and the projection of vector x onto vector y can be defined:

$$\mathrm{Pr}j_y x = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}}\sqrt{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} y_i^2}}. \tag{21}$$

Apparently, the larger the value of $\mathrm{Pr}j_y x$ is, the closer the vector x is to the vector y. For the MADM problem, if the vector y obtains a positive ideal solution and the vector x is used for each alternative, we can obtain the projection of each alternative on the positive ideal solution. The larger the projection value is, the better the alternative is.

Now, we extend the projection theory to the hesitant fuzzy sets and give the concepts of the hesitant fuzzy module, weighted module, projection, and weighted projection.

Definition 9 (see [34]). Suppose $H = (h_1, h_2, \ldots, h_n)$ be a vector of HFSs; then,

$$|H| = \sqrt{\sum_{j=1}^{n} h_j^2}, \tag{22}$$

is called the module of vector H.

In many cases, the weight of h_j should be considered. For example, in the multiple attribute decision-making, sometimes the evaluation experts are asked to provide the weight of every attribute, so we should define the hesitant fuzzy weighted module.

Definition 10 (see [34]). Let $H = (h_1, h_2, \ldots, h_n)$ be a vector of HFSs and $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of h_j $(j = 1, 2, \ldots, n)$, such that $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$. Then,

$$|H|_w = \sqrt{\sum_{i=1}^{n} \left(w_i |h_i|\right)^2}. \tag{23}$$

is called the weighted module of vector H.

Definition 11 (see [34]). Suppose $H = (h_1, h_2, \ldots, h_n)$ and $E = (e_1, e_2, \ldots, e_n)$ be two vectors of HFSs; then, the projection of vector H onto vector E can be defined:

$$\mathrm{Pr}j_E H = \frac{\sum_{i=1}^{n} h_i e_i}{|E|}. \tag{24}$$

Evidently, the larger the value of $\mathrm{Pr}j_E H$ is, the closer the vector H is to the vector E.

Definition 12. Suppose $H = (h_1, h_2, \ldots, h_n)$ and $E = (e_1, e_2, \ldots, e_n)$ be two vectors of HFSs. $|E|_w$ is the weighted module of vector E. Let w_i and t_i be the weight of h_i and e_i, such that $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1$, $t_i \in [0,1], \sum_{i=1}^{n} t_i = 1$. Then, the weighted projection of vector H onto vector E can be defined:

$$\mathrm{Pr}j_{(E)_w}(H)_w = \frac{\sum_{i=1}^{n} (w_i h_i)(t_i e_i)}{|E|_w}. \tag{25}$$

5. An Approach to Multiple Attribute Decision-Making with Hesitant Fuzzy Information under Confidence Levels

Based on the above models, we develop a practical approach to solve the group decision-making problems, where the information about attribute weights is not completely known or completely unknown, and the attribute values take the form of hesitant fuzzy information. The approach includes the following steps:

Step 1: we construct the decision matrix $A^{(k)} = (a_{ij})^{(k)}_{m\times n}$ with confidence levels, where all the arguments a_{ij} $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ are HFSs, given by the decision makers, for the alternative $A_i \in A$ with respect to the attribute $G_j \in G$, and the decision makers need to give out the confidence levels $l_k\,(0 \le l_k \le 1)\,(k = 1, 2, \ldots, p)$.

Step 2: we apply summary operators to aggregate decision makers' assessments into collective evaluation a_{ij} $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ on each alternative A_i $(i = 1, 2, \ldots, m)$.

Step 3: we can get the ideal of attribute values $A^* = \{a_1^*, a_2^*, \ldots, a_n^*\}$. If the information about the attribute weights is entirely unknown, then we can get the attribute weights $w = (w_1, w_2, \ldots, w_n)^T$ by using Model 3. If the information about the attribute weights is partly known, then we solve model 2 to obtain the attribute weights $w = (w_1, w_2, \ldots, w_n)^T$.

Step 4: apply formula (25) to calculate the relative weighted projection $\mathrm{Pr}j_{A_w^*} A_i$ of each alternative A_i:

$$\mathrm{Pr}j_{A_w^*} A_i = \frac{\sum_{j=1}^{n} w_j^2 \left(a_{ij} \cdot a_j^*\right)}{\left|A^*\right|_w}. \tag{26}$$

Step 5: rank the alternatives A_i $(i = 1, 2, \ldots, m)$ according to the relative weighted projection $\mathrm{Pr}j_{A_w^*} A_i$. The larger the $\mathrm{Pr}j_{A_w^*} A_i$ is, the better the alternative A_i is.

6. Numerical Example

In this section, we apply the proposed hesitant fuzzy multi-attribute decision-making method based on signed correlation to multisensor electronic reconnaissance decision problems. In these problems, multiattribute information about the enemy, such as pulse repetition frequency, carrier frequency, power, and pulse width, are detected using various sensors such as Radar, Electronic Support Measures (ESMs), and Infrared, and multiattribute decisions must be made regarding the kind of airborne platform the enemy is using based on these pieces of attribute information. However, in the actual measurement process under a complex electromagnetic environment, the electronic reconnaissance equipment is often interfered by air clutter and enemy signals, which causes a certain degree of information uncertainty that is in line with the characteristics of HFEs. Thus, it is more appropriate to describe these attribute decisions using HFE [35].

Assume that four types of airborne platform (i.e., A_i $(i = 1, 2, 3, 4)$) are present based on the information transmitted by each electronic reconnaissance sensor to the fusion center and that each type of airborne platform has four types of attribute, i.e., pulse repetition frequency (G_1), carrier frequency (G_2), power (G_3), and pulse width (G_4). The fusion center must perform multiattribute fusion judgment on the four types of airborne platform with uncertainty based on the four attribute types. The electronic reconnaissance equipment measures the value of each attribute $G_j \in G$ of each scheme $A_i \in A$, which forms a matrix based on hesitant fuzzy information. ($\mathbf{H} = (h_{ij})_{m\times n}$).

To obtain the type of airborne platform, the specific multiattribute decision-making steps are as follows:

Step 1: decision makers assess the alternatives A_i $(i = 1, 2, 3, 4)$ with respect to the attributes G_j $(j = 1, 2, 3, 4)$, and the decision makers give the confidence levels l_p $(0 \le l_p \le 1)$. Then, we set up the following hesitant fuzzy decision matrix $A^{(p)} = (a_{ij})^{(p)}_{4\times 4}$, andd the first components in the two types of information are the confidence levels (see Table 1).

Step 2: apply the CIHFHA operator to aggregate all the hesitant fuzzy decision matrix $A^{(p)} = (a_{ij})^{(p)}_{4\times 4}$ into the aggregate hesitant fuzzy decision matrix $A = a_{ij}$ $(i = 1, 2, 3, 4; j = 1, 2, 3, 4)$ (see Table 2).

Step 3: we can obtain the ideal of attribute values $A^* = \{0.36, 0.39, 0.38, 0.39\}$. Because the information about the attribute weights is entirely unknown, we can get the attribute weights $t = (0.24, 0.23, 0.27, 0.26)^T$ by using Model 3.

Step 4: use formula (25) to calculate the relative weighted projection $\mathrm{Pr}j_{A_w^*} A_i$ of each alternative A_i:

Table 1: Hesitant fuzzy decision matrix under confidence levels.

	G_1	G_2	G_3	G_4
A_1	{(0.7,0.5),(0.3,0.7)}	{(0.7,0.6),(0.5,0.7)}	{(0.6,0.5),(0.7,0.6),(0.5,0.8)}	{(0.8,0.4),(0.7,0.5)}
A_2	{(0.8,0.3),(0.6,0.4)}	{(0.6,0.4),(0.4,0.7)}	{(0.7,0.4),(0.6,0.6)}	{(0.9,0.4),(0.7,0.5),(0.6,0.6)}
A_3	{(0.7,0.6),(0.5,0.7),(0.4,0.8)}	{(0.7,0.4),(0.4,0.6)}	{(0.5,0.8),(0.3,0.9)}	{(0.4,0.8),(0.5,0.9)}
A_4	{(0.7,0.4),(0.5,0.6)}	{(0.6,0.5),(0.4,0.7),(0.3,0.9)}	{(0.5,0.4),(0.6,0.7)}	{(0.6,0.5),(0.4,0.6)}

Table 2: Comprehensive hesitant fuzzy decision matrix A.

	G_1	G_2	G_3	G_4
A_1	0.28	0.39	0.38	0.34
A_2	0.24	0.26	0.32	0.36
A_3	0.36	0.26	0.34	0.39
A_4	0.29	0.28	0.32	0.27

Table 3: Hesitant fuzzy decision matrix A (').

	G_1	G_2	G_3	G_4
A1	{0.5,0.7}	{0.6,0.7}	{0.5,0.6,0.8}	{0.4,0.5}
A2	{0.3,0.4}	{0.4,0.7}	{0.4,0.6}	{0.4,0.5,0.6}
A3	{0.6,0.7,0.8}	{0.4,0.6}	{0.8,0.9}	{0.8,0.9}
A4	{0.4,0.6}	{0.5,0.7,0.9}	{0.4,0.7}	{0.5,0.6}

$$\mathrm{Pr}j_{A_w^*}A_1 = 0.48, \mathrm{Pr}j_{A_w^*}A_2 = 0.41, \mathrm{Pr}j_{A_w^*}A_3 = 0.47, \mathrm{Pr}j_{A_w^*}A_4 = 0.40. \tag{27}$$

Step 5: rank the alternatives A_i $(i = 1, 2, 3, 4)$ according to the relative weighted projection $\mathrm{Pr}j_{A_w^*}A_i$:

$$\mathrm{Pr}j_{A_w^*}A_1 \succ \mathrm{Pr}j_{A_w^*}A_3 \succ \mathrm{Pr}j_{A_w^*}A_2 \succ \mathrm{Pr}j_{A_w^*}A_4, \tag{28}$$

so the airborne platform type is A_1.

If we do with the HFWA operator to aggregate the hesitant fuzzy decision matrix without considering the confidence levels, the main steps are as follows:

Step 1': decision makers evaluate the alternatives A_j' $(i = 1, 2, 3, 4)$ with respect to the attributes G_j' $(j = 1, 2, 3, 4)$ without considering the confidence levels. Then, we construct the following hesitant fuzzy decision matrix $A^{(\prime)} = (a_{ij})_{4\times4}^{(\prime)}$ (see Table 3).

Step 2': employ the HFWA operator to aggregate all the hesitant fuzzy decision matrices $A^{(\prime)} = (a_{ij})_{4\times4}^{(\prime)}$ into the collective hesitant fuzzy decision matrix $A\prime = a_{ij}'$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ (see Table 4).

Step 3': we can obtain the ideal of attribute values $A'^{*} = \{0.71, 0.75, 0.86, 0.86\}$. Because the information about the attribute weights is entirely unknown, we can get the attribute weights $t\prime = (0.25, 0.27, 0.25, 0.23)^T$ by using Model 3.

Step 4': use formula (25) to calculate the relative weighted projection $\mathrm{Pr}j_{A_w'^*}A_i'$ of each alternative A_i':

$$\mathrm{Pr}j_{A_w'^*}A_1' = 0.30, \mathrm{Pr}j_{A_w'^*}A_2' = 0.25, \mathrm{Pr}j_{A_w'^*}A_3' = 0.36, \mathrm{Pr}j_{A_w'^*}A_4' = 0.30. \tag{29}$$

Step 5': rank the alternatives A_i' $(i = 1, 2, 3, 4)$ according to the relative weighted projection $\mathrm{Pr}j_{A_w'^*}A_i'$:

$$\mathrm{Pr}j_{A_w'^*}A_3' \succ \mathrm{Pr}j_{A_w'^*}A_1' = \mathrm{Pr}j_{A_w'^*}A_4' \succ \mathrm{Pr}j_{A_w'^*}A_2', \tag{30}$$

so the airborne platform type is A_3.

If we do with the HFGA operator to aggregate the hesitant fuzzy decision matrix without considering the confidence levels, the main steps are as follows:

Step 1": decision makers evaluate the alternatives A_i' $(i = 1, 2, 3, 4)$ with respect to the attributes G_j' $(j = 1, 2, 3, 4)$ without considering the confidence levels. Then, we construct the following hesitant fuzzy decision matrix $A^{(\prime)} = (a_{ij})_{4\times4}^{(\prime)}$(see Table 3).

Step 2": employ the HFGA operator to aggregate all the hesitant fuzzy decision matrices $A^{(\prime)} = (a_{ij})_{4\times4}^{(\prime)}$ into the collective hesitant fuzzy decision matrix $A'' = a_{ij}''$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ (see Table 5).

Step 3": we can obtain the ideal of attribute values $A''^{*} = \{0.70, 0.68, 0.85, 0.85\}$. Because the information about the attribute weights is entirely unknown, we can get the attribute weights $t' = (0.25, 0.28, 0.24, 0.23)^T$ by using Model 3.

Step 4": use formula (25) to calculate the relative weighted projection $\mathrm{Pr}j_{A_w''^*}A_i''$ of each alternative A_i'':

$$\mathrm{Pr}j_{A_w''^*}A_1'' = 0.58, \mathrm{Pr}j_{A_w''^*}A_2'' = 0.46, \mathrm{Pr}j_{A_w''^*}A_3'' = 0.69, \mathrm{Pr}j_{A_w''^*}A_4'' = 0.56. \tag{31}$$

Table 4: Comprehensive hesitant fuzzy decision matrix A'.

	G_1	G_2	G_3	G_4
A_1	0.61	0.65	0.66	0.45
A_2	0.35	0.58	0.51	0.51
A_3	0.71	0.51	0.86	0.86
A_4	0.51	0.75	0.58	0.55

Table 5: Comprehensive hesitant fuzzy decision matrix A".

	G_1	G_2	G_3	G_4
A_1	0.59	0.65	0.62	0.45
A_2	0.35	0.53	0.49	0.49
A_3	0.70	0.49	0.85	0.85
A_4	0.49	0.68	0.53	0.55

Step 5": rank the alternatives A_i'' $(i = 1, 2, 3, 4)$ according to the relative weighted projection $\mathrm{Pr}j_{A_w''*} A_i''$:

$$\mathrm{Pr}j_{A_w''*} A_3'' \succ \mathrm{Pr}j_{A_w''*} A_1'' = \mathrm{Pr}j_{A_w''*} A_4'' \succ \mathrm{Pr}j_{A_w''*} A_2'', \tag{32}$$

so the airborne platform type is A_3.

If we do not consider the factor of confidence levels, our proposed operators are simplified to the existing hesitant fuzzy aggregation operators. Nevertheless, the experts' group is completely unaware of the alternatives so that the results are different. In view of these situations, the hesitant fuzzy aggregation operators at the confidence levels proposed in this paper are useful tools. As can be seen from the above analysis, compared with the traditional hesitant fuzzy decision-making methods, the main advantages of our methods are not only that they can adapt to the hesitant fuzzy environment but also that they take into account the confidence levels between decision makers, which makes more feasibility and practicality.

7. Conclusion

In this paper, a new decision analysis method under confidence levels is proposed for decision-making (MADM) problems where attribute values are in the form of hesitant fuzzy numbers and unknown attribute weights. Firstly, several hesitant fuzzy correlation coefficient based on confidence levels are proposed. Secondly, some simple optimization models based on the correlation coefficients under confidence levels for HFEs are used to determine the attribute weight, and the projection method for HFEs is proposed. Finally, a numerical example is used to illustrate the practicability and effectiveness of the method. The results show that the main advantages of the decision analysis methods under confidence levels are not only that they can adapt to the hesitant fuzzy environment but also that they take into account the confidence levels between decision makers, which makes the results more feasibility and practicality. However, expert credibility needs expert's subjective judgment, and there will be some errors in decision-making problems. In the future work, we will try our best to correct expert credibility and study probabilistic hesitant fuzzy environment under confidence levels [36, 37].

Data Availability

All data included in this study are available from the corresponding author upon request. Most of the data are already in the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Ship Target Detection Based on Improved YOLO Network

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Ship target detection is an important guarantee for the safe passage of ships on the river. However, the ship image in the river is difficult to recognize due to the factors such as clouds, buildings on the bank, and small volume. In order to improve the accuracy of ship target detection and the robustness of the system, we improve YOLOv3 network and present a new method, called Ship-YOLOv3. Firstly, we preprocess the inputting image through guided filtering and gray enhancement. Secondly, we use *k*-means++ clustering on the dimensions of bounding boxes to get good priors for our model. Then, we change the YOLOv3 network structure by reducing part of convolution operation and adding the jump join mechanism to decrease feature redundancy. Finally, we load the weight of PASCAL VOC dataset into the model and train it on the ship dataset. The experiment shows that the proposed method can accelerate the convergence speed of the network, compared with the existing YOLO algorithm. On the premise of ensuring real-time performance, the precision of ship identification is improved by 12.5%, and the recall rate is increased by 11.5%.

1. Introduction

With the vigorous development of shipping industry, the water traffic is more and more busy. Due to the frequent occurrence of collisions and other accidents between ships, it is necessary to detect the types of ships effectively to ensure the safety of water traffic. The ship detection technology based on computer vision is of great significance to improve port management and maritime inspection.

The traditional methods of ship detection are based on the automatic identification system and ship features [1, 2]. Li et al. proposed an improved dimensional space clustering algorithm to identify abnormal behavior of ships [3]. Zhang et al. used AIS data to identify ships with attempted collision [4, 5]. Zhou et al. proposed a detection method for classification and identification of bow and hull [6]. Zang et al. carried out ship target detection from the nonfixed platform [7, 8]. Although these studies have achieved good results, there are generally problems such as low recognition accuracy and human intervention. As a result, the traditional ship detection method is difficult to achieve the ideal detection effect.

In recent years, with the rapid development of artificial intelligence technology, the method based on deep learning has become the mainstream detection method. At present, there are two methods to solve the problem of target detection through deep learning: two-stage detection method and one-stage detection method. The two-stage algorithm uses region suggestion detection, mainly including AlexNet [9], VGG [10], ResNet [11], Fast-RCNN [12], and Faster-RCNN [13]. Although the detection accuracy is better than the traditional method, the detection speed is slightly insufficient. The feature extraction process takes a long time and is difficult to achieve the effect of real-time detection. In order to ensure the accuracy and improve the detection speed, one-stage algorithm is proposed. The one-stage detection does not use the idea of combining fine detection with rough detection, but directly detects the results in a single stage. The whole process does not need region suggestion detection and directly realizes end-to-end detection for inputting images. Therefore, the detection speed is greatly improved. The one-stage algorithm mainly includes SSD [14], YOLO [15], YOLOv2 [16], and YOLOv3 [17].

The YOLOv3 algorithm keeps the accuracy and improves the speed, which is favored by many researchers. Zhang et al. proposed multiscale ship target detection based on deep learning [18, 19]. Feng et al. improved the effect of ship classification and recognition by using spatial transform segmentation [20]. Li et al. used one-dimensional target to detect SAR image and used classifier to determine ship type [21, 22]. Chen et al. proposed automatic ship identification and behavior analysis detection based on video, which can accurately detect the ship and successfully identify the historical behavior of the ship [23]. Huang et al. proposed an improved YOLOv3 network for intelligent detection and classification of ship images/videos [24, 25].

Although the YOLOv3 algorithm has achieved good detection results, it has low recognition accuracy in complex scenes such as fog or night, and has phenomenon of missing detection for small target ships. Therefore, this paper proposes a Ship-YOLOv3 algorithm to solve these problems. We optimize the anchor boxes of YOLOv3 algorithm and improve the network structure of the algorithm. The experimental shows that this method can accelerate the convergence speed of the network under the premise of ensuring the real-time performance. Compared with the YOLOv3 algorithm, the precision of ship identification is improved by 12.5%, and the recall rate is increased by 11.5%.

Compared to other works on ship detection, our algorithm possesses the following advantages:

(1) Target box dimension clustering: the K-means++ algorithm is used to cluster the target box of self-made ship dataset. The optimal width and height value is calculated and the predefined anchor in YOLOv3 is modified accordingly.

(2) Improvement of network structure: aiming at the problem of false recognition rate of ship small target detection, the basic network Darknet53 in YOLOv3 algorithm is improved. Some redundant network layers of Darknet53 are reduced, and the jump join mechanism of residual network is added to enhance the detection of small ship targets.

2. Preliminaries

In this section, some essential concepts relating to YOLOv3 are briefly reviewed; these will be used in the rest of this work.

The target detection algorithm of YOLOv3 is proposed on the basis of YOLOv1 and YOLOv2. In order to achieve better classification effect, the residual network is used to realize jump connection. At the same time, the convolution with step size of 2 is used for down sampling to reduce the negative gradient effect caused by pooling. In addition, a batch normalization and activation function of each convolution layer are added to avoid overfitting in training. Based on the up sampling and fusion mechanism of FPN, three scales of output are designed to improve the accuracy of small target detection. The network structure of YOLOv3 is shown in Figure 1.

Firstly, the input image is divided into S × S grids. When the center of the object falls into the grid in the image, the grid is responsible for predicting the bounding box of the object. The corresponding score of this grid is 1, and the other grids are 0. Each grid predicts 3 bounding boxes, and each box represents (5 + N) values. The value 5 contains the location information of the bounding box: the center coordinates (x, y), width and height (w, h), and confidence of the bounding box. The value N represents the number of categories in the dataset.

The NMS algorithm filters the confidence level of the target in the grid and obtains the bounding box with the highest score as the object detection frame. Since a target may belong to multiple categories, the Softmax layer is replaced by a 1 × 1 convolution layer and a logistic regression activation function structure. The score of each category is predicted by logistic regression, and the target is predicted by a threshold. The category higher than the threshold is the real category of the bounding box.

The error between the predicted value and real value is usually calculated with the following crossentropy loss function:

$$\begin{aligned}\text{loss} = &\sum_{i=0}^{S^2}\sum_{j=0}^{B} I_{ij}^{\text{obj}}\left[\left(\sigma\left(t_x\right)_i^j - \sigma\left(\widehat{t}_x\right)_i^j\right)^2 + \left(\sigma\left(t_y\right)_i^j - \sigma\left(\widehat{t}_y\right)_i^j\right)^2\right] \\ &+ \sum_{i=0}^{S^2}\sum_{j=0}^{B} I_{ij}^{\text{obj}}\left[\left(t_{w_i^j} - \widehat{t}_{w_i^j}\right)^2 + \left(t_{hi}^j - \widehat{t}_{h_i^j}\right)^2\right] \\ &+ \sum_{i=0}^{S^2}\sum_{j=0}^{B} G_{ij}\left(C_i^j - \widehat{C}_i^j\right)^2 \\ &+ \sum_{i=0}^{S^2}\sum_{j=0}^{B}\sum_{c\in\text{classes}} I_{ij}^{\text{obj}}\left(p_i^j(c) - \widehat{p}_i^j(c)\right)^2,\end{aligned} \tag{1}$$

where I_{ij}^{obj} means that when the jth bounding box of the ith grid is responsible for predicting the target. The I_{ij}^{obj} value is 1, otherwise it is 0. The S^2 is the number of grids and B is the number of bounding boxes in each grid. The $\sigma(t_x)$ and $\sigma(t_y)$ are the offsets of the center position. The t_w and t_h are offsets for width and height. The parameter C is the confidence score. The p is the probability of ship class.

3. Materials and Methods

Although the YOLOv3 algorithm has a good detection effect on public dataset, the ship dataset used in this paper is obtained by monitoring video. The ship image is fuzzy at night or in foggy weather, and the gray level is uneven. The classification and detection of ships will be disturbed by these conditions. Therefore, it is necessary to improve the YOLOv3 algorithm to meet the requirements of ship classification and detection in complex scenes. The overall structure of the algorithm is shown in Figure 2, which mainly includes three modules: target box dimension

Figure 1: The algorithm structure of YOLOv3.

Figure 2: The algorithm structure of ship-YOLOv3.

Figure 3: The tensor information of different scales.

clustering, YOLOv3 network structure optimization, and data processing. Next, the implementation process of each module is introduced in detail.

3.1. Target Box Dimension Clustering. In the target detection task, selecting an appropriate anchor can significantly improve the speed and accuracy of target detection. Although the YOLOv3 algorithm is trained on MSCOCO and PASCAL VOC dataset, nine groups of anchors are obtained. The anchor values are (10, 13), (16, 30), (33, 23), (30, 61), (62, 45), (59, 119), (116, 90), (156, 198), and (373, 326). However, the anchor values used in COCO and VOC dataset are not suitable for the ship dataset used in this paper. Therefore, this paper uses the K-means++ algorithm to cluster the ship dataset and selects nine different anchors. The anchor values are (66, 13), (136, 22), (162, 30), (218, 30), (217, 38), (220, 47), (212, 64), (347, 40), and (444, 62). Compared with manual

Figure 4: Schematic diagram of ship IOU.

selection of the prior frame, clustering can accelerate the convergence of the network and effectively improve the gradient descent in the training process. The cluster evaluation criteria are as follows:

$$d(\text{bbox}, \text{center}) = 1 - \text{IOU}(\text{bbox}, \text{center}), \tag{2}$$

where $d(\text{bbox}, \text{center})$ is the distance between the bounding box and the center box and $\text{IOU}(\text{bbox}, \text{center})$ is the intersection over union between the two boxes. This clustering method can produce a larger intersection ratio and a smaller distance between the bounding boxes in the same cluster. Each grid in the ship feature map of each scale in Figure 3 predicts three prior frames. It contains $(4+1+N)$-dimensional vector, which represents the central coordinates (x, y), width and height (w, h), and confidence of the bounding box. The N represents the number of categories in the dataset:

$$\begin{aligned} b_x &= \sigma(t_x) + c_x, \\ b_y &= \sigma(t_y) + c_y, \\ b_w &= p_w e^{t_w}, \\ b_h &= p_h e^{t_h}, \\ p_r(\text{object}) * \text{IOU}(\text{b}, \text{object}) &= \sigma(t_0), \\ \text{IOU}(\text{b}, \text{object}) &= \frac{\text{area}(\text{b}) \cap \text{area}(g)}{\text{area}(\text{b}) \cup \text{area}(g)}, \end{aligned} \tag{3}$$

where b_x and b_y are the ensemble framework detection results and t_x and t_y are the predicted center coordinates of each ship's bounding box prior. After the activation function in the network, the offsets $\sigma(t_x)$ and $\sigma(t_y)$ corresponding to the center coordinates of the ship's bounding box will be obtained. The b_w and b_h are the width and height of the detected box, respectively. The p_w and p_h are the weight matrices on the width and height, respectively. The e^{t_w} and e^{t_h} are the rate of width and height of the bounding box. The $\sigma(t_o)$ is the offset of ship prediction score, and it is obtained by the product of ship prediction probability and IOU. The $\text{area}(\text{b})$ is the area of the ship's bounding box, and $\text{area}(g)$ is the area of ground truth. The ratio of the intersection area between $\text{area}(\text{b})$ and $\text{area}(g)$ represents IOU, as shown in Figure 4.

3.2. YOLOv3 Network Structure Optimization. For ships in different scenes, the feature extraction is very important for ship classification. The convolution layer of ship feature extraction network can effectively analyze the features of ships, as shown in Figures 5 and 6. The residual network is added to YOLOv3 network, and the hop connection of residual is used many times. The residual network can solve the problem that the gradient is difficult to descend in the process of feature extraction, accelerate the convergence speed, and reduce the error. The size of the image is increased to 448×448. The convolution with the step size of 2 in the network is used to down sample the image by 32, 16, and 8 times. The output of the last layer is lightweight improved. The convolution of the original output prediction layer 3×3 and 1×1 is pruned, and only the convolution of 3×3 is reserved for ship position and category prediction. This can reduce the network operation, avoid overfitting, and make the model have better generalization ability. Finally, the feature maps of three scales are obtained, and the feature information of the ship is shown in the matrix as follows:

$$f_a^b = f\left(\sum_t S_t^{b-1} * P_{ta}^b + W_a^b\right), \tag{4}$$

where f_a^b is the correspondence between the bth network and the ath output, f is the activation function of the layer b network, S_t^{b-1} is the mapping of the t ship by the $b-1$ convolution network layer, P_{ta}^b is the weight matrix between the tth and the ath ship feature layer, and parameter W_a^b is the bias of the ath output ship feature at the bth convolutional network layer.

3.3. Data Processing

3.3.1. Guided Filtering. In this paper, a guided filter is constructed to filter the images in the night or fog scene. It can be seen from Figure 7 that the guidance filter can protect the contour features of the ship while maintaining the smooth filtering of the ship. It can effectively solve the problem of blur and uneven gray level produced by night or foggy images and has good performance in denoising.

The ship image is filtered to get the output image through the guide image, and the weighted average value is filtered as follows:

$$\begin{aligned} q_i &= \sum_j W_{ij}(I) p_j, \\ q_i &= a_k I_i + b_k, \quad \forall i \in w_k, \\ &\arg\min \sum_{i \in w_k} (q_i - p_i)^2, \\ &\arg\min \sum_{i \in w_k} (a_k I_i + b_k - p_i)^2, \end{aligned} \tag{5}$$

Figure 5: Ship-YOLOv3 network structure.

Figure 6: Partial convolution block structure.

where q_i is the output image filtered by pixels at position i, W_{ij} is the filter core of guide image I, p_j is the input image of the ship with pixels at position j, And w_k is the window, a_k and b_k are the constant coefficients. When the guide image I has an edge, the output ship image will maintain the edge unchanged. The unsmoothed area of the input image is regarded as noise, and the noise is reduced to the minimum. The loss function generated by filtering is shown as the following formula:

$$\begin{aligned} E(a_k, b_k) &= \sum_{i \in w_k} \left((a_k I_i + b_k - p_i)^2 + \in a_k^2 \right), \\ a_k &= \frac{(1/|w|) \sum_{i \in w_k} I_i p_i - \mu_k \overline{p_k}}{\sigma_k^2 + \in}, \\ b_k &= \overline{p_k} - a_k \mu_k, \\ q_i &= \frac{1}{|w|} \sum_{k:\, i \in w_k} (a_k I_i + b_k) = \overline{a}_i I_i + \overline{b}_i, \\ \overline{p}_k &= \frac{1}{|w|} \sum i \in w_k p_i. \end{aligned} \tag{6}$$

The $|w|$ is the number of pixels, μ_k represents the average value of the guide image in the window, σ_k^2 represents the variance, and $\overline{p}_k$ is the average value of the image in the window. For different windows, average all q_i and finally establish the mapping of pixels from I to q.

3.3.2. Gray Scale Enhancement. The feature of the ship is weakened by the influence of illumination and environment. In this paper, the gray enhancement is used to improve the

(a)

(b)

Figure 7: Guided filtering for image processing. (a) Night or foggy images. (b) Guided filtered images.

Figure 8: The gray scale histogram. (a) Original image gray scale histogram. (b) Guided filter gray scale histogram. (c) CLAHE gray scale histogram.

situation. The histogram is the number of pixels in each gray level of the image, which reflects the contrast of the image. Contrast Limited Adaptive Histogram Equalization (CLAHE) is used to cut histogram by setting threshold. The cut part is divided into other histograms, and the contrast of each region is limited. At the same time, interpolation is used to improve the operation speed. The gray histogram is shown in Figure 8. After the processing of gray histogram, the image contrast is enhanced while the noise is suppressed. The gray range of the image becomes more uniform, which is conducive to the extraction of ship features.

Table 1: Sample parameters.

Sample label	Boat	Boat2	Boat3
Number of samples	2489	2096	330

(a) (b)

Figure 9: The loss curve (a) for our Ship-YOLOv3 network and (b) for YOLOv3 network.

(a)

(b)

Figure 10: Real-time detection results of (a) YOLOv3 algorithm and (b) Ship-YOLOv3 algorithm.

(a)

(b)

Figure 11: Real-time detection results of (a) the YOLOv3 algorithm and (b) Ship-YOLOv3 algorithm under normal conditions.

Table 2: Comparison of detection time.

Scenario	YOLOv3	Ship-YOLOv3
Night or fog	31.25 ms	22.65 ms
Conventional	26.12 ms	22.59 ms
Average	28.68 ms	22.62 ms

4. Experiment and Discussion

4.1. Experimental Environment and Dataset. Our framework was developed on the Win10 OS with 128G RAM and 3 GHz CPU. The GPU version is NVIDIA GeForce GTX 1080 Ti, which contains 11 GB RAM. The framework of deep learning is Tensorflow-GPU (1.9Version). The ship dataset is obtained by framing the surveillance video on the banks of Datong bridge and Hengmen waterway. The data scene is complex and changeable, and the time is also different. We use labeling software to label images. There are three types of ships: heavy bulk ships, empty bulk ships, passenger ship and management ship, respectively labeled boat, boat2, and boat3. The parameters are shown in Table 1. There are 4915 ship images, 4421 training images, 441 verification images, and 53 test images.

4.2. Training Parameter Selection. During the training process, the weights pretrained on the PASCAL VOC dataset are loaded for migration training. In the first stage, the weight of darknet53 was restored. When the learning rate is set to 0.0001, only the last regression layer is trained until the loss is reduced to a lower level. In the second stage, the weights are restored from the first stage, and then the learning rate is set to 0.00001 to train all network layers. The attenuation factor of learning rate was 0.96, which was decayed every 5 epochs. The total training epoch is 100, the batch is 6, and the optimizer is momentum. After 73700 iterations, the loss of Ship-YOLOv3 is reduced to about 0.09794. The loss of YOLOv3 algorithm is about 0.1120, and the loss curve is shown in Figure 9. The Ship-YOLOv3 algorithm has better convergence effect, improves the convergence speed, and reduces the loss value.

4.3. Experiment and Analysis

4.3.1. Algorithm Evaluation. In order to evaluate the effect of ship detection, this paper uses the precision rate and recall to test the Ship-YOLOv3 algorithm:

$$P = \frac{N_{tp}}{N_{tp} + N_{fp}}, \quad R = \frac{N_{tp}}{N_{tp} + N_{fn}}, \tag{7}$$

where P is the precision rate of the ship, the N_{tp} is the number of ship samples that are detected correctly, parameter N_{fp} is the number of samples with detection errors, R is the recall rate of the ship, and the parameter N_{fn} is the number of samples of the ship that missed detection. In addition, the NMS threshold is adjusted to 0.45. When the intersection ratio between the predicted bounding box and the actual position of the ship is greater than 0.45, the ship is detected correctly.

4.3.2. Contrast Experiment on Foggy Day or Night. The YOLOv3 algorithm and our Ship-YOLOv3 algorithm are used to detect the ship video in real time, and the key frames in the video are extracted respectively, as shown in Figure 10. At frame 106, frame 272, and frame 433, the YOLOv3 algorithm and the Ship-YOLOv3 algorithm have detected the target. For large target ships, the detection effect of Ship-YOLOv3 algorithm is better. The positioning is more accurate and the border is narrow. For small target ships, the Ship-YOLOv3 algorithm has better recognition rate and robustness. In this scene, the ship's feature extraction is greatly disturbed. In the training process, guided filtering and gray enhancement processing are carried out to improve the anti-interference ability of the improved algorithm. It can effectively detect the ship and overcome the adverse factors of light and fog.

4.3.3. Contrast Experiment under Normal Conditions. It can be seen from Figure 11 that the YOLOv3 algorithm only recognizes large-scale ships in frame 117. However, the inspection of small-scale ships is missing. The Ship-YOLOv3 algorithm can also accurately locate and identify small-scale ships, with remarkable effect. In 167 frames and 232 frames, the detection effect of Ship-YOLOv3 algorithm is better and the prediction frame deviates from the target ship in YOLOv3 algorithm.

Compared with YOLOv3 algorithm, the detection time of Ship-YOLOv3 algorithm in nighttime or foggy scene is reduced by 8.6 ms, as shown in Table 2. In the conventional scene, the detection time is reduced by 3.53 ms and 6.06 ms on average. In this paper, the Ship-YOLOv3 algorithm maintains a high recognition rate and improves the detection speed.

4.3.4. Classification and Recognition Experiment. It can be seen from Figure 12 that the Ship-YOLOv3 can extract different types of ship features. It can detect and locate three kinds of ships, and the recognition effect is good. And the recognition effect of small target ship is significant, which effectively reduces the probability of missing detection.

Through comparative analysis of Table 3, the Recall, Precision, and MAP of Ship-YOLOv3 are all higher than the YOLOv3 algorithm, but the total loss is lower, so the performance of our algorithm is more superior.

Figure 12: The Ship-YOLOv3 algorithm for real-time classification and detection of ships: (a) heavy bulk ships, (b) empty bulk ships, and (c) passenger and management ships.

Table 3: Comparative analysis of algorithms.

Algorithm	Recall (%)	Precision (%)	MAP (%)	Total loss (%)
YOLOv3	81.06	84.62	93.47	11.20
Ship-YOLOv3	92.56	97.12	96.52	9.794

5. Conclusion

In order to solve the problem of ship detection in complex scenes such as fog or night, a Ship-YOLOv3 algorithm is proposed to detect ship targets. It is mainly improved from two aspects: target box dimension clustering and YOLOv3 network structure optimization.

Firstly, the ship data is processed: the collected ship video is divided into frames to establish the ship dataset and guided filtering and gray enhancement processing are carried out on the night or foggy images to maintain the feature of the ship. Then, the self-made ship dataset is clustered into target box dimension: the size of prior box is improved to predict the position of ships. Lastly, the structure of YOLOv3 network is optimized: increase the input size of the image and reduce the convolution layers in the down sampling process. The multiscale output is used to adapt to different ship types and improve the detection accuracy.

The effectiveness of the improved algorithm in real-time detection and classification of ship dataset is verified by the comparative experiments in different scenarios. After 73700 step iteration, the loss of Ship-YOLOv3 is 0.01406 lower than that of YOLOv3, and the convergence effect is better. Compared with YOLOv3 algorithm, the detection time is reduced by 6.06 ms on average, which improves the detection speed while maintaining high recognition. The algorithm accelerates the convergence speed of the network and improves the precision of ship detection by 12.5% and recall rate by 11.5% on the premise of ensuring real-time performance.

Although the algorithm in this paper has achieved satisfactory results in real-time ship detection and classification, some remaining research work can be carried out to further improve our method performance. Firstly, the self-made ship dataset has less categories, so it is the focus of future research to increase the category of ships for detection. Secondly, we will continue to optimize the network structure of Ship-YOLOv3. The parameters of the network are simplified to adapt to ship classification and detection in different scenarios. Finally, the tracking identification of ships will be increased in the future, and the behavior of ships will be analyzed to ensure the safety of waterway transportation and improve the efficiency of port management and maritime inspection.

Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Vertex-Disjoint Paths in a 3-Ary n-Cube with Faulty Vertices

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The construction of vertex-disjoint paths (disjoint paths) is an important research topic in various kinds of interconnection networks, which can improve the transmission rate and reliability. The k-ary n-cube is a family of popular networks. In this paper, we determine that there are $m\,(2 \le m \le n)$ disjoint paths in 3-ary n-cube covering $Q_n^3 - F$ from S to T (many-to-many) with $|F| \le 2n - 2m$ and from s to T (one-to-many) with $|F| \le 2n - m - 1$ where s is in a fault-free cycle of length three.

1. Introduction

Vertex-disjoint paths (disjoint paths for short) are a set of paths in a graph that they do not share any vertices. A disjoint path cover of a graph is a disjoint path between two different vertices, and it covers all vertices of the graph. The disjoint path cover problem has been applied in many fields such as software testing, database design, and code optimization. Actually, vertex-disjoint paths can accelerate data transmission by providing parallel communication paths to avoid communication congestion. Besides, disjoint paths enhance the robustness of vertex failure and load balancing capability [1] and accelerate the transmission of large amounts of data by splitting data into disjoint communication paths of multiple vertices. The problem of many-to-many disjoint paths is an important problem in networks. It has attracted much attention because of its application in fault-tolerant routings for high-performance interconnection network.

Depending on the number of sources or sinks, there are one-to-one, one-to-many, and many-to-many disjoint paths problems. Among them, the one-to-one disjoint path is what we usually call the Hamilton path problem. A nonbipartite graph G is one-to-one m-disjoint path coverable if there is an m-disjoint path cover between any two vertices of G. A bipartite graph is one-to-one m-disjoint path coverable if there is an m-disjoint path cover between any two vertices in different partite sets. Shih and Kao [1] constructed the one-to-one m-disjoint path cover of k-ary n-cube for $1 \le m \le 2n$, $n \ge 2$ and $k \ge 3$. Li et al. [2] studied many-to-many k-disjoint paths in hypercubes with $f \le 2n - 2k - 2$, $1 \le k \le n - 1$ and each fault-free vertex has at least two fault-free neighbors. Chen in [3] considered many-to-many k disjoint (S, T)-paths in the hypercube with $f_v + f_e \le n - k - 1$, $1 \le k \le n - 1$ and any two sets S and T of k fault-free vertices in different parts. Zhang and Wang [4] proved that many-to-many m disjoint paths cover in k-ary n-cubes with even $k \ge 4$ and $1 \le m \le 2n - 1$. So, we consider some disjoint path problems about 3-ary n-cubes. There is also research on other graphs. Such as one-to-many disjoint paths of (n, k)-star graph [5] and hyper-star networks [6] and one-to-one disjoint path covers on alternating group graphs [7].

For a set $S = \{s_1, s_2, \ldots, s_k\}$ of k sources and a set $T = \{t_1, t_2, \ldots, t_k\}$ of k sinks in $V(G)$ with $S \cap T = \varnothing$, the many-to-many k-disjoint path problem is to determine whether there exist k disjoint paths each joining a source and a sink. In other words, we need to find k disjoint paths $P_1, P_2, \ldots, P_k$ in G joining S and T, which cover all the vertices in the graph; that is, $\cup_{1 \le i \le k} V(P_i) = V(G)$ and $V(P_i) \cap V(P_j) = \varnothing$ for $i \ne j$ and $1 \le i, j \le k$. In addition, this problem is also subclassified into paired and unpaired types. In paired type, each source should be joined to a specific sink; that is, s_i should be joined to t_i. Otherwise, each source can be joined to an arbitrary sink; that is, s_i could be joined to

t_j for $i \neq j$. We study the unpaired type in this paper. Specially, if $S = \{s\}$, then it is called one-to-many disjoint paths; that is, $V(P_i) \cap V(P_j) = \{s\}$ for all $i \neq j$ and $1 \leq i, j \leq k$.

With the development of science and the expansion of the scale of computer system, people have more higher requirements for the reliability of network. However, computer failure is inevitable in a system. So, the disjoint path cover problem is also extended to a graph with some faulty elements F (vertices and/or edges). A graph G is called f-fault hamiltonian-connected if each pair of vertices are joined by a hamiltonian path in $G - F$ for any faulty elements $|F| \leq f$.

In this paper, we study m disjoint paths of 3-ary n-cube covering $Q_n^3 - F$ with faulty vertices $|F| \leq 2n - 2m$ from S to T for $|S| = |T| = m$ in Theorem 1. Besides, we also consider m disjoint paths of 3-ary n-cube covering $Q_n^3 - F$ with faulty vertices $|F| \leq 2n - m - 1$ from s to T where $|T| = m$ and s is in a fault-free cycle of length three in Theorem 2.

2. Preliminaries

2.1. Notations. An interconnection network can be represented by a graph $G = (V, E)$, where V represents the vertex set and E represents the edge set. Let G be a simple undirected graph. For a vertex v, $N_G(v)$ is the set of vertices adjacent to v in G. The path $P = \langle v_1, v_2, \ldots, v_{n-1}, v_n \rangle$ is a sequence of adjacent vertices where all the vertices are distinct, which is denoted by $P = (v_1, v_n)$. If the path P satisfies $v_1 = v_n$, it is called a cycle. In addition, $P^{-1} = \langle v_n, v_{n-1}, \ldots, v_2, v_1 \rangle$. For $1 \leq i < j \leq n$, $P[v_i, v_j]$ denotes the section from v_i to v_j in P. For $uv_1, v_n w \in E$, $\{uv_1\} \cup P \cup \{v_n w\} = \langle u, v_1, \ldots, v_{n-1}, v_n, w \rangle$. A path or cycle which contains every vertex of a graph is called a Hamilton path (hamiltonian path) or Hamilton cycle (hamiltonian cycle) of the graph. A graph is hamiltonian if it contains a Hamilton cycle. A graph is hamiltonian-connected if there exists a Hamilton path between any two distinct vertices of the graph. For graph-theoretical terminology and notation not defined here, we follow [8].

2.2. The k-Ary n-Cube Q_n^k

Definition 1. The k-ary n-cube $Q_n^k (n \geq 1)$ is a graph with k^n vertices, each of which has the form $u = u_{n-1} u_{n-2} \ldots u_0$, where $0 \leq u_i \leq k - 1$ for every $0 \leq i \leq n - 1$. Two vertices $u = u_{n-1} u_{n-2} \ldots u_0$ and $v = v_{n-1} v_{n-2} \ldots v_0$ in Q_n^k are adjacent if and only if there exists an integer j, $0 \leq j \leq n - 1$, such that $u_j = v_j \pm 1 (\mathrm{mod} k)$ and $u_i = v_i$ for every $i \in \{0, 1, \ldots, n - 1\} \backslash \{j\}$. For clarity of presentation, we omit writing "(mod k)" in similar expressions for the remainder of the paper.

We discuss the 3-ary n-cube Q_n^3 with $2n$ regular. Obviously, Q_1^3 is a cycle of length 3, and Q_2^3 is a 3×3 wrap-around mesh. We can partition Q_n^3 along j-dimension, $0 \leq j \leq n - 1$, by deleting all the j-dimension edges, into three disjoint subcubes, $Q_n^3[0]$, $Q_n^3[1]$ and $Q_n^3[2]$ (abbreviated as $Q[0]$, $Q[1]$ and $Q[2]$, if there are no ambiguities). $Q[j]$ is isomorphic to Q_{n-1}^3 for $0 \leq j \leq 2$. For $0 \leq i, j \leq 2$ and $i \neq j$, the edge set between $Q[i]$ and $Q[j]$ is a perfect matching.

Lemma 1 (see [9]). *The k-ary n-cube is hamiltonian-connected when k is odd.*

Lemma 2 (see [10]). *Assume k is odd with $k \geq 3$. Let F be a set of faulty vertices and/or edges in a k-ary n-cube Q_n^k.*

(1) If $|F| \leq 2n - 2$, then the graph $Q_n^k - F$ is hamiltonian.

(2) If $|F| \leq 2n - 3$, then the graph $Q_n^k - F$ is hamiltonian-connected.

Lemma 3 (see [11]). *Let $n \geq 2$, $F \subset V(Q_n^3)$, $|F| \leq 2n - 4$. If s_1, t_1, s_2, t_2 are four vertices in $Q_n^3 - F$, then there are two vertex-disjoint paths $P_1 = (s_1, t_1)$ and $P_2 = (s_2, t_2)$ such that $V(P_1) \cup V(P_2) = V(Q_n^3 - F)$.*

3. Many-to-Many Disjoint Paths

Theorem 1. *Let Q_n^3 be a 3-ary n-cube for $n \geq 2$. Let F be a vertex faulty set of Q_n^3. Let $S, T \subset V(Q_n^3)$ with $2 \leq |S| = |T| = m \leq n$. If $|F| \leq 2n - 2m$, then there are m disjoint paths from S to T covering $Q_n^3 - F$.*

Proof. We prove the theorem by induction on m. When $m = 2$, $|S| = |T| = 2$ and $|F| \leq 2n - 4$. By Lemma 3, the theorem holds. Assume the theorem holds on $m - 1$. We prove the theorem holds on m for $n \geq m \geq 3$.

Suppose there is $s_i t_j \in E(Q_n^3)$ for $1 \leq i, j \leq m$. Note $|F \cup \{s_i, t_j\}| \leq 2n - 2m + 2 = 2n - 2(m - 1)$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $P_k (1 \leq k \leq m - 1)$ from $S \backslash \{s_i\}$ to $T \backslash \{t_j\}$ covering $Q_n^3 - F - \{s_i, t_j\}$. Let $P_m = \langle s_i, t_j \rangle$. Then, the theorem holds. Thus, we suppose $s_i t_j \notin E(Q_n^3)$ for all $1 \leq i, j \leq m$.

We decompose Q_n^3 into three isomorphic subgraphs $Q[0], Q[1], Q[2]$ along one dimension. Let $F_i = F \cap V(Q[i])$, $S_i = S \cap V(Q[i])$ and $T_i = T \cap V(Q[i])$ for $0 \leq i \leq 2$. Let $S = \{s_1, \ldots, s_m\}$ and $T = \{t_1, \ldots, t_m\}$. If $s_i, t_i \in V(Q[j])$ for $1 \leq i \leq m$ and $0 \leq j \leq 2$, then let $s_i^k = N(s_i) \cap V(Q[k])$ and $t_i^k = N(t_i) \cap V(Q[k])$ for $k \in (\{0, 1, 2\} \backslash \{j\})$.

Case 1. $|S_i| = |T_i|$ for all $0 \leq i \leq 2$.

Case 1.1. $|S_i| = |T_i| = m$ for some $0 \leq i \leq 2$.

Without loss of generality, let $|S_0| = |T_0| = m$. Then, $|S_1| = |S_2| = |T_1| = |T_2| = 0$. Let $S_0 = \{s_1, \ldots, s_m\}$ and $T_0 = \{t_1, \ldots, t_m\}$.

Case 1.1.1. $|F_0| = 2n - 2m$.

Then, $|F_1| = |F_2| = 0$. Let $S_0' = S_0 \backslash \{s_m\}$ and $T_0' = T_0 \backslash \{t_m\}$. Then, $|S_0'| = |T_0'| = m - 1$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, t_i) (1 \leq i \leq m - 1)$ from S_0' to T_0' covering $Q[0] - F_0 - s_m - t_m$. Then, $s_m, t_m \in \cup_{i=1}^{m-1} V(R_i)$.

Case 1.1.1.1. $s_m, t_m \in V(R_i)$ for some $1 \leq i \leq m - 1$.

Without loss of generality, let $s_m, t_m \in V(R_1)$. Suppose s_m is before t_m in R_1. Then, $R_1 = R_1[s_1, s_m] \cup R_1[s_m, t_m] \cup R_1[t_m, t_1]$. Let u (resp. v) be the before

vertex (resp. later vertex) of s_m (resp. t_m) in R_1. Let $u_1 = N(u) \cap V(Q[1])$ and $v_1 = N(v) \cap V(Q[1])$. By Lemma 1, there is a Hamilton path $L = (u_1, v_1)$ in $Q[1]$. Let $xy \in E(L)$, $x_2 = N(x) \cap V(Q[2])$ and $y_2 = N(y) \cap V(Q[2])$. By Lemma 1, there is a Hamilton path $M = (x_2, y_2)$ in $Q[2]$. Let $P_1 = R_1[s_1, u] \cup \{uu_1, vv_1\} \cup (L - xy) \cup \{xx_2, yy_2\} \cup M \cup R_1[v, t_1]$, $P_i = R_i (2 \le i \le m-1)$ and $P_m = R_1[s_m, t_m]$. Similarly, t_m is before s_m in R_1.

Case 1.1.1.2. $s_m \in V(R_i)$, $t_m \in V(R_j)$ for $1 \le i, j \le m-1$ and $i \ne j$.

Without loss of generality, let $s_m \in V(R_1)$ and $t_m \in V(R_2)$. Let u and v be the before and later vertex of s_m in R_1, respectively. Let $u_1 = N(u) \cap V(Q[1])$ and $v_1 = N(v) \cap V(Q[1])$. Let x and y be the before and later vertex of t_m in R_2, respectively. Let $x_1 = N(x) \cap V(Q[1])$ and $y_1 = N(y) \cap V(Q[1])$. Note $|F_1| = 0 \le 2n-6$ for $n \ge 3$. By Lemma 3, there are two disjoint paths $L_1 = (u_1, v_1)$ and $L_2 = (x_1, y_1)$ covering $Q[1] - F_1$. By Lemma 1, there is a Hamilton path $M = (s_m^2, t_m^2)$ in $Q[2]$. Let $P_1 = (R_1 - s_m) \cup \{uu_1, vv_1\} \cup L_1$, $P_2 = (R_2 - t_m) \cup \{xx_1, yy_1\} \cup L_2$, $P_i = R_i (3 \le i \le m-1)$ and $P_m = \{s_m s_m^2, t_m t_m^2\} \cup M$.

Case 1.1.2. $|F_0| = 2n - 2m - 1$.

Then, $|F_1 \cup F_2| \le 1$. Without loss of generality, let $|F_1| = 0$ and $|F_2| \le 1$. Let $S_0' = S_0 \backslash \{s_m\}$ and $T_0' = T_0 \backslash \{t_m\}$. Then, $|S_0'| = |T_0'| = m-1$. Note $|F_0 \cup \{t_m\}| = 2n - 2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $R_i = (s_i, t_i) (1 \le i \le m-1)$ from S_0' to T_0' covering $Q[0] - F_0 - t_m$. Then, $s_m \in V(R_i)$ for some $1 \le i \le m-1$. Without loss of generality, let $s_m \in V(R_1)$. Then, $R_1 = R_1[s_1, s_m] \cup R_1[s_m, t_1]$. Let u and v be the before and later vertex of s_m in R_1, respectively. Let $u_i = N(u) \cap V(Q[i])$ and $v_i = N(v) \cap V(Q[i])$ for $1 \le i \le 2$.

If $u_2 \in F_2$ or $v_2 \in F_2$, then $s_m^2 \notin F_2$ and $t_m^2 \notin F_2$. By Lemma 2, there is a path $L_1 = (u_1, v_1)$ and $M_1 = (s_m^2, t_m^2)$ covering $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_1 = (R_1 - s_m) \cup \{uu_1, vv_1\} \cup L_1$, $P_i = R_i (2 \le i \le m-1)$ and $P_m = \{s_m s_m^2, t_m t_m^2\} \cup M_1$.

Suppose $u_2 \notin F_2$ and $v_2 \notin F_2$. By Lemma 2, there is a path $L_2 = (s_m^1, t_m^1)$ and $M_2 = (u_2, v_2)$ covering $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_1 = (R_1 - s_m) \cup \{uu_2, vv_2\} \cup M_2$, $P_i = R_i (2 \le i \le m-1)$ and $P_m = \{s_m s_m^1, t_m t_m^1\} \cup L_2$.

Case 1.1.3. $2n - 2m - 3 \le |F_0| \le 2n - 2m - 2$.

In this case, $2n - 2m - 2 \ge 0$. We get $n \ge m + 1 \ge 4$ for $m \ge 3$. We have $|F_1 \cup F_2| \le 3$. Without loss of generality, let $|F_1| \le \lfloor 3/2 \rfloor = 1$. Since $|S| = |T| = m \ge 3 > 1 \ge |F_1|$, there are two vertices $s_i^1, t_i^1 \notin F_1$ for some $1 \le i \le m$. Without loss of generality, let $s_m^1, t_m^1 \notin F_1$. Let $S_0' = S_0 \backslash \{s_m\}$ and $T_0' = T_0 \backslash \{t_m\}$. Then, $|S_0'| = |T_0'| = m-1$. Note $|F_0 \cup \{s_m, t_m\}| \le 2n - 2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $P_i = (s_i, t_i) (1 \le i \le m-1)$ from S_0' to T_0' covering $Q[0] - F_0 - s_m - t_m$. Since $|V(Q[1])| - |F_1 \cup F_2| - |\{s_m^1, t_m^1\}| \ge 3^{n-1} - 3 - 2 \ge 4$ for $n \ge 3$, there are two edges $x_1 x_2$ and $y_1 y_2$ such that $\{x_i, y_i\} \subset V(Q[i] - F_i - \{s_m^1, t_m^1\})$ for $1 \le i \le 2$. Note $|F_1| \le 1 \le 2n - 6$ for $n \ge 4$. By Lemma 3, there are two disjoint paths $L_1 = (s_m^1, x_1)$ and $L_2 = (y_1, t_m^1)$ covering $Q[1] - F_1$. Note $|F_2| \le 3 \le 2n - 5$ for $n \ge 4$. By Lemma 2, there is a path $M = (x_2, y_2)$ covering $Q[2] - F_2$. Let $P_m = \{s_m s_m^1\} \cup L_1 \cup \{x_1 x_2\} \cup M \cup \{y_2 y_1\} \cup L_2 \cup \{t_m^1 t_m\}$.

Case 1.1.4. $|F_0| \le 2n - 2m - 4$.

Let $N_{Q[0]}(s_m) = \{u_1, \ldots, u_{2n-2}\}$ and $u_i^j = N(u_i) \cap V(Q[j])$ for $1 \le i \le 2n-2$ and $j = 1, 2$. Since $|N_{Q[0]}(s_m)| - |S \backslash \{s_m\}| = 2n - 2 - (m-1) > 2n - 2m \ge |F|$ for $m \ge 3$, there is an edge $u_i u_i^1$ such that $u_i \in V(Q[0] - F_0 - S)$ and $u_i^1 \in V(Q[1] - F_1)$ for $1 \le i \le 2n-2$. Without loss of generality, let $u_1 u_1^1$ be satisfied. Let $N_{Q[0]}(t_i) = \{v_1, \ldots, v_{2n-2}\}$ and $v_i^j = N(v_i) \cap V(Q[j])$ for $1 \le i \le 2n-2$ and $j = 1, 2$. Since $|N_{Q[0]}(t_m)| - |T \backslash \{t_m\}| - |\{u_1\}| \ge 2n - 2 - (m-1) - 1 > 2n - 2m \ge |F|$ for $m \ge 3$, there is an edge $v_i v_i^1$ such that $v_i \in V(Q[0] - F_0 - u_1)$ and $v_i^1 \in V(Q[1] - F_1 - u_1^1)$ for $1 \le i \le 2n-2$. Without loss of generality, let $v_1 v_1^1$ be satisfied.

Note $|F_0 \cup \{s_m, t_m, u_1, v_1\}| \le 2n - 2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $P_i = (s_i, t_i) (1 \le i \le m-1)$ covering. Since $|V(Q[1])| - |F| - |\{u_1^1, v_1^1\}| \ge 3^{n-1} - (2n - 2m) - 2 \ge 7$ for $3 \le m \le n$, there are two edges $x_1 x_2$, $y_1 y_2$ such that $\{x_i, y_i\} \subset V(Q[i] - F_i - \{u_1^1, v_1^1\})$ for $i = 1, 2$. Note $|F_1| \le 2n - 2m \le 2n - 6$ for $m \ge 3$. By Lemma 3, there are two disjoint paths $L_1 = (u_1^1, x_1)$ and $L_2 = (y_1, v_1^1)$ covering $Q[1] - F_1$. Note $|F_2| \le 2n - 2m < 2n - 5$ for $m \ge 3$. By Lemma 2, there is a path $M = (x_2, y_2)$ covering $Q[2] - F_2$. Let $P_m = \{s_m u_1, u_1 u_1^1, t_m v_1, v_1 v_1^1\} \cup L_1 \cup L_2 \cup \{x_1 x_2, y_1 y_2\} \cup M$.

Case 1.2. $|S_i| = |T_i| = m - 1$ for some $0 \le i \le 2$.

Without loss of generality, let $|S_0| = |T_0| = 1$, $|S_1| = |T_1| = m-1$ and $|S_2| = |T_2| = 0$. Let $S_0 = \{s_1\}$, $T_0 = \{t_1\}$, $S_1 = \{s_2, \ldots, s_m\}$ and $T_1 = \{t_2, \ldots, t_m\}$. By the induction hypothesis, there are $(m-1)$ disjoint paths $P_i = (s_i, t_i) (2 \le i \le m)$ covering $Q[1] - F_1$. Since $|V(Q[i])| - |F| - |S_0| - |T_0| \ge 3^{n-1} - (2n - 2m) - 2 \ge 7$ for $3 \le m \le n$, there are two edges $x_1 x_2$, $y_1 y_2$ such that $\{x_i, y_i\} \subset V(Q[i] - F_i - S_0 - T_0)$ for $i = 0, 2$. Note $|F_0| \le 2n - 2m \le 2n - 6$ for $m \ge 3$. By Lemma 3, there are two disjoint paths $R_1 = (s_1, x_0)$ and $R_2 = (y_0, t_1)$ covering $Q[0] - F_0$. By Lemma 2, there is a path $M = (x_2, y_2)$ covering $Q[2] - F_2$. Let $P_1 = R_1 \cup \{x_0 x_2\} \cup M \cup \{y_0 y_2\} \cup R_2$.

Case 1.3. $1 \le |S_i| = |T_i| \le m - 2$ for all $0 \le i \le 2$.

Let $|S_i| = |T_i| = p_i \le m - 2$ for $0 \le i \le 2$. Then, $p_0 + p_1 + p_2 = m$. Note $|F_i| \le 2n - 2m \le 2(n-1) - 2p_i$. By the induction hypothesis, there are p_i disjoint paths covering $Q[i] - F_i$ for $0 \le i \le 2$.

Case 1.4. $|S_i| = |T_i| = 0$ for only one $0 \le i \le 2$.

Let $|S_0| = |T_0| = 0$. If there is $|S_i| = |T_i| = m - 1$ or $|S_i| = |T_i| = 1$ for some $1 \le i \le 2$, then it is similar to Case 1.2.

Thus, suppose $2 \le |S_i| = |T_i| \le m-2$ for $1 \le i \le 2$. Let $|S_1| = |T_1| = p$. Then, $|S_2| = |T_2| = m-p$. Let $S_1 = \{s_1, \ldots, s_p\}$, $T_1 = \{t_1, \ldots, t_p\}$, $S_2 = \{s_{p+1}, \ldots, s_m\}$ and $T_2 = \{t_{p+1}, \ldots, t_m\}$. Note $|S_1| = |T_1| = p \le m-2$. By the induction hypothesis, there are p disjoint paths $P_i = (s_i, t_i)\,(1 \le i \le p)$ covering $Q[1] - F_1$.

Since $|V(Q[i])| - |F| - |S| - |T| \ge 3^{n-1} - (2n-2m) - 2m \ge 3$ for $n \ge 3$, there are two edges u_0u_2, v_0v_2 such that $u_i, v_i \in V(Q[i] - F - S - T)$ for $i = 0, 2$. Note $|S_2 \cup \{v_2\}| = |T_2 \cup \{u_2\}| = m-p+1 \le m-1$. By the induction hypothesis, there are $(m-p+1)$ disjoint paths $P_i = (s_i, t_i)\,(p+1 \le i \le m-1)$, $M_1 = (s_m, u_2)$ and $M_2 = (v_2, t_m)$ covering $Q[2] - F_2$. Note $|F_0| \le 2n-2m < 2n-5$ for $m \ge 3$. By Lemma 2, there is a path $R = (u_0, v_0)$ covering $Q[0] - F_0$. Let $P_m = M_1 \cup \{u_2u_0\} \cup R \cup \{v_0v_2\} \cup M_2$.

Case 2. $|S_i| = |T_i|$ for only one $0 \le i \le 2$.

Without loss of generality, let $|S_0| = |T_0|$. Then, $|S_1| \ne |T_1|$, $|S_2| \ne |T_2|$ and $|S_1| + |S_2| = |T_1| + |T_2|$. Without loss of generality, let $|S_1| > |T_1|$. Then, $|S_2| < |T_2|$ and $|S_1| - |T_1| = |T_2| - |S_2|$. Since $|V(Q[i])| - |F| - |S| - |T| \ge 3^{n-1} - (2n-2m) - 2m \ge m$ for $3 \le m \le n$, there are m cycles of length three $C_i = \langle w_i^0, w_i^1, w_i^2, w_i^0 \rangle$ for $1 \le i \le m$ such that $w_i^j \in V(Q[j] - F_j - S - T)$ for $0 \le j \le 2$.

Case 2.1. $|S_0| = |T_0| = 0$.

Then, $|S_1| + |S_2| = |T_1| + |T_2| = m$.

Case 2.1.1. $|S_1| = m$ and $|T_1| = 0$.

Then, $|S_2| = 0$ and $|T_2| = m$. Let $S_1 = \{s_1, \ldots, s_m\}$, $T_2 = \{t_1, \ldots, t_m\}$.

Case 2.1.1.1. $|F_0| = 0$.

Let $S_0' = \{s_i^0 \colon 1 \le i \le m-1\}$, $T_0' = \{t_i^0 \colon 1 \le i \le m-1\}$. Then, $|S_0'| = |T_0'| = m-1$. By the induction hypothesis, there are $(m-1)$ disjoint paths $R_i = (s_i^0, t_i^0)\,(1 \le i \le m-1)$ from S_0' to T_0' covering $Q[0] - F_0$. Let $P_i = \{s_is_i^0\} \cup R_i \cup \{t_i^0t_i\}\,(1 \le i \le m-1)$.

Let $W_1 = \{w_1^1\}$ and $W_2 = \{w_1^2\}$. Note $|S_1'| = |S_1 \backslash \{s_m\}| = m-1 = |T_2 \backslash \{t_m\}| = |T_2'|$ and $|F_1| + |S_1'| \le 2n-2m+(m-1) = 2n-m-1 \le 2n-5$ for $m \ge 4$. By Lemma 2, there is a path $L = (s_m, w_1^1)$ covering $Q[1] - F_1$. Similarly, there is a path $M = (w_1^2, t_m)$ covering $Q[2] - F_2$. Let $P_m = L \cup \{w_1^1w_1^2\} \cup M$.

Suppose $m = 3$. Then, $|F| \le 2n-2m = 2n-6$. Let $W_1 = \{w_1^1, w_2^1\}$ and $W_2 = \{w_1^2, w_2^2\}$. By Lemma 2, there are two paths $L_1 = (s_1, w_1^1)$ and $L_2 = (s_2, w_2^1)$ (resp. $M_1 = (w_2^1, t_1)$ and $M_2 = (w_2^2, t_2)$) covering $Q[1] - F_1$ (resp. $Q[2] - F_2$). Let $P_i' = L_i \cup \{w_i^1w_i^2\} \cup M_i$ for $i = 1, 2$. Then, $s_3, t_3 \in P_i'$ for $1 \le i \le 2$.

Suppose $s_3, t_3 \in P_1'$. Let $P_3 = P_1'[s_3, t_3]$. Let u and v be the before and later vertex of s_3 and t_3, respectively. Let $u_0 = N(u) \cap V(Q[0])$ and $v_0 = N(v) \cap V(Q[0])$.

If $u_0 = v_0$, then let $P_1 = P_1'[s_1, u] \cup \{uu_0, v_0v\} \cup P_1'[v, t_1]$. Let $wz \in P_2'$, $w_0 = N(w) \cap V(Q[0])$ and $z_0 = N(z) \cap V(Q[0])$. Note $|F_0 \cup \{u_0\}| = 1 \le 2n-5$ for $n \ge 3$. By Lemma 2, there is a path $R = (w_0, z_0)$ covering $Q[0] - F_0 - u_0$. Let $P_2 = (P_2' - wz) \cup \{ww_0, zz_0\} \cup R$.

Suppose $u_0 \ne v_0$. Note $|F_0| = 0$. By Lemma 1, there is a Hamilton path $R = (u_0, v_0)$ in $Q[0]$. Let $P_1 = P_1'[s_1, u] \cup \{uu_0, vv_0\} \cup R \cup P_1'[v, t_1]$ and $P_2 = P_2'$.

Suppose $s_3 \in P_1'$ and $t_3 \in P_2'$. Let u and v be the before and later vertex of s_3 and t_3 in P_1' and P_2', respectively. Let $u_0 = N(u) \cap V(Q[0])$ and $v_0 = N(v) \cap V(Q[0])$.

If $u_0 = v_0$, then let $P_1 = P_1'[s_1, u] \cup \{uu_0, v_0v\} \cup P_2'[v, t_2]$. Let $wz \in L_2$, $w_0 = N(w) \cap V(Q[0])$ and $z_0 = N(z) \cap V(Q[0])$. Note $|F_0 \cup \{u_0\}| = 1 \le 2n-5$ for $n \ge 3$. By Lemma 2, there is a path $R = (w_0, z_0)$ covering $Q[0] - F_0 - u_0$. Let $P_2 = (P_2'[s_2, t_3] - wz) \cup \{ww_0, zz_0\} \cup R$ and $P_3 = P_1'[s_3, t_1]$.

Suppose $u_0 \ne v_0$. Note $|F_0| = 0$. By Lemma 1, there is a Hamilton path $R = (u_0, v_0)$ in $Q[0]$. Let $P_1 = P_1'[s_1, u] \cup \{uu_0, vv_0\} \cup R \cup P_2'[v, t_2]$, $P_2 = P_2'[s_2, t_3]$ and $P_3 = P_1'[s_3, t_1]$.

Case 2.1.1.2. $|F_0| = 1$.

There is at most one vertex $s_i^0 \in F_0$ or $t_i^0 \in F_0$ for $1 \le i \le m$. Without loss of generality, let $s_i^0, t_i^0 \notin F_0$ for all $1 \le i \le m-1$. Let $S_0' = \{s_i^0 \colon 1 \le i \le m-2\}$, $T_0' = \{t_i^0 \colon 1 \le i \le m-2\}$. Then, $|S_0'| = |T_0'| = m-2$. Note $|F_0| = 1 \le 2(n-1) - 2(m-2)$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_i = (s_i^0, t_i^0)\,(1 \le i \le m-2)$ from S_0' to T_0' covering $Q[0] - F_0$. Let $P_i = \{s_is_i^0\} \cup R_i \cup \{t_i^0t_i\}\,(1 \le i \le m-2)$.

Let $W_1 = \{w_1^1, w_2^1\}$ and $W_2 = \{w_1^2, w_2^2\}$. Note $|S_1'| = |S_1 \backslash \{s_{m-1}, s_m\}| = m-2 = |T_2 \backslash \{t_{m-1}, t_m\}| = |T_2'|$ and $|F_1| + |S_1'| \le 2n-2m-1+(m-2) = 2n-m-m-3 \le 2n-6$ for $m \ge 3$. By Lemma 3, there are two disjoint paths $L_1 = (s_{m-1}, w_1^1)$ and $L_2 = (s_m, w_2^1)$ covering $Q[1] - F_1$. Similarly, there are two disjoint paths $M_1 = (w_1^2, t_{m-1})$ and $M_2 = (w_2^2, t_m)$ covering $Q[2] - F_2$. Let $P_{m-1} = L_1 \cup \{w_1^1w_1^2\} \cup M_1$ and $P_m = L_2 \cup \{w_2^1w_2^2\} \cup M_2$.

Case 2.1.1.3. $|F_0| \ge 2$.

Then, $|F_i| \le 2n-2m-2$ for $i = 1, 2$. Let $N_{Q[1]}(s_m) = \{u_1, \ldots, u_{2n-2}\}$ and $u_i^0 = N(u_i) \cap V(Q[0])$ for $1 \le i \le 2n-2$. Since $|N_{Q[1]}(s_m)| - |S \backslash \{s_m\}| = 2n-2-(m-1) > 2n-2m \ge |F|$ for $m \ge 3$, there is an edge $u_iu_i^0\,(1 \le i \le 2n-2)$ such that $u_i \in V(Q[1] - F_1 - S_1)$ and $u_i^0 \in V(Q[0] - F_0)$. Without loss of generality, let $u_1u_1^0$ be satisfied. Let $N_{Q[2]}(t_m) = \{v_1, \ldots, v_{2n-2}\}$ and $v_i^0 = N(v_i) \cap V(Q[0])$ for $1 \le i \le 2n-2$. Since $|N_{Q[2]}(t_m)| - |T \backslash \{t_m\}| = 2n-2-(m-1) > 2n-2m+1 \ge |F| + |\{u_1^0\}|$ for $m \ge 3$, there is an edge $v_iv_i^0$ such that $v_i \in V(Q[2] - F_2 - T_2)$ and $v_i^0 \in V(Q[0] - F_0 - \{u_1^0\})$. Without loss of generality, let $v_1v_1^0$ be satisfied. Let $S_1' = S_1 \backslash \{s_m\}$ and $T_2' = T_2 \backslash \{t_m\}$. Then, $|S_1'| = |T_2'| = m-1$. Let $W_1 = \{w_i^1 \colon 1 \le i \le m-1\}$ and $W_2 = \{w_i^2 \colon 1 \le i \le m-1\}$. Note $|F_1 \cup \{s_m, u_1\}| \le 2n-2m$ and $|F_2 \cup \{t_m, v_1\}| \le 2n-2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $L_i = (s_i, w_i^1)$ (resp. $M_i = (w_i^2, t_i)$) $(1 \le i \le m-1)$ from S_1' to W_1 (resp. from W_2 to T_2') covering $Q[1] - F_1 - s_m - u_1$ (resp. $Q[2] - F_2 - t_m -$

v_1). Note $|F_0| \le 2n-2m < 2n-5$ for $m \ge 3$. By Lemma 2, there is a path $R = (u_1^0, v_1^0)$ covering $Q[0]-F_0$. Let $P_i = L_i \cup \{w_i^1 w_i^2\} \cup M_i\,(1 \le i \le m-1)$ and $P_m = \{s_m u_1, u_1 u_1^0, t_m v_1, v_1 v_1^0\} \cup R$.

Case 2.1.2. $|S_1| = m$ and $1 \le |T_1| \le m-1$.

Let $S_1 = \{s_1, \ldots, s_m\}$. Let $|T_1| = p \le m-1$. Then, $|T_2| = m-p \le m-1$. Let $T_1 = \{t_1, \ldots, t_p\}$ and $T_2 = \{t_{p+1}, \ldots, t_m\}$. Let $W_2 = \{w_i^2\colon p+1 \le i \le m\}$. Note $|W_2| = |T_2| = m-p \le m-1$. By the induction hypothesis, there are $(m-p)$ disjoint paths $M_i = (w_i^2, t_i)\,(p+1 \le i \le m)$ covering $Q[2]-F_2$.

Case 2.1.2.1. $|F_0| = 0$.

Let $W_1 = \{w_i^1\colon p+1 \le i \le m-1\}$ for $p \le m-2$, $W_1 = \varnothing$ for $p = m-1$. Then, $|W_1| = m-p-1$. Note $|T_1| + |W_1| = p + (m-p-1) = m-1 = |S_1 \backslash \{s_m\}|$. By the induction hypothesis, there are $(m-1)$ disjoint paths $L_i = (s_i, t_i)\,(1 \le i \le p)$ and $L_i = (s_i, w_i^1)\,(p+1 \le i \le m-1)$ covering $Q[1]-F_1$. Let $P_i' = L_i\,(1 \le i \le p)$, $P_i' = L_i \cup \{w_i^1 w_i^2\} \cup M_i\,(p+1 \le i \le m-1)$. Then, $s_m \in V(P_j')$ for some $1 \le j \le m-1$. We get $P_j' = P_j'[s_j, s_m] \cup P_j'[s_m, t_j]$. Let u and v be the before and later vertex of s_m in P_j', respectively. Let $u_0 = N(u) \cap V(Q[0])$, $v_0 = N(v) \cap V(Q[0])$. Let $W_0 = \{w_m^0\}$.

Suppose $w_m^0 \ne u_0$ and $w_m^0 \ne v_0$. Note $|F_0| = 0$. By Lemma 3, there are two disjoint paths $R_1 = (u_0, v_0)$ and $R_2 = (s_m^0, w_m^0)$ covering $Q[0]-F_0$. Let $P_j = (P_j' - s_m) \cup \{uu_0, vv_0\} \cup R_1$, $P_i = P_i'(i \in \{1, \ldots, m-1\} \backslash \{j\})$ and $P_m = \{s_m s_m^0, w_m^2 w_m^0\} \cup R_2 \cup M_m$.

Suppose $w_m^0 = u_0$. Let z be the before vertex of u in P_j' and $z_0 = N(z) \cap V(Q[0])$. Note $|F_0 \cup \{u_0\}| = 1 \le 2n-5$ for $n \ge 3$. By Lemma 2, there is a path $R = (z_0, v_0)$ covering $Q[0]-F_0-u_0$. Let $P_j = P_j'[s_j, z] \cup \{zz_0, vv_0\} \cup R \cup P_j'[v, t_j]$, $P_i = P_i'(i \in \{1, \ldots, m-1\} \backslash \{j\})$ and $P_m = \{s_m u, uu_0, w_m^0 w_m^2\} \cup M_m$. Similar to $w_m^0 = v_0$.

Case 2.1.2.2. $|F_0| = 1$.

Then, $|F_i| \le 2n-2m-1$ for $i = 1, 2$. Since $|S_1| = m > 1 = |F_0|$, there is a vertex $s_i \in S_1$ such that $s_i^0 \notin F_0$ for $1 \le i \le m$. Without loss of generality, let $s_m^0 \notin F_0$. Let $W_1 = \{w_i^1\colon p+1 \le i \le m-1\}$ for $p \le m-2$, $W_1 = \varnothing$ for $p = m-1$. Then, $|W_1| = m-p-1$. Note $|T_1| + |W_1| = p + (m-p-1) = m-1 = |S_1 \backslash \{s_m\}|$ and $|F_1 \cup \{s_m\}| \le 2n-2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $L_i = (s_i, t_i)\,(1 \le i \le p)$ and $L_i = (s_i, w_i^1)\,(p+1 \le i \le m-1)$ covering $Q[1]-F_1-s_m$. Let $P_i = L_i\,(1 \le i \le p)$ and $P_i = L_i \cup \{w_i^1 w_i^2\} \cup M_i$ for $(p+1 \le i \le m-1)$.

Let $W_0 = \{w_m^0\}$. Note $|F_0| = 1 \le 2n-5$ for $n \ge 3$. By Lemma 2, there is a path $R = (s_m^0, w_m^0)$ covering $Q[0]-F_0$. Let $P_m = \{s_m s_m^0\} \cup R \cup \{w_m^0 w_m^2\} \cup M_m$.

Case 2.1.2.3. $|F_0| \ge 2$.

Then, $|F_i| \le 2n-2m-2$ for $i = 1, 2$. Let $N_{Q[1]}(s_m) = \{u_1, \ldots, u_{2n-2}\}$ and $u_i^0 = N(u_i) \cap V(Q[0])$ for $1 \le i \le 2n-2$. Since $|N_{Q[1]}(s_m)| - |S \backslash \{s_m\}| = 2n-2-(m-1) > 2n-2m+1 \ge |F \cup \{w_m^0\}|$ for $m \ge 3$, there is an edge $u_i u_i^0\,(1 \le i \le 2n-2)$ such that $u_i \in V(Q[1]-F_1-S_1)$ and $u_i^0 \in V(Q[0]-F_0-\{w_m^0\})$. Without loss of generality, let $u_1 u_1^0$ be satisfied.

Let $W_1 = \{w_i^1\colon p+1 \le i \le m-1\}$ for $p \le m-2$, $W_1 = \varnothing$ for $p = m-1$. Then, $|W_1| = m-p-1$. Note $|T_1| + |W_1| = p + (m-p-1) = m-1 = |S_1 \backslash \{s_m\}|$ and $|F_1 \cup \{s_m, u_1\}| \le 2n-2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $L_i = (s_i, t_i)\,(1 \le i \le p)$ and $L_i = (s_i, w_i^1)\,(p+1 \le i \le m-1)$ covering $Q[1]-F_1-s_m-u_1$. Let $P_i = L_i\,(1 \le i \le p)$ and $P_i = L_i \cup \{w_i^1 w_i^2\} \cup M_i\,(p+1 \le i \le m-1)$.

Let $W_0 = \{w_m^0\}$. Note $|F_0| \le 2n-2m < 2n-5$ for $m \ge 3$. By Lemma 2, there is a path $R = (u_1^0, w_m^0)$ covering $Q[0]-F_0$. Let $P_m = \{s_m u_1, u_1 u_1^0\} \cup R \cup \{w_m^0 w_m^2\} \cup M_m$.

Case 2.1.3. $1 \le |S_1| \le m-1$.

Let $|S_1| = p$. Then, $|S_2| = m-p$. Let $S_1 = \{s_1, \ldots, s_p\}$ and $S_2 = \{s_{p+1}, \ldots, s_m\}$. If $|T_1| = m$ or $|T_2| = m$, then it is similar to Case 2.1.2. Thus, $1 \le |T_i| \le m-1$ for $i = 1, 2$. Let $1 \le |T_1| = q < p = |S_1| \le m-1$. Then, $1 \le |T_2| = m-q \le m-1$. Let $T_1 = \{t_1, \ldots, t_q\}$ and $T_2 = \{t_{q+1}, \ldots, t_m\}$.

Let $W_j = \{w_i^j\colon q+1 \le i \le p-1\}$ for $p-q \ge 2$ and $W_j = \varnothing$ for $p-q = 1$ where $j = 1, 2$. Then, $|W_1| = |W_2| = p-q-1 \le m-3$. Since $|V(Q[i])| - |F| - |S| - |T| - |W_i| \ge 3^{n-1} - (2n-2m) - 2m - (m-3) \ge 3$ for $3 \le m \le n$, there are two edges $u_0 u_1$, $v_0 v_2$ such that $\{u_i, v_i\} \subset V(Q[i]-F-S-T-W_i)$ for $0 \le i \le 2$. Note $|F_0| = 2n-2m < 2n-5$ for $m \ge 3$. By Lemma 2, there is a path $R = (u_0, v_0)$ covering $Q[0]-F_0$.

Note $|W_2| + |\{v_2\}| + |S_2| = (p-q-1) + 1 + (m-p) = m-q = |T_2|$. By the induction hypothesis, there are $(m-q)$ disjoint paths $M_i = (w_i^2, t_i)\,(q+1 \le i \le p-1)$, $M_p = (v_2, t_p)$ and $M_i = (s_i, t_i)\,(p+1 \le i \le m)$ from $S_2 \cup W_2 \cup \{v_2\}$ to T_2 covering $Q[2]-F_2$.

Note $|S_1| = p = q + (p-q-1) + 1 = |T_1| + |W_1| + |\{u_1\}|$. By the induction hypothesis, there are p disjoint paths $L_i = (s_i, t_i)\,(1 \le i \le q)$, $L_i = (s_i, w_i^1)\,(q+1 \le i \le p-1)$, $L_p = (s_p, u_1)$ from S_1 to $T_1 \cup W_1 \cup \{u_1\}$ covering $Q[1]-F_1$.

Let $P_i = L_i\,(1 \le i \le q)$, $P_i = L_i \cup \{w_i^1 w_i^2\} \cup M_i\,(q+1 \le i \le p-1)$, $P_p = L_p \cup \{u_1 u_0\} \cup R \cup \{v_0 v_2\} \cup M_p$ and $P_i = M_i\,(p+1 \le i \le m)$.

Case 2.2. $1 \le |S_0| = |T_0| \le m-1$.

By the induction hypothesis, there are $|S_0|$ disjoint paths from S_0 to T_0 covering $Q[0]-F_0$. Then, $1 \le |S_1| - |T_1| = x = |T_2| - |S_2| \le m-1$. Let $W_j = \{w_i^j\colon 1 \le i \le x\}$ for $j = 1, 2$. Then, $|T_1| + |W_1| = |S_1| \le m-1$ and $|S_2| + |W_2| = |T_2| \le m-1$. By the induction hypothesis, there are $|S_1|$ (resp. $|T_2|$) disjoint paths from S_1 to $T_1 \cup W_1$ (resp. $S_2 \cup W_2$ to T_2) covering $Q[1]-F_1$ (resp. $Q[2]-F_2$).

Case 3. $|S_i| \ne |T_i|$ for all $0 \le i \le 2$.

By Case 2, there are m cycles of length three $C_i = \langle w_i^0, w_i^1, w_i^2, w_i^0 \rangle$ for $1 \le i \le m$ such that $w_i^j \in V(Q[j] - F_j - S - T)$ for $0 \le j \le 2$.

Case 3.1. $|S_i| > |T_i|$ for only one $0 \le i \le 2$.

Without loss of generality, let $|S_0| > |T_0|$. Then, $|S_1| < |T_1|$, $|S_2| < |T_2|$. We get $1 \le |S_0| - |T_0| = (|T_1| - |S_1|) + (|T_2| - |S_2|) \le m$.

Case 3.1.1. $|S_0| = m$ and $|T_0| = 0$.

Then, $|S_1| = |S_2| = 0$ and $1 \le |T_1|, |T_2| \le m - 1$. Let $|T_1| = p$ and $|T_2| = m - p$. Let $S_0 = \{s_1, \ldots, s_m\}$, $T_1 = \{t_1, \ldots, t_p\}$and $T_2 = \{t_{p+1}, \ldots, t_m\}$. Without loss of generality, let $|T_1| = p \ge m - p = |T_2|$. Then, $2p \ge m \ge 3$. We get $p \ge 2$ and $m - p \le m - 2$. Let $W_2 = \{w_i^1 \colon p + 1 \le i \le m\}$. By the induction hypothesis, there are $(m - p)$ disjoint paths $M_i = (w_i^2, t_i)(p + 1 \le i \le m)$ from W_2 to T_2 covering $Q[2] - F_2$.

Case 3.1.1.1. $|F_0| = 2n - 2m$.

Let $S_0' = S_0 \backslash \{s_1\}$. Then, $|S_0'| = m - 1$. Let $W_0 = \{w_i^0 \colon 2 \le i \le m\}$. Note $|W_0| = m - 1 = |S_0'|$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, w_i)(2 \le i \le m)$ from S_0' to W_0 covering $Q[0] - F_0$. Then, $s_1 \in R_i$ for some $2 \le i \le m$. Without loss of generality, let $s_1 \in R_2$. Let u be the before vertex of s_1 in R_2 and $u_1 = N(u) \cap V(Q[1])$.

Suppose $u_1 \in T_1$. Without loss of generality, let $u_1 = t_1$. Let $W_1 = \{w_i^1 \colon 2 \le i \le p\}$. Note $|W_1| = p - 1 = |T_1 \backslash \{t_1\}|$ and $|F_1 \cup \{t_1\}| = 1 \le 2(n - 1) - 2(m - 2) \le 2(n - 1) - 2(p - 1)$. By the induction hypothesis, there are $(p - 1)$ disjoint paths $L_i = (w_i^1, t_i)(2 \le i \le p)$ covering $Q[1] - F_1 - t_1$. Let $P_1 = R_2[s_1, w_2^0] \cup \{w_2^0 w_2^1\} \cup L_2$, $P_2 = R_2[s_2, u] \cup \{ut_1\}$, $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (3 \le i \le p)$ and $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (p + 1 \le i \le m)$.

Suppose $u_1 \notin T_1$. Note $|W_1 \cup \{u_1\}| = |T_1| = p$. By the induction hypothesis, there are p disjoint paths $L_1 = (u_1, t_1)$, $L_i = (w_i^1, t_i)(2 \le i \le p)$ from $W_1 \cup \{u_1\}$ to T_1 covering $Q[1] - F_1$. Let $P_1 = R_2[s_1, w_2^0] \cup \{w_2^0 w_2^1\} \cup L_2$, $P_2 = R_2[s_2, u] \cup \{uu_1\} \cup L_1$, $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (3 \le i \le p)$ and $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (p + 1 \le i \le m)$.

Case 3.1.1.2. $|F_0| = 2n - 2m - 1$.

Then, $|F_1 \cup F_2| \le 1$. Since $|S_0| = m > 1 \ge |F_1|$, there is a vertex s_i such that $s_i^1 \notin F_1$ for $1 \le i \le m$. Without loss of generality, let $s_1^1 \notin F_1$. Let $W_0 = \{w_i^0 \colon 2 \le i \le m\}$. Note $|S_0 \backslash \{s_1\}| = m - 1 = |W_0|$ and $|F_0 \cup \{s_1\}| = 2n - 2m$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, w_i)(2 \le i \le m)$ from $S_0 \backslash \{s_1\}$ to W_0 covering $Q[0] - F_0 - s_1$. Let $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (p + 1 \le i \le m)$.

Let $W_1 = \{w_i^1 \colon 2 \le i \le p\}$. Note $|W_1| + |\{s_1^1\}| = p = |T_1|$. By the induction hypothesis, there are p disjoint paths $L_1 = (s_1^1, t_1)$, $L_i = (w_i^1, t_i)(2 \le i \le p)$ from $\{s_1^1\} \cup W_1$ to T_1 covering $Q[1] - F_1$. Let $P_1 = \{s_1 s_1^1\} \cup L_1$ and $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (2 \le i \le p)$.

Case 3.1.1.3. $|F_0| \le 2n - 2m - 2$.

If $s_1^1 \notin F$, then it is similar to Case 3.1.1.2. Suppose $s_1^1 \in F$. Let $N_{Q[0]}(s_1) = \{u_1, \ldots, u_{2n-2}\}$ and $u_i^1 = N(u_i) \cap V(Q[1])$ for $1 \le i \le 2n - 2$. Since $|N_{Q[0]}(s_1)| - |S \backslash \{s_1\}| = 2n - 2 - (m - 1) > 2n - 2m - 1 + p \ge |F \backslash \{s_1^1\}| + |T_1|$ for $m > p$, there is a vertex $u_i \in V(Q[0] - F_0 - S)$ such that $u_i^1 \in V(Q[1] - F_1 - T_1)$. Without loss of generality, let u_1, u_1^1 be satisfied.

Let $W_0 = \{w_i^0 \colon 2 \le i \le m\}$ and $u_1 \notin W_0$. Then, $|F_0 \cup \{s_1, u_1\}| \le 2n - 2m$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, w_i)(2 \le i \le m)$ from $S_0 \backslash \{s_1\}$ to W_0 covering $Q[0] - F_0 - s_1 - u_1$. Let $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (p + 1 \le i \le m)$.

Let $W_1 = \{w_i^1 \colon 2 \le i \le p\}$. Note $|W_1| + |\{u_1^1\}| = p = |T_1|$. By the induction hypothesis, there are p disjoint paths $L_1 = (u_1^1, t_1)$, $L_i = (w_i^1, t_i)(2 \le i \le p)$ from $\{u_1^1\} \cup W_1$ to T_1 covering $Q[1] - F_1$. Let $P_1 = \{s_1 u_1, u_1 u_1^1\} \cup L_1$ and $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (2 \le i \le p)$.

Case 3.1.2. $|S_0| = m$ and $|T_0| \ge 1$.

Let $S_0 = \{s_1, \ldots, s_m\}$. Since $|S_i| < |T_i|$ for $i = 1, 2$, $1 \le |T_i| \le m - 2$ for $i = 0, 1, 2$. Let $1 \le |T_0| = m - p - q \le m - 2$, $1 \le |T_1| = p \le m - 2$ and $1 \le |T_2| = q \le m - 2$. Let $T_0 = \{t_1, \ldots, t_{m-p-q}\}$, $T_1 = \{t_{m-p-q+1}, \ldots, t_{m-q}\}$ and $T_2 = \{t_{m-q+1}, \ldots, t_m\}$.

Case 3.1.2.1. $|F_0| = 2n - 2m$.

Then, $|F_1| = |F_2| = 0$. Let $S_1' = \{s_i^1 \colon m - p - q + 1 \le i \le m - q\}$ and $S_2' = \{s_i^2 \colon m - q + 1 \le i \le m\}$. Note $|S_1'| = p = |T_1|$ and $|S_2'| = q = |T_2|$. By the induction hypothesis, there are p (resp. q) disjoint paths $L_i = (s_i^1, t_i)(m - p - q + 1 \le i \le m - q)$ (resp. $M_i = (s_i^2, t_i)(m - q + 1 \le i \le m)$) from S_1' to T_1 (resp. S_2' to T_2 covering $Q[1] - F_1$ (resp. $Q[2] - F_2$). Let $P_i = \{s_i s_i^1\} \cup L_i (m - p - q + 1 \le i \le m - q)$ and $P_i = \{s_i s_i^2\} \cup M_i (m - q + 1 \le i \le m)$.

Let $S_0' = \{s_i \colon 1 \le i \le m - p - q\}$. Note $|S_0'| = m - p - q = |T_0|$ and $|F_0| + |S_0 \backslash S_0'| = (2n - 2m) + (p + q) \le 2(n - 1) - 2(m - p - q)$ for $p + q \ge 2$. By the induction hypothesis, there are $(m - p - q)$ disjoint paths $P_i = (s_i, t_i)(1 \le i \le m - p - q)$ from S_0' to T_0 covering $Q[0] - F_0$.

Case 3.1.2.2. $|F_0| = 2n - 2m - 1$.

Since $|F_1 \cup F_2| \le 1 < m = |S_0|$, there is a vertex s_i such that $s_i^2 \notin F_2$ for $1 \le i \le m$. Without loss of generality, let $s_m^2 \notin F_2$. Let $W_0 = \{w_i^0 \colon m - p - q + 1 \le i \le m - 1\}$ for $p + q \ge 2$. Note $|W_0| + |T_0| = (p + q - 1) + (m - p - q) = m - 1 = |S_0 \backslash \{s_m\}|$ and $|F_0 \cup \{s_m\}| = 2n - 2m$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, t_i)(1 \le i \le m - p - q)$ and $R_i = (s_i, w_i)(m - p - q + 1 \le i \le m - 1)$ from $S_0' \backslash \{s_m\}$ to $T_0 \cup W_0$ covering $Q[0] - F_0 - s_m$. Let $P_i = R_i (1 \le i \le m - p - q)$.

Let $W_1 = \{w_i^1 \colon m - p - q + 1 \le i \le m - q\}$. Note $|T_1| = |W_1| = p \le m - 2$ and $|F_1| \le 1 \le 2(n - 1) - 2(m - 2) \le 2(n - 1) - 2p$. By the induction hypothesis, there are p disjoint paths $L_i = (w_i^1, t_i)(m - p - q + 1 \le$

$i \le m-q$) from W_1 to T_1 covering $Q[1]-F_1$. Let $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i\,(m-p-q+1 \le i \le m-q)$.

Let $W_2 = \{w_i^2 \colon m-q+1 \le i \le m-1\}$ for $q \ge 2$ and $W_2 = \varnothing$ for $q = 1$. Since $w_i^2 \in V(Q[2]-F-S-T)$, $s_m^2 \notin W_2$. Note $|T_2| = |W_2 \cup \{s_m^2\}| = q \le m-2$ and $|F_2| \le 1 \le 2(n-1) - 2(m-2) \le 2(n-1) - 2q$. By the induction hypothesis, there are q disjoint paths $M_i = (w_i^2, t_i)\,(m-q+1 \le i \le m-1)$ and $M_m = (s_m^2, t_m)$ from $W_2 \cup \{s_m^2\}$ to T_2 covering $Q[2]-F_2$. Let $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i\,(m-q+1 \le i \le m-1)$ and $P_m = \{s_m s_m^2\} \cup M_m$.

Case 3.1.2.3. $|F_0| \le 2n-2m-2$.

Let $N_{Q[0]}(s_m) = \{u_1, \ldots, u_{2n-2}\}$ and $u_i^2 = N(u_i) \cap V(Q[2])$ for $1 \le i \le 2n-2$. Since $|N_{Q[0]}(s_m)| - |S \backslash \{s_m\}| = 2n-2-(m-1) > (2n-2m) + q \ge |F| + |T_2|$ for $m-1 > q$, there is a vertex $u_i \in V(Q[0]-F_0-T_0)$ such that $u_i^2 \in V(Q[2]-F_2-T_2)$. Without loss of generality, let u_1 be satisfied.

Let $W_0 = \{w_i^0 \colon m-p-q+1 \le i \le m-1\}$ and $u_1 \notin W_0$. Note $|S_0'| = |S \backslash \{s_m\}| = m-1 = |T_0| + |W_0|$ and $|F_0 \cup \{s_m, u_1\}| = 2n-2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $R_i = (s_i, t_i)\,(1 \le i \le m-p-q)$ and $R_i = (s_i, w_i)\,(m-p-q+1 \le i \le m-1)$ from S_0' to $T_0 \cup W_0$ covering $Q[0]-F_0-s_m-u_1$. Let $P_i = R_i\,(1 \le i \le m-p-q)$.

Let $W_1 = \{w_i^1 \colon m-p-q+1 \le i \le m-q\}$. Note $|T_1| = |W_1| = p \le m-2$. By the induction hypothesis, there are p disjoint paths $L_i = (w_i^1, t_i)$ $(m-p-q+1 \le i \le m-q)$ from W_1 to T_1 covering $Q[1]-F_1$. Let $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i\,(m-p-q+1 \le i \le m-q)$.

Let $W_2 = \{w_i^2 \colon m-q+1 \le i \le m-1\}$ for $q \ge 2$ and $W_2 = \varnothing$ for $q = 1$. Note $|T_2| = |W_2 \cup \{u_1^2\}| = q \le m-2$. By the induction hypothesis, there are q disjoint paths $M_i = (w_i^2, t_i)\,(m-q+1 \le i \le m-1)$ and $M_m = (u_1^2, t_m)$ from $W_2 \cup \{u_1^2\}$ to T_2 covering $Q[2]-F_2$. Let $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i\,(m-q+1 \le i \le m-1)$ and $P_m = \{s_m u_1, u_1 u_1^2\} \cup M_m$.

Case 3.1.3. $1 \le |S_0| \le m-1$.

If $|T_i| = m$ for some $1 \le i \le 2$, then it is similar to Case 3.1.1 or 3.1.2. Since $|T_i| > |S_i| \ge 0$ for $1 \le i \le 2$, $1 \le |T_i| \le m-1$ for $1 \le i \le 2$. Let $W_0 = \{w_i^0 \colon 1 \le i \le |S_0| - |T_0|\}$, $W_1 = \{w_i^1 \colon 1 \le i \le |T_1| - |S_1|\}$ and $W_2 = \{w_i^2 \colon 1 \le i \le |T_2| - |S_2|\}$. Thus, $|W_0| = |S_0| - |T_0| = (|T_1| - |S_1|) + (|T_2| - |S_2|) = |W_1| + |W_2|$. Note $|S_0| = |T_0| + |W_0| \le m-1$, $|T_1| = |S_1| + |W_1| \le m-1$ and $|T_2| = |S_2| + |W_2| \le m-1$. By the induction hypothesis, there are $|S_0|$, $|T_1|$ and $|T_2|$ disjoint paths from S_0 to $W_0 \cup T_0$, from $S_1 \cup W_1$ to T_1 and from $S_2 \cup W_2$ to T_2 covering $Q[0]-F_0$, $Q[1]-F_1$ and $Q[2]-F_2$, respectively.

Case 3.2. $|S_i| > |T_i|$ for two $0 \le i \le 2$.

There is only one i such that $|S_i| < |T_i|$ for $0 \le i \le 2$. It is similar to Case 3.1. □

4. One-to-Many Disjoint Paths

Lemma 4. *Let $0 \le i \le 2$ be a 3-ary n-cube for $n \ge 2$. Let F be a faulty set of $V(Q_n^3)$. Let $\{s, t_1, t_2\}$ be the subset of $V(Q_n^3 - F)$. If $|F| \le 2n-3$, then there exist two disjoint paths from s to t_i $(i = 1, 2)$ covering $Q_n^3 - F$.*

Proof. Note $|F| \le 2n-3$ for $n \ge 2$. By Lemma 2, there is a path $P = (t_1, t_2)$ covering $Q_n^3 - F$. Then, $s \in V(P)$. Thus, there are two disjoint paths $P^{-1}[s, t_1]$ and $P[s, t_2]$ covering $Q_n^3 - F$. □

Lemma 5. *Let Q_n^3 be a 3-ary n-cube for $n \ge 3$. Let F be a faulty set of $V(Q_n^3)$. Let $C = \langle s_0, s_1, s_2, s_0 \rangle$ be a fault-free cycle of length three, $s \in V(C)$ and T be the subset of $V(Q_n^3 - F - V(C))$ with $|T| = 3$. If $|F| \le 2n-4$, then there are three disjoint paths from s to T covering $Q_n^3 - F$.*

Proof. Let $T = \{t_0, t_1, t_2\}$. We decompose Q_n^3 into three isomorphic subgraphs $Q[0], Q[1], Q[2]$ along one dimension such that $s_i \in V(Q[i])$ for $i \in \{0, 1, 2\}$. Suppose there is $st_i \in E(Q_n^3)$ for some $0 \le i \le 2$. Let $P_i = (s, t_i)$. Note $|F \cup \{t_i\}| \le 2n-3$. By Lemma 4, there are two disjoint paths $P_j = (s, t_j)$ for $j \in \{0, 1, 2\} \backslash \{i\}$. Thus, the lemma holds. So, suppose $st_i \notin E(Q_n^3)$ for all $0 \le i \le 2$. Let $T_i = T \cap V(Q[i])$ and $F_i = F \cap V(Q[i])$ for $0 \le i \le 2$. Without loss of generality, let $|T_0| > 0$ and $s = s_0$.

Case 1. $|T_0| = 1$, $|T_1| = 2$ and $|T_2| = 0$.

Let $T_0 = \{t_0\}$ and $T_1 = \{t_1, t_2\}$.

Case 1.1. $|F_i| \le 2n-5$ for $0 \le i \le 2$.

By Lemma 2, there is a Hamilton path $L = (t_1, t_2)$ in $Q[1]-F_1$. Since $|V(L)| - 2|F| \ge 3^{n-1} - 2(2n-4) > 0$ for $n \ge 3$, there is an edge $xy \in R$ such that $x_i, y_i \in V(Q[i]-F_i)$ for $i = 0, 2$ where $x_i = N(x) \cap V(Q[i])$ and $y_i = N(y) \cap V(Q[i])$. By Lemma 2, there is a Hamilton path $M = (s_2, y_2)$ in $Q[2]-F_2$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, x_0)$ covering $Q[0]-F_0$. Let $P_0 = R_1$, $P_1 = R_2 \cup \{x_0 x\} \cup L^{-1}[x, t_1]$ and $P_2 = \{ss_2\} \cup M \cup \{y_2 y\} \cup L[y, t_2]$.

Case 1.2. $|F_i| = 2n-4$ for some $0 \le i \le 2$.

Case 1.2.1. $|F_0| = 2n-4$.

Let $f \in F_0$ and $F_0' = F_0 \backslash \{f\}$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, f)$ covering $Q[0]-F_0$. Let u be the before vertex of f in R_2 and $u_1 = N(u) \cap V(Q[1])$.

$u_1 = t_1$ or $u_1 = t_2$. Without loss of generality, let $u_1 = t_1$. Note $|F_1 \cup \{t_1\}| = 1 \le 2n-5$ for $n \ge 3$. By Lemma 2, there is a Hamilton path $L = (s_1, t_2)$ in $Q[1]-F_1-t_1$. Let $xy \in L$, $x_2 = N(x) \cap V(Q[2])$ and $y_2 = N(y) \cap V(Q[2])$. By Lemma 2, there is a Hamilton path $M = (x_2, y_2)$ in $Q[2]-F_2$. Let $P_0 = R_1$, $P_1 =$

$R_2[s,u] \cup \{ut_1\}$ and $P_2 = \{ss_1\} \cup (L - xy) \cup \{xx_2, yy_2\} \cup M$.

$u_1 \neq t_1$ and $u_1 \neq t_2$. Let $t_2^2 = N(t_2) \cap V(Q[2])$. Note $|F_1 \cup \{t_2\}| = 1 \leq 2n-5$ and $|F_2| = 0$. By Lemma 2, there is a Hamilton path $L = (u_1, t_1)$ and $M = (s_2, t_2^2)$ in $Q[1] - F_1 - t_2$ and $Q[2] - F_2$, respectively. Let $P_0 = R_1$, $P_1 = R_2[s,u] \cup \{uu_1\} \cup L$ and $P_2 = \{ss_2\} \cup M \cup \{t_2^2 t_2\}$.

Case 1.2.2. $|F_1| = 2n-4$.

Let $f \in F_1$ and $F_1' = F_1 \backslash \{f\}$. By Lemma 4, there are two disjoint paths $L_1 = (f, t_1)$ and $L_2 = (f, t_2)$ covering $Q[1] - F_1'$. Let z_i be the next vertex of f in L_i for $i = 1,2$ and $z_i^j = N(u) \cap V(Q[j])$ for $j = 0,2$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, z_1^0)$ covering $Q[0] - F_0$. By Lemma 2, there is a Hamilton path $M = (s_2, z_2^2)$ in $Q[2] - F_2$. Let $P_0 = R_1$, $P_1 = R_2 \cup \{z_1^0 z_1\} \cup L_1[z_1, t_1]$ and $P_2 = \{ss_2\} \cup M \cup \{z_2^2 z_2\} \cup L_2[z_2, t_2]$.

Case 1.2.3. $|F_2| = 2n-4$.

By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0] - F_0$. Let $f \in F_2$ and $F_2' = F_2 \backslash \{f\}$. By Lemma 2, there is a Hamilton path $M = (s_2, f)$ in $Q[2] - F_2'$. Let u be the before vertex of f in M and $u_1 = N(u) \cap V(Q[1])$.

$u_1 = t_1$ or $u_1 = t_2$. Without loss of generality, let $u_1 = t_1$. Note $|F_1 \cup \{t_1\}| = 1 \leq 2n-5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $L = (s_1, t_2)$ in $Q[1] - F_1 - t_1$. Let $P_0 = R$, $P_1 = \{ss_2\} \cup M[s_2, u] \cup \{ut_1\}$ and $P_2 = \{ss_1\} \cup L$.

$u_1 \neq t_1$ and $u_1 \neq t_2$. Since $u \neq s_2$, $u_1 \neq s_1$. By Lemma 3, there are two disjoint paths $L_1 = (u_1, t_1)$ and $L_2 = (s_1, t_2)$ covering $Q[1] - F_1$. Let $P_0 = R$, $P_1 = \{ss_2\} \cup M[s_2, u] \cup \{uu_1\} \cup L_1$ and $P_2 = \{ss_1\} \cup L$.

Case 2. $|T_0| = 1$, $|T_1| = 1$ and $|T_2| = 1$.

Let $T_0 = \{t_0\}$, $T_1 = \{t_1\}$ and $T_2 = \{t_2\}$.

Case 2.1. $|F_i| \leq 2n-5$ for $0 \leq i \leq 2$.

By Lemma 2, there is a Hamilton path $R = (s, t_0)$, $L = (s_1, t_1)$ and $M = (s_2, t_2)$ in $Q[0] - F_0$, $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_0 = R$, $P_1 = \{ss_1\} \cup L$ and $P_2 = \{ss_2\} \cup M$.

Case 2.2. $|F_i| = 2n-4$ for some $0 \leq i \leq 2$.

Case 2.2.1. $|F_0| = 2n-4$.

Then, $|F_1| = |F_2| = 0$. Since $|F_0| = 2n-4 \geq 2$ for $n \geq 3$, let $f \in F_0$ and $F_0' = F_0 \backslash \{f\}$. Then, $|F_0'| = 2n-5$. By Lemma 4, there are two disjoint paths $P = (s_0, t_0)$ and $R = (s_0, f)$ covering $Q[0] - F_0$. Then, $|V(R)| \geq 2$.

Case 2.2.1.1. $|V(R)| = 2$.

Then, $R = \langle s_0, f \rangle$. By Lemma 2, there is a Hamilton path $P_i' = (s_i, t_i)$ in $Q[i]$ for $i = 1,2$. Let $P_0 = P$, $P_1 = \{ss_1\} \cup P_1'$ and $P_2 = \{ss_2\} \cup P_2'$.

Case 2.2.1.2. $|V(R)| \geq 3$.

Let u be the before vertex of f in R. Let $u_i = N(u) \cap V(Q[i])$ for $i \in \{1,2\}$. By the definition of Q_n^3, $u_i \in V(Q[i] - s_i)$ for $i \in \{1,2\}$.

Case 2.2.1.2.1. $u_1 = t_1$ or $u_2 = t_2$.

Without loss of generality, let $u_1 = t_1$. Since $|V(Q[i])| - |F_i| - |\{s_i, t_1, t_2\}| = 3^{n-1} - 3 > 1$ for $i = 1,2$, there is an edge $x_1 x_2$ such that $x_i \in V(Q[i] - F_i - \{s_i, t_1, t_2\})$ for $i = 1,2$. Note $|F_1 \cup \{t_1\}| = 1 \leq 2n-5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $L = (s_1, x_1)$ in $Q[1] - F_1 - t_1$. Note $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (x_2, t_2)$ in $Q[2]$. Let $P_0 = P$, $P_1 = R[s,u] \cup \{ut_1\}$, and $P_2 = \{ss_1\} \cup L \cup \{x_1 x_2\} \cup M$.

Case 2.2.1.2.2. $u_1 \neq t_1$ and $u_2 \neq t_2$.

By Lemma 2, there is a Hamilton path $L = (u_1, t_1)$ and $M = (s_2, t_2)$ in $Q[1]$ and $Q[2]$, respectively. Let $P_0 = P$, $P_1 = R[s,u] \cup \{uu_1\} \cup L$, $P_1 = \{ss_2\} \cup M$.

Case 2.2.2. $|F_1| = 2n-4$ or $|F_2| = 2n-4$.

Let $|F_1| = 2n-4$. Then, $|F_0| = |F_2| = 0$. Note $|F_1| = 2n-4 \geq 2$ for $n \geq 3$. Let $f \in F_1$. Then, $|F_1 \backslash \{f\}| = 2n-5$. By Lemma 4, there are two disjoint paths $L_1 = (s_1, t_1)$ and $L_2 = (s_1, f)$ covering $Q[1] - F_1 - f$.

Case 2.2.2.1. $|V(L_2)| = 2$.

Then, $L_2 = \langle s_1, f \rangle$. By Lemma 2, there is a Hamilton path $R = (s, t_0)$ and $M = (s_2, t_2)$ in $Q[0]$ and $Q[2]$, respectively. Let $P_0 = R$, $P_1 = \{ss_1\} \cup L_1$ and $P_2 = \{ss_2\} \cup M$.

Case 2.2.2.2. $|V(L_2)| = 3$.

Let $L_2 = \langle s_1, u, f \rangle$. Let $u_2 = N(u) \cap V(Q[2])$. Let w be the later vertex of s_1 in L_1 and $w_i = N(w) \cap V(Q[i])$ for $i = 0,2$.

Suppose $u_2 = t_2$. Note $|F_2 \cup \{t_2\}| = 1 \leq 2n-5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $M = (s_2, w_2)$ in $Q[2] - F_2 - t_2$. By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0]$. Let $P_0 = R$, $P_1 = \{ss_2\} \cup M \cup \{w_2 w\} \cup L_1[w, t_1]$ and $P_2 = \langle s, s_1, u, t_2 \rangle$.

Suppose $u_2 \neq t_2$. By Lemma 2, there is a Hamilton path $M = (u_2, t_2)$ in $Q[2]$. By the definition of Q_n^3, $w_0 \in N(s)$. Since $st_i \notin E(Q_n^3)$, $w_0 \neq t_0$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, w_0)$ covering $Q[0] - F_0$. Let $P_0 = R_1$, $P_1 = R_2 \cup \{w_0 w\} \cup L_1[w, t_1]$ and $P_2 = \langle s, s_1, u, u_2 \rangle \cup M$.

Case 2.2.2.3. $|V(L_2)| \geq 4$.

By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0]$. Let u and v be the later and before vertex of s_1 and f in L_2, respectively. Let $u_2 = N(u) \cap V(Q[2])$ and $v_2 = N(v) \cap V(Q[2])$.

Suppose $t_2 = u_2$ or $t_2 = v_2$. Without loss of generality, let $t_2 = u_2$. Note $|F_2 \cup \{t_2\}| = 1 \leq 2n-5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $M = (s_2, v_2)$ in $Q[2] - F_2 - t_2$. Let $P_0 = R$, $P_1 = \{ss_1\} \cup L_1$ and $P_2 = \{ss_2\} \cup M \cup \{v_2 v\} \cup L_2^{-1}[v, u] \cup \{ut_2\}$.

Suppose $t_2 \neq u_2$ and $t_2 \neq v_2$. By Lemma 3, there are two disjoint paths $M_1 = (s_2, u_2)$ and $M_2 = (v_2, t_2)$ covering $Q[2] - F_2$. Let $P_0 = R$, $P_1 = \{ss_1\} \cup L_1$ and $P_2 = \{ss_2\} \cup M_1 \cup \{u_2 u\} \cup L_2[u, v] \cup \{vv_2\} \cup M_2$.

Case 3. $|T_0| = 2$, $|T_1| = 1$ and $|T_2| = 0$.

Let $T_0 = \{t_0, t_1\}$ and $T_1 = \{t_2\}$.

Case 3.1. $|F_i| \le 2n-6$ for all $0 \le i \le 2$.

Since $|V(Q[i])| - |F| - |T| - |\{s\}| \ge 3^{n-1} - (2n-4) - 3 - 1 > 1$ for $n \ge 3$, there is an edge x_0x_2 such that $x_i \in V(Q[i] - F_i - T_i - s_i)$ for $i = 0, 2$. By Lemma 2, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (x_0, t_1)$ covering $Q[0] - F_0$. By Lemma 3, there is a Hamilton path $L = (s_1, t_2)$ and $M = (s_2, x_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_0 = R_1$, $P_1 = \{ss_2\} \cup M \cup \{x_2x_0\} \cup R_2$ and $P_2 = \{ss_1\} \cup L$.

Case 3.2. $|F_i| = 2n-5$ for some $0 \le i \le 2$.

We get $|F_j| \le 1 \le 2n-5$ for $n \ge 3$ and $j \in \{0, 1, 2\} \backslash \{i\}$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, t_1)$ covering $Q[0] - F_0$. Let $P_0 = R_1$, $P_1 = R_2$.

Since $|V(Q[i])| - |F| - |T| - |\{s_i\}| \ge 3^{n-1} - (2n-4) - 3 - 1 > 1$, there is an edge x_1x_2 such that $x_i \in V(Q[i] - F_i - T_i - s_i)$ for $i = 1, 2$. Note $|F_i| \le 2n-5$ for $i = 1, 2$. By Lemma 2, there is a Hamilton path $L = (x_1, t_2)$ and $M = (s_2, x_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_2 = \{ss_2\} \cup M \cup \{x_2x_1\} \cup L$.

Case 3.3. $|F_i| = 2n-4$ for some $0 \le i \le 2$.

Case 3.3.1. $|F_0| = 2n-4$.

Then, $|F_1| = |F_2| = 0$. By Lemma 2, there is a Hamilton path $L = (s_1, t_2)$ in $Q[1]$. Let $P_2 = \{ss_1\} \cup L$. Note $|F_0| = 2n-4 \ge 2$ for $n \ge 3$. Let $f \in F_0$. Then, $|F_0'| = |F_0 \backslash \{f\}| = 2n-5$. By Lemma 4, there are two disjoint paths $P = (s, t_0)$ and $R = (s, t_2)$ covering $Q[0] - F_0'$. Without loss of generality, let $f \in V(R)$. Then, $|V(R)| \ge 3$.

Case 3.3.1.1. $sf \in E(R)$ and $ft_1 \in E(R)$.

Then, $R = \langle s_0, f, t_1 \rangle$. Let $t_1^2 = N(t_1) \cap V(Q[2])$. Note that $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, t_1^2)$ in $Q[2]$. Let $P_0 = P$ and $P_1 = \{ss_2\} \cup M \cup \{t_1^2t_1\}$.

Case 3.3.1.2. $sf \in E(R)$ and $ft_1 \notin E(R)$.

Let u be the later vertex of f in R and $u_2 = N(u) \cap V(Q[2])$. Note that $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, u_2)$ in $Q[2]$. Let $P_0 = P$ and $P_1 = \{ss_2\} \cup M \cup \{u_2u\} \cup R[u, t_1]$.

Case 3.3.1.3. $sf \notin E(R)$ and $ft_1 \in E(R)$.

Let w and u be the later and before vertex of s_0 and f in P and R, respectively. Let $u_2 = N(u) \cap V(Q[2])$, $w_2 = N(w) \cap V(Q[2])$ and $t_1^2 = N(t_1) \cap V(Q[2])$. By the definition of Q_n^3, u_2, w_2, t_1^2 are different in $V(Q[2] - s_2)$. Note that $|F_2| = 0 \le 2n-6$ for $n \ge 3$. By Lemma 3, there are two disjoint paths $M_1 = (s_2, t_1^2)$ and $M_2 = (u_2, w_2)$ covering $Q[2] - F_2$. Let $P_0 = R[s, u] \cup \{uu_2\} \cup M_2 \cup \{w_2w\} \cup P[w, t_0]$ and $P_1 = \{ss_2\} \cup M_1 \cup \{t_1^2t_1\}$.

Case 3.3.1.4. $sf \notin E(R)$ and $ft_2 \notin E(R)$.

Let u and v be the before and later vertex of f in R, respectively. Let w be the later vertex of s_0 in P. Let $u_2 = N(u) \cap V(Q[2])$, $v_2 = N(v) \cap V(Q[2])$ and $w_2 = N(w) \cap V(Q[2])$. By the definition of Q_n^3, u_2, v_2, w_2 are different in $V(Q[2] - s_2)$. Note that $|F_2| = 0 \le 2n-6$ for $n \ge 3$. By Lemma 3, there are two disjoint paths $M_1 = (s_2, v_2)$ and $M_2 = (u_2, w_2)$ covering $Q[2] - F_2$. Let $P_0 = R[s, u] \cup \{uu_2\} \cup M_2 \cup \{w_2w\} \cup P[w, t_0]$ and $P_1 = \{ss_2\} \cup M_1 \cup \{v_2v\} \cup R[v, t_1]$.

Case 3.3.2. $|F_1| = 2n-4$.

Note $|F_1| = 2n-4 \ge 2$ for $n \ge 3$. Let $f \in F_1$. Then, $|F_1'| = |F_1 \backslash \{f\}| = 2n-5$. By Lemma 4, there are two disjoint paths $L_1 = (s_1, t_2)$ and $L_2 = (s_1, f)$ covering $Q[1] - F_1'$. Let $P_2 = \{ss_1\} \cup L_1$. Then, $|V(L_2)| \ge 2$.

Case 3.3.2.1. $|V(L_2)| = 2$.

Then, $L_2 = \langle s_1, f \rangle$. Let $t_1^2 = N(t_1) \cap V(Q[2])$. Note $|F_0 \cup \{t_1\}| = 1 \le 2n-5$ for $n \ge 3$. By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0] - F_0 - t_1$. Note $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, t_1^2)$ in $Q[2]$. Let $P_0 = R$ and $P_1 = \{ss_2\} \cup M \cup \{t_1^2t_1\}$.

Case 3.3.2.2. $|V(L_2)| = 3$.

Let $L_2 = \langle s_1, u, f \rangle$, $u_i = N(u) \cap V(Q[i])$ for $i = 0, 2$. Then, $u_i \in V(Q[i] - s_i)$ for $i = 0, 2$.

Suppose $u_0 = t_0$ or $u_0 = t_1$. Without loss of generality, let $u_0 = t_1$. Note $|F_0 \cup \{t_1\}| = 1 \le 2n-5$ for $n \ge 3$. By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0] - F_0 - t_1$. Note $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, u_2)$ in $Q[2]$. Let $P_0 = R$ and $P_1 = \{ss_2\} \cup M \cup \{u_2u, ut_1\}$.

Suppose $u_0 \ne t_0$ and $u_0 \ne t_1$. Note $|F_0| = 0 \le 2n-6$ for $n \ge 3$. By Lemma 3, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (u_0, t_1)$ covering $Q[0] - F_0$. Note $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, u_2)$ in $Q[2]$. Let $P_0 = R_1$ and $P_1 = \{ss_2\} \cup M \cup \{u_2u, uu_0\} \cup R_2$.

Case 3.3.2.3. $|V(L_2)| \ge 4$.

Let u and v be the later and before vertex of s_1 and f in L_2, respectively. Let $u_2 = N(u) \cap V(Q[2])$ and $v_0 = N(v) \cap V(Q[0])$. Note $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, u_2)$ in $Q[2]$.

Suppose $v_0 = t_0$ or $v_0 = t_1$. Without loss of generality, let $v_0 = t_1$. Note $|F_0 \cup \{t_1\}| = 1 \le 2n-5$ for $n \ge 3$. By Lemma 3, there is a Hamilton path $R = (s, t_0)$ in $Q[0] - F_0 - t_1$. Let $P_0 = R$ and $P_1 = \{ss_2\} \cup M \cup \{u_2u\} \cup L_2[u, v] \cup \{vt_1\}$.

Suppose $v_0 \ne t_0$ and $v_0 \ne t_1$. Note $|F_0| = 0 \le 2n-6$ for $n \ge 3$. By Lemma 2, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (v_0, t_1)$ covering $Q[0] - F_0$. Let $P_0 = R_1$ and $P_1 = \{ss_1\} \cup M \cup \{u_2u\} \cup L_2[u, v] \cup \{vv_0\} \cup R_2$.

Case 3.3.3. $|F_2| = 2n-4$.

Note $|F_1| = 0$. By Lemma 2, there is a Hamilton path $L = (s_1, t_2)$ in $Q[1]$. Let $P_2 = \{ss_1\} \cup L$. Note $|F_2| = 2n-4 \ge 2$ for $n \ge 3$. Let $f \in F_2$. Then, $|F_2'| = |F_2 \backslash \{f\}| = 2n-5$. By Lemma 2, there is a Hamilton path $M = (s_2, f)$ in $Q[2] - F_2'$. Let w be the before vertex of f in M. Let $w_0 = N(w) \cap V(Q[0])$. Then, $w_0 \in V(Q[0] - s_0)$.

Case 3.3.3.1. $w_0 = t_0$ or $w_0 = t_1$.

Without loss of generality, let $w_0 = t_1$. Note $|F_0 \cup \{t_1\}| = 1 \le 2n-5$ for $n \ge 3$. By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0] - F_0 - t_1$. Let $P_0 = R$ and $P_1 = \{ss_2\} \cup M[s_2, w] \cup \{wt_1\}$.

Case 3.3.3.2. $w_0 \ne t_0$ and $w_0 \ne t_1$.

Note $|F_0| = 0 \le 2n-6$ for $n \ge 3$. By Lemma 3, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (w_0, t_1)$ covering $Q[0] - F_0$. Let $P_0 = R_1$ and $P_1 = \{ss_2\} \cup M[s_2, w] \cup \{ww_0\} \cup R_2$.

Case 4. $|T_0| = 3$, $|T_1| = 0$ and $|T_2| = 0$.

Let $T_0 = \{t_0, t_1, t_2\}$. Since $|V(Q[i])| - |F| - |T| - |\{s\}| \ge 3^{n-1} - (2n-4) - 3 - 1 > 2$ for $n \ge 3$, there are two cycles of length three $C_i = \langle w_i^0, w_i^1, w_i^2, w_i^0 \rangle$ such that $w_i^j \in V(Q[j] - F_j - T_j - s_j)$ for $0 \le j \le 2$ and $i \in \{1, 2\}$.

Case 4.1. $|F_i| \le 2n-6$ for all $0 \le i \le 2$.

Let $t_i^j = N(t_i) \cap V(Q[j])$ for $0 \le i \le 2$ and $j \in \{1, 2\}$. Suppose $|F_i| \le 2$ for $i \in \{1, 2\}$. There is $t_i^1 \in V(Q[1] - F_1)$ for $0 \le i \le 2$. Let t_2 be satisfied. By Lemma 2, there is a Hamilton path $L = (w_1^1, t_2^1)$ and $M = (s_2, w_1^2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Note $|F_0 \cup \{t_2\}| \le 2n-5$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, t_1)$ covering $Q[0] - F_0$. Let $P_0 = R_1$, $P_1 = R_2$ and $P_2 = \{ss_2\} \cup M \cup \{w_1^2 w_1^1\} \cup L \cup \{t_2^1 t_2\}$.

Suppose $|F_i| \ge 3$ for $i \in \{1, 2\}$. Then, $|F_0| \le 2n-4-6 = 2n-10$. By Theorem 1, there are three disjoint paths $R_0 = (s, t_0)$ and $R_i = (w_i^0, t_i)$ $(i = 1, 2)$ covering $Q[0] - F_0$. By Lemma 2, there is a Hamilton path $L = (s_1, w_1^1)$ and $M = (s_2, w_2^2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_0 = R_0$, $P_1 = \{ss_1\} \cup L \cup \{w_1^1 w_1^0\} \cup R_1$ and $P_2 = \{ss_2\} \cup M \cup \{w_2^2 w_2^0\} \cup R_2$.

Case 4.2. $|F_i| = 2n-5$ for some $0 \le i \le 2$.

Case 4.2.1. $|F_0| = 2n-5$.

Then, $|F_1 \cup F_2| \le 1$. Without loss of generality, let $|F_1| = 0$ and $|F_2| \le 1$. By Lemma 4, there are two disjoint paths $R_0 = (s, t_0)$ and $R_1 = (s, t_1)$ covering $Q[0] - F_0$. Then, $t_2 \in R_i$ for some $0 \le i \le 1$. Without loss of generality, let $t_2 \in R_1$. Then, $R_1 = R_1[s, t_2] \cup R_1[t_2, t_1]$. Let z and u be the later vertex of s and t_2 in R_0 and R_1, respectively. Let $u_i = N(u) \cap V(Q[i])$ and $z_i = N(z) \cap V(Q[i])$ for $1 \le i \le 2$.

Suppose $u_2 \in F_2$. Then, $z_2 \notin F_2$. By Lemma 2, there is a Hamilton path $L = (s_1, u_1)$ and $M = (s_2, z_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_0 = \{ss_2\} \cup M \cup \{z_2 z\} \cup R_0[z, t_0]$, $P_1 = \{ss_1\} \cup L \cup \{u_1 u\} \cup R_1[u, t_1]$ and $P_2 = R_0[s, t_2]$.

Suppose $u_2 \notin F_2$. By Lemma 2, there is a Hamilton path $L = (s_1, z_1)$ and $M = (s_2, u_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_0 = \{ss_1\} \cup L \cup \{z_1 z\} \cup R_0[z, t_0]$, $P_1 = \{ss_2\} \cup M \cup \{u_2 u\} \cup R_1[u, t_1]$ and $P_2 = R_0[s, t_2]$.

Case 4.2.2. $|F_1| = 2n-5$ or $|F_2| = 2n-5$.

Let $|F_1| = 2n-5$. Then, $|F_0 \cup F_2| \le 1$.

Case 4.2.2.1. $|F_0| = 0$.

Since $|F_2| \le 1$, there is a vertex t_i such that $t_i^2 \notin F_2$ where $t_i^2 = N(t_i) \cap V(Q[2])$ for $0 \le i \le 2$. Let t_2 be satisfied. By Lemma 2, there is a Hamilton path $L = (s_1, w_1^1)$ and $M = (w_1^2, t_2^2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Note $|F_0 \cup \{t_2\}| = 1 \le 2n-5$. By Lemma 4, there are two disjoint paths $R_0 = (s, t_0)$ and $R_1 = (s_0, t_1)$ covering $Q[0] - F_0$. Let $P_0 = R_0$, $P_1 = R_1$ and $P_2 = \{ss1\} \cup L \cup \{w_1^1 w_1^2\} \cup M \cup \{t_2^2 t_2\}$.

Case 4.2.2.2. $|F_2| = 0$.

Then, $|F_0| \le 1 \le 2n-5$ for $n \ge 3$. By Lemma 4, there are two disjoint paths $R_1 = (s_0, t_1)$ and $R_2 = (s_0, t_2)$ covering $Q[0] - F_0$. Then, $t_0 \in R_i$ for some $1 \le i \le 2$. Without loss of generality, let $t_0 \in R_1$. Then, $R_1 = R_1[s_0, t_0] \cup R_1[t_0, t_1]$. Let u and v be the later vertex of t_0 and s_0 in R_1 and R_2, respectively. Let $u_2 = N(u) \cap V(Q[2])$ and $v_2 = N(v) \cap V(Q[2])$. Since $|V(Q[i])| - |F| - |\{s_i, u_i, v_i\}| \ge 3^{n-1} - (2n-4) - 3 \ge 1$ for $n \ge 3$, there is an edge $z_1 z_2$ such that $z_i \in V(Q[i] - F - \{s_i, u_2, v_2\})$ for $1 \le i \le 2$.

Note $|F_1| = 2n-5$. By Lemma 2, there is a Hamilton path $L = (s_1, z_1)$ in $Q[1] - F_1$. Note $|F_2| = 0 \le 2n-6$ for $n \ge 3$. By Lemma 3, there are two disjoint paths $M_1 = (z_2, u_2)$ and $M_2 = (s_2, v_2)$ covering $Q[2] - F_2$. Let $P_0 = R_1[s_0, t_0]$, $P_1 = \{ss_1\} \cup L \cup \{z_1 z_2\} \cup M_1 \cup \{u_2 u\} \cup R_1[u, t_1]$ and $P_2 = \{ss_2\} \cup M_2 \cup \{v_2 v\} \cup R_2[v, t_2]$.

Case 4.3. $|F_i| = 2n-4$ for some $0 \le i \le 2$.

Case 4.3.1. $|F_0| = 2n-4$.

Then, $|F_1| = |F_2| = 0$. Note $|F_0| = 2n-4 \ge 2$ for $n \ge 3$. Let $f_1, f_2 \in F_0$. Let $F_0' = F_0 \backslash \{f_1, f_2\} \cup \{t_2\}$. Then, $|F_0'| = 2n-5$. By Lemma 4, there are two disjoint paths $R_0 = (s, t_0)$ and $R_1 = (s, t_1)$ covering $Q[0] - F_0'$. Then, $f_1, f_2 \in R_i$ for $0 \le i \le 1$. Let $f_1, f_2 \in R_1$. Let u and v be the before and later vertex of f_1, $u_i = N(u) \cap V(Q[i])$ and $v_i = N(v) \cap V(Q[i])$ for $1 \le i \le 2$. Let x and y be the before and later vertex of f_2, $x_i = N(x) \cap V(Q[i])$ and $y_i = N(y) \cap V(Q[i])$ for $1 \le i \le 2$. Note $|F_1| = |F_2| = 0$. By Lemma 2, there is a Hamilton path $L = (u_1, v_1)$ in $Q[1] - F_1$. Let $t_2^2 = N(t_2) \cap V(Q[2])$. By Lemma 3, there are two disjoint paths $M_1 = (x_2, y_2)$ and $M_2 = (s_2, t_2^2)$ covering $Q[2] - F_2$. Let $P_0 = R_0$, $P_1 = (R_1 - \{f_1, f_2\}) \cup L \cup M$ and $P_2 = \{ss_2\} \cup M_2 \cup \{t_2^2 t_2\}$. Similarly, $f_1 \in R_0$ and $f_2 \in R_1$.

Case 4.3.2. $|F_1| = 2n-4$ or $|F_2| = 2n-4$.

Let $|F_1| = 2n-4$. Then, $|F_0| = |F_2| = 0$. Note $|F_1| = 2n-4 \ge 2$ for $n \ge 3$. Let $f \in F_1$. Then, $|F_1'| = |F_1 \backslash \{f\}| = 2n-5$. By Lemma 2, there is a Hamilton path $L = (s_1, f)$ in $Q[1] - F_1$. Let u be the before vertex of f in L and $u_2 = N(u) \cap V(Q[2])$. Since $|T| = 3 < 1$, there is a t_i such that $t_i^2 \ne u_2$ where $t_i^2 = N(t_i) \cap V(Q[2])$ for $0 \le i \le 2$. Let t_2 be satisfied. Note $|F_0 \cup \{t_2\}| = 1 \le 2n-5$. By Lemma 4, there are two disjoint paths $R_0 = (s, t_0)$ and $R_1 = (s, t_1)$ covering $Q[0] - F_0 - t_2$. By Lemma 2, there is a Hamilton path

$M = (u_2, t_2^2)$ in $Q[2] - F_2$. Let $P_0 = R_0$, $P_1 = R_1$ and $P_2 = \{ss_1\} \cup (L - f) \cup \{uu_2\} \cup M \cup \{t_2^2 t_2\}$. □

Theorem 2. *Let Q_n^3 be a 3-ary n-cube for $n \geq 2$. Let F be a vertex faulty set of Q_n^3. Let C be a fault-free cycle of length three. Let $s \in V(C)$ and $T \subset V(Q_n^3 - F - V(C))$ with $2 \leq |T| = m \leq n$. If $|F| \leq 2n - m - 1$, then there are m disjoint paths from s to T covering $Q_n^3 - F$.*

Proof. We prove the theorem by induction on m. If $m = 2$, then $|T| = 2$ and $|F| \leq 2n - 3$. By Lemma 4, there are two disjoint paths from s to T covering $Q_n^3 - F$. If $m = 3$, then $|T| = 3$ and $|F| \leq 2n - 4$. By Lemma 5, there are three disjoint paths from s to T covering $Q_n^3 - F$.

Assume the theorem is true for $m - 1$. Now we prove the theorem holds on m for $4 \leq m \leq n$. Let $T = \{t_1, \ldots, t_m\}$. Suppose there is $st_i \in E(Q_n^3)$ for some $1 \leq i \leq m$. Then, $|T \backslash \{t_i\}| = m - 1$ and $|F \cup \{t_i\}| \leq 2n - m = 2n - (m - 1) - 1$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $P_j = (s, t_j)$ $(j \in \{1, \ldots, m\} \backslash \{i\})$ covering $Q_n^3 - F - t_i$. Let $P_i = \langle s, t_i \rangle$. Thus, $P_1, \ldots, P_m$ are m disjoint paths covering $Q_n^3 - F$. So, we suppose $st_i \notin E(Q_n^3)$ for all $1 \leq i \leq m$.

Let $C = \langle s_0, s_1, s_2, s_0 \rangle$. So, we can decompose Q_n^3 into three subgraphs $Q[0], Q[1], Q[2]$ along one dimension such that $s_i \in V(Q[i])$ $(0 \leq i \leq 2)$. Let $T_i = T \cap V(Q[i])$ and $F_i = F \cap V(Q[i])$ for $0 \leq i \leq 2$. Without loss of generality, let $|T_0| \geq 1$ and $s = s_0$.

Case 1. $|T_0| = m$.

Let $T_0 = \{t_1, \ldots, t_m\}$ and $t_i^j = N(t_i) \cap V(Q[j])$ for $1 \leq i \leq m$ and $j = 1, 2$. Then, $|T_1| = |T_2| = 0$.

Case 1.1. $2n - m - 2 \leq |F_0| \leq 2n - m - 1$.

Then, $|F_1 \cup F_2| \leq 1$. Without loss of generality, let $|F_1| = 0$ and $|F_2| \leq 1$. Let $T_0' = T_0 \backslash \{t_{m-1}, t_m\}$. Then, $|T_0'| = m - 2$. Note $|F_0| \leq 2n - m - 1 = 2(n - 1) - (m - 2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i)$ $(1 \leq i \leq m - 2)$ covering $Q[0] - F_0$. Then, $t_{m-1}, t_m \in \cup_{i=1}^{m-2} V(R_i - t_i - s)$.

Case 1.1.1. $t_{m-1}, t_m \in V(R_i)$ for some $1 \leq i \leq m - 2$.

Without loss of generality, let $t_{m-1}, t_m \in V(R_1)$. Let t_{m-1} is before t_m in R_1. Then, $R_1 = R_1[s, t_{m-1}] \cup R_1[t_{m-1}, t_m] \cup R_1[t_m, t_1]$. Let u and v be the later vertex of t_{m-1} and t_m on R_1, respectively. Let $u_i = N(u) \cap V(Q[i])$ $v_i = N(v) \cap V(Q[i])$ for $i = 1, 2$. Since $|F_2| \leq 1$, $u_2 \notin F_2$ or $v_2 \notin F_2$.

Suppose $u_2 \notin F_2$. Note $|F_2| \leq 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $M = (s_2, u_2)$ in $Q[2] - F_2$. Note $|F_1| = 0$. By Lemma 2, there is a Hamilton path $L = (s_1, v_1)$ in $Q[1]$. Let $P_1 = \{ss_1\} \cup L \cup \{v_1 v\} \cup R_1[v, t_1]$, $P_i = R_i$ $(2 \leq i \leq m - 2)$, $P_{m-1} = R_1[s, t_{m-1}]$ and $P_m = \{ss_2\} \cup M \cup \{u_2 u\} \cup R_1[u, t_m]$. Similar to $v_2 \notin F_2$.

Similarly, t_m is before t_{m-1} on R_1.

Case 1.1.2. $t_{m-1} \in V(R_i)$, $t_m \in V(R_j)$ for $1 \leq i, j \leq m - 2$ and $i \neq j$.

Without loss of generality, let $t_{m-1} \in V(R_1)$ and $t_m \in V(R_2)$. Then, $R_1 = R_1[s, t_{m-1}] \cup R_1[t_{m-1}, t_1]$ and $R_2 = R_2[s, t_m] \cup R_2[t_m, t_2]$. Let u and v be the later vertex of t_{m-1} and t_m on R_1 and R_2, respectively. Let $u_i = N(u) \cap V(Q[i])$ and $v_i = N(v) \cap V(Q[i])$ for $i = 1, 2$. Since $|F_2| \leq 1$, $u_2 \notin F_2$ or $v_2 \notin F_2$.

Suppose $u_2 \notin F_2$. By Lemma 2, there is a Hamilton path $L = (s_1, v_1)$ and $M = (s_2, u_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_1 = \{ss_2\} \cup M \cup \{u_2 u\} \cup R_1[u, t_1]$, $P_2 = \{ss_1\} \cup L \cup \{v_1 v\} \cup R_2[v, t_2]$, $P_i = R_i$ $(3 \leq i \leq m - 2)$, $P_{m-1} = R_1[s, t_{m-1}]$ and $P_m = R_2[s, t_m]$. Similar to $v_2 \notin F_2$.

Case 1.2. $|F_0| = 2n - m - 3$.

Then, $|F_1 \cup F_2| \leq 2$. Without loss of generality, let $|F_1| \leq 1$. Since $|T_0| = m > 1 \geq |F_1|$, there is a vertex t_i such that $t_i^1 \notin F_1$ for $1 \leq i \leq m$. Without loss of generality, let $t_m^1 \notin F_1$. Note $|F_0 \cup \{t_m\}| = 2n - m - 2 = 2(n - 1) - (m - 1) - 1$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $P_i = (s, t_i)$ $(1 \leq i \leq m - 1)$ covering $Q[0] - F_0 - t_m$.

Since $|V(Q[i])| - |F| - |\{s_i, t_m^i\}| \geq 3^{n-1} - (2n - m - 1) - 2 \geq 5$ for $n \geq 3$, there is an edge $w_1 w_2$ such that $w_i \in V(Q[i] - F - \{s_i, t_m^i\})$ for $i = 1, 2$. Note $|F_1 \cup F_2| \leq 2 \leq 2n - 5$ for $n \geq m \geq 4$. By Lemma 2, there is a Hamilton path $R = (w_1, t_m^1)$ and $L = (s_2, w_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_m = \{ss_2\} \cup L \cup \{w_2 w_1\} \cup R \cup \{t_m^1 t_m\}$.

Case 1.3. $|F_0| \leq 2n - m - 4$.

If $t_m^1 \notin F_1$, then it is similar to Case 1.2. Suppose $t_m^1 \in F_1$. Let $T_0' = T \backslash \{t_m\}$. Let $N_{Q[0]}(t_m) = \{u_1, \ldots, u_{2n-2}\}$. Since $|N_{Q[0]}(t_m)| - |T_0'| = 2n - 2 - (m - 1) = 2n - m - 1 > 2n - m - 2 \geq |F \backslash \{t_m^1\}|$, there is a vertex $u_i \in V(Q[0] - F_0 - T_0 - s)$ such that $u_i^j \notin (F_j \backslash \{t_m^j\})$ where $u_i^j = N(u_i) \cap V(Q[j])$ for $1 \leq j \leq 2$ and $1 \leq i \leq 2n - 2$. Without loss of generality, let u_1 be satisfied. Note $|F_0| + |\{t_m, u_1\}| \leq 2n - m - 2 = 2(n - 1) - (m - 1) - 1$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $P_i = (s, t_i)$ $(1 \leq i \leq m - 1)$ covering $Q[0] - F_0 - \{t_m, u_1\}$. Since $|V(Q[i])| - |F| - |\{s_i, t_m^i, u_1^i\}| \geq 3^{n-1} - (2n - m - 1) - 3 > 1$ for $4 \leq m \leq n$, there is an edge $x_1 x_2$ such that $x_i \in V(Q[i] - F_i - \{s_i, t_m^i, u_1^i\})$ for $i = 1, 2$. Note $|F_i| \leq 2n - 5$ for $i = 1, 2$. By Lemma 2, there is a Hamilton path $L = (x_1, u_1^1)$ and $M = (s_2, x_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_m = \{ss_2\} \cup M \cup \{x_2 x_1\} \cup L \cup \{u_1^1 u_1, u_1 t_m\}$.

Case 2. $|T_0| = m - 1$.

Let $T_0 = \{t_1, \ldots, t_{m-1}\}$, $T_1 = \{t_m\}$ and $|T_2| = 0$.

Case 2.1. $|F_0| = 2n - m - 1$.

Then, $|F_1| = |F_2| = 0$. Let $T_0' = T_0 \backslash \{t_{m-1}\}$. Then, $|T_0'| = m - 2$. Note $|F_0| = 2n - m - 1 = 2(n - 1) - (m - 2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i)$ $(1 \leq i \leq m - 2)$ covering $Q[0] - F_0$. Then, $t_{m-1} \in V(R_i - t_i - s)$ for some $1 \leq i \leq m - 2$. Without loss of generality, let $t_{m-1} \in R_1$. Let u be the later vertex of t_m on R_1 and $u_2 = N(u) \cap V(Q[2])$. By Lemma 2, there is a

Hamilton path $L = (s_1, t_m)$ and $M = (s_2, u_2)$ in $Q[1]$ and $Q[2]$, respectively. Let $P_1 = \{ss_2\} \cup M \cup \{u_2 u\} \cup R_1[u, t_1]$, $P_i = R_i (2 \le i \le m-2)$, $P_{m-1} = R_1[s, t_{m-1}]$ and $P_m = \{ss_1\} \cup L$.

Case 2.2. $|F_0| \le 2n - m - 2$.

Note $|F_0| \le 2n - m - 2 = 2(n-1) - (m-1) - 1$. By the induction hypothesis, there are $(m-1)$ disjoint paths $P_i = (s, t_i)(1 \le i \le m-1)$ covering $Q[0] - F_0$. Since $|V(Q[i])| - |F| - |\{s_i, t_m\}| \ge 3^{n-1} - (2n - m - 1) - 2 > 1$ for $4 \le m \le n$, there is an edge $x_1 x_2$ such that $x_i \in V(Q[i] - F_i - \{s_i, t_m\})$ for $i = 1, 2$. Note $|F_i| \le 2n - 5$ for $i = 1, 2$. By Lemma 2, there is a Hamilton path $L = (x_1, t_m)$ and $M = (s_2, x_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_m = \{ss_2\} \cup M \cup \{x_2 x_1\} \cup L$.

Case 3. $1 \le |T_0| \le m - 2$.

Let $|T_0| = p$. Then, $1 \le p \le m-2$ and $2 \le |T_1 \cup T_2| = m - p \le m - 1$.

Case 3.1. $|T_1| = 0$ or $|T_2| = 0$.

Without loss of generality, let $|T_1| = 0$. Then, $2 \le |T_2| = m - p \le m - 1$. Let $T_0 = \{t_1, \ldots, t_p\}$ and $T_2 = \{t_{p+1}, \ldots, t_m\}$. Let $t_i^j = N(t_i) \cap V(Q[j])$ for $p + 1 \le i \le m$ and $j = 0, 1$. Since $|V(Q[i])| - |F| - |T| - |\{s_i\}| \ge 3^{n-1} - (2n - m - 1) - m - 1 > m - 1 \ge m - p$, there are $(m-p)$ cycles of length three $C_i = \langle w_i^0, w_i^1, w_i^2, w_i^0 \rangle$ such that $w_i^j \in V(Q[j] - F_j - T_j - s_j)$ for $0 \le j \le 2$ and $p + 1 \le i \le m$.

Case 3.1.1. $|T_0| > |T_2|$.

Since $|T_0| = p > m - p = |T_2|$, we get $m \le 2p - 1$. Let $W_2 = \{w_i^2 \colon p + 1 \le i \le m - 1\}$ for $m - p \ge 2$. Note $|W_2 \cup \{s_2\}| = m - p = |T_2|$ and $|F_2| \le 2n - m - 1 \le 2(n-1) - 2(m-p)$ for $m \le 2p - 1$. By Theorem 1, there are $(m-p)$ disjoint paths $M_i = (w_i^2, t_i)(p + 1 \le i \le m - 1)$ and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2 covering $Q[2] - F_2$.

Let $W_1 = \{w_{p+1}^1\}$. Note $|F_1| \le 2n - 5$. By Lemma 2, there is a Hamilton path $L = (s_1, w_{p+1}^1)$ in $Q[1] - F_1$.

Let $W_0 = \{w_i^0 \colon p + 2 \le i \le m - 1\}$ for $m - p \ge 3$ and $W_0 = \varnothing$ for $m - p = 2$. Then, $|W_0| = m - p - 2$. Note $|T_0| + |W_0| = p + (m - p - 2) = m - 2$ and $|F_0| \le 2n - m - 1 = 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_i = (s, t_i)$ $(1 \le i \le p)$ and $R_i = (s, w_i^0)(p + 2 \le i \le m - 1)$ from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i (1 \le i \le p)$, $P_{p+1} = \{ss_1\} \cup L \cup \{w_{p+1}^1 w_{p+1}^2\} \cup R_{p+1}$, $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i$ $(p + 2 \le i \le m - 1)$ and $P_m = \{ss_2\} \cup M_m$.

Case 3.1.2. $|T_0| = |T_2|$.

Since $|T_0| = p = m - p = |T_2|$, we get $m = 2p \ge 4$. Thus, $m - p = p \ge 2$.

Case 3.1.2.1. $|F_2| = 2n - m - 1$.

Let $T_2' = T_2 \backslash \{t_{p+1}\}$. Then, $|T_2'| = p - 1$. Note $|F_1| = 0$. By Lemma 2, there is a Hamilton path $L = (s_1, t_{p+1}^1)$ in $Q[1]$.

Let $W_2 = \{w_i^2 \colon p + 2 \le i \le m - 1\}$ for $m - p \ge 3$ and $W_2 = \varnothing$ for $m - p = 2$. Then, $|W_2| = m - p - 2 = p - 2$. Note $|W_2 \cup \{s_2\}| = |T_2'| = p - 1$ and $|F_2 \cup \{t_{p+1}\}| = 2n - m = 2(n-1) - 2(p-1)$ for $m = 2p$. By Theorem 1, there are $(p-1)$ disjoint paths $M_i = (w_i^2, t_i)(p + 2 \le i \le m - 1)$ and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2' covering $Q[2] - F_2 - t_{p+1}$.

Let $W_0 = \{w_i^0 \colon p + 2 \le i \le m - 1\}$ for $m - p \ge 3$ and $W_0 = \varnothing$ for $m - p = 2$. Then, $|W_0| = m - p - 2$. Note $|T_0| + |W_0| = p + (m - p - 2) = m - 2$ and $|F_0| \le 2n - m - 1 = 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_i = (s, t_i)(1 \le i \le p)$ and $R_i = (s, w_i^0)$ $(p + 2 \le i \le m - 1)$ from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i (1 \le i \le p)$, $P_{p+1} = \{ss_1\} \cup L \cup \{t_{p+1}^1 t_{p+1}\}$, $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (p + 2 \le i \le m - 1)$, and $P_m = \{ss_2\} \cup M_m$.

Case 3.1.2.2. $|F_2| \le 2n - m - 2$.

Let $W_2 = \{w_i^2 \colon p + 1 \le i \le m - 1\}$. Note $|\{s_2\} \cup W_2| = 1 + (m - p - 1) = m - p = p = |T_2|$ and $|F_2| \le 2n - m - 2 = 2(n-1) - 2p$ for $m = 2p$. By Theorem 1, there are p disjoint paths $M_i = (w_i^2, t_i)(p + 1 \le i \le m - 1)$ and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2 covering $Q[2] - F_2$.

Let $W_1 = \{w_{p+1}^1\}$. Note $|F_1| \le 2n - 5$. By Lemma 2, there is a Hamilton path $L = (s_1, w_{p+1}^1)$ in $Q[1] - F_1$.

Let $W_0 = \{w_i^0 \colon p + 2 \le i \le m - 1\}$ for $m - p \ge 3$ and $W_0 = \varnothing$ for $m - p = 2$. Then, $|W_0| = m - p - 2$. Note $|T_0| + |W_0| = p + (m - p - 2) = m - 2$ and $|F_0| \le 2n - m - 1 = 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_i = (s, t_i)$ $(1 \le i \le p)$ and $R_i = (s, w_i^0)$ $(p + 2 \le i \le m - 1)$ from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i$ $(1 \le i \le p)$, $P_{p+1} = \{ss_1\} \cup L \cup \{w_{p+1}^1 w_{p+1}^2\} \cup M_{p+1}$, $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i$ $(p + 2 \le i \le m - 1)$, and $P_m = \{ss_2\} \cup M_m$.

Case 3.1.3. $|T_0| < |T_2|$.

Since $|T_0| = p < m - p = |T_2|$, we get $m > 2p \ge 2$.

Case 3.1.3.1. $|F_2| \le 2n - 2m + 2p - 2$.

Note $|F_2| \le 2n - 2m + 2p - 2 = 2(n-1) - 2(m-p)$. Let $W_2 = \{w_i^2 \colon p + 1 \le i \le m - 1\}$. Note $|\{s_2\} \cup W_2| = 1 + (m - p - 1) = m - p = |T_2|$. By Theorem 1, there are $(m-p)$ disjoint paths $M_i = (w_i^2, t_i)$ $(p + 1 \le i \le m - 1)$ and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2 covering $Q[2] - F_2$.

Let $W_1 = \{w_{p+1}^1\}$. Note $|F_1| \le 2n - m - 1 \le 2n - 5$ for $m \ge 4$. By Lemma 2, there is a Hamilton path $L = (s_1, w_{p+1}^1)$ in $Q[1] - F_1$.

Let $W_0 = \{w_i^0 \colon p + 2 \le i \le m - 1\}$ for $m - p \ge 3$ and $W_0 = \varnothing$ for $m - p = 2$. Then, $|W_0| = m - p - 2$. Note

$|T_0| + |W_0| = p + (m - p - 2) = m - 2$ and $|F_0| \le 2n - m - 1 = 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_i = (s, t_i)$ $(1 \le i \le p)$ and $R_i = (s, w_i^0)$ $(p + 2 \le i \le m - 1)$ from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i (1 \le i \le p)$, $P_{p+1} = \{ss_1\} \cup L \cup \{w_{p+1}^1 w_{p+1}^2\} \cup M_{p+1}$, $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (p + 2 \le i \le m - 1)$, and $P_m = \{ss_2\} \cup M_m$.

Case 3.1.3.2. $2n - 2m + kp - 1 \le |F_2| \le 2n - 2m + (k+1)p - 2$ for $2p \le kp \le m$.

Then, $|F_1| \le 2n - m - 1 - (2n - 2m + kp - 1) = m - kp$. Since $|T_2| - |F_1| \ge (m - p) - (m - kp) = (k-1)p$, there are $(k-1)p$ vertices $t_i \in T_2$ such that $t_i^1 \notin F_1$. Without loss of generality, let $Z = \{t_i\colon p + 1 \le i \le kp\}$ satisfy above requirement. Let $Z_1 = \{t_i^1\colon p + 1 \le i \le kp\}$. Then, $|Z_1| = |Z| = (k-1)p$. Let $T_2' = T_2 \backslash Z$. Let $W_1 = \{w_i^1\colon p + 2 \le i \le kp\}$ for $(k-1)p \ge 2$ and $W_1 = \varnothing$ for $(k-1)p = 1$. Note $|F_1| \le m - kp \le 2(n-1) - 2(k-1)p$ for $2 \le 2p \le kp \le m \le n$. By Theorem 1, there are $(k-1)p$ disjoint paths $L_{p+1} = (s_1, t_{p+1}^1)$ and $L_i = (w_i^1, t_i^1)$ $(p + 2 \le i \le kp)$ from $W_1 \cup \{s_1\}$ to Z_1 covering $Q[1] - F_1$.

Case 3.1.3.2.1. $2p \le kp < m$.

Let $W_2 = \{w_i^2\colon kp + 1 \le i \le m - 1\}$ for $m - kp \ge 2$ and $W_2 = \varnothing$ for $m - kp = 1$. Note $|\{s_2\} \cup W_2| = m - kp = |T_2' \backslash Z|$ and $|F_2 \cup Z| \le (2n - 2m + (k+1)p - 2) + (k-1)p = 2n - 2m + 2kp - 2 = 2(n-1) - 2(m - kp)$. By Theorem 1, there are $(m - kp)$ disjoint paths $M_i = (w_i^2, t_i)$ $(kp + 1 \le i \le m - 1)$ and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2' covering $Q[2] - F_2 - Z$.

Let $W_0 = \{w_i^0\colon p + 2 \le i \le m - 1\}$ for $m - p \ge 3$ and $W_0 = \varnothing$ for $m - p = 2$. Then, $|W_0| = m - p - 2$. Note $|T_0| + |W_0| = p + (m - p - 2) = m - 2$ and $|F_0| \le 2n - m - 1 \le 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_i = (s, t_i)(1 \le i \le p)$ and $R_i = (s, w_i^0)(p + 2 \le i \le m - 1)$ from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i (1 \le i \le p)$, $P_{p+1} = \{ss_1\} \cup L_{p+1} \cup \{t_{p+1}^1 t_{p+1}\}$, $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i \cup \{t_i^1 t_i\} (p + 2 \le i \le kp)$, $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (kp + 1 \le i \le m - 1)$, and $P_m = \{ss_2\} \cup M_m$.

Case 3.1.3.2.2. $kp = m$.

$2n - m - 1 = 2n - 2m + kp - 1 \le |F_2| \le 2n - 2m + (k+1)p - 2 = 2n - m + p - 2 \le 2n - m - 1$. Thus, $|T_0| = p = 1$, $|T_2| = m - p = m - 1$, $|F_2| = 2n - m - 1$ and $|F_0| = |F_1| = 0$. Let $T_2' = T_2 \backslash \{t_m\}$. Then, $|T_2'| = m - 2$. Note $|F_2| = 2n - m - 1 = 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m-2)$ disjoint paths $M_i = (s_2, t_i)(2 \le i \le m - 1)$ from s_2 to T_2' covering $Q[2] - F_2$. Then, $t_m \in R_i$ for some $2 \le i \le m - 1$. Without loss of generality, let $t_m \in R_{m-1}$. Let u and z_i be the next vertex of t_m and s_2 in R_{m-1} and R_i for $2 \le i \le m - 2$, respectively. Let $u_1 = N(u) \cap V(Q[1])$ and $z_i^0 = N(z_i) \cap V(Q[0])$. By the definition of Q_n^3, $z_i^0 \in N(s)$. Since $st_i \notin E(Q_n^3)$, $z_i^0 \notin T_0$. Let $Z_0 = \{z_i^0\colon 2 \le i \le m - 2\}$. By Lemma 2, there is a Hamilton path $L = (s_1, u_1)$ in $Q[1]$. Note $|T_0| + |Z_0| = m - 2$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_1 = (s, t_1)$ and $R_i = (s, z_i^0)(2 \le i \le m - 2)$ from s to $T_0 \cup Z_0$ covering $Q[0] - F_0$.

Let $P_1 = R_1$, $P_i = R_i \cup \{z_i^0 z_i\} \cup M_i [z_i, t_i]$ $(2 \le i \le m - 2)$, $P_{m-1} = \{ss_1\} \cup L \cup \{u_1 u\} \cup M_{m-1} [u, t_{m-1}]$ and $P_m = \{ss_2\} \cup M_{m-1} [s_2, t_m]$.

Case 3.2. $|T_1| = |T_2| = 1$.

Then, $|T_0| = m - 2$. Let $T_0 = \{t_1, \ldots, t_{m-2}\}$, $T_1 = \{t_{m-1}\}$ and $T_2 = \{t_m\}$. Note $|F_i| \le 2n - 5$ for $1 \le i \le 2$. By Lemma 2, there is a Hamilton path $L = (s_1, t_{m-1})$ and $M = (s_2, t_m)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Note $|F_0| \le 2n - m - 1 = 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_i = (s, t_i)$ $(1 \le i \le m - 2)$ covering $Q[0] - F_0$.

Let $P_i = R_i (1 \le i \le m - 2)$, $P_{m-1} = \{ss_1\} \cup L$ and $P_m = \{ss_2\} \cup M$.

Case 3.3. $|T_i| \ge 2$ for $i = 1, 2$.

Let $|T_1| = p$ and $|T_2| = q$. Since $1 \le |T_0| = m - p - q \le m - 4$, $4 \le p + q \le m - 1$. Without loss of generality, let $|T_1| = p \ge q = |T_2|$. Then, $m - 3 \ge p \ge q \ge 2$. Since $1 \le m - p - q \le m - 2q$, we get $2q + 1 \le m$. Let $T_0 = \{t_1, \ldots, t_{m-p-q}\}$, $T_1 = \{t_{m-p-q+1}, \ldots, t_{m-q}\}$ and $T_2 = \{t_{m-q+1}, \ldots, t_m\}$.

Since $|V(Q[i])| - |F| - |T| - |\{s_i\}| \ge 3^{n-1} - (2n - m - 1) - m - 1 \ge m - 3 \ge p + q - 2$, there are $(p + q - 2)$ cycles $C_i = \langle w_i^0, w_i^1, w_i^2, w_i^0 \rangle$ where $w_i^j \in V(Q[j] - F - T - s_j)$ for $m - p - q + 2 \le i \le m - 1$ and $1 \le j \le 2$.

Let $W_2 = \{w_i^2\colon m - q + 1 \le i \le m - 1\}$ for $q \ge 2$. Then, $|W_2 \cup \{s_2\}| = q = |T_2|$. Note $|F_2| \le 2n - m - 1 \le 2(n-1) - 2q$ for $m \ge 2q + 1$ By Theorem 1, there are q disjoint paths $M_i = (w_i^2, t_i)(m - q + 1 \le i \le m - 1)$ and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2 covering $Q[2] - F_2$.

Case 3.3.1. $m \ge 2p + 1$.

Let $W_1 = \{w_i^1\colon m - p - q + 2 \le i \le m - q\}$ for $p \ge 2$. Then, $|W_1 \cup \{s_1\}| = p = |T_1|$. Note $|F_1| \le 2n - m - 1 \le 2(n-1) - 2p$ for $m \ge 2p + 1$. By Theorem 1, there are p disjoint paths $L_{m-p-q+1} = (s_1, t_{m-p-q+1})$ and $L_i = (w_i^1, t_i)(m - p - q + 2 \le i \le m - q)$ from $\{s_1\} \cup W_1$ to T_1 covering $Q[1] - F_1$.

Let $W_0 = \{w_i^0\colon m - p - q + 2 \le i \le m - 1\}$. Then, $|W_0| = p + q - 2$. Note $|T_0| + |W_0| = (m - p - q) + (p + q - 2) = m - 2$ and $|F_0| \le 2n - m - 1 = 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_i = (s, t_i)$ $(1 \le i \le m - p - q)$ and $R_i = (s, w_i^0)$ $(m - p - q + 2 \le i \le m - 1)$ from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i$ $(1 \le i \le m - p - q)$, $P_{m-p-q+1} = \{ss_1\} \cup L_{m-p-q+1}$, $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i$ $(m - p - q + 2 \le i \le m - q)$, $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i$ $(m - q + 1 \le i \le m - 1)$ and $P_m = \{ss_2\} \cup L_m$.

Case 3.3.2. $2p \ge m$.

Case 3.3.2.1. $|F_0| \ge 2p - m + 1$.

Note $|F_1| \le 2n - m - 1 - (2p - m + 1) = 2n - 2p - 2 = 2(n-1) - 2p$. Then, it is similar to Case 3.3.1.

Case 3.3.2.2. $2p - m - (k+1)q + 1 \le |F_0| \le 2p - m - kq$ for $0 \le kp \le 2p - m$.

Then, $|F_1| \le 2n - m - 1 - (2p - m - (k+1)q + 1) = 2n - 2p + (k+1)q - 2$ and $|F_0| + |T_0| \le (2p - m - kq) + (m - p - q) = p - (k+1)q < p = |T_1|$. Since $|T_2| - |T_0| - |F_0| \ge p - (p - (k+1)q) = (k+1)q$, there are $(k+1)q$ vertices $t_i \in T_1$ such that $t_i^0 \notin (F_0 \cup T_0)$. Let t_i $(m - p - q + 1 \le i \le m - p + kq)$ satisfy above requirement. Let $Z = \{t_i: m - p - q + 1 \le i \le m - p + kq\}$ and $Z_0 = \{t_i^0: m - p - q + 1 \le i \le m - p + kq\}$. Then, $|Z| = |Z_0| = (k+1)q$. Note $|T_1 \backslash Z| = p - (k+1)q \ge p - (2p - m) - q = m - p - q \ge 1$ and $|F_1| + |Z| \le (2n - 2p + (k+1)q - 2) + (k+1)q = 2n - 2p + 2(k+1)q - 2 = 2(n-1) - 2(p - (k+1)q)$.

Let $W_1 = \{w_i^1: m - p + kq + 2 \le i \le m - q\}$ for $(k+1)q \le 2p - m + 1 \le p - 2$. Then, $|W_1| = p - (k+1)q - 1$. By Theorem 1, there are $(p - (k+1)q)$ disjoint paths $L_{m-p+kq+1} = (s_1, t_{m-p+kq+1})$ and $L_i = (w_i^1, t_i)(m - p + kq + 2 \le i \le m - q)$ from $W_1 \cup \{s_1\}$ to $T_1 \backslash Z$ covering $Q[1] - F_1 - Z$.

Let $W_0 = \{w_i^0: m - p + kq + 2 \le i \le m - 1\}$ for $kq \le 2p - m \le p - 3$. Then, $|W_0| = p - kq - 2$. Note $|T_0| + |Z_0| + |W_0| = (m - p - q) + (k+1)q + (p - kq - 2) = m - 2$ and $|F_0| \le 2n - m - 1$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_i = (s, t_i)(1 \le i \le m - p - q)$ and $R_i = (s, t_i^0)(m - p - q + 1 \le i \le m - p + kq)$, $R_i = (s, w_i^0)(m - p + kq + 2 \le i \le m - 1)$ from s to $T_0 \cup Z_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i (1 \le i \le m - p - q)$, $P_i = R_i \cup \{t_i^0 t_i\}(m - p - q + 1 \le i \le m - p + kq)$, $P_{m-p+kq+1} = \{ss_1\} \cup L_{m-p+kq+1}$, $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (m - p + kq + 2 \le i \le m - q)$, $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (m - q + 1 \le i \le m - 1)$ and $P_m = \{ss_2\} \cup M_m$. □

5. Conclusions

In this paper, we studied m disjoint paths from S to T covering $Q_n^3 - F$ with $|F| \le 2n - 2m$ for $2 \le |S| = |T| = m \le n$. Furthermore, we determined there are m disjoint paths from s to T covering $Q_n^3 - F$ with $|F| \le 2n - m - 1$ where $2 \le |T| = m \le n$ and s is in a fault-free cycle of length three. In future research, scholars can continue to study vertex disjoint path problem of Q_n^k or other graphs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Prediction of Temperature Field and Thermal Damage of Multilayer Skin Tissues Subjected to Time-Varying Laser Heating and Fluid Cooling by a Semianalytical Method

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A semianalytical method based on the separation of variables is presented to solve the problem of heat transfer in multilayer skin tissues. This method is used to predict the temperature field and thermal damage of skin tissues, which are subjected to the time-varying laser heating and fluid cooling. The effect of convective heat transfer coefficient of the skin surface on temperature field and thermal damage is studied. The temperature field and thermal damage of skin tissue with different laser heating cycles are calculated. The relationship between the laser heating period and the temperature fluctuation is obtained. The effects of the peak time of laser heating on temperature and the thermal damage of skin tissues are investigated. The relationship between the peak time of laser and the temperature fluctuation of skin tissue is drawn. The results calculated in this paper can provide some guidance for hyperthermia of skin tissue. The presented semianalytical method can be applied to solve the general heat transfer problems of multilayer structures.

1. Introduction

Thermotherapy has been widely used in the treatment of skin tissue diseases. Laser, microwave, and radio frequency are often used to irradiate the surface of skin tissue during thermotherapy [1–3]. Accurate prediction of skin tissue temperature and thermal damage during hyperthermia plays an important role in thermotherapy [4]. More and more researchers have studied the heat transfer and thermal damage of the skin tissue. Important research can be summarized as follows.

Deng and Liu gave several closed-form analytical solutions to the bioheat transfer problems which are based on Green's function method [5]. Xu et al. studied the heat transfer of skin tissue by using pence model; they established a mathematical model of heat transfer of three-layer skin tissue and solved the model by using finitely different method [6]. Liu et al. established DPL model for heat transfer of multilayer skin tissue; the Laplace transform method was used to solve this model [7]. Tian et al. established a skin-fabric five-layer heat transfer model; a semianalytical method was employed to solve this model [8]. Ahmadikia et al. used analytical methods based on Laplace transform to study the distribution of temperature field in skin tissue under constant or transient heat flux conditions; parabolic and hyperbolic skin tissue heat transfer equations were studied [9]. In the study of Kumar et al., a dual-phase-lag model of bioheat transfer had been studied by using Gaussian distribution source term with the most generalized boundary condition during hyperthermia treatment [10]. Lin solved the analytical solution of heat transfer of single skin layer which was based on the separation of variable methods; Fourier and non-Fourier cases were studied [11]. Hooshmand et al. developed an analytical solution method for solving the problem of heat transfer in biological tissue, generalized dual-phase-lagging (DPL) model was studied, and the calculation results were compared with the classical DPL model [12]. Kumar et al. dealt with the study of heat

transfer and thermal damage in triple layer skin tissue using fractional bioheat model. An implicit finite difference scheme is obtained by a new method [13]. Kumar and Rai studied the thermal characteristics in skin tissue using time fractional dual-phase-lag bioheat transfer (DPLBHT) model subjected to Dirichlet boundary condition. They solved this problem using the finite element Legendre wavelet Galerkin method. They treated skin tissue as a single layer [14]. Li et al. predicted the temperature of different biological tissues using Fourier and non-Fourier heat transfer models. And the temperature response of different tissues was measured. The experimental results were compared with those predicted by the model [15]. Hobiny and Abbas used the Laplace transform method to solve the problem of biological tissues subjected to moving heat source [16]. Ghanmi and Abbas derived an analytical solution for the fractional transient heating within the skin tissue during hyperthermia [17]. Sur et al. used Laplace transform to study the thermal behavior of skin tissue when it was subjected to moving heat sources [18]. Sharma and Kumar established a nonlinear dual-phase-lag heat transfer model of biological tissue. The finite element Runge–Kutta method was used to solve this model. The temperature field and thermal damage of skin tissue were studied using the numerical solution of the model [19].

Skin tissue is the largest organ of the human body, including epidermis, dermis, and subcutaneous tissue [20]. Thermophysical parameters of different layers of skin tissue are different. So it is difficult to solve the heat transfer model of skin tissue by analytical method [9, 11, 21, 22]. Numerical methods were often used to solve this problem [23]. Several papers used analytical methods, but all of them treated skin tissue as a single layer [12]. In this paper, a semianalytical method is developed to solve heat transfer in multilayer skin tissues. A special method of separation of variables is used to derive the analytical solution. In the derivation process, the time domain is firstly divided into several small time periods. Then the temperature function at the interface of different layers of skin tissue is assumed to be linear in a small period of time. After that, the analytical solution of separated variables of skin tissue heat transfer is deduced in a small period of time. According to the condition that the temperature at the interface is equal and the heat flux at the interface is continuous, the temperature at the interface is calculated. Temperature field of skin tissue during laser heating and fluid cooling is obtained by the semianalytical solution. The effects of several parameters such as convective heat transfer coefficient, period, and peak time of laser heating on temperature field are studied. These parameters can be used to control the temperature of the skin tissue and the thermal damage in the appropriate range.

2. Mathematical Model

Human skin tissue consists of epidermis, dermis, and subcutaneous. The geometric model of skin tissue is shown in Figure 1. The corresponding mathematical model can be established by combining the boundary conditions. The governing equation can be expressed as (1) [2, 24–27]:

Figure 1: Geometric model of skin tissue.

$$k_i\frac{\partial^2 T}{\partial x^2} - \rho_i c_i \frac{\partial T}{\partial t} - \overline{w}_b \rho_b c_b (T - T_a) + q_{\text{met},i} + q_{\text{ext},i}, \quad L_{i-1} < x < L_i, t > 0. \tag{1}$$

The boundary conditions can be expressed as follows: At $x = 0$,

$$k_1 \frac{\partial T(0,t)}{\partial x} = h_0 (T - T_\infty) + q(t). \tag{2}$$

At $x = L_3$,

$$T(L_3, t) = T_a. \tag{3}$$

The initial condition is

$$T(x, 0) = T_a. \tag{4}$$

The corresponding dimensionless governing equation is

$$r_{ki}\frac{\partial^2 \theta}{\partial \xi^2} - \overline{w}_{bi}\theta - r_{\rho i} r_{ci}\frac{\partial \theta}{\partial \tau} + \overline{q}_{\text{met},i} + \overline{q}_{\text{ext},i} = 0, \quad \xi_{i-1} < x < \xi_i, \tau > 0. \tag{5}$$

The dimensionless boundary conditions are as follows: At $\xi = 0$,

$$\frac{\partial \theta(0,\tau)}{\partial \xi} - N_u \theta(0,\tau) = -N_{u0}\theta_\infty + \overline{f}_{1,1}(\tau) = f_{1,1}(\tau). \tag{6}$$

At $\xi = 1$,

$$\theta(1,\tau) = f_{3,2}(\tau). \tag{7}$$

The dimensionless initial condition is

$$\theta(\xi, 0) = 0. \tag{8}$$

3. Solution Method

If the value of $\Delta\tau$ is small enough, in time domain $[0\ \Delta\tau]$, $f_{1,1}(\tau)$ can be expressed as [28]

$$f_{1,1}(\tau) = \frac{f_{1,1}(\Delta\tau) - f_{1,1}(0)}{\Delta\tau}\tau = \nu_{1,1}\tau, \tag{9}$$

$$\theta(\xi_1, \tau) = \nu_{1,2}\tau = f_{1,2}(\tau) = f_{2,1}(\tau), \tag{10}$$

$$\theta(\xi_2,\tau) = v_{2,3}\tau = f_{2,2}(\tau) = f_{3,1}(\tau). \tag{11}$$

Equations (7) and (9)–(11) show that the boundary conditions on both sides of each layer of skin tissue are known in a short time interval $[0\ \Delta\tau]$. A extended shifting variable method is developed by Lee and Lin, which can be used to solve the temperature field of each layer of skin tissue [29].

Assume

$$\theta_i(\xi,\tau) = v_i(\xi,\tau) + \sum_{m=1}^{2} g_{i,m}(\xi) f_{i,m}(\tau). \tag{12}$$

Here, $g_{i,m}$ (ξ), $i=1,\ 2,\ 3$; $m=1,\ 2$ are the shifting functions to be specified and v_i $(\xi,\ \tau)$ is the transformed function. Substituting (12) into (5), the two following subsystems are obtained. The first subsystem is expressed in terms of the transforming variables $v_i(\xi,\tau)$ as follows; the transform governing equation is

$$r_{ki}\frac{\partial^2 v_i}{\partial \xi^2} - \overline{\omega}_b v_i - r_{\rho i} r_{ci}\frac{\partial v_i}{\partial \tau} + \overline{q}_{ts,i} = 0, \tag{13}$$

$$\begin{aligned}\overline{q}_{ts,i} = {} & -\overline{\omega}_b \sum_{m=1}^{2} g_{i,m} f_{1,m} - r_{\rho i} r_{ci} \sum_{m=1}^{2} g_{i,m}\frac{\mathrm{d} f_{i,m}}{\mathrm{d}\tau} \\ & + r_{ki}\sum_{m=1}^{2}\frac{\mathrm{d}^2 g_{i,m}}{\mathrm{d}\xi^2} f_{i,m}(\tau) + \overline{q}_{\mathrm{met},i} + \overline{q}_{\mathrm{ext},i}.\end{aligned} \tag{14}$$

If $i=1$, substitute (12) into (6) and (10):

$$\begin{aligned}&\frac{\partial v_1(0,\tau)}{\partial \xi} - N_u v_i(0,\tau) + \frac{\mathrm{d}\sum_{m=1}^{2} g_{i,m}(0) f_{i,m}(\tau)}{\mathrm{d}\xi} \\ &- N_u \sum_{m=1}^{2} g_{i,m}(0) f_{i,m}(\tau) = f_{1,1}(\tau),\end{aligned} \tag{15}$$

$$v_1(\xi_1,\tau) + \sum_{m=1}^{2} g_{1,m}(\xi_1) f_{1,m}(\tau) = f_{1,2}(\tau). \tag{16}$$

In order to homogenize the boundary conditions when separating variables, (17) and (18) need to be satisfied:

$$\frac{\partial v_1(0,\tau)}{\partial \xi} - N_u v_i(0,\tau) = 0, \tag{17}$$

$$v_1(\xi_1,\tau) = 0. \tag{18}$$

So,

$$\frac{\mathrm{d}\sum_{m=1}^{2} g_{i,m}(0) f_{i,m}(\tau)}{\mathrm{d}\xi} - N_u \sum_{m=1}^{2} g_{1,m}(0) f_{i,m}(\tau) = 0, \tag{19}$$

$$\sum_{m=1}^{2} g_{1,m}(\xi_1) f_{1,m}(\tau) = f_{1,2}(\tau). \tag{20}$$

Assume

$$g_{1,1}(\xi) = \begin{cases} \dfrac{\xi - \xi_1}{1 + N_{u0}\xi_1}, & m = 1, \\[2ex] \dfrac{N_{u0}\xi + 1}{1 + N_{u0}\xi}, & m = 2. \end{cases} \tag{21}$$

If $i=2,\ 3$, substitute (12) into (10) and (11):

$$v_i(\xi_{i-1},\tau) + \sum_{m=1}^{2} g_{i,m}(\xi_{i-1}) f_{i,m}(\tau) = f_{i,1}(\tau), \tag{22}$$

$$v_i(\xi_i,\tau) + \sum_{m=1}^{2} g_{i,m}(\xi_2) f_{i,m}(\tau) = f_{i,2}(\tau). \tag{23}$$

In order to homogenize the boundary conditions when separating variables, (24) and (25) need to be satisfied:

$$v_i(\xi_{i-1},\tau) = 0, \tag{24}$$

$$v_i(\xi_i,\tau) = 0. \tag{25}$$

So,

$$\sum_{m=1}^{2} g_{i,m}(\xi_{i-1}) f_{i,m}(\tau) = f_{i,1}(\tau), \tag{26}$$

$$\sum_{m=1}^{2} g_{i,m}(\xi_i) f_{i,m}(\tau) = f_{i,2}(\tau). \tag{27}$$

Assume

$$g_{i,m}(\xi) = \begin{cases} \dfrac{-\xi}{\xi_{i-1} - \xi} + \dfrac{\xi_{i-1}}{\xi_{i-1} - \xi_i}, & m = 1; \\[2ex] \dfrac{-\xi}{-\xi_{i-1} + \xi_i} + \dfrac{\xi_i}{-\xi_{i-1} + \xi_i}, & m = 2. \end{cases} \tag{28}$$

The characteristic equation of (13) is

$$r_{ki}\frac{\partial^2 v_i}{\partial \xi^2} - r_{\rho i} r_{ci}\frac{\partial v_i}{\partial \tau} = 0. \tag{29}$$

Assume

$$v_i = \psi_i(\xi)\phi_i(\tau). \tag{30}$$

Substitute (30) into (29):

$$\frac{\mathrm{d}^2\psi_i(\xi)}{\mathrm{d}\xi^2} - \lambda\psi_i(\xi) = 0, \tag{31}$$

$$\frac{\mathrm{d}\phi_i(\tau)}{\mathrm{d}\tau} - \lambda\frac{r_{ki}}{r_{\rho i} r_{ci}}\phi_i(\tau) = 0. \tag{32}$$

If $i=1$, substitute equation (30) into (17) and (18):

$$r_{k1}\frac{\mathrm{d}\psi_1(0)}{\mathrm{d}\xi} - N_u\psi_{11}(0) = 0, \tag{33}$$

$$\psi_1(\xi_1) = 0. \tag{34}$$

Equation (31) is a second-order homogeneous ordinary differential equation. Its characteristic function is as follows:

$$\psi_{1,j}(\xi) = \cos\sqrt{\lambda_j}\xi_1\xi + \frac{N_{u0}}{r_{k1}\sqrt{\lambda_j}}\sin\sqrt{\lambda_j}\xi_1\xi, \tag{35}$$

where λ_j is the positive solution of the following equation:

$$\frac{r_{k1}\sqrt{\lambda}}{N_u} + \tan\sqrt{\lambda}\xi_1 = 0. \tag{36}$$

If $i = 2, 3$, substitute (30) into (24) and (25):

$$\begin{aligned}\psi_{i,j}(\xi) = {} & \cos\left(\sqrt{\lambda_j}\,\xi\right)\sin\left(\sqrt{\lambda_j}\,\xi_{i-1}\right) \\ & - \cos\left(\sqrt{\lambda_j}\,\xi_{i-1}\right)\sin\left(\sqrt{\lambda_j}\,\xi\right),\end{aligned} \tag{37}$$

where λ_j is the positive solution of the following equation:

$$\cos\left[\sqrt{\lambda}\,\right]\sin\left[\sqrt{\lambda}\,\xi_{i-1}\right] - \cos\left[\sqrt{\lambda}\,\xi_{i-1}\right]\sin\left[\sqrt{\lambda}\,\right] = 0. \tag{38}$$

Obviously, $\psi_{i,j}(\xi)$ satisfies the following equation:

$$\int_{\xi_{i-1}}^{\xi_i}\psi_{i,j}(\xi)\psi_{i,k}(\xi)\mathrm{d}\xi = \begin{cases} 0, & \text{for } j \neq k; \\ \delta_{i,j}, & \text{for } j = k. \end{cases} \tag{39}$$

Assume

$$v_i(\xi,\tau) = \sum_{j=1}^{\infty}\psi_{i,j}(\xi)\gamma_{i,j}(\tau). \tag{40}$$

Substitute (40) into (13) and integrate it in $[\xi_{i-1}, \xi_i]$:

$$\frac{\mathrm{d}\gamma_{i,j}}{\mathrm{d}\tau} + \frac{\left(r_{ki}\lambda_j + \eta_i\right)}{r_{\rho i}r_{ci}}\gamma_{i,j} = q_{ts,i}, \tag{41}$$

$$q_{ts,i} = \frac{1}{r_{\rho i}r_{ci}}\frac{1}{\delta_{i,j}}\int_{\xi_{i-1}}^{\xi_i}\psi_{i,j}\overline{q}_{ts,i}\mathrm{d}\xi, \tag{42}$$

$$\eta_i = \begin{cases} N_u & i = 1; \\ \overline{\omega}_{bi} & i = 2, 3. \end{cases} \tag{43}$$

The solution of (41) is

$$\gamma_{i,j}(\tau) = e^{-\alpha_i\tau}\gamma_{i,j}(0) + e^{-\alpha_i\tau}\int_0^{\tau}e^{\alpha_1\tau}q_{tsi}\mathrm{d}\tau, \tag{44}$$

$$\alpha_i = \frac{\left(r_{ki}\lambda_j + \eta_i\right)}{r_{\rho i}r_{ci}}, \tag{45}$$

$$\theta_i(\xi,\tau) = \sum_{j=1}^{\infty}\lambda_{i,j}(\tau)\psi_{i,i}(\xi) + \sum_{m=1}^{2}g_{i,m}(\xi)f_{i,m}(\tau). \tag{46}$$

4. Thermal Damage

In addition to predicting the temperature field of skin tissue during thermotherapy, it is often necessary to predict its thermal damage. Formula (47) is developed by Henriques and Moritz which can be used to calculate thermal damage of skin tissue [1]:

$$\Omega(t) = \int_0^t A\exp\frac{-E_a}{RT}\mathrm{d}t, \tag{47}$$

where Ω represents the thermal damage factor of skin tissue. E_a represents the activation energy. T represents the temperature of the skin tissue. R represents the universal gas constant. According to the thermal damage factors, the thermal damage is divided into three levels in the literatures. $\Omega = 0.53$, the first level of thermal damage; $\Omega = 1$, the second level of thermal damage; $\Omega = 10^4$, the third level of thermal damage.

5. Numerical Results

When the surface of skin tissue is irradiated by pulsed laser, the variation of laser energy with time can be shown in Figure 2. Assume the initial temperature of the skin tissue $T(x, 0) = 37^\circ$C. At $x = L_3$, $T(L_3, 0) = 37^\circ$C. The thermophysical parameters of skin tissue can be shown in Table 1 [4].

Figure 3 shows the temperature history at the skin surface, ED interface, and DS interface in a laser heating cycle. It is found that the temperature rises first and then decreases with time at $x = 0$, $x = 0.1$ mm. If $h = 7$ W/m$^{2\circ}$C, the temperature peaks appear at $t = 2$ s and $t = 2.4$ s, respectively. If $h = 700$ W/m$^{2\circ}$C, the temperature peaks appear at $t = 1.4$ s and $t = 1.7$ s, respectively. So we can speculate that the larger the value of x, the shorter the time required for temperature peak. If $h = 7$ W/m$^{2\circ}$C, at $x = 1.6$ mm, the temperature first remains constant and then continues to rise. When $t = 10$ s, the peak value of temperature still does not appear. If $h = 700$ W/m$^{2\circ}$C, at $x = 1.6$ mm, because this location is far away from the heating source, temperature varies a little with time. It can be inferred from the comparison of solid and dotted lines that the larger the value of h, the lower the temperature. Investigating its reason we can know that the larger the value of h is, the more heat is taken away by water cooling on the surface of skin tissue, so the lower the temperature of skin tissue is.

Figure 4 shows the temperature distribution in the x direction at different times. As shown in the picture, if $h = 7$ W/m$^{2\circ}$C, $t = 1$ s, the temperature of the skin tissue surface is the highest, which is 65°C. The temperature distribution of skin tissue decreases in the x-axis. Comparing the temperature distribution curves of $t = 0.5$ s and $t = 1$ s, it is found that the temperature of $T = 1$ s is higher than that of $t = 1$ s. This is because the heat flux of laser heating keeps rising in 0 to 1 s. Because the heat flux of laser heating begins to decrease after $t = 1$ s, the temperature distribution at $t = 5$ s and 10 s is much lower than that at $t = 1$ s. Comparing the temperature distribution of $h = 7$ W/m$^{2\circ}$C and $h = 700$ W/m$^{2\circ}$C, it is found that, at the same time, the temperature distribution of skin tissue at $h = 700$ W/m$^{2\circ}$C is significantly lower than that at $h = 7$ W/m$^{2\circ}$C. Because the cooling effect is greater than the heating effect, the temperature of skin tissue surface is lower than 37°C at $t = 5$ s and $t = 10$ s.

Figure 5 demonstrates the relationship between heat flux and heating time at different locations of skin tissue. As can be seen from the figure, when $h = 7$ W/m$^{2\circ}$C, the heat flux density on the surface of skin tissue ($x = 0$) increases first and

Figure 2: Schematic diagram of pulsed laser heating and fluid cooling at skin tissue surface.

Table 1: Thermophysical parameters of skin tissue.

Parameters	$i=1$	$i=2$	$i=3$	References
L_i (mm)	0. 1	1.6	6	[30]
ρ_i (kg/m^3)	1190	1116	971	[31]
k_i (W/m-K)	0.235	0.445	0.185	[26]
c_i (J/kg-K)	3600	3300	2700	[26]
$\varpi_{b,i}$ (ml/ml/s)	0	0.1	0	[32]
$q_{met,i}$ (W/m^3)	368.1	368.1	368.3	[9]
$q_{ext,i}$ (W/m^3)	0	0	0	[9]

Figure 3: Temperature history at skin surface, ED interface, and DS interface ($T_C = 10$ s, $t_p = 1$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C).

then decreases, which is in accordance with the trend of surface heating heat flux. At the skin-dermis interface ($x = 0.1$ mm), the heat flux is slightly lower than that at the skin surface. At the epidermal-subcutaneous interface ($x = 1.6$ mm), the heat flux increases first and then decreases,

Figure 4: Temperature distribution along x direction at different time ($T_C = 10$ s, $t_p = 1$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C).

Figure 5: Relationship between heat flux and heating time at different locations of skin tissue ($T_C = 10$ s, $t_p = 1$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C).

but the range of variation is very small. On the left side of the skin tissue ($x = 6$ mm), the heat flux is 0 and remains unchanged. When $h = 700$ W/m^2°C, the heat flux of heating is less than that of cooling in the initial period of time. The heat flux of skin surface ($x = 0$) and dermal-epidermal interface ($x = 0.1$ mm) changed from negative to positive, continued to increase, and then gradually decreased. Because the boundary condition we set is that the right side of the skin tissue is adiabatic, on the left side of the skin tissue ($x = 6$ mm), the heat flux is 0 and remains unchanged too.

Figure 6 demonstrates the relationship between thermal damage and heating time at different locations of skin tissue.

FIGURE 6: Relationship between thermal damage and heating time at different locations of skin tissue ($T_C = 10$ s, $t_p = 1$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C).

As can be seen from the figure, thermal damage of skin tissue increases with time accumulation. The surface of skin tissue has the greatest thermal damage. If $h = 7$ W/m^{2}°C, the thermal damage value of skin tissue surface is close to 100, reaching the second-degree thermal damage, and the thermal damage value at the junction of epidermis and dermis exceeds 0.53, reaching the first-degree thermal damage. Because the greater the value of h, the better the heat dissipation and the smaller the thermal damage of skin tissue, the thermal damage at $h = 700$ W/m^{2}°C is six orders lower than that at $h = 7$ W/m^{2}°C. It can be seen from the figure that although the heat flux of laser heating increases first and then decreases, the heat damage increases continuously. As the heat flux of laser heating decreases, the trend of increasing skin tissue thermal damage slows down. In order to analyze the effect of R and H on thermal damage, a variance analysis was carried out. Table 2 shows the calculated results. By calculation, R^2 $(x) = 0.163812$, R^2 $(h) = 0.318886$, R^2 $_{(\text{interaction})} = 0.318877$. Therefore, it can be inferred that the influence level of x on thermal damage is lower than the average and the influence level of h on thermal damage is above average. These two factors and their interaction have a great influence on thermal damage.

Figure 7 demonstrates the effect of heating cycle Tc on the temperature history at skin surface. As can be seen from the figure, because heating and cooling are the same in the first five seconds, so the temperature curve coincides completely. If $T_C = 5$ s, the first pulse of laser heating is completed after 5 seconds, the heat flux of heating no longer decreases, and the second laser heating pulse starts. Therefore, the temperature on the surface of skin tissue starts to rise when $t = 5$ s, and the rising process is fluctuant, and the fluctuating period is $T_C = 5$ s. After five seconds, the shorter the heating cycle is, the faster the temperature rises. Temperature fluctuates with heating period. After 30 seconds, the amplitude of temperature fluctuation tends to be balanced. The shorter the heating period T_C is, the greater the temperature fluctuation amplitude $T_{s,max}$ is.

Figure 8 shows the effect of heating cycle T_C on the temperature history at the interface between epidermis and dermis. Comparing with Figure 7, it can be seen that the temperature fluctuation at the interface of epidermis and dermis is smaller than that at the skin surface. It is the same as the result shown in Figure 6, The shorter the heating period T_C is, the greater the temperature fluctuation amplitude $T_{s,max}$ is. When $T_C = 5$ s, the amplitude of fluctuation reaches the maximum value; at the junction of epidermis and dermis, $T_{ed1,max} = 82.5$°C, it is 9.5°C lower than the surface temperature of skin tissue. When $T_C = 15$ s, the amplitude of fluctuation reaches minimum value; at the junction of epidermis and dermis, $T_{ed3,max} = 65.2$°C, it is 7.9°C lower than the surface temperature of skin tissue.

Figure 9 shows the effect of heating cycle T_C on the temperature history at the interface between dermis and subcutaneous. Because the location is far from the heat source, the range of temperature fluctuation is much smaller than the skin surface and the interface between epidermis and dermis. It takes a long time for temperature fluctuation to reach a steady state. As can be seen from the figure, when T_C values are different, the time required for temperature fluctuation to reach stability is 75 s, 100 s, and 150 s, respectively. It can be inferred that the greater the value of temperature T_C is, the less time it takes for temperature fluctuation to reach stability. It can also be seen from the figure that the larger T_C value is, the larger temperature fluctuation range is.

Figure 10 shows the effect of heating cycles on the thermal damage at different locations of skin tissue. Because the surface of skin tissue is close to the heat source, the thermal damage on the surface of skin tissue ($x = 0$) is much greater than the interface between dermis and epidermis ($x = 0.1$). With the accumulation of heating time, the thermal damage of skin surface in all three cases exceeds 10^4, reaching the third-degree burn. The thermal damage with $T_C = 5$ s is four orders higher than the thermal damage with $T_C = 10$ s. It can be inferred from the figure that the smaller the heating period is, the more serious the thermal damage of skin tissue is. The thermal damage at the interface between epidermis ($x = 0.1$ mm) and dermis exceeds 10^4, reaching the third-degree burn when $T_C = 5$ s or $T_C = 10$ s. When $T_C = 15$ s, the thermal damage value at the interface between epidermis and dermis exceeds 100 to reach second-degree burn. Because the dermis-subcutaneous interface ($x = 1.6$ mm) is far away from the heat source, no burn occurs with $T_C = 10$ s and $T_C = 15$ s, and the first-degree burn occurs with $T_C = 5$ s. In order to study the influence of T_C and x on thermal damage, variance analysis was carried out. Table 3 shows the calculation results. By calculation, R^2 $(x) = 0.165575$, R^2 $(T_C) = 0.167528$, and R^2 $_{(\text{Interaction})} = 0.3308943$. Therefore, it can be inferred that the influence level of x and h on thermal damage is lower than the average, respectively. These two factors and their interaction have a significant effect on thermal damage.

Table 2: Analysis of variance about x and h ($\alpha = 0.01$).

Source of difference	SS	df	MS	F	P value	F crit
x	109139.8	1	109139.8	490.3853	$1.13E-79$	6.677735
h	212457.6	2	106228.8	477.3055	$2.5E-124$	4.641058
Interaction	212452	2	106226	477.2929	$2.5E-124$	4.641058
Internal	132200.2	594	222.5593	—	—	—
Total	666249.6	599	—	—	—	—

Figure 7: Effect of heating cycle T_C on the temperature history at skin surface ($t_p = 1$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C, $x = 0$).

Figure 11 indicates the relationship between laser heating cycle and temperature fluctuation amplitude. $T_{s,max}$ and $T_{s,min}$ represent the maximum and minimum values of temperature fluctuation amplitude of skin tissue surface, respectively. $T_{ed,max}$ and $T_{ed,min}$ represent the maximum and minimum values of temperature fluctuation amplitude of the interface between epidermis and dermis, respectively. $T_{ds,max}$ and $T_{ed,min}$ represent the maximum and minimum values of temperature fluctuation amplitude at the interface between dermis and subcutaneous, respectively. It can be seen from the figure that the amplitude of temperature fluctuation increases first and then decreases with the increase of laser heating period. The range of temperature fluctuation increases with the increase of laser heating period. Skin tissue surface temperature fluctuation range is the largest, the epidermis and dermis interface is the second, and the dermal and subcutaneous interface is the smallest. With the increase of laser period, the minimum amplitude of temperature fluctuation becomes closer and closer; when $T_C > 16$ s, the lowest temperature of skin tissue is equal at all locations.

Figure 12 shows the effect of time t_p on skin surface temperature history. It can be seen from the figure that the skin tissue surface temperature fluctuates and rises at the

Figure 8: Effect of heating cycle T_C on the temperature history at the interface between epidermis and dermis ($t_p = 1$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C, $x = 0.1$ mm).

beginning time and then tends to fluctuate steadily. If $t_p = 3$ s, the fluctuation range of skin surface temperature is 80–107°C. If $t_p = 1$ s, the fluctuation range of skin surface temperature is 45–75°C. If $t_p = 0.5$ s, the fluctuation range of skin surface temperature is 42–63°C. We can infer that the larger the t_p value is, the larger the maximum value of skin tissue surface temperature fluctuation is and the smaller the minimum value of skin tissue surface temperature fluctuation is.

Figure 13 shows the effect of peak time t_p on the temperature history at the interface between epidermis and dermis. Comparing with Figure 12, we can see that the temperature of this interface is lower than that of the skin surface. The trend of fluctuation is consistent with Figure 12.

Figure 14 shows the effect of peak time t_p on the temperature history at the interface between dermis and subcutaneous. Because this location is far away from the heat source, the temperature of this location is much lower than that of the skin tissue surface and the interface between the epidermis and dermis. The temperature at this position fluctuates with the heat flux heated by laser.

Figure 15 indicates the relation between peak time t_p and temperature fluctuation Amplitude. The meaning of $T_{s,max}$, $T_{s,min}$, $T_{ed,max}$, and $T_{ed,min}$ is the same as shown above. It can be seen from the figure that, with the increase of peak time t_p, the amplitude of temperature fluctuation becomes larger and larger at all positions of skin tissue. The temperature fluctuation range of skin surface and epidermis-dermis interface increases with the increase of peak time t_p. The temperature fluctuation range of dermal-subcutaneous interface increases first and then remains constant with the increase of t_p. The larger the t_p value is, the smaller the range of temperature fluctuation is.

Figure 16 indicates the effect of peak time t_p on the thermal damage history at different locations of skin tissue. It can be seen from the figure that the greater the t_p value, the more severe the thermal damage of the skin tissue. With $t_p = 3$ s thermal damage value of skin surface and epidermal-dermal interface exceeds 10^4, reaching the third-degree burn, and thermal damage value of dermal-subcutaneous interface exceeds 1 to reach a second-degree

FIGURE 9: Effect of heating cycle T_C on the temperature history at the interface dermis and subcutaneous ($t_p = 1$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20^\circ$C, $x = 1.6$ mm).

FIGURE 10: Effect of heating cycle T_C on the thermal damage history at different locations of skin tissue ($t_p = 1$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20^\circ$C, $x = 1.6$ mm).

Table 3: Analysis of variance about T_C and x ($\alpha = 0.01$).

Source of difference	SS	df	MS	F	P value	F crit
x	$2.04E+25$	2	$1.02E+25$	13309.32	0	4.605563007
h	$2.06E+25$	2	$1.03E+25$	13466.32	0	4.605563007
Interaction	$4.07E+25$	4	$1.02E+25$	13305.7	0	3.319522681
Internal	$4.13E+25$	53991	$7.65E+20$	—	—	—
Total	$1.23E+26$	53999	—	—	—	—

Figure 11: Relationship between laser heating period and temperature fluctuation amplitude ($t_p = 2$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C).

(a)

Figure 12: Continued.

(b)

Figure 12: Effect of peak time t_p on the temperature history at skin tissue surface ($T_C = 10$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C, $x = 0.1$ mm).

(a)

(b)

Figure 13: Effect of peak time t_p on the temperature history at the interface between epidermis and dermis ($T_C = 10$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C, $x = 0.1$ mm).

FIGURE 14: Effect of peak time t_p on the temperature history at the interface between dermis and subcutaneous ($T_C = 10$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C, $x = 0.1$ mm).

burn. If $t_p = 0.5$, the thermal damage on the skin surface ($x = 0$) exceeds 100 to reach a second-degree burn. The thermal damage on the skin surface ($x = 0.1$ mm) exceeds 1 to reach a first-degree burn. Because the dermal-subcutaneous interface is far away from the laser heating heat source, there is no burn. If $t_p = 1$, the thermal damage on the surface of skin tissue exceeds 10^6, reaching the third-degree burn. The thermal damage on the epidermis-dermis interface of skin tissue exceeds 10^6, reaching the second-degree burn. The thermal damage on the dermal-subcutaneous interface of skin tissue is less than 0.53, lower than the first-level burn standard. In order to study the influence of t_p and x on thermal damage, variance analysis was carried out. Table 4 shows the calculation results. By calculation, R^2 $(x) = 0.165721$, R^2 $(t_p) = 0.167244$, and R^2 $_{(\text{Interaction})} = 0.3325$. Therefore, it can be inferred that the influence level of x and t_p on thermal damage is lower than the average, respectively. These two factors and their interaction have a significant effect on thermal damage.

6. Discussion

It is of great significance to accurately predict the temperature field and control the thermal damage during skin tissue hyperthermia. In this paper, a special separation variable method is used to derive the semianalytical solution of temperature field in skin tissue hyperthermia. Compared with the traditional finite element method, finite difference method, Laplace transform method, and so on, this method has clearer physical meaning and higher calculation efficiency. The temperature field and thermal damage of skin tissue during laser hyperthermia are predicted by the analytical solution derived in this paper. Several parameters that can affect the temperature field and thermal damage of skin tissue during heating have been studied. Through analysis we can find that the convective heat transfer coefficient between the skin tissue surface and the cooling fluid has a greater effect on the temperature field and thermal damage. We can change the injection speed of the cooling fluid to change the convection heat transfer coefficient and then

Figure 15: Relation between peak time t_p and temperature fluctuation amplitude ($T_C = 2$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C).

Figure 16: Effect of peak time t_p on the thermal damage history at different locations of skin tissue ($T_C = 10$ s, $q_{peak} = 3$ W/cm^2, $T_\infty = 20$°C).

Table 4: Analysis of variance about t_p and x ($\alpha = 0.01$).

Source of difference	SS	df	MS	F	P value	F crit
x	$2E+32$	2	$9.98E+31$	13329.93	0	4.605563
t_p	$2.01E+32$	2	$1.01E+32$	13450.85	0	4.605563
Interaction	$3.99E+32$	4	$9.98E+31$	13329.93	0	3.319523
Internal	$4.04E+32$	53991	$7.48E+27$			
Total	$1.2E+33$	53999	—	—	—	—

control the temperature and thermal damage of the skin tissue in an appropriate range. In addition, the heating period and peak time of the laser also have a greater impact on the skin tissue temperature field and thermal damage during hyperthermia. Adjusting the parameters such as laser heating period or peak time can control the skin tissue

temperature to fluctuate within a stable range. This operation can achieve the desired effect of hyperthermia without thermal damage to skin tissue.

7. Conclusion

In this paper, a semianalytical method is presented to solve the heat transfer problem of multilayer skin tissues. The temperature field and thermal damage of skin tissue under time-varying laser heating and fluid cooling are obtained by the presented method. The effects of several parameters on skin tissue temperature and thermal damage are investigated. The main phenomena are revealed as follows:

(a) The smaller the convective heat transfer coefficient h is, the faster the temperature of skin tissue rises

(b) The larger the convective heat transfer coefficient h is, the smaller the heat flux of skin tissue is

(c) The smaller the heating period is, the more serious the thermal damage of skin tissue is

(d) The range of temperature fluctuation decreases with the increase of laser heating period T_C

(e) The greater the t_p value, the more severe the thermal damage of the skin tissue

In this paper, the Fourier model of heat transfer in biological tissue is used to study the heat transfer in skin tissue. In recent years, some literatures have used non-Fourier model to simulate the heat transfer in biological tissue. Our future research direction is to use the method of this paper to solve the non-Fourier heat transfer model of biological tissue and then to study the heat transfer of skin tissue.

Nomenclature

i: Layer number of skin tissue: $i = 1$, epidermis; $i = 2$, dermis; $i = 3$, subcutaneous
k_i: Thermal conductivity of i-th layer
T: Temperature
ρ_i: Density of i-th layer
c_i: Specific heat capacity of i-th layer
t: Time
δ_i: Thickness of i-th layer
L_i: x-axis coordinates, $L_i = \sum_{i=1}^{i} \delta_i$
ϖ_b: Blood perfusion rate
ρ_b: Density of epidermis blood
c_b: Specific heat capacity of blood
$q_{\mathrm{met},i}$: Metabolic heat generation in the skin of i-th layer
$q_{\mathrm{ext},i}$: Heat source due to external heating i-th layer
h_0: Heat convective coefficient
T_∞: Temperature of cooling fluid
T_a: Temperatures of arterial blood
$q(t)$: Heat flux function
r_{ki}: Dimensionless $k_i r_{ki} = (k_i/k_1)$
θ: Dimensionless $T\ \theta = (T - T_a)/T_a$
θ_∞: Dimensionless $\theta_\infty = (T_\infty - T_a)/T_a$
ξ: Dimensionless $\xi = (x/L_3)$
$\overline{\varpi}_{bi}$: Dimensionless $\varpi_b\ \overline{\varpi}_b = (L^2 \varpi_b \rho_b c_b/k_1)$
$r_{\rho i}$: Dimensionless $r_{\rho i}\ r_{\rho i} = (\rho_i/\rho_1)$
r_{ci}: Dimensionless $r_{ci} = (c_i/c_1)$
τ: Dimensionless $t\tau = (k_1 t/\rho c L^2)$
$\overline{q}_{\mathrm{met},i}$: Dimensionless $\overline{q}_{\mathrm{met},i}\ \overline{q}_{\mathrm{met},i} = (q_{\mathrm{met},i} L_3^2/k_1 T_a)$
$\overline{q}_{\mathrm{ext},i}$: Dimensionless $\overline{q}_{\mathrm{ext},i}\ \overline{q}_{\mathrm{ext},i} = (q_{\mathrm{ext},i} L_3^2/k_1 T_a)$
$\overline{N}_{u0}$: Dimensionless $\overline{N}_{u0} = (h_0 L_3/k_1)$
$\overline{f}_{1,1}(\tau)$: Dimensionless $\overline{f}_{1,1} = (L_3 q(t)/k_1 T_a)$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Dynamic Path Planning for Bevel-Tip Flexible Needle Insertion into Soft Tissue Based on a Real-Time Finite Element Model

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This paper presents a novel dynamic path planning methodology for needle steering into the soft tissue. A real-time finite element model is used to simulate the procedure of a flexible needle into the homogeneous soft tissue, which provides the dynamic deformation information for the path planning. The relationship between needle base and tip is formulated as the transformations of homogeneous matrix with quasi-static assumptions. Based on the reachability of the flexible needle, the real-time motions of obstacles and target are considered through the dynamic needle-tissue interactions. A testbed including a XY linear stage, one rotator, and a CCD camera is constructed, and the experiments are designed to validate the proposed method. The 23G PTC needle was inserted into the PVA phantom with markers, and the CCD camera was utilized to record the needle trajectories and motions of target and obstacles. The targeting errors between the experimental and planned paths are less than 1.20 mm, and the distance from the obstacle to needle is not smaller than 1.16 mm. The results demonstrate that the proposed algorithm is effective for online planning the paths in the needle-tissue interactive environment.

1. Introduction

The needle insertion procedures are one of the most common minimally invasive medical procedures, which are widely used in biopsy, anaesthesia, and radiotherapy [1–4]. Needles are inserted deep into the focus area of different organs such as lung, liver, or prostate for diagnosis, localized therapeutic drug delivery, and sample removal from tissues. Sensitive parts such as the vessels should avoid contact with needles at the same time. During the interaction process, the interaction force between the needle and tissue would lead to the movement of obstacles and goals. And the needle can deviate from its intended path due to the unbalanced resistance force acting on the needle tip. Hence, this research proposed a dynamic path planning methodology that can steer and control the needle to get to the focus area while avoiding the vessels or other sensitive parts inside the organ.

The needles used in the interventional procedures are classified into three types, that is, rigid needle, flexible needle, and soft needle [5]. In many diagnoses and treatment procedures, the soft and flexible needles are required to avoid obstacles or reach the targets in one insertion [6], which requires the needle possessing good flexibilities and steerability [7]. However, the needle-tissue interactions, which cause the obstacles and targets to move from their original position, contribute to the greater targeting challenges of the needle as the practice scenarios of needle insertions are more complicated. In the complicated scenarios, the percutaneous needles are required to hit the targets hidden by the anatomical obstacles or to get to multiple targets in different locations. Therefore, it is essential to research the needle steerability and path feasibility for improving the accuracy and successful rate of needle insertion operations.

To improve the accuracy of needle insertions, the needle trajectory should be planned and updated according to the dynamic needle-tissue interactions. Needle path planning is to control the insertion parameters and guide the tip for avoiding the obstacles and reaching the targets [8–10], which reflects the dynamic procedure of needle-tissue interactions [11]. DiMaio et al. first addressed the concept of needle steering, modeled the domain of target and obstacles with the artificial potential field (APF), and planned the effective trajectories [8, 9]. Jiang et al. improved the APF algorithm with the conjugate gradient method, which can escape from a local minimal point [12], and the discrete potential field was used for path planning of surgical needles in 3D anatomical environments [13]. The virtual spring was employed to model the tissue properties by Glozman, and the path planning problem is converted into the optimization of lateral force [14]. However, the artificial potential field and virtual spring method did not account for the steerability of flexible needles.

During insertion of the flexible needle, the tissue around the needle tip is compressed and the asymmetric resistance force against the compression due to the asymmetric needle tip can deflect the flexible needle, as shown in Figure 1. The direction of bending is determined by the direction of the bevel-tip. Rotating the needle base would change the direction of the bevel-tip, which would change the direction of the asymmetric force and tip bending direction; hence, by rotating the needle base, different needle trajectories would be generated.

Many groups have studied the steerability of bevel-tip needle. Webster's experiments showed that the trajectory of flexible needles follows a circular arc under the ideal conditions [15]. Based on the phenomena, the nonholonomic unicycle and bicycle models were presented for describing the kinematics of the bevel-tip needles using the Lie group [16]. Reed et al. discussed the instant angle discrepancy between the base and the tip for the soft needle [17], but the time lag is better for flexible needles.

Many path planning algorithms based on nonholonomic kinematic model have been proposed to obtain feasible needle trajectories [18–23]. Park et al. considered the probability of needle tip pose, which was approximated as the probability density function [18, 19]. The path planning problem is to solve the maximum probability of needle path [20]. Alterovitz et al. considered the uncertainties caused by the needle-tissue interactions [23]. The Markov decision model was used to depict the needle motion, and the dynamic programming is employed to obtain an optimal control sequence [22]. Alterovitz et al. also provided a targeting error model with Euclidean distance for optimizing the needle trajectories [10]. Duindam et al. presented the forward and inverse kinematic equations of flexible needle based on the nonholonomic model, and the null motion was employed to a three-dimensional needle path [21]. Wang et al. simplied Duindam's work and proposed the stop-and-turn strategy to control the needle, which could decrease the number of needle rotations [24]. The computer searching methods were also applied in the needle path planning. The exhaustive method, Nelder–Mead downhill simplex method, and Haystack were used in the path planning by Hamzé, and the comparisons showed that Haystack is faster than others [25]. Abayazid et al. proposed a path planning algorithm with rapid exploring random trees (RRTs) to guide the needle tip in real time [26]. Vrooijnk et al. studied needle path planning problem in the nonstatic environment with two-dimensional ultrasound images, but the obstacles and target moved with constant speed [27]. Morimoto et al. reported that the image-based path planning was employed to design and fabricate the concentric tube robots [28]. Korayem et al. proposed the dynamic modeling of flexible arms which simultaneously take large motions and small deflections into account [29, 30], and based on the precise model of the general flexible mobile manipulator, the path planning approach of mobile elastic robotic arms is proposed, which points out the importance of dynamic effect in path planning [31]. In these studies, the motions of obstacles and targets caused by needle-tissue interactions were not considered for adjusting the needle trajectories.

As it can be seen from the literature, various path planning algorithms have been developed for optimizing the needle trajectories. The traditional planning algorithm such as genetic, RRT, and annealing algorithm is in essence an optimization algorithm, which can only generate the optimal needle path in a rigid space. It is not sufficient and applicable to apply in the deformable environment because the tissue deformation cannot be considered. First, the targeting error involves two parts in the complicated environment with the moving targets and obstacles. One is the tip-target error, and the other is the needle-obstacle collision error. The two errors are affected by the needle-tissue interactions, which can lead to the offsets from their original locations of obstacles and targets inside the tissue. Second, the targeting error and the collisions with the obstacles cannot be predicted and corrected without a model-based control, even though there exist imaging devices in the needle insertion procedure. It is necessary to correct and compensate the insertion errors based on a predictive model-based control to ensure the convergence of a feasible and effective path in complicated circumstances.

The aim of this paper is to develop a novel algorithm for planning a feasible trajectory in the dynamic needle-tissue interactive environment. Unlike the existing algorithms, the dynamic information of the targets and obstacles is predicted and applied to the path planning procedure, and they can facilitate to correct the needle insertion errors. The advantages of the proposed planning algorithm are as follows. The targeting error and obstacle collision error are both considered in the path planning algorithm. The dynamic needle-tissue interactions are as the feedback to guide and control the needle to avoid the obstacles and get to the target. The experiments are designed to validate the effectiveness of the proposed algorithm. This work can be used for developing virtual training system and assisting the surgeon to plan pre- and intraoperative paths.

The rest of this paper is organized as follows: the problem of dynamic path planning is presented in Section 2; Section 3 provides the dynamic path planning with inverse

Figure 1: Asymmetric resistance force against needle tip: (a) translation of needle base; (b) rotation of needle base.

kinematics, finite element prediction model, and general insertion error, followed by the algorithm for the dynamic path generation introduced in Section 4; the testbed setup is introduced; the experimental results are discussed in Section 5; and finally, the conclusions and future work are provided.

2. Problem Definition and Assumptions

In this paper, obstacles represent the sensitive parts such as the vessels that should avoid contact with needles. Goals represent the focus area for tissue extraction or drug delivery. The soft tissue is a deformable medium. Under the interaction of needles, the internal structure would move dynamically, and the needle would bend at the same time, which would lead to needle targeting errors and mitigate the effectiveness of diagnosis and therapy. The objective of the path planning for needle insertion is to find feasible paths between given insertion location and the moving target and obstacles, whose movement is caused by needle insertion actions. A complicated case for needle insertion is illustrated in Figure 2.

In order to reach the targets inside the human body, the needle is deflected to avoid the obstacles (organs inside the body), such as the bones and vessels. It is necessary to plan a feasible path from the outside of the body to target inside the body.

The path planning problem is described as follows. Given the initial pose of needle tip q_{start}, an initial pose of the target q_{target}, and the initial poses of the obstacles q_{obstacle}. Since the motions of target and obstacles need to be considered, the parameters of needle-tissue interaction are also provided. The procedure of needle insertion is divided into n time sequences. The outputs of the algorithm are the adjusting angles of all the needle segments, as shown in equation (2):

$$\Phi = [\phi_1, \phi_2, \ldots, \phi_n]. \tag{1}$$

The solution of $g_n(\Phi)$ satisfies the pose of changed target, as follows:

$$g_n(\Phi) = g\left(q_{\text{target}}[t_n]\right), \tag{2}$$

and the paths should also avoid the pose of moved obstacles $g(q_{\text{obstacle}}[t_n])$.

The assumption of this work is as follows. The procedure of needle insertion is composed of a series of quasi-static processes. The flexible needle moves in the dynamic environment, and the obstacles and the target move in the homogeneous, linear-elastic material of soft tissue. The motion of the needle with bevel angle tip is fully determined by the motion of needle tip, and the motion of needle tip is steered by adjusting the needle base. The experimental settings in our group for needle insertion into the phantom are at 1 mm/s, and there would be pause of 10 seconds after the rotation of the needle base. Hence, the lag between the needle base and needle tip can be compensated, and the angle of needle tip is fully controlled by the rotating the needle base.

Figure 2: The complicated scenarios of flexible needle into soft tissue.

3. Proposed Methodology

The schematic diagram of the proposed path planning algorithm is shown in Figure 3. In clinical surgery, obstacles represent the sensitive parts such as the vessels that should avoid contact with needles. Goals represent the focus area for tissue extraction or drug delivery. The initial positions of obstacles and target can be predetermined by preoperative medical image. The dynamic path planning is converted as an optimal problem: the optimization objectives are to minimize the distance between the needle tip and the target and maximize the distance between the obstacle and needle trajectory. These two terms compose the general insertion

Figure 3: Flowchart for dynamic path planning algorithm. denotes the direction of data flow, and denotes the model or method employed.

error in artificial potential field (APF). The kinetic model that described the relationship between the needle base and needle tip and the needle-tissue interaction model that predicts the obstacles and goals' movement are both required: the Denavit–Hartenberg method is used to describe the motion from needle base to the tip and formulate the forward and inverse kinematic equations of bevel-tip flexible needle. A real-time finite element (FE) method is employed to estimate the needle-tissue interactions, which are used to predict the displacements of the obstacles and target inside the tissue when planning the needle paths. The feasible paths from the current position of needle tip to moved target can be generated by the inverse kinematics. After each adjusting action, the new locations of target and obstacles are calculated by the FE model. Then, according to the initial location of tip and updated location of target, the sets of adjusting angle are generated by the inverse kinematic equation. The feasible paths provided by the kinematic equation are used for calculating the general insertion error. If the tip reaches the target, the optimal path is found. If not, calculate the new locations and repeat the above steps.

3.1. Kinematics Model of Flexible Needle. In this paper, we take into account the path planning of flexible needles in the dynamic environment, and the obstacles and target move in the linear-elastic material of soft tissue. It is assumed that the motion of the needle is fully determined by the motion of needle tip, and the motion of needle tip is steered by adjusting the needle base [32]. The lag between the needle base and needle tip is neglected, and the angle of needle tip is fully controlled by rotating the needle base [17, 21]. Under the previous assumptions, the motion of needle is determined by two control inputs: the insertion depth d and the rotation angle θ. The motion of needle insertion includes one translation and one rotation of needle base, which is shown in Figure 1.

The translation of needle base transmitted to the needle tip is transformed into a combination of the translation along the z direction and the rotation about the y axis. Neglecting the needle torsion, the rotation of needle base is transformed into the rotation of needle tip, and $\omega_{\text{tip}} \equiv \omega_{\text{base}}$ [21].

By analyzing the deflection of flexible, the needle deflection is independent of the depth of insertion. When the same type of needles is inserted into the same material of soft tissue, the needle deflection is a constant over the small insertion depth [32]. The two control inputs are reduced to one input, i.e., the angle of rotation ϕ. According to the quasi-static thinking, the needle can be decomposed into finite segments, and the insertion procedure can be taken as the combination of the segments' insertions, shown in Figure 4.

The generalized coordinate is chosen to describe the pose of needle segment, written as

$$q = \left[q_1^T, q_2^T\right] = [x, y, z, \phi, \theta, \psi]^T. \tag{3}$$

The homogeneous matrix $g(q)$ of body frame fixed at q is written as

FIGURE 4: The motion from the ith segment to the $i+1^{\text{th}}$ segment. At O_i, the needle base is rotated $\phi_{i,i+1}$ angle about z_i first and then is translated along z_i.

$$g_i(q) = \begin{bmatrix} R(q) & P(q) \\ 0^T & 1 \end{bmatrix} \in \mathrm{SE}(3), \tag{4}$$

where $P(q) = [x, y, z]^T \in \mathbb{R}^3$ represents the relative position of the ith needle segment; $R(q) \in \mathrm{SO}(3)$ represents the orientation of the ith needle segment; and ϕ, θ, and ψ are the angles about X, Y, and Z axes in the reference frame. The initial body frame $Ox_0y_0z_0$ is created on the needle base, and the end body frame $Ox_ny_nz_n$ is created on the needle tip. Following the Denavit–Hartenberg method, the motion of flexible needles can be described by the transform of homogeneous matrix from the needle base to needle tip [32], written as follows:

$$g_n(\Phi) = g_{0,1}(\phi_1)g_{1,2}(\phi_2)\cdots g_{i,i+1}\cdots g_{n-1,n}(\phi_n), \tag{5}$$

where $g_{i,i+1}$ denotes the homogeneous transform from the ith segment to the $(i+1)^{\text{th}}$. Φ denotes the adjusting angle set of needle base. In the path planning algorithm, we formulate an expression to find an efficient adjusting angle set and construct a path from insertion location to the target.

3.2. Prediction Model for Needle-Tissue Interactions. The key to dynamic path planning is the real-time information of needle-tissue interactions, especially the movements of the target and obstacles. The finite element method can adopt the complex geometries and boundaries of linear and nonlinear soft tissue materials, and it is used for simulating the soft tissue deformation. In this paper, we consider the case that the flexible needle is inserted into the homogeneous, linear, and elastic soft tissue, which is modeled in the literature [33].

In the FE model, the needle-tissue interactive procedure is decomposed into four different processes [33]. The stiffness force of soft tissue is acted on the needle tip in process 1. The needle enters into the tissue at a certain depth in process 2, and the needle's displacement is constrained by the cutting and friction forces. The needle gets to the target and keeps its position in process 3. Finally, the needle is retreated from the tissue. After meshing the tissue model and initializing the boundaries and properties of needle and tissue, the calculation is implemented in Matlab. The iteration time for 1061 × 1061 2D grid mesh is 10 ms on an Intel i7-3.5 GHz dual-core machine running Linux OS. The simulation can provide real-time information for path planning in the dynamic environment.

The yield displacements of target and obstacles inside soft tissue are provided for updating the trajectories during a needle insertion procedure. For the sake of exposition, the displacement of tissue node from the FE model is defined as

$$u = \sqrt{u_x^2 + u_y^2}, \tag{6}$$

where u_x and u_y are the components in the x and y directions. Through the FE model, the displacements of tissue node can be related to the parameters of needle insertion, and it is formulated as

$$u = f(x_0, y_0, z_0, m_t, m_n, v, \alpha), \tag{7}$$

where x_0, y_0, and z_0 are the original coordination of the tissue node, for the two-dimensional case, $z_0 = 0$. m_t is the parameter of soft tissue, including its dimension, Young's modulus, Poisson ratio, and areal density. m_n is the parameter of needle material, including its Young's modulus and moment of inertia. v is the insertion velocity, and α is the angle of needle tip. For illustrating the validation of FE simulation, 10 markers were embedded inside the phantom, as shown in Figure 5.

The errors between the simulation and experiment are calculated as the absolute error, written as

$$e = \left| u_{\text{Sim}} - u_{\text{Exp}} \right|, \tag{8}$$

in which u_{Sim} and u_{Exp} are the displacements of simulation and experiment, respectively. The tissue nodes in Figure 5 are classified into three types, including the nodes constrained by the needle (nodes 1 and 3), the nodes close to the boundary (nodes 5 and 10), and the nodes far away from the needle (nodes 8 and 9). The errors of ten nodes in Figure 5 are calculated and listed in Table 1. In the table, $\overline{e}_x$ and $\overline{e}_y$ are the mean values of errors between the FE model and experimental results in the x and y directions, respectively. From the table, the errors of the nodes close to the needle are bigger than that of nodes far away from the needle, but the relative errors of the nodes close to the needle are smaller. The maximum value of the mean error between the simulated and experimental results is less than 0.50 mm. The simulation results agree well with the experimental results, and the finite element mode can reflect the interactions between flexible needle and linear-elastic soft tissue. The prediction results through the FE model can provide the real-time information of targets and obstacles inside the tissue for the path planning algorithm.

3.3. General Insertion Error. The objective of path planning is to avoid the obstacle and reach the target. The Euclidean distance is employed to describe the error of needle insertion into soft tissue. Figure 6 depicts the distances from needle body to the obstacle and from needle tip to the target.

In this figure, $d_{on}(t)$ is the Euclidean distance between needle body and the obstacle at time t and $d_{gt}(t)$ is the Euclidean distance between needle tip and the target at time t. The distances $d_{on}(t)$ and $d_{gt}(t)$ are formulated as

Figure 5: Comparisons between the FE simulation and experimental results. ∗ denotes the positions of the markers, and ○ denotes the positions of the markers.

Table 1: Errors between the FE model and experimental results.

Markers	$\bar{e}_x$	$\bar{e}_y$	$\bar{e}$
1	0.17 mm	0.37 mm	0.34 mm
2	0.23 mm	0.37 mm	0.35 mm
3	0.11 mm	0.31 mm	0.29 mm
4	0.21 mm	0.30 mm	0.27 mm
5	0.10 mm	0.19 mm	0.18 mm
6	0.08 mm	0.19 mm	0.19 mm
7	0.22 mm	0.17 mm	0.34 mm
8	0.27 mm	0.10 mm	0.22 mm
9	0.07 mm	0.10 mm	0.10 mm
10	0.08 mm	0.10 mm	0.10 mm

Figure 6: The errors for needle-tissue interactive procedure.

$$d_{gt}(t) = \left\| x_t(t) - x_g(t) \right\| \quad d_{on_i}(t) = \min \left\| x_n(t) - x_{o_i}(t) \right\|, \tag{9}$$

where $d_{on_i}(t)$ denotes the distance between flexible needle body and the ith obstacle at time t. $x_t(t)$, $x_g(t)$, and $x_{o_i}(t)$ are the positions of tip, target, and i^{th} obstacle at time t. $d_{gt}(t)$ can be calculated by

$$d_{gt} = \left\| x_t - x_g \right\| = \left\| (x_t - x_g) i + (y_t - y_g) j + (z_t - z_g) k \right\|. \tag{10}$$

Since the turning points on the path can be obtained using the inverse kinematic equation, the interpolation is employed to smooth the needle trajectory $S(x_n)$. The distance d_{on_i} is calculated with $S(x_n)$ and $x_{o_i}(t)$.

The artificial potential field is used to define the general insertion error as follows [34]:

$$\begin{aligned} \varepsilon_{gt}(t) &= \frac{1}{2}\kappa_g \cdot d_{gt}^2(t), \\ \varepsilon_{on}(t) &= \frac{1}{2}\kappa_o \cdot \sum_{i=1}^{N_o} \rho_i^2(t), \end{aligned} \tag{11}$$

where ε_{gt} is the targeting error and ε_{on} is the obstacle error. ρ_i is calculated with d_{on_i} and the security distance d_0, written as

$$\rho_i = \begin{cases} \dfrac{1}{d_{on_i}} - \dfrac{1}{d_0}, & \left(d_{on_i} \le d_0\right), \\ 0, & \left(d_{on_i} > d_0\right). \end{cases} \tag{12}$$

Equation (11) creates the attractive field with the target and the repulsive field with the obstacles. κ_g and κ_o are the factors of attractive and repulsive fields. N_o is the number of obstacles in the needle-tissue environment. The general insertion error $\varepsilon(t)$ is written as follows:

$$\varepsilon(t) = \varepsilon_{gt}(t) + \varepsilon_{on}(t). \tag{13}$$

The needle tip can move along the direction that the potential energy is the minimum; meanwhile, the needle body can avoid the obstacles.

3.4. Dynamic Path Planning with Inverse Kinematics. Planning the feasible paths is to solve the inverse kinematics of flexible needles. The initial insertion location needs to be determined considering the reachability. The initial insertion pose can be derived from equation (5) with the targeting pose, which is written as

$$g_{0,1} = g_n(\Phi) g_{n-1,n}^{-1} g_{n-2,n-1}^{-1}, \ldots, g_{1,2}^{-1}. \tag{14}$$

The needle insertion is an inverse procedure, and the reachable zone for the tip can help the surgeon to determine the insertion location when the target is given by the medical images. The feasible insertion locations can be calculated with the inverse kinematic equation [32]. Considering the anatomical structures around the target, the surgeon can determine an insertion location which can avoid the obstacles as much as possible.

Considering the displacements of the target and obstacles during the procedure of needle insertion, we employ the finite element model to describe the needle-tissue interactions and predict their displacements in real time. The general insertion error is defined with concept of artificial potential field (APF), and the dynamic path planning is converted as an optimal problem of general insertion error.

It is necessary to understand the real-time information of target and obstacles for the dynamic path planning. The needle-tissue procedure is divided into n continuous time sequence, and the corresponding adjusting angle is written as follows:

$$\begin{aligned} t &= [t_0, t_1, t_2, \ldots, t_n], \\ \Phi &= [\phi_0, \phi_1, \phi_2, \ldots, \phi_n]\,(\phi_0 = 0^\circ). \end{aligned} \tag{15}$$

Substituting the adjusting set Φ into the kinematic model, the position of needle tip can be calculated in the time sequence. The proposed real-time FE model is used to calculate the dynamic positions of target and each obstacle in the time sequence t, which is written as

$$\begin{aligned} x_g &= \left[x_g(t_1), x_g(t_2), \ldots, x_g(t_n)\right], \\ x_{o_j} &= \left[x_{o_j}(t_1), x_{o_j}(t_2), \ldots, x_{o_j}(t_n)\right], \quad j = 1, 2, \ldots, N_o. \end{aligned} \tag{16}$$

The general insertion error ε in the time sequence t can be obtained with the dynamic information of needle tip, target, obstacles, and needle trajectory. Since there exist many or no feasible paths to target with the avoidance of obstacles, an optimal trajectory is the one with a minimal insertion error. The dynamic path planning is converted as an optimal problem, which is written as

$$\begin{aligned} &\min \quad [\varepsilon(t_1), \varepsilon(t_2), \ldots, \varepsilon(t_n)] \\ &\text{S.t.} \quad \left\| g_{b,n}(\Phi) - g\left(\vec{x}_g(t_n)\right) \right\| \le \varepsilon \\ &\Phi = [\phi_1, \phi_2, \ldots, \phi_n] \\ &\quad -\pi \le \phi_i \le \pi, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{17}$$

where ϵ is an allowable small offset. According to the flowchart in Figure 3, the parameters are initialized firstly, including the properties of needle and tissue. The parameters of needle involve needle length L, bevel angle of needle tip α, needle deflection ω, and rotation angle θ. The insertion information is required, including initial insertion location $\vec{x}_t(t_0)$, the target position $\vec{x}_g(t_0)$, and locations of obstacles $x_o(t_0)$ at t_0. The security distance between needle and obstacles d_0, the attractive factor κ_g, and repulsive factor κ_o should also be initialized. If equation (17) is satisfied, the planned program follows YES or turns to NO path.

3.4.1. Online Rolling Prediction. In the optimization of feasible paths with equation (17), the information of target and obstacles should be updated in each iterative steps. The dynamic information of target and obstacles is used to calculate the general insertion error in the current iterative step, and the updated information predicted by the FE model is used for the next iterative step. The online rolling method is used to update the prediction information, as shown in Figure 7.

The adjusting angle ϕ_{k-1} in the previous step is the input of the current k^{th} iterative step. ϕ_{k-1} is substituted into the kinematic equation, and the location of needle tip $x_t(t_k)$ at the k step is calculated. The set of adjusting angles Φ_k is obtained with the future location of the target $x_g(t_{k+1})$ and $x_t(t_k)$ based on the inverse kinematics. The future locations of the obstacles $x_o(t_{k+1})$ are employed to update the general insertion error $\varepsilon(t_k)$. The candidate adjusting angle for the k step is selected by minimizing the general insertion errors. Repeating the above steps, the current adjusting angle ϕ_k is the input of next planning step until the needle tip gets to the target.

The detail planning process is shown as follows: first, run the FE simulation algorithm to calculate the position of target $x_g(t_1)$ and the locations of obstacles $x_o(t_1)$ at time t_1. According to the initial location of needle tip $x_t(t_0)$ and $x_g(t_1)$, the M feasible paths can be calculated with the static path planning algorithm and the candidate set of adjusting angle is $\Phi_1 = [\phi_{11}, \phi_{12}, \ldots, \phi_{1M}]$. Choosing the minimal general insertion error is to obtain an optimal adjusting angle ϕ_1. The specific update procedure of the needle tip is shown in Figure 8.

In the figure, $\Phi_1 = [\phi_{11}, \phi_{12}, \ldots, \phi_{1n_{t_1}}]$ is the set of adjusting angle, by which the needle tip gets to the position $x_g(t_1)$ at t_1. The green curve denotes the feasible trajectories to the new location $x_t(t_1)$ of needle tip at t_1, and green sphere denotes the location of needle tip at t_1. The purple curve denotes the feasible trajectories from $x_t(t_1)$ to $x_t(t_2)$ at t_2, and purple sphere is the location of needle tip at t_2.

The new location of needle tip $x_t(t_1)$ is calculated with the planned adjusting angle ϕ_1. Then, the new locations of target $x_g(t_2)$ and locations $x_o(t_2)$ are updated with the FE model. The M feasible trajectories from new tip location $x_t(t_1)$ to next target position $x_g(t_2)$ are updated with static path planning algorithm. The planned needle is decomposed into n_{t_2} segments, and the planned set of adjusting angle is calculated as $\Phi_2 = [\phi_{21}, \phi_{22}, \ldots, \phi_{2M}]$. The optimal set of adjusting angle is obtained with equation (17), and the second set of adjusting angle is obtained as ϕ_2. For the time t_i, the candidate set Φ_i of adjusting angle is calculated with the tip location $x_t(t_i)$ at t_i and the target location $x_g(t_{i+1})$ at t_{i+1}, and the adjusting angle is planned with the optimal general insertion error as $\phi_i \in \Phi_i$. Repeat the above steps, the optimal set adjusting angle and its corresponding set of segment length are written as

$$\begin{aligned} \Phi &= [\phi_1, \phi_2, \ldots, \phi_n], \\ L_{\text{seg}} &= [l_1, l_2, \ldots, l_n]. \end{aligned} \tag{18}$$

3.4.2. Updating the Number of Needle Segment. In the path planning, the length of needle segment determined the target reachability and obstacle avoidance of needle tip. For the given initial insertion location and target position, the longer needle segment can get to the target with few

Figure 7: Iterative scheme of dynamic path generation.

Figure 8: Diagram of dynamic update procedure of optimal needle trajectories.

rotations, which facilitates the operations of physicians and alleviates patient's suffering. However, the long needle segment cannot construct the complicated trajectories with the obstacle avoidance. Therefore, we select the longer segments to generate the feasible paths. If the longer segments fail, we select the shorter segments to generate the feasible paths. For the kth planning step, the planned needle length is written as

$$L_k = L - \sum_{i=1}^{k-1} l_i. \tag{19}$$

First, we select n_{tk_0} as the number of needle segments. If the planned paths can satisfy the general insertion error, the number of needle segments is chosen as $n_{tk} = n_{tk_0}$. If not, the number of needle segments is chosen as $n_{tk} = n_{tk_j} = 2j \times n_{tk_0}$ through verifying the general insertion error.

Because the needle path is limited to the radius and depth of needle insertion, it is possible to fail to plan the feasible paths. Increasing the number of needle segments infinitely will lead to the endless loop of iterative calculations. Therefore, the maximum number of needle segments is set in the algorithm; if the number of needle segments is bigger than 50, the program will stop calculating. For the clinical practice, the surgeons or robot cannot operate too many segments. In the path planning algorithm, the segment's length of initial planning step is bigger than that of later planning step, i.e., $l_{k-1} \geq l_k$. At the beginning of path planning algorithm, the initial segment's length is given. The segment's length of previous step l_{k-1} is selected as the initial value of segment's length for the k middle step. The number of needle segments is dynamic, not static for the path planning algorithm.

3.4.3. Convergence and Error Analysis. In the optimization of feasible paths, finding the minimum general insertion error is to search the zone with minimum potential energy. The local minimum of APF may lead to the failure of optimization and reaching the target [12]. For avoiding the local minimum of APF, finite initial values are given to calculate the sets of adjusting angles, in which the optimized path is searched from the finite feasible paths. The finite sets of inverse kinematic solution guarantees the convergence of optimization procedure. Because we do not search the optimized paths in the whole space, the optimized path is locally an optimal solution.

In order to obtain the optimal solution in the whole space, we select the uniformly distributed initial values to generate the feasible paths, as follows:

$$\phi_{0j} = \left[\left(0^\circ, j-1\right) \times \Delta\phi, \ldots, 180^\circ\right]^T, \quad j = 1, 2, \ldots, M. \tag{20}$$

Figure 9 shows an example of feasible paths filling the whole space.

From the figure, the uniformly distributed initial value can generate feasible paths filling the whole space, from which the optimal path is searched to guarantee the global optimization. If more optimal solution is required, the smaller interval $\Delta\phi$ is needed to generate the finer paths. In the path planning algorithm, we select $M = 21$ as the number of needle segments, which can search the optimal path with less computation.

The proposed algorithm does not include the feedback from the real-time information, which could lead to the error accumulation from the prediction models. According to the kinematics of flexible, the prediction error of needle tip is about ± 0.70 mm. The prediction error of target inside tissue with the finite element model is about ± 1.00 mm. Therefore, the needle tip may displace off the moved target, and the error is approximated as ± 1.70 mm. It is inevitable to eliminate the errors originated from the prediction models, so the real-time imaging information should be used for adjusting the paths online.

4. Experiments and Results

4.1. Experimental Setup. The experimental setup consists of a CCD camera, a tissue container, a needle clamper, a light source, a needle, two linear actuators, and one rotator, as shown in Figure 10.

The linear actuators (the resolution is $0.25\,\mu$m) are provided by Physik Instrumente (PI Shanghai) Co. Ltd. The

Figure 9: Feasible paths filling the whole space with the uniformly distributed initial values. Starting point is [0 cm, 0 cm, 0 cm], the goal is [0.8 cm, 1.2 cm, 9.8 cm] $\Delta\phi = 18^\circ$, and the number of segments is $M = 21$.

Figure 10: Needle insertion experimental setup. It consists of a CCD camera, a tissue container, a needle clamper, a light source, a needle, two linear actuators, and one rotator.

rotator (the resolution is 0.225°) is provided by HWHR Instruments Co. Ltd. A 22G × 150 mm PTC needle (the diameter of solid needle is 0.58 mm, and the tip is with a bevel angle of 30°) with a plastic hub (provided by Bard Peripheral Vascular, Inc.) is fixed by the clamper. The sample of artificial tissue (phantom) is made of polyvinyl alcohol (PVA) (60%) and dimethyl sulfoxide (6%), its size is 20 mm × 80 mm × 150 mm, and its Young's modulus is 10 kPa–50 kPa. The CCD camera (the maximum resolution is 1296 × 966) is provided to record the trajectory of flexible needles by the Baumer group. The frequency of image acquisition 12 Hz is used to record the needle trajectory and the motions of target and obstacles.

For investigating and recording the tissue's motion and needle trajectories, the markers are embedded inside the PVA phantom as the target and obstacles, shown in Figure 11.

In this figure, the markers 1–5 denote the obstacles, and the marker 6 denotes the target inside the phantom.

In the figure, the markers 1–5 obstacle and goal locations are shown in Table 2.

4.2. Experiment Design for Dynamic Path Planning. Two sets of experiments are designed to illustrate the effectiveness of the dynamic path planning algorithm. Three and four rotations of needle base are required to adjust the direction of bevel-tip angle, and the detail of experiments is shown as follows.

The needle is decomposed into 3 and 4 segments, and the continuous time sequences are $t_3 = [t_1, t_2, t_3]$ and $t_4 = [t_1, t_2, t_3, t_4]$ correspondingly. The motions of target and obstacles are predicted with the proposed FE model, the dynamic target locations are calculated as $x_{G_n} = [\vec{x}_{G_1}, \vec{x}_{G_2}, \ldots, \vec{x}_{G_n}]$, and the updated locations of obstacles are $x_{Oj_n} = [\vec{x}_{Oj_1}, \vec{x}_{Oj_2}, \ldots, \vec{x}_{Oj_n}]$ $(j = 1, 2, 3, 4, 5)$.

The turning points are calculated with the inverse and forward kinematics, and the spline curve is used to smooth the planned trajectory. The location of needle tip x_t in the time sequence is calculated with the planned adjusting angle. The CCD camera recorded the procedure of needle insertion into the phantom with a frequency of 12 HZ. The insertion time is about 20 s, in which 240–280 photos were stored for analyzing needle trajectory. Therefore, the time sequence is taken as $t = [t_1, t_2, \ldots, t_{250}]$, and the smoothed needle path in the time sequence is obtained as $x_n = [\vec{x}_{t_1}, \vec{x}_{t_2}, \ldots, \vec{x}_{t_{250}}]$ with the spline curve. The general insertion error $[\varepsilon_{t_1}, \varepsilon_{t_2}, \ldots, \varepsilon_{t_{250}}]$ is calculated with the dynamic needle-tissue information in the time sequence t_n.

For the insertion procedure with three and four rotations, the needle segments were taken as 33.33 mm and 25.00 mm. The attractive and repulsive factors are taken as $\kappa_g = 1$ and $\kappa_o = 10$, and the security distance $d_0 = 2.00$ mm. The initial insertion location of needle tip lied at $x_{t_0} = [53.00, 0.00, 5.00]$, which is above the target. With the dynamic path planning algorithm, the planned adjusting angle is $\Phi_{3D} = [0.20^\circ, 178.86^\circ, -0.24^\circ]$ and $\Phi_{4D} = [0.00^\circ, 175.70^\circ, -0.18^\circ, 0.12^\circ]$ for the insertions with three and four rotations, respectively.

In the experiments, the directions of bevel-tip angle were adjusted with the planned Φ_{3D} and its insertion depths were 34.00 mm, 33.00 mm, and 33.00 mm. With the set of adjusting angle Φ_{4D}, the insertion depth was 25.00 mm uniformly. The insertion velocity was 5 mm/s.

4.3. Experimental Results and Analysis. In the experiments, the 22G PTC needle bypassed the 2 and 5 obstacles and inserted into the PVA phantom. The CCD camera recorded the insertions with 3 and 4 rotations, shown in Figure 12.

From the figure, the needle was far away from 2 obstacles and approached 5 obstacles.

Figure 11: The markers of target and obstacles in the dynamic path planning experiments.

Table 2: Locations of obstacles 1–5 and the goal.

Obstacles and goal	x (mm)	y (mm)	z (mm)
O_1	20.83	38.18	4.98
O_2	55.38	39.87	5.13
O_3	19.25	65.00	5.20
O_4	20.75	68.00	4.89
O_5	53.60	70.11	5.15
G	52.66	97.64	5.08

The obstacles and target moved from their original positions because of the needle-tissue interactions. When the needle base is rotated 3 times, the displacements in X and Y directions are shown in Figure 13.

From the figure, all the obstacles and the target displaces from their original location. The obstacles 1 and 4 (marker 3 is similar to marker 4, and it was not plotted in Figure 13) are away from the needle, and their displacements are smaller. Obstacle 2 has bigger displacement in the Y direction, the maximum is 1.55 mm and has smaller displacement in X direction, and the maximum is 0.07 mm. The maximum of resultant displacement is 1.56 mm, and its final location is [55.34, 40.91]. The enlarged view shows the displacement of the obstacle 5 and demonstrates the trajectory off its original location [53.60, 70.11]. Obstacle 5 has bigger displacement in the X direction, and the maximum is 0.54 mm. The Y displacement is smaller than obstacle 2, and the maximum is 1.30 mm. The resultant displacement of obstacle 5 is 1.41 mm, and its final location is [53.18, 70.91]. The target's maximal displacement is 0.05 mm in the X direction, and its maximum is 1.42 mm in the Y direction. The final location of the target is [52.61, 98.68].

For the insertion with 4 rotations, the motions of obstacles and target are shown in Figure 14. The enlarged view shows the displacements of the obstacle 2. Obstacle 5 has the bigger displacement than obstacle 2. The maximal displacements in X and Y directions are 2.15 mm and 0.60 mm for obstacle 5. The maximal resultant displacement is 2.23 mm, and the final location of obstacle 5 is [53.07, 71.88]. For obstacle 2, the maximum is 0.02 mm in the X direction and 1.93 mm in the Y direction. Its maximal resultant displacement is 1.94 mm, and its final location is [55.37, 41.21]. The target's maximal displacements are 0.14 mm and 1.76 mm in the X and Y directions. The final location of target is [52.58, 99.00] in the XOY plane.

With the planned adjusting angles, the needle base was steered by the actuators to change the direction of bevel-tip's angle. The flexible needle inserts into the PVA phantom, and the experimental trajectory is shown in Figure 15.

The needle tip moves towards the target from the initial insertion location [53.00, 0.00] along the Y direction. With the 3 and 4 rotations, the tip's locations of planned path are [52.03, 98.23] and [52.52, 98.42], respectively. The tip's locations of experimental path are [52.03, 97.73] and [52.46, 97.92], respectively. There exists the difference between the planned and experimental paths, and the difference lies between 0.1 mm and 0.5 mm.

For illustrating the effectiveness of the planned path, the error $\|\varepsilon\|_G$ between needle tip and target is described as

$$\|\varepsilon\|_G = \sqrt{\left(x_g - x_t\right)^2 + \left(y_g - y_t\right)^2 + \left(z_g - z_t\right)^2}. \tag{21}$$

Because obstacles 1, 3, and 4 are far away from the needle, only the errors $\|\varepsilon\|_{O_2}$ and $\|\varepsilon\|_{O_5}$ of obstacles 2 and 5 are listed in Table 3. The distance between the needle and obstacles is bigger than 1.16 mm, and the distance between needle tip and target is less than 1.20 mm. The planned needle trajectory can avoid the dynamic obstacles and reach the moved target effectively.

From the experimental results, the proposed dynamic path planning algorithm facilitates to avoid obstacles and reaching target and searching the trajectory with the minimal general insertion error. The algorithm accounts for the reachability and steerability of flexible needles, which can be realized with the experiments.

5. Conclusions and Future Work

In this paper, a novel dynamic path planning algorithm is developed for needle insertion into soft tissue considering the dynamic motions of target and obstacles. The needle-

Figure 12: Experimental trajectories of flexible needle into PVA phantom.

Figure 13: The displacements of obstacles and target with 3 rotations.

Figure 14: The displacements of obstacles and target with 4 rotations.

tissue interaction model is based on a FE model with the modified local constraint method, which can provide real-time feedback for path planner. The forward and inverse kinematics of flexible needles are modeled based on the quasi-static approach and D-H method, in which the needle is steered by rotating the needle base to generate the desired trajectory.

Figure 15: Comparison of the planned path and experimental path. Obstacles 2 and 5 and target denote their final positions of needle insertion.

Table 3: Errors of target and obstacles for dynamic path planning.

Planned segments	$\lVert\varepsilon\rVert_{O_2}$	$\lVert\varepsilon\rVert_{O_5}$	$\lVert\varepsilon\rVert_{G}$
3 segments	2.05 mm	1.16 mm	1.13 mm
4 segments	2.15 mm	1.20 mm	1.09 mm

The existence of inverse kinematic solution is investigated to determine the reachability of flexible needle. An iterative approach is developed to generate the feasible path by minimizing the general insertion error defined by artificial potential field. The experimental results showed that the actual paths using proposed algorithm agree well with the simulation results. Future work includes developing a better FE model that can simulate the inhomogeneous and nonlinear tissue properties that are close to the realistic soft tissue. In addition, a robust control strategy will be studied to consider the stability and steering ability of the flexible needle.

Data Availability

The needle-tissue interaction simulation data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Murong Li and Dedong Gao contributed equally to this work.

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Research Article

Study of Dynamical Behavior and Stability of Iterative Methods for Nonlinear Equation with Applications in Engineering

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In this article, we first construct a family of optimal 2-step iterative methods for finding a single root of the nonlinear equation using the procedure of weight function. We then extend these methods for determining all roots simultaneously. Convergence analysis is presented for both cases to show that the order of convergence is 4 in case of the single-root finding method and is 6 for simultaneous determination of all distinct as well as multiple roots of a nonlinear equation. The dynamical behavior is presented to analyze the stability of fixed and critical points of the rational operator of one-point iterative methods. The computational cost, basins of attraction, efficiency, log of the residual, and numerical test examples show that the newly constructed methods are more efficient as compared with the existing methods in the literature.

1. Introduction

To solve the nonlinear equation

$$f(s) = 0, \tag{1}$$

is the oldest problem of engineering in general and in mathematics in particular. These nonlinear equations have diverse applications in many areas of science and engineering. To find the roots of (1), we look towards iterative schemes, which can be classified as to approximate single root and all roots of (1). In this article, we are going to work on both types of iterative schemes. A lot of iterative methods of different convergence orders already exist in the literature (see [1–11]) to approximate the roots of (1). Ostrowski [7] defined efficiency index I of these iterative methods in terms of their convergence order k and the number of function evaluations per iteration, say u, i.e.,

$$I = k^{1/u}. \tag{2}$$

An iterative method is said to be optimal according to Kung–Traub conjecture [1] if

$$k = 2^{u-1}, \tag{3}$$

holds. The aforementioned methods are used to approximate one root at a time. However, mathematicians are also interested in finding of all roots of (1) simultaneously. This is due to the fact that simultaneous iterative methods are very popular due to their wider region of convergence, are more stable as compared to single-root finding methods, and implemented for parallel computing as well. More detail on single as well as simultaneous determination of all roots can be found in [1, 12–24] and references cited therein.

The most famous of the single-root finding method is the classical Newton–Raphson method:

$$s_{i+1} = s_i - \frac{f(s_i)}{f'(s_i)}, \quad i = 1, 2, \ldots. \tag{4}$$

Method (4) requires one evaluation of the function and one of its first derivative to achieve optimal order 2 having efficiency 1.41 using the Traub conjecture. Using Weierstrass' correction [15],

$$\frac{f(s_i)}{f'(s_i)} = w(s) = \frac{f(s_i)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(s - s_j)}, \tag{5}$$

in (4), we get the classical Weierstrass—Dochive method to approximate all roots of nonlinear equation (1) given as

$$s_{i+1} = s_i - \frac{f(s_i)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(s_i - s_j)}. \tag{6}$$

Method (6) has convergence order 2. Later, Aberth-Ehrlich [14] presented the 3rd-order simultaneous method given as

$$s_{i+1} = s_i - \frac{1}{(1/N(s_i)) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/(s_i - s_j)\right)}, \tag{7}$$

where $N(s_i) = f'(s_i)/f(s_i)$.

The main aim of this paper is to construct the family of optimal fourth-order single-root finding methods using the procedure of weight function and then convert them into simultaneous iterative methods for finding all distinct as well as multiple roots of nonlinear equation (1). Using the complex dynamical system, we will be able to choose those values of parameters of iterative methods which give a wider convergence area on initial approximations.

2. Construction of the Method and Convergence Analysis

King [25] presented the following optimal fourth-order method (abbreviated as MM1):

$$\begin{aligned} y_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right), \\ z_i &= y_i - \frac{f(s_i) + \beta f(y_i)}{f(s_i) + (\beta - 2) f(y_i)} \frac{f(y_i)}{f'(s_i)}. \end{aligned} \tag{8}$$

Chun [26] gave the fourth-order optimal method as (abbreviated as MM2)

$$\begin{aligned} y_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right), \\ z_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(1 + u + 2u^2\right), \end{aligned} \tag{9}$$

$$\text{where } u = \left(\frac{f(y_i)}{f(s_i)}\right).$$

Cordero et al. [3] proposed the fourth-order optimal method as (abbreviated as MM3)

$$\begin{aligned} y_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right), \\ v_i &= s_i - \frac{f(s_i) + f(y_i)}{f'(s_i)}, \\ z_i &= v_i - \frac{(f(y_i))^2 (2f(s_i) + f(y_i))}{(f(s_i))^2 f'(s_i)}. \end{aligned} \tag{10}$$

Chun [27] introduced the fourth-order optimal method as (abbreviated as MM4)

$$\begin{aligned} y_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right), \\ z_i &= s_i - \left(\frac{fs_i)}{f's_i)}\right) - \left(\frac{fs_i)}{f's_i)}\right)\left(\frac{f(s_i)}{f(s_i) - f(y_i)}\right). \end{aligned} \tag{11}$$

Here, we propose the following family of iterative methods:

$$\begin{aligned} y_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right), \\ z_i &= y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\Gamma(u), \end{aligned} \tag{12}$$

$$\text{where } u = \frac{f(y_i)}{f(s_i)}.$$

For iterative scheme (12), we have the following convergence theorem:

Theorem 1. *Let $\zeta \in I$ be a simple root of a sufficiently differential function $f: I \subseteq \mathfrak{R} \longrightarrow \mathfrak{R}$ in an open interval I. If s_0 is sufficiently close to ζ and Γ be a real-valued function satisfying $\Gamma(0) = 0, \Gamma'(0) = 1, \Gamma''(0) = 4$, and $\Gamma'''(0) < \infty$, then the convergence order of the family of iterative method (12) is 4 and satisfies the following error equation:*

$$e_{i+1} = \left(5c_2^3 - c_2c_3 - \frac{1}{6}\Gamma'''(0)c_2^3\right)e_i^4 + O\left(e_i^5\right), \tag{13}$$

where $c_m = f^m(\zeta)/m!f'(\zeta); m \geq 2$.

Proof. Let ζ be a simple root of f, and $s_i = \zeta + e_i$. By Taylor's series expansion of $f(s_i)$ about $s = \zeta$ taking $f(\zeta) = 0$, we get

$$f(s_i) = f'(\zeta)\left(e_i + c_2e_i^2 + c_3e_i^3 + c_4e_i^4 + O\left(e_i^5\right)\right), \tag{14}$$

$$f'(s_i) = f'(\zeta)\left(1 + 2c_2e_i + 3c_3e_i^2 + O\left(e_i^3\right)\right) \tag{15}$$

Dividing (14) by (10), we have

$$\frac{f(s_i)}{f'(s_i)} = e_i + c_2e_i^2 + \left(2c_2^2 - 4c_3\right)e_i^3 + O\left(e_i^4\right). \tag{16}$$

This gives

$$y_i = \zeta + c_2 e_i^2 + \left(-2c_2^2 + 2c_3\right)e_i^3 + \cdots$$
$$u = \left(\frac{f(y_i)}{f(s_i)}\right) = c_2 e_i + \left(-3c_2^2 + 2c_3\right)e_i^2 + \left(8c_2^3 - 10c_2c_3 + 3c_4\right)e_i^3 + \cdots. \tag{17}$$

Thus, using Taylor series, we have

$$\Gamma(u) = \Gamma(0) + u\Gamma'(0) + \frac{u^2}{2!}\Gamma''(0) + \cdots$$
$$\Gamma(u) = \Gamma(0) + c_2\Gamma'(0)e_i + \left\{\frac{1}{2}c_2^2\left(\Gamma''(0) - 3\Gamma'(0)\right) + 2c_3\Gamma\prime(0)\right\}e_i^2 + \cdots. \tag{18}$$

Now, taking $\Gamma q = 0, \Gamma(0) = 1$, and $\Gamma''(0) = 4$ in equation (12) and on simplification gives

$$e_{i+1} = \left(5c_2^3 - c_2c_3 - \frac{1}{6}\Gamma'''(0)c_2^3\right)e_i^4 + O\left(e_i^5\right). \tag{19}$$

Hence, it proves fourth-order convergence. □

2.1. The Concrete Fourth-Order Family of Methods. We now construct some concrete forms of the family of fourth-order methods from the family of methods (12) by choosing weight function $\Gamma(u)$ containing an arbitrary real number β as provided in Table 1, satisfying the condition $\Gamma(0) = 0, \Gamma'(0) = 1, \Gamma''(0) = 4$ and $\Gamma'''(0) < \infty$ of Theorem 1 with β as a real number. Therefore, we get the following new five families of iterative methods:

Method-1 (abbreviated as BB1):

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$
$$z_i = y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(u + 2u^2 + \beta u^4\right). \tag{20}$$

Method-2 (abbreviated as BB2):

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$
$$z_i = y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(\frac{(u)}{1 - 2(u) + \beta(u)^2}\right). \tag{21}$$

Method-3 (abbreviated as BB3):

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$
$$z_i = y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(u + +2u^2 + \beta u^3\right). \tag{22}$$

Method-4 (abbreviated as BB4):

Table 1: Concrete forms of methods.

S. no	β	$\Gamma(u), u = (f(y_i)/f(s_i))$
B1	$\Re$	$\Gamma(u) = u + 2u^2 + \beta u^4$
B2	$\Re$	$\Gamma(u) = u/1 - 2u + \beta u^2$
B3	$\Re$	$\Gamma(u) = u + 2u^2 + \beta u^3$
B4	$\Re$	$\Gamma(u) = (u/1 - \beta u^2) + 2u^2$
B5	$\Re$	$\Gamma(u) = ((u + 2u^2)/(1 - \beta u^3))$

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$
$$z_i = y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(\frac{u}{1 - \beta u^2} + 2u^2\right). \tag{23}$$

Method-5 (abbreviated as BB5):

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$
$$z_i = y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(\frac{u + 2u^2}{1 - \beta u^3}\right), \tag{24}$$

where $u = \left(\frac{f(y_i)}{f(s_i)}\right)$.

3. Construction of Simultaneous Methods

Suppose nonlinear equation (1) has n roots. Then, $f(s)$ and $f'(s)$ can be approximated as

$$f(s) = \prod_{j=1}^{n}\left(s - s_j\right) \text{ and } f'(s) = \sum_{k=1}^{n}\prod_{\substack{j\neq k\\ j=1}}^{n}\left(s - s_j\right). \tag{25}$$

This implies that

$$\frac{f'(s)}{f(s)} = \sum_{j=1}^{n}\left(\frac{1}{\left(s - s_j\right)}\right) = \frac{1}{\left(1/(s - s_i)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/\left(s - s_j\right)\right)}. \tag{26}$$

Now, an approximation of $f(s_i)/f'(s_i)$ is formed by replacing s_j with z_{tj} as follows:

$$\frac{f(s_i)}{f'(s_i)} = \frac{1}{\left(1/N(s_i)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/\left(s_i - z_{tj}\right)\right)}. \tag{27}$$

Using (27) in (4), we have

$$s_{i+1} = s_i - \frac{1}{\left(1/N(s_i)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/\left(s_i - z_{tj}\right)\right)}, \quad \text{for } t = 1, \ldots, 5. \tag{28}$$

Using corrections z_{tj} from BB1 to BB5, we get the following five simultaneous iterative methods for extracting all distinct as well as multiple roots of nonlinear equation (1):

$$s_{i+1} = s_i - \frac{\sigma_i}{(\sigma_i/N(s_i)) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(\sigma_j/s_i - z_{tj}\right)}, \quad t = 1,\ldots,5,$$

$$\text{where } z_{1j} = y_j - \left(\frac{f(s_j)}{f'(s_j)}\right)\left(u + 2u^2 + \beta u^4\right),$$

$$z_{2j} = y_j - \left(\frac{f(s_j)}{f'(s_j)}\right)\left(\frac{u}{1 - 2u + \beta u^2}\right),$$

$$z_{3j} = y_j - \left(\frac{f(s_j)}{f'(s_j)}\right)\left(u + 2u^2 + \beta u^3\right),$$

$$z_{4j} = y_j - \left(\frac{f(s_j)}{f'(s_j)}\right)\left(\frac{u}{1 - \beta u^2} + 2u^2\right),$$

$$z_{5j} = y_j - \left(\frac{f(s_j)}{f'(s_j)}\right)\left(\frac{u + 2u^2}{1 - \beta u^3}\right),$$

$$\text{where } y_j = s_j - \left(\frac{f(s_j)}{f'(s_j)}\right),$$

$$u_1 = \left(\frac{f(y_j)}{f(s_j)}\right). \tag{29}$$

Thus, we constructed new five simultaneous iterative methods (29) abbreviated as M1–M5.

3.1. Convergence Analysis. In this section, the convergence analysis of a family of simultaneous methods (M1–M5) is given in the form of the following theorem. Obviously, convergence for the methods (28) will follow from the convergence of the methods (29) from theorem (2) when the multiplicities of the roots are one.

Theorem 2. *Let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be the simple roots of nonlinear equation (1). If $s_1^{(0)}, s_2^{(0)}, s_3^{(0)}, \ldots, s_n^{(0)}$. If be the initial approximations of the roots, respectively, and sufficiently close to actual roots, the order of convergence of method (58–62) is six.*

Proof. Let $\varepsilon_i = s_i - \zeta_i$and $\varepsilon_i' = s_{i+1} - \zeta_i$ be the errors in approximations s_i and s_{i+1}, respectively. Then, for distinct roots, we have

$$\begin{aligned}\frac{1}{N(s_i)} &= \left(\frac{f'(s_i)}{f(s_i)}\right) = \sum_{j=1}^{n}\left(\frac{1}{(s_i - \zeta_j)}\right)\\ &= \frac{1}{(s_i - \zeta_i)} + \sum_{\substack{j\neq i\\ j=1}}^{n}\left(\frac{1}{(s_i - \zeta_j)}\right).\end{aligned} \tag{30}$$

Thus, for multiple roots, we have from (29)

$$\begin{aligned}
s_{i+1} &= s_i - \frac{\sigma_i}{(\sigma_i/(s_i - \zeta_i)) + \sum_{\substack{j\neq i\\ j=1}}^{n}\left(\sigma_j/(s_i - \zeta_j)\right) - \sum_{\substack{j\neq i\\ j=1}}^{n}\left(\sigma_j/(s_i - z_{tj})\right)},\\
s_{i+1} - \zeta_i &= s_i - \zeta_i - \frac{\sigma_i}{(\sigma_i/(s_i - \zeta_i)) + \sum_{\substack{j\neq i\\ j=1}}^{n}\left(\sigma_j(s_i - z_{tj} - s_i + \zeta_j)/(s_i - \zeta_j)(s_i - z_{tj})\right)},\\
\text{this implies, } \varepsilon_i' &= \varepsilon_i - \frac{\sigma_i\varepsilon_i}{\sigma_i + \varepsilon_i\sum_{\substack{j\neq i\\ j=1}}^{n}\left(-\sigma_j(z_{tj} - \zeta_j)/(s_i - \zeta_j)(s_i - z_{tj})\right)},\\
&= \varepsilon_i - \frac{\sigma_i\varepsilon_i}{\sigma_i + \varepsilon_i\sum_{\substack{j\neq i\\ j=1}}^{n}\left(E_i\varepsilon_j^4\right)},\\
\text{where } z_{tj} - \zeta_j &= \varepsilon_j^4\\
E_i &= \left(\frac{-\sigma_j}{(s_i - \zeta_j)(s_i - z_{tj})}\right).\\
\text{Thus, } \varepsilon_i' &= \frac{\varepsilon_i^2\sum_{\substack{j\neq i\\ j=1}}^{n}\left(E_i\varepsilon_j^4\right)}{\sigma_i + \varepsilon_i\sum_{\substack{j\neq i\\ j=1}}^{n}\left(E_i\varepsilon_j^4\right)}.
\end{aligned} \tag{31}$$

If it is assumed that absolute values of all errors ε_j ($j = 1, 2, 3, \ldots$) are of the same order as, say, $|\varepsilon_j| = O|\varepsilon|$, then from (30), we have

$$\varepsilon_i' = O(\varepsilon)^6. \tag{32}$$

Thus, (32) shows the convergence order of methods M1–M5 which is six. Hence, the theorem is proved. □

4. Complex Dynamical Study of Families of Iterative Methods

Here, we discuss stability of the family of iterative method (BB1) only in the background contexture of complex dynamics. Rational map arising due to iterative method (BB1) is

$$\mathfrak{R}_f = y - \left(\frac{f(s)}{f'(s)}\right)\left(u + 2u^2 + \beta u^4\right), \tag{33}$$

where $y = s - (f(s)/f'(s))$ and $u = (f(y)/f(s))$.

Recalling some basic concepts of this theory, detailed information can be found in [2, 4, 6, 8]. Taking a rational function $\mathfrak{R}_f\colon C \longrightarrow C$, where C denotes the Riemann sphere, the orbit $s_0 \in C$ defines a set such as $\operatorname{orb}(s) = \left\{s_0, \mathfrak{R}_f(s_0), \mathfrak{R}_f^2(s_0), \ldots, \mathfrak{R}_f^m(s_0), \ldots\right\}$. A point $s_0 \in \mathrm{C}$ is called a fixed point if $\mathfrak{R}_f(s_0) = s_0$. In particular, a fixed point s_0 is called the strange fixed point if $f(s^r) = 0$ when $s_0 \neq s^r$. A T-periodic point is defined as a point $s_T \in C$ satisfying $\mathfrak{R}^T(s_T) = s_T$ but $\mathfrak{R}\prime(s_T) \neq s_T$ for $t < T$. If s_0 is the fixed point of $\mathfrak{R}$, then it is

(i) Superattracting if $|\mathfrak{R}_f'(s_0)| = 0$
(ii) Attracting if $|\mathfrak{R}_f'(s_0)| < 1$
(iii) Repulsive if $|\mathfrak{R}_f'(s_0)| > 1$
(iv) Neutral if $|\mathfrak{R}_f'(s_0)| = 1$
(v) A strange fixed point if it not associated to any root of nonlinear equation (1)

An attracting point $s^* \in \mathrm{C}$ defines the basin of attraction, $\mathfrak{R}(s^*)$, as the set of starting points whose orbit tends to s^*.

Furthermore, the implementation of the dynamical plane of the rational operator corresponding to iterative methods divides the complex plane into a mesh of values of real part along the x-axis and imaginary on the y-axis. The initial estimates are depicted in color depending on where their orbit converges, and thus, basins of attraction of the corresponding iterative methods are obtained. The scaling theorem allows the suitable change of the coordinate to reduce dynamics of iteration of general maps to study the specific family of iteration of similar maps.

Theorem 3. *Let us take an analytic function $f(s)$ and $T(s) = \alpha s + \beta$ be an affine map, with $\alpha \neq 0$. Take $g(s) = goT(s)$; then, $To\mathfrak{R}_g oT^{-1} = \mathfrak{R}_f(s)$, i.e., $\mathfrak{R}_g = \mathfrak{R}_f$ affine conjugate by T (Scaling theorem).*

As iterative method (33), holds scaling theorem and thus allows the dynamic studies of iterative function (BB1) for the polynomial $f(s) = (s-a)(s-b)$, where $a, b \in \mathfrak{R}$. One-point iterative method (BB1) has a universal Julia set if a rational map exists which conjugates by Mobius transformation.

Theorem 4. *For a rational map $\Omega_g(s)$ arising from (33) applied to $f(s) = (s-a)(s-b)$, where $a, b \in \mathfrak{R}, \Omega_g(s)$ is conjugate via Möbius transformation by $M(v) = (v-a)/(v-b)$ to*

$$\Omega_g(s,\beta) = \frac{s^4\left(-s^6 - 8s^5 - 27s^4 + A1\right)}{-5s^6 + \beta s^5 - 24s^5 - 47s^4 + A2}, \tag{34}$$

where $A_1 = -48s^3 + \beta s - 47s^2 - 24s - 5$ and $A_2 = -48s^3 - 27s^2 - 8s - 1$.

Proof. Let $f(s) = (s-a)(s-b)$, where $a, b \in \mathfrak{R}$. Möbius transformation is given by

$$M(v) = \frac{v-a}{v-b}, \quad \text{with inverse } [M(v)]^{-1} = \frac{vb-a}{v-1}, \tag{35}$$

which is considered as the map from $\mathrm{C} \cup \infty$. Then, we have

$$\Omega_g(s,\beta) = \frac{s^4\left(-s^6 - 8s^5 - 27s^4 + A_1\right)}{-5s^6 + \beta s^5 - 24s^5 - 47s^4 + A_2}, \tag{36}$$

where $A_1 = -48s^3 + \beta s - 47s^2 - 24s - 5$ and $A_2 = -48s^3 - 27s^2 - 8s - 1$.

Similarly, we can get the following conclusions. □

Theorem 5. *For a rational map $\Omega_g(s)$ arising from (BB2) to (BB5) applied to $f(s) = (s-a)(s-b)$, where $a, b \in \mathfrak{R}$, $\Omega_g(s)$ is conjugate via Möbius transformation by $M(v) = (v-a)/(v-b)$ to the following:*

$$\Omega_g(s,\beta) = \frac{s^4\left(s^2 + \beta + 2s + 1\right)}{(\beta+1)s^2 + 2s + 1}, \tag{37}$$

$$\Omega_g(s,\beta) = \frac{s^4\left(-s^4 - 6s^3 - 14s^2 + \beta - 14s - 5\right)}{(\beta-5)s^4 - 14s^3 - 14s^2 - 6s - 1}, \tag{38}$$

$$\Omega_g(s,\beta) = \frac{s^4\left(-s^6 - 8s^5 + (\beta-27)s^4 + B1\right)}{(\beta-5)s^6 + (2\beta-24)s^5 + B2}, \tag{39}$$

where $B_1 = -48s^3 + (6\beta-47)s^2 + (2\beta-24)s - 5$ and $B_2 = (6\beta-47)s^4 + (4\beta-48)s^3 + (\beta-27)s^2 - 8s - 1$.

$$\Omega_g(s,\beta) = \frac{s^4\left(-s^4 - 6s^3 + \beta s - 14s^2 - 14s - 5\right)}{-5s^4 + (\beta-14)s^3 - 14s^2 - 6s - 1}. \tag{40}$$

Theorem 6. *For a rational map $\Omega_g(s)$arising from (MM1) to (MM4) applied to $f(s) = (s-a)(s-b)$, where $a, b \in \mathfrak{R}$, $\Omega_g(s)$ is conjugate via Möbius transformation by $M(v) = (v-a)/(v-b)$ to the following:*

$$\Omega_g(s,\beta) = \frac{s^4\left(s^2+(\beta+2)s+2\beta+1\right)}{(2\beta+1)s^2+(\beta+2)s+1}, \tag{41}$$

$$\Omega_g(s,\beta) = \frac{s^4\left(s^2+4s+5\right)}{5s^2+4s+1}. \tag{42}$$

$$\Omega_g(s,\beta) = \frac{s^4\left(s^4+6s^3+14s^2+14s+4\right)}{4s^4+14s^3+14s^2+6s+1}, \tag{43}$$

$$\Omega_g(s,\beta) = \frac{s^4\left(s^2+2s+2\right)}{2s^2+2s+2}. \tag{44}$$

The unions of the respective stability functions of all the strange fixed points of rational maps (36)–(41) are graphed in Figures 1(a)–1(f).

The fixed points for rational function (36) are $s = 1, s = 0$, and $s = \infty$. For the stability of fixed points of iterative method (36), we calculate $\Omega_g'(s,\beta)$, i.e.,

$$\Omega_g'(s,\beta) = \frac{s^3(s+1)^8(A_3)}{(A_4)^2}, \tag{45}$$

where $A_3 = -20s^4 + (5\beta-80)s^3 + (-8\alpha^2-120)s^2 + (5\beta+80)s - 20$ and $A_4 = -5s^6 + (\beta-24)s^5 - 47s^4 - 48s^3 - 27s^2 - 8s - 1$.

It is evident from (36) that $s = 0$ and ∞ are always superattractive fixed points, but stability of other fixed points depends on the value of parameter β which is present here. The operator $\Omega_g'(s,\beta)$ for $s = 1$ gives

$$\left|\Omega_g'(1,\beta)\right| = \left|\frac{512}{\beta-160}\right|. \tag{46}$$

Analyzing (46), as $\beta \longrightarrow \pm\infty$, we obtained horizontal asymptotes in $|\Omega_g'(1,\beta)| = 0$.

In the following result, we present the stability of the strange fixed point for $s = 1$.

Theorem 7. *The character of the strange point $s = 1$ is as follows:*

(i) If $|512/\beta - 160| < 1$, then $s = 1$ is an attractor, and it can be superattractor for $\beta = 1$

(ii) When $|512/\beta - 160| = 1$, $s = 1$ is a parabolic point

(iii) If $\beta \neq 0$ and $|512/\beta - 160| > 1$, then $s = 1$ is repulsive

Proof. From (45), we have

$$\Omega_g'(1,\beta) = \frac{512}{\beta-160}. \tag{47}$$

So,

$$\left|\frac{512}{\beta-160}\right| \leq 1 \text{ is equivalent to } |512| \leq |\beta-160|. \tag{48}$$

Let us consider $\beta = a + ib$ as an arbitrary complex number. Then, $(a-160)^2 + b^2 \geq 512^2$. Therefore, $|\Omega_g'(1,\beta)| \leq 1 \text{iff} |\beta - 160 \geq 512|$. Finally, if β varies, then $|\Omega_g'(1,\beta)| > 1$ and $s = 1$ is repulsive, except if $\beta = 1$ and $\beta = 160$ for which $s = 1$ is not a fixed point.

The functions where we examine stability of iterative method (36) are given as

$$\Omega_g'(1,\beta) = \min\left\{\Omega_g'(1,\beta), 1\right\}. \tag{49}$$

□

4.1. Analysis of Critical Points. The critical points of (36) satisfy $\Omega_g'(s,\beta) = 0$, i.e., $s = 0, -1$ and $s = \infty$ for $\beta \neq 0$, and

$$\begin{aligned}
\text{cr1} &= \frac{1}{16}(-16+\beta) - \frac{\sqrt{\beta(5\beta-288)}}{16\sqrt{5}} - \frac{\sqrt{\beta_1}}{8\sqrt{10}},\\
\text{cr2} &= \frac{1}{16}(-16+\beta) - \frac{\sqrt{\beta(5\beta-288)}}{16\sqrt{5}} + \frac{\sqrt{\beta_1}}{8\sqrt{10}},\\
\text{cr3} &= \frac{\beta-16}{16} - \frac{\sqrt{\beta(5\beta-288)}}{16\sqrt{5}} - \frac{\sqrt{\beta_2}}{2},\\
\text{cr4} &= \frac{\beta-16}{16} - \frac{\sqrt{\beta(5\beta-288)}}{16\sqrt{5}} + \frac{\sqrt{\beta_2}}{2},
\end{aligned} \tag{50}$$

where $\beta_1 = -224\beta + 5\beta^2 - (4608\sqrt{5\beta}/\sqrt{5\beta-288}) + (268\sqrt{5}\beta^{2/3}/\sqrt{5\beta-288}) - (5\sqrt{5}\beta^{5/2}/\sqrt{5\beta-288})$, and $\beta_2 = -2 + ((16-\beta)^2/32) - (30+2\beta)/5 + (\beta\bullet/\sqrt{5\beta-288})$, where $\beta\bullet = 2\sqrt{5}((\beta-32) - ((16-\beta)^2/64) + (2(16-\beta)(15+\beta)/5))$. We observe cr1 = 1/cr2 and cr3 = 1/cr4. Figure 2 presents the zones of stability of strange fixed points. Fixed points 1 and 0 are represented by black-dotted lines, while critical points $(-1, \text{cr1}, \text{cr2}, \text{cr3}, \text{cr4})$ are represented by black, red, blue, green and orange-dotted lines, respectively (see Figure 3).

Theorem 8. *The only member of the family of iterative methods whose operator is always conjugated to the rational map s^2 is the element corresponding to $\beta = 0$.*

Proof. From (51), we represent

$$\begin{aligned}
nn(s) &= s^4\left(-s^6 - 8s^5 - 27s^4 - 48s^3 + \beta s - 47s^2 - 24s - 5\right),\\
dd(s) &= -5s^6 + \beta s^5 - 24s^5 - 47s^4 - 48s^3 - 27s^2 - 8s - 1.
\end{aligned} \tag{51}$$

The unique value of β for which $nn(s) = dd(s)$ is 0. □

4.2. Parametric Planes. Parametric planes are obtained by taking β over a mesh of 500×500 values in the complex plane in $R_e(\beta) \times \text{Im}(\beta) \in [-4,4] \times [-4,4]$. A critical point is taken as the initial approximation. The method is then iterated until it reaches to the maximum iterations or converges to any fixed point. Taking 10^{-3} as a tolerance used for stopping criteria $|f(s_i)|$. The complex value of the parameter in the complex plane is paint in red if the method converges to any of the roots (0 and ∞), black in the other case.

In Figures 4(a)–4(d) and 5(a)–5(h), red color shows the convergence region, and all those values of parameter β show stable behavior, while those values of the

Figure 1: (a–f) Unions of the respective stability function of all strange fixed points of the rational map (BB1–BB5 and MM1), respectively.

Figure 2: The zones of stability of strange fixed points of $\Omega'(1, \beta)$.

parameter which are taken from the black region (divergence region) show the unstable behavior of the iterative map (36). Stable and unstable behavior is shown in Figures 6 and 7, respectively.

4.3. Dynamical Planes. The generations of the dynamical planes are similar to the parametric plane. To execute them, the real and imaginary parts of the starting approximation are represented as two axes over a mesh of 250×250 in the complex plane. The stopping criteria are the same as in the parametric plane but assign different colors to indicate to which root the method converges and black in the other case.

Let us note that the iterative methods BB2–BB5, MM1–MM4, and BB1 satisfy Cayley's test [28–31] for all parameter values of β. It can be observed from Figure 8 that iterative methods BB1–BB5 and MM1–MM4 verifying Cayley's test [28] have the same dynamical properties as Newton's method. Figures 8(a)–8(e) clearly show quite stable behavior of iterative methods BB1–BB5 over MM1–MM4.

For unstable behavior, the value of parameter β is chosen from the black region. To generate basins of attractions, we take a square box of $[-1, 1] \times [-1, 1] \in \mathbb{C}$. To each root of (1), we assign color to which the corresponding orbit of iterative methods (BB1–BB5 and MM1–MM4) starts and convergences to a fixed point. Take color map as HSV. We use $|f(s_i)| < 10^{-3}$ as the stopping criteria, and the maximum iteration is taken as

Figure 3: Dynamical behavior of strange fixed points and critical points for $0 \le \beta \le 20$ (a) and $-20 \le \beta \le 0$ (b).

Figure 4: (a–d) Parametric planes associated with cr1, cr2, cr3, and cr4, respectively.

Figure 5: (a–h) Details of the parametric planes for cr1, cr2, cr3, and cr4.

Figure 6: (a–e): Dynamical planes for stable behavior of iterative method (18) for different values of β, i.e., $\beta = 0, 1, -1, 160$, and -20, respectively.

30. We make dark red points if the orbit of the iterative methods does not converge to a root after 30 iterations. Different colors are used for different roots. Iterative methods have different basins of attractions distinguished by their color. In basins, brightness in color represents less number of iterations to achieve the roots of (1). Figures 9 and 10 show the basins of attractions of iterative methods (BB1–BB5 and MM1–MM4) for nonlinear function $f_1(s) = s^4 + 4s^3 - 24s^2 + 16s + 16$ and $f_2(s) = \sin((s-1)/2)\sin((s-2)/2)\sin((s-2.5)/2)$, respectively. From Figures 9 and 10, divergent regions and brightness in color present that BB1–BB5 are better than MM1–MM4, respectively.

5. Computational Aspect

Here, we compare the computational efficiency of M. S. Petkovi´c method [32] and the new methods (M1–M5) given by (28). As presented in [32], the efficiency of an iterative method can be estimated using the efficiency index given by

$$EL(m) = \frac{\log \mathbf{u}}{\mathbf{D}}, \tag{52}$$

where $\mathbf{D}$ is the computational cost and $\mathbf{u}$ is the order of convergence of the iterative method. Using arithmetic operation per iteration with certain weight depending on the execution time of operation, the computational cost $\mathbf{D}$ is evaluated. The weights used for division, multiplication, and addition plus subtraction are w_d, w_m, and w_{as}, respectively. For a given polynomial of degree m and n roots, the number of division, multiplication, addition and subtraction per iteration for all roots is denoted by D_m, M_m, and AS_m. The cost of computation can be calculated as

$$\mathbf{D} = \mathbf{D}(m) = w_{as}AS_m + w_m M_m + w_d D_m. \tag{53}$$

Thus, (52) becomes

$$EL(m) = \left(\frac{\log \mathbf{r}}{w_{as}AS_m + w_m M_m + w_d D_m}\right). \tag{54}$$

Reducing the number of operations of a complex polynomial of degree m with real and complex roots to operations of real arithmetic, as given in Table 2. Applying

Figure 7: (a–e) Dynamical planes for unstable behavior of the iterative map (18) for different values of β, i.e., $\beta = 200$, $-140, 140, 500$, and -500, respectively.

(38) and data given in Table 2, we calculate the percentage ratio $\rho((M1-M5), PJ6M)$ [3] which is given by

$$\rho(PJ6M, (M1-M5)) = \left(\frac{EL(PJ6M)}{EL(M1-M5)} - 1\right) \times 100,$$
$$\rho((M1-M5), PJ6M) = \left(\frac{EL(M1-M5)}{EL(PJ6M)} - 1\right) \times 100, \tag{55}$$

where $PJ6M$ is the Petkovic method [32] of order six. Figures 11(a)–11(d) graphically illustrate these percentage ratios. It is evident from Figures 11(a)–11(d) that the newly constructed simultaneous methods (M1–M5) are more efficient as compared with the Petkovic method [32].

We also calculate the CPU execution time as all the calculations are done using Maple 18 on Processor Intel(R) Core(TM) i3-3110m CPU@2.4 GHz with 64 bit Operating System. We observe that CPU time of the methods M1–M5 is less than M. S. Petkovi´c methods [32], showing the dominance efficiency of our methods (M1–M5) as compared to them.

6. Numerical Results

Here, some numerical examples are considered in order to demonstrate the performance of our family of two-step fourth-order single-root finding methods (BB1–BB5) and six-order simultaneous methods (M1–M5), respectively. We compare our family of optimal fourth-order single-root finding methods (BB1–BB5) with MM1–MM4 methods. Family of simultaneous methods (M1–M5) of convergence order six is compared with M. S. Petkovi´c method [32] of the same order (abbreviated as the PJ6M method). All the computations are performed using CAS Maple 18 with 2500 (64-digit floating-point arithmetic in case of simultaneous methods) significant digits. For single-root finding methods, the stopping criteria are as follows:

(i) $|f(s_i)| < \epsilon$,

(ii) $|s_i^{(k)} - \alpha| < \epsilon$,

whereas $e_i = \|f(s_i)\|_2$ for simultaneous methods.

We take $\epsilon = 10^{-600}$ for the single-root finding method and $\epsilon = 10^{-30}$ for simultaneous determination of all roots of nonlinear equation (1).

Figure 8: (a–i) Dynamical planes of iterative methods (BB1–BB5, MM2, MM4, MM3, and MM1) for $\beta = 1$. (j) Dynamical plane of method MM1 for $\beta = 3$.

Numerical test examples from [22, 33–35] are provided in Tables 3–12. In Tables 3, 4, 6, 7, 9, and 11, we present the numerical results for simultaneous determination of all roots while Tables 5, 8, 10, and 12 for single-root finding methods. In all tables, CO represents the convergence order, n, the number of iterations, ρ, the computational order of convergence, and CPU, the computational time in seconds. The value of arbitrary parameter β used in iterative methods BB1–BB5 and M1–M5 is 1.5 for test Examples 1–4. We observe that numerical results of single-root finding methods (BB1–BB5) as well as all-root finding methods (M1–M5) are better than MM1–MM4 and PJ6M, respectively, on the same number of iterations. Figures 12(a)–12(d) represent the residual fall for the iterative methods (M1–M5 and PJ6M), and Figures 12(e)–12(h) represent the residual fall of iterative methods (BB1–BB5 and MM1–MM4) for given nonlinear functions. Tables 3–12 show that the family of methods BB1–BB5 and M1–M5 are more efficient as compared with MM1–MM4 and PJ6M, respectively.

We also calculate the CPU execution time as all the calculations are done using Maple 18 on Processor Intel (R) Core (TM) i3-3110m CPU@2.4GHz with 64 bit Operating System. We observe from the tables that CPU time of the methods M1–M5 is better than the PJ6M method, showing the efficiency of our methods (M1–M5) as compared to them.

6.1. Applications in Engineering. In this section, we discuss the applications in engineering.

Example 1 (for the beam designing model, see [33]).

Consider a problem of beam positioning, resulting in a nonlinear function which is given as

$$f_1(s) = s^4 + 4s^3 - 24s^2 + 16s + 16 = (s-2)^2\left(s^2 + 8s + 4\right). \tag{56}$$

The exact roots of $f_1(s)$ are $\zeta_{1,2} = 2, \zeta_3 = -4 - 2\sqrt{3}, \zeta_3 = 2, \zeta_4 = -4 + 2\sqrt{3}$ as shown in Figure 13.

Figure 9: (a–i) Basins of attraction of methods BB1–BB5 and MM1–MM4 for the nonlinear equation $f_1(s)$, respectively.

The initial estimates for $f_3(s)$ are taken as

$$\begin{aligned} \overset{(0)}{s}_{1,2} &= 1.17, \\ \overset{(0)}{s}_3 &= -7.4641, \\ \overset{(0)}{s}_4 &= -0.5359. \end{aligned} \tag{57}$$

For distinct roots, we used method (28), and for multiple roots, we used method (29).

$$f_{1-1}(s) = s^4 + 4s^3 - 24s^2 + 16s + 16. \tag{58}$$

Example 2 (for determination of all distinct and multiple roots, see [22]). Here, we consider another standard test function for the demonstration of convergence behavior of newly constructed methods.

Consider

$$f_2(s) = \sin^3\left(\frac{s-1}{2}\right)\sin^3\left(\frac{s-2}{2}\right)\sin^3\left(\frac{s-2.5}{2}\right), \tag{59}$$

with exact roots $\zeta_1 = 1, \zeta_2 = 2, \zeta_3 = 2.5$ as shown in Figure 14.

The initial guessed values have been taken as $\overset{(0)}{s}_1 = -0.2$, $\overset{(0)}{s}_2 = 1.7$, $\overset{(0)}{s}_3 = 3$.

For distinct roots, we used method (28), and for multiple roots, we used method (29).

$$f_{2-1}(s) = \sin\left(\frac{s-1}{2}\right)\sin\left(\frac{s-2}{2}\right)\sin\left(\frac{s-2.5}{2}\right). \tag{60}$$

Figure 10: (a–i) Basins of attraction of methods BB1–BB5 and MM1–MM4 for the nonlinear equation $f_2(s)$, respectively.

Table 2: The number of basic operations.

Methods	CO	AS_m	M_m	D_m
M1	6	$7m^2 + O(m)$	$5m^2 + O(m)$	$2m^2 + O(m)$
M2	6	$7m^2 + O(m)$	$2m^2 + O(m)$	$2m^2 + O(m)$
M3	6	$7m^2 + O(m)$	$4m^2 + O(m)$	$2m^2 + O(m)$
M4	6	$7m^2 + O(m)$	$4m^2 + O(m)$	$2m^2 + O(m)$
M5	6	$7m^2 + O(m)$	$4m^2 + O(m)$	$2m^2 + O(m)$
PJ6M	6	$8m^2 + O(m)$	$6m^2 + O(m)$	$2m^2 + O(m)$

Figure 11: (a–c) Computational efficiency of methods M1, M3–M5, and M2 w.r.t method PJ6M. (d) Computational efficiency of method PJ6M w.r.t M1–M5.

Table 3: Simultaneous finding of all district roots.

$f_{1-1}(s) = s^4 + 4s^3 - 24s^2 + 16s + 16$

$\zeta_{1,2} = 2, \zeta_3 = -4 - 2\sqrt{3}, \zeta_3 = 2, \zeta_4 = -4 + 2\sqrt{3}.$

$\overset{(0)}{s}_{1,2} = 1.17, \overset{(0)}{s}_3 = -7.4641, \overset{(0)}{s}_4 = -0.5359$

Method	CO	CPU	n	$e1$	$e2$	$e3$	$e4$
PJ6M	6	0.031	4	$3.5e-2$	$3.5e-2$	$6.0e-11$	0.0
M1–M5	6	0.015	4	$5.0e-13$	$5.0e-13$	0.0	0.0

Table 4: Simultaneous findings of all multiple roots.

$f_1(s) = s^4 + 4s^3 - 24s^2 + 16s + 16 = (s-2)^2(s^2 + 8s + 4)$

$\zeta_{1,2} = 2, \zeta_3 = -4 - 2\sqrt{3}, \zeta_3 = 2, \zeta_4 = -4 + 2\sqrt{3}.$

$\overset{(0)}{s}_{1,2} = 1.17, \overset{(0)}{s}_3 = -7.4641, \overset{(0)}{s}_4 = -0.5359$

Methods	CO	CPU	n	$e1$	$e2$	$e3$	$e4$
PJ6M	6	0.032	4	$3.1e-10$	$3.1e-10$	$3.2e-6$	0.0
M1–M5	6	0.016	4	0.0	0.0	0.0	0.0

Table 5: Comparison of optimal fourth-order methods.

Method	$\lvert s_i^{(6)} - \alpha \rvert$	$\lvert f_i^{(6)}(s_i) \rvert$	CPU	p
$f_{1-1}(s) = s^4 + 4s^3 - 24s^2 + 16s + 16, s_0 = -0.5$				
BB1	$1.4e-1399$	$3.5e-5594$	0.032	4.00
BB2	$1.9e-1825$	$1.7e-7299$	0.063	4.00
BB3	$1.8e-1829$	$1.5e-7315$	0.047	4.01
BB4	$6.4e-1420$	$9.9e-5676$	0.031	4.00
BB5	$8.9e-1420$	$3.7e-5675$	0.047	4.01
MM1	$4.4e-1295$	$5.3e-5176$	0.016	4.00
MM2	$4.9e-1569$	$1.3e-6272$	0.031	4.00
MM3	$1.7e-1458$	$4.7e-5830$	0.063	4.01
MM4	$1.0e-1458$	$4.0e-5830$	0.032	4.00

Example 3. (for uniform beam design, see [34]).

Figure 15 shows a uniform beam subject to a linearly increasing distributed load. The nonlinear equation for the resulting elastic curve is

$$f(s) = \frac{w_0}{120EIL}\left(-s^5 + 2L^2s^3 - L^4s\right). \tag{61}$$

Table 6: Simultaneous finding of all distinct roots.

$f_{2-1}(s) = \sin(s-1/2)\sin(s-2/2)\sin(s-2.5/2)$						
$\zeta_1 = 1, \zeta_2 = 2, \zeta_3 = 2.5,$						
$\overset{(0)}{s}_1 = -0.2, \overset{(0)}{s}_2 = 1.7, \overset{(0)}{s}_3 = 3$						
Method	CO	CPU	n	$e1$	$e2$	$e3$
PJ6M	6	0.047	4	0.019	$4.6e-3$	$9.9e-10$
M1–M5	6	0.042	4	$6.9e-13$	$3.0e-5$	0.0

Table 7: Simultaneous finding of all multiple roots.

$f_2(s) = \sin^3(s-1/2)\sin^3(s-2/2)\sin^3(s-2.5/2)$						
$\zeta_1 = 1, \zeta_2 = 2, \zeta_3 = 2.5$, each with multiplicity 3						
$\overset{(0)}{s}_1 = -0.2, \overset{(0)}{s}_2 = 1.7, \overset{(0)}{s}_3 = 3$						
Method	CO	CPU	n	$e1$	$e2$	$e3$
PJ6M	6	0.094	4	$5.0e-5$	$1.7e-2$	$3.3e-5$
M1–M5	6	0.062	4	0.0	0.0	0.0

Table 8: Comparison of optimal fourth-order methods.

Methods	$\lvert s_i^{(6)} - \alpha\rvert$	$\lvert f_i^{(6)}(s_i)\rvert$	CPU	ρ
$f_{2-1}(s) = \sin^3(s-1/2)\sin^3(s-2/2)\sin^3(s-2.5/2), s_0 0.5$				
BB1	$5.0e-573$	$4.8e-2003$	1.516	4.00
BB2	$3.0e-501$	$8.4e-2291$	1.125	4.00
BB3	$1.5e-572$	$6.8e-2289$	1.125	4.00
BB4	$8.8e-506$	$2.9e-2021$	1.531	4.00
BB5	$1.2e-505$	$1.3e-2020$	1.409	4.00
MM1	$4.9e-462$	$5.1e-1846$	1.406	4.00
MM2	$1.0e-563$	$2.7e-2145$	1.657	4.00
MM3	$9.7e-516$	$3.5e-2061$	1.421	4.00
MM4	$9.7e-516$	$3.5e-060$	1.546	4.00

Table 9: Simultaneous finding of all distinct roots.

$f_3(s) = (2.5/120 * 50000 * 30000 * 600)(-5s^4 + 2160000s^2 - (600)^4)$							
$\zeta_1 = -600, \zeta_2 = -268.328, \zeta_3 = 268.328, \zeta_4 = 600.$							
$\overset{(0)}{s}_1 = -590, \overset{(0)}{s}_2 = -250, \overset{(0)}{s}_3 = 255 \overset{(0)}{s}_4 = 590$							
Method	CO	CPU	n	$e1$	$e2$	$e3$	$e4$
PJ6M	6	0.031	4	$2.8e-3$	$2.3e-6$	$1.2e-6$	$7.2e-8$
M1–M5	6	0.015	4	0.0	$1.1e-30$	$1.1e-30$	0.0

We have to determine the point of maximum deflection which is the value of s where $f'(s) = 0$, i.e.,

$$\frac{w_0}{120EIL}\left(-5s^4 + 6L^2s^3 - L^4\right) = 0$$
$$\text{or } \frac{w_0}{120EIL}\left(-5s^4 + 6L^2s^3 - L^4\right) \tag{62}$$

where $L = 600\,\text{cm}$, $E = 50,000\text{KN/cm}^3$, $I = 30,000\,\text{cm}^4$, and $w_0 = 2.5\,\text{KN/cm}$.

Table 10: Comparison of optimal fourth-order methods.

Method	$\lvert s_i^{(6)} - \alpha\rvert$	$\lvert f_i^{(6)}(s_i)\rvert$	CPU	ρ
$f_3(s) = (2.5/120*50000*30000*600)(-5s^4 + 2160000s^2 - (600)^4)$, $s = -590$				
BB1	$5.7e-1050$	$3.1e-4208$	0.031	4.00
BB2	$3.1e-1332$	$1.4e-5338$	0.040	4.00
BB3	$3.6e-1342$	$2.5e-5378$	0.062	4.00
BB4	$1.5e-1060$	$1.3e-4250$	0.045	4.00
BB5	$1.0e-1059$	$2.6e-4247$	0.031	4.00
MM1	$5.4e-924$	$4.7e-3704$	0.047	4.00
MM2	$1.8e-1229$	$6.1e-4926$	1.734	4.00
MM3	$9.3e-1107$	$1.4e-4435$	0.046	4.00
MM4	$9.3e-1107$	$1.4e-4435$	0.032	4.00

Table 11: Simultaneous finding of all distinct roots.

$f_4(s) = s^4 + 11.50s^3 + 47.49s^2 + 83.06325s + 51.23266875$						
$\zeta_1 = -1.45, \zeta_2 = -2.85, \zeta_3 = -2.85, \zeta_4 = -4.45$						
$\overset{(0)}{s}_1 = -1.0, \overset{(0)}{s}_2 = -2.1, s_3^{(0)} = -1.8, s_4^{(0)} = -3.9$						
Method	CO	CPU	n	$e\ 1$	$e\ 2$	$e\ 3$	$e\ 4$
PJ6M	6	0.047	3	$9.0e-3$	$2.0e-3$	0.4	$2.0e-3$
M1–M5	6	0.032	3	0.0	$3.6e-7$	$2.4e-7$	0.0

Table 12: Comparison of optimal fourth-order methods.

Method	$\lvert s_i^{(6)} - \alpha\rvert$	$\lvert f_i^{(6)}(s_i)\rvert$	CPU	ρ
$f_3(s) = s^4 + 11.50s^3 + 47.49s^2 + 83.06325s + 51.23266875, s_0 = -4.33$				
BB1	$65e-1239$	$3.2e-4951$	0.031	4.00
BB2	$1.5e-1551$	$5.1e-6203$	0.062	4.13
BB3	$1.3e-1556$	$3.4e-6223$	0.047	4.49
BB4	$2.6e-1256$	$1.4e-5021$	0.045	4.49
BB5	5.6e−1257	$1.4e-5019$	0.031	4.01
MM1	$7.2e-1125$	$9.1e-5679$	0.047	4.00
MM2	$8.9e-1299$	$3.3e-5192$	0.035	4.01
MM3	$5.2e-1299$	$8.3e-5192$	0.046	4.11
MM4	$5.7e-1299$	$3.3e-5192$	0.047	4.00

The exact roots of (62) are $\zeta_1 = -600, \zeta_2 = -268.328, \zeta_3 = 268.328$, and $\zeta_4 = 600$ as shown in Figure 16. The initial estimates for $f_3(s)$ have been taken as

$$\begin{aligned} s_{1-}^{(0)} &= -590, \\ s_2^{(0)} &= -250, \\ s_3^{(0)} &= 255, \\ s_4^{(0)} &= 590. \end{aligned} \tag{63}$$

Example 4 (for continuous stirred tank reactor (CSTR), see [35]). We consider the isothermal stirred tank reactor (CSTR). Items A and R are fed to the reactor at rates of Q and

Figure 12: Continued.

Figure 12: (a–d) Residual fall of simultaneous iterative methods (M1–M5 and PJ6M). (e–h) Residual fall of iterative methods (BB1–BB5 and MM1–MM4) for nonlinear function $f_1(s)$, $f_2(s)$, $f_3(s)$, and $f_4(s)$, respectively.

Figure 13: Exact roots of nonlinear equation $f_1(s)$.

Figure 14: Exact roots of nonlinear equation $f_2(s)$.

Figure 15 (a) Uniform beam design subject to a linearly increasing distributed load. (b) Resulting elastic curve.

Figure 16: Exact roots of nonlinear equation $f(s)$.

Figure 17: Exact roots of nonlinear equation $f_4(s)$.

q-Q, respectively. The following complex reaction system develops in the reactor (see [35]):

$$\begin{aligned} A + R &\longrightarrow B, \\ B + R &\longrightarrow C, \\ C + R &\longrightarrow D, \\ C + R &\longrightarrow E. \end{aligned} \tag{64}$$

This problem was tested by Douglas (see [36]) in order to construct the simple feedback control systems. In his searching of this system, he designed the following equation for the transfer function of the reactor with a proportional control system:

$$H_c \frac{2.98(s + 2.25)}{(s + 1.45)(s + 2.85)^2 (s + 4.35)} = -1, \tag{65}$$

with H_c being the gain of the proportional controller. This control system is stable for values of H_c that yield roots of the transfer function having negative real parts. If we take $H_c = 0$, we get the nonlinear equation:

$$f_4(s) = s^4 + 11.50s^3 + 47.49s^2 + 83.06325s + 51.23266875 = 0. \tag{66}$$

The transfer function has four negative real roots as shown in Figure 17 that are

$$\begin{aligned} s_1 &= -1.45, \\ s_2 &= -2.85, \\ s_3 &= -2.85, \\ s_4 &= -4.45. \end{aligned} \tag{67}$$

These roots are known as poles of the open-loop transfer function.

The initial guessed values have been taken as

$$\begin{aligned} s_1^{(0)} &= -1.0, \\ s_2^{(0)} &= -2.1, \\ s_3^{(0)} &= -1.8, \\ s_3^{(0)} &= -3.9. \end{aligned} \tag{68}$$

7. Conclusion

Here, we have developed five families of two-step single-root finding methods of optimal convergence order four and five simultaneous iterative methods of order six, respectively. From Tables 3–10 and Figures 9, 10, and 12, we observe that our methods (BB1–BB5 and M1–M5) are superior in terms of efficiency, stability, CPU time, and residual error as compared with the methods MM1–MM4 and PJ6M, respectively [37].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally to the preparation of this manuscript.

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Research Article

Balancing Control of Two-Wheel Bicycle Problems

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In recent years, more and more scientists have been interested in research on driving two-wheel bicycles. The problems in two-wheel bicycle control problem are self-balancing, uncertain models, and the impact of noise. In the paper, to solve the self-balancing problem, we use the flywheel method according to the inverted pendulum principle. To overcome the effects of the uncertain model, the impact of noise, we designed the vehicle balance controller according to the robust control algorithm. However, robust controllers often have a high order, which affects the quality during real control. To simplify the robust controller, we propose the use of a model order reduction algorithm. The simulation and experimental results have proved the correctness of the solutions given in the paper.

1. Introduction

Although two-wheel bicycles have been invented for a long time, the dynamics and the way to control the balance of two-wheel bikes are still exciting and mysterious topics for researchers. In particular, a difficult problem is controlling the balance of a two-wheel bicycle. To solve the problem of balancing the two-wheel bike, there are three typical methods as follows:

(i) Controlling balance by controlling the steering angle or using centrifugal force as in the research of Tanaka and Murakami [1], Chen and Dao [2–4], Huang et al. [5], Vatanashevanopakorn Parnichkun [6], and Wang et al. [7]

(ii) Controlling balance by changing the center of mass such as Lee and Ham [8], Keo et al. [9], and Yamakita et al. [10]

(iii) Controlling balance by flywheel as in the studies of Suebsomran [11], Sikander and Prasad [12], Beznos et al. [13], Hwang et al. [14], Karnopp [15], Gallaspy [16], Xu et al. [17], Lam [18, 19], Lot and Fleming [20], Park and Yi [21], Stephen and Girard [22], Lee et al. [23], Suprapto [24], and Kim et al. [25]

The control method using centrifugal force has the disadvantage of not being able to control bicycle balance when the bicycle is stationary. The control method by changing the center of mass requires more weight, increasing the bicycle weight, while the response time of the system is slow. The control method using the flywheel has the advantage of being fast responsive and balanced even when the bicycle is not moving. Two-wheeled bicycle models using flywheels are based on the principle of gyroscope [11–16, 18–22, 24] to create a balanced moment for the bicycle; the flywheel usually has to rotate at high speed, so the flywheel dissipates a large amount of energy. The typical work of this group is the research of Lam [18, 19]; the author has built a self-balancing two-wheel bicycle based on a small bicycle model using flywheels operating under gyroscope principle. Simulation and experimental results prove that the vehicle can maintain balance when moving and is affected by external forces.

In contrast, two-wheel bicycle models use flywheels based on the principle of inverted pendulum [17, 23, 25]; to create a balanced torque for the bicycle, the flywheel usually rotates only with low speed, so the flywheel will dissipate a small amount of energy. This group's typical work is research [25]; the author has built a self-balancing two-wheeled

model based on the bike's chassis. The flywheel operates on the principle of inverted pendulum providing the ability to control the vehicle balance vertically when the vehicle is stationary as well as when the vehicle is moving. The vehicle model's experimental results show the vehicle's ability to balance when it moves in a curve. Two-wheel bicycle models are usually powered by a battery with limited capacity, so the two-wheel bicycle model using flywheels based on the principle of the inverted pendulum will save energy than the two-wheel bicycle model using the flywheel based on the principle of a gyroscope. Therefore, in this paper, we choose to build a self-balancing two-wheel bicycle model using flywheel based on the principle of an inverted pendulum.

There are many proposed balance control algorithms for two-wheel bicycle, such as nonlinear control of Lee and Ham [8], Beznos [13], compensator design using root locus approach of Gallaspy [16], PD control of Surpato [19, 24], PID control of Lee et al. [23], and fuzzy control of Chen and Dao [2–4]. However, these control algorithms are not robust; the two-wheel bicycle model cannot carry loads with varied weights and cannot work in disturbance environments. Therefore, a robust control algorithm, as in [12, 26], is suitable for the two-wheel bicycle model's real applications. However, the controller is designed according to RH_∞ robust control algorithm [27–38] which usually has high order (order of controller is defined as the order of the sample polynomial). The high-order controller has many disadvantages when we take control of the two-wheel bicycle model because the program code of complexity and calculation in a long time leads to a long system's response. Therefore, the reduced order of the controller, while ensuring quality, is a practical significance.

To obtain a low-order controller, we have two different methods. (i) Select a fixed structure of a low-order controller and then apply the optimal algorithm to find the low-order controller parameters so that the low-order controller still maintains robust control standards. (ii) Designing a robust controller for an uncertain object will obtain a high-order controller and then apply order reduction algorithms to reduce the order of a high-order controller to obtain a low-order controller. In our opinion, due to having to solve two optimization problems at the same time (robust control problem and the problem of finding parameters of the low-order controller), the first method [12, 26] has high complexity. If the choice of the low-order controller's structure is not appropriate, it may not be possible to determine the low-order controller (the optimal problem without solutions). With the second method, the order reduction problem is independent, and there will always be solutions like in [39], so to always find low-order controllers in all cases, the second method has the advantage of the first method.

In this paper, we focus our research on designing a low-order robust controller for the control problems of self-balancing two-wheel vehicles in two steps:

(i) Step 1: Design a RH_∞ robust controller to control the balance of two-wheel bicycle; the finder controller is called a full-order controller.

(ii) Step 2: Propose order reduction algorithm, using the proposed algorithm to reduce the order of the RH_∞ full-order controller to obtain a low-order controller while ensuring quality. The meaning of this order reduction is the reduced response time of the system.

The rest of this paper is organized as follows. The system composition of the two-wheel bicycle is described in Section 2.1, Part 2. Modeling two-wheel bike is presented in Section 2.2, Part 2. The balanced controller design is introduced in Section 2.3, Part 2. In Part 3, the model order reduction algorithm is presented. In Part 4, the model order reduction algorithm is applied to reduce the order of the controller, which is designed in Section 2.3, Part 2. Experimental results of two-wheel bicycle control are presented in Part 5. Then, the conclusion of this research is described in Part 6.

2. Balance Control Two-Wheel Bicycle Problem

2.1. Dynamic Model for Two-Wheel Bicycle. The two-wheel bicycle model is based on the principle of balance using flywheel under the inverted pendulum principle. Briefly, the balance principle is described as follows: without external torque effects on an object or system (or total torque impact to an object is zero), the total angular momentum of that object will be preserved. A bicycle is moved by two wheels. When deflected from the equilibrium position (corresponding to a vertical angle), the gravity of the bike will create a torque leading it to pour down. To maintain equilibrium in general, we put on the bike a flywheel operating based on the principle of "inverted pendulum." This flywheel will revolve around the axis (with the angular acceleration being a) and generate a torque that is balanced with the torque caused by the gravity the bicycle created. To control the flywheel's acceleration, we use a DC motor with the voltage U applied to the motor, and then the balance control problem will be put on the angle control problem (output) by controlling the voltage U (input) applied to the DC motor. The mission is set to design a controller to keep the bicycle balanced, that is, keeping the angle (output) always at zero. A detailed description of the self-balancing two-wheel bicycle constitutes in Figure 1.

Detailed parameters of the mechanical part of the bicycle are given as follows: bicycle 1,19 m long, 0,5 m high, 0,4 m wide, flywheel weighing 3,976 kg with a diameter of 0,26 m, to spin the flywheel using DC motor 100w-15v-3400 rev/min + H-bridge circuit, measuring speed of flywheel using Encoder Sharp 100 pulses, measuring tilt using angle sensor GY-521 MPU-6050, the backward and forward system including a DC motor, H-bridge, and the remote control. The entire hardware system is connected to circuit Adruino Uno under the following schematic structure in Figure 2.

2.2. Modeling Self-Balancing Two-Wheel Bicycle. The two-wheel bicycle model is shown in the Figure 3, in which m_1 is the weight of the bicycle (including the DC motor), m_2 is the weight of the flywheel, h_1 is the height of the center of the gravity of the bicycle (excluding the flywheel), h_2 is the height of the center of the gravity of the flywheel, I_1 is the moment of inertia of the bicycle, I_2 is the moment of inertia

Figure 1: Detailed model of self-balancing two-wheel bicycle.

Figure 2: Schematic structure of bicycle controller.

Figure 3: Self-balancing two-wheel bicycle model.

of the flywheel, θ is the tilt angle of the bicycle compared to the vertical, and φ is the angle of rotation of the flywheel.

We have the following:

Absolute velocity of A is $|v_A| = h_1\dot{\theta}$.

Absolute velocity of B is $|v_B| = h_2\dot{\theta}$.

In the study [16], we use the Lagrange equations to build dynamic models of the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial T}{\partial \dot{q}_i}\right\} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i. \tag{1}$$

T is the total kinetic energy of the system, V is the total potential energy of the system, Q_i is external forces, and q_i is generality coordinates.

Total the kinetic energy of the system is defined as $T = T_1 + T_2$.

T_1 is the total kinetic energy of bicycle and is defined by the following formula: $T_1 = (1/2)m_2|v_B|^2 + (1/2)I_1\dot{\theta}^2$; T_2 is the total potential energy of flywheel and is defined by the following formula: $T_2 = (1/2)m_2|v_B|^2 + (1/2)I_2(\dot{\theta} + t\dot{\varphi})$.

From here, we have

$$T = \frac{1}{2}m_1|v_A|^2 + \frac{1}{2}m_2|v_B|^2 + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\dot{\theta}^2 + \frac{1}{2}I_2\dot{\varphi}^2 + I_2\dot{\varphi}\dot{\theta}, \tag{2}$$

$$\Longrightarrow T = \frac{1}{2}\left(m_1h_1^2 + m_2h_2^2 + I_1 + I_2\right)\dot{\theta}^2 + \frac{1}{2}I_2\dot{\varphi}^2 + I_2\dot{\varphi}\dot{\theta}. \tag{3}$$

The total potential energy of the system is

$$V = g \cdot \cos\theta \cdot (m_1h_1 + m_2h_2), \tag{4}$$

with $q_i = \theta$, using formulas (1)–(4), we obtain the following formula:

$$\left(m_1h_1^2 + m_2h_2^2 + I_1 + I_2\right)\ddot{\theta} + I_2\ddot{\varphi} - g \cdot \sin\theta \cdot (m_1h_1 + m_2h_2) = 0, \tag{5}$$

with $q_i = \varphi$, using formulas (1)–(4), we obtain the following formula:

$$I_2\ddot{\varphi} + I_2\ddot{\theta} = T_m, \tag{6}$$

with T_m being torque if the motor shaft.

Consider a DC electric motor with a transmission ratio being a:1; the (torsion) torque of DC motor drives for the flywheel as follows:

$$T_m = aK_m\left[\frac{U - K_e\dot{\varphi}}{R}\right], \tag{7}$$

where K_m is the torque constant of the motor; K_e is the electromotive force constant of the motor; R is the resistance of the motor.

Replace (7) with (6); we have

$$I_2\ddot{\varphi} + I_2\ddot{\theta} = T_m = aK_m\left[\frac{U - K_e\dot{\varphi}}{R}\right]. \tag{8}$$

Equations (5) and (8) are the kinematic equations of the system. Obviously, with the above aerodynamics equations, the system is nonlinear.

2.3. Linearizing Model and Moving to the Form of the State-Space Model. Assuming that when the bicycle is operating, its angle is very small ($\theta < 10^0$); we linearized (5) around the equilibrium ($\theta = \varphi = 0$, $\sin\theta = \theta$) and obtained the following equations:

Table 1: The parameters of the self-balancing two-wheel bicycle.

Parameters	Value	Unit
I_1	0,1105	kg·m^2
h_1	0,105	M
I_2	0,03289	kg·m^2
h_2	0,205	M
m_1	10,024	kg
m_2	3,976	kg
K_e	0,045	V·s/rad
K_m	0,045	Nm/A
R	0,52	Ω
a	1 : 1	
g	9,81	m/s^2

$$\begin{aligned}
&\left(m_1h_1^2 + m_2h_2^2 + I_1 + I_2\right)\ddot{\theta} + I_2\ddot{\varphi} - g\cdot\theta\cdot\left(m_1h_1 + m_2h_2\right) = 0,\\
&I_2\ddot{\varphi} + I_2\ddot{\theta} = T_m\\
&\quad = aK_m\left[\frac{U - K_e\dot{\varphi}}{R}\right].
\end{aligned} \tag{9}$$

Set $A_1 = (m_1h_1^2 + m_2h_2^2 + I_1 + I_2)$; $B_1 = (m_1h_1 + m_2h_2)$.

Set $x = \begin{bmatrix} \theta = x_1 \\ \dot{\theta} = x_2 \\ \dot{\varphi} = x_3 \end{bmatrix}$ as state variables; $y = \theta, u = U$.

From here, we have the state system of equations describing the system as follows:

$$\begin{aligned}
\dot{x} &= \mathbf{A}x + \mathbf{B}u,\\
y &= \mathbf{C}x + \mathbf{D}u.
\end{aligned} \tag{10}$$

The parameters of the system are as follows:

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{B_1g}{(A_1 - I_2)} & 0 & \dfrac{aK_mK_e}{R(A_1 - I_2)} \\ -\dfrac{B_1g}{(A_1 - I_2)} & 0 & -aK_mK_e\dfrac{A_1}{I_2R(A_1 - I_2)} \end{bmatrix};\\
\mathbf{B} &= \begin{bmatrix} 0 \\ -\dfrac{aK_m}{R(A_1 - I_2)} \\ aK_m\dfrac{A_1}{I_2R(A_1 - I_2)} \end{bmatrix};\\
\mathbf{C} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix};\\
\mathbf{D} &= [0].
\end{aligned} \tag{11}$$

The nominal parameters of the two-wheel bicycle are shown in Table 1.

Replacing number into the system of (10), we obtained the parameters as follows:

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 0 & 1 & 0 \\ 47,2048 & 0 & 0,0100 \\ -47,2048 & 0 & -0,1248 \end{bmatrix},\\
\mathbf{B} &= \begin{bmatrix} 0 \\ -0,2230 \\ 2,8541 \end{bmatrix},\\
\mathbf{C} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{12}$$

Transfer bicycle model into a transfer function; we have results:

$$\mathbf{S}(s) = \frac{\boldsymbol{\theta}(s)}{\mathbf{U}(s)} = \frac{-0,223s}{s^3 + 0,1284s^2 - 47,2s - 5,589}. \tag{13}$$

2.4. Optimized Design RH_∞ for Balancing Two-Wheel Bicycle Problems. Commenting on the two-wheel bicycle model, modeling self-balancing two-wheel bicycle model shows that some parameters of a bicycle are uncertain, such as volume load changes leading to focus height of the bicycle and the moment of inertia of bicycle changes, ... simultaneous operation of the two-wheel bicycle model can be affected by uncertain external factors such as external forces, unknown noise due to the changing of moving topography, ... so that the two-wheel bicycle model is an uncertain object, in which we concerned the most to variability caused by the change of volume load and external force. Unknown factors may reduce the accuracy of the two-wheel bicycle model's mathematical model that would lead to lower controlling quality and even make the systems unstable. Because of the variability of the two-wheel bicycle model, robust control like in

Figure 4: Schematic structure of robust control system for self-balancing two-wheel bicycle model.

research [12, 26, 39] is the most suitable to control uncertain objects. Do design robust control system for self-balancing two-wheel bicycle model; we executed under the control structure diagram shown in Figure 4.

2.5. Building New Two-Wheel Bicycle Model $S_m(s)$. Assuming that when the bicycle is operating, its angle is very small; we linearized (5) around the equilibrium ($\theta = \varphi = 0$, $\sin\theta = \theta$) and obtained the following system of equations:

$$\begin{aligned}\left(m_1h_1^2 + m_2h_2^2 + I_1 + I_2\right)\ddot{\theta} + I_2\ddot{\varphi} - g\cdot\theta\cdot\left(m_1h_1 + m_2h_2\right) &= 0,\\ I_2\ddot{\varphi} + I_2\ddot{\theta} &= T_m\\ &= aK_m\left[\frac{U^* - \left(K_e + K_1\right)\dot{\varphi} + K_2\dot{\theta}}{R}\right].\end{aligned} \tag{14}$$

Set $A_1 = (m_1h_1^2 + m_2h_2^2 + I_1 + I_2)$; $B_1 = (m_1h_1 + m_2h_2)$.

Set $x = \begin{bmatrix}\theta = x_1\\ \dot{\theta} = x_2\\ \dot{\varphi} = x_3\end{bmatrix}$ as state variables; $y = \theta, u = U^*$.

From here, we have the status system of equations describing the system as follows:

$$\begin{aligned}\dot{x} &= \mathbf{A}x + \mathbf{B}u,\\ y &= \mathbf{C}x + \mathbf{D}u.\end{aligned} \tag{15}$$

The parameters of the system are as follows:

$$\begin{aligned}\mathbf{A} &= \begin{bmatrix}0 & 1 & 0\\ \dfrac{B_1g}{(A_1 - I_2)} & \dfrac{aK_mK_2}{R(A_1 - I_2)} & \dfrac{aK_m(K_e + K_1)}{R(A_1 - I_2)}\\ -\dfrac{B_1g}{(A_1 - I_2)} & -aK_mK_2\dfrac{A_1}{I_2R(A_1 - I_2)} & -aK_m(K_e + K_1)\dfrac{A_1}{I_2R(A_1 - I_2)}\end{bmatrix},\\ \mathbf{B} &= \begin{bmatrix}0\\ -\dfrac{aK_m}{R(A_1 - I_2)}\\ aK_m\dfrac{A_1}{I_2R(A_1 - I_2)}\end{bmatrix},\\ \mathbf{C} &= \begin{bmatrix}1 & 0 & 0\end{bmatrix},\\ \mathbf{D} &= [0].\end{aligned} \tag{16}$$

Select parameter $K_1 = 2$ and $K_2 = 5$ and replace the parameters in Table 1 in the system of (15); then transfer the bicycle model to form of transfer function obtaining the results:

$$\mathbf{S}_m(s) = \frac{\boldsymbol{\theta}(s)}{\mathbf{U}(s)} = \frac{-0,223s}{s^3 + 4,722s^2 - 47,2s - 254}. \tag{17}$$

To design a robust controller for self-balancing two-wheel bicycle model, we follow steps of the design RH_∞ robust controller under the documents [27–42]; the robust controller is obtained as follows:

$$\mathbf{R}(s) = \frac{\mathbf{H}(s)}{\mathbf{D}(s)}. \tag{18}$$

$$\begin{aligned}
\mathbf{H}(s) = {} & -2,23.10^{-7}s^{30} - 4,67.10^{-4}s^{29} - 0,266s^{28} - 22,96s^{27} - 1006s^{26} - 2,853.10^{4}s^{25} - 5,837.10^{5}s^{24} \\
& - 9,144.10^{6}s^{23} - 1,139.10^{8}s^{22} - 1,158.10^{9}s^{21} - 9,776.10^{9}s^{20} - 6,949.10^{10}s^{19} - 4,199.10^{11}s^{18} \\
& - 2,172.10^{12}s^{17} - 9,663.10^{12}s^{16} - 3,71.10^{13}s^{15} - 1,231.10^{14}s^{14} - 3,53.10^{14}s^{13} - 8,74.10^{14}s^{12} \\
& - 1,862.10^{15}s^{11} - 3,398.10^{15}s^{10} - 5,276.10^{15}s^{9} - 6,903.10^{15}s^{8} - 7,511.10^{15}s^{7} - 6,676.10^{15}s^{6} \\
& - 4,721.10^{15}s^{5} - 2,556.10^{15}s^{4} - 9,953.10^{14}s^{3} - 2,482.10^{14}s^{2} - 2,977.10^{13}s - 0,00439, \\
\mathbf{D}(s) = {} & 4,971.10^{-14}s^{30} + 2,032.10^{-10}s^{29} + 2,663.10^{-7}s^{28} + 1,221.10^{-4}s^{27} + 9,72.10^{-3}s^{26} \\
& + 0,3918s^{25} + 10,14s^{24} + 187.1s^{23} + 2612s^{22} + 2,862.10^{4}s^{21} + 1,088.10^{7}s^{18} + 2,523.10^{5}s^{20} \\
& + 1,82.10^{6}s^{19} + 5,428.10^{7}s^{17} + 2,273.10^{8}s^{16} + 8,005.10^{8}s^{15} + 2,372.10^{9}s^{14} + 5,9.10^{9}s^{13} \\
& + 1,225.10^{10}s^{12} + 2,107.10^{10}s^{11} + 2,962.10^{10}s^{10} + 3,341.10^{10}s^{9} + 2,941.10^{10}s^{8} + 1,931.10^{10}s^{7} \\
& + 8,743.10^{9}s^{6} + 2,286.10^{9}s^{5} + 1,519.10^{8}s^{4} - 5,226.10^{7}s^{3} + 3,6.10^{-6}s^{2} + 5,32.10^{-22}s.
\end{aligned} \tag{19}$$

When using the 30th-order robust controller (the original controller) to perform bicycle balance control, it will lead to a lot of unfavorable because the complex program codes will make the processing time increase and so speed of response of the control system is delayed and does not meet the real-time requirement of the controller and can cause the balance system instability. Therefore, to improve the quality of this controller, we should perform reducing order of controller for simpler the program code, to reduce processing time and to speed up a response that satisfies the requirement of sustainable stability of the system.

3. Model Order Reduction Algorithm for Unstable System

3.1. Problem of Model Order Reduction. A multiple inputs and outputs linear system is given with continuous-time constant parameters described in space stated in the following equations:

$$\begin{aligned}
\dot{x} &= \mathbf{A}x + \mathbf{B}u, \\
y &= \mathbf{C}x.
\end{aligned} \tag{20}$$

$x \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^q, \mathbf{A} \in \mathbb{R}^{nxn}, \mathbf{B} \in \mathbb{R}^{nxp}$, and $\mathbf{C} \in \mathbb{R}^{qxn}$.

The goal of the order reduction problem with the model described by (20) is to find models described by systems of equations:

$$\begin{aligned}
\dot{x}_r &= \mathbf{A}_r x_r + \mathbf{B}_r u, \\
y_r &= \mathbf{C}_r x_r.
\end{aligned} \tag{21}$$

$x_r \in \mathbb{R}^r, u_r \in \mathbb{R}^p, y_r \in \mathbb{R}^q, \mathbf{A}_r \in \mathbb{R}^{rxr}, \mathbf{B}_r \in \mathbb{R}^{rxp}, \mathbf{C}_r \in \mathbb{R}^{qxr}$ with $r \ll n$, so that the model described by (21) can be replaced by the model described in (20) to be applied in analysis, design, and control system.

3.2. Model Order Reduction Algorithm. In this section, we introduce a model order reduction algorithm for unstable system basic approaches as follows: Separation of unstable system $\mathbf{G}(s)$ into two parts is as follows: $\mathbf{G}(s) = \mathbf{G}^+(s) + \mathbf{G}^-(s)$ in which $\mathbf{G}^+(s)$ is unstable part and $\mathbf{G}^-(s)$ is stable part. Then performing reducing order for stable part $\mathbf{G}^-(s)$ obtained the stable compact system $\widehat{G}^-(s)$, resulting in reduced order of the unstable system $\mathbf{G}(s)$ which will be $\widehat{G}(s) = \widehat{G}^-(s) + \mathbf{G}^+(s)$.

The details of the algorithm are as follows:

Input: the system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is described as (20) (unstable system).

Step 1: decompose the unstable system into two subsystems: stable subsystem and unstable subsystem.

Step 1.1: transform this system to the triangle system; we obtain the system that has the form:

$$\mathbf{A}_t = \begin{bmatrix} \mathbf{A}_{t11} & \mathbf{A}_{t12} \\ 0 & \mathbf{A}_{t22} \end{bmatrix}, \mathbf{B}_t = \begin{bmatrix} \mathbf{B}_{t1} \\ \mathbf{B}_{t2} \end{bmatrix}, \mathbf{C}_t = \begin{bmatrix} \mathbf{C}_{t1} & \mathbf{C}_{t2} \end{bmatrix},$$

with $\mathbf{A}_{t11} \in \mathbb{R}^{m\times m}$ (m denotes the number of stable poles), $\mathbf{A}_{t12} \in \mathbb{C}^{m\times(n-m)}$, $\mathbf{A}_{t22} \in \mathbb{R}^{(n-m)\times(n-m)}$, $\mathbf{B}_{t1} \in \mathbb{C}^{m\times p}$, $\mathbf{B}_{t2} \in \mathbb{R}^{(n-m)\times p}$, $\mathbf{C}_{t1} \in \mathbb{R}^{q\times m}$, and $\mathbf{C}_{t2} \in \mathbb{R}^{q\times(n-m)}$.

Step 1.2: compute $\mathbf{S}$ from Lyapunov equation: $\mathbf{A}_{t11}\mathbf{S} - \mathbf{S}\mathbf{A}_{t22} + \mathbf{A}_{t12} = 0$.

Step 1.3: define a transition matrix

$$\mathbf{W} = \begin{bmatrix} \mathbf{I}_r & . & \mathbf{S} \\ \dots & . & \dots \\ 0 & . & \mathbf{I}_{n-r} \end{bmatrix},$$

where $\mathbf{I}_m$ and $\mathbf{I}_{n-m}$ are identity matrix of size $m \times m$ and $(n-m) \times (n-m)$.

Step 1.4: compute

$$(\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d) = (\mathbf{W}^{-1}\mathbf{A}_t\mathbf{W},\ \mathbf{W}^{-1}\mathbf{B}_t, \mathbf{C}_t\mathbf{W}).$$

Step 1.5: partition $(\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d)$ as
$\mathbf{A}_d = \begin{bmatrix} \mathbf{A}_{d11} & 0 \\ 0 & \mathbf{A}_{d22} \end{bmatrix}, \mathbf{B}_d = \begin{bmatrix} \mathbf{B}_{d1} \\ \mathbf{B}_{d2} \end{bmatrix}, \mathbf{C}_d = \begin{bmatrix} \mathbf{C}_{d1} & \mathbf{C}_{d2} \end{bmatrix}$,
with $\mathbf{A}_{d11} \in \mathbb{R}^{m\times m}$, $\mathbf{A}_{d22} \in \mathbb{R}^{(n-m)\times(n-m)}$
$\mathbf{B}_{d1} \in \mathbb{R}^{m\times p}$, $\mathbf{B}_{d2} \in \mathbb{R}^{(n-m)\times p}$, $\mathbf{C}_{d1} \in \mathbb{R}^{q\times m}$, $\mathbf{C}_{d2} \in \mathbb{R}^{q\times(n-m)}$,
stable subsystem $(\mathbf{A}_{d11}, \mathbf{B}_{d1}, \mathbf{C}_{d1})$,
unstable subsystem $(\mathbf{A}_{d22}, \mathbf{B}_{d2}, \mathbf{C}_{d2})$.

Step 2: triangle realization of the stable subsystem $(\mathbf{A}_{d11}, \mathbf{B}_{d1}, \mathbf{C}_{d1})$ in which $\mathbf{A}_{d11}$ has triangle form, according to the following steps:

Step 2.1: compute Schur decomposition of $\mathbf{A}_{d11}$: $\mathbf{A}_{d11} = \mathbf{U}\Delta\mathbf{U}^T$, where $\mathbf{U}$ is an unitary matrix and Δ is upper-triangle matrix.
Step 2.2: compute observability Gramian $\mathbf{Q}$ came from Lyapunov equation:

$$\Delta\mathbf{Q} + \mathbf{Q}\Delta + (\mathbf{CU})^T(\mathbf{CU}) = 0.$$

Step 2.3: compute Cholesky factorization of $\mathbf{Q}$: $\mathbf{Q} = \mathbf{R}^T\mathbf{R}$, where $\mathbf{R}$ is upper-triangle matrix.
Step 2.4: compute nonsingular transformation $\mathbf{T} = \mathbf{UR}^{-1}$.
Step 2.5: compute

$$(\widetilde{A}, t\widetilde{B}n, q\widetilde{C}) = (\mathbf{T}^{-1}\mathbf{A}_{d11}\mathbf{T}, \mathbf{T}^{-1}\mathbf{B}_{d1}, \mathbf{C}_{d1}\mathbf{T}).$$

Step 3: reorder the poles on the main diagonal of the upper-triangle matrix $\widetilde{A}$ by H_∞^-, H_2^-, or mixed-dominant index.

Input: the triangle realization $(\widetilde{A}, t\widetilde{B}n, q\widetilde{C})$.

Step 3.1: for each pole λ_i, with $i = 1, \ldots, n$ computing its H_∞-dominant index, $R_i = (\|\widetilde{C}_i\widetilde{B}_i\|_2/|\mathrm{Re}\lambda_i|)$ (or resp., its H_2 -dominant index, $S_i = \sqrt{\mathrm{trace}(\widetilde{B}_i^T\widetilde{B}_i)}$, or its mixed-dominant index, $J_i = \max\{R_i, S_i\}$).
Step 3.2: choose the largest H_∞-dominant index, R_{i_1}(resp., for H_2 -dominant index, mixed index).
Step 3.3: reorder the pole λ_{i_1} (and its conjugate $\overline{\lambda}_{i_1}$, if it appears) to the first position in the diagonal of $\widetilde{A}$ by unitary matrix $\mathbf{U}_1$:

$$\mathbf{U}_1^T\widetilde{A}\mathbf{U}_1 = \begin{bmatrix} \lambda_{i_1} & * & * & * & * \\ & \overline{\lambda}_{i_1} & * & * & * \\ & & * & * & * \\ & & & \ddots & \vdots \\ & & & & * \end{bmatrix}.$$

Step 3.4: compute a new equivalent realization $(\mathbf{U}_1^T\widetilde{A}\mathbf{U}_1, \mathbf{U}_1^T\widetilde{B}, \widetilde{C}\mathbf{U}_1)$.
Step 3.5: remove two first rows and columns of $(\mathbf{U}_1^T\widetilde{A}\mathbf{U}_1, \mathbf{U}_1^T\widetilde{B}, \widetilde{C}\mathbf{U}_1)$ to obtain a smaller realization $(\widehat{A}, t\widehat{B}n, q\widehat{C})$ with $n-2$ dimensions.
Step 3.6: do the same procedure from Step 1 to Step 5 for smaller realization $(\widehat{A}, t\widehat{B}n, q\widehat{C})$ and continue this loop until all poles are reordered.

Output: the equivalent system $(\breve{A}, t\breve{B}n, q\breve{C})$ with the poles arranged in descending of H_∞-, H_2-, and mixed-dominant index on the main diagonal of the upper-triangle matrix $\breve{A}$.

Step 4: reduced equivalent system $(\breve{A}, t\breve{B}n, q\breve{C})$.

Input: the equivalent system $(\breve{A}, t\breve{B}n, q\breve{C})$.

Step 4.1: choose reorder r so that $r << n$.
Step 4.2: $(\breve{A}, t\breve{B}n, q\breve{C})$ is partitioned as follows:

$$\breve{A} = \begin{bmatrix} \breve{A}_{11} & \breve{A}_{12} \\ 0 & \breve{A}_{22} \end{bmatrix}, \breve{B} = \begin{bmatrix} \breve{B}_1 \\ \breve{B}_2 \end{bmatrix}, \breve{C} = \begin{bmatrix} \breve{C}_1 & \breve{C}_2 \end{bmatrix},$$

where $\breve{A}_{11} \in \mathbb{R}^{rxr}, \breve{B}_1 \in \mathbb{R}^{rxp}, \breve{C}_1 \in \mathbb{R}^{qxr}$.

Output: the reduced system $(\breve{A}_{11}, \breve{B}_1, \breve{C}_1)$.

Step 5: compute reduced system $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r) = (\breve{A}_{11}, \breve{B}_1, \breve{C}_1) + (\mathbf{A}_{d22}, \mathbf{B}_{d2}, \mathbf{C}_{d2})$.

Output: the reduced system $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$.

4. Apply MOR Algorithm in Balance Control Two-Wheel Bicycle Problem

4.1. The Result of Order Reduction Controller of Two-Wheel Bicycle. The robust controller is designed as (18), which is the 30th-order controller (the original controller). Performing order reduction of the robust controller under the model order reduction algorithm proposed in Part 3, we obtain the results as in Table 2.

4.2. Applying the 5th- and 4th-Order Reduction Controller to Control Balance Two Wheelers. Use the 5th- and 4th-order reduction controller in Table 2 to control the balance system for the two-wheel bicycle having control object model as (17). To see the quality, we compare it with the original controller (30th-order controller).

Performing simulation with the parameter of two-wheel bicycle model is nominal and the bicycle is initial offset from a vertical angle $\theta = \pi/180\,(\mathrm{rad})$. The simulation is based on the Matlab/Simulink. Simulation results are shown in Figure 5.

When the bicycle carries load $m_t = 7\,\mathrm{kg}, h_1 = 0,155\,\mathrm{m}$, and $I_1 = 0,409\,\mathrm{kg{\cdot}m^2}$, it is initial offset from the vertical an angle $\theta = \pi/180\,(\mathrm{rad})$. The simulation results are shown in Figure 6.

Compare the result of the control system balancing two-wheel bicycle using the original controller, order reduction controller under the new model order reduction algorithm, and other model order reduction algorithms. We chose to compare the new model order reduction algorithm with the balanced truncation algorithm of Moore [43]; this is the most used model order reduction algorithm. Do perform order reduction under the balanced truncation algorithm in Matlab we use ***balancmr*** command, resulting in a 4th-order reduction controller as follows:

$$W_{\mathrm{cr}}(s) = \frac{-4,485.10^6 s^4 - 5,351.10^8 s^3 + 7,513.10^7 s^2 + 2,822.10^7 s + 1,307.10^7}{s^4 + 2000s^3 - 206,5s^2 + 1,258.10^{-10}s - 4,767.10^{-12}}. \tag{22}$$

Table 2: Results the order reduction of 30th-order controller.

Order	Reduced system-$W_{cr}(s)$
5	$(-4,485.10^6 s^5 - 6,804.10^7 s^4 - 4,123.10^8 s^3 - 1,235.10^9 s^2 - 1,816.10^9 s - 1,09.10^9)/$ $(s^5 + 2009s^4 + 1,833.10^4 s^3 - 1913s^2 + 2,165.10^{-13} s - 2,804.10^{-14})$
4	$(-4,485.10^6 s^4 - 2,65.10^7 s^3 - 1,141.10^8 s^2 - 1,833.10^8 s - 1,176.10^8)/$ $(s^4 + 2000s^3 - 206,5s^2 + 2,369.10^{-14} s - 3,026.10^{-15})$

Figure 5: The output response of the balanced control system of the two-wheel bicycle, while the bicycle is not carrying load.

Figure 6: The output response of the control system balancing two-wheel bicycle, while the bicycle is carrying load.

Doing simulation with parameters of the two-wheel bicycle model is nominal, and it is initial offset from the vertical an angle $\theta = \pi/180$ (rad). The simulation results are shown in Figure 7.

4.3. Reviews Result. Using 4th- and 5th-order reduction controller under the new algorithm can control balance for two-wheel bicycle model, in which the output response of 5th-order reduction controller absolutely fits with the original controller, and the output response of the 4th-order reduction controller is different from the original system. Comparing the quality of control system balancing two-wheel bicycle model using 4th-order reduction controller according to the new algorithm with the 4th-order reduction controller according to the balanced truncation algorithm, we find that the control system balancing two-wheel bicycle model using the 4th-order reduction controller under the new algorithm ensures stable equilibrium model, two-wheel bicycle model, while the bicycle is offset from the vertical and the parameters of the model change. Balance control system using the 4th-order reduction controller under the balanced truncation algorithm does not ensure stable for two-wheel bicycle model, while the bicycle is offset from the vertical and the parameters of the model change.

5. Experiment Results

To determine the results of empirical control self-balancing two-wheel bicycle, we follow these steps:

(1) Start the PC + Run the Matlab-Simulink program.

(2) Open the Simulink file.

(3) Connect the Adruino motherboard in self-balancing two-wheel bicycle with PC through I/O USB.

(4) Turn on the power supply for the two-wheel model.

(5) Build a Simulink model; this process will convert the Simulink model into the sdf form (describing the system file) and save it in the processor's mother-board Adruino.

(6) Run the Simulink file to control self-balancing two-wheel bicycle.

The experiment results as follows:

The operation of the two-wheel bicycle while not carrying a load is shown in Figures 8 and 9. As can be seen from

Figure 7: The output response of the control system balancing two-wheel bicycle using 4th-order reduction controller.

Figure 8: The output response of the self-balancing two-wheel bicycle while not carrying load.

(a) (b)

Figure 9: Self-balancing two-wheel bicycle while not carrying load. (a) View from the left. (b) View from the right.

these figures, when the bicycle does not carry a load, the control system keeps the tilt angle of bicycle balance around the angle of 0 degrees (from $-0{,}1 \div 0{,}1$ degree).

When the bicycle carries load with the volume load of $m_t = 1,25$ kg, putting on the height of 0,25 m is shown in the Figures 10 and 11. From these figures, we can see that when the bicycle carries a load (both sides' balance), after a short-time fluctuation (approximately 10 seconds) with a tilt angle margin of $-1{,}1 \div 0{,}8$ degree, the bicycle control system keeps its tilt angle around the 0 degrees.

When the vehicle carries an eccentric load with a weight of $m_t = 1,25$ kg, set at a height of 0,25 m, the result is shown in Figure 12. The figure shows that when the vehicle carries an eccentric load, after a short time of oscillation (about 15 seconds), with the amplitude of inclination angle ranging from 0 to 1,5 degrees, the control system maintains the vehicle balance at an angle of 1 degree (from 0,8 to 1,1 degrees). The new balanced tilt angle (different from the 0-degree angle) is the position that ensures the vehicle is balanced on both sides compared to the vertical.

When the vehicle carries an eccentric load with a weight of $m_t = 2,2$ kg, set at a height of 0,25 m, the result shown in Figure 13. The figure depicts that when the vehicle carries an eccentric load, after a short time of oscillation (about 9,5 seconds), with the amplitude of inclination angle ranging between 1,475 and $-3{,}75$ degrees, the control system

Figure 10: The output response of the self-balancing two-wheel bicycle while carrying load.

(a) (b)

Figure 11: Self-balancing two-wheel bicycle while carrying load. (a) View from the left. (b) View from the right.

Figure 12: Output response when the vehicle carries an eccentric load.

Figure 13: Output response when the vehicle carries an eccentric load.

maintains the vehicle balance at an angle of −1,88 degree. The new balanced tilt angle (different from the 0-degree angle) is the position that ensures the vehicle is balanced on both sides compared to the vertical.

Figure 14: Output response when the vehicle overcomes obstacles.

When the vehicle overcomes obstacles, the result is shown in Figure 14. As we can see from this figure, when the vehicle overcomes obstacles, the vehicle's tilt angle ranges between 1,43 and −1,37 degrees; then the control system maintains the vehicle balance at an angle of −0,2 degree. The new balanced tilt angle (different from the 0-degree angle) is the position that ensures the vehicle is balanced on both sides compared to the vertical.

6. Conclusions

The paper has built, modeled, and designed a robust controller for self-balancing two-wheel bicycle. At the same

time, the article introduced order reduction algorithms for stable and unstable systems. Applying this algorithm to reduce the order of the 30th-order robust controller of the control system balancing two-wheel bicycle showed that the 4th- and 5th-order controller can be used to replace the 30th-order controller that the system quality guarantees. Using the 4th- and 5th-order controller will create a simpler code of the program, higher calculation speed, and shorter processing duration and ensure the calculation of real-time self-balancing two-wheel bicycle control system. The simulation results and experiment results show the correctness of the proposed algorithms.

Data Availability

This publication is supported by multiple datasets, which are available at locations cited in the reference section.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Supplementary Materials

The supplementary materials consist of two appendices to explain in more detail the review question. Appendix 1: model of self-balancing two-wheel vehicle. Appendix 2: determination of the parameters of a two-wheel vehicle model. (*Supplementary Materials*)

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Research Article

Effect of Surface Roughness during Peristaltic Movement in a Nonuniform Channel

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In this study, the effect of the roughness parameter during the peristaltic transport of a Newtonian fluid in a nonuniform channel has been explored. The motivation of this study comes from various research studies in the area of life sciences and engineering, which reveal that the wall of living beings' arteries and all other surfaces have roughness to some extent. As peristalsis is a major mode of transporting biological fluids in various organs, the effect of surface roughness during peristaltic flow becomes very significant. The problem of peristaltic motion of a Newtonian fluid through a rough nonuniform channel having sinusoidal-shaped roughness has been investigated in the current work. To analyze the flow, analytic formulation of pressure rise, friction force, velocity, and pressure gradient has been carried out under the low Reynolds number and long-wavelength approximation. Results obtained for zero surface roughness from the current model are in complete agreement with previous studies available in the literature that have been carried out without considering the surface roughness of the wall. Numerical outcomes for the properties mentioned above have been plotted for analyzing the impact of roughness on the physical and flow parameters.

1. Introduction

For about five decades, peristaltic transport of fluids has captured the interest of researchers because of its physiological and industrial uses; as a result, extensive research has been undertaken in this area to interpret the process of peristaltic pumping in engineering and physiological applications. Peristaltic transport involves material (slurry, solid-liquid mixture, or two immiscible liquids) transfer from one point to another using a continuous wave of area contraction and expansion along the length of an expandable duct or tube. The movement of matter in many body organs such as food in the gastrointestinal tract, urine transport to the bladder by the ureter, fluid in the cervical canal, ova transportation from the ovary to the uterus, fluid flow in the ductus efferentes, lymph movement in lymphatic vessels, and the flow of blood in small blood arteries involves peristaltic transport. Many practical applications such as a heart-lung machine, which is used during bypass surgery, and hemodialysis system involve pumping of fluid using the above mechanism. Also, this pumping mechanism is used in the pumping of corrosive chemicals, solidus liquids, slurry, and pumping of such fluids where isolation of the fluid from the moving parts of the pump is required.

To understand the mechanism of peristaltic transport, several analytical and empirical studies have been undertaken; the first in the series of studies conducted was that by Latham [1]. Shapiro et al. [2] elaborated the mechanism of peristaltic transport in detail. Gupta and Seshadri [3] examined the facts related to the peristaltic transport in nonuniform channels as well as in tubes for spermatic fluid in the vas deferens. The results of the analysis revealed that an increase in the amplitude ratio after 0.9 does not increase the flow rate; thus, peristalsis is not the only mechanism of fluid transport in the vas deferens but also some other mechanisms also exist. Major studies on peristaltic transport until 1979 have been compiled by Rath [4]. The complications of peristaltic transportation of blood flow in the

uniform, nonuniform tube and the channel have been analyzed by Srivastava and Srivastava [5] with the presumption that blood behaves as a Casson fluid. Observation for heat transfer during transportation of the toxic liquid through pipes in a nuclear power plant has been documented by Radhakrishnamacharya and Murty [6]. Numerical analysis for stress variation during peristaltic transport was carried out by Kumar and Naidu [7]; from the study, it was concluded that the waves which possess high-amplitude and low-wave numbers produce peristaltic flows of high shear stress deviation, and such severe variation of stress can be controlled by applying an external magnitude field. Walker and Shelly [8] developed a variational technique to optimize the wave pattern for a peristaltic pump, especially for a two-dimensional channel. Tripathi and Anwar Bég [9] investigated the motion of chyme across a diseased intestine; from computations, it is concluded that pressure difference and mean volumetric flow rate show an inverse relation for all parameters, but the friction force shows direct proportionality to mean volumetric flow rate. An increase in the permeability of the porous medium minimizes the pressure difference for minor values of the mean flow rate and suppresses the development of a bolus. A significant contribution to the study of heat transfer along with the effect of Hall current during peristaltic transport of the particle-fluid suspension system under the influence of the magnetohydrodynamic wall has been made by Bhatti et al. [10]. From the results obtained, the researchers concluded that reduction in fluid velocity is observed at higher values of the Hartmann number, but the fluid velocity depicts a reverse character for variation in Hall effect, particle volume fraction, wall rigidity, and wall tension parameter. An increase in the temperature profile was observed for higher values of particle volume fraction, Eckert number, and Prandtl number. Riaz et al. [11] performed an extensive computational study to investigate the behavior of nanoparticles transported by the peristaltic mechanism through a porous rectangular duct and found that increasing the value of thermophoresis parameter, volumetric flow rate, aspect ratio, local temperature, Grashof number, and porous parameter results in velocity field enlargement. The temperature profile escalates with the intensification of the Brownian motion and the thermophoresis parameter. A reverse behavior is experienced with a variation in the Brownian motion on nanoparticle concentration.

Knowledge of the effect of surface roughness during transportation of physiological fluids is very useful in understanding various problems related to blood flow in coronary arteries (Park et al. [12] and Burton and Espino [13]). Effect on flow due to surface roughness has captured the interest of researchers in chemical engineering to understand the consequence of roughness of walls formed as a result of chemical erosion during transportation of chemicals. Darcy [14] conducted experiments on 21 pipes of asphalt-covered cast iron, glass, wrought iron, and cast iron to investigate the effect of roughness of the tube during turbulent flow. The study helped to conclude that the nature of the surface, the diameter, and the slope of the pipe have a significant influence on discharge. Furthermore, it was also observed that, with an increase in the Reynolds number, the friction factor decreases, and this decrement rate gets slower for higher values of relative roughness. Holf [15] carried out an analysis for rectangular channels of unequal depths having distinct values of surface roughness and found that, during turbulent flow in pipes, mainly two types of roughness are observed which are termed surface roughness and surface corrugation, which follow different laws of similarity. Nikuradse [16] was the first to study turbulent flow at different degrees of relative roughness (k/r) (where k = average projection of roughness and r = radius of the pipe); findings of this study helped to establish that, at a particular value of relative roughness, the velocity distribution depends slightly on the Reynolds number, but the form of this distribution strongly depends on the relative roughness. Many studies have been carried out in this area using mini channel, microchannels, and sand tubes (Kandlikar et al. [17], Wang et al. [18], Rawool et al. [19], Sun and Faghri [20], Wagner and Kandlikar [21], Lalegani et al. [22], and Ashmawy [23]). Koopaee and Zare [24] investigated the impact of interconnected and offset roughness arrangement on the fluid flow behavior along with the effect on heat transfer in microchannels with structured roughness having a sinusoidal shape.

From the above, it is concluded that though many investigations have been carried out to study the impact of surface roughness during fluid flow, no attention has been paid to find out the impact of surface roughness during peristaltic flow. In the current work, the effect of surface roughness during the peristaltic motion of a Newtonian fluid in a two-dimensional channel has been carried out under the low Reynolds number and long-wavelength approximation.

2. Analysis

A Newtonian fluid moving through a nonuniform channel is under investigation; furthermore, the channel wall is supposed to have sinusoidal-shaped roughness. A sinusoidal wave proliferates on channel walls with speed c. Assuming that the rectangular coordinate for the fixed frame is (X, Y), the peristaltic transport profile is shown in Figure 1.

The geometry of the wall surface is given as

$$H(X,t) = d(X) + b\sin\left(\frac{(2\pi(X-ct))}{\lambda}\right) - b_1\cos^4\left(\frac{\pi X}{\lambda_1}\right),$$
$$\text{with} \quad d(X) = a + KX, \tag{1}$$

where t is the time, b is the amplitude of the wave, λ is the wavelength, a is the half channel width at the inlet, K is the nonuniformity parameter, c is the speed of wave propagation, b_1 is the roughness height, λ_1 is the pitch, and X is the axial variable.

Continuity equation:

$$\frac{\partial U}{\partial X} + \frac{\partial U}{\partial Y} = 0. \tag{2}$$

Momentum equation:

Figure 1: Geometry of the problem.

$$\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial X}+\nu\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right). \quad (3)$$

$$\left(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial Y}+\nu\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right), \quad (4)$$

with boundary conditions

$$\begin{cases} U=0, & \text{at} \quad Y=H, \\ \dfrac{\partial U}{\partial Y}=0, & \text{at} \quad Y=0, \end{cases} \quad (5)$$

where U and $V$are velocity components for X and Y directions, respectively; in a fixed frame of reference, p is the pressure, ρ is the density, and t is the time.

Nondimensional parameters are presented as follows:

$$\begin{aligned} U' &= \frac{U}{c}, \\ V' &= \frac{V}{c\delta}, \\ X' &= \frac{X}{\lambda}, \\ Y' &= \frac{Y}{a}, \\ t' &= \frac{ct}{\lambda}, \\ p' &= \frac{a^2}{c\mu\lambda}p, \\ \delta &= \frac{a}{\lambda}, \\ \phi &= \frac{b}{a}, \\ \phi_1 &= \frac{b_1}{a}, \\ \mathrm{Re} &= \frac{\rho a c}{\mu}, \\ \lambda_1' &= \frac{\lambda_1}{\lambda}. \end{aligned} \quad (6)$$

Using equation (6) in equations (2) and (3) and equation (4) and dropping primes with the presumption of low Reynolds number and long-wavelength correspondence,

$$\frac{\partial U}{\partial X}+\frac{\partial U}{\partial Y}=0, \quad (7)$$

$$\frac{\partial^2 U}{\partial Y^2}=\frac{\partial p}{\partial X}, \quad (8)$$

$$\frac{\partial p}{\partial Y}=0, \quad (9)$$

with boundary conditions

$$\begin{cases} U=0, \quad \text{at}\, Y=h=\dfrac{H}{a}=1+\dfrac{K\lambda X}{a} \\ \qquad +\phi\sin(2\pi(X-t))-\phi_1\cos^4\left(\dfrac{\pi X}{\lambda_1}\right), \\ \dfrac{\partial U}{\partial Y}=0, \quad \text{at}\, Y=0, \end{cases} \quad (10)$$

with K which is the nonuniformity parameter of the wall, and ϕ is the amplitude ratio.

On solving equations (7)–(9) with given boundary condition equation (10), we get

$$U=\left(\frac{Y^2-h^2}{2}\right)\frac{\partial p}{\partial X}. \quad (11)$$

The flow flux $q(X,t)$ is computed as

$$q(X,t)=\int_0^h U\mathrm{d}Y=-\frac{h^3}{3}\frac{\mathrm{d}p}{\mathrm{d}X}, \quad (12)$$

$$\text{or} \quad \frac{\mathrm{d}p}{\mathrm{d}X}=\frac{-3q(X,t)}{h^3}. \quad (13)$$

As the roughness parameter $\phi_1 \longrightarrow 0$, the result is similar to that obtained by Gupta and Seshadri [3].

The pressure rise Δp as well as friction force F at the wall of the channel having length L, in their dimensionless forms, are found as

$$\Delta p=\int_0^{(L/\lambda)}\left(\frac{\mathrm{d}p}{\mathrm{d}X}\right)\mathrm{d}X, \quad (14)$$

$$F=\int_0^{(L/\lambda)}(h)\left(-\frac{\mathrm{d}p}{\mathrm{d}X}\right)\mathrm{d}X. \quad (15)$$

3. Results and Discussion

Mathematica software is used to calculate the analytic results achieved from equations (14) and (15) to investigate the qualitative and quantitative effect of different parameters on the behavior of flow, pressure rise, and friction force. The instantaneous flow rate $q(X, t)$, periodic in (X-t), is (Gupta and Seshadri [3])

$$q(X,t) = Q + \phi \sin 2\pi (X - t). \tag{16}$$

Following parameter values have been used: $a = 0.01\,\text{cm}, L = \lambda = 10\,\text{cm}, K = 0.5\,(a/L) = 0.0005$ [5, 25], and $\phi_1 = (b_1/a)$, where the value of b_1 is taken from [13].

By observing Figures 2(a) and 2(b), which depict the variation in pressure rise against time for various values of the roughness parameter for nonuniform and uniform channels, respectively, for the nonuniform channel initially, pressure rise has a significant value at the start of the peristaltic wave; however, in a narrow range of time ($t = 0.2$ to 0.4), a very sharp change occurs in pressure rise; this is mainly due to the result of the maximum occlusion occurring at $t = 0.3$ on the entrance of the channel. The maximum pressure rise visible in the nonuniform channel increases due to a gradual increase in the roughness parameter. The results may be interpreted as with an increase in the roughness parameter, the cross-sectional area decreases; hence, channels having higher surface roughness have lesser flow area, and at the cross section of the channel, just after full occlusion, the pressure experienced is higher for high values of surface roughness due to a reduction in flow area with increasing surface roughness. For both geometries, at any instant of time, the pressure rise goes on increasing with an increase in the roughness parameter of the channel wall. Correlating Figure 2(a) with 2(b), it is concluded that the pressure rise in the uniform channel is much larger than that in a nonuniform channel.

The pressure rise against the flow rate has been plotted for different values of the roughness parameter for nonuniform as well as uniform channels in Figures 3(a) and 3(b), respectively, showing a linear relationship between pressure rise and flow rate. It is a well-known fact that pressure rise should occur in the direction of flow ($\Delta p > 0$) to use the peristaltic wave as a pump. Pressure rise increases with the time-mean flow, maximum time-mean flow occurs at zero pressure rise, and maximum pressure rise occurs at zero flow rate. However, when considering the impact of the roughness parameter of the channel wall at any value of pressure rise, there is a gain in the time-mean flow rate with an increase in the roughness parameter; also, increase in the surface roughness parameter at any flow rate results in an increase in the pressure rise. For small flow rate, a large relative change in pressure rise has been detected with increment in the surface roughness parameter. The relative change in pressure rise keeps on decreasing with an increase in flow rate and finally coincides at zero pressure rise for a uniform channel. For the nonuniform channel, pressure rise keeps on decreasing with an increase in flow rate, though the pressure rise plotted for different surface parameters comes very close but do not coincide at zero pressure rise. Similarly, for a high-pressure rise, a large relative change in flow rate has been seen with an increase in the roughness parameter; unlike the nonuniform channel, the relative change in flow rate keeps on decreasing with a decrease in the pressure rise and finally coincides at zero pressure rise for the uniform channel. On the contrary, the observed pressure rise is much smaller for the nonuniform channel than that for the uniform channel. It is also analyzed that, for zero pressure rise

Figure 2: (a) Pressure rise versus time for various values of ϕ_1 at $K = 0.0005$, $Q = 0.3$, $\lambda_1 = 0.00011$, and $\phi = 0.8$. (b) Pressure rise versus time for various values of ϕ_1 at $K = 0$, $Q = 0.3$, $\lambda_1 = 0.00011$, and $\phi = 0.8$.

that is free pumping, the magnitude of the maximum rate of flow is much higher in a uniform channel than that in a nonuniform channel.

The pressure rise against the amplitude ratio has been plotted at different values of the roughness parameter in Figure 4. For a given value of surface roughness, at smaller values of the amplitude ratio, the magnitude of pressure rise increases slowly; however, at higher values of the amplitude ratio ($\phi > 0.8$), a rapid change is observed in the magnitude of pressure rise. At lower values of the amplitude ratio, a small change is observed in pressure rise with a gradual increase in the value of the roughness parameter, but rapid change in pressure rise is visible with a gradual increase in the roughness parameter at a higher magnitude of the amplitude ratio.

In Figures 5–7, it has been found that friction force shows a reverse nature to that of pressure rise.

The axial velocity distribution in the y-direction has been shown for a nonuniform channel in Figure 8. It can be

Figure 3: (a) Pressure rise versus flow rate for different values of ϕ_1 at $K = 0.0005$, $t = 0.2$, $\lambda_1 = 0.00011$, and $\phi = 0.8$. (b) Pressure rise versus flow rate for different values of ϕ_1 at $K = 0$, $t = 0.2$, $\lambda_1 = 0.00011$, and $\phi = 0.8$.

Figure 4: Pressure rise versus amplitude ratio for different values of ϕ_1 at $K = 0.0005$, $t = 0.2$, $\lambda_1 = 0.00011$, and $Q = 0.3$.

Figure 5: Friction force versus time for various values of ϕ_1 at $K = 0.0005$, $Q = 0.3$, $\lambda_1 = 0.00011$, and $\phi = 0.8$.

inferred that the maximum velocity is obtained at the centerline for uniform as well as nonuniform channels, and it becomes zero at the wall of the channel. However, the maximum velocity of flow is observed to be higher in a uniform channel than that in a nonuniform channel. The above results can be physically interpreted as follows: pressure rise for the uniform channel is much higher than that for the nonuniform channel because in the uniform channel, full occlusion occurs throughout the channel length, whereas in the diverging channel, full occlusion occurs only at the entrance. Velocity rises with a gradual increment in the value of the roughness parameter near the center of the nonuniform channel and depicts an adverse nature in the vicinity of the wall. Similar behavior has been observed for the uniform channel. When surface roughness increases, the velocity of flow in the vicinity of the wall of the channel reduces due to an increase in the contact area of the

FIGURE 6: (a) Friction force versus flow rate for different values of ϕ_1 at $K = 0.0005$, $t = 0.2$, $\lambda_1 = 0.00011$, and $\phi = 0.8$. (b) Friction force versus flow rate for different values of ϕ_1 at $K = 0$, $t = 0.2$, $\lambda_1 = 0.00011$, and $\phi = 0.8$.

FIGURE 7: Friction force versus amplitude ratio for different values of ϕ_1 at $K = 0.0005$, $t = 0.2$, $\lambda_1 = 0.00011$, and $Q = 0.3$.

FIGURE 8: (a) Velocity profile for different values of ϕ_1 at $K = 0.0005$, $t = 0.2$, $x = 0.75$, $\lambda_1 = 0.00011$, and $Q = 0.3$. (b) Velocity profile for different values of ϕ_1 at $K = 0$, $t = 0.2$, $x = 0.75$, $\lambda_1 = 0.00011$, and $Q = 0.3$.

fluid with the wall surface. Thus, it can be concluded that shear stress during flow increases with an increase in surface roughness. To compensate for the flow rate, there will be an increase in the magnitude of velocity close to the center of the channel.

4. Conclusions

The effect of surface roughness (sinusoidal structure) on peristaltic flow through a nonuniform channel for a Newtonian fluid has been studied under low Reynolds number and long-wavelength approximation, and results have been compared with those obtained for a smooth channel wall. The value of maximum pressure rise increases with a gradual increase in the roughness parameter of the wall. As roughness of the surface increases, the flow area reduces, resulting in an increment in the value of maximum pressure rise. The uniform geometry with surface roughness yields much higher values of pressure rise than those for a nonuniform geometry because in the uniform channel, full occlusion occurs throughout the channel length, whereas in the diverging channel, full occlusion occurs only at the entrance. With the gradual rise in the roughness parameter, the pressure rise in the nonuniform geometry increases with a higher intensity than that in the uniform geometry. With an increase in the value of surface roughness, a decrease in the velocity of the fluid is observed near the channel wall, which is primarily due to an increase in the contact area of the fluid with the wall. Friction force shows an inverse nature to that of pressure rise. The roughness parameter shows a considerable impact on the pressure rise in the region of a small rate of flow. For the peristaltic pumping region, the roughness parameter results in an increase of flow rate for any value of pressure rise. The roughness parameter affects the magnitude of axial velocity at any cross section. Due to roughness, the magnitude of velocity close to the middle of the channel gets increased, whereas it decreases near the channel wall, showing the reverse effect of the roughness parameter on velocity. The present investigation may be helpful in future studies on the transportation of physiological fluids as the inner surface of any biological organ is not fully smooth, for example, endothelium roughness in the coronary artery.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Elastostatic Stiffness Analysis for the US/UPS Parallel Manipulators

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The virtual joint method (VJM) cannot calculate the strain energy stored in each rod. In order to solve the problem, a modeling method of the elastostatic stiffness was investigated for the UP/UPS parallel manipulators (PMs), taking the example of the 6-SPS PM. The modeling method was based on screw theory, Castigliano's theorem, and strain energy (where U, P, and S, respectively, denote universal, prismatic, and spherical joints). First, the actuator and constraint wrenches of the mechanism were obtained by screw theory. Second, compact limb stiffness matrices were obtained in terms of strain energy and Castigliano's second theorem. Finally, analytic expressions for the overall stiffness matrix of the mechanism and the amplitudes of the actuator force were obtained by adopting the virtual work principle and the balance equation for the mobile platform. All relative errors between the results of the analytical model and the finite element model are below 2%, which validates the effectiveness of the elastostatic stiffness model. The virtual work index was adopted to evaluate the stiffness performance of the mechanism, and the results show that the stiffness is not only related to position and orientation but also closely related to the directions of external loads. It is also demonstrated that the method has general adaptability for the stiffness analysis for the US/UPS PMs, laying the foundation for further reasonable dynamic design and optimization of such manipulators.

1. Introduction

Compared with serial manipulators, PMs with closed kinematic structures perform well in terms of accuracy, rigidity, and payload capacity and show high potentials to deal with numerous tasks [1–3]. Therefore, the detailed analyses of PMs are extremely important, including the workspace [4], kinematics [5], dynamics [6], elastostatics [7], and motion/force transmissibility [8].

As an essential class of the UP/UPS PMs, the 6-SPS PM is an example of successful engineering applications [9], including parallel robotic manipulators [10], positioning devices [11], remote center-of-compliance devices [12], minimally invasive surgeries [13], and haptic interface mechanisms [14]. The 6 degrees of freedom (DOFs) of the Stewart PM based on the 6-SPS PM [15] are well known as a typical example to provide enough orientation capability and high stiffness, and it has been widely used in many industrial applications, for example, the flight simulators [16].

To meet the demand for high precision from the industry, the stiffness analysis of PMs is also essential in the pre-design stage, which should consider the influences of joints and links, and is also the basis of the multi-objective optimization and structural design of PMs [17, 18]. Considering the influence and importance of stiffness on the precision of PMs, many progresses have been presented by the academic community for the stiffness characteristics of PMs, which are focused on the various methods of the stiffness modeling and performance evaluation index [19–23].

Although the stiffness of the 6-SPS PM can be numerically simulated by using the finite element analysis (FEA)

software, from which the results with high accuracy can be obtained, the process using the FEA software is very time-consuming, and it has to rebuild the model under different configurations [24, 25]. Therefore, developing the analytical or semi-analytical models in the pre-design stage is a better choice for estimating the stiffness characteristics of the Stewart PMs in the whole workspace [26]. Some analytical studies about the stiffness analysis of the Stewart PM have been proposed. Using the kinematics error model of the Stewart robot, Ding et al. analyzed the stiffness performance of the Stewart PM equipped with decoupled sensors and actuators [27]. Additionally, the FEA method under the action of (nanoscale) ultrahigh accuracy positioning and the large external load was used to verify the stiffness results. Adam et al. analyzed and evaluated the stiffness matrix of a double Stewart PM truss adopting an analytical method [28]. Based on the principle of linear superposition, Li et al. studied and established the stiffness matrix of the Stewart platform by considering the deformation of the mechanism support [29]. Svinin et al. analyzed the stiffness and stability of the Stewart PM under the condition of internal forces preloading [30]. Huang et al. considered the flexibility factor of the Stewart PM using the law of the conservation of energy and proposed a new conservative stiffness mapping [31]. They captured the compliance characteristics of the mechanism using an additional stiffness matrix. Moreover, since the rod of two-force only subjected to tension and compression, the virtual joint method is used to analyze the elastostatic stiffness performance of PMs [32, 33].

Besides the stiffness modeling, the evaluation of stiffness performance is also essential for ensuring that the performance meets the requirement [34, 35]. The stiffness matrix of the PMs is a 6×6 mapping matrix, which can evaluate the relationships between the external wrenched and infinitesimal deformations. To evaluate the stiffness performance of the PMs, the stiffness matrix must be transformed into a quantifiable index [36, 37]. The most used stiffness indices at present include the determinant of the stiffness matrix [38], the average value of the eigenvalues [39], the maximum and minimum eigenvalues [40], and the ratio of the minimum and maximum eigenvalues [41]. However, owing to the different units used for position and orientation, there is a common shortcoming of all the above indices; i.e., the dimensions are not uniform, and the index values cannot be well illustrated. The virtual work index (VWI), used in this paper, is reciprocal of the virtual work done by the external load [35]. In contrast to the case for other indices, the units of position and orientation are uniformly used as virtual work within the VWI. Additionally, the VWI can be used to measure the deformation of PMs resisting external load in a specific direction.

In order to solve the problem of the fact that the virtual joint method cannot calculate the strain energy stored in each rod, the main content of this paper is to investigate a modeling method for elastostatic stiffness analysis for the UP/UPS PMs, taking the example of the 6-SPS PM, which is based on the screw theory, Castigliano's theorem, and strain energy method. Through calculating the strain energy stored in each rod, the stiffness matrices of each rod and PMs can be obtained, which have a clear physical meaning and concise expression.

The remainder of the paper is organized as follows. Section 2 adopts screw theory, strain energy, and Castigliano's second theorem to derive a general analytical expression for the stiffness matrix of the limb of the 6-SPS PM. The elastostatic stiffness of the 6-SPS PM using the proposed method was analyzed. Section 3 uses the finite element commercial software ANSYS to develop FEA models in different configurations with external wrenches and makes a verification to show the correctness of the theoretical method. Based on the virtual work, Section 4 introduces and evaluates the stiffness performance of the 6-SPS PM by using VWI. Section 5 discusses the results of the simulation. Section 6 presents the conclusions of the study and suggestions for future work.

2. Elastostatic Stiffness Modeling for the 6-SPS PM

2.1. Process of Elastostatic Stiffness Modeling for the 6-SPS PM. Based on the strain energy method, Castigliano's second theorem, and the screw system of the limb stiffness matrix for the Stewart PM and the deformation coordination equation (DCE), the global stiffness matrix of parallel mechanism is derived [42]. The calculation process is shown in Figure 1.

At the same time, the constrained reaction force exerted on the mobile platform by the supporting limb is obtained. The calculation steps are described in detail as follows:

Step 1: using reciprocal screw theory, the constraint wrenches exerted on the mobile platform were defined by limbs $\$_{i1} = (\$_{i1}\$_{i2}\ldots)$

Step 2: based on the assumption of small deformations and the mechanics of the materials, we calculate the strain energy $U_i = f \times (f_{i2})$ of each supporting limb, where $f_i = (f_{i1} f_{i2})$ are the magnitudes of the limb constraint wrenches and U_i a function of f_i

Step 3: based on Castigliano's second theorem, the elastic deformation at the end of each limb in the direction of the screw axis of the constraint wrenches is calculated as follows: $(\delta_i = \delta_{i1}\delta_{i2}\ldots)$, where $\delta_{ij} = \partial U_i / \partial f_{ij}$

Step 4: we map elastic deformation δ_i to constraint wrenches f_i, namely, $\delta_i = C_i f_i$; the stiffness matrix of the limb can be obtained easily, namely, $\mathbf{K}_i = C_i^{-1}$

Step 5: the equilibrium equation of the mobile platform in a specific coordinate system was formulated, where $\mathbf{W} = G_f^F f$

Step 6: using the results from steps 4 and 5 along with the virtual work principle, the DCEs are obtained as $\delta_i = J_i^T \mathbf{D}$, $\mathbf{J}_i = (\$_{i1}^r \$_{i2}^r \ldots)$, where $\mathbf{D}$ is the infinitesimal twist of the geometric center point of the mobile platform and J_i is the mapping matrix between δ_i and $\mathbf{D}$

Step 7: using the DCEs, the overall stiffness matrix K is obtained, where $\mathbf{K} = \mathbf{J}_i \sum \mathbf{K}_i J_i^T$ $\mathbf{D} = \mathbf{K}^{-1}\mathbf{W}$

Figure 1: Calculation flow chart of the global stiffness matrix K.

Step 8: from matrix K, the magnitudes of the constraint wrenches are obtained, where $f_i = K_i\delta_i = \mathbf{K}_i J_i^T K^{-1}\mathbf{W}$

2.2. Description for the UP/UPS PMs. In the following, the application of the stiffness modeling method for the UP/UPS PMs is studied. The UP/UPS PMs feature a static platform and mobile platform connected through both links (legs) of type US having constant length and legs of type UPS having controllable distance, as shown in Figure 2.

A typical characteristic of the US/UPS PMS is the same topological structure within the workspace. In the process of parameterization, generalized matrices with the same independent variables and similar structures are analyzed. A class of three-degree-of-freedom (3-DOF) US/UPS PMs, in which the relative motion between the base and platform is spherical, has been proposed, where mechanisms belonging to this class have at least three legs of type US and only three legs of type UPS [43]. A class of 6-DOF US/UPS PMs, in which the relative motion between the base and platform is general, has been proposed, where mechanisms belonging to this class have legs of type UPS only [44].

The diagram of the model and the coordinate system of the 6-SPS PM are shown in Figure 3. Six identical linkages of the 6-SPS PM connect the mobile platform and static platform at B_i and A_i through a spherical joint. Both the mobile platform and static platform are semi-regular hexagons. The circumferential radius of the mobile platform and static platform is, respectively, denoted by R_b and R_a.

Figure 2: PM with legs of types US and UPS.

Figure 3: Diagram of the model for the 6-SPS PM.

Figure 4: Force analysis chart of the mobile platform.

Figure 5: Free-body diagram and section view of the ith linear rod.

The center angles of the long sides A_1A_2 and B_2B_3 are β_0. As is well known, there are six SPS limbs for 6-SPS PM.

2.3. Stiffness Matrix of Each Limb for the 6-SPS PM. A force analysis diagram of the mobile platform of the 6-SPS PM is shown in Figure 4. The motion pairs of the limb from the base to the mobile platform are S pairs, P pairs, and S pairs. Each limb of the 6-SPS is subjected only to an axial force, resulting in the limb rod generating a tensile or compressive elastic deformation. According to the screw theory, there is a constrained wrench exerted on the SPS limb of the mobile platform along the center for the two S pairs. Therefore, the SPS rod can regard each limb as a two-force rod only subject to tension and compression. Force analyses of the SPS limb and mobile platform are shown in Figure 5:

$$f_{Ni} = f_i. \tag{1}$$

The strain energy of the SPS limb is given as

$$U_i = \frac{f_{Ni}^2 L_i}{2E_iA_i}, \tag{2}$$

where f_{Ni} is the axial force of the ith limb, L_i is the length of the ith limb, E_i is the elastic modulus of the ith limb, and A_i is the cross-sectional area of the ith limb. The total strain energy of the mechanism can be obtained as

$$U = \sum_{i=1}^{6} U_i = \sum_{i=1}^{6} \frac{f_{Ni}^2 L_i}{2E_iA_i}. \tag{3}$$

The elastic deformation of the limb end along the direction of the constraint wrench screw system is obtained from Castigliano's second theorem and the relation

$$\delta_i = \frac{\partial U_i}{\partial f_i} = \frac{L_i}{E_iA_i} f_i. \tag{4}$$

The screw flexibility matrix of the constraint system of the supporting limb is obtained as

$$C_i = \frac{L_i}{E_iA_i}. \tag{5}$$

And the screw stiffness matrix of the constraint system is further obtained as

$$K_i = C_i^{-1}. \tag{6}$$

According to the giveCcn position and attitude, the compliance matrix of the SPS limb is $C_i = L_i/E_iA_i$, as expressed by equation (5). The stiffness matrix is $\mathbf{K}_i = C_i^{-1}$, where $\$_i^r = (s_ib_i \times s_i)$ is the unit screw of the constraint wrench vector direction applied by the ith limb to the mobile platform. S_i is the unit vector along the path of the A_iB_i rod, $b_i = pB_i$, and $J_i = (\$_i^r)$. S_i and b_i are expressed in the definite coordinate system o–xyz. The overall stiffness matrix K_i can thus be obtained from equation (6).

2.4. Overall Stiffness Matrix. The mobile platform equilibrium equation is

$$\mathbf{W} = \left(\ldots \$_{i1}^r \ldots \$_{ij}^r \ldots\right) f = G_f^E f, \tag{7}$$

Table 1: Physical and structural parameters of the 6-SPS PM.

Parameters	Units	Data
R	mm	200
r	mm	150
d	mm	60
E	GPa	210
G	GPa	80

Table 2: Configuration and external loads of the 6-SPS PM.

Parameters (meters)	Units	Pose 1	Pose 2	Pose 3	Pose 4	Pose 5
X	mm	0	0	0	0	−70
Y	mm	0	0	0	0	60
Z	mm	430	430	430	430	430
α	rad	0	pi/9	0	pi/9	0
β	rad	0	pi/9	0	pi/9	0
γ	rad	0	0	pi/9	0	0
F_x	N	0	0	1	1	1
F_y	N	0	0	1	1	1
F_z	N	−1	−1	−1	−1	−1
M_x	Nm	0	0	1	1	1
M_y	Nm	0	0	1	1	1
M_z	Nm	0	0	1	1	1

where $\mathbf{W} = [\mathbf{F}\ \mathbf{M}]$ is an extra wrench acting on the mobile platform and $\$_{ij}^r$ is the unit screw of the jth constraint force that the ith limb exerts on the mobile platform. $G_f^E f = (\ldots \$_{i1}^r \ldots \$_{ij}^r \ldots)$ is a 6×6 matrix while $f = (\ldots f_{i1} \ldots f_{ij} \ldots)^T$ is a 6×1 matrix.

Under the action of the constraint wrench screw, elastic deformation at the end of each limb of the mechanism is generated along the axis of the constraint wrench screw. The deformation coordination between the infinitesimal displacements of the mobile platform under the action of external forces is expressed as

$$\left(\delta_{i1} \ldots \delta_{ij}\right)^T = \left(\$_{i1}^r \ldots \$_{ij}^r\right)^T \mathbf{D} = J_i^T \mathbf{D}, \tag{8}$$

where $J_i = (\$_{i1}^r \ldots \$_{ij}^r)^T$ and $\mathbf{D}$ is the tiny twist of the geometric center point of the mobile platform. According to the knowledge of material mechanics, we have

$$\left(f_{i1} \ldots f_{ij}\right)^T = K\left(\delta_{i1} \ldots \delta_{ij}\right)^T, \tag{9}$$

Then, by substituting equations (8) and (9) into equation (7), we have

$$\begin{aligned} W &= \left(\ldots \$_{i1}^r \ldots \$_{ij}^r\right) f = \sum_{i=1}^{n} \left[\left(\$_{i1}^r \ldots \$_{ij}^r\right) W = \left(f_{i1} \ldots f_{ij}\right)^T\right], \\ &= \sum_{i=1}^{n} J_i K_i J_i^T \mathbf{D} = K\mathbf{D}, \end{aligned} \tag{10}$$

$$K = \sum_{i=1}^{n} J_i K_i J_i^T, \tag{11}$$

where n is the number of limbs of the Stewart PM. Equation (11) is the general analytical expression of the overall

Table 3: Deformation of the geometric center point of the mobile platform of the 6-SPS PM (comparison of the results of the FEA and theoretical models).

Pose	Method	δx (mm) ($\times 10^{-4}$)	δy (mm) ($\times 10^{-4}$)	δz (mm) $\times 10^{-5}$)	$\delta\theta_x$ (rad) ($\times 10^{-7}$)	$\Delta\theta_y$ (rad) ($\times 10^{-7}$)	$\Delta\theta_z$ (rad) ($\times 10^{-7}$)
1	Theoretical model	$-1.066e-17$	$-3.272e-17$	-0.173	$8.338e-18$	$-4.254e-17$	$1.665e-16$
	FEA results	$-0.199e-7$	$0.345e-7$	-0.175	$0.212e-6$	$0.122e-6$	$-0.551e-13$
	Relative error (%)	—	—	1.297	—	—	—
2	Theoretical model	−0.0012691	0.0091930	−0.15982	0.014668	−0.0014667	−0.048271
	FEA results	−0.0012843	0.0093202	−0.16190	0.014880	−0.0014839	−0.048905
	Relative error (%)	1.1835	1.3648	1.2847	1.4247	1.1591	1.2964
3	Theoretical model	0.63848	0.63894	−0.17272	0.23141	−0.22817	0.014194
	FEA results	0.64688	0.64735	−0.17499	0.23445	−0.23117	0.014386
	Relative error (%)	1.2985	1.2991	1.1429	1.2821	1.2987	1.3889
4	Theoretical model	0.752	0.585	−0.239	0.0732	−0.388	−1.37
	FEA results	0.762	0.593	−0.242	0.0742	−0.393	−1.39
	Relative error (%)	1.3123	1.3491	1.2397	1.3477	1.2723	1.4388
5	Theoretical model	0.577	0.730	−0.261	0.248	−0.249	0.0431
	FEA results	0.585	0.739	−0.264	0.251	−0.253	0.0436
	Relative error (%)	1.3675	1.2179	1.1364	1.1952	1.5810	1.1468

stiffness matrix and is more concise. The analytical model of the stiffness matrix is given from the perspective of strain energy and Castigliano's second theorem.

2.5. Amplitude of the Driving Force. From equation (10), we further obtain

$$\mathbf{D} = K^{-1}\mathbf{W}. \tag{12}$$

Then, by substituting equations (8) and (12) into equation (9), we obtain the inverse constrained solution equation as

$$\left(f_{i1} \cdots f_{ij}\right)^T = K_i J_i^T K^{-1}\mathbf{W}. \tag{13}$$

2.6. Comparison with the Virtual Joint Method. According to the calculation requirements of the VJM, the rod is regarded as a rigid rod, supported by a virtual spring at both ends of the rod. The limb of the SPS can be treated as a two-force rod, and the virtual spring is a one-way spring along the axial direction of the rod. The stiffness coefficient of the spring is

$$\mathbf{K}_i = \frac{\mathbf{EA}}{\mathbf{L}_i}. \tag{14}$$

The overall stiffness matrix $\mathbf{K}$ is

$$\mathbf{K} = \sum_{i=1}^{6} \mathbf{K}_i \mathbf{J}_i \mathbf{J}_i^T. \tag{15}$$

Formally, equation (15) is the same as equation (11). Due to the fact that the VJM method cannot calculate the strain energy stored in each rod, the method proposed in this paper can calculate the strain energy stored in each rod and overall parallel manipulators. A new approach to solving the problem of each component for the stiffness performance of the PM is proposed.

3. Comparison of the Proposed Model and FEA Model

The finite-element commercial software ANSYS is used to develop a unified FEA model and verify the correctness and feasibility of the method investigated in this paper. Analytical expression theory and the FEA simulation model are implemented.

Physical parameters of the 6-SPS PM are given in Table 1, where R is the radius of the circumferential circle of the fixed platform, r is the radius of the circumferential circle of the mobile platform, d is the diameter of the cross section of the limb rod, and E and G are, respectively, the elastic modulus and shear modulus of the material.

At the same time, the five poses and five groups of external forces and their couples acting on the mobile platform of the 6-SPS PM are given in Table 2.

Five groups of data to make a theoretical analysis calculation and FEA simulation calculation for the established analytical model are taken. The results obtained using the two methods are compared one by one. The theoretical calculation values, FEA simulation values, and relative errors of each group are listed in Table 3. It shows that the maximum relative error is 1.581% for the fifth group. All relative errors between the results of the analytical model and the workbench finite element model are below 2%, which validates the effectiveness of the proposed elastostatic stiffness model of the 6-SPS PM.

Figure 6: FEA simulation calculation of (a) the stiffness for pose 2 angle change, (b) the stiffness for pose 3 displacement change, (c) the stiffness for pose 3 angle change, and (d) the stiffness for pose 4 displacement change.

4. Stiffness Index of the 6-SPS PM

The VWI is used to evaluate the stiffness of PMs. In contrast with other indices, this index unifies the units of displacement and angle relating to virtual work. By correlating the value of the stiffness index with the direction of the external load, the ability of the mechanism to resist the external load in one direction is measured.

Supposing that a unit wrench W is imposed on the mobile platform, the virtual work done by the unit wrench can be expressed as

$$\mathbf{W}_V = \mathbf{W}^T \delta\xi. \tag{16}$$

The infinitesimal twist is $\delta\xi = K^{-1}\mathbf{W}$, and the VWI of the stiffness for the 6-SPS can thus be expressed as

$$VW = \frac{1}{\mathbf{W}^T K^{-1} \mathbf{W}}. \tag{17}$$

The VWI of the stiffness for the 6-SPS PM is a measure of the ability of the mechanism to resist deformation under a given external wrench. It allows the stiffness matrix to be transformed into a single index value. To calculate the value of the VWI, the virtual work completed by the force and moment must have the same units.

5. Results and Discussion of the Simulation

For the convenience of research, the unit of virtual work in the calculation process is uniformly adopted as 0.001 J. The distribution of the VWI of the 6-SPS PM in the plane $\mathbf{z} = 430$ mm, when $\mathbf{W} = [0\ \ 0\ \ -1\ \ 0\ \ 0\ \ 0]^{\mathrm{T}}$ and $\mathbf{W} = [0.4082; 0.4082; 0.4082; 0.4082; 0.4082]^{\mathrm{T}}$. The VWI of the stiffness for the Stewart PM changes with x and y, as shown in Figures 6(a) and 6(b), and index value changes with pose, as shown in Figures 6(c) and 6(d).

Under the vertical load, Figures 7(a) and 7(c) show the VWI of the stiffness for the 6-SPS PM that has a symmetrical distribution. The results are consistent with the fact that the mechanism is symmetrical. The stiffness index presents a different distribution under the action of external loads in

Figure 7: Distribution of changes in the VWI with the values of x and y for the 6-SPS PM in the plane $z = 430$ when (a) $\mathbf{W} = [0\ 0\ -1\ 0\ 0\ 0]^{\mathrm{T}}$, (b) $\mathbf{W} = [0.4082;\ 0.4082;\ 0.4082;\ 0.4082;\ 0.4082]^{\mathrm{T}}$, (c) $\mathbf{W} = [0\ 0\ -1\ 0\ 0\ 0]^{\mathrm{T}}$, and (d) $\mathbf{W} = [0.4082;\ 0.4082;\ 0.4082;\ 0.4082;\ 0.4082]^{\mathrm{T}}$.

different directions. Figure 7(a) shows the results for a vertical load. The stiffness of the 6-SPS PM is best for the central position, and the stiffness performance of the mechanism decreases rapidly with deviation from this position.

Similarly, we can see from Figure 7(c) that the stiffness performance of the 6-SPS PM is best when the position is not deflected, and the stiffness performance of the mechanism decreases rapidly with deviation from this position.

Figures 7(b) and 7(d) show that the optimal position for the 6-SPS PM stiffness under random loading is not at the central position of the mechanism. The results show that the stiffness performance of the 6-SPS PM is closely related to the direction of the external load and different under different external loads. It is shown that the elastic deformation of the 6-SPS PM is closely associated with the stiffness, configuration, and external loads.

6. Conclusion

In this work, a modeling method is investigated for elastostatic stiffness analysis for the UP/UPS PMs, taking the example of the 6-SPS PM, which is based on the screw theory, Castigliano's theorem, and strain energy method. The global stiffness matrix was derived. The virtual work index was adopted to evaluate the stiffness of the 6-SPS PM, and the VWI was used to measure the deformation of the 6-SPS PM in resisting external loads in specific directions. The main conclusions of the study are as follows:

(1) Although the theoretical method used in this paper is much simplified and has many limitations, the results show that the relative errors of the theoretical model and finite element model are less than 2%, which verifies the correctness of the analytical model. During the predesign stage, the stiffness analysis model used in this paper can be regarded as an alternative to the finite element model in evaluating the stiffness performance of the 6-SPS PM.

(2) The VWI was used to evaluate the stiffness performance of the 6-SPS PM. The results show that the stiffness of the mechanism is related to the direction of the action of the external wrench.

(3) In contrast, VJM cannot calculate the strain energy stored in each rod, and the described method can be applied to not only the 6-SPS PM but also other US/UPS PMs, such as two-DOF PMs, three-DOF PMs, and other six-DOF PMs.

The modeling method for the elastostatic stiffness of the US/UPS PMs was derived in this paper. The limitation of the model is that the flexibility of a joint is ignored. The study did not establish an experimental model for verification and did not consider the problem of dynamics. These issues need further study in the future.

Data Availability

The MATLAB and ANSYS files data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

The Fuzzy PI Controller for PMSM's Speed to Track the Standard Model

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This paper proposes a fuzzy PI controller to control the speed of a permanent magnet synchronous motor (PMSM). The structure of the system includes the speed loop controller (SLC) and the current loop controller (CLC). The speed loop controller is the fuzzy PI and standard model (SM). The CLC includes vector control and the space vector pulse width modulation (SVPWM). It compiles two closed-loop control systems for the PMSM. This research uses a very high-speed integrated circuit hardware description language (VHDL) to implement the proposed algorithm and embed it into Matlab/Simulink for simulation. Based on the PMSM parameter, this article tests the controller's correctness with some of the load cases by changing the combined inertia and viscous friction of rotor and load. After success in simulation, the system is tested again by experiment on the FPGA kit. The simulation and experiment results show that when the load changes, the PMSM speed is still stable. The novelty of this research is that it compares two kinds of controllers between simulation and experiment results.

1. Introduction

The PMSMs are commonly used in systems requiring high precision such as the robot and mechanical processing. However, there are many uncertainties, for example, noise, external load, and friction force. They affect the performance quality of the system. Many intelligent control techniques such as fuzzy, neural network, and genetic algorithm have been developed to cope with those problems [1–4]. They helped to control precisely the speed and position of the motor. The FPGA supplied a solution for the implementation of the algorithm. Especially, FPGA has a programmable hard-wired feature, fast calculation ability, low power consumption, and higher density. There were many types of research on intelligent controllers such as fuzzy control, sliding fuzzy control, artificial intelligence, and adaptive control based on the FPGA platform to control multiaxis robots for improving precision in machining and assembly [5–9]. The fuzzy control method is an intelligent control system, with an inference mechanism from experts' control experience. However, obtaining fuzzy sets and optimal membership functions are not easy. To resolve this problem, the fuzzy knowledge base values were adjusted appropriately based on the feedback system parameter [10, 11]. This research proposes a drive system to overcome the slow response and tuning problems of the conventional PI speed controllers without any mathematical calculations [12, 13]. However, the aforementioned research studies do not check with the massive change of load. This research applied a fuzzy PI controller, current controller, and SVPWM for PMSM drive. The contents of the article include, firstly, the mathematical model of PMSM, SVPWM algorithm, vector control method, and fuzzy PI controller. Secondly, VHDL is used to program all systems. Finally, the article presents a simulation on Matlab/Simulink and experiment on FPGA kit results.

Figure 1 shows the block diagram of the proposed algorithm. The speed of the motor (ω_r) is fed back to the speed loop controller to compare with the output of the standard model (SM). The fuzzy PI controller generates the signal to minimize the error between rotor speed and SM. The adjustment mechanism keeps the motor speed stable when the load changes. The three-phase stator currents (i_a, i_b, i_c) and rotor flux angle (θ_e) are fed back to the current controller to

Figure 1: The simulation architecture of the fuzzy PI controller for PMSM drive.

calculate the generated voltage vector for SVPWM. The SVPWM generated the electric current to control three-phase inverter for providing power to the motor.

2. The Vector Controller Design for PMSM

2.1. The PMSM Mathematical Model.

$$\frac{\mathrm{d}i_d}{\mathrm{d}t} = \frac{1}{L_d}v_d - \frac{R}{L_d}i_d + \frac{L_q}{L_d}p * \omega_r * i_q,$$
$$\frac{\mathrm{d}i_q}{\mathrm{d}t} = \frac{1}{L_q}v_q - \frac{R}{L_q}i_q - \frac{L_d}{L_q}p * \omega_r * i_d - \frac{\lambda * p * \omega_r}{L_q}, \quad (1)$$

in which L_q and L_d are the inductances on q-axis and d-axis; R is the resistance of the stator windings; i_q and i_d are the currents on q-axis and d-axis; V_q and V_d are the voltages on q-axis and d-axis; λ is the permanent magnet flux linkage; p is the number of pole pairs; and ω_r is the rotational speed of the rotor.

2.2. The Vector Controller. The current controller of the PMSM drive in Figure 1 is based on a vector control approach. The current controller structure includes Clark, Park, inverse Park, and inverse Clark transformation, two PID controllers, and sine/cosine of flux angle. Three-phase current is fed back through vector control, enabling controlling current $i_d \approx 0$, which helps control the three-phase motor similar to a DC motor. The torque of the PMSM is controlled via current on the q-axis (i_q). The vector control is described below.

The Clark transformation converts the stationary a-b-c frame (i_a, i_b, i_c) to stationary α-β frame (i_α, i_β).

$$\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}. \quad (2)$$

The Park transformation converts the stationary α-β frame (i_α, i_β) to synchronously rotating d-q frame (i_d, i_q).

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e \\ -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix}. \quad (3)$$

The inverse Park transformation converts a rotating reference d-q frame (v_d, v_q to stationary α-β frame (v_α, v_β).

$$\begin{bmatrix} v_\alpha \\ v_\beta \end{bmatrix} = \begin{bmatrix} \cos\theta_e & -\sin\theta_e \\ \sin\theta_e & \cos\theta_e \end{bmatrix} \begin{bmatrix} v_d \\ v_q \end{bmatrix}. \quad (4)$$

The inverse Clark transformation converts a stationary α-β frame (v_α, v_β) to stationary a-b-c frame (v_{ref1}, v_{ref2}, v_{ref3}).

$$\begin{bmatrix} v_{ref1} \\ v_{ref2} \\ v_{ref3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} v_\alpha \\ v_\beta \end{bmatrix}. \quad (5)$$

Electromagnetic moment of the motor could be calculated using the following formula:

$$T_e = 1.5p\left[\lambda * i_q + \left(L_d - L_q\right)i_d * i_q\right] \approx K_t * i_q. \quad (6)$$

The mathematical equation of the motor under the load-bearing condition is

$$\frac{\mathrm{d}\omega_r}{\mathrm{d}t} = \frac{1}{J_m\left(T_e - F\omega_r - T_m\right)}, \quad (7)$$

where T_e is the motor torque; J_m and F are the inertia and viscous friction of rotor and load; T_m is the shaft mechanical torque; and K_t is the torque constant.

3. The SVPWM Algorithm Design

The SVPWM is a control technique widely used in power electronic equipment control. The idea of the method is to create a continuous shift of the space vector equivalent of the voltage vector of the inverter on the circle orbit (Figure 2). With the steady shift of the space vector on a circular orbit,

Figure 2: (a) The eight primary vectors and switching patterns. (b) PWM at S_3 Sector and its duty cycle.

the higher harmonics are removed, and the relationship between the control signal and output voltage amplitude became linear. In this article, the SVPWM technique is used to create a vector to control the inverter using IGBT power switches.

The phase-to-phase voltage and the phase-to-neutral voltage of the motor:

$$\begin{bmatrix} V_{AB} \\ V_{BC} \\ V_{CA} \end{bmatrix} = V_{DC} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$
$$\begin{bmatrix} V_{AN} \\ V_{BN} \\ V_{CN} \end{bmatrix} = \frac{1}{3} V_{DC} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \tag{8}$$

in which a, b, and c are switching vectors of IGBT power electronic switches.

There are six power IGBTs in the voltage-source inverter. When the upper switch is active, the corresponding lower switches must be turned off. Theremust be coordinated delay time (dead band) between the opening and closing of two switches on the same phase.

The eight primary space vectors $(V_0, V_1, V_2, V_3, V_4, V_5, V_6, V_7)$ are shown in Figure 2(a). The output voltage (V_{ref}) is located at one of the six sectors (S_1, S_2, S_3, S_4, S_5, S_6) at any given time.

$$V_{ref} = \frac{T_1}{T} U_x + \frac{T_2}{T} U_{x+1} + \frac{T_0 (V_0 \text{or} V_7)}{T}, \tag{9}$$

where $x \neq 0$, 7 and T is half of the PWM period, $T_0 = T - T_1 - T_2$. For example, in the S_3 sector, the output voltage can be calculated with V_1 and V_2 vectors (Figure 2(a)).

$$V_1 = \frac{2V_{DC}}{3} \vec{\alpha}, \tag{10}$$

$$V_2 = \frac{V_{DC}}{3} \vec{\alpha} + \frac{V_{DC}}{\sqrt{3}} \vec{\beta}. \tag{11}$$

From equations (9)–(11), the output voltage and the value of T_1 and T_2 can be obtained:

$$\begin{aligned} V_{ref} &= \frac{T_1 * V_1}{T} + \frac{T_2 * V_2}{T} = \frac{T_1}{T} \frac{2V_{DC}}{3} \vec{\alpha} \\ &\quad + \frac{T_2}{T} \left(\frac{V_{DC}}{3} \vec{\alpha} + \frac{V_{DC}}{\sqrt{3}} \vec{\beta} \right) \approx V_\alpha \vec{\alpha} + V_\beta \vec{\beta}, \\ T_1 &= \frac{T\left(3V_\alpha - \sqrt{3} V_\beta\right)}{2V_{DC}}, \\ T_2 &= \frac{\sqrt{3} T * V_\beta}{V_{DC}}. \end{aligned} \tag{12}$$

Similarly, the values of T_1 and T_2 at the other sectors shown in Table 1 with T_x, T_y, and T_z are

$$T_x = \frac{\sqrt{3} T * V_\beta}{V_{DC}}, \tag{13}$$

$$T_y = \frac{T\left(3V_\alpha + \sqrt{3} V_\beta\right)}{2V_{DC}}, \tag{14}$$

$$T_z = \frac{T\left(-3V_\alpha + \sqrt{3} V_\beta\right)}{2V_{DC}}, \tag{15}$$

Table 1: Time value in each section.

Value	S_1	S_2	S_3	S_4	S_5	S_6
T_1	T_z	T_y	$-T_z$	$-T_x$	T_x	$-T_y$
T_2	T_y	$-T_x$	T_x	T_z	$-T_y$	$-T_z$

with the condition $T_1 + T_2 > T$; T_1 and T_2 are modified as follows:

$$T_{1(\text{cond})} = \frac{T_1 * T}{T_1 + T_2}, \tag{16}$$

$$T_{2(\text{cond})} = \frac{T_2 * T}{T_1 + T_2}. \tag{17}$$

The duty cycles (Ta_{on}, Tb_{on}, Tc_{on}) in Figure 2(b) can calculated as follows:

$$Ta_{\text{on}} = \frac{T - T_1 - T_2}{2} = \frac{T_0}{2}, \tag{18}$$

$$Tb_{\text{on}} = Ta_{\text{on}} + T_1, \tag{19}$$

$$Tc_{\text{on}} = Tb_{\text{on}} + T_2. \tag{20}$$

PWM1, PWM2, and PWM3 with the duty time at V_1 and V_2 and zero vector (V_0 and V_7) are T_1, T_2, and T_0, respectively. The values of CMP_1, CMP_2, and CMP_3 at each sector are shown in Table 2. For quickly determining the sector, (5) (inverse Clark transformation) is modified as follows:

$$\begin{bmatrix} v_{ref1} \\ v_{ref2} \\ v_{ref3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} v_\beta \\ v_\alpha \end{bmatrix}. \tag{21}$$

We can determine the sectors S_1–S_6 (Figure 3) according to the rule below:

$$\begin{aligned} &\text{if } V_{ref1} > 0, \quad \text{then } a = 1 \text{ else } a = 0, \\ &\text{if } V_{ref2} > 0, \quad \text{then } b = 1 \text{ else } b = 0, \\ &\text{if } V_{ref3} > 0, \quad \text{then } c = 1 \text{ else } c = 0. \end{aligned} \tag{22}$$

The sector of SVPWM is defined by this equation: sector $= a + 2b + 4c$. From equations (13)–(15) and (21), we have

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \frac{\sqrt{3}T}{V_{DC}} \begin{bmatrix} v_{ref1} \\ -v_{ref3} \\ -v_{ref2} \end{bmatrix}. \tag{23}$$

The implementation of SVPWM by VHDL can be summarized below:

(i) From the value of V_{ref1}, V_{ref2}, and V_{ref3}, we determined the sector as (22).

Table 2: The values of CMP_1, CMP_2, and CMP_3 at each section.

Section	S_1	S_2	S_3	S_4	S_5	S_6
CMP_1	Tb_{on}	Ta_{on}	Ta_{on}	Tc_{on}	Tc_{on}	Tb_{on}
CMP_2	Ta_{on}	Tc_{on}	Tb_{on}	Tb_{on}	Ta_{on}	Tc_{on}
CMP_3	Tc_{on}	Tb_{on}	Tc_{on}	Ta_{on}	Tb_{on}	Ta_{on}

Figure 3: The voltage values supplying for the SVPWM.

(ii) Calculate the value of T_x, T_y, and T_z from (23).

(iii) Select T_1 and T_2, in Table 1.

(iv) Calculate the duty cycle (Ta_{on}, Tb_{on}, and Tc_{on}) from equations (18)–(20).

(v) Select comparative value (CMP_1, CMP_2, and CMP_3) at the duty cycles from Table 2.

4. Standard Model (SM)

The SM is the standard speed for PMSM. The output of the SM is increase and tangent to the set value at the input. To build the SM to meet the above requirements, we used the typical second-order system:

$$\frac{\omega_m(s)}{\omega_r^*(s)} = \frac{\omega_n^2}{s^2 + 2 * \eta * \omega_n * s + \omega_n^2}, \tag{24}$$

where ω_n and η are the natural frequency and damping ratio, respectively. Using the bilinear transformation, we obtained the discrete function in Z domain and difference equation:

$$\begin{aligned} \frac{\omega_m(z^{-1})}{\omega_r^*(z^{-1})} &= \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}, \\ \omega_m(k) &= -b_1 * \omega_m(k-1) - b_2 * \omega_m(k-2) \\ &\quad + a_0 * \omega_r^*(k) + a_1 * \omega_r^*(k-1) + a_2 * \omega_r^*(k-2), \end{aligned} \tag{25}$$

with coefficients: $a_0 = 0.0077$, $a_1 = 0.0153$, $a_2 = 0.0077$, $b_1 = -1.6496$, and $b_2 = 0.6803$.

The error between the output of SM and the rotor speed is supplied to the fuzzification and the fuzzy adjustment mechanism function of the fuzzy PI controller. Figure 4 shows the SM; it does not change the rising time, zero steady-state value, and overshoot when the system is running. Figures 5 and 6 show a finite state machine (FSM) and the VHDL code to describe the SM, respectively.

Figure 4: The response of the SM.

Figure 5: Seven steps to implement the SM by FSM with one adder and one multiplier.

5. Structure of Fuzzy PI Controller

The structure of the fuzzy PI controller includes fuzzification, fuzzy inference mechanism, defuzzification, fuzzy knowledge base, fuzzy adjusting mechanism, and the PI controller (Figure 1).

The rotor speed (ω_r) is fed back to SLC for comparison with the output of SM and an error signal (e) is generated, and the signal changes over time (de). Based on the e and de signal, the fuzzy PI controller generates the signal to control the motor's speed. This signal is defuzzified and then amplified by the PID controller. When the motor had a heavy load or light load, the difference between rotor speed and output of SM increased, the fuzzy adjustment mechanism (figure 1) changes the knowledge base of the fuzzy PI controller. With this method, the motor speed is still in the balance.

Fuzzy PI controller used singleton fuzzy rule, and the method of the average center solution with input linguist value includes

$$e(k) = \omega_m(k) - \omega_r(k), \tag{26}$$

$$de(k) = e(k) - e(k-1). \tag{27}$$

In Figure 7, the values e, de, and u_f are the input and output of fuzzy controller. Their linguist values E and dE are (A_1, A_2, A_3, A_4, A_5, A_6, A_7) and (B_1, B_2, B_3, B_4, B_5, B_6, B_7), respectively. During each cycle, there are only two E and dE values that are supplied to the fuzzy PI controller (Figure 7). From the values of E and dE, based on membership functions, we can calculate the values of $\mu_{A_i}(e)$ and $\mu_{B_i}(de)$ with the formula below:

$$\mu_{A_i}(E) = \frac{E_{i+1} - E}{2}, \tag{28}$$

$$\mu_{A_{i+1}}(E) = 1 - \mu_{A_i}(E), \tag{29}$$

$$\mu_{B_i}(dE) = \frac{dE_{i+1} - dE}{2}, \tag{30}$$

$$\mu_{B_{i+1}}(dE) == 1 - \mu_{B_i}(dE). \tag{31}$$

Membership function has the shape of the symmetrical triangle, and control rules have the form

$$\text{if } e \text{ is } A_m \text{ and } de \text{ is } B_n, \text{then } u_f \text{ is } C_{m,n}. \tag{32}$$

Crisp value in the output of the fuzzy controller:

```
LIBRARY IEEE;
USE IEEE.std_logic_1164.all;
Use IEEE.numeric_std.all;
USE IEEE.std_logic_arith.all;
USE IEEE.std_logic_signed.all;

LIBRARY lpm;
USE lpm.LPM_COMPONENTS.ALL;

ENTITY SM IS
port(CLK,CLK_40n      :IN  STD_LOGIC:='0';
          Wr          :IN STD_LOGIC_VECTOR(15 downto0);
          Wm          :BUFFER STD_LOGIC_VECTOR(15 downto0));
END SM;

ARCHITECTURE SM_archOF SM IS
SIGNAL a0,a1,a2,b1,b2          :STD_LOGIC_VECTOR(15 downto0);
SIGNAL Wm2,Wm1,wr1,wr2         :STD_LOGIC_VECTOR(15 downto0);
SIGNAL mula,mulb               :STD_LOGIC_VECTOR(15 downto0);
SIGNAL mulr                    :STD_LOGIC_VECTOR(31 downto0);
SIGNAL adda,addb,addr          :STD_LOGIC_VECTOR(15 downto0);
SIGNAL CNT                     :STD_LOGIC_VECTOR(7 downto0);

BEGIN

mull: lpm_mult
generic map(LPM_WIDTHA=>16,LPM_WIDTHB=>16,LPM_WIDTHS=>16,
LPM_WIDTHP=>32, LPM_REPRESENTATION=>"SIGNED",
LPM_PIPELINE=>1)
port map(dataa=> mula,datab=> mulb,clock=> clk,result=> mulr);

adder1: lpm_add_sub
generic map(lpm_width=>16,LPM_REPRESENTATION=>"SIGNED",
lpm_pipeline=>1)
port map(dataa=>adda,datab=>addb,clock=> clk,result=>addr);

--Q15
a0  <=  X"007E";
a1  <=  X"00FA";
a2  <=  X"007E";
b1  <=  X"966E";
b2  <=  X"2B89";

GEN:block
BEGIN
PROCESS(CLK_40N)
BEGIN
IF CLK_40N'EVENT and CLK_40N='1' THEN
          CNT <=CNT+1;
IF CNT=X"00" THEN
          mula<=  a0;
          mulb<=  wr;
ELSIF CNT=X"01" THEN
          adda<=  mulr(30 downto15);
          mula<=  a1;
          mulb<=  wr1;
ELSIF CNT=X"02" THEN
          addb<=  mulr(30 downto15);
          mula<=  a2;
          mulb<=  wr2;
ELSIF CNT=X"03" THEN
          adda<=  addr;
          addb<=  mulr(30 downto15);
          mula<=  b1;
          mulb<=  Wm1;
ELSIF CNT=X"04" THEN
          adda<=   addr;
          addb<=   mulr(30 downto15);
          mula<=   b2;
          mulb<=   Wm2;
ELSIF CNT=X"05" THEN
          adda<=   addr;
          addb<=   mulr(30 downto15);
ELSIF CNT=X"06" THEN
          Wm <=   (addr(15) & addr(13 downto0) & '0');
ELSIF CNT=X"08" THEN
          Wm1     <=  Wm;
          Wm2     <=  Wm1;
          wr2     <=  wr1;
          wr1     <=  wr;
          CNT     <=  X"00";
END IF;
END IF;
END PROCESS;
END BLOCK GEN;
END SM_arch;
```

Figure 6: The example of SM computation using VHDL.

$$u_f(E,dE) = \frac{\sum_{n=i}^{i+1}\sum_{m=j}^{j+1} c_{m,n}\left[\mu_{A_n}(E) * \mu_{B_m}(dE)\right]}{\sum_{n=i}^{i+1}\sum_{m=j}^{j+1} \mu_{A_n}(E) * \mu_{B_m}(dE)} \triangleq \frac{\sum_{n=i}^{i+1}\sum_{m=j}^{j+1} c_{m,n} * d_{n,m}}{\sum_{n=i}^{i+1}\sum_{m=j}^{j+1} d_{n,m}} = \sum_{n=i}^{i+1}\sum_{m=j}^{j+1} c_{m,n} * d_{n,m}, \tag{33}$$

where $d_{n,m} \triangleq \mu_{A_n}(E) * \mu_{B_m}(dE)$ and $c_{m,n}$ are adjustable parameters in the adaptation mechanism. Also, it can be obtained by the equations (28)–(31). Therefore, it is straightforward to obtain $\sum_{n=i}^{i+1}\sum_{m=j}^{j+1} d_{n,m} = 1$ in (33).

The purpose of the fuzzy adjustment mechanism is to create the signal to adjust the fuzzy knowledge base of the fuzzy PI controller so that the error between the speed of the rotor and the output of the SM is the smallest. We use the method of gradient descent to determine the adaptive control law for the system.

Definition of instantaneous value function:

$$J(k+1) = \frac{1}{2}[e(k+1)]^2 = \frac{1}{2}[\omega_m(k+1) - \omega_r(k+1)]^2. \tag{34}$$

Parameter $C_{m,n}$ is determined by the variation of the instantaneous value function.

$$\Delta C_{m,n}(k+1) = -\alpha \frac{\partial J(k+1)}{\partial C_{m,n}(k)}, \tag{35}$$

in which α shows the learning rate of the system.

By taking Laplace and bilinear transform of (7), we obtained a discrete function in the Z domain of PMSM.

$$\frac{\omega_r(k)}{i_q^*(k)} = \frac{K_t}{F}\frac{\left(1 - e^{(-FT/J_m)}\right)z^{-1}}{1 - e^{(-FT/J_m)}z^{-1}}, \tag{36}$$

in which T is a sampling cycle and z^{-1} is a stage of delay time. The equation describes the relationship between i_q^* and u_f:

$$i_q^*(k) = u_i(k-1) + \left(K_p + K_i\right) * u_f(k), \tag{37}$$

in which K_p and K_i are the coefficients of the PI controller and u_f is the output function of the fuzzy PI controller.

From (36) and (37), we get the function between ω_r and u_f:

dE \ E	A_1	A_2	A_3	A_4	A_5	A_6	A_7
B_1	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}
B_2	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}
B_3	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}
B_4	C_{41}	C_{42}	C_{43}	C_{44}	C_{45}	C_{46}	C_{47}
B_5	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}	C_{57}
B_6	C_{61}	C_{62}	C_{63}	C_{64}	C_{65}	C_{66}	C_{67}
B_7	C_{71}	C_{72}	C_{73}	C_{74}	C_{75}	C_{76}	C_{77}

FIGURE 7: The design of the fuzzy controller for PMSM drive.

$$\omega_r(k) = \Phi * \omega_r(k-1) + u_i(k-2) * \gamma + \left(K_p + K_i\right) * \gamma * u_f(k-1), \tag{38}$$

in which $\Phi = e^{(-FT/J_m)}, \gamma = K_t(1-\Phi)/F$.

Variation of the instantaneous value function $J(k+1)$ from (34) is

$$\frac{\partial J(k+1)}{\partial C_{m,n}(k)} = -\alpha * e(k-1)\frac{\partial \omega_r(k+1)}{\partial u_f(k)}\frac{\partial u_f(k)}{\partial C_{m,n}(k)}. \tag{39}$$

From (35), (38), and (39), the parameters of the fuzzy PI controller $C_{m,n}$ in (33) could be adjusted through the function.

$$\begin{aligned}\Delta C_{m,n}(k) &= \alpha * \left(K_p + K_i\right) * \gamma * e(k) * d_{m,n} \\ &\approx \alpha * \left(K_p + K_i\right) * Sign(\gamma) * e(k) * d_{m,n}.\end{aligned} \tag{40}$$

6. Implementation on FPGA

The proposed algorithm has been embedded in one FPGA chip. The architecture of FPGA is shown in Figure 8. The FSM sequential programming technique was used to describe control algorithms. The internal circuit of the proposed algorithm includes one circuit of the SLC and one circuit of the CLC.

Figure 8(a) shows the implementation of FPGA for the PMSM speed controller. CLK and CLK_40ns are the input clocks (50 MHz and 25 MHz) to supply all modules. The command speed and rotor speed (ω_r^*, ω_r) with 16 bit length are the input of the SLC. The three-phase currents (i_a, i_b, i_c) and rotor flux angle (θ_e) with 12 bit length are the input of the CLC. The 6 PWMs are the output of SVPWM.

In Figure 8(b), there are 33 steps to carry out the fuzzy PI controller. The steps s_0–s_5 compute SM, and the steps s_6-s_7 calculate the rotor speed error with SM and error change. The steps s_8–s_{11} compute the fuzzification. The step s_{12} looks up the fuzzy table to select four values of $c_{m,n}$ based on i, j obtained from s_{11}. The steps s_{13}–s_{21} compute the defuzzification function. The steps s_{22}–s_{32} turn the PI and turn the fuzzy rule parameters.

Figure 8(c) shows the design of SVPWM with 12 kHz frequency and 1.2 μs dead band. The input of SVPWM is V_{ref1}, V_{ref2}, and V_{ref3} with 12 bit length.

Figure 8(d) computes the current vector controller, and it takes 24 steps. The steps s_0-s_1 look up the table to take the value of sine and cosine functions. The steps s_2–s_8 calculate the Clark and Park transformations. The PID controller turns the current on d and q-axis from step s_9–s_{14}. The steps s_{15}–s_{23} calculate the inverse Park and modified inverse Clark transformations.

The data type in vector control and SVPWM for PMSM is 12 bits. The data type is designed with 16 bit length for the fuzzy PI controller and SM. Each clock pulse in the FPGA is 40 ns to complete a command. There are 33 steps (1.32 μs) and 24 steps (0.96 μs) to implement the SLC and the current vector controller, respectively. It does not lose any performance because the operation time of fuzzy PI with 1.32 μs is much less than the sampling time of the speed controller (2 kHz).

Although the control algorithm of this research by using the fuzzy PI controller is quite complicated, using the FSM description in step-by-step order reduced the execution time of coding. The simulation was performed on Matlab/Simulink/ModelSim for adjusting and verifying the controller's parameters and checking the accuracy of the

FIGURE 8: The implementation of the proposed algorithm on FPGA. The FSM was used to describe the VHDL code. (a) The architecture of the FPGA-based fuzzy PI controller for PMSM drive. (b) The speed loop circuit. (c) The SVPWM architecture. (d) The vector control circuit.

Figure 9: The simulation system for SVPWM generation. (a) The block diagram of step 1. (b) The Simulink model of step 1. (c) The simulation result of PWM with RC circuit connection (saddle waveform).

proposed algorithm. The ModelSim software was embedded in Matlab/Simulink for simulating the whole system. The feedback signals, PMSM model, and inverter are designed on Simulink. After success in simulation, the whole VHDL code is downloaded to the FPGA board for testing again by experiment.

7. Simulation Work

The simulation is demonstrated in this section to confirm the effectiveness of the proposed architecture. In simulation work, five steps are used to confirm the effectiveness of the system. In the first step (Figure 9), the generation of SVPWM function is firstly tested, and its block diagram includes inverse Clarke and inverse Park coordinate transformation, sin/cosine function, and SVPWM (Figure 9(a)). In Figure 9(b), the EDA simulator link for ModelSim executes the co-simulation using VHDL code, which describes the function of the block diagram in Figure 9(a). The switching frequency and dead band in SVPWM are designed with 16 kHz and 1.2 μs, respectively. The input signal i_q is set to a value varying from 0.1 to 0.4, i_d is chosen to be zero, and the address (θ_e) is generated from 0 to 360° with increment by one. The PWM1–PWM6 outputs can be generated through the computation in ModelSim. Further, PWM1, PWM3, and PWM5 are serially connected to the RC

Figure 10: Confirmation of the correctness of SVPWM on the PMSM model again. (a) The block diagram of step 2. (b) The rotor speed response. (c) The Simulink model.

circuit ($R = 10\,\Omega$, $C = 47$). The saddle waveform of PWMs is shown in Figure 9(c).

In step 2, the inverter and PMSM are added to step 1. A real rotor flux angle replaces the address. The block diagram of this step is shown in Figure 10(a), and the Simulink model is shown in Figure 10(c). This step confirms the correctness of SVPWM on PMSM again. The rotor speed is shown in Figure 10(b), and it follows the value of i_q at the input.

In step 3, the PI speed controller is applied to the system, and the real rotor speed is fed back to the speed controller. The block diagram of this step is shown in Figure 11(a), and the Simulink model is shown in Figure 11(c). In this step, we can check the speed response of PMSM. The rotor speed can follow well the step response at the input (Figure 11(b)).

In step 4, the three-phase stator currents are fed back and applied through Clack and Pack transformation. The results

Figure 11: The speed controller is applied to the system, and the real rotor speed is fed back to the speed controller. (a) The block diagram of step 3. (b) The rotor speed tracks the speed command perfectly. (c) Simulink model.

of that transformation are compared with i_d and i_q obtained from the motor. Then, i_d and i_q are fed back to the PI controller in the current loop to create the completed system. The block diagram of this step is shown in Figure 12(a), and the Simulink model is shown in Figure 12(b). In this step, the rotor speed also tracks the step response perfectly (Figure 12(c)). Figure 12(d) also validates the effectiveness of current vector control in Figure 1 ($i_d \approx 0$). The three-phase stator current is balanced (Figure 12(e)).

In step 5, the PI and fuzzy PI controllers are applied to check the overall system with three load cases. The Simulink model is shown in Figure 13. The VHDL code of the SLC is embedded in the ModelSim_1 block; the VHDL code of the CLC is embedded in the ModelSim_2 block; the inverter based on IGBT and PMSM is designed on Matlab/Simulink. The fuzzy PI rule table is designed in Figure 7, and its parameters are shown in Figure 14. The learning rate is set to 0.5. The load of motor is in light-load ($J = 0.000036$, $F = 0.00043$) and heavy-load ($J = 0.000324$, $F = 0.0039$) conditions. The heavy-load condition is nine times heavier than light-load condition.

The sampling frequency of the CLC and SLC is designed with 16 kHz and 2 kHz, respectively. The clocks of 50 MHz and 12.5 MHz supply to all modules of ModelSim. The FPGA resource usage of SLC and CLC blocks is 4915 LEs

(a)

(b)

(c)

Figure 12: Continued.

(d)

(e)

Figure 12: The three-phase stator currents are fed back. (a) The block diagram of step 4. (b) The Simulink model. (c) The rotor speed tracks the speed command without overshoot. (d) $i_d \approx 0$ and the PMSM can control as DC motor. (e) The three-phase stator current balances.

Figure 13: The simulation model of the proposed algorithm on Matlab/Simulink.

dE \ E	−600	−400	−200	0	200	400	600
−150	−600	−600	−600	−600	−400	−200	0
−100	−600	−600	−600	−400	−200	0	200
−50	−600	−600	−400	−200	0	200	400
0	−600	−400	−200	0	200	400	600
50	−400	−200	0	200	400	600	600
100	−200	0	200	400	600	600	600
150	0	200	400	600	600	600	600

Figure 14: The parameters of the fuzzy PI controller.

Table 3: The parameters of PMSM.

Parameters	Value
Stator resistor	1.3 Ω
Stator inductance	6.3 mH
Pole pairs	4
Inertia	$J = 0.000108$ kg·m^2
Friction factor	$F = 0.0013$ N·m·s
Voltage constant	52.2 V_peak/1000 rpm
Torque constant	0.43169 N·m/A_peak

(logic elements) (0 RAM bits) and 1742 LEs (196,608 RAM bits), respectively. The designed PMSM parameters are listed in Table 3.

The article also analyzes the case that just uses the PI controller. k_i and k_p are designed at a normal load; the rotor speed tracks the output of SM perfectly (Figure 15(b)). The setting value of PI speed controller and PID on d-axis and q-axis is shown in Table 4. After that, the PMSM is changed to light-load and heavy-load conditions for testing. At the light-load condition, the response of rotor speed is overshot to 43 rpm and the rotor runs faster than the normal-load condition (Figure 15(a)). At the heavy-load condition, the response of rotor speed is overshot to 47 rpm and the rotor runs slower than the normal-load condition (Figure 15(c)). The error between SM and rotor speed is maximum with 393 rpm (light-load condition) and 177 rpm (heavy-load condition).

The simulation results show that when using the fuzzy PI controller, the responding speed of the motor is obtained in the case of the light-load, normal-load, and heavy-load condition, and heavy-load (Figures 16–18). Although the load changed very largely, the rotor speed still tracked the output of the SM very well.

Also, from 0–0.1 s in Figure 19(a), the PI controller is used to control the speed of the motor in the light-load condition. The speed response has lag and overshoot. After 0.1 seconds, the fuzzy PI controller is applied, and the rotor speed tracks the SM perfectly, without overshoot. Similar to the heavy-load condition shown in Figure 19(b), from 0–0.15 s, the PI controller is used, and from 0.15 s–0.35 s, the fuzzy PI controller is used. The simulation result shows that with the proposed algorithm, the rotor speed can track the reference model well.

8. Experiment Work

After the successful simulation, we realize this code on the FPGA kit. The experimental system is shown in Figure 20, and the experimental results are shown in Figure 21–23.

Figure 15: The simulation results when the speed controller is the PI controller. (a) The light-load condition. (b) The normal-load condition. (c) The heavy-load condition.

Table 4: The parameters of PI speed controller and PID controller on d-axis and q-axis.

Parameters	Value
K_p speed controller	4600
K_i speed controller	27
K_p on d-axis	320
K_i on d-axis	5
K_d on d-axis	10
K_p on q-axis	224
K_i on q-axis	5
K_d on q-axis	10

The experimental system includes an FPGA (Cyclone IV EP4CE115F29C7N), a drive board, an encoder board, an ADC board, and a PMSM. The specifications of the motor are shown in Table 3. An encoder with a resolution of 2500 pps is attached to the motor as the rotor position sensor. The inverter has three sets of IGBT power transistors, and its specifications are shown in Table 5. The Toshiba IC (TLP250) is used in the gate driving circuit of IGBT. The PWM signals from the FPGA board are the inputs of the drive board. A Nios II embedded processor is set up in FPGA to construct a SOPC environment.

Figures 21–23 show the speed step response of the motor under an external load of 0 kg or 10 kg using the PI and fuzzy PI controller. The the speed command is square waves with period of 0.5 s and magnitude variation from 0 => 400 rpm => 1000 rpm => 1400 rpm => 1000 rpm => 1400 rpm => 1000 rpm => 1400 rpm => 1000 rpm.

In Figure 21(a), at the light-load condition (0 kg), the speed controller is PI; the step response shows an excellent dynamic response with a rising time of 0.1 s, without

Figure 16: The light-load condition and the fuzzy PI controller are applied in simulation. (a) The speed response. (b) The current control.

Figure 17: The normal-load condition and the fuzzy PI controller are applied in simulation. (a) The speed response. (b) The current control.

Figure 18: The heavy-load condition and the fuzzy PI controller are applied in simulation. (a) The speed response. (b) The current control.

Figure 19: The PI and fuzzy PI controllers are applied in simulation. (a) The light-load condition. (b) The heavy-load condition.

Figure 20: The experimental system.

Figure 21: The PI controller without external load (light-load condition). (a) The rotor speed. (b) The current control.

overshoot, and a zero steady state. However, when heavy-load condition (10 kg) is added and the same PI controller is used, the speed dynamic response worsens and exhibits a 48 rpm overshoot in Figure 22(a). In Figure 23(a), a fuzzy PI controller is applied to solve this problem. The speed responses are highly improved without overshoot. Due to the EMI effect, the measured current in Figures 21(b), 22(b), and 23(b) have a little ripple.

Nevertheless, the experiment result still shows that the measured current could track the current command well. Furthermore, it not only validates the function of the vector control but also could make the PMSM decouple. Therefore, the simulation and experiment results demonstrate the effectiveness and correctness of the proposed algorithm for PMSM.

9. Conclusion

The fuzzy PI controller has been studied to control the speed of PMSM. The adjustment mechanism produces a

Figure 22: The PI controller with external load = 10 kg. (a) The rotor speed. (b) The current control.

Figure 23: The fuzzy PI controller with external load = 10 kg. (a) The rotor speed. (b) The current control.

Table 5: The parameters of IGBT.

Parameters	Value
Collector-emitter voltage	600 V
Gate-emitter voltage	±20 V
DC collector current (rated)	25 A
DC collector current in a short time (1 ms)	50 A

compensation signal by using (40), which is added to the fuzzy controller's knowledge base to force the system to track the SM. The simulation and experiment results confirmed the efficiency of the proposed fuzzy PI and SVPWM for PMSM.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Sample Entropy Analysis of Horses with and without Lameness at Trot before and after Sedation

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The horses were trotted in a straight line, and the acceleration data were recorded from the movement of the horse's head. Inertial sensors were used to collect the information. The sample entropy was used to characterize the dynamics of horse gait. The gait of sound and lame horses was compared, and the sound horses had smaller values of the sample entropy. The effects of sedative detomidine on the gait of lame horses were also examined. The values of sample entropy for sedated horses were smaller than the values of the sample entropy for untreated horses. The sample entropy can provide important information about the dynamics of the equine gait.

1. Introduction

The concept of entropy originated from information technology and was introduced to quantify the amount of information obtained in communication signal [1]. The approximate entropy (ApEn) has been developed to classify complex systems that include both deterministic and stochastic processes [2]. The approximate entropy was not suitable for the analysis of the noisy data that are prevalent in biological studies, and inconsistent results were produced. As a modification of the ApEn, the sample entropy (SampEn) was designed to increase the accuracy of ApEn over a broad range of conditions [3]. The sample entropy has been useful in the study of the postural control and the center of pressure displacements [4, 5]. Stergiou et al. adopted entropy to measure the change in postural control for an infant with cerebral palsy and found that entropy values decreased significantly from prop sitting to beginning independent sitting [6]. Jia et al. reported that sample entropy of the functional connections of brain regions in healthy subjects decreased with advancing age [7]. A smaller value of the SampEn also indicates more self-similarity in dataset or less noise. The sample entropy has higher value for more complicated time series. For cardiovascular time series, one fundamental hypothesis is that healthy subjects have more complicated time series than cardiovascular patients [8].

Lameness in horses causes serious economic losses to the equine industry [9]. Detection of lameness is not difficult when lameness is moderate to severe, but confusion appears when lameness is subtle. There are disagreements between equine clinicians during subjectively evaluating lameness due to diversity of skills developed for lameness recognition. Clinicians are not able to capture small changes in locomotion pattern because of the relatively low image capturing frequency of human eye [10]. Detomidine, a commonly used sedative in horses, is often administered to unruly horses to facilitate lameness examination [11, 12]. However, there is a concern among veterinarians about the effect of sedation on movement patterns of horses [13, 14]. Many clinicians are reluctant to use sedative during lameness examination as they believe it might affect the identification of subtle lameness because of the inherent analgesic property of this drug [15, 16]. The effect of sedation on the gait of lame horses has been studied using vector sums of maximum and minimum head height and the sum of hip drop and hip hike [17].

Multiple methods have been initiated for studying the nonlinear dynamics of gait and human motion which may include model development, dynamic formulation, and

simulation solver [18]. The nonlinear dynamics equations are directly integrated or differentiated. The simulation solver demands experimental motion data and optimization-based methods. For the repeated motion of the hip during walking, an analytical approach has been proposed to obtain solution for nonlinear oscillators [19]. Optimal auxiliary functions are employed to approximate the analytical solution of the system. The deep learning technique has been used for the gait patterns [20]. With the help of deterministic learning algorithms, detailed information about gait dynamics can be produced. The authors applied the knowledge fusion strategy for gait aspects.

Equine locomotion involves rhythmic and coordinated patterns of body and limb segments [21]. An equine body system can be considered as a dynamic system, and the changes occurring in the body due to lameness can be analyzed according to dynamic laws. For the last two decades, using inertial sensors and high-speed camera capturing devices, researchers have studied the dynamic locomotion of the horse.

The concept of entropy was initiated from the energy loss during combustion and not converted into functional work. In thermodynamics, any event in an isolated system will make the entropy larger. The topological entropy estimates, in time, the progression of detectable orbits for an iterated function describing the dynamical system. For a stochastic system, the entropy is the average rate at which the information is produced. The proposed definition of entropy (SampEn) is a chosen option in our study.

The objective of this paper was to implement the SampEn for the equine gait. First, we studied the trot of sound and lame horses. Next, we compared the gait of unsedated lame horses with their gait when sedated with detomidine. The vertical movement data of the horse's head were used for this study. To our best knowledge, the use of SampEn on the analysis of horse gait was not studied. The SampEn can be utilized for the dynamic analysis of the horse gait.

2. Materials and Methods

Ten horses that were determined by subjective evaluation from the investigator to be lame and eight to be nonlame were enrolled in this study. The mean weight of the horses was 460 ± 50 kg. In the first study, the gait from sound horses and lame horses was compared by calculating SampEn. Horses were trotted for at least 25 consecutive strides during each of 10 random trials while wearing sensors.

In the second study, the effect of sedative detomidine on gait of ten lame horses was studied using two groups: Group 1 (no treatment, control group) and Group 2 (horses administered 10 mg of detomidine HCL). Horses were randomly assigned to two groups [17]. Lame horses were trotted about 25 consecutive strides every 5 minutes for 45 minutes. This study was approved by Auburn University's Animal Care and Use Committee.

Two inertial sensors were attached noninvasively to the horse as shown in Figure 1. A single-axis accelerometer was mounted on the head halter to measure the vertical acceleration of the head. A single-axis gyroscope was attached to dorsal surface of the region between the metacarpophalangeal joint and coronary band of the right forelimb to measure the rotation of the foot. The signal from sensors was transmitted to a tablet PC via Bluetooth. The raw data collected from the sensors were processed in MATLAB.

Figure 1: Inertial sensors attached noninvasively to the horse: (a) uniaxis vertical accelerometer head sensor and (b) uniaxial gyroscope right forelimb sensor.

The data were in tsv format with a unique 32-digit hex address. The raw output value in this file was digitized (8-bit) from the voltage value (+/−5 volts). A conversion factor was used to get the actual acceleration data. Figure 2 shows a simultaneous plot of vertical head acceleration and right forefoot angular velocity. Stride split was achieved by using the information from the gyroscope located at the right forelimb. Figure 3 shows the head acceleration data gathered from ten strides.

2.1. Sample Entropy. Acceleration data for each stride were time normalized to 50 data points. The whole motion cycle with 20 strides was time normalized with 1000 data points. This data length is considered sufficient for the entropy analysis [22]. After signal is segmented into strides and normalized, the entropy calculation was carried out. The SampEn gives us the probability of two data that are close (predetermined tolerance level) in m dimensional space and remain close in $m+1$ dimensional space. Larger is the probability less is the SampEn value and vice versa.

The sample entropy of a time series $x_1, x_2, \ldots x_N$ of length N with zero mean and unit standard deviation is defined by

$$\text{SampEn}(N, m, r) = -\log\left(\frac{A(r)}{B(r)}\right), \tag{1}$$

where m is the length of the subsequences of the time series; r is the maximum distance (or tolerance) between two subsequences of the same length m; $A(r)$ is the total number

Figure 2: Vertical head acceleration and right forefoot angular velocity.

Figure 3: Head acceleration for ten strides.

of subsequences in a $m+1$ dimensional phase space that are within the distance r; and $B(r)$ is the total number of subsequences in a m dimensional phase space that are within the tolerance r. The complexity level of two signals could be determined from this method. For the computation of the SampEn, the guideline detailed in Ramdani et al. was followed [23].

Sample entropy is found to be closely related to selection of m and r for short datasets. The embedding dimension m and the tolerance r are selected according to Lake et al. [24]. The procedure is based on the convergence criterion which estimates m leading to the selection of r. The median SampEn is calculated for all datasets as a function of r for $m = 1, 2, 3, 4, 5$.

Figure 4 shows that the entropy values start converging when m is equal to 3 which lead us to select $m = 3$. Next step is to compute the median of maximum relative error as a function of r for $m = 3$. The value of r is then estimated to give maximum relative error no higher than 0.05.

Figure 4: Median SampEn as a function of m and r.

2.2. Statistical Analysis. The difference of SampEn between sound and lame horses was compared with t test. The SampEn values were subjected to a log transformation to ensure the homogeneity of variance. To test the effect of detomidine on SampEn, linear mixed effects model $Y = X\beta + Z\mu + e$ was performed, where Y was the SampEn values, X was the treatment design matrix (treatment with detomidine or no treatment; data collection at 5, 10, 15, 20, 25, 30, 35, 40, and 45 minutes), β was the vector of fixed treatment effects, Z was the subject design matrix (horse), μ was a vector of random subject-specific effects, and e was a vector of random experimental error. The normality of the data was tested using the Shapiro–Wilk test. First-order autoregression has been used as the correlation structure. If a significant influence was found, paired t tests were performed to determine where significant differences were present. The significance level was set at $p < 0.05$. Statistical analyses were performed using SAS (SAS Studio 3.8, SAS Institute Inc., Cary, NC, USA.).

3. Results

When applied fast Fourier transform, no major differences were found between lame and sound horses (Figure 5). The phase planes are affected by the duration of the detomidine treatment but there are no limit cycles or clear attractors (Figure 6). For a periodic solution, the phase plane is a closed curve. For the return maps, there are differences between 5 min and 45 min following detomidine treatment but the systems are not periodic or quasiperiodic (Figure 7). The return map of a periodic system consists of a finite set of points. For a quasiperiodic system, a closed orbit tends to display on the return map. If a system has chaotic motion, an infinite number of points with a well-defined structure usually make up the return map. We cannot find distinct points, segments, or curves on the return maps.

Figure 5: Fast Fourier transform for (a) a sound horse and (b) a lame horse.

Figure 6: Phase plane for a lame horse at (a) 5 min and (b) 45 min following treatment with detomidine.

Figure 7: The return map for a lame horse at (a) 5 min and (b) 45 min following treatment with detomidine.

Gait from sound horses was associated with significantly smaller SampEn than lame horses ($p < 0.0001$) (Figure 8).

Treatment with detomidine had a significant effect on the SampEn of lame horses compared with no treatment ($p = 0.0225$) (Figure 9). At the 15 min data collection, the SampEn was significantly lower after treatment with detomidine compared with no treatment ($p = 0.0438$). No significant differences were found at other times of data collection.

Figure 8: Comparison of SampEn between sound and lame horses.

4. Discussion

Based on the results of this study, the sample entropies of lame horses are larger than that of sound horses. The sample entropies were significantly smaller at 15 min after the intravenous administration of 10 mg detomidine compared with no treatment. Our results are consistent with the results of Buchner et al. which observed a decrease of head height and increase of stride duration and stride length after detomidine sedation, with the maximum effect of detomidine at 15 min recording [11].

The investigation of sample entropy may reveal important information about the characteristics of locomotion pattern. Higher entropy values are associated with a functional decline of the locomotion control system, indicating a greater need for adaptability to perturbations in the system [5]. In contrast, the signal emanated from a system with low entropy values generates less additional information over time and is more predictable [25]. The less complex gait was observed in the sound horses in this study.

It has been reported that gait complexity of human adults walking on a treadmill increased with increasing walking speed, possibly related to the mechanism of

Figure 9: Mean SampEn of ten horses at 5, 10, and every 5 min over a total period of 45 min following no treatment and treatment with detomidine.

movement control [26]. In this study, however, horses were trotted by a single handler for consistency of speed. Evaluation of the effect of speed on sample entropy of horses should be considered for future study to help understanding the control system in horses. Furthermore, future studies of different gait of sound horses would benefit from using sample entropy as a new measurement variable.

5. Conclusion

In spite of achievements in the area, there is no common measure to evaluate the dynamics of animal and human locomotion. In the present study, we investigated the equine gait of sound and lame horses. By comparison, smaller SampEn was identified on sound horses. Moreover, we observed the effect of detomidine on gait of lame horses. The sample entropy values for horses with detomidine were smaller than the SampEn of horses that did not receive a treatment. In the present work, we have shown that, using an appropriate methodology, the SampEn algorithm was able to characterize the dynamic features of equine gait. The technique in this study gives the clinical team a useful measure of the gait. It is expected that this research will have straight implementation to the analysis of gait in humans and animals.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Flocking Control of Mobile Robots with Obstacle Avoidance Based on Simulated Annealing Algorithm

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Flocking control problem of mobile robots under environment with unknown obstacles is addressed in this paper. Based on the simulated annealing algorithm, a flocking behaviour for mobile robots is achieved which converges to alignment while avoiding obstacles. Potential functions are designed to evaluate the positional relationship between robots and obstacles. Unlike the existing analytical method, simulated annealing algorithm is utilized to search the quasi-optimal position of robots in order to reduce the potential functions. Motion control law is designed to drive the robot move to the desired position at each sampling period. Experiments are implemented, and the results illustrate the effectiveness of the proposed flocking control method.

1. Introduction

Flocking problems are studied and applied in many research fields, such as self-organized mobile sensor networks in [1], cooperative unmanned aerial vehicles (UAVs) in [2–5], and military reconnaissance in [6]. Research of flocking behaviour stems from the coordination phenomena in nature, from swarming of bacteria, biochemical cellular networks, up to flocking of birds, schooling of fish, and herds of land animals [7]. Reynolds proposed a computer model of coordinated animal motion named "boids" in [8] and defined three concepts to describe the behaviours of boids:

(1) Collision avoidance: avoid collisions with nearby flockmates

(2) Velocity matching: attempt to match velocity with nearby flockmates

(3) Flock centering: attempt to stay close to nearby flockmates

These behaviours are seen in astonishing amount of coordinated systems, which exhibit unbelievable efficient and robust coordination [9, 10]. These inspired a lot of work in statistical physics and control theories about stability. Flocking behaviour has been addressed in the context of nonequilibrium phenomena in many degree-of-freedom dynamical systems and self-organization in systems of self-propelled particles [7, 11, 12].

Booming-related work has been done worldwide in the last decades. Some works focus on the dynamic model of mobile robots [13]. In [14], a method based on the intrinsic properties of the Euclidean plane is proposed for the calculation of the moving mechanism on a trajectory. In [15], a direct analytical method for solving the linear transformation of the characteristic equation for industrial robots is proposed. A model predictive flocking control scheme for second-order multiagent systems with input constraints is designed in [16]. In [17], a dynamic pinning control algorithm (DPCA) is developed to generate stable flocking motion for all the agents without the assumption of connectivity or initial connectivity of the network. In [18], a distributed control protocol with individual local information is proposed for the second-order multiagent system with interference. The problem of flocking and velocity alignment in a group of kinematic

nonholonomic agents in 2 and 3 dimensions is addressed in [19]. A geodesic control law that minimizes a misalignment potential and results in velocity alignment and flocking is proposed. In [20], the flocking of multiple agents which have significant inertias and evolve on a balanced information graph is studied. Passive decomposition method is used to incorporate this inertial effect which can cause unstable group behaviour. The proposed flocking control law is proven to be stable and can ensure that the internal group shape is exponentially stabilized to a desired state.

In recent years, the flocking problem of mobile robots has become an attractive subject in the research field of the multiagent system and cooperative robots. In [21], the cooperative motion coordination of leader-follower formations of nonholonomic mobile robots under visibility and communication constraints in known polygonal obstacle environments is studied. Each robot can take care of converging to a desired configuration while maintaining visibility with its target. Flocking with obstacle avoidance is an emerging research interest in recent years [22–26]. A new distributed coordination algorithm with Voronoi partitions for multivehicle systems is presented in [22], which results not only in obstacle avoidance and motion to the goal but also in a desirable geographical distribution of the vehicles. A flocking algorithm in the presence of arbitrary shape obstacle avoidance is presented in [23] both in the 2D space and 3D space. In [27], learning-based model predictive control approach is applied to the flocking problem of UAVs in the presence of uncertainties and obstacles. A distributed framework that ensures reliably collision-free behaviours in large-scale multirobot systems with switching interaction topologies is presented in [28].

In practical applications, such as formulation of the mobile robot or surface vessels, the agent model is underactuated, and the kinematic motion is constricted with nonholonomic constraints. In our research project, AmigoBot is adopted as an $\alpha-$agent in the flocking control problem. The kinematic model of AmigoBot is a kind of nonholonomic system in which the number of control inputs is less than the degree of freedom and cannot be transformed into a second-order linear system. It is necessary to redesign the control algorithms for the flocking of AmigoBots.

In this paper, a control method for the flocking problem of mobile robots is proposed. Simulated annealing (SA) algorithm is used to design a simple behaviour for each robot while flocking can be self-established. SA algorithm is a probabilistic technique for approximating the global optimum of a given function. Specifically, it is a metaheuristic to approximate global optimization in a large search space for an optimization problem. It is often used when the search space is discrete (e.g., the traveling salesman problem). For problems where finding an approximate global optimum is more important than finding a precise local optimum in a fixed amount of time, simulated annealing may be preferable to alternatives such as gradient descent.

The paper is organized as follows. Section 2 gives the kinematic model of AmigoBot and the problem formulation. Design process of the flocking control law is presented in Section 3. Simulation is implemented to illustrate the effectiveness of the proposed control law, and results are shown in Section 4. Finally, the conclusions are made in Section 5.

2. Problem Formulation

2.1. Model of Flocking Agents. The first model which is widely considered as a flocking agent is constructed in [12], which was proposed by Vicsek et al. to interpret the flocking phenomenon such as clustering and convection:

$$\begin{aligned} \mathbf{x}_i(t+1) &= \mathbf{x}_i(t) + \mathbf{v}_i(t)\Delta t, \\ \theta(t+1) &= \langle \theta(t) \rangle_r + \Delta\theta, \end{aligned} \tag{1}$$

where $\mathbf{x}_i$ is the position of particle i on the plane, $\mathbf{v}_i$ is the velocity with an absolute value v and an angle given by $\theta(t+1), i = 1, 2, \cdots, n$, $\langle \theta(t) \rangle_r$ is the average direction of the particles in the neighbourhood of particle i, and $\Delta\theta$ is the random noise.

From the perspective of theoretical research, the dynamics of agents in flocking is also often modelled as a second-order linear system [29]:

$$\dot{x}_i = v_i, \dot{v}_i = u_i, i = 1, 2, \ldots, n, \tag{2}$$

where x_i, v_i, and u_i are the position, velocity, and control force, respectively.

The other second-order linear system of the flocking agent is proposed in [30, 31]. The agents moving on the plane are modelled with the following dynamics:

$$\dot{r}_i = v_i, \dot{v}_i = u_i, i = 1, 2, \ldots, n, \tag{3}$$

where $r_i = (x_i, y_i)^T$ is the position of agent i, $v_i = (\dot{x}_i, \dot{y}_i)^T$ is its velocity, and $u_i = (u_x, u_y)^T$ is its control inputs. The heading angle of agent i is defined with $\theta_i = \arctan(\dot{y}_i, \dot{x}_i)$.

2.2. Kinematic Model of the Mobile Robot. Different from the agent models proposed above which are treated with the linear system, the kinematic model of the mobile robot is much more complex. As shown in Figure 1, the mobile robot is controlled only by the velocities of two drive wheels. Assuming that there exists no slipping motion between the wheels and the ground, the dynamics of AmigoBot can be modelled as

$$\begin{aligned} \dot{x}_i &= v_i \cos\theta_i, \\ \dot{y}_i &= v_i \sin\theta_i, \\ \dot{\theta}_i &= \omega_i, \end{aligned} \tag{4}$$

where v_i and ω_i are the linear and angular velocity, respectively, (x_i, y_i) is the position of AmigoBot in Cartesian coordinates, and θ_i is the orientation angle of the AmigoBot.

(a) (b)

Figure 1: Mobile robot with sonar sensors. (a) AmigoBot. (b) Model of AmigoBot.

Unlike the models of (1)–(3), mobile robot like AmigoBot is usually considered as a nonholonomic system due to the fact that the system has less dimension of control inputs than its degrees of freedom. For AmigoBot, it has 3 degrees of freedom, i.e., the lateral and longitudinal motion on the horizontal plane and rotatory motion along the axis vertical to the horizontal plane but with only two control inputs.

For the differential wheel drive mobile robot, it has $v_i = (v_{Ri} + v_{Li})/2$ and $\omega_i = (v_{Ri} - v_{Li})/L$. v_{Ri} and v_{Li} are the velocities of the right and left wheel, respectively, and L is the distance between the two wheels as described in Figure 1. Constrained by limitations on the control inputs, the existing flocking laws or algorithms are not suited for the flocking control problem of mobile robots.

3. Design of the Flocking Control Law

3.1. Potential Function of Coordination. Each mobile robot is considered as an α – agent. Define the position vector of the ith AmigoBot with $r_i = [x_i, y_i]^T$. The relative positions between the ith and j^{th} AmigoBots are defined with $r_{ij} = r_i - r_j$. The distance between these two AmigoBots is then given with $\|r_{ij}\|$. A potential function V_{ij} is defined to evaluate the relative positions between any pair of neighbouring AmigoBots as follows:

$$V_{ij}\left(\|r_{ij}\|\right) = \frac{d_r^2}{\|r_{ij}\|^2} + \log\left(\|r_{ij}\|^2\right) + e^{\left(\|r_{ij}\| - d_r\right)^2}, \tag{5}$$

where $d_r > 0$ is a design parameter. The curve of function V_{ij} is shown in Figure 2. d_r determines the desired distance between the agents. When $\|r_{ij}\| = d_r$, it has $\min V_{ij}(\|r_{ij}\|) = 2 + \log(d_r^2)$. Unlike the potential function proposed in [31], also shown in Figure 2 as traditional V_{ij}, the third term in (5) is helpful to get a flexible behaviour such as split and squeezing maneuver in an environment with obstacles.

The potential function of AmigoBot i is then defined with the sum of all V_{ij}:

$$V_\alpha^i = \sum_{j \in \mathscr{A}} V_{ij}\left(\|r_{ij}\|\right), \tag{6}$$

where $\mathscr{A}$ is the set that includes all other AmigoBots that the ith AmigoBot can detect with sensors.

Figure 2: Potential function V_{ij}.

3.2. Potential Function of Obstacle Avoidance. Obstacles can be detected by AmigoBot with sonar sensors in an unknown environment. Define the position of the obstacle with $O_k = [x_k, y_k]^T$. To keep a safe distance from the obstacle O_k, it is assumed that the obstacle O_k has a circular boundary with radius d_s. β – agent is then defined with the projection of the α – agent on the boundary of obstacle O_k [32].

Define the projection of the position r_i of robot i on O_k as

$$q_{ik} = \arg\min_{q_k \in \Omega_k} \|r_i - q_k\|, \tag{7}$$

where Ω_k is the boundary of O_k.

Define the distance between robot i and its projection q_{ik} on the obstacle O_k with $\|r_{ik}\| = \|r_i - q_{ik}\|$. A potential function V_{ik} is designed to evaluate the relative position between robot i and obstacle O_k as follows:

$$V_{ik}\left(\|r_{ik}\|\right) = \begin{cases} \left(\dfrac{1}{\|r_{ik}\|} - \dfrac{1}{d_s}\right)^2, & \|r_{ik}\| < d_s, \\ 0, & \|r_{ik}\| \geq d_s. \end{cases} \tag{8}$$

Define

Input: objective function V_i and $r_i, i = 1, 2, \ldots, n$
Output: quasi – optimal solution $\longrightarrow r_i^*$
for robot i do
  Initialize SA (T, $T_{\min}$, α and N)
  $r_{\text{best}} \longleftarrow r_i$
  $\tilde{r}_i \longleftarrow r_i$
  While $T < T_{\min}$ **do**
    for $i = 1$: N **do**
      $r_{\text{new}} \longleftarrow \text{Neighbor}(\tilde{r}_i)$
      if $\Delta E < 0$ **then**
        $\tilde{r}_i \longleftarrow r_{\text{new}}$
        $r_{\text{best}} \longleftarrow \tilde{r}_i$
      else
        if $\exp(-\Delta E/T) > \text{Rand}$ **then**
        $\tilde{r}_i \longleftarrow r_{\text{new}}$
    $T \longleftarrow \alpha T$
  $r_i^* \longleftarrow r_{\text{best}}$
return r_i^*

Algorithm 1: Flocking behaviour search.

$$V_\beta^i = \sum_{k \in \mathcal{O}_i} V_{ik}(\|r_{ik}\|), \tag{9}$$

where $\mathcal{O}_i$ is the set of obstacles detected by robot i.

3.3. Flocking Behaviour via Random Search Algorithm. With the aforementioned designed potential functions V_α^i and V_β^i, the flocking control problem of the mobile robot is considered as an optimization problem in this paper. In [33], Cheng et al. designed a flocking control law that can drive the agents to evolve into the flock centering state. Motion control laws are proposed to synchronize the velocities and orientation angles via the Newton method in an environment without obstacles.

For the flocking problem of the mobile robot in a complex environment with unknown obstacles, define the total potential function for robot i as

$$V_i = V_\alpha^i + V_\beta^i. \tag{10}$$

Obviously, if all the robots move towards the direction of the gradient descent of V_i at each sample period, the robots will keep a desired distance among each other finally:

$$\theta_i^* = \arg \min_{\theta_i \in \Theta_i} V_i, \quad \forall t_k, \tag{11}$$

where Θ_i is the orientation space of robot i, t_k is the sample period, θ_i^* is the optimal orientation angle which has the maximal extent of the gradient descent of V_i at t_k. Due to that the space Θ_i is continuous, methods based on the gradient descent are generally adopted [30, 33]. Methods based on the random search algorithm provide another dependable solution. In this paper, SA algorithm is applied to searching of the minimal solution of V_i which is designed to regulate the flocking state.

Compared to other approximation methods, such as the Newton method, gradient descent method, and Levenberg–Marquardt method, SA can avoid being trapped in the local minimal in early iterations and is able to explore globally for better solutions in finite iterations [34]. SA is a method for solving unconstrained and bound-constrained optimization problems. The method models the physical process of heating a material and then slowly lowering the temperature to decrease defects, thus minimizing the system energy [35, 36]. There is an essential similarity between the flocking control problem and the simulated annealing course. Both of them aim to explore the optimal solution that has minimal energy or desired homogeneity.

The flocking control problem of mobile robots is considered as an approximation of the optimal solution of each V_i. With full information of all the other robots and the obstacles detected by sonar sensors, each robot tries to keep a quasi-optimal distance from each other in the motion while avoiding the obstacles. Firstly, the optimal position r_i^* with minimal V_i is approximated via the SA algorithm. Then, the optimal orientation angle θ_i^* is calculated with r_i^*. Finally, a motion control law is designed to drive the robot move to the optimal position r_i^* at each sampling period t_k.

The pseudo-code of the random search method of r_i^* via the SA algorithm is shown in Algorithm 1, where $r_i^* = [x_i^*, y_i^*]^T$ is the desired position vector for the ith AmigoBot. Define the change of energy level with

$$\Delta E = V_i(r_{\text{new}}) - V_i(\tilde{r}_i). \tag{12}$$

The initial position $\tilde{r}_i$ in Neighbor$(\tilde{r}_i)$ is not generated randomly but given with r_i. Also, the function Neighbor$(\tilde{r}_i)$ is defined with

$$r_{\text{new}} = \tilde{r}_i + \text{Step} * [\text{Rand}(), \text{Rand}()]^T, \tag{13}$$

where Step is a parameter decided by the maximal velocity of the robot and Rand is a random function which can generate a value randomly in the range between 0 and 1.

Table 1: Parameters in the flocking control method.

	Variables	Value
Potential function	d_r	1000 mm
	d_s	1000 mm
Motion control	k_w	0.0001
	k_v	0.003
SA algorithm	T	100
	$T_{\min}$	0.001
	α	0.5
	N	20
	Step	100

Figure 3: Experiment in MobileSim. (a) $t_k = 0$. (b) $t_k = 50$. (c) $t_k = 80$. (d) $t_k = 110$. (e) $t_k = 120$. (f) $t_k = 150$. (g) $t_k = 170$. (h) $t_k = 190$. (i) $t_k = 270$.

Obviously, if r_i^* is well tracked, V_i would converge to the minimal value, and the distance between the agents would also converge to a constant value, which in fact is determined by d_r. That is, the flocking centering state can be guaranteed.

Once the quasi-optimal position r_i^* is achieved, the optimal orientation angle can be calculated with $\theta_i^* = \arctan((y_i^* - y_i)/(x_i^* - x_i))$.

Define $\theta_i^* = \arctan((y_i^* - y_i)/(x_i^* - x_i))$, and it has

$$\frac{\partial \theta_i^*}{\partial x_i} = -\frac{y_i - y_i^*}{(x_i - x_i^*)^2 + (y_i - y_i^*)^2},$$
$$\frac{\partial \theta_i^*}{\partial y_i} = \frac{x_i - x_i^*}{(x_i - x_i^*)^2 + (y_i - y_i^*)^2}. \tag{14}$$

Define $\theta_e = \theta_i - \theta_i^*$, and design the control law for ω_i with

$$\omega_i = -k_w(\theta_i - \theta_i^*) + v_i \frac{\partial \theta_i^*}{\partial x_i} \cos\theta_i^* + v_i \frac{\partial \theta_i^*}{\partial y_i} \sin\theta_i^*, \tag{15}$$

where $k_w > 0$ is a design parameter. With the control input ω_i in (15), it has

$$\dot{\theta}_e = -k_w(\theta_i - \theta_i^*) = -k_w \theta_e. \tag{16}$$

Define a candidate function $V_{\theta_e} = (1/2)\theta_e^2$; then, it has $\dot{V}_{\theta_e} = -k_w \theta_e^2 \le 0$. With Lyapunov stability and LaSalle's invariant principle, it can be proven that the equilibrium $\theta_e = 0$ is asymptotically stable. That is, with the control law for ω_i in (15), the ith AmigoBot can be steered towards the desired position r_i^*.

Define $\delta r_i = [x_i - x_i^*, y_i - y_i^*]^T$. Let

$$v_i = \frac{1}{N-1} \sum_{j \in \mathscr{A}} v_j + k_v \|\delta r_i\| V_i, \tag{17}$$

Figure 4: Trajectories of robots.

Figure 5: Connection graph.

Figure 6: Potential function V_α^i.

where v_r is the linear speed of the $\gamma-$agent or the speed required for the flocking motion and $k_v > 0$ is a design parameter.

4. Simulation Results

The effectiveness of the proposed flocking control method is illustrated by experiments in MobileSim. MobileSim is software for simulating mobile robots and their environments, for debugging and experimentation with ARIA or other software that support mobile robot platforms. In this experiment, five AmigoBots are controlled to form a flocking centering state while avoiding the obstacles. The robots are connected by software with TCP ports from 8101 to 8105. Software is coded in VS2008. The data in the control process are stored in a text file at each sampling time, and the results are depicted in MATLAB figures.

The initial positions of the five AmigoBots are set with $(0, 2000\,\text{mm})$, $(0, 1000\,\text{mm})$, $(0, 0)$, $(0, -1000\,\text{mm})$, and $(0, -2000\,\text{mm})$. All the headings of the AmigoBots are set with 0°. Velocities of robots are set with 20mm/sec at the beginning.

The parameters in the method proposed above are designed as given in Table 1.

The initial and some intermediate processes of robots are shown in Figure 3. Obstacles are plotted with squares, hexagons, and triangles. With sonar sensors, the robot can detect the obstacles at each sampling period (500 ms). The positions of the other α-agents are shared at a period of 1 sec. From Figures 3(c)–3(f), it can be seen that the robots smartly avoid the obstacles. As shown in Figures 3(g)–3(i), all the robots attempt to assemble with a desired distance after passing through the obstacles, which brings out a flocking center state.

The detailed traces of the five AmigoBots during the flocking process are depicted in Figure 4. It proves that the designed potential function of V_α^i and V_β^i can regulate the relative position between robots and obstacles effectively. The robots keep a safe distance from obstacles while across the gap and a desired distance from each other in the obstacle-free space. Connection graph of the robots marked with the red circle is shown in Figure 5, which illustrates the evolution of the flocking center state during the whole process. It can be seen that the five AmigoBots are driven to a stable pattern of geometry. Each AmigoBot has a stable neighbouring relationship and keeps a stabilized motion.

For the 10 connection graphs in Figure 5, define $\overline{V}_{\alpha k}$ as the average of the 5 potential function V_α^i for each graph. It has $\overline{V}_{\alpha k} = [1605.2, 57.4, 88.9, 6.2, 148.0, 5.0, 3.9, 3.9, 3.6, 6.2]$. The potential function V_α^i reduces greatly from the beginning and increases highly while the robots go across the obstacles and then reduces to a low value when robots are in the obstacle-free space.

Figure 7: Potential function V_β^i.

Figure 8: Orientation angles of robots.

Figure 9: Velocities of robots.

The responses of potential functions V_α^i and V_β^i are depicted in Figures 6 and 7, which illustrate how V_α^i and V_β^i affect the behaviour of the robot. It can be seen that when the distances between robots are farther than the desired distance d_r or the distances to the obstacles are nearer than the safe distance d_s, V_α^i and V_β^i have large values, which impel the robot to move to a desired position to reduce them with the SA algorithm. It has t_0, $V_\alpha^i(t_0) = [7932.5, 658.0, 139.0, 664.0, 7931.7]$ $(i = 1, 2, 3, 4, 5)$ at the starting time t_0, and $V_\alpha^i(t_k) = [16.57, 12.38, 16.82, 20.80, 20.63]$ at the ending time $t_k = 350$. All the robots have approximated the minimal values of V_α^i.

Figures 8 and 9 show the flocking behaviour of alignment. The orientation angles and velocities of five robots almost converge to the same value.

5. Conclusion

In this paper, the flocking behaviour of the mobile robot in an unknown environment with obstacles is designed via the heuristic search algorithm. With the potential functions defined for the α – agent and β – agent, the mobile robot can acquire the quasi-optimal direction with the SA algorithm in finite search times. Control laws for v_i and ω_i are also designed. The system of mobile robots acts out separation

and alignment behaviours obviously. Moreover, the robots can avoid obstacles, detected with sonar sensors, by a safe distance. Future work will address the issue of flocking control of mobile robots in the field environment with this proposed method.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On Dynamics of Iterative Techniques for Nonlinear Equation with Applications in Engineering

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In this article, we construct an optimal family of iterative methods for finding the single root and then extend this family for determining all the distinct as well as multiple roots of single-variable nonlinear equations simultaneously. Convergence analysis is presented for both the cases to show that the optimal order of convergence is 4 in the case of single root finding methods and 6 for simultaneous determination of all distinct as well as multiple roots of a nonlinear equation. The computational cost, basins of attraction, efficiency, log of residual, and numerical test examples show that the newly constructed methods are more efficient as compared to the existing methods in the literature.

1. Introduction

To solve a nonlinear equation,

$$f(s) = 0 \tag{1}$$

is the oldest problem of science, in general, and mathematics, in particular. The nonlinear equations have diverse applications in many areas of science and engineering. In general, to find the roots of (1), we look towards iterative schemes, which can be further classified as to approximate the single root and all roots of (1). In this article, we are going to work on both types of iterative methods. Many iterative methods of different convergence orders already exist in the literature (see [1–11]) to approximate the roots of (1). Ostrowski [12] defined the efficiency index I to classify these iterative methods in terms of their convergence order k and number of function evaluations or its derivatives per iteration, say v, i.e.,

$$I = k^{1/v}. \tag{2}$$

An iterative method is said to be optimal according to the Kung–Traub conjecture [13], if

$$H = 2^{v-1}. \tag{3}$$

holds, where H is the optimal order of convergence. The aforementioned methods are used to approximate one root at a time. But, mathematicians are also interested in finding all the roots of (1) simultaneously. This is due to the fact that simultaneous iterative methods are very popular due to their wider region of convergence, are more stable as compared to single root finding methods, and can be implemented for parallel computing as well. More detail on single as well as simultaneous determination of all roots can be found in [1, 10–28] and the reference cited therein.

The most famous of the single root finding methods is the classical Newton–Raphson method:

$$s_{i+1} = s_i - \frac{f(s_i)}{f'(s_i)}, \quad i = 1, 2, \ldots. \tag{4}$$

Method (4) is optimal with an efficiency of 1.41 by the Traub conjecture. If we use Weierstrass' correction [26]:

$$\frac{f(s_i)}{f'(s_i)} = w(s) = \frac{f(s_i)}{\prod_{j=1}^{n}(s - s_j)}, \tag{5}$$

in (4), then, we get classical Weierstrass–Dochev methods to approximate all roots of nonlinear equation (1) as

$$s_{i+1} = s_i - \frac{f(s_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{n}(s_i - s_j)}, \quad i, j = 1, 2, 3, \ldots, n. \tag{6}$$

Method (6) has a convergence order 2. Later, Aberth [29] presented the 3rd order simultaneous method, given as

$$s_{i+1} = s_i - \frac{1}{(1/N(s_i)) - \sum_{\substack{j=1 \\ j \neq i}}^{n}(1/(s_i - s_j))}, \tag{7}$$

where $N(s_i) = (f'(s_i)/f(s_i))$.

First of all, we construct a family of optimal fourth-order methods using the procedure of weight function and then convert it into simultaneous iterative methods for finding all distinct as well as multiple roots of nonlinear equation (1).

2. Construction of Methods and Convergence Analysis

King [9] presented the following optimal fourth-order method (abbreviated as E1):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = y_i - \dfrac{f(s_i) + \beta f(y_i)}{f(s_i) + (\beta - 2) f(y_i)}\left(\dfrac{f(y_i)}{f'(s_i)}\right). \end{cases} \tag{8}$$

Cordero et al. [7] gave the fourth-order optimal method as follows (abbreviated as E2):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = s_i - \dfrac{f(s_i) + f(y_i)}{f'(s_i)} - \dfrac{(f(y_i))^2 (2f(s_i) + f(y_i))}{(f(s_i))^2 f'(s_i)}. \end{cases} \tag{9}$$

Chun [4] in 2008 gave the fourth-order optimal method as follows (abbreviated as E3):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right)\left(1 + \left(\dfrac{f(y_i)}{f(s_i)}\right) + 2\left(\dfrac{f(y_i)}{f(s_i)}\right)^2\right). \end{cases} \tag{10}$$

Maheshwari [11] gave the fourth-order optimal method as follows (abbreviated as E4):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = s_i - \dfrac{1}{f'(s_i)}\left(\dfrac{f(s_i)^2}{f(y_i) - f(s_i)} - \dfrac{(f(y_i))^2}{f(s_i)}\right). \end{cases} \tag{11}$$

Chun [3] gave the fourth-order optimal method as follows (abbreviated as E5):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right) - \left(\dfrac{f(y_i)}{f'(s_i)}\right)\left(\dfrac{f(s_i)}{f(s_i) - f(y_i)}\right)^2. \end{cases} \tag{12}$$

Kou et al. [10] gave the fourth-order optimal method as follows (abbreviated as E6):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = s_i - \dfrac{(f(s_i))^2 + (f(y_i))^2}{(f'(s_i))(f(s_i) - f(y_i))}. \end{cases} \tag{13}$$

Behzad [5] give the fourth-order optimal method as follows (abbreviated as E7):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right) - \left(\dfrac{f(s_i) + 2f(y_i)}{f(s_i) + f(y_i)}\right)^2 \left(\dfrac{f(y_i)}{f'(s_i)}\right). \end{cases} \tag{14}$$

Chun [2] in 2006 gave the fourth-order optimal method as follows (abbreviated as E8):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = y_i - \left(\dfrac{f(y_i)}{f'(s_i)}\right)\left(\dfrac{f(s_i) + 2f(y_i)}{f(s_i)}\right). \end{cases} \tag{15}$$

Ostrowski [12] give the fourth-order optimal method as follows (abbreviated as E9):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = y_i - \dfrac{(s_i - y_i) f(y_i)}{f(s_i) - 2f(y_i)}. \end{cases} \tag{16}$$

Here, we propose the following two families of iterative methods:

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = y_i - \left(\dfrac{f(y_i)}{f'(s_i)}\right)\left\{\beta(f(y_i))^2 + (\Psi(u))^3\right\}, \end{cases} \tag{17}$$

where $u = (f(y_i)/f(s_i))$ and β is a real number.

For the iteration schemes (17), we have the following convergence theorem by using CAS Maple 18 and error relation of the iterative schemes defined in (17) is found.

Theorem 1. *Let $\zeta \in I$ be a simple root of a sufficiently differential able function $f: I \subseteq R \longrightarrow R$ in an open interval I. If s_0 is sufficiently close to ζ and $\Psi(u)$ is a real-valued function satisfying $\Psi(0) = 1$, $\Psi\prime(0) = (2/3)$, and $\Psi\prime\prime(0) < \infty$, then the convergence order of the family of iterative method (17) is four and the error equation is given by*

$$e_{i+1} = \left(\frac{11}{3}c_2^3 - c_2c_3 - \frac{3}{2}c_2^3\Psi''(0)\right)e_i^4 + O\left(e_i^5\right), \tag{18}$$

where $c_m = (f^m(\zeta)/m!f'(\zeta))$, $m \geq 2$.

Proof. Let ζ be a simple root of f and $s_i = \zeta + e_i$. By Taylor's series expansion of $f(s_i)$ around $s = \zeta$, taking $f(\zeta) = 0$, we get

$$f(s_i) = f'(\zeta)\left(e_i + c_2e_i^2 + c_3e_i^3 + c_4e_i^4 + Oe_i^5\right), \tag{19}$$

and

$$f'(s_i) = f'(\zeta)\left(1 + 2c_2e_i + 3c_3e_i^2 + 4c_4e_i^3 + Oe_i^4\right). \tag{20}$$

Dividing (19) by (20), we have

$$\frac{f(s_i)}{f'(s_i)} = e - c_2e_i^2 + (2c_2 + 2c_3)e_i^3 + O\left(e_i^4\right), \tag{21}$$

so

$$\begin{aligned} y_i &= \zeta + c_2e_i^2 + \left(-2c_2^2 + 2c_3\right)e_i^3 + \cdots, \\ f(y_i) &= c_2e_i^2 + \left(-4c_2^2 + 2c_3\right)e_i^3 + \cdots. \end{aligned} \tag{22}$$

Now,

$$\begin{aligned} \left(\frac{f(y_i)}{f'(s_i)}\right) &= c_2e_i^2 + \left(-4c_2^2 + 2c_3\right)e_i^2 + \cdots, \\ \left(\frac{f(y_i)}{f(s_i)}\right) &= u = c_2e_i + \left(-3c_2^2 + 2c_3\right)e_i^2 + \cdots. \end{aligned} \tag{23}$$

Expanding $\Psi(u)$ about the origin, we have

$$\begin{aligned} \Psi(u) &= \Psi(0) + \Psi'(0)u + \frac{1}{2}\Psi''(0)u^2 + \cdots \\ &= \Psi(0) + \Psi'(0)c_2e_i + \left(-3\Psi'(0)c_2^2 + 2\Psi'(0)c_3 + \frac{1}{2}\Psi'(0)c_2^2\right)e_i^2 + \cdots, \end{aligned} \tag{24}$$

$$\beta(f(y_i))^2 + (\Psi(u))^3 = \Psi^3(0) + 3\Psi'(0)\Psi^2(0)c_2e_i + \cdots,$$

$$e_{i+1} = \left(-\Psi^3(0)c_2 + c_2\right)^2e_i^2 + \left(-3\Psi(0)\Psi'(0)c_2^2 + 4\Psi^3(0)c_2^2 - 2\Psi^3(0)c_3 - 2c_2^2 + 2c_3\right)e_i^3 + \cdots. \tag{25}$$

By putting $\Psi(0) = 1$ and $\Psi'(0) = (2/3)$ in (25), we have

$$e_{i+1} = \left(\frac{11}{3}c_2^3 - c_2c_3 - \frac{3}{2}c_2^3\Psi''(0)\right)e_i^4 + O\left(e_i^5\right). \tag{26}$$

Hence, it proves the theorem.

2.1. The Concrete Fourth-Order Methods. We now construct some concrete forms of the family of methods described by (17). Let us take the function $\Psi(u)$ satisfying the conditions of Theorem 1.

Therefore, we get the following new three iterative methods with arbitrary constant $\beta \in \mathfrak{R}$ and by choosing different weight functions given in Table 1:

Concrete method 1 (abbreviated as Q1):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = y_i - \left(\dfrac{f(y_i)}{f'(s_i)}\right)\left\{\beta(f(y_i))^2 + \left(1 + \frac{2}{3}u\right)^3\right\}. \end{cases} \tag{27}$$

Concrete method 2 (abbreviated as Q2):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = y_i - \left(\dfrac{f(y_i)}{f'(s_i)}\right)\left\{\beta(f(y_i))^2 + \left(1 + \frac{2}{3}u + \alpha u^2\right)^3\right\}. \end{cases} \tag{28}$$

Concrete method 3 (abbreviated as Q3):

$$\begin{cases} y_i = s_i - \left(\dfrac{f(s_i)}{f'(s_i)}\right), \\ z_i = y_i - \left(\dfrac{f(y_i)}{f'(s_i)}\right)\left\{\beta(f(y_i))^2 + \left(\frac{1}{1-(2/3)u}\right)^3\right\}, \end{cases} \tag{29}$$

where $u = (f(y_i)/f(s_i))$ and $\beta \in \mathfrak{R}$.

2.2. Complex Dynamical Study of Families of Iterative Methods. Here, we discuss the dynamical study of iterative methods (Q1–Q3 and E1–E9). We investigate the regions, from where we take the initial estimates to achieve the roots of the nonlinear equation. We actually numerically approximate the domain of attractions of the roots as a qualitative measure, that is, how the iterative methods depend on the choice of initial estimates? To answer this question on the dynamical behavior of the iterative methods, we investigate the dynamics of the methods (Q1–Q3) and compare it with (E1–E9). Let us recall some basic concepts of this study in the background contexture of complex dynamics. For more details on the dynamical behavior of the iterative methods, one can consult [6, 30, 31]. Taking a rational function $\mathfrak{R}_f : C \longrightarrow C$, where C denotes the complex plane, the orbit $s_0 \in C$ is defined as a set such as $\text{orb}(s) = \{s_0, \mathfrak{R}_f(s_0), \mathfrak{R}_f^2(s_0), \ldots, \mathfrak{R}_f^m(s_0), \ldots\}$. The convergence of $\text{orb}(s) \longrightarrow s^*$ is understood in the sense, if $\lim_{x\longrightarrow\infty} R^k(s) = s^*$ exists. A point $s_0 \in C$ is known as periodic with a minimal period m if $R^m(s_0) = s_0$ holds, where m is the smallest positive integer. A periodic point for $m = 1$ is known as fixed, attracting if $|R^{k'}(s)| < 1$, repelling if $|R^{k'}(s)| > 1$, and neutral otherwise. An attracting point $s_0 \in C$ defines the basin of attraction as the set of starting points whose orbit tends to s^*. The closure of the set of its repelling periodic points of a rational map is known as the Julia set denoted by $J(R)$, and its complement is the Fatou set denoted by $F(R)$. The iterative methods when applied to find the roots of (1) provide the rational map. But, we are interested in the basins of attraction of the roots of nonlinear function (1). Fatou set $F(Q)$ contains the basins of attraction of different roots is a well-known fact. In general, the Julia set (a fractal and rational map) behaves as unstable in this region. For the dynamical and graphical point of view, we take 2000×2000 grids of square $[-2.5, 2.5]^2 \in C^2$. To each root of (1), we assign a color to which the corresponding orbit of the iterative method starts and converges to a fixed point. Take color map as Jet. We use $|s_{i+1} - s_i| < 10^{-3}$ as a stopping criteria, and the maximum number of iterations is taken as 20. We mark a dark blue point, if the orbit of the iterative method does not converge to the root after 20 iterations which means it has a distance greater than to any other root. Different color is used for different roots. Iterative methods have different basins of attraction distinguished by their colors. In basins, brightness in color represents the number of iterations to achieve the root of (1). Note that the darkest blue regions denote the lack of convergence to any root of (1). Finally, in Tables 2–5, we present the elapsed time of basins of attraction corresponding to iterative maps (Q1–Q3 and E1–E9) using the tic-toc command in code using MATLAB (R2011b). Figures 1–4 show the basins of attraction of iterative methods (Q1–Q3 and E1–E9) for nonlinear functions $f_1(s) = s^3 + s - 1$, $f_2(s) = e^{-s} - 1$, $f_3(s) = ((s^3 + 2.87s^2 - 10.28)/4.62) - s$, and $f_4(s) = \sinh((s+2)/2)\sinh((s-3)/3)$, respectively. By observing the basins of attraction, we can easily judge the stability of iterative methods (Q1–Q3 and E1–E9). Elapsed time, divergent regions, and brightness in color presents that Q1–Q3 is better than E1–E9.

Table 1: Weight functions for methods.

S. no.	$\Psi(u)$, $u = (f(y_i)/f(s_i))$, $\Psi(0) = 1$, $\Psi'(0) = (2/3)$
1	$\Psi(u) = 1 + (2/3)u$
2	$\Psi(u) = 1 + (2/3)u + \alpha u^2$
3	$\Psi(u) = (1/1 - (2/3)u)$

Note. $u = (f(y_i)/f(s_i))$ and $\alpha \in \mathfrak{R}$.

Figures 1(a)–1(l) present the basins of attraction of methods (Q1–Q3 and E1–E9) for a nonlinear function $f_1(s) = s^3 + s - 1$. In Figures 1(a)–1(l), brightness of the color in basins shows the less number of iterations for convergence of iterative methods. Table 2 shows the elapsed time of Q1–Q3 and E1–E9.

Figures 2(a)–2(l) present the basins of attraction of methods (Q1–Q3 and E1–E9) for a nonlinear function $f_2(s) = e^{-s} - 1$. In Figures 2(a)–2(l), brightness of the color in basins shows the less number of iterations for convergence of iterative methods. Table 3 shows the elapsed time of Q1–Q3 and E1–E9.

Figures 3(a)–3(l) present the basins of attraction of methods (Q1–Q3 and E1–E9) for a nonlinear function $f_3(s) = ((s^3 + 2.87s^2 - 10.28)/4.62) - s$. In Figures 3(a)–3(l), brightness of the color in basins shows the less number of iterations for convergence of iterative methods. Table 4 shows the elapsed time of Q1–Q3 and E1–E9.

Table 2: Elapsed time for basins of attraction.

$f_1(s) = s^3 + s - 1$											
Q1	Q2	Q3	E1	E2	E3	E4	E5	E6	E7	E8	E9
17.4013	12.1801	11.8321	13.745	11.8612	11.6603	23.2042	12.5878	15.1456	12.6226	17.9360	12.9451

Table 3: Elapsed time for basins of attraction.

$f_2(s) = e^{-s} - 1$											
Q1	Q2	Q3	E1	E2	E3	E4	E5	E6	E7	E8	E9
12.5548	10.4487	9.4843	10.5134	9.0246	11.4686	22.6099	12.3507	12.5471	11.23315	8.8094	8.6127

Table 4: Elapsed time for basins of attraction.

$f_3(s) = ((s^3 + 2.87s^2 - 10.28)/4.62) - s$											
Q1	Q2	Q3	E1	E2	E3	E4	E5	E6	E7	E8	E9
20.2058	14.0826	12.7465	14.6167	13.4429	14.4602	22.1014	15.6955	19.3235	15.0689	20.4771	15.9503

Table 5: Elapsed time for basins of attraction.

$f_4(s) = \sinh((s+2)/2)\sinh((s-3)/3)$											
Q1	Q2	Q3	E1	E2	E3	E4	E5	E6	E7	E8	E9
35.6621	39.6453	40.0639	33.8435	39.1895	38.9645	67.9233	46.3621	56.9810	33.9914	42.2391	28.6589

Figures 4(a)–4(l) present the basins of attraction of methods (Q1–Q3 and E1–E9) for a nonlinear function $f_4(s) = \sinh((s+2)/2)\sinh((s-3)/3)$. In Figures 4(a)–4(l), brightness of the color in basins shows the less number of iterations for convergence of iterative methods. Table 5 shows the elapsed time of Q1–Q3 and E1–E9.

3. Generalization to Simultaneous Methods

Suppose nonlinear equation (1) has n roots. Then, $f(s)$ and $f'(s)$ can be approximated as

$$f(s) = \prod_{j=1}^{n}\left(s - s_j\right), f\prime(s) = \sum_{k=1}^{n}\prod_{\substack{j\neq k\\ j=1}}^{n}\left(s - s_j\right). \tag{30}$$

This implies

$$\frac{f'(s)}{f(s)} = \sum_{j=1}^{n}\left(\frac{1}{\left(s - s_j\right)}\right) = \frac{1}{\left(1/\left(s - s_i\right)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/\left(s - s_j\right)\right)}. \tag{31}$$

This gives the Aberth–Ehrlich method (7):

$$s_{i+1} = s_i - \frac{1}{\left(1/N\left(s_i\right)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/\left(s_i - s_j\right)\right)}, \quad \text{where } N\left(s_i\right) = \frac{f\left(s_i\right)}{f'\left(s_i\right)}. \tag{32}$$

Now from (31), an approximation of $(f(s_i)/f'(s_i))$ is formed by replacing s_j with z_{tj} as follows:

$$\frac{f\left(s_i\right)}{f'\left(s_i\right)} = \frac{1}{\left(1/N\left(s_i\right)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/\left(s_i - z_{tj}\right)\right)}, \quad t = 1, 2, 3. \tag{33}$$

Using (33) in (4), we have the following method for finding all the distinct roots:

$$s_{i+1} = s_i - \frac{1}{\left(1/N\left(s_i\right)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/\left(s_i - z_{tj}\right)\right)}. \tag{34}$$

In case of multiple roots, we have the following method:

$$s_{i+1} = s_i - \frac{1}{\left(1/N\left(s_i\right)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(\sigma_j/\left(s_i - z_{tj}\right)\right)}, \quad t = 1, 2, 3, \tag{35}$$

Figure 1: Basins of attraction of iterative methods ($Q1$–$Q3$ and $E1$–$E9$) for nonlinear function $f_1(s)$.

Figure 2: Basins of attraction of iterative methods ($Q1$–$Q3$ and $E1$–$E9$) for nonlinear function $f_2(s)$.

Figure 3: Basins of attraction of iterative methods ($Q1$–$Q3$ and $E1$–$E9$) for nonlinear function $f_3(s)$.

Figure 4: Basins of attraction of iterative methods ($Q1$–$Q3$ and $E1$–$E9$) for nonlinear function $f_4(s)$.

where σ_i is the multiplicity of the root s_i and

$$z_{1j} = y_j - \left(\frac{f(y_j)}{f'(s_j)}\right)\left(\beta\left(f(y_j)\right)^2 + \left(1 + \frac{2}{3}u^*\right)^3\right),$$

$$z_{2j} = y_j - \left(\frac{f(y_j)}{f'(s_j)}\right)\left(\beta\left(f(y_j)\right)^2 + \left(1 + \frac{2}{3}u^* + \alpha(u^*)^2\right)^3\right),$$

$$z_{3j} = y_j - \left(\frac{f(y_j)}{f'(s_j)}\right)\left(\beta\left(f(y_j)\right)^2 + \left(\frac{1}{1-(2/3)u^*}\right)^3\right), \tag{36}$$

in which $y_j = s_j - (f(s_j)/f'(s_j))$ and $u^* = (f(y_j)/f(s_j))$. Using correction z_{tj}, we get the following new family of simultaneous iterative methods for extracting multiple roots of nonlinear equation (1):

$$s_{i+1} = s_i - \frac{\sigma_i}{(1/N(s_i)) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(\sigma_j/\left(s_i - z_{tj}\right)\right)}, \quad t = 1, 2, 3. \tag{37}$$

Thus, we have constructed new three simultaneous iterative methods (37) abbreviated as SM1–SM3.

3.1. Convergence Analysis. In this section, we discuss the convergence analysis of a family of simultaneous methods (SM1–SM3) which is given in form of the following theorem. Obviously, convergence for the method (34) will follow from the convergence of the method (SM1–SM3) from Theorem 2 when the multiplicities of the roots are one.

Theorem 2 *Let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be the n number of simple roots of nonlinear equation (1). If $s_1^{(0)}$, $s_2^{(0)}$, $s_3^{(0)}$, …, $s_n^{(0)}$ be the initial approximations of the roots, respectively, and sufficiently close to actual roots, then the order of convergence of the method (SM1–SM3) equals six.*

Proof. Let $\varepsilon_i = s_i - \zeta_i$ and $\varepsilon_i' = s_{i+1} - \zeta_i$ be the errors in s_i and s_{i+1} approximations, respectively. Then, obviously for distinct roots

$$\frac{1}{N(s_i)} = \left(\frac{f'(s_i)}{f(s_i)}\right) = \sum_{j=1}^{n}\left(\frac{1}{(s_i - \zeta_j)}\right) = \frac{1}{(s_i - \zeta_i)} + \sum_{\substack{j\neq i\\ j=1}}^{n}\left(\frac{1}{(s_i - \zeta_j)}\right). \tag{38}$$

Thus, for multiple roots, we have from (37) that

$$\begin{aligned}
s_{i+1} - \zeta_i &= s_i - \zeta_i - \frac{\sigma_i}{(\sigma_i/(s_i - \zeta_i)) + \sum_{\substack{j\neq i\\ j=1}}^{n}\left(\sigma_j/(s_i - \zeta_i)\right) - \sum_{\substack{j\neq i\\ j=1}}^{n}\left(\sigma_j/\left(s_i - z_{tj}\right)\right)},\\
\varepsilon_i' &= \varepsilon_i - \frac{\sigma_i}{(\sigma_i/\varepsilon_i) + \sum_{\substack{j\neq i\\ j=1}}^{n}\left(-\sigma_j\left(z_{tj} - \zeta_j\right)/\left(s_i - \zeta_j\right)\left(s_i - z_{tj}\right)\right)}\\
&= \varepsilon_i - \frac{\sigma_i\varepsilon_i}{\sigma_i + \varepsilon_i\sum_{\substack{j\neq i\\ j=1}}^{n}\left(-\sigma_j\left(z_{tj} - \zeta_j\right)/\left(s_i - \zeta_j\right)\left(s_i - z_{tj}\right)\right)}\\
&= \varepsilon_i - \frac{\sigma_i\varepsilon_i}{\sigma_i + \varepsilon_i\sum_{\substack{j\neq i\\ j=1}}^{n}\left(E_i\varepsilon_j^4\right)},
\end{aligned} \tag{39}$$

where $z_{tj} - \zeta_j = \varepsilon_j^4$ from (18) and $E_i = (-\sigma_j/(s_i - \zeta_j)(s_i - z_{tj}))$. Thus,

$$\varepsilon_i' = \frac{\varepsilon_i^2\sum_{\substack{j\neq i\\ j=1}}^{n}\left(E_i\varepsilon_j^4\right)}{\sigma_i + \varepsilon_i\sum_{\substack{j\neq i\\ j=1}}^{n}\left(E_i\varepsilon_j^4\right)}. \tag{40}$$

If it is assumed that absolute values of all errors ε_j ($j = 1, 2, 3, \ldots$) are of the same order, say $|\varepsilon_j| = O|\varepsilon|$, then from (40), we have

$$\varepsilon_i' = O(\varepsilon_i)^6. \tag{41}$$

Hence, the theorem is proved.

4. Computational Aspect

Here, we compare the computational efficiencies of the Petkovic et al. [21] method and the new methods (SM1–SM3). As presented in [21], the efficiency of an iterative method can be estimated using the efficiency index given by

$$EL(m) = \frac{\log \mathbf{u}}{\mathbf{D}}, \tag{42}$$

where $\mathbf{D}$ is the computational cost and $\mathbf{u}$ is the order of convergence of the iterative method. Arithmetic operations per iteration with certain weight depending on the execution time of operation are used to evaluate the computational cost $\mathbf{D}$. The weights used for division, multiplication, and addition plus subtraction are w_d, w_m, and w_{AS}, respectively. For a given polynomial of degree m and n roots, the number of division, multiplication, and addition and subtraction per iteration for all roots are denoted by D_m, M_m, and AS_m. The cost of computation can be calculated as

$$\mathbf{D} = \mathbf{D}(m) = w_{as}\mathrm{AS}_m + w_m M_m + w_d D_m. \tag{43}$$

Thus, (42) becomes

$$EL(m) = \left(\frac{\log \mathbf{u}}{w_{as}\mathrm{AS}_m + w_m M_m + w_d D_m}\right). \tag{44}$$

Consider that the number of operations of a complex polynomial with real and complex roots reduces to the operation of real arithmetic, which is given in Table 6 as a polynomial degree m taking the dominant term of order (m^2). Applying (44) and the data given in Table 6, we calculate the percentage ratio $\rho((\mathrm{SM1–SM3}), (\mathrm{PJ6}))$ [21] given by

$$\begin{cases} \rho((\mathrm{SM1–SM3}), (\mathrm{PJ6})) = \left(\dfrac{\mathrm{EL}(\mathrm{SM1–SM3})}{\mathrm{EL}(\mathrm{PJ6})} - 1\right) \times 100, \\ \rho((\mathrm{PJ6}), (\mathrm{SM1–SM3})) = \left(\dfrac{\mathrm{EL}(\mathrm{PJ6})}{\mathrm{EL}(\mathrm{SM1–SM3})} - 1\right) \times 100, \end{cases} \tag{45}$$

where PJ6 is the Petkovic method [21] of order 4. Figures 5(a)–5(d) graphically illustrate these percentage ratios. Figures 5(a)–5(c) show the computational efficiency of methods (SM1–SM3) w.r.t the method PJ6, and Figure 5(d) shows the computational efficiency of the method PJ6 w.r.t (SM1–SM3). It is evident from Figures 5(a)–5(d) that the newly constructed simultaneous methods (SM1–SM3) are more efficient as compared to PJ6 [21].

5. Numerical Results

Here, some numerical examples are considered in order to demonstrate the performance of our family of one-step fourth-order single root finding methods (Q1–Q3) and sixth-order simultaneous methods (SM1–SM3), respectively. We compare our family of optimal fourth-order single root finding methods (Q1–Q3) with E1–E9 methods. The family of simultaneous methods (SM1–SM3) of order six is compared with the PJ6 method [21] of the same order. All the computations are performed using CAS Maple 18 with 9000 (64 digits floating point arithmetic in case of simultaneous methods) significant digits. For single root finding methods, the stopping criteria is as follows:

$$\begin{aligned} &\text{(i)} \left|f(s_i)\right| < \epsilon, \\ &\text{(ii)} \left|s_i^{(k)} - \alpha\right| < \epsilon, \end{aligned} \tag{46}$$

whereas

$$e_i = \left\|f(s_i)\right\|_2, \tag{47}$$

for simultaneous methods. We take $\epsilon = 10^{-600}$ for the single root finding method and $\epsilon = 10^{-30}$ for simultaneous determination of all roots of nonlinear equation (1).

Numerical tests examples from [32–34] are provided in Tables 7–13. In Tables 7, 9, 11, and 12, we present the numerical results for simultaneous determination of all roots, while Tables 8, 10, and 13 represent for single root finding methods. In all tables, CO represents the convergence order; n, the number of iterations; ρ, the computational order of convergence; and CPU, the computational time in seconds. Table 14 shows the values of arbitrary parameters α and β used in iterative methods Q1–Q3 for test Examples 1–3.

We also calculate the CPU execution time, as all the calculations are done using Maple 18 (Processor Intel(R) Core(TM) i3-3110m CPU@2.4 GHz with 64-bit operating system). We observe from tables that CPU time of the methods SM1–SM3 is comparable or better than that of thePJ6 method [21], showing the efficiency of our methods (SM1–SM3) as compared to them.

Table 6: The number of basic operations.

Methods	CO	AS_m	M_m	D_m
SM1	6	$6m^2 + O(m)$	$6m^2 + O(m)$	$2m^2 + O(m)$
SM2	6	$7m^2 + O(m)$	$5m^2 + O(m)$	$2m^2 + O(m)$
SM3	6	$6m^2 + O(m)$	$5m^2 + O(m)$	$2m^2 + O(m)$
PJ6	6	$8m^2 + O(m)$	$6m^2 + O(m)$	$2m^2 + O(m)$

6. Applications in Engineering

In this section, we consider two examples from engineering.

Example 1. (beam designing model, see [34]).

An engineer considers that a problem of embedment s of a sheet-pile wall results a nonlinear function given as

$$f_5(s) = \frac{s^3 + 2.87s^2 - 10.28}{4.62} - s. \tag{48}$$

The exact roots of (48) are $\zeta_1 = 2.0021$, $\zeta_2 = -3.3304$, and $\zeta_3 = -1.5417$ as shown in Figure 6.

The initial estimates for $f_5(s)$ are taken as

Figure 5: Computational efficiency of iterative methods SM1–SM3 w.r.t PJ6.

Table 7: Simultaneously finding of all roots.

$f_5(s) = ((s^3 + 2.87s^2 - 10.28)/4.62) - s$						
$\zeta_1 = 2.0021$, $\zeta_2 = -3.3304$, $\zeta_3 = -1.5417$						
(0) $S_{1,2} = 2.5$, (0) $S_3 = -7.4641$, (0) $S_4 = -0.5359$						
Method	CO	CPU	n	$e1$	$e2$	$e3$
PJ6	6	0.032	5	$2.0e-40$	$1.1e-38$	$1.5e-39$
SM1–SM3	6	0.015	5	0.0	$3.2e-27$	0.0

$$\begin{aligned} \overset{(0)}{s_1} &= 2.5, \\ \overset{(0)}{s_2} &= -7.4641, \\ \overset{(0)}{s_3} &= -0.5359. \end{aligned} \tag{49}$$

Example 2 (fractional conversion, see [32]).

As the expression described in [14, 35],

$$f_6(s) = s^4 - 7.79075s^3 + 14.7445s^2 + 2.511s - 1.674 \tag{50}$$

Table 8: Comparison of the optimal 4th order single root finding methods for $f_5(s)$.

Method	$\lvert s_i^{(6)} - \alpha \rvert$	$\lvert f_i^{(6)}(s_i) \rvert$	CPU	ρ
$f_5(s) = ((s^3 + 2.87s^2 - 10.28)/4.62) - S$, $s_0 = 2.5$, CO = 4				
Q1	$2.0e-58$	$1.0e-76$	0.062	4.0
Q2	$3.8e-63$	$2.0e-81$	0.031	4.0
Q3	$2.1e-60$	$1.1e-78$	0.031	4.0
E1	$1.6e-67$	$8.5e-86$	0.062	4.0
E2	$4.8e-66$	$2.4e-84$	0.031	4.0
E3	$1.8e-67$	$9.5e-86$	0.031	4.0
E4	$3.6e-66$	$1.8e-84$	0.063	4.0
E5	$3.2e-65$	$1.6e-83$	0.531	4.0
E6	$3.1e-67$	$1.9e-85$	0078	4.0
E7	$1.3e-64$	$7.1e-83$	0.046	4.0
E8	$3.2e-65$	$1.6e-83$	0.047	4.0
E9	$6.9e-69$	$3.5e-87$	0.062	4.0

is the fractional conversion of nitrogen, with hydrogen feed at 250 atm. and 227 k. The exact roots of (50) are $\zeta_1 = 3.9485 + 0.3161i$, $\zeta_2 = 3.9485 - 0.3161i$, $\zeta_3 = -0.3841$, and $\zeta_4 = 0.2778$. Real roots of (50) are shown in Figure 7. The initial estimates for $f_6(s)$ are taken as

Table 9: Simultaneously finding of all roots of $f_6(s)$.

$f_6(s) = s^4 - 7.7907s^3 + 2.511s - 1.674$							
$\zeta_1 = 3.9485 + 0.3161i, \zeta_2 = 3.9485 - 0.3161i, \zeta_3 = -0.3841, \zeta_4 = 0.2778$							
$S_1^{(0)} = 3.5 + 0.3i,\ S_2^{(0)} = 3.5 - 0.3i,\ S_3^{(0)} = 0.3 + 0.01i,\ S_4^{(0)} = 1.8 + 0.01i$							
Method	CO	CUP	n	$e1$	$e2$	$e3$	$e4$
PJ6	6	0.031	4	$4.6e-62$	$7.0e-62$	$3.7e-93$	0.0
SM1–SM3	6	0.015	4	0.0	0.0	$1.0e-137$	$1.0e-133$

Table 10: Comparison of the optimal 4th order single root finding methods for $f_6(s)$.

Method	$\lvert s_i^{(6)} - \alpha\rvert$	$\lvert f_i^{(6)}(s_i)\rvert$	CPU	ρ
$f_6(S) = s^4 - 7.79075s^3 + 14.744s^2 + 2.5111s - 1.674, s_0 = 0.27, \mathrm{CO} = 4$				
Q1	$9.5e-1947$	$3.0e-7783$	0.079	4.0
Q2	$1.3e-2120$	$2.9e-8477$	0.074	4.0
Q3	$5.9e-2002$	$3.2e-8004$	0.073	4.0
E1	$4.1e-2016$	$6.5e-8061$	0.078	4.0
E2	$1.4e-1934$	$1.6e-7734$	0.062	4.0
E3	$2.8e-2016$	$1.5e-8061$	0.047	4.0
E5	$2.1e-1934$	$9.1e-7734$	0.063	4.0
E5	$8.6e-1906$	$2.6e-7619$	0.062	4.0
E6	$7.4e-1970$	$8.4e-7876$	0.078	4.0
E7	$1.0e-1881$	$6.7e-7523$	0.079	4.0
E8	$8.6e-1906$	$2.6e-7619$	0.047	4.0
E9	$2.9e-2083$	$1.1e-8329$	0.078	4.0

Table 11: Simultaneously finding of all roots for $f_7(s)$.

$f_{7-1}(s) - \sin((s-1)/2)\sin((s-2)/2)\sin((s-2.5)/2)$						
$\zeta_1 - 1,\ \zeta_2 - 2,\ \zeta_3 - 2.5$						
$S_1^{(0)} = -0.2,\ S_2^{(0)} = 1.7,\ S_3^{(0)} = 3$						
Method	CO	CPU	n	$e1$	$e2$	$e3$
PJ6	6	0.047	3	0.01	0.01	$0.2e-5$
SM1–SM3	6	0.031	3	$8.8e-3$	$5.1e-5$	$4.3e-4$

Table 12: Simultaneously finding of all roots of $f_{7-1}(s)$.

$f_{7-2}(s) = \sin^3((s-1)/2)\sin^3((s-2)/2)\sin^3((s-2.5)/2)$						
$\zeta_1 - 1,\ \zeta_2 - 2,\ \zeta_3 - 2.5$, each with multiplicity 3						
$S_1^{(0)} = -0.2,\ S_2^{(0)} = 1.7,\ S_3^{(0)} = 3$						
Method	CO	CPU	n	$e1$	$e2$	$e3$
PJ6	6	0.047	3	$5.5e-16$	$2.7e-8$	$2.2e-17$
SM1–SM3	6	0.031	3	$1.8e-21$	$1.7e-13$	$3.9e-18$

$$\begin{aligned} \overset{(0)}{s_1} &= 3.5 + 0.3i, \\ \overset{(0)}{s_2} &= 3.5 - 0.3i, \\ \overset{(0)}{s_3} &= -0.3 + 0.01i, \\ \overset{(0)}{s_4} &= 1.8 + 0.01i. \end{aligned} \tag{51}$$

Example 3. (see [33], for simultaneous determination of distinct and multiple roots).

Here, we consider another standard test function for the demonstration of convergence behavior of newly constructed methods.

Consider

$$f_{7-1}(s) = \sin\left(\frac{s-1}{2}\right)\sin\left(\frac{s-2}{2}\right)\sin\left(\frac{s-2.5}{2}\right), \tag{52}$$

Table 13: Comparison of the optimal 4th order single root finding methods $f_{7-1}(s)$.

Method	$\lvert s_i^{(6)} - \alpha \rvert$	$\lvert f_i^{(6)}(s_i) \rvert$	CPU	ρ
$f_{7-1}(s) = \sin((s-1)/2)\sin((s-2)/2)\sin((s-2.5)/2)$, $s_0 = 1.68956$, CO = 4				
Q1	$6.7e-519$	$7.8e-2074$	0.422	4.0
Q2	$4.2e-557$	$5.1e-2227$	0.734	4.0
Q3	$8.8e-533$	$1.7e-2129$	1.328	4.0
E1	$1.2e-536$	$7.5e-2145$	1.328	4.0
E2	$97e-516$	$3.5e-2061$	1.344	4.0
E3	$1.0e-536$	$2.7e-2145$	1.344	4.0
E4	$1.6e-575$	$3.1e-2050$	1.375	4.0
E5	$8.0e-507$	$1.9e-2025$	0.906	4.0
E6	$2.2e-525$	$8.4e-2100$	1.375	4.0
E7	$8.2e-499$	$2.4e-1993$	0.0.891	4.0
E8	$8.0e-507$	$1.9e-2025$	1.1328	4.0
E9	$2.5e-550$	$8.3e-2200$	1.329	4.0

Table 14: Parameter values used in iterative methods Q1, Q2, and Q3 for Examples 1–3.

Method	Example 1		Example 2		Example 3	
	β	α	β	α	β	α
Q1	10.005	—	10.005	—	10.005	—
Q2	5.005	1.001	5.005	1.001	5.005	1.001
Q3	−5.005	—	15.0005	—	−5.005	—

Figure 6: Exact roots of nonlinear equation $f_5(s)$.

Figure 7: Exact roots of nonlinear equation $f_6(s)$.

with exact roots $\zeta_1 = 1$, $\zeta_2 = 2$, and $\zeta_3 = 2.5$ as shown in Figure 8. The initial guessed values have been taken as

$$\begin{aligned} s_1^{(0)} &= -0.2, \\ s_2^{(0)} &= 1.7, \\ s_3^{(0)} &= 3. \end{aligned} \tag{53}$$

For distinct roots, we use method (34), and for multiple roots, method (37):

$$f_{7-2}(s) = \sin^3\left(\frac{s-1}{2}\right)\sin^3\left(\frac{s-2}{2}\right)\sin^3\left(\frac{s-2.5}{2}\right). \tag{54}$$

Figures 9(e)–9(g) show the residual fall of iterative methods (Q1–Q3 and E1–E9), Figures 9(a), 9(b), and 9(d)

Figure 8: Exact real roots of nonlinear equation $f_7(s)$.

Figure 9: Continued.

Figure 9: (a) Residual fall of simultaneous methods SM1–SM3; PJ6 for nonlinear functions $f_5(s)$ for simple roots. (b) Residual fall of simultaneous methods SM1–SM3; PJ6 for nonlinear functions $f_6(s)$ for simple roots. (c) Residual fall of simultaneous methods SM1–SM3; PJ6 for nonlinear functions $f_7(s)$ for multiple roots. (d) Residual fall of simultaneous methods SM1–SM3; PJ6 for nonlinear functions $f_7(s)$ for simple roots. (e) Residual fall of iterative methods Q1–Q3; E1–E9 for nonlinear equation $f_5(s)$. (f) Residual fall of iterative methods Q1–Q3; E1–E9 for nonlinear equation $f_6(s)$. (g) Residual fall of iterative methods Q1–Q3; E1–E9 for nonlinear equation $f_7(s)$.

present the residual fall of simultaneous iterative methods (SM1–SM3 and PJ6) for nonlinear functions $f_5(s)$, $f_6(s)$, and $f_{7-1}(s)$ when the multiplicity of roots is simple, and Figure 9(c) presents the residual fall for multiple roots for a nonlinear function $f_{7-2}(s)$, respectively.

Table 13 shows the values of parameters α and β used in iterative methods Q1–Q3 for test Examples 1–3.

7. Conclusion

We have developed here three families of single-step single root finding methods of optimal convergence order four and three families of simultaneous methods of order six, respectively. From Tables 7–13 and Figures 1–5 and 8, we observe that our methods (Q1–Q3 and SM1–SM3) are superior in terms of efficiency, stability, CPU time, and residual error as compared to the methods E1–E9 and PJ6 method, respectively.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally in the preparation of this manuscript.

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Research Article

A Novel Collision-Free Navigation Approach for Multiple Nonholonomic Robots Based on ORCA and Linear MPC

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In the study of collision-free navigation methods of multirobots, much attention has been paid to the constraints of external environment. However, most of the wheeled mobile robots are subjected to nonholonomic constraints. A collision between robots may occur if the nonholonomic constraints are neglected. This paper presents an improved approach to collision-free navigation for multi-nonholonomic robots. This approach combines the Optimal Reciprocal Collision Avoidance (ORCA) algorithm and Model Predictive Control (MPC) strategy. ORCA used a simple robot model, in which kinematics and dynamics are ignored. To cope with this problem, the MPC controller is introduced. In each ORCA step, the reference trajectory, reference control inputs, and "safe zones" are generated based on the new velocity. Consequently, the derived safe zone is transformed into the constraints of decision variables for a MPC controller. Finally, quadratic programming is used to solve the MPC problem by successive linearization of an error model of the mobile robot. Simulation results illustrate the effectiveness of the proposed method.

1. Introduction

Motion planning/collision avoidance is concerned with computing a path or trajectory between two configurations embedded in cost field, while taking into account motion constraints, static obstacles, and moving obstacles [1]. The problem has been well studied for one robot avoiding static or moving obstacles [2–4].

Many classical approaches such as artificial potential field method [5], sampling-based algorithms [6–9], and dynamic window [10] are extended to dynamic environments [11]. These approaches assume the observed obstacles to be static in a period of time [12–14] and compute an immediate action to avert collisions with the consideration of kinematics and dynamics in many cases. When considering moving obstacles, such approaches typically repeatedly plan based on continuous cycle of sensing and acting. This may work well as if the obstacles move slower than the robot, but among fast obstacles, the static assumption will not lead a collision-free navigation. The methods such as multilayer fuzzy control method [15], reactive collision avoidance [16], and rolling window method [17] are also used in the dynamic environment. However, these methods based on a simple robot model ignore the nonholonomic constraints.

The problem of collision avoidance becomes even more complicated when the obstacles are also decision-making agents instead of entities moving arbitrarily without considering their environment [18]. This problem has many applications in the areas of robotics, such as multirobots navigation and coordination in the Container Terminal Logistics System [19] and crowd simulation for computer graphics and VR [20]. In the context of collision avoidance for multiple robots, similar approaches to those for the single robot case can be applied [21–23]. However, the increase in robot density may lead to oscillations [24]. Many motion-planning methods of multiple point-mass robots have been developed [24–26]. It means that applying these methods into practice directly is difficult because the mobility of the

robot is ignored. The Net-MPC strategy is also used to navigate in multirobots environment [27–30]. However, these methods relying on the communication between robots may increase the complexity of system.

A particularly successful method for mobile robot real-time navigation is the ORCA algorithm [31]. It is an effective decentralized collision avoidance method for a large number of decision-making agents. The ORCA infers for each robot a half-plane of velocities that are allowed to be selected in order to guarantee collision avoidance without communication among the robots or central coordination. This method works well if the robots act on holonomic constraints, since the robots must move along the direction of a new selected velocity by ORCA immediately. For many nonholonomic constrained robots, such as car-like robots, the set of feasible velocities at any instant is a specific velocity in the direction of the rear wheels [32, 33]. One way to work around these constraints is to use the set of velocities that can be achieved over some time interval τ [26]. An optimal method for distributed collision avoidance among multiple nonholonomic robots is presented in [32]. Furthermore, Jeon and Lee defined the wheel velocity obstacle, which is a set of all the left and right wheel velocity pairs that induce collisions with obstacles within a given time horizon [34]. However, these methods can only guarantee collision-free with the instant of each step time. To alleviate these issues, we can select a small value for τ, but this has some implications for navigation: as τ decreases, the set of velocities that are being considered by the robot becomes smaller, and the robot can miss feasible controls.

The Model Predictive Control (MPC) method is suitable for solving the multivariable constrained control problems due to its predictive nature. It means that the MPC method can handle the environment constraints and robot constraints in a single optimization problem [35, 36]. The applications of MPC for navigation of a single robot have been reported in [36–38].

Motivated by the features of MPC and ORCA methods, in this paper, we propose a Discrete-ORCA-MPC combined approach for the problem where collisions need to be avoided among multiple nonholonomic mobile robots with decision-making ability. Concretely, the ORCA algorithm is used by each agent to generate independently its new velocity at each cycle of sensing (ORCA step time τ). This new velocity is used to generate a discrete reference control input and trajectory. Meanwhile, a series of safe zones are constructed, which serve as constraints in a MPC problem. Unlike original ORCA, the construction of safe zones needs to know the new velocities of other agents in the neighbor region at each step. Furthermore, the proof for collision-free motion of this Discrete-ORCA-MPC approach is derived. Although Nonlinear Model Predictive Control (NMPC) has been well developed, the computational effort necessary is much higher than the linear model [39, 40]. This paper uses a successive linearization approach yielding a linear, time-varying description of the system which can be solved through linear MPC. Considering the control inputs as the decision variables, it is then used to transform the optimization problem into a Quadratic Programming (QP) problem. Since this is a convex problem, QP problems can be solved by commercial numerical solvers which lead to global optimal solutions. More importantly, it is easy to a real-time implementation. In addition, we focus on differential drive mobile robots (DDMRs) in the following work, even though our approach applies more generally for the class of feedback linearizable vehicles with nonholonomic kinematics, such as car-like robots or DDMR with trailer.

The main features of the proposed Discrete-ORCA-MPC combined approach are summarized as follows:

(1) Different from the traditional methods such as artificial potential functions based on a point-mass robot model, the proposed method considers the nonholonomic constraints of robots. The point-mass robot model cannot guarantee collision-free navigation, since a car-like robot will actually need to follow an arc to achieve the selected velocity, and this arc may lead to a collision with other robot. Consequently, the proposed approach solves this problem by using a dynamic model to predict the future system.

(2) Different from the network-based method such as Distributed Model Predictive Control method, the proposed method does not rely on communications among the robots or central coordination. Therefore, it is effective for multirobots to avoid obstacles real-time without a real-time communication environment.

The remainder of this paper is organized as follows: in the next section the brief definition of the ORCA is shown. The problem statement is illustrated in Section 3. Section 4 puts out the kinematic model of DDMR and the linear MPC algorithm. Simulation results in MATLAB and C++ are shown in Section 5. Section 6 presents some conclusions.

2. Optimal Reciprocal Collision Avoidance (ORCA)

ORCA is a rigorous approach for reciprocal n-body collision avoidance that allows each agent to compute independently the optimal moving velocities in each step [31]. In this section, we review the concept of velocity obstacles and discuss its application to DDMR with nonholonomic constraints.

2.1. Definition of ORCA. ORCA provides a sufficient condition for multiple robots to avoid collisions among one another and thus can guarantee collision-free navigation [31]. Let there be a set of n disc-shaped robots (this disc-shaped assumption can easily be extended to translate polygons) moving in the plane $\mathbb{R}^2$. Each robot has a current position $\mathbf{p}$, a current velocity $\mathbf{v}$, and a radius r. It is assumed that these parameters can be observed by other robots. Furthermore, each robot has a maximum speed $\mathbf{v}_{\max}$ and a preferred velocity $\mathbf{v}_{\text{pref}}$ (for instance, a velocity directed towards the robot's goal with a magnitude equal to the robot's preferred speed). These parameters are considered as

internal state of the robot, which cannot therefore be observed by other robots. The brief definitions are presented in the following.

Let $D(\mathbf{p}, r)$ denote an open disc of radius r centered at $\mathbf{p}$:

$$D(\mathbf{p}, r) = \{\mathbf{q} \mid \|\mathbf{q} - \mathbf{p}\| < r\}. \quad (1)$$

The velocity obstacle $\mathrm{VO}^{\tau}_{A|B}$ is the set of all relative velocities of A with respect to B that will result in a collision at some moment before time τ. The geometric interpretation of velocity obstacles is shown in Figure 1. And it can be formally represented as

$$\mathrm{VO}^{\tau}_{A|B} = \{\mathbf{v} \mid \exists t \in [0, \tau] :: t\mathbf{v} \in D(\mathbf{p}_B - \mathbf{p}_A, \mathbf{r}_A + \mathbf{r}_B)\}, \quad (2)$$

where $\mathbf{p}_A$ and $\mathbf{p}_B$ are the positions of agent A and agent B, respectively. r_A (r_B) is the radius of the safe zone of agent A (agent B) and is chosen slightly larger than the radius of the agent. Note that for agent B, its velocity obstacle is $\mathrm{VO}^{\tau}_{B|A} = -\mathrm{VO}^{\tau}_{A|B}$.

Let $\mathbf{v}^{\mathrm{opt}}_A$ and $\mathbf{v}^{\mathrm{opt}}_B$ be the optimization velocities of A and B, respectively. Nominally, the optimization velocities are equal to the current velocities, such that the robots have to deviate as little as possible from their current trajectories to avoid collisions. Let $\mathbf{u}$ be the vector from $\mathbf{v}^{\mathrm{opt}}_A - \mathbf{v}^{\mathrm{opt}}_B$ to the closest point on the boundary of the velocity obstacle:

$$\mathbf{u} = \left(\underset{\mathbf{v} \in \partial \mathrm{VO}^{\tau}_{A|B}}{\operatorname{argmin}} \left\| \mathbf{v} - \left(\mathbf{v}^{\mathrm{opt}}_A - \mathbf{v}^{\mathrm{opt}}_B\right\| \right) - \left(\mathbf{v}^{\mathrm{opt}}_A - \mathbf{v}^{\mathrm{opt}}_B\right), \quad (3)$$

and let $\mathbf{n}$ be the outward normal of the boundary of $\mathrm{VO}^{\tau}_{A|B}$ at point $(\mathbf{v}^{\mathrm{opt}}_A - \mathbf{v}^{\mathrm{opt}}_B) + \mathbf{u}$. Consequently, $\mathbf{u}$ is the smallest change required to the relative velocity of A and B to avert collision with τ time. To share the responsibility of avoiding collisions among the robots in a fair way, The increment $(1/2)\mathbf{u}$ means that agent A takes half the responsibility of avoiding potential collision with agent B, while the remaining half will be taken by agent B. Hence, the set $\mathrm{ORCA}^{\tau}_{A|B}$ of permitted velocities for A is the half-plane pointing in the direction of $\mathbf{n}$ starting at the point $\mathbf{v}^{\mathrm{opt}}_A + (1/2)\mathbf{u}$. More formally,

$$\mathrm{ORCA}^{\tau}_{A|B} = \left\{\mathbf{v} \mid \left(\mathbf{v} - \left(\mathbf{v}^{\mathrm{opt}}_A + \frac{1}{2}\mathbf{u}\right)\right)\mathbf{n} \geq 0\right\}. \quad (4)$$

This set is illustrated in Figure 2. By this definition, the chosen new relative velocities will not enter $\mathrm{VO}^{\tau}_{A|B}$, and consequently the velocities of the agent can be smoothed.

Each robot A performs a continuous cycle of sensing and acting with time step Δt. In each iteration, the robot acquires the radius, the current position, and the current optimization velocity of the other robots. Based on this information, the robot infers the permitted half-plane of velocities $\mathrm{ORCA}^{\tau}_{A|B}$ with respect to each other robot B. The set of ORCA velocities for agent A with respect to all robots is the intersection of the half-planes of $\mathrm{ORCA}^{\tau}_{A|B}$ induced by each other robot B:

$$\mathrm{ORCA}^{\tau}_A = D(\mathbf{0}, v^{\max}_A) \cap \bigcup_{B \neq A} \mathrm{ORCA}^{\tau}_{A|B}. \quad (5)$$

Note that $D(\mathbf{0}, v^{\max}_A)$ include the maximum speed constraint on the robot A.

Figure 1: The velocity obstacle for a short time horizon $\mathrm{VO}^{\tau}_{A|B}$.

Figure 2: $\mathrm{VO}^{\tau}_{A|B}$ and $\mathrm{ORCA}^{\tau}_{A|B}$ for agent A when $(\mathbf{v}_A - \mathbf{v}_B)$ lies in $\mathrm{VO}^{\tau}_{A|B}$.

At last, the new velocity $\mathbf{v}^{\mathrm{new}}_A$ is selected from ORCA^{τ}_A, which is the closest to its preferred velocity $\mathbf{v}^{\mathrm{pref}}_A$ amongst all velocities inside the region of permitted velocities:

$$\mathbf{v}^{\mathrm{new}}_A = \underset{\mathbf{v} \in \mathrm{ORCA}^{\tau}_A}{\operatorname{argmin}} \left\| \mathbf{v} - \mathbf{v}^{\mathrm{pref}}_A \right\|. \quad (6)$$

The ORCA algorithm in [31] used a simple robot model, in which kinematics are ignored. Consequently, the robot reaches its new position:

$$\mathbf{p}^{\mathrm{new}}_A = \mathbf{p}_A + \mathbf{v}^{\mathrm{new}}_A \Delta t. \quad (7)$$

In many of these applications, the robot of interest is a car-like robot and subject to nonholonomic kinematic constraints [33]. Hence, the constant velocity assumption rarely holds [41].

2.2. Nonholonomic Constraints. ORCA is a velocity obstacle based algorithm, which is a first-order method since it does not integrate velocities to yield positions as functions of time [26]. It means that the control input to the robot is velocity. By considering nonholonomic kinematic constraints, the navigation problem of a differential drive robot generally followed these two steps: first, the velocity $\mathbf{v}_H$ is generated by the ORCA based on the assumption that the robot is holonomic; second, the robot tracks this velocity by using the controller with nonholonomic constraints [34]. However, these will make a tracking error δ between the holonomic trajectory and the real trajectory inevitable, as shown in Figure 3.

The real nonholonomic trajectory is defined by two sections: an arc of circumference AB covered with constant linear speed $v(t)$ and decreasing angular velocity $\omega(t)$, followed by a straight line path BC with constant speed $v(t)$. In this case, a mobile robot will actually need to follow an arc to achieve the selected velocity $\mathbf{v}_H$, and the set of ORCA_A^τ provide no guarantee of collision-free.

3. Problem Statement

In this section, we define a new concept called the safe zone. This is an extended generalization of the ORCA concept and seeks to address the difficulty of using ORCA with nonholonomic constrained agents. The proof of collision-free navigation of safe zone is also derived. Note that, the actions of ORCA are computed for each robot independently, without communication among the robots or central coordination. Unlike original ORCA algorithm, the construction of safe zone needs to know the new velocities of other agents in neighbor region at each step. This is not a severe limitation since the multiagent system has been studied extensively.

3.1. Safe Zone. We consider multiple mobile robots sharing a common workspace where the robots have the same equations of motion and state spaces. Let the state space of robot be $\mathcal{X} \subset \mathbb{R}^n$. Let $\mathbb{R}^d$ be the robot's physical workspace. Let $\mathcal{O} \subset \mathbb{R}^d$ be the geometry of robot relative to its position. For simplicity, we restrict our analysis to circular robots and obstacles, which is not a severe limitation since general polygons can be represented by a number of circles [26]. We assume the position $p \in \mathbb{R}^d$ of robot can be derived from its state $\mathbf{x} \in \mathcal{X}$ by a potentially projection function $\mathbf{q} \in \mathcal{X} \longrightarrow \mathbb{R}^d$:

$$\mathbf{p}(t) = \mathbf{q}(\mathbf{x}(t)). \tag{8}$$

The control input space $\mathcal{U} \subset \mathbb{R}^m$ is assumed to be convex. Let the continuous-time equation of motion for robot be given by a potentially nonlinear function $\mathbf{f} \in \mathcal{X} \times \mathcal{U} \longrightarrow \mathbb{R}^n$:

$$\dot{x}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \tag{9}$$

where $\mathbf{x}(t)$ is the state and $\mathbf{u}(t)$ is the control input at time t.

Figure 3: The holonomic trajectory is tracked by the DDMR moving along the nonholonomic trajectory with a tracking error.

Given a current state $\mathbf{x}(0)$ of robot and some constant control input $\mathbf{u}(0)$ at time 0, the state of the robot at a given time $t > 0$ is given by

$$\mathbf{x}(t) = \mathbf{g}(t, \mathbf{x}(0), \mathbf{u}(0)), \tag{10}$$

where $\mathbf{g} \in \mathbb{R} \times \mathcal{X} \times \mathcal{U} \longrightarrow \mathcal{X}$ is the solution to the differential equation of equation (9), which can be obtained numerically.

Consider two circular robots A and B in an environment. Let circle A represent the robot, and let circle B represent the obstacle. Given the position of agent A at time $t = 0$ and a control $\mathbf{u}_A$, let us denote the position agent A will be later undertaking $\mathbf{u}_A$ for time interval t by $\mathbf{p}_A(t)$. Similarly, $\mathbf{p}_B(t)$ represents the state of agent B undertaking $\mathbf{u}_B$ for time interval t. Note that the control $\mathbf{u}$ is the control inputs to the robot that results in a change in the robot's configuration. Then, the control obstacle can be defined as

$$\left\{u_A, u_B \,\middle|\, \exists t \in [0, \tau] :: \left\|p_A(t) - p_B(t)\right\| < r_A + r_B\right\}, \tag{11}$$

where $p_A(t) = q_A(g_A(t, x_A(0), u_A))$ and $p_B(t) = q_B(g_B(t, x_B(0), u_B))$ by using equations (8) and (10). As shown in Figure 4, $\mathbf{v}_A(0) \notin \mathrm{ORCA}_{A|B}^\tau$, and symmetrically, $\mathbf{v}_B(0) \notin \mathrm{ORCA}_{B|A}^\tau$. Robots A and B chose constant velocities $\mathbf{v}_{AH}$ and $\mathbf{v}_{BH}$ outside the obstacle velocity set and generate trajectory $\mathbf{p}_{AH}(t)$ and $p_{AH}(t)$, respectively. Then A and B take controls u_A and $\mathbf{u}_B$ (for instance, $u = [v_H, k(\phi - \phi_d)]$), which generate trajectory $p_A(t)$ and $\mathbf{p}_B(t)$ (red dashed line), respectively. Note that this will lead to a collision, although the minimum distance between two holonomic trajectory (blue solid line) is greater than the radii sum of two robots ($d_{\min} > r_A + r_B$).

In fact, the ORCA algorithm can be transformed using a uniform sampling scheme to deal with the nonholonomic kinematic constraints. As illustrated in Figure 5, let $\{p_{AH}(0), p_{AH}(1), \ldots, p_{AH}(N)\}$ and $\{p_{BH}(0), p_{BH}(1), \ldots, p_{BH}(N)\}$ be the uniform sampling waypoints of holonomic trajectory $\mathbf{p}_{AH}(t)$ and $\mathbf{p}_{BH}(t)$, respectively. The distance between $\mathbf{p}_A(k)$ and $p_B(k)$ is $l_{AB}(k)$ for $k \in \{0, 1, 2, \ldots, N\}$. Note that $\mathbf{p}(N)$ is the terminal point of one step of ORCA; that is, $\mathbf{p}(N) = \mathbf{p}(\tau)$. Hence, the collision avoidance set of ORCA algorithm can be reformulated as

$$\mathrm{CA}_{A|B}^\tau = \left\{u_{As}, u_{Bs} \,\middle|\, \forall k \in \{0, 1, \ldots, N\} :: l_{AB}(k) > r_A + r_B\right\}, \tag{12}$$

where

Figure 4: Robots A and B collide in $p_A(t_{col})$ and $p_B(t_{coal})$ caused by the deviations for the nonholonomic constraints.

Figure 5: The discrete trajectories of robots A and B generated by ORCA.

$$\begin{aligned} l_{AB}(k) &= \|p_A(k) - p_B(k)\|, \\ u_{As} &= [u_A(0), u_A(1), \ldots, u_A(N-1)], \\ u_{Bs} &= [u_B(0), u_B(1), \ldots, u_B(N-1)]. \end{aligned} \tag{13}$$

Let $r_s^i(k)$ be the safe zone radius of robot i at time k, and it can be defined as

$$r_s^i(k) = \frac{1}{2} \min_{j \neq i} \left(l_{ij}(k) - r_i - r_j \right), \tag{14}$$

where r_i and r_j are the radii of robots i and j. The constructive process of safe zone can geometrically be illustrated in Figure 6. Note that $r_s^{ij}(k) = l_{ij}(k) - r_i - r_j$ and b is side length of the maximum inscribed square. This square is the approximate safe zone for linear constraints (see next section). If robot i applies the control input $\mathbf{u}_i(k-1)$ in time $k-1$ that "walk" to the safe zone $r_s^i(k)$ in time k will lead to collision avoidance.

3.2. Proof of Collision-Free Navigation

Theorem 1. *If the safe zone of robot i at control step k is equal to the relative safe zone between robots i and j, the safe zone of robot j must have been less than or equal to the safe zone of robot i in this moment. Then, the mathematical relationship can be described by* $r_s^i(k) = r_s^{ij}(k) \Rightarrow r_s^j(k) \leq r_s^i(k)$.

Figure 6: The safe zone for robot i with two neighbor robots j and k.

Proof. If $r_s^j(k) > r_s^i(k) \Rightarrow \exists m: : r_s^{jm}(k) = (1/2) \min_{m \neq j} (l_{jm}(k) - r_j - r_m) > r_s^i(k) = r_s^{ji}(k)$.

According to (14), $r_s^{ji}(k) \geq (1/2) \min_{m \neq j} (l_{jm}(k) - r_j - r_m)$.

By using reduction to absurdity, Theorem 1 is proved.

For robot i, there are n robots in its neighbor region. If the radius of safe zone at control step k is $r_s^i(k) = r_s^{ij}(k)$, it means that $r_s^{ij}(k)$ is the minimum dimension for robot i, but it does not mean that the $r_s^{ij}(k)$ is the minimum dimension for robot j. Theorem 1 gives the proof of this fact. □

Theorem 2. *In a multirobots environment, the collision-free navigation can be guaranteed when the safe zone constraints are maintained at every moment.*

Proof. Theorem 1 $\Rightarrow \forall i, j \in R_s$ (robots set): : $r_i + r_j + r_s^i(k) + r_s^j(k) \leq l_{ij}(k)$.

Consider any pair of robots in one neighbor region, the sum of the radii of these two robots and their safe zones is less than the distance between these two robots. This theorem proves that the collision-free navigation for Discrete-ORCA can be guaranteed. □

4. Discrete-ORCA-MPC Combined Collision Avoidance

In this section, the brief overview of the kinematic model of a DDMR is given. Then, a successive linearization approach is introduced, which yields a linear, time-varying description of the DDMR system. The optimal control input is solved through linear MPC. Meanwhile, the safe zone and control constraints are satisfied by transforming the optimization problem in a QP problem.

4.1. Kinematics and Dynamics of DDMR. As illustrated in Figure 7, the configuration of a DDMR is given by the position of its center point $\mathbf{p} = [x, y]$ and its orientation ϕ. The configuration and high-level control input transition equations [42] are given as

FIGURE 7: Structure of the differential drive mobile robot.

$$\begin{aligned} \dot{x} &= v\cos\phi, \\ \dot{y} &= v\sin\phi, \\ \dot{\phi} &= \omega, \end{aligned} \tag{15}$$

where $v = v_r + v_l/2$ is the linear velocity and $\omega = v_r - v_l/b$ is the angular velocity. b is the distance between the rear wheels of the robot, and v_r and v_l are the signed speeds of the left and right wheels, respectively. Generally, v_r and v_l are bounded such that $v_r, v_l \in [-v_{\max}, v_{\max}]$.

In general, the kinematic model can be written in the form of (9):

$$\dot{x} = f(\mathbf{x}, \mathbf{u}), \tag{16}$$

where $\mathbf{x} = [x, y, \phi]^T$ describes the configuration of point P (shown in Figure 7) and $\mathbf{u} = [v, \omega]^T$ is the control input.

Similarly, let $\mathbf{x}_r = [x_r, y_r, \phi_r]^T$ and $\mathbf{u}_r = [v_r, \omega_r]^T$ be the configuration and control input of reference system. Then, it can be described as

$$\dot{x}_r = f(\mathbf{x}_r, \mathbf{u}_r). \tag{17}$$

By expanding the right side of (16) in Taylor series around the reference point $(\mathbf{x}_r, \mathbf{u}_r)$, a linear model is obtained by discarding the high-order terms:

$$\dot{x} = f(\mathbf{x}_r, \mathbf{u}_r) + f'_{\mathbf{x}}(\mathbf{x} - \mathbf{x}_r) + f'_{\mathbf{u}}(\mathbf{u} - \mathbf{u}_r), \tag{18}$$

where $f'_{\mathbf{x}}$ and $f'_{\mathbf{u}}$ are the Jacobians of f with respect to $\mathbf{x}$ and $\mathbf{u}$, respectively, evaluated around the reference point $(\mathbf{x}_r, \mathbf{u}_r)$.

Then, the subtraction of (17) from (18) results in

$$\dot{\tilde{x}} = f'_{\mathbf{x}}\tilde{x} + f'_{\mathbf{u}}\tilde{u}, \tag{19}$$

where $\tilde{x} = \mathbf{x} - \mathbf{x}_r$ represents the error with respect to the reference robot configuration and $\tilde{u}$ is its error with respect to the reference input. Finally, the discrete-time system can be obtained:

$$\tilde{x}(k+1) = \mathbf{A}(k)\tilde{x}(k) + \mathbf{B}(k)\tilde{u}(k), \tag{20}$$

with

$$\begin{aligned} \mathbf{A}(k) &= \begin{bmatrix} 1 & 0 & -v_r(k)\sin(\phi_r)(k)T \\ 0 & 1 & v_r(k)\cos(\phi_r)(k)T \\ 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{B}(k) &= \begin{bmatrix} \cos(\phi_r)(k)T & 0 \\ \sin(\phi_r)(k)T & 0 \\ 0 & T \end{bmatrix}, \end{aligned} \tag{21}$$

where T is the sampling period and k is the sampling time.

In this paper, let $v_i^r = \|\mathbf{v}_i^{\text{new}}\|$ and $\phi_i^r = \arctan\mathbf{v}_i^{\text{new}}$ be the reference velocity and direction of robot i. Then, the reference trajectory of robot i is obtained as (6):

$$X_i^r = \begin{bmatrix} x(k) + v_i^r\cos(\phi_i^r)T & y(k) + v_i^r\sin(\phi_i^r)T & \phi_i^r \\ x(k) + 2v_i^r\cos(\phi_i^r)T & 2y(k) + v_i^r\sin(\phi_i^r)T & \phi_i^r \\ \vdots & \vdots & \vdots \\ x(k) + N_\tau v_i^r\cos(\phi_i^r)T & N_\tau y(k) + v_i^r\sin(\phi_i^r)T & \phi_i^r \end{bmatrix}, \tag{22}$$

where $N_\tau = (\tau/T)$. It means that robot i generates N_τ waypoints in the direction of ϕ_i^r uniformly. In [40], we notice that the controllability matrix has full rank if either v_r or ω_r is nonzero. This implies the tracking of a reference trajectory being possible with linear MPC.

In the field of mobile robotics, predictive approaches to trajectory tracking seem to be very promising because the reference trajectory is known beforehand [40]. The essence of MPC approach is to optimize predictions of process behavior over a sequence of future control inputs. At each sampling time, the model predictive controller generates an optimal control sequence by solving an optimization problem [39]. In this section, the trajectory-tracking control of a differentially driven mobile robot based on MPC is presented. The main idea is to minimize the future trajectory-following errors and control errors between the robot and reference waypoints generated by Discrete-ORCA.

4.2. MPC Algorithm. The ideal of MPC concept it to find the values of control variable that minimize the receding horizon quadratic cost function based on the predicted robot tracking error model (20). In general, MPC can be proven to be stable using a terminal cost and terminal constraints in its optimization problem [43]. The cost function at time k can be stated as

$$\begin{aligned} \Phi(k) = \sum_{i=1}^{h} &\tilde{x}^T(k+i\,|\,k)Q\tilde{x}(|k+i|k) \\ &+ \tilde{u}^T(|k+i-1|k)R\tilde{u}(k+i-1\,|\,k), \end{aligned} \tag{23}$$

where h is the prediction horizon and $\mathbf{Q}$ and $\mathbf{R}$ are the weighting matrices (diagonal matrix with $\mathbf{Q} \geq 0$ and $\mathbf{R} \geq 0$). Note that, the angle error $(\phi - \phi_r)$ must be wrapped in radians to $[-\pi, \pi]$. The notation $a(m\,|\,n)$ indicates the value of a at the predicted time m based on the current time n. In order to reformulate the optimization problem in a usual

quadratic programming form, the $\hat{x}(k+1)$ and $\hat{u}(k)$ are defined as

$$\hat{x}(k+1) = \begin{bmatrix} \tilde{x}(k+1\mid k) \\ \tilde{x}(k+2\mid k) \\ \vdots \\ \tilde{x}(k+h\mid k) \end{bmatrix}, \qquad \tilde{u}(k) = \begin{bmatrix} \tilde{u}(k\mid k) \\ \tilde{u}(k+1\mid k)) \\ \vdots \\ \tilde{u}(k+h-1\mid k) \end{bmatrix}. \tag{24}$$

Then, the original optimal problem (23) can be rewritten as

$$\Phi(k) = \hat{x}^T(k+1)\hat{Q}\hat{x}(k+1) + \hat{u}^T(k)\hat{R}\hat{u}(k), \tag{25}$$

where

$$\hat{Q} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q} \end{bmatrix}, \qquad \hat{R}(k) = \begin{bmatrix} \mathbf{R} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{R} \end{bmatrix}. \tag{26}$$

Therefore, (20) can be recasted as

$$\hat{x}(k+1) = \hat{A}(k)\hat{x}(k) + \hat{B}(k)\hat{u}(k), \tag{27}$$

with

$$\hat{A}(k) = \begin{bmatrix} \mathbf{A}(k\mid k) \\ \mathbf{A}(k\mid k)\mathbf{A}(k+1\mid k) \\ \vdots \\ \boldsymbol{\alpha}(k,0) \end{bmatrix}, \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}(k\mid k) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}(k+1\mid k)\mathbf{B}(k\mid k) & \mathbf{B}(k+1\mid k) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\alpha}(k,1)\mathbf{B}(k\mid k) & \boldsymbol{\alpha}(k,2)(\mathbf{B}k+1\mid k) & \cdots & \mathbf{B}(k+h-1\mid k) \end{bmatrix}, \tag{28}$$

where

$$\boldsymbol{\alpha}(k,j) = \prod_{i=j}^{h-1}(\mathbf{A}(k+i\mid k)). \tag{29}$$

Rearrange (25) and (27), the cost function (23) can be rewritten as

$$\Phi(k) = \frac{1}{2}\hat{u}^T(k)\mathbf{H}(k)\hat{u}(k) + \mathbf{f}^T(k)\hat{u}(k) + \mathbf{d}(k), \tag{30}$$

where

$$\begin{aligned} \mathbf{H}(k) &= 2\left(\hat{B}^T(k)\hat{Q}\hat{B}(k) + \hat{\mathbf{R}}\right), \\ \mathbf{f}(k) &= 2\hat{B}^T(k)\hat{Q}\hat{A}(k)\hat{x}(k\mid k), \\ \mathbf{d}(k) &= \tilde{x}^T(k\mid k)\hat{A}^T(k)\hat{Q}\hat{A}(k)\hat{x}(k\mid k). \end{aligned} \tag{31}$$

The $\mathbf{H}$ is a positive definite Hessian matrix. It is the quadratic part of the cost function while the vector $\mathbf{f}$ describes the linear part. Note that $\mathbf{f}(k)$ and $\mathbf{d}(k)$ contain $\tilde{x}(k\mid k)$ which is constant vector at time k. Hence, the optimization problem can be stated as to find $\hat{u}^*(k)$ at each time step k minimizing the cost function (30):

$$\hat{u}^*(k) = \underset{\hat{u}}{\operatorname{argmin}}\, \Phi(k). \tag{32}$$

The solution of (32) is a sequence of optimal control; that is $\hat{u}^*(k) = \{\tilde{u}^*(k\mid k), \tilde{u}^*(k+1\mid k), \ldots, \tilde{u}^*(k+h-1\mid k)\}$. The MPC scheme is implicitly given by the first control action of the sequence of optimal control, $\tilde{u}^*(k\mid k)$.

4.3. Collision-Free Constraints. In real world applications, the robots are inevitably subject to physical limitations. Without consideration of these limitations while computing the optimal control inputs, the robot will suffer from unpredictable mismatch. Hence, the constraints of upper and lower bounds should add to the optimization problem (32). And the control inputs constraint can be written as

$$\mathbf{u}_{\min}(k) \le \mathbf{u}(k) \le \mathbf{u}_{\max}(k). \tag{33}$$

Note that the free variable in the optimization (32) is $\hat{u}(k)$, so the constraint (33) must be rewritten as

$$\hat{u}_{\min}(k) \le \hat{u}(k) \le \hat{u}_{\max}(k), \tag{34}$$

where

$$\hat{u}_{\min}(k) = \begin{bmatrix} \mathbf{u}_{\min}(k\,|\,k) - \mathbf{u}_r(k\,|\,k) \\ \mathbf{u}_{\min}(k+1\,|\,k) - \mathbf{u}_r(k+1\,|\,k) \\ \vdots \\ \mathbf{u}_{\min}(k+h-1\,|\,k) - \mathbf{u}_r(k+h-1\,|\,k) \end{bmatrix},$$
$$\hat{u}_{\max}(k) = \begin{bmatrix} \mathbf{u}_{\max}(k\,|\,k) - \mathbf{u}_r(k\,|\,k) \\ \mathbf{u}_{\max}(k+1\,|\,k) - \mathbf{u}_r(k+1\,|\,k) \\ \vdots \\ \mathbf{u}_{\max}(k+h-1\,|\,k) - \mathbf{u}_r(k+h-1\,|\,k) \end{bmatrix}. \tag{35}$$

Furthermore, the collision avoidance requirement will be constructed as the constraint with respect to control inputs. Let us rewrite the safe zone (14) as

$$r_s^i(k+1) = \frac{1}{2}\min_{j\neq i}\left(l_{ij}(k+1) - r_i - r_j\right). \tag{36}$$

It means that the control inputs to robot i in time k must reach the safe zone $r_s^i(k+1)$. To make a linear constraint, the approximate safe zone is constructed as a square which is the maximum inscribed square as illustrated in Figure 6. Hence, the side length of this square b can be denoted as

$$b(k+1) = \frac{1}{\sqrt{2}} r_s^i(k+1). \tag{37}$$

Then, the collision avoidance manoeuvre can be represented as

$$-\frac{1}{2}b(k+1) \le x(k+1) - x_r(k+1) \le \frac{1}{2}b(k+1),$$
$$-\frac{1}{2}b(k+1) \le y(k+1) + y_r(k+1) \le \frac{1}{2}b(k+1), \tag{38}$$

where $x(k+1)$ and $y(k+1)$ can be obtained from the discrete time form of the kinematic equation (15):

$$x(k+1) = x(k) + T\cos(\phi(k))v(k),$$
$$y(k+1) = y(k) + T\sin(\phi(k))v(k). \tag{39}$$

Note that $\phi(k+1)$ is not the consideration of the navigation. Remember that $\mathbf{u} = [v, \omega]^T$, so we have $\tilde{v}(k) = v(k) - v_r(k)$. Then rearranging (38) to (39) can obtain the collision avoidance constraint in each step time k:

$$b_l(k) \le \tilde{v}(k) \le b_u(k), \tag{40}$$

where

$$b_l(k) = \min\left(\frac{-(1/2)b(k+1) + x_r(k+1) - x(k)}{T\cos(\phi(k))}, \frac{-(1/2)b(k+1) + y_r(k+1) - y(k)}{T\sin(\phi(k))}\right),$$
$$b_u(k) = \min\left(\frac{(1/2)b(k+1) + x_r(k+1) - x(k)}{T\cos(\phi(k))}, \frac{(1/2)b(k+1) + y_r(k+1) - y(k)}{T\sin(\phi(k))}\right). \tag{41}$$

Note that $b_l(k)$ and $b_u(k)$ are constants in each time step. Recalling (34), we can obtain

$$\tilde{b}_l(k) \le \tilde{v}(k) \le \tilde{b}_u(k), \tag{42}$$

with

$$\tilde{b}_l(k) = \min\left(b_l(k), v_{\min}(k) - v_r(k)\right),$$
$$\tilde{b}_u(k) = \min\left(b_u(k), v_{\max}(k) - v_r(k)\right). \tag{43}$$

Finally, the constraint of free variable $\hat{u}(k)$ for optimization problem (32) can be constructed by refreshing $\hat{u}_{\min}(k)$ and $\hat{u}_{\max}(k)$ in (34) as

$$\hat{u}_{\min}(k) = \begin{bmatrix} \mathbf{u}_{\min}(k\,|\,k) \\ \mathbf{u}_{\min}(k+1\,|\,k) - \mathbf{u}_r(k+1\,|\,k) \\ \vdots \\ \mathbf{u}_{\min}(k+h-1\,|\,k) - \mathbf{u}_r(k+h-1\,|\,k) \end{bmatrix},$$
$$\hat{u}_{\max}(k) = \begin{bmatrix} \mathbf{u}_{\max}(k\,|\,k) \\ \mathbf{u}_{\max}(k+1\,|\,k) - \mathbf{u}_r(k+1\,|\,k) \\ \vdots \\ \mathbf{u}_{\max}(k+h-1\,|\,k) - \mathbf{u}_r(k+h-1\,|\,k) \end{bmatrix}, \tag{44}$$

where

$$\tilde{u}_{\min}(k\,|\,k) = \begin{bmatrix} \tilde{b}_l(k) \\ \omega_{\min} - \omega_r \end{bmatrix},$$
$$\tilde{u}_{\max}(k\,|\,k) = \begin{bmatrix} \tilde{b}_u(k) \\ \omega_{\max} - \omega_r \end{bmatrix}. \tag{45}$$

4.4. Algorithm. The overall approach is summarized in Algorithm 1. In each ORCA step (τ), it has three things for every robot to do. First, robot i generates the discrete reference control inputs and trajectory. The number of waypoints is τ/T, and the corresponding reference control inputs are $[\|\mathbf{v}_i^{\text{new}}\|, 0]$. Second, robot i computes the safe zone $\mathbf{r}_s^i$ with respect to the neighbor region. Finally, robot i repeatedly computes the optimal control inputs at each sample time T to close the last waypoint. Then, it will go into the next ORCA step. This process will continue until every robot reaches its goal.

5. Experiments and Analysis

In this section, we conduct computer simulations to verify the proposed Discrete-ORCA-MPC algorithm. The following simulations are specifically designed to deal with multi-nonholonomic-agent navigation. These were carried

Input: R_s: list of robots; O: list of obstacles; τ: time horizon of ORCA; h: predict horizon of MPC; T: sampling period.
Output: optimal control $\mathbf{u}^*(k)$

```
repeat ORCA
    for each R_i ∈ R_s do
        compute v_i^new for R_i using ORCA (6)
        compute reference trajectory X_i^r using (22);
        compute safe zone r_s^i for robot i at each way point using (14)
    end for
    for k = 1; k < N_τ; k++ do
        for each R_i ∈ R_s do
            compute ũ_i^*(k) for robot i using MPC (27)
            compute control input u_i(k) = ũ_i^*(k) + u_i^r(k);
            compute the configuration of next step x_i(k+1) using (15)
        end for
    end for
until (all robots reach their goal zone)
```

Algorithm 1: Discrete-ORCA-MPC method for multi-nonholonomic robots collision avoidance.

out using MATLAB R2018a and Visual Studio 2017 on a laptop with Intel Quad Core i5 CPU and 16 GB RAM.

To simplify the configuration, it is assumed that all the DDMR have the same parameters. The radius of robots $r = 2$ and the control input constraints are set as $\mathbf{u}_{\min} = [0, -1]$ and $\mathbf{u}_{\max} = [2, 1]$, respectively. The preferred velocity $\mathbf{v}^{\text{pref}}$ was chosen as the velocity directed towards the robot's goal with a magnitude equal to 1. Meanwhile, the sampling time period of MPC is $T = 30$ ms and the time window of the ORCA algorithm is $\tau = 300$ ms. In the MPC controller formulation, the prediction horizon $h = 6$ and optimization problem is solved using interior point method. Furthermore, the free variable constraints are set as $\tilde{u}_{\min} = [-0.5, -0.5]$ and $\tilde{u}_{\max} = [0.5, 0.5]$, respectively. To keep the robot close to the ORCA trajectory, the weighting matrix $\mathbf{Q}$ should be larger than $\mathbf{R}$. In this paper, these are set as $\mathbf{Q} = 3\mathbf{I}_3$ and $\mathbf{R} = 0.1\mathbf{I}_3$.

5.1. Compare MPC and PID for 4 DDMRs. As mentioned in Section 2.2, the original ORCA algorithm is a first-order method. In other words, the control input to the robot is reference velocity but not the reference trajectory. In this simulation, a first-order method based on ORCA is formulated to compare with the proposed Discrete-ORCA-MPC approach. This method uses a proportional controller to control the angular velocity, and the linear velocity is equal to the new velocity generated by ORCA. It can be formulated as

$$v_{ik}(j) = v_{ik}^{\text{new}},$$
$$\omega_{ik}(j) = \begin{cases} \min\left(\mathrm{K}_{\text{gain}}\left(\phi_{ik}^{\text{new}} - \phi_{ik}(j)\right), \omega_{\max}\right), & \phi_{ik}^{\text{new}} \geq \phi_{ik}(j), \\ \max\left(\mathrm{K}_{\text{gain}}\left(\phi_{ik}^{\text{new}} - \phi_{ik}(j)\right), \omega_{\min}\right), & \phi_{ik}^{\text{new}} < \phi_{ik}(j), \end{cases} \tag{46}$$

where v_{ik}^{new} and ϕ_{ik}^{new} represent the 2-norm value and direction of the new velocity (equation (6)) of robot i generated by ORCA in step k; that is, $v_{ik}^{\text{new}} = \|\mathbf{v}_{ik}^{\text{new}}\|$ and $\phi_{ik}^{\text{new}} = \operatorname{atan2}(\mathbf{v}_{ik}^{\text{new}})$. $v_{ik}(j)$ and $\omega_{ik}(j)$ are the control input to robot i in control step j of ORCA step k. The proportional controller gain is set as 3. As a consequence, the constraints of angular velocity are also added to the controller. Note that we have no intention of constructing a perfect controller to following the reference velocity $\mathbf{v}^{\text{new}}$, since the tracking error to the holonomic trajectory always exists even for a "perfect" controller (see Figure 3). Furthermore, these tracking errors may lead to a collision. By contrast, the proposed MPC controller also has tracking errors; however, these tracking errors are formulated as constraints for the decision variables that the collision will not happen (see Theorem 2 and equation (45)).

In this simulation, 4 DDMRs are initially located on the four corners of a square, that is, [−25, −25] (Robot 1), [25, 25] (Robot 2), [−25, 25] (Robot 3), and [25, −25] (Robot 4), as illustrated in Figure 8. Robots 1, 2 and Robots 3, 4 at antipodal positions are expected to exchange positions with each other. Furthermore, the initial orientation of robots 1 to 4 is $(1/4)\pi$, $-(3/4)\pi$, $-(1/4)\pi$, and $(3/4)\pi$, respectively. Note that every 18 ORCA steps plot a circle as shown in Figure 8.

Figure 8 shows the motion trajectory of 4 DDMRs on the workspace without static obstacles based on the proposed Discrete-ORCA-MPC approach and a proportional controller. Comparing the trajectories of local enlarged drawing of Figures 8(a) and 8(b), at 37.5°s, Robot 2 $(-1.718, 1.56)$ collides with Robot 3 $(-1.421, -2.082)$ in Figure 8(b) (PID) while the collision did not happen in Figure 8(b) (Discrete-ORCA-MPC). Figure 9 depicts the relative distance between robots over the time where d_{ij} represents the relative distance between robots i and j. These results demonstrate the proposed method can generate a collision-free navigation for nonholonomic mobile robot effectively.

Figure 10 shows the control inputs (v and ω) to each robot. In most of the time of the navigation, the control inputs change a little with respect to time; however, these has been a great change near 28°s and 37°s. It can be explained that the robots took a sharp turn in 28°s and 37°s

Figure 8: Motion trajectory for (a) Discrete-ORCA-MPC and (b) PID.

Figure 9: The relative distance between robots. (a) Discrete-ORCA-MPC. (b) PID.

as shown in Figure 8. In particular, the rapid growth of angular velocity ω is inevitably accompanied by a decline in the linear velocity ω. In other words, the robot should slow down for taking a sharp turn. It also can be seen that the control inputs are inside the limits imposed by the constraints. In Figure 11, it demonstrates that the state errors

Figure 10: Control inputs of the 4 robots with respect to time. (a) Robot 1 control inputs. (b) Robot 2 control inputs. (c) Robot 3 control inputs. (d) Robot 1 control inputs.

Figure 11: Continued.

Figure 11: Robots state errors compared to the holonomic trajectory. (a) Robot 1 state inputs. (b) Robot 2 state inputs. (c) Robot 3 state inputs. (d) Robot 4 state inputs.

Figure 12: Motion trajectory of 4 DDMRs with static obstacles.

are compared with the trajectory generated by ORCA. It can be seen that maximum absolute errors of the position $[x_e, y_e]$ and orientation ϕ_e are 0.07 and 0.4, respectively. Obviously, position errors are much less than the orientation error ϕ_e. This is reasonable because the smaller $[x_e, y_e]$ shows a better performance of MPC controller. This simulation demonstrates that the proposed Discrete-ORCA-MPC algorithm has the ability to cope with the collision avoidance problem of multi-nonholonomic-agent.

Figure 13: The relative distance between robots.

Figure 14: Control inputs of the 4 robots with respect to time (consider static obstacles). (a) Robot 1 control inputs. (b) Robot 2 control inputs. (c) Robot 3 control inputs. (d) Robot 4 control inputs.

Figure 15: Robots state errors compared to the holonomic trajectory generated by ORCA (consider static obstacles). (a) Robot 1 state errors. (b) Robot 2 state errors. (c) Robot 3 state errors. (d) Robot 1 state errors.

5.2. 4 DDMRs with Static Obstacle. It is similar to the first simulation; 4 DDMRs are initially located on the four corners of a square also, that is, [−45,−45] (Robot 1), [45, 45] (Robot 2), [−45, 45] (Robot 3) and [45, −45] (Robot 4), as illustrated in Figure 12. Unlike the first one, this simulation considers static obstacle avoidance. These four static obstacles are located at [−22.5, −22.5], [22.5, 22.5], [−22.5, 22.5], and [22.5, −22.5] with sides that are 15 dm in length. Figure 13 depicts the relative distance between robots over the time. Furthermore, the initial orientations of Robots 1 to 4 are $(1/4)\pi + 0.1$, $-(3/4)\pi + 0.1$, $-(1/4)\pi + 0.1$, and $(3/4)\pi + 0.1$, respectively. This is why the trajectories of these four robots are asymmetrical. More sharp turn can be seen compared to the first simulation which leads to more sudden change in control inputs as shown in Figure 14. Meanwhile, the constraints of control inputs $[\mathbf{u}_{\min}, \mathbf{u}_{\max}]$ and the errors with respect to reference control inputs $[\hat{u}_{\min}, \hat{u}_{\max}]$ are satisfied. In Figure 15, it can also be seen that the position errors $[x_e, y_e]$ are much less than the orientation error ϕ_e. The maximum value of position errors is less than 0.1 which means that the performance of the obtained trajectory tracking is good enough.

This simulation demonstrates that the proposed Discrete-ORCA-MPC algorithm has the ability to cope with the navigation problem of multi-nonholonomic-agent with static obstacles.

5.3. Performance Results. In order to test the performance of Discrete-ORCA-MPC, we carried on large-scale simulations to compare the mean computation times of our method with Priority-Based Noncooperative Distributed Model Predictive Control (PB-NC-DMPC) [44]. PB-NC-DMPC is an effective NC-DMPC strategy that deals with the problem of

Figure 16: Comparison of computation times.

loss of the prediction consistency property while reducing the computation time as well. In the problem of collision avoidance of networked vehicles, PB-NC-DMPC showed better performance compared to NC-DMPC and Centralized MPC. The test scenario contains many robots whose goal is to move across a circle to the antipodal position. This scenario is the same as the published literature, Jur et al. [31]. In PB-NC-DMPC, we implemented the collision avoidance controller based on the same linear predictive model as (20). The controller parameters of PB-NC-DMPC are the same as Discrete-ORCA-MPC and remain unchanged for all robots in all simulations.

Figure 16 depicts the mean computation times of Discrete-ORCA-MPC and PB-NC-DMPC plotted over increasing number of DDMRs. It shows that running Discrete-ORCA-MPC is computationally cheaper than PB-NC-DMPC.

6. Conclusions

In this paper, we have described a method, namely, Discrete-ORCA-MPC, for collision avoidance among multiple nonholonomic robots in planar environments. In particular, we extended the traditional ORCA method to a Discrete-ORCA strategy which can generate a uniform distribution of safe zones in the direction of selected new velocity by ORCA. The idea was to introduce the collision-free constraints to MPC optimization problem. The presented Discrete-ORCA-MPC method allows for smooth and safe navigation and good performance was observed in simulation experimental tests.

The results of this paper show that our approach is capable of letting a homogeneous group of robots with nonlinear equations of motion safely avoid collisions at real-time computation rates. The collision-free navigation problem of nonhomogeneous group of robots will be solved through adjusting the safe zone. Many challenges for future development still remain. The future work should aim at further improving the adaptability in uncertain environment.

Data Availability

The data used to support the finds of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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