Abstract and Applied Analysis

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Editorial Board

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 515162, 2 pages
http://dx.doi.org/10.1155/2015/515162

Editorial

Qualitative Theory of Functional Differential and Integral Equations

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Received 10 November 2014; Accepted 10 November 2014

Functional differential equations arise in many areas of science and technology: whenever a deterministic relationship involving some varying quantities and their rates of change in space and/or time (expressed as derivatives or differences) is known or postulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as the time varies. In some cases, this differential equation (called an equation of motion) may be solved explicitly. In fact, differential equations play an important role in modelling virtually every physical, technical, biological, ecological, and epidemiological process, from celestial motion, to bridge design, to interactions between neurons, to interaction between species, to spread of diseases with a population, and so forth. Also many fundamental laws of chemistry can be formulated as differential equations and in economy differential equations are used to model the behavior of complex systems. However, the mathematical models can also take different forms depending on the time scale and space structure of the problem; it can be modeled by delay differential equations, difference equations, partial delay differential equations, partial delay difference equations, or the combination of these equations.

When necessary, random effects and sudden effects can also be considered in modelling problems. The mathematical theory of differential equations first developed, together with the sciences, where the equations had originated and where the results found applications. Differential and difference equations such as those used to solve real-life problems may not necessarily be directly solvable, that is, do not have closed form solutions. Only the simplest equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. In this case a recurrence relation is needed which is an equation that recursively defines a sequence: each term of the sequence is defined as a function of the preceding terms. A difference equation is a specific type of recurrence relations. Solving a recurrence relation means obtaining a closed form solution: a nonrecursive function.

However, diverse problems, sometimes originating in quite distinct scientific fields, may give rise to identical differential and difference equations. Whenever this happens, mathematical theory behind the equations can be viewed as a unifying principle behind diverse phenomena; see, for example, the books by Brauer and Castillo Chavize [1], Diekmann et al. [2], Gopalsamy [3], Gyori and Ladas [4], Kocic and Ladas [5], Kolmonovskii and Myshkis [6], Lakshmikantham et al. [7], May and Anderson [8], Murray [9], Sharkovsky et al. [10], and Wu [11].

A delay differential equation (DDE) is an equation for a function of a single variable, usually called time, in which the derivative of the function at a certain time is given in terms of the values of the function at earlier times. A functional equation (FE) is an equation involving an unknown function for different argument values. The equations $x(2t) + 2x(3t) = 1$, $x(t) = 2x(t+1) - [x(t-2)]^2$, and so forth are examples of this type. The differences between the argument values of an unknown function and t in a FE are called argument deviations. If all argument deviations are constants, then FE is called a difference equation. Combining the notions of differential and functional equations, we obtain the notion of functional differential equation (FDE) or equivalently differential equations with deviating argument. Thus, this is an equation concerning the unknown function and some of its derivatives for, in general, different argument values (present, past, or future). The order of a FDE is the order of the highest derivative of the unknown function entering in the equation. So, a FE may be regarded as FDE of order zero. Hence, the notion of FDE generalizes all equations of mathematical analysis for functions of a continuous argument.

The qualitative study of functional differential equations and difference equations is a wide field in pure and applied mathematics, physics, meteorology, engineering, and population dynamics. All of these disciplines are concerned with the properties of these equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions; for global existence and uniqueness theorems for differential equations, we refer to the books [6, 7, 12] and for basic theory of difference equations, we refer to the books [13]. On the other hand, applied mathematics emphasizes the rigorous justification of the qualitative behavior of solutions (oscillation, periodic orbits, persistence, permanence, stability, global attractivity, Hopf bifurcation, Floquet theory, control, synchronization, etc.) [2, 14, 15]. On the other hand the study of integral inequalities has received a lot of attention in the literature and has become a major field in pure and applied mathematics; we refer to the recent book [16].

The oscillation theory, stability theory, bifurcation theory, existence of periodic solutions and convergence of solutions as parts of the qualitative theory of differential and difference equations have been developed rapidly in the past thirty years and some interesting books have been written in these subjects. We refer the reader to the books [17–21]. In this special issue we will consider some papers in all the above different areas and hope that the reader will find in this special issue some important results.

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References

[1] F. Brauer and C. Castillo Chavize, *Mathematical Models in Population Biology and Epidemiology*, Springer, 2001.

[2] O. Diekmann, S. A. van Gils, S. M. V. Lunel, and H.-O. Walther, *Delay Equations, Functional, Complex and Nonlinear Analysis*, Springer, New York, NY, USA, 1995.

[3] K. Gopalsamy, *Stability and Oscillation in Delay Differential Equations of Population differentials*, Kluwer Academic, Dordrecht, The Netherlands, 1992.

[4] I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, UK, 1991.

[5] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic, Dordrecht, Germany, 1993.

[6] V. Kolmonovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic, Dordrecht, Germany, 1999.

[7] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[8] R. May and R. M. Anderson, *Infectious Diseases of Humans: Differentials and Control*, Oxford Science, 1995.

[9] J. D. Murray, *Mathematical Biology*, Springer, New York, NY, USA, 1993.

[10] A. N. Sharkovsky, Y. L. Miastrenko, and E. Y. Romamenko, *Difference Equations and Their Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[11] J. H. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer, New York, NY, USA, 1996.

[12] J. Hale, *Theory of Functional Different ial Equations*, Springer, New York, NY, USA, 1977.

[13] R. P. Agarwal, *Difference Equations and Inequalities, Theory, Methods and Applications*, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.

[14] S. N. Elaydi, *An Introduction to Difference Equations*, Springer, New York, NY, USA, 1999.

[15] I. T. Kiguradze and T. A. Chanturia, *Asy Mptotic Properties of Solutions of Nonatunomous Ordinary Differential Equations*, Kluwer Academic, Dordrecht, The Netherlands, 1993.

[16] R. Agarwal, D. O'Regan, and S. H. Saker, *Dynamic Inequalities on Time Scales*, Springer, 2014.

[17] R. P. Agarwal, S. R. Grace, and D. O'Regan, *Oscillation Theory for Second Order Dynamic Equations*, vol. 5 of *Series in Mathematical Analysis and Applications*, Taylor & Francis, London, UK, 2003.

[18] D. D. Bainov and D. P. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay*, Adam Hilger, New York, NY, USA, 1991.

[19] S. H. Saker, *Oscillation Theory of Delay Differential and Difference Equations: Second and Third Orders*, Verlag Dr. Müller, Saarbrücken, Germany, 2010.

[20] S. H. Saker, *Oscillation Theory of Dynamic Equations on Time Scales: Second and Third Orders*, Lambert Academic Publishing, 2010.

[21] R. P. Agarwal, D. O'Regan, and S. H. Saker, *Oscillation and Stability of Delay Models in Biology*, Springer, 2014.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 838396, 18 pages
http://dx.doi.org/10.1155/2014/838396

Research Article

Hopf Bifurcation and Stability of Periodic Solutions for Delay Differential Model of HIV Infection of $CD4^+$ T-cells

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Received 17 February 2014; Revised 13 June 2014; Accepted 19 June 2014; Published 31 August 2014

Academic Editor: Cemil Tunç

This paper deals with stability and Hopf bifurcation analyses of a mathematical model of HIV infection of $CD4^+$ T-cells. The model is based on a system of delay differential equations with logistic growth term and antiretroviral treatment with a discrete time delay, which plays a main role in changing the stability of each steady state. By fixing the time delay as a bifurcation parameter, we get a limit cycle bifurcation about the infected steady state. We study the effect of the time delay on the stability of the endemically infected equilibrium. We derive explicit formulae to determine the stability and direction of the limit cycles by using center manifold theory and normal form method. Numerical simulations are presented to illustrate the results.

1. Introduction

Since 1980, the human immunodeficiency virus (HIV) or the associated syndrome of opportunistic infections that causes acquired immunodeficiency syndrome (AIDS) has been considered as one of the most serious global public health menaces. When HIV enters the body, its main target is the CD4 lymphocytes, also called CD4 T-cells (including $CD4^+$ T-cells). When a CD4 cell is infected with HIV, the virus goes through multiple steps to reproduce itself and create many more virus particles. The AIDS term, which is known as the late stage of HIV, covers the range of infections and illnesses which can result from a weakened immune system caused by HIV. Based on the clinical studies, it is known that, for a normal person, the $CD4^+$ T-cells count is around $1000\,\text{mm}^{-3}$ and for HIV infected patient it gradually decreases to $200\,\text{mm}^{-3}$ or below, which leads to AIDS. However, this may take several years for the number of CD4 T-cells to reduce to a level where the immune system is weakened [1–6].

Mathematical models, usingdelay differential equations (DDEs), have provided insights in understanding the dynamics of HIV infection. Discrete or continuous time delays have been introduced to the models to describe the time between infection of a $CD4^+$ T-cell and the emission of viral particles on a cellular level [7–13]. In general, DDEs exhibit much more complicated dynamics than ODEs since the time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate [14–16]. In studying the viral clearance rates, Perelson et al. [17] assumed that there are two types of delays that occur between the administration of drug and the observed decline in viral load: a pharmacological delay that occurs between the ingestion of drug and its appearance within cells and an intracellular delay that is between initial infection of a cell by HIV and the release of new virion. In this paper, we incorporate an intracellular delay to the model to describe the time between infection of a $CD4^+$ T-cell and the emission of viral particles on a cellular level [18]. We study the impact of the presence of such time delay on the dynamics of the model.

The outline of the present paper is as follows. In Section 2, we describe the model. In Section 3, we study the qualitative behavior of the model via stability of the steady states and Hopf bifurcation when time delay is considered as a bifurcation parameter. In Section 4, we provide an explicit formula to determine the direction of bifurcating periodic solution by applying center manifold theory and normal form method. We provide some numerical simulations to demonstrate the effectiveness of the analysis in Section 5 and we conclude in Section 6.

2. Description of the Model

Let us start the analysis with some basic models of the dynamics of target (uninfected) cells and infected CD4$^+$ T-cells by HIV. As a first approximation, the dynamics between HIV and the macrophage population was described by the simplest model of infection dynamics presented in [19–21]. Denoting uninfected cells by $x(t)$ and infected cells by $y(t)$ and assuming that viruses are transmitted mainly by cell to cell contact, the model is given by

$$\begin{aligned} \dot{x}(t) &= \Lambda - \delta_1 x(t) - \beta x(t) y(t), \\ \dot{y}(t) &= \beta x(t) y(t) - \delta_2 y(t). \end{aligned} \tag{1}$$

The target (uninfected) CD4$^+$ T-cells are produced at a rate Λ, die at a rate δ_1, and become infected by virus at a rate β. The infected host cells die at a rate δ_2. The basic reproductive ratio of the virus is then given by $\mathscr{R}_0 = \Lambda\beta/\delta_1\delta_2$. If there is no infection or if $\mathscr{R}_0 < 1$, there is only trivial equilibrium ($\mathscr{E}_0 = (\Lambda/\delta_1, 0)$) with no virus-producing cells. Whereas if $\mathscr{R}_0 > 1$, the virus can establish an infection and the system converges to the equilibrium with both uninfected cells and infected cells, $\mathscr{E}_1 = (\delta_2/\beta, \Lambda/\delta_2 - \delta_1/\beta)$.

However, in most viral infections, the CTL response plays a crucial part in antiviral defence by attacking viral infected cells [22, 23]. As the the cytotoxic T-lymphocyte (CTL) immune response is necessary to eliminate or control the viral infection, we incorporated the antiviral CTL immune response into the basic model (1). Therefore, if we add CTL response, which is denoted by $z(t)$, into model (1) (see [19]), then the extended model is

$$\begin{aligned} \dot{x}(t) &= \Lambda - \delta_1 x(t) - \beta x(t) y(t), \\ \dot{y}(t) &= \beta x(t) y(t) - \delta_2 y(t) - p y(t) z(t), \\ \dot{z}(t) &= c q y(t) z(t) - h z(t). \end{aligned} \tag{2}$$

Thus, CTLs proliferate in response to antigen at a rate c, die at a rate h, and lyse infected cells at a rate p. We assume that the CTL pool consists of two populations: the precursors $w(t)$ and the effectors $z(t)$. In other words, we assume that there are primary and secondary responses to viral infections. Then, the model (2) becomes

$$\begin{aligned} \dot{x}(t) &= \Lambda - \delta_1 x(t) - \beta x(t) y(t), \\ \dot{y}(t) &= \beta x(t) y(t) - \delta_2 y(t) - p y(t) z(t), \\ \dot{w}(t) &= c(1-q) y(t) w(t) - b w(t), \\ \dot{z}(t) &= c q y(t) w(t) - h z(t). \end{aligned} \tag{3}$$

The infected cells are killed by CTL effector cells at a rate pyz. Upon contact with antigen, CTLp proliferate at a rate $cy(t)w(t)$ and differentiate into effector cells CTLe at a rate

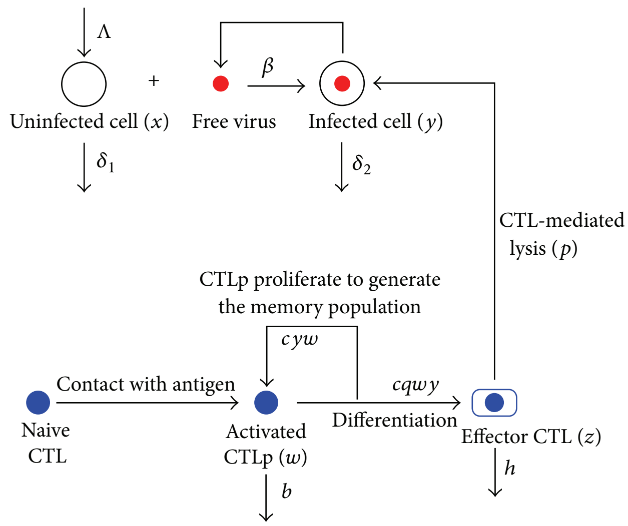

Figure 1: A simplified model of virus-CTL interaction. The virus dynamics is described by the basic model of Nowak and Bangham [19]. The uninfected target cells are produced at a rate Λ and die at a rate $\delta_1 x$. They become infected by the virus at a rate βxy. The infected cells produce new virus particle and die at a rate $\delta_2 y$. When CTL$_p$ recognize antigen on the surface of infected cells, they become activated and expand at a rate cyw, decay at a rate bw, and differentaite into efector cells at a rate $cqwy$. The effector cells lyse the infected cells at a rate pyz.

$cqy(t)w(t)$. CTL precursors die at a rate bw, and effectors die at a rate $hz(t)$; see Figure 1.

Since the proliferation of CD4$^+$ T-cells is density dependent, that is, the rate of proliferation decreases as T-cells increase and reach the carrying capacity, we then extend the above basic viral infection model to include the density dependent growth of the CD4$^+$ T-cell population (see [24–26]). It is also known that HIV infection leads to low levels of CD4$^+$ T-cells via three main mechanisms: direct viral killing of infected cells, increased rates of apoptosis in infected cells, and killing of infected CD4$^+$ T-cells by cytotoxic T-lymphocytes [26]. Hence, it is reasonable to include apoptosis of infected cells. An average of 10^{10} viral particles is produced by infected cells per day. The treatment with single antiviral drug is considered to be failed, so that the combination of antiviral drugs is needed for the better treatment [25]. Therefore, in the below revised model, we combine the antiretroviral drugs, namely, reverse transcriptase inhibitor (RTI) and protease inhibitor (PI) to make the model realistic (see [27–29]). RTIs can block the infection of target T-cells by infectious virus, and PIs cause infected cells to produce noninfectious virus particles. The modified model takes the form

$$\begin{aligned} \dot{x}(t) &= \Lambda - \delta_1 x(t) + r\left(1 - \frac{x(t)+y(t)}{T_{\max}}\right) x(t) \\ &\quad - (1-\epsilon)(1-\eta)\beta x(t) y(t), \\ \dot{y}(t) &= (1-\epsilon)(1-\eta)\beta x(t) y(t) \\ &\quad - \delta_2 y(t) - e_1 y(t) - p y(t) z(t), \end{aligned}$$

Table 1: Parameter definitions and estimations used in the underlying model.

Parameter	Notes	Estimated Value	Range	Source
Λ	Source of uninfected CD4$^+$ T-cells	10	0–10	[26]
β	Rate of infection	0.1	0.00001–0.5	[26]
$T_{\max}$	Total carrying capacity	1500	1500	[26]
r	Logistic growth term	0.03	0.03–3	[26]
δ_1	Mortality rate of CD4$^+$ T-cells	0.06	0.007–0.1	[26]
ϵ	Antiretroviral (RTI) therapy	0.9	0-1	see text
δ_2	Infected cells died out naturally	0.3	0.2–1.4	[26]
e_1	Apoptosis rate of infected cells	0.2	0.2	[26]
p	Clearance rate of infected cells	1	0.001–1	[26]
η	Protease inhibitor therapy	0.9	[0, 1]	see text
q	Rate of differentiation of CTLs	0.02	Assumed	—
b	Death rate of CTL precursors	0.02	0.005–0.15	[26]
c	Proliferation of CTLs responsiveness	0.1	0.001–1	[26]
h	Mortality rate or CTL effectors	0.1	0.005–0.15	[26]

$$\begin{aligned}\dot{w}(t) &= cy(t)w(t) - cqy(t)w(t) - bw(t),\\ \dot{z}(t) &= cqy(t)w(t) - hz(t).\end{aligned} \tag{4}$$

The first equation of model (4) represents the rate of change in the count of healthy CD4$^+$ T-cells that produced at rate Λ and become infected at rate β, with the mortality δ_1. We assume that the uninfected CD4$^+$ T-cells proliferate logistically, thus the growth rate r is multiplied by the term $(1-(x+y)/T_{\max})$ and this term approaches zero when the total number of T-cells approaches the carrying capacity $T_{\max}$. The effects of combination of RTI and PI antiviral drugs are represented by the term $(1-\epsilon)(1-\eta)\beta xy$, where $(1-\epsilon)$, $0<\epsilon<1$, represents the effects of RTI and $(1-\eta)$, $0<\eta<1$, represents the effects of PI. The second equation of model (4) denotes the rate of change in the count of infected CD4$^+$ T-cells. The infected CD4$^+$ T-cells decay at a rate δ_2 and e_1 denotes apoptosis rate of infected cell; infected cells are killed by CTL effectors at a rate p. The third equation of the model denotes the rate of change in the CTLp population; proliferation rate of the CTLp is given by c and is proportional to the infected cells y; CTLp die at a rate b and differentiate into CTL effectors at a rate cq. The last equation of the model represents the concentration of CTL effectors, which die at a rate h. In reality, the specific immune system is not immediately effective following invasion by a novel pathogen. There may be an explicit time delay between infection and immune initiation and there may be a gradual build-up in immune efficacy during which the immune response develops, before reaching maximal specificity to the pathogen ([8, 30, 31]). In order to make model (4) more realistic, time delay in the immune response should be included in the following model:

$$\begin{aligned}\dot{x}(t) &= \Lambda - (1-\epsilon)(1-\eta)\beta x(t)y(t)\\ &\quad + r\left(1-\frac{x(t)+y(t)}{T_{\max}}\right)x(t) - \delta_1 x(t),\\ \dot{y}(t) &= (1-\epsilon)(1-\eta)\beta x(t)y(t)\\ &\quad - (\delta_2+e_1)y(t) - py(t)z(t),\\ \dot{w}(t) &= c(1-q)y(t-\tau)w(t-\tau) - bw(t)\\ \dot{z}(t) &= cqy(t-\tau)w(t-\tau) - hz(t).\end{aligned} \tag{5}$$

The range of parameter values of the model are given in Table 1.

We start our analysis by presenting some notations that will be used in the sequel. Let $C = C([-\tau,0],\mathbb{R}_+^4)$ be the Banach space of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}_+^4$, where $\mathbb{R}_+^4 = (x,y,w,z)$; the initial conditions are given by

$$\begin{aligned}x(\theta) &= \varphi_1(\theta)\ge 0, \qquad y(\theta) = \varphi_2(\theta)\ge 0,\\ w(\theta) &= \varphi_3(\theta)\ge 0, \qquad z(\theta) = \varphi_4(\theta)\ge 0,\\ &\qquad \theta\in[-\tau,0],\end{aligned} \tag{6}$$

where $\varphi_i(\theta) \in \mathscr{C}^1$ are smooth functions, for all $i = 1,2,3,4$. From the fundamental theory of functional differential equations (see [32, 33]), it is easy to see that the solutions $(x(t),y(t),w(t),z(t))$ of system (5) with the initial conditions as stated above exist for all $t\ge 0$ and are unique. It can be shown that these solutions exist for all $t>0$ and stay nonnegative. In fact, if $x(0)>0$, then $x(t)>0$ for all $t>0$. The same argument is true for the y, w, and z components. Hence, the interior $\mathbb{R}_+^4$ is invariant for system (5).

3. Steady States

We can obtain the steady state values by setting $\dot{x} = \dot{y} = \dot{w} = \dot{z} = 0$. The steady state value of the infection-free steady sate $\mathscr{E}_0$ is given by $\mathscr{E}_0 = ((T_{\max}/2r)(r - \delta_1 + \sqrt{(r-\delta_1)^2 + 4r\Lambda/T_{\max}}), 0, 0, 0)$, while the infected steady state $\mathscr{E}_+ = (x^*, y^*, w^*, z^*)$ is given by

$$y^* = \frac{b}{c(1-q)}, \qquad w^* = \frac{h(1-q)z^*}{qb}, \qquad z^* = \frac{(1-\epsilon)(1-\eta)\beta x^* - (\delta_2 + e_1)}{p}, \tag{7}$$

and x^* is given by the following quadratic equation:

$$c_1 x^2 + c_2 x - c_3 = 0, \tag{8}$$

where $c_1 = c(1-q)r$, $c_2 = T_{\max} b\beta(1-\epsilon)(1-\eta) + br - c(1-q)T_{\max}(r-\delta_1)$, $c_3 = c(1-q)\Lambda T_{\max}$.

3.1. Stability and Hopf Bifurcation Analysis of Infected Steady State $\mathscr{E}_+$. In order to study full dynamics of model (4) by using time delay as a bifurcation parameter, we need to linearize the model around the steady state $\mathscr{E}_+$ and determine the characteristic equation of the Jacobian matrix. The roots of the characteristic equation determine the asymptotic stability and existence of Hopf bifurcation for the model. The characteristic equation of the linearized system is given by

$$\begin{vmatrix} -A_1 y^* + r - \frac{2r}{T_{\max}}x^* - \frac{r}{T_{\max}}y^* - \delta_1 - \lambda & -A_1 x^* - \frac{r}{T_{\max}}x^* & 0 & 0 \\ A_1 y^* & A_1 x^* - (\delta_2 + e_1) - pz^* - \lambda & 0 & -py^* \\ 0 & c(1-q)e^{-\lambda\tau}w^* & c(1-q)e^{-\lambda\tau}y^* - b - \lambda & 0 \\ 0 & cqe^{-\lambda\tau}w^* & cqe^{-\lambda\tau}y^* & -h-\lambda \end{vmatrix} = 0, \tag{9}$$

which is equivalent to the equation

$$\lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4 + e^{-\lambda\tau}\left(q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4\right) = 0, \tag{10}$$

where $A_1 = (1-\epsilon)(1-\eta)\beta$ and

$$\begin{aligned}
p_1 &= -a_1 - a_4 - a_8 - a_{11},\\
p_2 &= a_1a_8 + a_8a_{11} + a_1a_{11} + a_4a_8 + a_4a_{11} + a_1a_4 - a_2a_3,\\
p_3 &= a_2a_3a_8 + a_2a_3a_{11} - a_1a_8a_{11} - a_4a_8a_{11} - a_1a_4a_8 - a_1a_4a_{11},\\
p_4 &= a_1a_4a_8a_{11} - a_2a_3a_8a_{11},\\
q_1 &= -a_7,\\
q_2 &= a_1a_7 + a_7a_{11} + a_4a_7 - a_5a_9,\\
q_3 &= a_5a_8a_9 + a_1a_5a_9 + a_2a_3a_7 - a_1a_7a_{11} - a_4a_7a_{11} - a_1a_4a_7,\\
q_4 &= a_1a_4a_7a_{11} - a_1a_5a_8a_9 - a_2a_3a_7a_{11},\\
a_1 &= -(1-\epsilon)(1-\eta)\beta y^* + r - \frac{2rx^*}{T_{\max}} - \frac{ry^*}{T_{\max}} - \delta_1,\\
a_2 &= -(1-\epsilon)(1-\eta)\beta x^* - \frac{rx^*}{T_{\max}},\\
a_3 &= (1-\epsilon)(1-\eta)\beta y^*,\\
a_4 &= (1-\epsilon)(1-\eta)\beta x^* - (\delta_2 + e_1) - pz^*,\\
a_5 &= -py^*,\\
a_6 &= c(1-q)w^*,\\
a_7 &= c(1-q)y^*,\\
a_8 &= -b,\\
a_9 &= cqw^*,\\
a_{10} &= cqy^*,\\
a_{11} &= -h.
\end{aligned} \tag{11}$$

Let us consider the following equation:

$$\varphi(\lambda,\tau) = \lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4 + \left(q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4\right)e^{-\lambda\tau}. \tag{12}$$

For the nondelayed model (say $\tau = 0$), from (10), we have

$$\lambda^4 + D_1\lambda^3 + D_2\lambda^2 + D_3\lambda + D_4 = 0, \tag{13}$$

where

$$D_1 = p_1 + q_1, \qquad D_2 = p_2 + q_2, \qquad D_3 = p_3 + q_3, \qquad D_4 = p_4 + q_4. \tag{14}$$

Lemma 1. *For $\tau = 0$, the unique nontrivial equilibrium is locally asymptotically stable if the real parts of all the roots of* (13) *are negative.*

Proof. The proof of the above lemma is based on holding the following conditions: $D_1 > 0$, $D_3 > 0$, $D_4 > 0$, and $D_1D_2D_3 > D_1^2D_4 + D_3^2$, as proposed by Routh-Hurwitz criterion. We conclude that equilibrium $\mathscr{E}_+$ is locally asymptotically stable if and only if all the roots of the characteristic equation (13) have negative real parts which depends on the numerical values of parameters that are shown in the numerical exploration. □

3.2. Existence of Hopf Bifurcation. We here study the impact of the time-delay parameter on the stability of HIV infection of CD4$^+$ T-cells. We deduce criteria that ensure the asymptotic stability of infected steady state $\mathcal{E}_+$, for all $\tau > 0$. We arrive at the following theorem.

Theorem 2. *Necessary and sufficient conditions for the infected equilibrium $\mathcal{E}_+$ to be asymptotically stable for all delay $\tau \geq 0$ are as follows*

(i) *the real parts of all the roots of $\varphi(\lambda,\tau) = 0$ are negative;*

(ii) *for all ω and $\tau \geq 0$, $\varphi(i\omega,\tau) \neq 0$, where $i = \sqrt{-1}$.*

Proof. Assume that Lemma 1 is true. Now, for $\omega = 0$, we have

$$\varphi(0,\tau) = D_4 = p_4 + q_4 \neq 0. \quad (15)$$

Substituting $\lambda = i\omega$ ($\omega > 0$) into (5) and separating the real and imaginary parts of the equations yields

$$\begin{aligned} &\left(\omega^4 - p_2\omega^2 + p_4\right) + \left(-q_2\omega^2 + q_4\right)\cos(\omega\tau) \\ &\quad + \left(-q_1\omega^3 + q_3\omega\right)\sin(\omega\tau) = 0, \\ &\left(-p_1\omega^3 + p_3\omega\right) + \left(-q_1\omega^3 + q_3\omega\right)\cos(\omega\tau) \\ &\quad - \left(-q_2\omega^2 + q_4\right)\sin(\omega\tau) = 0. \end{aligned} \quad (16)$$

After some mathematical manipulations, we obtain the following equations

$$\begin{aligned} &\cos(\omega\tau) \\ &= \left((q_2 - p_1q_1)\,\omega^6 + (p_3q_1 - q_4 - p_2q_2 + p_1q_3)\,\omega^4 \right. \\ &\quad \left. + (p_2q_4 + p_4q_2 - p_3q_3)\,\omega^2 - p_4q_4\right) \\ &\quad \times \left(q_1^2\omega^6 + \left(q_2^2 - 2q_1q_3\right)\omega^4 + \left(q_3^2 - 2q_2q_4\right)\omega^2 + q_4^2\right)^{-1}, \\ &\sin(\omega\tau) \\ &= \left(q_1\omega^7 + (p_1q_2 - q_3 - p_2q_1)\,\omega^5 \right. \\ &\quad + (p_2q_3 + p_4q_1 - p_3q_2 - p_1q_4)\,\omega^3 \\ &\quad \left. + (p_3q_4 - p_4q_3)\,\omega\right) \\ &\quad \times \left(q_1^2\omega^6 + \left(q_2^2 - 2q_1q_3\right)\omega^4 + \left(q_3^2 - 2q_2q_4\right)\omega^2 + q_4^2\right)^{-1}. \end{aligned} \quad (17)$$

Let

$$\begin{gathered} b_1 = q_2 - p_1q_1, \qquad b_2 = p_3q_1 - q_4 - p_2q_2 + p_1q_3, \\ b_3 = p_2q_4 + p_4q_2 - p_3q_3, \qquad b_4 = -p_4q_4, \\ b_5 = q_1^2, \qquad b_6 = q_2^2 - 2q_1q_3, \\ b_7 = q_3^2 - 2q_2q_4, \qquad b_8 = q_4^2, \\ b_9 = q_1, \qquad b_{10} = p_1q_2 - q_3 - p_2q_1, \\ b_{11} = p_2q_3 + p_4q_1 - p_3q_2 - p_1q_4, \qquad b_{12} = p_3q_4 - p_4q_3. \end{gathered} \quad (18)$$

From (16), we have

$$\omega^8 + c_1\omega^6 + c_2\omega^4 + c_3\omega^2 + c_4 = 0, \quad (19)$$

where

$$\begin{gathered} c_1 = p_1^2 - 2p_2 - q_1^2, \qquad c_2 = p_2^2 - 2p_1p_3 + 2q_1q_3 + 2p_4 - q_2^2, \\ c_3 = p_3^2 - 2p_2p_4 + 2q_2q_4 - q_3^2, \qquad c_4 = p_4^2 - q_4^2. \end{gathered} \quad (20)$$

The conditions (i) and (ii) of Theorem 2 hold if and only if (19) has no real positive root. □

Let $m = \omega^2$; then (19) takes the form

$$m^4 + c_1m^3 + c_2m^2 + c_3m + c_4 = 0. \quad (21)$$

If $c_4 < 0$, then (19) has at least one positive root. In the case when (19) has four positive roots, we have

$$\begin{gathered} \omega_1 = \sqrt{m_1}, \qquad \omega_2 = \sqrt{m_2}, \\ \omega_3 = \sqrt{m_3}, \qquad \omega_4 = \sqrt{m_4}. \end{gathered} \quad (22)$$

From (16), we have

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{\arcsin\frac{b_9\omega_k^7 + b_{10}\omega_k^5 + b_{11}\omega_k^3 + b_{12}\omega_k}{b_5\omega_k^6 + b_6\omega_k^4 + b_7\omega_k^2 + b_8} + 2j\pi\right\}, \quad (23)$$

where $k = 1,2,3,4$ and $j = 0,1,2,\ldots$; we choose $\tau_0 = \min(\tau_k^{(j)})$.

To establish Hopf bifurcation at $\tau = \tau_0$, we need to show that

$$\Re\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_0} \neq 0. \quad (24)$$

By differentiating (10) with respect to τ, we can get

$$\begin{aligned} \frac{d\lambda}{d\tau} &= \lambda e^{-\lambda\tau}\left(q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4\right) \\ &\quad \times \left(\left(4\lambda^3 + 3p_1\lambda^2 + 2p_2\lambda + p_3\right) + e^{-\lambda\tau}\right. \\ &\quad \times \left[\left(3q_1\lambda^2 + 2q_2\lambda + q_3\right)\right. \\ &\quad \left.\left. - \tau\left(q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4\right)\right]\right)^{-1}. \end{aligned} \quad (25)$$

It follows that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \left(\left(4\lambda^3 + 3p_1\lambda^2 + 2p_2\lambda + p_3\right) + e^{-\lambda\tau} \times \left[\left(3q_1\lambda^2 + 2q_2\lambda + q_3\right) - \tau\left(q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4\right)\right]\right) \times \left(\lambda e^{-\lambda\tau}\left(q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4\right)\right)^{-1}. \quad (26)$$

Then, by combining (10), we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \left(\left(4\lambda^3 + 3p_1\lambda^2 + 2p_2\lambda + p_3\right) + e^{-\lambda\tau}\left(3q_1\lambda^2 + 2q_2\lambda + q_3\right)\right) \times \left(\lambda e^{-\lambda\tau}\left(q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4\right)\right)^{-1} - \frac{\tau}{\lambda}. \quad (27)$$

Substituting $\lambda = i\omega_0$ in (27) (where $\omega_0 > 0$ and $i = \sqrt{-1}$) yields

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_0} = \frac{d_1 + id_2}{d_3 + id_4} - \frac{\tau}{\lambda}, \quad (28)$$

where

$$\begin{aligned} d_1 &= \left(p_3 - 3p_1\omega_0^2\right) + \left(q_3 - 3q_1\omega_0^2\right)\cos\left(\omega_0\tau_0\right) + 2q_2\omega_0\sin\left(\omega_0\tau_0\right), \\ d_2 &= \left(2p_2\omega_0 - 4\omega^3\right) + 2q_2\omega_0\cos\left(\omega_0\tau_0\right) - \left(q_3 - 3q_1\omega_0^2\right)\sin\left(\omega_0\tau_0\right), \\ d_3 &= \left(q_1\omega_0^4 - q_3\omega_0^2\right)\cos\left(\omega_0\tau_0\right) + \left(q_4\omega_0 - q_2\omega_0^3\right)\sin\left(\omega_0\tau_0\right), \\ d_4 &= \left(q_4\omega_0 - q_2\omega_0^3\right)\cos\left(\omega_0\tau_0\right) - \left(q_1\omega_0^4 - q_3\omega_0^2\right)\sin\left(\omega_0\tau_0\right). \end{aligned} \quad (29)$$

Thus,

$$\Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_0} = \frac{d_1d_3 + d_2d_4}{d_3^2 + d_4^2}. \quad (30)$$

Notice that

$$\operatorname{sign}\left(\Re\frac{d\lambda(t)}{d\tau}\right)\Bigg|_{\tau=\tau_0} = \operatorname{sign}\left(\Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right)\Bigg|_{\tau=\tau_0}. \quad (31)$$

By summarizing the above analysis, we arrive at the following theorem.

Theorem 3. *The infected equilibrium $\mathscr{E}_+$ of the system (5) is asymptotically stable for $\tau \in [0, \tau_0)$ and it undergoes Hopf bifurcation at $\tau = \tau_0$.*

4. Direction and Stability of Bifurcating Periodic Solutions

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\tau_0 = \tau_k^{(j)}$, $j = 0, 1, 2, \ldots$. It is also important to derive explicit formulae from which we can determine the direction, stability, and period of periodic solutions bifurcating around the infected equilibrium $\mathscr{E}_+$ at the critical value τ_0. We use the cafeteria of normal forms and center manifold proposed by Hassard [34]. We assume that the model (5) undergoes Hopf bifurcation at the infected equilibrium $\mathscr{E}_+$ when $\tau_0 = \tau_k^{(j)}$, $j = 0, 1, 2, \ldots$, and then $\pm i\omega_0$ are the corresponding purely imaginary roots of the characteristic equation at the infected equilibrium $\mathscr{E}_+$. Assume also that

$$\begin{aligned} &\left(X_1(t), X_2(t), X_3(t), X_4(t)\right)^T \\ &= \left(x(t) - x^*, y(t) - y^*(t),\right. \\ &\quad \left. w(t) - w^*(t), z(t) - z^*(t)\right)^T; \end{aligned} \quad (32)$$

then using Taylors expansion for system (3) at the equilibrium point yields

$$\begin{aligned} \dot{X}_1 &= k_{11}X_1(t) + k_{12}X_2(t) + k_{13}X_1(t)X_1(t) + k_{14}X_1(t)X_2(t), \\ \dot{X}_2 &= k_{21}X_1(t) + k_{22}X_2(t) + k_{23}X_4(t) + k_{24}X_1(t)X_2(t) + k_{25}X_2(t)X_4(t), \\ \dot{X}_3 &= k_{31}X_3(t) + k_{32}X_2(t-\tau) + k_{33}X_3(t-\tau) + k_{34}X_2(t-\tau)X_3(t-\tau), \\ \dot{X}_4 &= k_{41}X_4(t) + k_{42}X_2(t-\tau) + k_{43}X_3(t-\tau) + k_{44}X_2(t-\tau)X_3(t-\tau). \end{aligned} \quad (33)$$

Here,

$$\begin{aligned} k_{11} &= -A_1y^* + r - \frac{2rx^*}{T_{\max}} - \frac{ry^*}{T_{\max}} - \delta_1, \\ k_{12} &= -A_1x^* - \frac{rx^*}{T_{\max}}, \\ k_{13} &= -\frac{2r}{T_{\max}}, \\ k_{14} &= -\frac{r}{T_{\max}} - A_1, \\ k_{21} &= A_1y^*, \\ k_{22} &= A_1x^* - A_2 - pz^*, \\ k_{23} &= -py^*, \\ k_{24} &= A_1, \\ k_{25} &= -p, \end{aligned}$$

$$k_{31} = -b,$$
$$k_{32} = c(1-q)w^*,$$
$$k_{33} = c(1-q)y^*,$$
$$k_{34} = c(1-q),$$
$$k_{41} = -h,$$
$$k_{42} = cqw^*,$$
$$k_{43} = cqy^*,$$
$$k_{44} = cq. \tag{34}$$

For convenience, let $\tau = \tau_0 + \mu$ and $u_t(\theta) = u(t+\theta)$ for $\theta \in [-\tau, 0]$. Denote $C^k([-\tau,0],\mathbb{R}^4) = \{\phi \mid \phi : [-\tau,0] \to \mathbb{R}^4\}$; ϕ has k-order continuous derivative. For initial conditions $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta))^T \in C([-\tau,0],\mathbb{R}^4)$, (33) can be rewritten as

$$\dot{u}(t) = L_\mu(u_t) + F(u_t, \mu), \tag{35}$$

where $u(t) = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in C$, $L_\mu : C \to \mathbb{R}^4$, and $F : C \to \mathbb{R}^4$ are given, respectively, by

$$\begin{gathered} L_\mu\phi = (\tau_0+\mu)G_1\phi(0) + (\tau_0+\mu)G_2\phi(-\tau), \\ F(\phi,\mu) = (\tau_0+\mu)(F_1, F_2, F_3, F_4)^T. \end{gathered} \tag{36}$$

L_μ is one parameter family of bounded linear operators in C and

$$G_1 = \begin{pmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{21} & k_{22} & 0 & k_{24} \\ 0 & 0 & k_{31} & 0 \\ 0 & 0 & 0 & k_{41} \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & k_{32} & k_{33} & 0 \\ 0 & k_{42} & k_{43} & 0 \end{pmatrix}, \tag{37}$$

$$F = \begin{pmatrix} k_{13}\phi_1(0)\phi_1(0) + k_{14}\phi_1(0)\phi_2(0) \\ k_{24}\phi_1(0)\phi_2(0) + k_{25}\phi_2(0)\phi_4(0) \\ k_{34}\phi_2(-\tau)\phi_3(-\tau) \\ k_{44}\phi_2(-\tau)\phi_3(-\tau) \end{pmatrix}.$$

From the discussion in the above section, we know that if $\mu = 0$, then model (5) undergoes a Hopf bifurcation at the infected equilibrium $\mathscr{E}_+$, and the associated characteristic equation of model (5) has a pair of purely imaginary roots $\pm i\tau_0\omega_0$. By Reisz representation, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-\tau, 0]$ such that

$$L_\mu\phi = \int_{-\tau}^{0} d\eta(\theta,\mu)\phi(\theta). \tag{38}$$

In fact, we can choose

$$\eta(\theta,\mu) = (\tau_0+\mu)G_1\delta(\theta) + (\tau_0+\mu)G_2\delta(\theta+\tau), \tag{39}$$

where $\delta(\theta)$ is Dirac delta function. Next, for $\phi \in C^1([-\tau, 0],\mathbb{R}^4)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi}{d\theta}, & \theta \in [-\tau, 0) \\ \displaystyle\int_{-\tau}^{0} d\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases} \tag{40}$$

$$R(\mu)\phi = \begin{cases} 0, & \theta = [-\tau, 0) \\ F(\phi,\mu), & \theta = 0. \end{cases} \tag{41}$$

Since $\dot{u}(t) = \dot{u}_t(\theta)$, (35) can be written as

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{42}$$

where $u_t = u(t+\theta)$, $\theta \in [-\tau, 0]$. For $\psi \in C^1([0,\tau],\mathbb{R}^4)$, the adjoint operator A^* of A can be defined as

$$A^*\psi(s)\phi = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (-\tau, 0] \\ \displaystyle\int_{-\tau}^{0} d\eta(\theta,\mu)\phi(\theta), & s = 0. \end{cases} \tag{43}$$

For $\phi \in C^1([-\tau,0],\mathbb{R}^4)$ and $\psi \in C^1([0,\tau],\mathbb{R}^4)$, in order to normalize the eigenvalues of operator A and adjoint operator A^*, the following bilinear form is defined by

$$\begin{aligned} \langle \psi, \phi \rangle &= \overline{\psi}(0)\phi(0) \\ &\quad - \int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)[d\eta(\theta)]\phi(\xi)\,d\xi, \end{aligned} \tag{44}$$

where $\eta(\theta) = \eta(\theta, 0)$ and $\overline{\psi}$ is complex conjugate of ψ. It can verify that A^* and $A(0)$ are adjoint operators with respect to this bilinear form.

We assume that $\pm i\omega_0$ are eigenvalues of $A(0)$ and the other eigenvalues have strictly negative real parts. Thus, they are also eigenvalues of A^*. Now we compute the eigenvector q of A corresponding to the eigenvalue $i\omega_0$ and the eigenvector q^* of A^* corresponding to the eigenvalue $-i\omega_0$. Suppose that $q(\theta) = (1, p_1, p_2, p_3)^T e^{i\omega_0\theta}$ is eigenvector of $A(0)$ associated with $i\omega_0$; then, $A(0)q(\theta) = i\omega_0 q(\theta)$. It follows from the definition of $A(0)$ and (36), (38), and (40) that

$$\begin{pmatrix} k_{11}-i\omega_0 & k_{12} & 0 & 0 \\ k_{21} & k_{22}-i\omega_0 & 0 & k_{23} \\ 0 & k_{32}e^{-i\omega_0\tau_0} & k_{31}+k_{33}e^{-i\omega_0\tau_0}-i\omega_0 & 0 \\ 0 & k_{42}e^{-i\omega_0\tau_0} & k_{43}e^{-i\omega_0\tau_0} & k_{41}-i\omega_0 \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (45)$$

Solving (45), we can easily obtain $q(0) = (1, p_1, p_2, p_3)^T$, where

$$p_1 = \frac{i\omega_0 - k_{11}}{k_{12}},$$

$$p_2 = \frac{k_{32}(k_{11}-i\omega_0)e^{-i\omega_0\tau_0}}{k_{12}(k_{31}+k_{33}e^{-i\omega_0\tau_0}-i\omega_0)},$$

$$p_3 = \frac{(k_{11}-i\omega_0)(k_{22}-i\omega_0)-k_{12}k_{21}}{k_{12}k_{23}}. \quad (46)$$

Similarly, suppose that the eigenvector q^* of A^* corresponding to $-i\omega_0$ is $q^*(s) = (1/D)(1, p_1^*, p_2^*, p_3^*)^T e^{i\omega_0 s}$, $s \in [0, \tau]$. By the definition of A^* and (36), (38), and (40), one gets

$$\begin{pmatrix} k_{11}+i\omega_0 & k_{21} & 0 & 0 \\ k_{12} & k_{22}+i\omega_0 & k_{32}e^{-i\omega_0\tau_0} & k_{42}e^{-i\omega_0\tau_0} \\ 0 & 0 & k_{31}+k_{33}e^{-i\omega_0\tau_0}+i\omega_0 & k_{43}e^{-i\omega_0\tau_0} \\ 0 & k_{23} & 0 & k_{41}+i\omega_0 \end{pmatrix} q^*(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (47)$$

Solving (47), we easily obtain $q^*(0) = (1/D)(1, p_1^*, p_2^*, p_3^*)^T$, where

$$\begin{aligned} p_1^* &= -\frac{k_{11}+i\omega_0}{k_{21}}, \\ p_2^* &= -\frac{k_{23}k_{43}(k_{11}+i\omega_0)e^{-i\omega_0\tau_0}}{k_{21}(k_{41}+i\omega_0)(k_{31}+k_{33}e^{-i\omega_0\tau_0}+i\omega_0)}, \\ p_3^* &= \frac{k_{23}(k_{11}+i\omega_0)}{k_{21}(k_{41}+i\omega_0)}. \end{aligned} \quad (48)$$

In order to assure that $\langle q^*, q\rangle = 1$, we need to determine the value of D. From (44), one gets

$$\begin{aligned} \langle q^*, q\rangle &= \overline{q^*}^T(0)\, q(0) - \int_{\theta=-\tau_0}^{0}\int_{\xi=0}^{\theta} \overline{q^*}^T(\xi-\theta)\,[d\eta(\theta)]\, q(\xi)\, d(\xi) \\ &= \frac{1}{\overline{D}}\left(1 + p_1\overline{p_1}^* + p_2\overline{p_2}^* + p_3\overline{p_3}^*\right) \\ &\quad - \int_{-\tau_0}^{0}\int_{\xi=0}^{\theta} \frac{1}{\overline{D}}\left(1, \overline{p_1}^*, \overline{p_2}^* \overline{p_3}^*\right) e^{-i\omega_0(\xi-\theta)} \\ &\quad \times [d\eta(\theta)]\,(1, p_1, p_2, p_3)^T e^{i\omega_0\xi} d\xi \\ &= \frac{1}{\overline{D}}\left(1 + p_1\overline{p_1}^* + p_2\overline{p_2}^* + p_3\overline{p_3}^*\right) \\ &\quad - \int_{-\tau_0}^{0} \frac{1}{\overline{D}}\left(1, \overline{p_1}^*, \overline{p_2}^*, \overline{p_3}^*\right)\theta e^{i\omega_0\theta} \\ &\quad \times [d\eta(\theta)]\,(1, p_1, p_2, p_3)^T \\ &= \frac{1}{\overline{D}}\Big(\left(1 + p_1\overline{p_1}^* + p_2\overline{p_2}^* + p_3\overline{p_3}^*\right) \\ &\quad + \tau_0 e^{-i\omega_0\tau_0}\left(1, \overline{p_1}^*, \overline{p_2}^*, \overline{p_3}^*\right) \\ &\quad \times G_2(1, p_1, p_2, p_3)^T\Big) \\ &= \frac{1}{\overline{D}}\Big(\left(1 + p_1\overline{p_1}^* + p_2\overline{p_2}^* + p_3\overline{p_3}^*\right) + \tau_0 e^{-i\omega_0\tau_0} \\ &\quad \times \left(\left(k_{32}\overline{p_2}^* + k_{42}\overline{p_3}^*\right) p_1 \right. \\ &\quad \left. + \left(k_{33}\overline{p_2}^* + k_{43}\overline{p_3}^*\right) p_2\right)\Big); \\ \overline{D} &= \left(1 + p_1\overline{p_1}^* + p_2\overline{p_2}^* + p_3\overline{p_3}^*\right) \\ &\quad + \tau_0 e^{-i\omega_0\tau_0}\left(\left(k_{32}\overline{p_2}^* + k_{42}\overline{p_3}^*\right) p_1 \right. \\ &\quad \left. + \left(k_{33}\overline{p_2}^* + k_{43}\overline{p_3}^*\right) p_2\right). \end{aligned} \quad (49)$$

Let

$$\begin{aligned} v(t) &= \langle q^*, u_t\rangle, \\ W(t,\theta) &= u_t - vq - \overline{vq} = u_t - 2\,\mathrm{Re}\,(v(t)\, q(\theta)). \end{aligned} \quad (50)$$

On the center manifold Ω_0, we have

$$W(t,\theta) = W(v(t), \overline{v}(t), \theta), \quad (51)$$

where

$$W(v,\overline{v},\theta) = W_{20}(\theta)\frac{v^2}{2} + W_{11}(\theta)v\overline{v} + W_{02}(\theta)\frac{\overline{v}v^2}{2} + \cdots. \tag{52}$$

v and $\overline{v}$ are local coordinates of the center manifold Ω_0 in the direction of q^* and $\overline{q}^*$, respectively. Note that W is real if u_t is real. So we only consider real solutions. From (50), we obtain

$$\begin{aligned}\langle q^*, W\rangle &= \langle q^*, u_t - vq - \overline{vq}\rangle \\ &= \langle q^*, u_t\rangle - v(t)\langle q^*, q\rangle - \overline{v}(t)\langle q^*, \overline{q}\rangle.\end{aligned} \tag{53}$$

For the solution $u_t \in \Omega_0$ of (35), from (41) and (44), since $\mu = 0$, we have

$$\begin{aligned}\dot{v}(t) &= \langle q^*, \dot{u}_t\rangle \\ &= \langle q^*, A(0)u_t + R(0)u_t\rangle \\ &= \langle q^*, A(0)u_t\rangle + \langle q^*, R(0)u_t\rangle \\ &= \langle A^*q^*, u_t\rangle + \overline{q^*}^T(0)F(u_t,0) \\ &= i\omega_0 v(t) + \overline{q^*}^T(0) f_0(v,\overline{v}).\end{aligned} \tag{54}$$

Rewrite (54) as

$$\dot{v}(t) = i\omega_0 v(t) + g(v,\overline{v}), \tag{55}$$

where

$$\begin{aligned}g(v,\overline{v}) &= \overline{q^*}^T(0) f_0(v,\overline{v}) \\ &= \overline{q^*}^T(0)F(W(v,\overline{v},\theta) + 2\operatorname{Re}\{v(t)q(\theta),0\}) \\ &= g_{20}\frac{v^2}{2} + g_{11}v\overline{v} + g_{02}\frac{\overline{v}^2}{2} + g_{21}\frac{v^2\overline{v}}{2}\cdots.\end{aligned} \tag{56}$$

Substituting (42) and (54) into (50) yields

$$\begin{aligned}\dot{W} &= \dot{u}(t) - \dot{v}q - \dot{\overline{v}}\overline{q} \\ &= Au_t + Ru_t - \left(i\omega_0 v + \overline{q^*}^T(0) f_0(v,\overline{v})\right)q \\ &\quad - \left(i\omega_0\overline{v} + \overline{q^*}^T(0)\overline{f_0(v,\overline{v})}\right)\overline{q} \\ &= Au_t + Ru_t - Avq - A\overline{v}\overline{q} \\ &\quad - 2\operatorname{Re}\left(\overline{q^*}^T(0) f_0(v,\overline{v})q\right),\end{aligned} \tag{57}$$

$$\dot{W} = \begin{cases} AW - 2\operatorname{Re}\left(\overline{q^*}^T(0) f_0(v,\overline{v})q\right), & \theta\in[-\tau,0) \\ AW - 2\operatorname{Re}\left(\overline{q^*}^T(0) f_0(v,\overline{v})q\right) + f_0(v,\overline{v}), & \theta = 0,\end{cases} \tag{58}$$

which can be written as

$$\dot{W} = AW + H(v,\overline{v},\theta), \tag{59}$$

where

$$H(v,\overline{v},\theta) = H_{20}(\theta)\frac{v^2}{2} + H_{11}(\theta)v\overline{v} + H_{02}(\theta)\frac{\overline{v}^2}{2} + \cdots. \tag{60}$$

On the center manifold Ω_0, we have

$$\dot{W} = W_v\dot{v} + W_{\overline{v}}\dot{\overline{v}}. \tag{61}$$

Substituting (52) and (55) into (61), one obtains

$$\begin{aligned}\dot{W} &= (W_{20}v + W_{11}\overline{v} + \cdots)(i\omega_0 v + g) \\ &\quad + (W_{11}v + W_{02}\overline{v} + \cdots)(-i\omega_0\overline{v} + \overline{g}).\end{aligned} \tag{62}$$

Substituting (52) and (60) into (59) yields

$$\begin{aligned}\dot{W} &= (AW_{20} + H_{20})\frac{v^2}{2} + (AW_{11} + H_{11})v\overline{v} \\ &\quad + (AW_{02} + H_{02})\frac{\overline{v}^2}{2} + \cdots.\end{aligned} \tag{63}$$

Comparing the coefficients of (62) and (63), one gets

$$\begin{aligned}(A - i2\omega_0)W_{20}(\theta) &= -H_{20}(\theta), \\ AW_{11}(\theta) &= -H_{11}(\theta), \\ (A + i2\omega_0)W_{02}(\theta) &= -H_{02}(\theta).\end{aligned} \tag{64}$$

Since $u_t = u(t+\theta) = W(v,\overline{v},\theta) + vq + \overline{v}q$, then we have

$$\begin{aligned}u_t &= \begin{pmatrix} u_1(t+\theta) \\ u_2(t+\theta) \\ u_3(t+\theta) \\ u_4(t+\theta)\end{pmatrix} \\ &= \begin{pmatrix} W^{(1)}(v,\overline{v},\theta) \\ W^{(2)}(v,\overline{v},\theta) \\ W^{(3)}(v,\overline{v},\theta) \\ W^{(4)}(v,\overline{v},\theta)\end{pmatrix} + v\begin{pmatrix}1 \\ p_1 \\ p_2 \\ p_3\end{pmatrix}e^{i\omega_0\theta} \\ &\quad + \overline{v}\begin{pmatrix}1 \\ \overline{p}_1 \\ \overline{p}_2 \\ \overline{p}_3\end{pmatrix}e^{-i\omega_0\theta}.\end{aligned} \tag{65}$$

Thus, we obtain

$$\begin{aligned}u_1(t+\theta) &= W^{(1)}(v,\overline{v},\theta) + ve^{i\omega_0\theta} + \overline{v}e^{-i\omega_0\theta} \\ &= \left(W_{20}^{(1)}(\theta)\frac{v^2}{2} + W_{11}^{(1)}(\theta)v\overline{v} + W_{02}^{(1)}(\theta)\frac{\overline{v}^2}{2} + \cdots\right) \\ &\quad + ve^{i\omega_0\theta} + \overline{v}e^{-i\omega_0\theta},\end{aligned}$$

$$u_2(t+\theta) = W^{(2)}(\nu,\overline{\nu},\theta) + \nu p_1 e^{i\omega_0\theta} + \overline{\nu}\overline{p}_1 e^{-i\omega_0\theta}$$
$$= \left(W_{20}^{(2)}(\theta)\frac{\nu^2}{2} + W_{11}^{(2)}(\theta)\nu\overline{\nu} + W_{02}^{(2)}(\theta)\frac{\overline{\nu}^2}{2} + \cdots\right)$$
$$+ \nu p_1 e^{i\omega_0\theta} + \overline{\nu}\overline{p}_1 e^{-i\omega_0\theta},$$
$$u_3(t+\theta) = W^{(3)}(\nu,\overline{\nu},\theta) + \nu p_2 e^{i\omega_0\theta} + \overline{\nu}\overline{p}_2 e^{-i\omega_0\theta}$$
$$= \left(W_{20}^{(3)}(\theta)\frac{\nu^2}{2} + W_{11}^{(3)}(\theta)\nu\overline{\nu} + W_{02}^{(3)}(\theta)\frac{\overline{\nu}^2}{2} + \cdots\right)$$
$$+ \nu p_2 e^{i\omega_0\theta} + \overline{\nu}\overline{p}_2 e^{-i\omega_0\theta},$$
$$u_4(t+\theta) = W^{(4)}(\nu,\overline{\nu},\theta) + \nu p_3 e^{i\omega_0\theta} + \overline{\nu}\overline{p}_3 e^{-i\omega_0\theta}$$
$$= \left(W_{20}^{(4)}(\theta)\frac{\nu^2}{2} + W_{11}^{(4)}(\theta)\nu\overline{\nu} + W_{02}^{(4)}(\theta)\frac{\overline{\nu}^2}{2} + \cdots\right)$$
$$+ \nu p_3 e^{i\omega_0\theta} + \overline{\nu}\overline{p}_3 e^{-i\omega_0\theta}. \tag{66}$$

It is obvious that

$$\phi_1(0) = \nu + \overline{\nu} + W_{20}^{(1)}(0)\frac{\nu^2}{2} + W_{11}^{(1)}(0)\nu\overline{\nu} + W_{02}^{(1)}(0)\frac{\overline{\nu}^2}{2} + \cdots,$$
$$\phi_2(0) = \nu p_1 + \overline{\nu}\overline{p}_1 + W_{20}^{(2)}(0)\frac{\nu^2}{2} + W_{11}^{(2)}(0)\nu\overline{\nu} + W_{02}^{(2)}(0)\frac{\overline{\nu}^2}{2} + \cdots, \tag{67}$$
$$\phi_4(0) = \nu p_3 + \overline{\nu}\,\overline{p}_3 + W_{20}^{(4)}(0)\frac{\nu^2}{2} + W_{11}^{(4)}(0)\nu\overline{\nu} + W_{02}^{(4)}(0)\frac{\overline{\nu}^2}{2} + \cdots.$$

So

$$\phi_1(0)\phi_1(0) = \nu^2 + \overline{\nu}^2 + 2\nu\overline{\nu} + \frac{1}{2}\left(4W_{11}^{(1)}(0) + 2W_{20}^{(1)}(0)\right)\nu^2\overline{\nu} + \cdots,$$
$$\phi_1(0)\phi_2(0) = p_1\nu^2 + \overline{p}_1\overline{\nu}^2 + \left(p_1 + \overline{p}_1\right)\nu\overline{\nu} + \frac{1}{2}\left(2W_{11}^{(2)}(0) + W_{20}^{(2)}(0) + W_{20}^{(1)}(0)\overline{p}_1 + 2W_{11}^{(1)}(0)p_1\right)\nu^2\overline{\nu} + \cdots,$$
$$\phi_2(0)\phi_4(0) = p_1p_3\nu^2 + \overline{p}_1\,\overline{p}_3\overline{\nu}^2 + \left[p_1\overline{p}_3 + \overline{p}_1 p_3\right]\nu\overline{\nu} + \frac{1}{2}\left(2W_{11}^{(4)}(0)p_1 + W_{20}^{(4)}(0)\overline{p}_1 + W_{20}^{(0)}(0)\overline{p}_3 + 2W_{11}^{(2)}(0)p_3\right)\nu^2\overline{\nu}\cdots; \tag{68}$$

also

$$\phi_2(-\tau) = \nu p_1 e^{-i\omega_0\tau} + \overline{\nu}\,\overline{p}_1 e^{i\omega_0\tau} + W_{20}^{(2)}(-\tau)\frac{\nu^2}{2} + W_{11}^{(2)}(-\tau)\nu\overline{\nu} + W_{02}^{(2)}(-\tau)\frac{\overline{\nu}^2}{2} + \cdots,$$
$$\phi_3(-\tau) = \nu p_2 e^{-i\omega_0\tau} + \overline{\nu}\,\overline{p}_2 e^{i\omega_0\tau} + W_{20}^{(3)}(-\tau)\frac{\nu^2}{2} + W_{11}^{(3)}(-\tau)\nu\overline{\nu} + W_{02}^{(3)}(-\tau)\frac{\overline{\nu}^2}{2} + \cdots \tag{69}$$

and hence

$$\phi_2(-\tau)\phi_3(-\tau) = p_1p_2e^{-2i\omega_0\tau_0}\nu^2 + \overline{p}_1\,\overline{p}_2e^{2i\omega_0\tau_0}\overline{\nu}^2 + \left(p_1\overline{p}_2 + \overline{p}_1p_2\right)\nu\overline{\nu} + \frac{1}{2}\left(2p_1e^{-i\omega_0\tau_0}W_{11}^{(3)}(-\tau) + \overline{p}_1e^{i\omega_0\tau}W_{20}^{(3)}(-\tau) + 2p_2e^{-i\omega_0\tau_0}W_{11}^{(2)}(-\tau)\right)\nu^2\overline{\nu} + \cdots. \tag{70}$$

It follows from (54) that

$$f_0(\nu,\overline{\nu}) = \begin{pmatrix} k_{13}\phi_1(0)\phi_1(0) + k_{14}\phi_1(0)\phi_2(0) \\ k_{24}\phi_1(0)\phi_2(0) + k_{25}\phi_2(0)\phi_4(0) \\ k_{34}\phi_2(-\tau)\phi_3(-\tau) \\ k_{44}\phi_2(-\tau)\phi_3(-\tau) \end{pmatrix} = \begin{pmatrix} F_{11}\nu^2 + F_{12}\overline{\nu}^2 + F_{13}\nu\overline{\nu} + F_{14}\nu^2\overline{\nu} \\ F_{21}\nu^2 + F_{22}\overline{\nu}^2 + F_{23}\nu\overline{\nu} + F_{24}\nu^2\overline{\nu} \\ F_{31}\nu^2 + F_{32}\overline{\nu}^2 + F_{33}\nu\overline{\nu} + F_{34}\nu^2\overline{\nu} \\ F_{41}\nu^2 + F_{42}\overline{\nu}^2 + F_{43}\nu\overline{\nu} + F_{44}\nu^2\overline{\nu} \end{pmatrix}, \tag{71}$$

where

$$F_{11} = k_{13} + k_{14}p_1,$$
$$F_{12} = k_{13} + k_{14}\overline{p}_1,$$
$$F_{13} = 2k_{13} + k_{14}\left(p_1 + \overline{p}_1\right),$$
$$F_{14} = k_{13}\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) + \frac{1}{2}k_{14}\left(2W_{11}^{(2)}(0) + W_{20}^{(2)}(0) + W_{20}^{(1)}(0)\overline{p}_1 + 2W_{11}^{(1)}(0)p_1\right),$$
$$F_{21} = k_{24}p_1 + k_{25}p_1p_3,$$
$$F_{22} = k_{24}\overline{p}_1 + k_{25}\overline{p}_1\overline{p}_3,$$
$$F_{23} = k_{24}\left(p_1 + \overline{p}_1\right) + k_{25}\left(p_1\overline{p}_3 + \overline{p}_1p_3\right),$$

$$F_{24} = \frac{1}{2}k_{24}\left(2W_{11}^{(2)}(0) + W_{20}^{(2)}(0) + W_{20}^{(1)}(0)\overline{p}_1 + 2W_{11}^{(2)}(0)p_1\right) + \frac{1}{2}k_{25}\left(2W_{11}^{(4)}(0)p_1 + W_{20}^{(4)}(0)\overline{p}_1 + W_{20}^{(2)}(0)\overline{p}_3 + 2W_{11}^{(2)}(0)p_3\right),$$
$$F_{31} = k_{34}\left(p_1p_2e^{-2i\omega_0\tau_0}\right),$$
$$F_{32} = k_{34}\left(\overline{p}_1\overline{p}_2e^{2i\omega_0\tau_0}\right),$$
$$F_{33} = k_{34}\left(p_1\overline{p}_2 + \overline{p}_1p_2\right),$$
$$F_{34} = \frac{1}{2}k_{34}\left(2p_1e^{-i\omega_0\tau_0}W_{11}^{(3)}(-\tau) + \overline{p}_1e^{i\omega_0\tau_0}W_{20}^{(3)}(-\tau) + 2p_2e^{-i\omega_0\tau_0}W_{11}^{(2)}(-\tau)\right),$$
$$F_{41} = k_{44}\left(p_1p_2e^{-2i\omega_0\tau_0}\right),$$
$$F_{42} = k_{44}\left(\overline{p}_1\overline{p}_2e^{2i\omega_0\tau_0}\right),$$
$$F_{43} = k_{44}\left(p_1\overline{p}_2 + \overline{p}_1p_2\right),$$
$$F_{44} = \frac{1}{2}k_{44}\left(2p_1e^{-i\omega_0\tau_0}W_{11}^{(3)}(-\tau) + \overline{p}_1e^{i\omega_0\tau_0}W_{20}^{(3)}(-\tau) + 2p_2e^{-i\omega_0\tau_0}W_{11}^{(2)}(-\tau)\right). \quad (72)$$

Since $\overline{q}^*(0) = (1/\overline{D})(1, \overline{p}_1^*, \overline{p}_2^*, \overline{p}_3^*)^T$, we have

$$g(\nu,\overline{\nu}) = \overline{q}^*(0)^T f_0(\nu,\overline{\nu}) = \frac{1}{\overline{D}}\left(1, \overline{p}_1^*, \overline{p}_2^*, \overline{p}_3^*\right) \times \begin{pmatrix} F_{11}\nu^2 + F_{12}\overline{\nu}^2 + F_{13}\nu\overline{\nu} + F_{14}\nu^2\overline{\nu} \\ F_{21}\nu^2 + F_{22}\overline{\nu}^2 + F_{23}\nu\overline{\nu} + F_{24}\nu^2\overline{\nu} \\ F_{31}\nu^2 + F_{32}\overline{\nu}^2 + F_{33}\nu\overline{\nu} + F_{34}\nu^2\overline{\nu} \\ F_{41}\nu^2 + F_{42}\overline{\nu}^2 + F_{43}\nu\overline{\nu} + F_{44}\nu^2\overline{\nu} \end{pmatrix}$$
$$= \frac{1}{\overline{D}}\Big(\left(F_{11} + F_{21}\overline{p}_1^* + F_{31}\overline{p}_2^* + F_{41}\overline{p}_3^*\right)\nu^2 + \left(F_{12} + F_{22}\overline{p}_1^* + F_{32}\overline{p}_2^* + F_{42}\overline{p}_3^*\right)\overline{\nu}^2 + \left(F_{13} + F_{23}\overline{p}_1^* + F_{33}\overline{p}_2^* + F_{43}\overline{p}_3^*\right)\nu\overline{\nu} + \left(F_{14} + F_{24}\overline{p}_1^* + F_{34}\overline{p}_2^* + F_{44}\overline{p}_3^*\right)\nu^2\overline{\nu}\Big). \quad (73)$$

Comparing the coefficients of the above equation with those in (61), we have

$$g_{20} = \frac{2}{\overline{D}}\left(F_{11} + F_{21}\overline{p}_1^* + F_{31}\overline{p}_2^* + F_{41}\overline{p}_3^*\right),$$
$$g_{11} = \frac{1}{\overline{D}}\left(F_{13} + F_{23}\overline{p}_1^* + F_{33}\overline{p}_2^* + F_{43}\overline{p}_3^*\right),$$
$$g_{02} = \frac{2}{\overline{D}}\left(F_{12} + F_{22}\overline{p}_1^* + F_{32}\overline{p}_2^* + F_{42}\overline{p}_3^*\right),$$
$$g_{21} = \frac{2}{\overline{D}}\left(F_{14} + F_{24}\overline{p}_1^* + F_{34}\overline{p}_2^* + F_{44}\overline{p}_3^*\right). \quad (74)$$

We need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta \in [-\tau, 0)$. Equations (62) and (63) imply that

$$H(\nu,\overline{\nu},\theta) = -2\,\mathrm{Re}\left\{\overline{q^*}^T(0)\,f_0(\nu,\overline{\nu})\,q(\theta)\right\} = -2\,\mathrm{Re}\left\{g(\nu,\overline{\nu})\,q(\theta)\right\} = -g(\nu,\overline{\nu})\,q(\theta) - \overline{g}(\nu,\overline{\nu})\,\overline{q}(\theta),$$
$$H(\nu,\overline{\nu},\theta) = -\left(g_{20}\frac{\nu^2}{2} + g_{11}\nu\overline{\nu} + g_{02}\frac{\overline{\nu}^2}{2} + g_{21}\frac{\nu^2\overline{\nu}}{2}\cdots\right)q(\theta) - \left(\overline{g}_{20}\frac{\overline{\nu}^2}{2} + \overline{g}_{11}\nu\overline{\nu} + \overline{g}_{02}\frac{\nu^2}{2} + \overline{g}_{21}\frac{\overline{\nu}^2\nu}{2}\cdots\right)\overline{q}(\theta). \quad (75)$$

Comparing the coefficients of the above equation with (60), we have

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta),$$
$$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta), \quad (76)$$
$$H_{02}(\theta) = -g_{02}q(\theta) - \overline{g}_{20}\overline{q}(\theta).$$

It follows from (40) and (64) that

$$\dot{W}(\theta) = AW_{20} = 2i\omega_0W_{20}(\theta) - H_{20}(\theta) = 2i\omega_0W_{20}(\theta) + g_{20}q(0)\,e^{i\omega_0\theta} + \overline{g}_{02}\overline{q}(0)\,e^{-i\omega_0\theta}. \quad (77)$$

By solving the above equation for $W_{20}(\theta)$ and for $W_{11}(\theta)$, one obtains

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0}q(0)\,e^{i\omega_0\theta} + \frac{i\overline{g}_{02}}{3\omega_0}\overline{q}(0)\,e^{-i\omega_0\theta} + E_1e^{2i\omega_0\theta},$$
$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0}q(0)\,e^{i\omega_0\theta} + \frac{i\overline{g}_{11}}{\omega_0}\overline{q}(0)\,e^{-i\omega_0\theta} + E_2, \quad (78)$$

where E_1 and E_2 can be determined by setting $\theta = 0$ in $H(\nu,\overline{\nu},\theta)$.

In fact, we have

$$\begin{aligned} H(v,\bar{v},0) &= -2\operatorname{Re}\left\{\overline{q^*}^T(0) f_0(v,\bar{v}q)\right\} + f_0(v,\bar{v}) \\ &= -\left(g_{20}\frac{v^2}{2} + g_{11}v\bar{v} + g_{02}\frac{\bar{v}^2}{2} + g_{21}\frac{v^2\bar{v}}{2}\cdots\right)q(0) \\ &\quad -\left(\bar{g}_{20}\frac{\bar{v}^2}{2} + \bar{g}_{11}v\bar{v} + \bar{g}_{02}\frac{v^2}{2} + \bar{g}_{20}\frac{\bar{v}^2 v}{2} + \cdots\right)\bar{q}(0) \\ &\quad + \begin{pmatrix} F_{11}v^2 + F_{12}\bar{v}^2 + F_{13}v\bar{v} + F_{14}v^2\bar{v} \\ F_{21}v^2 + F_{22}\bar{v}^2 + F_{23}v\bar{v} + F_{24}v^2\bar{v} \\ F_{31}v^2 + F_{32}\bar{v}^2 + F_{33}v\bar{v} + F_{34}v^2\bar{v} \\ F_{41}v^2 + F_{42}\bar{v}^2 + F_{43}v\bar{v} + F_{44}v^2\bar{v} \end{pmatrix}; \end{aligned} \tag{79}$$

comparing the coefficients of the above equations with those in (61), it follows that

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + (F_{11}, F_{21}, F_{31}, F_{41})^T, \\ H_{11}(0) &= -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + (F_{13}, F_{23}, F_{33}, F_{43})^T. \end{aligned} \tag{80}$$

By the definition of A and (40) and (64), we get

$$\begin{aligned} \int_{-\tau_0}^{0} d\eta(\theta) W_{20}(\theta) &= AW_{20}(0) = 2i\omega_0 W_{20}(0) - H_{20}(0), \\ \int_{-\tau_0}^{0} d\eta(\theta) W_{11}(\theta) &= AW_{11}(0) = -H_{11}(0). \end{aligned} \tag{81}$$

One can notice that

$$\begin{aligned} \left(i\omega_0 I - \int_{-\tau_0}^{0} e^{i\omega_0\theta} d\eta(\theta)\right) q(0) &= 0, \\ \left(-i\omega_0 I - \int_{-\tau_0}^{0} e^{-i\omega_0\theta} d\eta(\theta)\right) \bar{q}(0) &= 0. \end{aligned} \tag{82}$$

Thus, we obtain

$$\begin{aligned} \left(2i\omega_0 I - \int_{-\tau_0}^{0} e^{2i\omega_0\theta} d\eta(\theta)\right) E_1 &= (F_{11}, F_{21}, F_{31}, F_{41})^T \\ \left(\int_{-\tau_0}^{0} d\eta(\theta)\right) E_2 &= -(F_{13}, F_{23}, F_{33}, F_{43})^T, \end{aligned} \tag{83}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, E_1^{(4)})^T$, $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}, E_2^{(4)})^T$; the above equation can be written as

$$\begin{aligned} &\begin{pmatrix} 2i\omega_0 - k_{11} & -k_{12} & 0 & 0 \\ -k_{21} & 2i\omega_0 - k_{22} & 0 & -k_{23} \\ 0 & -k_{32}e^{-i\omega_0\tau_0} & 2i\omega_0 - k_{31} - k_{33}e^{-i\omega_0\tau_0} & 0 \\ 0 & -k_{42}e^{-i\omega_0\tau_0} & -k_{43}e^{-i\omega_0\tau_0} & 2i\omega_0 - k_{41} \end{pmatrix} E_1 = \begin{pmatrix} F_{11} \\ F_{21} \\ F_{31} \\ F_{41} \end{pmatrix}, \\ &\begin{pmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{21} & k_{22} & 0 & k_{23} \\ 0 & k_{32} & k_{31} & 0 \\ 0 & k_{42} & k_{43} & k_{41} \end{pmatrix} E_2 = \begin{pmatrix} F_{13} \\ F_{23} \\ F_{33} \\ F_{43} \end{pmatrix}. \end{aligned} \tag{84}$$

From (78), (84), we can calculate g_{21}, and we can derive the following parameters:

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}(C_1(0))}{\operatorname{Re}(\lambda'(\tau_0))}, \\ \beta_2 &= 2\operatorname{Re}C_1(0), \\ T_2 &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\lambda(\tau_0)}{\omega_0}. \end{aligned} \tag{85}$$

We arrive at the following theorem.

Theorem 4. *The periodic solution is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$); the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); the period of the bifurcating periodic solution increases (decreases) if $T_2 > 0$ ($T_2 < 0$).*

5. Numerical Simulations

In this section, we provide some simulations of model (4) to exhibit the impact of discrete time delay in the model. We consider the parameters values: $\Lambda = 10$, $\delta_1 = 0.06$, $\delta_2 = 0.3$, $e_1 = 0.2$, $\beta = 0.1$, $p = 1$, $c = 0.1$, $b = 0.02$, $q = 0.02$, $\eta \in [0,1]$, $h = 0.1$, $r = 0.03$, $\epsilon \in [0,1]$, and $T_{\max} = 1500$. According to the given parameters' values, the threshold critical value $\tau_0 = 0.4957$ from the formula (21) exists. The steady state $\mathscr{E}_+$ exists and is asymptotically stable (see Figure 1). We may notice that the solution converges to the equilibrium $\mathscr{E}_+$ with damping oscillations as the value of τ

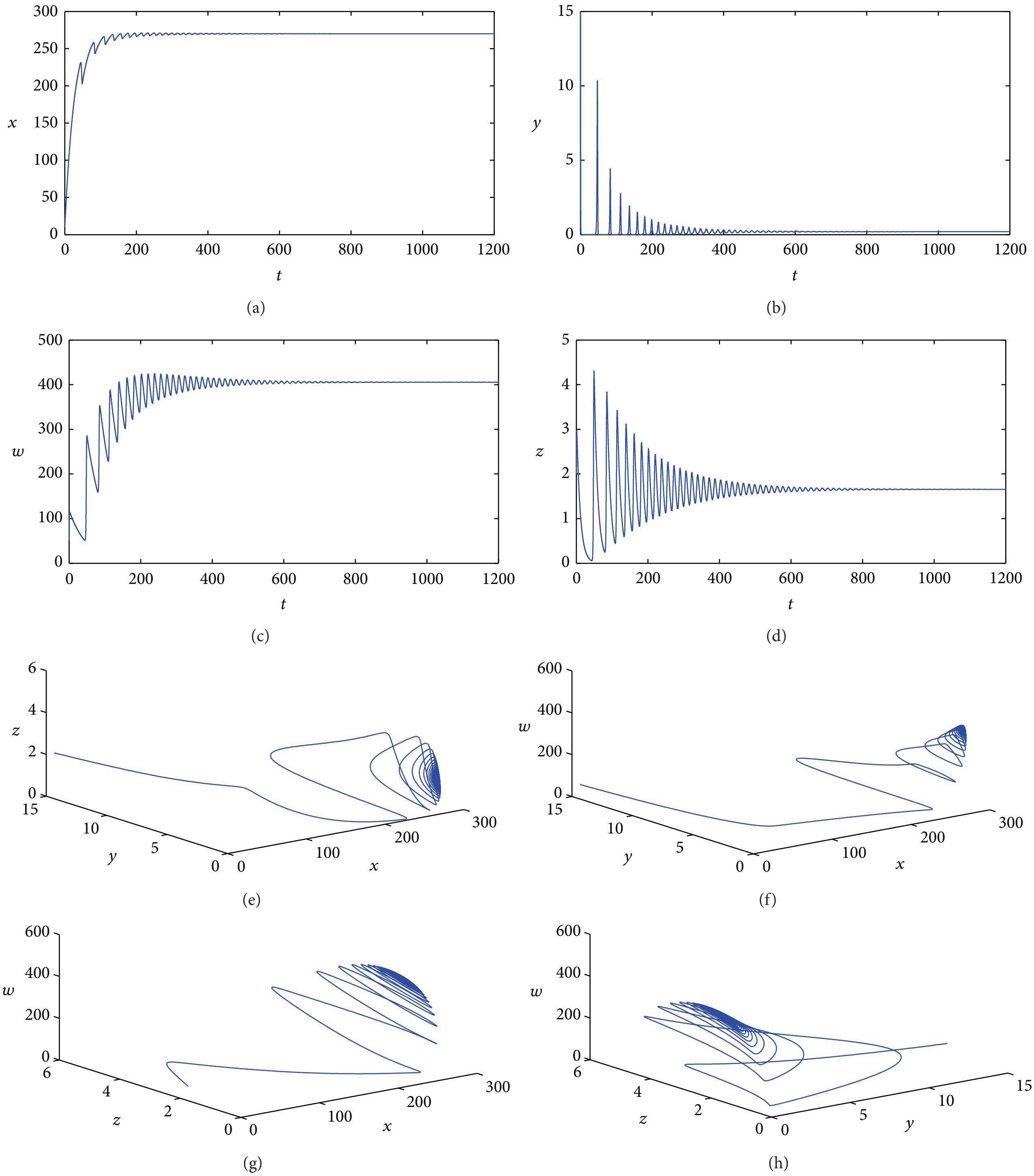

FIGURE 2: Each panel (from (a) to (h)) shows the time evolution and trajectory of model (4) when $\tau(=0.4) < \tau_0$ (critical value) and the effect of therapies is considered to be $\epsilon = 0.9$ and $\eta = 0.2$. It shows that the endemic steady state $\mathscr{E}_+$ of model is asymptotically stable.

increases. Once the delay τ crosses the critical value τ_0, then the model shows the existence of Hopf bifurcation which is depicted in the Figure 2. In Figure 3, we consider the efficacy of antiretroviral value is 0.9, which may be responsible for the loss of stability. The asymptotic behavior to the infection-free steady state, when we consider antiviral treatment (with $\epsilon = 0.9$, $\eta = 0.9$, and time delay $\tau = 15$), is shown in Figure 4. According to Theorem 4, the parameters $C_1 = -2.1108e+004+1.1224e+005i$, $\lambda' = -12.1371-0.6438i$, $\mu_2 = -1.7391e+003$, $\beta_2 = -4.2215e+004$, and $T_2 = -2.8052e+005$ are estimated. Based on these values one can conclude that bifurcating periodic solutions are unstable and decreases in

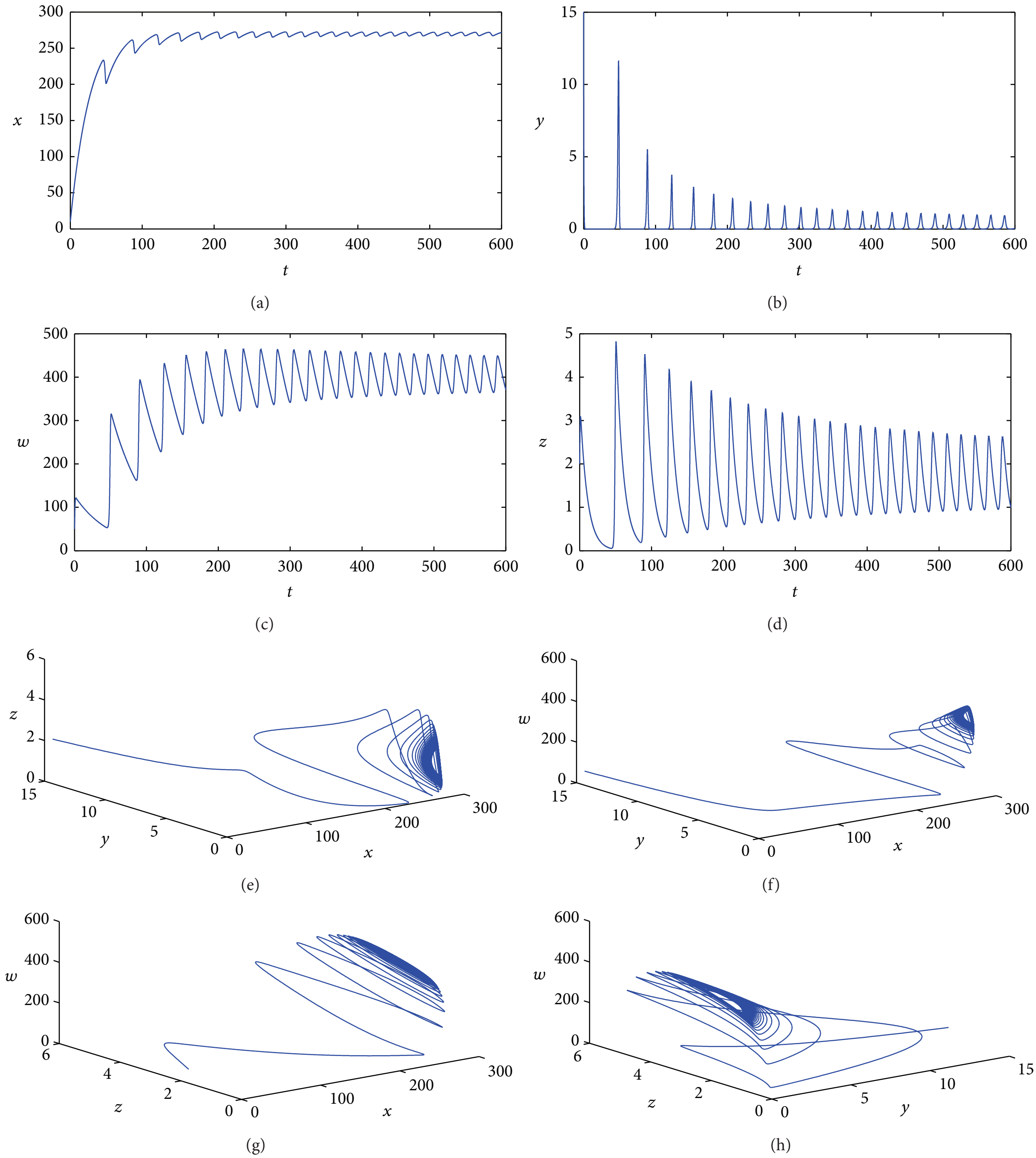

FIGURE 3: It shows the numerical simulations of model (4), when the time delay of immune activation exceeds the critical value, $\tau = 0.5 > \tau_0$. The endemic steady state $\mathscr{E}_+$ of the model undergoes Hopf bifurcation; stability switch and periodic solutions appear.

the period of bifurcating periodic solutions. The existence of periodic solution is subcritical. For numerical treatment of DDEs and related issues; we refer the readers to [35, 36].

Several packages and types of software are available for the numerical integration and/or the study of bifurcations in delay differential equations (see, e.g., [37, 38]. In this paper we utilize MIDDE code [39]) which is suitable to simulate stiff and nonstiff delay differential equations and Volterra delay integrodifferential equations, using monoimplicit RK methods.

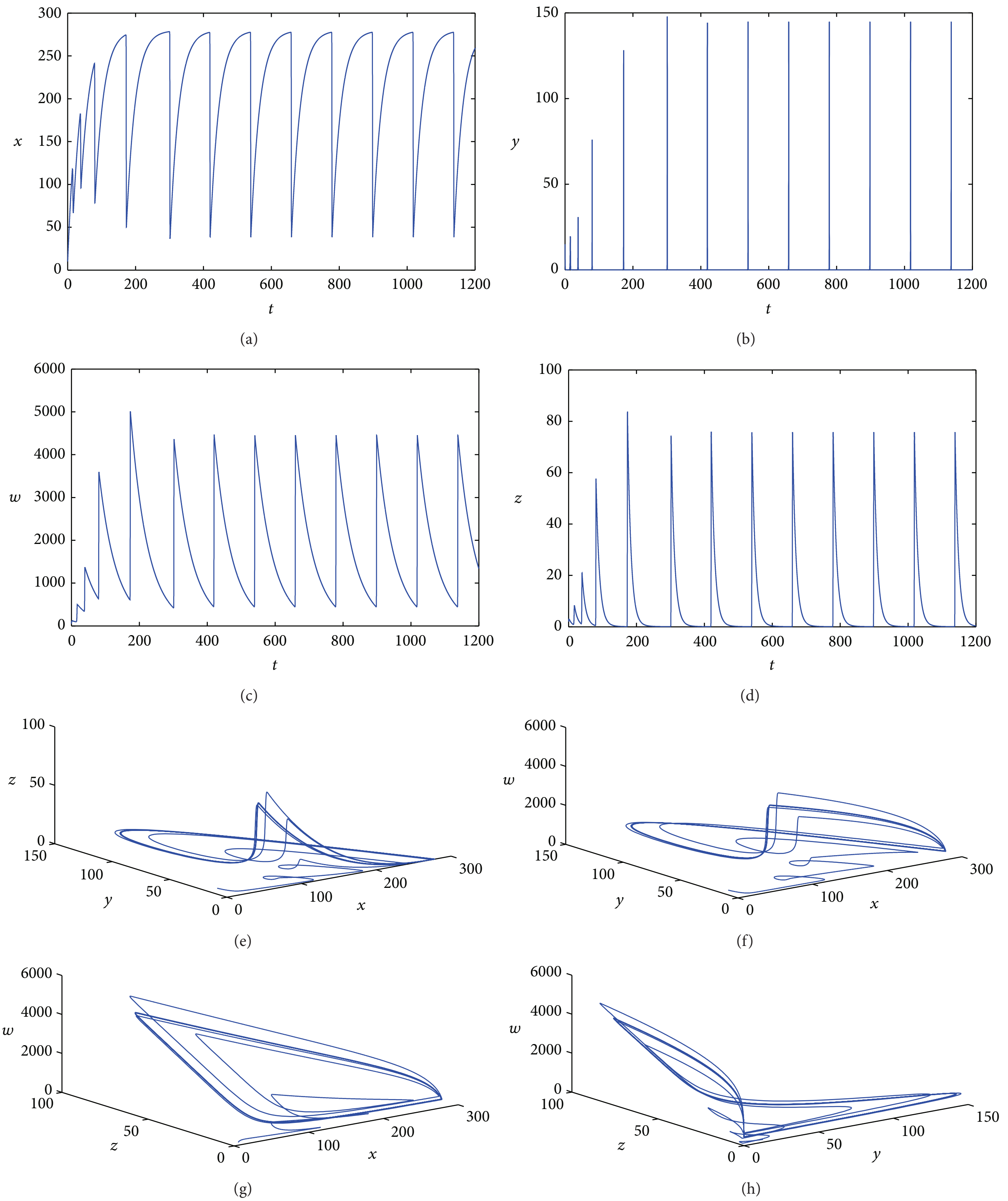

FIGURE 4: It shows the numerical simulations of model (4), when the efficacy rate of antiretroviral treatments is considered to be low; that is, $\epsilon = 0.2$ and $\eta = 0.2$. It shows that the equilibrium $\mathscr{E}_+$ of the model undergoes Hopf bifurcation with oscillatory behavior in solutions even though the delay value is less than the critical value ($\tau = 0.4 < \tau_0$).

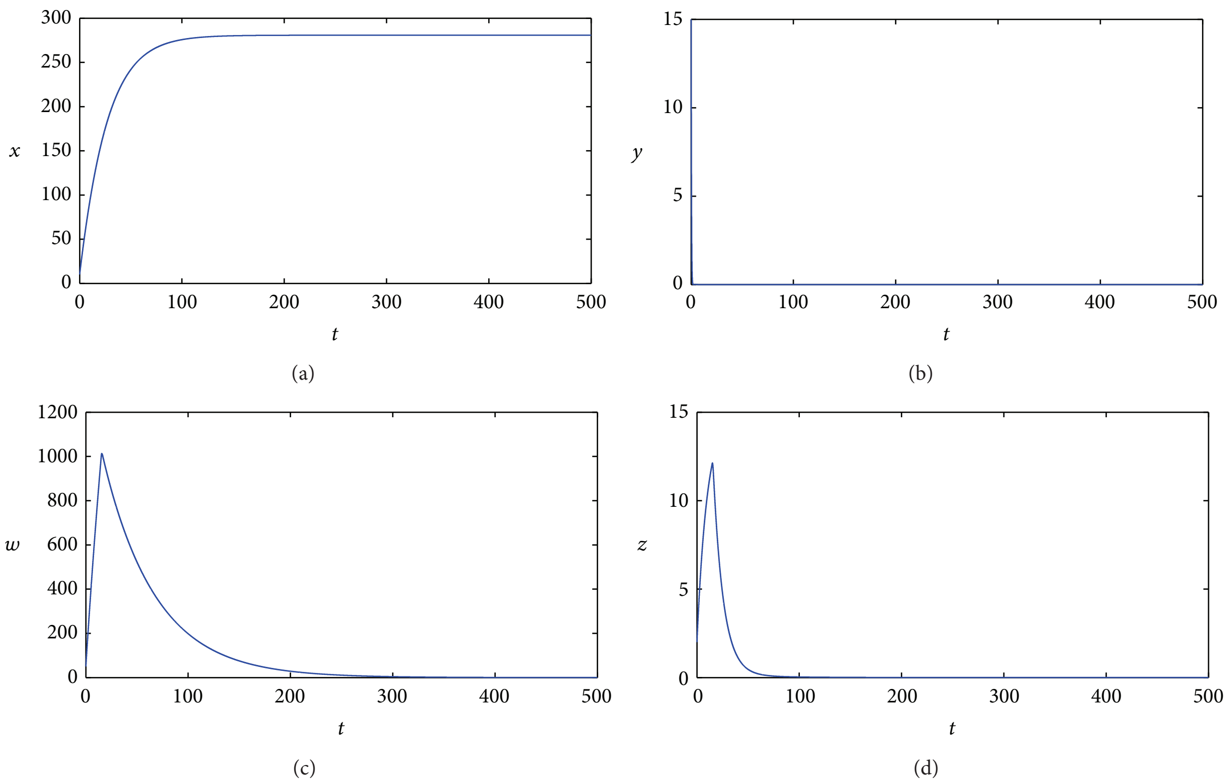

Figure 5: It shows the numerical simulations model (4) when the efficacy rate of antiretroviral treatment is at expected level, $\epsilon = 0.9$ and $\eta = 0.9$, and the delay value exceeds the critical value $\tau = 15 > \tau_0$. The solution always lies within the feasible region and the infection-free steady state $\mathscr{E}_0$ is asymptotically stable.

6. Concluding Remarks

In this manuscript, we provided a conceptual $CD4^+$ T-cell infection model which includes the logistic growth term along with two different types of antiretroviral drug therapies. The model includes a discrete time delay in the immune activation response, which plays an important role in the dynamics of the model. The infection-free and endemic steady states of the model are determined (Figure 5). The stability of steady states is analyzed. We deduced a formula that determines the critical value (branch value) τ_0. Necessary and sufficient conditions for the equilibrium to be asymptotically stable for all positive delay values are proved. We have seen that if the time delay exceeds the critical value τ_0, model (4) undergoes a Hopf bifurcation. The direction and stability of bifurcating periodic solutions are deduced in explicit formulae, using center manifold and normal forms. We also presented some numerical simulations to the underlying model to investigate the obtained results and theory. We have seen also that the antiretroviral treatments help to increase the level of uninfected $CD4^+$ T-cells. The theoretical results that were confirmed by the numerical simulations show that the delayed CTL response can lead to complex bifurcations, and, in particular, the coexistence of multiple stable periodic solutions. When the time delay exceeds the critical (threshold) value, we may get subcritical behaviour that leads to a loss of uninfected $CD4^+$ T-cells.

Conflict of Interests

The authors declare that they have no competing interests for this paper.

Acknowledgments

This work is supported by United Arab Emirates University, UAE, (Grant no. NRF-7-20886) and the National Board for Higher Mathematics, Mumbai (Grant no. 2/48(3)/2012/NBHM/R&D-II/11020). The authors would like to thank Professor Cemil Tunc and referees for their valuable comments.

References

[1] D. Liu and B. Wang, "A novel time delayed HIV/AIDS model with vaccination & antiretroviral therapy and its stability analysis," *Applied Mathematical Modelling*, vol. 37, no. 7, pp. 4608–4625, 2013.

[2] I. Ncube, "Absolute stability and Hopf bifurcation in a *Plasmodium falciparum* malaria model incorporating discrete immune response delay," *Mathematical Biosciences*, vol. 243, no. 1, pp. 131–135, 2013.

[3] X. Song, S. Wang, and J. Dong, "Stability properties and Hopf bifurcation of a delayed viral infection model with lytic immune

response," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 2, pp. 345–355, 2011.

[4] X. Song, S. Wang, and X. Zhou, "Stability and Hopf bifurcation for a viral infection model with delayed non-lytic immune response," *Journal of Applied Mathematics and Computing*, vol. 33, no. 1-2, pp. 251–265, 2010.

[5] T. Wang, Z. Hu, and F. Liao, "Stability and Hopf bifurcation for a virus infection model with delayed humoral immunity response," *Journal of Mathematical Analysis and Applications*, vol. 411, no. 1, pp. 63–74, 2014.

[6] T. Wang, Z. Hu, F. Liao, and W. Ma, "Global stability analysis for delayed virus infection model with general incidence rate and humoral immunity," *Mathematics and Computers in Simulation*, vol. 89, pp. 13–22, 2013.

[7] Z. Bai and Y. Zhou, "Dynamics of a viral infection model with delayed CTL response and immune circadian rhythm," *Chaos, Solitons & Fractals*, vol. 45, no. 9-10, pp. 1133–1139, 2012.

[8] E. Beretta, M. Carletti, D. E. Kirschner, and S. Marino, "Stability analysis of a mathematical model of the immune response with delays," in *Mathematics for Life Science and Medicine*, pp. 177–206, Springer, Berlin, Germany, 2007.

[9] Z. Hu, J. Zhang, H. Wang, W. Ma, and F. Liao, "Dynamics analysis of a delayed viral infection model with logistic growth and immune impairment," *Applied Mathematical Modelling*, vol. 38, no. 2, pp. 524–534, 2014.

[10] G. Huang, H. Yokoi, Y. Takeuchi, and T. Sasaki, "Impact of intracellular delay, immune activation delay and nonlinear incidence on viral dynamics," *Japan Journal of Industrial and Applied Mathematics*, vol. 28, no. 3, pp. 383–411, 2011.

[11] F. A. Rihan and D. H. Abdel Rahman, "Delay differential model for tumor-immune dynamics with HIV infection of $CD4^+$ T-cells," *International Journal of Computer Mathematics*, vol. 90, no. 3, pp. 594–614, 2013.

[12] J. Tam, "Delay effect in a model for virus replication," *Journal of Mathemathics Applied in Medicine and Biology*, vol. 16, no. 1, pp. 29–37, 1999.

[13] Z. Wang and R. Xu, "Stability and Hopf bifurcation in a viral infection model with nonlinear incidence rate and delayed immune response," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 2, pp. 964–978, 2012.

[14] J. E. Mittler, B. Sulzer, A. U. Neumann, and A. S. Perelson, "Influence of delayed viral production on viral dynamics in HIV-1 infected patients," *Mathematical Biosciences*, vol. 152, no. 2, pp. 143–163, 1998.

[15] F. A. Rihan, D. H. Abdel Rahman, and S. Lakshmanan, "A time delay model of tumour-immune system interactions: global dynamics, parameter estimation, sensitivity analysis," *Applied Mathematics and Computation*, vol. 232, pp. 606–623, 2014.

[16] L. Zhang, C. Zhang, and D. Zhao, "Hopf bifurcation analysis of integro-differential equation with unbounded delay," *Applied Mathematics and Computation*, vol. 217, no. 10, pp. 4972–4979, 2011.

[17] A. S. Perelson, A. U. Neumann, M. Markowitz, J. M. Leonard, and D. D. Ho, "HIV-1 dynamics in vivo: virion clearance rate, infected cell life-span, and viral generation time," *Science*, vol. 271, no. 5255, pp. 1582–1586, 1996.

[18] A. V. M. Herz, S. Bonhoeffer, R. M. Anderson, R. M. May, and M. A. Nowak, "Viral dynamics in vivo: limitations on estimates of intracellular delay and virus decay," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 93, no. 14, pp. 7247–7251, 1996.

[19] M. A. Nowak and C. R. M. Bangham, "Population dynamics of immune responses to persistent viruses," *Science*, vol. 272, no. 5258, pp. 74–79, 1996.

[20] R. M. Anderson and R. M. May, "Population biology of infectious diseases: part I," *Nature*, vol. 280, no. 5721, pp. 361–367, 1979.

[21] R. J. de Boer and A. S. Perelson, "Target cell limited and immune control models of HIV infection: a comparison," *Journal of Theoretical Biology*, vol. 190, no. 3, pp. 201–214, 1998.

[22] D. Wodarz and M. A. Nowak, "Specific therapy regimes could lead to long-term immunological control of HIV," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 96, no. 25, pp. 14464–14469, 1999.

[23] D. Wodarz, K. M. Page, R. A. Arnaout, A. R. Thomsen, J. D. Lifson, and M. A. Nowak, "A new theory of cytotoxic T-lymphocyte memory: implications for HIV treatment," *Philosophical Transactions of the Royal Society B: Biological Sciences*, vol. 355, no. 1395, pp. 329–343, 2000.

[24] C. Lv, L. Huang, and Z. Yuan, "Global stability for an HIV-1 infection model with Beddington-DeAngelis incidence rate and CTL immune response," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 1, pp. 121–127, 2014.

[25] A. S. Perelson and P. W. Nelson, "Mathematical analysis of HIV-1 dynamics in vivo," *SIAM Review*, vol. 41, no. 1, pp. 3–44, 1999.

[26] Y. Wang, Y. Zhou, F. Brauer, and J. M. Heffernan, "Viral dynamics model with CTL immune response incorporating antiretroviral therapy," *Journal of Mathematical Biology*, vol. 67, no. 4, pp. 901–934, 2013.

[27] M. Pitchaimani, C. Monica, and M. Divya, "Stability analysis for hiv infection delay model with protease inhibitor," *Biosystems*, vol. 114, no. 2, pp. 118–124, 2013.

[28] H. Shu and L. Wang, "Role of $cd4^+$ t-cell proliferation in HIV infection u nder antiretroviral therapy," *Journal of Mathematical Analysis and Applications*, vol. 394, no. 2, pp. 529–544, 2012.

[29] S. Wang and Y. Zhou, "Global dynamics of an in-host HIV 1-infection model with the long-lived infected cells and four intracellular delays," *International Journal of Biomathematics*, vol. 5, no. 6, Article ID 1250058, 2012.

[30] A. Fenton, J. Lello, and M. B. Bonsall, "Pathogen responses to host immunity: the impact of time delays and memory on the evolution of virulence," *Proceedings of the Royal Society B: Biological Sciences*, vol. 273, no. 1597, pp. 2083–2090, 2006.

[31] M. A. Nowak, R. M. May, and K. Sigmund, "Immune responses against multiple epitopes," *Journal of Theoretical Biology*, vol. 175, no. 3, pp. 325–353, 1995.

[32] J. Hale, *Theory of Functional Differential Equations*, Springer, New York, NY, USA, 1997.

[33] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, vol. 191 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1993.

[34] B. D. Hassard, *Theory and Applications of Hopf Bifurcation*, vol. 41, CUP Archive, 1981.

[35] C. T. H. Baker, G. Bocharov, A. Filiz et al., Numerical modelling by delay and volterra functional differential equations, Topics in Computer Mathematics and its Applications, LEA Athens, Hellas, 1999.

[36] F. A. Rihan, *Numerical treatment of delay differential equations in bioscience [Ph.D. thesis]*, The University of Manchester, Manchester, UK, 2000.

[37] L. F. Shampine and S. Thompson, "Solving DDEs in Matlab," *Applied Numerical Mathematics*, vol. 37, no. 4, pp. 441–458, 2001.

[38] C. Paul, "A user-guide to archi: an explicit runge-kutta code for so lving delay and neutral differential equations and parameter estimation problems," MCCM Techical Report 283, University of Manchester, 1997.

[39] F. A. Rihan, E. H. Doha, M. I. Hassan, and N. M. Kamel, "Numerical treatments for Volterra delay integro-differential equations," *Computational Methods in Applied Mathematics*, vol. 9, no. 3, pp. 292–308, 2009.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 956318, 17 pages
http://dx.doi.org/10.1155/2014/956318

Research Article

Approximate Solutions by Truncated Taylor Series Expansions of Nonlinear Differential Equations and Related Shadowing Property with Applications

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Received 5 December 2013; Accepted 11 June 2014; Published 7 July 2014

Academic Editor: Samir Saker

This paper investigates the errors of the solutions as well as the shadowing property of a class of nonlinear differential equations which possess unique solutions on a certain interval for any admissible initial condition. The class of differential equations is assumed to be approximated by well-posed truncated Taylor series expansions up to a certain order obtained about certain, in general nonperiodic, sampling points $t_i \in [t_0, t_J]$ for $i = 0, 1, \ldots, J$ of the solution. Two examples are provided.

1. Introduction

This paper investigates the errors of the solutions of nonlinear differential equations $\dot{y}(t) = f(y(t), t)$, where $f \in C^{(n+1)}(\mathbf{R}^n \times (t_0, t_J); \mathbf{R}^n)$, provided they exist and are unique for each given admissible initial condition $y(t_0) = y_0$, with respect to the solutions of its approximate differential equations $\dot{x}(t) = \sum_{k=0}^{\ell}(f^{(k)}(x(t_i), t_i)/k!)(x(t) - x(t_i))^k$; $x(t_0) = x_0$, for any given nonnegative integer $\ell \le n$, obtained from truncated Taylor expansions of the solutions about certain sampling points $t_i \in [t_0, t_J]$ for $i = 0, 1, \ldots, J$. It is assumed that if a unique solution exists on some interval $[t_0, t_J]$ and that the choice of the sampling points is such that the intersample intervals [1–4] are subject to a certain maximum allowable upper-bound then the error of the solution in the whole interval $[t_0, t_J]$ satisfies a prescribed norm bound. Using the obtained results, the shadowing property [5–10] of the true solution with respect to the approximate one is investigated in the sense that "shadowing" initial conditions of the true solution exist, for each initial condition of the approximate differential equation, such that any approximated solution trajectory on the interval of interest is arbitrarily close to the true one under prescribed allowable maximum norms of the error between both the true solution and the approximate solutions. The problem is extended to the case when the approximated solution is perturbed either by a sequence of a certain allowable size at the sampling points or with perturbation functions of a certain size in norm about the whole considered interval. The main tool involved in the analysis is an "ad hoc" use of a known preparatory theorem to the celebrated Bernstein's theorem, [11], which gives an upper-bound for the maximum norm of the error in between both the true and the approximate solutions. The results are potentially extendable to functional equations involving nonlinearities and the presence of delays subject to mixed types of uncertainties [12–18]. On the other hand, different characterizations of oscillatory solutions and limit oscillatory solutions (limit cycles) have received important interest in the literature concerning different types of nonlinear dynamic continuous-time, discrete and hybrid systems, and differential equations [19–28]. The shadowing property is naturally relevant for the characterization of limit oscillations. Therefore, the formulation is applied in the second example to the characterization of limit cycles generated as solutions to Van der Pol's equation.

2. Calculation of the Exact Solution from Taylor Series Expansion

Lemma 1. *Assume that $f \in C^{(n+1)}((a,b);\mathbf{R}^n)$ and divide the real interval (a,b) into J subintervals with points $y_n \in [a,b]$ such that*

$$a \equiv y_0 < y_1 < y_2 < \cdots < y_{J-1} < y_J \equiv b. \quad (1)$$

Then

$$\begin{aligned} f(y_{i+1}) &= f(y_0) + \int_0^{\bar{h}_i} f(y+y_0)\,dy \\ &= f(y_0) + \sum_{j=-1}^{i}\int_{\bar{h}_j}^{\bar{h}_{j+1}} f(y+y_0)\,dy \\ &= f(y_0) + \sum_{j=-1}^{i}\int_0^{h_j} f(y+y_0+\bar{h}_j)\,dy \\ &= f(y_0) \\ &\quad + \sum_{j=-1}^{i}\sum_{k=0}^{n}\frac{f^{(k)}(c_j)}{k!}\int_0^{h_j}(y+y_0+\bar{h}_j-c_{j+1})^k dy \\ &\quad + \frac{1}{n!}\sum_{j=-1}^{i}\int_0^{h_j}\int_{c_{j+1}}^{y_{j+1}}(y+y_0+\bar{h}_j-t)^n f^{(n+1)}(t)\,dt\,dy, \end{aligned} \quad (2)$$

where $h_n = y_{n+1} - y_n$, $\bar{h}_n = y_{n+1} - y_0 = \sum_{i=0}^{n} h_i$ for $n = 0,1,\ldots,J-1$ with $\bar{h}_{-1} = 0$, and

$$\begin{aligned} f\left[y_{i+1}(\tilde{h}_{i+1})\right] &= f(y_0) \\ &\quad + \sum_{j=-1}^{i}\sum_{k=0}^{n}\frac{f^{(k)}(c_j)}{k!}\int_0^{h_j}(y+y_0+\bar{h}_j-c_{j+1})^k dy \\ &\quad + \frac{1}{n!}\sum_{j=-1}^{i}\int_0^{h_j}\int_{c_{j+1}}^{y_{j+1}}(y+y_0+\bar{h}_j-t)^n f^{(n+1)}(t)\,dt\,dy \\ &\quad + \sum_{k=0}^{n}\frac{f^{(k)}(c_{i+1})}{k!}\int_0^{\tilde{h}_{i+1}}(y+y_0+\bar{h}_i+\tilde{h}_{i+1}-c_{i+2})^k dy \\ &\quad + \frac{1}{n!}\int_0^{\tilde{h}_{i+1}}\int_{c_{j+1}}^{y_{j+1}}(y+y_0+\bar{h}_i+\tilde{h}_{i+1}-t)^n f^{(n+1)}(t)\,dt\,dy; \end{aligned} \quad (3)$$

$\forall y \in [y_i, y_{i+1}]$ and any real $c_i \in [y_i, y_{i+1}]$ for $i = 0,1,\ldots,J-1$; $\forall \tilde{h}_i \in [0, h_i]$ for $i = 0,1,\ldots,J-2$.

Proof. It follows from a well-known preparatory theorem to Bernstein's theorem [5] that

$$f(y) = \sum_{k=0}^{n}\frac{f^{(k)}(c_i)}{k!}(y-c_i)^k + \frac{1}{n!}\int_{c_i}^{y}(y-t)^n f^{(n+1)}(t)\,dt. \quad (4)$$

□

Now, consider the nonlinear ordinary differential equation

$$\dot{y}(t) = f(y(t),t); \qquad y(t_0) = y_0 \quad (5)$$

in the real interval $\mathbf{R}^n \times [t_0, t_J]$ such that $f \in C^{(n+1)}(\mathbf{R}^n \times (t_0,t_J);\mathbf{R}^n)$ is Lipschitz-continuous in $[y(t_0)-\theta_0, y(t_0)+\theta_0] \times [t_0,t_J]$. The following result follows from Lemma 1.

Theorem 2. *The unique solution of (5) in $[t_0,t_J]$ is given by*

$$\begin{aligned} y(t) &= y(t_0) \\ &\quad + \sum_{j=0}^{i-2}\int_0^{h_j}\Bigg[\sum_{k=0}^{\ell}\frac{f^{(k)}(y(t_j),t_j)}{k!}\left(y(\tau+t_j)-y(t_j)\right)^k \\ &\qquad + \frac{1}{\ell!}\int_0^{\tau}\left(y(\sigma+t_j)-y(t_j)\right)^{\ell} \\ &\qquad \times f^{(\ell+1)}\left(y(\sigma+t_j),\sigma+t_j\right)d\sigma\Bigg]d\tau \\ &\quad + \int_{t_{i-1}}^{t}\Bigg[\sum_{k=0}^{\ell}\frac{f^{(k)}(y(t_j),t_j)}{k!}\left(y(\tau+t_{i-1})-y(t_{i-1})\right)^k \\ &\qquad + \frac{1}{\ell!}\int_0^{\tau}\left(y(\sigma+t_{i-1})-y(t_{i-1})\right)^{\ell} \\ &\qquad \times f^{(\ell+1)}\left(y(\sigma+t_{i-1}),\sigma+t_{i-1}\right)d\sigma\Bigg]d\tau; \end{aligned} \quad (6)$$

$\forall t \in [t_{i-1}, t_i]$; $\forall i \in \bar{J} = \{1,2,\ldots,J\}$ and $\forall \ell (\in \mathbf{Z}_{0+}) \le n$, where $t_i \in [t_0,t_J]$ are any arbitrary strictly ordered points such that $t_0 < t_1 < t_2 < \cdots < t_{J-1} < t_J$ with $h_i = t_{i+1} - t_i$ for $i = 0,1,\ldots,J-1$.

Proof. Note that $f \in C^{(n+1)}(\mathbf{R}^n \times (t_0,t_J);\mathbf{R}^n)$ is Lipschitz-continuous in $[y(t_0)-\theta_0, y(t_0)+\theta_0] \times [t_0,t_J]$ so that the solution $y(t)$ on $[t_0,t_J]$ is unique, provided that $t_J = t_J(\theta_0,t_0)$ for the given $t_0 \in \mathbf{R}$ and some $\theta_0 \in \mathbf{R}^n$ is such that $y(t) \in [y(t_0)-\theta_0, y(t_0)+\theta_0]$; $\forall t \in [t_0,t_J]$ and $t_J \in (t_0, t_{\bar{J}}]$ since $f : [y(t_0)-\theta_0, y(t_0)+\theta_0] \times [t_0,t_J] \to \mathbf{R}^n$ is local Lipschitz-continuous as a result. Such a unique solution is given by

$$y(t) = y_a + \int_a^t f(y(\tau),\tau)\,d\tau; \quad \forall t \in [t_0,t_J]. \quad (7)$$

Take any set of J strictly ordered points $t_n \in [t_0, t_J]$ satisfying $t_0 < t_1 < t_2 < \cdots < t_{J-1} < t_J$ with $h_i = t_{i+1} - t_i$ for $i = 0, 1, \ldots, J-1$, so that

$$y(t) = y(t_{i-1}) + \int_{t_{i-1}}^{t} f(y(\tau), \tau)\, d\tau = y(t_0) + \sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} f(y(\tau), \tau)\, d\tau + \int_{t_{i-1}}^{t} f(y(\tau), \tau)\, d\tau; \quad (8)$$

$\forall t \in [t_{i-1}, t_i]$; $\forall i \in \overline{J} = \{1, 2, \ldots, J\}$, with $y(t_0)$, so that, by choosing the real $c_i \in [t_{i-1}, t_i]$ as $c_i = t_i$ for $i = 0, 1, \ldots, J-1$, one gets from (3) in the proof of Lemma 1 into (4):

$$\begin{aligned} y(t) = y(t_0) &+ \sum_{j=0}^{i-2} \int_0^{h_j} \left[\sum_{k=0}^{n} \frac{f^{(k)}(y(t_j), t_j)}{k!} (y(\tau + t_j) - y(t_j))^k \right. \\ &\left. + \frac{1}{n!} \int_0^{\tau} (y(\sigma + t_j) - y(t_j))^n \times f^{(n+1)}(y(\sigma + t_j), \sigma + t_j)\, d\sigma \right] d\tau \\ &+ \int_0^{t - t_{i-1}} \left[\sum_{k=0}^{n} \frac{f^{(k)}(y(t_{i-1}), t_{i-1})}{k!} \times (y(\tau + t_{i-1}) - y(t_{i-1}))^k \right. \\ &\left. + \frac{1}{n!} \int_0^{\tau} (y(\sigma + t_{i-1}) - y(t_{i-1}))^n \times f^{(n+1)}(y(\sigma + t_{i-1}), \sigma + t_{i-1})\, d\sigma \right] d\tau; \end{aligned} \quad (9)$$

$\forall t \in [t_{i-1}, t_i]$; $\forall i \in \overline{J} = \{1, 2, \ldots, J\}$. Note that, since $f \in C^{(n+1)}(\mathbf{R}^n \times (t_0, t_J); \mathbf{R}^n)$, then $f \in C^{(\ell+1)}(\mathbf{R}^n \times (t_0, t_J); \mathbf{R}^n)$ for any nonnegative integer $\ell \le n$. Thus, we obtain the result from a similar expression of (9) by replacing n by $\ell (\le n)$ while truncating the Taylor series expansion by its $(\ell + 1)$th term. □

A consequence of Theorem 2 by using the same technique of the solution construction is as follows.

Corollary 3. *Consider the nonlinear ordinary differential equation* (5) *with initial condition $y(t_0)$ on the real interval $\mathbf{R}^n \times \mathbf{R}_{0+}$, with initial conditions $y^{(j)}(t_0)$ for $j = 0, 1, \ldots, n-1$, such that $f \in C^{(n+1)}(\mathbf{R}^n \times (t_0, t_J); \mathbf{R}^n)$ is Lipschitz-continuous in $[y(t_0) - \theta_0, y(t_0) + \theta_0] \times [t_0, t_J]$ for some $\theta_0 \in \mathbf{R}^n$, and consider also its ℓth order truncation*

$$\dot{x}(t) = \sum_{k=0}^{\ell} \frac{f^{(k)}(x(t_i), t_i)}{k!} (x(t) - x(t_i))^k; \qquad x(t_0) = x_0 \quad (10)$$

such that $f^{(k)}(y(t), t)$ are bounded in $[y(t_0) - \theta, y(t_0) + \theta] \times [t_0, t_J]$ for $k = 0, 1, \ldots, \ell + 1$ for some nonnegative integer $\ell \le n$ and some $\theta \in \overline{B}(\theta_0, R)$, where $\overline{B}(\theta, R) = \{z \in \mathbf{R}^n : \|z - \theta_0\| \le R\}$ for some positive real R with $x^{(j)}(t_0) = y^{(j)}(t_0)$ for $j = 0, 1, \ldots, \ell + 1$.

Since $f^{(k)}(y(t), t)$ are bounded in $[y(t_0) - \theta, y(t_0) + \theta] \times [t_0, \overline{t}_J]$ for $k = 0, 1, \ldots, \ell - 1$, then the right-hand-side of (10) is Lipschitz-continuous in $[y(t_0) - \theta, y(t_0) + \theta] \times [t_0, t_J] \subseteq [y(t_0) - \theta, y(t_0) + \theta] \times [t_0, t_J]$. Therefore, the unique solution of the truncated differential equation (10) in $[a, b]$ is

$$\begin{aligned} x(t) = x(t_0) &+ \sum_{k=0}^{\ell} \left\{ \sum_{j=0}^{i-2} \int_0^{h_j} \left[\frac{f^{(k)}(x(t_j), t_j)}{k!} (x(\tau + t_j) - x(t_j))^k \right] d\tau \right. \\ &\left. + \int_0^{t - t_{i-1}} \left[\frac{f^{(k)}(x(t_{i-1}), t_{i-1})}{k!} \times (x(\sigma + t_{i-1}) - x(t_{i-1}))^k \right] d\tau \right\}; \end{aligned} \quad (11)$$

$\forall t \in [t_{i-1}, t_i]$; $\forall i \in \overline{J} = \{1, 2, \ldots, J\}$, $\forall \ell (\in \mathbf{Z}_{0+}) \le n$, where $t_i \in [t_0, t_J]$ are arbitrary strictly ordered points such that $t_0 < t_1 < t_2 < \cdots < t_{J-1} < t_J$ with $h_i = t_{i+1} - t_i$ for $i = 0, 1, \ldots, J-1$. The error in between the exact solution of (10) and that of its truncated form (5) is

$$\begin{aligned} e(t) &= y(t) - x(t) \\ &= \sum_{j=0}^{i-2} \sum_{k=0}^{\ell} \int_0^{h_j} \left(\frac{f^{(k)}(y(t_j), t_j)}{k!} (y(\tau + t_j) - y(t_j))^k - \frac{f^{(k)}(x(t_j), t_j)}{k!} \times (x(\tau + t_j) - x(t_j))^k \right) d\tau \\ &+ \frac{1}{\ell!} \sum_{j=0}^{i-2} \int_0^{h_j} \int_0^{\tau} (y(\sigma + t_j) - y(t_j))^{\ell} \times f^{(\ell+1)}(y(\sigma + t_j), \sigma + t_j)\, d\sigma\, d\tau \\ &+ \sum_{k=0}^{\ell} \int_0^{t - t_{i-1}} \left(\frac{f^{(k)}(y(t_{i-1}), t_{i-1})}{k!} \times (y(\tau + t_{i-1}) - y(t_{i-1}))^k - \frac{f^{(k)}(x(t_{i-1}), t_{i-1})}{k!} \times (x(\tau + t_{i-1}) - x(t_{i-1}))^k \right) d\tau \end{aligned}$$

$$+\frac{1}{\ell!}\int_0^{t-t_{i-1}}\int_0^{\tau}\left(y\left(\sigma+t_{i-1}\right)-y\left(t_{i-1}\right)\right)^{\ell}\times f^{(\ell+1)}\left(y\left(\sigma+t_{i-1}\right),\sigma+t_{i-1}\right)d\sigma\,d\tau; \quad (12)$$

$\forall t\in[t_{i-1},t_i]$; $\forall i\in\overline{J}=\{1,2,\ldots,J\}$ and $\forall\ell(\in\mathbf{Z}_{0+})\leq n$.

Proof. Property (i) follows directly Theorem 2 applied to the truncated differential equation (10) leading to the solution (11) in $[t_0,t_J]$. Property (ii) follows from (6) and (11). □

Now, a preparatory result follows to be then used to guarantee sufficiency-type errors results in between the true and the approximate solutions in the interval $[a,b]$.

Lemma 4. *Assume that the following hypothesis holds.*

(A1) *$f(y(t),t)$ and its first $(\ell+1)$ derivatives are uniformly bounded from above on a bounded subset of their existence domain with the specific boundedness constraint:*

$$\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\|f(y(t),t)\|\leq K\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\|y(t)\|+K_1, \quad (13a)$$

$$\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\left\|f^{(j)}(y(t),t)\right\|\leq K\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\left\|f^{(j-1)}(y(t),t)\right\|+K_1, \quad (13b)$$

for $j=0,1,\ldots,\ell+1$ and some $K,K_1\in\mathbf{R}_{0+}$ with $K<1$ if $K_1\in\mathbf{R}_+$. Then, the following properties hold.

(i) *Assume that the intersample intervals $h_i=t_{i+1}-t_i$ for $i=0,1,\ldots,J-1$ fulfill the constraint*

$$h_i\leq h\leq\min\left(a_i,\frac{1-\rho_x/2}{\Lambda_x(1-\rho_x/2)^{\ell}},\frac{\rho_x}{2\Lambda_x(1-\rho_x/2)^{\ell+1}(J-1+\rho_x/2)}\right) \quad (14)$$

for $i=0,1,\ldots,J-1$ and any given real constant $\rho_x\in(0,2)$, where

$$\Lambda_x=\sum_{k=0}^{\ell}\frac{K^k}{k!}\left(\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\|f(y(t),t)\|\right)+\frac{K_1\left(1-K^{\ell+1}\right)}{1-K} \quad (15)$$

$$a_i:=\min\arg\left(t(\in\mathbf{R}_+)>t_i:\|x(t)-x(t_i)\|\leq\frac{\rho_x}{2}\right);\quad\forall t\in[t_i,t_{i+1}),\quad i=0,1,\ldots,J-1. \quad (16)$$

Then, the approximated solution fulfills $\sup_{t\in[t_0,t_J]}\|x(t)-x(t_0)\|\leq\rho_x/2$ provided that

$$t_1=\min\arg\left(t(\in\mathbf{R}_+)>t_0:\|x(t)-x(t_0)\|\leq\frac{\rho_x}{2}\right). \quad (17)$$

(ii) *Assume that the intersample intervals $h_i=t_{i+1}-t_i$ for $i=0,1,\ldots,J-1$ fulfill the constraint*

$$h_i\leq h\leq\min\left(b_i,\frac{1-\rho/2}{\Lambda(1-\rho/2)^{\ell}},\frac{\rho}{2\Lambda(1-\rho/2)^{\ell+1}(J-1+\rho/2)}\right) \quad (18)$$

for $i=0,1,\ldots,J-1$ and any given real constant $\rho\in(0,1)$, where

$$\Lambda=\sum_{k=0}^{\ell+1}\frac{K^k}{k!}\left[\left(\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\|f(y(t),t)\|\right)+\frac{K_1\left(1-K^k\right)}{1-K}\right] \quad (19)$$

$$b_i:=\min\arg\left(t>t_i:\|y(t)-y(t_i)\|\leq\frac{\rho}{2}\right);\quad\forall t\in[t_i,t_{i+1}),\quad i=0,1,\ldots,J-1. \quad (20)$$

Then, the true solution fulfills $\sup_{t\in[t_0,t_J]}\|y(t)-y(t_0)\|\leq\rho/2$ provided that

$$t_1=\min\arg\left(t(\in\mathbf{R}_+)>t_0:\|y(t)-y(t_0)\|\leq\frac{\rho}{2}\right). \quad (21)$$

(iii) *If $\rho_x=\rho\in(0,1)$ and, furthermore,*

$$h_i\leq h\leq\min\left(c_i,\frac{1-\rho/2}{\Lambda(1-\rho/2)^{\ell}},\frac{\rho}{2\Lambda(1-\rho/2)^{\ell+1}(J-1+\rho/2)}\right);\quad i=0,1,\ldots,J-1, \quad (22)$$

$$c_i:=\min\arg\left(t>t_i:\max\left(\|y(t)-y(t_i)\|,\|x(t)-x(t_i)\|\right)\leq\frac{\rho}{2}\right);\quad\forall t\in[t_i,t_{i+1}),\ i=0,1,\ldots,J-1, \quad (23)$$

$$t_1=\min\arg\left(t(\in\mathbf{R}_+)>t_0:\max\left(\|y(t)-y(t_0)\|,\|x(t)-x(t_0)\|\right)\leq\frac{\rho}{2}\right) \quad (24)$$

then the true, the approximated and the error solution fulfill

$$\sup_{t\in[t_0,t_J]}\left\|y(t)-y(t_0)\right\|\le\frac{\rho}{2},\qquad \sup_{t\in[t_0,t_J]}\left\|x(t)-x(t_0)\right\|\le\frac{\rho}{2}, \tag{25}$$

and the error in between them, $e(t)=y(t)-x(t)$, fulfills

$$\sup_{t\in[t_0,t_J]}\left\|e(t)-e(t_0)\right\|\le\rho. \tag{26}$$

Proof. Proceeding recursively, one gets from Assumption A1 that

$$\begin{aligned}
&\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\left\|f^{(j)}(y(t),t)\right\|\\
&\le K\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\left\|f^{(j-1)}(y(t),t)\right\|+K_1\\
&\le K^2\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\left\|f^{(j-1)}(y(t),t)\right\|\\
&\quad+K_1(1+K)\\
&\le K^kF_0+K_1\left(\sum_{i=0}^{k-1}K^i\right)\le K^kF_0+\frac{K_1(1-K^k)}{1-K}\\
&\le F_0+\frac{K_1}{1-K}
\end{aligned} \tag{27}$$

if $K<1$ and $K_1\ne 0$, and

$$\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\left\|f^{(k)}(y(t),t)\right\|\le K^kF_0, \tag{28}$$

if $K_1=0$, where

$$\begin{aligned}
F_0&=\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\left\|f^{(0)}(y(t),t)\right\|\\
&=\sup_{y(t)\in[y(t_0)-\theta,y(t_J)+\theta],t\in[t_0,t_J]}\left\|f(y(t),t)\right\|<+\infty.
\end{aligned} \tag{29}$$

Case (a). If $K<1$ and $K_1\ne 0$ proceed by complete induction by assuming that $\sup_{t\in[t_0,t_i]}\|x(t)-x(t_0)\|\le\rho_x/2$ since the condition $(t(\in\mathbf{R}_+)>t_i:\|x(t)-x(t_i)\|\le\rho_x/2)$ guarantees that $\sup_{t\in[t_0,t_1]}\|x(t)-x(t_0)\|\le\rho_x/2$. Thus, one gets from (11) that

$$\begin{aligned}
&\left\|x(t)-x(t_i)\right\|\\
&\le\sum_{k=0}^{\ell}\sum_{j=0}^{i}\frac{h_j}{k!}\left(K^kF_0+\frac{K_1(1-K^k)}{1-K}\right)\left(\frac{\rho_x}{2}\right)^k
\end{aligned} \tag{30}$$

$$\begin{aligned}
&=\sum_{k=0}^{\ell}\sum_{j=0}^{i-1}\frac{h_j}{k!}\left(K^kF_0+\frac{K_1(1-K^k)}{1-K}\right)\left(\frac{\rho_x}{2}\right)^k\\
&\quad+\sum_{k=0}^{\ell}\frac{h_i}{k!}\left(K^kF_0+\frac{K_1(1-K^k)}{1-K}\right)\left\|x(t)-x(t_i)\right\|^k\\
&\le\sum_{k=0}^{\ell}\sum_{j=0}^{i-1}\frac{h_j}{k!}\left(K^kF_0+\frac{K_1(1-K^k)}{1-K}\right)\left(\frac{\rho_x}{2}\right)^k\\
&\quad+\sum_{k=0}^{\ell}\frac{h_i}{k!}\left(K^kF_0+\frac{K_1(1-K^k)}{1-K}\right)\left\|x(t)-x(t_i)\right\|^k\\
&=\Lambda_x\left(\sum_{k=0}^{\ell}\left(\frac{\rho_x}{2}\right)^k\right)\left(\sum_{j=0}^{i-1}h_j\right)\\
&\quad+h_i\Lambda\left[\sum_{k=0}^{\ell}\left(\left\|x(t)-x(t_i)\right\|^k\right)\right]\\
&\le ih\Lambda_x\left(\sum_{k=0}^{\ell}\left(\frac{\rho_x}{2}\right)^k\right)\\
&\quad+h_i\Lambda_x\left(\sum_{k=0}^{\ell}\left(\left\|x(t)-x(t_i)\right\|^{k-1}\right)\right)\left\|x(t)-x(t_i)\right\|;\\
&\qquad\forall t\in[t_i,t_{i+1}),
\end{aligned} \tag{31}$$

where $\Lambda_x=\sum_{k=0}^{\ell}(1/k!)(K^kF_0+(K_1(1-K^k)/(1-K)))$ and $h\ge\max_{0\le i\le J-1}h_i$, with $h_i=t_{i+1}-t_i$, for $i=0,1,\ldots,J-1$, so that

$$\begin{aligned}
&\left[1-h_i\Lambda_x\left(\sum_{k=0}^{\ell}\left(\left\|x(t)-x(t_i)\right\|^{k-1}\right)\right)\right]\\
&\quad\times\left\|x(t)-x(t_i)\right\|\le ih\Lambda_x\left(\sum_{k=0}^{\ell}\left(\frac{\rho_x}{2}\right)^k\right)\\
&\quad=\frac{ih\Lambda_x\left(1-(\rho_x/2)^{\ell+1}\right)}{1-\rho_x/2},
\end{aligned} \tag{32}$$

or

$$\begin{aligned}
&\left\|x(t)-x(t_i)\right\|\\
&\le\frac{ih\Lambda_x\left(1-(\rho_x/2)^{\ell+1}\right)}{\left[1-h_i\Lambda_x\left(\sum_{k=0}^{\ell}\left(\left\|x(t)-x(t_i)\right\|^{k-1}\right)\right)\right](1-\rho_x/2)}\\
&\le\frac{ih\Lambda_x\left(1-(\rho_x/2)^{\ell+1}\right)}{\left[1-\left(\left(h_i\Lambda_x\left(1-(\rho_x/2)^{\ell}\right)\right)/(1-\rho_x/2)\right)\right](1-\rho_x/2)}\\
&\le\frac{\rho_x}{2}
\end{aligned} \tag{33}$$

provided that $0 < \rho_x < 2$, and $1 > h_i \Lambda_x (\sum_{k=0}^{\ell} (\|x(t) - x(t_i)\|^{k-1}))$ which is guaranteed if $h_i < \min(a_i, (1 - \rho_x/2)/(\Lambda_x (1-\rho_x/2)^{\ell}))$ holds with a_i for $i = 0, 1, \ldots, J-1$, defined in (16), provided that $\|x(t) - x(t_j)\| \leq \rho_x/2$; $\forall t \in [t_j, t_{j+1})$ for $j(\leq i) = 0, 1, \ldots, i-1$, and then (33) and $h_j < ((1-\rho_x/2)/(\Lambda_x(1-\rho_x/2)^{\ell}))$ for $j = 0, 1, \ldots, i-1$ are jointly guaranteed for the given $i = 0, 1, \ldots, J-1$ if

$$h_i < \min\left(\frac{1-\rho_x/2}{\Lambda_x (1-\rho_x/2)^{\ell}}, \frac{\rho_x}{2\Lambda_x (1-\rho_x/2)^{\ell+1} (J-1+\rho_x/2)}\right) \quad (34)$$

provided that $\|x(t) - x(t_j)\| \leq \rho_x/2$ for $t \in [t_j, t_{j+1})$ for $j = 0, 1, \ldots, i-1$, the last condition being identical to

$$t_{i+1} \leq a_i := \min \arg\left(t > t_i : \|x(t) - x(t_i)\| \leq \frac{\rho_x}{2}\right). \quad (35)$$

The above two conditions (34)-(35) reduce to (14). Then, one gets from complete induction from (31), if (14) holds, the following.

$\sup_{t \in [t_0, t_i]} \|x(t) - x(t_i)\| \leq \rho_x/2 \Rightarrow \sup_{t \in [t_0, t_{i+1})} \|x(t) - x(t_i)\| \leq \rho_x/2$ and one gets also by continuity extension,

$\sup_{t \in [t_0, t]} \|x(t) - x(t_0)\| \leq \rho_x/2$; $\forall t \in [t_0, t_J]$. Hence, we got the result for Case (a).

Case (b). If $K_1 = 0$ then

$$\begin{aligned} &\|x(t) - x(t_i)\| \\ &\quad \leq ih\Lambda_{x0} \left(\sum_{k=0}^{\ell} \left(\frac{\rho_x}{2}\right)^k\right) \\ &\quad + h_i \Lambda_{x0} \left(\sum_{k=0}^{\ell} \left(\|x(t) - x(t_i)\|^{k-1}\right)\right) \|x(t) - x(t_i)\|; \end{aligned} \quad (36)$$

$\forall t \in [t_i, t_{i+1})$, where $\Lambda_{x0} = \sum_{k=0}^{\ell} ((K^k F_0)/(k!)) \leq \Lambda_x$, so that $(1 - h\Lambda_{x0}) \|x(t) - x(t_i)\| \leq ih\Lambda_{x0} E$ for $i = 0, 1, \ldots, J$ and, thus, one gets the following.

$\|x(t) - x(t_i)\| \leq ((ih\Lambda_{x0} E)/(1 - h\Lambda_{x0})) \leq \rho_x/2$; $\forall t \in [t_i, t_{i+1})$ for $i = 0, 1, \ldots, J$ and one gets from complete induction the same conclusion $\sup_{t \in [t_0, t]} \|x(t) - x(t_0)\| \leq \rho_x/2$; $\forall t \in [t_0, t_J)$ as in Case (a) provided that (14) holds. Then, (14) guarantees Property (i) for both Cases (a) and (b). Then, Property (i) has been proven.

Property (ii) is proven "mutatis-mutandis" by noting that $\Lambda \geq \Lambda_x$ from (15) and (19) and noting also that a_i in (16) is replaced with b_i in (20) so that the admissible intersample interval satisfying the constraint (14) is replaced by such an interval satisfying the constraint (18). Finally, Property (iii) follows from Properties (i)-(ii) by equalizing ρ_x and ρ to take a maximum value less than 1/2. □

The following comments address the fact that it is not necessary to deal with the solution of the true differential equation (5) to calculate upper-bounds of the solution and error in Lemma 4.

Remark 5. Note that one gets by direct integration from (5) that

$$\begin{aligned} &\|y(t)\| \\ &\quad \leq \sup_{t_0 \leq \tau \leq t_J} \|y(\tau)\| \\ &\quad \leq \|y(t_0)\| + (t_J - t_0) K \sup_{t_0 \leq \tau \leq t_J} \|y(\tau)\| \end{aligned} \quad (37)$$

leading to

$$\sup_{t_0 \leq t \leq t_J} \|y(t)\| \leq \frac{K(t_J - t_0)}{1 - K(t_J - t_0)} \|y(t_0)\| \quad \text{if } \sum_{i=0}^{J-1} h_i < \frac{1}{K}. \quad (38)$$

Thus, (25)-(26) might be guaranteed with $\sup_{t_0 \leq t \leq t_J} \|y(t) - y(t_0)\| \leq (1/(1 - K(t_J - t_0))) \|y(t_0)\| \leq (\rho/2)$ if $\|y(t_0)\| \leq (\rho/2)(1 - K(t_J - t_0)) < (\rho/2)$. Thus, there is no need to compute the solution of the true differential equation (5) and $\sup_{t_i \leq t < t_{i+1}} \|y(t) - y(t_0)\| \leq (\rho/2)$ for $i = 0, 1, \ldots, J-1$ in (20) and (23) if $\|y(t_0)\| \leq (\rho/2)(1 - K(t_J - t_0))$.

One gets directly from Lemma 4 the subsequent result.

Theorem 6. *Assume the conditions* (13a) *and* (13b) *and* (22)–(24) *of Lemma 4(iii). Then*

$$\begin{aligned} &\max\left(\max_{0 \leq i \leq J-1} \|e(t_{i+1}) - e(t_i)\|, \|e(t) - e(t_0)\|\right) \leq \rho < 1; \\ &\max_{t \in [t_0, t_J]} \|e(t)\| \leq \|e(t_0)\| + \rho; \end{aligned} \quad (39)$$

$\forall t \in [t_0, t_J]$, *and*

$$\begin{aligned} &\max_{0 \leq i \leq J-1} \|e(t) - e(t_i)\| \leq \rho; \\ &\forall t \in [t_i, t_{i+1}] \text{ for } i = 0, 1, \ldots, J-1 \end{aligned} \quad (40)$$

for $i = 0, 1, \ldots, J-1$.

3. Orbits of the Exact Solution, Pseudo-Orbits of the Approximated Solution, and the Shadowing Property

Now, consider a perturbed solution (11) of the approximated differential equation (10) associated with a perturbation $x(t_i) = x(t_i^-) + g(t_i)$ with $\{g(t_i)\} \subset \mathbf{R}^n$ at $t = t_i$ fulfilling $\|g(t_i)\| \leq \overline{g}_i \leq \overline{g}$ for some given $\overline{g} \in \mathbf{R}$, $\forall i \in \overline{J}$. Note that a perturbation at the initial $t = t_0$ is considered by choosing $x(t_0) = y(t_0) + g_0$ for some nonzero $g_0 = g(t_0) \in \mathbf{R}$. The perturbed solution can be generated, in particular, from impulsive controls of amplitudes $g(t_i)$ at the sequence

$\{t_i : i \in \overline{J}\}$. The exact and approximate solutions (6) and (11) in $[t_0, t_J]$, provided that they exist, are

$$\begin{aligned} y(t) &= y(t_i) + \sum_{k=0}^{\ell} \int_0^{t-t_i} \frac{f^{(k)}(y(t_i), t_i)}{k!} (y(\tau + t_i) - y(t_i))^k d\tau \\ &+ \frac{1}{\ell!} \int_0^{t-t_i} \int_0^{\tau} \left(y(\sigma + t_j) - y(t_j)\right)^{\ell} \\ &\times f^{(\ell+1)}\left(y(\sigma + t_j), \sigma + t_j\right) d\sigma\, d\tau; \\ &\forall t \in [t_i, t_{i+1}],\ i = 0, 1, \dots, J-1, \end{aligned} \tag{41}$$

$$\begin{aligned} x(t) &= x(t_i) + \sum_{k=0}^{\ell} \int_0^{t-t_i} \frac{f^{(k)}(x(t_i), t_i)}{k!} (x(\tau + t_i) - x(t_i))^k d\tau \\ &+ U(t - t_{i+1}) g(t_{i+1}); \\ &\forall t \in [t_i, t_{i+1}],\ i = 0, 1, \dots, J-1, \end{aligned} \tag{42}$$

where $U(t)$ is the Heaviside function. The error between the exact and perturbed approximated solutions becomes

$$\begin{aligned} e(t) &= e(t_i) + \sum_{k=0}^{\ell} \int_0^{t-t_i} \left(\frac{f^{(k)}(y(t_i), t_i)}{k!} (y(\tau + t_i) - y(t_i))^k - \frac{f^{(k)}(x(t_i), t_i)}{k!} \times (x(\tau + t_i) - x(t_i))^k \right) d\tau \\ &+ \frac{1}{\ell!} \int_0^{t-t_i} \int_0^{\tau} \left(y(\sigma + t_j) - y(t_j)\right)^{\ell} \\ &\times f^{(\ell+1)}\left(y(\sigma + t_j), \sigma + t_j\right) d\sigma\, d\tau \\ &- U(t - t_{i+1}) g(t_{i+1}); \end{aligned} \tag{43}$$

$\forall t \in [t_i, t_{i+1}]$; $i = 0, 1, \dots, J-1$. Now, one gets from (25)-(26) of Lemma 4:

$$\begin{aligned} \|e(t) - e(t_i)\| &\le \sum_{k=0}^{\ell} \frac{2^{k+1}}{k!} (t - t_i) M_{ik} \left(\frac{\rho}{2}\right)^k \\ &+ \frac{1}{\ell!} (t - t_i)^2 2^{\ell} M_{i,\ell+1} \left(\frac{\rho}{2}\right)^{\ell} + \overline{g}_i; \quad \forall t \in [t_i, t_{i+1}], \end{aligned} \tag{44}$$

where

$$\begin{aligned} M_{ik} &= \sup_{y(t) \in [y(t_i) - \theta, y(t_i) + \theta], t \in [t_i, t_{i+1}]} \left\| f^{(j)}(y(t), t) \right\| \\ &\le K^k F_0 + \frac{K_1 (1 - K^k)}{1 - K} \\ &\le \frac{K^{k+1} \rho}{2} + \frac{K_1 (1 - K^k)}{1 - K} \le \frac{\rho}{2} + \frac{K_1}{1 - K} \end{aligned} \tag{45}$$

from (27) and one gets after using the triangle inequality,

$$\begin{aligned} \|e(t) - e(t_i)\| &\le \sum_{j=i}^{m} \left(\sum_{k=0}^{\ell} \frac{2^{k+1}}{k!} (t - t_j) \left[\frac{K^{k+1} \rho}{2} + \frac{K_1 (1 - K^k)}{1 - K} \right] \left(\frac{\rho}{2}\right)^k \right. \\ &+ \frac{1}{\ell!} (t - t_j)^2 2^{\ell} \\ &\left. \times \left[\frac{K^{\ell+2} \rho}{2} + \frac{K_1 (1 - K^{\ell+1})}{1 - K} \right] \left(\frac{\rho}{2}\right)^{\ell} + \overline{g}_j \right); \end{aligned} \tag{46}$$

$\forall t \in [t_{i+m}, t_{i+m+1}]$ for $m = 0, 1, \dots, J-i-1$; $i = 0, 1, \dots, J-1$. One obtains easily from (46), either by complete induction or via recursive calculation, that

$$\begin{aligned} \|e(t) - e(t_i)\| &\le \sum_{j=0}^{J-1} \left(\sum_{k=0}^{\ell} \frac{2^{k+1}}{k!} (t - t_j) \left[\frac{K^{k+1} \rho}{2} + \frac{K_1 (1 - K^k)}{1 - K} \right] \left(\frac{\rho}{2}\right)^k \right. \\ &+ \frac{1}{\ell!} (t - t_j)^2 2^{\ell} \\ &\left. \times \left[\frac{K^{\ell+2} \rho}{2} + \frac{K_1 (1 - K^{\ell+1})}{1 - K} \right] \left(\frac{\rho}{2}\right)^{\ell} + \overline{g}_j \right) \end{aligned} \tag{47a}$$

$$\begin{aligned} &\le Jh \left(\sum_{k=0}^{\ell} \frac{2^{k+1}}{k!} \left[\frac{K^{k+1} \rho}{2} + \frac{K_1 (1 - K^k)}{1 - K} \right] \left(\frac{\rho}{2}\right)^k \right. \\ &+ \frac{Jh}{\ell!} 2^{\ell} \\ &\left. \times \left[\frac{K^{\ell+2} \rho}{2} + \frac{K_1 (1 - K^{\ell+1})}{1 - K} \right] \left(\frac{\rho}{2}\right)^{\ell} \right) + J\overline{g}, \end{aligned} \tag{47b}$$

$$\|e(t)\| \le \|e(t_0)\| + \sum_{j=0}^{J-1}(t-t_j)\left(\sum_{k=0}^{\ell}\frac{2^{k+1}}{k!}\left[\frac{K^{k+1}\rho}{2}+\frac{K_1(1-K^k)}{1-K}\right]\left(\frac{\rho}{2}\right)^k + \frac{1}{\ell!}(t-t_j)2^\ell \times\left[\frac{K^{\ell+2}\rho}{2}+\frac{K_1(1-K^{\ell+1})}{1-K}\right] \times\left(\frac{\rho}{2}\right)^\ell + \overline{g}_j\right) \tag{47c}$$

$$\le \|e(t_0)\| + Jh\left(\sum_{k=0}^{\ell}\frac{2^{k+1}}{k!}\left[\frac{K^{k+1}\rho}{2}+\frac{K_1(1-K^k)}{1-K}\right]\left(\frac{\rho}{2}\right)^k + \frac{Jh}{\ell!}2^\ell\left[\frac{K^{\ell+2}\rho}{2}+\frac{K_1(1-K^{\ell+1})}{1-K}\right]\left(\frac{\rho}{2}\right)^\ell\right) + J\overline{g}; \tag{47d}$$

$\forall t \in [t_0, t_0+\sum_{i=0}^{J-1}h_i](\subseteq [t_0, t_0+Jh])$ with $h_i = t_{i+1}-t_i$ for $i = 0,1,\ldots,J-1$ and any given nonnegative integer $\ell \le n$. The following result follows directly from the above equations and Theorem 6.

Theorem 7. *Consider an approximated perturbed solution* (42) *under a forcing perturbation sequence* $\{g(t_i)\} \subset \mathbf{R}^n$ *at* $t = t_i$ *fulfilling* $\|g(t_i)\| \le \overline{g}_i \le \overline{g} \ge \|e(t_0)\|$ *for* $i = 1,2,\ldots$ *and some* $\overline{g} \in \mathbf{R}_+$. *Then, there are numbers* $h \in \mathbf{R}_+$, $J = J(h) \in \mathbf{Z}_+$, $\varepsilon_1 = \varepsilon_1(h,\overline{g}) \in \mathbf{R}_+$, *and* $\varepsilon = \varepsilon(\varepsilon_1, \|e(t_0)\|)$ *such that*

$$\max\left(\max_{0\le i\le J-1}\|e(t_{i+1})-e(t_i)\|, \max_{t\in\mathbf{R}_{0+}}\|e(t)-e(t_0)\|\right) \le \varepsilon_1; \quad \max_{t\in\mathbf{R}_{0+}}\|e(t)\| \le \varepsilon \tag{48}$$

on $[t_0, t_J]$ *for any strictly ordered sequence of* $(J+1)$ *nonnegative real numbers* $\{t_i : i = 0,1,\ldots,J\}$, *subject to the constraints*

$$t_J = t_0 + \sum_{i=0}^{J-1}h_i, \quad h_i = t_{i+1}-t_i \le h, \quad i = 0,1,\ldots,J-1 \tag{49}$$

satisfying the constraints (22)–(24) *of Lemma 4 subject to* (18).

Proof. Note that fixing $\sum_{i=0}^{J-1}h_i = t_J - t_0 \le Jh$, with $h = \max_{0\le i\le J-1}(t_{i+1}-t_i)$, and the use of (46), (47a), and (47b) leads to

$$\|e(t)-e(t_i)\| \le \sum_{i=0}^{J-1}\left(h_i\left(\sum_{k=0}^{\ell}\frac{2^{k+1}}{k!}\left[\frac{K^{k+1}\rho}{2}+\frac{K_1(1-K^k)}{1-K}\right]\left(\frac{\rho}{2}\right)^k + \frac{1}{\ell!}h2^\ell \times\left[\frac{K^{\ell+2}\rho}{2}+\frac{K_1(1-K^{\ell+1})}{1-K}\right]\left(\frac{\rho}{2}\right)^\ell\right)+\overline{g}_i\right) \le \varepsilon_1 = \rho + \sum_{i=0}^{J-1}\overline{g}_i, \tag{50a}$$

$$\|e(t)-e(t_i)\| \le Jh\left(\sum_{k=0}^{\ell}\frac{2^{k+1}}{k!}\left[\frac{K^{k+1}\rho}{2}+\frac{K_1(1-K^k)}{1-K}\right]\left(\frac{\rho}{2}\right)^k + \frac{1}{\ell!}h2^\ell\left[\frac{K^{\ell+2}\rho}{2}+\frac{K_1(1-K^{\ell+1})}{1-K}\right]\left(\frac{\rho}{2}\right)^\ell\right) + J\overline{g} \le \overline{\varepsilon}_1 = \overline{\rho} + J\overline{g}; \tag{50b}$$

$\forall t \in [t_0, t_J]$; $i = 0,1,\ldots,J-1$ by using Lemma 4 and Theorem 6 for any given prefixed $\rho \in \mathbf{R}_+$. The result then follows since $\overline{g}_0 = \|e(t_0)\| \le \overline{g}$ and either

$$\varepsilon = \varepsilon_1 + \|e(t_0)\|, \qquad \varepsilon_1 = \rho + \sum_{i=0}^{J-1}\overline{g}_i, \quad \sum_{i=0}^{J-1}h_i\left(\sum_{k=0}^{\ell}\frac{2^{k+1}}{k!}\left[\frac{K^{k+1}\rho}{2}+\frac{K_1(1-K^k)}{1-K}\right]\left(\frac{\rho}{2}\right)^k + \frac{h}{\ell!}2^\ell\left[\frac{K^{\ell+2}\rho}{2}+\frac{K_1(1-K^{\ell+1})}{1-K}\right]\left(\frac{\rho}{2}\right)^\ell\right) \le \rho \tag{51}$$

or

$$\varepsilon = \overline{\varepsilon}_1 + \|e(t_0)\|, \qquad \overline{\varepsilon}_1 = \overline{\rho} + J\overline{g}, \quad h\left(\sum_{k=0}^{\ell}\frac{2^{k+1}}{k!}\left[\frac{K^{k+1}\rho}{2}+\frac{K_1(1-K^k)}{1-K}\right]\left(\frac{\rho}{2}\right)^k + \frac{1}{\ell!}h2^\ell\left[\frac{K^{\ell+2}\rho}{2}+\frac{K_1(1-K^{\ell+1})}{1-K}\right]\left(\frac{\rho}{2}\right)^\ell\right) \le \overline{\rho}; \tag{52}$$

and the result has been proven. □

The following result extends Theorem 7 with results of Theorem 6 for the case when both the exact and approximated differential equations are subject to a piecewise-continuous bounded disturbance which might be dependent on the solution and also can have finite step discontinuities in the sequence $\{t_i : i = 0, 1, \ldots, J\}$.

Theorem 8. *Consider the forced versions of the differential equations* (5) *and* (10)*:*

$$\dot{y}(t) = f(y(t), t) + g(\tau, y(\tau)), \qquad y(t_0) = y_0, \quad (53)$$

$$\dot{x}(t) = \sum_{k=0}^{\ell} \frac{f^{(k)}(x(t_i), t_i)}{k!} (x(t) - x(t_i))^k + g(\tau, x(\tau)), \quad x(t_0) = x_0 \quad (54)$$

under the additive forcing perturbation $g \in C^{(n+1)}(\mathbf{R}^n \times (t_0, t_J); \mathbf{R}^n)$ *satisfying Assumption (A2) of Lemma 4 fulfilling* $g(y(t), t) = \lambda(t) y(t)$ *and* $g(x(t), t) = \lambda(t) x(t) + U(t - t_{i+1}) g_{i+1}$; $\forall t \in [t_i, t_{i+1}]$ *with* $\|g_{i+1}\| \leq \overline{g}$ *for* $i = 0, 1, \ldots, J-1$ *and some* $\overline{g} \in \mathbf{R}_+$ *and* $\lambda : [t_0, t_J] \to \mathbf{R}^n$ *being a bounded piecewise-continuous function. Then, there are numbers* $h \in \mathbf{R}_+$, $J = J(h) \in \mathbf{Z}_+$, $\varepsilon_1 = \varepsilon_1(h, \overline{g}) \in \mathbf{R}_+$ *and* $\varepsilon = \varepsilon(\varepsilon_1, \|e(t_0)\|)$ *such that*

$$\max\left(\max_{0 \leq i \leq J-1} \|e(t_{i+1}) - e(t_i)\|, \max_{t \in \mathbf{R}_{0+}} \|e(t) - e(t_0)\|\right) \leq \varepsilon_1, \quad \max_{t \in \mathbf{R}_{0+}} \|e(t)\| \leq \rho \quad (55)$$

on $[t_0, t_J]$ *for a strictly ordered finite set of* $(J+1)$ *nonnegative real numbers* $\{t_i : i = 0, 1, \ldots, J\}$*, subject to the constraints* $t_J = t_0 + \sum_{i=0}^{J-1} h_i$, $h_i = t_{i+1} - t_i \leq h$; $i = 0, 1, \ldots, J-1$*, the constraints* (22)–(24) *subject to* (18)*, and either*

$$\sum_{i=0}^{J-1} h_i \lambda_i < 1, \quad (56a)$$

$$\sum_{i=0}^{J-1} \overline{g}_i < \infty, \qquad \overline{g} \geq \frac{\sum_{i=0}^{J-1} h_i \lambda_i}{1 - \sum_{i=0}^{J-1} h_i \lambda_i} \|e(t_0)\|, \quad (56b)$$

or

$$Jh\lambda < 1, \quad (57a)$$

$$J\overline{g} < \infty, \qquad \overline{g} \geq \frac{Jh\lambda}{1 - Jh\lambda} \|e(t_0)\|. \quad (57b)$$

Proof. Fix $\sum_{i=0}^{J-1} h_i = t_J - t_0 \leq Jh$, with $h = \max_{0 \leq i \leq J-1}(t_{i+1} - t_i)$. Equations (53)-(54) have the following solutions:

$$\begin{aligned} y(t) = y(t_i) &+ \sum_{k=0}^{\ell} \int_0^{t-t_i} \Bigg(\frac{f^{(k)}(y(t_i), t_i)}{k!} (y(\tau + t_i) - y(t_i))^k \\ &+ \frac{1}{\ell!} \int_0^{\tau} \left(y(\sigma + t_j) - y(t_j)\right)^{\ell} \times f^{(\ell+1)}\left(y(\sigma + t_j), \sigma + t_j\right) d\sigma \\ &+ g(y(\tau), \tau) \Bigg) d\tau, \end{aligned} \quad (58)$$

$$\begin{aligned} x(t) = x(t_i) &+ \int_0^{t-t_i} \left(\sum_{k=0}^{\ell} \frac{f^{(k)}(x(t_i), t_i)}{k!} (x(\tau + t_i) - x(t_i))^k + g(x(\tau), \tau) \right) d\tau \\ &+ U(t - t_{i+1}) g_{i+1}; \end{aligned} \quad (59)$$

$\forall t \in [t_i, t_{i+1}]$, $i = 0, 1, \ldots, J-1$. Note that

$$g(y(t), t) - g(x(t), t) = \lambda(t)(y(t) - x(t)) = \lambda(t) e(t). \quad (60)$$

Thus, the error between both of them becomes

$$\begin{aligned} e(t) = e(t_i) &+ \sum_{k=0}^{\ell} \int_0^{t-t_i} \Bigg(\frac{f^{(k)}(y(t_i), t_i)}{k!} (y(\tau + t_i) - y(t_i))^k \\ &- \frac{f^{(k)}(x(t_i), t_i)}{k!} (x(\tau + t_i) - x(t_i))^k + g(e(\tau), \tau) \Bigg) d\tau \\ &+ \frac{1}{\ell!} \int_0^{t-t_i} \int_0^{\tau} \left(y(\sigma + t_j) - y(t_j)\right)^{\ell} \times f^{(\ell+1)}\left(y(\sigma + t_j), \sigma + t_j\right) d\sigma\, d\tau \\ &- U(t - t_{i+1}) g(x(t_{i+1}), t_{i+1}); \end{aligned} \quad (61)$$

$\forall t \in [t_i, t_{i+1}]$. Then, (45) leads to

$$\begin{aligned}
&\|e(t)\| \\
&\leq \|e(t_i)\| + (t - t_i) \\
&\quad \times \left(\sum_{k=0}^{\ell} \frac{2^{k+1}}{k!} \left[\frac{K^{k+1}\rho}{2} + \frac{K_1(1-K^k)}{1-K} \right] \left(\frac{\rho}{2} \right)^k \right. \\
&\quad\quad + \frac{1}{\ell!}(t - t_i) 2^{\ell} \\
&\quad\quad \left. \times \left[\frac{K^{\ell+2}\rho}{2} + \frac{K_1(1-K^{\ell+1})}{1-K} \right] \left(\frac{\rho}{2} \right)^{\ell} + g_{i+1} \right);
\end{aligned} \tag{62}$$

$\forall t \in [t_i, t_{i+1}]$, $i = 0, 1, \ldots, J-1$. Then

$$\begin{aligned}
&\sup_{t_i \leq t \leq t_{i+1}} (\|e(t)\|) \\
&\leq \|e(t_i)\| + (t - t_i) \\
&\quad \times \left(\sum_{k=0}^{\ell} \frac{2^{k+1}}{k!} \left[\frac{K^{k+1}\rho}{2} + \frac{K_1(1-K^k)}{1-K} \right] \left(\frac{\rho}{2} \right)^k \right. \\
&\quad\quad + \frac{1}{\ell!}(t - t_i) 2^{\ell} \\
&\quad\quad \left. \times \left[\frac{K^{\ell+2}\rho}{2} + \frac{K_1(1-K^{\ell+1})}{1-K} \right] \left(\frac{\rho}{2} \right)^{\ell} + g_{i+1} \right) \\
&\quad + h_i \lambda_i \sup_{t_i \leq \tau \leq t_{i+1}} (\|e(\tau)\|) + \overline{g}
\end{aligned} \tag{63}$$

so that, since $1 > h_i \lambda_i$, where $\lambda_i = \max_{t_i \leq \tau \leq t_{i+1}} |\lambda(\tau)|$ for $i = 0, 1, \ldots, J-1$, one gets

$$\begin{aligned}
&\sup_{t_i \leq t \leq t_{i+1}} (\|e(t)\|) \\
&\leq \frac{1}{1 - h_i \lambda_i} \\
&\quad \times \left(\|e(t_i)\| \right. \\
&\quad\quad + \left(\sum_{k=0}^{\ell} \frac{2^{k+1}}{k!} h_i \left[\frac{K^{k+1}\rho}{2} + \frac{K_1(1-K^k)}{1-K} \right] \left(\frac{\rho}{2} \right)^k \right. \\
&\quad\quad\quad + \frac{1}{\ell!}(t - t_i) 2^{\ell} \\
&\quad\quad\quad \left.\left. \times \left[\frac{K^{\ell+2}\rho}{2} + \frac{K_1(1-K^{\ell+1})}{1-K} \right] \left(\frac{\rho}{2} \right)^{\ell} + g_{i+1} \right) \right)
\end{aligned} \tag{64}$$

which implies

$$\begin{aligned}
&\left| \sup_{t_i \leq t \leq t_{i+1}} (\|e(t)\| - \|e(t_i)\|) \right| \\
&\leq \frac{h_i \lambda_i}{1 - h_i \lambda_i} \|e(t_i)\| + \frac{1}{1 - h_i \lambda_i} \\
&\quad \times \left(\sum_{k=0}^{\ell} \frac{2^{k+1}}{k!} h_i \left[\frac{K^{k+1}\rho}{2} + \frac{K_1(1-K^k)}{1-K} \right] \left(\frac{\rho}{2} \right)^k \right. \\
&\quad\quad + \frac{1}{\ell!} h_i^2 2^{\ell} \\
&\quad\quad \left. \times \left[\frac{K^{\ell+2}\rho}{2} + \frac{K_1(1-K^{\ell+1})}{1-K} \right] \left(\frac{\rho}{2} \right)^{\ell} + g_{i+1} \right).
\end{aligned} \tag{65}$$

If $\sum_{i=0}^{J-1} h_i \lambda_i < 1$, we also get (66)-(67) below from (65) as well as (68)-(69) if, in addition, $Jh\lambda < 1$:

$$\begin{aligned}
&\sup_{t_0 \leq t \leq t_J} (\|e(t)\|) \\
&\leq \frac{1}{1 - \sum_{i=0}^{J-1} h_i \lambda_i} \\
&\quad \times \left(\|e(t_0)\| \right. \\
&\quad\quad + \sum_{i=0}^{J-1} h_i \left(\sum_{k=0}^{\ell} \frac{2^{k+1}}{k!} \left[\frac{K^{k+1}\rho}{2} + \frac{K_1(1-K^k)}{1-K} \right] \left(\frac{\rho}{2} \right)^k \right. \\
&\quad\quad\quad + \frac{1}{\ell!} h_i 2^{\ell} \\
&\quad\quad\quad \left. \times \left[\frac{K^{\ell+2}\rho}{2} + \frac{K_1(1-K^{\ell+1})}{1-K} \right] \left(\frac{\rho}{2} \right)^{\ell} \right) \\
&\quad\quad \left. + \sum_{i=0}^{J-1} \overline{g}_i \right),
\end{aligned} \tag{66}$$

$$\begin{aligned}
&\left| \sup_{t_0 \leq t \leq t_J} (\|e(t)\| - \|e(t_0)\|) \right| \\
&\leq \frac{\sum_{i=0}^{J-1} h_i \lambda_i}{1 - \sum_{i=0}^{J-1} h_i \lambda_i} \\
&\quad \times \left(\|e(t_0)\| \right. \\
&\quad\quad + \sum_{i=0}^{J-1} h_i \left(\sum_{k=0}^{\ell} \frac{2^{k+1}}{k!} \left[\frac{K^{k+1}\rho}{2} + \frac{K_1(1-K^k)}{1-K} \right] \left(\frac{\rho}{2} \right)^k \right. \\
&\quad\quad\quad + \frac{1}{\ell!} h_i 2^{\ell} \\
&\quad\quad\quad \left. \times \left[\frac{K^{\ell+2}\rho}{2} + \frac{K_1(1-K^{\ell+1})}{1-K} \right] \left(\frac{\rho}{2} \right)^{\ell} \right) \\
&\quad\quad \left. + \sum_{i=0}^{J-1} \overline{g}_i \right)
\end{aligned} \tag{67}$$

$$\sup_{t_0\le t\le t_J}(\|e(t)\|)$$
$$\le \frac{1}{1-Jh\lambda}$$
$$\times\Bigg(\|e(t_0)\|$$
$$+Jh\Bigg(\sum_{k=0}^{\ell}\frac{2^{k+1}}{k!}\left[\frac{K^{k+1}\rho}{2}+\frac{K_1(1-K^k)}{1-K}\right]\left(\frac{\rho}{2}\right)^k$$
$$+\frac{1}{\ell!}h2^{\ell}$$
$$\times\left[\frac{K^{\ell+2}\rho}{2}+\frac{K_1(1-K^{\ell+1})}{1-K}\right]\left(\frac{\rho}{2}\right)^{\ell}\Bigg)$$
$$+J\overline{g}\Bigg), \tag{68}$$

$$\left|\sup_{t_0\le t\le t_J}(\|e(t)\|-\|e(t_0)\|)\right|$$
$$\le\frac{Jh\lambda}{1-Jh\lambda}$$
$$\times Jh\Bigg(\sum_{k=0}^{\ell}\frac{2^{k+1}}{k!}\left[\frac{K^{k+1}\rho}{2}+\frac{K_1(1-K^k)}{1-K}\right]\left(\frac{\rho}{2}\right)^k \tag{69}$$
$$+\frac{1}{\ell!}h2^{\ell}$$
$$\times\left[\frac{K^{\ell+2}\rho}{2}+\frac{K_1(1-K^{\ell+1})}{1-K}\right]\left(\frac{\rho}{2}\right)^{\ell}\Bigg)$$
$$+J\overline{g},$$

where $\lambda=\max_{0\le t\le t_J}|\lambda(t)|=\max_{0\le i\le J-1}\max_{t_i\le\tau\le t_{i+1}}|\lambda(\tau)|=\max_{0\le i\le J-1}\lambda_i$. Property (i) follows from (65)-(66) by defining ε_0, ε_1, and ε as in (51) since $\overline{g}\ge\max(\max_{0\le i\le J-1}\|g_{i+1}\|,((\sum_{i=0}^{J-1}h_i\lambda_i)/(1-\sum_{i=0}^{J-1}h_i\lambda_i))\|e(t_0)\|)$ and Property (ii) follows from (67)-(68) by defining ρ, ε_1, and ε as in (52) since $\overline{g}\ge\max(\max_{0\le i\le J-1}\|g_{i+1}\|,((Jh\lambda)/(1-Jh\lambda))\|e(t_0)\|)$. Thus, the result has been proven. □

Now, three definitions are given concerning the so-called pseudo-orbits, as a counterpart to the true sampled trajectory solution, or orbit, of finite size J of the approximate solutions and their perturbed version within the given classes of perturbations. The related concepts are relevant for then quantifying the maximum errors among the real and approximated solutions and parallel issues concerning their counterparts under perturbations of the studied types. More specifically refer to the following.

Definition 9. A sampling sequence $\widehat{t}_J=\{t_i : i=0,1,\dots,J\}$ of strictly ordered sampling points with $h_i=t_{i+1}-t_i\le h$; $i=0,1,\dots,J-1$ is said to be in the class $C_{Jh}=\{t_i\in\widehat{t}_J : t_{i+1}-t_i\le h; i=0,1,\dots,J-1\}$.

Note from Definition 9 that $h\le h'\Rightarrow C_{Jh}\subseteq C_{Jh'}$ and that $\widehat{t}_J\equiv\{t_i : i=0,1,\dots,J-1\}\subset C_{Jh}\Rightarrow t_J-t_0\le Jh$.

Definition 10. A sequence $\widehat{x}_J=\{x(t_i) : i=0,1,\dots,J-1\}$ of J samples of the solution of an approximate differential equation (10) is a δ-pseudo J-orbit of sampling sequence $\widehat{t}_J$ for some $\delta\in\mathbf{R}_+$ and is denoted by $O(\widehat{x}_J,\Gamma,\delta)$ if $\max_{t\in[t_0,t_J]}\|e(t)\|\le\delta$.

If the integer J and the real t_J are infinite, the corresponding trajectory solutions are referred to as complete pseudo-orbits and orbits. The solution of the true differential equation (5) is a J-orbit of sampling sequence $\widehat{t}_J$. The continuous approximate (resp., true) solution for $[t_0,t_J]$ is the δ-pseudo J-orbit (resp., J-orbit) of sampling sequence $\widehat{t}_J$. The perturbed solutions under the forcing perturbations of Theorems 7 and 8 are denoted in a similar way leading to the corresponding perturbed pseudo-orbits.

Definition 11. The set of all the δ-pseudo J-orbits $O(\widehat{x}_J,\Gamma,\delta)$ with $\max_{t\in[t_0,t_J]}\|e(t)\|\le\delta$, for some $\delta\in\mathbf{R}_+$, obtained for any sampling sequence $\widehat{t}_J\in C_{Jh}$, is said to be the class $CO(C_{Jh},\delta)$ of δ-pseudo J-orbits of sampling sequence $\widehat{t}_J$.

The mapping which generates the true solution sequences, for given initial conditions and sampling sequence, has the shadowing property if there is an arbitrarily close orbit for any given δ-pseudo-orbit $O(\widehat{x}_J,\Gamma,\delta)$ in the following precise sense.

Definition 12. The set $\widehat{Y}_J$ of true solution sequences $\widehat{y}_J=\{y(t_i) : t_i\in\widehat{t}_J, i=0,1,\dots,J-1\}$ of sampling sequence $\widehat{t}_J$ possesses the shadowing property on the corresponding set of approximate solutions if, for each given $\delta\in\mathbf{R}_+$, there is some $y_0=y_0(\delta)$ for which a $O(\widehat{x}_J,\Gamma,\delta)$ exists. It is said that $y_0=y_0(\delta)$ shadows $O(\widehat{x}_J,\Gamma,\delta)$.

The subsequent result establishes that if the set of true solution sequences has the shadowing property then the class $CO(C_{Jh},\delta)$ of δ-pseudo J-orbits of sampling sequence $\widehat{t}_J$ is nonempty for any $\delta\in\mathbf{R}_+$.

Proposition 13. *If the set $\widehat{Y}_J$ of true solution sequences of sampling sequence $\widehat{t}_J$ possesses the shadowing property then $CO(C_{Jh},\delta)$ is nonempty for any $\delta\in\mathbf{R}_+$.*

Note that $CO(C_{Jh},\rho)=\bigcup_{\Gamma\in C_{Jh}}O(\widehat{x}_J,\Gamma,\rho)$ and note also that $CO(C_{Jh},\rho)\subseteq CO(C_{Jh'},\rho)$ for any $h'\ge h$. The subsequent result relies on Theorem 7 and Definition 11 for a class of pseudo-orbits $CO(C_{Jh},\rho)$ defined by a sampling sequence class C_{Jh}. In fact, the characterization becomes global for all approximated solutions on a finite interval $[t_0,t_J]$ for sampling intervals $h_i=t_{i+1}-t_i\le h$; $i=0,1,\dots,J-1$ and initial conditions subject to a maximum allowable deviation with respect to the initial condition of the true solution provided that the approximate solution exists in a global (rather than local) definition domain.

The so-called shadowing properties, [5–9], of the true solutions with respect to the approximated ones rely on the physical meaning that for sets of appropriate initial conditions, the true solution is arbitrarily close to its approximate version on a certain interval $[t_0, t_J]$ where both solutions exist and are unique. Based on Theorems 6, 7, and 8, the shadowing properties of the true solution to the approximated solution, those ones being the nominal one or the perturbed ones under the class of perturbations of Theorems 7 and 8, are now discussed. It is seen that the shadowing properties at sampling points under Theorems 6, 7, and 8 guarantee the corresponding properties in $[t_0, t_J]$.

The shadowing properties of true solutions of pseudo-orbits for constrained sampling sequences according to the constraints of Theorem 6 are addressed in the subsequent result.

Proposition 14. *Consider the true and approximated solutions associated with the differential equations* (5) *and* (10) *satisfying the hypotheses and conditions of Theorem 6. Then, such a set of solutions lies in the class $CO(C_{Jh}, \varepsilon)$ of ε-pseudo J-orbits of sampling sequence $\hat{t}_J = \hat{t}_J(\rho)$ for $\rho \le \varepsilon$, subject to one of the constraints* (13a), (13b), (22)–(24) *(Lemma 4, Theorem 6), belonging to a sampling sequence class C_{Jh} for any $\rho, \varepsilon \in \mathbf{R}_+$ with arbitrary $\rho \le \varepsilon$ and any given ε. Also, there is a $y_0 = y_0(\varepsilon)$ which shadows each $O(\hat{x}_J, \Gamma, \varepsilon) \in CO(C_{Jh}, \varepsilon)$ for each given $\varepsilon \in \mathbf{R}_+$ and $\rho \le \varepsilon$.*

Proof. One gets from Theorem 6 that

$$\max_{t\in[t_0,t_J]} \|e(t)\| \le \varepsilon = \|e(t_0)\| + \rho \tag{70}$$

for an initial condition $y(t_0)$ of the true differential equation fulfilling $\|\|y(t_0)\| - \|x(t_0)\|\| \le \|e(t_0)\|$ and any given real constants $\varepsilon \ge \rho > 0$. This defines families of initial conditions $y_0 = y_0(\varepsilon)$ of the true differential equation which shadow each $O(\hat{x}_J, \Gamma, \varepsilon) \in CO(C_{Jh}, \varepsilon)$ for each given $\varepsilon \in \mathbf{R}_+$ and $\rho \le \varepsilon$. For any given $\varepsilon \in \mathbf{R}_+$, it suffices to take $0 < \rho \le \varepsilon$ to zero in (22)–(24) of Lemma 4 and (39)-(40) in Theorem 6 to fix an admissible sampling sequence $\hat{t}_J = \hat{t}_J(\varepsilon)$ and then to get the result. □

The perturbed approximated differential equations referred to in Theorems 8 and 7, which is a particular case of Theorem 8 for $g(x(t), t)$ being zero for $t \notin \hat{t}_J$, that is for nonsampling points, are analyzed in the subsequent result which generalizes Proposition 14.

Theorem 15. *Consider the true and approximated solutions* (58) *and* (59) *associated with the differential equations satisfying the hypotheses and conditions of Theorem 8. Then, such a set of solutions lies in the class $CO(C_{Jh}, \varepsilon)$ of ε-pseudo J-orbits of sampling sequence $\hat{t}_J = \hat{t}_J(\rho)$, for some $\rho > 0$, subject to one of the constraints* (13a), (13b), (22)–(24), *and* (39)-(40) *(Lemma 4, Theorem 6) and to either* (56a) *or* (56b) *(Theorem 8) with $\rho \le \varepsilon - \sum_{i=0}^{J-1} \overline{g}_i$ and any arbitrary $\varepsilon \in \mathbf{R}_+$, belonging to a sampling sequence class C_{Jh}. Also, there is an initial condition $y_0 = y_0(\varepsilon)$ which shadows each $O(\hat{x}_J, \Gamma, \varepsilon) \in CO(C_{Jh}, \varepsilon)$ for each given $\rho, \varepsilon \in \mathbf{R}_+$ with the perturbation fulfilling $\sum_{i=0}^{J-1} \overline{g}_i < \varepsilon$.*

Proof. One gets from (55) in Theorem 8 together with either (51) or (52) that

$$\max_{t\in[t_0,t_J]} \|e(t)\| \le \varepsilon = \|e(t_0)\| + \rho + \sum_{i=0}^{J-1} \overline{g}_i \tag{71}$$

with any arbitrary real constant $0 < \rho \le \varepsilon - \sum_{i=0}^{J-1} \overline{g}_i$, provided that $\sum_{i=0}^{J-1} \overline{g}_i < \varepsilon$, and any given real constant ε for an (shadowing) initial condition $y(t_0)$ of the true differential equation fulfilling

$$\begin{aligned} &\|\|y(t_0)\| - \|x(t_0)\|\| \\ &\le \|e(t_0)\| \\ &\le \min\left(\varepsilon - \rho - \sum_{i=0}^{J-1} \overline{g}_i, \frac{1 - \sum_{i=0}^{J-1} h_i\lambda_i}{\sum_{i=0}^{J-1} h_i\lambda_i} \overline{g}\right). \end{aligned} \tag{72}$$

Note that a sufficient condition guaranteeing (69) is

$$\|e(t_0)\| \le \min\left(\varepsilon - \varepsilon_0 - J\overline{g}, \frac{1 - \sum_{i=0}^{J-1} h_i\lambda_i}{\sum_{i=0}^{J-1} h_i\lambda_i} \overline{g}\right) \tag{73}$$

since $\overline{g} \ge ((\sum_{i=0}^{J-1} h_i\lambda_i)/(1 - \sum_{i=0}^{J-1} h_i\lambda_i))\|e(t_0)\|$ from (56b) in Theorem 8. Thus, it suffices to take $0 < \rho \le \varepsilon$ to zero in either (39) or (40) in Theorem 6 to fix an admissible sampling sequence $\hat{t}_J = \hat{t}_J(\varepsilon)$ so as to get the result. □

Remark 16. A particular case of Theorem 15 for the perturbations (42) which are defined only at sampling instants, which has been discussed in Theorem 7, is obtained by the particular constraint below obtained from (72) and (73):

$$\|e(t_0)\| \le \varepsilon - \rho - J\overline{g} \le \varepsilon - \rho - \sum_{i=0}^{J-1} \overline{g}_i. \tag{74}$$

Remark 17. Note that the condition $\sum_{i=0}^{\infty} h_i\lambda_i \le \chi < 1$ of applicability in Theorems 8 and 15 when J is infinity can be considered in certain cases when the perturbation vanishes asymptotically as, for instance, when it vanishes as an exponential rate.

4. Simulation Examples

This section contains two numerical examples regarding the theoretical results obtained in Sections 2 and 3.

Example 1. The first example is concerned with the nonlinear model describing the human heart rate during treadmill exercise [29], whose equations are given by

$$\begin{aligned} \dot{y}_1 &= -a_1 y_1 + a_2 y_2, \\ \dot{y}_2 &= -a_3 y_2 + a_4 \frac{y_1}{1 + e^{-(\overline{c} y_1 - a_5)}} \end{aligned} \tag{75}$$

with $\bar{c} = 1$, $a_1 = 2.2$, $a_2 = 19.96$, $a_3 = 0.0831$, $a_4 = 0.002526$, and $a_5 = 8.32$. This model, with an external control input, has been used to design training protocols for patients with cardiovascular problems [29]. Figure 1 shows the evolution of this system with initial conditions $y_1(0) = y_2(0) = 1$ on the time interval $[0, 50]$ seconds.

For the system (75), the nonlinear function $f(y)$ is given by

$$f(y) = f(y_1, y_2) = \begin{bmatrix} -a_1 y_1 + a_2 y_2 \\ -a_3 y_2 + a_4 \dfrac{y_1}{1 + e^{-(\bar{c} y_1 - a_5)}} \end{bmatrix}. \quad (76)$$

The first step to apply the results stated in Section 2 and obtain a truncated approximate model for (75) is to verify that conditions (13a) and (13b) hold. One way to check this fact is to depict the norms of the state vector of the function $f(y)$ and of its derivative $f'(y)$ and observe their behavior. Thus, the following Figures 2 and 3 show the time evolution of these norms. In particular, Figure 2 shows the values of the 2-norm of the state, $\|y\|_2$, and the function, $\|f(y)\|_2$. The supremum of these norms on this interval are $\sup_{t\in[0,50]}\|y\|_2 = 8.94$ and $\sup_{t\in[0,50]}\|f(y)\|_2 = 17.84$. On the other hand, Figure 3 shows the difference between the norm of the function, $\|f(y)\|_2$, and the norm of its derivative $\|f'(y)\|_2$. As it can be appreciated, this difference is positive implying that the linear approximation of the function is always bounded by the function itself. The supremum of these two norms on this interval is $\sup_{t\in[0,50]}\|f(y)\|_2 = 17.84$ and $\sup_{t\in[0,50]}\|f'(y)\|_2 = 17.76$. In this way, if we choose $K = 0.997$ and $K_1 = 9$, we have

$$\begin{aligned} 17.84 &= \sup_{t\in[0,50]} \left\|f(y)\right\|_2 < 0.997 \sup_{t\in[0,50]} \left\|y\right\|_2 + 9 = 17.91, \\ 17.76 &= \sup_{t\in[0,50]} \left\|f'(y)\right\|_2 < 0.997 \sup_{t\in[0,50]} \left\|f(y)\right\|_2 = 17.78. \end{aligned} \quad (77)$$

Hence, it is corroborated that both (13a) and (13b) hold. Notice that from a practical point of view, the analytical determination of the constants K and K_1 used in (13a) and (13b) is not necessary since a simple numerical experiment allows us to verify these upper-bounds. In consequence, the results stated in Section 2 can be applied in practical situations easily.

Once the basic conditions have been checked, a truncated approximate model (10) is generated for this problem by considering $\ell = 1 < 2 = n$. Thus, we have

$$\dot{x} = f(x_i) + J(x_i)(x(t) - x_i), \quad (78)$$

where

$$\begin{aligned} f'(x_i) &= J(x_i) = \begin{pmatrix} -a_1 & a_2 \\ J_{21} & -a_3 \end{pmatrix}, \\ J_{21} &= \frac{a_4\left(1 + e^{-(\bar{c}x_{i1} - a_5)}\right) + a_4\bar{c}x_{i1}e^{-(\bar{c}x_{i1} - a_5)}}{\left(1 + e^{-(\bar{c}x_{i1} - a_5)}\right)^2} \end{aligned} \quad (79)$$

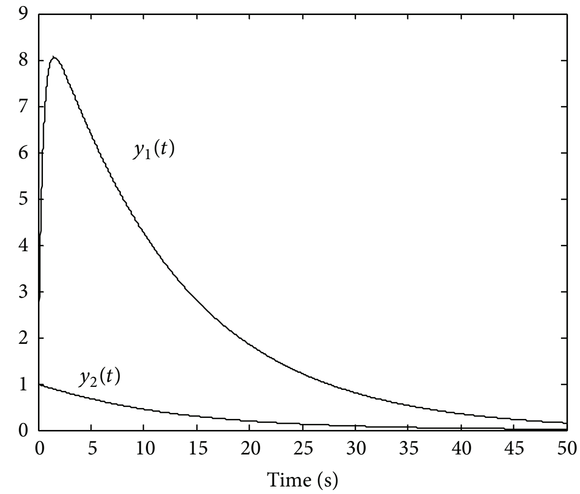

Figure 1: State evolution for the system (75).

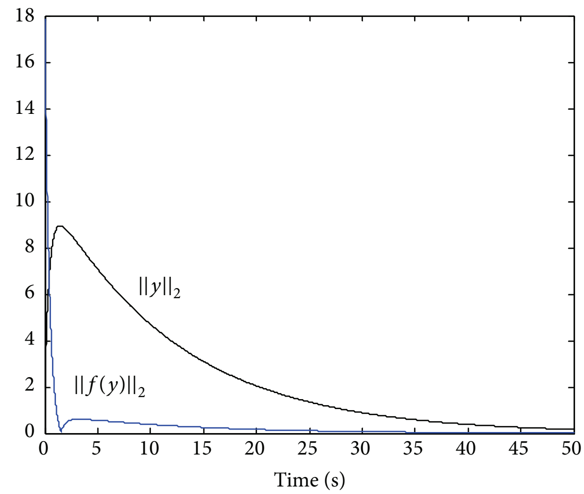

Figure 2: Relation between the 2-norms of the state, y, and the function, $f(y)$.

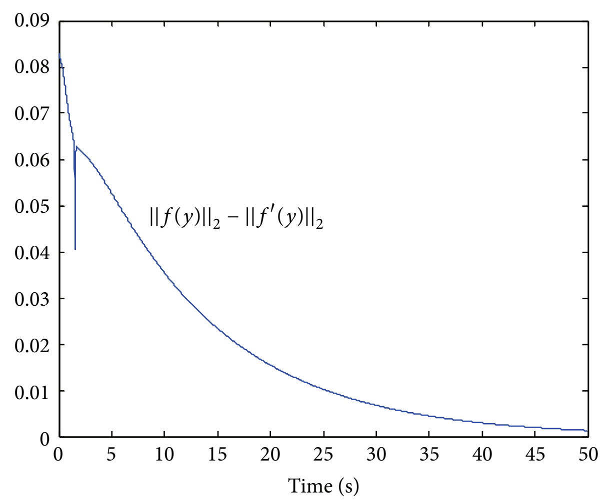

Figure 3: Relation between the 2-norms of the function, $f(y)$, and its derivative, $f'(y)$.

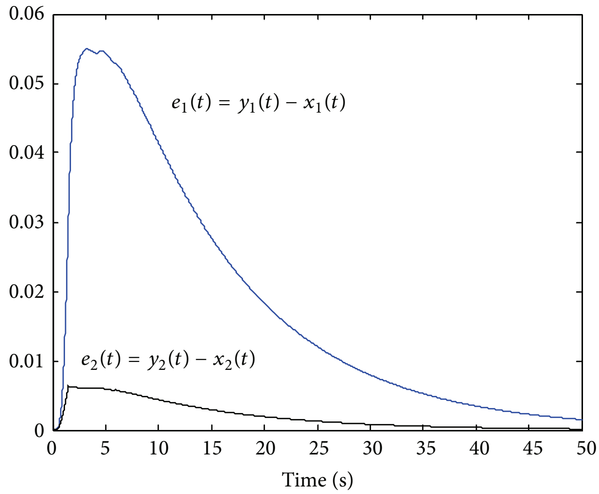

FIGURE 4: Error between the actual system and the approximate model.

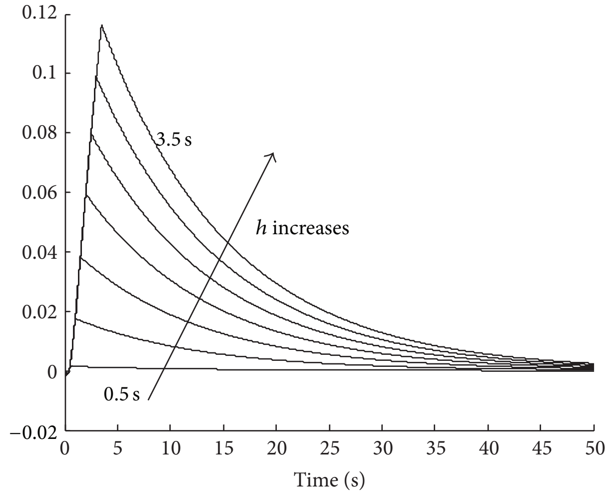

FIGURE 6: Effect of the variation of the sampling time, h, in the error of the second state variable, $x_2(t)$.

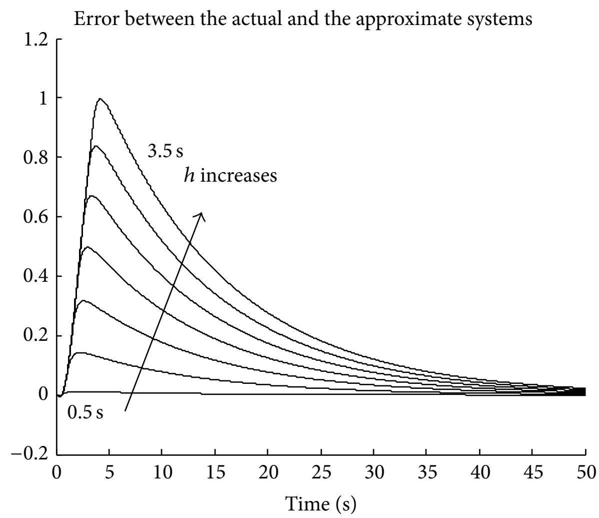

FIGURE 5: Effect of the variation of the sampling time, h, in the error of the first state variable, $x_1(t)$.

with initial conditions $y_1(0) = y_2(0) = x_1(0) = x_2(0) = 1$. The sampling instants $\{t_i\}$ have been chosen uniformly in time as $t_i - t_{i-1} = h = 1.5$ s. The error between the actual and the approximate model with this sampling time is depicted in Figure 4.

As it can be deduced from Figure 4, the error is very low and, therefore, the exact solution is shadowed by the solution of the approximate model. An important feature appears at this point which is how we should select the sampling time. Lemma 4 and Theorem 6 contain the analytical results providing the formal background on how to select it. However, from a practical point of view a trial-error procedure can be employed to obtain an appropriate sampling time. Thus, Figures 5 and 6 show how a variation in the sampling time impacts the error between the complete system and the approximate model. Figure 5 displays the impact on the first state variable while Figure 6 depicts the influence in the second one. As it is displayed in Figures 5 and 6, the larger the sampling time is, the larger the error between both systems is, as well. Hence, if we fix an upper-bound for the desired error, we may start with a tentative value for the sampling time and increase it if the maximum of the error is below that threshold or decrease it if the error exceeds the desired bound. This procedure allows us to tune an appropriate sampling time by just conducting a series of numerical experiments. Therefore, the mathematical results presented in Section 2 can be applied in a practical way with little effort since the computation of the bounds is not explicitly necessary to construct the approximate truncated model. Afterwards, this approximate model could be used for simulation or control design purposes. For instance, the obtained affine model could simplify the design of the controller with respect to the case when the original nonlinear model is used.

Example 2. The second example is related to the Van der Pol equation which exhibits a limit cycle as it is widely known. The equations are given by

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = \mu\left(1 - y_1^2\right) y_2 - y_1 \tag{80}$$

with $\mu = 1$, output $z(t) = y_1(t)$, initial conditions $y_1(0) = 4$ and $y_2(0) = -0.5$, and

$$f(y) = f(y_1, y_2) = \begin{bmatrix} y_2 \\ \mu\left(1 - y_1^2\right) y_2 - y_1 \end{bmatrix}. \tag{81}$$

The phase portrait of the Van der Pol equation is depicted in Figure 7.

In this case, the results introduced in Section 3 regarding the error between the actual and the approximate model in the presence of bounded perturbations will be used as a tool to analyze the stability of the limit cycle. Thus,

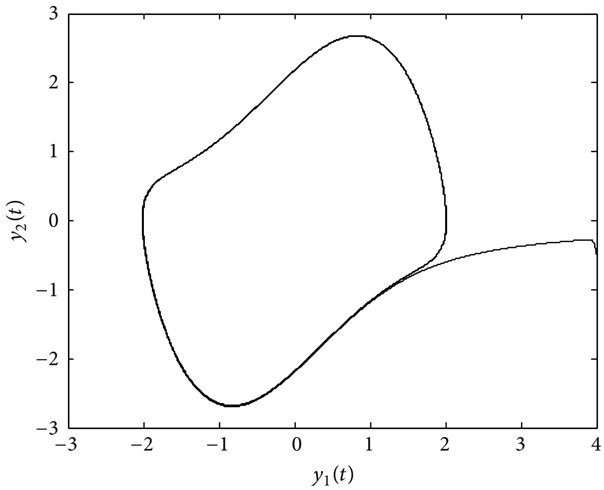

FIGURE 7: Phase portrait of the Van der Pol equation.

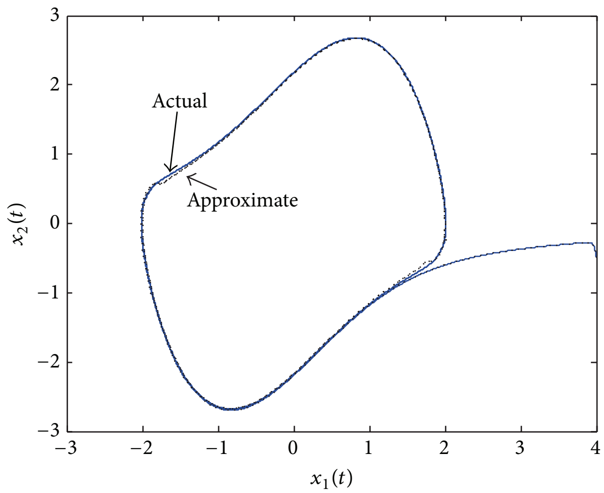

FIGURE 8: Phase portrait of the actual and approximate systems.

the approximate truncated model is given with $\ell = 1 < 2 = n$ by

$$\dot{x} = f(x_i) + J(x_i)(x(t) - x_i), \tag{82}$$

where

$$\begin{gathered} f'(x_i) = J(x_i) = \begin{pmatrix} 0 & 1 \\ J_{21} & J_{22} \end{pmatrix}, \\ J_{21} = -2\mu x_{i1} x_{i2} - 1, \\ J_{21} = \mu - \mu x_{i1}^2. \end{gathered} \tag{83}$$

In this example, the sampling points x_i will be generated by using the constant amplitude difference sampling criterion (CADSC) introduced in [4] as a method to generate sampling points in discretization procedures. This method is proposed as a way to generate the sequence of sampling points in a practical way, which shows that the application of the presented theories to real problems is feasible. Thus, the CADSC

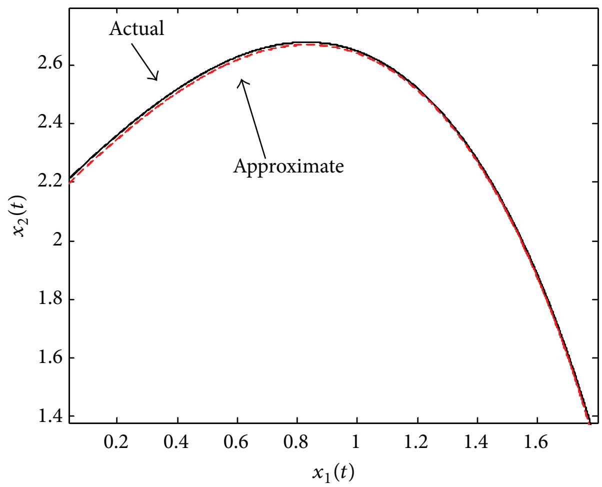

FIGURE 9: Zoom on the phase portrait of the actual and approximate systems.

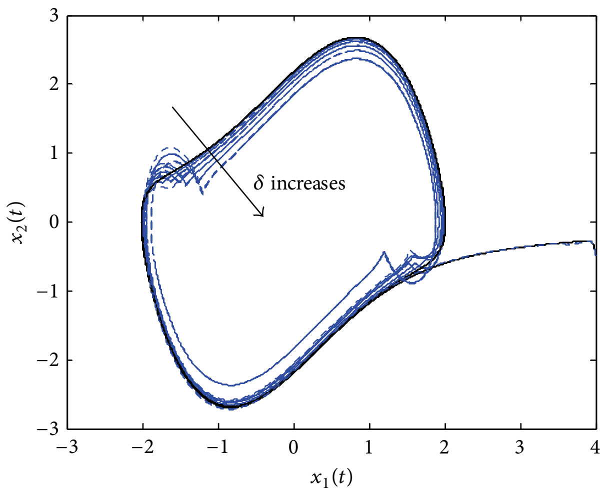

FIGURE 10: Effect of the variation of the sampling threshold δ in the approximate model.

method proposes to generate a new sampling point when the continuous-time output differs from the previous sampled one a certain threshold. Mathematically,

$$t_{i+1} = \arg\min \left(R_{0^+} \ni t > t_i : \left| x_1(t) - x_1(t_i) \right| = \delta_i \in R_+ \right), \tag{84}$$

where δ_i denotes the variation threshold. For this example, consider a constant threshold with a value of $\delta_i = \delta = 0.15$. Figures 8 and 9 display the solution of the actual and the approximate systems.

It can be appreciated in Figures 8 and 9 that the solution of the actual system is shadowed by one of the approximate models, confirming the results stated in Proposition 14. As the threshold δ on the sampling criterion enlarges, the sampling takes place in a more separate way, a fact that degrades the quality of the approximate solution as Figure 10 reveals.

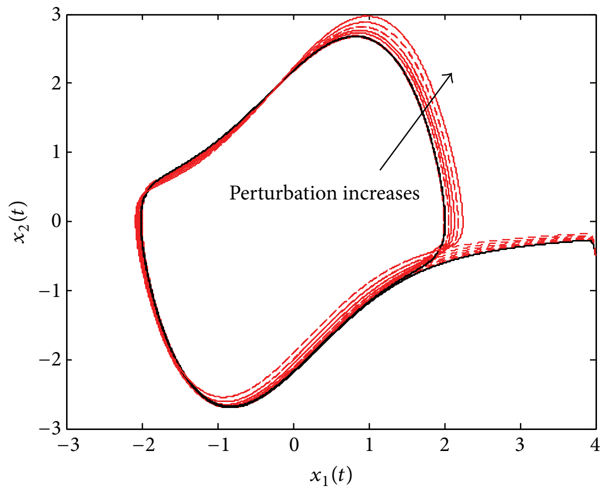

FIGURE 11: Stability of the limit cycle under increasing perturbations.

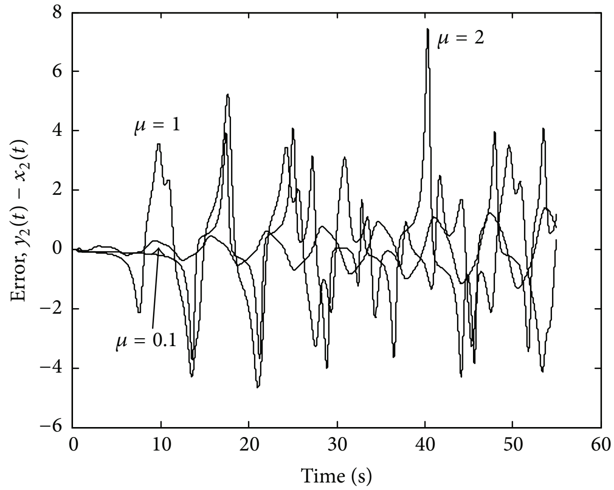

FIGURE 13: Error between $y_2(t)$ and $x_2(t)$ for different values for μ.

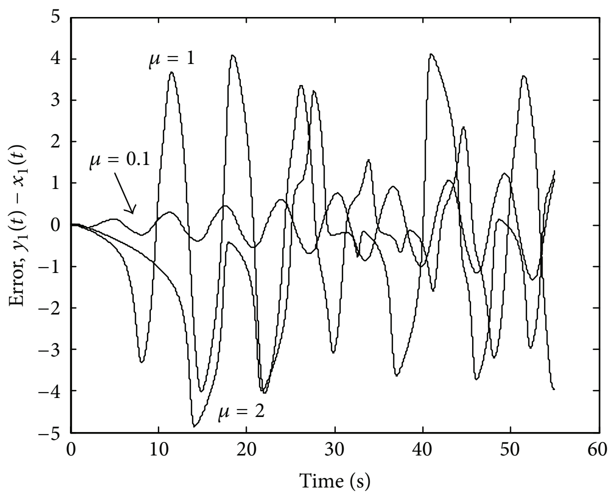

FIGURE 12: Error between $y_1(t)$ and $x_1(t)$ for different values for μ.

Figure 10 shows that the larger the threshold is, the smaller the approximation capabilities of the truncated model is. This feature is due to the fact that a larger threshold implies a greater separation between the sampling points. In Figure 10, δ is modified from 0.15 to 0.4. Thus, as Lemma 4 states, a large intersampling period might lead to higher errors in the approximated model. At this point we can introduce a bounded perturbation $g(t_i)$ at sampling points to analyze the stability of the limit cycle. For this, we can firstly select a value for the threshold δ in such a way that the solution of the approximate model shadows the one of the actual system. Afterwards, we can apply different perturbations to the system in an increasing way. If the limit cycle preserves its shape under this scheme, this would indicate that it is stable. This procedure has been applied in Figure 11.

Since the shape of the limit cycle is maintained, the stability of the original system is deduced from one of the approximate truncated models. Moreover, the shadowing property can be interpreted in terms of stability of the limit cycle in the following way. If the error between the solutions of the actual system and the approximate one under the same perturbation is less in a system A than in another system B, this means that A is more stable than B. Thus, the shadowing property can be viewed as a concept to measure the relative stability of systems by using its Taylor series expansion and construction of approximate models. For instance, consider the van del Pol equation with three different values of $\mu \in \{0.1, 1, 2\}$. The behavior of the van der Pol equation depends on the value of μ as it is widely recognized. Thus, the approximate perturbed model can be used to compute the error between the actual and reduced models in each case and determine for which value of μ the Van der Pol equation is "more stable," that is, has a greater relative stability. Thus, we fix the perturbation amplitude to $g(t_i) = 0.35$ and carry out some numerical experiments with the different values for μ. In this way, Figure 12 displays the error in the first state variable between the actual and approximate models while Figure 13 shows the error in the second state variable for each value of μ.

Figures 12 and 13 show that the larger μ is, the larger the peak error is. Therefore, in this case, systems with smaller μ have a greater relative stability. Finally, this approximate affine model could be used, as in the previous example, to design a control system based on a reduced model, rather than using the complete nonlinear one. In conclusion, the results presented in the previous sections have been applied in some case studies with little effort, a fact that backs up its potential practical applications.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are very grateful to the Spanish Government for its support of this research through Grant DPI2012-30651 and to the Basque Government for its support of this research through Grants IT378-10 and SAIOTEK S-PE12UN015. They are also grateful to the University of Basque Country for its financial support through Grant UFI 2011/07.

References

[1] M. de la Sen, "Application of the nonperiodic sampling to the identifiability and model matching problems in dynamic systems," *International Journal of Systems Science*, vol. 14, no. 4, pp. 367–383, 1983.

[2] M. de la Sen, "Stability of switched feedback time-varying dynamic systems based on the properties of the gap metric for operators," *Abstract and Applied Analysis*, vol. 2012, Article ID 612198, 17 pages, 2012.

[3] M. de la Sen, "About the stabilization of a nonlinear perturbed difference equation," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 320302, 19 pages, 2012.

[4] M. De la Sen, J. C. Soto, and A. Ibeas, "Stability and limit oscillations of a control event-based sampling criterion," *Journal of Applied Mathematics*, vol. 2012, Article ID 684292, 25 pages, 2012.

[5] R. Gu, "The average-shadowing property and topological ergodicity," *Journal of Computational and Applied Mathematics*, vol. 206, no. 2, pp. 796–800, 2007.

[6] R. Gu, "On ergodicity of systems with the asymptotic average shadowing property," *Computers & Mathematics with Applications*, vol. 55, no. 6, pp. 1137–1141, 2008.

[7] R. Gu, "Recurrence and the asymptotic pseudo-orbit tracing property," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 8, pp. 1698–1706, 2007.

[8] K. Lee and K. Sakai, "Various shadowing properties and their equivalence," *Discrete and Continuous Dynamical Systems A*, vol. 13, no. 2, pp. 533–540, 2005.

[9] E. S. van Vleck, "Numerical shadowing using componentwise bounds and a sharper fixed point result," *SIAM Journal on Scientific Computing*, vol. 22, no. 3, pp. 787–801, 2000.

[10] S. Reich, "Backward error analysis for numerical integrators," *SIAM Journal on Numerical Analysis*, vol. 36, no. 5, pp. 1549–1570, 1999.

[11] T. M. Apostol, *Mathematical Analysis*, Addison-Wesley, Reading, Mass, USA, 1958.

[12] Q. L. Zhang, G. Z. Dai, J. Lam, L. Q. Zhang, and M. De La Sen, "Asymptotic stability and stabilization of descriptor systems," *Acta Automatica Sinica*, vol. 24, no. 2, pp. 208–211, 1998.

[13] M. de la Sen, "Fundamental properties of linear control systems with after-effect. I. The continuous case," *Mathematical and Computer Modelling*, vol. 10, no. 7, pp. 473–489, 1988.

[14] R. A. Maher and R. Samir, "Robust stability of a class of unstable systems under mixed uncertainty," *Journal of Control Science and Engineering*, vol. 2011, Article ID 970962, 8 pages, 2011.

[15] D. Boutat, "Extended nonlinear observer normal forms for a class of nonlinear dynamical systems," *International Journal of Robust and Nonlinear Control*, 2013.

[16] M. Benchohra and M. Ziane, "Impulsive evolution inclusions with state-dependent delay and multivalued jumps," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 42, pp. 1–21, 2013.

[17] C. Tunç and M. Ateş, "Boundedness of solutions to differential equations of fourth order with oscillatory restoring and forcing terms," *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 758796, 5 pages, 2013.

[18] C. Tunc and M. Gozen, "Stability and uniform boundedness in multidelay functional differential equations of third order," *Abstract and Applied Analysis*, vol. 2013, Article ID 248717, 7 pages, 2013.

[19] J. Diblík, M. Fečkan, and M. Pospíšil, "Representation of a solution of the Cauchy problem for an oscillating system with multiple delays and pairwise permutable matrices," *Abstract and Applied Analysis*, vol. 2013, Article ID 931493, 10 pages, 2013.

[20] B. Baculikova, J. Dzurina, and Y. V. Rogovchenko, "Oscillation of third order trinomial delay differential equations," *Applied Mathematics and Computation*, vol. 218, no. 13, pp. 7023–7033, 2012.

[21] J. Baštinec, L. Berezansky, J. Diblík, and Z. Šmarda, "On the critical case in oscillation for differential equations with a single delay and with several delays," *Abstract and Applied Analysis*, vol. 2010, Article ID 417869, 20 pages, 2010.

[22] M. Hasanbulli and Y. V. Rogovchenko, "Oscillation criteria for second order nonlinear neutral differential equations," *Applied Mathematics and Computation*, vol. 215, no. 12, pp. 4392–4399, 2010.

[23] Z. Han, Y. Zhao, Y. Sun, and C. Zhang, "Oscillation for a class of fractional differential equation," *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 390282, 6 pages, 2013.

[24] J. C. Soto and M. Delasen, "Nonlinear oscillations in nonperiodic sampling systems," *Electronics Letters*, vol. 20, no. 20, pp. 816–818, 1984.

[25] M. de la Sen, "Oscillatory behavior in linear difference equations under unmodeled dynamics and parametrical errors," *Mathematical Problems in Engineering*, vol. 2007, Article ID 25692, 18 pages, 2007.

[26] J. C. Soto and M. Delasen, "On the derivation and analysis of a non-linear model for describing a class of adaptive sampling laws," *International Journal of Control*, vol. 42, no. 6, pp. 1347–1368, 1985.

[27] J. Chacón, J. Sánchez, A. Visioli, L. Yebra, and S. Dormido, "Characterization of limit cycles for self-regulating and integral processes with PI control and send-on-delta sampling," *Journal of Process Control*, vol. 23, no. 6, pp. 826–838, 2013.

[28] R. Memarbashi and H. Rasuli, "Notes on the dynamics of nonautonomous discrete dynamical systems," *Journal of Advanced Research in Dynamical and Control Systems*, vol. 6, no. 2, pp. 8–17, 2014.

[29] F. Mazenc, M. Malisoff, and M. de Querioz, "Tracking control and robustness analysis for a nonlinear model of human heart rate during exercise," *Automatica*, vol. 47, no. 5, pp. 968–974, 2011.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 725832, 6 pages
http://dx.doi.org/10.1155/2014/725832

Research Article

Nonlinear Variation of Parameters Formula for Impulsive Differential Equations with Initial Time Difference and Application

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Received 27 February 2014; Accepted 19 May 2014; Published 5 June 2014

Academic Editor: Samir Saker

This paper establishes variation of parameters formula for impulsive differential equations with initial time difference. As an application, one of the results is used to investigate stability properties of solutions.

1. Introduction

It is now well recognized that impulsive differential equations are suitable mathematical models for many processes and phenomena in biology, physics, technology, and so forth. That is why in recent years the mathematical theory of such systems has gained increasing significance. We notice that most of the studies about initial value problems of impulsive differential equations are investigated only for perturbation or change of dependent variable keeping the initial time unchanged. However, in dealing with real world phenomena, it is impossible not to make errors in the starting time. When we consider such a change of initial time for each solution, we need to deal with the problem of comparing between any two solutions which start at different times.

At present, the investigation of differential systems with initial time difference has attracted a lot of attention. There are two methods of comparing the differences of the two solutions. One is the differential inequalities technique and comparison principle; the other is variation of parameters. For the pioneering works in this area we can refer to the papers [1, 2]. Ever since then, many results for various differential and difference systems have been obtained. The results obtained by the former method can be seen in [3–10]; and those done by the latter can be found in [11–15]. However, up till now, to the best of our knowledge, there are few results for impulsive differential equations with initial time difference. To be specific, there are no results on variation of parameters formula for impulsive differential equations relative to initial time difference. The method of variation of parameters is an important and fruitful technique since it is a practical tool in the investigation of the properties of solutions. It has been applied to the study of the relations of unperturbed and perturbed systems with different initial conditions.

In this paper, we will develop variation of parameters formula for impulsive differential equations with initial time changed and investigate Lipschitz stability by using one of the results obtained. The remainder of this paper is organized in the following manner. Some preliminaries are presented in Section 2, and various types of nonlinear variation of parameters formulae are established in Section 3. Finally, as an application, one of the results is applied to impulsive differential equations and the stability properties are obtained.

2. Preliminaries

Let $R^+ = [0, +\infty)$ and let R^n denote the n-dimensional Euclidean space with appropriate norm $\|\cdot\|$.

Consider the following unperturbed impulsive differential equations

$$\begin{aligned} x' &= f(t,x), \quad t \neq t_k, \\ x(t_0^+) &= x_0, \\ x(t_k^+) &= x(t_k) + I_{t_k}(x(t_k)), \end{aligned} \tag{1}$$

$$\begin{aligned} x' &= f(t,x), \quad t \neq t_k, \\ x(\tau_0^+) &= y_0, \\ x(t_k^+) &= x(t_k) + I_{t_k}(x(t_k)), \quad \text{whenever } t_k \geq \tau_0, \end{aligned} \tag{2}$$

together with the perturbed ones of (2)

$$\begin{aligned} y' &= F(t,y), \quad t \neq t_k, \\ y(\tau_0^+) &= y_0, \\ y(t_k^+) &= y(t_k) + I_{t_k}(y(t_k)), \quad \text{whenever } t_k \geq \tau_0, \end{aligned} \tag{3}$$

where

(1) $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $\lim_{k\to\infty} t_k = \infty$, $k = 1, 2, \ldots$;
(2) $\tau_0 > t_0$, $\eta = \tau_0 - t_0$;
(3) $\bar{t}_k = t_k - \eta \geq 0$;
(4) $S_1 = \{t_k\}$, $S_2 = \{\bar{t}_k\}$, $S = S_1 \cup S_2$;
(5) $t \in R^+$, $x \in \Omega \subset R^n$, Ω – open;
(6) $f, F : R^+ \times \Omega \to R^n$;
(7) $I_{t_k} : \Omega \to R^n$;
(8) $f(t,0) = 0$, $I_{t_k}(0) = 0$, for all t_k.

We are concerned in this paper with the variation of parameters formula for impulsive differential equations relative to initial time difference. Before we can proceed, we will introduce the following lemmas [12], which are necessary for completing our main results.

Lemma 1. *Let the following conditions be fulfilled:*

(A_1) *the function $f : R^+ \times \Omega \to R^n$ is continuous in $(t_{k-1}, t_k] \times \Omega$, $k = 1, 2, \ldots$ and for every k and $x_0 \in R^n$, there exists a finite limit of $f(t,x)$ as $(t,x) \to (t_k, x_0)$, $t > t_k$;*

(A_2) *the function f is locally Lipschitzian in x on $R^+ \times \Omega$;*

(A_3) *for $k = 1, 2, \ldots$ the mapping $\psi_k : \Omega \to \Omega$, $x \to z$, $z = \psi_k(x) \equiv x + I_k(x)$ is a homeomorphism;*

(A_4) *the system* (1) *had a solution $\phi(t)$ defined in $[\alpha, \beta]$, $(\alpha, \beta \neq t_k,\ k = 1, 2, \ldots)$.*

Then there exist a number $\epsilon > 0$ and a set

$$V = \{(t,x) \in R^+ \times \Omega,\ \alpha \leq t \leq \beta,\ |x - \phi(t^+)| < \epsilon\}, \tag{4}$$

such that,

(i) *for every $(t_0, x_0) \in V$, there exists a unique solution $x(t, t_0, x_0)$ of the system* (1) *which is defined on $[\alpha, \beta]$;*

(ii) *the function $x(t, t_0, x_0)$ is continuous for*

$$t \in [\alpha, \beta], \qquad (t_0, x_0) \in V, \quad t, t_0 \notin S_1; \tag{5}$$

(iii) *for $k = 1, 2, \ldots$, $x_0 \in \Omega$, t, t_0 belonging to the interval of existence of solution $x(t, t_0, x_0)$ of* (1), *$t \notin S_1$,*

$$\lim_{\substack{\xi \to t_0 \\ \rho \to x_0}} x(t, \xi, \rho) = x(t, t_0, x_0). \tag{6}$$

Lemma 2. *Let the following conditions be fulfilled:*

(A_5) *the function $f : R^+ \times \Omega \to R^n$ is continuous in $(t_{k-1}, t_k] \times \Omega$, $k = 1, 2, \ldots$, and $f_x(t,x)$ is continuous in $(t_{k-1}, t_k) \times \Omega$, $k = 1, 2, \ldots$;*

(A_6) *for every $x_0 \in \Omega$, $k = 1, 2, \ldots$, there exist finite limits of functions f and f_x as $(t,x) \to (t_k, x_0)$, $t > t_k$;*

(A_7) *for $k = 1, 2, \ldots$ the mapping $\psi_k : \Omega \to \Omega$, $x \to z$, $z = \psi_k(x) \equiv x + I_k(x)$ is a diffeomorphism and for $x \in \Omega$*

$$\det\left(I + \frac{\partial I_k}{\partial x}(x)\right) \neq 0, \quad k = 1, 2, \ldots. \tag{7}$$

Then,

(i) *there exists $\delta > 0$ such that the solution $x(t, t_0, x_0)$ of* (1) *has continuous derivatives $\partial x/\partial t$, $\partial x/\partial t_0$, $\partial x/\partial x_0$, in the domain*

$$\begin{aligned} V : \alpha < t < \beta, \qquad \alpha < t_0 < \beta, \quad t, t_0 \neq t_k,\ k = 1, 2, \ldots \\ |x_0 - \phi(t_0^+)| < \delta; \end{aligned} \tag{8}$$

(ii) *the derivative $\Phi(t, t_0, x_0) = (\partial x/\partial x_0)(t, t_0, x_0)$ is a solution of the initial value problem*

$$\begin{aligned} u' &= f_x(t, \phi(t))\, u, \quad t \neq t_k, \\ \Delta u &= \frac{\partial I_k}{\partial x}(\phi(t_k))\, u, \quad t = t_k, \\ u(t_0^+) &= I, \end{aligned} \tag{9}$$

where $\phi(t)$ is the solution of (1) *in $[\alpha, \beta]$ $\alpha, \beta \neq t_k$, $k = 1, 2, \ldots$;*

(iii) *the derivative $\partial x/\partial t_0$ satisfies the relation*

$$\begin{aligned} \frac{\partial x}{\partial t_0}(t, t_0, x_0) &= -\frac{\partial x}{\partial x_0}(t, t_0, x_0)\, f(t_0, x_0) \\ &= -\Phi(t, t_0, x_0)\, f(t_0, x_0). \end{aligned} \tag{10}$$

3. Nonlinear Variation of Parameters Formula

We will present, in this section, the nonlinear variation of parameters formula for impulsive differential equations relative to initial time difference. It is very useful for investigating the stability properties of solutions.

Theorem 3. *Let the system* (1) *satisfy the conditions of Lemma 2 and let $x(t, t_0, x_0)$ be a solution of* (1). *Then for any solution $y(t) = y(t, \tau_0, y_0)$ of the system* (3). *The following formula is valid:*

$$\begin{aligned} y(t+\eta, \tau_0, y_0) &= x(t, t_0, x_0) + \int_0^1 \Phi(t, t_0, \sigma(s))(y_0 - x_0)\,ds \\ &+ \int_{t_0}^t \tilde{f}(s, \tilde{y}(s), \eta)\,ds \\ &+ \sum_{t_0<\bar{t}_k<t} \int_0^1 \Phi\left(t, \bar{t}_k, \tilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\right) ds \\ &\cdot I_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k)). \end{aligned} \tag{11}$$

Proof. Set $p(s) = x(t, s, \tilde{y}(s))$, where $\tilde{y}(s) = y(s+\eta, \tau_0, y_0)$, $t_0 < s < t$. Then for $s \notin S$, we have

$$\begin{aligned} p'(s) &= \frac{\partial x}{\partial s}(t, s, \tilde{y}(s)) \\ &+ \frac{\partial x}{\partial \tilde{y}}(t, s, \tilde{y}(s)) F(s+\eta, \tilde{y}(s)) \equiv \tilde{f}(s, \tilde{y}(s), \eta). \end{aligned} \tag{12}$$

When $s \in S$, we have two cases.

Case 1. Consider

$$\begin{aligned} \Delta p(s)\big|_{s=\bar{t}_k} &= x\left(t, \bar{t}_k^{+}, \tilde{y}\left(\bar{t}_k^{+}\right)\right) - x\left(t, \bar{t}_k^{-}, \tilde{y}\left(\bar{t}_k^{-}\right)\right) \\ &= x\left(t, \bar{t}_k, \tilde{y}(\bar{t}_k) + I_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\right) - x(t, \bar{t}_k, \tilde{y}(\bar{t}_k)) \\ &= \int_0^1 \Phi\left(t, \bar{t}_k, \tilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\right) ds \cdot I_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k)). \end{aligned} \tag{13}$$

Case 2. Consider

$$\Delta p(s)\big|_{s=t_k \in S_1 - S_2} = x(t, t_k^{+}, \tilde{y}(t_k^{+})) - x(t, t_k^{-}, \tilde{y}(t_k^{-})) = 0. \tag{14}$$

Integrating (12) from t_0 to t and using (13) and (14), we have

$$\begin{aligned} \tilde{y}(t) &= x(t, t_0, y_0) + \int_{t_0}^t \tilde{f}(s, \tilde{y}(s), \eta)\,ds \\ &+ \sum_{t_0<\bar{t}_k<t} \int_0^1 \Phi\left(t, \bar{t}_k, \tilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\right) ds \\ &\cdot I_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k)). \end{aligned} \tag{15}$$

Now let $q(s) = x(t, t_0, \sigma(s))$, where $\sigma(s) = y_0 s + (1-s)x_0$, $0 \le s \le 1$. Then we have

$$\frac{dq(s)}{ds} = \frac{\partial x}{\partial \sigma}(t, t_0, \sigma(s))(y_0 - x_0). \tag{16}$$

Integrating (16) from 0 to 1, we arrive at

$$x(t, t_0, y_0) = x(t, t_0, x_0) + \int_0^1 \frac{\partial x}{\partial \sigma}(t, t_0, \sigma(s))(y_0 - x_0)\,ds. \tag{17}$$

Combining (15) and (17) yields

$$\begin{aligned} y(t+\eta, \tau_0, y_0) &= x(t, t_0, x_0) + \int_0^1 \Phi(t, t_0, \sigma(s))(y_0 - x_0)\,ds \\ &+ \int_{t_0}^t \tilde{f}(s, \tilde{y}(s), \eta)\,ds \\ &+ \sum_{t_0<\bar{t}_k<t} \int_0^1 \Phi\left(t, \bar{t}_k, \tilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\right) ds \\ &\cdot I_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k)). \end{aligned} \tag{18}$$

The proof is complete. □

Corollary 4. *Suppose that the assumptions of Theorem 3 hold except that $F(t+\eta, y)$ being replaced with $f(t, y) + R(t+\eta, y)$; then the following formula is valid:*

$$\begin{aligned} y(t+\eta, \tau_0, y_0) &= x(t, t_0, x_0) + \int_0^1 \Phi(t, t_0, \sigma(s))(y_0 - x_0)\,ds \\ &+ \int_{t_0}^t \Phi(t, s, \tilde{y}(s)) R(s+\eta, \tilde{y}(s))\,ds \\ &+ \sum_{t_0<\bar{t}_k<t} \int_0^1 \Phi\left(t, \bar{t}_k, \tilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\right) ds \\ &\cdot I_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k)). \end{aligned} \tag{19}$$

Theorem 5. *Suppose that the assumptions of Theorem 3 hold; then the following formula is valid:*

$$\begin{aligned} y(t+\eta, \tau_0, y_0) - x(t, t_0, x_0) &= x(t, t_0, \eta) + \int_{t_0}^t \widetilde{H}(s, w(s), \eta)\,ds \\ &+ \sum_{\substack{t_0<\bar{t}_k<t \\ \bar{t}_k \in S_2 - S_1}} \int_0^1 \Phi\left(t, \bar{t}_k, y(\bar{t}_k+\eta) - x(\bar{t}_k)\right. \\ &\qquad \left. + sI_{\bar{t}_k+\eta}(y(\bar{t}_k+\eta))\right) ds \\ &\cdot I_{\bar{t}_k+\eta}(y(\bar{t}_k+\eta)) \end{aligned}$$

$$- \sum_{\substack{t_0<t_k<t\\ t_k\in S_1-S_2}} \int_0^1 \Phi\left(t, t_k, y\left(t_k+\eta\right)-x\left(t_k\right) - sI_{t_k}\left(x\left(t_k\right)\right)\right) ds \cdot I_{t_k}\left(x\left(t_k\right)\right) + \sum_{\substack{t_0<t_k<t\\ t_k\in S_1\cap S_2}} \int_0^1 \Phi\left(t, t_k, y\left(t_k+\eta\right)-x\left(t_k\right)+sI_{t_k+\eta}\left(y\left(t_k+\eta\right)\right) - sI_{t_k}\left(x\left(t_k\right)\right)\right) ds \cdot \left(I_{t_k+\eta}\left(y\left(t_k+\eta\right)\right)-I_{t_k}\left(x\left(t_k\right)\right)\right). \quad (20)$$

Proof. Set $p(s)=x(t,s,w(s))$, where $w(s)=y(s+\eta,\tau_0,y_0)-x(s,t_0,x_0)$, $t_0<s<t$. Then for $s\notin S$, we have

$$p'(s)=\frac{\partial x}{\partial s}(t,s,w(s)) + \frac{\partial x}{\partial w}(t,s,w(s))\,H(s,w(s),\eta) \equiv \widetilde{H}(s,w(s),\eta), \quad (21)$$

where $H(s,w(s),\eta)=F(s+\eta,w(s)+x(s))-f(s,x(s))$.

If $s\in S$, we have three cases.

Case 1. Consider

$$\begin{aligned}\Delta p(s)\big|_{s=\bar t_k\in S_2-S_1} &= x\left(t,\bar t_k^{+},w\left(\bar t_k^{+}\right)\right)-x\left(t,\bar t_k^{-},w\left(\bar t_k^{-}\right)\right)\\ &= x\left(t,\bar t_k,y\left(\bar t_k^{+}+\eta\right)-x\left(\bar t_k^{+}\right)\right)-x\left(t,\bar t_k^{-},y\left(\bar t_k^{-}+\eta\right)-x\left(\bar t_k^{-}\right)\right)\\ &= x\left(t,\bar t_k,y\left(\bar t_k+\eta\right)+I_{\bar t_k+\eta}\left(y\left(\bar t_k+\eta\right)\right)-x\left(\bar t_k\right)\right) - x\left(t,\bar t_k,y\left(\bar t_k+\eta\right)-x\left(\bar t_k\right)\right)\\ &= \int_0^1 \Phi\left(t,\bar t_k,y\left(\bar t_k+\eta\right)-x\left(\bar t_k\right)+sI_{\bar t_k+\eta}\left(y\left(\bar t_k+\eta\right)\right)\right)ds \cdot I_{\bar t_k+\eta}\left(y\left(\bar t_k+\eta\right)\right).\end{aligned} \quad (22)$$

Case 2. Consider

$$\begin{aligned}\Delta p(s)\big|_{s=t_k\in S_1-S_2} &= x\left(t,t_k^{+},w\left(t_k^{+}\right)\right)-x\left(t,t_k^{-},w\left(t_k^{-}\right)\right)\\ &= x\left(t,t_k,y\left(t_k^{+}+\eta\right)-x\left(t_k^{+}\right)\right)-x\left(t,t_k,y\left(t_k^{-}+\eta\right)-x\left(t_k^{-}\right)\right)\\ &= x\left(t,t_k,y\left(t_k+\eta\right)-I_{t_k}\left(x\left(t_k\right)\right)-x\left(t_k\right)\right) - x\left(t,t_k,y\left(t_k+\eta\right)-x\left(t_k\right)\right)\\ &= -\int_0^1 \Phi\left(t,t_k,y\left(t_k+\eta\right)-x\left(t_k\right)-sI_{t_k}\left(x\left(t_k\right)\right)\right)ds \cdot I_{t_k}\left(x\left(t_k\right)\right).\end{aligned} \quad (23)$$

Case 3. Consider

$$\begin{aligned}\Delta p(s)\big|_{s=t_k\in S_1\cap S_2} &= x\left(t,t_k^{+},w\left(t_k^{+}\right)\right)-x\left(t,t_k^{-},w\left(t_k^{-}\right)\right)\\ &= x\left(t,t_k,y\left(t_k^{+}+\eta\right)-x\left(t_k^{+}\right)\right)-x\left(t,t_k,y\left(t_k^{-}+\eta\right)-x\left(t_k^{-}\right)\right)\\ &= x\left(t,t_k,y\left(t_k+\eta\right)+I_{t_k+\eta}\left(y\left(t_k+\eta\right)\right)-I_{t_k}\left(x\left(t_k\right)\right)-x\left(t_k\right)\right) - x\left(t,t_k,y\left(t_k+\eta\right)-x\left(t_k\right)\right)\\ &= \int_0^1 \Phi\left(t,t_k,y\left(t_k+\eta\right)-x\left(t_k\right)+sI_{t_k+\eta}\left(y\left(t_k+\eta\right)\right)-sI_{t_k}\left(x\left(t_k\right)\right)\right)ds \cdot \left(I_{t_k+\eta}\left(y\left(t_k+\eta\right)\right)-I_{t_k}\left(x\left(t_k\right)\right)\right).\end{aligned} \quad (24)$$

Integrating (21) from t_0 to t and using (22) and (24), we have

$$\begin{aligned}y\left(t+\eta,\tau_0,y_0\right)-x\left(t,t_0,x_0\right) &= x\left(t,t_0,\eta\right)+\int_{t_0}^{t}\widetilde{H}\left(s,w(s),\eta\right)ds\\ &+ \sum_{\substack{t_0<\bar t_k<t\\ \bar t_k\in S_2-S_1}} \int_0^1 \Phi\left(t,\bar t_k,y\left(\bar t_k+\eta\right)-x\left(\bar t_k\right)+sI_{\bar t_k+\eta}\left(y\left(\bar t_k+\eta\right)\right)\right)ds \cdot I_{\bar t_k+\eta}\left(y\left(\bar t_k+\eta\right)\right)\\ &- \sum_{\substack{t_0<t_k<t\\ t_k\in S_1-S_2}} \int_0^1 \Phi\left(t,t_k,y\left(t_k+\eta\right)-x\left(t_k\right)-sI_{t_k}\left(x\left(t_k\right)\right)\right)ds \cdot I_{t_k}\left(x\left(t_k\right)\right)\\ &+ \sum_{\substack{t_0<t_k<t\\ t_k\in S_1\cap S_2}} \int_0^1 \Phi\left(t,t_k,y\left(t_k+\eta\right)-x\left(t_k\right)+sI_{t_k+\eta}\left(y\left(t_k+\eta\right)\right)-sI_{t_k}\left(x\left(t_k\right)\right)\right)ds\\ &\cdot \left(I_{t_k+\eta}\left(y\left(t_k+\eta\right)\right)-I_{t_k}\left(x\left(t_k\right)\right)\right).\end{aligned} \quad (25)$$

The proof is complete. □

4. Application

In this section, we turn to the Lipschitz stability of system (3):

$$\begin{aligned} y' &= F(t,y), \quad t\neq t_k,\\ y\left(\tau_0^{+}\right) &= y_0,\\ y\left(t_k^{+}\right) &= y\left(t_k\right)+I_{t_k}\left(y\left(t_k\right)\right), \quad \text{whenever } t_k\geq\tau_0,\end{aligned} \quad (26)$$

where $F(t+\eta,y)=f(t,y)+R(t+\eta,y)$.

Definition 6. The solution $y(t+\eta,\tau_0,y_0)$ of the system (3) is said to be initial time difference Lipschitz stable (ITDLS) with respect to the solution $x(t,t_0,x_0)$ for $t \geq t_0$, where $x(t,t_0,x_0)$ is any solution of the system (1), if and only if there exists an $M = M(\tau_0)$ such that

$$\left\| y(t+\eta,\tau_0,y_0) - x(t,t_0,x_0) \right\| \leq M\left(\left\| y_0 - x_0 \right\| + \tau_0 - t_0 \right). \tag{27}$$

Theorem 7. *Let the following conditions be fulfilled:*

(B_1) *the assumptions of Corollary 4 hold;*

(B_2) *the zero solution of (1) is Lipschitz stable;*

(B_3) $\|\Phi(t,s,\tilde{y}(s))R(s+\eta,\tilde{y}(s))\| \leq \gamma(s)\|\tilde{y}(s)\|$ *for* $t_0 < s \leq t$;

(B_4) $\|\Phi(t,t_0,\sigma(s))\| \leq M_1(\|y_0-x_0\|+\eta)/\|y_0-x_0\|$ *and* M_1 *is a constant;*

(B_5) $\|I_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\| \leq \beta_k\|\tilde{y}(\bar{t}_k)\|$ *and* $\beta_k \geq 0$ *are constants;*

(B_6) $\|\Phi(t,\bar{t}_k,\tilde{y}(\bar{t}_k)+sI_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k)))\| \leq \alpha_k$ *and* $\alpha_k \geq 0$ *are constants;*

(B_7) $\int_{t_0}^{\infty}\gamma(s)ds < \infty$, $\gamma(s) \in C[R^+,R^+]$ *and* $\prod_{t_0<\bar{t}_k<t}(1+\alpha_k\beta_k) < \infty$.

Then the solution $y(t+\eta,\tau_0,y_0)$ *of the system* (3) *is ITDLS with respect to the solution* $x(t,t_0,x_0)$.

Proof. From Corollary 4, it follows that

$$\begin{aligned} &\tilde{y}(t) - x(t,t_0,x_0) \\ &\quad = \int_0^1 \Phi(t,t_0,\sigma(s))(y_0-x_0)\,ds \\ &\qquad + \int_{t_0}^t \Phi(t,s,\tilde{y}(s))R(s+\eta,\tilde{y}(s))\,ds \\ &\qquad + \sum_{t_0<\bar{t}_k<t}\int_0^1 \Phi\left(t,\bar{t}_k,\tilde{y}(\bar{t}_k)+sI_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\right)ds \\ &\qquad \cdot I_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k)). \end{aligned} \tag{28}$$

Taking the norm and using the triangle inequality on both sides, we have

$$\begin{aligned} &\|\tilde{y}(t) - x(t,t_0,x_0)\| \\ &\quad \leq \int_0^1 \|\Phi(t,t_0,\sigma(s))\|\,\|(y_0-x_0)\|\,ds \\ &\qquad + \int_{t_0}^t \|\Phi(t,s,\tilde{y}(s))R(s+\eta,\tilde{y}(s))\|\,ds \\ &\qquad + \sum_{t_0<\bar{t}_k<t}\int_0^1 \left\|\Phi\left(t,\bar{t}_k,\tilde{y}(\bar{t}_k)+sI_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\right)\right\|ds \\ &\qquad \cdot \left\|I_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\right\|. \end{aligned} \tag{29}$$

From conditions (B_2)–(B_5), we obtain

$$\begin{aligned} &\|\tilde{y}(t) - x(t,t_0,x_0)\| \\ &\quad \leq M_1(\|y_0-x_0\|+\eta) + \int_{t_0}^t \gamma(s)\|\tilde{y}(s)\|\,ds \\ &\qquad + \sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k\|\tilde{y}(\bar{t}_k)\|. \end{aligned} \tag{30}$$

Setting $M^*(t) = \|\tilde{y}(t) - x(t,t_0,x_0)\|$, we have

$$\begin{aligned} M^*(t) &\leq M_1(\|y_0-x_0\|+\eta) + \int_{t_0}^t \gamma(s)M^*(s)\,ds \\ &\quad + \int_{t_0}^t \gamma(s)\|x(s,t_0,x_0)\|\,ds \\ &\quad + \sum_{t_0<\bar{t}_k<t}\alpha_k\beta_kM^*(\bar{t}_k) + \sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k\|x(\bar{t}_k)\|. \end{aligned} \tag{31}$$

Since $\|x(t,t_0,x_0)\| \leq M_2\|x_0\|$, as long as $\|x_0\| < \varepsilon$, then we have

$$\|x(t,t_0,x_0)\| \leq M_2\varepsilon,$$

$$\begin{aligned} M^*(t) &\leq M_1(\|y_0-x_0\|+\eta) + \int_{t_0}^t \gamma(s)M^*(s)\,ds \\ &\quad + M_2\varepsilon\int_{t_0}^t \gamma(s)\,ds + \sum_{t_0<\bar{t}_k<t}\alpha_k\beta_kM^*(\bar{t}_k) \\ &\quad + M_2\varepsilon\sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k. \end{aligned} \tag{32}$$

Applying Gronwall's inequality to (32), we get

$$\begin{aligned} &M^*(t) \\ &\quad \leq \left\{ M_1(\|y_0-x_0\|+\eta) \right. \\ &\qquad \left. + M_2\varepsilon\int_{t_0}^t \gamma(s)\,ds + M_2\varepsilon\sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k \right\} \\ &\qquad \cdot \prod_{t_0<\bar{t}_k<t}(1+\alpha_k\beta_k)\exp\left\{\int_{t_0}^t \gamma(s)\,ds\right\}. \end{aligned} \tag{33}$$

Setting $M_3 = \{M_1 + (M_2\varepsilon\int_{t_0}^t \gamma(s)ds/\|y_0-x_0\|+\eta) + (M_2\varepsilon\sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k/\|y_0-x_0\|+\eta)\}\prod_{t_0<\bar{t}_k<t}(1+\alpha_k\beta_k)\exp\{\int_{t_0}^t \gamma(s)ds\}$, we have

$$M^*(t) \leq M_3(\|y_0-x_0\|+\eta). \tag{34}$$

From condition (B_7), it follows that the solution $y(t+\eta,\tau_0,y_0)$ of the system (3) is ITDLS with respect to the solution $x(t,t_0,x_0)$.

The proof is complete. □

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors completed the paper together. All authors read and approved the final paper.

Acknowledgments

The authors would like to thank the reviewers for their valuable suggestions and comments. This paper is supported by the National Natural Science Foundation of China (11271106) and the Natural Science Foundation of Hebei Province of China (A2013201232).

References

[1] V. Lakshmikantham and A. S. Vatsala, "Differential inequalities with initial time difference and applications," *Journal of Inequalities and Applications*, vol. 3, no. 3, pp. 233–244, 1999.

[2] V. Lakshmikantham, S. Leela, and J. V. Devi, "Another approach to the theory of differential inequalities relative to changes in the initial times," *Journal of Inequalities and Applications*, vol. 4, no. 2, pp. 163–174, 1999.

[3] T. Jankowski, "Systems of differential inequalities with initial time difference," *Ukrainian Mathematical Journal*, vol. 56, no. 1, pp. 139–145, 2004.

[4] T. Jankowski, "Integro-differential inequalities with initial time difference and applications," *Acta Mathematica Hungarica*, vol. 100, no. 4, pp. 329–342, 2003.

[5] T. Jankowski, "Differential inequalities with initial time difference," *Applicable Analysis*, vol. 81, no. 3, pp. 627–635, 2002.

[6] F. A. McRae, "Perturbing Lyapunov functions and stability criteria for initial time difference," *Applied Mathematics and Computation*, vol. 117, no. 2-3, pp. 313–320, 2001.

[7] Y. Zhang and B. Zhang, "Impulsive differential equations with initial time difference and applications," *Dynamics of Continuous, Discrete & Impulsive Systems A. Mathematical Analysis*, vol. 9, no. 3, pp. 439–447, 2002.

[8] C. Yakar and M. D. Shaw, "Practical stability in terms of two measures with initial time difference," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 71, no. 12, pp. 781–785, 2009.

[9] C. Yakar, "Fractional differential equations in terms of comparison results and Lyapunov stability with initial time difference," *Abstract and Applied Analysis*, vol. 2010, Article ID 762857, 16 pages, 2010.

[10] J. Y. Bao, P. G. Wang, and C. X. Gao, "Stability criteria for differential equations with initial time difference," *Acta Mathematicae Applicatae Sinica*, vol. 35, no. 4, pp. 608–616, 2012.

[11] M. D. Shaw and C. Yakar, "Generalized variation of parameters with initial time difference and a comparison result in term Lyapunov-like functions," *International Journal of Non-Linear Differential Equations-Theory Methods and Applications*, vol. 5, pp. 86–108, 1999.

[12] C. Yakar and S. G. Deo, "Variation of parameters formulae with initial time difference for linear integrodifferential equations," *Applicable Analysis*, vol. 85, no. 4, pp. 333–343, 2006.

[13] M. D. Shaw and C. Yakar, "Stability criteria and slowly growing motions with initial time difference," *Problems of Nonlinear Analysis in Engineering Systems*, vol. 1, pp. 50–66, 2000.

[14] C. Yakar and M. D. Shaw, "Initial time difference stability in terms of two measures and a variational comparison result," *Dynamics of Continuous, Discrete & Impulsive Systems A. Mathematical Analysis*, vol. 15, no. 3, pp. 417–425, 2008.

[15] V. Lakshmikantham, S. Leela, and J. V. Devi, "Stability criteria for solutions of differential equations relative to initial time difference," *International Journal of Nonlinear Differential Equations*, vol. 5, pp. 109–114, 1999.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 316093, 7 pages
http://dx.doi.org/10.1155/2014/316093

Research Article

Multiple Positive Periodic Solutions for a Functional Difference System

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Received 20 January 2014; Accepted 12 May 2014; Published 22 May 2014

Academic Editor: Samir Saker

We obtain two existence results about multiple positive periodic solutions for a class of functional difference system. Two examples are given to illustrate our results.

1. Introduction and Preliminaries

Throughout this paper, we denote by $\mathbb{Z}$ the set of all integers, by $\mathbb{R}$ the set of all real numbers, and by X a real Banach space. Moreover, let

$$\mathbb{R}_+^n = \{(x_1, \dots, x_n) : x_1, \dots, x_n \geq 0\} \tag{1}$$

and let $l_T^\infty(\mathbb{Z}, \mathbb{R}^n)$ ($l_T^\infty(\mathbb{Z}, \mathbb{R}_+^n)$) be the space of all T-periodic functions $f : \mathbb{Z} \to \mathbb{R}^n$ ($f : \mathbb{Z} \to \mathbb{R}_+^n$), where $T > 1$ is fixed positive integer. It is well known that $l_T^\infty(\mathbb{Z}, \mathbb{R}^n)$ is a Banach space under the norm

$$\|f\| = \max_{1\leq k\leq T} \max_{1\leq j\leq n} |f_j(k)|, \tag{2}$$

where $f = (f_1, f_2, \dots, f_n)^T$.

The aim of this paper is to investigate the existence of multiple positive periodic solutions to the following functional difference system:

$$x(k+1) - x(k) = A(k)x(k) + f(k, x_k), \quad k \in \mathbb{Z}, \tag{3}$$

where $x : \mathbb{Z} \to \mathbb{R}^n$ is an n-dimensional vector function

$$A(k) = \operatorname{diag}[a_1(k), a_2(k), \dots, a_n(k)], \tag{4}$$

a_j, $j = 1, 2, \dots, n$, are T-periodic functions from $\mathbb{Z}$ to $\mathbb{R}$, f is a function from $\mathbb{Z} \times l_T^\infty(\mathbb{Z}, \mathbb{R}^n)$ to $\mathbb{R}^n$, and x_k is defined by $x_k(m) = x(k+m)$ for all $m \in \mathbb{Z}$.

The existence of periodic solutions has been an important topic in the qualitative theory of functional differential equations and functional difference equations. There is a large body of literature on this interesting topic. We refer the reader to [1–17] and references therein for some recent contributions. Especially, the existence of periodic solutions for system (3) and its variants has been of great interest for many authors (see, e.g., [5, 6, 8, 9, 17] and references therein).

It is needed to note that Raffoul [8] and Raffoul and Tisdell [9] have made an important contribution to this topic. In fact, Raffoul constructed Green function for system (3) and transformed system (3) into an equivalent system. This enables us to use some suitable fixed point theorems to investigate the existence of periodic solutions for system (3). In addition, we would like to draw the reader's attention to [6], where Dix et al. initiated the study on the *multiple* periodic solutions for a variant of system (3) in a 1-dimensional case.

Stimulated by [6, 8, 9], in this paper, we will make further study on this topic for an n-dimensional case. Next, we recall two fixed point theorems, which will be used in the proof of our main results. We first recall some definitions and notations.

A closed convex set K in X is called a cone if the following conditions are satisfied:

(i) if $x \in K$, then $\lambda x \in K$ for any $\lambda \geq 0$,

(ii) if $x \in K$ and $-x \in K$, then $x = 0$.

A nonnegative continuous functional ψ is said to be a concave on K if ψ is continuous and

$$\psi(\mu x + (1-\mu) y) \geq \mu\psi(x) + (1-\mu)\psi(y), \quad x, y \in K, \quad \mu \in [0,1]. \tag{5}$$

Letting c_1, c_2, c_3 be three positive constants and letting ϕ be a nonnegative continuous functional on K, we denote

$$\begin{aligned} K_{c_1} &= \{y \in K : \|y\| < c_1\}, \\ K(\phi, c_1) &:= \{x \in K : \phi(x) < c_1\}, \\ \overline{K(\phi, c_1)} &:= \{x \in K : \phi(x) \leq c_1\}, \\ \partial K(\phi, c_1) &:= \{x \in K : \phi(x) = c_1\}, \\ K(\phi, c_2, c_3) &= \{y \in K : c_2 \leq \phi(y), \|y\| < c_3\}. \end{aligned} \tag{6}$$

In addition, we call that ϕ is increasing on K if $\phi(x) \geq \phi(y)$ for all $x, y \in K$ with $x - y \in K$.

Lemma 1 (see [18]). *Let K be a cone in X, let α and φ be increasing, nonnegative, continuous functionals on K, and let ρ be a nonnegative continuous functional on K with $\rho(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\varphi(u) \leq \rho(u) \leq \alpha(u), \qquad \|u\| \leq M\varphi(u) \tag{7}$$

for all $u \in \overline{K(\varphi, c)}$. Suppose that there exists a completely continuous operator $\Phi : \overline{K(\varphi, c)} \to K$ and $0 < a < b < c$ such that

$$\rho(\lambda u) \leq \lambda\rho(u), \quad \text{for } 0 \leq \lambda \leq 1,\ u \in \partial K(\rho, b), \tag{8}$$

and

(i) *$\varphi(\Phi u) > c$, for all $u \in \partial K(\varphi, c)$;*

(ii) *$\rho(\Phi u) < b$, for all $u \in \partial K(\rho, b)$;*

(iii) *$K(\alpha, a) \neq \emptyset$ and $\alpha(\Phi x) > a$, for all $u \in \partial K(\alpha, a)$.*

Then Φ has at least two fixed points u_1 and u_2 belonging to $\overline{K(\varphi, c)}$ such that

$$\begin{aligned} a &< \alpha(u_1), \quad \text{with } \rho(u_1) < b, \\ b &< \rho(u_2), \quad \text{with } \varphi(u_2) < c. \end{aligned} \tag{9}$$

Lemma 2 (see [19]). *Let K be a cone in X, let c_4 be a positive constant, let $\Phi : \overline{K}_{c_4} \to \overline{K}_{c_4}$ be a completely continuous mapping, and let ψ be a concave nonnegative continuous functional on K with $\psi(u) \leq \|u\|$ for all $u \in \overline{K}_{c_4}$. Suppose that there exist three constants c_1, c_2, c_3 with $0 < c_1 < c_2 < c_3 \leq c_4$ such that*

(i) *$\{u \in K(\psi, c_2, c_3) : \psi(u) > c_2\} \neq \emptyset$ and $\psi(\Phi u) > c_2$ for all $u \in K(\psi, c_2, c_3)$;*

(ii) *$\|\Phi u\| < c_1$ for all $u \in \overline{K}_{c_1}$;*

(iii) *$\psi(\Phi u) > c_2$ for all $u \in K(\psi, c_2, c_4)$ with $\|\Phi u\| > c_3$.*

Then Φ has at least three fixed points u_1, u_2, u_3 in $\overline{K}_{c_4}$. Furthermore, $\|u_1\| \leq c_1 < \|u_2\|$ and $\psi(u_2) < c_2 < \psi(u_3)$.

2. Main Results

Throughout the rest of this paper, we assume that the following assumptions for system (3) hold.

(H0) For every $j \in \{1, 2, \ldots, n\}, 0 < 1 + a_j(k) \leq 1$ for all $k \in \mathbb{Z}$ and

$$\prod_{k=1}^{T} [1 + a_j(k)] \neq 1. \tag{10}$$

(H1) $k \to f(k, x_k)$ belongs to $l_T^\infty(\mathbb{Z}, \mathbb{R}_+^n)$ whenever $x \in l_T^\infty(\mathbb{Z}, \mathbb{R}_+^n)$.

(H2) For every $L > 0$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|f(k, \phi_k) - f(k, \psi_k)\| < \varepsilon, \quad k = 1, 2, \ldots, T, \tag{11}$$

for all $\phi, \psi \in l_T^\infty(\mathbb{Z}, \mathbb{R}_+^n)$ with $\|\phi\| \leq L, \|\psi\| \leq L$, and $\|\phi - \psi\| < \delta$.

Now, we define

$$G_j(k, s) = \frac{\prod_{m=s-T+1}^{k-1} [1 + a_j(m)]}{1 - \prod_{m=1}^{T} [1 + a_j(m)]}, \quad j = 1, 2, \ldots, n, \tag{12}$$

for $(k, s) \in \mathbb{Z} \times \mathbb{Z}$ with $k \leq s \leq k + T - 1$.

Then, by a proof similar to [8], we can transform (3) into the following equivalent equation:

$$x(k) = \sum_{s=k}^{k+T-1} G(k, s) f(s, x_s), \quad k \in \mathbb{Z}, \tag{13}$$

where

$$G(k, s) = \operatorname{diag}[G_1(k, s), G_2(k, s), \ldots, G_n(k, s)]. \tag{14}$$

It is easy to see that

$$G(k, s) = G(k + T, s + T) \tag{15}$$

for all $(k, s) \in \mathbb{Z} \times \mathbb{Z}$ with $k \leq s \leq k + T - 1$. In addition, it follows from (H0)–(H2) that, for every $j \in \{1, 2, \ldots, n\}, G_j(\cdot, \cdot)$ has a positive denominator, while the numerator is a positive and increasing function of $s \in [k, k + T - 1]$. Thus, for $(k, s) \in \mathbb{Z} \times \mathbb{Z}$ with $k \leq s \leq k + T - 1$, we have

$$\begin{aligned} \frac{\prod_{m=k-T+1}^{k-1} [1 + a_j(m)]}{1 - \prod_{m=1}^{T} [1 + a_j(m)]} &= G_j(k, k) \leq G_j(k, s), \\ G_j(k, s) \leq G_j(k, k + T - 1) &= \frac{1}{1 - \prod_{m=1}^{T} [1 + a_j(m)]}. \end{aligned} \tag{16}$$

Letting

$$p = \min_{1\le k\le T}\min_{1\le j\le n} G_j(k,k), \quad q = \max_{1\le k\le T}\max_{1\le j\le n} G_j(k,k+T-1), \tag{17}$$

we have

$$p \le G_j(k,s) \le q, \quad (k,s)\in\mathbb{Z}\times\mathbb{Z},\ k\le s\le k+T-1, \quad j=1,2,\ldots,n. \tag{18}$$

Next, we introduce a set

$$K = \left\{x\in l_T^\infty(\mathbb{Z},\mathbb{R}^n) : \min_{1\le k\le T} x_j(k) \ge \sigma\left\|x_j\right\|,\ j=1,2,\ldots,n\right\}, \tag{19}$$

where $\sigma = p/q$. It is not difficult to verify that K is a cone in $l_T^\infty(\mathbb{Z},\mathbb{R}^n)$. Finally, we define an operator Φ on K by

$$(\Phi x)(k) = \sum_{s=k}^{k+T-1} G(k,s) f(s,x_s), \quad x\in K,\ k\in\mathbb{Z}. \tag{20}$$

Lemma 3. *Φ is an operator from K to K.*

Proof. Let $x\in K$. By (H1) and $G(k,s)=G(k+T,s+T)$, we get

$$\begin{aligned}(\Phi x)(k+T) &= \sum_{s=k+T}^{k+2T-1} G(k+T,s) f(s,x_s)\\ &= \sum_{s=k}^{k+T-1} G(k+T,s+T) f(s+T,x_{s+T})\\ &= \sum_{s=k}^{k+T-1} G(k,s) f(s,x_s) = (\Phi x)(k),\end{aligned} \tag{21}$$

for all $k\in\mathbb{Z}$. So $\Phi x\in l_T^\infty(\mathbb{Z},\mathbb{R}^n)$.

In addition, for $j=1,2,\ldots,n$, we have

$$\left\|(\Phi x)_j\right\| = \max_{1\le k\le T}\sum_{s=k}^{k+T-1} G_j(k,s) f_j(s,x_s) \le q\sum_{s=1}^{T} f_j(s,x_s), \tag{22}$$

where f_j is the jth component of f. Then, we obtain

$$\begin{aligned}(\Phi x)_j(k) &= \sum_{s=k}^{k+T-1} G_j(k,s) f_j(s,x_s)\\ &\ge p\sum_{s=1}^{T} f_j(s,x_s) \ge \frac{p}{q}\left\|(\Phi x)_j\right\|\end{aligned} \tag{23}$$

for all $k\in\mathbb{Z}$. Thus, $\Phi x\in K$. This completes the proof. □

2.1. Existence of Two Positive Periodic Solutions of System (3). In this section, we apply Lemma 1 to establish an existence result about two positive periodic solutions of system (3). For convenience, we list some assumptions.

(H3) There exists a constant $c>0$ such that

$$p\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1} f_j(s,x_s) > c \quad \text{for } x\in K \text{ with } \|x\| = c. \tag{24}$$

(H4) There exists a constant $b>0$ such that

$$q\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1} f_j(s,x_s) < b \quad \text{for } x\in K \text{ with } \|x\| = b. \tag{25}$$

(H5) There exists a constant $a>0$ such that

$$p\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1} f_j(s,x_s) > a \quad \text{for } x\in K \text{ with } \|x\| = a. \tag{26}$$

Theorem 4. *Assume that there exist three constants a,b,c with $0<a<b<c$ such that (H0)–(H5) hold. Then system* (3) *has at least two positive T-periodic solutions.*

Proof. Firstly, by Lemma 3, Φ is an operator from K to K. Secondly, by a proof similar to [9, Lemma 2.5], one can show that $\Phi : K\to K$ is completely continuous.

Now, we begin to verify that all the assumptions of Lemma 1 hold. Let

$$\varphi(x) = \rho(x) = \alpha(x) = \|x\|, \quad x\in K. \tag{27}$$

It is clear that α, ρ, and φ are increasing, nonnegative, continuous functionals on K with $\rho(0)=0$. Moreover, we have

$$\|x\| \le \sigma^{-1}\varphi(x), \qquad \rho(\lambda x) = \lambda\rho(x), \tag{28}$$

for all $x\in K$ and $0\le\lambda\le 1$.

Next, we proceed to show that conditions (i)–(iii) of Lemma 1 are also satisfied. For every $x\in\partial K(\varphi,c)$, noting that $\|x\|=\varphi(x)=c$, by (H3), we conclude that

$$\begin{aligned}\varphi(\Phi x) &= \max_{1\le k\le T}\max_{1\le j\le n}\sum_{s=k}^{k+T-1} G_j(k,s) f_j(s,x_s)\\ &\ge p\cdot\max_{1\le k\le T}\max_{1\le j\le n}\sum_{s=k}^{k+T-1} f_j(s,x_s)\\ &= p\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1} f_j(s,x_s) > c;\end{aligned} \tag{29}$$

that is, condition (i) of Lemma 1 holds. For every $x \in \partial K(\rho, b)$, since $\|x\| = \rho(x) = b$, by (H4), we get

$$\begin{aligned}\rho(\Phi x) &= \max_{1\le k\le T}\max_{1\le j\le n}\sum_{s=k}^{k+T-1} G_j(k,s) f_j(s,x_s)\\ &\le q\cdot\max_{1\le k\le T}\max_{1\le j\le n}\sum_{s=k}^{k+T-1} f_j(s,x_s)\\ &= q\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1} f_j(s,x_s) < b;\end{aligned} \tag{30}$$

that is, condition (ii) of Lemma 1 holds. Finally, it is easy to see that

$$K(\alpha, a) = \{x \in K : \|x\| < a\} \neq \emptyset, \tag{31}$$

and for every $x \in K(\alpha, a)$, it follows from (H5) that

$$\alpha(\Phi x) \ge p\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1} f_j(s,x_s) > a. \tag{32}$$

Thus, condition (iii) of Lemma 1 holds.

Now, by applying Lemma 1, there exist two fixed points $u_1, u_2 \in \overline{K(\varphi, c)}$, which are just two T-periodic solutions to system (3). This completes the proof. □

Remark 5. In Theorem 4, the two T-periodic solutions u_1, u_2 do not equal zero. In fact, according to Lemma 1, we have

$$a < \|u_1\| < b < \|u_2\| < c. \tag{33}$$

Corollary 6. *Assume that (H0)–(H2) and (H4) hold. Moreover,*

$$\limsup_{\|x\|\to+\infty, x\in K}\max_{1\le j\le n}\sum_{s=0}^{T-1}\frac{f_j(s,x_s)}{\|x\|} > \frac{1}{p}, \tag{34}$$

$$\limsup_{\|x\|\to 0^+, x\in K}\max_{1\le j\le n}\sum_{s=0}^{T-1}\frac{f_j(s,x_s)}{\|x\|} > \frac{1}{p}. \tag{35}$$

Then system (3) *has at least two positive T-periodic solutions.*

Proof. By (34), there exists a constant $c > b$ such that (H3) holds. By (35), there exists a constant $a \in (0, b)$ such that (H5) holds. Then, by applying Theorem 4, we complete the proof. □

Next, we present a simple example, which does not aim at generality but illustrates how to use our existence theorem.

Example 7. Consider the following system:

$$\begin{aligned}x_1(k+1) - x_1(k) &= -\frac{1}{2}\left|\sin\frac{\pi k}{2}\right| x_1(k) + f_1(k, x_k),\\ x_2(k+1) - x_2(k) &= -\frac{1}{2}\left|\cos\frac{\pi k}{2}\right| x_2(k) + f_2(k, x_k),\end{aligned} \tag{36}$$

where

$$\begin{aligned}f_1(k,x_k) &= f_2(k,x_k)\\ &= \frac{4[x_1(k)+x_2(k)]\exp((1/384)[x_1(k)+x_2(k)])}{1+x_1(k)+x_2(k)}.\end{aligned} \tag{37}$$

We have $n = T = 2$,

$$\begin{aligned}&a_1(k) = -\frac{1}{2}\left|\sin\frac{\pi k}{2}\right|, \qquad a_2(k) = -\frac{1}{2}\left|\cos\frac{\pi k}{2}\right|,\\ &G_1(k,s) = \frac{\prod_{m=s-1}^{k-1}[1+a_1(m)]}{1-\prod_{m=1}^{2}[1+a_1(m)]} = 2\prod_{m=s-1}^{k-1}\left[1-\frac{1}{2}\left|\sin\frac{\pi m}{2}\right|\right],\\ &G_2(k,s) = \frac{\prod_{m=s-1}^{k-1}[1+a_2(m)]}{1-\prod_{m=1}^{2}[1+a_2(m)]} = 2\prod_{m=s-1}^{k-1}\left[1-\frac{1}{2}\left|\cos\frac{\pi m}{2}\right|\right],\\ &p = \min_{k\in\mathbb{Z}}\min_{1\le j\le 2}\frac{1+a_j(k-1)}{1-\prod_{m=1}^{2}[1+a_j(m)]} = 1,\\ &q = \max_{k\in\mathbb{Z}}\max_{1\le j\le 2}\frac{1}{1-\prod_{m=1}^{2}[1+a_j(m)]} = 2,\\ &\sigma = \frac{p}{q} = \frac{1}{2},\\ &K = \left\{x \in l_2^\infty(\mathbb{Z}, \mathbb{R}_+^2) : \min_{k\in\mathbb{Z}} x_j(k) \ge \frac{1}{2}\|x_j\|, j = 1, 2\right\}.\end{aligned} \tag{38}$$

It is easy to verify that conditions (H0)–(H2) hold. Since, for $x \in K$,

$$\begin{aligned}&\max_{1\le j\le 2}\sum_{s=0}^{1}\frac{f_j(s,x_s)}{\|x\|}\\ &\quad= \frac{4[x_1(0)+x_2(0)]\exp((1/384)[x_1(0)+x_2(0)])}{\|x\|[1+x_1(0)+x_2(0)]}\\ &\qquad+ \frac{4[x_1(1)+x_2(1)]\exp((1/384)[x_1(1)+x_2(1)])}{\|x\|[1+x_1(1)+x_2(1)]}\\ &\quad\ge \frac{4\exp((1/384)\|x\|)}{1+2\|x\|},\end{aligned}$$

$$\lim_{\|x\|\to+\infty}\frac{4\exp((1/384)\|x\|)}{1+2\|x\|}=+\infty,$$
$$\lim_{\|x\|\to 0^+}\frac{4\exp((1/384)\|x\|)}{1+2\|x\|}=4>1, \tag{39}$$

we conclude that (34) and (35) are satisfied. It remains to verify (H4). Letting $b=192$, for all $x\in K$ with $\|x\|=192$, we have

$$\begin{aligned}2\max_{1\le j\le 2}\sum_{s=0}^{1}f_j(s,x_s) &= \frac{8[x_1(0)+x_2(0)]\exp((1/384)[x_1(0)+x_2(0)])}{1+x_1(0)+x_2(0)} \\ &\quad+\frac{8[x_1(1)+x_2(1)]\exp((1/384)[x_1(1)+x_2(1)])}{1+x_1(1)+x_2(1)} \\ &\le 16\exp\left(\frac{1}{192}\|x\|\right)=16e<b,\end{aligned} \tag{40}$$

which means that (H4) holds. Therefore, by Corollary 6, we know that system (36) has at least two positive 2-periodic solutions.

Remark 8. In the above example, 0 is obviously a trivial periodic solution for system (36). But by Remark 5, we know that the two positive 2-periodic solutions do not equal zero.

2.2. Existence of Three Nonnegative Periodic Solutions of System (3). In [6], Dix et al. investigated the existence of multiple nonnegative periodic solutions for a first order functional difference equation by the Leggett-Williams fixed point theorem. In this section, we will investigate the existence of multiple nonnegative periodic solutions for system (3) by using an idea similar to that of [6]. For convenience, we also list some assumptions.

(H6) There exists a constant $c_1>0$ such that

$$q\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1}f_j(s,x_s)<c_1 \quad \text{for } x\in K \text{ with } \|x\|\le c_1. \tag{41}$$

(H7) There exists a constant $c_2>c_1>0$ such that

$$\frac{p}{n}\cdot\sum_{j=1}^{n}\sum_{s=0}^{T-1}f_j(s,x_s)>c_2 \quad \text{for } x\in K \text{ with } c_2\le\|x\|<\frac{nc_2}{\sigma}. \tag{42}$$

(H8) There exists a constant $c_4>nc_2/\sigma:=c_3$ such that

$$q\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1}f_j(s,x_s)\le c_4 \quad \text{for } x\in K \text{ with } \|x\|\le c_4. \tag{43}$$

Theorem 9. *Assume that (H0)–(H2) and (H6)–(H8) hold. Then system* (3) *has at least three nonnegative T-periodic solutions.*

Proof. By the proof of Theorem 4, we know that Φ is an operator from K to K and completely continuous. Let

$$\psi(x)=\min_{1\le k\le T}\frac{\sum_{j=1}^{n}x_j(k)}{n}, \quad x\in K. \tag{44}$$

It is easy to see that ψ is a concave nonnegative continuous functional on K and $\psi(x)\le\|x\|$.

Firstly, we show that Φ maps $\overline{K}_{c_4}$ into $\overline{K}_{c_4}$. For every $x\in\overline{K}_{c_4}$, we have $\|x\|\le c_4$. Combining this with (H8), we get

$$\begin{aligned}\|\Phi x\| &= \max_{k\in\mathbb{Z}}\max_{1\le j\le n}\sum_{s=k}^{k+T-1}G_j(k,s)f_j(s,x_s) \\ &\le q\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1}f_j(s,x_s)\le c_4.\end{aligned} \tag{45}$$

Secondly, let us verify condition (i) of Lemma 2. Since $\sigma<1$, $c_3>c_2$, then, it is easy to see that the set

$$\{x\in K(\psi,c_2,c_3):\psi(x)>c_2\}\ne\emptyset. \tag{46}$$

In addition, for every $x\in K(\psi,c_2,c_3)$, we have $c_2\le\psi(x)\le\|x\|<c_3=nc_2/\sigma$. Then, by (H7), we have

$$\begin{aligned}\psi(\Phi x) &= \frac{1}{n}\cdot\min_{1\le k\le T}\sum_{j=1}^{n}\sum_{s=k}^{k+T-1}G_j(k,s)f_j(s,x_s) \\ &\ge\frac{p}{n}\cdot\sum_{j=1}^{n}\sum_{s=0}^{T-1}f_j(s,x_s)>c_2\end{aligned} \tag{47}$$

which means that condition (i) of Lemma 2 holds.

Thirdly, for every $x\in\overline{K}_{c_1}$, since $\|x\|\le c_1$, it follows from (H6) that

$$\|\Phi x\|\le q\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1}f_j(s,x_s)<c_1; \tag{48}$$

that is, condition (ii) of Lemma 2 holds.

Finally, for every $x\in K(\psi,c_2,c_4)$ with $\|\Phi x\|>c_3$, we have $c_2\le\|x\|<c_4$ and

$$q\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1}f_j(s,x_s)\ge\|\Phi x\|>c_3, \tag{49}$$

which yields that

$$\sum_{j=1}^{n}\sum_{s=0}^{T-1}f_j(s,x_s)\ge\max_{1\le j\le n}\sum_{s=0}^{T-1}f_j(s,x_s)>\frac{c_3}{q}=\frac{nc_2}{p}. \tag{50}$$

Then, we have

$$\psi(\Phi x)\ge\frac{p}{n}\cdot\sum_{j=1}^{n}\sum_{s=0}^{T-1}f_j(s,x_s)>c_2; \tag{51}$$

that is, condition (iii) of Lemma 2 holds.

Now, by Lemma 2, we know that Φ has at least three fixed points in $\overline{K}_{c_4}$, and thus system (3) has at least three nonnegative T-periodic solutions. □

Corollary 10. *Assume that (H0)–(H2) hold and there exists a constant $c_2 > 0$ such that*

$$\frac{p}{n}\cdot\sum_{j=1}^{n}\sum_{s=0}^{T-1} f_j(s,x_s) > c_2 \quad \text{for} \quad c_2 \le \|x\| < \frac{nc_2}{\sigma}. \tag{52}$$

Moreover, there hold

$$\limsup_{\|x\|\to+\infty, x\in K} \max_{1\le j\le n}\sum_{s=0}^{T-1}\frac{f_j(s,x_s)}{\|x\|} < \frac{1}{q},$$
$$\limsup_{\|x\|\to 0^+, x\in K} \max_{1\le j\le n}\sum_{s=0}^{T-1}\frac{f_j(s,x_s)}{\|x\|} < \frac{1}{q}. \tag{53}$$

Then system (3) *has at least three nonnegative T-periodic solutions.*

Proof. We only need to verify that (H6) and (H8) hold. Let

$$\alpha = q\cdot \limsup_{\|x\|\to 0^+, x\in K} \max_{1\le j\le n}\sum_{s=0}^{T-1}\frac{f_j(s,x_s)}{\|x\|},$$
$$\beta = q\cdot \limsup_{\|x\|\to+\infty, x\in K} \max_{1\le j\le n}\sum_{s=0}^{T-1}\frac{f_j(s,x_s)}{\|x\|}. \tag{54}$$

Then $\alpha,\beta\in[0,1)$. There exists a constant $\delta\in(0,c_2)$ such that, for all $x\in K$ with $\|x\|\le\delta$, there holds

$$q\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1} f_j(s,x_s) < \frac{\alpha+1}{2}\|x\|. \tag{55}$$

Taking $c_1=\delta$, (H6) holds. In addition, there exists a constant $M > nc_2/\sigma$ such that, for all $x\in K$ with $\|x\|\ge M$, there holds

$$q\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1} f_j(s,x_s) < \frac{\beta+1}{2}\|x\|. \tag{56}$$

Taking

$$c_4 = M + \sup_{x\in K,\|x\|\le M}\left[q\cdot\max_{1\le j\le n}\sum_{s=0}^{T-1} f_j(s,x_s)\right], \tag{57}$$

(H8) holds. □

Next, we also provide a simple example to illustrate our existence theorem.

Example 11. Let $n=T=2$, a_1,a_2 be the same as in Example 7, and

$$f_1(k,x_k) = f_2(k,x_k) = \frac{64[x_1(k)+x_2(k)]^2}{1+[x_1(k)+x_2(k)+x_1(k+1)+x_2(k+1)]^4}. \tag{58}$$

By Example 7, we have $p=1, q=2,$ and $\sigma=1/2$, and (H0)–(H2) hold.

By a direct calculation, we get

$$\max_{1\le j\le 2}\sum_{s=0}^{1}\frac{f_j(s,x_s)}{\|x\|} = \frac{64[x_1(0)+x_2(0)]^2+64[x_1(1)+x_2(1)]^2}{\|x\|\cdot[1+(x_1(0)+x_2(0)+x_1(1)+x_2(1))^4]} \le \frac{512\|x\|}{1+\|x\|^4}. \tag{59}$$

Then, it is easy to see that (53) holds.

Let $c_2 = 1/16$. Then, for all

$$\frac{1}{16}\le\|x\|\le\frac{1}{4}=\frac{nc_2}{\sigma}, \tag{60}$$

we have

$$\frac{p}{n}\cdot\sum_{j=1}^{2}\sum_{s=0}^{1} f_j(s,x_s) = \frac{64[x_1(0)+x_2(0)]^2+64[x_1(1)+x_2(1)]^2}{1+[x_1(0)+x_2(0)+x_1(1)+x_2(1)]^4} \ge \frac{64\|x\|^2}{2} = 32\|x\|^2 \ge \frac{1}{8} > c_2. \tag{61}$$

Thus, all the assumptions of Corollary 10 hold. Then, we know that the considered functional difference system has at least three nonnegative 2-periodic solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

Hui-Sheng Ding acknowledges support from the NSF of China (11101192), the Program for Cultivating Young Scientist of Jiangxi Province (20133BCB23009), and the NSF of Jiangxi Province.

References

[1] B. Liu, "The existence and uniqueness of positive periodic solutions of Nicholson-type delay systems," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3145–3151, 2011.

[2] B. Liu and S. Gong, "Periodic solution for impulsive cellar neural networks with time-varying delays in the leakage terms," *Abstract and Applied Analysis*, vol. 2013, Article ID 701087, 10 pages, 2013.

[3] J. O. Alzabut and C. Tunç, "Existence of periodic solutions for Rayleigh equations with state-dependent delay," *Electronic Journal of Differential Equations*, vol. 77, pp. 1–8, 2012.

[4] J. O. Alzabut, "Existence of periodic solutions for a type of linear difference equations with distributed delay," *Advances in Difference Equations*, p. 2012, article 53, 2012.

[5] H.-S. Ding and J. G. Dix, "Multiple Periodic Solutions for Discrete Nicholson's Blowflies Type System," *Abstract and Applied Analysis*, vol. 2014, Article ID 659152, 6 pages, 2014.

[6] J. G. Dix, S. Padhi, and S. Pati, "Multiple positive periodic solutions for a nonlinear first order functional difference equation," *Journal of Difference Equations and Applications*, vol. 16, no. 9, pp. 1037–1046, 2010.

[7] W. Long, X. J. Zheng, and L. Li, "Existence of periodic solutions for a class of functional integral equations," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 57, pp. 1–11, 2012.

[8] Y. N. Raffoul, "Positive periodic solutions of nonlinear functional difference equations," *Electronic Journal of Differential Equations*, vol. 55, pp. 1–8, 2002.

[9] Y. N. Raffoul and C. C. Tisdell, "Positive periodic solutions of functional discrete systems and population models," *Advances in Difference Equations*, no. 3, pp. 369–380, 2005.

[10] E. Braverman and S. H. Saker, "Permanence, oscillation and attractivity of the discrete hematopoiesis model with variable coefficients," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 10, pp. 2955–2965, 2007.

[11] E. Braverman and S. H. Saker, "Periodic solutions and global attractivity of a discrete delay host macroparasite model," *Journal of Difference Equations and Applications*, vol. 16, no. 7, pp. 789–806, 2010.

[12] E. Braverman and S. H. Saker, "On the Cushing-Henson conjecture, delay difference equations and attenuant cycles," *Journal of Difference Equations and Applications*, vol. 14, no. 3, pp. 275–286, 2008.

[13] E. Braverman and S. H. Saker, "On a difference equation with exponentially decreasing nonlinearity," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 147926, 17 pages, 2011.

[14] E. M. Elabbasy and S. H. Saker, "Periodic solutions and oscillation of discrete non-linear delay population dynamics model with external force," *IMA Journal of Applied Mathematics*, vol. 70, no. 6, pp. 753–767, 2005.

[15] S. H. Saker, "Qualitative analysis of discrete nonlinear delay survival red blood cells model," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 2, pp. 471–489, 2008.

[16] S. H. Saker, "Periodic solutions, oscillation and attractivity of discrete nonlinear delay population model," *Mathematical and Computer Modelling*, vol. 47, no. 3-4, pp. 278–297, 2008.

[17] Z. C. Hao, T. J. Xiao, and J. Liang, "Multiple positive periodic solutions for delay differential system," *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 239209, 18 pages, 2009.

[18] R. I. Avery and J. Henderson, "Two positive fixed points of nonlinear operators on ordered Banach spaces," *Communications on Applied Nonlinear Analysis*, vol. 8, no. 1, pp. 27–36, 2001.

[19] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 673–688, 1979.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 618434, 11 pages
http://dx.doi.org/10.1155/2014/618434

Research Article

Existence and Characterization of Solutions of Nonlinear Volterra-Stieltjes Integral Equations in Two Variables

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Received 7 February 2014; Accepted 29 April 2014; Published 18 May 2014

Academic Editor: Cemil Tunç

The paper is devoted mainly to the study of the existence of solutions depending on two variables of a nonlinear integral equation of Volterra-Stieltjes type. The basic tool used in investigations is the technique of measures of noncompactness and Darbo's fixed point theorem. The results obtained in the paper are applicable, in a particular case, to the nonlinear partial integral equations of fractional orders.

1. Introduction

The theory of differential and integral equations of fractional order creates nowadays a large subject of mathematics which found in the last three decades numerous applications in physics, mechanics, engineering, bioengineering, viscoelasticity, electrochemistry, control theory, porous media, and other fields connected with real world problems [1–5]. Let us mention that recently there have appeared a few important and expository monographs covering both the theory and applications of differential and integral equations of fractional order (cf. [1, 4, 6–8]).

It turns out that a lot of results of the theory of differential and integral equations of fractional order can be considered from a unified point of view with help of the theory of the so-called Volterra-Stieltjes integral equations (cf. [9, 10]). The approach applied in those papers allows us not only to consider the mentioned theories of differential and integral equations of fractional order from one point of view but also to obtain deeper results with help of less complicated tools of nonlinear analysis.

Such an approach can be also applied to investigations associated with the theory of differential and integral equations of fractional orders in two variables. That subject of nonlinear analysis was recently studied in a few papers [11–16]. It seems that results obtained in those papers are not sufficiently general. The tools and methods associated with the theory of nonlinear Volterra-Stieltjes integral equations which will be applied in this paper are more convenient and allow us to obtain more applicable results. Indeed, further on we obtain, as particular cases, the existence theorems concerning both nonlinear integral equations of fractional orders, nonlinear integral equations of Volterra-Chandrasekhar type, and nonlinear equations of mixed type. Moreover, we indicate possible generalizations of our results to the situation of nonlinear Volterra-Stieltjes integral equations in n variables. Additionally, we indicate also a few open problems appearing in our theory.

2. Notation, Definitions, and Auxiliary Results

This section is devoted to provide the notation, definitions, and other auxiliary facts which will be needed in our further study.

At the beginning let us assume that f is a real function defined on the interval $[a,b]$. Then the symbol $\bigvee_a^b f$ will denote the variation of the function f on the interval $[a,b]$. In the case when $\bigvee_a^b f < \infty$ we say that f is of bounded variation on $[a,b]$. If we have a function $u(t,x) = u : [a,b] \times [c,d] \to \mathbb{R}$, then we denote by $\bigvee_{t=p}^q u(t,x)$ the variation of the function

$t \to u(t,x)$ on the interval $[p,q] \subset [a,b]$. Similarly we define the quantity $\bigvee_{x=p}^{q} u(t,x)$.

For the properties of functions of bounded variation we refer to [17].

If f and φ are two real functions defined on the interval $[a,b]$, then under some additional conditions [17, 18] we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b f(t)\, d\varphi(t) \tag{1}$$

of the function f with respect to the function φ. In this case we say that f is Stieltjes integrable on the interval $[a,b]$ with respect to φ.

It is worthwhile mentioning that several conditions ensuring Stieltjes integrability may be found in [17]. One of the most frequently used requires f to be continuous and φ to be of bounded variation on $[a,b]$.

In the sequel we will utilize a few properties of the Stieltjes integral contained in the below quoted lemmas (cf. [17]).

Lemma 1. *If f is Stieltjes integrable on the interval $[a,b]$ with respect to a function φ of bounded variation, then*

$$\left| \int_a^b f(t)\, d\varphi(t) \right| \le \int_a^b |f(t)|\, d\left(\bigvee_a^t \varphi \right). \tag{2}$$

Lemma 2. *Let f_1, f_2 be Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function φ such that $f_1(t) \le f_2(t)$ for $t \in [a,b]$. Then*

$$\int_a^b f_1(t)\, d\varphi(t) \le \int_a^b f_2(t)\, d\varphi(t). \tag{3}$$

In what follows we will also consider Stieltjes integrals having the form

$$\int_a^b f(t)\, d_s g(t,s), \tag{4}$$

where $g : [a,b] \times [a,b] \to \mathbb{R}$ and the symbol d_s indicates the integration with respect to the variable s. The details concerning the integral of this type will be given later.

Even more, in our considerations we will use the double Stieltjes integrals of the form

$$\int_c^d \int_c^d f(t,x)\, d_y g_2(x,y)\, d_s g_1(t,s), \tag{5}$$

where $g_i : [a,b] \times [c,d] \to \mathbb{R}$ $(i = 1,2)$. Obviously, the double Stieltjes integral (5) is understood as the following double iterated Stieltjes integral:

$$\int_c^d \left(\int_c^d f(t,x)\, d_y g_2(x,y) \right) d_s g_1(t,s). \tag{6}$$

Now, let us assume that $u = u(t,x)$ is a real function defined on the Cartesian product $[a,b] \times [c,d]$. Denote by $\omega(u,\varepsilon)$ the modulus of continuity of the function u; that is,

$$\omega(u,\varepsilon) = \sup \{ |u(t,x) - u(s,y)| : t,s \in [a,b],\, x,y \in [c,d],\ |t-s| \le \varepsilon,\, |x-y| \le \varepsilon \}. \tag{7}$$

Obviously, we can also consider the modulus of continuity of the function $u(t,x)$ with respect to each variable separately. For example,

$$\omega(u(t,\cdot),\varepsilon) = \sup \{ |u(t,x) - u(t,y)| : x,y \in [c,d],\ |x-y| \le \varepsilon \}, \tag{8}$$

where t is a fixed number in the interval $[a,b]$.

Further on, in order to simplify our investigations, we will always assume that $[a,b] = [c,d] = [0,1]$ and we will denote by I the unit interval $[0,1]$; that is, $I = [0,1]$.

Now, we recall some facts concerning measures of noncompactness, which will be applied in the sequel. To this end assume that E is an infinite dimensional Banach space with the norm $\|\cdot\|$ and zero element θ. Denote by $B(x,r)$ the closed ball centered at x and radius r. The symbol B_r stands for the ball $B(\theta,r)$.

Next, for a given nonempty bounded subset X of E, we denote by $\chi(X)$ the so-called Hausdorff measure of noncompactness of the set X [19]. This quantity is defined by the formula

$$\chi(X) = \inf \{ \varepsilon > 0 : X \text{ has a finite } \varepsilon - \text{net in } E \}. \tag{9}$$

Let us mention that the function χ has several useful properties and is often applied in nonlinear analysis, operator theory, and the theories of differential and integral equations [19, 20].

Notice that the concept of a measure of noncompactness may be defined in a more general way [19, 21], but for our purposes the Hausdorff measure of noncompactness defined by (9) will be thoroughly sufficient.

In fact, in our further considerations, we will work in the space $C(I^2)$ consisting of all functions $u = u(t,x)$ defined and continuous on the Cartesian product $I^2 = I \times I$ with real values. The space $C = C(I^2)$ will be furnished with the standard maximum norm

$$\|u\|_C = \max \left\{ |u(t,x)| : (t,x) \in I^2 \right\}. \tag{10}$$

It can be shown [19] that if U is a nonempty and bounded subset of $C(I^2)$, then the Hausdorff measure of noncompactness of U can be expressed by the following formula:

$$\chi(U) = \frac{1}{2}\omega_0(U), \tag{11}$$

where

$$\omega_0(U) = \lim_{\varepsilon \to 0} \omega(U,\varepsilon). \tag{12}$$

The symbol $\omega(U,\varepsilon)$ used above denotes the modulus of continuity of the set U and is defined as follows:

$$\omega(U,\varepsilon) = \sup \{ \omega(u,\varepsilon) : u \in U \}, \tag{13}$$

while $\omega(u,\varepsilon)$ stands for the modulus of continuity of the function u defined by (7).

Now we recall a fixed point theorem of Darbo type which will be utilized in our investigations (cf. [19]).

Theorem 3. *Let Ω be a nonempty, bounded, closed, and convex subset of the Banach space E and let $Q : \Omega \to \Omega$ be a continuous operator such that there exists a constant $k \in [0, 1)$ for which $\chi(QU) \le k\chi(U)$ provided U is an arbitrary nonempty subset of Ω. Then Q has at least one fixed point in the set Ω.*

Next we recall a few facts concerning the so-called superposition operator [19]. To this end assume that $D = I^2$, where $I = [0, 1]$. Let $f : D \times \mathbb{R} \to \mathbb{R}$ be a given function. Then, to every function u acting from I^2 into $\mathbb{R}$ we may assign the function Fu defined by the formula

$$(Fu)(t, x) = f(t, x, u(t, x)), \tag{14}$$

for $(t, x) \in I^2$. The operator F defined in such a way is called *the superposition operator* generated by the function $f = f(t, x, u)$.

The properties of the superposition operator may be found in [22]. For our further purposes we will only need the below quoted result concerning the behaviour of the superposition operator F in the space $C(I^2)$.

Lemma 4. *The superposition operator F generated by the function $f : I^2 \times \mathbb{R} \to \mathbb{R}$ transforms the space $C(I^2)$ into itself and is continuous if and only if the function f is continuous on the set $I^2 \times \mathbb{R}$.*

Remark 5. Let us notice that in our considerations concerning the superposition operator F generated by the function f we may replace the set $D = I^2$ by an arbitrary Cartesian product $D = I^2$ with $I = [a, b]$ or even by $D = [a, b] \times [c, d]$ (cf. [22] for further possible generalizations).

Finally, we recall some fundamental facts associated with fractional calculus (cf. [7, 8, 23]). To this end denote by $L^1(a, b)$ the space of all real functions defined and Lebesgue integrable on the interval (a, b). The space $L^1(a, b)$ is equipped with the standard norm. Further, fix a number $\alpha > 0$ and take an arbitrary function $u \in L^1(a, b)$. *The Riemann-Liouville fractional integral of order α* of the function $u = u(t)$ is defined by the formula

$$I^{-\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{u(s)}{(t-s)^{1-\alpha}} ds, \quad t \in (a, b), \tag{15}$$

where $\Gamma(\alpha)$ denotes the gamma function.

It may be shown that the fractional integral operator $I^{-\alpha}$ transforms the space $L^1(a, b)$ into itself and has some additional properties [7, 8, 23].

3. Main Result

Investigations of this paper are connected mainly with the solvability of the following nonlinear quadratic integral equation of Volterra-Stieltjes type having the form

$$u(t, x) = h(t, x) + f(t, x, u(t, x)) \times \int_0^t \int_0^x v(t, s, x, y, u(s, y))\, d_y g_2(x, y)\, d_s g_1(t, s), \tag{16}$$

for $(t, x) \in I^2$, where $I = [0, 1]$.

Let us recall that details concerning the notation used in (16) were presented in the previous section.

In order to formulate the assumptions under which (16) will be investigated, let us denote by Δ_i $(i = 1, 2)$ the following triangles:

$$\Delta_1 = \{(t, s) : 0 \le s \le t \le 1\}, \quad \Delta_2 = \{(x, y) : 0 \le y \le x \le 1\}. \tag{17}$$

We will study (16) assuming the following hypotheses:

(i) $h \in C(I^2)$;

(ii) the function $f(t, x, u) = f : I^2 \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition with respect to the variable u; that is, there exists a constant $k > 0$ such that

$$|f(t, x, u) - f(t, x, w)| \le k|u - w| \tag{18}$$

for all $t, x \in I$ and $u, w \in \mathbb{R}$;

(iii) the function $g_i(w, z) = g_i : \Delta_i \to \mathbb{R}$ is continuous on the triangle Δ_i for $i = 1, 2$;

(iv) the function $z \to g_i(w, z)$ is of bounded variation on the interval $[0, w]$ for each fixed $w \in I$ $(i = 1, 2)$;

(v) for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $w_1, w_2 \in I$, $w_1 < w_2$, and $w_2 - w_1 \le \delta$, the following inequality is satisfied:

$$\bigvee_{z=0}^{w_1} [g_i(w_2, z) - g_i(w_1, z)] \le \varepsilon \tag{19}$$

for $i = 1, 2$;

(vi) $g_i(w, 0) = 0$ for each $w \in I$ $(i = 1, 2)$;

(vii) $v : \Delta_1 \times \Delta_2 \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$|v(t, s, x, y, u)| \le \phi(|u|) \tag{20}$$

for all $(t, s) \in \Delta_1$, $(x, y) \in \Delta_2$ and for each $u \in \mathbb{R}$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function.

Before formulating further assumptions concerning (16) we provide a few lemmas proved in [9] which will be utilized in our investigations.

Lemma 6. *The function*

$$p \longrightarrow \bigvee_{z=0}^{p} g_i(w, z) \tag{21}$$

is continuous on the interval $[0, w]$ for any $w \in I$ $(i = 1, 2)$.

Lemma 7. *Let assumptions (iii)–(v) be satisfied. Then, for arbitrarily fixed number $w_2 \in I$ ($w_2 > 0$) and for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $w_1 \in I$, $w_1 < w_2$, and $w_2 - w_1 \leq \delta$, then*

$$\bigvee_{z=w_1}^{w_2} g_i(w_2, z) \leq \varepsilon \tag{22}$$

($i = 1, 2$).

Lemma 8. *Under assumptions (iii)–(v) the function*

$$w \longrightarrow \bigvee_{z=0}^{w} g_i(w, z) \tag{23}$$

is continuous on the interval I ($i = 1, 2$).

As an immediate consequence of the above lemma we derive the following corollary.

Corollary 9. *There exists a finite positive constant K_i such that*

$$K_i = \sup\left\{\bigvee_{z=0}^{w} g_i(w, z) : w \in I\right\} \tag{24}$$

($i = 1, 2$).

In what follows let us denote by F_1 the constant defined by the formula

$$F_1 = \max\{|f(t, x, 0)| : t, x \in I\}. \tag{25}$$

Obviously, in view of assumption (ii) we have that $F_1 < \infty$.

Now, we can formulate the last assumption used further on.

(viii) There exists a positive solution r_0 of the inequality

$$\|h\|_C + (kr + F_1) K_1 K_2 \phi(r) \leq r \tag{26}$$

such that

$$kK_1K_2\phi(r_0) < 1. \tag{27}$$

Our main result is contained in the following theorem.

Theorem 10. *Under assumptions (i)–(viii) there exists at least one solution $u = u(t, x)$ of (16) in the space $C = C(I^2)$.*

Proof. We start with the following notation:

$$M_i(\varepsilon) = \sup\left\{\bigvee_{z=0}^{w_1} [g_i(w_2, z) - g_i(w_1, z)] : w_1, w_2 \in I, w_1 < w_2, w_2 - w_1 \leq \varepsilon\right\} \tag{28}$$

for $i = 1, 2$. Notice that in view of assumption (v) we infer that $M_i(\varepsilon) \to 0$ as $\varepsilon \to 0$ for $i = 1, 2$.

Moreover, for further purposes let us define the function $N_i = N_i(\varepsilon)$ ($i = 1, 2$) by putting

$$N_i(\varepsilon) = \sup\left\{\bigvee_{z=w_1}^{w_2} g_i(w_2, z) : w_1, w_2 \in I, w_1 < w_2, w_2 - w_1 \leq \varepsilon\right\}. \tag{29}$$

Observe that in virtue of Lemma 7 we have that $N_i(\varepsilon) \to 0$ as $\varepsilon \to 0$ ($i = 1, 2$).

Further, for a fixed function $u \in C$ and $t, x \in I$, let us denote

$$\begin{aligned}
(Fu)(t, x) &= f(t, x, u(t, x)), \\
(Vu)(t, x) &= \int_0^t \int_0^x v(t, s, x, y, u(s, y))\, d_y g_2(x, y)\, d_s g_1(t, s), \\
(Qu)(t, x) &= h(t, x) + (Fu)(t, x)(Vu)(t, x).
\end{aligned} \tag{30}$$

Next, fix arbitrarily $\varepsilon > 0$ and choose $t_1, t_2, x_1, x_2 \in I$ such that $t_1 \leq t_2$, $x_1 \leq x_2$, $t_2 - t_1 \leq \varepsilon$, $x_2 - x_1 \leq \varepsilon$. Then, keeping in mind our assumptions, for a fixed function $u \in C(I^2)$, we get

$$\begin{aligned}
&|(Vu)(t_2, x_2) - (Vu)(t_1, x_1)| \\
&\leq \left| \int_0^{t_2} \int_0^{x_2} v(t_2, s, x_2, y, u(s, y))\, d_y g_2(x_2, y)\, d_s g_1(t_2, s) \right. \\
&\quad \left. - \int_0^{t_1} \int_0^{x_1} v(t_1, s, x_1, y, u(s, y))\, d_y g_2(x_1, y)\, d_s g_1(t_1, s) \right| \\
&\leq \left| \int_0^{t_2} \int_0^{x_2} v(t_2, s, x_2, y, u(s, y))\, d_y g_2(x_2, y)\, d_s g_1(t_2, g) \right. \\
&\quad \left. - \int_0^{t_1} \int_0^{x_2} v(t_2, s, x_2, y, u(s, y))\, d_y g_2(x_2, y)\, d_s g_1(t_2, s) \right| \\
&\quad + \left| \int_0^{t_1} \int_0^{x_2} v(t_2, s, x_2, y, u(s, y))\, d_y g_2(x_2, y)\, d_s g_1(t_2, s) \right. \\
&\quad \left. - \int_0^{t_1} \int_0^{x_2} v(t_1, s, x_1, y, u(s, y))\, d_y g_2(x_2, y)\, d_s g_1(t_2, s) \right| \\
&\quad + \left| \int_0^{t_1} \int_0^{x_2} v(t_1, s, x_1, y, u(s, y))\, d_y g_2(x_2, y)\, d_s g_1(t_2, s) \right. \\
&\quad \left. - \int_0^{t_1} \int_0^{x_1} v(t_1, s, x_1, y, u(s, y))\, d_y g_2(x_2, y)\, d_s g_1(t_2, s) \right| \\
&\quad + \left| \int_0^{t_1} \int_0^{x_1} v(t_1, s, x_1, y, u(s, y))\, d_y g_2(x_2, y)\, d_s g_1(t_2, s) \right.
\end{aligned}$$

$$- \int_0^{t_1}\int_0^{x_1} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_1,y)\, d_s g_1(t_2,s)\Bigg| + \Bigg|\int_0^{t_1}\int_0^{x_1} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_1,y)\, d_s g_1(t_2,s) - \int_0^{t_1}\int_0^{x_1} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_1,y)\, d_s g_1(t_1,s)\Bigg|. \tag{31}$$

Further, estimating step by step the terms occurring on the right-hand side of inequality (31), on the basis of Lemmas 1 and 2, we obtain

$$\begin{aligned}
&\Bigg|\int_0^{t_2}\int_0^{x_2} v(t_2,s,x_2,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) - \int_0^{t_1}\int_0^{x_2} v(t_2,s,x_2,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s)\Bigg| \\
&\le \Bigg|\int_{t_1}^{t_2}\int_0^{x_2} v(t_2,s,x_2,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s)\Bigg| \\
&\le \int_{t_1}^{t_2}\Bigg|\int_0^{x_2} v(t_2,s,x_2,y,u(s,y))\, d_y g_2(x_2,y)\Bigg| d_s \left(\bigvee_{p=0}^{s} g_1(t_2,p)\right) \\
&\le \int_{t_1}^{t_2}\int_0^{x_2} |v(t_2,s,x_2,y,u(s,y))|\, d_y \left(\bigvee_{q=0}^{y} g_2(x_2,q)\right) d_s \left(\bigvee_{p=0}^{s} g_1(t_2,p)\right) \\
&\le \int_{t_1}^{t_2}\int_0^{x_2} \phi(\|u\|_C)\, d_y \left(\bigvee_{q=0}^{y} g_2(x_2,q)\right) d_s \left(\bigvee_{p=0}^{s} g_1(t_2,p)\right) \\
&= \phi(\|u\|_C)\left(\bigvee_{y=0}^{x_2} g_2(x_2,y)\right)\left(\bigvee_{s=t_1}^{t_2} g_1(t_2,s)\right) \\
&\le K_2\phi(\|u\|_C)\bigvee_{s=t_1}^{t_2} g_1(t_2,s) \le K_2\phi(\|u\|_C)\, N_1(\varepsilon),
\end{aligned} \tag{32}$$

where the function $N_1(\varepsilon)$ is defined by (29).

Next, evaluating similarly as above, in view of Lemmas 1 and 2, we have

$$\begin{aligned}
&\Bigg|\int_0^{t_1}\int_0^{x_2} v(t_2,s,x_2,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) - \int_0^{t_1}\int_0^{x_2} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s)\Bigg| \\
&\le \Bigg|\int_0^{t_1}\int_0^{x_2} [v(t_2,s,x_2,y,u(s,y)) - v(t_1,s,x_1,y,u(s,y))] \times d_y g_2(x_2,y)\, d_s g_1(t_2,s)\Bigg| \\
&\le \int_0^{t_1}\int_0^{x_2} |v(t_2,s,x_2,y,u(s,y)) - v(t_1,s,x_1,y,u(s,y))|\, d_y \left(\bigvee_{q=0}^{y} g_2(x_2,q)\right) d_s \left(\bigvee_{p=0}^{s} g_1(t_2,p)\right) \\
&\le \int_0^{t_1}\int_0^{x_2} \omega_{1,3}(v,\varepsilon)\, d_y \left(\bigvee_{q=0}^{y} g_2(x_2,q)\right) d_s \left(\bigvee_{p=0}^{s} g_1(t_2,p)\right) \\
&\le \omega_{1,3}(v,\varepsilon)\left(\bigvee_{y=0}^{x_2} g_2(x_2,y)\right)\left(\bigvee_{s=0}^{t_1} g_1(t_2,p)\right) \\
&\le K_1K_2\omega_{1,3}(v,\varepsilon),
\end{aligned} \tag{33}$$

where we denoted

$$\omega_{1,3}(v,\varepsilon) = \sup\{|v(t_2,s,x_2,y,u) - v(t_1,s,x_1,y,u)| : t_1,t_2,x_1,x_2,s,y \in I, |t_2-t_1| \le \varepsilon, |x_2-x_1| \le \varepsilon,\ u \in [-\|u\|_C, \|u\|_C]\}. \tag{34}$$

Moreover, the constants K_1, K_2 are defined by (24).

Observe that taking into account the fact that the function $v = v(t,s,x,y,u)$ is uniformly continuous on the set $I^4 \times [-\|u\|_C, \|u\|_C]$ we deduce that $\omega_{1,3}(v,\varepsilon) \to 0$ as $\varepsilon \to 0$.

Further, using the imposed hypotheses, in light of Lemmas 1 and 2, we derive the following estimate:

$$\begin{aligned}
&\Bigg|\int_0^{t_1}\int_0^{x_2} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) - \int_0^{t_1}\int_0^{x_1} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s)\Bigg| \\
&\le \int_0^{t_1}\Bigg|\int_{x_1}^{x_2} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_2,y)\Bigg| d_s \left(\bigvee_{p=0}^{s} g_1(t_2,p)\right) \\
&\le \int_0^{t_1}\int_{x_1}^{x_2} |v(t_1,s,x_1,y,u(s,y))|\, d_y \left(\bigvee_{q=0}^{y} g_2(x_2,q)\right) d_s \left(\bigvee_{p=0}^{s} g_1(t_2,p)\right)
\end{aligned}$$

$$\leq \phi(\|u\|_C) \int_0^{t_1} \int_{x_1}^{x_2} d_y \left(\bigvee_{q=0}^{y} g_2(x_2, q) \right) d_s \left(\bigvee_{p=0}^{s} g_1(t_2, p) \right)$$
$$\leq \phi(\|u\|_C) \left(\bigvee_{s=0}^{t_1} g_1(t_2, s) \right) \left(\bigvee_{y=x_1}^{x_2} g_2(x_2, y) \right)$$
$$\leq K_1 \phi(\|u\|_C) N_2(\varepsilon), \tag{35}$$

where the constant K_1 is defined by (24) and the function $N_2(\varepsilon)$ is defined by (29).

Next, using similar reasonings, we arrive at the following estimate:

$$\left| \int_0^{t_1} \int_0^{x_1} v(t_1, s, x_1, y, u(s, y)) \, d_y g_2(x_2, y) \, d_s g_1(t_2, s) \right.$$
$$\left. - \int_0^{t_1} \int_0^{x_1} v(t_1, s, x_1, y, u(s, y)) \, d_y g_2(x_1, y) \, d_s g_1(t_2, s) \right|$$
$$= \left| \int_0^{t_1} \int_0^{x_1} v(t_1, s, x_1, y, u(s, y)) \, d_y \right.$$
$$\left. \times [g_2(x_2, y) - g_2(x_1, y)] \, d_s g_1(t_2, s) \right|$$
$$\leq \int_0^{t_1} \int_0^{x_1} |v(t_1, s, x_1, y, u(s, y))| \, d_y$$
$$\times \left(\bigvee_{q=0}^{y} [g_2(x_2, q) - g_2(x_1, q)] \right) d_s \left(\bigvee_{p=0}^{s} g_1(t_2, p) \right)$$
$$\leq \phi(\|u\|_C) \int_0^{t_1} \int_0^{x_1} d_y \left(\bigvee_{q=0}^{y} [g_2(x_2, q) - g_2(x_1, q)] \right) d_s$$
$$\times \left(\bigvee_{p=0}^{s} g_1(t_2, p) \right)$$
$$= \phi(\|u\|_C) \left(\bigvee_{y=0}^{x_1} [g_2(x_2, y) - g_2(x_1, y)] \right) \bigvee_{s=0}^{t_1} g_1(t_2, s)$$
$$\leq K_1 \phi(\|u\|_C) \bigvee_{y=0}^{x_1} [g_2(x_2, y) - g_2(x_1, y)]$$
$$\leq K_1 \phi(\|u\|_C) M_2(\varepsilon), \tag{36}$$

where the function $M_2(\varepsilon)$ was defined in (28).

Now, we estimate the last term appearing on the right-hand side of inequality (31). Arguing similarly as above, we get

$$\left| \int_0^{t_1} \int_0^{x_1} v(t_1, s, x_1, y, u(s, y)) \, d_y g_2(x_1, y) \, d_s g_1(t_2, s) \right.$$
$$\left. - \int_0^{t_1} \int_0^{x_1} v(t_1, s, x_1, y, u(s, y)) \, d_y g_2(x_1, y) \, d_s g_1(t_1, s) \right|$$
$$= \left| \int_0^{t_1} \int_0^{x_1} v(t_1, s, x_1, y, u(s, y)) \, d_y g_2(x_1, y) \, d_s \right.$$
$$\left. \times [g_1(t_2, s) - g_1(t_1, s)] \right|$$
$$\leq \int_0^{t_1} \int_0^{x_1} |v(t_1, s, x_1, y, u(s, y))| \, d_y \left(\bigvee_{q=0}^{y} g_2(x_1, q) \right) d_s$$
$$\times \left(\bigvee_{p=0}^{s} [g_1(t_2, p) - g_1(t_1, p)] \right)$$
$$\leq \phi(\|u\|_C) \int_0^{t_1} \int_0^{x_1} d_y \left(\bigvee_{q=0}^{y} g_2(x_1, q) \right) d_s$$
$$\times \left(\bigvee_{p=0}^{s} [g_1(t_2, p) - g_1(t_1, p)] \right)$$
$$= \phi(\|u\|_C) \left(\bigvee_{y=0}^{x_1} g_2(x_1, y) \right) \left(\bigvee_{s=0}^{t_1} [g_1(t_2, p) - g_1(t_1, s)] \right)$$
$$\leq K_2 \phi(\|u\|_C) M_1(\varepsilon), \tag{37}$$

where the constant K_2 and the function $M_1(\varepsilon)$ were defined in (24) and (28), respectively.

Now, linking estimates (31)–(33) and (34)–(37) as well as taking into account the properties of the functions $M_i = M_i(\varepsilon)$ and $N_i = N_i(\varepsilon)$ $(i = 1, 2)$ and the comment given after estimate (33), we conclude that the operator V transforms the space $C(I^2)$ into itself.

On the other hand, keeping in mind Lemma 4 we infer that the operator F defined in (30) transforms also the space $C(I^2)$ into itself. Consequently we obtain that the operator Q defined in (30) is a self-mapping of the space $C(I^2)$.

In the sequel we show that the operator Q is continuous on the space $C(I^2)$. To this end let us first observe that in view of the properties of the superposition operator F expressed in Lemma 4 it is sufficient to show that the operator V defined by (30) is continuous on $C(I^2)$. To prove this fact fix arbitrarily $\varepsilon > 0$ and $u \in C(I^2)$. Further, take an arbitrary function $w \in C(I^2)$ with $\|u - w\|_C \leq \varepsilon$. Then, keeping in mind Lemma 1, for arbitrarily fixed $t, x \in I$, we obtain

$$|(Vu)(t, x) - (Vw)(t, x)|$$
$$\leq \int_0^{t} \int_0^{x} |v(t, s, x, y, u(s, y)) - v(t, s, x, y, w(s, y))| \, d_y$$
$$\times \left(\bigvee_{q=0}^{y} g_2(x, q) \right) d_s \left(\bigvee_{p=0}^{s} g_1(t, p) \right). \tag{38}$$

Next, let us denote

$$P = \|u\|_C + \varepsilon \tag{39}$$

and let us define

$$\omega_P(v,\varepsilon)=\sup\{|v(t,s,x,y,u_1)-v(t,s,x,y,u_2)|:\ (t,s)\in\Delta_1,(x,y)\in\Delta_2,\ u_1u_2\in[-P,P],|u_1-u_2|\le\varepsilon\}. \quad (40)$$

Then, from (38) we derive the following estimates:

$$\begin{aligned}|(Vu)(t,x)-(Vw)(t,x)| &\le \int_0^t\int_0^x \omega_P(v,\varepsilon)\,d_y\left(\bigvee_{q=0}^{y}g_2(x,q)\right)d_s\left(\bigvee_{p=0}^{s}g_1(t,p)\right)\\ &\le \omega_P(v,\varepsilon)\left(\bigvee_{y=0}^{x}g_2(x,y)\right)\left(\bigvee_{s=0}^{t}g_1(t,s)\right)\\ &\le K_1K_2\omega_P(v,\varepsilon).\end{aligned} \quad (41)$$

Hence, in view of the uniform continuity of the function v on the set $\Delta_1\times\Delta_2\times[-P,P]$, we infer that the operator V is continuous on the space $C(I^2)$. According to the above remark this implies that the operator Q is continuous on the space $C(I^2)$.

In what follows let us fix an arbitrary function $u\in C(I^2)$. Utilizing the imposed assumptions and Lemmas 1 and 2, for fixed $t,x\in I$, we obtain

$$\begin{aligned}|(Qu)(t,x)| &\le |h(t,x)|+|f(t,x,u(t,x))|\\ &\quad\times\int_0^t\left|\int_0^x v(t,s,x,y,u(s,y))\,d_yg_2(x,y)\right|d_s\left(\bigvee_{p=0}^{s}g_1(t,p)\right)\\ &\le \|h\|_C+[|f(t,x,u(t,x))-f(t,x,0)|+|f(t,x,0)|]\\ &\quad\times\int_0^t\int_0^x|v(t,s,x,y,u(s,y))|\,d_y\left(\bigvee_{q=0}^{y}g_2(x,q)\right)d_s\left(\bigvee_{p=0}^{s}g_1(t,p)\right)\\ &\le \|h\|_C+[k|u(t,x)|+|f(t,x,0)|]\\ &\quad\times\int_0^t\int_0^x\phi(|u(s,y)|)\,d_y\left(\bigvee_{q=0}^{y}g_2(x,q)\right)d_s\left(\bigvee_{p=0}^{s}g_1(t,p)\right)\\ &\le \|h\|_C+(k\|u\|_C+F_1)\phi(\|u\|_C)\\ &\quad\times\int_0^t\int_0^x d_y\left(\bigvee_{q=0}^{y}g_2(x,q)\right)d_s\left(\bigvee_{p=0}^{s}g_1(t,p)\right)\\ &= \|h\|_C+(k\|u\|_C+F_1)\phi(\|u\|_C)\left(\bigvee_{y=0}^{x}g_2(x,y)\right)\times\left(\bigvee_{s=0}^{t}g_1(t,s)\right).\end{aligned} \quad (42)$$

Hence, in light of Corollary 9, we obtain the following estimate:

$$|(Qu)(t,x)|\le\|h\|_C+(k\|u\|_C+F_1)K_1K_2\phi(\|u\|_C). \quad (43)$$

Consequently, we get

$$\|Qu\|_C\le\|h\|_C+(k\|u\|_C+F_1)K_1K_2\phi(\|u\|_C). \quad (44)$$

Now, taking into account assumption (viii), from estimate (44), we derive that there exists a number $r_0>0$ such that Q transforms the ball B_{r_0} into itself and $kK_1K_2\phi(r_0)<1$.

Next, let us take a nonempty subset X of the ball B_{r_0} and choose arbitrarily a function $u\in X$. Then, for a fixed $\varepsilon>0$ and for arbitrary $(t_1,x_1),(t_2,x_2)\in I^2$ such that $t_1\le t_2$, $x_1\le x_2$ (cf. Remark 11) and $t_2-t_1\le\varepsilon$, $x_2-x_1\le\varepsilon$, using the standard tools, we get

$$\begin{aligned}&|(Qu)(t_2,x_2)-(Qu)(t_1,x_1)|\\ &\le |h(t_2,x_2)-h(t_1,x_1)|\\ &\quad+|(Fu)(t_2,x_2)(Vu)(t_2,x_2)-(Fu)(t_2,x_2)(Vu)(t_1,x_1)|\\ &\quad+|(Fu)(t_2,x_2)(Vu)(t_1,x_1)-(Fu)(t_1,x_1)(Vu)(t_1,x_1)|\\ &\le \omega(h,\varepsilon)+|(Fu)(t_2,x_2)|\,|(Vu)(t_2,x_2)-(Vu)(t_1,x_1)|\\ &\quad+|(Vu)(t_1,x_1)|\,|(Fu)(t_2,x_2)-(Fu)(t_1,x_1)|\\ &\le \omega(h,\varepsilon)+[|f(t_2,x_2,u(t_2,x_2))-f(t_2,x_2,0)|\\ &\qquad+|f(t_2,x_2,0)|]\\ &\quad\times\{|(Vu)(t_2,x_2)-(Vu)(t_1,x_1)|\}+|(Vu)(t_1,x_1)|\\ &\quad\times\{|f(t_2,x_2,u(t_2,x_2))-f(t_2,x_2,u(t_1,x_1))|\\ &\qquad+|f(t_2,x_2,u(t_1,x_1))-f(t_1,x_1,u(t_1,x_1))|\}\\ &\le \omega(h,\varepsilon)+[k|u(t_2,x_2)|+|f(t_2,x_2,0)|]\\ &\quad\times\{|(Vu)(t_2,x_2)-(Vu)(t_1,x_1)|\}\\ &\quad+|(Vu)(t_1,x_1)|\{k|u(t_2,x_2)-u(t_1,x_1)|+\overline{\omega}(f,\varepsilon)\}\\ &\le \omega(h,\varepsilon)+(k\|u\|_C+F_1)\{|(Vu)(t_2,x_2)-(Vu)(t_1,x_1)|\}\\ &\quad+|(Vu)(t_1,x_1)|(k\omega(u,\varepsilon)+\overline{\omega}(f,\varepsilon)),\end{aligned} \quad (45)$$

where we denoted

$$\overline{\omega}(f,\varepsilon) = \sup\{|f(t_2,x_2,u) - f(t_1,x_1,u)| : t_1,t_2,x_1,x_2 \in I, u \in [-r,r], |t_2-t_1| \le \varepsilon, |x_2-x_1| \le \varepsilon\}. \quad (46)$$

Further, based on estimates (31)–(33) and (35)–(37) we obtain

$$\begin{aligned}|(Vu)(t_2,x_2) - (Vu)(t_1,x_1)| &\le K_2\phi(\|u\|_C)\bigvee_{s=t_1}^{t_2} g_1(t_2,s) \\ &+ K_1K_2\omega_{1,3}(v,\varepsilon) + K_1\phi(\|u\|_C)\bigvee_{y=x_1}^{x_2} g_2(x_2,y) \\ &+ K_1\phi(\|u\|_C)M_2(\varepsilon) + K_2\phi(\|u\|_C)M_1(\varepsilon).\end{aligned} \quad (47)$$

Now, from (47) and (29) we derive the following estimate:

$$\begin{aligned}\omega(Vu,\varepsilon) &\le K_2\phi(r_0)N_1(\varepsilon) + K_1\phi(r_0)N_2(\varepsilon) \\ &+ K_1\phi(r_0)M_2(\varepsilon) + K_2\phi(r_0)M_1(\varepsilon).\end{aligned} \quad (48)$$

The above estimate yields the following one:

$$\begin{aligned}\omega(VX,\varepsilon) &\le K_2\phi(r_0)N_1(\varepsilon) + K_1\phi(r_0)N_2(\varepsilon) \\ &+ K_1K_2\omega_{1,3}(v,\varepsilon) + K_1\phi(r_0)M_2(\varepsilon) \\ &+ K_2\phi(r_0)M_1(\varepsilon).\end{aligned} \quad (49)$$

On the other hand, using our assumptions and applying Lemmas 1 and 2, we get

$$\begin{aligned}|(Vu)(t_1,x_1)| &\le \int_0^{t_1}\int_0^{x_1} |v(t_1,s,x_1,y,u(s,y))|\, d_y\left(\bigvee_{q=0}^{y} g_2(x_1,q)\right) d_s \times \left(\bigvee_{p=0}^{s} g_1(t_1,p)\right) \\ &\le \phi(\|u\|_C)\int_0^{t_1}\int_0^{x_1} d_y\left(\bigvee_{q=0}^{y} g_2(x_1,q)\right) d_s\left(\bigvee_{p=0}^{s} g_1(t_1,p)\right) \\ &\le \phi(r_0)\left(\bigvee_{y=0}^{x_1} g_2(x_1,y)\right)\left(\bigvee_{s=0}^{t_1} g_1(t_1,p)\right) \le K_1K_2\phi(r_0).\end{aligned} \quad (50)$$

Finally, combining estimates (45), (49), and (50), we get

$$\begin{aligned}\omega(QX,\varepsilon) &\le \omega(h,\varepsilon) \\ &+ (kr_0 + F_1)\{K_2\phi(r_0)N_1(\varepsilon) + K_1\phi(r_0)N_2(\varepsilon) \\ &+ K_1K_2\omega_{1,3}(v,\varepsilon) + K_1\phi(r_0)M_2(\varepsilon) \\ &+ K_2\phi(r_0)M_1(\varepsilon)\} \\ &+ K_1K_2\phi(r_0)\{k\omega(X,\varepsilon) + \overline{\omega}(f,\varepsilon)\}.\end{aligned} \quad (51)$$

Hence, taking into account the properties of the functions M_i, N_i $(i = 1,2)$ and the functions $\varepsilon \to \omega(h,\varepsilon)$, $\varepsilon \to \omega_{1,3}(v,\varepsilon)$, and $\varepsilon \to \overline{\omega}(f,\varepsilon)$, we obtain

$$\omega_0(QX) \le kK_1K_2\phi(r_0)\omega_0(X). \quad (52)$$

Keeping in mind assumption (viii) and Theorem 3 and taking into account formula (11), from (52) we deduce that there exists at least one function $u = u(t,x)$ belonging to the ball B_{r_0} which is a solution of (16).

The proof is complete. □

Remark 11. In considerations conducted in the above proof, taking two points $(t_1,x_1),(t_2,x_2) \in I^2$, we assumed that $t_1 \le t_2$ and $x_1 \le x_2$.

Observe that all possible cases can be always converted to that indicated above. For example, if we assume that $t_1 \le t_2$ and $x_1 > x_2$, then, taking an arbitrary function $w = w(t,x,z)$ with real variables (z denotes an arbitrary real number), we get

$$\begin{aligned}|w(t_2,x_2,z) - w(t_1,x_1,z)| &\le |w(t_2,x_2,z) - w(t_1,x_2,z)| \\ &+ |w(t_1,x_2,z) - w(t_1,x_1,z)| \\ &= |w(t_2,x_2,z) - w(t_1,x_2,z)| \\ &+ |w(t_1,x_1,z) - w(t_1,x_2,z)|.\end{aligned} \quad (53)$$

Now, we can repeat all estimates of the above proof under requirements concerning the choice of the points (t_1,x_1), (t_2,x_2).

4. Applications to Functional Integral Equations of Fractional Order and to Other Types of Functional Integral Equations

We start with providing some facts concerning assumption (v) imposed in investigations conducted in the preceding section (cf. also [9]). At the beginning we formulate a condition which is handy and convenient in applications and which guarantees that the functions g_1, g_2 appearing in (16) satisfy assumption (v).

In order to formulate the announced condition assume (as we have done previously) that $g_i(w,z) = g_i : \Delta_i \to \mathbb{R}$ is a given function $(i = 1,2)$. Further we assume that the function $g_i(w,z)$ $(i = 1,2)$ satisfies the following condition:

(v′) for arbitrary $w_1, w_2 \in I$, $w_1 < w_2$, the function $z \to g_i(w_2,z) - g_i(w_1,z)$ is monotone on the interval $[0,w_1]$ $(i = 1,2)$.

From results proved in [9] the following lemma immediately follows.

Lemma 12. *Suppose the function $g_i = g_i(w,z)$ satisfies assumptions (iii), (v′), and (vi) for $i = 1,2$. Then g_i satisfies assumption (v) $(i = 1,2)$.*

Indeed, in the case when g_i is nonincreasing, this result was proved as Theorem 3 in [9], while the case when g_i is nondecreasing is covered by Theorem 5 in [9] ($i = 1, 2$).

Further, based on results obtained in [9] we provide two examples of functions $g_i = g_i(w, z)$ satisfying assumption (v′) and being essential in our considerations.

Example 13. Let us fix i ($i = 1, 2$) and take the function $g_i(w, z) = g_i : \Delta_i \to \mathbb{R}$ defined by the formula

$$g_i(w, z) = \frac{1}{\alpha}\left[w^{\alpha} - (w - z)^{\alpha}\right], \tag{54}$$

where α is a fixed number from the interval $(0, 1)$. If we fix arbitrary numbers $w_1, w_2 \in I$ such that $w_1 < w_2$, then it is easily seen that the function $z \to g_i(w_2, z) - g_i(w_1, z)$ is nonincreasing on the interval $[0, w_1]$. This means that g_i satisfies assumption (v′). Moreover, we can verify that the function $g_i(w, z)$ satisfies also assumptions (iii), (iv), and (vi).

Example 14. Similarly as above fix $i \in \{1, 2\}$. Consider the function $g_i(w, z) = g_i : \Delta_i \to \mathbb{R}$ defined by the formula

$$g_i(w, z) = \begin{cases} w \ln \dfrac{w + z}{w} & \text{for } 0 < z \le w \le 1 \\ 0 & \text{for } w = 0. \end{cases} \tag{55}$$

Using the standard methods of mathematical analysis (cf. [9]) it can be easily shown that the function $z \to g_i(w_2, z) - g_i(w_1, z)$ is nondecreasing on the interval $[0, w_1]$ for $w_1 < w_2$ and satisfies assumptions (iii), (iv), and (vi) formulated in the preceding section.

In what follows we will consider the fractional integral equation with functions involved depending on two variables, which has the form

$$u(t, x) = h(t, x) + \frac{f(t, x, u(t, x))}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^x \frac{v(t, s, x, y, u(s, y))}{(t - s)^{1-\alpha}(x - y)^{1-\beta}}\, ds\, dy, \tag{56}$$

where $t, x \in I$ and α, β are fixed numbers from the interval $(0, 1)$. Moreover, the symbol $\Gamma(\gamma)$ indicates the gamma function.

Let us mention that (56) represents the so-called partial singular integral equation of Volterra type in two variables. Recently, equations of such a type were intensively investigated in some papers [11–16].

Obviously, (56) creates a generalization of the classical Volterra integral equation of fractional order in one variable which is studied in several papers and monographs and finds numerous applications (cf. [1–10, 23–27] and references therein).

Now, we show that the functional integral equation of fractional orders (56) can be treated as a particular case of the Volterra-Stieltjes functional integral equation (16) studied in Section 3.

In fact, take the functions $g_i(w, z)$ considered in Example 13, which have the form

$$g_1(t, s) = \frac{1}{\alpha}\left[t^{\alpha} - (t - s)^{\alpha}\right], \qquad g_2(x, y) = \frac{1}{\beta}\left[x^{\beta} - (x - y)^{\beta}\right], \tag{57}$$

for $(t, s) \in \Delta_1$ and $(x, y) \in \Delta_2$.

Then, it can be easily seen that (56) can be written in the form of (16). Thus, we can apply Theorem 10 in order to obtain an existence result concerning (56).

To formulate such a result let us first calculate the constants K_1, K_2 from Corollary 9. Indeed, we have (cf. [9])

$$K_1 = \sup\left\{\bigvee_{z=0}^{w} g_1(w, z) : w \in I\right\} = \frac{1}{\alpha}, \qquad K_2 = \sup\left\{\bigvee_{z=0}^{w} g_2(w, z) : w \in I\right\} = \frac{1}{\beta}. \tag{58}$$

Now, we present the above announced result.

Theorem 15. *Assume that the function h involved in (56) satisfies assumption (i) and the function $v = v(t, s, x, y, u)$ satisfies assumption (vii) of Theorem 10. Moreover, one assumes that the following condition is satisfied.*

(viii′) *There exists a positive solution r_0 of the inequality*

$$\|h\|_C + \frac{kr + F_1}{\Gamma(\alpha + 1)\Gamma(\beta + 1)}\phi(r) \le r \tag{59}$$

such that $k\phi(r_0) < \Gamma(\alpha + 1)\Gamma(\beta + 1)$.

Then there exists at least one solution $u = u(t, x)$ of (56) in the space $C(I^2)$, belonging to the ball B_{r_0}.

Remark 16. In the above conducted calculations we used the well-known formula $\delta\Gamma(\delta) = \Gamma(\delta + 1)$ (cf. [28]).

In what follows we consider the functional integral equation of the so-called Volterra-Chandrasekhar type in two variables having the form

$$u(t, x) = h(t, x) + f(t, x, u(t, x)) \int_0^t \int_0^x \frac{tx}{(t + s)(x + y)} \times v(t, s, x, y, u(s, y))\, ds\, dy \tag{60}$$

for $t, x \in I$. We refer to [4, 9, 29, 30] for the case of the Chandrasekhar and Volterra-Chandrasekhar equations in one variable.

Observe that taking the function $g = g_i = g_i(y,z)$ ($i = 1,2$) appearing in Example 14 we can represent (60) in the form (16). Obviously, in this case we take

$$g_1(t,s) = \begin{cases} t\ln\dfrac{t+s}{t} & \text{for } 0 < s \le t \le 1 \\ 0 & \text{for } t = 0, \end{cases}$$
$$g_2(x,y) = \begin{cases} x\ln\dfrac{x+y}{x} & \text{for } 0 < y \le x \le 1 \\ 0 & \text{for } x = 0. \end{cases} \tag{61}$$

Further, using the fact that the function g satisfies assumption (v$'$), we get

$$\sup\left\{\bigvee_{z=0}^{w} g(w,z) : w \in I\right\} = \sup\{g(w,w) : w \in I\} = \ln 2. \tag{62}$$

Now, we can formulate an existence theorem concerning (60).

Theorem 17. *Assume that the function h appearing in (60) satisfies assumption (i), the function $f = f(t,x,u)$ satisfies assumption (ii), and the function $v = v(t,s,x,y,u)$ satisfies assumption (vii) of Theorem 10. Apart from this one assumes that the following condition is satisfied.*

(viii$''$) *There exists a positive solution r_0 of the inequality*

$$\|h\|_C + (kr + F_1)\,\phi(r)\ln^2 2 \le r \tag{63}$$

such that $k\phi(r_0)\ln^2 2 < 1$.

Then there exists at least one solution $u = u(t,x)$ of (60) in the space $C(I^2)$, belonging to the ball B_{r_0}.

Finally, we consider the functional integral equation having the form linking equations (56) and (60), that is, the following integral equation:

$$u(t,x) = h(t,x) + \frac{f(t,x,u(t,x))}{\Gamma(\alpha)} \times \int_0^t \int_0^x \frac{x v(t,s,x,y,u(s,y))}{(x+y)(t-s)^{1-\alpha}}\,ds\,dy, \tag{64}$$

where, as above, α is a fixed number in the interval $(0,1)$ and $\Gamma(\alpha)$ denotes the gamma function.

An existence theorem concerning (64) can be formulated almost in the same way as Theorems 15 and 17. We need only to replace assumptions (viii$'$) and (viii$''$) by the following one.

($\overline{\text{viii}}$) There exists a positive solution r_0 of the following inequality:

$$\|h\|_C + \frac{(kr + F_1)\ln 2}{\Gamma(\alpha+1)}\phi(r) \le r \tag{65}$$

such that $k\phi(r_0)\ \ln 2 < \Gamma(\alpha+1)$.

We omit other details.

5. Final Remarks concerning Possible Generalizations

In this section we focus briefly on possible generalizations of results presented in previous sections.

First of all let us notice that instead of the functional integral equation of Volterra-Stieltjes type in two variables (16) we can consider the general case of the functional integral equation of Volterra-Stieltjes type in n variables which has the form

$$u(t) = h(t) + f(t,u(t))(Vu)(t), \tag{66}$$

where we denoted

$$t = (t_1, t_2, \ldots, t_n),$$
$$(Vu)(t) = \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_n} v(t_1,s_1,t_2,s_2,\ldots,t_n,s_n,u(s_1,s_2,\ldots s_n)) \times d_{s_1}g_1(t_1,s_1)\,d_{s_2}g_2(t_2,s_2) \cdots d_{s_n}g_n(t_n,s_n). \tag{67}$$

We assume here that $t = (t_1,t_2,\ldots,t_n) \in I^n$ and the functions involved in (66) satisfy assumptions similar to those formulated in Theorem 10. For example, $g_i : \Delta_i \to \mathbb{R}$ is a continuous function on the triangle $\Delta_i = \{(t_i,s_i) : 0 \le s_i \le t_i \le 1\}$ for $i = 1,2,\ldots,n$. Apart from this the function g_i satisfies assumptions (iv)–(vi) of Theorem 10 ($i = 1,2,\ldots,n$). Regarding the function $v = v(t_1,s_1,t_2,s_2,\ldots,t_n,s_n,u)$ we assume that $v : \Delta_1 \times \Delta_2 \times \cdots \times \Delta_n \times \mathbb{R} \to \mathbb{R}$ is continuous and such that

$$|v(t_1,s_1,t_2,s_2,\ldots,t_n,s_n,u)| \le \phi(u) \tag{68}$$

for all $(t_i,s_i) \in \Delta_i$ ($i = 1,2,\ldots,n$) and for each $u \in \mathbb{R}$, where (similarly as in assumption (vii)) $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function.

Other assumptions concerning (66), that is, assumptions (i), (ii), and (viii), can be easily adapted to our case.

Let us mention that we can also investigate equations considered precedingly, that is, (16), (56), (60), and (64), in the case when we replace the bounded interval I by an unbounded interval, for example, by $\mathbb{R}_+$. Such a situation was considered in [10] in the case of the functional integral equation of Volterra-Stieltjes type in one variable. Obviously, even in the case of one variable considered in [10], investigations are very extensive and complicated. On the other hand, those investigations allow us to study some qualitative aspects concerning solutions of the equation in question, such as asymptotic behaviour, stability, and asymptotic stability.

One can expect that in the case of (16), (56), and so forth, we can also investigate qualitative aspects of solutions of those equations. Investigations of such a type will appear elsewhere.

Conflict of Interests

The authors declare that there is no conflict of interests in the submitted paper.

Acknowledgment

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant no. (363-001-D1434). The authors, therefore, acknowledge with thanks DSR technical and financial support.

References

[1] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific, New York, NY, USA, 2012.

[2] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer, Berlin, Germany, 2010.

[3] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, River Edge, NJ, USA, 2000.

[4] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Nonlinear Physical Science, Springer, Heidelberg, Germany; Higher Education Press, Beijing, China, 2010.

[5] N. T. Dung, "Fractional stochastic differential equations with applications to finance," *Journal of Mathematical Analysis and Applications*, vol. 397, no. 1, pp. 334–348, 2013.

[6] S. Abbas, M. Benchohra, and G. M. N'Guérékata, *Topic in Fractional Differential Equations*, vol. 27 of *Developments in Mathematics*, Springer, New York, NY, USA, 2012.

[7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[8] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[9] J. Banaś and T. Zając, "A new approach to the theory of functional integral equations of fractional order," *Journal of Mathematical Analysis and Applications*, vol. 375, no. 2, pp. 375–387, 2011.

[10] T. Zając, "Solvability of fractional integral equations on an unbounded interval through the theory of Volterra-Stieltjes integral equations," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 33, no. 1, pp. 65–85, 2014.

[11] S. Abbas, D. Baleanu, and M. Benchohra, "Global attractivity for fractional order delay partial integro-differential equations," *Advances in Difference Equations*, vol. 2012, article 62, 2012.

[12] S. Abbas, M. Benchohra, and J. J. Nieto, "Global attractivity of solutions for nonlinear fractional order Riemann-Liouville Volterra-Stieltjes partial integral equations," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 81, pp. 1–15, 2012.

[13] S. Abbas and M. Benchohra, "Existence and stability of nonlinear, fractional order Riemann-Liouville Volterra-Stieltjes multi-delay integral equations," *Journal of Integral Equations and Applications*, vol. 25, no. 2, pp. 143–158, 2013.

[14] S. Abbas and M. Benchohra, "Fractional order integral equations of two independent variables," *Applied Mathematics and Computation*, vol. 227, pp. 755–761, 2014.

[15] A. N. Vityuk and A. V. Golushkov, "Existence of solutions of systems of partial differential equations of fractional order," *Nonlinear Oscillations*, vol. 7, no. 3, pp. 328–335, 2004.

[16] B. Ahmad, J. J. Nieto, A. Alsaedi, and H. Al-Hutami, "Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions," *Journal of the Franklin Institute*, vol. 351, no. 5, pp. 2890–2909, 2014.

[17] J. Appell, J. Banaś, and N. Merentes, *Bounded Variation and Around*, vol. 17 of *Series in Nonlinear Analysis and Applications*, Walter de Gruyter, Berlin, Germany, 2014.

[18] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, NY, USA, 1970.

[19] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, vol. 60 of *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1980.

[20] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*, vol. 55 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 1992.

[21] M. A. Toledano, T. D. Benavides, and G. L. Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, vol. 99 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 1997.

[22] J. Appell and P. P. Zabrejko, *Nonlinear Superposition Operators*, vol. 95 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 1990.

[23] H. M. Srivastava and R. K. Saxena, "Operators of fractional integration and their applications," *Applied Mathematics and Computation*, vol. 118, no. 1, pp. 1–52, 2001.

[24] M. A. Darwish, "Nondecreasing solutions of a fractional quadratic integral equation of Urysohn-Volterra type," *Dynamic Systems and Applications*, vol. 20, no. 4, pp. 423–438, 2011.

[25] M. A. Darwish and S. K. Ntouyas, "On a quadratic fractional Hammerstein-Volterra integral equation with linear modification of the argument," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 11, pp. 3510–3517, 2011.

[26] S. Abbas, M. Benchohra, and J. Henderson, "Global asymptotic stability of solutions of nonlinear quadratic Volterra integral equations of fractional order," *Communications on Applied Nonlinear Analysis*, vol. 19, no. 1, pp. 79–89, 2012.

[27] J. Caballero, M. A. Darwish, and K. Sadarangani, "Solvability of a fractional hybrid initial value problem with supremum by using measures of noncompactness in Banach algebras," *Applied Mathematics and Computation*, vol. 224, pp. 553–563, 2013.

[28] G. M. Fichtenholz, *Differential and Integral Calculus*, vol. 2, Wydawnictwo Naukowe PWN, Warsaw, Poland, 2007 (Polish).

[29] I. K. Argyros, "On a class of quadratic integral equations with perturbation," *Functiones et Approximatio Commentarii Mathematici*, vol. 20, pp. 51–63, 1992.

[30] S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, London, UK, 1950.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 818732, 11 pages
http://dx.doi.org/10.1155/2014/818732

Research Article

On Eventually Positive Solutions of Quasilinear Second-Order Neutral Differential Equations

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Received 17 February 2014; Accepted 16 April 2014; Published 13 May 2014

Academic Editor: Samir Saker

We study the second-order neutral delay differential equation $[r(t)\Phi_\gamma(z'(t))]' + q(t)\Phi_\beta(x(\sigma(t))) = 0$, where $\Phi_\alpha(t) = |t|^{\alpha-1}t$, $\alpha \geq 1$ and $z(t) = x(t) + p(t)x(\tau(t))$. Based on the conversion into a certain first-order delay differential equation we provide sufficient conditions for nonexistence of eventually positive solutions of two different types. We cover both cases of convergent and divergent integral $\int^\infty r^{-1/\gamma}(t)dt$. A suitable combination of our results yields new oscillation criteria for this equation. Examples are shown to exhibit that our results improve related results published recently by several authors. The results are new even in the linear case.

1. Introduction

In the paper we study the equation

$$\left[r(t)\Phi_\gamma\left(z'(t)\right)\right]' + q(t)\Phi_\beta(x(\sigma(t))) = 0, \qquad z(t) = x(t) + p(t)x(\tau(t)), \tag{1}$$

where $\Phi_\alpha(t) = |t|^{\alpha-1}t$, $\alpha \geq 1$, is the power type nonlinearity. The coefficients r and p are subject to the usual conditions $r \in C^1([t_0,\infty),\mathbb{R}^+)$, $p \in C^1([t_0,\infty),\mathbb{R}_0^+)$ and the coefficient q is positive, $q \in C([t_0,\infty),\mathbb{R}^+)$.

We assume that $\lim_{t\to\infty}\tau(t) = \infty = \lim_{t\to\infty}\sigma(t)$,

$$\sigma(\tau(t)) = \tau(\sigma(t)), \tag{2}$$

and there exist numbers $p_0 \geq 0$ and $\tau_0 > 0$ such that $p(t) \leq p_0$ and $\tau'(t) \geq \tau_0$.

Under the solution of (1) we understand any differentiable function $x(t)$ which does not identically equal zero eventually, such that $r(t)\Phi_\gamma(z'(t))$ is differentiable and (1) holds for large t.

Following the widely accepted terminology, the solution of (1) is said to be oscillatory if it has infinitely many zeros tending to infinity. Equation (1) is said to be oscillatory if all its solutions are oscillatory. In the opposite case, that is, if there exists an eventually positive solution of (1), (1) is said to be nonoscillatory.

In the paper we study nonoscillatory solutions of (1). Since $x(t)$ is a solution of (1) if and only if $-x(t)$ is a solution of (1), we can focus our attention on positive solutions.

The paper is organized as follows. In the remaining part of the current section we summarize selected important facts related to (1) and trends in the oscillation theory of this equation. In Section 2 we summarize tools like inequalities and oscillation criteria used in the proofs of main results. The main results are presented in the next three sections. Results on eventually positive solutions are separated into Sections 3 and 4 according to different asymptotic behavior: $z'(t) > 0$ in Section 3 and $z'(t) < 0$ in Section 4. In both cases we provide an efficient condition which ensures that solutions of this type do not exist. Note that under some additional conditions (namely, divergence of integral (3) below) the results from Section 3 immediately yield also oscillation criteria. If (3) fails, we can formulate oscillation criteria using a suitable combination of results from Sections 3 and 4, as shown in Section 5. The results of the paper improve several recently published results even in the linear case. We discuss these improvements in detail in remarks and examples accompanying the main theorems.

Neutral differential equation (1) as well as other related equations have been studied frequently in the literature. There are two main methods in the oscillation theory of (1). One of them is based on a modification of the classical Riccati substitution which is known to be a powerful tool in theory of second-order linear differential equations. Following this method, neutral equation (1) is in some sense considered as a perturbation of some second-order ordinary differential equation. An alternative approach, used for example, in a series of papers by Baculíková et al. [1–4] and Li [5] is based on the fact that it is possible to derive neutral first-order differential inequality for quasiderivative from (1) and the resulting inequality can be studied in the scope of theory elaborated for first-order delay differential inequalities. In this paper we use the later approach. The resulting theorems are sometimes referred to as comparison theorems for neutral differential equations.

Two main approaches are used to put the shift $\tau(t)$ in the differential term under the control. If $p(t) < 1$, then (1) can be "majorized" (in the sense of the classical Sturm comparison theory, which however has no extension to delay equations) by a delay equation of the form (1) with $p(t) = 0$. Oscillation criteria for second-order delay differential equations can be then used to conclude results for neutral equation (1) (see, e.g., [6–8]). An alternative approach deals with a suitable combination of (1) and the same equation with independent variable shifted from t to $\tau(t)$. This approach, which is used also in our paper, does not require $p(t) < 1$ but yields other restrictions, such as commutativity of the composition of delays (2).

Neglecting which method is used to study the oscillation of (1), it turns out that it is necessary to distinguish two cases: either

$$\int^{\infty} \frac{1}{r^{1/\gamma}(t)} \mathrm{d}t = \infty \tag{3}$$

or

$$\int^{\infty} \frac{1}{r^{1/\gamma}(t)} \mathrm{d}t < \infty. \tag{4}$$

The absolute majority of oscillation results in the literature concerns case (3), since in this case the positive solutions of (1) exhibit simpler behavior than in case (4); see Lemma 5 below. Case (4) has been studied, for example, in [9–16]. Note that for this case it is typical that the oscillation criterion consists of two relatively independent conditions. One of them is used to eliminate positive solutions with $z'(t) < 0$; the other one to eliminate positive solutions with $z'(t) > 0$. There are also results which treat both cases $z'(t) > 0$ and $z'(t) < 0$ in one unified approach. However, following this approach a typical conclusion is weaker: the equation is proved to be almost oscillatory (all nonoscillatory solutions, if exist any, tend to zero). Note also that the paper [16] does not satisfy these rules (makes use of unified approach to both cases but concludes oscillation), but there are several inaccuracies in this paper; see [10, 17] for corrected version of [16].

In this paper we essentially use the method from [1, 2] with a modification for case (4) presented in [12]. However, to keep the influence of each condition as transparent as possible we used different organization of the paper, as we explained above. The main improvement with respect to these papers is that we replace inequalities and estimates used in these papers by suitable parametrized versions depending on parameters l and φ (see below). This yields criteria with some degree of freedom and optimization with respect to the parameters which yields sharper results, as we carefully explain on examples of equations with proportional delay. A similar method where we use parameters l and φ to refine the widely used inequalities has been used in the recent paper [18].

Finally, note that [12] in fact deals with linear equations and the extension to nonlinear equations is suggested in Remark 11 at the end of the paper [12]. However here we use an advanced technique rather than the method suggested in [12].

2. Preliminary Results

In the paper we derive results related to the existence or nonexistence of certain equations and inequalities in terms of several parameters. The following two lemmas allow to find the values of the parameters, which yield sharpest results.

The function h introduced in the following lemma plays a role in a formulation of oscillation criteria in the case $\beta \geq 1$.

Lemma 1. *Let $\beta \geq 1$. The function*

$$h(x, y) = x^{\beta-1} + y\left(\frac{x}{x-1}\right)^{\beta-1} \tag{5}$$

satisfies

$$h(x, y) \geq h\left(1 + y^{1/\beta}, y\right) = \left(1 + y^{1/\beta}\right)^{\beta}, \tag{6}$$

for every $x > 1$ and $y > 0$.

Proof. It follows from the fact that

$$\frac{\partial}{\partial x} h(x, y) = (\beta - 1) x^{\beta-2} \left[1 - \frac{y}{(x-1)^{\beta}}\right] \tag{7}$$

and h as the function of x on $(1, \infty)$ attains its minimal value at the point $x = 1 + y^{1/\beta}$ and

$$\begin{aligned} h\left(1 + y^{1/\beta}, y\right) &= \left(1 + y^{1/\beta}\right)^{\beta-1} + y^{1/\beta}\left(1 + y^{1/\beta}\right)^{\beta-1} \\ &= \left(1 + y^{1/\beta}\right)^{\beta}. \end{aligned} \tag{8}$$

□

The following functions appear in the examples and allow to find the optimal values of the parameters which yield the sharpest result.

Lemma 2. *Let c_1, c_2 be positive numbers.*

(i) *The function $f(x) = x^{c_1}\ln(c_2/x)$ is increasing on $(0, c_2e^{-1/c_1})$, decreasing on $(c_2e^{-1/c_1}, \infty)$, and satisfies*

$$f(x) \le \frac{1}{c_1 e} c_2^{c_1} \tag{9}$$

on $(0,\infty)$ with the equality, if and only if $x = c_2e^{-1/c_1}$.

(ii) *The function $g(x) = x^{-c_1}\ln(x/c_2)$ is increasing on $(0, c_2e^{1/c_1})$, decreasing on $(c_2e^{1/c_1}, \infty)$, and satisfies*

$$g(x) \le \frac{1}{c_1 e} c_2^{-c_1} \tag{10}$$

on $(0,\infty)$ with the equality, if and only if $x = c_2e^{1/c_1}$.

Proof. By a direct computation

$$f'(x) = x^{c_1-1}\left(c_1 \ln\frac{c_2}{x} - 1\right). \tag{11}$$

Hence $f(x)$ has a local maximum at the point $x_0 = c_2e^{-1/c_1}$ and the value of this local maximum is $c_2^{c_1}/c_1e$. Similarly,

$$g'(x) = x^{-1-c_1}\left(1 - c_1 \ln\frac{x}{c_2}\right), \tag{12}$$

and hence $g(x)$ has a local maximum at the point $x_1 = c_2e^{1/c_1}$ and the maximal value is $(1/c_1e)c_2^{-c_1}$. □

Lemma 3. *Let $A \ge 0$, $B \ge 0$, $\beta \ge 1$, $l > 1$, $l^* = l/(l-1)$. Then*

$$(A+B)^\beta \le l^{\beta-1}A^\beta + (l^*)^{\beta-1}B^\beta. \tag{13}$$

Proof. From the fact that the function x^β is a convex function for $\beta \ge 1$ we have

$$\left(\frac{1}{l}a + \frac{1}{l^*}b\right)^\beta \le \frac{1}{l}a^\beta + \frac{1}{l^*}b^\beta \tag{14}$$

for nonnegative a and b. From here we obtain the desired inequality for $A = a/l$ and $B = b/l^*$. □

Lemma 4. *Let $\beta \ge 1$. The inequality*

$$l^{\beta-1}x^\beta(\sigma(t)) + (l^*)^{\beta-1}p^\beta(\sigma(t))\,x^\beta(\sigma(\tau(t))) \ge z^\beta(\sigma(t)) \tag{15}$$

holds for positive mutually conjugate numbers l, l^* and every t which satisfies $x(\sigma(t)) \ge 0$ and $x(\sigma(\tau(t))) \ge 0$.

Proof. It follows from Lemma 3, from the definition of $z(t)$ and from condition (2). □

The following lemma is well known in theory of neutral differential equations. It states (among others) that if x is an eventually positive solution, then z' is eventually of one sign and the negative sign of z' is excluded if (3) holds.

Lemma 5. *Let $x(t)$ be an eventually positive solution of* (1). *The corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ satisfies*

$$z(t) > 0, \qquad z'(t) > 0, \qquad \left(r(t)\Phi_\gamma\left(z'(t)\right)\right)' < 0 \tag{16}$$

eventually if (3) *holds and either* (16) *or*

$$z(t) > 0, \qquad z'(t) < 0, \qquad \left(r(t)\Phi_\gamma\left(z'(t)\right)\right)' < 0 \tag{17}$$

eventually if (4) *holds.*

Proof. It follows from [7, Lemma 10] and from the proof of that lemma. □

In the following lemma we summarize effective oscillation criteria for delay and advanced first-order equation which appear in the analysis of (1). Note that (iii) is sharper version of the related condition from [2, Lemma 4].

Lemma 6. *Let $q(t) \ge 0$.*

(i) *If $\sigma(t) < t$ and*

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} q(s)\,ds > \frac{1}{e}, \tag{18}$$

then

$$y'(t) + q(t)\,y(\sigma(t)) \le 0 \tag{19}$$

has no eventually positive solution.

(ii) *If $\sigma(t) > t$ and*

$$\liminf_{t\to\infty}\int_{t}^{\sigma(t)} q(s)\,ds > \frac{1}{e}, \tag{20}$$

then

$$y'(t) - q(t)\,y(\sigma(t)) \ge 0 \tag{21}$$

has no eventually positive solution.

(iii) *Let $\sigma(t) < t$, $\alpha \in (0,1)$. If*

$$\int_{t_0}^{\infty} q(s)\,ds = \infty, \tag{22}$$

then

$$y'(t) + q(t)\,y^\alpha(\sigma(t)) \le 0 \tag{23}$$

has no eventually positive solution.

(iv) *Let $\sigma(t) > t$, $\alpha \in (1,\infty)$. If*

$$\int_{t_0}^{\infty} q(s)\,ds = \infty, \tag{24}$$

then

$$y'(t) - q(t)\,y^\alpha(\sigma(t)) \ge 0 \tag{25}$$

has no eventually positive solution.

Proof. See [9, Lemmas 2.1–2.4] and [19, Lemma 2.2.9]. Note that the original proof of condition (i) is due to [20] and the proofs of conditions (iii) and (iv) for equations are due to [21]. □

3. Positive Solutions with $z'(t) > 0$ Eventually

In this section we give sufficient conditions which exclude the possibility that the equation possesses an eventually positive solution $x(t)$ such that the corresponding function $z(t)$ is eventually increasing. Note that Lemma 5 excludes other types of eventually positive solutions if (3) holds. Hence if (3) holds as well, then the criteria from this section guarantee oscillation of (1).

Denote

$$Q(t;\varphi) = \min\{q(t), \varphi q(\tau(t))\}, \tag{26}$$

$$Q_\eta^*(t;\varphi,t_1) = Q(t;\varphi)\left[\int_{t_1}^{\eta(t)} r^{-1/\gamma}(s)\,\mathrm{d}s\right]^\beta. \tag{27}$$

The following theorem allows us to relate positive solutions of (1) with a certain first-order neutral equation. This neutral equation can be further compared with a certain nonneutral differential equation. The form of this nonneutral differential equation depends on the fact whether the deviating argument $\tau(t)$ in the differential term is delay or advanced argument. If $\varphi = 1$, $\eta \equiv \sigma$, and $l = 2$, then Theorem 7 reduces to [2, Theorems 4, 5 and 6].

Theorem 7. *Let $\beta \geq 1$, $\varphi > 0$, $l > 1$ and $\eta(t)$ a function which satisfies $\eta(t) \leq \sigma(t)$ and $\lim_{t\to\infty}\eta(t) = \infty$. Suppose that there exists a number $T > t_0$ and a solution $x(t)$ of* (1) *which satisfy*

$$x(t) > 0, \quad z'(t) > 0 \quad \text{for } t \geq T. \tag{28}$$

Let $t_1 > T$ be such that

$$\min\{\eta(\tau(t)), \eta(t)\} > T \tag{29}$$

for every $t \geq t_1$ and let $t_2 \geq t_1$ be such that $\eta(t) \geq t_1$ for $t \geq t_2$. Then the following statements are true.

(i) *The inequality*

$$\left[l^{\beta-1}w(t) + \frac{p_0^\beta\varphi}{\tau_0}(l^*)^{\beta-1}w(\tau(t))\right]' + Q_\eta^*(t;\varphi,t_1)\,w^{\beta/\gamma}(\eta(t)) \leq 0 \tag{30}$$

has a positive decreasing solution on (t_2,∞).

(ii) *If $\tau(t) \geq t$, then*

$$y' + Q_\eta^*(t;\varphi,t_1)\,h^{-\beta/\gamma}\left(l, \frac{p_0^\beta\varphi}{\tau_0}\right) y^{\beta/\gamma}(\eta(t)) \leq 0 \tag{31}$$

has a positive solution on (t_2,∞).

(iii) *If $\tau(t) \leq t$, then*

$$y' + Q_\eta^*(t;\varphi,t_1)\,h^{-\beta/\gamma}\left(l, \frac{p_0^\beta\varphi}{\tau_0}\right) y^{\beta/\gamma}\left(\tau^{-1}(\eta(t))\right) \leq 0 \tag{32}$$

has a positive solution on (t_2,∞).

Proof. Let $x(t)$ be a solution of (1) which satisfies $x(t) > 0$ and $z'(t) > 0$ for $t \geq T$. Inequalities (15), $p(t) \leq p_0$, and $\eta(t) \leq \sigma(t)$ imply

$$l^{\beta-1}x^\beta(\sigma(t)) + (l^*)^{\beta-1}p_0^\beta x^\beta(\sigma(\tau(t))) \geq z^\beta(\eta(t)) \tag{33}$$

for $t \geq t_1$.

We shift (1) from t to $\tau(t)$ and get

$$\begin{aligned} 0 &= \frac{1}{\tau'(t)}\left[r(\tau(t))\Phi_\gamma\left(z'(\tau(t))\right)\right]' + q(\tau(t))\Phi_\beta(x(\sigma(\tau(t)))) \\ &\geq \frac{1}{\tau_0}\left[r(\tau(t))\Phi_\gamma\left(z'(\tau(t))\right)\right]' + q(\tau(t))\Phi_\beta(x(\sigma(\tau(t)))). \end{aligned} \tag{34}$$

Substituting $\Phi_\beta(x(\sigma(\tau(t))))$ from this inequality and $\Phi_\beta(x(\sigma(t)))$ from (1) to (33) and using (26) we obtain

$$\begin{aligned} 0 \geq{}& l^{\beta-1}\left[r(t)\Phi_\gamma\left(z'(t)\right)\right]' + \frac{(l^*)^{\beta-1}p_0^\beta\varphi}{\tau_0} \\ &\times\left[r(\tau(t))\Phi_\gamma\left(z'(\tau(t))\right)\right]' + Q(t;\varphi)z^\beta(\eta(t)), \end{aligned} \tag{35}$$

for $t \geq t_1$. Denoting $w(t) = r(t)\Phi_\gamma(z'(t))$ and using the obvious fact that w is positive and decreasing on (t_1,∞) we have

$$z(t) = \int_{t_1}^{t} w^{1/\gamma}(s)r^{-1/\gamma}(s)\,\mathrm{d}s \geq w^{1/\gamma}(t)\int_{t_1}^{t} r^{-1/\gamma}(s)\,\mathrm{d}s \tag{36}$$

for $t \geq t_1$. Thus w is an eventually positive and eventually decreasing solution of (30) and claim (i) is proved.

Denote

$$y(t) = l^{\beta-1}w(t) + \frac{p_0^\beta\varphi}{\tau_0}(l^*)^{\beta-1}w(\tau(t)). \tag{37}$$

Since w is a positive decreasing function, we have $w(t) \geq w(\tau(t))$ for $t \leq \tau(t)$ and $w(t) \leq w(\tau(t))$ for $t \geq \tau(t)$. Hence if $t \leq \tau(t)$ we have

$$y(t) \leq w(t)\left(l^{\beta-1} + \frac{p_0^\beta\varphi}{\tau_0}(l^*)^{\beta-1}\right) = w(t)\,h\left(l, \frac{p_0^\beta\varphi}{\tau_0}\right), \tag{38}$$

where the function h is defined by (5), and, if $t \geq \tau(t)$, then similarly

$$y(t) \leq w(\tau(t))\,h\left(l, \frac{p_0^\beta\varphi}{\tau_0}\right). \tag{39}$$

Hence we have

$$w^{\beta/\gamma}(\eta(t)) \geq h^{-\beta/\gamma}\left(l, \frac{p_0^\beta\varphi}{\tau_0}\right) y^{\beta/\gamma}(\eta(t)), \tag{40}$$

if $t \le \tau(t)$, and

$$w^{\beta/\gamma}(\eta(t)) \ge h^{-\beta/\gamma}\left(l, \frac{p_0^\beta \varphi}{\tau_0}\right) y^{\beta/\gamma}\left(\tau^{-1}(\eta(t))\right) \quad (41)$$

if $t \ge \tau(t)$. This and claim (i) prove claims (ii) and (iii) since in each case we have found an eventually positive solution y of the corresponding inequality. □

Remark 8. Note that in the proof of Theorem 7 we constructed the solutions of the inequalities (30), (31), and (32).

In the following corollary we give an efficient condition for nonexistence of the solutions mentioned in the points (ii) and (iii) of Theorem 7. According to Lemma 6 we distinguish the cases $\gamma = \beta$ and $\gamma \ne \beta$.

Corollary 9. *Let $\gamma \ge \beta \ge 1$. Equation* (1) *has no solution $x(t)$ which satisfies*

$$x(t) > 0, \quad z'(t) > 0 \quad \text{eventually} \quad (42)$$

if there exists number $\varphi > 0$ and a function $\eta(t)$ satisfying $\eta(t) \le \sigma(t)$ and $\lim_{t\to\infty}\eta(t) = \infty$ such that one of the following conditions holds.

(i) *$\eta(t) < t \le \tau(t)$ and for every T there exists $t_1 > T$ such that*

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} Q_\eta^*(s;\varphi,t_1)\,ds > \frac{1}{e}\left(1 + \left(\frac{\varphi}{\tau_0}\right)^{1/\beta} p_0\right)^{\beta^2/\gamma} \quad (43)$$

if $\beta = \gamma$, and

$$\int_{t_0}^{\infty} Q_\eta^*(t;\varphi,t_1)\,dt = \infty \quad (44)$$

if $\beta < \gamma$.

(ii) *$\eta(t) < \tau(t) \le t$ and for every T there exists $t_1 > T$ such that*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\eta(t))}^{t} Q_\eta^*(s;\varphi,t_1)\,ds > \frac{1}{e}\left(1 + \left(\frac{\varphi}{\tau_0}\right)^{1/\beta} p_0\right)^{\beta^2/\gamma} \quad (45)$$

if $\beta = \gamma$, and (44) *holds if $\beta < \gamma$.*

Proof. It follows from Theorem 7 and Lemmas 6 and 1. □

In the following example we compare our results with the results of [1]. Note that in this example (3) holds and hence the results of this section ensure oscillation of the equation.

Example 10 (linear equation). Baculíková and Džurina studied in [1] differential equation

$$\left(\sqrt{t}[x(t) + p_0 x(\alpha t)]'\right)' + \frac{a}{t^{3/2}} x(\mu t) = 0, \quad (46)$$

where $0 < \mu < 1$, $\alpha > 0$, $a > 0$, and obtained that the equation is oscillatory if either

$$\alpha \ge 1, \qquad 2a\sqrt{\mu}\ln\frac{1}{\mu} > \frac{1}{e}\left(\alpha^{3/2} + p_0\sqrt{\alpha}\right) \quad (47)$$

or

$$0 < \mu < \alpha \le 1, \qquad 2a\sqrt{\mu}\ln\frac{\alpha}{\mu} > \frac{\alpha + p_0}{\alpha e}. \quad (48)$$

We have $\gamma = \beta = 1$, $\tau(t) = \alpha t$, $\sigma(t) = \mu t$, $r(t) = \sqrt{t}$, $q(t) = at^{-3/2}$, $q(\tau(t)) = a(\alpha t)^{-3/2}$, and

$$\int r^{-1/\gamma}(t)\,dt = \int t^{-1/2}dt = 2\sqrt{t}, \quad (49)$$

and hence (3) holds. Using Corollary 9 with $\eta(t) = \lambda t$, $\lambda \le \mu$ and $\varphi = \alpha^{3/2}$ we have $Q(t) = q(t)$ and consequently,

$$Q_\eta^*(t;\varphi,t_1) = \frac{a}{t^{3/2}}\left[2\sqrt{\lambda}\sqrt{t} - 2\sqrt{t_1}\right],$$
$$\liminf_{t\to\infty}\int_{\eta(t)}^{t} Q_\eta^*(s;\varphi,t_1)\,ds = 2a\sqrt{\lambda}\ln\frac{1}{\lambda}, \quad (50)$$
$$\liminf_{t\to\infty}\int_{\tau^{-1}(\eta(t))}^{t} Q_\eta^*(s;\varphi,t_1)\,ds = 2a\sqrt{\lambda}\ln\frac{\alpha}{\lambda}.$$

Hence, (46) is oscillatory if either

$$\alpha \ge 1, \qquad 2a\sqrt{\lambda}\ln\frac{1}{\lambda} > \frac{1}{e}\left(1 + \sqrt{\alpha}p_0\right) \quad (51)$$

or

$$0 < \lambda < \alpha \le 1, \qquad 2a\sqrt{\lambda}\ln\frac{\alpha}{\lambda} > \frac{1}{e}\left(1 + \sqrt{\alpha}p_0\right). \quad (52)$$

Even in case $\mu = \lambda$, if we view the right hand sides as linear functions of p_0, we easily see from the slope and y-intercept of these lines that these conditions are sharper than those from [1]. Note that the fact that the equation is linear causes that the parameter l does not have any influence on the sharpenes of these conditions, since the function h from Lemma 1 is a constant function with respect to the first variable for $\beta = 1$. Hence, the improvement with respect to the results from [1] is purely in the presence of the parameter φ in the definition of the function Q. Figure 1 reveals also different asymptotic behavior of the right hand sides of (51) and (52) with respect to the corresponding constants from (47) and (48). Based on this fact we see that the improvement is significant especially if α is sufficiently far from 1.

When looking for optimal conditions for oscillation of (46) it is easy to ensure that the case $\lambda = \mu$ is not optimal for every μ. Really, if we replace inequality signs in (51) and (52) by equality signs and view the resulting equality as a formula which defines a as a function of λ, we get U-shaped function with one local minimum (see Figure 2). Since λ can be any positive number smaller than μ, it turns out that the optimal choice for λ is $\lambda = \mu$ on the decreasing part and $\lambda = \lambda_{\max}$ on the increasing part, where $\lambda_{\max}$ is the point where

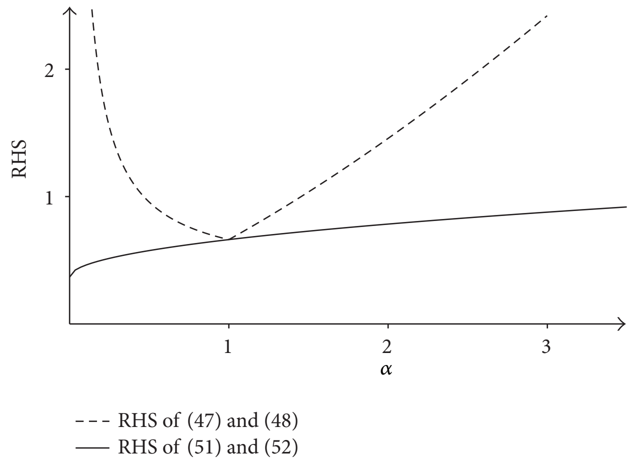

FIGURE 1: The graph of right hand sides (RHS) of (47) and (48) for $p_0 = 0.8$ compared to (51) and (52) as functions of α.

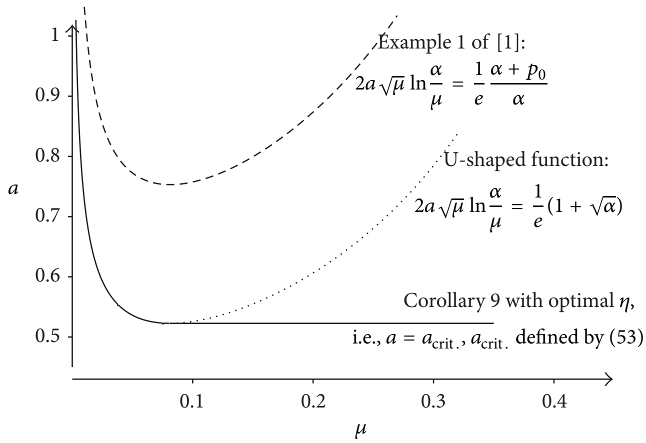

FIGURE 2: A comparison of the lower bounds for the coefficient a which guarantee oscillation of (46) for different values of μ. Parameters used for the graphs are $\alpha = 0.6$ and $p_0 = 0.8$.

the function $\sqrt{\lambda}\ln(\alpha/\lambda)$ attains its global maximum. Using Lemma 2, we have $\lambda_{\max} = \alpha/e^2$ and $\sqrt{\lambda}\ln(\alpha/\lambda) \le (2\sqrt{\alpha}/e)$. Hence we get that the equation is oscillatory if

$$a > a_{\text{crit.}} := \begin{cases} \frac{1}{4}\left(1+\sqrt{\alpha}p_0\right) & \text{for } \alpha \ge 1,\ \mu \ge \frac{1}{e^2}, \\ \frac{1}{4\sqrt{\alpha}}\left(1+\sqrt{\alpha}p_0\right) & \text{for } \alpha \le 1,\ \mu \ge \frac{\alpha}{e^2}, \\ \frac{1}{2e\sqrt{\mu}\ln(1/\mu)}\left(1+\sqrt{\alpha}p_0\right) & \text{for } \alpha \ge 1,\ \mu \le \frac{1}{e^2}, \\ \frac{1}{2e\sqrt{\mu}\ln(\alpha/\mu)}\left(1+\sqrt{\alpha}p_0\right) & \text{for } \alpha \le 1,\ \mu \le \frac{\alpha}{e^2}. \end{cases} \tag{53}$$

From the graphical point of view these conditions arise from (51) and (52) by isolating a and replacing the increasing part of the resulting curve by a constant function; see Figure 2 for more details and for comparison with the oscillation constant resulting from (48).

Example 11 (half-linear equation). Consider the differential equation

$$\left(t^2\left[\left(x(t) + (3+\sin t)\,x(\alpha t)\right)'\right]^3\right)' + \frac{b}{t^2}x^3(\mu t) = 0, \qquad \mu < 1, \quad a > 0. \tag{54}$$

We have $\beta = \gamma = 3$, $\tau(t) = \alpha t$, $\sigma(t) = \mu t$, $p_0 = 4$, $\tau_0 = \alpha$, $r(t) = t^2$, $q(t) = b/t^2$, $q(\tau(t)) = b/(\alpha t)^2$ and,

$$\int r^{-1/\gamma}(t)\,dt = \int t^{-2/3}dt = 3t^{1/3}, \tag{55}$$

and hence condition (3) holds. Using results of Baculíková and Džurina [2, Corollaries 3 and 4] we obtain that (54) is oscillatory if

$$\alpha \ge 1, \qquad 27eb\mu\ln\frac{1}{\mu} > 4\alpha^2\left(1+\frac{4^3}{\alpha}\right) \tag{56}$$

or

$$0 < \mu < \alpha \le 1, \qquad 27eb\mu\ln\frac{\alpha}{\mu} > 4\left(1+\frac{4^3}{\alpha}\right). \tag{57}$$

Using Corollary 9 with $\eta(t) = \lambda t$, $\lambda \le \mu$ and $\varphi = \alpha^2$ we have $Q(t;\varphi) = q(t)$ and consequently,

$$Q^*_\eta(t;\varphi,t_1) = \frac{27b}{t^2}\left[\lambda^{1/3}t^{1/3} - t_1^{1/3}\right]^3. \tag{58}$$

Hence,

$$\liminf_{t\to\infty}\int_{\eta(t)}^{t} Q^*_\eta(s;\varphi,t_1)\,ds = 27b\lambda\ln\frac{1}{\lambda}, \qquad \liminf_{t\to\infty}\int_{\tau^{-1}(\eta(t))}^{t} Q^*_\eta(s;\varphi,t_1)\,ds = 27b\lambda\ln\frac{\alpha}{\lambda} \tag{59}$$

and (54) is oscillatory if

$$\alpha \ge 1, \quad 27eb\lambda\ln\frac{1}{\lambda} > \left(1+4\alpha^{1/3}\right)^3 \tag{60}$$

or

$$0 < \lambda < \alpha \le 1, \quad 27eb\lambda\ln\frac{\alpha}{\lambda} > \left(1+4\alpha^{1/3}\right)^3. \tag{61}$$

If $\lambda = \mu$, the comparison of our lower bound $(1+4\alpha^{1/3})^3$ and the upper bound $4(1+4^3/\alpha)$ if $\alpha < 1$ and $4\alpha^2(1+4^3/\alpha)$ if $\alpha > 1$ is on Figure 3. Note that in contrast to the linear case, both curves do not intersect at $\alpha = 1$.

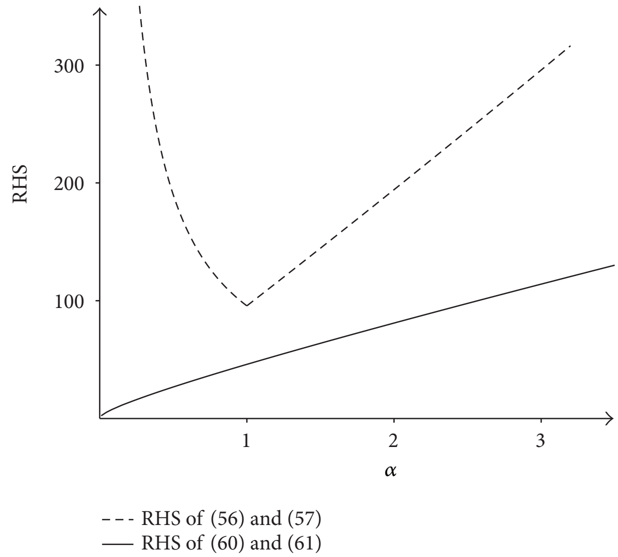

FIGURE 3: The graph of right hand sides (RHS) of (56) and (57) for $p_0 = 4$ compared to (60) and (61) as functions of α.

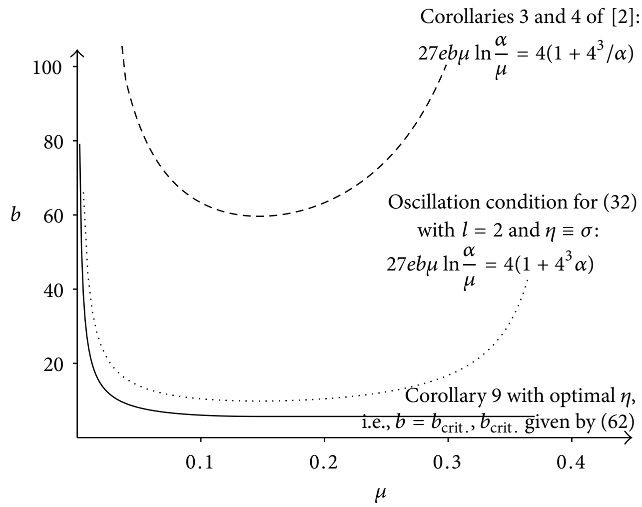

FIGURE 4: A comparison of the lower bounds for the coefficient b which guarantee oscillation of (54) for different values of μ. Parameters are $\alpha = 0.4$ and $p_0 = 4$.

The function $f(\lambda) = \lambda \ln(\alpha/\lambda)$ is positive on the interval $(0, \alpha)$ and it follows from Lemma 1 that the maximal value of $f(\lambda)$ is α/e at the point $\lambda = \alpha/e$. Hence, (54) is oscillatory if

$$b > b_{\text{crit.}} := \begin{cases} \frac{1}{27}\left(1+4\alpha^{1/3}\right)^3 & \text{for } \alpha \ge 1,\ \mu \ge \frac{1}{e}, \\ \frac{1}{27\alpha}\left(1+4\alpha^{1/3}\right)^3 & \text{for } \alpha \le 1,\ \mu \ge \frac{\alpha}{e}, \\ \frac{1}{27e\mu \ln(1/\mu)}\left(1+4\alpha^{1/3}\right)^3 & \text{for } \alpha \ge 1,\ \mu \le \frac{1}{e}, \\ \frac{1}{27e\mu \ln(\alpha/\mu)}\left(1+4\alpha^{1/3}\right)^3 & \text{for } \alpha \le 1,\ \mu \le \frac{\alpha}{e}. \end{cases} \quad (62)$$

Figure 4 shows how the critical constant which ensures oscillation of (54) is improved with respect to the results of [2] (dashed line) for various values of the delay μ. For readers convenience we graphed also a dotted curve which is only partial improvement of [2]: the values of l and η are chosen as in [2] and the value of φ (which plays role in Q) is choosen to equal to $q(t)/q(\tau(t))$.

The following theorem and corollary are variants of Theorem 7 and Corollary 9 for sublinear case $\beta \le 1$.

Theorem 12. *Let $\beta \le 1$, $\varphi > 0$, and $\eta(t)$ a function which satisfies $\eta(t) \le \sigma(t)$ and $\lim_{t\to\infty}\eta(t) = \infty$. Suppose that there exists a number $T > t_0$ and a solution $x(t)$ of* (1) *which satisfy*

$$x(t) > 0, \quad z'(t) > 0 \quad \text{for } t \ge T. \quad (63)$$

Let $t_1 > T$ be such that

$$\min\{\eta(\tau(t)), \eta(t)\} > T \quad (64)$$

for every $t \ge t_1$ and let $t_2 \ge t_1$ be such that $\eta(t) \ge t_1$ for $t \ge t_2$. Then the following statements are true.

(i) *The inequality*

$$\left[w(t) + \frac{p_0^\beta \varphi}{\tau_0} w(\tau(t))\right]' + Q_\eta^*(t;\varphi,t_1)\, w^{\beta/\gamma}(\eta(t)) \le 0 \quad (65)$$

has a positive decreasing solution on (t_2, ∞).

(ii) *If $\tau(t) \ge t$, then*

$$y' + Q_\eta^*(t;\varphi,t_1)\left(\frac{\tau_0}{\tau_0 + p_0^\beta \varphi}\right)^{\beta/\gamma} y^{\beta/\gamma}(\eta(t)) \le 0 \quad (66)$$

has a positive solution on (t_2, ∞).

(iii) *If $\tau(t) \le t$, then*

$$y' + Q_\eta^*(t;\varphi,t_1)\left(\frac{\tau_0}{\tau_0 + p_0^\beta \varphi}\right)^{\beta/\gamma} y^{\beta/\gamma}\left(\tau^{-1}(\eta(t))\right) \le 0 \quad (67)$$

has a positive solution on (t_2, ∞).

Proof. The proof is the same as the proof of Theorem 7 where we formally put $l = l^* = 1$ and use

$$(A+B)^\beta \le A^\beta + B^\beta \quad (68)$$

instead of Lemma 3. □

Corollary 13. *Let $\beta \le 1$, $\gamma \ge \beta$. Equation* (1) *has no solution $x(t)$ which satisfies*

$$x(t) > 0, \quad z'(t) > 0 \quad \text{eventually} \quad (69)$$

if there exists $\varphi > 0$ and a function $\eta(t)$ satisfying $\eta(t) \le \sigma(t)$ and $\lim_{t\to\infty}\eta(t) = \infty$ such that one of the following conditions holds.

(i) $\eta(t) < t \le \tau(t)$ *and for every* T *there exists* $t_1 > T$ *such that*

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} Q_{\eta}^{*}(s;\varphi,t_1)\, ds > \frac{1}{e}\left(\frac{\tau_0 + p_0^{\beta}\varphi}{\tau_0}\right)^{\beta/\gamma} \quad (70)$$

if $\beta = \gamma$, *and* (44) *holds if* $\beta < \gamma$.

(ii) $\eta(t) < \tau(t) \le t$ *and for every* T *there exists* $t_1 > T$ *such that*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\eta(t))}^{t} Q_{\eta}^{*}(s;\varphi,t_1)\, ds > \frac{1}{e}\left(\frac{\tau_0 + p_0^{\beta}\varphi}{\tau_0}\right)^{\beta/\gamma} \quad (71)$$

if $\beta = \gamma$, *and* (44) *holds if* $\beta < \gamma$.

Proof. It follows from Theorem 12 and Lemmas 6 and 1. □

Remark 14. If $\varphi = 1$ and $\eta(t) = \sigma(t)$, then Theorem 12 reduces to [2, Theorems 1, 2, and 3] and Corollary 13 reduces to [2, Corollaries 1 and 2].

4. Positive Solutions with $z'(t) < 0$ Eventually

In this section we modify the methods from previous section for positive solutions $x(t)$ which satisfy $z'(t) < 0$ eventually. Troughout this section we will suppose that (4) holds, since if it fails, then evetually positive solutions with $z'(t) < 0$ eventually do not exist.

The function $Q_{\zeta}^{\star}$ defined by the relation

$$Q_{\zeta}^{\star}(t;\varphi) = Q(t;\varphi)\left[\int_{\zeta(t)}^{\infty} r^{-1/\gamma}(s)\, ds\right]^{\beta} \quad (72)$$

and the following Theorem 15 are the corresponding modifications of the function Q_{η}^{*} and Theorem 7.

Theorem 15. *Let* $\beta \ge 1$, $\varphi > 0$, $l > 1$, *and let* $\zeta(t)$ *be a function which satisfies* $\zeta(t) \ge \sigma(t)$. *Suppose that there exists a number* $T > t_0$ *and a solution* $x(t)$ *of* (1) *which satisfy*

$$x(t) > 0, \quad z'(t) < 0 \quad \text{for } t \ge T. \quad (73)$$

Let $t_1 > T$ *be such that*

$$\min\{\sigma(t), \sigma(\tau(t))\} > T \quad (74)$$

for every $t \ge t_1$. *Then the following statements are true.*

(i) *The inequality*

$$\left[l^{\beta-1}u(t) + \frac{p_0^{\beta}\varphi}{\tau_0}(l^{*})^{\beta-1}u(\tau(t))\right]' - Q_{\zeta}^{\star}(t;\varphi)\, u^{\beta/\gamma}(\zeta(t)) \ge 0 \quad (75)$$

has a positive increasing solution on (t_1,∞).

(ii) *If* $\tau(t) \le t$, *then*

$$y' - Q_{\zeta}^{\star}(t;\varphi)\, h^{-\beta/\gamma}\left(l, \frac{p_0^{\beta}\varphi}{\tau_0}\right) y^{\beta/\gamma}(\zeta(t)) \ge 0 \quad (76)$$

has a positive solution on (t_1,∞).

(iii) *If* $\tau(t) \ge t$, *then*

$$y' - Q_{\zeta}^{\star}(t;\varphi)\, h^{-\beta/\gamma}\left(l, \frac{p_0^{\beta}\varphi}{\tau_0}\right) y^{\beta/\gamma}\left(\tau^{-1}(\zeta(t))\right) \ge 0 \quad (77)$$

has a positive solution on (t_1,∞).

Proof. Let x be a solution of (1) which satisfies $x(t) > 0$ and $z'(t) < 0$ for $t > T$. Inequalities (15), $p(t) \le p_0$, and $\zeta(t) \ge \sigma(t)$ imply

$$l^{\beta-1}x^{\beta}(\sigma(t)) + (l^{*})^{\beta-1}p_0^{\beta}x^{\beta}(\sigma(\tau(t))) \ge z^{\beta}(\zeta(t)). \quad (78)$$

Combining this inequality with (1) and (1) shifted from t to $\tau(t)$, similarly as in the proof of Theorem 7, we obtain

$$\begin{aligned} 0 \ge l^{\beta-1}\left[r(t)\Phi_{\gamma}\left(z'(t)\right)\right]' &+ \frac{(l^{*})^{\beta-1}p_0^{\beta}\varphi}{\tau_0} \\ &\times \left[r(\tau(t))\Phi_{\gamma}\left(z'(\tau(t))\right)\right]' + Q(t;\varphi)\, z^{\beta}(\zeta(t)) \end{aligned} \quad (79)$$

for $t \ge t_1$. The function w defined by $w(t) = r(t)\Phi_{\gamma}(z'(t))$ is negative and decreasing. Hence for $s \ge t$ we have

$$\begin{aligned} r(s)\Phi_{\gamma}\left(z'(s)\right) &\le r(t)\Phi_{\gamma}\left(z'(t)\right), \\ z'(s) &\le z'(t)\,\Phi_{\gamma}^{-1}\left(\frac{r(t)}{r(s)}\right) \end{aligned} \quad (80)$$

and hence

$$z(l) - z(t) \le \Phi_{\gamma}^{-1}(r(t))\, z'(t) \int_{t}^{l} r^{-1/\gamma}(s)\, ds. \quad (81)$$

Since $\lim_{l\to\infty} z(l) \ge 0$ we have

$$-z(t) \le \Phi_{\gamma}^{-1}(r(t))\, z'(t) \int_{t}^{\infty} r^{-1/\gamma}(s)\, ds, \quad (82)$$

which implies

$$z^{\beta}(\zeta(t)) \ge (-w(\zeta(t)))^{\beta/\gamma}\left[\int_{\zeta(t)}^{\infty} r^{-1/\gamma}(s)\, ds\right]^{\beta}. \quad (83)$$

Combining this inequality with (79) and multiplying by -1 we find that $u(t) = -w(t)$ is a positive and increasing solution of (75). Claim (i) is proved. Denote

$$y(t) = l^{\beta-1}w(t) + \frac{p_0^{\beta}\varphi}{\tau_0}(l^{*})^{\beta-1}w(\tau(t)). \quad (84)$$

Since $u(t)$ is positive and increasing, we have $u(\tau(t)) \le u(t)$ for $\tau(t) \le t$ and $u(\tau(t)) \ge u(t)$ for $\tau(t) \ge t$. Hence, if $\tau(t) \le t$, we have

$$y(t) \le u(t)\, h\left(l, \frac{p_0^{\beta}\varphi}{\tau_0}\right), \quad (85)$$

which implies

$$u^{\beta/\gamma}(\zeta(t)) \ge h^{-\beta/\gamma}\left(l, \frac{p_0^{\beta}\varphi}{\tau_0}\right) y^{\beta/\gamma}(\zeta(t)). \quad (86)$$

Analogously, if $\tau(t) \geq t$,

$$y(t) \leq u(\tau(t))\, h\left(l, \frac{p_0^{\beta}\varphi}{\tau_0}\right), \tag{87}$$

which implies

$$u^{\beta/\gamma}(\zeta(t)) \geq h^{-\beta/\gamma}\left(l, \frac{p_0^{\beta}\varphi}{\tau_0}\right) y^{\beta/\gamma}\left(\tau^{-1}(\zeta(t))\right). \tag{88}$$

Claims (ii) and (iii) then follow from (i) and positivity of y. ☐

Corollary 16. *Let $\beta \geq 1$ and $\beta \geq \gamma$. Equation* (1) *has no solution $x(t)$ which satisfies*

$$x(t) > 0, \quad z'(t) < 0 \quad \text{eventually} \tag{89}$$

if there exists $\varphi > 0$ and a function $\zeta(t)$ satisfying $\zeta(t) \geq \sigma(t)$ such that one of the following conditions holds.

(i) *$\tau(t) \leq t < \zeta(t)$ and*

$$\liminf_{t\to\infty} \int_t^{\zeta(t)} Q_\zeta^{\star}(s;\varphi)\, ds > \frac{1}{e}\left(1 + \left(\frac{\varphi}{\tau_0}\right)^{1/\beta} p_0\right)^{\beta^2/\gamma} \tag{90}$$

if $\beta = \gamma$, and

$$\int_{t_0}^{\infty} Q_\zeta^{\star}(t;\varphi)\, dt = \infty \tag{91}$$

if $\beta > \gamma$.

(ii) *$t \leq \tau(t) < \zeta(t)$ and*

$$\liminf_{t\to\infty} \int_t^{\tau^{-1}(\zeta(t))} Q_\zeta^{\star}(s;\varphi)\, ds > \frac{1}{e}\left(1 + \left(\frac{\varphi}{\tau_0}\right)^{1/\beta} p_0\right)^{\beta^2/\gamma} \tag{92}$$

if $\beta = \gamma$, and (91) *holds if $\beta > \gamma$.*

Proof. It follows from Theorem 15 and Lemmas 6 and 1. ☐

In a similar way as in Theorem 12 and Corollary 13, if we suppose $\beta \leq 1$ in Theorem 15 and Corollary 16 and use inequality (68) instead of inequality from Lemma 3, we get the following statements.

Theorem 17. *Let $\beta \leq 1$, $\varphi > 0$, and $\zeta(t)$ a function which satisfies $\zeta(t) \geq \sigma(t)$. Suppose that there exists a number $T > t_0$ and a solution $x(t)$ of* (1) *which satisfy*

$$x(t) > 0, \quad z'(t) < 0 \quad \text{for } t \geq T. \tag{93}$$

Let $t_1 > T$ be such that

$$\min\{\sigma(t), \sigma(\tau(t))\} > T \tag{94}$$

for every $t \geq t_1$. Then the following statements are true.

(i) *The inequality*

$$\left[u(t) + \frac{p_0^{\beta}\varphi}{\tau_0} u(\tau(t))\right]' - Q_\zeta^{\star}(t;\varphi)\, u^{\beta/\gamma}(\zeta(t)) \geq 0 \tag{95}$$

has a positive increasing solution on (t_1, ∞).

(ii) *If $\tau(t) \leq t$, then*

$$y' - Q_\zeta^{\star}(t;\varphi)\left(\frac{\tau_0}{\tau_0 + p_0^{\beta}\varphi}\right)^{\beta/\gamma} y^{\beta/\gamma}(\zeta(t)) \geq 0 \tag{96}$$

has a positive solution on (t_1, ∞).

(iii) *If $\tau(t) \geq t$, then*

$$y' - Q_\zeta^{\star}(t;\varphi)\left(\frac{\tau_0}{\tau_0 + p_0^{\beta}\varphi}\right)^{\beta/\gamma} y^{\beta/\gamma}\left(\tau^{-1}(\zeta(t))\right) \geq 0 \tag{97}$$

has a positive solution on (t_1, ∞).

Corollary 18. *Let $\beta \leq 1$, $\gamma \leq \beta$. Equation* (1) *has no solution $x(t)$ which satisfies*

$$x(t) > 0 \quad z'(t) < 0 \quad \text{eventually} \tag{98}$$

if there exists $\varphi > 0$ and a function $\zeta(t)$ satisfying $\eta(t) \geq \sigma(t)$ such that one of the following conditions holds.

(i) *$\tau(t) \leq t < \zeta(t)$ and*

$$\liminf_{t\to\infty} \int_t^{\zeta(t)} Q_\zeta^{\star}(s;\varphi)\, ds > \frac{1}{e}\left(\frac{\tau_0 + p_0^{\beta}\varphi}{\tau_0}\right)^{\beta/\gamma} \tag{99}$$

if $\beta = \gamma$, and (91) *holds if $\beta > \gamma$.*

(ii) *$t \leq \tau(t) < \zeta(t)$ and*

$$\liminf_{t\to\infty} \int_t^{\tau^{-1}(\zeta(t))} Q_\zeta^{\star}(s;\varphi)\, ds > \frac{1}{e}\left(\frac{\tau_0 + p_0^{\beta}\varphi}{\tau_0}\right)^{\beta/\gamma} \tag{100}$$

if $\beta = \gamma$, and (91) *holds if $\beta > \gamma$.*

5. Oscillation Criteria If (4) Holds

As we explained before, it follows from Lemma 5 that if (3) holds, then the criteria from Section 3 are in fact oscillation criteria. If (3) fails and (4) holds, then the set of all possible eventually positive solutions is more comprehensive and may contain also solution which satisfy $z'(t) < 0$ eventually. Hence to ensure oscillation of (1) in the case (4) we have to eliminate both cases; criteria from both Sections 3 and 4 apply. For example, in the half-linear case $\beta = \gamma$, (1) is oscillatory if either conditions

$$(4),\ (43),\ \text{and}\ (92)\ \text{hold if}\ \tau(t) \geq t, \tag{101}$$

or

$$(4),\ (45)\ \text{and}\ (90)\ \text{hold if}\ \tau(t) \leq t; \tag{102}$$

see the example below.

Example 19. Consider the equation

$$\left(t^{3/2}\left(x(t)+p_0x(\alpha t)\right)'\right)'+\frac{a}{\sqrt{t}}x(\mu t)=0,\quad a>0. \tag{103}$$

We have $\gamma=\beta=1$, $\tau(t)=\alpha t$, $\sigma(t)=\mu t$, $r(t)=t^{3/2}$, $q(t)=at^{-1/2}$, $q(\tau(t))=a(\alpha t)^{-1/2}$. We will apply Corollaries 9 and 16 with $\eta(t)=\lambda_1 t$, $\zeta(t)=\lambda_2 t$, $\lambda_1\le\mu\le\lambda_2$ and $\varphi=\alpha^{1/2}$. We have $Q(t)=q(t)$. Since

$$\int r^{-1/\gamma}(t)\,\mathrm{d}t=\int t^{-3/2}\mathrm{d}t=-2t^{-1/2}, \tag{104}$$

condition (4) holds. Next we have

$$\begin{aligned}Q_\eta^*(t;\varphi,t_1)&=-2a\left(\sqrt{\lambda_1}t^{-1}-\sqrt{t_1}t^{-1/2}\right),\\ Q_\zeta^*(t;\varphi)&=-2at^{-1/2}\left(\lim_{t\to\infty}t^{-1/2}-(\lambda_2 t)^{-1/2}\right)\\ &=2a\lambda^{-1/2}t^{-1}.\end{aligned} \tag{105}$$

Consequently, if $\lambda_1<1\le\alpha<\lambda_2$, then (43) and (92) give

$$\begin{aligned}&\liminf_{t\to\infty}\int_{\eta(t)}^{t}Q_\eta^*(s;\varphi,t_1)\,\mathrm{d}s\\ &\quad=-2a\sqrt{\lambda_1}\ln\frac{1}{\lambda_1}+2a\sqrt{t_1}\left(1-\sqrt{\lambda_1}\right)\lim_{t\to\infty}\sqrt{t}\\ &\quad=\infty,\\ &\liminf_{t\to\infty}\int_t^{\tau^{-1}(\zeta(t))}Q_\zeta^*(s;\varphi)\,\mathrm{d}s=\frac{2a}{\sqrt{\lambda_2}}\ln\frac{\lambda_2}{\alpha}.\end{aligned} \tag{106}$$

If $\lambda_1<\alpha\le 1<\lambda_2$, then (45) and (90) give

$$\begin{aligned}&\liminf_{t\to\infty}\int_{\tau^{-1}(\eta(t))}^{t}Q_\eta^*(s;\varphi,t_1)\,\mathrm{d}s\\ &\quad=-2a\sqrt{\lambda_1}\ln\frac{\alpha}{\lambda_1}+2a\sqrt{t_1}\left(1-\sqrt{\frac{\lambda_1}{\alpha}}\right)\lim_{t\to\infty}\sqrt{t}\\ &\quad=\infty,\\ &\liminf_{t\to\infty}\int_t^{\zeta(t)}Q_\zeta^*(s;\varphi)\,\mathrm{d}s=\frac{2a}{\sqrt{\lambda_2}}\ln\lambda_2.\end{aligned} \tag{107}$$

Hence, (103) is oscillatory if

$$\lambda_1<1\le\alpha<\lambda_2,\qquad \frac{2a}{\sqrt{\lambda_2}}\ln\frac{\lambda_2}{\alpha}>\frac{1}{e}\left(1+\frac{p_0}{\sqrt{\alpha}}\right) \tag{108}$$

or

$$\lambda_1<\alpha\le 1<\lambda_2,\qquad \frac{2a}{\sqrt{\lambda_2}}\ln\lambda_2>\frac{1}{e}\left(1+\frac{p_0}{\sqrt{\alpha}}\right). \tag{109}$$

The function $g(\lambda_2)=\lambda_2^{-1/2}\ln(\lambda_2/\alpha)$ is positive on the interval (α,∞) and it follows from Lemma 1 that the maximal value of $g(\lambda_2)$ is $2/(e\sqrt{\alpha})$ at the point αe^2. Hence, (103) is oscillatory if

$$a>a_{\text{crit.}}:=\begin{cases}\dfrac{\sqrt{\alpha}}{4}\left(1+\dfrac{p_0}{\sqrt{\alpha}}\right) & \text{for } \alpha\ge 1,\ \mu\le\alpha e^2,\\ \dfrac{1}{4}\left(1+\dfrac{p_0}{\sqrt{\alpha}}\right) & \text{for } \alpha\le 1,\ \mu\le e^2,\\ \dfrac{\sqrt{\mu}}{2e\ln\left(\dfrac{\mu}{\alpha}\right)}\left(1+\dfrac{p_0}{\sqrt{\alpha}}\right) & \text{for } \alpha\ge 1,\ \mu\ge\alpha e^2,\\ \dfrac{\sqrt{\mu}}{2e\ln\mu}\left(1+\dfrac{p_0}{\sqrt{\alpha}}\right) & \text{for } \alpha\le 1,\ \mu\ge e^2.\end{cases} \tag{110}$$

6. Conclusion

In the paper we derived asymptotic results for neutral quasilinear equation (1). Note that this equation covers several types of second-order differential equations studied in the literature, namely, the linear and half-linear second-order differential equations.

Using the comparison method we derived sufficient conditions for nonexistence of eventually positive solutions with various asymptotic behaviors. Additional assumptions (such as (3)) or suitable combinations of the results yield oscillation criteria for this equation. The novelty of the presented results is in the fact that we used parametrized versions of inequalities used typically in comparison theory of neutral differential equations. Despite the fact that we introduced three parameters (l, φ, and η), the results remain simple and effective. We have shown on several examples that effective oscillation criteria can be formulated for particular equations by establishing the optimal values for these parameters.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the referee for valuable comments to the paper. Research supported by the Grant P201/10/1032 of the Czech Science Foundation.

References

[1] B. Baculíková and J. Džurina, "Oscillation theorems for second order neutral differential equations," *Computers & Mathematics with Applications*, vol. 61, no. 1, pp. 94–99, 2011.

[2] B. Baculíková and J. Džurina, "Oscillation theorems for second-order nonlinear neutral differential equations," *Computers & Mathematics with Applications*, vol. 62, no. 12, pp. 4472–4478, 2011.

[3] B. Baculíková, T. Li, and J. Džurina, "Oscillation theorems for second order neutral differential equations," *Electronic Journal*

of Qualitative Theory of Differential Equations, no. 74, pp. 1–13, 2011.

[4] B. Baculíková, T. Li, and J. Džurina, "Oscillation theorems for second-order superlinear neutral differential equations," *Mathematica Slovaca*, vol. 63, no. 1, pp. 123–134, 2013.

[5] T. Li, "Comparison theorems for second-order neutral differential equations of mixed type," *Electronic Journal of Differential Equations*, vol. 2010, no. 167, pp. 1–7, 2010.

[6] J.-G. Dong, "Oscillation behavior of second order nonlinear neutral differential equations with deviating arguments," *Computers & Mathematics with Applications*, vol. 59, no. 12, pp. 3710–3717, 2010.

[7] S. Fišnarová and R. Mařík, "Oscillation of half-linear differential equations with delay," *Abstract and Applied Analysis*, vol. 2013, Article ID 583147, 6 pages, 2013.

[8] M. Hasanbulli and Y. V. Rogovchenko, "Oscillation criteria for second order nonlinear neutral differential equations," *Applied Mathematics and Computation*, vol. 215, no. 12, pp. 4392–4399, 2010.

[9] R. P. Agarwall, M. Bohner, T. Li, and Ch. Zhuang, "Oscillation of second order Emden-Fowler neutral delay differential equations," in *Annali di Matematica Pura ed Applicata*, pp. 1–15, 2013.

[10] Z. Han, T. Li, S. Sun, and Y. Sun, "Remarks on the paper," *Applied Mathematics and Computation*, vol. 207, pp. 388–396, 2009, Applied Mathematics and Computation, vol. 215, pp. 3998–4007, 2010.

[11] T. Li, Z. Han, Ch. Zhang, and H. Li, "Oscillation criteria for second-order superlinear neutral differential equations," *Abstract and Applied Analysis*, vol. 2011, Article ID 367541, 17 pages, 2011.

[12] T. Li, Y. V. Rogovchenko, and Ch. Zhang, "Oscillation of second-order neutral differential equations," *Funkcialaj Ekvacioj*, vol. 56, no. 1, pp. 111–120, 2013.

[13] Y. G. Sun and F. W. Meng, "Note on the paper of Džurina and Stavroulakis," *Applied Mathematics and Computation*, vol. 174, no. 2, pp. 1634–1641, 2006.

[14] S. Sun, T. Li, Z. Han, and Y. Sun, "Oscillation of second-order neutral functional differential equations with mixed nonlinearities," *Abstract and Applied Analysis*, vol. 2011, Article ID 927690, 15 pages, 2011.

[15] S. Sun, T. Li, Z. Han, and H. Li, "Oscillation theorems for second-order quasilinear neutral functional differential equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 819342, 17 pages, 2012.

[16] L. Ye and Z. Xu, "Oscillation criteria for second order quasilinear neutral delay differential equations," *Applied Mathematics and Computation*, vol. 207, no. 2, pp. 388–396, 2009.

[17] J. R. Graef and S. H. Saker, "New oscillation criteria for generalized second-order nonlinear neutral functional differential equations," *Dynamic Systems and Applications*, vol. 19, no. 3-4, pp. 455–472, 2010.

[18] S. Fišnarová and R. Mařík, "Oscillation criteria for neutral second-order half-linear differential equations with applications to Euler type equations," *Boundary Value Problems*, vol. 2014, article 83, 2014.

[19] R. P. Agarwall, S. R. Grace, and D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[20] R. G. Koplatadze and T. A. Chanturiya, "Oscillating and monotone solutions of first-order differential equations with deviating argument," *Differentsial'nye Uravneniya*, vol. 18, no. 8, pp. 1463–1472, 1982 (Russian).

[21] Y. Kitamura and T. Kusano, "Oscillation of first-order nonlinear differential equations with deviating arguments," *Proceedings of the American Mathematical Society*, vol. 78, no. 1, pp. 64–68, 1980.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 593210, 7 pages
http://dx.doi.org/10.1155/2014/593210

Research Article

Symmetry and Nonexistence of Positive Solutions for Weighted HLS System of Integral Equations on a Half Space

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Received 28 January 2014; Revised 15 April 2014; Accepted 15 April 2014; Published 30 April 2014

Academic Editor: Bingwen Liu

We consider system of integral equations related to the weighted Hardy-Littlewood-Sobolev (HLS) inequality in a half space. By the Pohozaev type identity in integral form, we present a Liouville type theorem when the system is in both supercritical and subcritical cases under some integrability conditions. Ruling out these nonexistence results, we also discuss the positive solutions of the integral system in critical case. By the method of moving planes, we show that a pair of positive solutions to such system is rotationally symmetric about x_n-axis, which is much more general than the main result of Zhuo and Li, 2011.

1. Introduction

In [1], Jin and Li studied the weighted HLS system of nonlinear equations in R^n:

$$u(x) = \frac{1}{|x|^{\alpha}} \int_{R^n} \frac{1}{|x-y|^{\lambda}} \frac{v^q(y)}{|y|^{\beta}} dy,$$
$$v(x) = \frac{1}{|x|^{\beta}} \int_{R^n} \frac{1}{|x-y|^{\lambda}} \frac{u^p(y)}{|y|^{\alpha}} dy, \tag{1}$$

where $0 < \lambda < n$ and $1/(p+1) + 1/(q+1) = (\lambda + \alpha + \beta)/n$.

By the method of moving planes in integral forms they derived symmetry and monotonicity of positive solutions of (1) under some integrability conditions.

Theorem 1 (see [1]). *Let the pair (u, v) be a positive solution of system* (1) *with $u \in L^{p+1}(R^n)$, $v \in L^{q+1}(R^n)$ and $p, q \geq 1$, $pq \neq 1$, and $\alpha, \beta \geq 0$. Then u and v are radially symmetric and decreasing about some point x_0.*

Jin and Li [2] and Chen et al. [3] also discussed the regularity of solutions to (1).

Let R^n_+ be the upper half Euclidean space

$$R^n_+ = \{x = (x_1, \ldots, x_n) \in R^n \mid x_n > 0\}. \tag{2}$$

In this paper, we want to consider the similar integral system in the half space R^n_+ as (1). More precisely, we discuss the following weighted HLS type system of nonlinear equations in R^n_+:

$$u(x) = \frac{1}{|x|^{\alpha}} \int_{R^n_+} G(x, y, \gamma) \frac{v^q(y)}{|y|^{\beta}} dy,$$
$$v(x) = \frac{1}{|x|^{\beta}} \int_{R^n_+} G(x, y, \gamma) \frac{u^p(y)}{|y|^{\alpha}} dy, \tag{3}$$

where $u, v \geq 0$, $0 < p, q < \infty$, $0 < \gamma < n$, $\alpha + \beta \geq 0$, $\alpha/n < 1/(p+1) < (n - \gamma + \alpha)/n$, and

$$G(x, y, \gamma) = \frac{1}{|x-y|^{n-\gamma}} - \frac{1}{|x^* - y|^{n-\gamma}}; \tag{4}$$

here x^* is the reflection point of x about the plane ∂R^n_+.

Similar to some integral systems or PDEs systems, the integral system (3) is usually divided into three cases according to the value of exponents (p, q). We say that system (3) is in critical case when the pair (p, q) satisfies the relation

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n - \gamma + \alpha + \beta}{n}. \tag{5}$$

It is in supercritical case when "<" holds; and in subcritical case when ">" holds; that is

$$\frac{1}{p+1}+\frac{1}{q+1}>\frac{n-\gamma+\alpha+\beta}{n}. \tag{6}$$

In the special case, where $\alpha = 0$ and $\beta = 0$, system (3) reduces to

$$\begin{aligned} u(x) &= \int_{R_+^n} G(x,y,\gamma)\, v^q(y)\, dy, \\ v(x) &= \int_{R_+^n} G(x,y,\gamma)\, u^p(y)\, dy, \end{aligned} \tag{7}$$

and system (7) is closely related to the following system of PDEs with Navier boundary conditions:

$$\begin{aligned} &(-\Delta)^{\gamma/2} u = v^q, \quad \text{in } R_+^n; \\ &(-\Delta)^{\gamma/2} v = u^p, \quad \text{in } R_+^n; \\ &(-\Delta)^{k} u = 0 \quad \text{on } \partial R_+^n; \\ &(-\Delta)^{k} v = 0, \quad \text{on } \partial R_+^n; \\ &\quad k = 0, 1, \ldots, \frac{\gamma}{2}-1. \end{aligned} \tag{8}$$

In particular, when γ is an even number, the authors ([4]) proved the equivalence between the two systems (7) and (8) under some mild growth condition.

Symmetry of solutions to integral system (8) was established by Zhuo and Li [5]. They proved that in critical case $1/(p+1) + 1/(q+1) = (n-\alpha)/n$, any pair of positive solutions of (7) with $u \in L^{p+1}(R_+^n)$ and $v \in L^{q+1}(R_+^n)$ is rotationally symmetric about some line parallel to x_n-axis. Under the same integrability conditions, in [6], we obtained the nonexistence of positive solutions of (7).

The general case is that, for $\alpha \neq 0$ and $\beta \neq 0$ in (3), there are few results concerning symmetry and nonexistence for this doubled weighted system. In this paper, by the Pohozaev type identity in integral form, we present a Liouville type theorem when the system (3) is in both supercritical and subcritical cases under some integrability conditions. Based on these nonexistence results, we discuss the positive solutions of (3) in critical case. By the method of moving planes, we show that a pair of positive solutions to such system is rotationally symmetric about x_n-axis. To carry on the moving of planes, we explore global features of the integral equations and estimate certain integral norms. This is the essence of the method of moving planes in integral forms. The readers who are interested in the integral system and the applications of this method may consult [7–10] and the references therein.

The paper is organized as follows.

In Section 2, by the Pohozaev type identity in integral forms, we prove the following nonexistence results.

Theorem 2. *Suppose that $(u(x), v(x)) \in C^1(R^n)$ are nonnegative solutions of (3) with $u \in L^{p+1}(R_+^n)$, $v \in L^{q+1}(R_+^n)$.*

(i) *If p and q are both supercritical, that is,*

$$\frac{1}{p+1}<\frac{n-\gamma}{2n}+\frac{\alpha}{n}, \qquad \frac{1}{q+1}<\frac{n-\gamma}{2n}+\frac{\beta}{n}, \tag{9}$$

or

(ii) *if p and q are both subcritical, that is,*

$$\begin{aligned} \frac{1}{p+1} &\in \left(\frac{n-\gamma}{2n}+\frac{\alpha}{n}, \frac{n-\gamma+\alpha}{n}\right), \\ \frac{1}{q+1} &\in \left(\frac{n-\gamma}{2n}+\frac{\beta}{n}, \frac{n-\gamma+\beta}{n}\right), \end{aligned} \tag{10}$$

then $u \equiv 0$ and $v \equiv 0$.

Based on these results and ruling out cases where there are no solutions, we are only interested in critical case (5). In Section 3, by means of method of moving planes in integral form, we establish rotational symmetry of solutions of (3) in critical case (5) as follows.

Theorem 3. *Assume that $u \in L^{p+1}(R_+^n)$, $v \in L^{q+1}(R_+^n)$ and p, q satisfy (5). If (u, v) is a pair of positive solutions of (3), then (u, v) is rotationally symmetric about x_n-axis.*

Remark 4. When $\alpha = \beta = 0$, Theorem 3 is coincident with the result in [5].

2. Proof of Theorem 2

In this section we will prove the nonexistence of positive solutions to the weighted HLS type system (3). These nonexistence results, known as Liouville type theorems, are useful in deriving existence, a priori estimate, regularity, and asymptotic analysis of solutions.

A celebrated result of S. I. Pohozaev is known as the Pohozaev identity. This classical result has many consequences, the most immediate one being the nonexistence of nontrivial bounded solutions to PDE. Here we apply the Pohozaev type identity in integral forms to the integral system (3) (see in [9, 11]).

For any $\rho \neq 0$, there holds

$$u(\rho x)=\frac{1}{|\rho x|^{\alpha}}\int_{R_+^n}\left(\frac{1}{|\rho x-y|^{n-\gamma}}-\frac{1}{|\rho x^*-y|^{n-\gamma}}\right)\frac{v^q(y)}{|y|^{\beta}}dy. \tag{11}$$

By an elementary calculation,

$$\begin{aligned} &\frac{d\left(|\rho x|^{-\alpha}\right)}{d\rho} \\ &\quad = -\frac{\alpha}{2}|\rho x|^{-\alpha-2}\cdot(2\rho x\cdot x) \\ &\quad = (-\alpha\rho)\,|\rho x|^{-\alpha-2}|x|^2. \end{aligned}$$

$$\begin{aligned}
&\frac{d\left(|\rho x-y|^{\gamma-n}\right)}{d\rho}\\
&\quad=\frac{\gamma-n}{2}|\rho x-y|^{\gamma-n-2}\\
&\qquad\times\frac{d}{d\rho}\left[(\rho x_1-y_1)^2+\cdots+(\rho x_n-y_n)^2\right]\\
&\quad=(\gamma-n)\,|\rho x-y|^{\gamma-n-2}x\cdot(\rho x-y),\\
&\frac{d\left(|\rho x^*-y|^{\gamma-n}\right)}{d\rho}\\
&\quad=\frac{\gamma-n}{2}|\rho x^*-y|^{\gamma-n-2}\\
&\qquad\times\frac{d}{d\rho}\left[(\rho x_1-y_1)^2+\cdots+(\rho x_{n-1}-y_{n-1})^2\right.\\
&\qquad\qquad\left.+(-\rho x_n-y_n)^2\right]\\
&\quad=(\gamma-n)\,|\rho x-y|^{\gamma-n-2}x^*\cdot(\rho x^*-y).
\end{aligned}\tag{12}$$

Noting $u\in C^1(R^n)$, differentiating both sides of (11) with respect to ρ and letting $\rho=1$, we have

$$\begin{aligned}
x\cdot\nabla u(x)=&\;(-\alpha)\,u(x)\\
&+(\gamma-n)\frac{1}{|x|^\alpha}\int_{R^n_+}\left[\frac{x\cdot(x-y)}{|x-y|^{n-\gamma+2}}\right.\\
&\qquad\left.-\frac{x^*\cdot(x^*-y)}{|x^*-y|^{n-\gamma+2}}\right]\frac{v^q(y)}{|y|^\beta}dy.
\end{aligned}\tag{13}$$

Let $B_r^+(0)=B_r(0)\cap R^n_+$ be the upper half ball in the half space in R^n_+. Multiplying left side of (13) by $u^p(x)$ and integrating on B_r^+ yields

$$\begin{aligned}
&\int_{B_r^+}u^p(x)\,(x\cdot\nabla u(x))\,dx\\
&\quad=\frac{1}{p+1}\int_{B_r^+}x\cdot\nabla\left(u^{p+1}(x)\right)dx\\
&\quad=\frac{1}{p+1}\int_{\partial B_r^+}ru^{p+1}(x)\,d\sigma-\frac{n}{p+1}\int_{B_r^+}u^{p+1}(x)\,dx.
\end{aligned}\tag{14}$$

Similarly, we also have

$$\begin{aligned}
&\int_{B_r^+}v^q(x)\,(x\cdot\nabla v(x))\,dx\\
&\quad=\frac{1}{q+1}\int_{\partial B_r^+}rv^{q+1}(x)\,d\sigma-\frac{n}{q+1}\int_{B_r^+}v^{q+1}(x)\,dx.
\end{aligned}\tag{15}$$

Since

$$\int_{R^n_+}u^{p+1}(x)\,dx<\infty,\qquad\int_{R^n_+}v^{q+1}(x)\,dx<\infty.\tag{16}$$

Thus, there exists a sequence $\{r_m\}$ such that

$$\begin{gathered}
r_m\int_{\partial B_{r_m}^+}u^{p+1}(x)\,d\sigma\longrightarrow 0,\\
\int_{\partial B_{r_m}^+}r_m v^{q+1}(x)\,d\sigma\longrightarrow 0,\\
r_m\longrightarrow\infty.
\end{gathered}\tag{17}$$

Let $r_m\to\infty$; by (14), (15), and (17), we have

$$\begin{aligned}
&\int_{R^n_+}u^p(x)\,(x\cdot\nabla u(x))\,dx+\int_{R^n_+}v^q(x)\,(x\cdot\nabla v(x))\,dx\\
&\quad=-\frac{n}{p+1}\int_{R^n_+}u^{p+1}(x)\,dx-\frac{n}{q+1}\int_{R^n_+}v^{q+1}(x)\,dx<\infty.
\end{aligned}\tag{18}$$

On the other hand,

$$\begin{aligned}
&\int_{R^n_+}u^p(x)\,(x\cdot\nabla u(x))\,dx\\
&\quad=(\gamma-n)\int\int_{R^n_+}\left[\frac{x\cdot(x-y)}{|x-y|^{n-\gamma+2}}-\frac{x^*\cdot(x^*-y)}{|x^*-y|^{n-\gamma+2}}\right]\\
&\qquad\times\frac{u^p(x)\,v^q(y)}{|x|^\alpha|y|^\beta}dx\,dy\\
&\qquad+(-\alpha)\int_{R^n_+}u^{p+1}(x)\,dx\\
&\quad=\frac{\gamma-n}{2}\int\int_{R^n_+}\left[\frac{x\cdot(x-y)}{|x-y|^{n-\gamma+2}}-\frac{x^*\cdot(x^*-y)}{|x^*-y|^{n-\gamma+2}}\right]\\
&\qquad\times\frac{u^p(x)\,v^q(y)}{|x|^\alpha|y|^\beta}dx\,dy\\
&\qquad+\frac{\gamma-n}{2}\int\int_{R^n_+}\left[\frac{y\cdot(y-x)}{|x-y|^{n-\gamma+2}}-\frac{y^*\cdot(y^*-x)}{|y^*-x|^{n-\gamma+2}}\right]\\
&\qquad\times\frac{u^p(y)\,v^q(x)}{|y|^\alpha|x|^\beta}dx\,dy\\
&\qquad+(-\alpha)\int_{R^n_+}u^{p+1}(x)\,dx.
\end{aligned}\tag{19}$$

There also holds

$$\begin{aligned}
&\int_{R^n_+}v^q(x)\,(x\cdot\nabla v(x))\,dx\\
&\quad=(\gamma-n)\int\int_{R^n_+}\left[\frac{x\cdot(x-y)}{|x-y|^{n-\gamma+2}}-\frac{x^*\cdot(x^*-y)}{|x^*-y|^{n-\gamma+2}}\right]\\
&\qquad\times\frac{u^p(y)\,v^q(x)}{|y|^\alpha|x|^\beta}dx\,dy\\
&\qquad+(-\beta)\int_{R^n_+}v^{q+1}(x)\,dx
\end{aligned}$$

$$= \frac{\gamma - n}{2} \int\int_{R_+^n} \left[\frac{x \cdot (x-y)}{|x-y|^{n-\gamma+2}} - \frac{x^* \cdot (x^*-y)}{|x^*-y|^{n-\gamma+2}} \right] \times \frac{u^p(y) v^q(x)}{|y|^\alpha |x|^\beta}$$
$$+ \frac{\gamma - n}{2} \int\int_{R_+^n} \left[\frac{y \cdot (y-x)}{|x-y|^{n-\gamma+2}} - \frac{y^* \cdot (y^*-x)}{|y^*-x|^{n-\gamma+2}} \right] \times \frac{u^p(x) v^q(y)}{|x|^\alpha |y|^\beta} dx\, dy$$
$$+ (-\beta) \int_{R_+^n} v^{q+1}(x)\, dx. \tag{20}$$

Using

$$x \cdot (x-y) + y \cdot (y-x) = |x-y|^2,$$
$$x^* \cdot (x^*-y) + y^* \cdot (y^*-x) = |x^*-y|^2. \tag{21}$$

Combining the fact $|x^* - y| = |y^* - x|$, (19), and (20), we have

$$\int_{R_+^n} u^p(x)(x \cdot \nabla u(x))\, dx + \int_{R_+^n} v^q(x)(x \cdot \nabla v(x))\, dx$$
$$= \left(\frac{\gamma - n}{2} - \alpha\right) \int_{R_+^n} u^{p+1}(x)\, dx + \left(\frac{\gamma - n}{2} - \beta\right) \int_{R_+^n} v^{q+1}(x)\, dx. \tag{22}$$

By (18) and (22), we have

$$\left(\frac{\gamma - n}{2} - \alpha + \frac{n}{p+1}\right) \int_{R_+^n} u^{p+1}(x)\, dx + \left(\frac{\gamma - n}{2} - \beta + \frac{n}{q+1}\right) \int_{R_+^n} v^{q+1}(x)\, dx = 0. \tag{23}$$

Hence, if

$$\frac{\gamma - n}{2} - \alpha + \frac{n}{p+1} > 0, \qquad \frac{\gamma - n}{2} - \beta + \frac{n}{q+1} > 0 \tag{24}$$

or

$$\frac{\gamma - n}{2} - \alpha + \frac{n}{p+1} < 0, \qquad \frac{\gamma - n}{2} - \beta + \frac{n}{q+1} < 0, \tag{25}$$

hold, it follows that $u \equiv 0$ and $v \equiv 0$.

This completes the proof of Theorem 2.

Remark 5. In [11], the authors consider another weighted HLS type integral system

$$u(x) = \int_{R_+^n} G(x, y, \gamma) |y|^{-s} v^q(y)\, dy,$$
$$v(x) = \int_{R_+^n} G(x, y, \gamma) |y|^{-t} u^p(y)\, dy, \qquad \forall x \in R_+^n \tag{26}$$

and showed the Liouville type theorem as follows.

Theorem 6 (see [11]). *Suppose that $u(x), v(x) \in C^1(R^n)$ are positive solutions of (26) when p and q are both subcritical; that is $1/(p+1) > (n-\gamma)/2(n-t)$ and $1/(q+1) > (n-\gamma)/2(n-s)$. If $\int_{R_+^n}(u^{p+1}/|x|^t)dx < \infty$, $\int_{R_+^n}(v^{q+1}/|x|^s)dx < \infty$ and $\gamma - s > 1$, $\gamma - t > 1$, then $u \equiv 0$ and $v \equiv 0$.*

When $s = t = 0$ in system (26) or $\alpha = \beta = 0$ in system (3), the two systems reduce to the simple integral system (7). In this special case, we can find that Theorem 6 is coincident with case (ii) in Theorem 2.

3. Proof of Theorem 3

In this section, we will consider rotational symmetry of weighted HLS type system (3) in critical case (5).

Firstly, we need the following weighted HLS inequality.

Lemma 7 (see [12]). *Let $1 < l, m < \infty$, $0 < \gamma < n$, $\tau + \beta \geq 0$, $1/l + 1/m + (\gamma + \tau + \beta)/n = 2$, and $1 - 1/m - \gamma/n < \tau/n < 1 - 1/m$. Then*

$$\left| \int\int_{R^n} \frac{f(x) g(y)}{|x|^\tau |x-y|^\gamma |y|^\beta} dx\, dy \right| \leq C \|f\|_m \|g\|_l. \tag{27}$$

One can also write the weighted HLS inequality in another form. Let

$$Tg(x) = \int_{R^n} \frac{g(y)}{|x|^\tau |x-y|^\gamma |y|^\beta} dy. \tag{28}$$

Then

$$\|Tg(x)\|_{L^\mu} = \sup_{\|f\|_m = 1} < Tg(x), \quad f(x) > \leq \|g\|_{L^l}, \tag{29}$$

where $1/l + (\gamma + \tau + \beta)/n = 1 + 1/\mu$, $1/\mu + 1/m = 1$.

For a given real number λ, define

$$\Sigma_\lambda = \{x = (x_1, x_2, \ldots, x_n) \in R_+^n \mid x_1 < \lambda\},$$
$$T_\lambda = \{x \in R_+^n \mid x_1 = \lambda\}. \tag{30}$$

Let $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_{n-1}, x_n)$ be the reflection of the point $x = (x_1, x_2, \ldots, x_n)$ about the plane T_λ. Set

$$u_\lambda(x) = u(x^\lambda), \qquad v_\lambda(x) = v(x^\lambda). \tag{31}$$

Lemma 8 (see [8, 13]). *For $x, y \in \Sigma_\lambda$, $x \neq y$, one has*

$$G(x, y, \gamma) \geq G(x^\lambda, y, \gamma). \tag{32}$$

Lemma 9. *Let (u, v) be any pair of positive solutions of* (3) *in critical case* (5)*; for any $x \in \Sigma_\lambda$ and $|x| > |x^\lambda|$, one has*

$$u(x) - u_\lambda(x) \leq \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma)\right] \frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta} dy,$$
$$v(x) - v_\lambda(x) \leq \frac{1}{|x|^\beta} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma)\right] \frac{u^p(y) - u_\lambda^p(y)}{|y|^\alpha} dy. \tag{33}$$

Proof. Through the calculation, we have

$$u(x) = \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} G(x, y, \gamma) \frac{v^q(y)}{|y|^\beta} dy + \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} G(x^\lambda, y, \gamma) \frac{v_\lambda^q(y)}{|y^\lambda|^\beta} dy,$$
$$u_\lambda(x) = \frac{1}{|x^\lambda|^\alpha} \int_{\Sigma_\lambda} G(x^\lambda, y, \gamma) \frac{v^q(y)}{|y|^\beta} dy + \frac{1}{|x^\lambda|^\alpha} \int_{\Sigma_\lambda} G(x, y, \gamma) \frac{u_\lambda^p(y)}{|y^\lambda|^\beta} dy. \tag{34}$$

By the assumption $|x| > |x^\lambda|$, we have

$$\begin{aligned} u(x) - u_\lambda(x) &\leq \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma)\right] \times \left(\frac{v^q(y)}{|y|^\beta} - \frac{v_\lambda^q(y)}{|y^\lambda|^\beta}\right) dy \\ &\leq \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma)\right] \frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta} dy. \end{aligned} \tag{35}$$

Similarly, we have

$$v(x) - v_\lambda(x) \leq \frac{1}{|x|^\beta} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma)\right] \times \frac{u^p(y) - u_\lambda^p(y)}{|y|^\alpha} dy. \tag{36}$$

□

Proof of Theorem 3. Step 1. We will show that for sufficiently negative λ,

$$u_\lambda(x) \geq u(x), \quad v_\lambda(x) \geq v(x), \quad \text{a.e. } \forall x \in \Sigma_\lambda. \tag{37}$$

Define

$$\Sigma_\lambda^u = \{x \in \Sigma_\lambda, u(x) > u_\lambda(x)\}, \quad \Sigma_\lambda^v = \{x \in \Sigma_\lambda, v(x) > v_\lambda(x)\}. \tag{38}$$

We prove that, for sufficiently negative λ, both Σ_λ^u and Σ_λ^v must be empty and thus (37) holds.

In fact, by Lemma 9 and the mean value theorem, we have, for $x \in \Sigma_\lambda^u$,

$$\begin{aligned} 0 &< u(x) - u_\lambda(x) \\ &\leq \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma)\right] \frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta} dy \\ &\leq \int_{\Sigma_\lambda^v} \frac{1}{|x|^\alpha} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma)\right] \frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta} dy \\ &\leq \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda^v} G(x, y, \gamma) \frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta} dy \\ &\leq \frac{q}{|x|^\alpha} \int_{\Sigma_\lambda^v} \frac{1}{|x-y|^{n-\gamma}} \psi_\lambda^{q-1}(y) \frac{v(y) - v_\lambda(y)}{|y|^\beta} dy \\ &\leq q \int_{\Sigma_\lambda^v} \frac{1}{|x|^\alpha |x-y|^{n-\gamma} |y|^\beta} v^{q-1}(y) \left[v(y) - v_\lambda(y)\right] dy, \end{aligned} \tag{39}$$

where $\psi_\lambda(y)$ is valued between $v(y)$ and $v_\lambda(y)$; therefore, on Σ_λ^v, we have

$$0 \leq v_\lambda(y) \leq \psi_\lambda(y) \leq v(y). \tag{40}$$

By Lemma 7 and the Hölder inequality, we have

$$\begin{aligned} \|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} &\leq C \left\|v^{q-1}(v_\lambda - v)\right\|_{L^{T_1}(\Sigma_\lambda^v)} \\ &\leq C \|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1} \|v_\lambda - v\|_{L^{q+1}(\Sigma_\lambda^v)}, \end{aligned} \tag{41}$$

$$\begin{aligned} \|v_\lambda - v\|_{L^{q+1}(\Sigma_\lambda^v)} &\leq C \left\|u^{p-1}(u_\lambda - u)\right\|_{L^{T_2}(\Sigma_\lambda^u)} \\ &\leq C \|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}, \end{aligned} \tag{42}$$

where $T_1 = n(p+1)/(n + (\gamma - \alpha - \beta)(p+1))$ and $T_2 = n(q+1)/(n + (\gamma - \alpha - \beta)(q+1))$. It easy to show that $T_1, T_2 > 1$. Combining (41) and (42), we arrive

$$\begin{aligned} &\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} \\ &\quad \leq C \|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1} \|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}. \end{aligned} \tag{43}$$

The conditions $u \in L^{p+1}(R_+^n)$ and $v \in L^{q+1}(R_+^n)$ make us able to choose sufficiently negative λ, so that

$$C \|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1} \|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \leq \frac{1}{2}. \tag{44}$$

Now inequality (43) implies

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} = 0, \tag{45}$$

and therefore Σ_λ^u must be measure zero. Similarly, one can show that Σ_λ^v is measure zero. Therefore (37) holds.

Step 2. Inequality (37) provides a starting point to move the plane $T_\lambda = \{x \in R_+^n \mid x_1 = \lambda\}$. Now we start from the neighborhood of $x_1 = -\infty$ and move the plane to the right as long as (37) holds to the limiting position. More precisely, define

$$\lambda_0 = \sup \{\lambda \mid u(x) \le u_\mu(x), \quad v(x) \le v_\mu(x), \mu \le \lambda, \forall x \in \Sigma_\mu\}. \tag{46}$$

We will prove that $\lambda_0 = 0$. On the contrary, we suppose $\lambda_0 < 0$. We show that $u(x)$ and $v(x)$ are symmetric about the plane T_{λ_0}; that is

$$u_{\lambda_0}(x) \equiv u(x), \quad v_{\lambda_0}(x) \equiv v(x), \quad \text{a.e. } \forall x \in \Sigma_{\lambda_0}. \tag{47}$$

Otherwise, on Σ_{λ_0},

$$u(x) \le u_{\lambda_0}(x), \qquad v(x) \le v_{\lambda_0}(x), \quad \text{but } u(x) \not\equiv u_{\lambda_0}(x) \quad \text{or} \quad v(x) \not\equiv v_{\lambda_0}(x). \tag{48}$$

We show that the plane can be moved further to the right. More precisely, there exists an $\epsilon > 0$ such that, for $\forall\lambda \in [\lambda_o, \lambda_o + \epsilon)$,

$$u(x) \le u_\lambda(x), \quad v(x) \le v_\lambda(x), \quad \text{a.e.} \forall x \in \Sigma_\lambda. \tag{49}$$

Without loss of generality, we assume

$$v(x) \not\equiv v_{\lambda_0}(x), \quad \text{on } \Sigma_{\lambda_0}. \tag{50}$$

by Lemma 9, we have in fact $u(x) < u_{\lambda_0}(x)$ in the interior of Σ_{λ_0}. Let

$$\overline{\Sigma_{\lambda_0}^u} = \{x \in \Sigma_{\lambda_0} \mid u(x) \ge u_{\lambda_0}(x)\}, \quad \overline{\Sigma_{\lambda_0}^v} = \{x \in \Sigma_{\lambda_0} \mid v(x) \ge v_{\lambda_0}(x)\}. \tag{51}$$

Then obviously $\overline{\Sigma_{\lambda_0}^u}$ has measure zero and $\lim_{\lambda\to\lambda_0}\Sigma_\lambda^u \subset \overline{\Sigma_{\lambda_0}^u}$. The same argument above is also true for the other solution v of (3). From (41) and (42), we deduce

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} \le C\|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1}\|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1}\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}. \tag{52}$$

Again the conditions that $u \in L^{p+1}(R_+^n)$ and $v \in L^{q+1}(R_+^n)$ ensure that one can choose ϵ sufficiently small, so that, for all λ in $[\lambda_o, \lambda_o + \epsilon)$,

$$C\|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1}\|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \le \frac{1}{2}. \tag{53}$$

The method to verify this inequality is standard and the proofs of the rest are similar to the proof in paper [6, 11, 14].

Now by (52) and (53), we have $\|u_\lambda - u\|_{L^p(\Sigma_\lambda^u)} = 0$, and therefore Σ_λ^u must be measure zero. Similarly, Σ_λ^v must also be measure zero. Hence, for these values of $\lambda > \lambda_o$, we have

$$u_\lambda(x) \ge u(x), \quad v_\lambda(x) \ge v(x), \quad \text{a.e. } \forall x \in \Sigma_{\lambda_0}. \tag{54}$$

This (47) must hold and therefore both $u(x)$ and $v(x)$ are symmetric about the plane T_{λ_0}.

Now we show that the plane cannot stop before hitting the origin. Otherwise, assume that the plane stops at $x_1 = \lambda_0 < 0$. By the fact that $|y| > |y^{\lambda_0}|$, we have

$$\begin{aligned} u(x) - u_{\lambda_0}(x) &\le \frac{1}{|x|^\alpha}\int_{\Sigma_{\lambda_0}}\left[G(x,y,\gamma) - G(x^{\lambda_0},y,\gamma)\right] \times \left[\frac{v^q(y)}{|y|^\beta} - \frac{v_{\lambda_0}^q(y)}{|y^{\lambda_0}|^\beta}\right]dy \\ &< \frac{1}{|x|^\alpha}\int_{\Sigma_{\lambda_0}}\left[G(x,y,\gamma) - G(x^{\lambda_0},y,\gamma)\right] \times \left[\frac{v^q(y) - v_{\lambda_0}^q(y)}{|y|^\beta}\right]dy = 0. \end{aligned} \tag{55}$$

This contradicts with (47).

As the direction of x_1 can be chosen arbitrarily, we derive that $(u(x), v(x))$ is rotationally symmetric about x_n-axis. This completes the proof of Theorem 3. ☐

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank Professor Wenxiong Chen and the referees for their valuable suggestions and comments. This work is supported by Grant (nos. U1304101 and 11171091) of NSFC and NSF of Henan Province (no. 132300410141).

References

[1] C. Jin and C. Li, "Symmetry of solutions to some systems of integral equations," *Proceedings of the American Mathematical Society*, vol. 134, no. 6, pp. 1661–1670, 2006.

[2] C. Jin and C. Li, "Quantitative analysis of some system of integral equations," *Calculus of Variations and Partial Differential Equations*, vol. 26, no. 4, pp. 447–457, 2006.

[3] W. Chen, C. Jin, C. Li, and J. Lim, "Weighted Hardy-Littlewood-Sobolev inequalities and systems of integral equations," *Discrete and Continuous Dynamical Systems A*, vol. 2005, supplement, pp. 164–172, 2005.

[4] W. Zhao, J. Yang, and S. Zheng, "Liouville type theorem for higher-orderelliptic system with Navier boundary condition," 2013.

[5] R. Zhuo and D. Li, "A system of integral equations on half space," *Journal of Mathematical Analysis and Applications*, vol. 381, no. 1, pp. 392–401, 2011.

[6] L. Cao and Z. Dai, "A Liouville-type theorem for an integral system on a half-space R_+^n," *Journal of Inequalities and Applications*, vol. 2013, article 37, 2013.

[7] Y. Fang and J. Zhang, "Nonexistence of positive solution for an integral equation on a half-space R_+^n," *Communications on Pure and Applied Analysis*, vol. 12, no. 2, pp. 663–678, 2013.

[8] L. Cao and W. Chen, "Liouville type theorems for poly-harmonic Navier problems," *Discrete and Continuous Dynamical Systems A*, vol. 33, no. 9, pp. 3937–3955, 2013.

[9] C. Li and Y. Lei, "Sharp criteria of Liouville thpe for somenonlinear systems," http://arxiv.org/abs/1301.6235.

[10] Y. Zhao, "Regularity and symmetry for solutions to a system of weighted integral equations," *Journal of Mathematical Analysis and Applications*, vol. 391, no. 1, pp. 209–222, 2012.

[11] D. Li, P. Niu, and R. Zhuo, "Symmetry and non-existence of positive solutions for PDE system with Navier boundary conditions on a half space," *Complex Variables and Elliptic Equations*.

[12] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, USA, 1971.

[13] D. Li and R. Zhuo, "An integral equation on half space," *Proceedings of the American Mathematical Society*, vol. 138, no. 8, pp. 2779–2791, 2010.

[14] L. Cao and Z. Dai, "A Liouville-type theorem for an integral equation on a half-space R_+^n," *Journal of Mathematical Analysis and Applications*, vol. 389, no. 2, pp. 1365–1373, 2012.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 252579, 8 pages
http://dx.doi.org/10.1155/2014/252579

Research Article

Almost Periodic Solution of a Modified Leslie-Gower Predator-Prey Model with Beddington-DeAngelis Functional Response and Feedback Controls

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Received 30 January 2014; Revised 19 February 2014; Accepted 26 February 2014; Published 22 April 2014

Academic Editor: Bingwen Liu

We consider a modified Leslie-Gower predator-prey model with the Beddington-DeAngelis functional response and feedback controls as follows: $\dot{x}(t) = x(t)(a_1(t) - b(t)x(t) - c(t)y(t)/(\alpha(t) + \beta(t)x(t) + \gamma(t)y(t)) - e_1(t)u(t))$, $\dot{u}(t) = -d_1(t)u(t) + p_1(t)x(t-\tau)$, $\dot{y}(t) = y(t)(a_2(t) - r(t)y(t)/(x(t) + k(t)) - e_2(t)v(t))$, and $\dot{v}(t) = -d_2(t)v(t) + p_2(t)y(t-\tau)$. Sufficient conditions which guarantee the permanence and existence of a unique globally attractive positive almost periodic solution of the system are obtained.

1. Introduction

In recent years, the modified predator-prey systems with periodic or almost periodic coefficients have been studied extensively.

Leslie [1] proposed the famous Leslie predator-prey system as follows:

$$\begin{aligned} \dot{x}(t) &= x(a - bx) - p(x)y, \\ \dot{y} &= y\left(e - f\frac{y}{x}\right), \end{aligned} \tag{1}$$

where x and y stand for the population of the prey and the predator at time t, respectively, and $p(x)$ is the so-called predator functional response to the prey. The term y/x is the Leslie-Gower term which measures the loss in the predator population due to rarity of its favorite food.

Global stability of the positive locally asymptotically stable equilibrium in a class of predator-prey systems has been introduced by Hsu and Huang [2], and the system is as follows:

$$\begin{gathered} \frac{dx}{dt} = rx\left(1 - \frac{x}{k} - yp(x)\right), \\ \frac{dy}{dx} = y\left[s\left(1 - \frac{hy}{s}\right)\right], \\ x(0) > 0, \quad y(0) > 0, \quad r, s, k, h > 0. \end{gathered} \tag{2}$$

When the functional response $p(x)$ equals mx, then (2) turns into a Leslie-Gower system [3].

On the other hand, the periodic solution (almost periodic solution) and some other properties of Leslie-Gower predator-prey models were studied (see [4–9]). In particular, Zhang [10] discussed the almost periodic solution of a modified Leslie-Gower predator-prey model with the Beddington-DeAngelis function response as follows:

$$\begin{aligned} \dot{x}(t) &= x(t)\Bigg(r_1(t) - b(t)x(t) \\ &\quad - \frac{c(t)y(t)}{\alpha(t) + \beta(t)x(t) + \gamma(t)y(t)}\Bigg), \\ \dot{y}(t) &= y(t)\left(r_2(t) - \frac{d(t)y(t)}{x(t) + k(t)}\right), \end{aligned} \tag{3}$$

where $x(t)$ is the size of prey population and $y(t)$ is the size of predator population.

Stimulated by the above reasons, in this paper, we incorporate the feedback control into model (3) and consider the following model:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(a_1(t) - b(t)x(t) - \frac{c(t)y(t)}{\alpha(t)+\beta(t)x(t)+\gamma(t)y(t)} - e_1(t)u(t)\right), \\ \dot{u}(t) &= -d_1(t)u(t) + p_1(t)x(t-\tau), \\ \dot{y}(t) &= y(t)\left(a_2(t) - \frac{r(t)y(t)}{x(t)+k(t)} - e_2(t)v(t)\right), \\ \dot{v}(t) &= -d_2(t)v(t) + p_2(t)y(t-\tau), \end{aligned} \tag{4}$$

where $\tau > 0$ and all the coefficients $b(t)$, $c(t)$, $r(t)$, $k(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $a_i(t)$, $d_i(t)$, $p_i(t)$, and $e_i(t)$ $(i = 1, 2)$ are all continuous, almost periodic functions on R.

Associated with (4), we consider a group of initial conditions with the following form (we assume, without loss of generality, that the initial time $t_0 = 0$):

$$\begin{aligned} x(s) &= \phi(s) \geq 0, \quad s \in [-\tau, 0], \ \phi(0) > 0, \\ y(s) &= \varphi(s) \geq 0, \quad s \in [-\tau, 0], \ \varphi(0) > 0, \\ u(0) &> 0, \quad v(0) > 0. \end{aligned} \tag{5}$$

Let f be a continuous bounded function on R and we set

$$f^l = \inf_{t\in R} f(t), \qquad f^u = \sup_{t\in R} f(t). \tag{6}$$

Throughout this paper, we assume that the coefficients of the almost periodic system (4) satisfy

$$\begin{aligned} &\min_{i=1,2}\left\{b^l, c^l, \alpha^l, \beta^l, \gamma^l, r^l, k^l, a_i^l, d_i^l, p_i^l, e_i^l\right\} > 0, \\ &\max_{i=1,2}\left\{b^u, c^u, \alpha^u, \beta^u, \gamma^u, r^u, k^u, a_i^u, d_i^u, p_i^u, e_i^u\right\} < +\infty. \end{aligned} \tag{7}$$

By constructing a suitable Lyapunov functional, we obtain some sufficient conditions for the existence of a globally attractive positive almost periodic solution of system (4) with initial conditions (5).

2. Permanence

In this section, we give some definitions and results that we will use in the rest of the paper.

Lemma 1 (see [11]). *If $a > 0$, $b > 0$, and $\dot{x} \geq (\leq)\, x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, one has*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}, \quad \left(\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}\right). \tag{8}$$

Lemma 2 (see [11]). *If $a > 0$, $b > 0$, and $\dot{x} \geq (\leq)\, b - ax$, when $t \geq 0$ and $x(0) > 0$, one has*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}, \quad \left(\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}\right). \tag{9}$$

Set the following:

$$\begin{aligned} M_1 &= \frac{a_1^u}{b^l}, \qquad L_1 = \frac{p_1^u M_1}{d_1^l}, \\ M_2 &= \frac{a_2^u\left(M_1 + k^u\right)}{r^l}, \qquad L_2 = \frac{p_2^u M_2}{d_2^l}, \\ m_1 &= \frac{a_1^l - c^u/r^l - e_1^u L_1}{b^u}, \qquad l_1 = \frac{p_1^l m_1}{d_1^u}, \\ m_2 &= \frac{1}{r^u}\left(a_2^l - e_2^u L_2\right)\left(m_1 + k^l\right), \qquad l_2 = \frac{p_2^l m_2}{d_2^u}. \end{aligned} \tag{10}$$

Theorem 3. *Suppose that system* (4) *with initial condition* (5) *satisfies the following condition:*

$$a_1^l - \frac{c^u}{r^l} - e_1^u L_1 > 0, \qquad a_2^l - e_2^u L_2 > 0. \tag{11}$$

Then system (4) *is permanent; that is, any positive solution $(x(t), u(t), y(t), v(t))^T$ of the system* (4) *satisfies*

$$\begin{aligned} 0 < m_1 &\leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M_1, \\ 0 < l_1 &\leq \liminf_{t\to+\infty} u(t) \leq \limsup_{t\to+\infty} u(t) \leq L_1, \\ 0 < m_2 &\leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq M_2, \\ 0 < l_2 &\leq \liminf_{t\to+\infty} v(t) \leq \limsup_{t\to+\infty} v(t) \leq L_2. \end{aligned} \tag{12}$$

Proof. From the first equation of (4), we have the following:

$$\dot{x}(t) \leq x(t)\left(a_1^u - b^l x(t)\right). \tag{13}$$

Applying Lemma 1 to (13) leads to

$$\limsup_{t\to+\infty} x(t) \leq \frac{a_1^u}{b^l} = M_1. \tag{14}$$

From (14), we know that there exists an enough large $T_1 > 0$ such that

$$x(t) \leq M_1, \quad t \geq T_1 > 0, \tag{15}$$

so there exists an enough large $T_2 = T_1 + \tau$ such that

$$x(t-\tau) \leq M_1, \quad t \geq T_2 > 0. \tag{16}$$

It follows from (16) and the second equation of system (4) that, for $t \geq T_2$,

$$\dot{u}(t) \leq -d^l u(t) + p_1^u M_1. \tag{17}$$

Applying Lemma 2 to (17) leads to

$$\limsup_{t\to+\infty} u(t) \leq \frac{p_1^u M_1}{d_1^l} = L_1. \tag{18}$$

By using a similar argument as that in the proof of (14) and (18), we can get the following:

$$\limsup_{t\to+\infty} y(t) \le \frac{a_2^u (M_1 + k^u)}{r^l} = M_2,$$
$$\limsup_{t\to+\infty} v(t) \le \frac{p_2^u M_2}{d_2^l} = L_2. \quad (19)$$

From (18) and the first equation of system (4) we know

$$\dot{x}(t) \ge x(t)\left(a_1^l - \frac{c^u}{\gamma^l} - e_1^u L_1 - b^u x(t)\right). \quad (20)$$

Applying Lemma 1 and (11) to the above leads to

$$\liminf_{t\to+\infty} x(t) \ge \frac{a_1^l - c^u/r^l - e_1^u L_1}{b^u} = m_1. \quad (21)$$

Therefore, we know that there exists an enough large T_3 such that

$$x(t) \ge m_1, \quad t \ge T_3 > 0. \quad (22)$$

From the second equation of system (4) we have the following:

$$\dot{u}(t) \ge -d_1^u u(t) + p_1^l m_1. \quad (23)$$

Applying Lemma 2 to the above, we obtain the following:

$$\liminf_{t\to+\infty} u(t) \ge \frac{p_1^l m_1}{d_1^u} = l_1. \quad (24)$$

By using a similar method as that in the proof of (21) and (24), it follows that

$$\liminf_{t\to+\infty} y(t) \ge \frac{1}{r^u}\left(a_2^l - e_2^u L_2\right)\left(m_1 + k^l\right) = m_2$$
$$\liminf_{t\to+\infty} v(t) \ge \frac{p_2^l m_2}{d_2^u} = l_2. \quad (25)$$

This completes the proof. □

We denote by Ω the set of all solutions $z(t) = (x(t), u(t), y(t), v(t))^T$ of system (4) satisfying $m_1 \le x(t) \le M_1$, $l_1 \le u(t) \le L_1$, $m_2 \le y(t) \le M_2$, and $l_2 \le v(t) \le L_2$ for all $t > 0$.

Theorem 4. *Consider the following:* $\Omega \ne \emptyset$.

Proof. From the properties of almost periodic function there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ such that

$$\begin{aligned}
&a_i(t+t_n) \longrightarrow a_i(t), \quad d_i(t+t_n) \longrightarrow d_i(t),\\
&e_i(t+t_n) \longrightarrow e_i(t), \quad p_i(t+t_n) \longrightarrow p_i(t), \quad (i = 1, 2),\\
&b(t+t_n) \longrightarrow b(t), \quad c(t+t_n) \longrightarrow c(t),\\
&r(t+t_n) \longrightarrow r(t), \quad k(t+t_n) \longrightarrow k(t),\\
&\alpha(t+t_n) \longrightarrow \alpha(t), \quad \beta(t+t_n) \longrightarrow \beta(t),\\
&\gamma(t+t_n) \longrightarrow \gamma(t),
\end{aligned} \quad (26)$$

as $n \to \infty$ uniformly on R. Let $z(t) = (x(t), u(t), y(t), v(t))^T$ be a solution of system (4) satisfying $m_1 \le x(t) \le M_1$, $l_1 \le u(t) \le L_1$, $m_2 \le y(t) \le M_2$, and $l_2 \le v(t) \le L_2$ for $t > T$. Clearly, the sequence $z(t + t_n)$ is uniformly bounded and equicontinuous on each bounded subset of R. Therefore, by the Arzelà-Ascoli theorem, there exists a subsequence $z(t + t_k)$ which converges to a continuous function $z^*(t) = (x^*(t), u^*(t), y^*(t), v^*(t))^T$ as $k \to +\infty$ uniformly on each bounded subset of R. Let $T_0 \in R$ be given. We may assume that $t_k + T_0 \ge T$ for all k. For $t \ge 0$, we have the following:

$$\begin{aligned}
x(t+t_k+T_0) &= x(t_k+T_0) + \int_{T_0}^{t+T_0} x(s+t_k)\big(a_1(s+t_k) - b(s+t_k)x(s+t_k)\\
&\quad - (c(s+t_k)y(s+t_k))\\
&\quad \times (\alpha(s+t_k) + \beta(s+t_k)x(s+t_k) + \gamma(s+t_k)y(s+t_k))^{-1}\\
&\quad - e_1(s+t_k)u(s+t_k)\big)\,ds,\\
u(t+t_k+T_0) &= u(t_k+T_0) - \int_{T_0}^{t+T_0} d_1(s+t_k)u(s+t_k) + p_1(s+t_k)x(s+t_k-\tau)\,ds,\\
y(t+t_k+T_0) &= y(t_k+T_0) + \int_{T_0}^{t+T_0} y(s+t_k)\left(a_2(s+t_k) - \frac{r(s+t_k)y(s+t_k)}{x(s+t_k)+k(s+t_k)} - e_2(s+t_k)v(s+t_k)\right)ds,\\
\dot{v}(t+t_k+T_0) &= v(t_k+T_0) + \int_{T_0}^{t+T_0} -d_2(s+t_k)v(s+t_k) + p_2(s+t_k)y(s+t_k-\tau)\,ds.
\end{aligned} \quad (27)$$

Applying Lebesgue's dominated convergence theorem and letting $k \to +\infty$ in (27), we obtain the following:

$$\begin{aligned}
x^*(t+T_0) &= x^*(T_0) + \int_{T_0}^{t+T_0} x^*(s)\Big(a_1(s) - b(s)x^*(s)\\
&\quad - \frac{c(s)y^*(s)}{\alpha(s)+\beta(s)x^*(s)+\gamma(s)y^*(s)}\\
&\quad - e_1(s)u^*(s)\Big)\,ds,
\end{aligned}$$

$$u^*(t+T_0) = u^*(T_0) - \int_{T_0}^{t+T_0} d_1(s)\,u^*(s) + p_1(s)\,x^*(s-\tau)\,ds,$$
$$y^*(t+T_0) = y^*(T_0) + \int_{T_0}^{t+T_0} y^*(s)\left(a_2(s) - \frac{r(s)\,y^*(s)}{x^*(s)+k(s)} - e_2(s)\,v^*(s)\right)ds,$$
$$v^*(t+T_0) = v^*(T_0) + \int_{T_0}^{t+T_0} -d_2(s)\,v^*(s) + p_2(s)\,y^*(s-\tau)\,ds. \quad (28)$$

Since $T_0 \in R$ is arbitrarily given, $z^*(t) = (x^*(t), u^*(t), y^*(t), v^*(t))^T$ is a solution of system (4) on R. It is clear that $m_1 \le x^*(t) \le M_1, l_1 \le u^*(t) \le L_1, m_2 \le y^*(t) \le M_2, l_2 \le v^*(t) \le L_2$ for $t \in R$. Thus $z^*(t) \in \Omega$. This completes the proof. □

3. Existence of a Unique Almost Periodic Solution

Now let us state several definitions and lemmas which will be useful in the proving of the main result of this section.

Definition 5 (see [12]). A function $f(t,x)$, where f is an m-vector, t is a real scalar, and x is an n-vector, is said to be almost periodic in t uniformly with respect to $x \in S \subset R^n$, if $f(t,x)$ is continuous in $t \in R$ and $x \in S$ and if, for any $\varepsilon > 0$, there is a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a ς such that the inequality

$$|f(t+\varsigma, x) - f(t,x)| < \varepsilon \quad (29)$$

is satisfied for all $t \in (-\infty, +\infty)$, $x \in S$. The number ς is called an ε-*translation* number of $f(t,x)$.

Definition 6 (see [12]). A function $f : R \to R$ is said to be asymptotically almost periodic function, if there exists an almost periodic function $q(t)$ and a continuous function $r(t)$ such that $f(t) = q(t) + r(t)$, $t \in R$ and $r(t) \to 0$ as $t \to \infty$.

Lemma 7 (see [13]). *Let f be a nonnegative, integral, and uniformly continuous function defined on $[0,+\infty)$; then* $\lim_{t\to+\infty} f(t) = 0$.

Theorem 8. *Suppose that all conditions of Theorem 3 hold; furthermore assume that*

(H) $\Theta > 0$, *where* $\Theta = \min\{\Theta_1, \Theta_2, \Theta_3, \Theta_4\}$,

$$\Theta_1 = b^l m_1 - p_1^u M_1 - \frac{c^u \beta^u M_1 M_2}{(\alpha^l + \beta^l m_1 + \gamma^l m_2)^2} - \frac{r^u M_1 M_2}{(m_1 + k^l)^2} > 0,$$
$$\Theta_2 = \frac{\gamma^l}{M_1 + k^u} - \frac{c^l m_2}{\alpha^u + \beta^u M_1 + \gamma^u M_2} - \frac{c^u \gamma^u M_2^2}{(\alpha^l + \beta^l m_1 + \gamma^l m_2)^2} - p_2^u M_2 > 0,$$
$$\Theta_3 = d_1^l - e_1^u, \qquad \Theta_4 = d_2^l - e_2^u. \quad (30)$$

Then system (4) *with initial conditions* (5) *is globally attractive.*

Proof. Let $x(t) = e^{x_1(t)}$, $y(t) = e^{y_1(t)}$, and then system (4) is transformed into

$$\dot{x}_1(t) = a_1(t) - b(t)\,e^{x_1(t)} - \frac{c(t)\,e^{y_1(t)}}{\alpha(t) + \beta(t)\,e^{x_1(t)} + \gamma(t)\,e^{y_1(t)}} - e_1(t)\,u(t),$$
$$\dot{u}(t) = -d_1(t)\,u(t) + p_1(t)\,e^{x_1(t-\tau)},$$
$$\dot{y}_1(t) = a_2(t) - \frac{r(t)\,e^{y_1(t)}}{e^{x_1(t)} + k(t)} - e_2(t)\,v(t),$$
$$\dot{v}(t) = -d_2(t)\,v(t) + p_2(t)\,e^{y_1(t-\tau)}. \quad (31)$$

Suppose that $z_1(t) = (x_1(t), u(t), y_1(t), v(t))^T$ and $z_1^*(t) = (x_1^*(t), u^*(t), y_1^*(t), v^*(t))^T$ are any two positive solutions of (31).

Let $V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t)$, where

$$V_1(t) = |x_1(t) - x_1^*(t)|,$$
$$V_2(t) = |u(t) - u^*(t)| + p_1^u \int_{t-\tau}^{t} \left|e^{x_1(t)} - e^{x_1^*(t)}\right| ds,$$
$$V_3(t) = |y_1(t) - y_1^*(t)|,$$
$$V_4(t) = |v(t) - v^*(t)| + p_2^u \int_{t-\tau}^{t} \left|e^{y_1(t)} - e^{y_1^*(t)}\right| ds. \quad (32)$$

Calculating the right derivative $D^+V_1(t)$ of $V_1(t)$ along the solution of (31), we have the following:

$$D^+V_1(t) = \operatorname{sgn}(x_1(t) - x_1^*(t)) \times \left[-b(t)\left(e^{x_1(t)} - e^{x_1^*(t)}\right) - \frac{c(t)\,e^{y_1(t)}}{\alpha(t) + \beta(t)\,e^{x_1(t)} + \gamma(t)\,e^{y_1(t)}} + \frac{c(t)\,e^{y_1^*(t)}}{\alpha(t) + \beta(t)\,e^{x_1^*(t)} + \gamma(t)\,e^{y_1^*(t)}} - e_1(t)(u(t) - u^*(t))\right]$$

$$
\begin{aligned}
&= \operatorname{sgn}\left(x_1(t) - x_1^*(t)\right) \\
&\quad \times \Big[-b(t) e^{\xi(t)}\left(x_1(t) - x_1^*(t)\right) \\
&\qquad - \frac{c(t) e^{y_1(t)}}{\alpha(t) + \beta(t) e^{x_1(t)} + \gamma(t) e^{y_1(t)}} \\
&\qquad + \frac{c(t) e^{y_1^*(t)}}{\alpha(t) + \beta(t) e^{x_1(t)} + \gamma(t) e^{y_1(t)}} \\
&\qquad - \frac{c(t) e^{y_1^*(t)}}{\alpha(t) + \beta(t) e^{x_1(t)} + \gamma(t) e^{y_1(t)}} \\
&\qquad + \frac{c(t) e^{y_1^*(t)}}{\alpha(t) + \beta(t) e^{x_1^*(t)} + \gamma(t) e^{y_1^*(t)}} \\
&\qquad - e_1(t)\left(u(t) - u^*(t)\right)\Big] \\
&\le \operatorname{sgn}\left(x_1(t) - x_1^*(t)\right) \\
&\quad \times \left[-b^l m_1\left(x_1(t) - x_1^*(t)\right)\right] \\
&\quad - \frac{c(t)}{\alpha(t) + \beta(t) e^{x_1(t)} + \gamma(t) e^{y_1(t)}} \\
&\quad \cdot e^{\eta(t)}\left(y_1(t) - y_1^*(t)\right) \\
&\qquad + \Big(c(t) e^{y_1^*(t)}\Big[\beta(t)\left(e^{x_1(t)} - e^{x_1^*(t)}\right) \\
&\qquad\qquad + \gamma(t)\left(e^{y_1(t)} - e^{y_1^*(t)}\right)\Big]\Big) \\
&\qquad \times \Big(\left(\alpha(t) + \beta(t) e^{x_1^*(t)} + \gamma(t) e^{y_1^*(t)}\right) \\
&\qquad\quad \times \left(\alpha(t) + \beta(t) e^{x_1^*(t)} + \gamma(t) e^{y_1^*(t)}\right)\Big)^{-1} \\
&\qquad - e_1(t)\left(u(t) - u^*(t)\right) \\
&\le \left(\frac{c^u \beta^u M_1 M_2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} - b^l m_1\right) \\
&\quad \times \left|x_1(t) - x_1^*(t)\right| \\
&\quad + \left(\frac{c^u \gamma^u M_2^2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2}\right. \\
&\qquad \left. + \frac{c^l m_2}{\alpha^u + \beta^u M_1 + \gamma^u M_2}\right) \\
&\quad \times \left|y_1(t) - y_1^*(t)\right| \\
&\quad + e_1^u \left|u(t) - u^*(t)\right|.
\end{aligned}
\tag{33}
$$

Further, it follows that

$$
\begin{aligned}
D^+ V_2(t) &= \operatorname{sgn}\left(u(t) - u^*(t)\right) \\
&\quad \times \Big(- d_1(t)\left(u(t) - u^*(t)\right) \\
&\qquad + p_1(t)\left(e^{x_1(t-\tau)} - e^{x_1^*(t-\tau)}\right) \\
&\qquad + p_1^u\left(e^{x_1(t)} - e^{x_1^*(t)}\right) \\
&\qquad - p_1(t)\left(e^{x_1(t-\tau)} - e^{x_1^*(t-\tau)}\right)\Big) \\
&\le -d_1^l \left|u(t) - u^*(t)\right| + p_1^u M_1 \left|x_1(t) - x_1^*(t)\right|, \\
D^+ V_3(t) &= \operatorname{sgn}\left(y_1(t) - y_1^*(t)\right) \\
&\quad \times \Bigg[-\frac{r(t) e^{y_1(t)}}{e^{x_1(t)} + k(t)} + \frac{r(t) e^{y_1^*(t)}}{e^{x_1^*(t)} + k(t)} \\
&\qquad - e_2(t)\left(v(t) - v^*(t)\right)\Bigg] \\
&\le -\frac{r^l m_2}{M_1 + k^u}\left|y_1(t) - y_1^*(t)\right| \\
&\quad + \frac{r^u M_1 M_1}{\left(m_1 + k^l\right)^2}\left|x_1(t) - x_1^*(t)\right| \\
&\quad + e_2^u \left|v(t) - v^*(t)\right|, \\
D^+ V_4(t) &\le -d_2^l \left|v(t) - v^*(t)\right| + p_2^u M_2 \left|y_1(t) - y_1^*(t)\right|.
\end{aligned}
\tag{34}
$$

Therefore, we have the following:

$$
\begin{aligned}
&D^+ V(t) \\
&= D^+ V_1(T) + D^+ V_2(T) + D^+ V_3(T) + D^+ V_4(T) \\
&\le -\left(b^l m_1 - p_1^u M_1 - \frac{c^u \beta^u M_1 M_2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} - \frac{r^u M_1 M_1}{\left(m_1 + k^l\right)^2}\right) \\
&\quad \times \left|x_1(t) - x_1^*(t)\right| \\
&\quad - \left(\frac{r^l m_2}{M_1 + k^u} - \frac{c^l m_2}{\alpha^u + \beta^u M_1 + \gamma^u M_2}\right. \\
&\qquad \left. - \frac{c^u \gamma^u M_2^2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} - p_2^u M_2\right)\left|y_1(t) - y_1^*(t)\right| \\
&\quad - \left(d_1^l - e_1^u\right)\left|u(t) - u^*(t)\right| - \left(d_2^l - e_2^u\right)\left|v(t) - v^*(t)\right| \\
&\le -\Theta\Big(\left|x_1(t) - x_1^*(t)\right| + \left|y_1(t) - y_1^*(t)\right| \\
&\qquad + \left|u(t) - u^*(t)\right| + \left|v(t) - v^*(t)\right|\Big).
\end{aligned}
\tag{35}
$$

Integrating the above inequality on internal $[0, t]$, it follows that, for $t \ge 0$,

$$
\begin{aligned}
V(t) + \Theta \int_0^t &\left|x_1(t) - x_1^*(t)\right| + \left|y_1(t) - y_1^*(t)\right| \\
&+ \left|u(t) - u^*(t)\right| + \left|v(t) - v^*(t)\right| ds \\
\le V(0) &< +\infty.
\end{aligned}
\tag{36}
$$

Then, for $t > 0$, we obtain that

$$\int_0^t |x_1(t) - x_1^*(t)| + |y_1(t) - y_1^*(t)| + |u(t) - u^*(t)| + |v(t) - v^*(t)|\, ds \le \frac{V(0)}{\Theta} < +\infty. \quad (37)$$

By Lemma 7, we obtain

$$\lim_{t\to+\infty} |x_1(t) - x_1^*(t)| = 0, \qquad \lim_{t\to+\infty} |y_1(t) - y_1^*(t)| = 0,$$
$$\lim_{t\to+\infty} |u(t) - u^*(t)| = 0, \qquad \lim_{t\to+\infty} |v(t) - v^*(t)| = 0. \quad (38)$$

Then the solution of systems (4) and (5) is globally attractive. □

Theorem 9. *Suppose that all conditions of Theorem 8 hold; then there exists a unique almost periodic solution of systems* (4) *and* (5).

Proof. According to Theorem 4, there exists a bounded positive solution $W(t) = (w_1(t), w_2(t), w_3(t), w_4(t))^T$ of (4) and (5). Then there exists a sequence $\{t_k'\}$, $t_k' \to \infty$ as $k \to \infty$, such that $(w_1(t+t_k'), w_2(t+t_k'), w_3(t+t_k'), w_4(t+t_k'))^T$ is a solution of the following system:

$$\begin{aligned}
\dot{x}(t) &= x(t)\Big(a_1(t+t_k') - b(t+t_k')x(t) - \frac{c(t+t_k')y(t)}{\alpha(t+t_k') + \beta(t+t_k')x(t) + \gamma(t+t_k')y(t)} - e_1(t+t_k')u(t)\Big),\\
\dot{u}(t) &= -d_1(t+t_k')u(t) + p_1(t+t_k')x(t-\tau),\\
\dot{y}(t) &= y(t)\Big(a_2(t+t_k') - \frac{r(t+t_k')y(t)}{x(t) + k(t+t_k')} - e_2(t+t_k')v(t)\Big),\\
\dot{v}(t) &= -d_2(t+t_k')v(t) + p_2(t+t_k')y(t-\tau).
\end{aligned} \quad (39)$$

According to Theorem 3, we get that not only $\{(w_1(t+t_k'), w_2(t+t_k'), w_3(t+t_k'), w_4(t+t_k'))^T\}$ but also $\{(\dot{w}_1(t+t_k'), \dot{w}_2(t+t_k'), \dot{w}_3(t+t_k'), \dot{w}_4(t+t_k'))^T\}$ are uniformly bounded and equicontinuous. By Ascoli's theorem there exists a uniformly convergent subsequence $w_i(t+t_k) \subseteq w_i(t+t_k')(i = 1,2,3,4)$ such that, for any $\varepsilon > 0$, there exists a $K(\varepsilon) > 0$ with the property that if $m, k \ge K(\varepsilon)$, then

$$|w_i(t+t_m) - w_i(t+t_k)| < \varepsilon, \quad (i = 1,2,3,4). \quad (40)$$

This is to say, $w_i(t+t_k)$ $(i = 1,2,3,4)$ are asymptotically almost periodic functions; hence there exist four almost periodic functions $P_i(t+t_k)$ $(i = 1,2,3,4)$ and four continuous functions $F_i(t+t_k)$ $(i = 1,2,3,4)$ such that

$$w_i(t+t_k) = P_i(t+t_k) + F_i(t+t_k), \quad t \in R,\ i = 1,2,3,4, \quad (41)$$

where

$$\lim_{k\to+\infty} P_i(t+t_k) = P_i(t), \qquad \lim_{k\to+\infty} F_i(t+t_k) = 0, \quad i = 1,2,3,4, \quad (42)$$

$P_i(t)$ $(i = 1,2,3,4)$ are an almost periodic function.

Therefore,

$$\lim_{k\to+\infty} w_i(t+t_k) = P_i(t), \quad (i = 1,2,3,4). \quad (43)$$

On the other hand,

$$\begin{aligned}
\lim_{k\to+\infty} \dot{w}_i(t+t_k) &= \lim_{k\to+\infty}\lim_{h\to0} \frac{w_i(t+t_k+h) - w_i(t+t_k)}{h}\\
&= \lim_{h\to+\infty}\lim_{k\to0} \frac{w_i(t+t_k+h) - w_i(t+t_k)}{h}\\
&= \lim_{h\to0} \frac{P_i(t+h) - P_i(t)}{h}, \quad (i = 1,2,3,4).
\end{aligned} \quad (44)$$

So $\dot{P}_i(t)$ $(i = 1,2,3,4)$ exist. Now we will prove that $(P_1(t), P_2(t), P_3(t), P_4(t))^T$ is an almost periodic solution of system (4).

From properties of almost periodic function, there exits a sequence $\{t_n\}$, $\{t_n\} \to \infty$ as $n \to \infty$, such that

$$\begin{aligned}
&a_i(t+t_n) \longrightarrow a_i(t), \qquad d_i(t+t_n) \longrightarrow d_i(t),\\
&e_i(t+t_n) \longrightarrow e_i(t), \qquad p_i(t+t_n) \longrightarrow p_i(t), \quad (i = 1,2),\\
&b(t+t_n) \longrightarrow b(t), \qquad c(t+t_n) \longrightarrow c(t),\\
&r(t+t_n) \longrightarrow r(t), \qquad k(t+t_n) \longrightarrow k(t),\\
&\alpha(t+t_n) \longrightarrow \alpha(t), \qquad \beta(t+t_n) \longrightarrow \beta(t),\\
&\gamma(t+t_n) \longrightarrow \gamma(t),
\end{aligned} \quad (45)$$

as $n \to \infty$ uniformly on R.

It is easy to know that $w_i(t+t_n) \to P_i(t)$ $(i = 1,2,3,4)$ as $n \to \infty$, and then we have the following:

$$\begin{aligned}
&\dot{P}_1(t) \\
&= \lim_{n\to+\infty} \dot{w}_1(t+t_n) \\
&= \lim_{n\to+\infty} \Big[w_1(t+t_n)\big(a_1(t+t_n) - b(t+t_n)w_1(t+t_n) \\
&\qquad - (c(t+t_n)w_3(t+t_n)) \\
&\qquad \times (\alpha(t+t_n) + \beta(t+t_n)w_1(t+t_n) \\
&\qquad + \gamma(t+t_n)w_3(t+t_n))^{-1} \\
&\qquad - e_1(t+t_n)w_2(t+t_n)\big)\Big] \\
&= P_1(t)\Bigg(a_1(t) - b(t)P_1(t) \\
&\qquad - \frac{c(t)P_3(t)}{\alpha(t) + \beta(t)P_1(t) + \gamma(t)P_3(t)} - e_1(t)P_2(t)\Bigg).
\end{aligned} \tag{46}$$

By using a similar argument as that in the above, we have the following:

$$\begin{aligned}
\dot{P}_2(t) &= -d_1(t)P_2(t) + p_1(t)P_1(t-\tau), \\
\dot{P}_3(t) &= P_3(t)\left(a_2(t) - \frac{r(t)P_3(t)}{P_1(t)+k(t)} - e_2(t)P_4(t)\right), \\
\dot{P}_4(t) &= -d_2(t)P_4(t) + p_2(t)P_3(t-\tau).
\end{aligned} \tag{47}$$

This proves that $P_i(t)$ $(i = 1,2,3,4)$ is a nonnegative almost periodic solution of systems (4) and (5); by Theorem 8, it follows that there exists a globally asymptotically stable nonnegative almost periodic solution of system (4). The proof is complete. □

4. An Example

Consider the following system:

$$\begin{aligned}
\dot{x}(t) &= x(t)\left(4 - 2x(t) - \frac{10y(t)}{2+20x(t)+20y(t)} - 2u(t)\right), \\
\dot{u}(t) &= -3u(t) + \frac{1}{5}x(t-\tau), \\
\dot{y}(t) &= y(t)\left(\frac{1}{10} - \frac{20y(t)}{x(t)+23} - \frac{2}{5}v(t)\right), \\
\dot{v}(t) &= -2v(t) + 2y(t-\tau).
\end{aligned} \tag{48}$$

By a simple calculation, we check that all conditions in Theorems 8 and 9 are fulfilled. Therefore, by Theorems 8 and 9, system (48) has a unique globally asymptotically stable nonnegative almost periodic solution (see Figure 1).

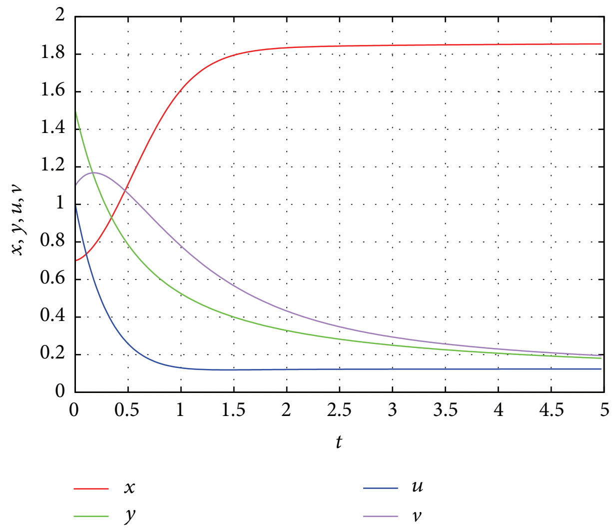

Figure 1: Dynamic behavior of system (48) with the initial $(x(0), y(0), u(0), v(0))^T = (0.7, 1.5, 1.0, 1.1)^T$, for $\tau = 0$, $t \in [0,5]$. From the figure, we could easily see that the solution $(x(t), y(t), u(t), v(t))^T$ is asymptotic to the unique almost periodic solution of the system (48).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is supported by the NNSF of China (no. 11171085) and Hunan Provincial Natural Science Foundation of China (no. 10JJ6002).

References

[1] P. H. Leslie, "Some further notes on the use of matrices in population mathematics," *Biometrika*, vol. 35, no. 3-4, pp. 213–249, 1948.

[2] S. B. Hsu and T. W. Huang, "Global stability for a class of predator-prey systems," *SIAM Journal on Applied Mathematics*, vol. 55, no. 3, pp. 763–783, 1995.

[3] P. H. Leslie and J. C. Gower, "The properties of a stochastic model for the predator-prey type of interaction between two species," *Biometrika*, vol. 47, pp. 219–234, 1960.

[4] A. Korobeinikov, "A Lyapunov function for Leslie-Gower predator-prey models," *Applied Mathematics Letters*, vol. 14, no. 6, pp. 697–699, 2001.

[5] M. A. Aziz-Alaoui and M. Daher Okiye, "Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes," *Applied Mathematics Letters*, vol. 16, no. 7, pp. 1069–1075, 2003.

[6] H. F. Huo and W. T. Li, "Periodic solutions of delayed Leslie-Gower predator-prey models," *Applied Mathematics and Computation*, vol. 155, no. 3, pp. 591–605, 2004.

[7] A. F. Nindjin, M. A. Aziz-Alaoui, and M. Cadivel, "Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay," *Nonlinear Analysis: Real World Applications*, vol. 7, no. 5, pp. 1104–1118, 2006.

[8] L. Chen and F. Chen, "Global stability of a Leslie-Gower predator-prey model with feedback controls," *Applied Mathematics Letters*, vol. 22, no. 9, pp. 1330–1334, 2009.

[9] T. W. Zhang and X. R. Gan, "Existence and permanence of almost periodic solutions for Leslie-Gower predatorCprey model with variable delays," *Electronic Journal of Differential Equations*, vol. 2013, pp. 1–21, 2013.

[10] Z. M. Zhang, "Almost periodic solution of a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response," *Journal of Applied Mathematics*, vol. 2013, Article ID 834047, 9 pages, 2013.

[11] F. Chen, Z. Li, and Y. Huang, "Note on the permanence of a competitive system with infinite delay and feedback controls," *Nonlinear Analysis: Real World Applications*, vol. 8, no. 2, pp. 680–687, 2007.

[12] X. Y. Dou and Y. K. Li, "Almost periodic solution for a food-limited population model with delay and feedback control," *International Journal of Computational and Mathematical Sciences*, vol. 5, no. 4, pp. 174–179, 2011.

[13] M. Fan, K. Wang, and D. Jiang, "Existence and global attractivity of positive periodic solutions of periodic n-species Lotka-Volterra competition systems with several deviating arguments," *Mathematical Biosciences*, vol. 160, no. 1, pp. 47–61, 1999.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 413037, 9 pages
http://dx.doi.org/10.1155/2014/413037

Research Article

Stability to a Kind of Functional Differential Equations of Second Order with Multiple Delays by Fixed Points

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Received 13 January 2014; Revised 29 March 2014; Accepted 30 March 2014; Published 17 April 2014

Academic Editor: Samir Saker

We discuss the stability of solutions to a kind of scalar Liénard type equations with multiple variable delays by means of the fixed point technique under an exponentially weighted metric. By this work, we improve some related results from one delay to multiple variable delays.

1. Introduction

For more than one hundred years, Lyapunov's direct (second) method has been very effectively used to investigate the stability problems in ordinary and functional differential equations. This method is one of the highly effective methods to determine the stability properties of solutions of ordinary and functional differential equations of higher order in the literature. However, till now, constructing or defining Lyapunov functions or functionals which give a meaningful discussion remains a general problem in the literature. In recent years, many researchers discussed that the fixed point theory has an important advantage over Lyapunov's direct method. While Lyapunov's direct method usually requires pointwise conditions, fixed point theory needs average conditions; see Burton [1].

In 2001, Burton and Furumochi [2] observed some difficulties that occur in studying the stability theory of ordinary and functional differential equations by Lyapunov's second (direct) method. Rather than inventing new modifications of the standard Lyapunov function(al) method to overcome the difficulties, the authors demonstrate by various examples that the contraction mapping principle can do the magic in many circumstances.

Later, in 2005, by using contraction mappings, Burton [3] investigated the scalar Liénard type equation with constant delay, $L(> 0)$:

$$\ddot{x} + f(t, x, \dot{x})\,\dot{x} + b(t)\,g(x(t - L)) = 0. \tag{1}$$

Burton [3] obtained conditions for each solution $x(t)$ to satisfy $(x(t), x'(t)) \to (0, 0)$ as $t \to \infty$.

Later, in 2011, Pi [4] studied stability properties of solutions to a scalar functional Liénard type equation with variable delay, $\tau(t)$ (> 0):

$$\ddot{x} + f(t, x, \dot{x})\,\dot{x} + b(t)\,g(x(t - \tau(t))) = 0. \tag{2}$$

By using fixed point theory under an exponentially weighted metric, Pi [4] obtained some interesting sufficient conditions ensuring that the zero solution of this equation is stable and asymptotically stable.

On the other hand, some recent relative results proceeded on the qualitative behaviors of delay differential equations, neutral differential equations, neutral Volterra integrodifferential equations, and certain nonlinear differential equations of second order with and without delay can be summarized as follows.

In [5], Fan et al. studied delay differential equations of the form

$$\begin{aligned} \dot{x} &= -a(t, x_t)\,x(t) + f(t, x_t), \\ \dot{x} &= -g(t, x(t)) + f(t, x_t), \end{aligned} \tag{3}$$

and the authors established sufficient and necessary criteria for the asymptotic stability by using two different approaches, the contraction mapping principle and Schauder's fixed point theorem.

Raffoul [6] dealt with the stability of the zero solution of a scalar neutral differential equation. The author established sufficient conditions for the stability of the zero solution on the base of the contraction mapping principle.

In [7], Jin and Luo aimed to study the asymptotic stability for some scalar differential equations of retarded and neutral type by using a fixed point approach. The authors did not use Lyapunov's method; they got interesting results for the stability even when the delay is unbounded. The authors also obtained necessary and sufficient conditions for the asymptotic stability.

Zhang and Liu [8] considered a nonlinear neutral differential equation. By using fixed point theory, they gave some conditions to ensure that the zero solution to the equation is asymptotically stable. Hence, some existing results were improved and generalized by this work.

Ardjouni and Djoudi [9] used the contraction mapping theorem to obtain an asymptotic stability result of the zero solution of a nonlinear neutral Volterra integrodifferential equation with variable delays. The asymptotic stability theorem with a necessary and sufficient condition was proved, which improves and extends the results in the literature.

In 2010, Tunç [10] considered the following Liénard type equation with multiple variable deviating arguments, $\tau_j(t)$:

$$\begin{aligned} &\ddot{x}(t) + f_1(x(t),\dot{x}(t))\dot{x}(t) + f_2(x(t))\dot{x}(t) \\ &\quad + g_0(x(t)) + \sum_{j=1}^{m} g_j\left(x\left(t-\tau_j(t)\right)\right) \\ &= p\left(t, x(t), x\left(t-\tau_1(t)\right), \dots, \right. \\ &\qquad \left. x\left(t-\tau_m(t)\right), \dots, \dot{x}\left(t-\tau_m(t)\right)\right). \end{aligned} \tag{4}$$

The author studied the problems of stability and boundedness of the solutions of this equation by using the Lyapunov second method and made a comparison with some earlier works in the literature.

In [11], the author considered the nonlinear differential equation of second order with a constant delay, r:

$$\begin{aligned} &\ddot{x}(t) + \{f(t,x(t),x(t-r),\dot{x}(t),\dot{x}(t-r)) \\ &\quad + g(t,x(t),x(t-r),\dot{x}(t),\dot{x}(t-r))\dot{x}(t)\}\dot{x}(t) \\ &\quad + b(t)h(x(t-r)) = e(t,x(t),x(t-r),\dot{x}(t),\dot{x}(t-r)). \end{aligned} \tag{5}$$

The author discussed the stability of the zero solution of this equation, when $e(\cdot) = 0$, and established two new results on the boundedness and uniform-boundedness of the solutions of the same equation, when $e(\cdot) \neq 0$. By this work, Tunç [11] improved the existing results on the stability and boundedness of the solutions of the differential equations of second order without a delay by imposing a few new criteria to the second order nonlinear and nonautonomous delay differential equations of the above form.

Further, Tunç [12] took into consideration the vector Liénard equation with the multiple constant deviating arguments, $\tau_i > 0$:

$$\begin{aligned} &\ddot{X}(t) + F\left(X(t),\dot{X}(t)\right)\dot{X}(t) + G(X(t)) \\ &\quad + \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right) = P(t). \end{aligned} \tag{6}$$

Based on the Lyapunov-Krasovskii functional approach, the asymptotic stability of the zero solution and the boundedness of all solutions of this equation, when $P(t) = 0$ and $P(t) \neq 0$, respectively, are discussed.

More recently, by using Lyapunov's function and functional approach, Tunç [13, 14] and Tunç and Yazgan [15] discussed some problems on stability, the boundedness, and the existence of periodic solutions of a certain second order vector and scalar nonlinear differential equations without and with delay. In [16], Tunç also gave certain sufficient conditions for the existence of periodic solutions to a Rayleigh-type equation with state-dependent delay.

By the mentioned papers, the authors contributed to the subject for a class of ordinary and functional differential equations.

In this paper, instead of the mentioned equations, we consider the scalar Liénard type equation with multiple variable delays:

$$\ddot{x} + f(t,x,\dot{x})\dot{x} + \sum_{j=1}^{n} b_j(t)\, g_j\left(x\left(t-\tau_j(t)\right)\right) = 0, \tag{7}$$

where $t \in \mathfrak{R}^+$, $\mathfrak{R}^+ = [0,\infty)$, $b_j : \mathfrak{R}^+ \to \mathfrak{R}^+$ are bounded and continuous functions, $g_j : \mathfrak{R} \to \mathfrak{R}$, $g_j(0) = 0$, $f : \mathfrak{R}^+ \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}^+$, and $\tau_j : \mathfrak{R}^+ \to \mathfrak{R}^+$ are all continuous functions such that $t - \tau_j(t) \geq 0$.

We can write (7) as follows:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -f(t,x,y)\,y - \sum_{j=1}^{n} b_j(t)\, g_j\left(x\left(t-\tau_j(t)\right)\right). \end{aligned} \tag{8}$$

For each $t_0 \geq 0$, we define $m(t_0) = \inf\{s - \tau_1(s), \dots, s - \tau_n(s) : s \geq t_0\}$ and $C(t_0) = C([m(t_0), t_0], R)$ with the continuous function norm $\|\cdot\|$, where

$$\|\psi\| = \sup\left\{|\psi(s)| : m(t_0) \leq s \leq t_0\right\}. \tag{9}$$

It will cause no confusion even if we use $\|\phi\|$ as the supremum on $[m(t_0), \infty)$. It can be seen from [9] that, for a given continuous function ϕ and a number y_0, there exists a solution of system (8) on an interval $[t_0, T)$; if the solution remains bounded, then $T = \infty$. Let $(x(t), y(t))$ denote the solution $(x(t,\phi,y_0), y(t,\phi,y_0))$.

Definition 1. The zero solution of system (8) is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\phi \in C(t_0), y_0 \in R, \|\phi\| + |y_0| < \delta]$ implies that $|x(t,\phi,y_0)| + |y(t,\phi,y_0)| < \varepsilon$ for $t \geq t_0$.

We make the following basic assumptions on the delay functions $\tau_j(t)$: (A_1). Let $t-\tau_j(t)$ be strictly increasing and $\lim_{t\to\infty}(t-\tau_j(t))=\infty$. The inverses of $t-\tau_j(t)$ exist, denoted by $P_j(t)$ and $0\le b_j(t)\le M_j$, $j=1,2,\dots,n$. Let $M=\max\{M_1,\dots,M_n\}$. Hence, $0\le b_j(t)\le M$.

It is also worth mentioning that throughout the papers [10–15] the authors discussed the qualitative behavior of solutions of certain scalar and vector ordinary and functional differential equations of second order by means of the Lyapunov function or functional approach. In this paper, instead of the mentioned methods, we use the fixed point technique under an exponentially weighted metric to discuss stability of solutions to a kind of scalar Liénard type equations with multiple variable delays. This approach has a contribution to the topic in the literature, and it may be useful for researchers to work on the qualitative behaviors of solutions.

2. Main Result

In this section, sufficient conditions for stability are presented by the fixed point theory. We give some results on stability of the zero solution of (7). Before giving our main result, we introduce some auxiliary results.

Lemma 2. *Let $\psi:[m(t_0),t_0]\to R$ be a given continuous function. If $(x(t),y(t))$ is the solution of system* (8) *on $[t_0,T_1)$ satisfying $(t)=\psi(t)$, $t\in[m(t_0),t_0]$, and $y(t_0)=x'(t_0)$, then $x(t)$ is the solution of the following integral equation:*

$$\begin{aligned}x(t)&=\psi(t_0)e^{-\int_{t_0}^{t}K(s)ds}+\int_{t_0}^{t}e^{-\int_u^t K(s)ds}B(u)\,du\\&+\sum_{j=1}^{n}\int_{t_0}^{t}e^{-\int_u^t K(s)ds}\widehat{D}_j(u)\left[x(u)-g_j(x(u))\right]du\\&+\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\widehat{D}_j(s)\,g_j(x(s))\,ds\\&-\sum_{j=1}^{n}e^{-\int_{t_0}^{t}K(s)ds}\int_{t_0-\tau_j(t_0)}^{t_0}\widehat{D}_j(s)\,g_j(x(s))\,ds\\&-\sum_{j=1}^{n}\int_{t_0}^{t}\left[\int_{u-\tau_j(u)}^{u}\widehat{D}_j(s)\,g_j(x(s))\,ds\right]e^{-\int_u^t K(s)ds}K(u)\,du\\&+\sum_{j=1}^{n}\int_{t_0}^{t}E_j(t,s)\,g_j\left(x\left(s-\tau_j(s)\right)\right)ds\\&-\sum_{j=1}^{n}\int_{t_0}^{t}\left[\int_{t_0}^{u}E_j(u,s)\right.\\&\qquad\left.\times g_j\left(x\left(s-\tau_j(s)\right)\right)ds\right]e^{-\int_u^t K(s)ds}K(u)\,du.\end{aligned}\tag{10}$$

Conversely, if the continuous function $x(t)=\psi(t)$, $t\in[m(t_0),t_0]$ is the solution of (10) *on $[t_0,T_2]$, then $(x(t),y(t))$ is the solution of system* (8) *on $[t_0,T_2]$.*

Proof. Let $f(t,x(t),y(t))=A(t)$. Then, (8) can be written as the following system:

$$\begin{aligned}\dot{x}&=y,\\\dot{y}&=-A(t)\,y-\sum_{j=1}^{n}b_j(t)\,g_j\left(x\left(t-\tau_j(t)\right)\right)\end{aligned}\tag{11}$$

so that

$$\dot{y}+A(t)\,y+\sum_{j=1}^{n}b_j(t)\,g_j\left(x\left(t-\tau_j(t)\right)\right)=0.\tag{12}$$

Multiplying both sides of (12) by $e^{\int_{t_0}^{t}A(s)ds}$ and then integrating from t_0 to t, we obtain

$$\begin{aligned}y(t)&=y(t_0)\,e^{-\int_{t_0}^{t}A(s)ds}\\&\quad-\int_{t_0}^{t}e^{-\int_u^t A(s)ds}\sum_{j=1}^{n}b_j(u)\,g_j\left(x\left(u-\tau_j(u)\right)\right)du,\end{aligned}\tag{13}$$

and hence

$$\begin{aligned}\dot{x}(t)&=\dot{x}(t_0)\,e^{-\int_{t_0}^{t}A(s)ds}\\&\quad-\int_{t_0}^{t}e^{-\int_u^t A(s)ds}\sum_{j=1}^{n}b_j(u)\,g_j\left(x\left(u-\tau_j(u)\right)\right)du.\end{aligned}\tag{14}$$

If we choose $\dot{x}(t_0)e^{-\int_{t_0}^{t}A(s)ds}=B(t)$, then it follows that

$$\dot{x}(t)=B(t)-\sum_{j=1}^{n}\int_{t_0}^{t}e^{-\int_u^t A(s)ds}b_j(u)\,g_j\left(x\left(u-\tau_j(u)\right)\right)du.\tag{15}$$

Let

$$\begin{aligned}\sum_{j=1}^{n}e^{-\int_u^t A(s)ds}b_j(u)&=\sum_{j=1}^{n}C_j(t,u),\\\sum_{j=1}^{n}\int_{t_0}^{\infty}C_j(u+t-t_0,t)\,du&=\sum_{j=1}^{n}D_j(t),\\\sum_{j=1}^{n}\frac{D_j(t)}{1-\tau'_j(t)}&=\sum_{j=1}^{n}\widetilde{D}_j(t),\\\sum_{j=1}^{n}\widetilde{D}_j\left(p_j(t)\right)&=\sum_{j=1}^{n}\widehat{D}_j(t),\\\sum_{j=1}^{n}\int_{t_0+t-s}^{\infty}C_j(u+s-t_0,s)\,du&=\sum_{j=1}^{n}E_j(t,s).\end{aligned}\tag{16}$$

Then, (15) can be written in the form of

$$\dot{x}(t) = B(t) - \sum_{j=1}^{n} g_j\left(x\left(t-\tau_j(t)\right)\right)\int_{t_0}^{\infty} C_j\left(u+t-t_0,t\right)du + \sum_{j=1}^{n}\frac{d}{dt}\int_{t_0}^{t} E_j(t,s)\, g_j\left(x\left(s-\tau_j(s)\right)\right)ds. \quad (17)$$

Hence

$$\dot{x}(t) = B(t) - \sum_{j=1}^{n} g_j\left(x\left(t-\tau_j(t)\right)\right)D_j(t) + \sum_{j=1}^{n}\frac{d}{dt}\int_{t_0}^{t} E_j(t,s)\, g_j\left(x\left(s-\tau_j(s)\right)\right)ds \quad (18)$$

so that

$$\dot{x}(t) = B(t) - \sum_{j=1}^{n}\widetilde{D}_j\left(p_j(t)\right)g_j(x(t)) + \sum_{j=1}^{n}\frac{d}{dt}\int_{t-\tau_j(t)}^{t}\widetilde{D}_j\left(p_j(s)\right)g_j(x(s))\,ds + \sum_{j=1}^{n}\frac{d}{dt}\int_{t_0}^{t} E_j(t,s)\, g_j\left(x\left(s-\tau_j(s)\right)\right)ds. \quad (19)$$

Thus, it can be written that

$$\dot{x}(t) = B(t) - \sum_{j=1}^{n}\widehat{D}_j(t)\,x(t) + \sum_{j=1}^{n}\widehat{D}_j(t)\left[x(t)-g_j(x(t))\right] + \sum_{j=1}^{n}\frac{d}{dt}\int_{t-\tau_j(t)}^{t}\widehat{D}_j(s)\,g_j(x(s))\,ds + \sum_{j=1}^{n}\frac{d}{dt}\int_{t_0}^{t} E_j(t,s)\, g_j\left(x\left(s-\tau_j(s)\right)\right)ds. \quad (20)$$

Let $\sum_{j=1}^{n}\widehat{D}_j(t) = K(t)$. Then,

$$\dot{x}(t) + K(t)\,x(t) = B(t) + \sum_{j=1}^{n}\widehat{D}_j(t)\left[x(t)-g_j(x(t))\right] + \sum_{j=1}^{n}\frac{d}{dt}\int_{t-\tau_j(t)}^{t}\widehat{D}_j(s)\,g_j(x(s))\,ds + \sum_{j=1}^{n}\frac{d}{dt}\int_{t_0}^{t} E_j(t,s)\, g_j\left(x\left(s-\tau_j(s)\right)\right)ds. \quad (21)$$

Multiplying both sides of (21) by $e^{\int_{t_0}^{t}K(s)ds}$ and then integrating from t_0 to t, then

$$\int_{t_0}^{t}\left[x(u)e^{\int_{t_0}^{t}K(s)ds}\right]' du = \int_{t_0}^{t} e^{\int_{t_0}^{u}K(s)ds}B(u)\,du + \sum_{j=1}^{n}\int_{t_0}^{t} e^{\int_{t_0}^{u}K(s)ds}\widehat{D}_j(u)\left[x(u)-g_j(x(u))\right]du + \sum_{j=1}^{n}\int_{t_0}^{t} e^{\int_{t_0}^{u}K(s)ds}\frac{d}{dt}\int_{t-\tau_j(t)}^{t}\widehat{D}_j(s)\,g_j(x(s))\,ds\,du + \sum_{j=1}^{n}\int_{t_0}^{t} e^{\int_{t_0}^{u}K(s)ds}\frac{d}{dt}\int_{t_0}^{t}E_j(t,s)\,g_j\left(x\left(s-\tau_j(s)\right)\right)ds\,du \quad (22)$$

so that

$$x(t) = \psi(t_0)\,e^{-\int_{t_0}^{t}K(s)ds} + \int_{t_0}^{t} e^{-\int_{u}^{t}K(s)ds}B(u)\,du + \sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_{u}^{t}K(s)ds}\widehat{D}_j(u)\left[x(u)-g_j(x(u))\right]du + \sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_{u}^{t}K(s)ds}\left[\frac{d}{du}\int_{u-\tau_j(u)}^{u}\widehat{D}_j(s)\,g_j(x(s))\,ds\right]du + \sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_{u}^{t}K(s)ds} \times\left[\frac{d}{du}\int_{t_0}^{u}E_j(u,s)\,g_j\left(x\left(s-\tau_j(s)\right)\right)ds\right]du. \quad (23)$$

Applying the integration by parts formula for the last two terms, we have

$$x(t) = \psi(t_0)\,e^{-\int_{t_0}^{t}K(s)ds} + \int_{t_0}^{t} e^{-\int_{u}^{t}K(s)ds}B(u)\,du + \sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_{u}^{t}K(s)ds}\widehat{D}_j(u)\left[x(u)-g_j(x(u))\right]du + \sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\widehat{D}_j(s)\,g_j(x(s))\,ds - \sum_{j=1}^{n} e^{-\int_{t_0}^{t}K(s)ds}\int_{t_0-\tau_j(t_0)}^{t_0}\widehat{D}_j(s)\,g_j(x(s))\,ds$$

$$-\sum_{j=1}^{n}\int_{t_0}^{t}\left[\int_{u-\tau_j(u)}^{u}\widehat{D}_j(s)\,g_j(x(s))\,ds\right]e^{-\int_u^t K(s)ds}K(u)\,du$$
$$+\sum_{j=1}^{n}\int_{t_0}^{t}E_j(t,s)\,g_j\left(x\left(s-\tau_j(s)\right)\right)ds$$
$$-\sum_{j=1}^{n}\int_{t_0}^{t}\left[\int_{t_0}^{u}E_j(u,s)\times g_j\left(x\left(s-\tau_j(s)\right)\right)ds\right]e^{-\int_u^t K(s)ds}K(u)\,du. \tag{24}$$

Conversely, we assume that a continuous function $x(t)=\psi(t)$ for $t\in[m(t_0),t_0]$ and satisfies the integral equation on $t\in[t_0,T_2]$. Then, it is differentiable on $[t_0,T_2]$. Hence, it is only needed to differentiate the integral equation. When we differentiate the integral equation, we can conclude the desired result.

Let $(C,\|\cdot\|)$ be the Banach space of bounded continuous functions on $[m(t_0),\infty)$ with the supremum norm $\|\phi\| = \sup\{|\phi(t)| : t\in[m(t_0),\infty)\}$ for $\phi\in C$. Let ρ denote the supremum metric and $\rho(\phi_1,\phi_2)=\|\phi_1-\phi_2\|$, where $\phi_1,\phi_2\in C$. Next, let $\psi:[m(t_0),t_0]\to R$ be a given continuous initial function.

Define the set $S\subset C$ by

$$S=\{\phi:[m(t_0),\infty)\longrightarrow R\mid\phi\in C,\ \phi(t)=\psi(t),t\in[m(t_0),t_0]\} \tag{25}$$

and its subset

$$S'=\{\phi:[m(t_0),\infty)\longrightarrow R\mid\phi\in C,\phi(t)=\psi(t),t\in[m(t_0),t_0],|\phi(t)|\le l,t\ge m(t_0)\}, \tag{26}$$

where $\psi:[m(t_0),t_0]\to[-l,l]$ is a given initial function and l is a positive constant. Define the mapping $P:S'\to S'$ by

$$(P\phi)(t)=\psi(t),\quad \text{if } t\in[m(t_0),t_0], \tag{27}$$

and if $t>t_0$, then

$$(P\phi)(t)=\psi(t_0)e^{-\int_{t_0}^{t}K(s)ds}+\int_{t_0}^{t}e^{-\int_u^t K(s)ds}B(u)\,du$$
$$+\sum_{j=1}^{n}\int_{t_0}^{t}e^{-\int_u^t K(s)ds}\widehat{D}_j(u)\left[\phi(u)-g_j(\phi(u))\right]du$$
$$+\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\widehat{D}_j(s)\,g_j(\phi(s))\,ds$$
$$-\sum_{j=1}^{n}e^{-\int_{t_0}^{t}K(s)ds}\int_{t_0-\tau_j(t_0)}^{t_0}\widehat{D}_j(s)\,g_j(\psi(s))\,ds$$
$$+\sum_{j=1}^{n}\int_{t_0}^{t}E_j(t,s)\,g_j(\phi(s))\,ds$$
$$-\sum_{j=1}^{n}\int_{t_0}^{t}\left[\int_{u-\tau_j(u)}^{u}\widehat{D}_j(s)\times g_j(\phi(s))\,ds\right]e^{-\int_u^t K(s)ds}K(u)\,du$$
$$-\sum_{j=1}^{n}\int_{t_0}^{t}\left[\int_{t_0}^{u}E_j(u,s)\times g_j\left(\phi\left(s-\tau_j(s)\right)\right)ds\right]e^{-\int_u^t K(s)ds}\times K(u)\,du. \tag{28}$$

Since $g_j(x)$ satisfy the Lipschitz condition, let $L_1,\ldots,L_n$ denote the common Lipschitz constants for $g_j(x)$ and $x-g_j(x)$.

It is also clear that

$$\int_{t_0}^{t}e^{-\int_u^t K(s)ds}K(u)\,du=e^{-\int_u^t K(s)ds}\Big|_{t_0}^{t}=1-e^{-\int_{t_0}^{t}K(s)ds}\approx 1,\quad\text{for large } t. \tag{29}$$

But since $g_j(x)$ are nonlinear, then L_j may not be small enough. Hence, P may not be a contracting mapping. We can solve this problem by giving an exponentially weight metric via the next lemma. □

Lemma 3. *We suppose that there exists a constant $l>0$ such that $g_j(x)$ satisfy the Lipschitz condition on $[-l,l]$. Then there exists a metric on S' such that*

(i) *the metric space (S',d) is complete,*

(ii) *P is a contraction mapping on (S',d) if P maps S' into itself.*

Proof. (i) We change the supremum norm to an exponentially weighted norm $|\phi|_h$, which is defined on S'. Let X be the space of all continuous functions $\phi:[m(t_0),\infty)\to R$ such that

$$|\phi|_h=\sup_n\left\{|\phi(t)|\,e^{-h(t)}:t\in[m(t_0),\infty)\right\}<\infty, \tag{30}$$

where $h(t)=k\sum_{j=1}^{n}L_j\int_{t_0}^{t}[\widehat{D}_j(s)+D_j(s)]ds$, k is a constant, and L_j are the common Lipschitz constants for $x-g_j(x)$ and $g_j(x)$. Then $(X,|\cdot|_h)$ is a Banach space. Thus (X,d) is a complete metric space with $d(\phi,\varphi) = |\phi-\varphi|_h$, where $\phi,\varphi \in S$. Under this metric, the space S' is a closed subset of X. Thus the metric space (S', d) is complete.

(ii) Let $P : S' \to S'$. It is clear that $\sum_{j=1}^{n} \widehat{D}_j(t) \geq 0$ and $\sum_{j=1}^{n} E_j(t,s) \geq 0$. Then, for $\phi,\varphi \in S'$, we can get

$$\begin{aligned}
&\left|(P\phi)(t) - (P\varphi)(t)\right| e^{-h(t)} \\
&\quad \leq \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_u^t K(s)ds} \widehat{D}_j(u) \left|\left[\phi(u) - g_j(\phi(u))\right]\right. \\
&\qquad\qquad \left. - \left[\varphi(u) - g_j(\varphi(u))\right]\right| e^{-h(t)} du \\
&\qquad + \sum_{j=1}^{n} \int_{t_0}^{t} E_j(t,s) \left|g_j(\phi(s)) - g_j(\varphi(s))\right| e^{-h(t)} ds \\
&\qquad + \sum_{j=1}^{n} \int_{t-\tau_j(t)}^{t} \widehat{D}_j(s) \left|g_j(\phi(s)) - g_j(\varphi(s))\right| e^{-h(t)} ds \\
&\qquad + \sum_{j=1}^{n} \int_{t_0}^{t} \left[\int_{u-\tau_j(u)}^{u} \widehat{D}_j(s) ds \left|g_j(\phi(s))\right.\right. \\
&\qquad\qquad \left.\left. - g_j(\varphi(s))\right| e^{-h(t)} ds\right] \\
&\qquad\quad \times e^{-\int_u^t K(s)ds} K(u) du \\
&\quad + \sum_{j=1}^{n} \int_{t_0}^{t} \left[\int_{t_0}^{u} E_j(u,s) \left|g_j\left(\phi\left(s-\tau_j(s)\right)\right)\right.\right. \\
&\qquad\qquad \left.\left. - g_j\left(\varphi\left(s-\tau_j(s)\right)\right)\right| e^{-h(t)} ds\right] \\
&\qquad\quad \times e^{-\int_u^t K(s)ds} K(u) du.
\end{aligned} \tag{31}$$

For $u \leq t$, since $D_j(t) \geq 0$, we have

$$\begin{aligned}
h(u) - h(t) &= \sum_{j=1}^{n} kL_j \int_{t_0}^{u} \left[\widehat{D}_j(s) + D_j(s)\right] ds \\
&\quad - \sum_{j=1}^{n} kL_j \int_{t_0}^{t} \left[\widehat{D}_j(s) + D_j(s)\right] ds \\
&= \sum_{j=1}^{n} (-k) L_j \int_{u}^{t} \left[\widehat{D}_j(s) + D_j(s)\right] ds \\
&\leq \sum_{j=1}^{n} (-k) L_j \int_{u}^{t} \widehat{D}_j(s) ds.
\end{aligned} \tag{32}$$

Further for $s \leq t$, it follows that

$$\begin{aligned}
&h\left(s-\tau_j(s)\right) - h(t) \\
&\quad = \sum_{j=1}^{n} kL_j \int_{t_0}^{s-\tau_j(s)} \left[\widehat{D}_j(s) + D_j(s)\right] ds \\
&\qquad - \sum_{j=1}^{n} kL_j \int_{t_0}^{t} \left[\widehat{D}_j(s) + D_j(s)\right] ds \\
&\quad = \sum_{j=1}^{n} (-k) L_j \int_{s-\tau_j(s)}^{t} \left[\widehat{D}_j(u) + D_j(u)\right] du \\
&\quad \leq \sum_{j=1}^{n} (-k) L_j \int_{s}^{t} D_j(u) du.
\end{aligned} \tag{33}$$

Since $E(t,s) \geq 0$, then we have

$$\begin{aligned}
\sum_{j=1}^{n} E_j(t,s) &= \sum_{j=1}^{n} \int_{t_0+t-s}^{\infty} c_j(u+s-t_0, s) du \\
&\leq \sum_{j=1}^{n} \int_{t_0}^{\infty} c_j(u+s-t_0, s) du = \sum_{j=1}^{n} D_j(s).
\end{aligned} \tag{34}$$

Hence

$$\begin{aligned}
&\left|(P\phi)(t) - (P\varphi)(t)\right| e^{-h(t)} \\
&\quad \leq |\phi-\varphi|_h \\
&\qquad \times \left\{\sum_{j=1}^{n} L_j \int_{t_0}^{t} e^{-\int_u^t K(s)ds} \widehat{D}_j(u) e^{h(u)-h(t)} du\right. \\
&\qquad\quad + \sum_{j=1}^{n} L_j \int_{t_0}^{t} E_j(t,s) e^{h(s)-h(t)} ds \\
&\qquad\quad + \sum_{j=1}^{n} L_j \int_{t-\tau_j(t)}^{t} \widehat{D}_j(s) e^{h(s)-h(t)} ds \\
&\qquad\quad + \sum_{j=1}^{n} L_j \int_{t_0}^{t} \left[\int_{u-\tau_j(t)}^{u} \widehat{D}_j(s) e^{h(s)-h(t)} ds\right] \\
&\qquad\qquad \times e^{-\int_u^t K(s)ds} K(u) du \\
&\qquad\quad + \sum_{j=1}^{n} L_j \int_{t_0}^{t} \left[\int_{t_0}^{u} E_j(u,s) e^{h(s-\tau_j(s))-h(t)} ds\right] \\
&\qquad\qquad \left. \times e^{-\int_u^t K(s)ds} K(u) du\right\}.
\end{aligned} \tag{35}$$

Therefore,

$$\begin{aligned}
&\sum_{j=1}^{n} L_j \int_{t_0}^{t} e^{-\int_u^t K(s)ds} \widehat{D}_j(u)\, e^{h(u)-h(t)} du \\
&= \sum_{j=1}^{n} L_j \int_{t_0}^{t} e^{-\sum_{j=1}^{n}\int_u^t \widehat{D}_j(s)ds} \widehat{D}_j(u)\, e^{h(u)-h(t)} du \\
&\le \sum_{j=1}^{n} L_j \int_{t_0}^{t} \frac{e^{-\sum_{j=1}^{n}\int_u^t \widehat{D}_j(s)ds} \widehat{D}_j(u)}{e^{\sum_{j=1}^{n} kL_j \int_u^t \widehat{D}_j(s)ds}} du \\
&= \sum_{j=1}^{n} L_j \int_{t_0}^{t} e^{-\sum_{j=1}^{n}(kL_j+1)\int_u^t \widehat{D}_j(s)ds} \widehat{D}_j(u)\, du \\
&\le \sum_{j=1}^{n} L_j \frac{1}{\sum_{j=1}^{n}(kL_j+1)} e^{-\sum_{j=1}^{n}(kL_j+1)\int_u^t \widehat{D}_j(s)ds}\Bigg|_{t_0}^{t} \qquad (36)\\
&\le \sum_{j=1}^{n} L_j \frac{1}{\sum_{j=1}^{n} kL_j} \le \frac{1}{k},
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{n} L_j \int_{t_0}^{t} E_j(t,s)\, e^{h(s)-h(t)} ds \\
&\le \sum_{j=1}^{n} L_j \int_{t_0}^{t} D_j(s)\, e^{\sum_{j=1}^{n}(-k)L_j \int_s^t D_j(s)ds} ds \\
&\le \sum_{j=1}^{n} L_j \frac{1}{\sum_{j=1}^{n} kL_j} e^{-\sum_{j=1}^{n} kL_j \int_u^t D_j(s)ds}\Bigg|_{t_0}^{t} \le \frac{1}{k}.
\end{aligned}$$

Thus, we have

$$\left|(P\phi)(t) - (P\varphi)(t)\right| e^{-h(t)} \le \frac{5}{k}|\phi-\varphi|_h, \quad t > t_0. \qquad (37)$$

For $t \in [m(t_0), t_0]$, $(P\phi)(t) = (P\varphi)(t) = \theta(t)$. Thus,

$$d(P\phi, P\varphi) \le \frac{5}{k} d(\phi-\varphi), \quad (k > 5). \qquad (38)$$

Therefore, P is contraction mapping on (S', d). □

Theorem 4. *We suppose that the assumption (A_1) holds. Moreover, we assume the following.*

(i) *There exists a positive constant l such that g_j satisfy the Lipschitz condition on $[-l, l]$ and g_j are odd and they are strictly increasing on $[-l, l]$, and $x - g_j(x)$ are nondecreasing on $[-l, l]$.*

(ii) *There exist an $\alpha_j \in (0,1)$ and a continuous function $a(t) : [0,\infty) \to [0,\infty)$ such that $f(t,x,y) \ge a(t)$ for $t \ge 0$, $x \in R$, $y \in R$,*

$$\begin{aligned}
&2\sup_{t\ge 0} \int_t^{P_j(t)} \int_0^{\infty} e^{-\int_s^{w+s} a(v)dv} b_j(s)\, dw\, ds \\
&\quad + 2\sup_{t\ge 0} \int_0^{t} \int_{t-s}^{\infty} e^{-\int_s^{w+s} a(v)dv} b_j(s)\, dw\, ds \le \alpha_j.
\end{aligned} \qquad (39)$$

(iii) *There exist constants $a_0 > 0$ and $Q > 0$ such that, for each $t \ge 0$, if $J \ge Q$, then*

$$\int_t^{t+J} a(v)\, dv \ge a_0 J. \qquad (40)$$

Then there exists $\delta \in (0, l)$ such that, for each initial function $\psi : [m(t_0), t_0] \to R$ and $\dot{x}(t_0)$ satisfying $|\dot{x}(t_0)| + \|\psi\| \le \delta$, there is a unique continuous function $x : [m(t_0), \infty) \to R$ satisfying $x(t) = \psi(t)$, which is a solution of (7) on $[t_0, \infty)$. Moreover, the zero solution of (7) is stable.

Proof. Choose $\psi : [m(t_0), t_0] \to R$ and $\dot{x}(t_0)$ such that

$$\begin{aligned}
&\left(Q + \frac{e^{-a_0\cdot Q}}{a_0}\right)\left|\dot{x}(t_0)\right| + \delta + \sum_{j=1}^{n} g_j(\delta) \int_{t_0-\tau_j(t_0)}^{t_0} \widehat{D}_j(s)\, ds \\
&\quad \le \left[1 - (\alpha_1 + \alpha_2 + \cdots + \alpha_n)\right] \sum_{j=1}^{n} g_j(l).
\end{aligned} \qquad (41)$$

In view of the assumptions and $g_j(0) = 0$, it follows that $g_j(l) \le l$. Since $g_j(x)$ satisfies Lipschitz condition on $[-l, l]$, thus $g_j(x)$ is continuous function on $[-l, l]$. Then, there exists a constant δ such that $\delta < l$.

Thus, we can get

$$\begin{aligned}
&\left|(P\phi)(t)\right| \\
&\le \delta + \int_{t_0}^{t} e^{-\int_u^t K(s)ds} \left|\dot{x}(t_0)\right| e^{-\int_{t_0}^{u} A(s)ds} du \\
&\quad + \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_u^t K(s)ds} \widehat{D}_j(u)\left(l - g_j(l)\right) du \\
&\quad + \sum_{j=1}^{n} \int_{t-\tau_j(t)}^{t} \widehat{D}_j(s)\, g_j(l)\, ds \\
&\quad + \sum_{j=1}^{n} \int_{t_0}^{t} \left[\int_{u-\tau_j(u)}^{u} \widehat{D}_j(s)\, g_j(l)\, ds\right] e^{-\int_u^t K(s)ds} K(u)\, du \\
&\quad + \sum_{j=1}^{n} \int_{t_0-\tau_j(t_0)}^{t_0} \widehat{D}_j(s)\, g_j(\delta)\, ds \\
&\quad + \sum_{j=1}^{n} \int_{t_0}^{t} E_j(t,s)\, g_j(l)\, ds \\
&\quad + \sum_{j=1}^{n} \int_{t_0}^{t} \left[\int_{t_0}^{u} E_j(u,s)\, g_j(l)\, ds\right] e^{-\int_u^t K(s)ds} K(u)\, du.
\end{aligned} \qquad (42)$$

It also follows that

$$\sum_{j=1}^{n}\int_{t_0}^{t} E_j(t,s)\,ds = \sum_{j=1}^{n}\int_{t_0}^{t}\int_{t_0+t-s}^{\infty} C_j(u+s-t_0,s)\,du\,ds$$
$$= \sum_{j=1}^{n}\int_{t_0}^{t}\int_{t_0+t-s}^{\infty} e^{-\int_s^{u+s-t_0} A(v)dv} b_j(s)\,du\,ds$$
$$= \sum_{j=1}^{n}\int_{t_0}^{t}\int_{t-s}^{\infty} e^{-\int_s^{u+s} A(v)dv} b_j(s)\,du\,ds$$
$$\leq \sup_{t\geq 0}\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_s^{u+s} a(v)dv} b_1(s)\,du\,ds$$
$$+\cdots+\sup_{t\geq 0}\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_s^{u+s} a(v)dv} b_n(s)\,du\,ds,$$
$$\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} \widehat{D}_j(s)\,ds = \sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} \widetilde{D}_j(P_j(s))\,ds$$
$$= \sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} \frac{D_j(P_j(s))}{1-\tau_j'(s)}ds = \sum_{j=1}^{n}\int_{t}^{P_j(t)} D_j(s)\,ds$$
$$= \sum_{j=1}^{n}\int_{t}^{P_j(t)}\int_{0}^{\infty} e^{-\int_s^{w+s} a(v)dv}\cdot b_j(s)\,dw\,ds$$
$$\leq \sup_{t\geq 0}\int_{t}^{P_1(t)}\int_{0}^{\infty} e^{-\int_s^{w+s} a(v)dv} b_1(s)\,dw\,ds$$
$$+\cdots+\sup_{t\geq 0}\int_{t}^{P_n(t)}\int_{0}^{\infty} e^{-\int_s^{w+s} a(v)dv} b_n(s)\,dw\,ds. \tag{43}$$

From assumption (ii), we have

$$\sum_{j=1}^{n}\int_{t_0}^{t} E_j(t,s)\,g_j(l)\,ds$$
$$+\sum_{j=1}^{n}\int_{t_0}^{t}\left[\int_{u-\tau_j(u)}^{u}\widehat{D}_j(s)\,g_j(l)\,ds\right]e^{-\int_u^t K(s)ds}K(u)\,du$$
$$+\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\widehat{D}_j(s)\,g_j(l)\,ds$$
$$+\sum_{j=1}^{n}\int_{t_0}^{t}\left[\int_{t_0}^{u}E_j(u,s)\,g_j(l)\,ds\right]e^{-\int_u^t K(s)ds}K(u)\,du$$
$$\leq \sum_{j=1}^{n} g_j(l)\left\{2\sup_{t\geq 0}\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_s^{u+s} a(v)dv} b_1(s)\,du\,ds\right.$$
$$+\cdots+2\sup_{t\geq 0}\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_s^{u+s} a(v)dv} b_n(s)\,du\,ds$$
$$+\,2\sup_{t\geq 0}\int_{t}^{P_1(t)}\int_{0}^{\infty} e^{-\int_s^{w+s} a(v)dv} b_1(s)\,dw\,ds$$
$$\left.+\cdots+2\sup_{t\geq 0}\int_{t}^{P_n(t)}\int_{0}^{\infty} e^{-\int_s^{w+s} a(v)dv} b_n(s)\,dw\,ds\right\}$$
$$\leq (\alpha_1+\alpha_2+\cdots+\alpha_n)\sum_{j=1}^{n} g_j(l). \tag{44}$$

Hence

$$|(P\phi)(t)| \leq \delta + \sum_{j=1}^{n} g_j(\delta)\int_{t_0-\tau_j(t_0)}^{t_0}\widehat{D}_j(s)\,ds + \sum_{j=1}^{n}\left(l-g_j(l)\right)$$
$$+(\alpha_1+\alpha_2+\cdots+\alpha_n)\sum_{j=1}^{n} g_j(l)$$
$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\,|\dot{x}(t_0)|\,e^{-\int_{t_0}^{u} A(s)ds}du$$
$$\leq \delta + \sum_{j=1}^{n} g_j(\delta)\int_{t_0-\tau_j(t_0)}^{t_0}\widehat{D}_j(s)\,ds + \sum_{j=1}^{n}\left(l-g_j(l)\right)$$
$$+(\alpha_1+\alpha_2+\cdots+\alpha_n)\sum_{j=1}^{n} g_j(l)$$
$$+\int_{t_0}^{t} |\dot{x}(t_0)|\,e^{-\int_{t_0}^{u} A(s)ds}du. \tag{45}$$

Using condition (iii) of the theorem, we get

$$\int_{t_0}^{t} e^{-\int_{t_0}^{u} A(s)ds}du = \int_{t_0}^{t_0+Q} e^{-\int_{t_0}^{u} A(s)ds}du + \int_{t_0+Q}^{t} e^{-\int_{t_0}^{u} A(s)ds}du$$
$$\leq Q + \frac{e^{-a_0\cdot Q}}{a_0}. \tag{46}$$

Thus,

$$|(P\phi)(t)|$$
$$\leq \delta + \sum_{j=1}^{n} g_j(\delta)\int_{t_0-\tau_j(t_0)}^{t_0}\widetilde{D}_j(s)\,ds + \sum_{j=1}^{n}\left(l-g_j(l)\right)$$
$$+(\alpha_1+\alpha_2+\cdots+\alpha_n)\sum_{j=1}^{n} g_j(l) + |\dot{x}(t_0)|\left(Q+\frac{e^{-a_0\cdot Q}}{a_0}\right) \tag{47}$$

and so

$$|(P_2\phi)(t)| \leq \sum_{j=1}^{n} l. \tag{48}$$

It is obvious that if $t \in [m(t_0), t_0]$, then $(P_2\phi)(t) = \psi(t)$. Moreover, for $t \in [m(t_0), \infty)$, we get $|(P_2\phi)(t)| \leq \sum_{j=1}^{n} l$.

Therefore, $P\phi : S' \to S'$. Since P is a contraction mapping, then P has unique fixed point $x(t)$ such that $|x(t)| \leq \sum_{j=1}^{n} l$.

From (14), we have

$$|y(t)| \leq |\dot{x}(t_0)| + \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_u^t A(s)ds} b_j(u) \left| g_j\left(x\left(u-\tau_j(u)\right)\right)\right| du. \tag{49}$$

Since, for $t \in [0,\infty)$, $0 \leq b_j(t) \leq M_j$, then

$$\begin{aligned} |y(t)| &\leq |\dot{x}(t_0)| + \sum_{j=1}^{n} M_j \int_{t_0}^{t} e^{-\int_u^t A(s)ds} \left| x\left(u-\tau_j(u)\right)\right| du \\ &\leq \sum_{j=1}^{n} l\left(1 + M_j \int_{t_0}^{t} e^{-\int_u^t A(s)ds} du\right) \\ &< \sum_{j=1}^{n} l\left[1 + M_j\left(Q + \frac{e^{-a_0 Q}}{a_0}\right)\right]. \end{aligned} \tag{50}$$

Hence

$$|x(t)| + |y(t)| < \sum_{j=1}^{n} l\left[2 + M_j\left(Q + \frac{e^{-a_0 Q}}{a_0}\right)\right]. \tag{51}$$

If we replace ε by l, then we show that the zero solution of (7) is stable. This result completes the proof of the theorem. □

3. Conclusion

A kind of scalar Liénard type equations with multiple variable delays is considered. The stability of the zero solution of this equation is investigated. In proving our main result, we use the fixed points theory by giving an exponentially weight metric. Our result extends and improves some recent results in the literature.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors of this paper would like to expresses their sincere appreciation to the anonymous referees for their valuable comments and suggestions which have led to an improvement in the presentation of the paper.

References

[1] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover Publications, Mineola, NY, USA, 2006.

[2] T. A. Burton and T. Furumochi, "Fixed points and problems in stability theory for ordinary and functional differential equations," *Dynamic Systems and Applications*, vol. 10, no. 1, pp. 89–116, 2001.

[3] T. A. Burton, "Fixed points, stability, and exact linearization," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 5, pp. 857–870, 2005.

[4] D. Pi, "Study the stability of solutions of functional differential equations via fixed points," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 2, pp. 639–651, 2011.

[5] M. Fan, Z. Xia, and H. Zhu, "Asymptotic stability of delay differential equations via fixed point theory and applications," *Canadian Applied Mathematics Quarterly*, vol. 18, no. 4, pp. 361–380, 2010.

[6] Y. N. Raffoul, "Stability in neutral nonlinear differential equations with functional delays using fixed-point theory," *Mathematical and Computer Modelling*, vol. 40, no. 7-8, pp. 691–700, 2004.

[7] C. Jin and J. Luo, "Fixed points and stability in neutral differential equations with variable delays," *Proceedings of the American Mathematical Society*, vol. 136, no. 3, pp. 909–918, 2008.

[8] R. Zhang and G. Liu, "Stability of nonlinear neutral differential equation via fixed point," *Annals of Differential Equations*, vol. 28, no. 4, pp. 488–493, 2012.

[9] A. Ardjouni and A. Djoudi, "Fixed points and stability in nonlinear neutral Volterra integro-differential equations with variable delays," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 20–37, pp. 1–13, 2013.

[10] C. Tunç, "New stability and boundedness results of Liénard type equations with multiple deviating arguments," *Journal of Contemporary Mathematical Analysis*, vol. 45, no. 4, pp. 47–56, 2010.

[11] C. Tunç, "Stability, boundedness and uniform boundedness of solutions of nonlinear delay differential equations," *Discrete and Continuous Dynamical Systems A*, pp. 1395–1403, 2011.

[12] C. Tunç, "Stability and boundedness in multi delay Lienard equation," *Filomat*, vol. 27, no. 3, pp. 437–447, 2013.

[13] C. Tunç, "Stability to vector Liénard equation with constant deviating argument," *Nonlinear Dynamics*, vol. 73, no. 3, pp. 1245–1251, 2013.

[14] C. Tunç, "A note on the bounded solutions to $x'' + c(t, x, x') + q(t)b(x) = f(t)$," *Applied Mathematics & Information Sciences*, vol. 8, no. 1, pp. 393–399, 2014.

[15] C. Tunç and R. Yazgan, "Existence of periodic solutions to multidelay functional differential equations of second order," *Abstract and Applied Analysis*, vol. 2013, Article ID 968541, 5 pages, 2013.

[16] C. Tunç, "New results on the existence of periodic solutions for Rayleigh equation with state-dependent delay," *Journal of Mathematical & Fundamental Sciences*, vol. 45, no. 2, pp. 154–162, 2013.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 120781, 7 pages
http://dx.doi.org/10.1155/2014/120781

Research Article

Existence and Estimates of Positive Solutions for Some Singular Fractional Boundary Value Problems

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Received 25 December 2013; Accepted 14 February 2014; Published 1 April 2014

Academic Editor: Samir Saker

We establish the existence and uniqueness of a positive solution u for the following fractional boundary value problem: $D^{\alpha}u(x) = -a(x)u^{\sigma}(x)$, $x \in (0,1)$ with the conditions $\lim_{x\to 0^+} x^{2-\alpha}u(x) = 0$, $u(1) = 0$, where $1 < \alpha \le 2$, $\sigma \in (-1,1)$, and a is a nonnegative continuous function on $(0,1)$ that may be singular at $x = 0$ or $x = 1$. We also give the global behavior of such a solution.

1. Introduction

Recently, the theory of fractional differential equations has been developed very quickly and the investigation for the existence of solutions of these differential equations has attracted considerable attention of researchers in the last few years (see [1–11] and the references therein).

Fractional differential equations arise in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, and electromagnetism. They also serve as an excellent tool for the description of hereditary properties of various materials and processes (see [12–14]). In consequence, the subject of fractional differential equations is gaining much importance. Motivated by the surge in the development of this subject, we consider the following singular Dirichlet problem:

$$D^{\alpha}u(x) = -a(x)u^{\sigma}(x), \quad x \in (0,1), \qquad \lim_{x\to 0^+} x^{2-\alpha}u(x) = 0, \quad u(1) = 0, \tag{1}$$

where $1 < \alpha \le 2$, $-1 < \sigma < 1$, and a is a nonnegative continuous function on $(0,1)$ that may be singular at $x = 0$ or $x = 1$. Then we study the existence and exact asymptotic behavior of positive solutions for this problem.

We recall that, for a measurable function ν, the Riemann-Liouville fractional integral $I_{\beta}\nu$ and the Riemann-Liouville derivative $D^{\beta}\nu$ of order $\beta > 0$ are, respectively, defined by

$$\begin{aligned} I_{\beta}\nu(x) &= \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}\nu(t)\,dt, \\ D^{\beta}\nu(x) &= \frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\beta-1}\nu(t)\,dt \\ &= \left(\frac{d}{dx}\right)^n I_{n-\beta}\nu(x), \end{aligned} \tag{2}$$

provided that the right hand sides are pointwise defined on $(0,1]$. Here $n = [\beta] + 1$ and $[\beta]$ means the integer part of the number β and Γ is the Euler Gamma function.

Moreover, we have the following well-known properties (see [3, 13, 15]):

(i) $I_{\beta}I_{\gamma}\nu(x) = I_{\beta+\gamma}\nu(x)$ for $x \in [0,1]$, $\nu \in L^1((0,1])$, $\beta + \gamma \ge 1$;

(ii) $D^{\beta}I_{\beta}\nu(x) = \nu(x)$ for a.e. $x \in [0,1]$, where $\nu \in L^1((0,1])$, $\beta > 0$;

(iii) if $v \in C((0,1)) \cap L^1((0,1))$ and $D^\beta v(x) = 0$, then $v(x) = \sum_{j=1}^n c_j t^{\beta-j}$, where $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ and n is the smallest integer greater than or equal to β.

Several results are obtained for fractional differential equations with different boundary conditions, but none of them deal with the existence of a positive solution to problem (1).

Our aim in this paper is to establish the existence and uniqueness of a positive solution $u \in C_{2-\alpha}([0,1])$ for problem (1) with a precise asymptotic behavior, where $C_{2-\alpha}([0,1])$ is the set of all functions f such that $t \to t^{2-\alpha} f(t)$ is continuous on $[0,1]$.

To state our result, we need some notations. We will use $\mathcal{K}$ to denote the set of Karamata functions L defined on $(0,\eta]$ by

$$L(t) := c \exp\left(\int_t^\eta \frac{z(s)}{s} ds\right), \tag{3}$$

for some $\eta > 1$, where $c > 0$ and $z \in C([0,\eta])$ such that $z(0) = 0$. It is clear that a function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1((0,\eta])$ such that

$$\lim_{t \to 0^+} \frac{tL'(t)}{L(t)} = 0. \tag{4}$$

For two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that $(1/c)f(x) \le g(x) \le cf(x)$ for all $x \in S$. We denote by $x^+ = \max(x, 0)$ for $x \in \mathbb{R}$ and by $B^+((0,1))$ the set of all nonnegative measurable functions on $(0,1)$.

Throughout this paper, we assume that a is nonnegative on $(0,1)$ and satisfies the following condition:

(H_0) $a \in C((0,1))$ such that for $t \in (0,1)$

$$a(t) \approx t^{-\lambda} L_1(t)(1-t)^{-\mu} L_2(1-t), \tag{5}$$

where $\lambda \le \alpha + (2-\alpha)(1-\sigma)$, $\mu \le \alpha$, $L_1, L_2 \in \mathcal{K}$ satisfying

$$\int_0^\eta \frac{L_1(t)}{t^{\lambda+(2-\alpha)\sigma-1}} dt < \infty, \qquad \int_0^\eta \frac{L_2(t)}{t^{\mu-\alpha+1}} dt < \infty. \tag{6}$$

In the sequel, we introduce the function θ defined on $(0,1)$ by

$$\theta(x) = x^{\min(1,(2-\lambda+(\alpha-2)\sigma)/(1-\sigma))} \left(\widetilde{L}_1(x)\right)^{1/(1-\sigma)} \times (1-x)^{\min(1,(\alpha-\mu)/(1-\sigma))} \left(\widetilde{L}_2(1-x)\right)^{1/(1-\sigma)}, \tag{7}$$

where

$$\widetilde{L}_1(x) = \begin{cases} 1, & \text{if } \lambda < \alpha - (\alpha-1) \times (1-\sigma), \\ \displaystyle\int_x^\eta \frac{L_1(s)}{s} ds, & \text{if } \lambda = \alpha - (\alpha-1) \times (1-\sigma), \\ L_1(x), & \text{if } \alpha - (\alpha-1)(1-\sigma) < \lambda < \alpha + (2-\alpha)(1-\sigma), \\ \displaystyle\int_0^x \frac{L_1(s)}{s} ds, & \text{if } \lambda = \alpha + (2-\alpha) \times (1-\sigma), \end{cases}$$

$$\widetilde{L}_2(x) = \begin{cases} 1, & \text{if } \mu < \alpha + \sigma - 1, \\ \displaystyle\int_x^\eta \frac{L_2(s)}{s} ds, & \text{if } \mu = \alpha + \sigma - 1, \\ L_2(x), & \text{if } \alpha + \sigma - 1 < \mu < \alpha, \\ \displaystyle\int_0^x \frac{L_2(s)}{s} ds, & \text{if } \mu = \alpha. \end{cases} \tag{8}$$

Our main result is the following.

Theorem 1. *Let $\sigma \in (-1,1)$ and assume that a satisfies (H_0). Then problem (1) has a unique positive solution $u \in C_{2-\alpha}([0,1])$ satisfying for $x \in (0,1)$,*

$$u(x) \approx x^{\alpha-2}\theta(x). \tag{9}$$

Remark 2. Note that, for $x \in (0,1)$, we have

$$x^{\alpha-2}\theta(x) \approx x^{\min(\alpha-1,(\alpha-\lambda)/(1-\sigma))} \times \left(\widetilde{L}_1(x)\right)^{1/(1-\sigma)} (1-x)^{\min(1,(\alpha-\mu)/(1-\sigma))} \times \left(\widetilde{L}_2(1-x)\right)^{1/(1-\sigma)}. \tag{10}$$

This implies in particular that, for $1 < \alpha < 2$ and $\alpha < \lambda \le \alpha + (2-\alpha)(1-\sigma)$, the solution u blows up at $x = 0$ and for $\lambda < \alpha$, $\lim_{x \to 0^+} u(x) = 0$.

This paper is organized as follows. Some preliminary lemmas are stated and proved in the next section, involving some already known results on Karamata functions. In Section 3, we give the proof of Theorem 1.

2. Technical Lemmas

To let the paper be self-contained, we begin this section by recapitulating some properties of Karamata regular variation theory. The following is due to [16, 17].

Lemma 3. *The following hold.*

(i) *Letting $L \in \mathcal{K}$ and $\varepsilon > 0$, then one has*

$$\lim_{t \to 0^+} t^\varepsilon L(t) = 0. \tag{11}$$

(ii) *Let $L_1, L_2 \in \mathcal{K}$ and $p \in \mathbb{R}$. Then one has $L_1 + L_2 \in \mathcal{K}$, $L_1 L_2 \in \mathcal{K}$, and $L_1^p \in \mathcal{K}$.*

Example 4. Let m be a positive integer. Let $c > 0$, $(\mu_1, \mu_2, \dots, \mu_m) \in \mathbb{R}^m$, and d be a sufficiently large positive real number such that the function

$$L(t) = c\prod_{k=1}^{m}\left(\log_k\left(\frac{d}{t}\right)\right)^{-\mu_k} \tag{12}$$

is defined and positive on $(0, \eta]$, for some $\eta > 1$, where $\log_k x = \log \circ \log \circ \cdots \circ \log x$ (k times). Then $L \in \mathcal{K}$.

Applying Karamata's theorem (see [16, 17]), we get the following.

Lemma 5. *Let $\mu \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0, \eta]$. One has the following:*

(i) *if $\mu < -1$, then $\int_0^\eta s^\mu L(s)ds$ diverges and $\int_t^\eta s^\mu L(s)ds \sim_{t\to 0^+} -t^{1+\mu}L(t)/(\mu+1)$;*

(ii) *if $\mu > -1$, then $\int_0^\eta s^\mu L(s)ds$ converges and $\int_0^t s^\mu L(s)ds \sim_{t\to 0^+} t^{1+\mu}L(t)/(\mu+1)$.*

Lemma 6. *Let $L \in \mathcal{K}$ be defined on $(0, \eta]$. Then one has*

$$\lim_{t\to 0^+}\frac{L(t)}{\int_t^\eta (L(s)/s)\,ds} = 0. \tag{13}$$

If further $\int_0^\eta (L(s)/s)ds$ converges, then one has

$$\lim_{t\to 0^+}\frac{L(t)}{\int_0^t (L(s)/s)\,ds} = 0. \tag{14}$$

Proof. We distinguish two cases.

Case 1. We suppose that $\int_0^\eta (L(s)/s)ds$ converges. Since the function $t \to L(t)/t$ is nonincreasing in $(0, \omega]$, for some $\omega < \eta$, it follows that, for each $t \le \omega$, we have

$$L(t) \le \int_0^t \frac{L(s)}{s}ds. \tag{15}$$

It follows that $\lim_{t\to 0^+} L(t) = 0$. So we deduce (13).

Now put

$$\varphi(t) = \frac{L(t)}{t}, \quad \text{for } t \in (0, \eta). \tag{16}$$

Using that $\lim_{t\to 0^+}(t\varphi'(t)/\varphi(t)) = -1$, we obtain

$$\int_0^t \varphi(s)\,ds \sim_{t\to 0^+} -\int_0^t s\varphi'(s)\,ds = -t\varphi(t) + \int_0^t \varphi(s)\,ds. \tag{17}$$

This implies that

$$\int_0^t \frac{L(s)}{s}ds \sim_{t\to 0^+} -L(t) + \int_0^t \frac{L(s)}{s}ds. \tag{18}$$

So (14) holds.

Case 2. We suppose that $\int_0^\eta (L(s)/s)ds$ diverges. We have, for some $\omega < \eta$,

$$\int_t^\omega \varphi(s)\,ds \sim_{t\to 0^+} t\varphi(t) - \omega\varphi(\omega) + \int_t^\omega \varphi(s)\,ds. \tag{19}$$

This implies that

$$\int_t^\omega \frac{L(s)}{s}ds \sim_{t\to 0^+} L(t) - \omega\varphi(\omega) + \int_t^\omega \frac{L(s)}{s}ds. \tag{20}$$

This proves (13) and completes the proof. □

Remark 7. Let $L \in \mathcal{K}$ be defined on $(0, \eta]$; then using (4) and (13), we deduce that

$$t \longrightarrow \int_t^\eta \frac{L(s)}{s}ds \in \mathcal{K}. \tag{21}$$

If further $\int_0^\eta (L(s)/s)ds$ converges, we have by (13) that

$$t \longrightarrow \int_0^t \frac{L(s)}{s}ds \in \mathcal{K}. \tag{22}$$

Lemma 8. *Given $1 < \alpha \le 2$ and $\varphi \in C([0,1])$, then the unique continuous solution of*

$$\begin{gathered} D^\alpha u(x) = -\varphi(x), \quad x \in (0,1), \\ \lim_{x\to 0} x^{2-\alpha}u(x) = 0, \quad u(1) = 0 \end{gathered} \tag{23}$$

is given by

$$u(x) = G_\alpha\varphi(x) := \int_0^1 G_\alpha(x,t)\,\varphi(t)\,dt, \tag{24}$$

where

$$G_\alpha(x,t) = \frac{1}{\Gamma(\alpha)}\left[x^{\alpha-1}(1-t)^{\alpha-1} - \left((x-t)^+\right)^{\alpha-1}\right] \tag{25}$$

is Green's function for the boundary value problem (23).

Proof. Since $\varphi \in C([0,1])$, then $u_0 = -I_\alpha\varphi$ is a solution of the equation $D^\alpha u = -\varphi$. Hence $D^\alpha(u + I_\alpha\varphi) = 0$. Consequently there exist two constants $c_1, c_2 \in \mathbb{R}$ such that $u(x) + I_\alpha\varphi(x) = c_1x^{\alpha-1} + c_2x^{\alpha-2}$. Using the fact that $\lim_{x\to 0}x^{2-\alpha}u(x) = 0$ and $u(1) = 0$, we obtain $c_2 = 0$ and $c_1 = I_\alpha\varphi(1)$. So

$$\begin{aligned} u(x) &= \frac{1}{\Gamma(\alpha)}x^{\alpha-1}\int_0^1 (1-t)^{\alpha-1}\varphi(t)\,dt \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\varphi(t)\,dt \\ &= \int_0^1 G_\alpha(x,t)\,\varphi(t)\,dt. \end{aligned} \tag{26}$$

□

In the following, we give some estimates on the Green function $G_\alpha(x, y)$. So, we need the following lemma.

Lemma 9. *For $\lambda, \mu \in (0,\infty)$ and $t \in [0,1]$ one has*

$$\min\left(1, \frac{\mu}{\lambda}\right)\left(1-t^{\lambda}\right) \leq 1-t^{\mu} \leq \max\left(1, \frac{\mu}{\lambda}\right)\left(1-t^{\lambda}\right). \quad (27)$$

Proposition 10. *On $(0,1)\times(0,1)$, one has*

(i) $G_\alpha(x,t) \approx x^{\alpha-2}(1-t)^{\alpha-2}\ \min(x,t)\ (1-\max(x,t))$;

(ii) *there exist two constants $c_1, c_2 > 0$ such that*

$$c_1 x^{\alpha-1} t(1-t)^{\alpha-1}(1-x) \leq G_\alpha(x,t) \leq c_2 x^{\alpha-2} t(1-t)^{\alpha-1}. \quad (28)$$

Proof. (i) For $x, t \in (0,1)\times(0,1)$ we have

$$G_\alpha(x,t) = \frac{(1-t)^{\alpha-1}x^{\alpha-1}}{\Gamma(\alpha)}\left[1-\left(\frac{(x-t)^{+}}{x(1-t)}\right)^{\alpha-1}\right]. \quad (29)$$

Since $(x-t)^{+}/x(1-t) \in (0,1)$ for $x, t \in (0,1)$, then by applying Lemma 9 with $\mu = \alpha-1$ and $\lambda = 1$, we obtain

$$\begin{aligned} G_\alpha(x,t) &\approx x^{\alpha-1}(1-t)^{\alpha-1}\left(1-\frac{(x-t)^{+}}{x(1-t)}\right) \\ &= x^{\alpha-2}(1-t)^{\alpha-2}\min(x,t)(1-\max(x,t)). \end{aligned} \quad (30)$$

(ii) Using the following inequalities for $x, t \in [0,1]$,

$$x(1-x)t(1-t) \leq \min(x,t)(1-\max(x,t)) \leq t(1-t), \quad (31)$$

we deduce the result from (i). □

As a consequence of Proposition 10, we obtain the following.

Corollary 11. *Let $f \in B^{+}((0,1))$ and put $G_\alpha f(x) := \int_0^1 G_\alpha(x,t)f(t)dt$ for $x \in (0,1]$. Then*

$$\begin{aligned} &G_\alpha f(x) < \infty \text{ for } x \in (0,1) \\ &\text{iff } \int_0^1 t(1-t)^{\alpha-1}f(t)\,dt < \infty. \end{aligned} \quad (32)$$

Proposition 12. *Given $1<\alpha<2$ and f such that the function $t \to t(1-t)^{\alpha-1}f(t)$ is continuous and integrable on $(0,1)$, then $G_\alpha f$ is the unique solution in $C_{2-\alpha}([0,1])$ of the following boundary value problem:*

$$\begin{aligned} &D^{\alpha}u(x) = -f(x), \quad x \in (0,1), \\ &\lim_{x\to 0^{+}} x^{2-\alpha}u(x) = 0, \quad u(1) = 0. \end{aligned} \quad (33)$$

Proof. From Corollary 11, the function $G_\alpha f$ is defined on $(0,1)$ and by Proposition 10, we have

$$G_\alpha|f|(x) \leq c_2 x^{\alpha-2}\int_0^1 t(1-t)^{\alpha-1}|f(t)|\,dt, \quad (34)$$

which implies that $I_{2-\alpha}(G_\alpha|f|)$ is bounded on $(0,1)$. Now, using Fubini's theorem, we have

$$\begin{aligned} I_{2-\alpha}&(G_\alpha f)(x) \\ &= \frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}G_\alpha f(t)\,dt \\ &= \frac{1}{\Gamma(2-\alpha)}\int_0^1\left(\int_0^x (x-t)^{1-\alpha}G_\alpha(t,s)\,dt\right)f(s)\,ds. \end{aligned} \quad (35)$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}G_\alpha(t,s)\,dt \\ &\quad= \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)}\left[(1-s)^{\alpha-1}\int_0^x (x-t)^{1-\alpha}t^{\alpha-1}dt \right.\\ &\qquad\left. -\int_0^x (x-t)^{1-\alpha}\left((t-s)^{+}\right)^{\alpha-1}dt\right] \\ &\quad= x(1-s)^{\alpha-1} \\ &\qquad -\frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)}\int_0^x (x-t)^{1-\alpha} \\ &\qquad \times\left((t-s)^{+}\right)^{\alpha-1}dt. \end{aligned} \quad (36)$$

Now, suppose that $s \leq x$; then we have

$$\begin{aligned} &\int_0^x (x-t)^{1-\alpha}\left((t-s)^{+}\right)^{\alpha-1}dt \\ &\quad= \int_s^x (x-t)^{1-\alpha}(t-s)^{\alpha-1}dt. \end{aligned} \quad (37)$$

By considering the substitution $t = s+\theta(x-s)$, we obtain

$$\int_s^x (x-t)^{1-\alpha}(t-s)^{\alpha-1}dt = \Gamma(\alpha)\Gamma(2-\alpha)(x-s). \quad (38)$$

Moreover if $x \leq s$ and $t \in (0,x)$, we have $\int_0^x (x-t)^{1-\alpha}\left((t-s)^{+}\right)^{\alpha-1}dt = 0$.

So, it follows that

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}G_\alpha(t,s)\,dt \\ &\quad= x(1-s)^{\alpha-1}-(x-s)^{+}. \end{aligned} \quad (39)$$

This implies that

$$\begin{aligned} I_{2-\alpha}(G_\alpha f)(x) &= \int_0^1 \left[x(1-s)^{\alpha-1} - (x-s)^+\right] f(s)\,ds \\ &= x\int_0^x \left((1-s)^{\alpha-1} - 1\right) f(s)\,ds \\ &\quad + \int_0^x s f(s)\,ds + x\int_x^1 (1-s)^{\alpha-1} f(s)\,ds, \\ D^\alpha (G_\alpha f)(x) &= \frac{d^2}{dx^2}\left(I_{2-\alpha}(G_\alpha f)\right)(x) \\ &= -f(x), \quad \text{for } x \in (0,1). \end{aligned} \tag{40}$$

Moreover, using part (i) of Proposition 10 and the dominated convergence theorem, we conclude that $\lim_{x\to 0^+} x^{2-\alpha} G_\alpha f(x) = 0$ and $G_\alpha f(1) = 0$.

Finally, we prove the uniqueness. Let $u, v \in C_{2-\alpha}([0,1])$ be two solutions of (33) and put $w = v - u$. Then $w \in C_{2-\alpha}([0,1]) \subset L^1((0,1)) \cap C((0,1))$ and $D^\alpha w = 0$. Hence, it follows that $w(x) = c_1 x^{\alpha-1} + c_2 x^{\alpha-2}$. Using the fact that $\lim_{x\to 0^+} x^{2-\alpha} w(x) = w(1) = 0$, we conclude that $w = 0$ and so $u = v$.

In the sequel, we assume that $\beta \le 2$ and $\gamma \le \alpha$ and we put

$$b(t) = t^{-\beta} L_3(t)(1-t)^{-\gamma} L_4(1-t), \tag{41}$$

where $L_3, L_4 \in \mathcal{K}$ satisfy

$$\int_0^\eta \frac{L_3(t)}{t^{\beta-1}}\,dt < \infty, \qquad \int_0^\eta t^{\alpha-1-\gamma} L_4(t)\,dt < \infty. \tag{42}$$

So, we aim to give some estimates on the potential function $G_\alpha b(x)$.

We define the Karamata functions ψ_β, ϕ_γ by

$$\psi_\beta(x) = \begin{cases} \int_0^x \frac{L_3(t)}{t}\,dt, & \text{if } \beta = 2, \\ L_3(x), & \text{if } 1 < \beta < 2, \\ \int_x^\eta \frac{L_3(t)}{t}\,dt, & \text{if } \beta = 1, \\ 1, & \text{if } \beta < 1, \end{cases} \tag{43}$$

$$\phi_\gamma(x) = \begin{cases} \int_0^x \frac{L_4(t)}{t}\,dt, & \text{if } \gamma = \alpha, \\ L_4(x), & \text{if } \alpha - 1 < \gamma < \alpha, \\ \int_x^\eta \frac{L_4(t)}{t}\,dt, & \text{if } \gamma = \alpha - 1, \\ 1, & \text{if } \gamma < \alpha - 1. \end{cases} \tag{44}$$

□

Then, we have the following.

Proposition 13. *For $x \in (0,1)$,*

$$G_\alpha b(x) \approx x^{\min(\alpha-1,\alpha-\beta)}(1-x)^{\min(1,\alpha-\gamma)} \psi_\beta(x)\,\phi_\gamma(1-x). \tag{45}$$

Proof. Using Proposition 10, we have

$$\begin{aligned} x^{2-\alpha} G_\alpha b(x) &\approx \int_0^1 (1-t)^{\alpha-2-\gamma} t^{-\beta} \min(x,t) \\ &\qquad \times (1-\max(x,t)) L_3(t) L_4(1-t)\,dt \\ &\approx (1-x)\int_0^x (1-t)^{\alpha-2-\gamma} t^{1-\beta} \\ &\qquad \times L_3(t) L_4(1-t)\,dt \\ &\quad + x\int_x^1 (1-t)^{\alpha-1-\gamma} t^{-\beta} \\ &\qquad \times L_3(t) L_4(1-t)\,dt \\ &= (1-x) I(x) + xJ(x). \end{aligned} \tag{46}$$

For $0 < x \le 1/2$, we have $I(x) \approx \int_0^x t^{1-\beta} L_3(t)dt$. So, using Lemma 5 and hypothesis (42), we deduce that

$$I(x) \approx \begin{cases} x^{2-\beta} L_3(x), & \text{if } \beta < 2, \\ \int_0^x \frac{L_3(t)}{t}\,dt, & \text{if } \beta = 2. \end{cases} \tag{47}$$

Now, we have

$$\begin{aligned} J(x) &\approx \int_x^{1/2} t^{-\beta} L_3(t)\,dt \\ &\quad + \int_{1/2}^1 (1-t)^{\alpha-1-\gamma} L_4(1-t)\,dt \\ &\approx 1 + \int_x^{1/2} t^{-\beta} L_3(t)\,dt, \end{aligned} \tag{48}$$

which implies by Lemma 5 that

$$J(x) \approx \begin{cases} x^{1-\beta} L_3(x), & \text{if } 1 < \beta \le 2, \\ \int_x^\eta \frac{L_3(t)}{t}\,dt, & \text{if } \beta = 1, \\ 1, & \text{if } \beta < 1. \end{cases} \tag{49}$$

Hence, it follows by Lemma 6 and hypothesis (42) that, for $0 < x \le 1/2$, we get

$$x^{2-\alpha} G_\alpha b(x) \approx \begin{cases} \int_0^x \frac{L_3(t)}{t}\,dt & \text{if } \beta = 2, \\ x^{2-\beta} L_3(x) & \text{if } 1 < \beta < 2, \\ x\int_x^\eta \frac{L_3(t)}{t}\,dt & \text{if } \beta = 1, \\ x & \text{if } \beta < 1. \end{cases} \tag{50}$$

That is

$$G_\alpha b(x) \approx x^{\min(\alpha-1,\alpha-\beta)} \psi_\beta(x). \tag{51}$$

Now, for $1/2 \le x < 1$, we use again Lemma 5 and hypothesis (42) to deduce that

$$\begin{aligned} I(x) &\approx \int_0^{1/2} t^{1-\beta} L_3(t)\,dt \\ &\quad + \int_{1/2}^{x} (1-t)^{\alpha-2-\gamma} L_4(1-t)\,dt \\ &\approx 1 + \int_{1/2}^{x} (1-t)^{\alpha-2-\gamma} L_4(1-t)\,dt \\ &\approx \begin{cases} (1-x)^{\alpha-1-\gamma} L_4(1-x), & \text{if } \alpha-1<\gamma\le\alpha, \\ \displaystyle\int_{1-x}^{\eta} \frac{L_4(t)}{t}\,dt, & \text{if } \gamma=\alpha-1, \\ 1, & \text{if } \gamma<\alpha-1, \end{cases} \\ J(x) &\approx \int_0^{1-x} t^{\alpha-1-\gamma} L_4(t)\,dt \\ &\approx \begin{cases} (1-x)^{\alpha-\gamma} L_4(1-x), & \text{if } \gamma<\alpha, \\ \displaystyle\int_0^{1-x} \frac{L_4(t)}{t}\,dt, & \text{if } \gamma=\alpha. \end{cases} \end{aligned} \tag{52}$$

Hence, it follows from Lemma 3 that, for $x \in [1/2, 1)$, we get

$$x^{2-\alpha} G_\alpha b(x) \approx \begin{cases} \displaystyle\int_0^{1-x} \frac{L_4(t)}{t}\,dt, & \text{if } \gamma=\alpha, \\ (1-x)^{\alpha-\gamma} L_4(1-x), & \text{if } \alpha-1<\gamma<\alpha, \\ (1-x)\displaystyle\int_{1-x}^{\eta} \frac{L_4(t)}{t}\,dt, & \text{if } \gamma=\alpha-1, \\ 1-x, & \text{if } \gamma<\alpha-1. \end{cases} \tag{53}$$

That is

$$x^{2-\alpha}\, G_\alpha b(x) \approx (1-x)^{\min(1,\alpha-\gamma)} \phi_\gamma(1-x). \tag{54}$$

This together with (51) implies that, for $x \in (0,1)$, we have

$$G_\alpha b(x) \approx x^{\min(\alpha-1,\alpha-\beta)} (1-x)^{\min(1,\alpha-\gamma)} \psi_\beta(x)\, \phi_\gamma(1-x). \tag{55}$$

□

3. Proof of Theorem 1

In order to prove Theorem 1, we need the following Lemma.

Lemma 14. *Assume that the function a satisfies (H_0) and put $\omega(t) = a(t)t^{(\alpha-2)\sigma}(\theta(t))^\sigma$ for $t \in (0,1)$. Then one has, for $x \in (0,1)$,*

$$G_\alpha \omega(x) \approx x^{\alpha-2}\theta(x). \tag{56}$$

Proof. Put $r = \min(\alpha-1, (\alpha-\lambda)/(1-\sigma))$ and $s = \min(1, (\alpha-\mu)/(1-\sigma))$. Then for $t \in (0,1)$, we have

$$\begin{aligned} \omega(t) &= t^{-\lambda+r\sigma} L_1(t) \left(\widetilde{L}_1(t)\right)^{\sigma/(1-\sigma)} \\ &\quad \times (1-t)^{-\mu+s\sigma} L_2(1-t) \left(\widetilde{L}_2(1-t)\right)^{\sigma/(1-\sigma)}. \end{aligned} \tag{57}$$

Let $\beta = \lambda - r\sigma$, $\gamma = \mu - s\sigma$, $L_3(t) = L_1(t)(\widetilde{L}_1(t))^{\sigma/(1-\sigma)}$, and $L_4(t) = L_2(t)(\widetilde{L}_2(t))^{\sigma/(1-\sigma)}$. Then, using Proposition 13, we obtain by a simple computation that

$$x^{2-\alpha} G_\alpha(\omega)(x) \approx \theta(x). \tag{58}$$

□

Proof of Theorem 1. From Lemma 14, there exists $M > 1$ such that, for each $x \in (0,1)$,

$$\frac{1}{M}\theta(x) \le x^{2-\alpha} G_\alpha \omega(x) \le M\theta(x), \tag{59}$$

where $\omega(t) = a(t)t^{(\alpha-2)\sigma}\theta^\sigma(t)$.

Put $c_0 = M^{1/(1-|\sigma|)}$ and let

$$\Lambda = \left\{ v \in C([0,1]) : \frac{1}{c_0}\theta \le v \le c_0\theta \right\}. \tag{60}$$

In order to use a fixed point theorem, we denote $\widetilde{a}(t) = a(t)t^{(\alpha-2)\sigma}$ and we define the operator T on Λ by

$$Tv(x) = x^{2-\alpha} G_\alpha(\widetilde{a}v^\sigma)(x). \tag{61}$$

For this choice of c_0, we can easily prove that, for $v \in \Lambda$, we have $Tv \le c_0\theta$ and $Tv \ge (1/c_0)\theta$.

Now, we have

$$\begin{aligned} Tv(x) &= \frac{x^{2-\alpha}}{\Gamma(\alpha)} \int_0^1 G_\alpha(x,t)\, \widetilde{a}(t)\, v^\sigma(t)\,dt \\ &= \frac{x^{2-\alpha}}{\Gamma(\alpha)} \int_0^1 \Big[x^{\alpha-1}(1-t)^{\alpha-1} \\ &\qquad - \left((x-t)^+\right)^{\alpha-1} \Big] \widetilde{a}(t)\, v^\sigma(t)\,dt. \end{aligned} \tag{62}$$

Since the function $(x,t) \to x^{\alpha-1}(1-t)^{\alpha-1} - ((x-t)^+)^{\alpha-1}$ is continuous on $[0,1]\times[0,1]$ and by Proposition 10, Corollary 11, and Lemma 14, the function $t \to t(1-t)^{\alpha-1}\,\widetilde{a}(t)\theta^\sigma(t)$ is integrable on $(0,1)$, we deduce that the operator T is compact from Λ to itself. It follows by the Schauder fixed point theorem that there exists $v \in \Lambda$ such that $Tv = v$. Put $u(x) = x^{\alpha-2}v(x)$. Then $u \in C_{2-\alpha}([0,1])$ and u satisfies the equation

$$u(x) = G_\alpha(au^\sigma)(x). \tag{63}$$

Since the function $t \to t(1-t)^{\alpha-1}a(t)u^\sigma(t)$ is continuous and integrable on $(0,1)$, then by Proposition 12, the function u is a positive continuous solution of problem (1).

Finally, let us prove that u is the unique positive continuous solution satisfying (9). To this aim, we assume that (1) has two positive solutions $u, v \in C_{2-\alpha}([0,1])$ satisfying (9) and consider the nonempty set $J = \{m \ge 1 : 1/m \le u/v \le m\}$ and put $c = \inf J$. Then $c \ge 1$ and we have $(1/c)v \le u \le cv$. It follows that $u^\sigma \le c^{|\sigma|}v^\sigma$ and consequently

$$\begin{aligned} &-D^\alpha\left(c^{|\sigma|}v - u\right) = a\left(c^{|\sigma|}v^\sigma - u^\sigma\right) \ge 0, \\ &\lim_{t\to 0^+} x^{2-\alpha}\left(c^{|\sigma|}v - u\right)(t) = 0, \\ &\left(c^{|\sigma|}v - u\right)(1) = 0, \end{aligned} \tag{64}$$

which implies by Proposition 12 that $c^{|\sigma|}v - u = G_\alpha(a(c^{|\sigma|}v^\sigma - u^\sigma)) \geq 0$. By symmetry, we also obtain that $v \leq c^{|\sigma|}u$. Hence, $c^{|\sigma|} \in J$ and $c \leq c^{|\sigma|}$. Since $|\sigma| < 1$, then $c = 1$ and consequently $u = v$. □

Example 15. Let $\sigma \in (-1, 1)$ and a be a positive continuous function on $(0, 1)$ such that

$$a(t) \approx t^{-\lambda}(1-t)^{-\mu}\log\left(\frac{2}{1-t}\right), \tag{65}$$

where $\lambda < \alpha + (2-\alpha)(1-\sigma)$ and $\mu < \alpha$. Then, using Theorem 1, problem (1) has a unique positive continuous solution u satisfying the following estimates:

$$\begin{aligned} u(x) \approx{}& x^{\min(\alpha-1,(\alpha-\lambda)/(1-\sigma))}\left(\widetilde{L}_1(x)\right)^{1/(1-\sigma)} \\ &\times (1-x)^{\min(1,(\alpha-\mu)/(1-\sigma))}\left(\widetilde{L}_2(1-x)\right)^{1/(1-\sigma)}, \end{aligned} \tag{66}$$

where

$$\widetilde{L}_1(x) = \begin{cases} 1, & \text{if } \lambda \neq \alpha - (\alpha-1)(1-\sigma), \\ \log\left(\dfrac{2}{x}\right), & \text{if } \lambda = \alpha - (\alpha-1)(1-\sigma), \end{cases}$$

$$\widetilde{L}_2(x) = \begin{cases} 1, & \text{if } \mu < \alpha + \sigma - 1, \\ \left(\log\left(\dfrac{2}{x}\right)\right)^2, & \text{if } \mu = \alpha + \sigma - 1, \\ \log\left(\dfrac{2}{x}\right), & \text{if } \alpha + \sigma - 1 < \mu < \alpha. \end{cases} \tag{67}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

References

[1] S. Abbas, M. Benchohra, and G. M. N'Guérékata, *Topics in Fractional Differential Equations*, Developments in Mathematics, Springer, New York, NY, USA, 2012.

[2] R. P. Agarwal, D. O'Regan, and S. Staněk, "Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 371, no. 1, pp. 57–68, 2010.

[3] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[4] J. Caballero, J. Harjani, and K. Sadarangani, "Positive solutions for a class of singular fractional boundary value problems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1325–1332, 2011.

[5] J. Deng and L. Ma, "Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations," *Applied Mathematics Letters*, vol. 23, no. 6, pp. 676–680, 2010.

[6] N. Kosmatov, "A singular boundary value problem for nonlinear differential equations of fractional order," *Journal of Applied Mathematics and Computing*, vol. 29, no. 1-2, pp. 125–135, 2009.

[7] Y. Liu, w. Zhang, and X. Liu, "A sufficient condition for the existence of a positive solution for a nonlinear fractional differential equation with the Riemann-Liouville derivative," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1986–1992, 2012.

[8] T. Qiu and Z. Bai, "Existence of positive solutions for singular fractional differential equations," *Electronic Journal of Differential Equations*, vol. 149, p. 19, 2008.

[9] S. Staněk, "The existence of positive solutions of singular fractional boundary value problems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1379–1388, 2011.

[10] S. Zhang, "Positive solutions to singular boundary value problem for nonlinear fractional differential equation," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1300–1309, 2010.

[11] Y. Zhao, S. Sun, Z. Han, and Q. Li, "Positive solutions to boundary value problems of nonlinear fractional differential equations," *Abstract and Applied Analysis*, vol. 2011, Article ID 390543, 16 pages, 2011.

[12] K. Diethelm and A. D. Freed, "On the solution of nonlinear fractional order differential equations used in the modelling of viscoplasticity," in *SCientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties*, F. Keil, W. Mackens, and H. Voss, Eds., pp. 217–224, Springer, Heidelberg, Germany, 1999.

[13] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.

[14] W. Lin, "Global existence theory and chaos control of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 1, pp. 709–726, 2007.

[15] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[16] V. Marić, *Regular Variation and Differential Equations*, vol. 1726 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.

[17] E. Seneta, *Regularly Varying Functions*, vol. 508 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1976.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 657926, 5 pages
http://dx.doi.org/10.1155/2014/657926

Research Article

Existence and Global Behavior of Positive Solutions for Some Fourth-Order Boundary Value Problems

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Received 10 November 2013; Accepted 27 February 2014; Published 31 March 2014

Academic Editor: Samir Saker

We establish the existence and uniqueness of a positive solution to the following fourth-order value problem: $u^{(4)}(x) = a(x)u^{\sigma}(x)$, $x \in (0,1)$ with the boundary conditions $u(0) = u(1) = u'(0) = u'(1) = 0$, where $\sigma \in (-1,1)$ and a is a nonnegative continuous function on $(0,1)$ that may be singular at $x = 0$ or $x = 1$. We also give the global behavior of such a solution.

1. Introduction

The purpose of this paper is to study the existence and uniqueness with a precise global behavior of a positive solution $u \in C^4((0,1)) \cap C([0,1])$ for the following fourth-order two-point boundary value problem:

$$u^{(4)}(x) = a(x)u^{\sigma}(x), \quad x \in (0,1), \qquad u(0) = u(1) = u'(0) = u'(1) = 0, \tag{1}$$

where $-1 < \sigma < 1$ and a is a nonnegative continuous function on $(0,1)$ that may be singular at $x = 0$ or $x = 1$ and satisfies some hypotheses related to the class of Karamata regularly varying functions.

There have been extensive studies on fourth-order boundary value problems with diverse boundary conditions via many methods; see, for example, [1–9] and the references therein.

A naturel motivation for studying higher order boundary value problems lies in their applications. For example, it is well known that the deformation of an elastic beam in equilibrium state, whose both ends clamped, can be described by fourth-order boundary value problem

$$u^{(4)}(x) = g(x, u(x)), \quad x \in (0,1), \qquad u(0) = u(1) = u'(0) = u'(1) = 0. \tag{2}$$

Our aim in this paper is to give a contribution to the study of these problems by exploiting the properties of the Karamata class of functions.

To state our result, we need some notations. We denote by $C([0,1])$ the set of all continuous functions f on $[0,1]$, and we will use $\mathcal{K}$ to denote the set of Karamata functions L defined on $(0,\eta]$ by

$$L(t) := c\exp\left(\int_t^{\eta} \frac{z(s)}{s}ds\right), \tag{3}$$

for some $\eta > 1$, where $c > 0$ and $z \in C([0,\eta])$ such that $z(0) = 0$. It is clear that a function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1((0,\eta])$ such that

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0. \tag{4}$$

For two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that $(1/c)f(x) \le g(x) \le cf(x)$, for all $x \in S$. We denote by $x^+ = \max(x,0)$, $x \wedge t = \min(x,t)$, $x \vee t = \max(x,t)$, for $x,\ t \in \mathbb{R}$, and $B^+((0,1))$ the set of all measurable functions on $(0,1)$.

Throughout this paper, we assume that a is nonnegative on $(0,1)$ and satisfies the following condition:

(H_0) $a \in C((0,1))$ such that for $t \in (0,1)$

$$a(t) \approx t^{-\lambda}L_1(t)(1-t)^{-\mu}L_2(1-t), \tag{5}$$

where $\lambda \leq 3+\sigma$, $\mu \leq 3+\sigma$, $L_1, L_2 \in \mathcal{K}$ satisfying

$$\int_0^\eta t^{2+\sigma-\lambda} L_1(t)\,dt < \infty, \qquad \int_0^\eta t^{2+\sigma-\mu} L_2(t)\,dt < \infty. \quad (6)$$

In the sequel, we introduce the function $\theta_{\lambda,\mu}$ defined on $(0,1)$ by

$$\begin{aligned}\theta_{\lambda,\mu}(x) &= x^{\min(2,(4-\lambda)/(1-\sigma))}\left(\widetilde{L}_1(x)\right)^{1/(1-\sigma)} \\ &\quad \times (1-x)^{\min(2,(4-\lambda)/(1-\sigma))}\left(\widetilde{L}_2(1-x)\right)^{1/(1-\sigma)},\end{aligned} \quad (7)$$

where

$$\widetilde{L}_1(x) = \begin{cases} 1 & \text{if } \lambda < 2(1+\sigma), \\ \int_x^\eta \frac{L_1(s)}{s} ds & \text{if } \lambda = 2(1+\sigma), \\ L_1(x) & \text{if } 2(1+\sigma) < \lambda < 3+\sigma, \\ \int_0^x \frac{L_1(s)}{s} ds & \text{if } \lambda = 3+\sigma, \end{cases}$$
$$\widetilde{L}_2(x) = \begin{cases} 1 & \text{if } \mu < 2(1+\sigma), \\ \int_x^\eta \frac{L_2(s)}{s} ds & \text{if } \mu = 2(1+\sigma), \\ L_2(x) & \text{if } 2(1+\sigma) < \mu < 3+\sigma, \\ \int_0^x \frac{L_2(s)}{s} ds & \text{if } \mu = 3+\sigma. \end{cases} \quad (8)$$

Our main result is the following.

Theorem 1. *Let $\sigma \in (-1,1)$ and assume that a satisfies (H_0). Then, problem (1) has a unique positive solution $u \in C^4((0,1)) \cap C([0,1])$ satisfying for $x \in (0,1)$*

$$u(x) \approx \theta_{\lambda,\mu}(x). \quad (9)$$

This paper is organized as follows. Some preliminary lemmas are stated and proved in the next section, involving some already known results on Karamata functions. In Section 3, we give the proof of Theorem 1.

2. Technical Lemmas

To let the paper be self-contained, we begin this section by recapitulating some properties of Karamata regular variation theory. The following is due to [10, 11].

Lemma 2. *The following assertions hold.*

(i) *Let $L \in \mathcal{K}$ and $\varepsilon > 0$; then, one has*

$$\lim_{t\to 0^+} t^\varepsilon L(t) = 0. \quad (10)$$

(ii) *Let $L_1, L_2 \in \mathcal{K}$ and let $p \in \mathbb{R}$. Then, one has $L_1 + L_2 \in \mathcal{K}$, $L_1 L_2 \in \mathcal{K}$, and $L_1^p \in \mathcal{K}$.*

Example 3. Let m be a positive integer. Let $c > 0$, let $(\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m$, and let d be a sufficiently large positive real number such that the function

$$L(t) = c\prod_{k=1}^m \left(\log_k\left(\frac{d}{t}\right)\right)^{\mu_k} \quad (11)$$

is defined and positive on $(0,\eta]$, for some $\eta > 1$, where $\log_k x = \log \circ \log \circ \cdots \circ \log x$ (k times). Then, $L \in \mathcal{K}$.

Applying Karamata's theorem (see [10, 11]), we get the following.

Lemma 4. *Let $\mu \in \mathbb{R}$ and let L be a function in $\mathcal{K}$ defined on $(0,\eta]$. One has the following:*

(i) *if $\mu < -1$, then $\int_0^\eta s^\mu L(s)ds$ diverges and $\int_t^\eta s^\mu L(s)ds \sim_{t\to 0^+} -(t^{1+\mu}L(t)/(\mu+1))$;*

(ii) *if $\mu > -1$, then $\int_0^\eta s^\mu L(s)ds$ converges and $\int_0^t s^\mu L(s)ds \sim_{t\to 0^+} (t^{1+\mu}L(t)/(\mu+1))$.*

Lemma 5 (see [12] or [13]). *Let $L \in \mathcal{K}$ be defined on $(0,\eta]$. Then, one has*

$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta (L(s)/s)\,ds} = 0. \quad (12)$$

If further $\int_0^\eta (L(s)/s)ds$ converges, then one has

$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t (L(s)/s)\,ds} = 0. \quad (13)$$

Remark 6. Let $L \in \mathcal{K}$ be defined on $(0,\eta]$; then, using (4) and (12), we deduce that

$$t \longrightarrow \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}. \quad (14)$$

If further $\int_0^\eta (L(s)/s)ds$ converges, we have by (12) that

$$t \longrightarrow \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}. \quad (15)$$

Lemma 7. *Given that $f \in C([0,1])$, then the unique continuous solution of*

$$\begin{aligned} u^{(4)}(x) &= f(x), \quad x \in (0,1), \\ u(0) = u(1) &= u'(0) = u'(1) = 0 \end{aligned} \quad (16)$$

is given by

$$u(x) = Gf(x) := \int_0^1 G(x,t) f(t)\,dt, \quad (17)$$

where

$$G(x,t) = \frac{1}{6}(x\wedge t)^2(1-x\vee t)^2\left[3(x\vee t) - (x\wedge t)(1+2(x\vee t))\right] \quad (18)$$

is Green's function for the boundary value problem (16).

Remark 8. For $x, t \in (0,1)$, we have $G(1-x, 1-t) = G(x,t)$.

In the following, we give some estimates on the Green function $G(x,t)$ that will be used later.

Proposition 9. *On $(0,1)\times(0,1)$, one has the following:*

(i) $(1/3)(x\wedge t)^2(1-x\vee t)^2(x\vee t)(1-x\wedge t) \le G(x,t) \le (1/2)(x\wedge t)^2(1-x\vee t)^2(x\vee t)(1-x\wedge t)$;

(ii) $(1/3)x^2(1-x)^2t^2(1-t)^2 \le G(x,t) \le (1/2)x(1-x)t^2(1-t)^2$.

Proof. (i) It follows from the fact that for $x, t \in (0,1)\times(0,1)$ we have

$$\begin{aligned} 2(x\vee t)(1-x\wedge t) &\le 3(x\vee t) - (x\wedge t)(1+2x\vee t) \\ &\le 3(x\vee t)(1-x\wedge t). \end{aligned} \tag{19}$$

(ii) Since for $x, t \in (0,1)$ we have $x^2(1-x)^2t^2(1-t)^2 \le (x\wedge t)^2(1-x\vee t)^2(x\vee t)(1-x\wedge t)$, the result follows from (i).

As a consequence of the assertion (ii) of Proposition 9, we obtain the following. □

Corollary 10. *Let $f \in B^+((0,1))$ and put $Gf(x) := \int_0^1 G(x,t)f(t)dt$, for $x \in (0,1]$.*

Then,

$$\begin{gathered} Gf(x) < \infty \\ \text{for } x \in (0,1) \quad \text{iff} \quad \int_0^1 t^2(1-t)^2 f(t)\,dt < \infty. \end{gathered} \tag{20}$$

Proposition 11. *Let f be a measurable function such that the function $t \to t^2(1-t)^2 f(t)$ is continuous and integrable on $(0,1)$. Then, Gf is the unique solution in $C^4((0,1))\cap C([0,1])$ of the problem*

$$\begin{gathered} u^{(4)}(x) = f(x), \quad x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{gathered} \tag{21}$$

Proof. From Corollary 10, the function Gf is defined on $(0,1)$ and, by Proposition 9, we have

$$G(f)(x) \le \frac{1}{2}\, x(1-x)\int_0^1 t^2(1-t)^2 |f(t)|\,dt. \tag{22}$$

Now, since $t \to t^2 f(t)$ is integrable near 0 and $t \to (1-t)^2 f(t)$ is integrable near 1, then, for $x \in (0,1)$, we have

$$\begin{aligned} Gf(x) = {} & \frac{1}{2}x(1-x)^2 \int_0^x t^2 f(t)\,dt \\ & + \frac{1}{2}x^2 \int_x^1 t(1-t)^2 f(t)\,dt \\ & - \frac{1}{6}(1+2x)(1-x)^2 \int_0^x t^3 f(t)\,dt \\ & - \frac{1}{6}x^3 \int_x^1 (1+2t)(1-t)^2 f(t)\,dt. \end{aligned} \tag{23}$$

This gives

$$\begin{aligned} (Gf)'(x) = {} & \frac{1}{2}(1-3x)(1-x)\int_0^x t^2 f(t)\,dt \\ & + x\int_x^1 t(1-t)^2 f(t)\,dt + x(1-x)\int_0^x t^3 f(t)\,dt \\ & - \frac{1}{2}x^2 \int_x^1 (1+2t)(1-t)^2 f(t)\,dt, \\ (Gf)''(x) = {} & (3x-2)\int_0^x t^2 f(t)\,dt \\ & + \int_x^1 t(1-t)^2 f(t)\,dt + (1-2x)\int_0^x t^3 f(t)\,dt \\ & - x\int_x^1 (1+2t)(1-t)^2 f(t)\,dt, \\ (Gf)'''(x) = {} & \int_0^x \left(3t^2 - 2t^3\right) f(t)\,dt \\ & - \int_x^1 (1+2t)(1-t)^2 f(t)\,dt, \\ (Gf)^{(4)}(x) = {} & f(x). \end{aligned} \tag{24}$$

Moreover, we have $Gf(0) = Gf(1) = (Gf)'(0) = (Gf)'(1) = 0$.

Finally, we prove the uniqueness. Let $u, v \in C^4((0,1))\cap C([0,1])$ be two solutions of (21) and put $w = v - u$. Then, $w \in C^4((0,1))\cap C([0,1])$ and $w^{(4)} = 0$. Hence, it follows that $w(x) = ax^3 + bx^2 + cx + d$. Using the fact that $w(0) = w(1) = w'(0) = w'(1) = 0$, we conclude that $w = 0$ and so $u = v$.

In the sequel, we assume that $\beta \le 3$ and $\gamma \le 3$ and we put

$$b(t) = t^{-\beta} L_3(t)(1-t)^{-\gamma} L_4(1-t), \tag{25}$$

where $L_3, L_4 \in \mathscr{K}$ satisfy

$$\int_0^\eta t^{2-\beta} L_3(t)\,dt < \infty, \qquad \int_0^\eta t^{2-\gamma} L_4(t)\,dt < \infty. \tag{26}$$

So, we aim to give some estimates on the potential function $Gb(x)$.

We define the Karamata functions ψ_β, ϕ_γ by

$$\psi_\beta(x) = \begin{cases} \int_0^x \frac{L_3(t)}{t} dt & \text{if } \beta = 3, \\ L_3(x) & \text{if } 2 < \beta \le 3, \\ \int_x^\eta \frac{L_3(t)}{t} dt & \text{if } \beta = 2, \\ 1 & \text{if } \beta < 2, \end{cases} \quad (27)$$

$$\phi_\gamma(x) = \begin{cases} \int_0^x \frac{L_4(t)}{t} dt & \text{if } \gamma = 3, \\ L_4(x) & \text{if } 2 < \gamma < 3, \\ \int_x^\eta \frac{L_4(t)}{t} dt & \text{if } \gamma = 2, \\ 1 & \text{if } \gamma < 2. \end{cases}$$

Then, we have the following. □

Proposition 12. *For* $x \in (0,1)$,

$$Gb(x) \approx x^{\min(2,4-\beta)}(1-x)^{\min(2,4-\gamma)}\psi_\beta(x)\phi_\gamma(1-x). \quad (28)$$

Proof. Using Proposition 9, we have

$$\begin{aligned} Gb(x) &\approx \int_0^1 (1-t)^{-\gamma} t^{-\beta} (x \wedge t)^2 \\ &\quad \times (1 - x \vee t)^2 (x \vee t)(1 - x \wedge t) L_3(t) L_4(1-t)\, dt \\ &\approx x(1-x)^2 \int_0^x (1-t)^{1-\gamma} t^{2-\beta} L_3(t) L_4(1-t)\, dt \\ &\quad + x^2(1-x) \int_x^1 (1-t)^{2-\gamma} t^{1-\beta} L_3(t) L_4(1-t)\, dt \\ &= x(1-x)^2 I(x) + x^2(1-x) J(x). \end{aligned} \quad (29)$$

For $0 < x \le 1/2$, we have $I(x) \approx \int_0^x t^{2-\beta} L_3(t) dt$. So, using Lemma 4 and hypothesis (26), we deduce that

$$I(x) \approx \begin{cases} \int_0^x \frac{L_3(t)}{t} dt & \text{if } \beta = 3, \\ x^{3-\beta} L_3(x) & \text{if } \beta < 3. \end{cases} \quad (30)$$

Now, we have

$$\begin{aligned} J(x) &\approx \int_x^{1/2} t^{1-\beta} L_3(t)\, dt + \int_{1/2}^1 (1-t)^{2-\gamma} L_4(1-t)\, dt \\ &\approx 1 + \int_x^{1/2} t^{1-\beta} L_3(t)\, dt. \end{aligned} \quad (31)$$

This implies by Lemma 4 that

$$J(x) \approx \begin{cases} x^{2-\beta} L_3(x) & \text{if } 2 < \beta \le 3, \\ \int_x^\eta \frac{L_3(t)}{t} dt & \text{if } \beta = 2, \\ 1 & \text{if } \beta < 2. \end{cases} \quad (32)$$

Hence, it follows by Lemma 5 and hypothesis (26) that, for $0 < x \le 1/2$, we get

$$Gb(x) \approx \begin{cases} x \int_0^x \frac{L_3(t)}{t} dt & \text{if } \beta = 3, \\ x^{4-\beta} L_3(x) & \text{if } 2 < \beta < 3, \\ x^2 \int_x^\eta \frac{L_3(t)}{t} dt & \text{if } \beta = 2 \\ x^2 & \text{if } \beta < 2, \end{cases} \quad (33)$$

That is, for $0 < x \le 1/2$,

$$Gb(x) \approx x^{\min(2,4-\beta)} \psi_\beta(x). \quad (34)$$

Now, since $G(1-x, 1-t) = G(x,t)$, we use similar arguments as above applied to L_4 instead of L_3 to obtain

$$Gb(x) \approx (1-x)^{\min(2,4-\gamma)} \phi_\gamma(1-x) \quad \text{for } \frac{1}{2} \le x \le 1. \quad (35)$$

This together with (34) implies that, for $x \in (0,1)$, we have

$$Gb(x) \approx x^{\min(2,4-\beta)} (1-x)^{\min(2,4-\gamma)} \psi_\beta(x) \phi_\gamma(1-x). \quad (36)$$

□

3. Proof of Theorem 1

In order to prove Theorem 1, we need the following lemma.

Lemma 13. *Assume that the function a satisfies* (H_0) *and put* $\omega(t) = a(t)(\theta_{\lambda,\mu}(t))^\sigma$ *for* $t \in (0,1)$. *Then, one has, for* $x \in (0,1)$,

$$G\omega(x) \approx \theta_{\lambda,\mu}(x). \quad (37)$$

Proof. Put $r = \min(2, (4-\lambda)/(1-\sigma))$ and $s = \min(2, (4-\mu)/(1-\sigma))$. Then, for $t \in (0,1)$, we have

$$\begin{aligned} \omega(t) &= t^{-\lambda + r\sigma} L_1(t) \left(\widetilde{L}_1(t)\right)^{\sigma/(1-\sigma)} (1-t)^{-\mu + s\sigma} \\ &\quad \times L_2(1-t) \left(\widetilde{L}_2(1-t)\right)^{\sigma/(1-\sigma)}. \end{aligned} \quad (38)$$

Let $\beta = \lambda - r\sigma$, $\gamma = \mu - s\sigma$, $L_3(t) = L_1(t)(\widetilde{L}_1(t))^{\sigma/(1-\sigma)}$, and $L_4(t) = L_2(t)(\widetilde{L}_2(t))^{\sigma/(1-\sigma)}$. Then, using Proposition 12, we obtain by a simple computation that

$$G(\omega)(x) \approx \theta_{\lambda,\mu}(x). \quad (39)$$

□

Proof of Theorem 1. From Lemma 13, there exists $M > 1$ such that for each $x \in (0,1)$

$$\frac{1}{M} \theta_{\lambda,\mu}(x) \le G\omega(x) \le M \theta_{\lambda,\mu}(x), \quad (40)$$

where $\omega(t) = a(t)(\theta_{\lambda,\mu}(t))^\sigma$.

Put $c_0 = M^{1/(1-|\sigma|)}$ and let

$$\Lambda = \left\{u \in C([0,1]) : \frac{1}{c_0}\,\theta_{\lambda,\mu} \le u \le c_0\theta_{\lambda,\mu}\right\}. \quad (41)$$

In order to use a fixed point theorem, we define the operator T on Λ by

$$Tu(x) = G(au^{\sigma})(x) = \int_0^1 G(x,t)\,a(t)\,u^{\sigma}(t)\,dt. \quad (42)$$

For this choice of c_0, we can easily prove that, for $u \in \Lambda$, we have $Tu \le c_0\theta_{\lambda,\mu}$ and $Tu \ge (1/c_0)\theta_{\lambda,\mu}$.

Now, since the function $(x,t) \to G(x,t)$ is continuous on $[0,1] \times [0,1]$ and, by Proposition 9, Corollary 10, and Lemma 13, the function $t \to t^2(1-t)^2\, a(t)\theta^{\sigma}_{\lambda,\mu}(t)$ is integrable on $(0,1)$, we deduce that the operator T is compact from Λ to itself. It follows by the Schauder fixed point theorem that there exists $u \in \Lambda$ such that $Tu = u$. Then, $u \in C([0,1])$ and u satisfies the equation

$$u(x) = G(au^{\sigma})(x). \quad (43)$$

Since the function $t \to t^2(1-t)^2 a(t)u^{\sigma}(t)$ is continuous and integrable on $(0,1)$, then by Proposition 11, the function u is a positive solution in $C^4((0,1)) \cap C([0,1])$ of problem (1).

Finally, let us prove that u is the unique positive continuous solution satisfying (9). To this aim, we assume that (1) has two positive solutions $u, v \in C^4((0,1)) \cap C([0,1])$ satisfying (9) and consider the nonempty set $J = \{m \ge 1 : 1/m \le u/v \le m\}$ and put $c = \inf J$. Then, $c \ge 1$ and we have $(1/c)v \le u \le cv$. It follows that $u^{\sigma} \le c^{|\sigma|}v^{\sigma}$ and consequently

$$\begin{aligned} \left(c^{|\sigma|}v - u\right)^{(4)} &= a\left(c^{|\sigma|}v^{\sigma} - u^{\sigma}\right) := f \ge 0, \\ \left(c^{|\sigma|}v - u\right)(0) &= \left(c^{|\sigma|}v - u\right)(1) \\ &= \left(c^{|\sigma|}v - u\right)'(0) \\ &= \left(c^{|\sigma|}v - u\right)'(1) = 0. \end{aligned} \quad (44)$$

Since the function $t \to t^2(1-t)^2 f(t)$ is continuous and integrable on $(0,1)$, it follows by Proposition 11 that $c^{|\sigma|}v - u = G(a(c^{|\sigma|}v^{\sigma} - u^{\sigma})) \ge 0$. By symmetry, we obtain also that $v \le c^{|\sigma|}\, u$. Hence, $c^{|\sigma|} \in J$ and $c \le c^{|\sigma|}$. Since $|\sigma| < 1$, then $c = 1$ and consequently $u = v$. □

Example 14. Let $\sigma \in (-1,1)$ and let a be a positive continuous function on $(0,1)$ such that

$$a(t) \approx t^{-\lambda}(1-t)^{-\mu}\log\left(\frac{2}{1-t}\right), \quad (45)$$

where $\lambda < 3+\sigma$ and $\mu < 3+\sigma$. Then, using Theorem 1, problem (1) has a unique positive continuous solution u satisfying the following estimates:

$$\begin{aligned} u(x) &\approx x^{\min(2,(4-\lambda)/(1-\sigma))}\left(\widetilde{L}_1(x)\right)^{1/(1-\sigma)} \\ &\quad \times (1-x)^{\min(2,(4-\mu)/(1-\sigma))}\left(\widetilde{L}_2(1-x)\right)^{1/(1-\sigma)}, \end{aligned} \quad (46)$$

where

$$\widetilde{L}_1(x) = \begin{cases} 1 & \text{if } \lambda \ne 2(1+\sigma), \\ \log\left(\frac{2}{x}\right) & \text{if } \lambda = 2(1+\sigma), \end{cases}$$

$$\widetilde{L}_2(x) = \begin{cases} 1 & \text{if } \mu < 2(1+\sigma), \\ \left(\log\left(\frac{2}{x}\right)\right)^2 & \text{if } \mu = 2(1+\sigma), \\ \log\left(\frac{2}{x}\right) & \text{if } 2(1+\sigma) < \mu < 3+\sigma. \end{cases} \quad (47)$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] C. Bai, "Existence of positive solutions for fourth-order three-point boundary value problems," *Boundary Value Problems*, vol. 2007, Article ID 68758, 10 pages, 2007.

[2] Y. Cui and Y. Zou, "Positive solutions of singular fourth-order boundary-value problems," *Electronic Journal of Differential Equations*, vol. 2006, no. 39, pp. 1–10, 2006.

[3] N. Kosmatov, "Countably many solutions of a fourth order boundary value problem," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 12, pp. 1–15, 2004.

[4] R. Ma and C. C. Tisdell, "Positive solutions of singular sublinear fourth-order boundary value problems," *Applicable Analysis*, vol. 84, no. 12, pp. 1199–1220, 2005.

[5] D. O'Regan, "Solvability of some fourth (and higher) order singular boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 161, no. 1, pp. 78–116, 1991.

[6] M. Pei and S. K. Chang, "Monotone iterative technique and symmetric positive solutions for a fourth-order boundary value problem," *Mathematical and Computer Modelling*, vol. 51, no. 9-10, pp. 1260–1267, 2010.

[7] Q. Yao, "Positive solutions for eigenvalue problems of fourth-order elastic beam equations," *Applied Mathematics Letters*, vol. 17, no. 2, pp. 237–243, 2004.

[8] Y. Liu and G. Weigao, "Double positive solutions of fourth-order nonlinear boundary value problems," *Applicable Analysis*, vol. 82, no. 4, pp. 369–380, 2003.

[9] K. Zhang and C. Wang, "The existence of positive solutions of a class of fourth-order singular boundary value problems," *Acta Mathematica Scientia. Series A*, vol. 29, no. 1, pp. 127–135, 2009.

[10] V. Maric, *Regular Variation and Differential Equations*, vol. 1726 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.

[11] E. Seneta, *Regularly Varying Functions*, vol. 508 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1976.

[12] R. Chemmam, H. Mâagli, S. Masmoudi, and M. Zribi, "Combined effects in nonlinear singular elliptic problems in a bounded domain," *Advances in Nonlinear Analysis*, vol. 1, no. 4, pp. 301–318, 2012.

[13] N. Zeddini, R. Alsaedi, and H. Maagli, "Exact boundary behavior of the unique positive solution to some singular elliptic problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 89, pp. 146–156, 2013.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 659152, 6 pages
http://dx.doi.org/10.1155/2014/659152

Research Article

Multiple Periodic Solutions for Discrete Nicholson's Blowflies Type System

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Received 5 January 2014; Accepted 27 February 2014; Published 27 March 2014

Academic Editor: Samir Saker

This paper is concerned with the existence of multiple periodic solutions for discrete Nicholson's blowflies type system. By using the Leggett-Williams fixed point theorem, we obtain the existence of three nonnegative periodic solutions for discrete Nicholson's blowflies type system. In order to show that, we first establish the existence of three nonnegative periodic solutions for the n-dimensional functional difference system $y(k+1) = A(k)y(k) + f(k, y(k-\tau)), k \in \mathbb{Z}$, where $A(k)$ is not assumed to be diagonal as in some earlier results. In addition, a concrete example is also given to illustrate our results.

1. Introduction and Preliminaries

In 1954 Nicholson [1] and later in 1980 Gurney et al. [2] proposed the following delay differential equation model:

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-\gamma x(t-\tau)}, \tag{1}$$

where $x(t)$ is the size of the population at time t, p is the maximum per capita daily egg production, $1/\gamma$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time.

Now, Nicholson's blowflies model and its various analogous equations have attracted more and more attention. There is large literature on this topic. Recently, the study on Nicholson's blowflies type systems has attracted much attention (cf. [3–8] and references therein). In particular, several authors have made contribution on the existence of periodic solutions for Nicholson's blowflies type systems (see, e.g., [6, 7]). In addition, discrete Nicholson's blowflies type models have been studied by several authors (see, e.g., [9–12] and references therein).

Stimulated by the above works, in this paper, we consider the following discrete Nicholson's blowflies type system:

$$\begin{aligned}
x_1(k+1) = a_{11}(k)x_1(k) &+ a_{12}(k)x_2(k) + b(k) \\
&\times [x_1(k-\tau) + x_2(k-\tau)]^m \\
&\times e^{-c(k)[x_1(k-\tau)+x_2(k-\tau)]}, \\
x_2(k+1) = a_{21}(k)x_1(k) &+ a_{22}(k)x_2(k) + b(k) \\
&\times [x_1(k-\tau) + x_2(k-\tau)]^m \\
&\times e^{-c(k)[x_1(k-\tau)+x_2(k-\tau)]},
\end{aligned} \tag{2}$$

where $m > 1$ is a constant, τ is a nonnegative integer, and a_{ij}, $i, j = 1, 2$, b, and c are all N-periodic functions from $\mathbb{Z}$ to $\mathbb{R}$.

In fact, there are seldom results concerning the existence of *multiple* periodic solutions for Nicholson's blowflies type equations. It seems that the only results on this topic are due to Padhi et al. [13–15], where they established several existence theorems about multiple periodic solutions of

Nicholson's blowflies type equations. In addition, recently, several authors have investigated the existence of almost periodic solutions for Nicholson's blowflies type equations (see, e.g., [11, 16, 17] and references therein). However, to the best our knowledge, there are few results concerning the existence of multiple periodic solutions for Nicholson's blowflies type systems. That is the main motivation of this paper.

Next, let us recall the Leggett-Williams fixed point theorem, which will be used in the proof of our main results.

Let X be a Banach space. A closed convex set K in X is called a cone if the following conditions are satisfied: (i) if $x \in K$, then $\lambda x \in K$ for any $\lambda \geq 0$; (ii) if $x \in K$ and $-x \in K$, then $x = 0$.

A nonnegative continuous functional ψ is said to be concave on K if ψ is continuous and

$$\psi(\mu x + (1-\mu) y) \geq \mu\psi(x) + (1-\mu)\psi(y), \quad x, y \in K, \quad \mu \in [0,1]. \tag{3}$$

Letting c_1, c_2, and c_3 be three positive constants and letting ϕ be a nonnegative continuous functional on K, we denote

$$K_{c_1} = \{y \in K : \|y\| < c_1\}, \quad K(\phi, c_2, c_3) = \{y \in K : c_2 \leq \phi(y), \|y\| < c_3\}. \tag{4}$$

In addition, we call that ϕ is increasing on K if $\phi(x) \geq \phi(y)$ for all $x, y \in K$ with $x - y \in K$.

Lemma 1 (see [18]). *Let K be a cone in a Banach space X, let c_4 be a positive constant, let $\Phi : \overline{K}_{c_4} \to \overline{K}_{c_4}$ be a completely continuous mapping, and let ψ be a concave nonnegative continuous functional on K with $\psi(u) \leq \|u\|$ for all $u \in \overline{K}_{c_4}$. Suppose that there exist three constants c_1, c_2, and c_3 with $0 < c_1 < c_2 < c_3 \leq c_4$ such that*

(i) *$\{u \in K(\psi, c_2, c_3) : \psi(u) > c_2\} \neq \emptyset$, and $\psi(\Phi u) > c_2$ for all $u \in K(\psi, c_2, c_3)$;*

(ii) *$\|\Phi u\| < c_1$ for all $u \in \overline{K}_{c_1}$;*

(iii) *$\psi(\Phi u) > c_2$ for all $u \in K(\psi, c_2, c_4)$ with $\|\Phi u\| > c_3$.*

Then Φ has at least three fixed points u_1, u_2, and u_3 in $\overline{K}_{c_4}$. Furthermore, $\|u_1\| \leq c_1 < \|u_2\|$, and $\psi(u_2) < c_2 < \psi(u_3)$.

Throughout the rest of this paper, we denote by $\mathbb{Z}$ the set of all integers, by $\mathbb{R}$ the set of all real numbers, and by $l_N^\infty(\mathbb{Z}, \mathbb{R}^n)$ the space of all N-periodic functions $x : \mathbb{Z} \to \mathbb{R}^n$, where N is a fixed positive integer. It is easy to see that $l_N^\infty(\mathbb{Z}, \mathbb{R}^n)$ is a Banach space under the norm

$$\|x\| = \max_{1\leq k\leq N} \max_{1\leq i\leq n} |x_i(k)|, \tag{5}$$

where $x = (x_1, x_2, \ldots, x_n)^T$. In addition, we denote

$$\mathbb{R}_+^n := \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \geq 0\}. \tag{6}$$

2. Main Results

To study the existence of multiple periodic solutions for system (2), we first consider the following more general n-dimensional functional difference system:

$$y(k+1) = A(k) y(k) + f(k, y(k-\tau)), \quad k \in \mathbb{Z}, \tag{7}$$

where, for every $k \in \mathbb{Z}$, $A(k)$ is N-periodic and nonsingular $n \times n$ matrix, and $f = (f_1, \ldots, f_n)^T : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is N-periodic in the first argument and continuous in the second argument.

To note that the existence of periodic solutions for system (7) and its variants had been of great interest for many authors (see, e.g., [19–25] and references therein) is needed. However, in some earlier works (see, e.g., [21]) on the existence of periodic solutions for system (7), the matrix $A(k)$ is assumed to be diagonal. In this paper, we will remove this restrictive condition by utilizing an idea in [22], where the authors studied the existence of periodic solutions for a class of nonlinear neutral systems of differential equations.

Let $\Phi(0) = I$,

$$\Phi(k) = \prod_{i=0}^{k-1} A(i) = A(k-1)\cdots A(0), \quad k \geq 1,$$

$$\Phi(k) = \prod_{i=k}^{-1} [A(i)]^{-1} = [A(k)]^{-1} \cdots [A(-1)]^{-1}, \quad k \leq -1,$$

$$G(k,s) = \Phi(k)\left[\Phi^{-1}(N) - I\right]^{-1} \Phi^{-1}(s+1), \quad k \in \mathbb{Z}, \quad k \leq s \leq k+N-1. \tag{8}$$

We first present some basic results about $\Phi(k)$ and $G(k,s)$.

Lemma 2. *For all $k, s \in \mathbb{Z}$ with $k \leq s \leq k + N - 1$, the following assertions hold:*

(i) $\Phi(k+1) = A(k)\Phi(k)$,

(ii) $\Phi(k+N) = \Phi(k)\Phi(N)$,

(iii) $G(k+1, s) = A(k)G(k,s)$,

(iv) $G(k+N, s+N) = G(k,s)$.

Proof. One can show (i) and (ii) by some direct calculations and noting that $A(k+N) = A(k)$. So we omit the details. In addition, the assertion (iii) follows from the assertion (i) and the assertion (iv) follows from the assertion (ii). □

By using Lemma 2, we can get the following result.

Lemma 3. *A function $y : \mathbb{Z} \to \mathbb{R}^n$ is a N-periodic solution of system (7) if and only if y is a N-periodic function satisfying*

$$y(k) = \sum_{s=k}^{k+N-1} G(k,s) f(s, y(s-\tau)), \quad k \in \mathbb{Z}. \tag{9}$$

Proof. Sufficiency. Assume that $y : \mathbb{Z} \to \mathbb{R}^n$ is a N-periodic function satisfying (9); that is,

$$y(k) = \sum_{s=k}^{k+N-1} G(k,s) f(s, y(s-\tau)), \quad k \in \mathbb{Z}. \tag{10}$$

Then, we have

$$\begin{aligned} y(k+1) &= \sum_{s=k+1}^{k+N} G(k+1,s) f(s, y(s-\tau)) \\ &= \sum_{s=k+1}^{k+N-1} G(k+1,s) f(s, y(s-\tau)) \\ &\quad + G(k+1,k+N) f(k+N, y(k+N-\tau)) \\ &= \sum_{s=k+1}^{k+N-1} A(k) G(k,s) f(s, y(s-\tau)) \\ &\quad + G(k+1,k+N) f(k, y(k-\tau)) \\ &= \sum_{s=k}^{k+N-1} A(k) G(k,s) f(s, y(s-\tau)) \\ &\quad - A(k) G(k,k) f(k, y(k-\tau)) \\ &\quad + G(k+1,k+N) f(k, y(k-\tau)) \\ &= A(k) y(k) - A(k) G(k,k) f(k, y(k-\tau)) \\ &\quad + G(k+1,k+N) f(k, y(k-\tau)) \\ &= A(k) y(k) + f(k, y(k-\tau)), \end{aligned} \tag{11}$$

where

$$\begin{aligned} G&(k+1,k+N) - A(k) G(k,k) \\ &= \Phi(k+1) \left[\Phi^{-1}(N) - I\right]^{-1} \Phi^{-1}(k+N+1) \\ &\quad - A(k) \Phi(k) \left[\Phi^{-1}(N) - I\right]^{-1} \Phi^{-1}(k+1) \\ &= \Phi(k+1) \left[\Phi^{-1}(N) - I\right]^{-1} \Phi^{-1}(N) \Phi^{-1}(k+1) \\ &\quad - \Phi(k+1) \left[\Phi^{-1}(N) - I\right]^{-1} \Phi^{-1}(k+1) \\ &= \Phi(k+1) \Phi^{-1}(k+1) = I. \end{aligned} \tag{12}$$

Thus, we conclude that y is a N-periodic solution of system (7).

Necessity. Let $y : \mathbb{Z} \to \mathbb{R}^n$ be a N-periodic solution of system (7). Then, we have

$$y(s+1) = A(s) y(s) + f(s, y(s-\tau)), \quad s \in \mathbb{Z}, \tag{13}$$

which yields

$$\begin{aligned} \Phi^{-1}&(s+1) y(s+1) - \Phi^{-1}(s) y(s) \\ &= \Phi^{-1}(s+1) \left[A(s) y(s) + f(s, y(s-\tau))\right] \\ &\quad - \Phi^{-1}(s) y(s) = \Phi^{-1}(s+1) f(s, y(s-\tau)), \\ &\qquad s \in \mathbb{Z}. \end{aligned} \tag{14}$$

For all $l \geq k$, we have

$$\begin{aligned} \Phi^{-1}&(l+1) y(l+1) - \Phi^{-1}(k) y(k) \\ &= \sum_{s=k}^{l} \left[\Phi^{-1}(s+1) y(s+1) - \Phi^{-1}(s) y(s)\right] \\ &= \sum_{s=k}^{l} \Phi^{-1}(s+1) f(s, y(s-\tau)), \end{aligned} \tag{15}$$

which yields

$$\begin{aligned} \Phi^{-1}&(l+1) y(l+1) \\ &= \Phi^{-1}(k) y(k) + \sum_{s=k}^{l} \Phi^{-1}(s+1) f(s, y(s-\tau)). \end{aligned} \tag{16}$$

Letting $l = k + N - 1$ and noting that y is N-periodic, we get

$$\begin{aligned} \Phi^{-1}&(k+N) y(k) \\ &= \Phi^{-1}(k+N) y(k+N) \\ &= \Phi^{-1}(k) y(k) + \sum_{s=k}^{k+N-1} \Phi^{-1}(s+1) f(s, y(s-\tau)). \end{aligned} \tag{17}$$

Noting that

$$\Phi^{-1}(k+N) - \Phi^{-1}(k) = \left[\Phi^{-1}(N) - I\right] \Phi^{-1}(k), \tag{18}$$

we conclude

$$\begin{aligned} y(k) &= \Phi(k) \left[\Phi^{-1}(N) - I\right]^{-1} \\ &\quad \times \sum_{s=k}^{k+N-1} \Phi^{-1}(s+1) f(s, y(s-\tau)) \\ &= \sum_{s=k}^{k+N-1} G(k,s) f(s, y(s-\tau)). \end{aligned} \tag{19}$$

That is, (9) holds. This completes the proof. □

Let

$$\begin{aligned} G(k,s) &= \left[G_{ij}(k,s)\right], \\ p &= \min_{1\leq i\leq n} \min_{1\leq k\leq N} \min_{k\leq s\leq k+N-1} \sum_{j=1}^{n} G_{ij}(k,s), \\ q &= \max_{1\leq i\leq n} \max_{1\leq k\leq N} \max_{k\leq s\leq k+N-1} \sum_{j=1}^{n} G_{ij}(k,s). \end{aligned} \tag{20}$$

Now, we introduce a set

$$K = \{x \in l_N^\infty(\mathbb{Z}, \mathbb{R}^n) : x_i(k) \geq 0, \quad k = 1, 2, \ldots, N, i = 1, 2, \ldots, n\}. \tag{21}$$

It is not difficult to verify that K is a cone in $l_N^\infty(\mathbb{Z}, \mathbb{R}^n)$. Finally, we define an operator Φ on K by

$$(\Phi x)(k) = \sum_{s=k}^{k+N-1} G(k,s) f(s, x(s-\tau)), \quad x \in K, \quad k \in \mathbb{Z}. \tag{22}$$

Theorem 4. *Assume that $f_1 = f_2 = \cdots = f_n$ and the following assumptions hold.*

(H0) $q > p > 0$, $f_1(s,x) \geq 0$ for all $s \in \mathbb{Z}$ and $x \in \mathbb{R}_+^n$, and $\sum_{j=1}^n G_{ij}(k,s) \geq 0$ for all $k \in \mathbb{Z}$, $k \leq s \leq k+N-1$, and $i = 1, 2, \ldots, n$.

(H1) There exist two constants $c_4 > c_1 > 0$ such that

$$q \cdot \sum_{s=1}^N f_1(s,x) < c_1 \quad \text{for } x \in \mathbb{R}_+^n \text{ with } \|x\| \leq c_1,$$
$$q \cdot \sum_{s=1}^N f_1(s,x) \leq c_4 \quad \text{for } x \in \mathbb{R}_+^n \text{ with } \|x\| \leq c_4. \tag{23}$$

(H2) There exists a constant $c_2 \in (c_1, c_4)$ such that $qc_2 \leq pc_4$, and

$$p \cdot \sum_{s=1}^N f_1(s,x) > c_2 \quad \text{for } x \in \mathbb{R}_+^n \text{ with } \|x\| < \frac{q}{p} c_2, \quad \sum_{i=1}^n x_i \geq nc_2. \tag{24}$$

Then system (7) has at least three nonnegative N-periodic solutions.

Proof. Firstly, by (H0) and noting that $G(k+N, s+N) = G(k,s)$, Φ is an operator from K to K. Secondly, noting that f is continuous for the second argument, by similar proof to [21, Lemma 2.5], one can show that $\Phi : K \to K$ is completely continuous.

Let

$$\psi(x) = \min_{1\leq k\leq N} \frac{\sum_{i=1}^n x_i(k)}{n}, \quad x \in K. \tag{25}$$

It is easy to see that ψ is a concave nonnegative continuous functional on K and $\psi(x) \leq \|x\|$.

Now, we show that Φ maps $\overline{K}_{c_4}$ into $\overline{K}_{c_4}$. For every $x \in \overline{K}_{c_4}$, we have $x(s-\tau) \in \mathbb{R}_+^n$ and $\|x(s-\tau)\| \leq c_4$ for all $s \in \mathbb{Z}$. Then, by (H1), we have

$$\begin{aligned} \|\Phi x\| &= \max_{1\leq k\leq N} \max_{1\leq i\leq n} \sum_{s=k}^{k+N-1} \sum_{j=1}^n G_{ij}(k,s) f_j(s, x(s-\tau)) \\ &\leq q \cdot \sum_{s=1}^N f_1(s, x(s-\tau)) \leq c_4. \end{aligned} \tag{26}$$

Similarly, for every $x \in \overline{K}_{c_1}$, it follows from (H1) that

$$\|\Phi x\| \leq q \cdot \sum_{s=1}^N f_1(s, x(s-\tau)) < c_1. \tag{27}$$

That is, condition (ii) of Lemma 1 holds.

Let $c_3 = (q/p)c_2$. Next, let us verify condition (i) of Lemma 1. It is easy to see that the set

$$\{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \emptyset. \tag{28}$$

In addition, for every $x \in K(\psi, c_2, c_3)$, we have $x(s-\tau) \in \mathbb{R}_+^n$, $\|x(s-\tau)\| < c_3 = (q/p)c_2$, and $\sum_{i=1}^n x_i(s-\tau) \geq nc_2$ for all $s \in \mathbb{Z}$. Then, by (H2), we get

$$\begin{aligned} \psi(\Phi x) &= \frac{1}{n} \cdot \min_{1\leq k\leq N} \sum_{i=1}^n \sum_{s=k}^{k+N-1} \sum_{j=1}^n G_{ij}(k,s) f_j(s, x(s-\tau)) \\ &\geq p \cdot \sum_{s=1}^N f_1(s, x(s-\tau)) > c_2 \end{aligned} \tag{29}$$

which means that condition (i) of Lemma 1 holds.

It remains to verify that condition (iii) of Lemma 1 holds. Let $x \in K(\psi, c_2, c_4)$ with $\|\Phi x\| > c_3$; we have $c_2 \leq \|x\| < c_4$ and

$$q \cdot \sum_{s=1}^N f_1(s, x(s-\tau)) \geq \|\Phi x\| > c_3, \tag{30}$$

which yields

$$\sum_{s=1}^N f_1(s, x(s-\tau)) > \frac{c_3}{q} = \frac{c_2}{p}. \tag{31}$$

Then, we have

$$\psi(\Phi x) \geq p \cdot \sum_{s=1}^N f_1(s, x(s-\tau)) > c_2. \tag{32}$$

Then, by Lemma 1, we know that Φ has at least three fixed points in $\overline{K}_{c_4}$. Then, it follows from Lemma 3 that system (7) has at least three nonnegative N-periodic solutions. □

Now, we apply Theorem 4 to Nicholson's blowflies system (2). Let $n = 2$,

$$A(k) = \begin{pmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{pmatrix},$$
$$f_1(k,x) = f_2(k,x) = b(k)\,[x_1 + x_2]^m e^{-c(k)[x_1+x_2]}, \tag{33}$$

and let $\Phi(k)$, $G(k,s)$, p, q, and K be as in Theorem 4.

Corollary 5. *Assume that $q > p > 0$, and $\sum_{j=1}^{2} G_{ij}(k,s)$ ($i = 1, 2$), $b(k)$, and $c(k)$ are all nonnegative for $k \in \mathbb{Z}$ and $k \le s \le k+N-1$. Then the system (2) has at least three nonnegative N-periodic solutions provided that $c^+ := \max_{1\le s\le N} c(s) \ge c^- := \min_{1\le s\le N} c(s) > 0$, and*

$$p \cdot 2^m \cdot \sum_{s=1}^{N} b(s) > e^{m-1} \cdot \left[\frac{2c^+ q}{p(m-1)}\right]^{m-1}. \tag{34}$$

Proof. We only need to verify that all the assumptions of Theorem 4 are satisfied. Firstly, it is easy to see that (H0) holds. Let

$$c_2 = \frac{p(m-1)}{2c^+ q}. \tag{35}$$

Secondly, let us check (H1). In fact, one can choose sufficiently small $c_1 \in (0, c_2)$ such that, for all $x \in \mathbb{R}_+^n$ with $\|x\| \le c_1$, there holds

$$\begin{aligned} q \cdot \sum_{s=1}^{N} f_1(s,x) &= q \cdot \sum_{s=1}^{N} b(s)\,[x_1+x_2]^m e^{-c(s)[x_1+x_2]} \\ &\le q \cdot \sum_{s=1}^{N} b(s) \cdot 2^m \cdot \|x\|^m \\ &= \left(2^m q \cdot \sum_{s=1}^{N} b(s)\right) \cdot \|x\|^m < \|x\| \le c_1. \end{aligned} \tag{36}$$

In addition, for all $x \in \mathbb{R}_+^n$, we have

$$\begin{aligned} q \cdot \sum_{s=1}^{N} f_1(s,x) &\le q \cdot \sum_{s=1}^{N} b(s)\,[x_1+x_2]^m e^{-c^-[x_1+x_2]} \\ &\le q \cdot \sum_{s=1}^{N} b(s) \cdot \left(\frac{m}{c^-}\right)^m e^{-m}. \end{aligned} \tag{37}$$

So, letting

$$c_4 = \max\left\{\frac{qc_2}{p}, q \cdot \sum_{s=1}^{N} b(s) \cdot \left(\frac{m}{c^-}\right)^m e^{-m}\right\}, \tag{38}$$

we conclude that (H1) holds.

It remains to verify (H2). For all $x \in \mathbb{R}_+^n$ with $\|x\| < (q/p)c_2$ and $\sum_{i=1}^{2} x_i \ge 2c_2$, by using (34), we have

$$\begin{aligned} p \cdot \sum_{s=1}^{N} f_1(s,x) &= p \cdot \sum_{s=1}^{N} b(s)\,[x_1+x_2]^m e^{-c(s)[x_1+x_2]} \\ &\ge p \cdot \sum_{s=1}^{N} b(s) \cdot 2^m c_2^m \cdot e^{-(2c^+q/p)c_2} \\ &= \left(p \cdot 2^m \cdot \sum_{s=1}^{N} b(s)\right) \cdot c_2^{m-1} \cdot e^{-(2c^+q/p)c_2} \cdot c_2 \\ &= \left(p \cdot 2^m \cdot \sum_{s=1}^{N} b(s) \cdot e^{-(m-1)} \cdot \left[\frac{p(m-1)}{2c^+q}\right]^{m-1}\right) \cdot c_2 \\ &> c_2. \end{aligned} \tag{39}$$

This completes the proof. □

Next, we give a concrete example for Nicholson's blowflies type system (2).

Example 6. Let $m = N = 2$, $\tau = 1$, $b(k) = 100 + \sin^2(\pi k/2)$, $c(k) = 1 + \cos^2(\pi k/2)$, and

$$A(0) = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad A(1) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}. \tag{40}$$

By a direct calculation, we can get

$$\begin{aligned} G(1,1) &= \begin{pmatrix} 0 & \frac{3}{8} \\ \frac{2}{3} & 0 \end{pmatrix}, \qquad G(1,2) = \begin{pmatrix} \frac{9}{8} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}, \\ G(2,2) &= \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{3}{8} & 0 \end{pmatrix}, \qquad G(2,3) = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{9}{8} \end{pmatrix}. \end{aligned} \tag{41}$$

Then, we have $p = 3/8$ and $q = 4/3$. In addition, we have $c^+ = 2 > c^- = 1 > 0$ and

$$\begin{aligned} p \cdot 2^m \cdot \sum_{s=1}^{N} b(s) &= \frac{3}{8} \cdot 4 \cdot 201 \\ &= \frac{603}{2} > \frac{128}{9} e = e^{m-1} \cdot \left[\frac{2c^+q}{p(m-1)}\right]^{m-1}. \end{aligned} \tag{42}$$

So, by Corollary 5, we know that system (2) has at least three nonnegative 2-periodic solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are grateful to the referees for valuable suggestions and comments. Hui-Sheng Ding acknowledges support from the NSF of China (11101192), the Program for Cultivating Young Scientist of Jiangxi Province (20133BCB23009), and the NSF of Jiangxi Province.

References

[1] A. J. Nicholson, "An outline of the dynamics of animal populations," *Australian Journal of Zoology*, vol. 2, pp. 9–65, 1954.

[2] W. S. Gurney, S. P. Blythe, and R. M. Nisbet, "Nicholson's blowflies revisited," *Nature*, vol. 287, pp. 17–21, 1980.

[3] B. Liu, "The existence and uniqueness of positive periodic solutions of Nicholson-type delay systems," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3145–3151, 2011.

[4] B. Liu and S. Gong, "Permanence for Nicholson-type delay systems with nonlinear density-dependent mortality terms," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 4, pp. 1931–1937, 2011.

[5] L. Berezansky, L. Idels, and L. Troib, "Global dynamics of Nicholson-type delay systems with applications," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 436–445, 2011.

[6] W. Chen and L. Wang, "Positive periodic solutions of Nicholson-type delay systems with nonlinear density-dependent mortality terms," *Abstract and Applied Analysis*, vol. 2012, Article ID 843178, 13 pages, 2012.

[7] Q. Zhou, "The positive periodic solution for Nicholson-type delay system with linear harvesting terms," *Applied Mathematical Modelling: Simulation and Computation for Engineering and Environmental Systems*, vol. 37, no. 8, pp. 5581–5590, 2013.

[8] S. H. Saker and S. Agarwal, "Oscillation and global attractivity in a periodic Nicholson's blowflies model," *Mathematical and Computer Modelling*, vol. 35, no. 7-8, pp. 719–731, 2002.

[9] S. H. Saker, "Oscillation of continuous and discrete diffusive delay Nicholson's blowflies models," *Applied Mathematics and Computation*, vol. 167, no. 1, pp. 179–197, 2005.

[10] J. W.-H. So and J. S. Yu, "On the stability and uniform persistence of a discrete model of Nicholson's blowflies," *Journal of Mathematical Analysis and Applications*, vol. 193, no. 1, pp. 233–244, 1995.

[11] J. Alzabut, Y. Bolat, and T. Abdeljawad, "Almost periodic dynamics of a discrete Nicholson's blowflies model involving a linear harvesting term," *Advances in Difference Equations*, vol. 2012, article 158, 2012.

[12] J. O. Alzabut, "Existence and exponential convergence of almost periodic solutions for a discrete Nicholson's blowflies model with a nonlinear harvesting term," *Mathematical Sciences Letters*, vol. 2, pp. 201–207, 2013.

[13] S. Padhi, C. Qian, and S. Srivastava, "Multiple periodic solutions for a first order nonlinear functional differential equation with applications to population dynamics," *Communications in Applied Analysis*, vol. 12, no. 3, pp. 341–351, 2008.

[14] S. Padhi, S. Srivastava, and J. G. Dix, "Existence of three nonnegative periodic solutions for functional differential equations and applications to hematopoiesis," *Panamerican Mathematical Journal*, vol. 19, no. 1, pp. 27–36, 2009.

[15] S. Padhi and S. Pati, "Positive periodic solutions for a nonlinear functional differential equation," *Georgian Academy of Sciences A. Razmadze Mathematical Institute: Memoirs on Differential Equations and Mathematical Physics*, vol. 51, pp. 109–118, 2010.

[16] J. O. Alzabut, "Almost periodic solutions for an impulsive delay Nicholson's blowflies model," *Journal of Computational and Applied Mathematics*, vol. 234, no. 1, pp. 233–239, 2010.

[17] H.-S. Ding and J. J. Nieto, "A new approach for positive almost periodic solutions to a class of Nicholson's blowflies model," *Journal of Computational and Applied Mathematics*, vol. 253, pp. 249–254, 2013.

[18] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 673–688, 1979.

[19] J. G. Dix, S. Padhi, and S. Pati, "Multiple positive periodic solutions for a nonlinear first order functional difference equation," *Journal of Difference Equations and Applications*, vol. 16, no. 9, pp. 1037–1046, 2010.

[20] Y. N. Raffoul, "Positive periodic solutions of nonlinear functional difference equations," *Electronic Journal of Differential Equations*, no. 55, pp. 1–8, 2002.

[21] Y. N. Raffoul and C. C. Tisdell, "Positive periodic solutions of functional discrete systems and population models," *Advances in Difference Equations*, no. 3, pp. 369–380, 2005.

[22] M. N. Islam and Y. N. Raffoul, "Periodic solutions of neutral nonlinear system of differential equations with functional delay," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 1175–1186, 2007.

[23] D. Jiang, J. Wei, and B. Zhang, "Positive periodic solutions of functional differential equations and population models," *Electronic Journal of Differential Equations*, no. 71, pp. 1–13, 2002.

[24] Z. Zeng, L. Bi, and M. Fan, "Existence of multiple positive periodic solutions for functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 2, pp. 1378–1389, 2007.

[25] W. Zhang, D. Zhu, and P. Bi, "Existence of periodic solutions of a scalar functional differential equation via a fixed point theorem," *Mathematical and Computer Modelling*, vol. 46, no. 5-6, pp. 718–729, 2007.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 424512, 6 pages
http://dx.doi.org/10.1155/2014/424512

Research Article

Convergence of Solutions to a Certain Vector Differential Equation of Third Order

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Received 12 December 2013; Accepted 18 January 2014; Published 26 February 2014

Academic Editor: Bingwen Liu

We give some sufficient conditions to guarantee convergence of solutions to a nonlinear vector differential equation of third order. We prove a new result on the convergence of solutions. An example is given to illustrate the theoretical analysis made in this paper. Our result improves and generalizes some earlier results in the literature.

1. Introduction

This paper is concerned with the following nonlinear vector differential equation of third order:

$$\dddot{X} + F(\ddot{X}) + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \quad (1)$$

where $X \in \mathfrak{R}^n$ and $F, G, H : \mathfrak{R}^n \to \mathfrak{R}^n$ and $P : \mathfrak{R} \times \mathfrak{R}^n \times \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ are continuous functions in their respective arguments.

It should be noted that, in 2005, Afuwape and Omeike [1] considered the following nonlinear vector differential equation of third order:

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \quad (2)$$

where A is real symmetric $n \times n$-matrix. The author established a new result on the convergence of solutions of (2) under different conditions on the function P. For some related papers on the convergence of solutions to certain vector differential equations of third order, the readers can referee to the papers of Afuwape [2], Afuwape and Omeike [3], and Olutimo [4]. Further, it is worth mentioning that in a sequence of results Afuwape [2, 5, 6], Afuwape and Omeike [3], Afuwape and Ukpera [7], Ezeilo [8], Ezeilo and Tejumola [9, 10], Meng [11], Olutimo [4], Reissig et al. [12], Tiryaki [13], Tunç [14–16], Tunç and Ateş [17], C. Tunç and E. Tunç [18], and Tunç and Karakas [19] investigated the qualitative behaviors of solutions, stability, boundedness, uniform boundedness and existence of periodic solutions, and so on, except convergence of solutions, for some kind of vector differential equations of third order.

The Lyapunov direct method was used with the aid of suitable differentiable auxiliary functions throughout the mentioned papers. However, to the best of our knowledge, till now, the convergence of the solutions to (1) has not been discussed in the literature. Thus, it is worthwhile to study the topic for (1). It should be noted that the result to be established here is different from that in Afuwape [2], Afuwape and Omeike [1, 3], Olutimo [4], and the above mentioned papers. This paper is an extension and generalization of the result of Afuwape and Omeike [3]. It may be useful for the researchers working on the qualitative behaviors of solutions (see, also, Tunç and Gözen [20]).

It should be noted that throughout the paper R^n will denote the real Euclidean space of n-vectors and $\|X\|$ will denote the norm of the vector X in R^n.

Definition 1. Any two solutions $X_1(t)$, $X_2(t)$ of (1) in R^n will be said to converge to each other if

$$\|X_2(t) - X_1(t)\| \longrightarrow 0, \qquad \|\dot{X}_2(t) - \dot{X}_1(t)\| \longrightarrow 0,$$
$$\|\ddot{X}_2(t) - \ddot{X}_1(t)\| \longrightarrow 0 \quad (3)$$

as $t \to \infty$.

2. Main Result

The main result of this paper is the following theorem.

Theorem 2. *We assume that there are positive constants δ_g, δ_h, δ_f, Δ_g, Δ_h, Δ_f, and Δ_1 such that the following conditions hold:*

(i) *the Jacobian matrices $J_g(Y) = \partial g_i/\partial y_j$, $J_h(X) = \partial h_i/\partial x_j$, and $J_f(Z) = \partial f_i/\partial z_j$ exist and are symmetric and their eigenvalues satisfy*

$$0 < \delta_g \le \lambda_i\left(J_g(Y)\right) \le \Delta_g,$$
$$0 < \delta_h \le \lambda_i\left(J_h(X)\right) \le \Delta_h, \quad (4)$$
$$0 < \delta_f \le \lambda_i\left(J_f(Z)\right) \le \Delta_f, \quad (i = 1, 2, \ldots, n),$$

for all X, Y, Z in R^n;

(ii) *$P(t, X, Y, Z)$ satisfies*

$$\|P(t, X_2, Y_2, Z_2) - P(t, X_1, Y_1, Z_1)\| \le \Delta_1\left\{\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2\right\}^{1/2} \quad (5)$$

for any X_i, Y_i, Z_i, $(i = 1, 2)$, in R^n.

If

$$\Delta_1 < \varepsilon,$$
$$\Delta_h \le \min\left\{3^{-1}\beta(1-\beta)\delta_g^2; 6^{-1}\alpha(1-\beta)\delta_g\delta_f(1+\alpha)^{-2}\right\} = k\delta_g\delta_f, \quad (6)$$

then any two solutions $X_1(t)$, $X_2(t)$ of (1) *necessarily converge, where α, ε, k, β are some positive constants with $0 < \beta < 1$ and $k(< 1)$,*

$$k = \min\left\{3^{-1}\beta(1-\beta)\delta_g\delta_f^{-1}; 6^{-1}\alpha(1-\beta)(1+\alpha)^{-2}\right\}. \quad (7)$$

Remark 3. The mentioned theorem itself still holds valid with (5) replaced by the much weaker condition

$$\|P(t, X_2, Y_2, Z_2) - P(t, X_1, Y_1, Z_1)\| \le \phi(t)\left\{\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2\right\}^{1/2} \quad (8)$$

for arbitrary t any X_i, Y_i, Z_i, $(i = 1, 2)$, in R^n, where it is assumed that $\int_0^t \phi^v(s)ds \le \Delta_1 t$ for $1 \le v \le 2$.

The following lemma is needed in our later analysis.

Lemma 4. *Let A be a real symmetric $n \times n$-matrix and*

$$\overline{a} \ge \lambda_i(A) \ge a > 0, \quad (i = 1, 2, \ldots, n), \quad (9)$$

where $\overline{a}$ and a are constants.

Then

$$\overline{a}\langle X, X\rangle \ge \langle AX, X\rangle \ge a\langle X, X\rangle,$$
$$\overline{a}^2\langle X, X\rangle \ge \langle AX, AX\rangle \ge a^2\langle X, X\rangle. \quad (10)$$

Proof (see Afuwape [5]). Our main tool in the proof of our result is the continuous function $V = V(X, Y, Z)$ defined for any triple vectors X, Y, Z in R^n, by

$$2V = \left\langle \beta(1-\beta)\delta_g^2 X, X\right\rangle + \left\langle \beta\delta_g Y, Y\right\rangle + \left\langle \alpha\delta_g Y, Y\right\rangle + \langle \alpha Z, Z\rangle + \left\langle Z + Y + (1-\beta)\delta_g X, Z + Y + (1-\beta)\delta_g X\right\rangle. \quad (11)$$

This function can be rearranged as

$$2V = \beta(1-\beta)\delta_g^2\|X\|^2 + \beta\delta_g\|Y\|^2 + \alpha\delta_g\|Y\|^2 + \alpha\|Z\|^2 + \left\|Z + Y + (1-\beta)\delta_g X\right\|^2, \quad (12)$$

where $0 < \beta < 1$ and $\alpha > 0$

The following result is immediate from the estimate (11). □

Lemma 5. *Assume that all the conditions on the vectors $F(Z)$, $H(X)$, and $G(Y)$ in the theorem hold. Then, there exist positive constants δ_1 and δ_2 such that*

$$2V(X, Y, Z) \ge \delta_1\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right),$$
$$2V(X, Y, Z) \le \delta_2\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right) \quad (13)$$

for arbitrary X, Y, Z in R^n.

Proof. Let

$$\delta_1 = \min\left\{\beta(1-\beta)\delta_g^2, \delta_g(\beta+\alpha), \alpha\right\},$$
$$\delta_2 = \max\left\{\delta_g(1-\beta)\left(1+\delta_g\right), \delta_g(\beta+\alpha) + 1 + \delta_g(1-\beta), 1 + \alpha + \delta_g(1-\beta)\right\}. \quad (14)$$

Then the proof can be easily completed by using Lemma 4. Therefore, we omit the details of the proof. □

Proof of the Theorem. Let X in R^n be any solution of (1). For such a solution, let $\dot{X}$ and $\ddot{X}$ be denoted, respectively, by Y and Z. Then, we can rewrite (1) in the following equivalent system form:

$$\dot{X} = Y, \qquad \dot{Y} = Z,$$
$$\dot{Z} = -F(Z) - G(Y) - H(X) + P(t, X, Y, Z). \quad (15)$$

Let $X_1(t)$, $X_2(t)$ in R^n be any solution of (1), define $W = W(t)$ by

$$W(t) = V\left(X_2(t) - X_1(t), Y_2(t) - Y_1(t), Z_2(t) - Z_1(t)\right), \quad (16)$$

where V is the function defined in (11) with X, Y, Z replaced by $X_2 - X_1$, $Y_2 - Y_1$ and $Z_2 - Z_1$, respectively.

By Lemma 5, it follows that there exist $\delta_3 > 0$ and $\delta_4 > 0$ such that

$$\delta_3 \left(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2\right) \leq W \leq \delta_4 \left(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2\right). \tag{17}$$

When we differentiate the function $W(t)$ with respect to t along the system (15), it follows, after simplification, that

$$\dot{W}(t) = -W_1 - W_2 - W_3 - W_4 - W_5 + W_6, \tag{18}$$

where

$$\begin{aligned}
W_1 &= \frac{1}{2}\delta_g (1-\beta) \langle X_2 - X_1, H(X_2) - H(X_1)\rangle \\
&\quad + \frac{1}{2}\beta\delta_g \langle Y_2 - Y_1, Y_2 - Y_1\rangle \\
&\quad + \frac{1}{2}\alpha \langle Z_2 - Z_1, F(Z_2) - F(Z_1)\rangle, \\
W_2 &= \frac{1}{6}\delta_g (1-\beta) \langle X_2 - X_1, H(X_2) - H(X_1)\rangle \\
&\quad + \langle Y_2 - Y_1, H(X_2) - H(X_1)\rangle \\
&\quad + \frac{1}{2}\beta\delta_g \langle Y_2 - Y_1, Y_2 - Y_1\rangle \\
&\quad + \langle Z_2 - Z_1, F(Z_2) - F(Z_1)\rangle, \\
W_3 &= \frac{1}{6}\delta_g (1-\beta) \langle X_2 - X_1, H(X_2) - H(X_1)\rangle \\
&\quad + \frac{1}{4}\alpha \langle F(Z_2) - F(Z_1), Z_2 - Z_1\rangle \\
&\quad + \langle (1+\alpha)(Z_2 - Z_1), H(X_2) - H(X_1)\rangle, \\
W_4 &= \frac{1}{6}\delta_g (1-\beta) \langle X_2 - X_1, H(X_2) - H(X_1)\rangle \\
&\quad + \left\langle \delta_g (1-\beta)(X_2 - X_1), G(Y_2) - G(Y_1) \right. \\
&\qquad \left. -\delta_g (Y_2 - Y_1)\right\rangle \\
&\quad + \frac{1}{2}\left\langle Y_2 - Y_1, G(Y_2) - G(Y_1) - \delta_g (Y_2 - Y_1)\right\rangle \\
&\quad + \langle F(Z_2) - F(Z_1) - (Z_2 - Z_1), Y_2 - Y_1\rangle \\
&\quad + \left\langle F(Z_2) - F(Z_1) - (Z_2 - Z_1), \right. \\
&\qquad \left. (1-\beta)\delta_g (X_2 - X_1)\right\rangle, \\
W_5 &= \frac{1}{4}\alpha \langle F(Z_2) - F(Z_1), Z_2 - Z_1\rangle \\
&\quad + \left\langle (1+\alpha)(Z_2 - Z_1), G(Y_2) - G(Y_1) - \delta_g (Y_2 - Y_1)\right\rangle \\
&\quad + \frac{1}{2}\left\langle Y_2 - Y_1, G(Y_2) - G(Y_1) - \delta_g (Y_2 - Y_1)\right\rangle, \\
W_6 &= \left\langle \delta_g (1-\beta)(X_2 - X_1) + Y_2 - Y_1 + (1+\alpha)(Z_2 - Z_1), \right. \\
&\qquad \left. P(t, X_2, Y_2, Z_2) - P(t, X_1, Y_1, Z_1)\right\rangle \\
&\quad + \langle Z_2 - Z_1, Z_2 - Z_1\rangle.
\end{aligned} \tag{19}$$

Note that the existence of the following estimates is clear (see Afuwape and Omeike [1]):

$$\begin{aligned}
H(X_2) - H(X_1) &= \int_0^1 J_h(\xi)(X_2 - X_1)\, ds, \\
G(Y_2) - G(Y_1) &= \int_0^1 J_g(\tau)(Y_2 - Y_1)\, dt, \\
F(Z_2) - F(Z_1) &= \int_0^1 J_f(\eta)(Z_2 - Z_1)\, d\mu,
\end{aligned} \tag{20}$$

where $\xi = sX_2 + (1-s)X_1$, $0 \leq s \leq 1$, $\tau = tY_2 + (1-t)Y_1$, $0 \leq t \leq 1$, $\eta = \mu Z_2 + (1-\mu)Z_1$, $0 \leq \mu \leq 1$.

Subject to the assumptions, it can be easily obtained that

$$W_j \geq 0, \quad (j = 3, 4, 5). \tag{21}$$

In view of the assumptions of the theorem, it is also clear that

$$\begin{aligned}
&\langle Y_2 - Y_1, H(X_2) - H(X_1)\rangle \\
&\quad = \left\|k_1 (Y_2 - Y_1) + 2^{-1}k_1^{-1}(H(X_2) - H(X_1))\right\|^2 \\
&\qquad - \left\langle k_1^2 (Y_2 - Y_1), Y_2 - Y_1\right\rangle \\
&\qquad - \left\langle 4^{-1}k_1^{-2}(H(X_2) - H(X_1)), H(X_2) - H(X_1)\right\rangle, \\
&\langle Z_2 - Z_1, F(Z_2) - F(Z_1)\rangle \geq \delta_f \langle Z_2 - Z_1, Z_2 - Z_1\rangle.
\end{aligned} \tag{22}$$

Hence,

$$\begin{aligned}
W_2 &\geq \left\|k_1 (Y_2 - Y_1) + 2^{-1}k_1^{-1}(H(X_2) - H(X_1))\right\|^2 \\
&\quad + \left\langle Y_2 - Y_1, \left(2^{-1}\beta\delta_g - k_1^2\right)(Y_2 - Y_1)\right\rangle \\
&\quad + \left\langle H(X_2) - H(X_1), \left(6^{-1}\delta_g (1-\beta)\right)(X_2 - X_1) \right. \\
&\qquad \left. -4^{-1}k_1^{-2}(H(X_2) - H(X_1))\right\rangle \\
&\quad + \delta_f \langle Z_2 - Z_1, Z_2 - Z_1\rangle.
\end{aligned} \tag{23}$$

Using the estimate $0 < \delta_h \leq \lambda_i(J_h(X)) \leq \Delta_h$, it follows that

$$\begin{aligned}
W_2 &\geq \left\|k_1 (Y_2 - Y_1) + 2^{-1}k_1^{-1}(H(X_2) - H(X_1))\right\|^2 \\
&\quad + \left\langle Y_2 - Y_1, \left(2^{-1}\beta\delta_g - k_1^2\right)(Y_2 - Y_1)\right\rangle \\
&\quad + \left\langle H(X_2) - H(X_1), \left(6^{-1}\delta_g (1-\beta)\right)(X_2 - X_1) \right. \\
&\qquad \left. -4^{-1}k_1^{-2}\delta_h (X_2 - X_1)\right\rangle + \delta_f \langle Z_2 - Z_1, Z_2 - Z_1\rangle
\end{aligned}$$

$$\begin{aligned}
&\geq \left\| k_1 (Y_2 - Y_1) + 2^{-1} k_1^{-1} (H(X_2) - H(X_1)) \right\|^2 \\
&\quad + \left\langle Y_2 - Y_1, \left(2^{-1}\beta\delta_g - k_1^2\right)(Y_2 - Y_1) \right\rangle \\
&\quad + \left\langle \delta_h (X_2 - X_1), 6^{-1}\delta_g (1-\beta)(X_2 - X_1) \right\rangle \\
&\quad - \left\langle \Delta_h (X_2 - X_1), -4^{-1} k_1^{-2} \delta_h (X_2 - X_1) \right\rangle \\
&\quad + \delta_f \langle Z_2 - Z_1, Z_2 - Z_1 \rangle \\
&= \left\| k_1 (Y_2 - Y_1) + 2^{-1} k_1^{-1} (H(X_2) - H(X_1)) \right\|^2 \\
&\quad + \left(\frac{1}{2}\beta\delta_g - k_1^2 \right) \|Y_2 - Y_1\|^2 \\
&\quad + \left(\frac{1}{6}\delta_h\delta_g (1-\beta) - \frac{1}{4k_1^2}\Delta_h\delta_h \right) \|X_2 - X_1\|^2 \\
&\quad + \delta_f \|Z_2 - Z_1\|^2.
\end{aligned} \tag{24}$$

Then

$$W_2 \geq 0 \quad \forall X, Y, Z \text{ in } R^n \tag{25}$$

if $k_1^2 \leq (1/2)\beta\delta_g$ with $\Delta_h \leq 3^{-1}\beta\delta_g^2(1-\beta)$.

Further, since

$$\begin{aligned}
&\left\langle 2^{-1}(1-\beta)\delta_g (X_2 - X_1), H(X_2) - H(X_1) \right\rangle \\
&\qquad \geq \frac{1}{2}(1-\beta)\delta_g\delta_h \|X_2 - X_1\|^2,
\end{aligned} \tag{26}$$

$$\langle F(Z_2) - F(Z_1), Z_2 - Z_1 \rangle \geq \delta_f \|Z_2 - Z_1\|^2,$$

then

$$\begin{aligned}
W_1 &\geq \frac{1}{2}(1-\beta)\delta_g\delta_h \|X_2 - X_1\|^2 + \frac{1}{2}\beta\delta_g \|Y_2 - Y_1\|^2 \\
&\quad + \frac{1}{2}\alpha\delta_f \|Z_2 - Z_1\|^2 \\
&\geq 2\delta_5 \left(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2 \right),
\end{aligned} \tag{27}$$

where $\delta_5 = (1/4)\min\{(1-\beta)\delta_g\delta_h, \beta\delta_g, \alpha\delta_f\}$.

Moreover, it is obvious that

$$\begin{aligned}
|W_6| \leq &\left\{ (1-\beta)\delta_g \|X_2 - X_1\| + \|Y_2 - Y_1\| \right. \\
&\left. + (\alpha + 1)\|Z_2 - Z_1\| \right\} \|\theta\|,
\end{aligned} \tag{28}$$

where $\theta = P(t, X_2, Y_2, Z_2) - P(t, X_1, Y_1, Z_1)$.

Hence,

$$\begin{aligned}
|W_6| \leq \delta_6 &\left\{ \|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 \right. \\
&\left. + \|Z_2 - Z_1\|^2 \right\}^{1/2} \|\theta\|.
\end{aligned} \tag{29}$$

Using the assumption (5), we get

$$|W_6| \leq \delta_6\Delta_1 \left\{ \|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2 \right\}, \tag{30}$$

so that

$$\begin{aligned}
\dot{W}(t) \leq &-(2\delta_5 - \delta_6\Delta_1) \\
&\times \left\{ \|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2 \right\}.
\end{aligned} \tag{31}$$

There exists a constant $\delta_7 > 0$ such that

$$\dot{W}(t) \leq -\delta_7 \left\{ \|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2 \right\}, \tag{32}$$

provided that $\Delta_1 < \varepsilon$, where ε is a sufficiently small positive constant.

In view of (17), the last estimate implies that

$$\dot{W}(t) \leq -\delta_8 W(t) \tag{33}$$

for some positive constant δ_8.

The conclusion of the theorem is immediate if, provided that $\Delta_1 < \varepsilon$, on integrating $\dot{W}(t)$ in (33) between t_0 and t, we have

$$W(t) \leq W(t_0)\exp[-\delta_8(t - t_0)], \quad t \geq t_0, \tag{34}$$

which implies that

$$W(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{35}$$

By (17), this shows that

$$\begin{aligned}
&\|X_2(t) - X_1(t)\| \longrightarrow 0, \qquad \|Y_2(t) - Y_1(t)\| \longrightarrow 0, \\
&\qquad \|X_2(t) - X_1(t)\| \longrightarrow 0, \quad \text{as } t \longrightarrow \infty.
\end{aligned} \tag{36}$$

This completes the proof of the theorem. □

Example 6. Let us consider (1),

$$\dddot{X} + F(\ddot{X}) + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \quad X \in R^2 \tag{37}$$

with

$$\begin{aligned}
&F = \begin{pmatrix} \ddot{x}_1 + \arctan \ddot{x}_1 \\ \ddot{x}_2 + \arctan \ddot{x}_2 \end{pmatrix}, \qquad G = \begin{pmatrix} \tan^{-1}\dot{x}_1 + 0.00004\dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, \\
&H = \begin{pmatrix} 0.001\tan^{-1}x_1 + 0.0001x_1 \\ 0.0001x_2 \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \\
&P(t) = \begin{pmatrix} e^{-t} \\ \sin t \end{pmatrix},
\end{aligned} \tag{38}$$

where e^{-t}, $\sin t$ are bounded continuous functions on $[0, \infty)$.

Then, it can be easily seen that

$$J_f(\ddot{X}) = \begin{pmatrix} 1 + \dfrac{1}{1+\ddot{x}_1^2} & 0 \\ 0 & 1 + \dfrac{1}{1+\ddot{x}_2^2} \end{pmatrix},$$

$$\lambda_1(J_f) = 1 + \frac{1}{1+\ddot{x}_1^2}, \qquad \lambda_2(J_f) = 1 + \frac{1}{1+\ddot{x}_2^2},$$

$$J_g(\dot{X}) = \begin{pmatrix} \dfrac{1}{1+\dot{x}_1^2} + 0.00004 & 0 \\ 0 & 1 \end{pmatrix}, \tag{39}$$

$$\lambda_1(J_g) = 1, \qquad \lambda_2(J_g) = \frac{1}{1+\dot{x}_1^2} + 0.00004,$$

$$J_h(X) = \begin{pmatrix} \dfrac{0.001}{1+x_1^2} + 0.0001 & 0 \\ 0 & 0.0001 \end{pmatrix},$$

$$\lambda_1(J_h) = \frac{0.001}{1+x_1^2} + 0.0001, \qquad \lambda_2(J_h) = 0.00001.$$

Thus, $\delta_f = 1$, $\Delta_f = 2$, $\delta_g = 1$, $\Delta_g = 1.00004$, $\delta_h = 0.0001$, and $\Delta_h = 0.0011$.

Let us choose

$$\begin{aligned} \alpha &= 3, \\ \beta &= \frac{1}{2}\text{in}\left(\Delta_h \le \min\left\{3^{-1}\beta(1-\beta)\delta_g^2;\right.\right. \\ &\qquad \left.\left. 6^{-1}\alpha(1-\beta)\delta_g\delta_f(\alpha+1)^{-2}\right\} \right. \\ &\left. = k\delta_g\delta_f\right). \end{aligned} \tag{40}$$

Then,

$$k = \frac{1}{64} < 1. \tag{41}$$

Since $0.0011 < 1/64$, then all the conditions of Theorem 2 hold. Therefore, all solutions of the equation considered converge (see, also, [1]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] A. U. Afuwape and M. O. Omeike, "Convergence of solutions of certain system of third order non-linear ordinary differential equations," *Annals of Differential Equations*, vol. 21, no. 4, pp. 533–540, 2005.

[2] A. U. Afuwape, "Convergence of solutions of some third order systems of non-linear ordinary differential equations," *Analele Ştiinţifice ale Universitatii 'Alexandru Ioan Cuza'*, vol. 55, no. 1, pp. 11–20, 2009.

[3] A. U. Afuwape and M. O. Omeike, "Convergence of solutions of certain third order systems of non-linear ordinary differential equations," *Journal of the Nigerian Mathematical Society*, vol. 25, pp. 1–12, 2006.

[4] A. L. Olutimo, "Convergence results for solutions of certain third-order nonlinear vector differential equations," *Indian Journal of Mathematics*, vol. 54, no. 3, pp. 299–311, 2012.

[5] A. U. Afuwape, "Ultimate boundedness results for a certain system of third-order nonlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 97, no. 1, pp. 140–150, 1983.

[6] A. U. Afuwape, "Further ultimate boundedness results for a third-order nonlinear system of differential equations," *Analisi Funzionale e Applicazioni*, vol. 6, no. 1, pp. 348–360, 1985.

[7] A. U. Afuwape and A. S. Ukpera, "Existence of solutions of periodic boundary value problems for some vector third order differential equations," *Journal of the Nigerian Mathematical Society*, vol. 20, pp. 1–17, 2001.

[8] J. O. C. Ezeilo, "*n*-dimensional extensions of boundedness and stability theorems for some third order differential equations," *Journal of Mathematical Analysis and Applications*, vol. 18, no. 3, pp. 395–416, 1967.

[9] J. O. C. Ezeilo and H. O. Tejumola, "Boundedness and periodicity of solutions of a certain system of third-order non-linear differential equations," *Annali di Matematica Pura ed Applicata*, vol. 74, no. 1, pp. 283–316, 1966.

[10] J. O. C. Ezeilo and H. O. Tejumola, "Further results for a system of third order differential equations," *Atti della Accademia Nazionale dei Lincei*, vol. 58, no. 2, pp. 143–151, 1975.

[11] F. W. Meng, "Ultimate boundedness results for a certain system of third order nonlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 177, no. 2, pp. 496–509, 1993.

[12] R. Reissig, G. Sansone, and R. Conti, *Non-Linear Differential Equations of Higher Order*, Noordhoff, Groningen, The Netherlands, 1974.

[13] A. Tiryaki, "Boundedness and periodicity results for a certain system of third order nonlinear differential equations," *Indian Journal of Pure and Applied Mathematics*, vol. 30, no. 4, pp. 361–372, 1999.

[14] C. Tunç, "On the boundedness and periodicity of the solutions of a certain vector differential equation of third-order," *Applied Mathematics and Mechanics*, vol. 20, no. 2, pp. 163–170, 1999.

[15] C. Tunç, "On the boundedness of solutions of certain nonlinear vector differential equations of third order," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 49, no. 3, pp. 291–300, 2006.

[16] C. Tunç, "On the stability and boundedness of solutions of nonlinear vector differential equations of third order," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 6, pp. 2232–2236, 2009.

[17] C. Tunç and M. Ateş, "Stability and boundedness results for solutions of certain third order nonlinear vector differential equations," *Nonlinear Dynamics*, vol. 45, no. 3-4, pp. 273–281, 2006.

[18] C. Tunç and E. Tunç, "New ultimate boundedness and periodicity results for certain third-order nonlinear vector differential equations," *Mathematical Journal of Okayama University*, vol. 48, pp. 159–172, 2006.

[19] C. Tunç and B. Karakas, "On boundedness of solutions to nonlinear vector differential equations of third-order," *Nonlinear Studies*, vol. 18, no. 1, pp. 63–73, 2011.

[20] C. Tunç and M. Gözen, "Stability and uniform boundedness in multidelay functional differential equations of third order," *Abstract and Applied Analysis*, vol. 2013, Article ID 248717, 7 pages, 2013.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 378968, 8 pages
http://dx.doi.org/10.1155/2014/378968

Research Article

Behaviors and Numerical Simulations of Malaria Dynamic Models with Transgenic Mosquitoes

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Received 21 November 2013; Accepted 4 January 2014; Published 20 February 2014

Academic Editor: Bingwen Liu

The release of transgenic mosquitoes to interact with wild ones is a promising method for controlling malaria. How to effectively release transgenic mosquitoes to prevent malaria is always a concern for researchers. This paper investigates two methods of releasing transgenic mosquitoes and proposes two epidemic models involving malaria patients, anopheles, wild mosquitoes, and transgenic mosquitoes based on system of continuous differential equations. A basic reproduction number $\mathbf{R}_0$ is defined for the models and it serves as a threshold parameter that predicts whether malaria will spread. By theoretical analysis of the dynamic behaviors of the models and numerical simulations, it is verified that malaria can be effectively controlled by the opportune release of transgenic mosquitoes; that is, when $\mathbf{R}_0 \le 1$, malaria will disappear; when $\mathbf{R}_0 > 1$, malaria will become an endemic disease in the target field.

1. Introduction

Malaria is an infectious mosquito-borne disease. No vaccine and no specific drugs are available for malaria because plasmodium which causes malaria have become increasingly resistant to drugs. An effective way to prevent malaria is to control mosquitoes. Therefore, scientists hope to use genetic engineering technology to release transgenic mosquitoes which cannot transmit malaria to cut off malaria transmission chains [1–4].

But function laws of transgenic mosquitoes released to prevent malaria transmission can only be acquired from a large number of experimental data in consideration of the influence on other species, ecological environment, possible risks, and involvement [4–6]. Therefore, it has a practical significance to establish dynamic models which can reflect the change laws of many factors and study the dynamic behaviors of the models to recognize the role of transgenic mosquitoes in malaria transmission in terms of all statuses of releasing transgenic mosquitoes.

Many researchers have made a lot of theoretical researches on the role of transgenic mosquitoes in decreasing anopheles and preventing malaria transmission. Some wonderful mathematical models are presented. For example, the possibilities of replacing wild mosquitoes with transgenic ones released in different ways were considered, malaria transmission model was established, and the existence and stability of the disease-free equilibrium points were discussed with the aid of Floquet theory in [7]. The epidemic dynamic models of malaria for sawtooth animals were established and behaviors of each infection stage were discussed with numerical simulations in [8]. The detailed analysis and discussion on how to prevent malaria by mathematical models were presented in [9]. Li has long been engaged in the research in this field. He mainly investigated the impact of the environment, wild mosquitoes, genetically modified mosquitoes, and so forth and formulated the stage-structured discrete-time and continuous-time mathematical models for interacting wild and transgenic mosquito populations [10–14].

However, at present, many existing models are generally limited to discuss the role of population characters of two classes of mosquitoes, technology development, and the environmental factors that prevent malaria and do not consider other more factors, such as infected patients. For the analysis of the infected population controlled by transgenic mosquitoes in malaria transmission, these works are not enough. In order to better investigate the actual role of transgenic mosquitoes in malaria transmission and make the models more realistic, according to the current international research results which have been obtained, we establish a population dynamic model and an epidemic dynamic model involving patients, anopheles, wild mosquitoes, and transgenic mosquitoes released in two different ways based on systems of differential equations. With the aid of qualitative theory, we study behaviors of these models with numerical simulations and provide the conditions under which equilibriums of these models are asymptotically stable. Based on the researches on modeling dynamical behaviors, the impacts of various parameters changes in the models on malaria transmission and the two methods of releasing transgenic mosquitoes on controlling the amount of malaria patients in practice are investigated with numerical simulations.

This paper is organized as follows. In Section 2, we firstly assume that transgenic mosquitoes are released at a fixed proportion and interact with wild mosquitoes. Secondly, we assume that transgenic mosquitoes are released at a changeable proportion. Then the existence of all possible equilibriums is investigated and their stabilities are studied. In Section 3, numerical simulations are supplemented to demonstrate the results in Section 2. In Section 4, the brief discussions of our findings and prospects are presented.

2. The Models and Stability Analyses

In 1927, Kermack and Mckendrick established an epidemic model, namely, KM model [15]. In this paper, based on KM assumptions and the epidemic model in [16, 17], we will formulate and discuss two dynamic models with male transgenic mosquitoes which do not suck blood [18] released in two different ways.

2.1. The Dynamic Model with Transgenic Mosquitoes Released at a Fixed Proportion. For simplicity, we first consider the model with transgenic mosquitoes released into the field of mosquitoes at a fixed proportion. We assume that the population size in the target field is large and KM assumptions of epidemic model are met. The proportion at which transgenic mosquitoes are released is a constant a $(0 \leq a < 1)$. Then, we can establish a system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= \beta y(1-x) - \gamma x, \\ \frac{dy}{dt} &= \alpha x(1-a-y) - \mu y - cay, \end{aligned} \tag{1}$$

where $\beta, \gamma, \alpha, \mu, c$ are positive constants. The descriptions of these parameters in system (1) are shown in Table 1.

Table 1: Descriptions of the parameters in system (1).

Parameter	Description
a	Proportion at which transgenic mosquitoes are released
β	Incidence rate of malaria due to biting
α	Efficiency of infection in mosquitoes by biting patient
μ	Death rate of anopheles
γ	Recovery rate of patients
c	Decrement rate of anopheles due to transgenic mosquitoes bred by transgenic mosquitoes and anopheles
$x(t)$	Proportion of patients at t time
$y(t)$	Proportion of anopheles at t time

Let $\mathbf{R}_0^1 = \alpha\beta(1-a)/\gamma(\mu+ca)$. System (1) has a unique equilibrium $(0,0)$ for $\mathbf{R}_0^1 \leq 1$ and the endemic equilibrium (x_0, y_0) exists if and only if $\mathbf{R}_0^1 > 1$; here

$$x_0 = \frac{\gamma(ca+\mu)(\mathbf{R}_0^1 - 1)}{\alpha\beta(1-a)+\alpha\gamma}, \qquad y_0 = \frac{\gamma(ca+\mu)(\mathbf{R}_0^1 - 1)}{\beta(ac+\alpha+\mu)}. \tag{2}$$

For system (1), we can obtain the following conclusion.

Theorem 1. *(i) The equilibrium* $(0,0)$ *is locally asymptotically stable if* $\mathbf{R}_0^1 < 1$. *(ii) The endemic equilibrium* (x_0, y_0) *is locally asymptotically stable and* $(0,0)$ *is unstable if* $\mathbf{R}_0^1 > 1$*; here* (x_0, y_0) *is the same as that in* (2).

Proof. (i) The linearization form of system (1) about $(0,0)$ is

$$\begin{aligned} \frac{dx}{dt} &= -\gamma x + \beta y, \\ \frac{dy}{dt} &= \alpha(1-a)x - (\mu+ca)y. \end{aligned} \tag{3}$$

The corresponding characteristic equation of (3) is

$$\lambda^2 + (\gamma+\mu+ca)\lambda + \gamma(ac+\mu)(1-\mathbf{R}_0^1) = 0. \tag{4}$$

Solving (4), we can get two characteristic roots:

$$\lambda_{1,2} = \frac{-(ac+\gamma+\mu) \pm \sqrt{(ac+\gamma+\mu)^2 - 4\gamma(ac+\mu)(1-\mathbf{R}_0^1)}}{2}. \tag{5}$$

From (5), it is easy to see when $\mathbf{R}_0^1 < 1$, the real parts of (5) are all negative. Therefore, the equilibrium $(0,0)$ is locally asymptotically stable. It implies that malaria will eventually disappear. When $\mathbf{R}_0^1 > 1$, $(0,0)$ is saddle point; it is unstable.

(ii) Linearizing system (1) about (x_0, y_0) yields

$$\begin{aligned} \frac{dx}{dt} &= (-\beta y_0 - \gamma)(x-x_0) + \beta(1-x_0)(y-y_0), \\ \frac{dy}{dt} &= \alpha(1-a-y_0)(x-x_0) - (\alpha x_0 + \mu + ca)(y-y_0). \end{aligned} \tag{6}$$

The corresponding characteristic equation of (6) is

$$\begin{aligned}(\lambda + \beta y_0 + \gamma)(\lambda + \alpha x_0 + \mu + ca) \\ - \alpha\beta(1 - x_0)(1 - a - y_0) = 0.\end{aligned} \quad (7)$$

Solving (7), we can get two characteristic roots

$$\lambda_{1,2} = \frac{-(\beta y_0 + \gamma + \alpha x_0 + \mu + ca) \pm \sqrt{\Delta}}{2}, \quad (8)$$

where

$$\begin{aligned}\Delta &= (\beta y_0 + \gamma + \alpha x_0 + \mu + ca)^2 - 4K, \\ K &= (\beta y_0 + \gamma)(\alpha x_0 + \mu + ca) - \alpha\beta(1 - x_0)(1 - a - y_0).\end{aligned} \quad (9)$$

From (2) and (9), it follows that

$$K = (ac\gamma + \gamma\mu)(\mathbf{R}_0^1 - 1). \quad (10)$$

Therefore, if $\mathbf{R}_0^1 > 1$, the real parts of (8) are all negative, which indicates that the endemic equilibrium (x_0, y_0) is locally asymptotically stable. It implies that malaria will be popular in the target field, but it will not be massively diffusive. Hence, Theorem 1 is completed. □

Remark 2. When $a = 0$, system (1) becomes the model in [16]. From above results, when $\mathbf{R}_0^1 < 1$, we can see that malaria are prevented effectively by releasing transgenic mosquitoes. From $\mathbf{R}_0^1 = \alpha\beta(1-a)/\gamma(\mu+ca)$, as long as we increase the value of a or c, that is, increase transgenic mosquitoes or decrease anopheles, malaria can be prevented in the target field.

2.2. The Dynamic Model with Transgenic Mosquitoes Released at a Changeable Proportion. Releasing transgenic mosquitoes at a fixed proportion is more difficult than being carried out in reality and also does not accord with the actual situation. Therefore, in this section we introduce a changeable proportion $z(t)$, which is the proportion of transgenic mosquitoes to mosquito population at time t. At time t, the proportion of anopheles is $y(t)$ and the proportion of susceptible mosquitoes is $1 - y(t) - z(t)$. $x(t)$ is the proportion of infected persons; then $1 - x(t)$ is the proportion of susceptible persons. Similarly, we can establish the following model:

$$\begin{aligned}\frac{dx}{dt} &= \beta y(1 - x) - \gamma x, \\ \frac{dy}{dt} &= \alpha x(1 - y - z) - \mu y - czy, \\ \frac{dz}{dt} &= \delta_1 zy + \delta_2 z(1 - y - z) - \omega z,\end{aligned} \quad (11)$$

where $\beta, \gamma, \alpha, \mu, c, \delta_1, \delta_2, \omega$ are constants. The description of these parameters in system (11) is shown in Table 2.

Assume that $\delta_1 = \delta_2 = \delta$, system (11) has four fixed points which are malaria-free equilibriums $E_0(0, 0, 0)$, $E_1(0, 0, z_1)$, and endemic equilibriums $E_2(x_2, y_2, z_2)$ and $E_3(x_3, y_3, 0)$, respectively; here

$$\begin{aligned}x_2 &= \frac{\gamma c\omega + \alpha\beta\omega - \gamma\mu\delta - \gamma c\delta}{\alpha\beta\omega + \alpha\delta\gamma}, \\ y_2 &= \frac{\gamma c\omega + \alpha\beta\omega - \gamma\mu\delta - \gamma c\delta}{\alpha\beta\omega + \mu\delta\beta + c\delta\beta - c\beta\omega},\end{aligned} \quad (12)$$

$$z_1 = z_2 = 1 - \frac{\omega}{\delta}, \qquad x_3 = \frac{\alpha\beta - \gamma\mu}{\alpha(\beta + \gamma)}, \qquad y_3 = \frac{\alpha\beta - \gamma\mu}{\beta(\alpha + \mu)}. \quad (13)$$

Remark 3. When $\delta_1 \neq \delta_2$, we can still obtain (13), but $z_1 = z_2 = 1 - \omega/\delta_2$ and the expressions of (12) are very complicated.

For system (11), since the equilibrium $E_0(0, 0, 0)$ and endemic equilibrium $E_3(x_3, y_3, 0)$ are not meaningful for reality, we just consider the stability of the equilibrium $E_1(0, 0, z_1)$ and $E_2(x_2, y_2, z_2)$. Taking $\mathbf{R}_0^2 = \alpha\beta\omega/\gamma(c\delta + \delta\mu - \omega)$, for system (11), we can obtain the following conclusion.

Theorem 4. *Assuming that $\delta_1 = \delta_2 = \delta$ and $\omega < \delta$ which implies the birth rate of the transgenic mosquitoes is larger than the death rate, one has*

(i) *the equilibrium $E_1(0, 0, z_1)$ is locally asymptotically stable if $\mathbf{R}_0^2 < 1$,*

(ii) *the endemic equilibrium $E_2(x_2, y_2, z_2)$ is locally asymptotically stable if $\mathbf{R}_0^2 > 1$; here z_1, x_2, y_2, z_2 are the same as those in (12) and (13).*

Proof. The matrix corresponding to the linearization form of system (11) on $(0, 0, z_1)$ is

$$A = \begin{pmatrix} -\gamma & \beta & 0 \\ \alpha(1 - z_1) & -\mu - cz_1 & 0 \\ 0 & \delta z_1 - \delta z_1 & \delta - 2\delta z_1 - \omega \end{pmatrix}. \quad (14)$$

Its characteristic equation is

$$|\lambda E - A| = \begin{vmatrix} \lambda + \gamma & -\beta & 0 \\ \alpha(z_1 - 1) & \lambda + \mu + cz_1 & 0 \\ 0 & \delta z_1 - \delta z_1 & \lambda - \delta + 2\delta z_1 + \omega \end{vmatrix} = 0. \quad (15)$$

Solving the above equation, we can get its characteristic roots

$$\begin{aligned}\lambda_1 &= -\omega + (1 - 2z_1)\delta_2, \\ \lambda_{2,3} &= \Big(-\gamma - \mu - cz_1 \\ &\quad \pm \sqrt{(\gamma + \mu + cz_1)^2 - 4(-\alpha\beta + \gamma\mu + (\alpha\beta + c\gamma) z_1)} \Big) \\ &\quad \times (2)^{-1}.\end{aligned} \quad (16)$$

From (13) and $\omega < \delta$, we have

$$\lambda_1 = -\omega + (1 - 2z_1)\delta = \omega - \delta < 0. \quad (17)$$

Table 2: Description of the parameters in system (11).

Parameter	Description
β	Incidence rate of malaria due to biting
α	Efficiency of infection in mosquitoes by biting patients
μ	Death rate of anopheles
γ	Recovery rate of patients
c	Decrement rate of anopheles due to transgenic mosquitoes bred by transgenic mosquitoes and anopheles
ω	Death rate of transgenic mosquitoes
δ_1	Birth rate of transgenic mosquitoes bred by transgenic mosquitoes and wild anopheles
δ_2	Birth rate of transgenic mosquitoes bred by transgenic mosquitoes and wild susceptible mosquitoes
$x(t)$	Proportion of patients at t time
$y(t)$	Proportion of anopheles at t time
$z(t)$	Proportion of transgenic mosquitoes released at time t

If $\mathrm{Re}\lambda_{2,3} < 0$, system (11) is locally asymptotically stable on $(0, 0, z_1)$. In fact, from (13) and $\lambda_{2,3}$ in (16), we have

$$-\alpha\beta + \gamma\mu + (\alpha\beta + c\gamma) z_1 = \alpha\beta\frac{\omega}{\delta}\left(1 - \mathbf{R}_0^2\right) > 0. \tag{18}$$

That is to say, $\mathrm{Re}\lambda_{2,3} < 0$. Therefore, system (11) is locally asymptotically stable on $(0, 0, z_1)$. It implies that malaria and anopheles will eventually disappear.

Now we consider behaviors of the local asymptotical stability of system (11) on (x_2, y_2, z_2) which is a positive root under the condition $\mathbf{R}_0^2 > 1$.

The matrix corresponding to the linearization form of system (11) on (x_2, y_2, z_2) is

$$A = \begin{pmatrix} -\beta y_2 - \gamma & \beta(1 - x_2) & 0 \\ \alpha(1 - y_2 - z_2) & -\alpha x_2 - \mu - c z_2 & -\alpha x_2 - c y_2 \\ 0 & 0 & \delta - \omega - 2\delta z_2 \end{pmatrix}. \tag{19}$$

The characteristic equation of this matrix is

$$|\lambda E - A| = \begin{vmatrix} \lambda + \beta y_2 + \gamma & -\beta(1 - x_2) & 0 \\ \alpha(y_2 + z_2 - 1) & \lambda + \alpha x_2 + \mu + c z_2 & \alpha x_2 + c y_2 \\ 0 & 0 & \lambda - \delta + \omega + 2\delta z_2 \end{vmatrix} = 0. \tag{20}$$

Solving the characteristic equation, we can get its characteristic roots:

$$\lambda_1 = \delta - \omega - 2\delta z_2 = \omega - \delta, \qquad \lambda_{2,3} = \frac{-\gamma - \mu - \alpha x_2 - \beta y_2 - c z_2 \pm \sqrt{\Delta}}{2}, \tag{21}$$

where

$$\Delta = (\gamma + \mu + \alpha x_2 + \beta y_2 + c z_2)^2 - 4\left((\beta y_2 + \gamma)(\alpha x_2 + \mu + c z_2) - \alpha\beta(1 - x_2)(1 - y_2 - z_2)\right). \tag{22}$$

From (13) and (21), we have

$$\lambda_1 = \delta - \omega - 2\delta z_2 = \omega - \delta < 0. \tag{23}$$

$\mathrm{Re}\lambda_{2,3} < 0$ is equivalent to

$$(\beta y_2 + \gamma)(\alpha x_2 + \mu + c z_2) - \alpha\beta(1 - x_2)(1 - y_2 - z_2) > 0. \tag{24}$$

By the condition $\omega < \delta$ and the above inequality, we have

$$\begin{aligned} &(\beta y_2 + \gamma)(\alpha x_2 + \mu + c z_2) - \alpha\beta(1 - x_2)(1 - y_2 - z_2) \\ &= \alpha\beta\frac{\omega}{\delta} - \gamma\left(\mu + c - \frac{c\omega}{\delta}\right) \\ &= \gamma\left(\mu + c - \frac{c\omega}{\delta}\right)\left(\mathbf{R}_0^2 - 1\right) > 0. \end{aligned} \tag{25}$$

Therefore, system (11) is locally asymptotically stable about (x_2, y_2, z_2). The proof of Theorem 4 is completed. □

Remark 5. When $\delta_1 \neq \delta_2$, taking $\mathbf{R}_0^2 = \alpha\beta\omega/(c\gamma\delta_2 + \gamma\delta_2\mu - c\gamma\omega)$, for system (11), the conclusions of Theorem 4 are still valid, but the proofs are more complicated. So we will verify them with numerical simulations in the following section.

Remark 6. From the above discussion, we can see that releasing transgenic mosquitoes into wild mosquitoes in the target field can prevent malaria. When $\mathbf{R}_0^2 > 1$, we can make the proportions of patients and anopheles steady and malaria will not be massively popular. Most importantly, when $\mathbf{R}_0^2 < 1$, there will be only transgenic mosquitoes; patients and anopheles will disappear. That is to say, we can eventually eradicate malaria. It is easy to see when we are unable to change the traditional infection rate and death rate; as long as we increase the amount of transgenic mosquitoes and the birth rate δ of transgenic mosquitoes bred by wild mosquitoes, then malaria can be eliminated.

3. Numerical Simulations

Here mainly for system (11), especially for $\delta_1 \neq \delta_2$, we verify its results with numerical simulations. Assume that the amounts

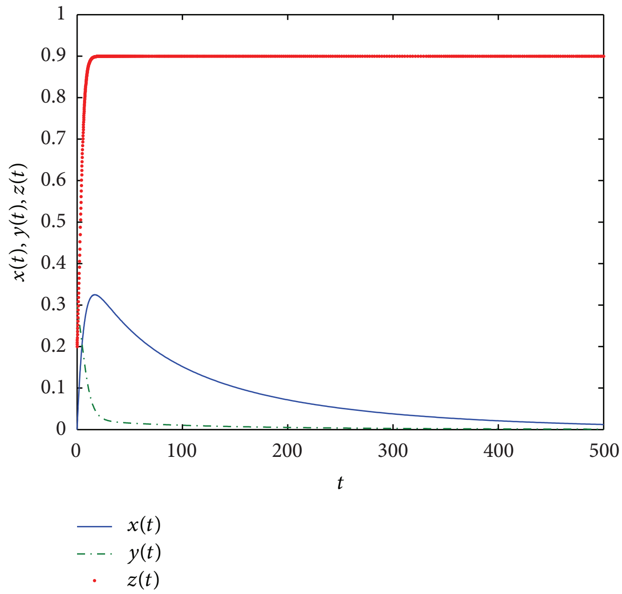

Figure 1: Parameters: $\beta = 0.2, \alpha = 0.1, \gamma = 0.02, \mu = 0.05, \omega = 0.05, c = 0.1$, and $\delta_1 = \delta_2 = \delta = 0.5$.

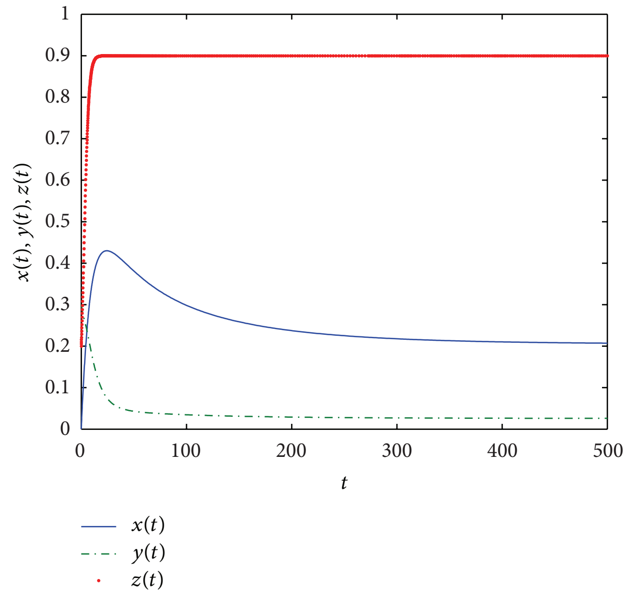

Figure 2: Parameters: $\beta = 0.2$, $\alpha = 0.1, \gamma = 0.02, \mu = 0.05, \omega = 0.05, c = 0.01$, and $\delta_1 = \delta_2 = \delta = 0.5$.

of persons and mosquitoes in an isolated target field are constants. According to the discussions in [8, 11, 19–21] and empirical data, we let the parameters in system (11) take the following ranges as in Table 3.

At the beginning, we assume that there is no patients; the proportion of anopheles and the proportion of transgenic mosquitoes are 0.3 and 0.2, respectively. That is to say, the initial values of system (11) are $(0, 0.3, 0.2)$. We take parameter values as $\beta = 0.2, \alpha = 0.1, \gamma = 0.02, \mu = 0.05, \omega = 0.05, c = 0.1$, and $\delta_1 = \delta_2 = \delta = 0.5$. According to these parameter values, we have

$$\mathbf{R}_0^2 = \frac{\alpha\beta\omega}{c\gamma\delta + \gamma\delta\mu - c\gamma\omega} \approx 0.714286 < 1. \tag{26}$$

The corresponding result of numerical simulation is shown in Figure 1.

From Theorem 4 and the above parameter values, it is easy to know that system (11) should be stable on $(0, 0, 0.9)$. From Figure 1, we can see that the simulation result is consistent with the result of Theorem 4. From the curve for anopheles, we can see that releasing the transgenic mosquitoes can effectively reduce anopheles. It follows that releasing the transgenic mosquitoes suppresses the outbreak of malaria in a relatively short time. As to why the patient and anopheles do not disappear immediately after a sharp drop in a short time, that is, it takes a long time that their amounts reach to zero, we think that the male transgenic mosquitoes cannot compete with some excellent wild male mosquitoes and they cannot capture the "heart" of all female anopheles in a short time or mating is a probability event. There is always mating between wild male mosquitoes and female anopheles. But after transgenic mosquitoes released control the rest of anopheles to a very small amount, the anopheles will die themselves one month later due to their limited lifetime. It can be considered that anopheles have been eliminated.

Take $c = 0.01$ and keep other parameter values unchanged. That is, we assume that the decrement rate of anopheles due to transgenic male mosquito bred by transgenic mosquito and anopheles is decreased. According to the values, we have

$$\mathbf{R}_0^2 = \frac{\alpha\beta\omega}{c\gamma\delta + \gamma\delta\mu - c\gamma\omega} \approx 1.69492 > 1. \tag{27}$$

The corresponding result of numerical simulation is shown in Figure 2.

From Theorem 4 and the above parameter values, it is easy to know that system (11) should be stable about $(0.205, 0.026, 0.9)$ which is an endemic equilibrium. From Figure 2, we can see that the simulation result is also consistent with result of Theorem 4 and parameter c has a distinct impact on the reduction of anopheles. When we decrease c, the reduction speed of anopheles becomes significantly slow.

When $\delta_1 \neq \delta_2$, take $\delta_2 = 0.25$ and keep other parameter values in the first numerical simulation unchanged. We have

$$\mathbf{R}_0^2 = \frac{\alpha\beta\omega}{c\gamma\delta_2 + \gamma\delta_2\mu - c\gamma\omega} \approx 1.53846 > 1. \tag{28}$$

The corresponding result of numerical simulation is shown in Figure 3.

Take $\delta_1 = 0.1$ and $\delta_2 = 0.5$ and keep other values unchanged; we have

$$\mathbf{R}_0^2 = \frac{\alpha\beta\omega}{c\gamma\delta_2 + \gamma\delta_2\mu - c\gamma\omega} \approx 0.714286 < 1. \tag{29}$$

The corresponding result of numerical simulation is shown in Figure 4.

Table 3: Ranges of the parameters in system (11).

Parameter	Description	Range (per day)
β	Incidence rate of malaria due to biting	0.1–0.5
α	Efficiency of infection in mosquitoes by biting patients	0.01–0.2
μ	Death rate of anopheles	0.05–0.5
γ	Recovery rate of patients	0.01–0.1
c	Decrement rate of anopheles due to transgenic mosquitoes bred by transgenic mosquitoes and anopheles	0.01–0.2
ω	Death rate of transgenic mosquitoes	0.05–0.5
δ_1	Birth rate of transgenic mosquitoes bred by transgenic mosquitoes and wild anopheles	0.1–0.5
δ_2	Birth rate of transgenic mosquitoes bred by transgenic mosquitoes and wild susceptible mosquitoes	0.1–0.5

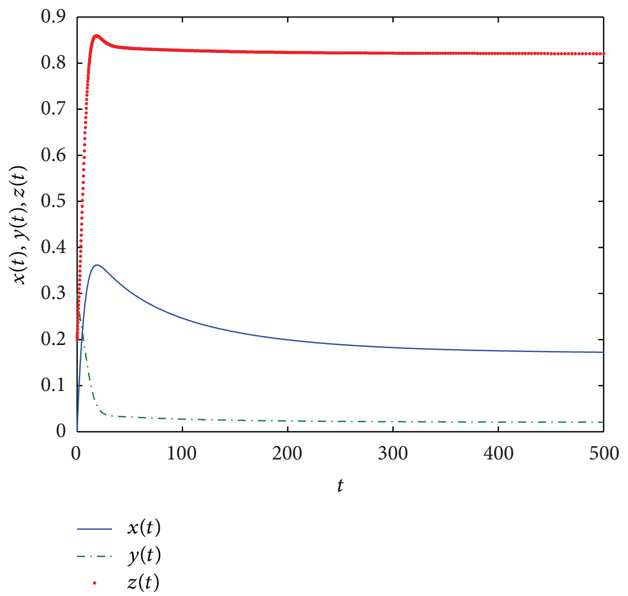

Figure 3: Parameters: $\beta = 0.2, \alpha = 0.1, \gamma = 0.02, \mu = 0.05, \omega = 0.05, c = 0.1, \delta_1 = 0.5$, and $\delta_2 = 0.25$.

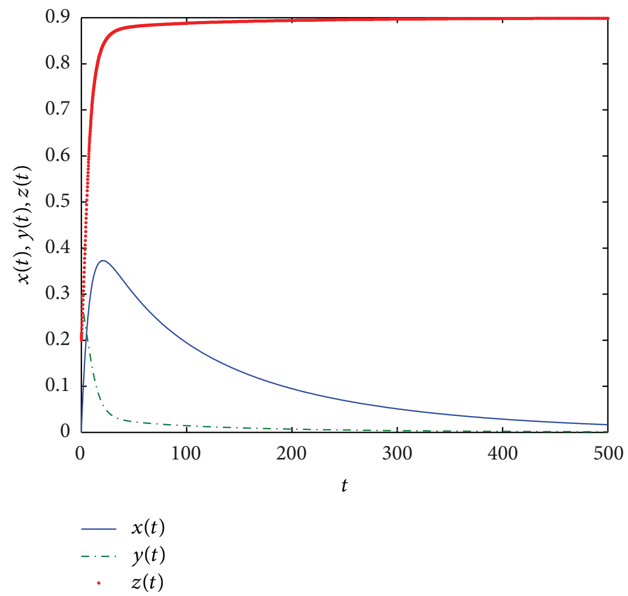

Figure 4: Parameters: $\beta = 0.2, \alpha = 0.1, \gamma = 0.02, \mu = 0.05, \omega = 0.05, c = 0.1, \delta_1 = 0.1$, and $\delta_2 = 0.5$.

From Figure 3, we can see that $x(t) \to x_2, y(t) \to y_2$, and $z(t) \to z_2$ as $t \to \infty$. From Figure 4 we can see that $x(t) \to 0, y(t) \to 0, z(t) \to z_1$ as $t \to \infty$. That is to say, when $\delta_1 \neq \delta_2$, the results of Theorem 4 remain valid.

Take $\alpha = 0.14, \delta_1 = 0.1$, and $\delta_2 = 0.5$ and keep other values unchanged; we have

$$\mathbf{R}_0^2 = \frac{\alpha\beta\omega}{c\gamma\delta_2 + \gamma\delta_2\mu - c\gamma\omega} = 1. \quad (30)$$

The corresponding result of numerical simulation is shown in Figure 5.

Set $\alpha = 0.14$ and $\delta_1 = \delta_2 = \delta = 0.5$ and keep other values unchanged; we have

$$\mathbf{R}_0^2 = \frac{\alpha\beta\omega}{c\gamma\delta + \gamma\delta\mu - c\gamma\omega} = 1. \quad (31)$$

The corresponding result of numerical simulation is shown in Figure 6.

For the above two numerical simulations, we increase the value of α; that is to say, we assume that the efficiency of patient infecting mosquito by biting is greater. From Figures 5 and 6, we can see that anopheles will become more and the decreasing speed of patients will become relatively slow. But malaria and anopheles will still disappear eventually. That is to say, for $\delta_1 \neq \delta_2$ and $\delta_1 = \delta_2$, when $\mathbf{R}_0^2 = 1$, $x(t) \to 0$, $y(t) \to 0$, and $z(t) \to z_1$ as $t \to \infty$.

4. Conclusion and Prospect

In this paper we firstly establish system (1) with transgenic mosquitoes released at a fixed proportion and then establish system (11) with transgenic mosquitoes released at a changeable proportion. For these models, we obtain a disease-free equilibrium and an endemic equilibrium. We prove theoretically and verify our conclusions of Theorems 1 and 4 with numerical simulations. For $\delta_1 \neq \delta_2$ and $\mathbf{R}_0^2 = 1$, we

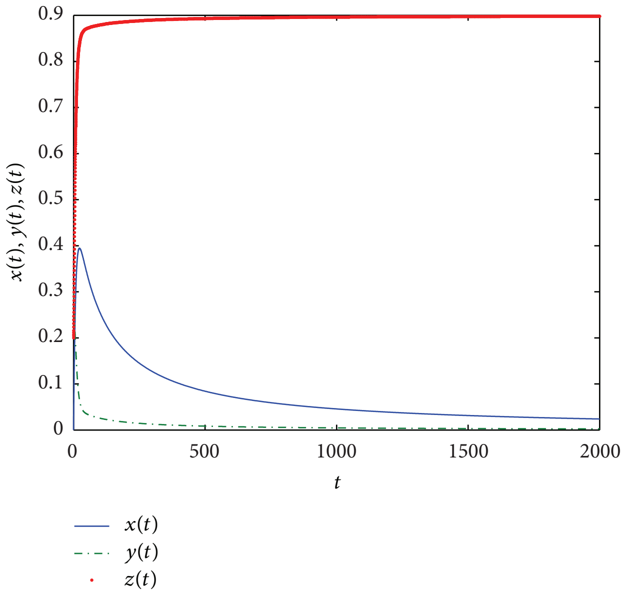

FIGURE 5: Parameters: $\beta = 0.2, \alpha = 0.14, \gamma = 0.02, \mu = 0.05, \omega = 0.05, c = 0.1, \delta_1 = 0.1$, and $\delta_2 = 0.5$.

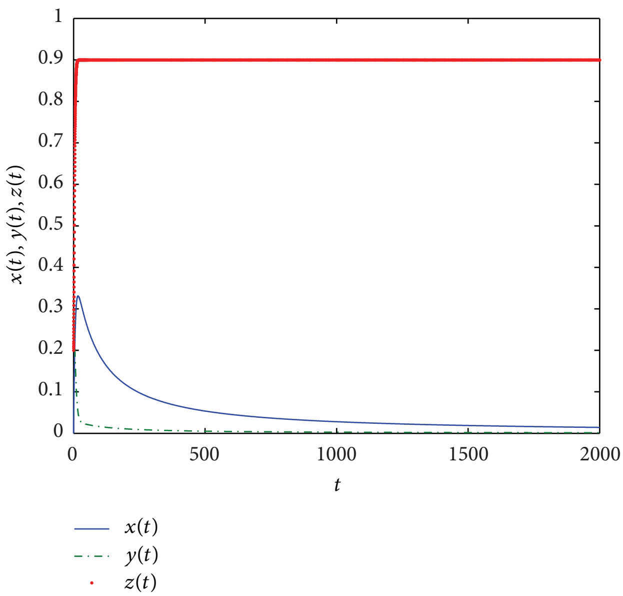

FIGURE 6: Parameters: $\beta = 0.2, \alpha = 0.14, \gamma = 0.02, \mu = 0.05, \omega = 0.05, c = 0.1, \delta_1 = 0.1$, and $\delta_2 = 0.5$.

do not theoretically prove the conclusions of Theorem 4, but we verify them with simulations. From Figures 3, 4, 5, and 6, we can see our results are also valid. We also have a unified conclusion for system (1) and system (11). That is, if $\mathbf{R}_0^1 < 1$ or $\mathbf{R}_0^2 < 1$, it implies that malaria will be eliminated; if $\mathbf{R}_0^1 > 1$ or $\mathbf{R}_0^2 > 1$, malaria will become an epidemic disease in the target field.

The models in this paper are simpler and more ideal. For example, we do not take the incubation period of malaria into account and only consider the one-time delivery of transgenic mosquitoes. The factors involved in our models are incomplete and we only considered the local asymptotic stabilities of systems. In order to make up for these deficiencies and establish more realistic models, we think that we can choose different birth function according to the actual situation and take more factors into account, such as susceptible, wild mosquito population, recovered patients, latency period, and environmental factor.

Although our dynamic models have many disadvantages, they are continuous differential equations compared with the existing models which were discrete-differential equations that only considered the competition between two classes of mosquito populations. In addition, our epidemic models for transgenic mosquitoes released at changeable proportion are considered by few researchers. Two models in this paper are consistent with the actual situation and they are relatively complete, various, and comprehensive for the research and applications of transgenic mosquito in malaria transmission.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by Hunan Provincial Natural Science Foundation of China (12jj4012) and Research Project of National University of Defense Technology (JC12-02-01).

References

[1] F. A. Collins and A. A. James, "Genetic modifications of mosquitoes," *Science Medicine*, vol. 3, pp. 52–61, 1996.

[2] G. K. Christophides, "Transgenic mosquitoes and malaria transmission," *Cellular Microbiology*, vol. 7, no. 3, pp. 325–333, 2005.

[3] F. Gould, K. Magori, and Y. X. Huang, "Genetic strategies for controlling mosquito-borne diseases," *American Scientist*, vol. 94, no. 3, pp. 238–246, 2006.

[4] S. Levy, "Mosquito modifications: new approaches to controlling malaria," *BioScience*, vol. 57, no. 10, pp. 816–821, 2007.

[5] J. V. Lavery, L. C. Harrington, and T. W. Scott, "Ethical, social, and cultural considerations for site selection for research with genetically modified mosquitoes," *American Journal of Tropical Medicine and Hygiene*, vol. 79, no. 3, pp. 312–318, 2008.

[6] B. G. J. Knols, H. C. Bossin, W. R. Mukabana, and A. S. Robinson, "Transgenic mosquitoes and the fight against malaria: managing technology push in a turbulent GMO world," *The American Journal of Tropical Medicine and Hygiene*, vol. 77, no. 6, pp. 232–242, 2007.

[7] L. M. Dai, *The influence of resistant mosquitoes to malaria dynamical system [M.S. thesis]*, Chongqing, China, 2009, Southwest University.

[8] R. Antia, A. Yates, and J. C. de Roode, "The dynamics of acute malaria infections. I. Effect of the parasite's red blood cell preference," *Proceedings of the Royal Society B*, vol. 275, no. 1641, pp. 1449–1458, 2008.

[9] N. R. Chitnis, *Using mathematical models in controlling the spread of malaria [Ph.D. thesis]*, the university of Arizona, Tucson, Arizona, 2005.

[10] J. Li, "Simple stage-structured models for wild and transgenic mosquito populations," *Journal of Difference Equations and Applications*, vol. 15, no. 4, pp. 327–347, 2009.

[11] J. Li, "Simple mathematical models for interacting wild and transgenic mosquito populations," *Mathematical Biosciences*, vol. 189, no. 1, pp. 39–59, 2004.

[12] J. Li, "Discrete time models with mosquitoes carrying genetically modified bacteria," *Mathematical Biosciences*, vol. 240, no. 1, pp. 35–44, 2012.

[13] J. Li, S. B. Ai, and J. L. Lu, "Mosquito-stage-structuried Malaria models and their global dynamics," *SIAM Journal of Applied Mathematics*, vol. 27, pp. 1223–1237, 2012.

[14] J. Li, "Modelling of transgenic mosquitoes and impact on malaria transmission," *Journal of Biological Dynamics*, vol. 5, no. 5, pp. 474–494, 2011.

[15] W. O. Kermack and A. G. Mckendrick, "A contribution to the mathematical theory of epidemics," *Proceeding of the Royal Society of London A*, vol. 115, no. 772, pp. 700–721, 1927.

[16] V. Lakshmikantham, *Nonlinear Systems and Applications*, Princeton University Press, Princeton, NJ, USA, 1976.

[17] L. S. Chen, X. Z. Meng, and J. J. Jiao, *Biodynamics*, Science Press, Beijing, China, 2009.

[18] B. H. Kim, H. K. Kim, and S. J. Lee, "Experimental analysis of the blood-sucking mechanism of female mosquitoes," *The Journal of Experimental Biology*, vol. 214, no. 7, pp. 1163–1169, 2011.

[19] F. Oduro, G. A. Okyere, and G. T. Azu-Tungmah, "Transmission dynamics of Malaria in ghana," *Journal of Mathematics Research*, vol. 4, no. 6, pp. 22–33, 2012.

[20] S. Ruan, D. Xiao, and J. C. Beier, "On the delayed Ross-Macdonald model for malaria transmission," *Bulletin of Mathematical Biology*, vol. 70, no. 4, pp. 1098–1114, 2008.

[21] A. F. Harris, D. Nimmo, A. R. McKemey et al., "Field performance of engineered male mosquitoes," *Nature Biotechnology*, vol. 29, no. 11, pp. 1034–1037, 2011.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 157140, 4 pages
http://dx.doi.org/10.1155/2013/157140

Research Article

Affine-Periodic Solutions for Dissipative Systems

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Received 4 November 2013; Accepted 10 December 2013

Academic Editor: Bingwen Liu

As generalizations of Yoshizawa's theorem, it is proved that a dissipative affine-periodic system admits affine-periodic solutions. This result reveals some oscillation mechanism in nonlinear systems.

1. Introduction

Consider the system

$$x' = f(t, x), \quad ' = \frac{d}{dt}, \tag{1}$$

where $f : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and ensures the uniqueness of solutions with respect to initial values. Fix $T > 0$. The system (1) is said to be T-periodic if $f(t+T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R}^1 \times \mathbb{R}^n$. For this T-periodic system, a major problem is to seek the existence of T-periodic solutions. Actually, some physical systems also admit the certain affine-periodic invariance. For example, let $Q \in GL(n)$, and

$$f(t+T, x) = Qf\left(t, Q^{-1}x\right), \quad \forall (t, x). \tag{2}$$

This affine-periodic invariance exhibits two characters: periodicity in time and symmetry in space. Obviously, when $Q = id$, the invariance is just the usual periodicity; when $Q = -id$, the invariance implies the usual antisymmetry in space. When $Q \in SO(n)$, the invariance shows the rotating symmetry in space. Hence, the invariance also reflects some properties of solutions in geometry. Now, (2) is said to possess the affine-periodic structure. For this affine-periodic system, we are concerned with the existence of affine-periodic solutions $x(t)$ with

$$x(t+T) = Qx(t), \quad \forall t. \tag{3}$$

In the qualitative theory, it is a basic result that the dissipative periodic systems admit the existence of periodic solutions. The related topics had ever captured the main field in periodic solutions theory from the 1960s to the 1990s. For some litratures, see, for example, [1–12].

In the present paper, we will see whether (1) admits affine-periodic solutions or not if (1) is affine-dissipative. Here, (1) is said to be affine-dissipative if $Q^{-m}x(t+mT)$ are ultimately bounded. Our main result is the following.

Theorem 1. *Let $Q \in GL(n)$. If the system* (1) *is Q-affine-periodic, that is,*

$$f(t+T, x) = Qf\left(t, Q^{-1}x\right), \tag{4}$$

and affine-dissipative, then it admits a Q-affine-periodic solution $x_*(t)$; that is,

$$x_*(t+T) = Qx_*(t), \quad \forall t. \tag{5}$$

The paper is organized as follows. In Section 2, we use the asymptotic fixed-point theorem, for example, Horn's fixed-point theorem to prove Theorem 1. Section 3 deals with the case of functional differential equations, where an anagolous version is given and the proof is sketched. Finally, in Section 4, we illustrate some applications.

2. Proof of Theorem 1

In order to prove Theorem 1, we first recall some preliminaries.

Lemma 2 (Horn's fixed-point theorem [13]). *Let X be a Banach space, and let $S_0 \subset S_1 \subset S_2 \subset X$ be convex sets, where*

S_0 is compact, S_1 relatively open with respect to S_2, and S_2 closed. Assume that $P : S_0 \to X$ is continuous and satisfies

$$P^j(\overline{S}_1) \subset S_2, \quad j = 0, 1, \dots, N-1, \\ P^j(\overline{S}_1) \subset S_0, \quad j = N, \dots, 2N-1. \tag{6}$$

Then, P has a fixed point in S_0.

The following is a usual definition.

Definition 3. The system (1) is said to be dissipative or ultimately bounded, if there is $B_0 > 0$ and for any $B > 0$, there are $M = M(B) > 0$ and $L = L(B) > 0$ such that for $|x_0| \le B$,

$$|x(t, x_0)| \le M, \quad \forall t \in [0, L], \\ |x(t, x_0)| \le B_0, \quad \forall t \in [L, \infty), \tag{7}$$

where $x(t, x_0)$ denotes the solution of (1) with the initial value $x(0) = x_0$.

For the affine-periodic system (1), we have the following.

Definition 4. The system (1) is said to be Q-affine-dissipative, if there is $B_0 > 0$ and for any $B > 0$, there are $M = M(B) > 0$ and $L = L(B) > 0$ such that

$$|x(t, x_0)| \le M, \quad \forall t \in [0, L], \\ |Q^{-m} x(t + mT, x_0)| \le B_0, \quad \forall t \in [L, \infty),\ m \in \mathbb{Z}^1_+, \tag{8}$$

whenever $|x_0| \le B$.

Proof of Theorem 1. Define the map $P : \mathbb{R}^n \to \mathbb{R}^n$ by

$$P(x_0) = Q^{-1} x(T, x_0), \quad \forall x_0 \in \mathbb{R}^n, \tag{9}$$

and set

$$S_0 = \{y \in \mathbb{R}^n : |y| \le B_0\}, \\ S_1 = \{y \in \mathbb{R}^n : |y| < B_1\}, \\ S_2 = \{y \in \mathbb{R}^n : |y| \le B_2\}, \tag{10}$$

where

$$B_1 = B_0 + 1, \\ B_2 = \sup\{|Q^{-m} x(mT, x_0)| : m \in \{0, \dots, N\}, \\ |x_0| \le B_0 + 1\} + B_0 + 2, \\ N = [L(B_1)] + 1. \tag{11}$$

By uniqueness and the affine periodicity of $f(t, x)$, $Q^{-m}x(t + mT, x_0)$ is still the solution of (1) for each $m \in \mathbb{Z}^1_+$. Therefore,

$$P^i(x_0) = Q^{-i} x(iT, x_0), \quad i = 0, 1, \dots. \tag{12}$$

It follows from (8) that

$$P^j(\overline{S}_1) \subset S_2, \quad j = 0, \dots, N-1, \\ P^j(\overline{S}_1) \subset S_0, \quad j = N, \dots, 2N-1. \tag{13}$$

Thus, Horn's fixed-point theorem implies that P has a fixed point $\overline{x}_0$ in S_0; that is, $Q^{-1}x(T, \overline{x}_0) = \overline{x}_0$. Also, uniqueness yields

$$Q^{-1} x(t + T, \overline{x}_0) = x(t, \overline{x}_0), \\ \Longleftrightarrow x(t + T, \overline{x}_0) = Qx(t, \overline{x}_0), \quad \forall t. \tag{14}$$

This completes the proof of Theorem 1. □

3. A Version to Functional Differential Equations

Consider the functional differential equation (FDE)

$$x' = F(t, x_t), \tag{15}$$

where $F : \mathbb{R}^1 \times \mathbb{C} \to \mathbb{R}^n$ is continuous, takes any bounded set in $\mathbb{C}$ to a bounded set in $\mathbb{R}^n$, and ensures the uniqueness of solutions with respect to initial values, where $\mathbb{C} = \mathbb{C}([-r, 0], \mathbb{R}^n)$, $x_t(s) = x(t + s)$, and $s \in [-r, 0]$. Moreover, F is Q-affine-periodic; that is,

$$F(t + T, \varphi) = QF(t, Q^{-1}\varphi), \quad \forall (t, \varphi) \in \mathbb{R}^1 \times \mathbb{C}. \tag{16}$$

Definition 5. The system (15) is said to be Q-affine-dissipative; if there is $B_0 > 0$ and for any $B > 0$, there are $M = M(B) > 0$ and $L = L(B) > 0$ such that

$$|x(t, \varphi)| \le M, \quad \forall t \in [0, L], \\ |Q^{-m} x(t + mT, \varphi)| \le B_0, \quad \forall t \in [L, \infty), \tag{17}$$

whenever $\|\varphi\| = \max_{[-r,0]} |\varphi(s)| \le B$; here, $x(t, \varphi)$ denotes the solution of (15) at initial value $x_0 = \varphi$.

We are in position to state another main result.

Theorem 6. *If the system (15) is Q-affine-periodic-dissipative, then it admits a Q-affine-periodic solution $x(t)$; that is,*

$$x(t + T) = Qx(t), \quad \forall t. \tag{18}$$

Proof. Define the map $P : \mathbb{C} \to \mathbb{C}$ by

$$P(\varphi) = Q^{-1} x_T(\cdot, \varphi), \quad \forall \varphi \in \mathbb{C}, \tag{19}$$

and set

$$S_0 = \{\varphi \in \mathbb{C} : \|\varphi\| \le B_0, \\ |\varphi(s_1) - \varphi(s_2)| \le h|s_1 - s_2|,\ \forall s_1, s_2 \in [-r, 0]\}, \\ S_1 = \{\varphi \in \mathbb{C} : \|\varphi\| < B_1, \\ |\varphi(s_1) - \varphi(s_2)| < h_1|s_1 - s_2|,\ \forall s_1, s_2 \in [-r, 0]\}, \\ S_2 = \{\varphi \in \mathbb{C} : \|\varphi\| \le B_2, \\ |\varphi(s_1) - \varphi(s_2)| \le h_2|s_1 - s_2|,\ \forall s_1, s_2 \in [-r, 0]\}, \tag{20}$$

where

$$h = \sup\left\{|F(t,\varphi)| : t \in \mathbb{R}^1, \|\varphi\| \le B_0\right\},$$

$$h_1 = \sup\left\{|F(t,\varphi)| : t \in \mathbb{R}^1, \|\varphi\| \le B_0 + 1\right\}, \quad (21)$$

$$B_1 = B_0 + 1,$$

$$B_2 = \sup\{\|Q^{-m} x_{mT}(\cdot,\varphi)\| : m \in \{0,1,\dots,N\},$$
$$\|\varphi\| \le B_0 + 1\} + B_0 + 2, \quad (22)$$

$$h_2 = \sup\left\{|F(t,\varphi)| : t \in \mathbb{R}^1, \|\varphi\| \le B_2\right\},$$

where $N = [L(B_1) + r] + 2$. Then, (17) and the constructions imply that

$$P^j\left(\bar{S}_1\right) \subset S_2, \quad j = 0,\dots,N-1,$$
$$P^j\left(\bar{S}_1\right) \subset S_0, \quad j = N,\dots,2N-1. \quad (23)$$

Hence, P has a fixed point $\varphi_* \in S_0$ via Horn's theorem. The uniqueness implies that $x(t,\varphi_*)$ is the desired affine-periodic solution of (15). The proof is complete. □

4. Some Applications

First, we observe a simple example to show the meanings of affine-periodic solutions.

Example 7. Consider the equation

$$x' + 2x = e^{-t}. \quad (24)$$

Put $f(t,x) = -2x + e^{-t}$. The general solution of (24) is

$$x(t) = e^{-2t}c + e^{-t} \quad (c \text{ is any constant}). \quad (25)$$

Obviously, for given $\tau > 0$,

$$f(t+\tau, x) = e^{-\tau} f(t, e^{\tau}x), \quad (26)$$

and any solution $x(t)$ satisfies

$$\left|\left(e^{-\tau}\right)^{-m} x(t+m\tau)\right| = \left|e^{m\tau}e^{-2(t+m\tau)}c + e^{m\tau}e^{-(t+m\tau)}\right|$$
$$\le e^{-(2t+m\tau)}|c| + 1 \quad (27)$$
$$\le e^{-2t}|c| + 1,$$

which implies that (24) is $e^{-\tau}$-periodic-dissipative. By Theorem 1, (24) has an $e^{-\tau}$-affine-periodic solution. This solution is just $x(t) = e^{-t}$ and different from the usual periodic solutions!

As usual, Lyapunov's method is flexible in studying the existence of affine-periodic solutions. The following results illustrate applications in this aspect.

Theorem 8. *Assume that there exists a Lyapunov's function $V : \mathbb{R}^1_+ \times \mathbb{R}^n \to \mathbb{R}^1_+$ such that*

(i) *$V(t,x)$ is of $\mathbb{C}^1$;*

(ii) *$V'(t,x) \le -W(t,x)$, $|x| \ge M > 0$, where $W(t,x)$ is continuous in $\mathbb{R}^1_+ \times \{|x| \le M\}$, and $W(t,x) \ge \alpha > 0$, $|x| \ge M$;*

(iii) *Uniformly in t,*

$$\liminf_{|x|\to\infty} V(t,x) > \sup\left\{V(t,x) : t \in \mathbb{R}^1_+, |x| \le M\right\}. \quad (28)$$

Then, the system (1) *has a Q-affine-periodic solution.*

Proof. Let $x(t,x_0)$ denote the solution of (1) with the initial value $x(0) = x_0$. Put

$$K = \sup\left\{V(t,x) : t \in \mathbb{R}^1_+, |x| \le M\right\},$$
$$G = \left\{x \in \mathbb{R}^n : V(t,x) \le K\right\}. \quad (29)$$

By assumption (iii), G is bounded and closed. In the following, we will prove that for each $B > 0$, there are $M = M(B) > 0$ and $N = N(B) > 0$ such that

$$\left|x(t,x_0)\right| \le M, \quad \forall t \in [0,N],$$
$$x(t,x_0) \in G, \quad \forall t \ge N, \quad (30)$$

whenever $|x_0| \le B$.

In fact, given that $x_0 \in \mathbb{R}^n$, $|x_0| > M$ implies on the maximal interval $[0,L)$ that $|x(t,x_0)| > M$; we have

$$0 \le V(t, x(t,x_0)) \le V(0,x_0) - \int_0^t W(s, x(s,x_0))\,ds$$
$$\le V(0,x_0) - \alpha t. \quad (31)$$

This shows that there is $t_1 \in (0,\infty)$ such that

$$\left|x(t,x_0)\right| > M, \quad \forall t \in [0,t_1),$$
$$\left|x(t_1,x_0)\right| = M. \quad (32)$$

Note that

$$V(t, x(t,x_0)) \le V(t_1, x(t_1,x_0)), \quad \text{if } \left|x(t_1,x_0)\right| \ge M, \quad (33)$$

which together with the construction of G yields

$$x(t,x_0) \in G, \quad t \in [t_1,\infty). \quad (34)$$

If $|x_0| < M$, and there is a $\bar{t} \in (0,\infty)$ such that

$$\left|x(t,x_0)\right| < M, \quad t \in (0,\bar{t}), \quad \left|x(\bar{t},x_0)\right| = M, \quad (35)$$

then we also have

$$x(t,x_0) \in G, \quad \forall t \in [\bar{t},\infty). \quad (36)$$

Of course, in case of $|x_0| = M$, we have

$$x(t,x_0) \in G, \quad \forall t \in [0,\infty). \quad (37)$$

Taking these cases into account, we choose

$$N = t_1. \quad (38)$$

Now, the existence of affine-periodic solutions is an immediate consequence. The proof is complete. □

Theorem 9. *Assume that*

$$\langle x, f(t,x)\rangle \le -a(t)|x|^2, \tag{39}$$

where $a \in \mathrm{Loc}(\mathbb{R}^1_+)$ *satisfies*

$$\int_0^\infty a(s)\,\mathrm{d}s = \infty, \qquad \int_0^\infty a^-(s)\,\mathrm{d}s < \infty. \tag{40}$$

Then, (1) *has an affine-periodic solution.*

Proof. Let

$$V(t,x) = \frac{1}{2}|x|^2. \tag{41}$$

Then,

$$\begin{aligned} &V'(t,x) = \langle x, f(t,x)\rangle \le -2a(t)V(t,x) \\ &\Longrightarrow V(t, x(t,x_0)) \le e^{-\int_0^t 2a(s)\mathrm{d}s} V(0,x_0), \quad \forall t \ge 0. \end{aligned} \tag{42}$$

By assumption, $\int_0^\infty a(s)\mathrm{d}s = \infty$, there is $t_1 \in (0,\infty)$ such that

$$e^{-\int_0^{t_1} 2a(s)\mathrm{d}s}\frac{1}{2}|x|^2 \le 1. \tag{43}$$

Thus,

$$\begin{aligned} &V(t, x(t,x_0)) \le e^{-\int_{t_1}^\infty 2a^-(s)\mathrm{d}s}, \quad \forall t \ge t_1, \\ &\Longrightarrow |x(t,x_0)| \le \sqrt{2}e^{-\int_{t_1}^\infty a^-(s)\mathrm{d}s}, \quad \forall t \ge t_1. \end{aligned} \tag{44}$$

By Theorem 1, (1) has an affine-periodic solution. This finishes the proof. □

Example 10. Consider the system

$$x' = \pm|x|^{2\beta}x + \left(e^{\sqrt{-1}2\pi\Theta t}\right) \equiv f(t,x), \qquad (*)_\pm$$

where $\beta \ge 0$; $x \in \mathbb{C}^n$; $\Theta = (\theta_1, \theta_2, \dots, \theta_n)^T$, $\theta_i > 0$, $i = 1, 2, \dots, n$. Let

$$Q = e^{\sqrt{-1}2\pi\Theta T}, \quad T > 0. \tag{45}$$

Then

$$f(t+T, x) = Qf\left(t, Q^{-1}x\right). \tag{46}$$

In the following, we only consider the case $(*)_-$. Otherwise, set $t \to -t$ for $(*)_+$. Take $V(t,x) = (1/2)|x|^2$. Notice that for $|x| \ge \sqrt{2} = M$, $\alpha = \sqrt{2}$,

$$\begin{aligned} V'(t,x) &= \langle x, f(t,x)\rangle = x^T\overline{f}(t,x) \\ &= -|x|^{2\beta+2} + x^T e^{-\sqrt{-1}2\pi\Theta t} \\ &\le -|x|^{2\beta+2} + |x| \\ &\le -|x| = -W(t,x) \le -\alpha. \end{aligned} \tag{47}$$

Hence, by Theorem 8, $(*)_-$ has a Q-affine T-periodic solution. Now, if leting p/q be a reduced fraction and $\theta_i T = p/q$, $i = 1, 2, \dots, n$, then the Q-affine T-periodic solutions are just q-subharmonic ones; if $\Theta T \in \mathbb{Q}^n$ (the set of rational vectors), then there is a K such that these affine T-periodic solutions are K-periodic ones; if $\Theta T \in \mathbb{R}^n \setminus \mathbb{Q}^n$, then these solutions are quasiperiodic ones with frequency ΘT.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (Grants nos. 11171132 and 11201173).

References

[1] O. A. Arino, T. A. Burton, and J. R. Haddock, "Periodic solutions to functional differential equations," *Proceedings of the Royal Society of Edinburgh A*, vol. 101, pp. 253–271, 1985.

[2] T. A. Burton and S. Zhang, "Unified boundedness, periodicity, and stability in ordinary and functional differential equations," *Annali di Matematica Pura ed Applicata*, vol. 145, no. 1, pp. 129–158, 1986.

[3] J. Chu, J. Sun, and P. J. Y. Wong, "Existence for singular periodic problems: a survey of recent reaults," *Abstract and Applied Analysis*, Article ID 420835, 17 pages, 2013.

[4] H. Ergoren and C. Tunc, "Anti-periodic solutions for a class of fourth-order nonlinear differential equations with multiple deviating arguments," *Azerbaijan Journal of Mathematics*, vol. 3, no. 1, pp. 111–126, 2013.

[5] H. Gao, Y. Li, and B. Zhang, "A fixed point theorem of Krasnoselskii-Schaefer type and its applications in control and periodicity of integral equations," *Fixed Point Theory*, vol. 12, no. 1, pp. 91–112, 2011.

[6] J. K. Hale and O. Lopes, "Fixed point theorems and dissipative processes," *Journal of Differential Equations*, vol. 13, no. 2, pp. 391–402, 1973.

[7] T. Küpper, Y. Li, and B. Zhang, "Periodic solutions for dissipative-repulsive systems," *Tohoku Mathematical Journal*, vol. 52, no. 3, pp. 321–329, 2000.

[8] Y. Li, Q. D. Zhou, and X. R. Lu, "Periodic solutions and equilibrium states for functional differential inclusions with nonconvex right-hand side," *Quarterly of Applied Mathematics*, vol. 55, no. 1, pp. 57–68, 1997.

[9] G. Makay, "Periodic solutions of dissipative functional differential equations," *Tohoku Mathematical Journal*, vol. 46, pp. 417–426, 1994.

[10] T. I. Seidman and O. Klein, "Periodic solutions of isotone hybrid systems, Discrete Contin," *Dynamic Systems Service B*, vol. 18, no. 2, pp. 483–493, 2013.

[11] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan, Tokyo, Japan, 1966.

[12] Y. Yuan and Z. Guo, "On the existence and stability of periodic solutions for a nonlinear neutral functional differential equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 175479, 8 pages, 2013.

[13] W. A. Horn, "Some fixed point theorems for compact maps and flows in Banach spaces," *Transactions of the American Mathematical Society*, vol. 149, pp. 391–404, 1970.