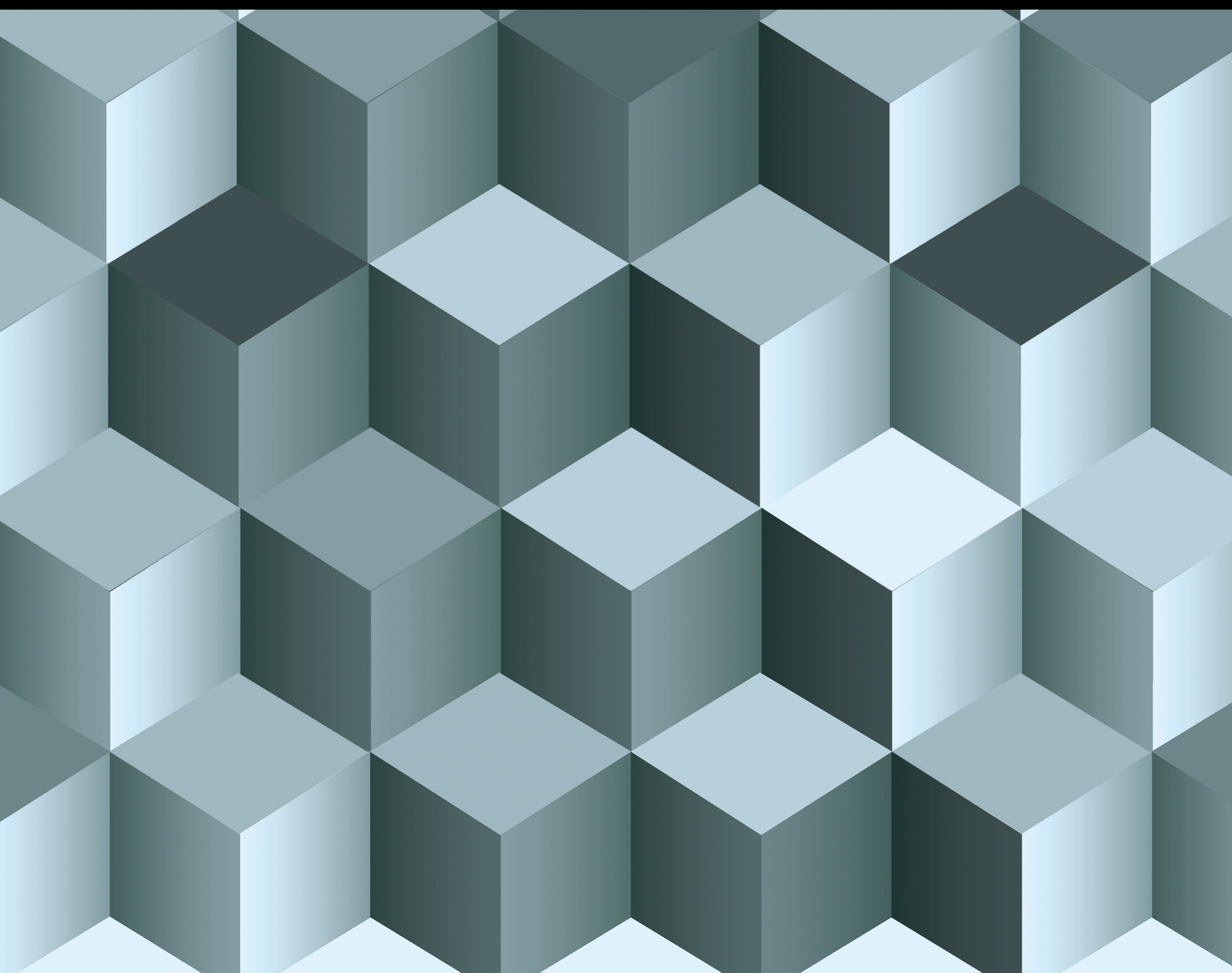
Journal of Function Spaces

Developments in Functions and Operators of Complex Variables 2022

Lead Guest Editor: Sarfraz Nawaz Malik
Guest Editors: Sibel Yalçın, Mohsan Raza, and Wasim Ul-Haq



Developments in Functions and Operators of Complex Variables 2022

Journal of Function Spaces

Developments in Functions and Operators of Complex Variables 2022

Lead Guest Editor: Sarfraz Nawaz Malik
Guest Editors: Sibel Yalçın, Mohsan Raza, and Wasim Ul-Haq

Contents

Hindawi
Journal of Function Spaces
Volume 2023, Article ID 5205867, 10 pages
https://doi.org/10.1155/2023/5205867

Research Article

Applications and Properties for Bivariate Bell-Based Frobenius-Type Eulerian Polynomials

Waseem Ahmad Khan,[1] **Maryam Salem Alatawi**,[2] **and Ugur Duran**[3]

[1]*Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia*
[3]*Department of Basic Sciences of Engineering, Faculty of Engineering and Natural Sciences, İskenderun Technical University, Hatay 31200, Turkey*

Correspondence should be addressed to Waseem Ahmad Khan; wkhan1@pmu.edu.sa
and Maryam Salem Alatawi; msoalatawi@ut.edu.sa

Received 2 September 2022; Accepted 15 October 2022; Published 25 August 2023

Academic Editor: Sarfraz Nawaz Malik

In this study, we introduce sine and cosine Bell-based Frobenius-type Eulerian polynomials, and by presenting several relations and applications, we analyze certain properties. Our first step is to obtain diverse relations and formulas that cover summation formulas, addition formulas, relations with earlier polynomials in the literature, and differentiation rules. Finally, after determining the first few zero values of the Eulerian polynomials, we draw graphical representations of these zero values.

1. Introduction

In recent times, the use of sine and cosine polynomials has led to the definition and construction of generating functions for new families of special polynomials, such as Bernoulli, Euler, and Genocchi; see [1–4]. Fundamental properties and diverse applications for these polynomials have been provided by these types of studies. For instance, not only various implicit and explicit summation formulas, differentiation-integration formulas, symmetric identities, and a lot of relationships with the well-known polynomials have been deeply investigated but also graphical representations of the zero values of these polynomials are drawn after determining them. Moreover, the aforementioned polynomials allow us to investigate worthwhile properties from a very basic procedure and assist to define novel types of special polynomials. Motivated by the above, in this paper, we define the cosine and sine Bell-based Frobenius-type Eulerian polynomials and examine several properties and applications. Our first step is to obtain diverse relations and formulas that cover summation formulas, addition formulas, relations with earlier polynomials in the literature, and differentiation rules. Finally, after determining the first few zero values of the Eulerian polynomials, we draw graphical representations of these zero values.

Let $\xi \in \mathbb{R}$ denotes the set of all real numbers and $\lambda \in \mathbb{C}$ denotes the set of all complex numbers with $\lambda \neq 1$. The Frobenius-type Eulerian polynomial of order $\alpha \in \mathbb{C}$ is introduced as follows (see [5–7]):

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{\xi z} = \sum_{j=0}^{\infty} \mathbb{A}_j^{(\alpha)}(\xi|\lambda)\frac{z^j}{j!}, \left|\frac{\log \lambda}{\lambda-1}\right| > |z|. \tag{1}$$

The Frobenius-type Eulerian polynomials have worked by many mathematicians; see [6–11].

Upon setting $\xi = 0$, $\mathbb{A}_j^{(\alpha)}(\lambda) = \mathbb{A}_j^{(\alpha)}(0|\lambda)$ are termed the Frobenius-type Eulerian numbers of order α. In view of (1), it can be readily observed that

$$\begin{aligned}\mathbb{A}_j^{(\alpha)}(\xi|\lambda) &= \sum_{\nu=0}^{j}\binom{j}{\nu}\mathbb{A}_\nu^{(\alpha)}(\lambda)\xi^{j-\nu},\\ \mathbb{A}_j^{(\alpha)}(\xi|\lambda) &= (\lambda-1)^j \mathbb{H}_j^{(\alpha)}\left(\frac{\xi}{\lambda-1}\middle|\lambda\right),\end{aligned} \tag{2}$$

where $\mathbb{H}_j^{(\alpha)}(\xi|\lambda)$ are the Frobenius-Euler polynomials of order α (cf. [11, 12]).

The Stirling numbers of the first kind are introduced for $j \geq 0$ as follows (cf. [13–15]):

$$(\xi)_j = \sum_{p=0}^{j} S_1(j, p)\xi^p, \tag{3}$$

where $(\xi)_0 := 1$ and $(\xi)_j := (\xi - j + 1)(\xi - j + 2) \cdots (\xi - 1)\xi$, $(j \geq 1)$. By (3), we acquire that (see [14, 16, 17])

$$\frac{1}{r!}(\log(1 + z))^r = \sum_{j=r}^{\infty} S_1(j, r)\frac{z^j}{j!}, (r \geq 0). \tag{4}$$

The Stirling numbers of the second kind are given for $j \geq 0$ as follows (see [5, 18]):

$$\xi^j = \sum_{q=0}^{j} S_2(j, q)(\xi)_q. \tag{5}$$

In terms of (5), it is easily shown that

$$\sum_{j=r}^{\infty} S_2(j, r)\frac{z^j}{j!} = \frac{(e^z - 1)^r}{r!}. \tag{6}$$

For any nonnegative integer q, the q-Stirling numbers $S_q(j, k)$ of the second kind are introduced as follows (see [19]):

$$\frac{1}{k!}e^{qz}(e^z - 1)^k = \sum_{j=k}^{\infty} S_2(j + q, k + q)\frac{z^j}{j!}. \tag{7}$$

Let q be any nonnegative integer. The Bell-based Stirling polynomials of the second kind are provided as follows (see [13]):

$$\frac{1}{k!}e^{q(e^z-1)}(e^z - 1)^k = \sum_{j=k}^{\infty} {}_{Bel}S_2(j, k : q)\frac{z^j}{j!}. \tag{8}$$

The Apostol types of the Bernoulli $\mathbb{B}_j^{(\alpha)}(\xi|\lambda)$, the Euler $\mathbb{E}_j^{(\alpha)}(\xi|\lambda)$, and the Genocchi polynomials $\mathbb{G}_j^{(\alpha)}(\xi|\lambda)$ of order α are introduced as follows (cf. [11, 17, 20]):

$$\begin{aligned}
\left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} e^{\xi z} &= \sum_{j=0}^{\infty} \mathbb{B}_j^{(\alpha)}(\xi|\lambda)\frac{z^j}{j!} (|z + \log \lambda| < 2\pi), \\
\left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} e^{\xi z} &= \sum_{j=0}^{\infty} \mathbb{E}_j^{(\alpha)}(\xi|\lambda)\frac{z^j}{j!} (|z + \log \lambda| < \pi), \\
\left(\frac{2z}{\lambda e^z + 1}\right)^{\alpha} e^{\xi z} &= \sum_{j=0}^{\infty} \mathbb{G}_j^{(\alpha)}(\xi|\lambda)\frac{z^j}{j!}, (|z + \log \lambda| < \pi).
\end{aligned} \tag{9}$$

Also, their corresponding numbers are determined by

$$\mathbb{B}_j^{(\alpha)}(0|\lambda) := \mathbb{B}_j^{(\alpha)}(\lambda), \mathbb{E}_j^{(\alpha)}(0|\lambda) := \mathbb{E}_j^{(\alpha)}(\lambda), \mathbb{G}_j^{(\alpha)}(0|\lambda) := \mathbb{G}_j^{(\alpha)}(\lambda), \tag{10}$$

respectively. In addition, their familiar polynomials and numbers are determined by just choosing $\lambda = \alpha = 1$ in their definitions and shown by $B_j(\xi)$ and $E_j(\xi)$.

The Bell polynomials $\mathrm{Bel}_j(\xi)$ are introduced as follows (see [18, 21–25]):

$$e^{\xi(e^z-1)} = \sum_{j=0}^{\infty} \mathrm{Bel}_j(\xi)\frac{z^j}{j!}. \tag{11}$$

Also, the corresponding Bell numbers are determined by $\mathrm{Bel}_j(1) := \mathrm{Bel}_j$, $(j \geq 0)$. In terms of (6) and (11), it is seen that

$$\mathrm{Bel}_j(\xi) = \sum_{k=0}^{j} S_2(j, k)\xi^k (j \geq 0). \tag{12}$$

In recent years, Duran et al. [13] considered the Bell-based Bernoulli polynomials of order α ${}_{Bel}\mathbb{B}_j^{(\alpha)}(\xi; \eta)$ given by

$$\sum_{j=0}^{\infty} {}_{Bel}\mathbb{B}_j^{(\alpha)}(\xi; \eta)\frac{z^j}{j!} = e^{\xi z + \eta(e^z-1)}\left(\frac{z}{e^z - 1}\right)^{\alpha}, \tag{13}$$

which also provides that

$${}_{Bel}\mathbb{B}_j^{(\alpha)}(\xi; \eta) = \sum_{r=0}^{j} \binom{j}{r} \mathbb{B}_{j-r}^{(\alpha)}(\xi)\mathrm{Bel}_r(\eta). \tag{14}$$

Also, in [13], the authors proved several properties and relations for the aforesaid polynomials. In addition, they gave many quirky formulas arising from the theory of umbral calculus.

Kim et al. [3] and Jamei et al. [1] considered the Bernoulli polynomials and the Euler polynomials based on the cosine and sine polynomials as follows:

$$\frac{1}{2}\sum_{j=0}^{\infty}(\mathbb{B}_j(\xi + i\eta) + \mathbb{B}_j(\xi - i\eta))\frac{z^j}{j!} = \sum_{j=0}^{\infty} \mathbb{B}_j^{(c)}(\xi, \eta)\frac{z^j}{j!} = e^{\xi z} \cos \eta z \frac{z}{e^z - 1}, \tag{15}$$

$$\frac{1}{2i}\sum_{j=0}^{\infty}(\mathbb{B}_j(\xi + i\eta) - \mathbb{B}_j(\xi - i\eta))\frac{z^j}{j!} = \sum_{j=0}^{\infty} \mathbb{B}_j^{(s)}(\xi, \eta)\frac{z^j}{j!} = e^{\xi z} \sin \eta z \frac{z}{e^z - 1}, \tag{16}$$

$$\frac{1}{2}\sum_{j=0}^{\infty}(\mathbb{E}_j(\xi + i\eta) + \mathbb{E}_j(\xi - i\eta))\frac{z^j}{j!} = \sum_{j=0}^{\infty} \mathbb{E}_j^{(c)}(\xi, \eta)\frac{z^j}{j!} = e^{\xi z} \cos \eta z \frac{2}{e^z + 1}, \tag{17}$$

$$\frac{1}{2i}\sum_{j=0}^{\infty}\left(\mathbb{E}_j(\xi+i\eta)-\mathbb{E}_j(\xi-i\eta)\right)\frac{z^j}{j!}=\sum_{j=0}^{\infty}\mathbb{E}_j^{(s)}(\xi,\eta)\frac{z^j}{j!}=e^{\xi z}\sin\eta z\frac{2}{e^z+1}, \tag{18}$$

respectively. In addition, they investigated many relations for the polynomials given above.

The trigonometric polynomials, cosine, and sine polynomials are introduced as follows (see [2–4, 7]):

$$\begin{aligned} e^{\xi z}\cos\eta z &= \sum_{r=0}^{\infty}C_r(\xi,\eta)\frac{z^r}{r!},\\ e^{\xi z}\sin\eta z &= \sum_{r=0}^{\infty}S_r(\xi,\eta)\frac{z^r}{r!}, \end{aligned} \tag{19}$$

which satisfy the following expansion formulas:

$$\begin{aligned} C_r(\xi,\eta) &= \sum_{j=0}^{\lfloor \frac{r}{2}\rfloor}(-1)^j\binom{r}{2j}\xi^{r-2j}\eta^{2j},\\ S_r(\xi,\eta) &= \sum_{j=0}^{\lfloor \frac{r-1}{2}\rfloor}\binom{r}{2j+1}(-1)^j\xi^{r-2j-1}\eta^{2j+1}, \end{aligned} \tag{20}$$

where the value of $\lfloor y\rfloor$ is the largest integer that is equal or less than y.

2. Cosine and Sine Bell-Based Frobenius-Type Eulerian Polynomials

Here, we introduce the cosine and sine Bell-based Frobenius-type Eulerian numbers and polynomials, and then we derive several properties and identities for the above polynomials.

Motivated and inspired by the definitions (13) and (15)–(18), we first consider the Bell-based Frobenius-type Eulerian polynomials defined as follows:

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{z\xi}e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi,\zeta|\lambda)\frac{z^j}{j!}. \tag{21}$$

By (21) and the following well-known formula

$$e^{i\eta z}=(\cos\eta z+i\sin\eta z), \tag{22}$$

thus, we have

$$\begin{aligned} \sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi+i\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{(\xi+i\eta)z}e^{\zeta(e^z-1)}\\ &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\zeta(e^z-1)e^{\xi z}(\cos\eta z+i\sin\eta z)}, \end{aligned} \tag{23}$$

$$\begin{aligned} \sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi-i\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{(\xi-i\eta)z}e^{\zeta(e^z-1)}\\ &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}(\cos\eta z-i\sin\eta z)e^{\xi z}e^{\zeta(e^z-1)}. \end{aligned} \tag{24}$$

From (23) and (24), we get

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\cos\eta z e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}\left(\frac{{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi+i\eta,\zeta\mid\lambda)+{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi-i\eta,\zeta\mid\lambda)}{2}\right)\frac{z^j}{j!}, \tag{25}$$

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\sin\eta z e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}\left(\frac{{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi+i\eta,\zeta\mid\lambda)-{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi-i\eta,\zeta\mid\lambda)}{2i}\right)\frac{z^j}{j!}. \tag{26}$$

Hence, here is our definition.

Definition 1. We consider the cosine and sine Bell-based Frobenius-type Eulerian polynomials of order α, for nonnegative integer j, as follows:

$$\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\cos\eta z e^{\zeta(e^z-1)}, \tag{27}$$

$$\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\sin\eta z e^{\zeta(e^z-1)}, \tag{28}$$

respectively.

Note that ${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,0,0|\lambda)\coloneqq\mathbb{A}_j^{(\alpha)}(\xi|\lambda)$ and ${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,0,0|\lambda)=0(j\geq 0)$.

From (25)–(28), we have

$$\begin{aligned} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) &= \frac{1}{2}\left({}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi+i\eta,\zeta|\lambda)+{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi-i\eta,\zeta|\lambda)\right),\\ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) &= \frac{1}{2i}\left({}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi+i\eta,\zeta|\lambda)-{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi-i\eta,\zeta|\lambda)\right). \end{aligned} \tag{29}$$

Remark 2. For $\zeta=\xi=0$ in (27) and (28), we get novel type of the polynomials $\mathbb{A}_j^{(\alpha,c)}(\eta|\lambda)$ and $\mathbb{A}_j^{(\alpha,s)}(\eta|\lambda)$ as

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}\cos\eta z=\sum_{j=0}^{\infty}\mathbb{A}_j^{(\alpha,c)}(\eta|\lambda)\frac{z^j}{j!}, \tag{30}$$

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}\sin\eta z=\sum_{j=0}^{\infty}\mathbb{A}_j^{(\alpha,s)}(\eta|\lambda)\frac{z^j}{j!}. \tag{31}$$

It is readily observed that (for $j\geq 0$)

$$\mathbb{A}_j^{(\alpha,c)}(0|\lambda)=\mathbb{A}_j^{(\alpha,c)}(\lambda)\text{ and }\mathbb{A}_j^{(\alpha,s)}(0|\lambda)=0. \tag{32}$$

Remark 3. Putting $\zeta=0$ in (27) and (28), we attain cosine $\mathbb{A}_j^{(\alpha,c)}(\xi,\eta|\lambda)$ and sine $\mathbb{A}_j^{(\alpha,s)}(\xi,\eta|\lambda)$ Frobenius-type Eulerian polynomials:

$$\sum_{j=0}^{\infty}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta|\lambda)\frac{z^j}{j!}=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\cos\eta z,$$
$$\sum_{j=0}^{\infty}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta|\lambda)\frac{z^j}{j!}=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\sin\eta z, \tag{33}$$

respectively.

Remark 4. Letting $\xi=0$ in (27) and (28), we have novel kind cosine and sine Bell-based Frobenius-type Eulerian polynomials as

$$\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\eta,\zeta|\lambda)\frac{z^j}{j!}=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}\cos\eta z e^{\zeta(e^z-1)},$$
$$\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\eta,\zeta|\lambda)\frac{z^j}{j!}=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}\sin\eta z e^{\zeta(e^z-1)}, \tag{34}$$

respectively.

Remark 5. On setting $\xi=\eta=0$ in (27) and (28), we attain the Bell-based Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_j^{(\alpha)}(\zeta|\lambda)$ as

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha)}(\zeta|\lambda)\frac{z^j}{j!}. \tag{35}$$

Theorem 6. *Let $j\geq 0$. We acquire that*

$${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)=\sum_{\nu=0}^{[j/2]}\binom{j}{2\nu}(-1)^{\nu}\eta^{2\nu}{}_{Bel}\mathbb{A}_{j-2\nu}^{(\alpha)}(\xi,\zeta|\lambda), \tag{36}$$

$${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)=\sum_{\nu=0}^{[(j-1)/2]}\binom{j}{2\nu+1}(-1)^{\nu}\eta^{2\nu+1}{}_{Bel}\mathbb{A}_{j-2\nu-1}^{(\alpha)}(\xi,\zeta|\lambda). \tag{37}$$

Proof. It is seen from (30) and (31) that

$$\begin{aligned}\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}&=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\cos\eta z e^{\zeta(e^z-1)}\\&=\sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{A}_j^{(\alpha)}(\xi,\zeta|\lambda)\frac{z^j}{j!}\sum_{\nu=0}^{\infty}(-1)^{\nu}\eta^{2\nu}\frac{z^{\nu}}{(2\nu)!}\\&=\sum_{j=0}^{\infty}\left(\sum_{\nu=0}^{[j/2]}(-1)^{\nu}\eta^{2\nu}\binom{j}{2\nu}{}_{\mathrm{Bel}}\mathbb{A}_{j-2\nu}^{(\alpha)}(\xi,\zeta|\lambda)\right)\frac{z^j}{j!},\end{aligned} \tag{38}$$

$$\begin{aligned}\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}&=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\sin\eta z e^{\zeta(e^z-1)}\\&=\sum_{j=0}^{\infty}\left(\sum_{\nu=0}^{[(j-1)/2]}\binom{j}{2\nu+1}(-1)^{\nu}\eta^{2\nu+1}{}_{Bel}\mathbb{A}_{j-2\nu-1}^{(\alpha)}(\xi,\zeta|\lambda)\right)\frac{z^j}{j!}.\end{aligned} \tag{39}$$

We acquire the asserted results (36) and (37) in accordance with (38) and (39). □

Theorem 7. *Let $j\geq 0$. We acquire that*

$${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta+x|\lambda)=\sum_{k=0}^{j}{}_{Bel}\mathbb{A}_k^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)Bel_{j-k}(x), \tag{40}$$

$${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta+x|\lambda)=\sum_{k=0}^{j}{}_{Bel}\mathbb{A}_k^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)Bel_{j-k}(x). \tag{41}$$

Proof. By using (11), (23), and (24), we can readily derive (40) and (41) by utilizing series methods. Therefore, we exclude the proofs. □

Theorem 8. *For $j\geq 0$, we have*

$${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)=\sum_{u=0}^{j}\binom{j}{u}{}_{Bel}\mathbb{A}_{j-u}^{(\alpha)}(\zeta|\lambda)C_u(\xi,\eta), \tag{42}$$

$${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)=\sum_{u=0}^{j}\binom{j}{u}{}_{Bel}\mathbb{A}_{j-u}^{(\alpha)}(\zeta|\lambda)S_u(\xi,\eta). \tag{43}$$

Proof. Utilizing the Cauchy product rule

$$\left(\sum_{j=0}^{\infty}a_j\frac{z^j}{j!}\right)\left(\sum_{\nu=0}^{\infty}b_{\nu}\frac{z^{\nu}}{\nu!}\right)=\sum_{j=0}^{\infty}\left(\sum_{\nu=0}^{j}\binom{j}{\nu}a_{j-\nu}b_{\nu}\right)\frac{z^j}{j!}, \tag{44}$$

we investigate

$$\begin{aligned}\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}&=e^{\xi z}\cos\eta z e^{\zeta(e^z-1)}\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}\\&=\left(\sum_{\nu=0}^{\infty}C_{\nu}(\xi,\eta)\frac{z^{\nu}}{\nu!}\right)\left(\sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{A}_j^{(\alpha)}(\zeta|\lambda)\frac{z^j}{j!}\right)\\&=\sum_{j=0}^{\infty}\left(\sum_{h=0}^{j}\binom{j}{h}{}_{\mathrm{Bel}}\mathbb{A}_{j-h}^{(\alpha)}(\zeta|\lambda)C_h(\xi,\eta)\right)\frac{z^j}{j!},\end{aligned} \tag{45}$$

which implies (42). The other proof (43) can be done similarly. □

Theorem 9. *For $j \geq 0$, we attain*

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = \sum_{k=0}^{j} Bel_{j-k}(\zeta)\mathbb{A}_k^{(\alpha,c)}(\xi,\eta|\lambda), \tag{46} $$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \sum_{k=0}^{j} Bel_{j-k}(\zeta)\mathbb{A}_k^{(\alpha,s)}(\xi,\eta|\lambda), \tag{47} $$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = \sum_{k=0}^{j} \xi^{j-k}{}_{Bel}\mathbb{A}_k^{(\alpha,c)}(\eta,\zeta|\lambda), \tag{48} $$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \sum_{k=0}^{j} \xi^{j-k}{}_{Bel}\mathbb{A}_k^{(\alpha,s)}(\eta,\zeta|\lambda). \tag{49} $$

Proof. Using (27) and (28), the proofs of (46)–(49) can be shown similarly to the proofs of the above theorems. Therefore, we exclude the proof. □

Theorem 10. *For $j \geq 0$, we have*

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(s+\xi,\eta,\zeta|\lambda) = \sum_{u=0}^{j} \binom{j}{u} {}_{Bel}\mathbb{A}_u^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)s^{j-u}, \tag{50} $$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi+s,\eta,\zeta|\lambda) = \sum_{u=0}^{j} \binom{j}{u} {}_{Bel}\mathbb{A}_u^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)s^{j-u}. \tag{51} $$

Proof. By (27), we attain

$$ \begin{aligned} \sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi+s,\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{\xi z} e^{\zeta(e^z-1)} \cos \eta z e^{sz} \\ &= \left(\sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}\right)\left(\sum_{u=0}^{\infty} s^u \frac{z^u}{u!}\right) \\ &= \sum_{j=0}^{\infty}\left(\sum_{u=0}^{j}\binom{j}{u} {}_{\mathrm{Bel}}\mathbb{A}_u^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)s^{j-u}\right)\frac{z^j}{j!}, \end{aligned} \tag{52} $$

which completes the proof (50). The result (51) can be done similarly. □

Theorem 11. *For $j \geq 0$, we have*

$$ \frac{\partial}{\partial \xi}{}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = j_{Bel}\mathbb{A}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta|\lambda), \tag{53} $$

$$ \frac{\partial}{\partial \eta}{}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = -j_{Bel}\mathbb{A}_{j-1}^{(\alpha,s)}(\xi,\eta,\zeta|\lambda), \tag{54} $$

$$ \frac{\partial}{\partial \xi}{}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = j_{Bel}\mathbb{A}_{j-1}^{(\alpha,s)}(\xi,\eta,\zeta|\lambda), \tag{55} $$

$$ \frac{\partial}{\partial \eta}{}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = j_{Bel}\mathbb{A}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta|\lambda). \tag{56} $$

Proof. By means of (27), we compute that

$$ \begin{aligned} \sum_{j=1}^{\infty}\frac{\partial}{\partial \xi}{}_{\mathrm{Bel}}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} z e^{\xi z} \cos \eta z e^{\zeta(e^z-1)} \\ &= \sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^{j+1}}{j!} \\ &= \sum_{j=1}^{\infty} {}_{Bel}\mathbb{A}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{(j-1)!} \\ &= \sum_{j=1}^{\infty} j_{\mathrm{Bel}}\mathbb{A}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}, \end{aligned} \tag{57} $$

which means (53). The formulas (54), (55), and (56) can be derived similarly. □

Theorem 12. *For $j \geq 0$, we attain*

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = \sum_{u=0}^{j}\sum_{k=0}^{u}\binom{j}{u} {}_{Bel}\mathbb{A}_{j-u}^{(\alpha,c)}(\eta,\zeta|\lambda)(\xi)_k S_2(u,k), \tag{58} $$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \sum_{u=0}^{j}\sum_{k=0}^{u}\binom{j}{u} {}_{Bel}\mathbb{A}_{j-u}^{(\alpha,s)}(\eta,\zeta|\lambda)(\xi)_k S_2(u,k). \tag{59} $$

Proof. Using (6) and (27), we find

$$ \begin{aligned} \sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} (e^z-1+1)^{\xi} \cos \eta z e^{\zeta(e^z-1)} \\ &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} \cos \eta z e^{\zeta(e^z-1)} \sum_{k=0}^{\infty}(\xi)_k \frac{(e^z-1)^k}{k!} \\ &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} \cos \eta z e^{\zeta(e^z-1)} \sum_{k=0}^{\infty}(\xi)_k \sum_{u=k}^{\infty} S_2(u,k)\frac{z^u}{u!} \\ &= \sum_{j=0}^{\infty} {}_{\mathrm{Bel}}\mathbb{A}_j^{(\alpha,c)}(\eta;\zeta|\lambda)\frac{z^j}{j!}\sum_{u=0}^{\infty}\sum_{k=0}^{u}(\xi)_k S_2(u,k)\frac{z^u}{u!} \\ &= \sum_{j=0}^{\infty}\sum_{u=0}^{j}\sum_{k=0}^{u}\binom{j}{u} {}_{\mathrm{Bel}}\mathbb{A}_{j-u}^{(\alpha,c)}(\eta,\zeta|\lambda)(\xi)_k S_2(u,k)\frac{z^j}{j!}. \end{aligned} \tag{60} $$

In view of (27) and (60), we attain the claimed result (58). Also, we can easily obtain (59) in a similar way. □

We give a relation with the Bell-Stirling polynomials of the second kind as follows.

Table 1: Real and complex zeros of ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi, 6, 5\,; 2)$.

Degree	Real zeros	Complex zeros
1	-9	—
2	-13.79583, -4.20417	—
3	-17.50929, -8.57862, -0.91208	—
4	-21.37817, 2.60420	−8.61301 + 2.46740 *i*, −8.61301 − 2.46740 *i*
5	-25.2312, -10.1799, 6.23115	−7.91002 − 4.42772 *i*, −7.91002 + 4.42772 *i*
6	-29.0774, 9.91421	−10.306 − 1.41814 *i*, -10.306 + 1.41814 *i*, −7.11246 − 6.21942 *i*, −7.11246 + 6.21942 *i*
7	-32.918, -12.0177, 13.6309	−9.56347 − 3.17179 *i*, −9.56347 + 3.17179 *i*, −6.2841 − 7.82954 *i*, −6.2841 + 7.82954 *i*
8	-36.7547, -12.8049, -11.0126, 17.3697	−8.97353 − 4.95183 *i*, -8.97353 + 4.95183 *i*, −5.42516 − 9.29329 *i*, -5.42516 + 9.29329 *i*,
9	-40.5884, -14.3645, 21.1239	−10.6868 − 2.38861 *i*, -10.6868 + 2.38861 *i*, −8.35989 − 6.57959 *i*, -8.35989 + 6.57959 *i*, −4.53883 − 10.6389 *i*, -4.53883 + 10.6389 *i*
10	-44.4198, -15.6243, -11.5504, 4.8894	−10.3149 − 4.14379 *i*, -10.3149 + 4.14379 *i*, -7.70471 − 8.09496 *i*, -7.70471 + 8.09496 *i*, −3.62785 − 11.8863 *i*, −3.62785 + 11.8863 *i*
11	-48.24939, -16.94854, 28.66337	−11.660956 − 1.92837 *i*, −11.660956 + 1.92837 *i*, −9.86402 − 5.74486 *i*, −9.86402 + 5.74486 *i*, −7.01359 − 9.51912 *i*, −7.01359 + 9.51912 *i*, −2.6941 − 13.0504 *i*, −2.6941 + 13.0504 *i*
12	-52.0776, -18.24958, -12.42031, 32.444	−11.46025 − 3.57703 *i*, −11.46025 + 3.57703 *i*, 5694 − 7.25491 *i*, −9.35694 + 7.25491 *i*, −6.292007 − 10.86579 *i*, −6.292007 + 10.86579 *i*, −1.73903 − 14.14248 *i*, −1.73903 + 14.14248 *i*
13	-55.90467, -19.5518, 36.22987	−12.64336 − 1.62677 *i*, −12.64336 + 1.62677 *i*, −11.12894 − 5.13136 *i*, −11.12894 + 5.13136 *i*, −8.80659 − 8.69073 *i*, −8.80659 + 8.69073 *i*, −5.54426 − 12.14505 *i*, −5.54426 + 12.14505 *i*, −0.76349 − 15.17179 *i*, −0.76349 + 15.17179 *i*
14	-59.7308, -20.84998, -13.40199, 40.0199	−12.52359 − 3.14914 *i*, −12.52359 + 3.14914 *i*, −10.73302 − 6.62043 *i*, −10.73302 + 6.62043 *i*, −8.2199 − 10.0621 *i*, −8.2199 − 10.0621 *i*, −4.77378 − 13.36466 *i*, −4.77378 + 13.36466 *i*, 0.23173 − 16.145982 *i*, 0.23173 + 16.145982 *i*
15	-63.55622, -22.14583, 43.81356 −3.98323 − 14.53071 *i*, -3.98323 + 14.53071 *i*,	−13.65616 − 1.38254 *i*, −13.65616 + 1.38254 *i*, −12.27016 − 4.65428 *i*, −12.27016 + 4.65428 *i*, −10.29033 − 8.04947 *i*, −10.29033 + 8.04947 *i*, −7.60183 − 11.37668 *i*, −7.60183 + 11.37668 *i* 1.24599 − 17.07154 *i*, 1.24599 + 17.07154 *i*
16	-67.381, -23.4394, -14.4645, 47.61	−13.5438 − 2.79653 *i*, −13.5438 + 2.79653 *i*, −11.9582 − 6.11878 *i*, −11.9582 + 6.11878 *i*, −9.80817 − 9.42427 *i*, −9.80817 + 9.42427 *i*, −6.95627 − 12.6403 *i*, −6.95627 + 12.6403 *i*, −3.17465 − 15.6481 *i*, −3.17465 + 15.6481 *i*, 2.27862 − 17.9541 *i*, 2.27862 + 17.9541 *i*

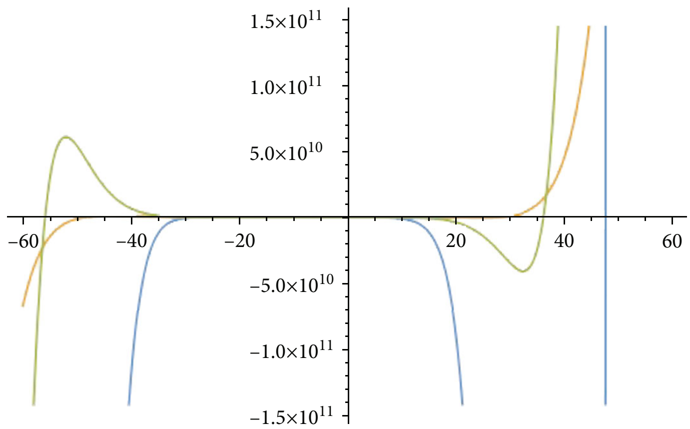


FIGURE 1: Parametric cosine Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi,6,5\,;2)$ for $j=11$ (orange),13 (green), and16 (blue).

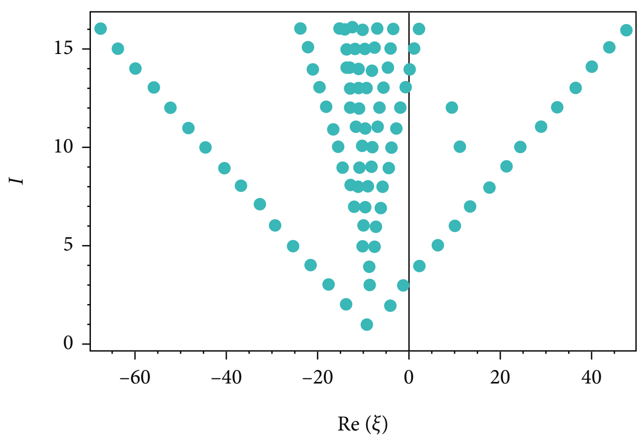


FIGURE 2: Structure of real zeros of ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi,6,5\,;2)$, $j\le 1\le 16$.

Theorem 13. *For $j\ge 0$, we attain*

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)=\sum_{u=0}^{j}\sum_{k=0}^{u}\binom{j}{u}\mathbb{A}_{j-u}^{(\alpha,c)}(\eta|\lambda)\,{}_{Bel}S_2(u,k:\zeta)(\xi)_k, \tag{61}$$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)=\sum_{u=0}^{j}\sum_{k=0}^{u}\binom{j}{u}\mathbb{A}_{j-u}^{(\alpha,s)}(\eta|\lambda)\,{}_{Bel}S_2(u,k:\zeta)(\xi)_k. \tag{62}$$

Proof. Using (8), (27), and (28), the proofs of (61) and (62) can be shown similar to the proofs of Theorem 12. So, we omit the proofs. □

3. Some Values with Graphical Representations and Zeros of Sine and Cosine Bell-Based Frobenius-Type Eulerian Polynomials

Here, we indicate the first few sine and cosine Bell-based Frobenius-type Eulerian polynomials with beautiful graphical representations and examine some zero values of these polynomials ${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)$ and ${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)$.

It is not difficult to check that the first five parametric kinds of ${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)$ are

$$ {}_{Bel}\mathbb{A}_0^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)=1,$$

$$ {}_{Bel}\mathbb{A}_1^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)=\xi+\zeta+\alpha,$$

$$\begin{aligned} {}_{Bel}\mathbb{A}_2^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)&=\frac{\zeta}{2}+\frac{\zeta^2}{2}+\alpha\zeta+\zeta\xi+\frac{1}{2}\xi^2\\ &\quad+\alpha\xi+\frac{1}{2}\lambda\alpha+\frac{1}{2}\alpha^2-\frac{1}{2}\eta^2,\end{aligned}$$

$$\begin{aligned} {}_{Bel}\mathbb{A}_3^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)&=\frac{1}{6}\zeta+\frac{1}{2}\xi\alpha^2-\frac{1}{2}\eta^2\xi+\frac{1}{2}\zeta^2\xi+\frac{1}{2}\zeta\xi^2+\frac{1}{2}\zeta\xi\\ &\quad+\frac{1}{2}\xi^2\alpha+\frac{1}{2}\xi\alpha\lambda+\alpha\zeta\xi+\frac{1}{6}\xi^3+\frac{1}{2}\alpha\zeta^2\\ &\quad+\frac{1}{2}\alpha^2\zeta-\frac{1}{2}\zeta\eta^2+\frac{1}{2}\zeta\alpha+\frac{1}{2}\alpha\zeta\lambda+\frac{1}{6}\alpha^3\\ &\quad+\frac{1}{6}\alpha\lambda^2+\frac{1}{2}\alpha^2\lambda+\frac{1}{6}\lambda\alpha-\frac{1}{2}\eta^2\alpha+\frac{1}{6}\zeta^3+\frac{1}{2}\zeta^2,\end{aligned}$$

$$\begin{aligned} {}_{Bel}\mathbb{A}_4^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)&=\frac{1}{24}\zeta+\frac{1}{24}\xi^4+\frac{1}{6}\zeta\xi+\frac{1}{2}\zeta^2\xi+\frac{1}{4}\zeta\xi^2+\frac{1}{4}\xi^2\alpha^2\\ &\quad+\frac{1}{6}\zeta^3\xi+\frac{1}{4}\zeta^2\xi^2+\frac{1}{6}\zeta\xi^3+\frac{1}{6}\alpha^3\xi-\frac{1}{4}\eta^2\xi^2\\ &\quad+\frac{1}{6}\xi^3\alpha+\frac{1}{6}\alpha\lambda^2\xi+\frac{1}{2}\alpha^2\lambda\xi-\frac{1}{2}\zeta\eta^2\xi+\frac{1}{2}\alpha\zeta\xi^2\\ &\quad+\frac{1}{2}\alpha\zeta^2\xi+\frac{1}{2}\alpha^2\zeta\xi+\frac{1}{4}\xi^2\alpha\lambda-\frac{1}{2}\alpha\eta^2\xi+\frac{1}{6}\xi\alpha\lambda\\ &\quad+\frac{1}{2}\alpha\zeta\xi+\frac{1}{4}\alpha\zeta^2\lambda+\frac{1}{2}\alpha^2\zeta\lambda+\frac{1}{6}\alpha\zeta\lambda^2-\frac{1}{2}\alpha\zeta\eta^2\\ &\quad-\frac{1}{4}\alpha\lambda\eta^2+\frac{1}{6}\alpha\zeta^3+\frac{1}{4}\alpha^2\zeta^2-\frac{1}{4}\zeta^2\eta^2+\frac{1}{6}\alpha^3\zeta\\ &\quad+\frac{1}{24}\alpha\lambda^3+\frac{7}{24}\alpha^2\lambda^2+\frac{1}{4}\alpha^3\lambda-\frac{1}{4}\alpha^2\eta^2+\frac{1}{2}\alpha\zeta^2\\ &\quad+\frac{1}{4}\alpha^2\zeta-\frac{1}{4}\zeta\eta^2+\frac{1}{6}\zeta\alpha+\frac{1}{2}\alpha\zeta\lambda+\frac{5}{12}\alpha\zeta\lambda\\ &\quad+\frac{1}{24}\alpha^4+\frac{1}{24}\zeta^4+\frac{1}{6}\alpha\lambda^2+\frac{1}{6}\alpha^2\lambda+\frac{1}{24}\lambda\alpha\\ &\quad+\frac{1}{4}\zeta^3+\frac{7}{24}\zeta^2+\frac{1}{24}\eta^4,\end{aligned}$$

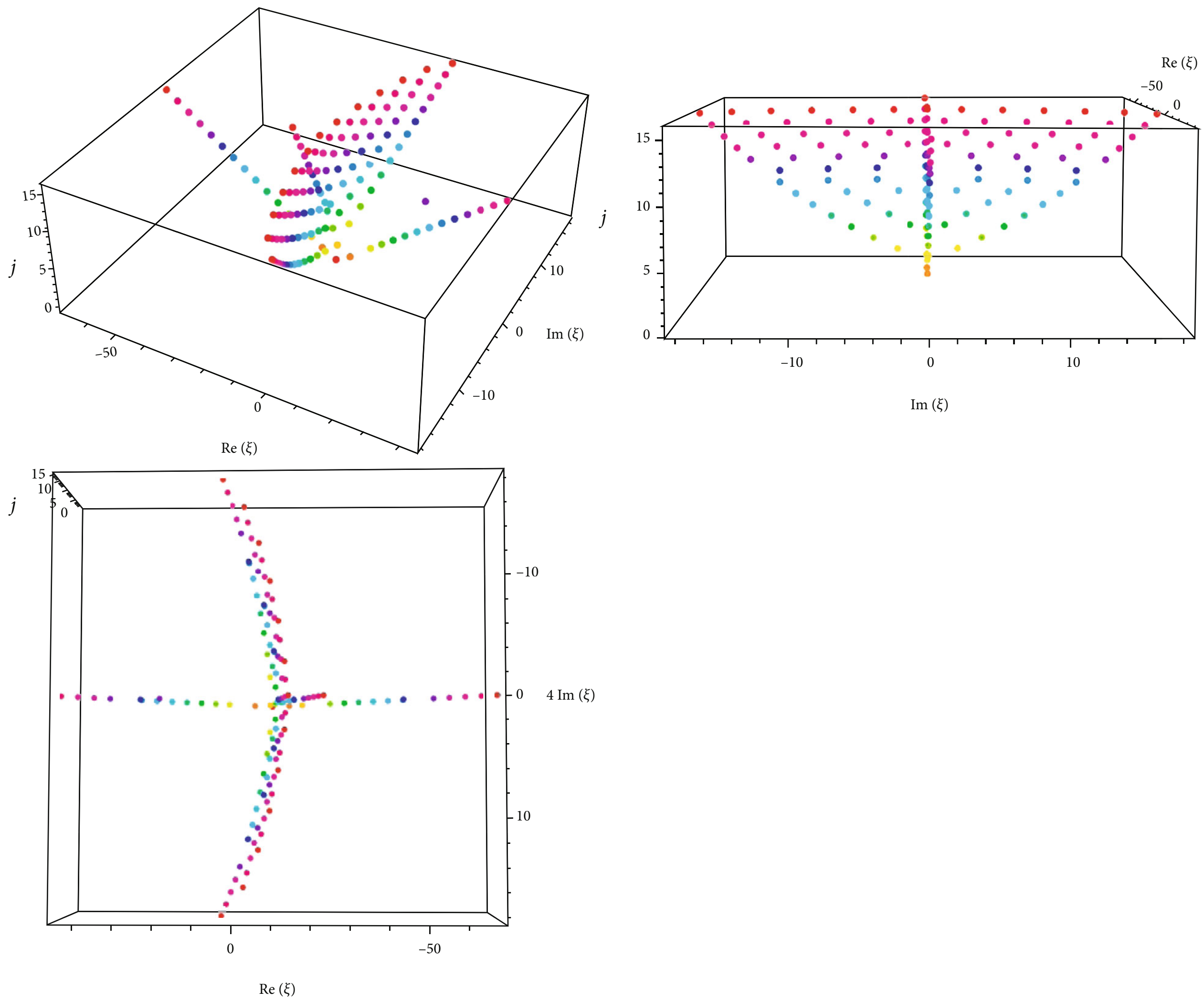
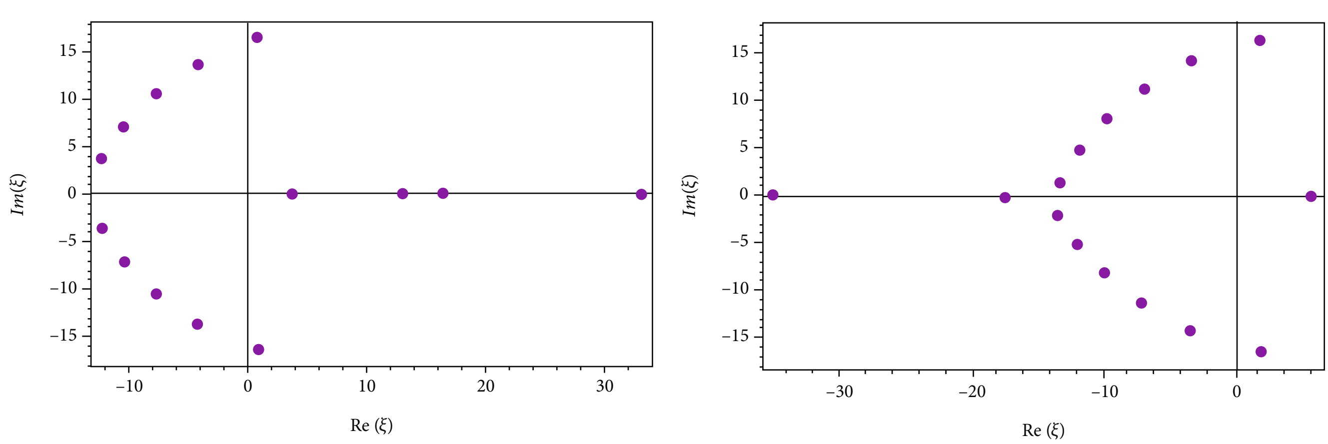


Figure 3: Stacking structure zeros of ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi, 6, 5; 2)$, $j \le 1 \le 16$.



(a) Zeros of ${}_{Bel}\mathbb{A}_{15}^{(4,s)}(\xi, 6, 5; 2)$

(b) Zeros of ${}_{Bel}\mathbb{A}_{16}^{(4,s)}(\xi, 6, 5; 2)$

Figure 4: Graphic behavior of the zeros of ${}_{Bel}\mathbb{A}_j^{(4,s)}(\xi, 6, 5; 2)$ for $j = 15, 16$.

$$
\begin{aligned}
{}_{Bel}\mathbb{A}_5^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = {} & \frac{1}{120}\zeta + \frac{1}{120}\xi^5 + \frac{1}{24}\zeta\xi + \frac{7}{24}\zeta^2\xi + \frac{1}{12}\zeta\xi^2 \\
& + \frac{1}{4}\zeta^3\xi + \frac{1}{4}\zeta^2\xi^2 + \frac{1}{12}\zeta\xi^3 + \frac{1}{12}\xi^3\alpha^2 + \frac{1}{24}\eta^4\xi \\
& + \frac{1}{24}\zeta^4\xi + \frac{1}{12}\zeta^3\xi^2 + \frac{1}{12}\zeta^2\xi^3 + \frac{1}{24}\zeta\xi^4 + \frac{1}{12}\alpha^3\xi^2 \\
& + \frac{1}{24}\alpha^4\xi - \frac{1}{12}\eta^2\xi^3 + \frac{1}{24}\xi^4\alpha + \frac{1}{4}\alpha^2\zeta\xi^2 + \frac{1}{6}\alpha^3\zeta\xi \\
& - \frac{1}{4}\zeta^2\eta^2\xi + \frac{1}{4}\alpha\zeta^2\xi^2 + \frac{1}{6}\alpha\zeta^3\xi + \frac{1}{4}\alpha^2\zeta^2\xi + \frac{1}{12}\xi^3\alpha\lambda \\
& + \frac{7}{24}\alpha^2\lambda^2\xi + \frac{1}{4}\alpha^3\lambda\xi - \frac{1}{4}\alpha^2\eta^2\xi + \frac{1}{24}\alpha\lambda^3\xi + \frac{1}{6}\alpha\zeta\xi^3 \\
& - \frac{1}{4}\zeta\eta^2\xi^2 + \frac{1}{4}\alpha^2\lambda\xi^2 + \frac{1}{12}\alpha\lambda^2\xi^2 - \frac{1}{4}\alpha\eta^2\xi^2 + \frac{1}{6}\alpha\lambda^2\xi \\
& + \frac{1}{6}\alpha^2\lambda\xi - \frac{1}{4}\zeta\eta^2\xi + \frac{1}{4}\alpha\zeta\xi^2 + \frac{1}{2}\alpha\zeta^2\xi + \frac{1}{4}\alpha^2\zeta\xi \\
& + \frac{1}{12}\xi^2\alpha\lambda + \frac{1}{24}\xi\alpha\lambda + \frac{1}{6}\alpha\zeta\xi + \frac{1}{24}\alpha\zeta^4 + \frac{1}{12}\alpha^2\zeta^3 \\
& - \frac{1}{12}\zeta^3\eta^2 + \frac{1}{12}\alpha^3\zeta^2 + \frac{1}{24}\alpha^4\zeta + \frac{1}{24}\zeta\eta^4 + \frac{1}{120}\alpha\lambda^4 \\
& + \frac{1}{8}\alpha^2\lambda^3 + \frac{5}{24}\alpha^3\lambda^2 + \frac{1}{12}\alpha^4\lambda - \frac{1}{12}\alpha^3\eta^2 + \frac{1}{12}\alpha\zeta^3\lambda \\
& + \frac{1}{4}\alpha^2\zeta^2\lambda + \frac{1}{12}\alpha\zeta^2\lambda^2 - \frac{1}{4}\alpha\zeta^2\eta^2 + \frac{1}{4}\alpha^3\zeta\lambda \\
& + \frac{7}{24}\alpha^2\zeta\lambda^2 - \frac{1}{4}\alpha^2\zeta\eta^2 + \frac{1}{24}\alpha\zeta\lambda^3 - \frac{1}{4}\alpha^2\lambda\eta^2 \\
& - \frac{1}{12}\alpha\lambda^2\eta^2 + \frac{1}{3}\alpha\zeta^2\lambda + \frac{5}{12}\alpha^2\beta\lambda + \frac{1}{4}\alpha\zeta\lambda^2 - \frac{1}{4}\alpha\zeta\eta^2 \\
& - \frac{1}{12}\alpha\lambda\eta^2 + \frac{1}{4}\alpha\zeta^3 + \frac{1}{4}\alpha^2\zeta^2 - \frac{1}{4}\zeta^2\eta^2 \\
& + \frac{1}{12}\alpha^3\zeta + \frac{11}{120}\alpha\lambda^3 + \frac{1}{4}\alpha^2\lambda^2 + \frac{1}{12}\alpha^3\lambda \\
& + \frac{7}{24}\alpha\zeta^2 + \frac{1}{12}\alpha^2\zeta - \frac{1}{12}\zeta\eta^2 + \frac{1}{24}\alpha\zeta \\
& + \frac{1}{4}\alpha\zeta^2\lambda\xi + \frac{1}{2}\alpha^2\zeta\lambda\xi + \frac{1}{6}\alpha\zeta\lambda^2\xi + \frac{1}{4}\alpha\zeta\lambda\xi^2 \\
& - \frac{1}{2}\alpha\zeta\eta^2\xi - \frac{1}{4}\alpha\lambda\eta^2\xi + \frac{5}{12}\alpha\zeta\lambda\xi - \frac{1}{4}\alpha\zeta\lambda\eta^2 \\
& + \frac{5}{24}\alpha\zeta\lambda + \frac{1}{120}\alpha^5 + \frac{1}{120}\zeta^5 + \frac{1}{12}\zeta^4 + \frac{1}{24}\eta^4\alpha \\
& + \frac{11}{120}\alpha\lambda^2 + \frac{1}{24}\alpha^2\lambda + \frac{1}{120}\lambda\alpha + \frac{5}{24}\zeta^3 + \frac{1}{8}\zeta^2,
\end{aligned} \tag{63}
$$

while the first five parametric kinds of ${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)$ are

$$
{}_{Bel}\mathbb{A}_0^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = 0,
$$

$$
{}_{Bel}\mathbb{A}_1^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \eta,
$$

$$
{}_{Bel}\mathbb{A}_2^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = (\xi+\zeta+\alpha)\eta,
$$

$$
{}_{Bel}\mathbb{A}_3^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \frac{1}{6}\eta\left(3\alpha^2 + 6\zeta\alpha + 3\alpha\lambda + 6\xi\alpha + 3\zeta^2 + 6\zeta\xi - \eta^2 + 3\xi^2 + 3\zeta\right),
$$

$$
\begin{aligned}
{}_{Bel}\mathbb{A}_4^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = {} & \frac{1}{6}\eta\Big(\alpha^3 + 3\alpha^2\zeta + 3\alpha^2\lambda + 3\xi\alpha^2 + 3\alpha\zeta^2 \\
& + 3\alpha\zeta\lambda + 6\alpha\zeta\xi + \alpha\lambda^2 + 3\xi\alpha\lambda - \eta^2\alpha + 3\xi^2\alpha \\
& + \zeta^3 + 3\zeta^2\xi - \zeta\eta^2 + 3\zeta\xi^2 - \eta^2\xi + \xi^3 + 3\zeta\alpha \\
& + \alpha\lambda + 3\zeta^2 + 3\zeta\xi + \zeta\Big),
\end{aligned}
$$

$$
\begin{aligned}
{}_{Bel}\mathbb{A}_5^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = {} & \frac{1}{120}\eta\big(5\zeta + 20\alpha\lambda^2\xi + 60\alpha^2\lambda\xi - 20\zeta\eta^2\xi \\
& + 60\alpha\zeta\xi^2 + 60\alpha\zeta^2\xi + 60\alpha^2\zeta\xi + 30\xi^2\alpha\lambda \\
& - 20\alpha\eta^2\xi + 5\xi^4 + 5\alpha^4 + 20\zeta\xi + 60\zeta^2\xi \\
& + 30\zeta\xi^2 + 30\xi^2\alpha^2 + 20\zeta^3\xi + 30\zeta^2\xi^2 \\
& + 20\zeta\xi^3 + 20\alpha^3\xi - 10\eta^2\xi^2 + 20\xi^3\alpha \\
& + 20\alpha\zeta^3 + 30\alpha^2\zeta^2 - 10\zeta^2\eta^2 + 20\alpha^3\zeta \\
& + 5\alpha\lambda^3 + 35\alpha^2\lambda^2 + 30\alpha^3\lambda - 10\alpha^2\eta^2 \\
& + 60\alpha\zeta^2 + 30\alpha^2\zeta - 10\zeta\eta^2 + 20\alpha\lambda^2 \\
& + 20\alpha^2\lambda + 20\zeta\alpha + 5\alpha\lambda + 20\xi\alpha\lambda + 60\alpha\zeta\xi \\
& + 30\alpha\zeta^2\lambda + 60\alpha^2\zeta\lambda + 20\alpha\zeta\lambda^2 - 20\alpha\zeta\eta^2 \\
& - 10\alpha\lambda\eta^2 + 50\alpha\zeta\lambda + 5\zeta^4 + 30\zeta^3 + 35\zeta^2 \\
& + \eta^4 + 60\alpha\zeta\lambda\xi\big).
\end{aligned} \tag{64}
$$

For $1 \le j \le 16$, the complex and real zero values of ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi, 6, 5\,; 2)$ are showed in Table 1.

Figure 1 shows the plots for some parametric cosine Frobenius-type Eulerian polynomials.

Figure 2 shows the structure of real zeros of the parametric cosine Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi, 6, 5\,; 2)$, with $j \le 1 \le 16$.

Figure 3 shows the stacking structure zeros of the parametric cosine Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_j^{(4,c)}$ $(\xi, 6, 5\,; 2)$, with $j \le 1 \le 16$.

Finally, Figure 4 shows the graphic behavior of the zeros of the parametric sine Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_j^{(4,s)}(\xi, 6, 5\,; 2)$ for $j = 15, 16$.

4. Conclusion

Our paper introduced sine and cosine Bell-based Frobenius-type Eulerian polynomials and analyzed their properties by providing several relations and applications. Also, various formulas and properties including differentiation rules, addition formulas, relations, and summation formulas have been investigated. Moreover, after determining the first few zero values of the Eulerian polynomials, we have drawn graphical representations of these zero values.

It is possible that this paper's idea can be applied to polynomials that are similar and these polynomials have potential applications in other fields of science in addition to the applications at the end of the article. We will continue to explore this opinion in various directions in our next scientific works to advance the purpose of this article.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. M. Jamei, M. R. Beyki, and W. Koepf, "A new type of Euler polynomials and numbers," *Mediterranean Journal of Mathematics*, vol. 15, no. 3, pp. 1–17, 2018.

[2] T. Kim and C. S. Ryoo, "Some identities for Euler and Bernoulli polynomials and their zeros," *Axioms*, vol. 7, no. 3, p. 56, 2018.

[3] D. S. Kim, T. Kim, and H. Lee, "A note on degenerate Euler and Bernoulli polynomials of complex variable," *Symmetry*, vol. 11, no. 9, p. 1168, 2019.

[4] G. Muhiuddin, W. A. Khan, and D. Al-Kadi, "Construction on the degenerate poly-Frobenius-Euler polynomials of complex variable," *Journal of Function Spaces*, vol. 2021, Article ID 3115424, 9 pages, 2021.

[5] D. S. Kim, T. Kim, W. J. Kim, and D. V. Dolgy, "A note on Eulerian polynomials," *Abstract and Applied Analysis*, vol. 2012, Article ID 269640, 10 pages, 2012.

[6] W. A. Khan and D. Srivastava, "Certain properties of Apostol-type Hermite-based Frobenius-Genocchi polynomials," *Kragujevac Journal of Mathematics*, vol. 45, no. 6, pp. 859–872, 2021.

[7] H. M. Srivastava, M. A. Boutiche, and M. Rahmani, "A class of Frobenius-type Eulerian polynomials," *The Rocky Mountain Journal of Mathematics*, vol. 48, pp. 1003–1013, 2018.

[8] L. Carlitz, "Eulerian numbers and polynomials," *Mathematics Magazine*, vol. 32, no. 5, pp. 247–260, 1959.

[9] L. Carlitz, "Eulerian numbers and polynomials of higher order," *Duke Mathematical Journal*, vol. 27, no. 3, pp. 401–423, 1960.

[10] W. A. Khan, K. S. Nisar, A. Mehmet, and U. Duran, "A novel kind of Hermite-based Frobenius type Eulerian polynomials," *Proceedings of the Jangjeon Mathematical Society*, vol. 22, no. 4, pp. 551–563, 2019.

[11] W. A. Khan and D. Srivastava, "On the generalized Apostol-type Frobenius-Genocchi polynomials," *Filomat Journal*, vol. 33, no. 7, pp. 1967–1977, 2019.

[12] A. U. Ashraf, T. N. Tran, and M. Yoshinaga, "Eulerian polynomials for subarrangements of Weyl arrangements," *Advances in Applied Mathematic*, vol. 120, article 102064, 2020.

[13] U. Duran, A. Mehmet, and S. Araci, "Bell-based Bernoulli polynomials with applications," *Axioms*, vol. 10, no. 1, p. 29, 2021.

[14] W. A. Khan, G. Muhiuddin, A. Muhyi, and D. Al-Kadi, "Analytical properties of type 2 degenerate poly-Bernoulli polynomials associated with their applications," *Advances in Difference Equations*, vol. 2021, no. 1, 2021.

[15] C. S. Ryoo and W. A. Khan, "On two bivariate kinds of poly-Bernoulli and poly-Genocchi polynomials," *Mathematics*, vol. 8, no. 3, p. 417, 2020.

[16] M. A. Pathan and W. A. Khan, "A new class of generalized Apostol-type Frobenius-Euler-Hermite polynomials," *Honam Mathematical Journal*, vol. 42, no. 3, pp. 477–499, 2020.

[17] I. Mezö, "A new formula for the Bernoulli polynomials," *Results in Mathematics*, vol. 58, no. 3-4, pp. 329–335, 2010.

[18] D. S. Kim and T. Kim, "Some identities of degenerate Bell polynomials," *Mathematics*, vol. 8, no. 1, p. 40, 2020.

[19] A. Z. Border, "The r-Stirling numbers," *Discrete Mathematics*, vol. 49, no. 3, pp. 241–259, 1984.

[20] M. A. Pathan and W. A. Khan, "On λ-Changhee–Hermite polynomials," *Analysis*, vol. 42, no. 2, pp. 57–69, 2022.

[21] E. T. Bell, "Exponential polynomials," *The Annals of Mathematics*, vol. 35, no. 2, pp. 258–277, 1934.

[22] W. A. Khan, M. Kamarujjama, and Daud, "Construction of partially degenerate Bell–Bernoulli polynomials of the first kind and their certain properties," *Analysis*, vol. 42, no. 3, pp. 171–184, 2022.

[23] N. Alam, W. A. Khan, and C. S. Ryoo, "A note on Bell-based Apostol-type Frobenius-Euler polynomials of complex variable with its certain applications," *Mathematics*, vol. 10, no. 12, p. 2109, 2022.

[24] N. Alam, W. A. Khan, S. Obeidat et al., "A note on Bell-based Bernoulli and Euler polynomials of complex variable," *Computer Modeling in Engineering and Sciences*, vol. 135, no. 1, pp. 187–209, 2023.

[25] R. B. Corcino, C. B. Corcino, J. M. Ontolan, C. M. Perez, and E. R. Cantallopez, "Hankel transform of q-noncentral Bell numbers," *International Journal of Mathematics and Mathematical Sciences*, vol. 2015, Article ID 417327, 10 pages, 2015.

Hindawi
Journal of Function Spaces
Volume 2023, Article ID 4759187, 10 pages
https://doi.org/10.1155/2023/4759187

Research Article

Some Interesting Inequalities for the Class of Generalized Convex Functions of Higher Order

Limei Liu,[1] **Muhammad Shoaib Saleem**,[2] **Faisal Yasin**,[3] **Kiran Naseem Aslam**,[2] **and Pengfei Wang**[4]

[1]*General Studies Teaching Department, Shenyang Institute of Engineering, Shenyang 110136, China*
[2]*Department of Mathematics, University of Okara, Okara, Pakistan*
[3]*Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan*
[4]*Department of Mathematics, Xinzhou Teachers University, Xinzhou, Shanxi 034000, China*

Correspondence should be addressed to Limei Liu; liulimei_2022@sina.com and Faisal Yasin; faisal.yasin@math.uol.edu.pk

Received 3 August 2022; Revised 12 October 2022; Accepted 14 October 2022; Published 22 August 2023

Academic Editor: Mohsan Raza

In this paper, we study a generalized version of strongly reciprocally convex functions of higher order. Firstly, we prove some basic properties for addition, scalar multiplication, and composition of functions. Secondly, we establish Hermite-Hadamard and Fejér type inequalities for the generalized version of strongly reciprocally convex functions of higher order. We also include some fractional integral inequalities concerning with this class of functions. Our results have applications in optimization theory and can be considered extension/generalization of many existing results.

1. Introduction

Convexity is a very simple and ordinary concept. Due to its massive applications in industry and business, convexity has a great influence on our daily life. In the solution of many real-world problems, the concept of convexity is very decisive. Problems faced in constrained control and estimation are convex. Geometrically, a real-valued function is said to be convex if the line segment joining any two of its points lies on or above the graph of the function in Euclidean space.

Convexity of a function in classical sense is defined as a function $f_1 : M \longrightarrow \mathbb{R}, f_1$ is convex if we have

$$f_1(\mathrm{j}x + (1-j)y) \leq \mathrm{j}f_1(x) + (1-j)f_1(y), \forall j \in [0,1]. \quad (1)$$

If the above inequality is reversed, then the function is said to be concave.

Using different techniques, the notion of convexity is being extended day by day [1–3]. Many extensions and generalizations are made speedily due to its applications in modern engineering, optimization, economics, and nonlinear programming [4–7]. For recent generalizations, one can see [8, 9] and the references therein.

Using the definition of convex functions, several important inequalities can be proved, and the Hermite-Hadamard inequality is one of them. The Hermite-Hadamard inequality is for any convex function $f_1 : M \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ with $a_1, b_1 \in M$ and $a_1 b_1$, the Hermite-Hadamard double inequality is

$$f_1\left(\frac{a_1+b_1}{2}\right) \leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1} f_1(x)dx \leq \frac{f_1(a_1)+f_1(b_1)}{2}. \quad (2)$$

In [9], using the weight function $w(x)$, Fejér gave a generalization of the Hermite-Hadamard inequality as follows:

Let $f_1 : [a_1, b_1] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $w : [a_1, b_1] \longrightarrow \mathbb{R}$ is nonnegative, integrable, and symmetric about $(a_1 + b_1)/2$, then we have

$$f_1\left(\frac{a_1+b_1}{2}\right)\int_{a_1}^{b_1} w(x)\mathrm{d}x \leq \int_{a_1}^{b_1} f_1(x)w(x)\mathrm{d}x \leq \frac{f_1(a_1)+f_1(b_1)}{2}\int_{a_1}^{b_1} w(x)\mathrm{d}x. \quad (3)$$

In [10], the notations of p-convex set and p-convex functions are introduced. The strongly convex functions of modulus μ are introduced in [11]. In [12, 13], the strongly p-convex and harmonic convex functions were introduced, respectively. The p-harmonic convex set and p-harmonic convex functions were studied in [14], and in [15], the strongly reciprocally convex of modulus μ are introduced. The strongly reciprocally p-convex and h-convex functions were introduced in [16, 17], respectively. The (p,h)-convex functions are introduced in [18], and the higher-order strongly convex with modulus μ are introduced in [19]. Now, we present the notation of strongly reciprocally (p,h)-convex functions of higher order (SRHO).

Definition 1. Let $\mu \in (0,\infty)$ and M is any interval. Then the function $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is SRHO of modulus μ on the interval M, if we have

$$f_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \leq h(1-j)f_1(x) + h(j)f_1(y) - \mu\phi(j)\left\|\frac{1}{x^p} - \frac{1}{y^p}\right\|^l, \quad (4)$$

for all $x, y \in M$, $j \in [0,1]$, and $l \geq 1$, where $\phi(j) = j(1-j)$.

Remark 2. Inserting $l = 2$ in Def. 1 with same $\phi(j)$ as defined above, we obtain strongly reciprocally (p,h)-convex functions. Similarly, inserting $l = 2$ and $h(j) = j$ in Def. 1, we obtain strongly reciprocally p-convex functions, and for $l = 2$, $h(j) = j$, and $p = 1$, Def. 1 reduces to the strongly reciprocally convex function of modulus μ.

As we know that $\mathbb{R}$ is a Norm space under the usual modulus norm, thus, for any $x \in \mathbb{R}$,

$$\|x\| = |x|. \quad (5)$$

Using (5), the inequality 1 can be written as

$$f_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \leq h(1-j)f_1(x) + h(j)f_1(y) - \mu\phi(j)\left|\frac{1}{x^p} - \frac{1}{y^p}\right|^l, \quad (6)$$

$\forall x, y \in M$ and $j \in [0,1]$ with $l \geq 1$, where $\phi(j)$ is same as in Definition 1.

The aim of this paper is to study a generalized version of strongly reciprocally convex functions of higher order and establish the Hermite-Hadamard and Fejér type inequalities for this new class of convex functions. We also presented fractional versions of the above mentioned inequalities for the strongly reciprocally (p,h)-convex of higher order. It is worthy to mention here that the results presented in this paper are more generalized and can be considered extensions of many existing results.

2. Basic Results

Now, we present some basic properties for strongly reciprocally (p,h)-convex of higher order.

Proposition 3. *For any two SRHO $f_1, g_1 : M \longrightarrow \mathbb{R}$ with modulus μ on the interval M, the $f_1 + g_1 : M \longrightarrow \mathbb{R}$ is also SRHO with modulus μ^* on the interval M, where $1/2\mu^* = \mu$.*

Proof. By definition, we have

$$\begin{aligned} f_1 + g_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] &= f_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \\ &+ g_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \leq h(j)f_1(x) + h(1-j)f_1(y) \\ &- \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l + h(j)g_1(x) + h(1-j)g_1(y) - \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l, \end{aligned} \quad (7)$$

which in turns implies that

$$\begin{aligned} &= h(j)(f_1 + g_1)(x) + h(1-j)(f_1 + g_1)(y) - 2\mu j(1-j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l \\ &= h(j)(f_1 + g_1)(x) + h(1-j)(f_1 + g_1)(y) - \mu^*\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l, \end{aligned} \quad (8)$$

where $\mu^* = 2\mu$, $\mu \geq 0$ and $\phi(j) = j(1-j)$.

This completes the proof. □

Proposition 4. *For any SRHO $f_1 : M \longrightarrow \mathbb{R}$ with modulus $\mu \geq 0$ and any $\lambda \geq 0$, λf_1 is also SRHO with modulus ν^* on the interval M, where $1/\lambda\nu^* = \mu$.*

Proof. Let $\lambda \geq 0$, then by definition of f_1, we obtain

$$\begin{aligned} &\lambda f_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \\ &= \lambda\left[f_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right]\right] \\ &\leq \lambda\left[h(j)f_1(x) + h(1-j)f_1(y) - \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l\right] \\ &= h(j)\lambda f_1(x) + h(1-j)\lambda f_1(y) - \lambda\mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l \\ &= h(j)\lambda f_1(x) + h(1-j)\lambda f_1(y) - \nu^*\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l, \end{aligned} \quad (9)$$

where $\nu^* = \lambda\mu$, $\mu \geq 0$ and $\phi(j) = j(1-j)$. This completes the proof. □

Proposition 5. *Consider a sequence of SRHO, f_{1i} are defined on an interval M, provide $1 \leq i \leq n$, then for positive constants λ_i, the function $f_1 = \sum_{i=1}^n \lambda_i f_{1i}$ is SRHO with nonnegative modulus $\gamma\sum_{i=1}^n \lambda_i\mu$.*

Proof. For a p-harmonic convex set M, we have

$$\begin{aligned}
f_1\left[\left(\frac{x^p y^p}{jx^p+(1-j)y^p}\right)^{1/p}\right]
&= \sum_{i=1}^{n}\lambda_i f_{1i}\left[\left(\frac{x^p y^p}{jx^p+(1-j)y^p}\right)^{1/p}\right]\\
&\le \sum_{i=1}^{n}\lambda_i\left[h(j)f_{1i}(x)+h(1-j)f_{1i}(y)-\mu\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l\right]\\
&= h(j)\sum_{i=1}^{n}\lambda_i f_{1i}(x)+h(1-j)\sum_{i=1}^{n}\lambda_i f_{1i}(y)-\sum_{i=1}^{n}\lambda_i\left[\mu\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l\right]\\
&= h(j)f_1(x)+h(1-j)f_1(y)-\gamma\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l,
\end{aligned} \tag{10}$$

$\forall x, y \in M$ and $j \in [0,1]$, where $\gamma = \sum_{i=1}^{n}\lambda_i\mu$. Hence, the result is proved. □

Proposition 6. *Consider a sequence of SRHO, f_{1i} are defined on an interval M, provide $1 \le i \le n$, then for positive constants λ_i, the function $f_1 = \max\{f_{1i}, i = 1, 2, \cdots, n\}$, is SRHO of modulus μ.*

Proof. Let M be a p-harmonic convex set. Then, $\forall x, y \in M$ and $j \in [0,1]$, we have

$$\begin{aligned}
f_1\left[\left(\frac{x^p y^p}{jx^p+(1-j)y^p}\right)^{1/p}\right]
&= \max\left\{f_{1i}\left[\left(\frac{x^p y^p}{jx^p+(1-j)y^p}\right)^{1/p}\right], i = 1,2,3,\cdots,n\right\}\\
&= f_c\left(\left[\frac{x^p y^p}{jx^p+(1-j)y^p}\right]\right)\\
&\le h(j)f_c(x)+h(1-j)f_c(y)-\mu\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l\\
&= h(j)\max\{f_{1i}(x)\}+h(1-j)\max\{f_{1i}(y)\}-\mu\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l\\
&= h(j)f_1(x)+h(1-j)f_1(y)-\mu\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l.
\end{aligned} \tag{11}$$

This completes the proof. □

3. Hermite-Hadamard Type Inequality

In this section, we establish Hermite-Hadamard's type inequality for the function belonging to $SR(ph)$.

Theorem 7. *Consider an interval M not containing zero and SRHO $f_1 : M \longrightarrow \mathbb{R}$ of nonnegative modulus μ and $f_1 \in L[a_1, b_1]$, then for $h(1/2) \ne 0$, we have*

$$\begin{aligned}
&\frac{1}{2h(1/2)}\left[f_1\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p}+\mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l\left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right]\right]\\
&\quad\le \frac{p\left(a_1^p b_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\\
&\quad\le \int_0^1[h(1-j)f_1(a_1)+h(j)f_1(b_1)]dj-\mu\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|\int_0^1\phi(j)dj.
\end{aligned} \tag{12}$$

Proof. Substituting $j = 1/2$ in Definition 1, gives

$$f_1\left[\left(\frac{2x^p y^p}{x^p+y^p}\right)^{1/p}\right] \le h\left(\frac{1}{2}\right)f_1(x)+h\left(\frac{1}{2}\right)f_1(y)-\mu\phi\left(\frac{1}{2}\right)\left\|\frac{1}{x^p}-\frac{1}{y^p}\right\|^l. \tag{13}$$

Considering $x = [(a_1^p b_1^p / j a_1^p + (1-j)b_1^p)^{1/p}]$ and $y = [(a_1^p b_1^p / j b_1^p + (1-j)a_1^p)^{1/p}]$ and integrating (13)

$$\begin{aligned}
f_1\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p} &\le h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\\
&\quad+h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^p b_1^p}{jb_1^p+(1-j)a_1^p}\right)^{1/p}\right]\\
&\quad-\mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l|1-2j|^l,
\end{aligned}$$

$$\begin{aligned}
\int_0^1 f_1\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p}dj &\le \int_0^1 h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\
&\quad+\int_0^1 h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^p b_1^p}{jb_1^p+(1-j)a_1^p}\right)^{1/p}\right]dj\\
&\quad-\mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l\int_0^1|1-2j|^l dj,
\end{aligned}$$

$$\begin{aligned}
f_1\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p} &\le 2h\left(\frac{1}{2}\right)\frac{p\left(a_1^p b_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\\
&\quad-\mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l\left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right],
\end{aligned}$$

$$\begin{aligned}
&f_1\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p}+\mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l\left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right]\\
&\quad\le 2h\left(\frac{1}{2}\right)\frac{p\left(a_1^p b_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx,
\end{aligned}$$

$$\frac{1}{2h(1/2)}\left[f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}+\mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right|^l\left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right]\right]$$
$$\le\frac{p\left(a_1^pb_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx, \tag{14}$$

which is left side of the inequality (12).

Finally, for the right side of the inequality (12), setting $x=a_1$ and $y=b_1$ in Definition 1 gives

$$\begin{aligned}f_1&\left[\left(\frac{a_1^pb_1^p}{ta_1^p+(1-t)b_1^p}\right)^{1/p}\right]\\&\le h(1-j)f_1(a_1)+h(j)f_1(b_1)\\&\quad-\mu\phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l.\end{aligned} \tag{15}$$

Integrating (15)

$$\begin{aligned}\int_0^1 f_1&\left[\left(\frac{a_1^pb_1^p}{ta_1^p+(1-t)b_1^p}\right)^{1/p}\right]dj\\&\le\int_0^1h(1-j)f_1(a_1)dj\\&\quad+\int_0^1h(j)f_1(b_1)dj-\mu\left|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right|^l\int_0^1\phi(j)dj,\end{aligned}$$

$$\begin{aligned}\frac{p\left(a_1^pb_1^p\right)}{b_1^p-a_1^p}&\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\\&\le\int_0^1[h(1-j)f_1(a_1)+h(j)f_1(b_1)]dj\\&\quad-\mu\left|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right|^l\int_0^1\phi(j)dj,\end{aligned} \tag{16}$$

that is right hand side of (12) and proof is completed. □

4. Fejér Type Inequality

Now, we are going to develop the Fejér type inequality for the function belonging to $SR(ph)$.

Theorem 8. *Consider an interval M not containing zero and real-valued SRHO f_1 defined on M of nonnegative modulus μ, then for $h(1/2)\neq 0$, we have*

$$\begin{aligned}\frac{1}{2h(1/2)}&\left[f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\int_{a_1}^{b_1}\frac{w(x)}{x^{1+p}}dx+\frac{\mu}{\left|a_1^pb_1^p\right|^l}\phi\left(\frac{1}{2}\right)\right.\\&\left.\cdot\int_{a_1}^{b_1}\frac{\left|2a_1^pb_1^p-\left(a_1^p+b_1^p\right)x^p\right|^lw(x)}{\left|x^p\right|^lx^{1+p}}dx\right]\\&\le\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}dx\\&\le[f_1(a_1)+f_1(b_1)]\int_{a_1}^{b_1}h\left(\frac{a_1^p\left(b_1^p-x^p\right)}{x^p\left(b_1^p-a_1^p\right)}\right)\frac{w(x)}{x^{1+p}}dx\\&\quad-\mu\left\|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right\|^l\int_{a_1}^{b_1}\phi\left(\frac{a_1^p\left(b_1^p-x^p\right)}{x^p\left(b_1^p-a_1^p\right)}\right)\frac{w(x)}{x^{1+p}}dx,\end{aligned} \tag{17}$$

holds for $a_1,b_1\in M$ with $a_1\le b_1$ and $f_1\in L[a_1,b_1]$, where the nonnegative real-valued function w defined on M satisfies

$$w\left(\frac{a_1^pb_1^p}{x^p}\right)^{1/p}=w\left[\left(\frac{a_1^pb_1^p}{a_1^p+b_1^p-x^p}\right)^{1/p}\right]. \tag{18}$$

Proof. Substituting $j=1/2$ in Definition 1, yields

$$f_1\left[\left(\frac{2x^py^p}{x^p+y^p}\right)^{1/p}\right]\le h\left(\frac{1}{2}\right)f_1(x)+h\left(\frac{1}{2}\right)f_1(y)-\mu\phi\left(\frac{1}{2}\right)\left\|\frac{1}{x^p}-\frac{1}{y^p}\right\|^l. \tag{19}$$

Considering $x=[(a_1^pb_1^p/ja_1^p+(1-j)b_1^p)^{1/p}]$ and $y=[(a_1^pb_1^p/jb_1^p+(1-j)a_1^p)^{1/p}]$ and integrating (19),

$$\begin{aligned}f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}&\le h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\\&\quad+h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{jb_1^p+(1-j)a_1^p}\right)^{1/p}\right]\\&\quad-\mu\phi\left(\frac{1}{2}\right)\left|\frac{ja_1^p+(1-j)b_1^p}{a_1^pb_1^p}-\frac{jb_1^p+(1-j)a_1^p}{a_1^pb_1^p}\right|^l.\end{aligned} \tag{20}$$

By the properties of w,

$$\begin{aligned}f_1&\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\\&\le h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\\&\quad+h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{jb_1^p+(1-j)a_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right],\end{aligned} \tag{21}$$

$$-\mu\phi\left(\frac{1}{2}\right)\left|\frac{ja_1^p+(1-j)b_1^p}{a_1^pb_1^p}-\frac{jb_1^p+(1-j)a_1^p}{a_1^pb_1^p}\right|^l w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]. \quad (22)$$

Integrating inequality (21),

$$\begin{aligned}&\int_0^1 f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p} w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\&\le\int_0^1 h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\&+\int_0^1 h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{jb_1^p+(1-j)a_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\&-\mu\phi\left(\frac{1}{2}\right)\left|\frac{ja_1^p+(1-j)b_1^p}{a_1^pb_1^p}-\frac{jb_1^p+(1-j)a_1^p}{a_1^pb_1^p}\right|^l w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj,\end{aligned}$$

$$\begin{aligned}&f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\int_{a_1}^{b_1}\frac{w(x)}{x^{1+p}}dx+\frac{\mu}{\left|a_1^pb_1^p\right|^l}\phi\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{\left|2a_1^pb_1^p-\left(a_1^p+b_1^p\right)x^p\right|^l w(x)}{\left|x^p\right|^l x^{1+p}}dx\\&\le 2h\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}dx\,\frac{1}{2h(1/2)}\left[f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\int_{a_1}^{b_1}\frac{w(x)}{x^{1+p}}dx\right.\\&\left.+\frac{\mu}{\left|a_1^pb_1^p\right|^l}\phi\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{\left|2a_1^pb_1^p-\left(a_1^p+b_1^p\right)x^p\right|^l w(x)}{\left|x^p\right|^l x^{1+p}}dx\right]\\&\le\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}dx,\end{aligned} \quad (23)$$

which is left side of the inequality (17).

Finally, for the right side of the inequality (17), setting $x=a_1$ in Definition 1 gives

$$f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\le h(1-j)f_1(a_1)+h(j)f_1(b_1)-\mu\phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l. \quad (24)$$

By the properties of w,

$$\begin{aligned}&f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\\&\le h(1-j)f_1(a_1)w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\\&+h(j)f_1(b_1)w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\\&-\mu\phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right].\end{aligned} \quad (25)$$

Integrating inequality (25),

$$\begin{aligned}&\int_0^1 f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\&\le\int_0^1 h(1-j)f_1(a_1)w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\&+\int_0^1 h(j)f_1(b_1)w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\&-\mu\int_0^1\phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj,\end{aligned}$$

$$\begin{aligned}\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}dx&\le\left[f_1(a_1)+f_1(b_1)\right]\int_{a_1}^{b_1}h\left(\frac{a_1^p\left(b_1^p-x^p\right)}{x^p\left(b_1^p-a_1^p\right)}\right)\frac{w(x)}{x^{1+p}}dx\\&-\mu\left\|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right\|^l\int_{a_1}^{b_1}\phi\left(\frac{a_1^p\left(b_1^p-x^p\right)}{x^p\left(b_1^p-a_1^p\right)}\right)\frac{w(x)}{x^{1+p}}dx,\end{aligned} \quad (26)$$

that is right hand side of (17) and the proof is completed. □

5. Fractional Integral Inequalities

Lemma 9 ([20], Lemma 2.1). *Let $f_1: M=[a_1,b_1]\subseteq\mathbb{R}$ be a differentiable function on the interior M of M. If $f_1'\in L[a_1,b_1]$ and $\lambda\in[0,1]$, then*

$$\begin{aligned}&(1-\lambda)f_1\left[\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\right]+\lambda\left(\frac{f_1(a_1)+f_1(b_1)}{2}\right)\\&-\frac{p\left(a_1^pb_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\\&=\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^pb_1^p\right)}\left[\int_0^{1/2}(2j-\lambda)\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\right.\\&\cdot f_1'\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\&+\int_{1/2}^1(2j-2+\lambda)\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\\&\left.\cdot f_1'\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\right].\end{aligned} \quad (27)$$

Theorem 10. *Let $M=[a_1,b_1]\subset\mathbb{R}\setminus\{0\}$ be a p-harmonic convex set and let $f_1: M=[a_1,b_1]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ be a differentiable function on the interior M of M. If $f_1'\in L[a_1,b_1]$ and $|f_1'|^q$ are strongly reciprocally (p,h)-convex function of higher order on M, $q\ge 1$, and $\lambda\in[0,1]$, then*

$$\left|(1-\lambda)f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p}\right]+\lambda\left(\frac{f_1(a_1)+f_1(b_1)}{2}\right)-\frac{p\left(a_1^p b_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right|$$
$$\leq\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\left[C_1(p,a_1,b_1)^{1-1/q}\left[C_3(p,a_1,b_1)\left|f_1'(a_1)\right|^q+C_5(p,a_1,b_1)\left|f_1'(b_1)\right|^q+C_7(p,a_1,b_1)\mu\right]^{1/q}+C_2(p,b_1,a_1)^{1-1/q}\left[C_6(p,b_1,a_1)\left|f_1'(a_1)\right|^q+C_4(p,b_1,a_1)\left|f_1'(b_1)\right|^q+C_8(p,b_1,a_1)\mu\right]^{1/q}\right], \tag{28}$$

where

$$C_1(p,a_1,b_1)=\int_0^{1/2}|2j-\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{29}$$

$$C_2(p,b_1,a_1)=\int_{1/2}^{1}|2j-2+\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{30}$$

$$C_3(p,a_1,b_1)=\int_0^{1/2}h(1-j)|2j-\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{31}$$

$$C_4(p,b_1,a_1)=\int_{1/2}^{1}h(j)|2j-2+\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{32}$$

$$C_5(p,a_1,b_1)=\int_0^{1/2}h(j)|2j-\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{33}$$

$$C_6(p,b_1,a_1)=\int_{1/2}^{1}h(1-j)|2j-2+\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{34}$$

$$C_7(p,a_1,b_1)=-\int_0^{1/2}\phi(j)|2j-\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\cdot\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l dj, \tag{35}$$

$$C_8(p,b_1,a_1)=-\int_{1/2}^{1}\phi(j)|2j-2+\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\cdot\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l dj. \tag{36}$$

Proof. Using Lemma 9, we have

$$\left|(1-\lambda)f\left[\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p}\right]+\lambda\left(\frac{f(a_1)+f(b_1)}{2}\right)-\frac{p\left(a_1^p b_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f(x)}{x^{1+p}}dx\right|$$
$$\leq\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\left[\int_0^{1/2}\left|(2j-\lambda)\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\right|\left|f_1'\cdot\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\right|dj+\int_{1/p}^{1}|(2j-2+\lambda)\cdot\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\left|\left|f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\right|dj\right]. \tag{37}$$

Using power mean inequality,

$$\leq\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\left[\left(\int_0^{1/2}|(2j-\lambda)|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj\right)^{1-1/q}\cdot\left(\int_0^{1/2}|(2j-\lambda)|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\left|f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}+\left(\int_{1/2}^{1}|(2j-2+\lambda)|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj\right)^{1-1/q}\cdot\left(\int_{1/2}^{1}|(2j-2+\lambda)|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\left|f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\right]. \tag{38}$$

Since $\left|f_1'(x)\right|^q$ is in $SR(ph)$, so

$$\leq\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\left(\int_0^{1/2}\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{q+q/p}\cdot\left[h(1-j)\left|f_1'(a_1)\right|^q+h(j)\left|f_1'(b_1)\right|^q-\mu\phi(j)\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l\right]dj\right)^{1/q}+\left(\int_{1/2}^{1}|(2j-2+\lambda)|^r dj\right)^{1/r}\left(\int_{1/2}^{1}\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{q+q/p}\cdot\left[h(1-j)\left|f_1'(a_1)\right|^q+h(j)|(b_1)|^q-\mu\phi(j)\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l\right]dj\right)^{1/q}\right]$$
$$\leq\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\cdot\left(C_9(q,p;a_1,b_1)\left|f_1'(a_1)\right|^q+C_{11}(q,p;a_1,b_1)\left|f_1'(b_1)\right|^q+C_{13}(q,p;a_1,b_1)\mu\right)^{1/q}+\left(\int_0^{1/2}|(2j-2+\lambda)|^r dj\right)^{1/r}\cdot\left(C_{12}(q,p;b_1,a_1)\left|f_1'(a_1)\right|^q+C_{10}(q,p;b_1,a_1)\left|f_1'(b_1)\right|^q+C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right]$$
$$\leq\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\times\left(\frac{\lambda^{r+1}+(1-\lambda)^{r+1}}{2(r+1)}\right)^{1/r}\cdot\left[\left(C_9(q,p;a_1,b_1)\left|f_1'(a_1)\right|^q+C_{11}(q,p;a_1,b_1)\left|f_1'(b_1)\right|^q+C_{13}(q,p;a_1,b_1)\mu\right)^{1/q}+\left(C_{12}(q,p;b_1,a_1)\left|f_1'(a_1)\right|^q+C_{10}(q,p;b_1,a_1)\left|f_1'(b_1)\right|^q+C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right]. \tag{39}$$

Hence, the desired result is obtained. □

For $q = 1$, Theorem 10 reduces to the following result.

Corollary 11. *Let $M = [a_1, b_1] \subset \mathbb{R} \setminus \{0\}$ be a p-harmonic convex and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ are in SR(ph) on M and $\lambda \in [0, 1]$, then*

$$\begin{aligned}
&\left|(1-\lambda) f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{1/p}\right] + \lambda\left(\frac{f_1(a_1) + f_1(b_1)}{2}\right) - \frac{p\left(a_1^p b_1^p\right)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right| \\
&\le \frac{\left(b_1^p - a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\left[\left(C_3(p, a_1, b_1) + C_6(p, b_1, a_1)\right)\left|f'_1(a_1)\right| + \left(C_5(p, b_1, a_1) + C_4(p, a_1, b_1)\right)\left|f'_1(b_1)\right| + \left(C_7(p, a_1, b_1) + C_8(p, b_1, a_1)\right)\mu\right],
\end{aligned} \tag{40}$$

where C_3, C_4, C_5, C_6, C_7, and C_8 are given by (31) to (36).

Theorem 12. *Let $M = [a_1, b_1] \subset \mathbb{R} \setminus \{0\}$ be a p-harmonic convex and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ are strongly reciprocally (p,h)-convex function of higher order on M, $r, q > 1$, $1/r + 1/q = 1$ and $\lambda \in [0, 1]$, then*

$$\begin{aligned}
&\left|(1-\lambda) f_1\left(\left[\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right]^{1/p}\right) + \lambda\left(\frac{f_1(a_1) + f_1(b_1)}{2}\right) - \frac{p\left(a_1^p b_1^p\right)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right| \\
&\le \frac{\left(b_1^p - a_1^p\right)}{2p\left(a_1^p b_1^p\right)} \times \left(\frac{\lambda^{r+1} + (1-\lambda)^{r+1}}{2(r+1)}\right)^{1/r} \cdot \left[\left(C_9(q, p; a_1, b_1)\left|f'_1(a_1)\right|^q + C_{11}(q, p; a_1, b_1)\left|f'_1(b_1)\right|^q + C_{13}(q, p; a_1, b_1)\mu\right)^{1/q} \right.\\
&\left. + \left(C_{12}(q, p; b_1, a_1)\left|f'_1(a_1)\right|^q + C_{10}(q, p; b_1, a_1)\left|f'_1(b_1)\right|^q + C_{14}(q, p; b_1, a_1)\mu\right)^{1/q}\right],
\end{aligned} \tag{41}$$

where

$$C_9(q, p; a_1, b_1) = \int_0^{1/2} h(1-j)\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{q+q/p} dj, \tag{42}$$

$$C_{10}(q, p; b_1, a_1) = \int_{1/2}^{1} h(j)\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{q+q/p} dj, \tag{43}$$

$$C_{11}(q, p; a_1, b_1) = \int_0^{1/2} h(j)\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{q+q/p} dj, \tag{44}$$

$$C_{12}(q, p; b_1, a_1) = \int_{1/2}^{1} h(1-j)\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{q+q/p} dj, \tag{45}$$

$$C_{13}(q, p; a_1, b_1) = -\int_0^{1/2} \phi(j)\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{q+q/p}\left\|\frac{1}{b_1^p} - \frac{1}{a_1^p}\right\|^l dj, \tag{46}$$

$$C_{14}(q, p; b_1, a_1) = -\int_{1/2}^{1} \phi(j)|2j - 2 + \lambda|\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{q+q/p} \cdot \left\|\frac{1}{b_1^p} - \frac{1}{a_1^p}\right\|^l dj. \tag{47}$$

Proof. Using Lemma 9, we have

$$\begin{aligned}
&\left|(1-\lambda) f_1\left(\left[\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right]\right)^{1/p} + \lambda\left(\frac{f_1(a_1) + f_1(b_1)}{2}\right) - \frac{p\left(a_1^p b_1^p\right)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right| \\
&\le \frac{\left(b_1^p - a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\left[\int_0^{1/2}\left|(2j-\lambda)\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{1+1/p}\right|\left|f'_1 \cdot \left[\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{1/p}\right]\right| dj + \int_{1/2}^{1}\left|(2j-2+\lambda) \cdot \left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{1+1/p}\right|\left|f'_1\left[\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{1/p}\right]\right| dj\right].
\end{aligned} \tag{48}$$

Applying Hölder's integral inequality,

$$\begin{aligned}
&\le \frac{\left(b_1^p - a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\left[\left(\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\left(\int_0^{1/2}\left|\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{1+1/p} f'_1 \cdot \left[\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q} + \left(\int_{1/2}^{1}|(2j-2+\lambda)|^r dj\right)^{1/r} \right.\right.\\
&\left.\left. \cdot \left(\int_{1/2}^{1}\left|\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{1+1/p} f'_1 \cdot \left[\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\right]\right] = \frac{\left(b_1^p - a_1^p\right)}{2p\left(a_1^p b_1^p\right)} \\
&\cdot \left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\left(\int_0^{1/2}\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{q+q/p}\left|f'_1 \cdot \left[\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q} + \left(\int_{1/2}^{1}|(2j-2+\lambda)|^r dj\right)^{1/r} \right.\\
&\left. \cdot \left(\int_{1/2}^{1}\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{q+q/p}\left|f'_1\left[\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\right].
\end{aligned} \tag{49}$$

Since $|f_1'(x)|^q$ is in $SR(ph)$, so

$$\begin{aligned}
&\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\left(\int_0^{1/2}\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+q/p}\right.\right.\\
&\quad\cdot\left.\left[h(1-j)|f_1'(a_1)|^q + h(j)|f_1'(b_1)|^q - \mu\phi(j)\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l\right]dj\right)^{1/q}\\
&\quad+\left(\int_{1/2}^{1}|(2j-2+\lambda)|^r dj\right)^{1/r}\left(\int_{1/2}^{1}\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+q/p}\right.\\
&\quad\cdot\left.\left.\left[h(1-j)|f_1'(a_1)|^q + h(j)|(b_1)|^q - \mu\phi(j)\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l\right]dj\right)^{1/q}\right]\\
&\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\left(C_9(q,p;a_1,b_1)|f_1'(a_1)|^q\right.\right.\\
&\quad\left.+C_{11}(q,p;a_1,b_1)|f_1'(b_1)|^q + C_{13}(q,p;a_1,b_1)\mu\right)^{1/q}\\
&\quad+\left(\int_0^{1/2}|(2j-2+\lambda)|^r dj\right)^{1/r}\left(C_{12}(q,p;b_1,a_1)|f_1'(a_1)|^q\right.\\
&\quad\left.\left.+C_{10}(q,p;b_1,a_1)|f_1'(b_1)|^q + C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right]\\
&\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\times\left(\frac{\lambda^{r+1}+(1-\lambda)^{r+1}}{2(r+1)}\right)^{1/r}\\
&\quad\cdot\left[\left(C_9(q,p;a_1,b_1)|f_1'(a_1)|^q + C_{11}(q,p;a_1,b_1)|f_1'(b_1)|^q\right.\right.\\
&\quad\left.+C_{13}(q,p;a_1,b_1)\mu\right)^{1/q} + \left(C_{12}(q,p;b_1,a_1)|f_1'(a_1)|^q\right.\\
&\quad\left.\left.+C_{10}(q,p;b_1,a_1)|f_1'(b_1)|^q + C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right].
\end{aligned} \tag{50}$$

Hence, proved. □

For $\lambda = 0$, Theorem 12 reduces to the following result.

Corollary 13. *Let $M = [a_1, b_1] \subset \mathbb{R}\setminus\{0\}$ be a p-harmonic convex set and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ are in $SR(ph)$ on M, $r, q > 1$, $1/r + 1/q = 1$, and $\lambda \in [0, 1]$, then*

$$\begin{aligned}
&\left|f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{1/p}\right] - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right|\\
&\quad\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\times\left(\frac{1}{2(r+1)}\right)^{1/r}\left[\left(C_9(q,p;a_1,b_1)|f_1'(a_1)|^q\right.\right.\\
&\quad\left.+C_{11}(q,p;a_1,b_1)|f_1'(b_1)|^q + C_{14}(q,p;a_1,b_1)\mu\right)^{1/q}\\
&\quad+\left(C_{12}(q,p;b_1,a_1)|f_1'(a_1)|^q + C_{10}(q,p;b_1,a_1)|f_1'(b_1)|^q\right.\\
&\quad\left.\left.+C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right],
\end{aligned} \tag{51}$$

where C_9, C_{10}, C_{11}, C_{12}, C_{14},and C_{14} are given by (42)–(47). For $\lambda = 1$, Theorem 12 reduces to the following result.

Corollary 14. *Let $M = [a_1, b_1] \subset \mathbb{R}\setminus\{0\}$ be a p -harmonic convex set and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ are strongly reciprocally (p,h)-convex function of higher order on M, $r, q > 1$, $1/r + 1/q = 1$ and $\lambda \in [0, 1]$ then,*

$$\begin{aligned}
&\left|\frac{f_1(a_1)+f_1(b_1)}{2} - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right|\\
&\quad\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\times\left(\frac{1}{2(r+1)}\right)^{1/r}\\
&\quad\cdot\left[\left(C_9(q,p;a_1,b_1)|f_1'(a_1)|^q + C_{11}(q,p;a_1,b_1)|f_1'(b_1)|^q\right.\right.\\
&\quad\left.+C_{14}(q,p;a_1,b_1)\mu\right)^{1/q} + \left(C_{12}(q,p;b_1,a_1)|f_1'(a_1)|^q\right.\\
&\quad\left.\left.+C_{10}(q,p;b_1,a_1)|f_1'(b_1)|^q + C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right],
\end{aligned} \tag{52}$$

where C_9, C_{10}, C_{11}, C_{12}, C_{14},and C_{14} are given by (42)–(47). For $\lambda = 1/3$, Theorem 12 reduces to the following result.

Corollary 15. *Let $M = [a_1, b_1] \subset \mathbb{R}\setminus\{0\}$ be a p-harmonic convex set and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ is strongly reciprocally (p,h)-convex function of higher order on M, $r, q > 1$, $1/r + 1/q = 1$ and $\lambda \in [0, 1]$ then,*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f_1(a) + 4f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{1/p}\right] + f_1(b)\right] - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right|\\
&\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\times\left(\frac{1+2^{r+1}}{6.3^r(r+1)}\right)^{1/r}\\
&\quad\cdot\left[\left(C_9(q,p;a_1,b_1)|f_1'(a_1)|^q + C_{11}(q,p;a_1,b_1)|f_1'(b_1)|^q\right.\right.\\
&\quad\left.+C_{14}(q,p;a_1,b_1)\mu\right)^{1/q} + \left(C_{12}(q,p;b_1,a_1)|f_1'(a_1)|^q\right.\\
&\quad\left.\left.+C_{10}(q,p;b_1,a_1)|f_1'(b_1)|^q + C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right],
\end{aligned} \tag{53}$$

where C_9, C_{10}, C_{11}, C_{12}, C_{14}, and C_{14} are given by (42)–(47). For $\lambda = 1/2$, Theorem 12 reduces to the following result.

Corollary 16. *Let $M = [a_1, b_1] \subset \mathbb{R}\setminus\{0\}$ be a p-harmonic convex set and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ is strongly reciprocally (p,h)-convex function of higher order on M, $r, q > 1$, $1/r + 1/q = 1$ and $\lambda \in [0, 1]$ then,*

$$\left|\frac{1}{4}\left[f_1(a)+2f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p}\right]+f_1(b)\right]-\frac{p\left(a_1^p b_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right|$$
$$\le \frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\times\left(\frac{2}{4.2^r(r+1)}\right)^{1/r}$$
$$\cdot\left[\left(C_9(q,p;a_1,b_1)\left|f_1'(a_1)\right|^q+C_{11}(q,p;a_1,b_1)\left|f_1'(b_1)\right|^q\right.\right.$$
$$\left.+C_{14}(q,p;a_1,b_1)\mu\right)^{1/q}+\left(C_{12}(q,p;b_1,a_1)\left|f_1'(a_1)\right|^q\right.$$
$$\left.\left.+C_{10}(q,p;b_1,a_1)\left|f_1'(b_1)\right|^q+C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right], \tag{54}$$

where C_9, C_{10}, C_{11}, C_{12}, C_{14}, and C_{14} are given by (42)–(47).

Remark 17. Inserting $h(j)=j, \mu=0$ and $l=2$ with $\phi(j)=j(1-j)$ in Corollary 16, we obtained ([20], Corollary 3.8).

6. Conclusion

In this paper, a new definition of convex functions "namely strongly reciprocally (p,h)-convex functions of higher order" is introduced. This new definition extends almost all the existing versions of convex functions. For the strongly reciprocally (p,h)-convex functions of higher order, we established several interesting inequalities which have applications in optimization theory, probability theory, as well as pure and applied mathematics. We also established several fractional versions of the Hermite-Hadamard type inequalities. The remarks presented in the paper justify the validity of our results and prove that our results are more general than almost every existing result.

Data Availability

All data required for this research is included within this paper.

Conflicts of Interest

The authors declare that they do not have any competing interests.

Authors' Contributions

Limei Liu proposed and proved the main results, verified and analyzed the results, and arranged the funding. Muhammad Shoaib Saleem also verified and analyzed the results, arranged the funding for this paper, and prepared the final version of the manuscript. Faisal Yasin wrote the introduction and related the results of this paper to the existing literature. In particular, he showed that several results in the literature can be obtained directly from the results of this paper by suitable substitution of the involved parameters. Kiran Naseem Aslam supervised this work and proved some of its main results. Pengfei Wang proposed the problem and supervised this work.

Acknowledgments

This research is supported by Department of Mathematics, University of Okara, Okara Pakistan.

References

[1] L. Geng, M. S. Saleem, K. N. Aslam, and R. Bano, "Fractional version of Hermite–Hadamard and Fejér Type inequalities for a generalized class of convex functions," *Journal of Function Spaces*, vol. 2022, Article ID 2935740, 10 pages, 2022.

[2] B. Definetti, "Sulle stratificazioni convesse," *Annali di Matematica Pura ed Applicata*, vol. 30, no. 1, pp. 173–183, 1949.

[3] O. L. Manasarian, "Pseudo-convex functions," *SIAM Journal on Control*, vol. 3, pp. 281–290, 1965.

[4] X. Zhang and D. Alahmadi, "Study on the maximum value of flight distance based on the fractional differential equation for calculating the best path of shot put," *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 151–162, 2021.

[5] H. Hao and A. Albarakati, "Sports intensity and energy consumption based on fractional linear regression equation," *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 115–126, 2022.

[6] C. Li, N. Alhebaishi, and R. E. Omer, "Calculating university education model based on finite element fractional differential equations and macro-control analysis," *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 229–238, 2022.

[7] Y. Xiao, J. Liu, and A. Alkhathlan, "Informatisation of educational reform based on fractional differential equations," *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 79–90, 2022.

[8] A. R. Chouikha, "Sharp inequalities related to Wilker results," *Open Journal of Mathematical Sciences*, vol. 7, no. 1, pp. 19–34, 2023.

[9] S. Kermausuor, "New Simpson's type inequalities via (α_1, m_1)-(α_2, m_2)-preinvexity on the coordinates in both the first and second sense," *Open Journal of Mathematical Sciences*, vol. 7, no. 1, pp. 135–147, 2023.

[10] K. S. Zhang, "P-convex functions and their properties," *Pure and Applied Mathematics*, vol. 1, no. 23, pp. 130–133, 2007.

[11] B. T. Páolyak, "Existence theorems and convergence of minimizing sequences in extremum problems with restrictions," *Soviet Mathematics - Doklady*, vol. 7, pp. 72–75, 1966.

[12] S. Maden, S. Turhan, and Í. Íscan, "Hermite-Hadamard inequality for strongly p-convex functions," *Turkish Journal of Mathematics and Computer Science*, vol. 10, pp. 184–189, 2018.

[13] H. N. Shi and J. Zhang, "Some new judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.

[14] M. A. Noor, K. I. Noor, and S. Iftikhar, "Hermite-Hadamard inequalities for harmonic nonconvex functions," *MAGNT Research Report*, vol. 4, no. 1, pp. 24–40, 2016.

[15] M. Bracamonte, J. Medina, and M. Vivas, "On some inequalities for strongly reciprocally convex functions," *Extracta Mathematicae*, vol. 33, pp. 109–126, 2018.

[16] S. Varosanec, "On h-convexity," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 303–311, 2007.

[17] H. Li, M. S. Saleem, I. Hussain, and M. Imran, "Strongly Reciprocally p-Convex Functions and Some Inequalities," *Journal of Mathematics*, vol. 2020, Article ID 4957141, 9 pages, 2020.

[18] Z. B. Fang and R. Shi, "On the (p,h)-convex function and some integral inequalities," *Journal of Inequalities and Applications*, vol. 2014, no. 1, 2014.

[19] G. H. Lin and M. Fukushima, "Some exact penalty results for nonlinear programs and mathematical programs with equilibrium constraints," *Journal of Optimization Theory and Applications*, vol. 118, no. 1, pp. 67–80, 2003.

[20] M. A. Noor and K. I. Noor, "Integral inequalities for differentiable p-harmonic convex functions," *Faculty of Sciences and Mathematics*, vol. 31, no. 20, pp. 6575–6584, 2017.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 9742133, 12 pages
https://doi.org/10.1155/2022/9742133

Research Article

New Developments on Ostrowski Type Inequalities via q-Fractional Integrals Involving s-Convex Functions

Xiaoming Wang,[1] **Khuram Ali Khan**,[2] **Allah Ditta**,[3] **Ammara Nosheen**,[2] **Khalid Mahmood Awan**,[2] **and Rostin Matendo Mabela**[4]

[1]School of Mathematics & Computer Science, Shangrao Normal University, Shangrao 334001, China
[2]Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan
[3]Department of Mathematics, University of Jhang, Pakistan
[4]Department of Maths and Computer Science, Faculty of Science, University of Kinshasa, Congo

Correspondence should be addressed to Rostin Matendo Mabela; rostin.mabela@unikin.ac.cd

Received 23 August 2022; Accepted 27 September 2022; Published 13 October 2022

Academic Editor: Mohsan Raza

In the present paper, q-fractional integral operators are used to construct quantum analogue of Ostrowski type inequalities for the class of s-convex functions. The limiting cases include the nonfractional existing cases from literature. Specially, Ostrowski type inequalities for q-integrals and Ostrowski type inequalities for convex functions are deduced.

1. Introduction

In mid of twentieth century, Jackson (1910) has begun a symmetric investigation on q-integrals. The subject of quantum analysis, depending upon q-integrals, has different applications in different branches of mathematics and material sciences like number hypothesis, combinatorics, symmetrical polynomials, essential hypermathematical capacities, quantum hypothesis, mechanics, and in the hypothesis of relativity. The perusers are suggested to Set [1], Gauchman [2], and Kac and Cheung [3] for q-analogues of fractional calculus.

In numerous pragmatic issues, convexity hypothesis has stayed as a significant device in formation of vital imbalances. In many practical problems, it is important to bound one quantity by another quantity. The classical inequalities such as Ostrowski's inequalities are very useful for this purpose. Ostrowski type inequalities are well known to study the upper bounds for approximation of the integral average by the value of function and definition of s-convex function. Some new Ostrowski type inequalities for Riemann-Liouville fractional integral are established. Fractional calculus has been a well-known topic since it was initiated in the seventeenth century and studied by many great mathematicians of the time. Some classical inequalities including Ostrowski's inequality are examples of it.

2. Preliminary Results

In 1938, Ostrowski [4] established the following well-known and useful integral inequality:

Theorem 1. *Suppose $\psi : I \longrightarrow \mathbb{R}$ is the function differentiable in open interval of I, where $I \subseteq \mathbb{R}$ and let $\xi, \eta \in I^\circ$ with $\xi < \eta$. If $|\psi'(\theta)| \leq M$ for all $\theta \in [\xi, \eta]$, then the following inequality holds*

$$\left|\psi(\theta) - \frac{1}{\eta - \xi}\int_\xi^\eta \psi(u)du\right| \leq M(\eta - \xi)\left[\frac{1}{4} + \frac{(\theta - ((\xi + \eta)/2))^2}{(\eta - \xi)^2}\right], \tag{1}$$

for all $\theta \in [\xi, \eta]$. The least value of constant on R.H.S of (1) is 1/4.

Inequality (1) gives an approximate upper bound for the deviation of integral arithmetic mean $1/(\eta - \xi)\int_\xi^\eta \psi(\theta)d\theta$ to the function $\psi(\cdot)$ at point $\theta \in [\xi, \eta]$. In recent years, this inequality is studied extensively by different researchers, and its different variants can be seen in number of research papers including [5–11]. Recently, in [12, 13], Ostrowski type inequalities are studied for q-integrals.

The following notion of s-convex function in the second sense is from [14]:

A mapping $\psi : [0,\infty) \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense if

$$\psi(u\xi + (1-u)\eta) \le u^s\psi(\xi) + (1-u)^s\psi(\eta), \tag{2}$$

for all $\xi, \eta \in [0,\infty)$, $u \in [0, 1]$, and some static $s \in (0, 1]$.

The following Lemma is established by Alomari et al. (see [8]).

Lemma 2. *If $\psi' \in L[\xi, \eta]$, then we have the equality*

$$\psi(\theta) - \frac{1}{\eta-\xi}\int_\xi^\eta \psi(u)du = \frac{(\theta-\xi)^2}{\eta-\xi}\int_0^1 uf'(u\theta + (1-u)\xi)du - \frac{(\eta-\theta)^2}{\eta-\xi}\int_0^1 uf'(u\theta + (1-u)\eta)du, \tag{3}$$

for each $\theta \in [\xi, \eta]$.

By using Lemma 2, Alomari et al. in [8] proved the following results of Ostrowski type inequalities:

Theorem 3. *Suppose $\psi' \in L[\xi, \eta]$. If $|\psi'|$ in term of second sense is s-convex on $[\xi, \eta]$, for unique $s \in (0, 1]$ and $|\psi'(\theta)| \le M$,$\theta \in [\xi, \eta]$, then*

$$\left|\psi(\theta) - \frac{1}{\eta-\xi}\int_\xi^\eta \psi(u)du\right| \le \frac{M}{\eta-\xi}\left[\frac{(\theta-\xi)^2 + (\eta-\theta)^2}{s+1}\right] \tag{4}$$

holds, for each $\theta \in [\xi, \eta]$.

Theorem 4. *Suppose $\psi' \in L[\xi, \eta]$. If $|\psi'|^m$ is a s-convex in second sense in $[\xi, \eta]$ for unique $s \in (0, 1]$,$m > 1$,$n = m/(m-1)$ and $|\psi'(\theta)| \le M$,$\theta \in [\xi, \eta]$, then*

$$\left|\psi(\theta) - \frac{1}{\eta-\xi}\int_\xi^\eta \psi(u)du\right| \le \frac{M}{(1+n)^{1/n}}\left(\frac{2}{s+1}\right)^{1/m} \cdot \left[\frac{(\theta-\xi)^2 + (\eta-\theta)^2}{\eta-\xi}\right] \tag{5}$$

holds, for each $\theta \in [\xi, \eta]$.

Theorem 5. *Suppose $\psi' \in L[\xi, \eta]$. If $|\psi'|^m$ is s-convex in second sense on $[\xi, \eta]$ for static $s \in (0, 1]$,$m \ge 1$ and $|\psi'(\theta)| \le M$,$\theta \in [\xi, \eta]$, then*

$$\left|\psi(\theta) - \frac{1}{\eta-\xi}\int_\xi^\eta \psi(u)du\right| \le M\left(\frac{2}{s+1}\right)^{1/m}\left[\frac{(\theta-\xi)^2 + (\eta-\theta)^2}{2(\eta-\xi)}\right] \tag{6}$$

holds, for each $\theta \in [\xi, \eta]$.

Further some existing results on s-convex functions can be seen in [11], and some results involving fractional operators can be found in [15–21]. In case of fractional integrals, see the following lemma from [1].

Lemma 6. *If $\psi' \in L[\xi, \eta]$, then for all $\theta \in [\xi, \eta]$ and $\beta > 0$*

$$\left(\frac{(\theta-\xi)^\beta + (\eta-\theta)^\beta}{\eta-\xi}\right)\psi(\theta) - \frac{\Gamma(\beta+1)}{(\eta-\xi)}\left[J^\beta_{\theta-}\psi(\xi) + J^\beta_{\theta+}\psi(\eta)\right] = \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta\psi'(u\theta + (1-u)\xi)du - \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta\psi'(u\theta + (1-u)\eta)du \tag{7}$$

holds, where $\Gamma(\beta) = \int_0^\infty e^{-u}u^{\beta-1}du$ is Euler Gamma function.

By using Lemma 6, Set in [1] proved the following:

Theorem 7. *For $\psi' \in L[\xi, \eta]$. If $|\psi'(\theta)|$ is s-convex in second sense on $[\xi, \eta]$ for fix $s \in (0, 1]$ and $|\psi'(\theta)| \le M$,$\theta \in [\xi, \eta]$, then the following fractional integrals inequality, for $\beta > 0$*

$$\left|\left(\frac{(\theta-\xi)^\beta + (\eta-\theta)^\beta}{\eta-\xi}\right)\psi(\theta) - \frac{\Gamma(\beta+1)}{(\eta-\xi)}\left[J^\beta_{\theta-}\psi(\xi) + J^\beta_{\theta+}\psi(\eta)\right]\right| \le \frac{M}{\eta-\xi}\left(1 + \frac{\Gamma(\beta+1)\Gamma(s+1)}{\Gamma(\beta+s+1)}\right)\left[\frac{(\theta-\xi)^{\beta+1} + (\eta-\theta)^{\beta+1}}{\beta+s+1}\right] \tag{8}$$

holds.

Theorem 8. *Suppose $\psi' \in L[\xi, \eta]$. If $|\psi'|^m$ is s-convex in second sense on $[\xi, \eta]$ for some fixed $s \in (0, 1]$, $n, m > 1$ and $|\psi'(\theta)| \le M$,$\theta \in [\xi, \eta]$, then*

$$\left|\left(\frac{(\theta-\xi)^\beta + (\eta-\theta)^\beta}{\eta-\xi}\right)\psi(\theta) - \frac{\Gamma(\beta+1)}{(\eta-\xi)}\left[J^\beta_{\theta-}\psi(\xi) + J^\beta_{\theta+}\psi(\eta)\right]\right| \le \frac{M}{(1+n\beta)^{1/n}}\left(\frac{2}{s+1}\right)^{1/m}\left[\frac{(\theta-\xi)^{\beta+1} + (\eta-\theta)^{\beta+1}}{\eta-\xi}\right] \tag{9}$$

holds, where $(1/n) + (1/m) = 1$,$\beta > 0$.

Theorem 9. *Suppose $\psi' \in L[\xi,\eta]$, and assume that $|\psi'|^m$ is s-convex in second sense on $[\xi,\eta]$ for some fix $s \in (0,1]$, $m \geq 1, |\psi'(\theta)| \leq M$, and $\theta \in [\xi,\eta]$. Then*

$$\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\psi(\theta)-\frac{\Gamma(\beta+1)}{(\eta-\xi)}\left[J^\beta_{\theta-}\psi(\xi)+J^\beta_{\theta+}\psi(\eta)\right]\right| \leq M\left(\frac{1}{1+\beta}\right)^{1-(1/m)}\left(\frac{1}{\beta+s+1}\right)^{1/m} \times\left(1+\frac{\Gamma(\beta+1)\Gamma(s+1)}{\Gamma(\beta+s+1)}\right)^{1/m}\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right] \tag{10}$$

holds for each $\beta > 0$.

Theorem 10. *Suppose $\psi' \in L[\xi,\eta]$. If $|\psi'|^m$ is s-convex in second sense on $[\xi,\eta]$ for some fixed $s \in (0,1]$ and $n, m > 1$. Then*

$$\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\psi(\theta)-\frac{\Gamma(\beta+1)}{(\eta-\xi)}\left[J^\beta_{\theta-}\psi(\xi)+J^\beta_{\theta+}\psi(\eta)\right]\right| \leq \frac{2^{(s-1)/m}}{(1+n\beta)^{1/n}(\eta-\xi)}\left[(\theta-\xi)^{\beta+1}\left|\psi'\left(\frac{\theta+\xi}{2}\right)\right| +(\eta-\theta)^{\beta+1}\left|\psi'\left(\frac{\eta+\theta}{2}\right)\right|\right] \tag{11}$$

holds, for $(1/n)+(1/m)=1$ and $\beta>0$.

Note that if $s=1$, the definition of s-convexity reduces to classical convexity of functions defined on $\mathbb{R}^+$.

3. Preliminaries about q-Integrals and Related Inequalities

The following properties of q-derivatives are recalled from [3].

3.1. q-Derivative. For $\phi \in C[\xi,\eta]$, q-derivative of ϕ at $\theta \in [\xi,\eta]$ is given by

$$_\xi D_q\phi(\theta)=\frac{\phi(\theta)-\phi(q\theta+(1-q)\xi)}{(1-q)(\theta-\xi)}\theta \neq \xi. \tag{12}$$

For $n \geq 1$, we have the following relation:

$$(\theta-\xi)^n_q=\prod_{J=0}^{n}(\theta-q^J\xi), \quad (\xi-\theta)^n_q=\prod_{J=0}^{n}(\xi-q^J\theta). \tag{13}$$

Respective derivatives are

$$D_q(\theta-\xi)^n_q=[n](\theta-\xi)^{n-1}_q, \quad D_q(\xi-\theta)^n_q=-[n](\xi-q\theta)^{n-1}_q, \quad (\xi-q\theta)^n_q=-\frac{1}{[n+1]}D_q(\xi-\theta)^{n+1}_q, \tag{14}$$

where $[n]=(q^n-1)/(q-1)$. The fractional calculus is a generalization of classical calculus concerned with operations of integration and differentiation of noninteger fractional order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The first known reference can be found in the correspondence of G. W. Leibniz and Marquis de l'Hospital in 1695 where the question of meaning of the semiderivative has been raised. This question consequently attracted the interest of many well-known mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, and Letnikov.

3.2. Fractional Integral from [8]. Let $\psi \in L^1[\xi,\eta]$. The Riemann-Liouville integrals $J^\beta_{\xi+}\psi$ and $J^\beta_{\eta-}\psi$ of order $\beta>0$ for $\xi \geq 0$ are defined by

$$J^\beta_{\xi^+}\psi(\theta)=\frac{1}{\Gamma(\beta)}\int_\xi^\theta(\theta-u)^{\beta-1}\psi(u)du, \quad \theta>\xi, \quad J^\beta_{\eta^-}\psi(\theta)=\frac{1}{\Gamma(\beta)}\int_\theta^\eta(u-\theta)^{\beta-1}\psi(u)du, \quad \theta<\eta. \tag{15}$$

3.3. q-Antiderivative. q-Antiderivative along with its properties can be studied in [22]. Suppose that $\psi \in C[\xi,\eta]$. Then q-definite integral for $\theta \in [\xi,\eta]$ is defined as

$$\int_\xi^\theta \psi(u)_\xi d_q u=(1-q)(\theta-\xi)\sum_{n=0}^{\infty}q^n\psi(q^n\theta+(1-q^n)\xi), \tag{16}$$

which gives

$$\int(\xi-\theta)^\beta_q d_q\theta=-\frac{q(\xi-q^{-1}\theta)^{\beta+1}_q}{[\beta+1]} \quad (\beta \neq -1), \quad \int_0^1 u^{\beta+s}d_q u=\frac{1}{[\beta+s+1]}, \quad \int_0^1 \mathrm{u}^\beta(1-\mathrm{u})^\mathrm{s}\mathrm{d}_q\mathrm{u}=\frac{\Gamma_q(\beta+1)\Gamma_q(\mathrm{s}+1)}{\Gamma_q(\beta+\mathrm{s}+2)}. \tag{17}$$

Formula for q-integration by parts [23]: Let $\psi, g \in C[\xi,\eta]$, and $\theta \in [\xi,\eta]$, and then

$$\int_c^\theta \psi(u)_\xi D_q g(u) d_q u = \psi(\theta)g(\theta) - \psi(c)g(c) - \int_c^\theta g(qu + (1-q)\xi)_\xi D_q \psi(u) d_q u. \quad (18)$$

The following q-integral inequality is from [23]:

Theorem 11. *Suppose $\psi : [\xi, \eta] \longrightarrow \mathbb{R}$ is a q-differentiable mapping. If $|D_q \psi(\theta)| \le M$ for all $\theta \in [\xi, \eta]$ and $0 < q < 1$, then*

$$\left| \psi(\theta) - \frac{1}{\eta - \xi} \int_\xi^\eta \psi(u)_\xi d_q u \right| \le M \left[\frac{2q}{1+q} \left(\frac{\theta - (((3q-1)\xi + (1+q)\eta)/4q)}{\eta - \xi} \right)^2 + \left(\frac{(-q^2 + 6q - 1)}{8q(1+q)} \right) \right] \quad (19)$$

holds for all $\theta \in [\xi, \eta]$.

The least value of constant on R.H.S of (19) is $(-q^2 + 6q - 1)/8q(1+q)$.

q-Hölder inequality [24]. Let ψ, Φ be q-integrable on $[\xi, \eta]$ and $0 < q < 1$ and $(1/n) + (1/m) = 1$ with $m > 1$, and then we state as

$$\int_\xi^\eta |\psi(u)\Phi(u)| d_q u \le \left\{ \int_\xi^\eta |\psi(u)|^n d_q u \right\}^{1/n} \left\{ \int_\xi^\eta |\Phi(u)|^m d_q u \right\}^{1/m}. \quad (20)$$

The following inequalities are derived from q-Hölder inequality:

q-Minkowski's inequality. Let $\xi, \eta \in \mathbb{R}$ and $n > 1$, for continuous functions $\psi, \Phi : [\xi, \eta] \longrightarrow \mathbb{R}$, we have stated

$$\left\{ \int_\xi^\eta |(\psi(u) + \Phi(u))|^n d_q u \right\}^{1/n} \le \left\{ \int_\xi^\eta |\psi(u)|^n d_q u \right\}^{1/n} + \left\{ \int_\xi^\eta |\Phi(u)|^n d_q u \right\}^{1/n}. \quad (21)$$

q-Power mean inequality. Let $(1/n) + (1/m) = 1$ with $n, m > 1$, and let $\xi, \eta \in \mathbb{R}$ and for continuous functions $\psi, g : [\xi, \eta] \longrightarrow \mathbb{R}$, and then

$$\int_\xi^\eta |\psi(u)\Phi(u)| d_q u \le \left\{ \int_\xi^\eta |\psi(u)| d_q u \right\}^{1-(1/m)} \left\{ \int_\xi^\eta |\psi(u)||\Phi(u)|^m d_q u \right\}^{1/m}. \quad (22)$$

Theorem 12 (see [25]). *Let $\psi : [\xi, \eta] \longrightarrow \mathbb{R}$ be a function and $0 < q < 1$. Then*

$$\int_0^1 \psi(u\eta + (1-u)\xi) d_q u = \frac{1}{\eta - \xi} \int_\xi^\eta \psi(t) d_q t. \quad (23)$$

Example 1.

$$\begin{aligned} \int_\xi^\theta u(u-\xi) d_q u &= \frac{1}{q+1} \left(\theta(\theta - \xi)^2 - \int_\xi^\theta (u-\xi)^2 d_q u \right) \\ &= \frac{1}{q+1} \left(\theta(\theta-\xi)^2 - \frac{(\theta-\xi)^3}{1+q+q^2} \right) \\ &= \frac{(\theta-\xi)^2}{1+q} \left(\frac{\xi(1+q) + q^2 a}{1+q+q^2} \right). \end{aligned} \quad (24)$$

Now, we have

$$\begin{aligned} \int_\xi^\theta u(u-\xi) d_q u &= \int_0^{\theta-\xi} (t+\xi) t d_q t \\ &= \frac{1}{q+1} \left(\theta(\theta-\xi)^2 - \int_0^{\theta-\xi} (qt)^2 d_q t \right) \\ &= \frac{(\theta-\xi)^2}{1+q} \left(\frac{\xi(1+q) + q^2 a}{1+q+q^2} \right). \end{aligned} \quad (25)$$

Proposition 13. *For each $k, r \in \mathbb{N}$(or $\mathbb{Z}\, q \in \mathbb{R}^+$) [2], we have*

$$[k+r]_q = [k]_q + q^k [r]_q. \quad (26)$$

Exponential functions and Taylor series (q-analogues) from [26]:

$$E_q^\theta = \sum_{n=0}^\infty q^{n(n-1)/2} \frac{\theta^n}{[n]_q!} = (1 + (1-q)\theta)_q^\infty. \quad (27)$$

q-Gamma and q-Beta functions:

For any $u > 0$

$$\Gamma_q(u) = \int_0^\infty \theta^{u-1} \mathrm{E}_q^{-q\theta} d_q \theta \quad (28)$$

is called q-Gamma Euler function and for any $u, p > 0$

$$\beta_q(u,p) = \int_0^1 \theta^{u-1} (1 - q\theta)_q^{p-1} d_q \theta \quad (29)$$

is called q-Beta function.

Relation between q-Gamma and q-Beta Function:

$$\beta_q(\theta, \sigma) = \frac{\Gamma_q(\theta)\Gamma_q(\sigma)}{\Gamma_q(\theta + \sigma)}, \quad (30)$$

$$\Gamma_q(\mathrm{m}+1) = [m]\Gamma_q(\mathrm{m})\ (\mathrm{m} > 0). \quad (31)$$

The following definition is introduced by Agarwal in [27] when $\xi = 0$ and by Rajkovic et al. [28] for $\xi \ne 0$.

q-Fractional integral from [29]. Let $\psi \in L^1[\xi, \eta]$. The Riemann-Liouville q-integrals $J^{\beta}_{\xi+}\psi$ and $J^{\beta}_{\eta-}\psi$ of order $\beta > 0$ for $\xi \geq 0$ are defined by

$$J^{\beta}_{q,\xi^+}\psi(\theta) = \frac{1}{\Gamma_q(\beta)}\int_{\xi}^{\theta}(\theta - qu)^{(\beta-1)}\psi(u)d_q u, \theta > \xi,$$
$$J^{\beta}_{q,\eta^-}\psi(\theta) = \frac{1}{\Gamma_q(\beta)}\int_{q\theta}^{\eta}(u - q\theta)^{(\beta-1)}\psi(u)d_q u, \theta < \eta, \tag{32}$$

respectively, where $\Gamma_q(\beta) = \int_0^{\infty} u^{(\beta-1)}\mathrm{E}_q^{-qu}d_q u$. Here $J^0_{\xi+}\psi(\theta) = J^0_{\eta-}\psi(\theta) = \psi(\theta)$.

4. Ostrowski Type Inequalities via q-Fractional Integrals

In order to prove our main results, we have to prove the following lemma with the help of ([30], Lemma 1), which can be seen in [31]).

Lemma 14. *Suppose that $\psi : [\xi, \eta] \longrightarrow \mathbb{R}$ is q-differentiable mapping. If $D_q\psi \in L[\xi, \eta]$, then for all $\theta \in [\xi, \eta]$ and $\beta > 0$, we have*

$$\begin{aligned}\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}&\int_0^1 u^{\beta}D_q\psi(u\theta+(1-u)\xi)d_q u\\ &-\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\int_0^1 u^{\beta}D_q\psi(u\theta+(1-u)\eta)d_q u\\ &=\left(\frac{(\theta-\beta)^{\beta}+(\eta-\theta)^{\beta}}{\eta-\xi}\right)\left(\frac{q^{\beta}+[\beta](1-q)}{q^{\beta}}\right)\psi(\theta)\\ &-\frac{\Gamma_q(\beta+1)}{q^{\beta}(\eta-\xi)}\left[J^{\beta}_{q,\theta^-}\psi(\xi)+J^{\beta}_{q,\theta^+}\psi(\eta)\right],\end{aligned} \tag{33}$$

where $\Gamma_q(\beta) = \int_0^{\infty} u^{(\beta-1)}E_q^{-qu}d_q u$.

Proof. By using the formula of q-integration by parts

$$\begin{aligned}\int_0^1 &u^{\beta}D_q\psi(u\theta+(1-u)\xi)d_q u\\ &=u^{\beta}\frac{\psi(u\theta+(1-u)\xi)}{\theta-\xi}\bigg|_0^1-\int_0^1[\beta]u^{\beta-1}\frac{\psi(qu\theta+(1-qu)\xi)}{\theta-\xi}d_q u\\ &=\frac{\psi(\theta)}{\theta-\xi}-\frac{[\beta]}{\theta-\xi}\int_{\xi}^{q\theta-q\xi+\xi}\left(\frac{(u-q\xi)}{q(\theta-\xi)}\right)^{\beta-1}\frac{\psi(u)}{q(\theta-\xi)}d_q u\\ &=\frac{\psi(\theta)}{\theta-\xi}+\frac{[\beta]}{q^{\beta}(\theta-\xi)^{\beta+1}}\int_{\xi}^{q\theta-q\xi+\xi}(u-\xi)^{\beta-1}\psi(u)d_q u,\end{aligned} \tag{34}$$

where

$$\begin{aligned}\int_{\xi}^{q\theta-q\xi+\xi}&(u-\xi)^{\beta-1}\psi(u)d_q u=(q\theta-q\xi+\xi-\xi)(1-q)\sum_{n=0}^{\infty}q^n\\ &\cdot(q^n(q\theta-q\xi+\xi)+(1-q^n)\xi-\xi)^{\beta-1}\\ &\times\psi(q^n(q\theta-q\xi+\xi)+(1-q^n)\xi)\\ &=\frac{\psi(\theta)}{\theta-\xi}-\frac{[\beta]q(\theta-\xi)(1-q)}{q^{\beta}(\theta-\xi)^{\beta+1}q}\sum_{n=1}^{\infty}q^n(q^n\theta-q^n\xi)^{\beta-1}\\ &\cdot\psi(q^n\theta+(1-q^n)\xi)+\frac{[\beta](\theta-\xi)(\theta-\xi)^{\beta-1}(1-q)}{q^{\beta}(\theta-\xi)^{\beta+1}}\\ &\cdot\psi(\theta)-\frac{[\beta](\theta-\xi)(\theta-\xi)^{\beta-1}(1-q)}{q^{\beta}(\theta-\xi)^{\beta+1}}\psi(\theta)\\ &=\frac{\psi(\theta)}{\theta-\xi}-\left[\frac{[\beta](\theta-\xi)(1-q)}{q^{q}(\theta-\xi)^{\beta+1}}\sum_{n=1}^{\infty}q^n(q^n\theta-q^n\xi)^{\beta-1}\right.\\ &\left.\cdot\psi(q^n\theta+(1-q^n)\xi)+\frac{[\beta](\theta-\xi)(\theta-\xi)^{\beta-1}(1-q)}{q^{\beta}(\theta-\xi)^{\beta+1}}\psi(\theta)\right]\\ &+\frac{[\beta](1-q)}{q^{\beta}(\theta-\xi)}\psi(\theta)=\frac{\psi(\theta)}{\theta-\xi}\\ &-\left[\frac{[\beta](\theta-\xi)(1-q)}{q^{\beta}(\theta-\xi)^{\beta+1}}\sum_{n=0}^{\infty}q^n(q^n\theta-q^n\xi)^{\beta-1}\right.\\ &\left.\cdot\psi(q^n\theta+(1-q^n)\xi)\right]+\frac{[\beta](1-q)}{q^{\beta}(\theta-\xi)}\psi(\theta).\end{aligned} \tag{35}$$

$$\begin{aligned}\int_0^1 u^{\beta}D_q\psi(u\theta+(1-u)\xi)d_q u&=\frac{\psi(\theta)}{\theta-\xi}-\frac{[\beta]\Gamma_q(\beta)}{q^{\beta}(\theta-\xi)^{\beta+1}\Gamma_q(\beta)}\\ &\cdot\int_{\xi}^{q\theta}(u-q\xi)^{\beta-1}\psi(u)d_q u+\frac{[\beta](1-q)}{q^{\beta}(\theta-\xi)}\psi(\theta).\end{aligned} \tag{36}$$

Multiply both sides of (36) by $(\theta-\xi)^{\beta+1}/(\eta-\xi)$

$$\begin{aligned}\frac{(\theta-\xi)^{\beta}}{\eta-\xi}&\int_0^1 u^{\beta}D_q\psi(u\theta+(1-u)\xi)d_q u\\ &=\frac{(\theta-\xi)^{\beta}}{\eta-\xi}\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^{\beta}(\eta-\xi)}J^{\beta}_{q,\theta^-}(\xi)\\ &+\frac{[\beta](1-q)(\theta-\xi)^{\beta}}{q^{\beta}(\eta-\xi)}\psi(\theta).\end{aligned} \tag{37}$$

Similar, calculation gives

$$\begin{aligned}\int_0^1 u^{\beta}D_q\psi(u\theta+(1-u)\eta)d_q u&=\frac{\psi(\theta)}{\theta-\eta}+\frac{[\beta]\Gamma_q(\beta)}{q^{\beta}(\eta-\theta)^{\beta+1}\Gamma_q(\beta)}\\ &\cdot\int_{\eta}^{\theta}(\eta-qu)^{\beta-1}\psi(u)d_q u-\frac{[\beta](1-q)}{q^{\beta}(\eta-\theta)}\psi(\theta).\end{aligned} \tag{38}$$

Multiply both sides of (38) by $(\eta-\theta)^{\beta+1}/(\eta-\xi)$ to get

$$\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta D_q\psi(u\theta+(1-u)\eta)d_qu=\frac{\psi(\theta)}{\theta-\eta}+\frac{[\beta]\Gamma_q(\beta)}{q^\beta(\eta-\theta)^{\beta+1}\Gamma_q(\beta)}\int_\eta^\theta(\eta-qu)^{\beta-1}\psi(u)d_qu-\frac{[\beta](1-q)}{q^\beta(\eta-\theta)}\psi(\theta). \tag{39}$$

By combining (37) and (39), we obtain the following desired result.

$$\begin{aligned}&\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta D_q\psi(u\theta+(1-u)\xi)d_qu-\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\cdot\int_0^1 u^\beta D_q\psi(u\theta+(1-u)\eta)d_qu=\frac{(\theta-\xi)^\beta}{\eta-\xi}\psi(\theta)\\&-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}J^\beta_{q,\theta^-}(\xi)+\frac{[\beta](1-q)(\theta-\xi)^\beta}{q^\beta(\eta-\xi)}\psi(\theta)+\frac{(\eta-\theta)^\beta}{\eta-\xi}\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}J^\beta_{q,\theta^+}(\eta)\\&+\frac{[\beta](1-q)(\eta-\theta)^\beta}{q^\beta(\eta-\xi)}\psi(\theta)=\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)\\&-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\left[J^\beta_{q,\theta^-}(\xi)+J^\beta_{q,\theta^+}(\eta)\right].\end{aligned} \tag{40}$$

□

Remark 15.

(a) Taking $q=1$, Lemma 14 becomes Lemma 6

(b) Taking $\beta=1$, Lemma 14 reduces to ([32], Lemma 3.1)

$$\begin{aligned}&\frac{1}{q}\left[\psi(\theta)-\frac{1}{\eta-\xi}\int_\xi^\eta\psi(u)_a d_qu\right]\\&=\frac{(\theta-\xi)^2}{\eta-\xi}\int_0^1 uD_q\psi(u\theta+(1-u)\xi)_0d_qu-\frac{(\eta-\theta)^2}{\eta-\zeta}\int_0^1 uD_q\psi(u\theta+(1-u)\eta)_0d_qu.\end{aligned} \tag{41}$$

By using Lemma 14, we established some Ostrowski type q-fractional integral inequalities.

Theorem 16. *Suppose $\psi:[\xi,\eta]\subset\mathbb{R}^+\longrightarrow\mathbb{R}$ is a q-differentiable mapping in such a way that $D_q\psi\in L[\xi,\eta]$. If $|D_q\psi|$ is $s-$convex in second sense on $[\xi,\eta]$ for some static $s,q\in(0,1]$ and $|D_q\psi(\theta)|\le M,\theta\in[\xi,\eta]$, subsequently, the following integral inequality for q-fractional integrals is valid.*

$$\begin{aligned}&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\cdot\left[J^\beta_{q,\theta^-}\psi(\xi)+J^\beta_{q,\theta^+}\psi(\eta)\right]\right|\le\frac{M}{\eta-\xi}\\&\cdot\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{[\beta+s+1]}\right].\end{aligned} \tag{42}$$

Proof. Consider Lemma 14, and since $|D_q\psi|$ is $s-$convex mapping on $[\xi,\eta]$, we can write

$$\begin{aligned}&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\cdot\left[J^\beta_{q,\theta^-}(\xi)+J^\beta_{q,\theta^+}(\eta)\right]\right|\le\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\\&\cdot\int_0^1 u^\beta|D_q\psi(u\theta+(1-u)\xi)|d_qu+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\cdot\int_0^1 u^\beta|D_q\psi(u\theta+(1-u)\eta)|d_qu\le\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\\&\cdot\int_0^1 u^{\beta+s}|D_q\psi(\theta)|d_qu+u^\beta(1-u)^s|D_q\psi(\xi)|d_qu+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\int_0^1 u^{\beta+s}|D_q\psi(\theta)|d_qu+u^\beta(1-u)^s|D_q\psi(\eta)|d_qu\\&\le\frac{(\theta-\xi)^{\beta+1}M}{\eta-\xi}\left(\int_0^1 u^{\beta+s}+u^\beta(1-u)^s\right)d_qu+\frac{(\eta-\theta)^{\beta+1}M}{\eta-\xi}\cdot\left(\int_0^1 u^{\beta+s}+u^\beta(1-u)^s\right)d_qu\le\frac{(\theta-\xi)^{\beta+1}M}{\eta-\xi}\\&\cdot\left[\frac{1}{[\beta+s+1]}+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+2)}\right]+\frac{(\eta-\theta)^{\beta+1}M}{\eta-\xi}\cdot\left[\frac{1}{[\beta+s+1]}+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+2)}\right]\\&=\frac{M}{\eta-\xi}\left[\frac{1}{[\beta+s+1]}+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+2)}\right]\cdot\left[(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}\right]=\frac{M}{\eta-\xi}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\\&\cdot\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{[\beta+s+1]}\right],\end{aligned} \tag{43}$$

where we have used the fact that

$$\int_0^1 u^{\beta+s} d_q u = \frac{1}{[\beta+s+1]},$$
$$\int_0^1 u^{\beta}(1-u)^{s} d_q u = \frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+2)}. \quad (44)$$

Therefore by applying the reduction formula $\Gamma_q(m+1) = [m]\Gamma_q(m)(n>0)$ for Euler gamma function, it completes the proof. □

Corollary 17. *Suppose that $\psi : [\xi,\eta] \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$, is q-differentiable function in such a way that $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|$ is convex on $[\xi,\eta]$ for some fixed $s, q \in (0,1]$ and $|D_q\psi(\theta)| \le M$, $\theta \in [\xi,\eta]$, then we have the following q-fractional integral inequality*

$$\left|\left(\frac{(\theta-\xi)^{\beta}+(\eta-\theta)^{\beta}}{\eta-\xi}\right)\left(\frac{q^{\beta}+[\beta](1-q)}{q^{\beta}}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^{\beta}(\eta-\xi)} \cdot \left[J^{\beta}_{q,\theta^-}\psi(\xi) + J^{\beta}_{q,\theta^+}\psi(\eta)\right]\right| \le \frac{M}{\eta-\xi}\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{[\beta+1]}\right] \quad (45)$$

is valid for $\beta > 0$.

Proof. Taking $s=1$ in (42), the required result follows. □

Remark 18.

(a) q-analogue of Theorem 3, (4) is followed by taking $\beta = 1$ and $q=1$ in (42)

(b) Taking $q=1$ in (42), it follows the inequality (8) of Theorem 7

Theorem 19. *Suppose that $\psi : [\xi,\eta] \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is q-differentiable function in such a way that $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|^m$ is s-convex in second sense on $[\xi,\eta]$ for some static $s, q \in (0,1]$, $n, m > 1$ and $|D_q\psi(\theta)| \le M, \theta \in [\xi,\eta]$, then*

$$\left|\left(\frac{(\theta-\xi)^{\beta}+(\eta-\theta)^{\beta}}{\eta-\xi}\right)\left(\frac{q^{\beta}+[\beta](1-q)}{q^{\beta}}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^{\beta}(\eta-\xi)} \cdot \left[J^{\beta}_{q,\theta^-}\psi(\xi) + J^{\beta}_{q,\theta^+}\psi(\eta)\right]\right| \le \frac{M}{(1+n\beta)^{1/n}} \cdot \left[\frac{1+q\left\{1-\left(1-q^{-1}\right)^{s+1}\right\}}{[s+1]}\right]^{1/m}\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right], \quad (46)$$

where $(1/n)+(1/m)=1$, and $\beta > 0$.

Proof. From Lemma 14,

$$\left|\left(\frac{(\theta-\xi)^{\beta}+(\eta-\theta)^{\beta}}{\eta-\xi}\right)\left(\frac{q^{\beta}+[\beta](1-q)}{q^{\beta}}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^{\beta}(\eta-\xi)} \cdot \left[J^{\beta}_{q,\theta^-}(\xi) + J^{\beta}_{q,\theta^+}(\eta)\right]\right| \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi} \cdot \int_0^1 u^{\beta}\left|D_q\psi(u\theta+(1-u)\xi)\right| d_q u + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi} \cdot \int_0^1 u^{\beta}\left|D_q\psi(u\theta+(1-u)\eta)\right| d_q u. \quad (47)$$

By virtue of Hölder's inequality, we have

$$\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^{\beta}\left|D_q\psi(u\theta+(1-u)\xi)\right| d_q u + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi} \cdot \int_0^1 u^{\beta}\left|D_q\psi(u\theta+(1-u)\eta)\right| d_q u \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi} \cdot \left(\int_0^1 u^{\beta n} d_q u\right)^{1/n}\left(\int_0^1 \left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_q u\right)^{1/m} + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\left(\int_0^1 u^{\beta n} d_q u\right)^{1/n} \cdot \left(\int_0^1 \left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_q u\right)^{1/m}. \quad (48)$$

Since $|D_q\psi|^m$ is $s-$ convex in the second sense on $[\xi,\eta]$ and $D_q\psi(\theta)$ is bounded by number M,

$$\int_0^1 u^{\beta n} d_q u = \frac{1}{[\beta n+1]} \quad (49)$$

$$\begin{aligned}\int_0^1 \left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_q u &\le \int_0^1 u^{s}\left|D_q\psi(\theta)\right|^m d_q u + \int_0^1 (1-u)^{s}\left|D_q\psi(\xi)\right|^m d_q u \le \int_0^1 u^{s} d_q u + \int_0^1 (1-u)^{s} d_q u \\ &= M^m\left(\left|\frac{u^{s+1}}{[s+1]}\right|_0^1 + \left|\frac{-q\left(1-q^{-1}u\right)^{s+1}}{[s+1]}\right|_0^1\right) \\ &= M^m\left(\frac{1}{[s+1]} - \frac{q\left(1-q^{-1}\right)^{s+1}}{[s+1]} + \frac{q}{[s+1]}\right) \\ &= M^m\left(\frac{1+q-q\left(1-q^{-1}\right)^{s+1}}{[s+1]}\right).\end{aligned} \quad (50)$$

Similarly, we have

$$\int_0^1 \left|D_q\psi(u\theta + (1-u)\eta)\right|^m d_qu \leq \int_0^1 u^s\left|D_q\psi(\theta)\right|^m d_qu + \int_0^1 (1-u)^s\left|D_q\psi(\eta)\right|^m d_qu, \tag{51}$$

$$\int_0^1 \left|D_q\psi(u\theta + (1-u)\eta)\right|^m d_qu \leq M^m\left(\frac{1+q\left(1-\left(1-q^{-1}\right)^{s+1}\right)}{[s+1]}\right). \tag{52}$$

Substitute (49), (50), and (52) in (48) to get,

$$\begin{aligned}&\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\left(\frac{1}{[\beta n+1]}\right)^{1/n}\left(\frac{M^m\left(1+q-q\left(1-q^{-1}\right)^{s+1}\right)}{[s+1]}\right)^{1/m} + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\left(\frac{1}{[\beta n+1]}\right)^{1/n}\left(\frac{1+q-q\left(1-q^{-1}\right)^{s+1}}{[s+1]}\right)^{1/m}\\ &= \frac{M}{([\beta n+1])^{1/n}}\left(\frac{1+q\left(1-\left(1-q^{-1}\right)^{s+1}\right)}{[s+1]}\right)^{1/m} \cdot \left(\frac{(\theta-\beta)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right).\end{aligned} \tag{53}$$

Hence, this completes the proof. □

Corollary 20. *Suppose $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|^m$ is convex on $[\xi,\eta]$, for some static $s, q \in (0,1]$, $m, n > 1$, and $|D_q\psi(\theta)| \leq M, \theta \in [\xi,\eta]$, then subsequently, we have*

$$\begin{aligned}&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)} \cdot \left[J^\beta_{q,\theta^-}\psi(\xi) + J^\beta_{q,\theta^+}\psi(\eta)\right]\right|\\ &\leq \frac{M}{([1+n\beta])^{1/n}}\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right],\end{aligned} \tag{54}$$

where $(1/n)+(1/m)=1$ and $\beta>0$.

Proof. Taking $s=1$ in (46), the required result follows. □

Remark 21.

(a) Taking $\beta=1$ and $q=1$ in (46), it follows the q-analogue of (5), Theorem 4

(b) Taking $q=1$ in (46), it follows the inequality (9) of Theorem 8

Theorem 22. *Suppose $\psi : [\xi,\eta] \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is q-differentiable mapping in such a way that $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|^m$ is $s-$convex in second sense on $[\xi,\eta]$ for some static $s, q \in (0,1]$, $m \geq 1$, and $|D_q\psi(\theta)| \leq M, \theta \in [\xi,\eta]$, then for $\beta > 0$, we have*

$$\begin{aligned}&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)} \cdot \left[J^\beta_{q,\theta^-}(\xi) + J^\beta_{q,\theta^+}(\eta)\right]\right| \leq M\left(\frac{1}{[\beta+1]}\right)^{1-(1/m)}\left(\frac{1}{[\beta+s+1]}\right)^{1/m}\\ &\cdot\left(\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\right)^{1/m}\left(\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right).\end{aligned} \tag{55}$$

Proof. From Lemma 14,

$$\begin{aligned}&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)} \cdot \left[J^\beta_{q,\theta^-}\psi(\xi) + J^\beta_{q,\theta^+}\psi(\eta)\right]\right| \leq \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\\ &\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|d_qu + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi} \cdot \int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right|d_qu.\end{aligned} \tag{56}$$

Now applying familiar power mean inequality, we have

$$\begin{aligned}&\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|d_qu + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi} \cdot \int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right|d_qu\\ &\leq \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\left(\int_0^1 u^\beta d_qu\right)^{1-(1/m)} \cdot \left(\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_qu\right)^{1/m}\\ &+ \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\left(\int_0^1 u^\beta\right)^{1-(1/m)} \cdot d_qu\left(\int_0^1\left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_qu\right)^{1/m}.\end{aligned} \tag{57}$$

Since $|D_q\psi|^m$ is s-convex in the second sense on $[\xi,\eta]$ and $|D_q\psi(\theta)| \leq M$, we get

$$\begin{aligned}
&\int_0^1 u^\beta \left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_qu \\
&\le \int_0^1 u^{\beta+s}\left|D_q\psi(\theta)\right|^m d_qu + \int_0^1 u^\beta(1-u)^s\left|D_q\psi(\xi)\right|^m d_qu \\
&\le M^m\int_0^1 u^{\beta+s}d_qu + M^m\int_0^1 u^\beta(1-u)^s d_qu \\
&= M^m\left(\int_0^1 u^{\beta+s}d_qu + \int_0^1 u^\beta(1-u)^s d_qu\right),
\end{aligned}$$

$$\int_0^1 u^{\beta+s}d_qu = \frac{1}{[\beta+s+1]},$$

$$\int_0^1 u^\beta d_qu = \frac{1}{[\beta+1]},$$

$$\begin{aligned}
\int_0^1 u^\beta(1-u)^s d_qu &= \frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+2)} \\
&= \frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{[\beta+s+1]\Gamma_q(\beta+s+1)},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_qu \\
&\le \frac{M^m}{[\beta+s+1]}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right),
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_qu \\
&\le \int_0^1 u^{\beta+s}\left|D_q\psi(\theta)\right|^m d_qu + \int_0^1 u^\beta(1-u)^s\left|D_q\psi(\eta)\right|^m d_qu \\
&\le \frac{M^m}{[\beta+s+1]}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right).
\end{aligned} \tag{58}$$

By using (30), we have

$$\begin{aligned}
&\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_qu \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\left(\frac{1}{[\beta+1]}\right)^{1-(1/m)} \\
&\cdot\left(\frac{M^m}{[\beta+s+1]}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\right)^{1/m} + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi} \\
&\cdot\left(\frac{1}{[\beta+1]}\right)^{1-(1/m)}\left(\frac{M^m}{[\beta+s+1]}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\right)^{1/m} \\
&= M\left(\frac{1}{[\beta+1]}\right)^{1-(1/m)}\left(\frac{1}{[\beta+s+1]}\right)^{1/m}\left(\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\right)^{1/m} \\
&\cdot\left(\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right),
\end{aligned} \tag{59}$$

which completes the proof. □

Corollary 23. *Suppose $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|^m$ is convex on $[\xi,\eta]$ and for some static $s, q \in (0,1]$, $m \ge 1$, and $|D_q\psi(\theta)| \le M, \theta \in [\xi,\eta]$, then for $\beta > 0$, we have*

$$\begin{aligned}
&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right. \\
&\left.\cdot\left[J^\beta_{q,\theta^-}\psi(\xi)+J^\beta_{q,\theta^+}\psi(\eta)\right]\right| \\
&\le M\left(\frac{1}{[\beta+1]}\right)\left(\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right).
\end{aligned} \tag{60}$$

Proof. Taking $s=1$ in (55), the required result follows. □

Remark 24.

(a) Taking $\beta=1$ and $q=1$, in (55), it follows the q -analogue of (6), Theorem 5

(b) Taking $q=1$, in (55), it follows formula (10) of Theorem 9

Theorem 25. *Suppose that $\psi : [\xi,\eta] \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is q-differentiable mapping and $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|^m$ is s-convex in second sense on $[\xi,\eta]$ for some static $s \in (0,1]$ and $m, n > 1$, therefore, the following integral inequality for q-fractional integrals is valid.*

$$\begin{aligned}
&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right. \\
&\left.\cdot\left[J^\beta_{q,\theta^-}\psi(\xi)+J^\beta_{q,\theta^+}\psi(\eta)\right]\right| \le \frac{2^{((s-1)/m)}}{(1+n\beta)^{1/n}(\eta-\xi)} \\
&\cdot\left[(\theta-\xi)^{\beta+1}\left|D_q\psi\left(\frac{\theta+\xi}{2}\right)\right| + (\eta-\theta)^{\beta+1}\left|D_q\psi\left(\frac{\eta+\theta}{2}\right)\right|\right],
\end{aligned} \tag{61}$$

where $(1/n)+(1/m)=1$, and $\beta > 0$.

Proof. From Lemma 14 and keeping the familiar Hölder inequality in use, it follows

$$\begin{aligned}
&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right. \\
&\left.\cdot\left[J^\beta_{q,\theta^-}(\xi)+J^\beta_{q,\theta^+}(\eta)\right]\right| \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi} \\
&\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|d_qu + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi} \\
&\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right|d_qu \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\left(\int_0^1 u^{\beta n}\right)^{1/n} \\
&\cdot d_qu\left(\int_0^1\left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_qu\right)^{1/m} \\
&+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\left(\int_0^1 u^{\beta n}\right)^{1/n} d_qu\left(\int_0^1\left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_qu\right)^{1/m}.
\end{aligned} \tag{62}$$

Since $|D_q\psi|^m$ is $s-$concave, we have

$$\int_0^1 \left|D_q\psi(u\theta+(1-u)\xi)\right|^m du \le 2^{s-1}\left|D_q\psi\left(\frac{\theta+\xi}{2}\right)\right|^m,$$
$$\int_0^1 \left|D_q\psi(u\theta+(1-u)\eta)\right|^m du \le 2^{s-1}\left|D_q\psi\left(\frac{\eta+\theta}{2}\right)\right|^m. \quad (63)$$

Therefore,

$$\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|d_qu+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right|d_qu \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\left(\frac{1}{[\beta n+1]}\right)^{1/n}\cdot\left(2^{s-1}\left|D_q\psi\left(\frac{\theta+\xi}{2}\right)\right|^m\right)^{1/m}+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\left(\frac{1}{[\beta n+1]}\right)^{1/n}\cdot\left(2^{s-1}\left|D_q\psi\left(\frac{\eta+\theta}{2}\right)\right|^m\right)^{1/m}=\frac{2^{(s-1)/m}}{([\beta n+1])^{1/n}(\eta-\xi)}\cdot\left((\theta-\xi)^{\beta+1}\left|D_q\psi\left(\frac{\theta+\xi}{2}\right)\right|+(\eta-\theta)^{\beta+1}\left|D_q\psi\left(\frac{\eta+\theta}{2}\right)\right|\right), \quad (64)$$

which completes the proof. □

Remark 26. Taking $q=1$ in (61), it follows the inequality (11) of Theorem 10.

5. Applications and Examples

Example 2. Let $\psi(u)=1-u$ and $q\in(0,1)$, and we fixed $\theta=1/2$; $\xi=0$, $\eta=1$; $\beta=1$; $q=1/2$, and $s=1$, and then we get verification of Theorem 16.

$$\left|\left(\frac{((1/2)-0)^1+(1-(1/2))^1}{1-0}\right)\left(\frac{(1/2)^1+[1](1-(1/2))}{1/2}\right)\cdot\psi\left(\frac{1}{2}\right)-\frac{\Gamma_{1/2}(2)}{1/2(1)}\left(J^1_{1/2,1/2^-}\psi(0)+J^1_{1/2,1/2^+}\psi(1)\right)\right| \le \frac{1}{1-0}\left(1+\frac{\Gamma_{1/2}(2)\Gamma_{1/2}(2)}{\Gamma_{1/2}(3)}\right)\left(\frac{((1/2)-0)^2+(1-(1/2))^2}{[3]}\right),$$

$$\left|(0.5+0.5)2(0.5+0.5)\left(\frac{1}{2}\right)-2\left(\frac{1}{\Gamma_{1/2}(1)}\int_0^{1/2}(1-u)d_qu\right)\right| \le\left(1+\frac{2}{3}\right)\left(\frac{0.5}{1.75}\right),$$

$$\left|1-\frac{2}{3}\right|\le(1.6667)(0.2857),$$

$$0.3333\le 0.4762. \quad (65)$$

For any $q\in(0,1)$ and $s\in(0,1]$ result holds.

Example 3. Let $\psi(u)=1-u$ and $q\in(0,1)$, and we fixed $\theta=1/2$; $\xi=0,\eta=1$; $\beta=1$; $q=1/2$; $m=n=1/2$, and $s=1$, and then we get verification of Theorem 19.

$$\left|\left(\frac{((1/2)-0)^1+(1-(1/2))^1}{1-0}\right)\left(\frac{(1/2)^1+[1](1-(1/2))}{1/2}\right)\cdot\psi\left(\frac{1}{2}\right)-\frac{\Gamma_{1/2}(2)}{1/2(1)}\left(J^1_{1/2,1/2^-}\psi(0)+J^1_{1/2,1/2^+}\psi(1)\right)\right| \le\left(\frac{1}{[1+(1/2)]}\right)^{1/2}\left(\frac{1+(1/2)\{1-(1-(1/(1/2)))^2_{1/2}\}}{[2]}\right)^{0.5}\cdot\left(\frac{((1/2)-0)^2+(1-(1/2))^2}{1-0}\right),$$

$$\left|(0.5+0.5)2(0.5+0.5)\left(\frac{1}{2}\right)-2\left(\frac{1}{\Gamma_{1/2}(1)}\int_0^{1/2}(1-u)d_qu+\frac{1}{\Gamma_{1/2}(1)}\int_{1/2}^1(1-u)d_qu\right)\right|\le\left(\frac{1-q}{1-q^{3/2}}\right)^{0.5}\cdot\left(\frac{1+.5\{1-0\}}{q+1}\right)^{0.5}(0.5),$$

$$\left|1-\frac{2}{3}\right|\le\left(\frac{0.5}{1-0.3536}\right)^{0.5}\left(\frac{3/2}{1+(1/2)}\right)^{0.5}(0.5),$$

$$0.3333\le(0.7735)^{0.5}(0.5),$$

$$0.3333\le 0.4398. \quad (66)$$

For any $q\in(0,1)$ and $s\in(0,1]$ result holds.

Example 4. Let $\psi(u)=1-u$ and $q\in(0,1)$, and we fixed $\theta=1/2$; $\xi=0,\eta=1$; $\beta=1$; $q=1/2$; $m=n=1/2$, and $s=1$, and then we get verification of Theorem 22.

$$\left|\left(\frac{((1/2)-0)^1+(1-(1/2))^1}{1-0}\right)\left(\frac{(1/2)^1+[1](1-(1/2))}{1/2}\right)\cdot\psi\left(\frac{1}{2}\right)-\frac{\Gamma_{1/2}(2)}{1/2(1)}\left(J^1_{1/2,1/2^-}\psi(0)+J^1_{1/2,1/2^+}\psi(1)\right)\right| \le(1)\left(\frac{1}{[2]}\right)^{1/2}\left(\frac{1}{[3]}\right)^{1/2}\left(1+\frac{1}{\Gamma_{1/2}(3)}\right)^{1/2}\cdot\left(\frac{((1/2)-0)^2+(1-(1/2))^2}{1-0}\right),$$

$$\left|(0.5+0.5)2(0.5+0.5)\left(\frac{1}{2}\right)-2\left(\frac{1}{\Gamma_{1/2}(1)}\int_0^{1/2}(1-u)d_qu+\frac{1}{\Gamma_{1/2}(1)}\int_{1/2}^{1}(1-u)d_qu\right)\right|\leq\left(\frac{1}{(1/2)+1}\right)^{0.5}\cdot\left(\frac{1}{1+(1/2)+(1/2)^2}\right)^{0.5}\left(1+\frac{2}{3}\right)^{1/2}(0.5),$$

$$\left|1-\frac{2}{3}\right|\leq\left(\frac{2}{3}\right)^{0.5}\left(\frac{1}{1.75}\right)^{0.5}(1.6667)^{0.5}(0.5),$$

$$0.3333\leq(0.6667)^{0.5}(0.5414)^{0.5}(1.6667)^{0.5}(0.5),$$

$$0.3333\leq 0.3984. \quad (67)$$

For any $q\in(0,1)$ and $s\in(0,1]$ result holds.

Example 5. Let $\psi(u)=1-u$ and $q\in(0,1)$, and we fixed $\theta=1/2$; $\xi=0,\eta=1$; $\beta=1$; $q=1/2$; and $s=1$, and then we get verification of Theorem 25.

$$\left|\left(\frac{((1/2)-0)^1+(1-(1/2))^1}{1-0}\right)\left(\frac{(1/2)^1+[1](1-(1/2))}{1/2}\right)\cdot\psi\left(\frac{1}{2}\right)-\frac{\Gamma_{1/2}(2)}{1/2(1)}\left(J^1_{1/2,1/2^-}\psi(0)+J^1_{1/2,1/2^+}\psi(1)\right)\right|\leq\left(\frac{1}{[(1/2)+1]}\right)^{0.5}\left(\left(\frac{1}{2}\right)^2|(-1)|\right)+\left(\left(\frac{1}{2}\right)^2|(-1)|\right),$$

$$\left|(0.5+0.5)2(0.5+0.5)\left(\frac{1}{2}\right)-2\left(\frac{1}{\Gamma_{1/2}(1)}\int_0^{1/2}(1-u)d_qu+\frac{1}{\Gamma_{1/2}(1)}\int_{1/2}^{1}(1-u)d_qu\right)\right|\leq\left(\frac{1-q}{1-q^{3/2}}\right)^{0.5}\left(\left(\frac{1}{2}\right)^2|(-1)|\right)+\left(\left(\frac{1}{2}\right)^2|(-1)|\right),$$

$$\left|1-\frac{2}{3}\right|\leq\left(\frac{0.5}{1-0.3536}\right)^{0.5}(0.5),$$

$$0.3333\leq 0.4398. \quad (68)$$

Result holds for several example such as every $q\in(0,1)$ and $\psi(u)=(1-u)^n,\psi(s)=u^n,\psi(u)=u(1-u)^n,\psi(u)=u-\xi,\psi(u)=(u-\xi)^n,\psi(s)=(\xi-u)$, and $\psi(u)=(\xi-u)^n$ $n\in\mathbb{R}^+$ have good approximation. At the end, we compare Theorems 16, 19, and 22. We have seen that Theorem 19 has good approximation than Theorem 16, and Theorem 22 has better approximation than Theorems 16 and 19.

$$\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\cdot\left[J^\beta_{q,\theta^-}\psi(\xi)+J^\beta_{q,\theta^+}\psi(\eta)\right]\right|\leq M\left(\frac{1}{[\beta+1]}\right)^{1-(1/m)}\cdot\left(\frac{1}{[\beta+s+1]}\right)^{1/m}\left(\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\right)^{1/m}\cdot\left(\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right),\leq\frac{M}{(1+n\beta)^{1/n}}\cdot\left[\frac{1+q\left\{1-\left(1-q^{-1}\right)^{s+1}\right\}}{[s+1]}\right]^{1/m}\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right]\leq\frac{M}{\eta-\xi}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{[\beta+s+1]}\right]. \quad (69)$$

6. Conclusion

The major goal of this paper is to prove fractional quantum integral identities in order to establish some new quantum Ostrowski type inequalities involving q-fractional integrable inequalities. By the virtue of discrete fractional q-calculus, Ostrowski type inequalities are generalized for q-fractional integrals, which provide a method to study some properties of q-fractional integrals via other classes of integral inequalities. Similar method can be applied to other inequalities, like Sipmson's and Newton's type inequalities.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

X. Wang provided the main idea of the article. K. A. Khan wrote the initial draft and investigated the results. A. Ditta contributed to editing of the original draft and methodology. A. Nosheen dealt with the conceptualization and handled the latex work. K. M. Awan performed the validation and formal analysis. R. M. Mabela performed review and editing along with the submission of manuscript. All authors read and approved the final manuscript.

Acknowledgments

The research of 1st author was supported by the National Natural Science Foundation of China (Grant No. 11861053).

References

[1] E. Set, "New inequalities of Ostrowski type for mappings whose derivatives ares-convex in the second sense via fractional integrals," *Computers & Mathematcs with Applications*, vol. 63, no. 7, pp. 1147–1154, 2012.

[2] H. Gauchman, "Integral inequalities in q-calculus," *Computers & Mathematics with Applications*, vol. 47, no. 2-3, pp. 281–300, 2004.

[3] V. Kac and P. Cheung, *Quantum Calculus*, Springer-Verlag, New York, 2002.

[4] A. M. Ostrowski, "Über die Absolutabweichung einer differentierbaren Funktion von ihren Integralmittelwert," *Commentarii Mathematici Helvetici*, vol. 10, pp. 226-227, 1937.

[5] R. N. Eze, K. Nurhan, G. T. Fatma, and T. Adnan, "Some new inequalities of the Ostrowski-Gruss, Chebyshev, and Trapezoid types on time scales," *Journal of Non-Linear Science and Applications*, vol. 12, pp. 192–205, 2019.

[6] S. Erhan, O. A. Ahmet, G. Abdurrahman, and J. Fahd, "Ostrowski type inequalities via new fractional conformable integrals," *AIMS Mathematics*, vol. 4, no. 6, pp. 1684–1697, 2019.

[7] S. S. Dragomir, "On the Ostrowski's integral inequality for mappings with bounded variation and applications," *Mathematical Inequalities & Applications*, vol. 4, pp. 59–66, 2001.

[8] M. Alomari, M. Darus, S. S. Dragomir, and P. Cerone, "Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense," *Applied Mathematics Letters*, vol. 23, no. 9, pp. 1071–1076, 2010.

[9] S. S. Dragomir, "The Ostrowski's integral inequality for Lipschitzian mappings and applications," *Computers & Mathematcs with Applications*, vol. 38, no. 11-12, pp. 33–37, 1999.

[10] Z. Liu, "Some companions of an Ostrowski type inequality and application," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 2, p. 12, 2009.

[11] M. Alomari and M. Darus, "Some Ostrowski type inequalities for quasi-convex functions with applications to special means," *RGMIA*, vol. 13, no. 1, p. 3, 2010.

[12] A. Ditta, K. A. Khan, K. M. Awan, A. Nosheen, and H. R. Moradi, "Extensions of Ostrowski type inequalities via h-integrals and s-convexity," *Journal of Mathematics*, vol. 2021, Article ID 2416819, 10 pages, 2021.

[13] K. A. Khan, A. Ditta, M. Nosheen, K. M. Awan, and R. M. Mabela, "Ostrowski type inequalities for s-convex functions via q-integrals," *Journal of Function Spaces*, vol. 2022, Article ID 8063803, 8 pages, 2022.

[14] H. Hudzik and L. Maligranda, "Some remarks ons-convex functions," *Aequationes Mathematicae*, vol. 48, no. 1, pp. 100–111, 1994.

[15] M. Çakmak, "Refinements of bullen-type inequalities for s-convex functions via Riemann-Liouville fractional integrals involving Gauss hypergeometric function," *Journal of Interdisciplinary Mathematics*, vol. 22, no. 6, pp. 975–989, 2019.

[16] B. M. Abdellatif, K. Mohamed, A. H. Mohamed, and B. Dumitru, "Henry-Gronwall type q-fractional integral inequalities," *Mathematical Methods in the Applied Sciences*, vol. 44, pp. 2033–2039, 2021.

[17] S. Belarbi and Z. Dahmani, "On some new fractional integral inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 3, p. 86, 2009.

[18] Z. Dahmani, "New inequalities in fractional integrals," *International Journal of Nonlinear Science*, vol. 9, no. 4, pp. 493–497, 2010.

[19] Z. Dahmani, L. Tabharit, and S. Taf, "New generalizations of Gruss inequality usin Riemann-Liouville fractional integrals," *Bulletin of Mathematical Analysis and Applications*, vol. 2, no. 3, pp. 93–99, 2010.

[20] S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 1993.

[21] M. Z. Sarikaya and H. Ogunmez, "On new inequalities via Riemann-Liouville fractional integration," *Abstract and Applied Analysis*, vol. 2012, Article ID 428983, 10 pages, 2012.

[22] N. Alp, M. Z. Sarikaya, M. Kunt, and I. Iscan, "q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions," *Journal of King Saud University - Science*, vol. 30, no. 2, pp. 193–203, 2018.

[23] J. Tariboon and S. K. Ntouyas, "Quantum integral inequalities on finite intervals," *Journal of Inequalities and Applications*, vol. 2014, no. 1, Article ID 121, 2014.

[24] S. Shaimardan, *Hardy-Type Inequalities Quantum Calculus*, Luleå University of Technology, Graphic Production, Sweden, 2018.

[25] N. Alp and M. Z. Sarikaya, "A new definition and properties of quantum integral which call $q-$ integral," *Konuralp Journal of Mathematics*, vol. 5, no. 2, pp. 146–159, 2017.

[26] L. S. Lima, "Expression of some special functions through q -exponentials of the nonadditive statistical mechanics," *Journal of Modern Physics*, vol. 11, no. 1, pp. 81–90, 2020.

[27] R. Agarwal, "Certain fractional q-integrals and q-derivatives," *Proceedings. Cambridge Philological Society*, vol. 66, no. 2, pp. 365–370, 1969.

[28] P. Rajkovic, S. Marinkovic, and M. Stankovic, "A generalization of the concept of q-fractional integrals," *Acta Mathematica Sinica, English Series*, vol. 25, no. 10, pp. 1635–1646, 2009.

[29] Z. S. I. Mansour, "On fractional q-Sturm-Liouville problems," *Journal of Fixed Point Theory and Applications*, vol. 19, no. 2, pp. 1591–1612, 2017.

[30] J. Soontharanon, M. A. Ali, H. Budak, K. Nonlaopon, and Z. Abdullah, "Simpson's and Newton's type inequalities for $(\alpha, m)-$ convex functions via quantum calculus," *Symmetry*, vol. 14, no. 736, 2022.

[31] I. B. Sial, S. Mai, M. A. Ali, and K. Nonlaopon, "On some generalized Simpson's and Newton's inequalities for (α, m) -convex functions in $q-$calculus," *Mathematics*, vol. 9, no. 24, p. 3266, 2021.

[32] M. A. Noor, M. U. Awan, and K. I. Noor, "Quantum Ostrowski inequalities for q-differentiable convex functions," *Journal of Mathematical Inequalities*, vol. 4, no. 10, pp. 1013–1018, 2007.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 5495011, 7 pages
https://doi.org/10.1155/2022/5495011

Research Article

A Study of Spiral-Like Harmonic Functions Associated with Quantum Calculus

Shujaat Ali Shah,[1] **Luminita-Ioana Cotirla**,[2] **Adriana Catas**,[3] **Calin Dubau**,[4] **and Gabriel Cheregi**[4]

[1]*Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology (QUEST) Nawabshah, 67450, Pakistan*
[2]*Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania*
[3]*Department of Mathematics and Computer Science, University of Oradea, 1 University St., 410087 Oradea, Romania*
[4]*Faculty of Environmental Protection, University of Oradea, 26 Gen. Magheru St., 410048 Oradea, Romania*

Correspondence should be addressed to Shujaat Ali Shah; shahglike@yahoo.com

Received 9 June 2022; Revised 16 September 2022; Accepted 21 September 2022; Published 11 October 2022

Academic Editor: Wasim Ul-Haq

This article introduces new subclasses of harmonic univalent functions associated with q-difference operator. Modified q-multiplier transformation is defined, and certain geometric properties such as the sufficient condition, distortion result, extreme points, and invariance of convex combination of the elements of the subclasses are discussed by employing the newly defined q-operator. Also, various well-known results already proved in the literature are pointed out.

1. Introduction

A function $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ is known to be real harmonic function in domain $\mathscr{D}$ if θ_{xx} and θ_{yy} are continuous in $\mathscr{D}$ and satisfies

$$\theta_{xx}(x, y) + \theta_{yy}(x, y) = 0. \tag{1}$$

Continuous function $h : \Omega(\subset\mathbb{C}) \longrightarrow \mathbb{C}$ defined by $h(z) = \theta_1(x, y) + i\theta_2(x, y)$ is harmonic if both $\theta_1(x, y)$ and $\theta_2(x, y)$ are real harmonic in Ω. We found that, in any simply connected domain Ω, every harmonic function $h(z)$ can be expressed by $h(z) = h_1(z) + \bar{h}_2(z)$, where h_1 and h_2 are analytic in Ω, and are called, respectively, the analytic and coanalytic parts of h.

The class of complex-valued harmonic functions $h = h_1 + \bar{h}_2$ defined in the open unit disc $\mathscr{U} = \{z : |z| < 1\}$ and normalized by $h_1(0) = h_2(0) = h_1'(0) - 1 = 0$ is denoted by $\mathscr{H}$. The function in the class $\mathscr{H}$ has the following power series representation:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} \mathrm{b}_n z^n}. \tag{2}$$

It is clear that when $h_2(z)$ is identically zero, the class $\mathscr{H}$ coincides with the class $\mathscr{A}$ of normalized analytic functions in $\mathscr{U}$. Due to Lewy [1], a function $h \in \mathscr{H}$ is locally univalent and sense-preserving in $\mathscr{U}$ if and only if

$$\left|h_1'(z)\right| > \left|h_2'(z)\right|, \text{for} z \in \mathscr{U}. \tag{3}$$

We indicate by $\mathscr{S}_{\mathscr{H}}$ the subclass of $\mathscr{H}$ consisting of all sense-preserving univalent harmonic functions h. Firstly, Clunie and Sheil-Small [2] discussed certain geometric properties of the class $\mathscr{S}_{\mathscr{H}}$ and its subclasses. Later on, several authors contributed in the study of subclasses of the class $\mathscr{S}_{\mathscr{H}}$, for example, see [3–9]. The most prominent author Jahangiri [10] investigated various interesting properties of the class $\mathscr{S}^*_{\mathscr{H}}(\varsigma)$ of starlike harmonic functions of order ς, $(0 \le \varsigma < 1)$, defined by

$$\Re\left(\frac{zh'(z)}{h(z)}\right)=\Re\left(\frac{zh_1'(z)-z\overline{h_2'(z)}}{h_1(z)+\overline{h_2(z)}}\right)\geq\varsigma. \tag{4}$$

For the convenience, we present the notion of q-difference operator briefly. Jackson [11] introduced the q-difference operator and is defined by

$$\partial_q h_1(z)=\frac{h_1(z)-h_1(qz)}{(1-q)z}\ ;q\neq 1,z\neq 0, \tag{5}$$

for $q\in(0,1)$ and $h_1\in\mathscr{A}$ with $h_1(z)=z+\sum_{n=2}^{\infty}a_nz^n$. We note that $\lim_{q\longrightarrow 1^-}\partial_q h_1(z)=h_1'(z)$, where $h_1'(z)$ is the ordinary derivative of the function. It is clear that

$$\partial_q\left\{\sum_{n=1}^{\infty}a_nz^n\right\}=\sum_{n=1}^{\infty}[n]_q z^{n-1}, \tag{6}$$

where

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\cdots, \tag{7}$$

for $n\in\mathbb{N}=\{1,2,3,..\}$ and $z\in\mathscr{U}$. For some recent investigations involving q-calculus, we may refer the interested reader to [12–17]. Recently, in [18], Shah and Noor introduced the q-analogue of multiplier transformation $I_{q,\tau}^s:\mathscr{A}\longrightarrow\mathscr{A}$ by

$$I_{q,\tau}^s h_1(z)=z+\sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s a_nz^n, \tag{8}$$

where $h_1\in\mathscr{A}$, $s\in\mathbb{R}$ and $\tau>-1$. It is noted that for nonnegative integer s and $\tau=0$, the operator $I_{q,\tau}^s$ coincides with the Salagean q-differential operator defined in [19]. Moreover, if $q\longrightarrow 1^-$ in (8), then the multiplier transformation studied by the Cho and Kim in [20] is deduced. Nowadays, several subclasses of $\mathscr{S}_{\mathscr{H}}$ associated with operators and q-operators were discussed by the prominent researchers, like [21–26]. In motivation of the above said literature, first, we modify the q-multiplier transformation, and then we define certain new subclasses of $\mathscr{S}_{\mathscr{H}}$. For $h=h_1+\overline{h_2}$ given by (2), we define the modified q-multiplier transformation of h as

$$I_{q,\tau}^s h(z)=I_{q,\tau}^s h_1(z)+(-1)^s I_{q,\tau}^s\overline{h_2}(z), \tag{9}$$

where $I_{q,\tau}^s h_1(z)$ is given by (8) and

$$I_{q,\tau}^s h_2(z)=\sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s b_nz^n. \tag{10}$$

It is observed that, for $h_2=0$, the modified q-multiplier transformation defined by (9) turns out to be the $q-$multiplier transformation introduced in [18]. For $h=h_1+\overline{h_2}\in\mathscr{S}_{\mathscr{H}}$, we define a new class $\mathscr{H}\mathscr{S}\mathscr{T}_q(\zeta,\varsigma)$ as the following.

Definition 1. Let $h\in\mathscr{S}_{\mathscr{H}}$. Then $h\in\mathscr{H}\mathscr{S}\mathscr{T}_q(\zeta,\varsigma)$ if

$$\Re\left\{1+e^{i\zeta}\left(\frac{z\partial_q h(z)}{h(z)}-1\right)\right\}\geq\varsigma\cos\zeta, \tag{11}$$

where $\varsigma\in[0,1)$, $|\zeta|<\pi/2$ and $q\in(0,1)$.

Particularly, for $q\longrightarrow 1^-$, the class $\mathscr{H}\mathscr{S}\mathscr{T}_q(\zeta,\varsigma)$ reduces to the class denoted by $\mathscr{S}_{\mathscr{H}}^*(\zeta,\varsigma)$ of functions $h\in\mathscr{S}_{\mathscr{H}}$ that satisfies

$$\Re\left\{1+e^{i\zeta}\left(\frac{zh'(z)}{h(z)}-1\right)\right\}\geq\varsigma\cos\zeta, \tag{12}$$

where $\varsigma\in[0,1)$ and $|\zeta|<\pi/2$. Moreover, if $\zeta=0$, then the class $\mathscr{S}_{\mathscr{H}}^*(\zeta,\varsigma)$ coincides with the class $\mathscr{S}_{\mathscr{H}}^*(\varsigma)$ introduced by Jahangiri [10]. We further define $\mathscr{H}\overline{\mathscr{S}}\mathscr{T}_q(\zeta,\varsigma)=\mathscr{H}\mathscr{S}\mathscr{T}_q(\zeta,\varsigma)\cap\overline{\mathscr{S}_{\mathscr{H}}}$, where $\overline{\mathscr{S}_{\mathscr{H}}}$ denotes the subclass of $\mathscr{S}_{\mathscr{H}}$ consisting of functions of the type $h_s(z)=h_1(z)+\overline{h_{2,s}}(z)$, where

$$h_1(z)=z-\sum_{n=2}^{\infty}|a_n|z^n \text{and} h_{2,s}(z)=(-1)^s\sum_{n=2}^{\infty}|b_n|z^n. \tag{13}$$

Now, by using modified q-multiplier transformation given by (9), we define the following.

Definition 2. Let $h\in\mathscr{S}_{\mathscr{H}}$. Then $h\in\mathscr{H}\mathscr{S}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$ if

$$\Re\left\{1+e^{i\zeta}\left(\frac{I_{q,\tau}^{s+1}h(z)}{I_{q,\tau}^{s}h(z)}-1\right)\right\}\geq\varsigma\cos\zeta, \tag{14}$$

where $s\in\mathbb{R}$, $\tau>-1$, $\varsigma\in[0,1)$, $|\zeta|<\pi/2$, and $q\in(0,1)$.

Also, we define $\mathscr{H}\overline{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)=\mathscr{H}\mathscr{S}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)\cap\overline{\mathscr{S}_{\mathscr{H}}}$, where $\overline{\mathscr{S}_{\mathscr{H}}}$ denotes the subclass of $\mathscr{S}_{\mathscr{H}}$ consisting of functions given by (13). It is noted that, for $s=\tau=0$, we have $\mathscr{H}\mathscr{S}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)=\mathscr{H}\mathscr{S}\mathscr{T}_q(\zeta,\varsigma)$ and $\mathscr{H}\overline{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)=\mathscr{H}\overline{\mathscr{S}}\mathscr{T}_q(\zeta,\varsigma)$. In particular, if we take $\zeta=\tau=0$ and $s=m\in\mathbb{N}$ in above definitions, then we have well-known classes $\mathscr{H}_q^m(\zeta,\varsigma)$ and $\overline{\mathscr{H}}_q^m(\zeta,\varsigma)$ introduced by Jahangiri [22].

The next section presents the main investigations such as the sufficient condition, distortion result, extreme points, and invariance of convex combination of the elements of the subclasses defined as above.

2. Main Results

Theorem 3. *Let $h=h_1+\overline{h_2}\in\mathscr{S}_{\mathscr{H}}$ is given by (2) and satisfies*

$$\left[\sum_{n=1}^{\infty}\left\{\left(\frac{[n+\tau]_q}{[1+\tau]_q}-\varsigma\cos\zeta\right)|a_n|+\left(\frac{[n+\tau]_q}{[1+\tau]_q}+\varsigma\cos\zeta\right)|b_n|\right\}\right]\times\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s\leq 2(1-\varsigma\cos\zeta), \tag{15}$$

where $s \in \mathbb{R}$, $\tau > -1$, $\varsigma \in [0, 1)$, $|\zeta| < \pi/2$, *and* $q \in (0, 1)$. *Then,* $h \in \mathscr{H}\mathscr{S}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$.

Proof. We need to prove that if the coefficients of the harmonic function $h = h_1 + \overline{h_2} \in \mathscr{S}_{\mathscr{H}}$ given by (2) satisfy the inequality (15), then it also satisfies (14). It is known that $\Re(w) \geq \xi$ if and only if $|1 - \xi + w| \geq |1 + \xi - w|$. So, it suffices to prove that

$$\begin{aligned} &\left|1 - \varsigma\cos\zeta + 1 + e^{i\zeta}\left(\frac{I_{q,\tau}^{s+1}h(z)}{I_{q,\tau}^{s}h(z)} - 1\right)\right| \\ &\geq \left|1 + \varsigma\cos\zeta - 1 - e^{i\zeta}\left(\frac{I_{q,\tau}^{s+1}h(z)}{I_{q,\tau}^{s}h(z)} - 1\right)\right|, \end{aligned} \tag{16}$$

or equivalently,

$$\begin{aligned} &\left|(2 - \varsigma\cos\zeta)I_{q,\tau}^{s}h(z) + e^{i\zeta}\left(I_{q,\tau}^{s+1}h(z) - I_{q,\tau}^{s}h(z)\right)\right| \\ &- \left|(\varsigma\cos\zeta)I_{q,\tau}^{s}h(z) - e^{i\zeta}\left(I_{q,\tau}^{s+1}h(z) - I_{q,\tau}^{s}h(z)\right)\right| \geq 0. \end{aligned} \tag{17}$$

From the left hand side,

$$\begin{aligned} &\left|(2 - \varsigma\cos\zeta)I_{q,\tau}^{s}h(z) + e^{i\zeta}\left(I_{q,\tau}^{s+1}h(z) - I_{q,\tau}^{s}h(z)\right)\right| \\ &- \left|(\varsigma\cos\zeta)I_{q,\tau}^{s}h(z) - e^{i\zeta}\left(I_{q,\tau}^{s+1}h(z) - I_{q,\tau}^{s}h(z)\right)\right| \\ &= \left|\begin{array}{c} (2 - \varsigma\cos\zeta)\left\{I_{q,\tau}^{s}h_1(z) + (-1)^s I_{q,\tau}^{s}\overline{h_2}(z)\right\} \\ -e^{i\zeta}\left\{I_{q,\tau}^{s}h_1(z) + (-1)^s I_{q,\tau}^{s}\overline{h_2}(z) - I_{q,\tau}^{s+1}h_1(z) + (-1)^s I_{q,\tau}^{s+1}\overline{h_2}(z)\right\} \end{array}\right| \\ &- \left|\begin{array}{c} (\varsigma\cos\zeta)\left\{I_{q,\tau}^{s}h_1(z) + (-1)^s I_{q,\tau}^{s}\overline{h_2}(z)\right\} \\ +e^{i\zeta}\left\{I_{q,\tau}^{s}h_1(z) + (-1)^s I_{q,\tau}^{s}\overline{h_2}(z) - I_{q,\tau}^{s+1}h_1(z) + (-1)^s I_{q,\tau}^{s+1}\overline{h_2}(z)\right\} \end{array}\right| \\ &\geq \left|\begin{array}{c} (2 - \varsigma\cos\zeta)z + (2 - \varsigma\cos\zeta) \\ \times\left\{\sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s a_n z^n + (-1)^s\sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s b_n\overline{z}^n\right\} \\ -e^{i\zeta}\left\{\begin{array}{c} \sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s a_n z^n + (-1)^s\sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s b_n\overline{z}^n \\ -\sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^{s+1} a_n z^n + (-1)^s\sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^{s+1} b_n\overline{z}^n \end{array}\right\} \end{array}\right| \\ &- \left|\begin{array}{c} (\varsigma\cos\zeta)z + (\varsigma\cos\zeta) \\ \times\left\{\sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s a_n z^n + (-1)^s\sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s b_n\overline{z}^n\right\} \\ +e^{i\zeta}\left\{\begin{array}{c} \sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s a_n z^n + (-1)^s\sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s b_n\overline{z}^n \\ -\sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^{s+1} a_n z^n + (-1)^s\sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^{s+1} b_n\overline{z}^n \end{array}\right\} \end{array}\right| \end{aligned}$$

$$\begin{aligned} &\geq (2 - \varsigma\cos\zeta)|z| - \sum_{n=2}^{\infty}\left[2 - \varsigma\cos\zeta + \frac{[n+\tau]_q}{[1+\tau]_q} - 1\right]\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s \\ &\cdot |a_n||z|^n - \sum_{n=1}^{\infty}\left[\varsigma\cos\zeta + \frac{[n+\tau]_q}{[1+\tau]_q} - 1\right]\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s |b_n||z|^n \\ &- \sum_{n=2}^{\infty}\left[\frac{[n+\tau]_q}{[1+\tau]_q} - 1 - \varsigma\cos\zeta\right]\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s |a_n||z|^n \\ &- \sum_{n=2}^{\infty}\left[\frac{[n+\tau]_q}{[1+\tau]_q} + \varsigma\cos\zeta + 1\right]\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s |b_n||z|^n \geq 2(1 - \varsigma\cos\zeta) \\ &\times\left[1 - \frac{1}{(1 - \varsigma\cos\zeta)}\left\{\begin{array}{c} \sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q} - \varsigma\cos\zeta\right)|a_n| \\ + \sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q} + \varsigma\cos\zeta\right)|b_n| \end{array}\right\}\right]\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s. \end{aligned} \tag{18}$$

The above expression is nonnegative by (15). Hence, $h \in \mathscr{H}\mathscr{S}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$.

The harmonic function is

$$\begin{aligned} h(z) = z &+ \sum_{n=2}^{\infty}\frac{(1 - \varsigma\cos\zeta)[1+\tau]_q}{\left\{[n+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}\psi_n} t_n z^n \\ &+ \sum_{n=1}^{\infty}\frac{(1 - \varsigma\cos\zeta)[1+\tau]_q}{\left\{[n+\tau]_q + [1+\tau]_q\varsigma\cos\zeta\right\}\psi_n}\overline{\upsilon}_n\overline{z}^n, \end{aligned} \tag{19}$$

where $\psi_n = ([n+\tau]_q/[1+\tau]_q)^s$ and $\sum_{n=2}^{\infty}|t_n| + \sum_{n=1}^{\infty}|\upsilon_n| = 1$ show that the coefficient bound given by (15) is sharp. For different choices of parameters, we deduce certain results as follows. If $s = \tau = 0$ in Theorem 3, then we have a following new result. □

Corollary 4. *Let a function* $h(z) = h_1(z) + \overline{h_2}(z) \in \mathscr{S}_{\mathscr{H}}$ *given by (2) and satisfies*

$$\left[\sum_{n=1}^{\infty}\left\{\left(\frac{[n]_q - \varsigma\cos\zeta}{1 - \varsigma\cos\zeta}\right)|a_n| + \left(\frac{[n]_q + \varsigma\cos\zeta}{1 - \varsigma\cos\zeta}\right)|b_n|\right\}\right] \leq 2, \tag{20}$$

where $\varsigma \in [0, 1)$, $|\zeta| < \pi/2$, *and* $q \in (0, 1)$. *Then,* $h \in \mathscr{H}\mathscr{S}\mathscr{T}_q(\zeta, \varsigma)$.

If $q \longrightarrow 1^-$, then Corollary 4 reduces to a new result as follows:

Corollary 5. *Let a function* $h = h_1 + \overline{h_2} \in \mathscr{S}_{\mathscr{H}}$ *given by (2) and satisfies*

$$\left[\sum_{n=1}^{\infty}\left\{\left(\frac{n - \varsigma\cos\zeta}{1 - \varsigma\cos\zeta}\right)|a_n| + \left(\frac{n + \varsigma\cos\zeta}{1 - \varsigma\cos\zeta}\right)|b_n|\right\}\right] \leq 2, \tag{21}$$

where $\varsigma \in [0, 1)$ *and* $|\zeta| < \pi/2$. *Then,* $h \in \mathscr{S}_{\mathscr{H}}^{*}(\zeta, \varsigma)$.

If we take $\zeta=\tau=0$ and $s=m\in\mathbb{N}$, then we have well-known result.

Corollary 6 (see [22]). *Let a function $h=h_1+\overline{h_2}\in\mathcal{S}_{\mathcal{H}}$ given by (2) and satisfies*

$$\left[\sum_{n=2}^{\infty}[n]_q^m\left([n]_q-\varsigma\right)|a_n|+\sum_{n=1}^{\infty}[n]_q^m\left([n]_q+\varsigma\right)|b_n|\right]\leq 1-\varsigma, \tag{22}$$

where $\varsigma\in[0,1)$ and $q\in(0,1)$. Then, $f\in\mathcal{H}_q^m(\zeta,\varsigma)$.

When $\zeta=0$ in Corollary 5, we get the sufficient condition for f in $\mathcal{S}_{\mathcal{H}}^*(\varsigma)$ proved by Jahangiri [10]. Moreover, for $\zeta=\varsigma=0$ in Corollary 5, the sufficient condition for function in the class of starlike harmonic univalent mappings is obtained, see [4]. Now, we state and prove the necessary and sufficient conditions for the harmonic functions $h=h_1+\overline{h_2}$ to be in $\mathcal{H}\bar{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$ as follows.

Theorem 7. *Let $h_s=h+\overline{h_{2,s}}\in\mathcal{S}_{\mathcal{H}}$ given by (13). Then, $h_s\in\mathcal{H}\bar{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$ if and only if*

$$\sum_{n=1}^{\infty}\left\{\left(\frac{[n+\tau]_q}{[1+\tau]_q}-\varsigma\cos\zeta\right)|a_n|+\left(\frac{[n+\tau]_q}{[1+\tau]_q}+\varsigma\cos\zeta\right)|b_n|\right\}\cdot\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s\leq 2(1-\varsigma\cos\zeta), \tag{23}$$

where $s\in\mathbb{R}$, $\tau>-1$, $\varsigma\in[0,1)$, $|\zeta|<\pi/2$, and $q\in(0,1)$.

Proof. The sufficient condition is obvious from the Theorem 3, because $\mathcal{H}\bar{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)\subset\mathcal{H}\mathcal{S}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$. We need to prove the necessary condition only; that is, if $h_s\in\mathcal{H}\bar{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$, then the coefficients of the function $h_s=h+\overline{h_{2,s}}$ satisfy the inequality (23). Let $h_s\in\mathcal{H}\bar{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$. Then, by the definition of $\mathcal{H}\bar{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$, we have

$$\Re\left\{1+e^{i\zeta}\left(\frac{I_{q,\tau}^{s+1}h_s(z)}{I_{q,\tau}^{s}h_s(z)}-1\right)-\varsigma\cos\zeta\right\}\geq 0, \tag{24}$$

where $s\in\mathbb{R}$, $\tau>-1$, $\varsigma\in[0,1)$, $|\zeta|<\pi/2$, and $q\in(0,1)$. Equivalently, we can write (24) as

$$\Re\left\{\frac{(1-\varsigma\cos\zeta)I_{q,\tau}^{s}h_s(z)+e^{i\zeta}\left(I_{q,\tau}^{s+1}h_s(z)-I_{q,\tau}^{s}h_s(z)\right)}{I_{q,\tau}^{s}h_s(z)}\right\}\geq 0. \tag{25}$$

Substituting $h_s=h+\overline{h_{2,s}}$ in (25) and employing (8) along with (13), and also some computation yields

$$\Re\left[\frac{(1-\varsigma\cos\zeta)z-\sum_{n=2}^{\infty}\left([n+\tau]_q/[1+\tau]_q-\varsigma\cos\zeta\right)\left([n+\tau]_q/[1+\tau]_q\right)^s|a_n||z|^n-\sum_{n=1}^{\infty}\left(\left([n+\tau]_q/[1+\tau]_q\right)+\varsigma\cos\zeta\right)\left([n+\tau]_q/[1+\tau]_q\right)^s|b_n||z|^n}{z-\sum_{n=2}^{\infty}\left([n+\tau]_q/[1+\tau]_q\right)^s|a_n||z|^n-\sum_{n=1}^{\infty}\left([n+\tau]_q/[1+\tau]_q\right)^s|b_n||z|^n}\right]\geq 0. \tag{26}$$

For all values of z in $\mathcal{U}$ above required condition must hold. Selecting z on the positive real axis where $0\leq z=r<1$, we obtain

$$\frac{(1-\varsigma\cos\zeta)-\sum_{n=2}^{\infty}\left([n+\tau]_q/[1+\tau]_q-\varsigma\cos\zeta\right)\left([n+\tau]_q/[1+\tau]_q\right)^s|a_n|r^{n-1}-\sum_{n=1}^{\infty}\left(\left([n+\tau]_q/[1+\tau]_q\right)+\varsigma\cos\zeta\right)\left([n+\tau]_q/[1+\tau]_q\right)^s|b_n|r^{n-1}}{z-\sum_{n=2}^{\infty}\left([n+\tau]_q/[1+\tau]_q\right)^s|a_n|r^{n-1}-\sum_{n=1}^{\infty}\left([n+\tau]_q/[1+\tau]_q\right)^s|b_n|r^{n-1}}\geq 0. \tag{27}$$

The numerator in (27) is negative for r sufficiently close to 1 whenever the inequality (23) does not hold. Hence, there exists $z_0=r_0$ in $(0,1)$ for which the quotient in (27) is negative. This contradicts the required condition for $h_s\in\mathcal{H}\bar{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$, and so the proof is complete. □

Next, we want to discuss the distortion bounds for the function $h\in\mathcal{H}\bar{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$, which yields a covering result for this class.

Theorem 8. *If $h\in\mathcal{H}\bar{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$ and $|z|=r<1$, then*

$$(1-|b_1|)r-Tr^2\leq|h(z)|\leq(1+|b_1|)r+Tr^2, \tag{28}$$

with

$$T=\frac{[1+\tau]_q^{s+1}}{[2+\tau]_q^s}\left[\frac{1-\varsigma\cos\zeta}{[2+\tau]_q-[1+\tau]_q\varsigma\cos\zeta}-\frac{1+\varsigma\cos\zeta}{[2+\tau]_q-[1+\tau]_q\varsigma\cos\zeta}|b_1|\right]. \tag{29}$$

Proof. Let $h\in\mathcal{H}\bar{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$. Taking absolute value of h, we get

$$\begin{aligned}|h(z)|&\leq(1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2\leq(1+|b_1|)r+\frac{(1-\varsigma\cos\zeta)[1+\tau]_q^{s+1}}{\left\{[2+\tau]_q-[1+\tau]_q\varsigma\cos\zeta\right\}[2+\tau]_q^s}\\&\quad\times\sum_{n=2}^{\infty}\left\{\left(\frac{\left\{[2+\tau]_q-[1+\tau]_q\varsigma\cos\zeta\right\}[2+\tau]_q^s}{(1-\varsigma\cos\zeta)[1+\tau]_q^{s+1}}\right)(|a_n|+|b_n|)\right\}r^2\\&\leq(1+|b_1|)r+\frac{(1-\varsigma\cos\zeta)[1+\tau]_q^{s+1}}{\left\{[2+\tau]_q-[1+\tau]_q\varsigma\cos\zeta\right\}[2+\tau]_q^s}\\&\quad\cdot\sum_{n=2}^{\infty}\left\{\left(\frac{\left\{[n+\tau]_q-[1+\tau]_q\varsigma\cos\zeta\right\}}{(1-\varsigma\cos\zeta)[1+\tau]_q}\right)\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s(|a_n|+|b_n|)\right\}r^2\\&\leq(1+|b_1|)r+\frac{(1-\varsigma\cos\zeta)[1+\tau]_q^{s+1}}{\left\{[2+\tau]_q-[1+\tau]_q\varsigma\cos\zeta\right\}[2+\tau]_q^s}\\&\quad\cdot\left\{1-\frac{(1-\varsigma\cos\zeta)}{(1-\varsigma\cos\zeta)}|b_1|\right\}r^2,\ (\text{by}(2.2))\leq(1+|b_1|)r+Tr^2,\end{aligned} \tag{30}$$

where T is given by (29). Hence, this is the required right hand inequality. Similarly, one can easily prove the required left hand inequality. □

Letting $r \longrightarrow 1$ and by making use of the left hand inequality of the above theorem, we obtain the following.

Corollary 9 (covering result). *If $h \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$, then*

$$\left\{ w : |w| < \frac{L - M(1 - \varsigma \cos \zeta)}{L} - \frac{L - M(1 + \varsigma \cos \zeta)}{L} |b_1| \right\} \cdot \subset f(\mathscr{E}), \tag{31}$$

where $L = \{[2+\tau]_q - [1+\tau]_q \varsigma \cos \zeta\}[2+\tau]_q^s$ *and* $M = [1+\tau]_q^{s+1}$.

In particular, we obtain the covering results for the newly defined classes and well-known classes of harmonic functions by choosing suitable choices of parameters.

Now, our task is to examine $clco\mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$, the extreme points of closed convex hulls of $\mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$.

Theorem 10. *A function $h_s \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$ if and only if*

$$h_s(z) = \sum_{n=1}^{\infty} \left(\nu_n h_n(z) + \omega_n g_{s_n}(z) \right), \tag{32}$$

where $h_1(z) = z$,

$$h_n(z) = z - \frac{(1 - \varsigma \cos \zeta)[1+\tau]_q}{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos \zeta\right\}\psi_n} z^n \,; (n = 2, 3, \cdots),$$
$$g_{s_n}(z) = z + (-1)^s \frac{(1 - \varsigma \cos \zeta)[1+\tau]_q}{\left\{[n+\tau]_q + [1+\tau]_q \varsigma \cos \zeta\right\}\psi_n} \bar{z}^n \,; (n = 1, 2, 3, \cdots), \tag{33}$$

with $\sum_{n=1}^{\infty}(\nu_n + \Omega_n) = 1$, $\nu_n, \Omega_n \geq 0$, *and* $\psi_n = ([n+\tau]_q/[1+\tau]_q)^s$. *Particularly,* $\{h_n\}$ *and* $\{g_{s_n}\}$ *are the extreme points of $\mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$.*

Proof. We assume function h_s as given by (32)

$$h_s(z) = \sum_{n=1}^{\infty} \left(\nu_n h_n(z) + \Omega_n g_{s_n}(z) \right) = \sum_{n=1}^{\infty} (\nu_n + \Omega_n) z - \sum_{n=2}^{\infty} \nu_n R_n z^n + (-1)^s \sum_{n=1}^{\infty} \Omega_n R_n^* \bar{z}^n, \tag{34}$$

where $R_n = (1 - \varsigma \cos \zeta)[1+\tau]_q / \{[n+\tau]_q - [1+\tau]_q \varsigma \cos \zeta\}\psi_n$ and $R_n^* = (1 - \varsigma \cos \zeta)[1+\tau]_q / \{[n+\tau]_q + [1+\tau]_q \varsigma \cos \zeta\}\psi_n$.

Equating (34) with (13), we get

$$|a_n| = \nu_n R_n \text{and} |b_n| = \Omega_n R_n^*. \tag{35}$$

Now,

$$\sum_{n=1}^{\infty} \left[\frac{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos \zeta\right\}\psi_n}{(1 - \varsigma \cos \zeta)[1+\tau]_q} |a_n| + \frac{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos \zeta\right\}\psi_n}{(1 - \varsigma \cos \zeta)[1+\tau]_q} |b_n| \right] = 1 - X_1 + \sum_{n=1}^{\infty} (\nu_n + \Omega_n) = 2 - X_1 \leq 2. \tag{36}$$

Thus, by Theorem 7, $h_s \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$. Conversely, let $h_s \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$. We take

$$\nu_n = \frac{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos \zeta\right\}\psi_n}{(1 - \varsigma \cos \zeta)[1+\tau]_q} |a_n| \,; (n = 2, 3, \cdots), \tag{37}$$

$$\Omega_n = \frac{\left\{[n+\tau]_q + [1+\tau]_q \varsigma \cos \zeta\right\}\psi_n}{(1 - \varsigma \cos \zeta)[1+\tau]_q} |b_n| \,; (n = 1, 2, \cdots), \tag{38}$$

with $\sum_{n=1}^{\infty}(\nu_n + \Omega_n) = 1$. We follow our required result by substituting the values of $|a_n|$ and $|b_n|$ from the above relations in (13).

Finally, we wish to show that the class $\mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$ is closed under the convex combination of its elements. □

Theorem 11. *The class $\mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$ is closed under the convex combination.*

Proof. Let $h_{i_s} \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$, $(i = 1, 2, \cdots)$, with

$$h_{i_s} = z - \sum_{n=2}^{\infty} |a_{i,n}| z^n + (-1)^s \sum_{n=2}^{\infty} |b_{i,n}| \bar{z}^n. \tag{39}$$

Making use of Theorem 7, we have

$$\sum_{n=2}^{\infty} \frac{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos \zeta\right\}\psi_n}{(1 - \varsigma \cos \zeta)[1+\tau]_q} |a_{i,n}| + \sum_{n=1}^{\infty} \frac{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos \zeta\right\}\psi_n}{(1 - \varsigma \cos \zeta)[1+\tau]_q} |b_{i,n}| \leq 1, \tag{40}$$

with $\psi_n = ([n+\tau]_q/[1+\tau]_q)^s$.

Now,

$$\sum_{i=1}^{\infty} \mathrm{u}_i h_{i_s} = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} \mathrm{u}_i |a_{i,n}|\right) z^n + (-1)^s \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} \mathrm{u}_i |b_{i,n}|\right)\bar{z}^n. \tag{41}$$

To prove our result, we use (40) and (41)

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos\zeta\right\}\psi_n}{(1-\varsigma\cos\zeta)[1+\tau]_q}\left(\sum_{i=1}^{\infty} \mathrm{u}_i |a_{i,n}|\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos\zeta\right\}\psi_n}{(1-\varsigma\cos\zeta)[1+\tau]_q}\left(\sum_{i=1}^{\infty} \mathrm{u}_i |b_{i,n}|\right) \\
&\quad \le \sum_{i=1}^{\infty} \mathrm{u}_i = 1.
\end{aligned} \tag{42}$$

Therefore, $\sum_{i=1}^{\infty} \mathrm{u}_i h_{i_s} \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$. □

3. Conclusions

In this research, we have defined some new subclasses of harmonic univalent functions related to the q-difference operator. Also, we have introduced and studied the modified q-multiplier transformation. Several geometric properties such as sufficient condition, necessary conditions, distortion results, and invariance of classes under convex combination and extreme points are investigated. It is also noted that our investigations deduced various well-known results. In addition, this work can be extend for multivalent functions and (p, q)-calculus.

Data Availability

No data were used to support this study.

Conflicts of Interest

There is no conflict of interest regarding the publication of this article.

Authors' Contributions

All authors equally contributed to this manuscript and approved the final version.

References

[1] H. Lewy, "On the non-vanishing of the Jacobian in certain one-to-one mappings," *Bulletin of the American Mathematical Society*, vol. 42, no. 10, pp. 689–692, 1936.

[2] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiae Scientiarum Fennicae Mathematica*, vol. 9, pp. 3–25, 1984.

[3] O. P. Ahuja and J. M. Jahangiri, "Noshiro-type harmonic univalent functions," *Scientiae Mathematicae Japonicae*, vol. 56, pp. 1–7, 2002.

[4] Y. Avci and E. Zlotkiewicz, "On harmonic univalent mappings," *Annales Universitatis Mariae Curie-Sklodowska, Sectio A*, vol. 44, pp. 1–7, 1990.

[5] M. Darus and K. Al-Shaqsi, "On certain subclass of harmonic univalent functions," *Journal of Analysis and Applications*, vol. 6, pp. 17–28, 2008.

[6] P. L. Duren, "A survey of harmonic mappings in the plane," *Mathematics Series Visiting Scholars Lectures*, vol. 18, pp. 1–15, 1992.

[7] J. Dziok, M. Darus, J. Sokół, and T. Bulboaca, "Generalisations des fonctions harmoniques etoilees," *Comptes Rendus de l'Académie des Sciences*, vol. 354, no. 1, pp. 13–18, 2016.

[8] T. Sheil-Small, "Constants for planar harmonic mappings," *Journal of the London Mathematical Society*, vol. 2, pp. 237–248, 1990.

[9] H. Silverman, "Harmonic univalent functions with negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 1, pp. 283–289, 1998.

[10] J. M. Jahangiri, "Harmonic Functions Starlike in the Unit Disk," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 2, pp. 470–477, 1999.

[11] F. H. Jackson, "On q-functions and a certain difference operator," *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, vol. 46, pp. 253–281, 1908.

[12] B. Khan, L. Zhi-Guo, T. G. Shaba, S. Araci, N. Khan, and M. G. Khan, "Applications of -derivative operator to the subclass of Bi-univalent functions involving -Chebyshev polynomials," *Journal of Mathematics*, vol. 2022, Article ID 8162182, 7 pages, 2022.

[13] S. A. Shah, A. A. Maitlo, M. A. Soomro, and K. I. Noor, "On new subclass of harmonic univalent functions associated with modified q-operator," *International Journal of Analysis and Applications*, vol. 19, no. 6, pp. 826–835, 2021.

[14] B. Khan, H. M. Srivastava, S. Arjika, S. Khan, N. Khan, and Q. Z. Ahmad, "A certain q-Ruscheweyh type derivative operator and its applications involving multivalent functions," *Advances in Difference Equations*, vol. 2021, no. 1, 2021.

[15] S. Khan, S. Hussain, M. Naeem, M. Darus, and A. Rasheed, "A subclass of q-starlike functions defined by using a symmetric q-derivative operator and related with generalized symmetric conic domains," *Mathematics*, vol. 9, no. 9, p. 917, 2021.

[16] H. M. Srivastava, A. K. Wanas, and R. Srivastava, "Applications of the q-Srivastava-Attiya operator involving a certain family of bi-univalent functions associated with the Horadam polynomials," *Symmetry*, vol. 13, no. 7, p. 1230, 2021.

[17] Q. Hu, H. M. Srivastava, B. Ahmad et al., "A subclass of multivalent Janowski type q-starlike functions and its consequences," *Symmetry*, vol. 13, no. 7, p. 1275, 2021.

[18] S. A. Shah and K. I. Noor, "Study on the q-analogue of a certain family of linear operators," *Turkish Journal of Mathematics*, vol. 43, no. 6, pp. 2707–2714, 2019.

[19] M. Govindaraj and S. Sivasubramanian, "On a class of analytic functions related to conic domains involving q-calculus," *Analysis Mathematica*, vol. 43, no. 3, pp. 475–487, 2017.

[20] N. E. Cho and T. H. Kim, "Multiplier transformations and strongly close-to-convex functions," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 3, pp. 399–410, 2003.

[21] J. Dziok, "Classes of harmonic functions associated with Ruscheweyh derivatives," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, no. 2, pp. 1315–1329, 2019.

[22] J. M. Jahangiri, "Harmonic univalent functions defined by q-calculus operators," *International Journal of Applied Mathematical Analysis and Applications*, vol. 5, pp. 39–43, 2018.

[23] J. M. Jahangiri, G. Murugusundaramoorthy, and K. Vijaya, "Starlikeness of Rucheweyh type harmonic univalent functions," *Indian Academy of Mathematics*, vol. 26, pp. 191–200, 2004.

[24] G. Murugusundramoorthy and J. M. Jahangiri, "Ruscheweyh-type harmonic functions defined by q-differential operators," *Khayyam Journal of Mathematics*, vol. 5, pp. 79–88, 2019.

[25] K. I. Noor, B. Malik, and S. Z. H. Bukhari, "Harmonic functions defined by a generalized fractional differential operator," *Journal of Advanced Mathematical Studies*, vol. 2, pp. 41–52, 2009.

[26] S. Yalcin, "A new class of Salagean-type harmonic univalent functions," *Applied Mathematics Letters*, vol. 18, no. 2, pp. 191–198, 2005.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2621811, 10 pages
https://doi.org/10.1155/2022/2621811

Research Article

The Second Hankel Determinant of Logarithmic Coefficients for Starlike and Convex Functions Involving Four-Leaf-Shaped Domain

Azzh Saad Alshehry,[1] **Rasool Shah**,[2] **and Abdul Bariq**[3]

[1]*Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia*
[2]*Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan*
[3]*Department of Mathematics, Laghman University, Mehtarlam, Laghman, Afghanistan*

Correspondence should be addressed to Rasool Shah; rasoolshah1988@gmail.com and Abdul Bariq; abdulbariq.maths@lu.edu.af

Received 10 August 2022; Accepted 2 September 2022; Published 28 September 2022

Academic Editor: Sarfraz Nawaz Malik

In this particular research article, we take an analytic function $Q_{4\mathscr{L}} = 1 + 5/6z + 1/6z^5$, which makes a four-leaf-shaped image domain. Using this specific function, two subclasses, $\mathscr{S}^*_{4\mathscr{L}}$ and $\mathscr{C}_{4\mathscr{L}}$, of starlike and convex functions will be defined. For these classes, our aim is to find some sharp bounds of inequalities that consist of logarithmic coefficients. Among the inequalities to be studied here are Zalcman inequalities, the Fekete-Szegö inequality, and the second-order Hankel determinant.

1. Introduction and Definitions

To properly comprehend the findings provided in the paper, certain important literature on geometric function theory must first be discussed. In this regard, the letters $\mathscr{S}$ and $\mathscr{A}$ stand for the normalized univalent (or schlicht) functions class and the normalized holomorphic (or analytic) functions class, respectively. These primary notions are defined in the disc $\mathbb{U}_d = \{z \in \mathbb{C} : |z| < 1\}$ by

$$\mathscr{A} = \left\{ F \in \mathscr{G}(\mathbb{U}_d) \colon F(z) = \sum_{l=1}^{\infty} b_l z^l \right\}, \tag{1}$$

where $\mathscr{G}(\mathbb{U}_d)$ expresses holomorphic functions class, and

$$\mathscr{S} = \{F \in \mathscr{A} : F \text{ is Schlicht in } \mathbb{U}_d\}. \tag{2}$$

This class $\mathscr{S}$ evolved as the foundational component of cutting-edge research in this area. In his paper [1], Koebe established the presence of a "covering constant" ζ, demonstrating that if F is holomorphic and Schlicht in $\mathbb{U}_d$ with $F'(0) = 1$ and $F(0) = 0$, then $F(\mathbb{U}_d) = \{w : |w| < \zeta\}$. Many mathematicians were intrigued by this beautiful result. Within a few years, the wonderful article by Bieberbach [2], which gave rise to the renowned coefficient hypothesis, was published.

The below expression provided the coefficients λ_n of logarithmic function $J_F(z)$ for $F \in \mathscr{S}$

$$J_F(z) = \frac{1}{2} \log \left(\frac{F(z)}{z} \right) = \lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \forall z \in \mathbb{U}_d. \tag{3}$$

The above coefficients have a considerable impact on the theory of Schlicht functions in many estimations. De Branges [3] achieved that $n \geq 1$ in1985,

$$\sum_{l=1}^{n} l(n-l+1)|\lambda_n|^2 \leq \sum_{l=1}^{n} \frac{n-l+1}{l}, \tag{4}$$

and equality will be achieved if F has the form $z/(1-e^{i\varphi}z)^2$ for some $\varphi \in \mathbb{R}$. It is obvious that this inequality provides the most general version of the well-known Bieberbach-Robertson-Milin conjectures concerning the Taylor coefficients of $F \in \mathcal{S}$. We quote [4–6] for further information on the demonstration of de Brange's conclusion. By taking into account, the logarithmic coefficients, in 2005, Kayumov [7] established Brennan's conjecture for conformal mappings. The major contributions to study the bounds of logarithmic coefficients for various holomorphic univalent functions are due to Alimohammadi et al. [8], Obradović et al. [9], Ye [10], Deng [11], Girela [12], Roth [13], and Andreev and Duren [14].

For the prescribed functions $Q_1, Q_2 \in \mathcal{A}$, the relation of subordination between Q_1 and Q_2 is as follows (mathematically as $Q_1 \prec Q_2$), if an holomorphic function u comes in $\mathbb{U}_d$ with the limitation $|u(z)| < 1$ and $u(0) = 0$ in a manner that $Q_1(z) = Q_2(u(z))$ satisfy. Consequently, the following relation applies if $Q_2 \in \mathcal{S}$ in $\mathbb{U}_d$:

$$Q_1(z) \prec Q_2(z), (z \in \mathbb{U}_d) \tag{5}$$

if and only if

$$Q_1(0) = Q_2(0) \& Q_1(\mathbb{U}_d) \subset Q_2(\mathbb{U}_d). \tag{6}$$

By applying the notion of subordination, Ma and Minda [15] proposed a consolidated version of the set $\mathcal{S}^*(\psi)$ in 1992, and the following is a description of it:

$$\mathcal{S}^*(\psi) = \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec \psi(z) \text{ for } z \in \mathbb{U}_d \right\}, \tag{7}$$

with the Schlicht function ψ that satisfies

$$\psi'(0) > 0 \& \Re\psi > 0. \tag{8}$$

Various subclasses of the set $\mathcal{S}$ have been examined in the past few years as particular choices for family $\mathcal{S}^*(\psi)$. For instance,

(i) $\mathcal{S}^*_{\cos} \equiv \mathcal{S}^*(\cos z)$ (see [16]) and $\mathcal{S}^*_{\cosh} \equiv \mathcal{S}^*(\cosh z)$ (see [17])

(ii) $\mathcal{S}^*_{\tanh} \equiv \mathcal{S}^*(1 + \tanh z)$ (see [18, 19])

(iii) $\mathcal{S}^*_{e} \equiv \mathcal{S}^*(e^z)$ (see [20, 21]) and $\mathcal{S}^*_{\rho} \equiv \mathcal{S}^*(1 + \sinh^{-1} z)$ (see [22])

(iv) $\mathcal{S}\mathcal{S}^*(\xi) \equiv \mathcal{S}^*(\psi(z))$ with $\psi(z) = (1 + z/1 - z)^\xi$ and $0 < \xi \le 1$ (see [23])

(v) $\mathcal{S}^*_{\mathcal{L}} \equiv \mathcal{S}^*(\sqrt{1+z})$ (see [24]) and $\mathcal{S}^*_{car} \equiv \mathcal{S}^*(1 + 4/3z + 2/3\, z^2)$ (see [25, 26])

For given $q, n \in \mathbb{N} = \{1, 2, \cdots\}$, $b_1 = 1$, and $F \in \mathcal{S}$ with the series representation (1), the Hankel determinant $H_{q,n}(F)$ is expressed by

$$H_{q,n}(F) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+q-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n+q-1} & b_{n+q} & \cdots & b_{n+2q-2} \end{vmatrix}, \tag{9}$$

and it was established by Pommerenke and Pommerenke [27, 28]. For several subcollections of Schlicht functions, the determinant $H_{q,n}(F)$ has been examined. In specific, the sharp estimate of the functional $|H_{2,2}(F)| = |b_2 b_4 - b_3^2|$ for sets $\mathcal{C}$(convexfunctions), $\mathcal{S}^*$(starlikefunctions), and $\mathcal{R}$ (boundedturningfunctions) were determined in [29, 30]. However, for the class of close-to-convex functions, the exact bounds of this determinant remain open [31]. The researchers were inspired by the works of Babalola [32], Bansal, et al. [33], Zaprawa [34], Kwon et al. [35], Kowalczyk et al. [36], and Lecko et al. [37].

It is easy to deduce from equation (2) that, for $F \in \mathcal{S}$, the logarithmic coefficients are computed by

$$\lambda_1 = \frac{1}{2} b_2, \tag{10}$$

$$\lambda_2 = \frac{1}{2}\left(b_3 - \frac{1}{2} b_2^2\right), \tag{11}$$

$$\lambda_3 = \frac{1}{2}\left(b_4 - b_2 b_3 + \frac{1}{3} b_2^3\right), \tag{12}$$

$$\lambda_4 = \frac{1}{2}\left(b_5 - b_2 b_4 + b_2^2 b_3 - \frac{1}{2} b_3^2 - \frac{1}{4} b_2^4\right). \tag{13}$$

Currently, Lecko and Kowalczyk and Kowalczyk and Lecko [38, 39] studied the following Hankel determinant $H_{q,n}(J_F/2)$ of logarithmic coefficients

$$H_{q,n}\left(\frac{J_F}{2}\right) = \begin{vmatrix} \lambda_n & \lambda_{n+1} & \cdots & \lambda_{n+q-1} \\ \lambda_{n+1} & \lambda_{n+2} & \cdots & \lambda_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_{n+q-1} & \lambda_{n+q} & \cdots & \lambda_{n+2q-2} \end{vmatrix}. \tag{14}$$

It has been noted that

$$H_{2,1}\left(\frac{J_F}{2}\right) = \lambda_1 \lambda_3 - \lambda_2^2, \quad H_{2,2}\left(\frac{J_F}{2}\right) = \lambda_2 \lambda_4 - \lambda_3^2. \tag{15}$$

By the virtue of the function $Q_{4\mathcal{L}} = 1 + 5/6z + 1/6z^5$, we define the following classes:

$$\mathcal{S}^*_{4\mathcal{L}} = \left\{ F \in \mathcal{S} : \frac{zF'(z)}{F(z)} \prec Q_{4\mathcal{L}}, (z \in \mathbb{U}_d) \right\}, \tag{16}$$

$$\mathscr{C}_{4\mathscr{L}} = \left\{ F \in \mathcal{S} : 1 + \frac{zF''(z)}{F'(z)} \prec Q_{4\mathscr{L}}, (z \in \mathbb{U}_d) \right\}. \quad (17)$$

Alternatively, $F \in \mathcal{S}^*_{4\mathscr{L}}$ if and only if an analytic function q occurs that satisfies $q(z) \prec Q_{4\mathscr{L}}$ in such that

$$F(z) = z \exp\left(\int_0^z \frac{q(t)-1}{t} dt \right). \quad (18)$$

By taking $q(z) = Q_{4\mathscr{L}}$ in (18), we achieve the following function, which serves as an extremal in many of the class $\mathcal{S}^*_{4\mathscr{L}}$ problems.

$$F_0(z) = z \exp\left(\int_0^z \left(\frac{5}{6} + \frac{1}{6} t^4 \right) dt \right) = z + \frac{5}{6} z^2 + \cdots. \quad (19)$$

The following Alexander-type connection-related two classes were mentioned above. The above two families are interlinked by the following Alexander-type relation

$$F \in \mathscr{C}_{4\mathscr{L}} \Leftrightarrow zF' \in \mathcal{S}^*_{4\mathscr{L}}. \quad (20)$$

From (19) and (20), we easily obtain the following extremal functions in various problems of the class $\mathscr{C}_{4\mathscr{L}}$

$$g_0(z) = z + \frac{5}{12} z^2 + \cdots. \quad (21)$$

Clearly, $g_0(z)$, $g_0(z^2)$, $g_0(z^3)$, and $g_0(z^4)$ belong to the class $\mathscr{C}_{4\mathscr{L}}$. That is,

$$\begin{aligned} g_1(z) &= g_0(z) = z + \frac{5}{12} z^2 + \cdots, \\ g_2(z) &= g_0(z^2) = z + \frac{5}{36} z^3 + \cdots, \\ g_3(z) &= g_0(z^3) = z + \frac{5}{72} z^4 + \cdots, \\ g_4(z) &= g_0(z^4) = z + \frac{5}{120} z^5 + \cdots. \end{aligned} \quad (22)$$

In the present paper, our core objective is to find the sharp coefficient type problems of logarithmic functions for the families $\mathcal{S}^*_{4\mathscr{L}}$ and $\mathscr{C}_{4\mathscr{L}}$. Among the inequalities to be studied here are Zalcman inequalities, the Fekete-Szegö inequality, and the second-order Hankel determinant $H_{2,1}(J_F/2)$.

2. A Set of Lemmas

We must first create the class $\mathscr{P}$ in the below set-builder form in order to declare the Lemmas that are employed in our primary findings.

$$\mathscr{P} = \{ q \in \mathscr{G}(\mathbb{U}_d) : q(0) = 1 \& \Re e q > 0, (z \in \mathbb{U}_d) \}. \quad (23)$$

That is, if $q \in \mathscr{P}$, then q has the below series expansion

$$q(z) = \sum_{n=0}^{\infty} e_n z^n, (z \in \mathbb{U}_d). \quad (24)$$

The following Lemma consists of the widely used e_2 formula [40], the e_3 formula [41], and the e_4 formula illustrated in [42].

Lemma 1. *Let $q \in \mathscr{P}$ be given in the form (24), then for , ρ, $\delta \in \bar{\mathbb{U}}_d = \mathbb{U}_d \cup \{1\}$.*

Lemma 2. *Let $q \in \mathscr{P}$ be of the form (24), then for $x, \delta, \rho \in \bar{\mathbb{U}}_d = \mathbb{U}_d \cup \{1\}$*

$$2e_2 = e_1^2 - (e_1^2 - 4)x, \quad (25)$$

$$4e_3 = e_1^3 - 2(e_1^2 - 4)e_1 x + e_1(e_1^2 - 4)x^2 - 2(e_1^2 - 4)(1 - |x|^2)\rho, \quad (26)$$

$$\begin{aligned} 8e_4 = e_1^4 &- (e_1^2 - 4)x[e_1^2(x^2 - 3x + 3) + 4x] \\ &+ 4(e_1^2 - 4)(1 - |x|^2)[e(x-1)\rho + \bar{x}\rho^2 - (1 - |\rho|^2)\delta]. \end{aligned} \quad (27)$$

Lemma 3. *Let $q \in \mathscr{P}$ and has the expansion (24). Then,*

$$|e_{n+l} - \mu e_n e_l| \le 2 \max(1, |2\mu - 1|), \quad (28)$$

$$|e_n| \le 2 \text{ for } n \ge 1, \quad (29)$$

$$|e_{n+l} - \mu e_n e_l| \le 2, 0 \le \mu \le 1. \quad (30)$$

The inequalities (28)–(30) are taken from [40, 43] and [26, 44, 45], respectively.

Lemma 4 (see [40]). *If $q \in \mathscr{P}$ has the representation (24), then*

$$\frac{1}{2}|Je_1^3 - Ke_1e_2 + Le_3| \le (|J| + |K - 2J| + |K - J + L|). \quad (31)$$

Lemma 5 [46]. *Let γ, τ, ψ and ς satisfy that $\tau, \varsigma \in (0,1)$ and*

$$\begin{aligned} 8(1-\varsigma)\varsigma&[(\tau(\varsigma + \tau) - \psi)^2 + (\tau\psi - 2\gamma)^2] \\ &+ \tau(\psi - 2\varsigma\tau)^2(1 - \tau) \le 4\tau^2\varsigma(1 - \varsigma)(1 - \tau)^2. \end{aligned} \quad (32)$$

If $q \in \mathscr{P}$ has the expansion (24), then

$$\left| \gamma e_1^4 + \varsigma e_2^2 + 2\tau e_1 e_3 - \frac{3}{2}\psi e_1^2 e_2 - e_4 \right| \le 2. \quad (33)$$

3. Coefficient Inequalities for the Class $\mathcal{S}^*_{4\mathscr{L}}$

We start by establishing out the class $\mathcal{S}^*_{4\mathscr{L}}$'s initial coefficient bounds.

Theorem 6. *Let F be the series form (1) and if $F \in \mathcal{S}^*_{4\mathcal{L}}$, then*

$$\begin{aligned} |\lambda_1| &\le \frac{5}{12}, \\ |\lambda_2| &\le \frac{5}{24}, \\ |\lambda_3| &\le \frac{5}{36}, \\ |\lambda_4| &\le \frac{5}{48}. \end{aligned} \tag{34}$$

These bounds are sharp.

Proof. Let $F \in \mathcal{S}^*_{4\mathcal{L}}$. Then, Schwarz function u may therefore be used to express (16) as

$$\frac{zF'(z)}{F(z)} = 1 + \frac{5}{6}u(z) + \frac{1}{6}(u(z))^5 = \alpha(z). \tag{35}$$

From the use of Schwarz function u and if $q \in \mathcal{P}$, we have

$$q(z) = \frac{1+(u(z))}{1-(u(z))} := 1 + e_1 z + e_2 z^2 + \cdots, \tag{36}$$

and by simple computation, we get

$$\begin{aligned} u(z) = {}& \frac{1}{2}e_1 z + \left(\frac{1}{2}e_2 - \frac{1}{4}e_1^2\right)z^2 + \left(\frac{1}{8}e_1^3 - \frac{1}{2}e_1 e_2 + \frac{1}{2}e_3\right)z^3 \\ &+ \left(\frac{1}{2}e_4 - \frac{1}{2}e_1 e_3 - \frac{1}{4}e_2^2 - \frac{1}{16}e_1^4 + \frac{3}{8}e_1^2 e_2\right)z^4 + \cdots. \end{aligned} \tag{37}$$

Using (1), we attain

$$\begin{aligned} \frac{zF'(z)}{F(z)} := {}& 1 + b_2 z + \left(-b_2^2 + 2b_3\right)z^2 + \left(-3b_2 b_3 + 3b_4 + b_2^3\right)z^3 \\ &+ \left(-2b_3^2 + 4b_5 - 4b_2 b_4 + 4b_2^2 b_3 - b_2^4\right)z^4 + \cdots. \end{aligned} \tag{38}$$

By some calculation and using the series expansion of (37), we get

$$\begin{aligned} \alpha(z) = {}& 1 + \frac{5}{12}e_1 z + \left(\frac{5}{12}e_2 - \frac{5}{24}e_1^2\right)z^2 \\ &+ \left(\frac{5}{48}e_1^3 - \frac{5}{12}e_1 e_2 + \frac{5}{12}e_3\right)z^3 \\ &+ \left(\frac{5}{12}e_4 - \frac{5}{96}e_1^4 + \frac{5}{16}e_1^2 e_2 - \frac{5}{12}e_1 e_3 - \frac{5}{24}e_2^2\right)z^4 + \cdots. \end{aligned} \tag{39}$$

Now, by comparing (38) and (39), we get

$$b_2 = \frac{5}{12}e_1, \tag{40}$$

$$b_3 = \frac{5}{24}e_2 - \frac{5}{288}e_1^2, \tag{41}$$

$$b_4 = \frac{5}{36}e_3 + \frac{35}{10368}e_1^3 - \frac{5}{96}e_1 e_2, \tag{42}$$

$$b_5 = \frac{5}{48}e_4 - \frac{455}{497664}e_1^4 + \frac{115}{6912}e_1^2 e_2 - \frac{35}{1152}e_2^2 - \frac{5}{108}e_1 e_3. \tag{43}$$

Utilizing (40) and (10), (11), (12), and (13), we have

$$\lambda_1 = \frac{5}{24}e_1, \tag{44}$$

$$\lambda_2 = \frac{5}{48}e_2 - \frac{5}{96}e_1^2, \tag{45}$$

$$\lambda_3 = \frac{5}{288}e_1^3 - \frac{5}{72}e_1 e_2 + \frac{5}{72}e_3, \tag{46}$$

$$\lambda_4 = \frac{5}{96}e_4 - \frac{5}{768}e_1^4 + \frac{5}{128}e_1^2 e_2 - \frac{5}{96}e_1 e_3 - \frac{5}{192}e_2^2. \tag{47}$$

From (44), using triangle inequality and (29), we get

$$|\lambda_1| \le \frac{5}{12}. \tag{48}$$

Also, from (45), application (30), and triangle inequality, we get

$$|\lambda_2| \le \frac{5}{24}. \tag{49}$$

By rearranging (46), we have

$$|\lambda_3| = \frac{5}{288}\left|e_1^3 - 4e_1 e_2 + 4e_3\right|. \tag{50}$$

By Lemma 4 and triangle inequality, we obtain

$$|\lambda_3| \le \frac{5}{36}. \tag{51}$$

By rearranging (47), we have

$$\lambda_4 = -\frac{5}{96}\left(\left(\frac{1}{2}\right)e_2^2 + \left(\frac{1}{8}\right)e_1^4 - \left(\frac{3}{4}\right)e_2 e_1^2 + e_1 e_3 - e_4\right). \tag{52}$$

Comparing the equation of (52) right side with

$$\left|\gamma e_1^4 + \varsigma e_2^2 + 2\tau e_1 e_3 - \frac{3}{2}\psi e_1^2 e_2 - e_4\right|, \tag{53}$$

we get $\gamma = 1/8$, $\varsigma = 1/2$, $\tau = 1/2$, $\psi = 1/2$, and

$$\begin{aligned} &8(1-\varsigma)\varsigma\left[(\tau(\varsigma+\tau)-\psi)^2 + (\tau\psi - 2\gamma)^2\right] \\ &\quad + \tau(\psi - 2\varsigma\tau)^2(1-\tau) \le 4\tau^2\varsigma(1-\varsigma)(1-\tau)^2. \end{aligned} \tag{54}$$

Thus, Lemma 5's requirements are all met. Hence,

$$|\lambda_4| \leq \frac{5}{96}(2) = \frac{5}{48}. \tag{55}$$

These are sharp outcomes. Equality is determined by using (10)–(13) and

$$\begin{aligned} F_1(z) &= z \ \exp\left(\int_0^z \left(\frac{5}{6} + \frac{1}{6}t^4\right)dt\right) = z + \frac{5}{6}z^2 + \cdots, \\ F_2(z) &= z \ \exp\left(\int_0^z \left(\frac{5}{6}t + \frac{1}{6}t^9\right)dt\right) = z + \frac{5}{12}z^3 + \cdots, \\ F_3(z) &= z \ \exp\left(\int_0^z \left(\frac{5}{6}t^2 + \frac{1}{6}t^{14}\right)dt\right) = z + \frac{5}{18}z^4 + \cdots, \\ F_4(z) &= z \ \exp\left(\int_0^z \left(\frac{5}{6}t^3 + \frac{1}{6}t^{19}\right)dt\right) = z + \frac{5}{24}z^5 + \cdots. \end{aligned} \tag{56}$$

□

Theorem 7. *If $F \in \mathcal{S}^*_{4\mathcal{L}}$, then*

$$\left|\lambda_2 - \mu\lambda_1^2\right| \leq \max\left\{\frac{5}{24}, \frac{5}{48}\left|\frac{5\mu}{3}\right|\right\}. \tag{57}$$

The above stated inequality is best possible.

Proof. By utilizing (44) and (45), we have

$$\left|\lambda_2 - \mu\lambda_1^2\right| = \frac{5}{48}\left|e_2 - \frac{e_1^2}{2}\left(\frac{6+5\mu}{6}\right)\right|. \tag{58}$$

Implementation of (28) and triangle inequality, we get

$$\left|\lambda_2 - \mu\lambda_1^2\right| \leq \max\left\{\frac{5}{24}, \frac{5}{48}\left|\frac{5\mu}{3}\right|\right\}. \tag{59}$$

Equality is determined by using (10), (11), and

$$F_2(z) = z \ \exp\left(\int_0^z \left(\frac{5}{6}t + \frac{1}{6}t^9\right)dt\right) = z + \frac{5}{12}z^3 + \cdots. \tag{60}$$

□

Corollary 8. *If $F \in \mathcal{S}^*_{4\mathcal{L}}$, then*

$$\left|\lambda_2 - \lambda_1^2\right| \leq \frac{5}{24}. \tag{61}$$

This inequality is sharp and can be obtained by using (10), (11), and

$$F_2(z) = z \ \exp\left(\int_0^z \left(\frac{5}{6}t + \frac{1}{6}t^9\right)dt\right) = z + \frac{5}{12}z^3 + \cdots. \tag{62}$$

Theorem 9. *Let F be the expansion (1) and if $F \in \mathcal{S}^*_{4\mathcal{L}}$, then*

$$|\lambda_1\lambda_2 - \lambda_3| \leq \frac{5}{36}. \tag{63}$$

The above stated result is the best possible.

Proof. From (44)–(46), we easily attain

$$|\lambda_1\lambda_2 - \lambda_3| = \frac{65}{2304}\left|-e_1^3 + \frac{42}{13}e_1e_2 - \frac{32}{13}e_3\right|. \tag{64}$$

By using Lemma 4 and triangle inequality, we obtain

$$|\lambda_1\lambda_2 - \lambda_3| \leq \frac{65}{2304}\left(\frac{64}{13}\right) = \frac{5}{36}. \tag{65}$$

Equality is determined by using (10), (11), (12), and

$$F_3(z) = z \ \exp\left(\int_0^z \left(\frac{5}{6}t^2 + \frac{1}{6}t^{14}\right)dt\right) = z + \frac{5}{18}z^4 + \cdots. \tag{66}$$

□

Theorem 10. *Let F be the expansion (1) and if $F \in \mathcal{S}^*_{4\mathcal{L}}$, then*

$$\left|\lambda_4 - \lambda_2^2\right| \leq \frac{5}{48}. \tag{67}$$

The last stated inequality is the finest.

Proof. From the use (45) and (47), we get

$$\left|\lambda_4 - \lambda_2^2\right| = -\frac{5}{96}\left|\left(\frac{17}{24}\right)e_2^2 - \left(\frac{23}{44}\right)e_2e_1^2 + \left(\frac{17}{96}\right)e_1^4 + e_1e_3 - e_4\right|. \tag{68}$$

Comparing the right side of (68) with

$$\left|\gamma e_1^4 + \varsigma e_2^2 + 2\tau e_1e_3 - \frac{3}{2}\psi e_1^2e_2 - e_4\right|, \tag{69}$$

we get $\gamma = 17/96$, $\varsigma = 17/24$, $\tau = 1/2$, $\psi = 23/36$, and

$$\begin{aligned} &8(1-\varsigma)\varsigma\left[(\tau(\varsigma+\tau)-\psi)^2 + (\tau\psi - 2\gamma)^2\right] \\ &\quad + \tau(\psi - 2\varsigma\tau)^2(1-\tau) = 0.0051909, \\ &4\tau^2\varsigma(1-\varsigma)(1-\tau)^2 = 0.051649. \end{aligned} \tag{70}$$

Thus, Lemma 5's requirements are all met. Hence,

$$\left|\lambda_4 - \lambda_2^2\right| \leq \frac{5}{96}(2) = \frac{5}{48}. \tag{71}$$

Equality is determined by using (11), (13), and

$$F_4(z) = z \ \exp\left(\int_0^z \left(\frac{5}{6}t^3 + \frac{1}{6}t^{19}\right)dt\right) = z + \frac{5}{24}z^2 + \cdots. \tag{72}$$

□

Theorem 11. *Let $F \in \mathcal{S}^*_{4\mathcal{L}}$ be the representation (1). Then,*

$$\left|H_{2,1}(J_F/2)\right| \le \frac{25}{576}. \tag{73}$$

This result is sharp.

Proof. We can write the $H_{2,1}(J_F/2)$ as

$$H_{2,1}(J_F/2) = \left|\lambda_1\lambda_3 - \lambda_2^2\right|. \tag{74}$$

From (44)–(46), we have

$$\left|\lambda_1\lambda_3 - \lambda_2^2\right| = \frac{25}{27648}\left|e_1^4 - 4e_1^2e_2 + 16e_1e_3 - 12e_2^2\right|. \tag{75}$$

Using (25) and (26) to express e_2 and e_3 in terms of e_1 and also $e_1 = e$, with $0 \le e \le 2$, we obtain

$$\left|\lambda_1\lambda_3 - \lambda_2^2\right| = \frac{25}{27648}\Big|-4e^2x^2\left(4-e^2\right) + 8e\left(1-|x|^2\right)\left(4-e^2\right)\delta - 3x^2\left(4-e^2\right)^2\Big|. \tag{76}$$

By changing $|\delta| \le 1$ and $|x| = c$, where $c \le 1$ and utilizing triangle inequality and pickings $e \in [0, 2]$, so

$$\left|\lambda_1\lambda_3 - \lambda_2^2\right| \le \frac{25}{27648}\left\{4e^2c^2\left(4-e^2\right) + 8e\left(1-c^2\right)\left(4-e^2\right) + 3c^2\left(4-e^2\right)^2\right\} \coloneqq \Xi(e,c). \tag{77}$$

Differentiate with respect to c, we have

$$\frac{\partial\Xi(e,c)}{\partial c} = \frac{25}{27648}\left(-2ce^4 + 16ce^3 - 16ce^2 - 64ce + 96c\right). \tag{78}$$

It is easy exercise to show that $\Xi'(e,c) \ge 0$ on $[0, 1]$, so that $\Xi(e,c) \le \Xi(e,1)$. Putting $c = 1$, we get

$$\left|\lambda_1\lambda_3 - \lambda_2^2\right| \le \frac{25}{27648}\left(4e^2\left(4-e^2\right) + 3\left(4-e^2\right)^2\right) \coloneqq \Theta(e). \tag{79}$$

As $\Theta'(e) \le 0$, so $\Theta(e)$ is a decreasing function, so that it gives a maximum value at $e = 0$

$$\left|H_{2,1}\left(\frac{J_F}{2}\right)\right| \le \frac{25}{27648}(48) = \frac{25}{576}. \tag{80}$$

Equality is determined by using (10), (11), (12), and

$$F_2(z) = z\ \exp\left(\int_0^z\left(\frac{5}{6}t + \frac{1}{6}t^9\right)dt\right) = z + \frac{5}{12}z^3 + \cdots. \tag{81}$$

□

4. Coefficient Inequalities for the Class $\mathcal{C}_{4\mathcal{L}}$

For the function of class $\mathcal{C}_{4\mathcal{L}}$, we start this portion by determining the absolute values of the first four initial logarithmic coefficients.

Theorem 12. *Let F be given by (1) and if $F \in \mathcal{C}_{4\mathcal{L}}$, then*

$$\begin{aligned}|\lambda_1| &\le \frac{5}{24},\\ |\lambda_2| &\le \frac{5}{72},\\ |\lambda_3| &\le \frac{5}{144},\\ |\lambda_4| &\le \frac{1}{48}.\end{aligned} \tag{82}$$

These bounds are sharp.

Proof. Let $F \in \mathcal{C}_{4\mathcal{L}}$. Then, (17) can be written in the form of Schwarz function as

$$1 + \frac{zF''(z)}{F'(z)} = 1 + \frac{5}{6}u(z) + \frac{1}{6}(u(z))^5 = \psi(z). \tag{83}$$

Using (1), we obtain

$$1 + \frac{zF''(z)}{F'(z)} \coloneqq 1 + 2b_2z + \left(6b_3 - 4b_2^2\right)z^2 + \left(8b_2^3 - 18b_2b_3 + 12b_4\right)z^3 + \left(20b_5 - 16b_2^4 + 48b_2^2b_3 - 32b_2b_4 - 18b_3^2\right)z^4 + \cdots. \tag{84}$$

Now, by comparing (84) and (39), we get

$$\begin{aligned}b_2 &= \frac{5}{24}e_1,\\ b_3 &= \frac{5}{72}e_2 - \frac{5}{864}e_1^2,\\ b_4 &= \frac{5}{144}e_3 + \frac{35}{41472}e_1^3 - \frac{5}{384}e_1e_2,\\ b_5 &= \frac{1}{48}e_4 - \frac{91}{497664}e_1^4 + \frac{23}{6912}e_1^2e_2 - \frac{7}{1152}e_2^2 - \frac{1}{108}e_1e_3.\end{aligned} \tag{85}$$

Utilizing (85) and (10), (11), (12), and (13) we have

$$\lambda_1 = \frac{5}{24}e_1, \tag{86}$$

$$\lambda_2 = \frac{5}{48}e_2 - \frac{5}{96}e_1^2, \tag{87}$$

$$\lambda_3 = \frac{5}{288}e_1^3 - \frac{5}{72}e_1e_2 + \frac{5}{72}e_3, \tag{88}$$

$$\lambda_4 = \frac{5}{96}e_4 - \frac{5}{768}e_1^4 + \frac{5}{128}e_1^2e_2 - \frac{5}{96}e_1e_3 - \frac{5}{192}e_2^2. \tag{89}$$

From (86), using triangle inequality and (29), we get

$$|\lambda_1| \le \frac{5}{24}. \tag{90}$$

Also, from (87), application (30), and triangle inequality, we get

$$|\lambda_2| \le \frac{5}{72}. \tag{91}$$

By rearranging (88), we have

$$|\lambda_3| = \frac{5}{288}\left|\frac{7}{48}e_1^3 - \frac{19}{24}e_1e_2 + e_3\right|. \tag{92}$$

By Lemma 4 and triangle inequality, we obtain

$$|\lambda_3| \le \frac{5}{144}. \tag{93}$$

By rearranging (89), we have

$$\lambda_4 = -\frac{1}{96}\left(\frac{11}{27}e_2^2 + \frac{13109}{248832}e_1^4 - \frac{2353}{5184}e_2e_1^2 + \frac{19}{24}e_1e_3 - e_4\right). \tag{94}$$

Comparing the right side of (94) with

$$\left|\gamma e_1^4 + \varsigma e_2^2 + 2\tau e_1e_3 - \frac{3}{2}\psi e_1^2e_2 - e_4\right|, \tag{95}$$

we get $\gamma = 13109/248832$, $\varsigma = 11/27$, $\tau = 19/48$, and $\psi = 2353/7776$. Thus, all the conditions of Lemma 5 are satisfied. Hence, we have

$$|\lambda_4| \le \frac{1}{96}(2) = \frac{1}{48}. \tag{96}$$

These are sharp outcomes. Equality is determined by using (10), (11), (12), and (13) along with (22). □

Theorem 13. *Let $F \in \mathscr{C}_{4\mathscr{L}}$ be the series form (1). Then,*

$$\left|\lambda_2 - \mu\lambda_1^2\right| \le \max\left\{\frac{5}{72}, \frac{5}{72}\left|\frac{7+15\mu}{12}\right|\right\}, \text{for } \mu \in \mathbb{C}. \tag{97}$$

This inequality is sharp.

Proof. By utilizing (86) and (87), we have

$$\left|\lambda_2 - \mu\lambda_1^2\right| = \frac{5}{144}\left|e_2 - \frac{e_1^2}{2}\left(\frac{19+15\mu}{24}\right)\right|. \tag{98}$$

Implementation of (28) and triangle inequality, we get

$$\left|\lambda_2 - \mu\lambda_1^2\right| \le \max\left\{\frac{5}{72}, \frac{5}{72}\left|\frac{7+15\mu}{12}\right|\right\}. \tag{99}$$

Equality is determined by using (10), (11), and (22). □

For $\lambda = 1$, we get the below corollary.

Corollary 14. *Let $F \in \mathscr{C}_{4\mathscr{L}}$, and it has the form (1). Then,*

$$\left|\lambda_2 - \lambda_1^2\right| \le \frac{5}{72}. \tag{100}$$

This inequality is sharp and can be obtained by using (10), (11), and (22).

Theorem 15. *Let F be the form (1) and if $F \in \mathscr{C}_{4\mathscr{L}}$, then*

$$|\lambda_1\lambda_2 - \lambda_3| \le \frac{5}{144}. \tag{101}$$

This result is sharp.

Proof. By using (86)–(88), we obtain

$$|\lambda_1\lambda_2 - \lambda_3| = \frac{5}{288}\left|-\frac{263}{1152}e_1^3 + e_1e_2 - e_3\right|. \tag{102}$$

By using Lemma 4 and triangle inequality, we obtain

$$|\lambda_1\lambda_2 - \lambda_3| \le \frac{5}{288}(2) = \frac{5}{144}. \tag{103}$$

Equality is determined by using (10), (11), (12), and (22). □

Theorem 16. *Let F be the form (1) and $F \in \mathscr{C}_{4\mathscr{L}}$. Then,*

$$\left|\lambda_4 - \lambda_2^2\right| \le \frac{1}{48}. \tag{104}$$

This result is sharp.

Proof. By using (87) and (89), we obtain

$$\left|\lambda_4 - \lambda_2^2\right| = -\frac{1}{96}\left|\frac{113}{216}e_2^2 - \frac{707}{1296}e_2e_1^2 + \frac{35243}{497664}e_1^4 + \frac{19}{24}e_1e_3 - e_4\right|. \tag{105}$$

Comparing the right side of (68) with

$$\left|\gamma e_1^4 + \varsigma e_2^2 + 2\tau e_1e_3 - \frac{3}{2}\psi e_1^2e_2 - e_4\right|, \tag{106}$$

we get $\gamma = 35243/497664$, $\varsigma = 113/216$, $\tau = 19/24$, $\psi = 707/1944$, and

$$8(1-\varsigma)\varsigma\left[(\tau(\varsigma+\tau)-\psi)^2+(\tau\psi-2\gamma)^2\right]+\tau(\psi-2\varsigma\tau)^2(1-\tau)=0.00062010, \quad 4\tau^2\varsigma(1-\varsigma)(1-\tau)^2=0.057070. \tag{107}$$

Thus, all the conditions of Lemma 5 are satisfied. Hence, we have

$$\left|\lambda_4-\lambda_2^2\right| \le \frac{1}{96}(2)=\frac{1}{48}. \tag{108}$$

Equality is determined by using (11), (13), and (22). □

Theorem 17. *Let F be given the form (1) and* $F\in\mathscr{C}_{4\mathscr{L}}$. *Then,*

$$\left|H_{2,1}\left(\frac{J_F}{2}\right)\right| \le \frac{25}{576}. \tag{109}$$

This result is sharp.

Proof. We can write the $H_{2,1}(F_F/2)$ as;

$$H_{2,1}\left(\frac{J_F}{2}\right)=\left|\lambda_1\lambda_3-\lambda_2^2\right|. \tag{110}$$

From (86)–(88), we have

$$\left|\lambda_1\lambda_3-\lambda_2^2\right|=\frac{1}{47775744}\left|3575e_1^4-22800e_1^2e_2+86400e_1e_3-57600e_2^2\right|. \tag{111}$$

Using (25) and (26) to express e_2 and e_3 in terms of e_1 and also $e_1=e$, with $0\le e\le 2$, we obtain

$$\left|\lambda_1\lambda_3-\lambda_2^2\right|=\frac{1}{47775744}\left|-21600e^2x^2\left(4-e^2\right)+43200e\left(1-|x|^2\right)\left(4-e^2\right)\delta-14400x^2\left(4-e^2\right)^2+3000e^2x\left(4-e^2\right)-625e^4\right|. \tag{112}$$

By replacing $|\delta|\le 1$ and $|x|=c$, where $c\le 1$ and using triangle inequality and taking $e\in[0,2]$, so

$$\left|\lambda_1\lambda_3-\lambda_2^2\right| \le \frac{1}{47775744}\left\{21600e^2c^2\left(4-e^2\right)+43200e\left(1-c^2\right)\left(4-e^2\right)+14400c^2\left(4-e^2\right)^2+3000e^2c\left(4-e^2\right)+625e^4\right\}:=\Omega(e,c). \tag{113}$$

Differentiate with respect to c, we have

$$\frac{\partial\Omega(e,c)}{\partial c}=\frac{1}{47775744}\left(-600(e-2)(e+2)\cdot\left(24ce^2-144ce+5e^2+192c\right)\right). \tag{114}$$

It is a simple exercise to show that $\Omega'(e,c)\ge 0$ on $[0,1]$, so that $\Omega(e,c)\le\Omega(e,1)$. Putting $c=1$ gives

$$\left|\lambda_1\lambda_3-\lambda_2^2\right| \le \frac{1}{47775744}\left(24600e^2\left(4-e^2\right)+625e^4+14400\left(4-e^2\right)^2\right):=\Theta(e). \tag{115}$$

As $\Theta'(e)\le 0$, so $\Theta(e)$ is a decreasing function, so that it gives a maximum value at $e=0$

$$\left|H_{2,1}\left(\frac{J_F}{2}\right)\right| \le \frac{1}{47775744}(230400)=\frac{25}{5184}. \tag{116}$$

Equality is determined by using (10), (11), (12), and (22). □

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

This research was supported by the Princess Nourah Bint Abdulrahman University Researchers Supporting Project number (PNURSP2022R183), Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia.

References

[1] P. Koebe, "Über die Uniformisierung der algebraischen Kurven. II," *Mathematische Annalen*, vol. 69, no. 1, pp. 1–81, 1910.

[2] L. Bieberbach, "Über einige Extremalprobleme im Gebiete der konformen abbildung," *Mathematische Annalen*, vol. 77, no. 2, pp. 153–172, 1916.

[3] L. De Branges, “A proof of the Bieberbach conjecture,” *Acta Mathematica*, vol. 154, no. 1-2, pp. 137–152, 1985.

[4] F. G. Avkhadiev and K. J. Wirths, *Schwarz-Pick Type Inequalities*, Springer Science & Business Media, 2009.

[5] C. H. FitzGerald and C. Pommerenke, “The de Branges theorem on univalent functions,” *Transactions of the American Mathematical Society*, vol. 290, no. 2, pp. 683–690, 1985.

[6] C. H. FitzGerald and C. Pommerenke, “A theorem of de Branges on univalent functions,” *Serdica*, vol. 13, no. 1, pp. 21–25, 1987.

[7] I. P. Kayumov, “On Brennan's conjecture for a special class of functions,” *Mathematical Notes*, vol. 78, no. 3-4, pp. 498–502, 2005.

[8] D. Alimohammadi, E. Analouei Adegani, T. Bulboaca, and N. E. Cho, “Logarithmic coefficient bounds and coefficient conjectures for classes associated with convex functions,” *Journal of Function Spaces*, vol. 2021, Article ID 6690027, 7 pages, 2021.

[9] M. Obradović, S. Ponnusamy, and K. J. Wirths, “Logarithmic coefficients and a coefficient conjecture for univalent functions,” *Monatshefte für Mathematik*, vol. 185, no. 3, pp. 489–501, 2018.

[10] Z. Ye, “The logarithmic coefficients of close-to-convex functions,” *Bulletin of the Institute of Mathematics, Academia Sinica*, vol. 3, no. 3, pp. 445–452, 2008.

[11] Q. Deng, “On the logarithmic coefficients of Bazilevic functions,” *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5889–5894, 2011.

[12] D. Girela, “Logarithmic coefficients of univalent functions,” *Annales-Academiae Scientiarum Fennicae Series A1 Mathematica*, vol. 25, no. 2, pp. 337–350, 2000.

[13] O. Roth, “A sharp inequality for the logarithmic coefficients of univalent functions,” *Proceedings of the American Mathematical Society*, vol. 135, no. 7, pp. 2051–2054, 2007.

[14] V. V. Andreev and P. L. Duren, “Inequalities for logarithmic coefficients of univalent functions and their derivatives,” *Indiana University Mathematics journal*, vol. 37, no. 4, pp. 721–733, 1988.

[15] W. C. Ma and D. Minda, “A unified treatment of some special classesof univalent functions,” in *Proceedings of the Conference on Complex Analysis*, pp. 157–169, Tianjin, China, 1992.

[16] K. Bano and M. Raza, “Starlike functions associated with cosine function,” *Bulletin of Iranian Mathematical Society*, vol. 47, no. 11, p. 20, 2021.

[17] A. Alotaibi, M. Arif, M. A. Alghamdi, and S. Hussain, “Starlikness associated with cosine hyperbolic function,” *Mathematics*, vol. 8, no. 7, 2020.

[18] K. Ullah, S. Zainab, M. Arif, M. Darus, and M. Shutaywi, “Radius problems for starlike functions associated with the tan hyperbolic function,” *Journal of Function Spaces*, vol. 2021, Article ID 9967640, 15 pages, 2021.

[19] K. Ullah, H. M. Srivastava, A. Rafiq, M. Arif, and S. Arjika, “A study of sharp coefficient bounds for a new subfamily of starlike functions,” *Journal of Inequalities and Applications*, vol. 2021, no. 1, 2021.

[20] R. Mendiratta, S. Nagpal, and V. Ravichandran, “On a subclass of strongly starlike functions associated with exponential function,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 38, no. 1, pp. 365–386, 2015.

[21] L. Shi, H. M. Srivastava, M. Arif, S. Hussain, and H. Khan, “An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function,” *Symmetry*, vol. 11, no. 5, p. 598, 2019.

[22] S. S. Kumar and K. Arora, “Starlike functions associated with a petal shaped domain,” 2020, https://arxiv.org/abs/2010.10072.

[23] D. A. Brannan and W. E. Kirwan, “On some classes of bounded univalent functions,” *Journal of the London Mathematical Society*, vol. 2, no. 1, pp. 431–443, 1969.

[24] J. Sokół and J. Stankiewicz, “Radius of convexity of some subclasses of strongly starlike functions,” *Zeszyty naukowe Politechniki Rzeszowskiej Matematyka*, vol. 19, pp. 101–105, 1996.

[25] K. Sharma, N. K. Jain, and V. Ravichandran, “Starlike functions associated with a cardioid,” *Afrika Matematika*, vol. 27, no. 5-6, pp. 923–939, 2016.

[26] L. Shi, I. Ali, M. Arif, N. E. Cho, S. Hussain, and H. Khan, “A study of third Hankel determinant problem for certain subfamilies of analytic functions involving cardioid domain,” *Mathematics*, vol. 7, no. 5, p. 418, 2019.

[27] C. Pommerenke, “On the coefficients and Hankel determinants of univalent functions,” *Journal of the London Mathematical Society*, vol. 1, no. 1, pp. 111–122, 1966.

[28] C. Pommerenke, “On the Hankel determinants of univalent functions,” *Mathematika*, vol. 14, no. 1, pp. 108–112, 1967.

[29] A. Janteng, S. A. Halim, and M. Darus, “Coefficient inequality for a function whose derivative has a positive real part,” *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 2, pp. 1–5, 2006.

[30] A. Janteng, S. A. Halim, and M. Darus, “Hankel determinant for starlike and convex functions,” *International Journal of Mathematical Analysis*, vol. 1, no. 13, pp. 619–625, 2007.

[31] D. Raducanu and P. Zaprawa, “Second Hankel determinant for close-to-convex functions,” *Comptes Rendus Mathematique*, vol. 355, no. 10, pp. 1063–1071, 2017.

[32] K. O. Babalola, “On $H_{3,1}(F)$ Hankel determinant for some classes of univalent functions,” *Inequality Theory and Applications*, vol. 6, pp. 1–7, 2010.

[33] D. Bansal, S. Maharana, and J. K. Prajapat, “Third order Hankel Determinant for certain univalent functions,” *Journal of Korean Mathematical Society*, vol. 52, no. 6, pp. 1139–1148, 2015.

[34] P. Zaprawa, “Third Hankel determinants for subclasses of univalent functions,” *Mediterranean Journal of Mathematics*, vol. 141, no. 19, p. 10, 2017.

[35] O. S. Kwon, A. Lecko, and Y. J. Sim, “The bound of the Hankel determinant of the third kind for starlike functions,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 2, pp. 767–780, 2019.

[36] B. Kowalczyk, A. Lecko, and Y. J. Sim, “The sharp bound FOR the Hankel determinant of the third kind for convex functions,” *Bulletin of the Australian Mathematical Society*, vol. 97, no. 3, pp. 435–445, 2018.

[37] A. Lecko, Y. J. Sim, and B. Smiarowska, “The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2,” *Complex Analysis and Operator Theory*, vol. 13, no. 5, pp. 2231–2238, 2019.

[38] B. Kowalczyk and A. Lecko, “Second Hankel determinant of logarithmic coefficients of convex and starlike functions,” *Bulletin of the Australian Mathematical Society*, vol. 105, no. 3, pp. 458–467, 2022.

[39] B. Kowalczyk and A. Lecko, “Second Hankel determinant of logarithmic coefficients of convex and starlike functions of

order alpha," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 45, pp. 727–740, 2022.

[40] C. Pommerenke, *Univalent Function*, Göttingen, Germany, Vanderhoeck & Ruprecht, 1975.

[41] R. J. Libera and E. J. Złotkiewicz, "Early coefficients of the inverse of a regular convex function," *Proceedings of the American Mathematical Society*, vol. 85, no. 2, pp. 225–230, 1982.

[42] O. S. Kwon, A. Lecko, and Y. J. Sim, "On the fourth coefficient of functions in the Carathéodory class," *Computational Methods and Function Theory*, vol. 18, no. 2, pp. 307–314, 2018.

[43] F. Keough and E. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proceedings of the American Mathematical Society*, vol. 20, no. 1, pp. 8–12, 1969.

[44] M. Arif, M. Raza, H. Tang, S. Hussain, and H. Khan, "Hankel determinant of order three for familiar subsets of analytic functions related with sine function," *Open Mathematics*, vol. 17, no. 1, pp. 1615–1630, 2019.

[45] M. Arif, S. Umar, M. Raza, T. Bulboaca, M. U. Farooq, and H. Khan, "On fourth Hankel determinant for functions associated with Bernoulli's lemniscate," *Hacettepe Journal of Mathematics and Statistics*, vol. 49, no. 5, pp. 1–11, 2020.

[46] V. Ravichandran and S. Verma, "Borne pour le cinquieme coefficient des fonctions etoilees," *Comptes Rendus Mathematique*, vol. 353, no. 6, pp. 505–510, 2015.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2046173, 11 pages
https://doi.org/10.1155/2022/2046173

Research Article

Computing Wiener and Hyper-Wiener Indices of Zero-Divisor Graph of $\mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}$

Yonghong Liu[1], **Abdul Rauf**[2], **AdnanAslam**[3], **Saira Ishaq**[2] **and Abudulai Issa**[4]

[1]*School of Computer Science, Chengdu University, Chengdu, China*
[2]*Department of Mathematics, Air University Islamabad, Multan Campus, Pakistan*
[3]*Department of Natural Sciences and Humanities, University of Engineering and Technology, Lahore, Pakistan (RCET), Pakistan*
[4]*Department of Mathematics, University of Lome, 1 PoBox 1515 Lome 1, Togo*

Correspondence should be addressed to Abudulai Issa; issaabudulai13@gmail.com

Received 29 July 2022; Revised 27 August 2022; Accepted 29 August 2022; Published 19 September 2022

Academic Editor: Sarfraz Nawaz Malik

Let $S = \mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}$ be a commutative ring where $\wp$, $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are positive prime integers with $\mathfrak{I}_1 \neq \mathfrak{I}_2$. The zero-divisor graph assigned to S is an undirected graph, denoted as $Y(S)$ with vertex set $V(Y(S))$ consisting of all Zero-divisor of the ring S and for any c, d $\in V(Y(S))$, $cd \in E(Y(S))$ if and only if cd =0. A topological index/descriptor is described as a topological-invariant quantity that transforms a molecular graph into a mathematical real number. In this paper, we have computed distance-based polynomials of $Y(R)$ i-e Hosoya polynomial, Harary polynomial, and the topological indices related to these polynomials namely Wiener index, and Hyper-Wiener index.

1. Introduction

The Zero-divisor graph is the undirected graph on the set of the Zero-divisors of a commutative ring. In a Zero-divisor graph, the set of zero-divisors are considered as vertices and pairs of elements whose product is zero as its edges. The Zero-divisor graph is very helpful to study the algebraic properties of rings using graph-theoretical tools.

In 1988 Beck was the first who gives the idea of interlinking two main mathematics topics Algebra, and Graph theory [1]. First, he presented the concept of a Zero-divisor graph of Commutative Ring R, in which all the elements of ring R were considered as the vertices of a Zero-divisor graph, and those two distinct vertices c and d are connected if and only if cd =0. Beck's main objective in his work was to show the coloring of the Commutative ring. Naseer and Anderson extended the work by Beck's in [2]. In [3], Anderson-Livingston worked on the Zero-divisor graphs in which only non-zero Zero-divisors are considered as the vertices. Anderson-Livingston discussed the relations between ring theoretic properties of R and graph-theoretic properties of the $Y(R)$. Furthermore, this study presents some important results of zero-divisor graphs. Later Anderson, Frazier, Lauve, Levy, Livingston, and Shapiro [4, 5] worked on translating the algebraic properties of rings into graphical language.

Redmond developed the idea of the Zero-divisor graphs associated to non-commutative rings [6]. He defined a Zero-divisor graph associated with a non-commutative ring in many more ways, including both directed and undirected graphs. Redmond [7] carried on this work by extending the concept of Zero-divisor graph of a Commutative ring to an ideal-based Zero-divisor graph of a Commutative ring by replacing elements whose product is zero with elements whose product lies in some ideal I of ring R. Since then, many researchers have worked on it and defined graphs such as unit graphs, the equivalence class of Zero-divisor graphs, total graphs, ideal-based Zero-divisor graphs, the Jacobson graphs, and so on (see, [7–11]).

Let G be graph with vertex set V(G) and edge set E(G). The degree of a vertex v is the number of edges attached to v. For any two vertices u and v, the distance between them is denoted by d(u,v) and is defined as the length of the shortest path joining them. For instance, d (u, v) = 0 If and only if u = v and d (u, v) = ∞ if there does not exist any possible shortest path. For more details on the basic definition related to graph theory, the readers can see the book by [12].

Let Z(S) be the set of Zero-divisors of the Commutative ring 'S'. The Zero-divisor graph of S is denoted by Y (S) = (V(Y (S)), E(Y (S))), which is an undirected graph with V(Y (S)) = Z(S) \ {0} as vertex set, and for different c, d ∈ V(Y(S)), the vertices c and d have an edge $cd \in$ E(Y(S)) between them if and only if cd = 0 [13]. It is very interesting to translate the algebraic properties of algebraic graphs into numerical molecular descriptors. A topological index/descriptor is described as a topological-invariant quantity that transforms a molecular graph into a mathematical real number. QSPR/QSAR studies are majorly concerned with the application of these molecular descriptors or Topological indices. Many molecular descriptors have been introduced in the last decade, demonstrating their importance. A molecular structure is denoted as a graph with atoms as vertices and bonds as edges. Then, using various forms of topological indices, one can study both the algebraic and chemical aspects of the compounds. There are two main types of topological indices, the first based on degree [14–19] and the second based on distance. Randic Connectivity index, Zagreb Indices, the Harmonic index, Atom bond Connectivity, and Geometric Arithmetic index are degree-based topological indices. Some well-known distance-based topological indices are the Wiener index, Hosoya index, and Estrada index [20, 21]. For details related to the computation of Hosoya and Harary polynomial of zero divisor graphs associated to some commutative rings, the readers can see [22–24].

Eccentricity-based indices have been effectively used to construct a variety of mathematical models for the prediction of biological activities of various types. Several authors have examined the uses and mathematical properties of these indices. Further, the readers can see [25–28] for more details about eccentricity based-indices. In 1988 Hosoya polynomial was introduced by Haruo Hosoya [29]. With its vast application in graph theory and chemistry, it is proved to be an effective distance-based topological index [30]. The Hosoya polynomial has many chemical applications [31]. Almost all distance-based topological indices can be computed from this polynomial [32–34]. The Hosoya polynomial is related to a variety of topological indices, the most well-known of which is the Wiener, Hyper Wiener, and Harary polynomial. For further study see [21, 35–37]. Hosoya polynomial of a graph H is defined as:

$$H(H,x) = \sum_{v\epsilon V(H)} \sum_{u\epsilon V(H)} x^{d(u,v)}. \quad (1)$$

The Harary polynomial was introduced in 1985 and is denoted by $h(H,x)$ and is defined as:

$$h(H,x) = \sum_{v\epsilon V(H)} \sum_{u\epsilon V(H)} \frac{x^{d(u,v)}}{d(u,v)}. \quad (2)$$

The generalized Harary index of graph H is denoted by$h_t(H)$ and defined as:

$$h_t(H) = \sum_{v\epsilon V(H)} \sum_{u\epsilon V(H)} \frac{1}{d(u,v)+t}. \quad (3)$$

Where t= 1,2, 3, ... The Wiener index is the oldest topological index introduced by Harold Wiener in 1947 for the study of boiling points of paraffin [38]. It plays a very important role in inverse structure-property relationship problems [39]. For applications and mathematical properties of Weiner index see [40–46]. The Wiener index is defined as:

$$W(H) = \frac{1}{2} \sum_{v\epsilon V(H)} \sum_{u\epsilon V(H)} d(u,v) \quad (4)$$

In 1993 another distance-based topological index was introduced by Randic known as the Hyper-Wiener index [35]. This index is used for predicting physicochemical properties of organic compounds [47], and is defined as:

$$WW(H) = \frac{1}{2} \sum_{v\epsilon V(H)} \sum_{u\epsilon V(H)} \left(d(u,v)^2 + d(u,v)\right) \quad (5)$$

It is easy to observe that there is an effective relation between Hosoya Polynomial, Wiener Index, and Hyper-Wiener Index.

$$W(H) = H'(H:1), WW(H) = H'(H;1) + \frac{1}{2}H''(H;1) \quad (6)$$

2. Result and Discussion

Let $\mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}$ be a commutative ring where $\wp$, $\mathfrak{I}_1$and $\mathfrak{I}_2$ are positive prime integers with$\mathfrak{I}_1 \neq \mathfrak{I}_2$. The total number of elements in this ring is$\wp^3\mathfrak{I}_1\mathfrak{I}_2$. We are only concerned with the set of zero-divisors of $\mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}$ denoted by $V(Y(\mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}))$ having cardinality$(\wp^3 - \wp^2)(\ \mathfrak{I}_1 + \mathfrak{I}_2 - 1) + \wp^2\mathfrak{I}_1\mathfrak{I}_2 - 1$. We made the degree-based disjoint partition for vertices of the zero-divisor graph $Y(\mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ of the commutative ring, which are given as:

$$W_1 = \{(0,\mathrm{v})\colon \mathrm{v} \in Z_{\mathfrak{I}_1\mathfrak{I}_2} \text{ with } v \neq 0, k_1\mathfrak{I}_1, k_2\mathfrak{I}_2, 1 \le k_1 \le \mathfrak{I}_2 - 1, 1 \le k_2 \le \mathfrak{I}_1 - 1\}, \quad (7)$$

$$W_2=\{(0,\mathrm{v}):\ \mathrm{v}=k_1\mathfrak{I}_1, 1\le k_1\le \mathfrak{I}_2-1\}, \quad (8)$$

$$W_3=\{(0,\mathrm{v}):\ \mathrm{v}=k_2\mathfrak{I}_2, 1\le k_2\le \mathfrak{I}_1-1\}, \quad (9)$$

$$W_4=\left\{(u,0):\ u\epsilon\mathbb{Z}_{g^3},\quad \text{with}\quad u\ne 0, t_1 g, 1\le t_1\le g^2-1\right\}, \quad (10)$$

$$W_5=\left\{(u,v):\ u\ne 0, t_1 g, 1\le t_1\le g^2-1, \mathrm{v} = k_1\mathfrak{I}_1, 1\le k_1\le \mathfrak{I}_2-1\right\}, \quad (11)$$

$$W_6=\left\{(u,v):\ u\ne 0, t_1 g, 1\le t_1\le g^2-1, \mathrm{v}=k_2\mathfrak{I}_2, 1\le k_2\le \mathfrak{I}_1-1\right\}, \quad (12)$$

$$W_7=\left\{(u,0):\ u=t_1 g, 1\le t_1\le g^2-1\quad \text{with}\, t_1\ne lg, 1\le t_1\le g-1\right\}, \quad (13)$$

$$W_8=\left\{(u,v):\ u=t_1 g, 1\le t_1\le g^2-1\quad \text{with}\quad t_1\ne lg, 1\le t_1\le g-1, \mathrm{v}\ne 0, k_1\mathfrak{I}_1, k_2\mathfrak{I}_2, 1\le k_1\le \mathfrak{I}_2-1, 1\le k_2\le \mathfrak{I}_1-1\right\}, \quad (14)$$

$$W_9=\left\{(u,v):\ u=t_1 g, 1\le t_1\le g^2-1\quad \text{with}\quad t_1\ne lg, 1\le t_1\le g-1, \mathrm{v}=k_1\mathfrak{I}_1, 1\le k_1\le \mathfrak{I}_2-1\right\}, \quad (15)$$

$$W_{10}=\left\{(u,v):\ u=t_1 g, 1\le t_1\le g^2-1\quad \text{with}\quad t_1\ne lg, 1\le t_1\le g-1, \mathrm{v}=k_1\mathfrak{I}_2, 1\le k_2\le \mathfrak{I}_1-1\right\}, \quad (16)$$

$$W_{11}=\left\{(u,0):\ u=t_2 g^2, 1\le t_2\le g-1\right\}, \quad (17)$$

$$W_{12}=\left\{(u,v):\ u=t_2 g^2, 1\le t_2\le g-1, \mathrm{v}\ne 0, k_1\mathfrak{I}_1, k_2\mathfrak{I}_2, 1\le k_1\le \mathfrak{I}_2-1, 1\le k_2\le \mathfrak{I}_1-1\right\}, \quad (18)$$

$$W_{13}=\left\{(u,v):\ u=t_2 g^2, 1\le t_2\le g-1, \mathrm{v}=k_1\mathfrak{I}_1, 1\le k_1\le \mathfrak{I}_2-1,\right\}, \quad (19)$$

$$W_{14}=\left\{(u,v):\ u=t_2 g^2, 1\le t_2\le g-1, \mathrm{v}k_2\mathfrak{I}_2, 1\le k_2\le \mathfrak{I}_1-1\right\}. \quad (20)$$

From the above, it is clear that $\mathrm{V}(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})) = \bigcup_{i=1}^{14} W_i$. Observe that $|W_1|=(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)$, $|W_2|=\mathfrak{I}_2-1$, $|W_3|=\mathfrak{I}_1-1$, $|W_4|=(g^3-g^2)$, $|W_5|=(g^3-g^2)(\mathfrak{I}_2-1)$, $|W_6|=(g^3-g^2)(\mathfrak{I}_1-1)$, $|W_7|=g^2-g$, $|W_8|=(g^2-g)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)$, $|W_9|=(g^2-g)(\mathfrak{I}_2-1)$, $|W_{10}|=(g^2-g)(\mathfrak{I}_1-1)$, $|W_{11}|=g-1$, $|W_{12}|=(g-1)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)$, $|W_{13}|=g\mathfrak{I}_2-g-\mathfrak{I}_2+1$, $|W_{14}|=g\mathfrak{I}_1-g-\mathfrak{I}_1+1$. Furthermore $d_v=g^3-1$ for $v\epsilon W_1$, $d_v=\mathfrak{I}_1 g^3-1$ for $v\epsilon W_2$, $d_v=\mathfrak{I}_1 g^3-1$ for $v\epsilon W_3$, $d_v=\mathfrak{I}_1\mathfrak{I}_2-1$ for $v\epsilon W_4$, $d_v=\mathfrak{I}_1-1$ for $v\epsilon W_5$, $d_v=\mathfrak{I}_2-1$ for $v\epsilon W_6$, $d_v=g\mathfrak{I}_1\mathfrak{I}_2-1$ for $v\epsilon W_7$, $d_v=g-1$ for $v\epsilon W_8$, $d_v=\mathfrak{I}_1 g-1$ for $v\epsilon W_9$, $d_v=\mathfrak{I}_1 g-1$ for $v\epsilon W_{10}$, $d_v=g^2\mathfrak{I}_1\mathfrak{I}_2-1$ for $v\epsilon W_{11}$, $d_v=g^2-1$ for $v\epsilon W_{12}$, $d_v=\mathfrak{I}_1 g^2-1$ for $v\epsilon W_{13}$, $d_v=\mathfrak{I}_2 g^2-1$ for $v\epsilon W_{14}$.

Let $1\le i\le 14$be a fixed integer and$u\in W_i$. From the above partition one can observe that the distance $d(u,v)$ is same for any $v\in W_j$, where $1\le j\le 14$. This observation is depicted in Figure 1. If there is an edge between any two partition sets W_i and W_j, then it means that $d(u,v)=1$ for any $u\in W_i$and $v\in W_j$. It can be seen that the maximum distance between any two vertices of Zero-divisor graph $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ is atmost three. This important fact is stated in Lemma 1.

Lemma 1. *The maximum diameter of the Zero-divisor graph $Y(S)$ of a commutative ring is 3 [3].*

We have summarized the eccentricity and degree of each vertex of Zero-divisor graph $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ in Table 1:

3. Hosoya and Harary Polynomial of Zero Divisor Graph $\mathrm{Y}(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$

Now we compute the Harary and Hosoya polynomial of zero divisor graph $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$. First, we find the distance $d(u,v)$ for each pair of vertices u,v of $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$. For this, we consider the partitionsW_i, $1\le i\le 14$ as vertices. Now we compute the distance $d(u,v)$ between any two vertices such that $u\in W_i$ and $v\in W_j$. Let $x\in\mathbb{Z}, x>0$ and $DP_x(W_i,W_j)=|\{(u,v)\in(W_i,W_j)|d(u,v)=x\}|$. If $i=j$, we simply use the notation $DP_x(W_i)$. The cardinality of the set of ordered pair of vertices which are adjacent to each other is denoted by $DP_1 Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$.

Lemma 2. *The cardinality of the set of ordered pair of vertices in $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ which are adjacent to each other is:*

$$DP_1 Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})=\frac{g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)(g^2+g+1)-14}{2} \quad (21)$$

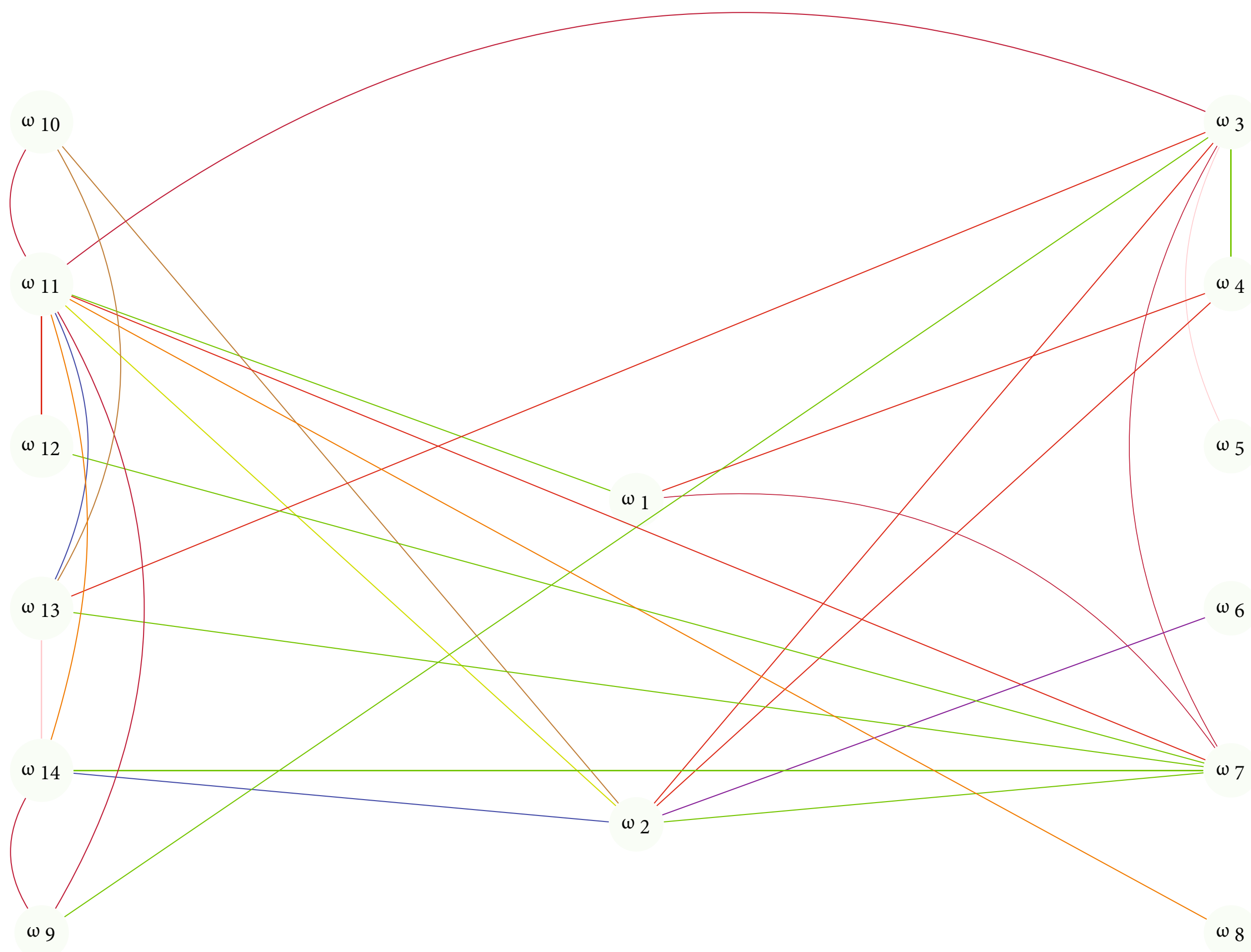


FIGURE 1: Zero-divisor graph of $\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}$.

TABLE 1: Eccentricity and degree of each vertex of Zero-divisor graph $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$.

Vertices	Eccentricity	Degree	Frequency
W_1	3	$g^3 - 1$	$\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1$
W_2	2	$\mathfrak{I}_1 g^3 - 1$	$\mathfrak{I}_2 - 1$
W_3	2	$\mathfrak{I}_2 g^3 - 1$	$\mathfrak{I}_1 - 1$
W_4	3	$\mathfrak{I}_1\mathfrak{I}_2 - 1$	$g^3 - g^2$
W_5	3	$\mathfrak{I}_1 - 1$	$(g^3 - g^2)(\mathfrak{I}_2 - 1)$
W_6	3	$\mathfrak{I}_2 - 1$	$(g^3 - g^2)(\mathfrak{I}_1 - 1)$
W_7	2	$g\mathfrak{I}_1\mathfrak{I}_2 - 1$	$g^2 - g$
W_8	3	$g - 1$	$(g^2 - g)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)$
W_9	3	$\mathfrak{I}_1 g - 1$	$(g^2 - g)(\mathfrak{I}_2 - 1)$
W_{10}	3	$\mathfrak{I}_2 g - 1$	$(g^2 - g)(\mathfrak{I}_1 - 1)$
W_{11}	2	$g^2\mathfrak{I}_1\mathfrak{I}_2 - 1$	$g - 1$
W_{12}	3	$g^2 - 1$	$(g - 1)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1)$
W_{13}	3	$\mathfrak{I}_1 g^2 - 1$	$g\mathfrak{I}_2 - g - \mathfrak{I}_2 + 1$
W_{14}	3	$\mathfrak{I}_2 g^2 - 1$	$g\mathfrak{I}_1 - g - \mathfrak{I}_1 + 1$

Proof. By Handshaking lemma and using table 1 we concluded

$$DP_1Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)=\frac{1}{2}\left[g^3-1+\mathfrak{I}_1g^3-1+\mathfrak{I}_2g^3-1+\mathfrak{I}_1\mathfrak{I}_2-1+\mathfrak{I}_1-1+\mathfrak{I}_2-1+g\mathfrak{I}_1\mathfrak{I}_1-1+g-1+\mathfrak{I}_1g-1+\mathfrak{I}_2g-1+g^2\mathfrak{I}_1\mathfrak{I}_2-1+g^2-1+\mathfrak{I}_1g^2-1+\mathfrak{I}_2g^2-1\right]=\frac{g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)\left(g^2+g+1\right)-14}{2} \quad (22)$$

□

Lemma 3. *The cardinality of the set of ordered pair of vertices in $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ which are distance 2 to each other is*

$$DP_2Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)=\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7. \quad (23)$$

Where

$$\Omega_1=\left[(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2)!+(\mathfrak{I}_2-2)!+(\mathfrak{I}_1-2)!+\left(g^3-g^2-1\right)!+\left(\left(g^3-g^2\right)(\mathfrak{I}_2-1)-1\right)!+\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)-1\right)!+\left(g^2-g-1\right)!+\left[\left(g^2-g\right)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1\right]!+\left[\left(g_1{}^2\mathfrak{I}_2-g_1{}^2-g\mathfrak{I}_2+g\right)-1\right]!+\left[\left(g^2-g\right)(\mathfrak{I}_1-1)-1\right]!+(g-2)!+[(g-1)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1]!+(g\mathfrak{I}_1-g-\mathfrak{I}_1)!+(g\mathfrak{I}_2-g-\mathfrak{I}_2)!\right], \quad (24)$$

$$\Omega_2=(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\left\{(\mathfrak{I}_1\mathfrak{I}_2-1)\left(g^2-1\right)+\mathfrak{I}_1+\mathfrak{I}_2\right\}, \quad (25)$$

$$\Omega_3=(\mathfrak{I}_2-1)^2\left\{\left(g^3-g^2\right)+\left(g^2-g\right)(\mathfrak{I}_1-1)+\left(g^2-g\right)+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\right\}+(\mathfrak{I}_1-1)^2\left\{\left(g^3-g^2\right)+\left(g^2-g\right)(\mathfrak{I}_1-1)+\left(g^2-g\right)+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\right\}, \quad (26)$$

$$\Omega_4=\left(g^3-g^2\right)\left\{(\mathfrak{I}_1+\mathfrak{I}_2-2)\left(g^3-g^2\right)+\mathfrak{I}_1\mathfrak{I}_2\left(g^2-g\right)+g(-1+\mathfrak{I}_1+\mathfrak{I}_2)+1-\mathfrak{I}_1-\mathfrak{I}_2\right\}, \quad (27)$$

$$\Omega_5=\left(g^3-g^2\right)(\mathfrak{I}_2-1)\left\{\left(g^2-g\right)+\left(g^2-g\right)(1+\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2)+\left(g^2-g\right)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_2-1)\right\}+\left(g^3-g^2\right)(\mathfrak{I}_1-1)\left\{\left(g^2-g\right)+\left(g^2-g\right)(\mathfrak{I}_1-1)+\left(g^2-g\right)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_1-1)\right\}, \quad (28)$$

$$\Omega_6=\left(g^2-g\right)^2(\mathfrak{I}_1\mathfrak{I}_2-1)+\left(g^2-g\right)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\cdot\left\{(\mathfrak{I}_1+\mathfrak{I}_2)\left(g^2-g\right)+(g-1)(1+\mathfrak{I}_1\mathfrak{I}_2)-2g^2\right\}+\left(g^2\mathfrak{I}_2-g^2-g\mathfrak{I}_2+g\right)\cdot\left\{g^2\mathfrak{I}_1-g^2-2g\mathfrak{I}_1+g\mathfrak{I}_1\mathfrak{I}_2+g-\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1\right\}, \quad (29)$$

$$\Omega_7=\left(g^2\mathfrak{I}_1-g^2-g\mathfrak{I}_1+g\right)\{g\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1\mathfrak{I}_2-g+1\}+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)(\mathfrak{I}_1+\mathfrak{I}_2-2\}. \quad (30)$$

Proof. From Figure 1. We conclude that W_1 is at distance 2 from $W_1, W_2, W_3, W_8, W_9, W_{10}, W_{12}, W_{13}$ and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_1)=(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2)! \quad (31)$$

$$DP_2(W_1,W_2)=(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)^2 \quad (32)$$

$$DP_2(W_1,W_3)=(\mathfrak{I}_1-1)^2(\mathfrak{I}_2-1) \quad (33)$$

$$DP_2(W_1,W_8)=\left(g^2-g\right)(\mathfrak{I}_1-1)^2(\mathfrak{I}_2-1)^2 \quad (34)$$

$$DP_2(W_1,W_9)=\left(g^2-g\right)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)^2 \quad (35)$$

$$DP_2(W_1,W_{10})=\left(g^2-g\right)(\mathfrak{I}_1-1)^2(\mathfrak{I}_1-1) \quad (36)$$

$$DP_2(W_1,W_{12})=(g-1)(\mathfrak{I}_1-1)^2(\mathfrak{I}_2-1)^2 \quad (37)$$

$$DP_2(W_1,W_{13})=(g-1)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)^2 \quad (38)$$

$$DP_2(W_1,W_{14})=(g-1)(\mathfrak{I}_1-1)^2(\mathfrak{I}_2-1) \quad (39)$$

From Figure 1, we conclude that W_2 is at distance 2 from $W_2, W_5, W_8, W_9, W_{12}$ and W_{13}. Hence by using the values from table 1, we have:

$$DP_2(W_2)=(\mathfrak{I}_2-2)! \quad (40)$$

$$DP_2(W_2,W_5)=\left(g^3-g^2\right)(\mathfrak{I}_2-1)^2 \quad (41)$$

$$DP_2(W_2,W_8)=\left(g^2-g\right)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)^2 \quad (42)$$

$$DP_2(W_2,W_9)=\left(g^2-g\right)(\mathfrak{I}_2-1)^2 \quad (43)$$

$$DP_2(W_2,W_{12})=(g-1)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)^2 \quad (44)$$

$$DP_2(W_2, W_{13}) = (\mathscr{g} - 1)(\mathfrak{I}_2 - 1)^2 \tag{45}$$

From Figure 1, we conclude that W_3 is at distance 2 from $W_3, W_6, W_8, W_{10}, W_{12}$ and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_3) = (\mathfrak{I}_1 - 2)! \tag{46}$$

$$DP_2(W_3, W_6) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)(\mathfrak{I}_1 - 1)^2 \tag{47}$$

$$DP_2(W_3, W_8) = \left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_1 - 1)^2(\mathfrak{I}_2 - 1) \tag{48}$$

$$DP_2(W_3, W_{10}) = \left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_1 - 1)^2 \tag{49}$$

$$DP_2(W_3, W_{12}) = (\mathscr{g} - 1)(\mathfrak{I}_1 - 1)^2(\mathfrak{I}_2 - 1) \tag{50}$$

$$DP_2(W_3, W_{14}) = (\mathscr{g} - 1)(\mathfrak{I}_1 - 1)^2 \tag{51}$$

From Figure 1. We conclude that W_4 is at distance 2 from $W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{13}$ and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_4) = \left(\mathscr{g}^3 - \mathscr{g}^2 - 1\right)! \tag{52}$$

$$DP_2(W_4, W_5) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)^2(\mathfrak{I}_2 - 1) \tag{53}$$

$$DP_2(W_4, W_6) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)^2(\mathfrak{I}_1 - 1) \tag{54}$$

$$DP_2(W_4, W_7) = \left(\mathscr{g}^2 - \mathscr{g}\right)\left(\mathscr{g}^3 - \mathscr{g}^2\right) \tag{55}$$

$$DP_2(W_4, W_8) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)\left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1) \tag{56}$$

$$DP_2(W_4, W_9) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)\left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_2 - 1) \tag{57}$$

$$DP_2(W_4, W_{10}) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)\left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_1 - 1) \tag{58}$$

$$DP_2(W_4, W_{11}) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)(\mathscr{g} - 1) \tag{59}$$

$$DP_2(W_4, W_{13}) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)(\mathscr{g} - 1)(\mathfrak{I}_2 - 1) \tag{60}$$

$$DP_2(W_4, W_{14}) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)(\mathscr{g} - 1)(\mathfrak{I}_1 - 1) \tag{61}$$

From Figure 1, we conclude that $W_5$is at distance 2 from $W_5, W_7, W_8, W_9, W_{11}$ and W_{13}. Hence by using the values from table 1, we have:

$$DP_2(W_5) = \left(\left(\mathscr{g}^3 - \mathscr{g}^2\right)(\mathfrak{I}_2 - 1) - 1\right)! \tag{62}$$

$$DP_2(W_5, W_7) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)\left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_2 - 1) \tag{63}$$

$$DP_2(W_5, W_8) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)\left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)^2 \tag{64}$$

$$DP_2(W_5, W_9) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)\left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_2 - 1)^2 \tag{65}$$

$$DP_2(W_5, W_{11}) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)(\mathscr{g} - 1)(\mathfrak{I}_2 - 1) \tag{66}$$

$$DP_2(W_5, W_{13}) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)(\mathscr{g} - 1)(\mathfrak{I}_2 - 1)^2 \tag{67}$$

From Figure 1, we conclude that W_6 is at distance 2 from W_6, W_7, W_{10}, W_{11} and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_5) = \left(\left(\mathscr{g}^3 - \mathscr{g}^2\right)(\mathfrak{I}_1 - 1) - 1\right)! \tag{68}$$

$$DP_2(W_5, W_7) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)\left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_1 - 1) \tag{69}$$

$$DP_2(W_6, W_{10}) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)\left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_1 - 1)^2 \tag{70}$$

$$DP_2(W_6, W_{11}) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)(\mathscr{g} - 1)(\mathfrak{I}_1 - 1) \tag{71}$$

$$DP_2(W_6, W_{14}) = \left(\mathscr{g}^3 - \mathscr{g}^2\right)(\mathscr{g} - 1)(\mathfrak{I}_1 - 1)^2 \tag{72}$$

From Figure 1, we conclude that W_7 is at distance 2 from W_7, W_8, W_9 and W_{10}. Hence by using the values from table 1, we have:

$$DP_2(W_7) = \left(\mathscr{g}^2 - \mathscr{g} - 1\right)! \tag{73}$$

$$DP_2(W_7, W_8) = \left(\mathscr{g}^2 - \mathscr{g}\right)^2(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1) \tag{74}$$

$$DP_2(W_7, W_9) = \left(\mathscr{g}^2 - \mathscr{g}\right)^2(\mathfrak{I}_2 - 1) \tag{75}$$

$$DP_2(W_7, W_{10}) = \left(\mathscr{g}^2 - \mathscr{g}\right)^2(\mathfrak{I}_1 - 1) \tag{76}$$

From Figure 1, we conclude that W_8 is at distance 2 from $W_8, W_9, W_{10}, W_{12}, W_{13}$ and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_8) = \left[\left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1) - 1\right]! \tag{77}$$

$$DP_2(W_8, W_9) = \left(\mathscr{g}^2 - \mathscr{g}\right)^2(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)^2 \tag{78}$$

$$DP_2(W_8, W_{10}) = \left(\mathscr{g}^2 - \mathscr{g}\right)^2(\mathfrak{I}_1 - 1)^2(\mathfrak{I}_2 - 1) \tag{79}$$

$$DP_2(W_8, W_{12}) = \left(\mathscr{g}^2 - \mathscr{g}\right)^2(\mathscr{g} - 1)(\mathfrak{I}_1 - 1)^2(\mathfrak{I}_2 - 1)^2 \tag{80}$$

$$DP_2(W_8, W_{13}) = \left(\mathscr{g}^2 - \mathscr{g}\right)^2(\mathscr{g} - 1)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)^2 \tag{81}$$

$$DP_2(W_8, W_{14}) = \left(\mathscr{g}^2 - \mathscr{g}\right)^2(\mathscr{g} - 1)(\mathfrak{I}_1 - 1)^2(\mathfrak{I}_2 - 1) \tag{82}$$

From Figure 1, we conclude that W_9 is at distance 2 from W_9, W_{10}, W_{12}, and W_{13}. Hence by using the values from table 1, we have:

$$DP_2(W_9) = \left[\left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_2 - 1) - 1\right]! \tag{83}$$

$$DP_2(W_9, W_{10}) = \left(\mathscr{g}^2 - \mathscr{g}\right)^2(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1) \tag{84}$$

$$DP_2(W_9, W_{12}) = \left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_2 - 1)^2(\mathscr{g} - 1)(\mathfrak{I}_1 - 1) \tag{85}$$

$$DP_2(W_9, W_{13}) = \left(\mathscr{g}^2 - \mathscr{g}\right)(\mathfrak{I}_2 - 1)^2(\mathscr{g} - 1) \tag{86}$$

From Figure 1, we conclude that W_{10} is at distance 2 from W_{10}, W_{12}, W_{13}, and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_{10}) = \left[\left(g^2 - g\right)(\mathfrak{I}_1 - 1) - 1\right]! \tag{87}$$

$$DP_2(W_{10}, W_{12}) = \left(g^2 - g\right)(\mathfrak{I}_1 - 1)^2(\mathfrak{I}_2 - 1)(g - 1) \tag{88}$$

$$DP_2(W_{10}, W_{13}) = \left(g^2 - g\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)(g - 1) \tag{89}$$

$$DP_2(W_{10}, W_{14}) = \left(g^2 - g\right)(\mathfrak{I}_1 - 1)^2(g - 1) \tag{90}$$

From Figure 1, we conclude that W_{11} is at distance 2 from W_{11}. Hence by using the values from table 1, we have:

$$DP_2(W_{11}) = (g - 2)! \tag{91}$$

From Figure 1, we conclude that W_{12} is at distance 2 from W_{12}, W_{13}, and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_{12}) = [(g - 1)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1) - 1]! \tag{92}$$

$$DP_2(W_{12}, W_{13}) = (g - 1)^2(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)^2 \tag{93}$$

$$DP_2(W_{12}, W_{14}) = (g - 1)(\mathfrak{I}_1 - 1)^2(\mathfrak{I}_2 - 1) \tag{94}$$

From Figure 1, we conclude that W_{13} is at distance 2 from W_{13}. Hence by using the values from table 1, we have:

$$DP_2(W_{13}) = [(g - 1)(\mathfrak{I}_2 - 1) - 1]! \tag{95}$$

From Figure 1, we conclude that W_{14} is at distance 2 from W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_{14}) = [(g - 1)(\mathfrak{I}_2 - 1) - 1]! \tag{96}$$

Finally, we add all the values computed in equations (31)–(96), and we get the required formula for $DP_2Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$:

$$\begin{aligned}
DP_2Y\left(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right) &= [(\mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1 + \mathfrak{I}_2)! + (\mathfrak{I}_2 - 2)! \\
&+ (\mathfrak{I}_1 - 2)! + \left(g^3 - g^2 - 1\right)! + \left(\left(g^3 - g^2\right)(\mathfrak{I}_2 - 1) - 1\right)! \\
&+ \left(\left(g^3 - g^2\right)(\mathfrak{I}_1 - 1) - 1\right)! + \left(g^2 - g - 1\right)! \\
&+ \left[\left(g^2 - g\right)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1) - 1\right]! \\
&+ \left[\left({g_1}^2\mathfrak{I}_2 - {g_1}^2 - g\mathfrak{I}_2 + g\right) - 1\right]! + \left[\left(g^2 - g\right)(\mathfrak{I}_1 - 1) - 1\right]! \\
&+ (g - 2)! + [(g - 1)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1) - 1]! \\
&+ (g\mathfrak{I}_1 - g - \mathfrak{I}_1)! + (g\mathfrak{I}_2 - g - \mathfrak{I}_2)!] \\
&+ (\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1)\left\{(\mathfrak{I}_1\mathfrak{I}_2 - 1)\left(g^2 - 1\right) + \mathfrak{I}_1 + \mathfrak{I}_2\right\} \\
&+ (\mathfrak{I}_2 - 1)^2\left\{\left(g^3 - g^2\right) + \left(g^2 - g\right)(\mathfrak{I}_1 - 1) + \left(g^2 - g\right)\right. \\
&+ (g\mathfrak{I}_1 - g - \mathfrak{I}_1 + 1) + (g - 1)\big\} + (\mathfrak{I}_1 - 1)^2\left\{\left(g^3 - g^2\right)\right. \\
&+ \left(g^2 - g\right)(\mathfrak{I}_1 - 1) + \left(g^2 - g\right) + (g - 1)(\mathfrak{I}_1 - 1) + (g - 1)\big\} \\
&+ \left(g^3 - g^2\right)\left\{(\mathfrak{I}_1 + \mathfrak{I}_2 - 2)\left(g^3 - g^2\right) + \mathfrak{I}_1\mathfrak{I}_2\left(g^2 - g\right)\right. \\
&+ g(\mathfrak{I}_1 + \mathfrak{I}_2 - 1) - \mathfrak{I}_1 - \mathfrak{I}_2 + 1\big\} + \left(g^3 - g^2\right)(\mathfrak{I}_2 - 1)\left\{\left(g^2 - g\right)\right. \\
&+ \left(g^2 - g\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1) + \left({g_1}^2\mathfrak{I}_2 - {g_1}^2 - g\mathfrak{I}_2 + g\right) + (g - 1) \\
&+ (g - 1)(\mathfrak{I}_2 - 1)\big\} + \left(g^3 - g^2\right)(\mathfrak{I}_1 - 1)\left\{\left(g^2 - g\right)\right. \\
&+ \left({g_1}^2\mathfrak{I}_1 - {g_1}^2 - g\mathfrak{I}_1 + g\right) + \left({g_1}^2\mathfrak{I}_2 - {g_1}^2 - g\mathfrak{I}_2 + g\right) \\
&+ (g - 1) + (g - 1)(\mathfrak{I}_1 - 1)\big\} + \left(g^2 - g\right)^2(\mathfrak{I}_1\mathfrak{I}_2 - 1) \\
&+ \left(g^2 - g\right)(1 + \mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2)\left\{(\mathfrak{I}_1 + \mathfrak{I}_2)\left(g^2 - g\right)\right. \\
&+ (g - 1)(1 + \mathfrak{I}_1\mathfrak{I}_2) - 2g^2\big\} + \left(g^2\mathfrak{I}_2 - g^2 - g\mathfrak{I}_2 + g\right) \\
&\cdot \left\{g^2\mathfrak{I}_1 - g^2 - 2g\mathfrak{I}_1 + g\mathfrak{I}_1\mathfrak{I}_2 + g - \mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1\right\} \\
&+ \left(g^2\mathfrak{I}_1 - g^2 - g\mathfrak{I}_1 + g\right)\{1 - g + g\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1\mathfrak{I}_2\} \\
&+ (g - 1)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)(\mathfrak{I}_1 + \mathfrak{I}_2 - 2\} = \Omega_1 + \Omega_2 \\
&+ \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7
\end{aligned} \tag{97}$$

□

Lemma 4. *The cardinality of the set of ordered pair of vertices in $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ which are distance 3 to each other is*

$$\begin{aligned}
&DP_3\left(Y\left(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right) \\
&\quad = \left(g^3 - g^2\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)\left[g^3 + \mathfrak{I}_1 g + \mathfrak{I}_2 g^2 - 3\right]
\end{aligned} \tag{98}$$

Proof. By using the figure, we conclude that:

$$\begin{aligned}
DP_3\left(Y\left(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right) = |W_1|(|W_5| + |W_6|) + |W_4||W_{12}| \\
+ |W_5|(|W_6| + |W_{10}| + |W_{12}| + |W_{14}|) \\
+ |W_6|(|W_8| + |W_9| + |W_{12}| + |W_{13}|)
\end{aligned} \tag{99}$$

Now by putting values from table 1, we obtain:

$$\begin{aligned}
&DP_3\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)\\
&\quad=(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\left(\left(g^3-g^2\right)(\mathfrak{I}_2-1)\right.\\
&\qquad+\left.\left(g^3-g^2\right)(\mathfrak{I}_2-1)\right)+\left(g^3-g^2\right)(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)\\
&\qquad+\left(g^3-g^2\right)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\left(\left(g^3-g^2\right)\right.\\
&\qquad+\left(g^2-g\right)(\mathfrak{I}_1-1)+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)\\
&\qquad+\left.(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)\right)+\left(g^3-g^2\right)(\mathfrak{I}_1-1)\left(\left(g^2-g\right)\right.\\
&\qquad\times(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)+\left(g^2-g\right)(\mathfrak{I}_2-1)\\
&\qquad+\left.(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)+(g\mathfrak{I}_2-g-\mathfrak{I}_2+1)\right)\\
&\quad=\left(g^3-g^2\right)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)\left[g^3+\mathfrak{I}_1g+\mathfrak{I}_2g^2-3\right]
\end{aligned}\tag{100}$$

□

Theorem 5. *The Hosoya polynomial $H(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}))$ of the graph $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ is:*

$$\begin{aligned}
&H\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)=\left(g^3-g^2\right)(\mathfrak{I}_1+\mathfrak{I}_2-1)+g^2\mathfrak{I}_1\mathfrak{I}_2-1\\
&\quad+\left(\frac{g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)\left(g^2+g+1\right)-14}{2}\right)\\
&\quad x+(\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7)x^2\\
&\quad+\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)\left[g^3+\mathfrak{I}_1g+\mathfrak{I}_2g^2-3\right]\right)x^3
\end{aligned}\tag{101}$$

Where

$$\begin{aligned}
\Omega_1=&\left[(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)!+(\mathfrak{I}_2-2)!+(\mathfrak{I}_1-2)!+\left(g^3-g^2-1\right)!\right.\\
&+\left(\left(g^3-g^2\right)(\mathfrak{I}_2-1)-1\right)!+\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)-1\right)!\\
&+\left(g^2-g-1\right)!+\left[\left(g^2-g\right)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1\right]!\\
&+\left[\left(g_1{}^2\mathfrak{I}_2-g_1{}^2-g\mathfrak{I}_2+g\right)-1\right]!+\left[\left(g^2-g\right)(\mathfrak{I}_1-1)-1\right]!\\
&+(g-2)!+[(g-1)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1]!\\
&+\left.(g\mathfrak{I}_1-g-\mathfrak{I}_1)!+(g\mathfrak{I}_2-g-\mathfrak{I}_2)!\right],
\end{aligned}\tag{102}$$

$$\Omega_2=(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\left\{(\mathfrak{I}_1\mathfrak{I}_2-1)\left(g^2-1\right)+\mathfrak{I}_1+\mathfrak{I}_2\right\},\tag{103}$$

$$\begin{aligned}
\Omega_3=&(\mathfrak{I}_2-1)^2\left\{\left(g^3-g^2\right)+\left(g^2-g\right)(\mathfrak{I}_1-1)+\left(g^2-g\right)\right.\\
&+\left.(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\right\}+(\mathfrak{I}_1-1)^2\left\{\left(g^3-g^2\right)\right.\\
&+\left.\left(g^2-g\right)(\mathfrak{I}_1-1)+\left(g^2-g\right)+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\right\},
\end{aligned}\tag{104}$$

$$\begin{aligned}
\Omega_4=&\left(g^3-g^2\right)\left\{(\mathfrak{I}_1+\mathfrak{I}_2-2)\left(g^3-g^2\right)+\mathfrak{I}_1\mathfrak{I}_2\left(g^2-g\right)\right.\\
&+\left.g(-1+\mathfrak{I}_1+\mathfrak{I}_2)+1-\mathfrak{I}_1-\mathfrak{I}_2\right\},
\end{aligned}\tag{105}$$

$$\begin{aligned}
\Omega_5=&\left(g^3-g^2\right)(\mathfrak{I}_2-1)\left\{\left(g^2-g\right)+\left(g^2-g\right)(1+\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2)\right.\\
&+\left.\left(g^2-g\right)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_2-1)\right\}\\
&+\left(g^3-g^2\right)(\mathfrak{I}_1-1)\left\{\left(g^2-g\right)+\left(g^2-g\right)(\mathfrak{I}_1-1)\right.\\
&+\left.\left(g^2-g\right)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_1-1)\right\},
\end{aligned}\tag{106}$$

$$\begin{aligned}
\Omega_6=&\left(g^2-g\right)^2(\mathfrak{I}_1\mathfrak{I}_2-1)+\left(g^2-g\right)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\\
&\cdot\left\{(\mathfrak{I}_1+\mathfrak{I}_2)\left(g^2-g\right)+(g-1)(1+\mathfrak{I}_1\mathfrak{I}_2)-2g^2\right\}\\
&+\left(g^2\mathfrak{I}_2-g^2-g\mathfrak{I}_2+g\right)\left\{g^2\mathfrak{I}_1-g^2-2g\mathfrak{I}_1\right.\\
&+\left.g\mathfrak{I}_1\mathfrak{I}_2+g-\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1\right\},
\end{aligned}\tag{107}$$

$$\begin{aligned}
\Omega_7=&\left(g^2\mathfrak{I}_1-g^2-g\mathfrak{I}_1+g\right)\left\{g\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1\mathfrak{I}_2-g+1\right\}\\
&+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)(\mathfrak{I}_1+\mathfrak{I}_2-2\}.
\end{aligned}\tag{108}$$

Proof. The Hosoya polynomial of zero divisor graph $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ can be calculated by using lemma 2, lemma 3 and lemma 4 as follows:

$$\begin{aligned}
&H\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right),x\right)=DP_0\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)\\
&\quad+DP_1\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)x+DP_2\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)x^2\\
&\quad+DP_3\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)x^3=\left(g^3-g^2\right)(\mathfrak{I}_1+\mathfrak{I}_2-1)+g^2\mathfrak{I}_1\mathfrak{I}_2-1\\
&\quad+\left(\frac{g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)\left(g^2+g+1\right)-14}{2}\right)\\
&\quad x+(\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7)x^2\\
&\quad+\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)\left[g^3+\mathfrak{I}_1g+\mathfrak{I}_2g^2-3\right]\right)x^3
\end{aligned}\tag{109}$$

□

Theorem 6. *The Harary polynomial $h(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}),x)$ of the graph $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ is*

$$\begin{aligned}
&h\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right),x\right)\\
&\quad=3\left(g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)\right.\\
&\qquad\times\left.\left(g^2+g+1\right)-14\right)x+3(\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5\\
&\qquad+\Omega_6+\Omega_7)x^2+2\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)\right.\\
&\qquad\times\left.(\mathfrak{I}_2-1)\left[g^3+\mathfrak{I}_1g+\mathfrak{I}_2g^2-3\right]\right)x^3
\end{aligned}\tag{110}$$

Where

$$\begin{aligned}
\Omega_1=&[(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)!+(\mathfrak{I}_2-2)!+(\mathfrak{I}_1-2)!\\
&+\left(g^3-g^2-1\right)!+\left(\left(g^3-g^2\right)(\mathfrak{I}_2-1)-1\right)!\\
&+\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)-1\right)!+\left(g^2-g-1\right)!\\
&+\left[\left(g^2-g\right)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1\right]!\\
&+\left[\left(g_1{}^2\mathfrak{I}_2-g_1{}^2-g\mathfrak{I}_2+g\right)-1\right]!+\left[\left(g^2-g\right)(\mathfrak{I}_1-1)-1\right]!\\
&+(g-2)!+[(g-1)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1]!\\
&+(g\mathfrak{I}_1-g-\mathfrak{I}_1)!+(g\mathfrak{I}_2-g-\mathfrak{I}_2)!],
\end{aligned}\tag{111}$$

$$\Omega_2=(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\left\{(\mathfrak{I}_1\mathfrak{I}_2-1)\left(g^2-1\right)+\mathfrak{I}_1+\mathfrak{I}_2\right\},\tag{112}$$

$$\Omega_3 = (\mathfrak{I}_2 - 1)^2\left\{\left(g^3 - g^2\right) + \left(g^2 - g\right)(\mathfrak{I}_1 - 1) + \left(g^2 - g\right) + (g\mathfrak{I}_1 - g - \mathfrak{I}_1 + 1) + (g - 1)\right\} + (\mathfrak{I}_1 - 1)^2\left\{\left(g^3 - g^2\right) + \left(g^2 - g\right)(\mathfrak{I}_1 - 1) + \left(g^2 - g\right) + (g\mathfrak{I}_1 - g - \mathfrak{I}_1 + 1) + (g - 1)\right\}, \quad (113)$$

$$\Omega_4 = \left(g^3 - g^2\right)\left\{(\mathfrak{I}_1 + \mathfrak{I}_2 - 2)\left(g^3 - g^2\right) + \mathfrak{I}_1\mathfrak{I}_2\left(g^2 - g\right) + g(-1 + \mathfrak{I}_1 + \mathfrak{I}_2) + 1 - \mathfrak{I}_1 - \mathfrak{I}_2\right\}, \quad (114)$$

$$\Omega_5 = \left(g^3 - g^2\right)(\mathfrak{I}_2 - 1)\left\{\left(g^2 - g\right) + \left(g^2 - g\right)(1 + \mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2) + \left(g^2 - g\right)(\mathfrak{I}_2 - 1) + (g - 1) + (g - 1)(\mathfrak{I}_2 - 1)\right\} + \left(g^3 - g^2\right)(\mathfrak{I}_1 - 1)\left\{\left(g^2 - g\right) + \left(g^2 - g\right)(\mathfrak{I}_1 - 1) + \left(g^2 - g\right)(\mathfrak{I}_2 - 1) + (g - 1) + (g - 1)(\mathfrak{I}_1 - 1)\right\}, \quad (115)$$

$$\Omega_6 = \left(g^2 - g\right)^2(\mathfrak{I}_1\mathfrak{I}_2 - 1) + \left(g^2 - g\right)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1) \cdot \left\{(\mathfrak{I}_1 + \mathfrak{I}_2)\left(g^2 - g\right) + (g - 1)(1 + \mathfrak{I}_1\mathfrak{I}_2) - 2g^2\right\} + \left(g^2\mathfrak{I}_2 - g^2 - g\mathfrak{I}_2 + g\right)\left\{g^2\mathfrak{I}_1 - g^2 - 2g\mathfrak{I}_1 + g\mathfrak{I}_1\mathfrak{I}_2 + g - \mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1\right\}, \quad (116)$$

$$\Omega_7 = \left(g^2\mathfrak{I}_1 - g^2 - g\mathfrak{I}_1 + g\right)\{g\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1\mathfrak{I}_2 - g + 1\} + (g\mathfrak{I}_1 - g - \mathfrak{I}_1 + 1)(\mathfrak{I}_2 - 1)(\mathfrak{I}_1 + \mathfrak{I}_2 - 2\}. \quad (117)$$

Proof. The Harary polynomial of $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ can be calculated by using lemma 2, lemma 3 and lemma 4 as follows:

$$h\left(Y\left(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right), x\right) = 6DP_1\left(Y\left(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)x + 3DP_2\left(Y\left(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)x^2 + 2DP_3\left(Y\left(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)x^3 = 3\left(g^3 + g^2 + g + g^3(\mathfrak{I}_1 + \mathfrak{I}_2) + (\mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1 + \mathfrak{I}_2) \cdot \left(g^2 + g + 1\right) - 14\right)x + 3(\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7)x^2 + 2\left(\left(g^3 - g^2\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)\left[g^3 + \mathfrak{I}_1 g + \mathfrak{I}_2 g^2 - 3\right]\right)x^3 \quad (118)$$

□

Corollary 7. *The Wiener index and Hyper-Wiener index for zero divisor graph $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ can be expressed as:*

$$W\left(Y\left(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right) = \frac{g^3 + g^2 + g + g^3(\mathfrak{I}_1 + \mathfrak{I}_2) + (\mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1 + \mathfrak{I}_2)\left(g^2 + g + 1\right) - 14}{2} + (\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7) + 3\left(\left(g^3 - g^2\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)\left[g^3 + \mathfrak{I}_1 g + \mathfrak{I}_2 g^2 - 3\right]\right) \quad (119)$$

$$WW\left(Y\left(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right) = \left(g^3 + g^2 + g + g^3(\mathfrak{I}_1 + \mathfrak{I}_2) + (\mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1 + \mathfrak{I}_2)\left(g^2 + g + 1\right) - 14\right) + 3(\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7) + 6\left(\left(g^3 - g^2\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)\left[g^3 + \mathfrak{I}_1 g + \mathfrak{I}_2 g^2 - 3\right]\right) \quad (120)$$

Where

$$\Omega_1 = [(\mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1 + \mathfrak{I}_2)! + (\mathfrak{I}_2 - 2)! + (\mathfrak{I}_1 - 2)! + \left(g^3 - g^2 - 1\right)! + \left(\left(g^3 - g^2\right)(\mathfrak{I}_2 - 1) - 1\right)! + \left(\left(g^3 - g^2\right)(\mathfrak{I}_1 - 1) - 1\right)! + \left(g^2 - g - 1\right)! + \left[\left(g^2 - g\right)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1) - 1\right]! + \left[\left(g_1^{\,2}\mathfrak{I}_2 - g_1^{\,2} - g\mathfrak{I}_2 + g\right) - 1\right]! + \left[\left(g^2 - g\right)(\mathfrak{I}_1 - 1) - 1\right]! + (g - 2)! + [(g - 1)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1) - 1]! + (g\mathfrak{I}_1 - g - \mathfrak{I}_1)! + (g\mathfrak{I}_2 - g - \mathfrak{I}_2)!], \quad (121)$$

$$\Omega_2 = (\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1)\left\{(\mathfrak{I}_1\mathfrak{I}_2 - 1)\left(g^2 - 1\right) + \mathfrak{I}_1 + \mathfrak{I}_2\right\}, \quad (122)$$

$$\Omega_3 = (\mathfrak{I}_2 - 1)^2\left\{\left(g^3 - g^2\right) + \left(g^2 - g\right)(\mathfrak{I}_1 - 1) + \left(g^2 - g\right) + (g\mathfrak{I}_1 - g - \mathfrak{I}_1 + 1) + (g - 1)\right\} + (\mathfrak{I}_1 - 1)^2\left\{\left(g^3 - g^2\right) + \left(g^2 - g\right)(\mathfrak{I}_1 - 1) + \left(g^2 - g\right) + (g\mathfrak{I}_1 - g - \mathfrak{I}_1 + 1) + (g - 1)\right\}, \quad (123)$$

$$\Omega_4 = \left(g^3 - g^2\right)\left\{(\mathfrak{I}_1 + \mathfrak{I}_2 - 2)\left(g^3 - g^2\right) + \mathfrak{I}_1\mathfrak{I}_2\left(g^2 - g\right) + g(-1 + \mathfrak{I}_1 + \mathfrak{I}_2) + 1 - \mathfrak{I}_1 - \mathfrak{I}_2\right\}, \quad (124)$$

$$\Omega_5 = \left(g^3 - g^2\right)(\mathfrak{I}_2 - 1)\left\{\left(g^2 - g\right) + \left(g^2 - g\right)(1 + \mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2) + \left(g^2 - g\right)(\mathfrak{I}_2 - 1) + (g - 1) + (g - 1)(\mathfrak{I}_2 - 1)\right\} + \left(g^3 - g^2\right)(\mathfrak{I}_1 - 1)\left\{\left(g^2 - g\right) + \left(g^2 - g\right)(\mathfrak{I}_1 - 1) + \left(g^2 - g\right)(\mathfrak{I}_2 - 1) + (g - 1) + (g - 1)(\mathfrak{I}_1 - 1)\right\}, \quad (125)$$

$$\Omega_6 = \left(g^2 - g\right)^2(\mathfrak{I}_1\mathfrak{I}_2 - 1) + \left(g^2 - g\right)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1) \cdot \left\{(\mathfrak{I}_1 + \mathfrak{I}_2)\left(g^2 - g\right) + (g - 1)(1 + \mathfrak{I}_1\mathfrak{I}_2) - 2g^2\right\} + \left(g^2\mathfrak{I}_2 - g^2 - g\mathfrak{I}_2 + g\right)\left\{g^2\mathfrak{I}_1 - g^2 - 2g\mathfrak{I}_1 + g\mathfrak{I}_1\mathfrak{I}_2 + g - \mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1\right\}, \quad (126)$$

$$\Omega_7 = \left(g^2\mathfrak{I}_1 - g^2 - g\mathfrak{I}_1 + g\right)\{g\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1\mathfrak{I}_2 - g + 1\} + (g\mathfrak{I}_1 - g - \mathfrak{I}_1 + 1)(\mathfrak{I}_2 - 1)(\mathfrak{I}_1 + \mathfrak{I}_2 - 2\}. \quad (127)$$

4. Conclusion

In this paper, we computed some distance-based polynomial of zero divisor graph $Y(\mathbb{Z}_{g_1{}^2 g_2} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ namely Hosoya polynomial, Harary polynomial, Wiener index, and hyper Wiener index. We have seen that the eccentricity of any vertex of a zero divisor graph is always 2 or 3, which helps for the computation of the above polynomial. These computations can be helpful in computation of eccentricity based topological indices of the considered zero divisor graph.

Data Availability

No data is required to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Funding

This work is carried out as a part of employment of the authors. The authors do not receive any funding.

References

[1] I. Beck, "Coloring of commutative rings," *Journal of Algebra*, vol. 116, no. 1, pp. 208–226, 1988.

[2] D. D. Anderson and M. Naseer, "Beck′ s coloring of a commutative ring," *Journal of Algebra*, vol. 159, no. 2, pp. 500–514, 1993.

[3] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring," *Journal of Algebra*, vol. 217, no. 2, pp. 434–447, 1999.

[4] S. Hayat and M. Imran, "Computation of topological indices of certain networks," *Applied Mathematics and Computation*, vol. 240, pp. 213–228, 2014.

[5] D. F. Anderson and A. Badawi, "On the zero-divisor graph of a ring," *Communications in Algebra*, vol. 36, no. 8, pp. 3073–3092, 2008.

[6] S. P. Redmond, "The zero-divisor graph of a non-commutative ring," *International Journal of Commutative Rings*, vol. 1, no. 4, pp. 203–211, 2002.

[7] S. P. Redmond, "An ideal-based zero-divisor graph of a commutative ring," *Communications in Algebra*, vol. 31, no. 9, pp. 4425–4443, 2003.

[8] N. Ashrafi, H. R. Maimani, M. R. Pournaki, and S. Yassemi, "Unit graphs associated with rings," *Communications in Algebra®*, vol. 38, no. 8, pp. 2851–2871, 2010.

[9] S. Spiroff and C. Wickham, "A zero divisor graph determined by equivalence classes of zero divisors," *Communications in Algebra*, vol. 39, no. 7, pp. 2338–2348, 2011.

[10] D. F. Anderson and A. Badawi, "The total graph of a commutative ring," *Journal of Algebra*, vol. 320, no. 7, pp. 2706–2719, 2008.

[11] A. Azimi, A. Erfanian, and D. G. M. Farrokhi, "The Jacobson graph of commutative rings," *Journal of Algebra and its Applications*, vol. 12, no. 3, article 1250179, 2013.

[12] F. Harary, *Graph Theory*, Addison-Wesley publishing company, 1969.

[13] K. Samei, "The zero-divisor graph of a reduced ring," *Journal of Pure and Applied Algebra*, vol. 209, no. 3, pp. 813–821, 2007.

[14] A. Ahmad, "On the degree-based topological indices of benzene ring embedded in P-type-surface in the 2D network," *Hacettepe Journal of Mathematics and Statistics*, vol. 47, no. 1, pp. 9–18, 2018.

[15] A. Ahmad, "Computation of certain topological properties of the Para-line graph of honeycomb networks and graphene," *Discrete Mathematics, Algorithms, and Applications*, vol. 9, no. 5, article 1750064, 2017.

[16] S. Akhter, M. Imran, G. A. O. Wei, and M. R. Farahani, "On topological indices of honeycomb networks and graphene networks," *Hacettepe Journal of Mathematics and Statistics*, vol. 47, no. 1, pp. 19–35, 2018.

[17] M. F. Nadeem, S. Zafar, and Z. Zahid, "On topological properties of the line graphs of subdivision graphs of certain nanostructures," *Applied Mathematics and Computation*, vol. 273, pp. 125–130, 2016.

[18] X. Zhang, X. Wu, S. Akhter, M. K. Jamil, J.-B. Liu, and M. R. Farahani, "Edge-Version Atom-Bond Connectivity and Geometric Arithmetic Indices of Generalized Bridge Molecular Graphs," *Symmetry*, vol. 10, no. 12, p. 751, 2018.

[19] X. Zhang, H. M. Awais, M. Javaid, and M. K. Siddiqui, "Multiplicative Zagreb Indices of Molecular Graphs," *Journal of Chemistry*, vol. 2019, Article ID 5294198, 19 pages, 2019.

[20] S. Wang, M. R. Farahani, M. R. Kanna, M. K. Jamil, and R. P. Kumar, "The Wiener index and the Hosoya polynomial of the Jahangir graphs," 2016, https://arxiv.org/abs/1607.00402.

[21] H. Wiener, "Structural determination of paraffin boiling points," *Journal of the American Chemical Society*, vol. 69, no. 1, pp. 17–20, 1947.

[22] N. A. Ali, "Computation of eccentric topological indices of zero-divisor graphs based on their edges," *AIMS Mathematics*, vol. 7, no. 7, pp. 11509–11518, 2022.

[23] A. Ahmad and S. C. López, "Distance-Based Topological Polynomials Associated with Zero-Divisor Graphs," *Mathematical Problems in Engineering*, vol. 2021, Article ID 4959559, 8 pages, 2021.

[24] Z. H. Hui, A. Rauf, M. M. Abbas, and A. Aslam, "Analysis of Eccentricity-Based Topological Invariants with Zero-Divisor Graphs," *Journal of Function Spaces*, vol. 2022, Article ID 6911654, 10 pages, 2022.

[25] M. R. Farahani, "Hosoya, Schultz, modified Schultz polynomials and their topological indices of benzene molecules: first members of polycyclic aromatic hydrocarbons (PAHs)," *International Journal of Theoretical Chemistry*, vol. 1, no. 2, pp. 9–16, 2013.

[26] M. V. Diudea, "Wiener and hyper-Wiener numbers in a single matrix," *Journal of Chemical Information and Computer Sciences*, vol. 36, no. 4, pp. 833–836, 1996.

[27] H. Hosoya, "On some counting polynomials in chemistry," *Discrete Applied Mathematics*, vol. 19, no. 1-3, pp. 239–257, 1988.

[28] E. Estrada, O. Ivanciuc, I. Gutman, A. Gutierrez, and L. Rodríguez, "Extended Wiener indices. A new set of descriptors for quantitative structure-property studies," *New Journal of Chemistry*, vol. 22, no. 8, pp. 819–822, 1998.

[29] B. Zhou and Z. Du, "On eccentric connectivity index," 2010, https://arxiv.org/abs/1007.2235.

[30] I. Gutman, S. Klavzar, M. Petkovšek, and P. Zigert, "On Hosoya polynomials of benzenoid graphs," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 43, pp. 49–66, 2001.

[31] G. G. Cash, "Relationship between the Hosoya polynomial and the hyper-Wiener index," *Applied Mathematics Letters*, vol. 15, no. 7, pp. 893–895, 2002.

[32] D. Stevanović, "Hosoya polynomial of composite graphs," *Discrete Mathematics*, vol. 235, no. 1-3, pp. 237–244, 2001.

[33] F. Harary, "Conditional colorability in graphs," in *Graphs and applications (Boulder, Colo., 1982)*, pp. 127–136, Wiley-Interscience Publisher, 1985.

[34] H. Wiener, "Relation of the physical properties of the isomeric alkanes to molecular structure. Surface tension, specific dispersion, and critical solution temperature in aniline," *The Journal of Physical Chemistry*, vol. 52, no. 6, pp. 1082–1089, 1948.

[35] M. Randić, "Novel molecular descriptor for structure—property studies," *Chemical Physics Letters*, vol. 211, no. 4-5, pp. 478–483, 1993.

[36] V. Sharma, R. Goswami, and A. K. Madan, "Eccentric connectivity index: a novel highly discriminating topological descriptor for structure– property and structure– activity studies," *Journal of Chemical Information and Computer Sciences*, vol. 37, no. 2, pp. 273–282, 1997.

[37] A. A. Dobrynin, R. Entringer, and I. Gutman, "Wiener index of trees: theory and applications," *Acta Applicandae Mathematica*, vol. 66, no. 3, pp. 211–249, 2001.

[38] D. J. Klein, Z. Mihalic, D. Plavsic, and N. Trinajstic, "Molecular topological index: a relation with the Wiener index," *Journal of Chemical Information and Computer Sciences*, vol. 32, no. 4, pp. 304-305, 1992.

[39] M. Randic, "Characterization of molecular branching," *Journal of the American Chemical Society*, vol. 97, no. 23, pp. 6609–6615, 1975.

[40] W. Gao, M. Younas, A. Farooq, A. U. R. Virk, and W. Nazeer, "Some reverse degree-based topological indices and polynomials of dendrimers," *Mathematics*, vol. 6, no. 10, p. 214, 2018.

[41] A. A. Dobrynin and I. Gutman, "The Wiener index for trees and graphs of hexagonal systems," *Diskretnyi Analiz i Issledovanie Operatsii*, vol. 5, no. 2, pp. 34–60, 1998.

[42] R. C. Entringer, A. Meir, J. W. Moon, and L. A. Székely, "On the Wiener index of trees from certain families," *Australasian Journal of Combinatorics*, vol. 10, no. 1994, pp. 211–224, 1994.

[43] A. Graovac and T. Pisanski, "On the Wiener index of a graph," *Journal of Mathematical Chemistry*, vol. 8, no. 1, pp. 53–62, 1991.

[44] W. Gao, M. Younas, A. Farooq, A. Mahboob, and W. Nazeer, "M-polynomials and degree-based topological indices of the crystallographic structure of molecules," *Biomolecules*, vol. 8, no. 4, p. 107, 2018.

[45] L. Yang, "Wiener index and traceable graphs," *Bulletin of the Australian Mathematical Society*, vol. 88, no. 3, pp. 380–383, 2013.

[46] D. J. Klein, I. Lukovits, and I. Gutman, "On the definition of the hyper-Wiener index for cycle-containing structures," *Journal of Chemical Information and Computer Sciences*, vol. 35, no. 1, pp. 50–52, 1995.

[47] I. Lukovits and W. Linert, "A novel definition of the hyper-Wiener index for cycles," *Journal of Chemical Information and Computer Sciences*, vol. 34, no. 4, pp. 899–902, 1994.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 5317797, 5 pages
https://doi.org/10.1155/2022/5317797

Research Article

Convolution and Coefficient Estimates for (p,q)-Convex Harmonic Functions Associated with Subordination

Hasan Bayram and Sibel Yalçın

Faculty of Arts and Sciences, Department of Mathematics, Bursa Uludag University, 16059, Görükle, Bursa, Turkey

Correspondence should be addressed to Hasan Bayram; hbayram@uludag.edu.tr

Received 30 June 2022; Revised 19 August 2022; Accepted 30 August 2022; Published 15 September 2022

Academic Editor: Jia-Bao Liu

We preface and examine classes of (p, q)-convex harmonic locally univalent functions associated with subordination. We acquired a coefficient characterization of (p, q)-convex harmonic univalent functions. We give necessary and sufficient convolution terms for the functions we will introduce.

1. Introduction

First, let us give the basic definitions and notations that we will use in our article. In order not to spoil the generality of this study, let us denote the continuous complex valued harmonic functions class with $\mathscr{H}$ which are harmonic in the open unit disk $\mathscr{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathscr{A}$ be the subclass of $\mathscr{H}$ which consists of functions that are analytic in $\mathscr{U}$. A function harmonic in $\mathscr{U}$ can be written in the form $f = h + \bar{g}$, where h and g are analytic in $\mathscr{U}$. Here, the analytic part of the f function is h and the co-analytical part is g. The necessary and sufficient condition for f to be locally univalent and the sense-preserving in $\mathscr{U}$ is that $|h'(z)| > |g'(z)|$ ([1]). In the light of this information, we can write without losing generality as follows:

$$h(z) = z + \sum_{\jmath=2}^{\infty} a_\jmath z^\jmath \text{ and } g(z) = \sum_{\jmath=1}^{\infty} b_\jmath z^\jmath. \tag{1}$$

Let us denote the class of functions satisfying $f = h + \bar{g}$ which are harmonic, univalent, and sense-preserving in $\mathscr{U}$ for which $h(0) = h'(0) - 1 = 0 = g(0)$ conditions with $\mathscr{SH}$. From this point of view, we can easily say that $|b_1| < 1$ if there is a sense-preserving feature.

In 1984 Clunie and Sheil-Small [1] defined and analyzed characteristic features of the class $\mathscr{SH}$. Over the years, many articles on the class of $\mathscr{SH}$ and its subclasses have been made by many researchers by referring to this article.

Many studies have been done on quantum calculus. As the importance of this subject can be understood from its multidisciplinary nature, it is known to be innovative and important in many fields. The quantum calculus is also known as q-calculus. We can roughly define this calculus as the traditional infinitesimal calculus. In fact, Euler and Jacobi first started to study the subject of q-calculus, they are also people who find many attractive implementations in several fields of mathematics and other sciences.

In the last study by Sahai and Yadav [2], the quantum calculus was based on two parameters (p, q). In fact, this two-parameter definition is the postquantum calculus denoted by (p, q)-calculus, which is the generalization of q-calculus. We will use the definition of (p, q)-calculus in this article as the basis of the article published by Chakrabarti and Jagannathan in 1991 [3]. Let $p > 0, q > 0$, for any non-negative integer $\jmath$, the (p, q)-integer number $\jmath$, denoted by $[\jmath]_{p,q}$ is

$$[\jmath]_{p,q} = \frac{p^\jmath - q^\jmath}{p - q}, (\jmath = 1, 2, 3, \cdots), \quad [0]_{p,q} = 0. \tag{2}$$

It can be seen that this twin-basic number defined above is a generalization of the q-number defined as follows:

$$[j]_q = \frac{1-q^j}{1-q}, (j = 1, 2, 3, \cdots), \quad q \neq 1. \tag{3}$$

In like manner, the (p, q)-differential operator of a function f, analytic in $|z| < 1$ is defined by

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, p \neq q, z \in \mathcal{U}. \tag{4}$$

It may be easily shown that $D_{p,q}f(z) \longrightarrow f'(z)$ as $p \longrightarrow 1^-$ and $q \longrightarrow 1^-$. We can easily see that

$$\lim_{q\longrightarrow 1^-} \lim_{p\longrightarrow 1^-} [j]_{p,q} = j. \tag{5}$$

For more information and details on q-calculus and (p, q)-calculus, [2, 4] can be used as references. Apart from these, different studies have also been carried out [5–7].

Ismail et al. [8] and Ahuja et al. and Ahuja and Çetinkaya [6, 9]z) used q- calculus in the theory of analytic univalent functions. The q-difference operator's definition is

$$D_{p,q}h(z) = \begin{cases} \dfrac{h(pz) - h(qz)}{(p-q)z}, & z \neq 0, \\ h'(0), & z = 0, \end{cases}$$
$$D_{p,q}g(z) = \begin{cases} \dfrac{g(pz) - g(qz)}{(p-q)z}, & z \neq 0, \\ g'(0), & z = 0, \end{cases} \tag{6}$$

where h and g are of the form (1) which given in [4] and we get the following result for same h and g

$$D_{p,q}h(z) = 1 + \sum_{j=2}^{\infty} [j]_{p,q} a_j z^{j-1} \text{ and } D_{p,q}g(z) = \sum_{j=1}^{\infty} [j]_{p,q} b_j z^{j-1}. \tag{7}$$

For $f \in \mathcal{SH}$, let

$$\mathfrak{D}^1_{p,q}f(z) = zD_{p,q}h(z) - z\overline{D_{p,q}g(z)}, \tag{8}$$

$$\begin{aligned} \mathfrak{D}^2_{p,q}f(z) &= zD_{p,q}\left(zD_{p,q}h(z)\right) + z\overline{D_{p,q}\left(zD_{p,q}g(z)\right)} \\ &= z + \sum_{j=2}^{\infty} [j]^2_{p,q} a_j z^j + \sum_{j=1}^{\infty} [j]^2_{p,q} \overline{b_j z^j}. \end{aligned} \tag{9}$$

We say that an analytic function f is subordinate to an analytic function F and write $f \prec F$, if there exists a complex valued function ϖ which maps $\mathcal{U}$ into oneself with $\varpi(0) = 0$, such that $f(z) = F(\varpi(z))(z \in \mathcal{U})$.

Additionally, if F is univalent in $\mathcal{U}$, then we can give the following result:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}). \tag{10}$$

Denote by $\mathcal{SHC}_{p,q}(A, B)$ the subclass of $\mathcal{H}$ consisting of functions f of the form (1) that satisfy the condition

$$\frac{\mathfrak{D}^2_{p,q}f(z)}{\mathfrak{D}^1_{p,q}f(z)} \prec \frac{1+Az}{1+Bz}, \tag{11}$$

$$(0 < p, q < 1, p \neq q, z \in \mathcal{U} \text{ and } -B \leq A < B \leq 1), \tag{12}$$

where is $\mathfrak{D}^1_{p,q}f(z)$ and $\mathfrak{D}^2_{p,q}f(z)$ are defined by (8) and (9).

By suitably specializing the parameters, the classes $\mathcal{SHC}_{p,q}(A, B)$ reduce to the various subclasses of harmonic univalent functions. That is, by assigning special values instead of p,q, A, and B, we are saying that they become classes that used to be studied. This is an indication that this article is a general subclass that includes other classes in harmonic functions. Such as

(i) $\mathcal{SHC}_{p,q}(A, B) = \mathcal{SHC}_q(A, B)$ for $p \longrightarrow 1^-$([10])

(ii) $\mathcal{SHC}_{p,q}(A, B) = \mathcal{SHC}(A, B)$ for $p \longrightarrow 1^-$ and $q \longrightarrow 1^-$([11]),

(iii) $\mathcal{SHC}_{p,q}((p+q)\vartheta - 1, p+q-1) = \mathcal{A}_{p,q}\mathcal{C}_H(\vartheta)$ for $0 \leq \vartheta < 1$ ([12]),

(iv) $\mathcal{SHC}_{p,q}((p+q)\alpha - 1, p+q-1) = \mathcal{SHC}_q(\alpha)$ for $p \longrightarrow 1^-$and $0 \leq \alpha < 1$ ([6, 13]),

(v) $\mathcal{SHC}_{p,q}((p+q)\alpha - 1, p+q-1) = \mathcal{SHC}(\alpha)$ for $p \longrightarrow 1^-$, $q \longrightarrow 1^-$ and $0 \leq \alpha < 1$([14, 15]),

(vi) $\mathcal{SHC}_{p,q}((p+q)\alpha - 1, p+q-1) = \mathcal{SHC}(0)$ for $p \longrightarrow 1^-$, $q \longrightarrow 1^-$ and $\alpha = 0$ ([16]).

Using the method that used by Dziok et al. [11, 17–19] we find necessary and sufficient conditions for the above defined class $\mathcal{SHC}_{p,q}(A, B)$.

2. Main Results

For functions $f_1$and $f_2 \in \mathcal{H}$ of the form

$$f_j(z) = z + \sum_{j=2}^{\infty} a_{j,j} z^j + \sum_{j=1}^{\infty} \overline{b_{j,j} z^j}, (j = 1, 2), \tag{13}$$

we define the Hadamard product of $f_1$and f_2 by

$$(f_1 * f_2)(z) = z + \sum_{j=2}^{\infty} a_{j,1} a_{j,2} z^j + \sum_{j=1}^{\infty} b_{j,1} \overline{b_{j,2} z^j}, z \in \mathcal{U}. \tag{14}$$

In the first theorem, we introduce a sufficient coefficient bound for harmonic functions in $\mathcal{SHC}_{p,q}(A, B)$.

Theorem 1. *Let us first assume that $f \in \mathcal{H}$. Then, $f \in \mathcal{SHC}_{p,q}(A, B)$ if and only if*

$$f(z) * \Xi_{p,q}(z;\zeta) \neq 0, (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathcal{U} \setminus \{0\}), \tag{15}$$

where

$$\begin{aligned}
&\Xi_{p,q}(z;\zeta)\\
&= \frac{(B-A)\zeta z + \left[(p+q)(p+q-1) + \zeta\left(A(p+q)^2 - B(p+q) + pq(B-A)\right)\right]z^2}{(1-pz)(1-qz)(1-p^2z)(1-q^2z)(1-pqz)}\\
&+ \frac{1+pq-(p+q)(p+q+1)+(B+Apq-(p+q)[(p+q)A+B])\zeta}{(1-pz)(1-qz)(1-p^2z)(1-q^2z)(1-pqz)}pqz^3\\
&+ \frac{1+pq+(B+Apq)\zeta}{(1-pz)(1-qz)(1-p^2z)(1-q^2)z(1-pqz)}p^2q^2z^4\\
&+ \frac{[2+(A+B)\zeta]\bar{z} - \left[\left(p^2+q^2\right)(1+A\zeta)+(A-B)pq\zeta+(p+q)(1+B\zeta)\right]\bar{z}^2}{(1-p\bar{z})(1-q\bar{z})(1-p^2\bar{z})(1-q^2\bar{z})(1-pq\bar{z})}\\
&+ \frac{\left[\left(p^2+q^2+pq\right)(1+A\zeta)+(1-p-q)(1+B\zeta)\right]}{(1-p\bar{z})(1-q\bar{z})(1-p^2\bar{z})(1-q^2\bar{z})(1-pq\bar{z})}pq\bar{z}^3\\
&+ \frac{1+B\zeta-pq(1+A\zeta)}{(1-p\bar{z})(1-q\bar{z})(1-p^2\bar{z})(1-q^2\bar{z})(1-pq\bar{z})}p^2q^2\bar{z}^4.
\end{aligned} \tag{16}$$

Proof. Let $f \in \mathscr{H}$ be of the form (1). Then $f \in \mathscr{SHC}_{p,q}(A,B)$ if and only if it satisfies (11) or equivalently

$$\frac{\mathfrak{D}^2_{p,q}f(z)}{\mathfrak{D}^1_{p,q}f(z)} \neq \frac{1+A\zeta}{1+B\zeta}, \tag{17}$$

where $\zeta \in \mathbb{C}$, $|\zeta| = 1$ and $z \in \mathscr{U} \setminus \{0\}$. Since

$$\begin{aligned}
zD_{p,q}h(z) &= h(z) * \frac{z}{(1-pz)(1-qz)},\\
zD_{p,q}g(z) &= g(z) * \frac{z}{(1-pz)(1-qz)},\\
zD_{p,q}\left(zD_{p,q}h(z)\right) &= h(z) * \frac{z(1+pqz)}{(1-p^2z)(1-q^2z)(1-pqz)},\\
zD_{p,q}\left(zD_{p,q}g(z)\right) &= g(z) * \frac{z(1+pqz)}{(1-p^2z)(1-q^2z)(1-pqz)},
\end{aligned} \tag{18}$$

the inequality (17) yields

$$\begin{aligned}
&(1+B\zeta)\mathfrak{D}^2_{p,q}f(z) - (1+A\zeta)\mathfrak{D}^1_{p,q}f(z)\\
&= (1+B\zeta)\left[zD_{p,q}\left(zD_{p,q}h(z)\right) + zD_{p,q}\left(z\bar{D}_{p,q}g(z)\right)\right]\\
&\quad - (1+A\zeta)\left[zD_{p,q}h(z) - zD_{p,q}\bar{g}(z)\right]\\
&= h(z) * \left\{\frac{(1+B\zeta)z(1+pqz)}{(1-p^2z)(1-q^2z)(1-pqz)} - \frac{(1+A\zeta)z}{(1-pz)(1-qz)}\right\}\\
&\quad + g(\bar{z}) * \left\{\frac{(1+B\zeta)\bar{z}(1+pq\bar{z})}{(1-p^2\bar{z})(1-q^2\bar{z})(1-pq\bar{z})} + \frac{(1+A\zeta)\bar{z}}{(1-p\bar{z})(1-q\bar{z})}\right\}\\
&= f(z) * \Xi_{p,q}(z;\zeta) \neq 0.
\end{aligned} \tag{19}$$

□

Theorem 2. *Let $f = h + \bar{g}$ be given by (1). If*

$$\begin{aligned}
\sum_{j=1}^{\infty} [j]_{p,q} \Big\{ &\left(\left|[j]_{p,q} - 1\right| + \left|[j]_{p,q}B - A\right|\right)|a_j|\\
&+ \left([j]_{p,q}(1+B) + 1 + A\right)|b_j| \Big\} \le 2(B-A),
\end{aligned} \tag{20}$$

where $a_1 = 1$, $0 < p, q < 1$, $-B \le A < B \le 1$, then, $f \in \mathscr{SHC}_{p,q}(A,B)$.

Proof. $f \in \mathscr{SHC}_{p,q}(A,B)$ if and only if there exists a complex valued function ϖ; $\varpi(0) = 0$, $|\varpi(z)| < 1$ $(z \in \mathscr{U})$ such that

$$\frac{zD_{p,q}\left(zD_{p,q}h(z)\right) + zD_{p,q}\left(z\bar{D}_{p,q}g(z)\right)}{zD_{p,q}h(z) - zD_{p,q}\bar{g}(z)} = \frac{1+A\varpi(z)}{1+B\varpi(z)}, \tag{21}$$

or equivalently

$$\left|\frac{\mathfrak{D}^2_{p,q}f(z) - \mathfrak{D}^1_{p,q}f(z)}{B\mathfrak{D}^2_{p,q}f(z) - A\mathfrak{D}^1_{p,q}f(z)}\right| < 1. \tag{22}$$

The above inequality (20) holds, since for $|z| = r(0 < r < 1)$, we obtain

$$\begin{aligned}
&\left|zD_{p,q}\left(zD_{p,q}h(z)\right) - zD_{p,q}h(z) + zD_{p,q}\left(z\bar{D}_{p,q}g(z)\right) + zD_{p,q}\bar{g}(z)\right|\\
&\quad - \Big|BD_{p,q}\left(zD_{p,q}h(z)\right) + BzD_{p,q}\left(z\bar{D}_{p,q}g(z)\right)\\
&\quad - AD_{p,q}h(z) + AD_{p,q}\bar{g}(z)\Big|\\
&= \left|\sum_{j=2}^{\infty}[j]_{p,q}\left([j]_{p,q}-1\right)a_jz^j + \sum_{j=1}^{\infty}[j]_{p,q}\left([j]_{p,q}+1\right)b_j\bar{z}^j\right|\\
&\quad - \Big|(B-A)z + \sum_{j=2}^{\infty}[j]_{p,q}\left([j]_{p,q}B - A\right)a_jz^j\\
&\quad + \sum_{j=1}^{\infty}[j]_{p,q}\left([j]_{p,q}B + A\right)b_j\bar{z}^j\Big|\\
&\le \sum_{j=2}^{\infty}[j]_{p,q}\left(\left|[j]_{p,q}-1\right|\right)|a_j|r^j\\
&\quad + \sum_{j=1}^{\infty}[j]_{p,q}\left([j]_{p,q}+1\right)|b_j|r^j - (B-A)r\\
&\quad + \sum_{j=2}^{\infty}[j]_{p,q}\left|[j]_{p,q}B - A\right||a_j|r^j + \sum_{j=1}^{\infty}[j]_{p,q}\left([j]_{p,q}B + A\right)|b_j|r^j\\
&\le r\Big\{\sum_{j=2}^{\infty}[j]_{p,q}\left(\left|[j]_{p,q}-1\right| + \left|[j]_{p,q}B - A\right|\right)|a_j|r^{j-1}\\
&\quad + \sum_{j=1}^{\infty}[j]_{p,q}\left([j]_{p,q}(1+B) + 1 + A\right)|b_j|r^{j-1} - (B-A)\Big\} < 0.
\end{aligned} \tag{23}$$

The harmonic function

$$f(z)=z+\sum_{\jmath=2}^{\infty}\frac{(B-A)x_\jmath}{[\jmath]_{p,q}\left(\left|[\jmath]_{p,q}-1\right|+\left|[\jmath]_{p,q}B-A\right|\right)}z^\jmath+\sum_{\jmath=1}^{\infty}\frac{(B-A)y_\jmath}{[\jmath]_{p,q}\left([\jmath]_{p,q}(1+B)+1+A\right)}\bar{z}^\jmath, \tag{24}$$

where

$$\sum_{\jmath=1}^{\infty}\left|x_\jmath\right|+\sum_{\jmath=1}^{\infty}\left|y_\jmath\right|=1 \tag{25}$$

shows that the coefficient bound given by (20) is sharp. The functions of the form (8) are in $\mathcal{SHC}_{p,q}(A,B)$ because

$$\sum_{\jmath=1}^{\infty}\frac{[\jmath]_{p,q}\left(\left|[\jmath]_{p,q}-1\right|+\left|[\jmath]_{p,q}B-A\right|\right)}{2(B-A)}\left|a_\jmath\right|+\frac{[\jmath]_{p,q}\left([\jmath]_{p,q}(1+B)+1+A\right)}{2(B-A)}\left|b_\jmath\right|=\sum_{\jmath=1}^{\infty}\left(\left|x_\jmath\right|+\left|y_\jmath\right|\right)=1. \tag{26}$$

□

Denote by $\mathcal{TSHC}_{p,q}(A,B)$ the subclass of $\mathcal{H}$ consisting of functions f of the form (1) that satisfy the inequality (20). It is clear that $\mathcal{TSHC}_{p,q}(A,B)\subset\mathcal{SHC}_{p,q}(A,B)$.

Theorem 3. *The class $\mathcal{TSHC}_{p,q}(A,B)$ is closed under convex combination.*

Proof. For $\jmath=1,2,\cdots$, suppose that $f_j\in\mathcal{TSHC}_{p,q}(A,B)$, where

$$f_\jmath(z)=z+\sum_{\jmath=2}^{\infty}\left|a_{\jmath_\jmath}\right|z^\jmath+\sum_{\jmath=1}^{\infty}\left|b_{\jmath_\jmath}\right|\bar{z}^\jmath. \tag{27}$$

Then, by Theorem 2.,

$$\sum_{\jmath=2}^{\infty}\frac{[\jmath]_{p,q}\left(\left|[\jmath]_{p,q}-1\right|+\left|[\jmath]_{p,q}B-A\right|\right)}{B-A}\left|a_{\jmath_\jmath}\right|+\sum_{\jmath=1}^{\infty}\frac{[\jmath]_{p,q}\left([\jmath]_{p,q}(1+B)+1+A\right)}{B-A}\left|b_{\jmath_\jmath}\right|\le 1. \tag{28}$$

For $\sum_{\jmath=1}^{\infty}t_\jmath=1, 0\le t_\jmath\le 1$, we can write the convex combination of $f_\jmath$ as follows:

$$\sum_{\jmath=1}^{\infty}t_\jmath f_\jmath(z)=z+\sum_{\jmath=2}^{\infty}\left(\sum_{\jmath=1}^{\infty}t_\jmath\left|a_{\jmath_\jmath}\right|\right)z^\jmath+\sum_{\jmath=1}^{\infty}\left(\sum_{\jmath=1}^{\infty}t_\jmath\left|b_{\jmath_\jmath}\right|\right)\bar{z}^\jmath. \tag{29}$$

Then, by (9),

$$\begin{aligned}&\sum_{\jmath=1}^{\infty}\left\{\frac{[\jmath]_{p,q}\left(\left|[\jmath]_{p,q}-1\right|+\left|[\jmath]_{p,q}B-A\right|\right)}{2(B-A)}\sum_{\jmath=1}^{\infty}t_\jmath\left|a_{\jmath_\jmath}\right|+\frac{[\jmath]_{p,q}\left([\jmath]_{p,q}(1+B)+1+A\right)}{2(B-A)}\sum_{\jmath=1}^{\infty}t_\jmath\left|b_{\jmath_\jmath}\right|\right\}\\&=\sum_{\jmath=1}^{\infty}t_\jmath\sum_{\jmath=1}^{\infty}\left(\frac{[\jmath]_{p,q}\left(\left|[\jmath]_{p,q}-1\right|+\left|[\jmath]_{p,q}B-A\right|\right)}{2(B-A)}\left|a_{\jmath_\jmath}\right|+\frac{[\jmath]_{p,q}\left([\jmath]_{p,q}(1+B)+1+A\right)}{2(B-A)}\left|b_{\jmath_\jmath}\right|\right)\\&\le\sum_{\jmath=1}^{\infty}t_\jmath=1,\end{aligned} \tag{30}$$

hence

$$\sum_{\jmath=1}^{\infty}t_\jmath f_\jmath(z)\in\mathcal{TSHC}_{p,q}(A,B). \tag{31}$$

□

3. Conclusions

As a result, a general subclass has been defined in this article. Thus, with this study, which will be a good reference for the new results to be obtained, a subclass study has been made for harmonic functions using the (p,q) derivative, which is still popular today.

Data Availability

There is no data available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academie Scientiarum Fennice Series A. I. Mathematica*, vol. 9, pp. 3–25, 1984.

[2] V. Sahai and S. Yadav, "Representations of two parameter quantum algebras and p,q-special functions," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 1, pp. 268–279, 2007.

[3] R. Chakrabarti and R. Jagannathan, "A (p, q)-oscillator realization of two-parameter quantum algebras," *Journal of Physics A*, vol. 24, no. 13, pp. L711–L718, 1991.

[4] F. H. Jackson, "XI—on q-functions and a certain difference operator," *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2, pp. 253–281, 1908.

[5] M. A. Ali, M. Abbas, H. Budak, P. Agarval, G. Murtaza, and Y. M. Chu, "New quantum boundaries for quantum Simpson's

and quantum Newton's type inequalities for preinvex functions," *Advances in Difference Equations*, vol. 2021, no. 1, 2021.

[6] O. P. Ahuja, A. Çetinkaya, and Y. Polatoglu, "Harmonic univalent convex functions using a quantum calculus approach," *Acta Universitatis Apulensis*, vol. 58, pp. 67–81, 2019.

[7] J. B. Liu, S. I. Butt, J. Nasir, A. Aslam, A. Fahad, and J. Soontharanon, "Jensen-Mercer variant of Hermite-Hadamard type inequalities via Atangana-Baleanu fractional operator," *AIMS Math*, vol. 7, no. 2, pp. 2123–2141, 2022.

[8] M. E. H. Ismail, E. Merkes, and D. Steyr, "A generalization of starlike functions," *Complex Variables, Theory and Application: An International Journal*, vol. 14, no. 1-4, pp. 77–84, 1990.

[9] O. P. Ahuja and A. Çetinkaya, "Connecting quantum calculus and harmonic starlike functions," *Univerzitet u Nišu*, vol. 34, no. 5, pp. 1431–1441, 2020.

[10] P. Sharma and O. Mishra, "A class of harmonic functions associated with a qs Al Agean Operator," *U.P.B. Sci. Bull., Series*, vol. 82, no. 3, pp. 3–12, 2020.

[11] J. Dziok, J. M. Jahangiri, and H. Silverman, "Harmonic functions with varying coefficients," *Journal of Inequalities and Applications*, vol. 2016, no. 1, 2016.

[12] A. Alsoboh and M. Darus, "On subclasses of harmonic univalent functions defined by Jackson (p, q) derivative," *Journal of Analysis*, vol. 10, no. 3, pp. 123–130, 2019.

[13] J. M. Jahangiri, "Harmonic univalent functions defined by q-calculus operators, international," *Journal of Mathematical Analysis and Applications*, vol. 5, no. 2, pp. 39–43, 2018.

[14] J. M. Jahangiri, "Harmonic functions starlike in the unit disk," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 2, pp. 470–477, 1999.

[15] M. Öztürk and S. Yalçin, "On univalent harmonic functions," *Journal of Inequalities in Pure andApplied Mathematics*, vol. 3, no. 4, article 61, 2002.

[16] H. Silverman and E. M. Silvia, "Subclasses of harmonic univalent functions," *New Zealand Journal of Mathematics*, vol. 28, pp. 275–284, 1999.

[17] J. Dziok, "Classes of harmonic functions defined by subordination," *Abstract and Applied Analysis*, vol. 2015, Article ID 756928, 9 pages, 2015.

[18] J. Dziok, "On Janowski harmonic functions," *Journal of Applied Analysis*, vol. 21, no. 2, pp. 99–107, 2015.

[19] J. Dziok, S. Yalçın, and Ş. Altınkaya, "Subclasses of harmonic univalent functions associated with generalized Ruscheweyh operator," *Publications de l'Institut Mathematique*, vol. 106, no. 120, pp. 19–28, 2019.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2229960, 12 pages
https://doi.org/10.1155/2022/2229960

Research Article

The Sharp Upper Bounds of the Hankel Determinant on Logarithmic Coefficients for Certain Analytic Functions Connected with Eight-Shaped Domains

Pongsakorn Sunthrayuth,[1] **Naveed Iqbal**,[2] **Muhammad Naeem**,[3] **Yousef Jawarneh**,[2] **and Sallieu K. Samura**[4]

[1]*Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT) Thanyaburi, Pathum Thani, Thailand*
[2]*Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia*
[3]*Deanship of Joint First Year Umm Al-Qura University Makkah, P.O. Box 715, Saudi Arabia*
[4]*Department of Mathematics and Statistics, Fourah Bay College, University of Sierra Leone, Sierra Leone*

Correspondence should be addressed to Muhammad Naeem; mfaridoon@uqu.edu.sa and Sallieu K. Samura; sallieu.samura@usl.edu.sl

Received 20 June 2022; Accepted 24 August 2022; Published 9 September 2022

Academic Editor: Mohsan Raza

The present study's intention is to produce exact estimations of some problems involving logarithmic coefficients for functions belonging to the considered subcollection $\mathscr{BT}_{\sin}$ of the bounded turning class. Furthermore, for the class $\mathscr{BT}_{\sin}$, we look into the accurate bounds of the Zalcman inequality, Fekete-Szegö inequality along with $\mathscr{D}_{2,1}(G_g/2)$ and $\mathscr{D}_{2,2}(G_g/2)$. Importantly, all of these bounds are shown to be sharp.

1. Introduction and Definitions

To properly understand the findings provided in the article, certain important literature on Geometric Function Theory must first be discussed. In this regard, the letters $\mathscr{S}$ and $\mathscr{A}$ stand for the normalized univalent functions class and the normalized holomorphic (or analytic) functions class, respectively. These primary notions are defined in the region $\mathbb{E}_d = \{z \in \mathbb{C} : |z| < 1\}$ by

$$\mathscr{A} = \left\{ g \in \mathscr{H}(\mathbb{E}_d) \colon g(z) = z + \sum_{k=2}^{\infty} b_k z^k (z \in \mathbb{E}_d) \right\}, \tag{1}$$

where $\mathscr{H}(\mathbb{E}_d)$ symbolizes the holomorphic functions class, and

$$\mathscr{S} = \{ g \in \mathscr{A} : g \text{ is univalent in } \mathbb{E}_d \}. \tag{2}$$

The following formula defines the logarithmic coefficients β_n of g that belong to $\mathscr{S}$

$$G_g(z) \coloneqq \log\left(\frac{g(z)}{z}\right) = 2\sum_{n=1}^{\infty} \beta_n z^n \text{ for } z \in \mathbb{E}_d. \tag{3}$$

In many estimations, these coefficients provide a significant contribution to the concept of univalent functions. In 1985, De Branges [1] proved that

$$\sum_{k=1}^{n} k(n-k+1)|\beta_n|^2 \le \sum_{k=1}^{n} \frac{n-k+1}{k} \forall n \ge 1, \tag{4}$$

and equality will be achieved if g has the form $z/(1-e^{i\theta}z)^2$ for some $\theta \in \mathbb{R}$. In its most comprehensive version, this inequality offers the famous Bieberbach-Robertson-Milin conjectures regarding Taylor coefficients of $g \in \mathscr{S}$. We refer to [2–4] for further details on the proof of De Branges' finding. By considering the logarithmic coefficients, Kayumov [5] was able to prove Brennan's conjecture for conformal mappings in 2005. For your reference, we mention a few works that have made major contributions to the research of the logarithmic coefficients. Andreev and Duren [6], Alimohammadi et al. [7], Deng [8], Roth [9], Ye [10], Obradović et al. [11], and finally the work of Girela [12] are the major contributions to the study of logarithmic coefficients for different subclasses of holomorphic univalent functions.

As stated in the definition, it is simple to determine that for $g \in \mathcal{S}$, the logarithmic coefficients are computed by

$$\beta_1 = \frac{1}{2} b_2, \tag{5}$$

$$\beta_2 = \frac{1}{2}\left(b_3 - \frac{1}{2} b_2^2\right), \tag{6}$$

$$\beta_3 = \frac{1}{2}\left(b_4 - b_2 b_3 + \frac{1}{3} b_2^3\right), \tag{7}$$

$$\beta_4 = \frac{1}{2}\left(b_5 - b_2 b_4 + b_2^2 b_3 - \frac{1}{2} b_3^2 - \frac{1}{4} b_2^4\right). \tag{8}$$

For given $q, n \in \mathbb{N} = \{1, 2, \cdots\}$, $b_1 = 1$, and $g \in \mathcal{S}$ with the series expansion (1), the Hankel determinant $\mathcal{D}_{q,n}(g)$ is represented by

$$\mathcal{D}_{q,n}(g) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+q-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n+q-1} & b_{n+q} & \cdots & b_{n+2q-2} \end{vmatrix}. \tag{9}$$

It was defined by Pommerenke [13, 14]. This determinant has indeed been investigated for a number of univalent function subclasses. In specific, the sharp estimate of the functional $|\mathcal{D}_{2,2}(g)| = |b_2 b_4 - b_3^2|$ for the sets $\mathcal{C}$ (convex functions), $\mathcal{S}^*$(starlike functions), and $\mathcal{R}$ (bounded turning functions) has been effectively established in [15, 16]. Later, numerous scholars published their findings on the upper bounds of $|\mathcal{D}_{2,2}(g)|$ for various subcollections of holomorphic functions; see [17–23]. However, for the class of close-to-convex functions, the exact estimation of this determinant is yet unknown [24].

Analogous to the determinant $\mathcal{D}_{q,n}(g)$ mentioned above, Kowalczyk and Lecko [25, 26] considered to examine the following determinant $\mathcal{D}_{q,n}(G_g/2)$ with entries from logarithmic coefficients of g

$$\mathcal{D}_{q,n}\left(\frac{G_g}{2}\right) = \begin{vmatrix} \beta_n & \beta_{n+1} & \cdots & \beta_{n+q-1} \\ \beta_{n+1} & \beta_{n+2} & \cdots & \beta_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_{n+q-1} & \beta_{n+q} & \cdots & \beta_{n+2q-2} \end{vmatrix}. \tag{10}$$

It is observed that

$$\begin{aligned} \mathcal{D}_{2,1}\left(\frac{G_g}{2}\right) &= \beta_1\beta_3 - \beta_2^2, \\ \mathcal{D}_{2,2}\left(\frac{G_g}{2}\right) &= \beta_2\beta_4 - \beta_3^2. \end{aligned} \tag{11}$$

For the given functions $G_1, G_2 \in \mathcal{A}$, the subordination between G_1 and G_2 (mathematically written as $G_1 \prec G_2$), if we get a Schwarz function ν with $\nu(0) = 0$ and $|\nu(z)| < 1$ for $z \in \mathbb{E}_d$ in a way such that $G_1(z) = G_2(\nu(z))$ hold true. Additionally, the following relation applies if G_2 in $\mathbb{E}_d$ is univalent:

$$G_1(z) \prec G_2(z), (z \in \mathbb{E}_d), \tag{12}$$

if and only if

$$\begin{aligned} G_1(0) &= G_2(0), \\ G_1(\mathbb{E}_d) &\subset G_2(\mathbb{E}_d). \end{aligned} \tag{13}$$

In 1992, Ma and Minda [27] developed a consolidated version of the collection $\mathcal{S}^*(\pi)$ by using the principle of subordination, and the following is a description of it:

$$\mathcal{S}^*(\pi) := \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec \pi(z), (z \in \mathbb{E}_d) \right\}, \tag{14}$$

where the univalent function π satisfies

$$\begin{aligned} \pi'(0) &> 0, \\ \Re e\pi &> 0. \end{aligned} \tag{15}$$

The area $\pi(\mathbb{E}_d)$ is also symmetric about x-axis and has a star-shaped form around the point $\pi(0) = 1$. In recent years, a wide variety of the collection $\mathcal{S}$'s subfamilies have been looked into as particular alternatives for the class $\mathcal{S}^*(\pi)$. As an illustration:

(i) $\mathcal{S}\mathcal{S}^*(\xi) \equiv \mathcal{S}^*(\pi(z))$ with $\pi(z) = ((1+z)/(1-z))^{\xi}$ and $0 < \xi \le 1$ (see [28])

(ii) $\mathcal{S}^*_{\mathcal{L}} \equiv \mathcal{S}^*((1+z)^{1/2})$ (see [29]), and $\mathcal{S}^*_{c\varsigma r} \equiv \mathcal{S}^*(1 + (4/3)z + (2/3)z^2)$ (see [30, 31])

(iii) $\mathcal{S}^*_{\rho} \equiv \mathcal{S}^*(1 + \sinh^{-1} z)$ (see [32]), and $\mathcal{S}^*_{e} \equiv \mathcal{S}^*(e^z)$ (see [33, 34])

(iv) $\mathcal{S}^*_{\cos} \equiv \mathcal{S}^*(\cos z)$ (see [35]), and $\mathcal{S}^*_{\cosh} \equiv \mathcal{S}^*(\cosh z)$ (see [36])

(v) $\mathcal{S}^*_{\tanh} \equiv \mathcal{S}^*(1 + \tanh z)$ (see [37, 38])

In [39], Cho et al. developed a novel subfamily of starlike function described by

$$\mathcal{S}^*_{\sin} := \left\{ g \in \mathcal{A} : \frac{zg'(z)}{g(z)} \prec 1 + \sin z\, (z \in \mathbb{E}_d) \right\}. \tag{16}$$

From the definition of the family $\mathcal{S}^*_{\sin}$, the authors [39] deduced that

$$g \in \mathcal{S}^*_{\sin} \Leftrightarrow g(z) = z \exp\left(\int_0^z \frac{u(t)-1}{t}dt\right), \quad (17)$$

for some $u(z) \prec u_0(z) = 1 + \sin z$. By substituting

$$u(z) = u_0(z) = 1 + \sin z \quad (18)$$

in (17), we acquire the function

$$g_0(z) = z \exp\left(\int_0^z \frac{\sin t}{t}dt\right) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{9}z^4 \cdots, \quad (19)$$

which acts as the extremal function in a variety of $\mathcal{S}^*_{\sin}$-family problems. In [40], the authors defined the following subfamily $\mathcal{BT}_{\sin}$ of holomorphic functions by using (18):

$$\mathcal{BT}_{\sin} = \left\{g \in \mathcal{S} : g'(z) \prec 1 + \sin z \, (z \in \mathbb{E}_d)\right\}. \quad (20)$$

Our primary objective in the current paper is to compute the problems involving the sharp logarithmic coefficients for the class $\mathcal{BT}_{\sin}$ of bounded turning functions connected to an eight-shaped domain. The sharp bounds of the Zalcman inequality, the Fekete-Szegö type inequality, along with the determinants $\mathcal{D}_{2,1}(G_g/2)$ and $\mathcal{D}_{2,2}(G_g/2)$ for the family $\mathcal{BT}_{\sin}$ are found using logarithmic coefficient entries.

2. Preliminary Lemmas

We must first create the class $\mathcal{P}$ in the below set-builder form in order to declare the Lemmas that are employed in our primary findings:

$$\mathcal{P} = \{p \in \mathcal{H}(\mathbb{E}_d) : p(0) = 1 \& \mathfrak{Re}\, p > 0, (z \in \mathbb{E}_d)\}. \quad (21)$$

That is, if $p \in \mathcal{P}$, then it has the series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n (z \in \mathbb{E}_d). \quad (22)$$

Lemma 1 (see [41]). *Let $p \in \mathcal{P}$ and has the series form (22). Then for $x, \delta, \rho \in \bar{\mathbb{E}}_d = \mathbb{E}_d \cup 1\{1\}$*

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right), \quad (23)$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1 x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)\delta, \quad (24)$$

$$8p_4 = p_1^4 + \left(4 - p_1^2\right)x\left[p_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4\left(4 - p_1^2\right)\left(1 - |x|^2\right), \quad (25)$$

$$\left[p(x-1)\delta + \bar{x}\delta^2 - \left(1 - |\delta|^2\right)\rho\right]. \quad (26)$$

Lemma 2. *If $p \in \mathcal{P}$ and has the expansion (22), then*

$$|p_n| \le 2(n \ge 1), \quad (27)$$

and if $Q \in [0, 1]$ and $Q(2Q-1) \le R \le Q$, then

$$\left|p_3 - 2Qp_1p_2 + Rp_1^3\right| \le 2. \quad (28)$$

Also,

$$|p_{n+k} - \mu p_n p_k| \le 2 \max\{1, |2\mu - 1|\} = 2\begin{cases} 1, & \text{for } 0 \le \mu \le 1, \\ |2\mu - 1|, & \text{otherwise}. \end{cases} \quad (29)$$

The inequalities (27), (28) and (29) are taken from [42, 43], and [44], respectively.

Lemma 3 (see [45]). *Let τ, ψ, ρ, and ς satisfy the inequalities $0 < \tau < 1$, $0 < \varsigma < 1$ and*

$$8\varsigma(1-\varsigma)\left((\tau\psi - 2\rho)^2 + (\tau(\varsigma + \tau) - \psi)^2\right) + \tau(1-\tau)(\psi - 2\varsigma\tau)^2 \le 4\varsigma\tau^2(1-\tau)^2(1-\varsigma). \quad (30)$$

If $p \in \mathcal{P}$ has the form (22), then

$$\left|\rho p_1^4 + \varsigma p_2^2 + 2\tau p_1 p_3 - \frac{3}{2}\psi p_1^2 p_2 - p_4\right| \le 2. \quad (31)$$

3. Coefficient Inequalities for the Class $\mathcal{BT}_{\sin}$

Theorem 4. *If $g \in \mathcal{BT}_{\sin}$ and has the series representation (1), then*

$$|\beta_1| \le \frac{1}{4}, \quad |\beta_2| \le \frac{1}{6}, \quad |\beta_3| \le \frac{1}{8}, \quad |\beta_4| \le \frac{1}{10}. \quad (32)$$

These bounds are sharp and can be obtained from the following extremal functions

$$\begin{aligned} g_0(z) &= \int_0^z (1 + \sin(t))dt = z + \frac{1}{2}z^2 + \cdots, \\ g_1(z) &= \int_0^z \left(1 + \sin\left(t^2\right)\right)dt = z + \frac{1}{3}z^3 + \cdots, \\ g_2(z) &= \int_0^z \left(1 + \sin\left(t^3\right)\right)dt = z + \frac{1}{4}z^4 + \cdots, \\ g_3(z) &= \int_0^z \left(1 + \sin\left(t^4\right)\right)dt = z + \frac{1}{5}z^5 + \cdots. \end{aligned} \quad (33)$$

Proof. Let $g \in \mathcal{BT}_{\sin}$. Consequently, (20) may be expressed using the Schwarz function as

$$g'(z) = 1 + \sin(w(z)), (z \in \mathbb{E}_d). \quad (34)$$

The Schwarz function w may be used to express it if $p \in \mathcal{P}$ as follows

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots, \quad (35)$$

equivalently,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{p_1 z + p_2 z^2 + p_3 z^3 + \cdots}{2 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots}. \quad (36)$$

From (1), we obtain

$$g'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots. \quad (37)$$

By simplification and using the series expansion of (36), we get

$$\begin{aligned} 1 + \sin(w(z)) = 1 &+ \frac{1}{2}p_1 z + \left(-\frac{1}{4}p_1^2 + \frac{1}{2}p_2\right)z^2 \\ &+ \left(-\frac{1}{2}p_1 p_2 + \frac{5}{48}p_1^3 + \frac{1}{2}p_3\right)z^3 \\ &+ \left(\frac{1}{2}p_4 - \frac{1}{32}p_1^4 + \frac{5}{16}p_1^2 p_2 - \frac{1}{2}p_1 p_3 - \frac{1}{4}p_2^2\right)z^4 + \cdots. \end{aligned} \quad (38)$$

Comparing (37) and (38), we obtain

$$a_2 = \frac{1}{4}p_1, \quad (39)$$

$$a_3 = -\frac{1}{12}p_1^2 + \frac{1}{6}p_2, \quad (40)$$

$$a_4 = -\frac{1}{8}p_1 p_2 + \frac{5}{192}p_1^3 + \frac{1}{8}p_3, \quad (41)$$

$$a_5 = \frac{1}{10}p_4 - \frac{1}{160}p_1^4 + \frac{5}{80}p_1^2 p_2 - \frac{1}{10}p_1 p_3 - \frac{1}{20}p_2^2. \quad (42)$$

Putting (42) in (5), (6), (7), and (8), we obtain

$$\beta_1 = \frac{1}{8}p_1, \quad (43)$$

$$\beta_2 = -\frac{11}{192}p_1^2 + \frac{1}{12}p_2, \quad (44)$$

$$\beta_3 = -\frac{1}{12}p_1 p_2 + \frac{5}{192}p_1^3 + \frac{1}{16}p_3, \quad (45)$$

$$\beta_4 = \frac{1}{20}p_4 - \frac{1033}{92160}p_1^4 + \frac{17}{288}p_1^2 p_2 - \frac{23}{720}p_2^2 - \frac{21}{320}p_1 p_3. \quad (46)$$

For β_1, using (27), in (43), we obtain

$$|\beta_1| \le \frac{1}{4}. \quad (47)$$

For β_2, putting (29) in (44), we obtain

$$|\beta_2| \le \frac{1}{6}. \quad (48)$$

For β_3, we can rewrite (45) as

$$|\beta_3| = \frac{1}{16}\left|\left(p_3 - \frac{4}{3}p_1 p_2 + \frac{5}{12}p_1^3\right)\right|. \quad (49)$$

Using (28) we get

$$|\beta_3| \le \frac{1}{8}. \quad (50)$$

For β_4, we can rewrite (46) as

$$\beta_4 = -\frac{1}{20}\left(\frac{1033}{4608}p_1^4 + \frac{23}{36}p_2^2 + 2\left(\frac{21}{32}\right)p_1 p_3 - \frac{3}{2}\left(\frac{85}{108}\right)p_1^2 p_2 - p_4\right). \quad (51)$$

Comparing the right side of (51) with

$$\left|\varrho p_1^4 + \varsigma p_2^2 + 2\tau p_1 p_3 - \frac{3}{2}\psi p_1^2 p_2 - p_4\right|, \quad (52)$$

where

$$\begin{aligned} \varrho &= \frac{1033}{4608}, \\ \varsigma &= \frac{23}{36}, \\ \tau &= \frac{21}{32}, \\ \psi &= \frac{85}{108}. \end{aligned} \quad (53)$$

It follows that

$$\begin{aligned} &8\varsigma(1-\varsigma)\left((\tau\psi - 2\rho)^2 + (\tau(\varsigma+\tau) - \psi)^2\right) \\ &\quad + \tau(1-\tau)(\psi - 2\varsigma\tau)^2 = 0.01647, \\ &4\varsigma\tau^2(1-\tau)^2(1-\varsigma) = 0.04696. \end{aligned} \quad (54)$$

Using (30) we deduce that

$$|\beta_4| \le \frac{1}{10}. \quad (55)$$

□

Theorem 5. *If g has the series form (1) and belongs to $\mathscr{B}\mathscr{T}_{\sin}$, then*

$$\left|\beta_2 - \eta\beta_1^2\right| \le \max\left\{\frac{1}{6}, \left|\frac{1}{48}(3+3|\eta|)\right|\right\}. \quad (56)$$

Equality will be attained by using (5), (6), and

$$g_1(z) = \int_0^z \left(1+\sin\left(t^2\right)\right)dt = z + \frac{1}{3}z^3 + \cdots. \quad (57)$$

Proof. From (43) to (44), we get

$$\left|\beta_2 - \eta\beta_1^2\right| = \left|-\frac{11}{192}p_1^2 + \frac{1}{12}p_2 - \frac{\eta}{64}p_1^2\right|. \quad (58)$$

Using (29), we have

$$\left|\beta_2 - \eta\beta_1^2\right| \le \frac{1}{12}\max\left\{2, 2\left|2\left(\frac{11+3\eta}{16}\right) - 1\right|\right\}. \quad (59)$$

After the simplification, we get

$$\left|\beta_2 - \eta\beta_1^2\right| \le \max\left\{\frac{1}{6}, \left|\frac{1}{48}(3+3|\eta|)\right|\right\}. \quad (60)$$

□

Theorem 6. *If g has the series expansion (1) and belongs to $\mathscr{B}\mathscr{T}_{\sin}$, then*

$$|\beta_1\beta_2 - \beta_3| \le \frac{1}{8}. \quad (61)$$

Equality can be attained by applying (5), (6), (7), and

$$g_2(z) = \int_0^z \left(1+\sin\left(t^3\right)\right)dt = z + \frac{1}{4}z^4 + \cdots. \quad (62)$$

Proof. From (43), (44), and (45), we obtain

$$|\beta_1\beta_2 - \beta_3| = \left|-\frac{17}{512}p_1^3 + \frac{3}{32}p_1p_2 - \frac{1}{16}p_3\right|. \quad (63)$$

After the simplification, we obtain

$$|\beta_1\beta_2 - \beta_3| = \frac{1}{16}\left|p_3 - \frac{3}{2}p_1p_2 + \frac{17}{32}p_1^3\right|. \quad (64)$$

Using (28), we have

$$|\beta_1\beta_2 - \beta_3| \le \frac{1}{8}. \quad (65)$$

□

Theorem 7. *If $g \in \mathscr{B}\mathscr{T}_{\sin}$ has given by (1), then*

$$\left|\beta_4 - \beta_2^2\right| \le \frac{1}{10}. \quad (66)$$

This result is sharp and equality can be achieved by applying (6), (8), and

$$g_3(z) = \int_0^z \left(1+\sin\left(t^4\right)\right)dt = z + \frac{1}{5}z^5 + \cdots. \quad (67)$$

Proof. From (44) to (46), we obtain

$$\left|\beta_4 - \beta_2^2\right| = \left|-\frac{7}{180}p_2^2 + \frac{1}{20}p_4 + \frac{79}{1152}p_1^2p_2 - \frac{2671}{184320}p_1^4 - \frac{21}{320}p_1p_3\right|. \quad (68)$$

After the simplification, we obtain

$$\left|\beta_4 - \beta_2^2\right| = -\frac{1}{20}\left|\frac{2671}{9216}p_1^4 + \frac{7}{9}p_2^2 + 2\left(\frac{21}{32}\right)p_1p_3 - \frac{3}{2}\left(\frac{395}{432}\right)p_1^2p_2 - p_4\right|. \quad (69)$$

Comparing the right side of (69)with

$$\left|\varrho p_1^4 + \varsigma p_2^2 + 2\tau p_1p_3 - \frac{3}{2}\psi p_1^2p_2 - p_4\right|, \quad (70)$$

where

$$\varrho = \frac{2671}{9216}, \quad \varsigma = \frac{7}{9}, \quad \tau = \frac{21}{32}, \quad \psi = \frac{395}{432}. \quad (71)$$

It follows that

$$8\varsigma(1-\varsigma)\left((\tau\psi - 2\rho)^2 + (\tau(\varsigma+\tau) - \psi)^2\right) + \tau(1-\tau)(\psi - 2\varsigma\tau)^2 = 0.004121,$$

$$4\varsigma\tau^2(1-\tau)^2(1-\varsigma) = 0.03518. \quad (72)$$

Using (30) we deduce that

$$\left|\beta_4 - \beta_2^2\right| \le \frac{1}{10}. \quad (73)$$

□

4. Hankel Determinant with Logarithmic Coefficients

Theorem 8. *Let $g \in \mathcal{BT}_{\sin}$ and be of the form (1). Then*

$$\left|\mathcal{D}_{2,1}\left(\frac{G_g}{2}\right)\right| = \left|\beta_1\beta_3 - \beta_2^2\right| \leq \frac{1}{36}. \tag{74}$$

The above stated result is sharp. Equality can be attained with the use of (5), (6), (7), and

$$g_1(z) = \int_0^z \left(1 + \sin\left(t^2\right)\right)dt = z + \frac{1}{3}z^3 + \cdots. \tag{75}$$

Proof. Employing (43), (44), and (45), we obtain

$$\mathcal{D}_{2,1}\left(\frac{G_g}{2}\right) = \frac{1}{128}p_1p_3 - \frac{1}{36864}p_1^4 - \frac{1}{144}p_2^2 - \frac{1}{1152}p_1^2p_2. \tag{76}$$

Using (23) and (24) along with the assumption that $p_1 = p, p \in [0, 2]$, we get

$$\begin{aligned}\left|\mathcal{D}_{2,1}\left(\frac{G_g}{2}\right)\right| = \Bigg|&-\frac{1}{576}\left(4-p^2\right)^2x^2 - \frac{1}{4096}p^4 \\ &- \frac{1}{512}p^2\left(4-p^2\right)x^2 \\ &+ \frac{1}{256}\left(4-p^2\right)\left(1-|x|^2\right)p\delta\Bigg|.\end{aligned} \tag{77}$$

Applying triangle inequality and assuming $|\delta| \leq 1$, $|x| = J, J \leq 1$ and also setting $p \in [0, 2]$, we have

$$\begin{aligned}\left|\mathcal{D}_{2,1}\left(\frac{G_g}{2}\right)\right| \leq &\frac{1}{576}\left(4-p^2\right)^2J^2 + \frac{1}{4096}p^4 \\ &+ \frac{1}{512}p^2\left(4-p^2\right)J^2 \\ &+ \frac{1}{256}\left(4-p^2\right)\left(1-J^2\right)p \coloneqq \phi(p, J).\end{aligned} \tag{78}$$

A little exercise can verify that $\phi'(p, J) \geq 0$ in $[0, 1]$, and this implies $\phi(p, J) \leq \phi(p, 1)$. Thus, by choosing $J = 1$, we achieve

$$\begin{aligned}\left|\mathcal{D}_{2,1}\left(\frac{G_g}{2}\right)\right| \leq &\frac{1}{576}\left(4-p^2\right)^2 + \frac{1}{4096}p^4 + \frac{1}{512}p^2\left(4-p^2\right) \\ &\coloneqq \phi(p, 1).\end{aligned} \tag{79}$$

Now, since $\phi'(p, 1) < 0$, we see that $\phi(p, 1)$ is a decreasing function, and so its maximum value appears at the lowest point $p = 0$, which is

$$\left|\mathcal{D}_{2,1}\left(\frac{G_g}{2}\right)\right| \leq \frac{1}{36}. \tag{80}$$

□

Theorem 9. *If $g \in \mathcal{BT}_{\sin}$ and has the form (1), then*

$$\left|\mathcal{D}_{2,2}\left(\frac{G_g}{2}\right)\right| \leq \frac{1}{64}. \tag{81}$$

The inequality is sharp and can be obtained by using (6), (7), (8), and

$$g_2(z) = \int_0^z \left(1 + \sin\left(t^3\right)\right)dt = z + \frac{1}{4}z^4 + \cdots. \tag{82}$$

Proof. The determinant $\mathcal{D}_{2,2}(G_g/2)$ can be written as

$$\mathcal{D}_{2,2}\left(\frac{G_g}{2}\right) = \beta_2\beta_4 - \beta_3^2. \tag{83}$$

Putting (44), (45), and (46), with $p_1 = p$, we obtain

$$\begin{aligned}\mathcal{D}_{2,2}\left(\frac{G_g}{2}\right) = \frac{1}{17694720}\big(&432p^4p_2 - 3456p^2p_2^2 - 637p^6 \\ &- 50688p^2p_4 + 8928p^3p_3 - 69120p_3^2 \\ &- 47104p_2^3 + 73728p_2p_4 + 87552pp_2p_3\big).\end{aligned} \tag{84}$$

Let $v = 4 - p^2$ in (23), (24), and (25). Now, applying the simplest version of the given lemma, we get

$$432p^4p_2 = 216\left(p^6 + p^4vx\right),$$

$$3456p^2p_2^2 = 1728p^4vx + 864p^2v^2x^2 + 864p^6,$$

$$\begin{aligned}50688p^2p_4 = &6336p^6 - 25344p^3v\left(1-|x|^2\right)\delta x \\ &- 25344p^2v\left(1-|x|^2\right)\bar{x}\delta^2 + 25344vp^2x^2 \\ &+ 25344p^3v\left(1-|x|^2\right)\delta + 19008p^4xv \\ &+ 19008p^4xv + 25344p^2v\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho \\ &+ 6336p^4vx^3,\end{aligned}$$

$$\begin{aligned}8928p^3p_3 = &4464p^4xv - 2232p^4vx^2 + 2232p^6 \\ &+ 4464p^3v\left(1-|x|^2\right)\delta,\end{aligned}$$

$$\begin{aligned}69120p_3^2 = &-17280x^2v^2\left(1-|x|^2\right)p\delta - 8640p^4vx^2 \\ &+ 4320p^6 + 4320x^4v^2p^2 + 17280p^3v\left(1-|x|^2\right)\delta \\ &+ 17280v^2\left(1-|x|^2\right)^2\delta^2 + 17280p^2x^2v^2 \\ &+ 17280p^4xv + 34560pxv^2\left(1-|x|^2\right)\delta \\ &- 17280x^3v^2p^2,\end{aligned}$$

$$47104p_2^3 = 5888\left(v^3x^3 + p^6\right) + 17664\left(p^4vx + p^2v^2x^2\right),$$

$$73728p_2p_4 = 18432x^3v^2 + 4608p^6 + 18432vp^2x^2 + 4608p^4vx^3 + 4608x^4v^2p^2 + 18432p^4xv + 18432pxv^2\left(1-|x|^2\right)\delta - 13824p^4vx^2 - 13824x^3v^2p^2 - 18432xv^2\bar{x}\left(1-|x|^2\right)\bar{x}\delta^2 - 18432p^2v\left(1-|x|^2\right)\bar{x}\delta^2 + 13824p^2x^2v^2 - 18432p^3v\left(1-|x|^2\right)\delta x + 18432p^3v\left(1-|x|^2\right)\delta + 18432xv^2\left(1-|x|^2\right),$$
$$\left(1-|\delta|^2\right)\rho + 18432p^2v\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho - 18432x^2v^2\left(1-|x|^2\right)p\delta,$$
$$87552pp_2p_3 = 21888pxv^2\left(1-|x|^2\right)\delta - 10944x^3v^2p^2 + 32832p^4xv - 10944p^4vx^2 + 21888p^3v\left(1-|x|^2\right)\delta + 10944p^6 + 21888p^2x^2v^2. \quad (85)$$

Putting the above expressions in (84), we get,

$$\mathscr{D}_{2,2}\left(\frac{G_g}{2}\right) = \frac{1}{17694720}\left\{-6912vp^2x^2 - 5888x^3v^3 - 45p^6 + 2160p^3v\left(1-|x|^2\right)\delta + 264p^4xv - 1728p^4vx^3 + 288x^4v^2p^2 - 96p^2x^2v^2 + 18432x^3v^2 - 7488x^3v^2p^2 + 6912p^3v\left(1-|x|^2\right)\delta x + 6912p^2v\left(1-|x|^2\right)\bar{x}\delta^2 + 5760pxv^2\left(1-|x|^2\right)\delta + 648p^4vx^2 + 18432xv^2\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho - 1152x^2v^2\left(1-|x|^2\right)p\delta - 18432xv^2 \cdot \left(1-|x|^2\right)\bar{x}\delta^2 - 17280v^2\left(1-|x|^2\right)^2\delta^2 - 6912p^2v\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho\right\}. \quad (86)$$

Since $v = 4 - p^2$,

$$\mathscr{D}_{2,2}\left(\frac{G_g}{2}\right) = \frac{1}{17694720}\left(q_1(p,x) + q_2(p,x)\delta + q_3(p,x)\delta^2 + q_4(p,x,\delta)\rho\right), \quad (87)$$

where $\rho, x, \delta \in \bar{\mathbb{U}}_d$, and

$$q_1(p,x) = \left(4-p^2\right)\left[\left(4-p^2\right)\left(288x^4p^2 - 1600x^3p^2 - 5120x^3 - 96x^2p^2\right) - 1728x^3p^4 + 264xp^4 - 6912x^2p^2 + 648x^2p^4\right] - 45p^6,$$
$$q_2(p,x) = \left(4-p^2\right)\left(1-|x|^2\right)\left[\left(4-p^2\right)\left(5760xp - 1152x^2p\right) + 2160p^3 + 6912xp^3\right],$$
$$q_3(p,x) = \left(4-p^2\right)\left(1-|x|^2\right)\left[\left(4-p^2\right)\left(-17280 - 1152|x|^2\right) + 6912\bar{x}p^2\right],$$
$$q_4(p,x,\delta) = \left(4-p^2\right)\left(1-|x|^2\right)\left(1-|\delta|^2\right) \cdot \left[18432x\left(4-p^2\right) - 6912p^2\right]. \quad (88)$$

Now, by the virtue of $|\delta| = y, |x| = x$, and $|\rho| \le 1$, we get

$$\left|\mathscr{D}_{2,2}\left(\frac{G_g}{2}\right)\right| \le \frac{1}{17694720}\left(|q_1(p,x)| + |q_2(p,x)|y + |q_3(p,x)|y^2 + |q_4(p,x,\delta)|\right) \le \frac{T(p,x,y)}{17694720}, \quad (89)$$

where

$$T(p,x,y) = m_1(p,x) + m_2(p,x)y + m_3(p,x)y^2 + m_4(p,x)\left(1-y^2\right), \quad (90)$$

with

$$m_1(p,x) = \left(4-p^2\right)\left[\left(4-p^2\right)\left(288x^4p^2 + 1600x^3p^2 + 5120x^3 + 96x^2p^2\right) + 1728x^3p^4 + 264xp^4 + 6912x^2p^2 + 648x^2p^4\right] + 45p^6,$$

$$m_2(p,x) = \left(4-p^2\right)\left(1-x^2\right)\left[\left(4-p^2\right)\left(5760xp + 1152x^2p\right) + 2160p^3 + 6912xp^3\right],$$

$$m_3(p,x) = \left(4-p^2\right)\left(1-x^2\right)\left[\left(4-p^2\right)\left(17280 + 1152x^2\right) + 6912xp^2\right],$$

$$m_4(p,x) = \left(4-p^2\right)\left(1-x^2\right)\left[18432x\left(4-p^2\right) + 6912p^2\right]. \quad (91)$$

To illustrate the sharp bounds of the given problem, we must maximize $T(p,x,y)$ in the closed cuboid $Y : [0,2] \times [0,1] \times [0,1]$.

(1) Interior points of cuboid Y

Let us choose $(p,x,y) \in (0,2) \times (0,1) \times (0,1)$. Then simple calculation yields

$$\frac{\partial T}{\partial y} = 144\left(4-p^2\right)\left(1-x^2\right)\left[16y(x-1)\left(6p^2 + (x-15)\left(4-p^2\right)\right) + 8p\left(x(x+5)\left(4-p^2\right) + p^2\left(6x + \frac{15}{8}\right)\right)\right]. \quad (92)$$

Putting $\partial T/\partial y = 0$, we obtain

$$y = \frac{8p\left(x(x+5)\left(4-p^2\right) + p^2(6x + 15/8)\right)}{16(x-1)(-6p^2 + (15-x)(4-p^2))} = y_0. \quad (93)$$

If $y_0 \in Y$ is a critical point, then $y_0 \in (0,1)$, and it is applicable only if

$$8px(x+5)\left(4-p^2\right)+p^3(48x+15)+16(1-x)(15-x)\cdot\left(4-p^2\right)<96p^2(1-x). \tag{94}$$

$$p^2>\frac{4(15-x)}{21-x}. \tag{95}$$

To check critical points existence, we must find solutions that fulfill both constraints (94) and (95).

Let $k(x)=(4(15-x))/(21-x)$. As $k'(x)<0$ for all $x\in(0,1)$, it is evident that $k(x)$ is decreasing in $(0,1)$. Hence $p^2>14/5$. It is easy to showcase that the inequality (94) does not hold in this scenario for all $x\in[2/5,1)$. As a result, $T(p,x,y)$ does not have a critical point in $(0,2)\times[2/5,1)\times(0,1)$. Assume a critical point $(\tilde{p},\tilde{x},\tilde{y})$ of T exists inside the interior of the cuboid Y, it must unquestionably fulfil that $\tilde{x}<2/5$.

From the arguments above, it is undeniable that $\tilde{p}^2\geq 292/103$ and $\tilde{y}\in(0,1)$. Now let us establish that $T(\tilde{p},\tilde{x},\tilde{y})<276480$. For $(p,x,y)\in((292/103)^{1/2},2)\times(0,2/5)\times(0,1)$, by invoking $x<2/5$ and $1-x^2<1$, it is not hard to observe that

$$\begin{aligned}
m_1(p,x)\leq{}&\left(4-p^2\right)\Bigg[\left(4-p^2\right)\Bigg(288\left(\frac{2}{5}\right)^4p^2+1600\left(\frac{2}{5}\right)^3p^2\\
&+5120\left(\frac{2}{5}\right)^3+96\left(\frac{2}{5}\right)^2p^2+1728\left(\frac{2}{5}\right)^3p^4\\
&+264\left(\frac{2}{5}\right)p^4+6912\left(\frac{2}{5}\right)^2p^2+648\left(\frac{2}{5}\right)^2p^4\Bigg)\Bigg]\\
&+45p^6=\left(4-p^2\right)\left(\frac{799232}{625}p^2+\frac{121712}{625}p^4\right.\\
&\left.+\frac{819200}{625}\right)+45p^6:=\Theta_1(p),\\
m_2(p,x)\leq{}&\left(4-p^2\right)\Bigg[\left(4-p^2\right)\left(5760\left(\frac{2}{5}\right)p+1152\left(\frac{2}{5}\right)^2p\right)\\
&+2160p^3+6912\left(\frac{2}{5}\right)p^3\Bigg]\\
={}&\left(4-p^2\right)\left(\frac{248832}{25}p+\frac{60912}{25}p^3\right):=\Theta_2(p),\\
m_3(p,x)\leq{}&\left(4-p^2\right)\Bigg[\left(4-p^2\right)\left(17280+1152\left(\frac{2}{5}\right)^2\right)\\
&+6912\left(\frac{2}{5}\right)p^2\Bigg]\\
={}&\left(4-p^2\right)\left(\frac{1746432}{25}-\frac{367488}{25}p^2\right):=\Theta_3(p),\\
m_4(p,x)\leq{}&\left(4-p^2\right)\left[18432\left(\frac{2}{5}\right)\left(4-p^2\right)+6912p^2\right]\\
={}&\left(4-p^2\right)\left(\frac{147456}{5}-\frac{2304}{5}p^2\right):=\Theta_4(p).
\end{aligned} \tag{96}$$

Therefore, we have

$$T(p,x,y)\leq\Theta_1(p)+\Theta_4(p)+\Theta_2(p)y+[\Theta_3(p)-\Theta_4(p)]y^2:=\Gamma(p,y). \tag{97}$$

Obviously, it can be seen that

$$\begin{aligned}
\frac{\partial\Gamma}{\partial y}&=\Theta_2(p)+2y[\Theta_3(p)-\Theta_4(p)],\\
\frac{\partial^2\Gamma}{\partial y^2}&=2[\Theta_3(p)-\Theta_4(p)]=2\left(4-p^2\right)\left(\frac{1009152}{25}-\frac{355968}{25}p^2\right).
\end{aligned} \tag{98}$$

Since $\Theta_3(p)-\Theta_4(p)\leq 0$ for $p\in((292/103)^{1/2},2)$, we obtain that $\partial^2\Gamma/\partial y^2\leq 0$ for $y\in(0,1)$, and thus, it follows that

$$\frac{\partial\Gamma}{\partial y}\geq\left.\frac{\partial\Gamma}{\partial y}\right|_{y=1}=\left(4-p^2\right)\left(-\frac{711936}{25}p^2+\frac{60912}{25}p^3+\frac{2018304}{25}+\frac{248832}{25}p\right)\geq 0. \tag{99}$$

Therefore, we have

$$\Gamma(p,y)\leq\Gamma(p,1)=\Theta_1(p)+\Theta_2(p)+\Theta_3(p):=\iota(p). \tag{100}$$

It is easy to be calculated that $\iota(p)$ attains its maximum value 74510.30 at $p\approx 1.68373$. Thus, we have

$$T(p,x,y)<276480,\ (p,x,y)\in\left(\sqrt{\frac{292}{103}},2\right)\times\left(0,\frac{2}{5}\right)\times(0,1). \tag{101}$$

Hence, $T(\tilde{p},\tilde{x},\tilde{y})<276480$. This implies that T is less than 276480 at all the critical points in the interior of Y. Therefore, T has no optimal solution in the interior of Y.

(2) Interior of all the six faces of cuboid Y :

(i) On the face $p=0$, $T(p,x,y)$ yields

$$b_1(x,y)=T(0,x,y)=2048\left(9\left(1-x^2\right)\cdot\left(16x+(x-15)(x-1)y^2\right)+40x^3\right),\ x,y\in(0,1). \tag{102}$$

Differentiating $b_1(x,y)$ with respect to y, we have

$$\frac{\partial b_1}{\partial y}=36864y\left(1-x^2\right)(x-15)(x-1),\ x,y\in(0,1). \tag{103}$$

Thus, $b_1(x,y)$ has no critical point in the interval $(0,1)\times(0,1)$.

(ii) On the face $p = 2, T(p, x, y)$ becomes

$$T(2, x, y) = 2880. \quad (104)$$

(iii) On the face $x = 0, T(p, x, y)$ reduces to

$$b_2(p, y) = T(p, 0, y) = (4 - p^2)(6912p^2 + 2160p^3 y - 24192y^2p^2 + 69120y^2) + 45p^6. \quad (105)$$

Differentiating $b_2(p, y)$ partially with respect to y, we have

$$\frac{\partial b_2}{\partial y} = (4 - p^2)(2160p^3 - 48384yp^2 + 138240y). \quad (106)$$

Solving $\partial b_2/\partial y = 0$, we obtain

$$y = \frac{5p^3}{16(7p^2 - 20)} = y_1. \quad (107)$$

For the given range of y,y_1 should belong to $(0, 1)$, which is possible only if $p > p_0, p_0 \approx 1.7609$. Also derivative of $b_2(p, y)$ partially with respect to p is

$$\frac{\partial b_2}{\partial p} = -4320p^4y - 13824p^3 + (4 - p^2) \cdot (-48384y^2p + 6480yp^2 + 13824p) + 48384y^2p^3 - 138240y^2p + 270p^5. \quad (108)$$

Putting the value of y in (108), with $\partial b_2/\partial p = 0$ and simplifying, we obtain

$$\frac{\partial b_2}{\partial p} = -27(49576p^7 + 35p^9 - 385072p^5 - 819200p + 983040p^3) = 0. \quad (109)$$

A calculation gives the solution of (109) in the interval $(0, 1)$, that is, $p \approx 1.3851$. Thus, $b_2(p, y)$ has no optimal point in the interval $(0, 2) \times (0, 1)$.

(iv) On the face $x = 1, T(p, x, y)$ becomes

$$b_3(p, y) = T(p, 1, y) = 45p^6 + (4 - p^2) \cdot ((4 - p^2)(1984p^2 + 5120) + 6912p^2 + 2640p^4). \quad (110)$$

Then

$$\frac{\partial b_3}{\partial p} = -3666p^5 - 28416p^3 + 36864p. \quad (111)$$

By setting $\partial b_3/\partial p = 0$, we get the critical point $p \approx 1.0639$ at which $b_3(p, y)$ attains its maximum value, which is given below

$$T(p, 1, y) \leq 92795.48842. \quad (112)$$

(v) On the face $y = 0, T(p, x, y)$ yields

$$b_4(p, x) = T(p, x, 0) = -128p^6x^3 + 288p^6x^4 - 552p^6x^2 + 19488p^4x - 264p^6x - 147456p^2x + 45p^6 + 4608p^2x^4 + 294912x - 19200p^4x^3 + 1536p^2x^2 + 132096p^2x^3 + 1824p^4x^2 - 6912p^4 + 27648p^2 - 2304p^4x^4 - 212992x^3. \quad (113)$$

A numerical computation shows that the solution for the system of equations

$$\frac{\partial b_4}{\partial p} = 0, \quad \frac{\partial b_4}{\partial x} = 0 \quad (114)$$

does not exists in the interval $(0, 2) \times (0, 1)$. Hence, $b_4(p, x)$ has no optimal solution in the interval $(0, 2) \times (0, 1)$.

(vi) On the face $y = 1, T(p, x, y)$ reduces to

$$b_5(p, x) = T(p, x, 1) = 45p^6 + 288p^6x^4 + 9216p^3x^4 + 1152p^5x^3 - 2160p^5 - 264p^6x - 1152p^5x^4 + 18432p^3x^3 + 3312p^5x^2 + 6144p^4x^3 - 128p^6x^3 - 43008p^2x^3 - 3456p^4x^4 - 552p^6x^2 - 1152p^5x - 138240p^2 - 21216p^4x^2 + 276480 + 13824p^2x^4 + 81920x^3 - 5856p^4x - 92160px^3 - 17856p^3x^2 - 18432px^4 - 258048x^2 + 92160px - 18432x^4 - 18432p^3x + 8640p^3 + 18432px^2 + 17280p^4 + 27648p^2x + 158208p^2x^2. \quad (115)$$

As in the above case, we conclude the same result for the face $y = 0$, that is, system of equations

$$\frac{\partial b_5}{\partial p} = 0, \quad \frac{\partial b_5}{\partial x} = 0 \quad (116)$$

has no solution in the interval $(0, 2) \times (0, 1)$.

(3) On the edges of cuboid Y :

(i) On the edge $x=0$ and $y=0, T(p,x,y)$ reduces to

$$T(p,0,0) = -6912p^4 + 45p^6 + 27648p^2 = b_6(p). \quad (117)$$

It follows that

$$b_6'(p) = -27648p^3 + 270p^5 + 55296p. \quad (118)$$

We see that $b_6'(p)=0$ for the critical point $p_0 \approx 1.4285$ at which $b_6(p)$ obtain its maximum value, which is given by

$$T(p,0,0) \le 28018.97. \quad (119)$$

(ii) On the edge $x=0$ and $y=1, T(p,x,y)$ becomes

$$T(p,0,1) = -2160p^5 - 138240p^2 + 17280p^4 + 45p^6 + 8640p^3 + 276480 = b_7(p). \quad (120)$$

Differentiating $b_7(p)$ with respect to p, we have

$$b_7'(p) = -10800p^4 - 276480p + 69120p^3 + 270p^5 + 25920p^2. \quad (121)$$

We know that $b_7'(p) < 0$ in $[0,2]$ follows that $b_7(p)$ is decreasing over $[0,2]$. Therefore, $b_7(p)$ gets its maxima at $p=0$. Hence

$$T(p,0,1) \le 276480. \quad (122)$$

(iii) On the edge $p=0$ and $x=0, T(p,x,y)$ reduces to

$$T(0,0,y) = 276480y^2 = b_8(y). \quad (123)$$

Noting that $b_8'(y) > 0$ in $[0,1]$ shows that $b_8(y)$ is increasing over $[0,1]$. Thus, $b_8(y)$ gets its maxima at $y=1$. Thus, we have

$$T(0,0,y) \le 276480. \quad (124)$$

(iv) On the edges $T(p,1,1)$ and $T(p,1,0)$

Since $T(p,1,y)$ is free of y, therefore

$$T(p,1,1) = T(p,1,0) = -7104p^4 - 611p^6 + 81920 + 18432p^2 = b_9(p). \quad (125)$$

Then

$$b_9'(p) = -28416p^3 - 3666p^5 + 36864p. \quad (126)$$

By putting $b_9'(p)=0$, we obtain the critical point $p_0 \approx 1.0639$ at which $b_9(p)$ attains its maximum value, which is given by

$$T(p,1,1) = T(p,1,0) \le 92795.48. \quad (127)$$

(v) On the edge $p=0$ and $x=1, T(p,x,y)$ becomes

$$T(0,1,y) = 81920. \quad (128)$$

(vi) On the edge $p=2, T(p,x,y)$ reduces to

$$T(2,x,y) = 2880. \quad (129)$$

$T(2,x,y)$ is independent of x and y; therefore

$$T(2,x,0) = T(2,x,1) = T(2,0,y) = T(2,1,y) = 2880. \quad (130)$$

(vii) On the edge $p=0$ and $y=1, T(p,x,y)$ takes the form

$$T(0,x,1) = 81920x^3 - 18432x^4 + 276480 - 258048x^2 = b_{10}(x). \quad (131)$$

It is clear that

$$b_{10}'(x) = 245760x^2 - 73728x^3 - 516096x. \quad (132)$$

We see that $b_{10}'(x) < 0$ in $[0,1]$ shows that $b_{10}(x)$ is decreasing over $[0,1]$. Thus, $b_{10}(x)$ gets its maxima at $x=0$. Hence, we have

$$T(0,x,1) \le 276480. \quad (133)$$

(viii) On the edge $p=0$ and $y=0, T(p,x,y)$ yields

$$T(0,x,0) = 294912x - 212992x^3 = b_{11}(x). \quad (134)$$

It follows that

$$b_{11}'(x) = 294912 - 638976x^2. \quad (135)$$

By taking $b'_{11}(x) = 0$, we obtain the critical point $x_0 \approx 0.6793$ at which $b_{11}(x)$ attains its maximum value, which is given by

$$T(0, x, 0) \leq 133568.833. \tag{136}$$

Hence, from the above cases we deduce that

$$T(p, x, y) \leq 276480 \text{ on } [0, 2] \times [0, 1] \times [0, 1]. \tag{137}$$

From (89)we have

$$\left| \mathcal{D}_{2,2}\left(\frac{G_g}{2}\right) \right| \leq \frac{T(p, x, y)}{17694720} \leq \frac{1}{64}. \tag{138}$$

If $g \in \mathcal{BT}_{\sin}$, then sharp bound for this Hankel determinant is determined by

$$\left| \mathcal{D}_{2,2}\left(\frac{G_g}{2}\right) \right| = \frac{1}{64} \approx 0.0156. \tag{139}$$

Thus, we have completed the proof. □

5. Conclusion

In our current investigation, we have considered a class $\mathcal{BT}_{\sin}$ of bounded turning functions associated with an eight-shaped domain. For such a class, we studied some interesting problems involving logarithmic coefficients. The Zalcman inequality, the Fekete-Szegö inequality, and the determinants $\mathcal{D}_{2,2}(G_g/2)$ and $\mathcal{D}_{2,1}(G_g/2)$ for the family $\mathcal{BT}_{\sin}$ have been studied here in this article. All the obtained results are proven to be the best possible.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work under Grant Code number: 22UQU4310396DSR23.

References

[1] L. De Branges, “A proof of the Bieberbach conjecture,” *Acta Mathematica*, vol. 154, no. 1-2, pp. 137–152, 1985.

[2] F. G. Avkhadiev and K. J. Wirths, *Schwarz-Pick Type Inequalities*, Springer Science & Business Media, 2009.

[3] C. H. FitzGerald and C. Pommerenke, “The de Branges theorem on univalent functions,” *Transactions of the American Mathematical Society*, vol. 290, no. 2, pp. 683–690, 1985.

[4] C. H. FitzGerald and C. Pommerenke, “A theorem of de Branges on univalent functions,” *Serdica*, vol. 13, no. 1, pp. 21–25, 1987.

[5] I. P. Kayumov, “On Brennan’s conjecture for a special class of functions,” *Mathematical Notes*, vol. 78, no. 3-4, pp. 498–502, 2005.

[6] V. V. Andreev and P. L. Duren, “Inequalities for logarithmic coefficients of univalent functions and their derivatives,” *Indiana University Mathematics journal*, vol. 37, no. 4, pp. 721–733, 1988.

[7] D. Alimohammadi, E. Analouei Adegani, T. Bulboacă, and N. E. Cho, “Logarithmic coefficient bounds and coefficient conjectures for classes associated with convex functions,” *Journal of Function Spaces*, vol. 2021, Article ID 6690027, 7 pages, 2021.

[8] Q. Deng, “On the logarithmic coefficients of Bazilevicˇ functions,” *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5889–5894, 2011.

[9] O. Roth, “A sharp inequality for the logarithmic coefficients of univalent functions,” *Proceedings of the American Mathematical Society*, vol. 135, no. 7, pp. 2051–2054, 2007.

[10] Z. Ye, “The logarithmic coefficients of close-to-convex functions,” *Bulletin of the Institute of Mathematics Academia Sinica (New Series)*, vol. 3, no. 3, pp. 445–452, 2008.

[11] M. Obradović, S. Ponnusamy, and K. J. Wirths, “Logarithmic coefficients and a coefficient conjecture for univalent functions,” *Monatshefte für Mathematik*, vol. 185, no. 3, pp. 489–501, 2018.

[12] D. Girela, “Logarithmic coefficients of univalent functions,” in *Annales-Academiae Scientiarum Fennicae Series A1 Mathematica*, vol. 25, no. 2pp. 337–350, Academia Scientiarum Fennica, 2000.

[13] C. Pommerenke, “On the coefficients and Hankel determinants of univalent functions,” *Journal of the London Mathematical Society*, vol. s1-41, no. 1, pp. 111–122, 1966.

[14] C. Pommerenke, “On the Hankel determinants of univalent functions,” *Mathematika*, vol. 14, no. 1, pp. 108–112, 1967.

[15] A. Janteng, S. A. Halim, and M. Darus, “Coefficient inequality for a function whose derivative has a positive real part,” *JIPAM - Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 2, pp. 1–5, 2006.

[16] A. Janteng, S. A. Halim, and M. Darus, “Hankel determinant for starlike and convex functions,” *International Journal of Mathematical Analysis*, vol. 1, no. 13, pp. 619–625, 2007.

[17] S. Altinkaya and S. Yalcin, “Third Hankel determinant for Bazilevic functions,” *Advances in Mathematics*, vol. 5, pp. 91–96, 2016.

[18] B. Kowalczyk, A. Lecko, and Y. J. Sim, “The sharp bound FOR the Hankel determinant of the third kind for convex functions,” *Bulletin of the Australian Mathematical Society*, vol. 97, no. 3, pp. 435–445, 2018.

[19] A. Lecko, Y. J. Sim, and B. Śmiarowska, “The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2,” *Complex Analysis and Operator Theory*, vol. 13, no. 5, pp. 2231–2238, 2019.

[20] O. S. Kwon, A. Lecko, and Y. J. Sim, “The bound of the Hankel determinant of the third kind for starlike functions,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 2, pp. 767–780, 2019.

[21] P. Zaprawa, “Third Hankel determinants for subclasses of univalent functions,” *Mediterranean Journal of Mathematics*, vol. 14, no. 1, 2017.

[22] O. M. Barukab, M. Arif, M. Abbas, and S. A. Khan, "Sharp bounds of the coefficient results for the family of bounded turning functions associated with a Petal-shaped domain," *Journal of Function Spaces*, vol. 2021, Article ID 5535629, 9 pages, 2021.

[23] Z.-G. Wang, M. Raza, M. Arif, and K. Ahmad, "On the third and fourth Hankel determinants for a subclass of analytic functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 45, no. 1, pp. 323–359, 2022.

[24] D. Răducanu and P. Zaprawa, "Deuxieme determinant de Hankel pour les fonctions presque convexes," *Comptes Rendus Mathematique*, vol. 355, no. 10, pp. 1063–1071, 2017.

[25] B. Kowalczyk and A. Lecko, "Second Hankel determinant of logarithmic coefficients of convex and starlike functions," *Bulletin of the Australian Mathematical Society*, vol. 105, no. 3, pp. 458–467, 2022.

[26] B. Kowalczyk and A. Lecko, "Second Hankel determinant of logarithmic coefficients of convex and starlike functions of order alpha," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 45, no. 2, pp. 727–740, 2022.

[27] W. C. Ma and D. Minda, "A unified treatment of some special classesof univalent functions," *Proceedings of the Conference on Complex Analysis*, vol. I, 1994, pp. 157–169, Tianjin, China, 1992.

[28] D. A. Brannan and W. E. Kirwan, "On some classes of bounded univalent functions," *Journal of the London Mathematical Society*, vol. 2, no. 1, pp. 431–443, 1969.

[29] J. Sokół and J. Stankiewicz, "Radius of convexity of some subclasses of strongly starlike functions," *Zeszyty naukowe Politechniki Rzeszowskiej. Matematyka*, vol. 19, pp. 101–105, 1996.

[30] K. Sharma, N. K. Jain, and V. Ravichandran, "Starlike functions associated with a cardioid," *Afrika Matematika*, vol. 27, no. 5-6, pp. 923–939, 2016.

[31] L. Shi, I. Ali, M. Arif, N. E. Cho, S. Hussain, and H. Khan, "A study of third Hankel determinant problem for certain subfamilies of analytic functions involving cardioid domain," *Mathematics*, vol. 7, no. 5, p. 418, 2019.

[32] S. S. Kumar and K. Arora, "Starlike functions associated with a petal shaped domain," 2020, https://arxiv.org/abs/2010.10072.

[33] R. Mendiratta, S. Nagpal, and V. Ravichandran, "On a subclass of strongly starlike functions associated with exponential function," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 38, no. 1, pp. 365–386, 2015.

[34] L. Shi, H. M. Srivastava, M. Arif, S. Hussain, and H. Khan, "An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function," *Symmetry*, vol. 11, no. 5, p. 598, 2019.

[35] K. Bano and M. Raza, "Starlike functions associated with cosine functions," *Bulletin of the Iranian Mathematical Society*, vol. 47, no. 5, pp. 1513–1532, 2021.

[36] A. Alotaibi, M. Arif, M. A. Alghamdi, and S. Hussain, "Starlikness associated with cosine hyperbolic function," *Mathematics*, vol. 8, no. 7, p. 1118, 2020.

[37] K. Ullah, S. Zainab, M. Arif, M. Darus, and M. Shutaywi, "Radius problems for starlike functions associated with the tan hyperbolic function," *Journal of Function Spaces*, vol. 2021, Article ID 9967640, 15 pages, 2021.

[38] K. Ullah, H. M. Srivastava, A. Rafiq, M. Arif, and S. Arjika, "A study of sharp coefficient bounds for a new subfamily of starlike functions," *Journal of Inequalities and Applications*, vol. 2021, no. 1, Article ID 194, p. 20, 2021.

[39] N. E. Cho, V. Kumar, S. S. Kumar, and V. Ravichandran, "Radius problems for starlike functions associated with the sine function," *Bulletin of the Iranian Mathematical Society*, vol. 45, no. 1, pp. 213–232, 2019.

[40] M. Arif, M. Raza, H. Tang, S. Hussain, and H. Khan, "Hankel determinant of order three for familiar subsets of analytic functions related with sine function," *Open Mathematics*, vol. 17, no. 1, pp. 1615–1630, 2019.

[41] O. S. Kwon, A. Lecko, and Y. J. Sim, "On the fourth coefficient of functions in the Carathéodory class," *Computational Methods and Function Theory*, vol. 18, no. 2, pp. 307–314, 2018.

[42] C. Carathéodory, "Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen," *Rendiconti del Circolo Matematico di Palermo*, vol. 32, no. 1, pp. 193–217, 1911.

[43] R. J. Libera and E. J. Zlotkiewicz, "Coefficient bounds for the inverse of a function with derivative in P," *Proceedings of the American Mathematical Society*, vol. 87, no. 2, pp. 251–257, 1983.

[44] C. Pommerenke, *Univalent Function, Math*, Lehrbucher, vandenhoeck and Ruprecht, Gottingen, 1975.

[45] V. Ravichandran and S. Verma, "Borne pour le cinquieme coefficient des fonctions etoilees," *Comptes Rendus Mathematique*, vol. 353, no. 6, pp. 505–510, 2015.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 8525804, 9 pages
https://doi.org/10.1155/2022/8525804

Research Article

On Some Numerical Radius Inequalities Involving Generalized Aluthge Transform

Javariya Hyder,[1] Muhammad Saeed Akram,[2] Abdulaziz S. Alofi,[3] and Dilruba Akter[4]

[1] *Department of Mathematics, Khwaja Fareed University of Engineering and Information Technology, Rahim Yar Khan, Pakistan*
[2] *Department of Mathematics, Faculty of Science, Ghazi University, 32200 Dera Ghazi Khan, Pakistan*
[3] *Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[4] *Department of Applied Mathematics, Gono Bishwabidyalay University, Dhaka, Bangladesh*

Correspondence should be addressed to Dilruba Akter; dilruba.amath@gonouniversity.edu.bd

Received 23 June 2022; Revised 25 July 2022; Accepted 5 August 2022; Published 1 September 2022

Academic Editor: Sarfraz Nawaz Malik

Let S be any bounded linear operator defined on a complex Hilbert space $\mathscr{H}$. In this paper, we present some numerical radius inequalities involving the generalized Aluthge transform to attain upper bounds for numerical radius. Numerical computations are carried out for some particular cases of generalized Aluthge transform.

1. Introduction

Mathematical inequalities play an essential role in developing various areas of pure and applied mathematics. The usefulness of mathematical inequalities is to estimate the solutions of real-life problems in engineering and other fields of science. In mathematics, particularly in functional analysis, the study of numerical radius inequalities has become the attention of many researchers due to the applications of numerical radius in operator theory and numerical analysis, etc. (see [1–4]). Various mathematicians have developed number of numerical radius inequalities to estimate the upper and lower bounds for numerical radius. It is interesting for researchers to get the refinements and generalization of these inequalities. The aim of this paper is to study the generalization and refinements of existing inequalities for numerical radius. Now, we recall some notions to proceed our work.

Let $\mathscr{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators defined on a complex Hilbert space $\mathscr{H}$. For $S \in \mathscr{B}(\mathscr{H})$, the usual operator norm is defined as

$$\|S\| = \sup \{\|Sx\| : x \in \mathscr{H}, \|x\| = 1\}, \tag{1}$$

and the numerical radius is defined as

$$w(S) = \sup \{|\langle Sx, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}. \tag{2}$$

It is well known that the numerical radius defines an equivalent operator norm on $\mathscr{B}(\mathscr{H})$, and for $S \in \mathscr{B}(\mathscr{H})$, we have

$$\frac{1}{2}\|S\| \le w(S) \le \|S\|. \tag{3}$$

Many authors worked on the refinement of inequality (3) (see [5–7]). Kittaneh developed the following upper bound of numerical radius:

$$w(S) \le \frac{1}{2}\left(\|S\| + \left\|S^2\right\|^{1/2}\right), \tag{4}$$

which is a refinement of inequality (3) (see [5]). For $S \in \mathscr{B}(\mathscr{H})$ having polar decomposition $S = U|S|$ where U is a partial isometry and $|S| = (S^* S)^{1/2}$, the Aluthge transform is defined as

$$\tilde{S} = |S|^{1/2} U |S|^{1/2}, \tag{5}$$

see [8]. Yamazaki developed an upper bound of numerical radius involving Aluthge transform given by

$$w(S) \leq \frac{1}{2}\left(\|S\| + w\left(\tilde{S}\right)\right), \tag{6}$$

which is an improvement of bounds (3) and (4) (see [9]). Bhunia et al. developed a bound of the numerical radius given by

$$w^2(S) \leq \frac{1}{4}\left(\|S\|^2 + w^2\left(\tilde{S}\right) + w\left(|S|\tilde{S} + \tilde{S}|S|\right)\right), \tag{7}$$

and proved that it is a refinement of bound (6) (see [10]). Okubo introduced a generalization of Aluthge transform which is defined as

$$\widetilde{S_\lambda} = |S|^\lambda U |S|^{1-\lambda}, \tag{8}$$

for $\lambda \in [0, 1]$, known as λ-Aluthge transform (see [11]). After that, a number of numerical radius inequalities were established involving λ-Aluthge transform (see [12–14]). Abu Omar and Kittaneh using λ-Aluthge transform generalized bound (6) given by

$$w(S) \leq \frac{1}{2}\left(\|S\| + w\left(\widetilde{S_\lambda}\right)\right), \tag{9}$$

see [12]. Shebrawi and Bakherad introduced another generalization of Aluthge transform which is defined as

$$\widetilde{S_{f,g}} = f(|S|)Ug(|S|), \tag{10}$$

where f and g are nonnegative and continuous functions such that $f(x)g(x) = x (x \geq 0)$, known as generalized Aluthge transform. The authors generalized inequality (9) given by

$$w(S) \leq \frac{1}{2}\left(\|S\| + w\left(\widetilde{S_{f,g}}\right)\right), \tag{11}$$

$$w(S) \leq \frac{1}{4}\left\|f^2(|S|) + g^2(|S|)\right\| + \frac{1}{2} w\left(\widetilde{S_{f,g}}\right), \tag{12}$$

see [15].

In this paper, we establish some new inequalities of the numerical radius using generalized Aluthge transform. Specifically, we generalize inequality (7) and improve the inequalities (3), (4), and (12). Some examples of operators are presented for which the bounds of numerical radius are computed from these inequalities for some choices of f, g in (10).

2. Main Results

Now, we recall a lemma that will be used to achieve our goals.

Lemma 1 (see [9]). *Let $S \in \mathcal{B}(\mathcal{H})$; then, for $\theta \in \mathbb{R}$, we have*

$$w(S) = \sup_{\theta\in\mathbb{R}} \|H_\theta\| = \sup_{\theta\in\mathbb{R}} \left\|\operatorname{Re}\left(e^{i\theta}S\right)\right\|, \tag{13}$$

where $H_\theta = (e^{i\theta}S + e^{-i\theta}S^*)/2$.

Polarization identity: [15] For each $x_1, y_1 \in \mathcal{H}$, we have

$$\begin{aligned}\langle x_1, y_1\rangle = \frac{1}{4}\big(&\|x_1 + y_1\|^2 - \|x_1 - y_1\|^2 \\ &+ i\|x_1 + iy_1\|^2 - i\|x_1 - iy_1\|^2\big).\end{aligned} \tag{14}$$

Now, we establish an inequality of numerical radius which is a generalization of inequality (7) and a refinement of inequality (12).

Theorem 2. Let $S \in \mathcal{B}(\mathcal{H})$. *Then,*

$$w^2(S) \leq \frac{1}{16}\|Q\|^2 + \frac{1}{4} w^2\left(\widetilde{S_{f,g}}\right) + \frac{1}{8} w\left(Q\widetilde{S_{f,g}} + \widetilde{S_{f,g}}Q\right), \tag{15}$$

where $Q = (f(|S|))^2 + (g(|S|))^2$ and f, g is nonnegative continuous functions defined on $[0, \infty)$ such that $f(t)g(t) = t$.

Proof. Let $S = U|S|$ be the polar decomposition of S. Then, by polarization identity, we have

$$\begin{aligned}\left\langle e^{i\theta}Sx, x\right\rangle &= \left\langle e^{i\theta}U|S|x, x\right\rangle = \left\langle e^{i\theta}Ug(|S|)f(|S|)x, x\right\rangle \\ &= \left\langle e^{i\theta}f(|S|)x, g(|S|)U^*x\right\rangle \\ &= \frac{1}{4}\left(\left\|e^{2i\theta}f(|S|)x + g(|S|)U^*x\right\|^2\right. \\ &\quad - \left\|e^{2i\theta}f(|S|)x - g(|S|)U^*x\right\|^2 \\ &\quad + i\left\|e^{2i\theta}f(|S|)x + ig(|S|)U^*x\right\|^2 \\ &\quad \left.- i\left\|e^{2i\theta}f(|S|)x - ig(|S|)U^*x\right\|^2\right).\end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned}&\operatorname{Re}\left\langle e^{i\theta}Sx, x\right\rangle \\ &\quad= \frac{1}{4}\left\|e^{2i\theta}f(|S|) + g(|S|)U^*x\right\|^2 \\ &\qquad - \frac{1}{4}\left\|e^{2i\theta}f(|S|) - g(|S|)U^*x\right\|^2 \\ &\quad\leq \frac{1}{4}\left\|e^{2i\theta}f(|S|) + g(|S|)U^*\right\|^2 \\ &\quad= \frac{1}{4}\left\|\left(e^{2i\theta}f(|S|) + g(|S|)U^*\right)\left(e^{2i\theta}f(|S|) + g(|S|)U^*\right)^*\right\| \\ &\quad= \frac{1}{4}\left\|\left(e^{2i\theta}f(|S|) + g(|S|)U^*\right)\left(e^{-2i\theta}f(|S|) + Ug(|S|)\right)\right\|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\Big\| (f(|S|))^2 + (g(|S|))^2 + e^{2i\theta} f(|S|)Ug(|S|) \\
&\quad + e^{-2i\theta} g(|S|)U^* f(|S|) \Big\| \\
&= \frac{1}{4}\Big\| (f(|S|))^2 + (g(|S|))^2 + e^{2i\theta}\widetilde{S_{f,g}} + e^{-2i\theta}\left(\widetilde{S_{f,g}}\right)^* \Big\| \\
&= \frac{1}{4}\Big\| Q + 2\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right) \Big\| \\
&= \frac{1}{4}\Big\| \left(Q + 2\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right)\right)^2 \Big\|^{1/2} \\
&= \frac{1}{4}\Big\| Q^2 + 4\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right)^2 + 2Q\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right) \\
&\quad + 2\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right)Q \Big\|^{1/2} \\
&= \frac{1}{4}\Big\| Q^2 + 4\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right)^2 \\
&\quad + 2\,\mathrm{Re}\left(e^{2i\theta}\left(Q\widetilde{S_{f,g}} + \widetilde{S_{f,g}}Q\right)\right) \Big\|^{1/2} \\
&\le \frac{1}{4}\left(\|Q\|^2 + 4\Big\|\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right)\Big\|^2 \right. \\
&\quad \left. + 2\Big\|\mathrm{Re}\left(e^{2i\theta}\left(Q\widetilde{S_{f,g}} + \widetilde{S_{f,g}}Q\right)\right)\Big\| \right)^{1/2}.
\end{aligned} \tag{17}
$$

Now taking supremum over $\theta \in \mathbb{R}$ in the last inequality and then applying Lemma 1, we obtain

$$
w^2(S) \le \frac{1}{16}\|Q\|^2 + \frac{1}{4}w^2\left(\widetilde{S_{f,g}}\right) + \frac{1}{8}w\left(Q\widetilde{S_{f,g}} + \widetilde{S_{f,g}}Q\right), \tag{18}
$$

as desired. □

Theorem 2 includes some particular cases of generalized Aluthge transform for different choices of continuous functions f and g in (10) as follows.

Corollary 3. Let $S \in \mathscr{B}(\mathscr{H})$. *Then, for* $\lambda \in [0, 1]$, *we have*

$$
w^2(S) \le \frac{1}{16}\|Q\|^2 + \frac{1}{4}w^2\left(\widetilde{S}_e\right) + \frac{1}{8}w\left(Q\widetilde{S}_e + \widetilde{S}_e Q\right), \tag{19}
$$

where $Q = \left(e^{|S|^\lambda}\right)^2 + \left(|S|e^{-|S|^\lambda}\right)^2$ *and* $\widetilde{S}_e = e^{|S|^\lambda}U|S|e^{|S|^{-\lambda}}$.

Corollary 4. Let $S \in \mathscr{B}(\mathscr{H})$. *Then, for* $\lambda \in [0, 1]$, *we have*

$$
w^2(S) \le \frac{1}{16}\|Q\|^2 + \frac{1}{4}w^2\left(\widetilde{S}_e\right) + \frac{1}{8}w\left(Q\widetilde{S}_e + \widetilde{S}_e Q\right), \tag{20}
$$

where $Q = \left(|S|e^{-|S|^\lambda}\right)^2 + \left(e^{|S|^\lambda}\right)^2$ *and* $\widetilde{S}_e = |S|e^{|S|^{-\lambda}}Ue^{|S|^\lambda}$.

Corollary 5. Let $S \in \mathscr{B}(\mathscr{H})$. *Then, for* $\lambda \in [0, 1]$, *we have*

$$
w^2(S) \le \frac{1}{16}\|Q\|^2 + \frac{1}{4}w^2\left(\widetilde{S}_\lambda\right) + \frac{1}{8}w\left(Q\widetilde{S}_\lambda + \widetilde{S}_\lambda Q\right), \tag{21}
$$

where $Q = |S|^{2\lambda} + |S|^{2(1-\lambda)}$. *In particular,*

$$
w^2(S) \le \frac{1}{4}\left(\|S\|^2 + w^2\left(\tilde{S}\right) + w\left(|S|\tilde{S} + \tilde{S}|S|\right)\right). \tag{22}
$$

Remark 6. By using the inequality

$$
w(YA + AY^*) \le 2\|Y\|w(A), \tag{23}
$$

for all $Y, A \in \mathscr{B}(\mathscr{H})$ (see [16]) in inequality (15) obtained in Theorem 2, we have

$$
\begin{aligned}
w^2(S) &\le \frac{1}{16}\|Q\|^2 + \frac{1}{4}w^2\left(\widetilde{S_{f,g}}\right) + \frac{1}{8}\left(2\|Q\|w\left(\widetilde{S_{f,g}}\right)\right) \\
&= \left(\frac{1}{4}\|Q\| + \frac{1}{2}w\left(\widetilde{S_{f,g}}\right)\right)^2.
\end{aligned} \tag{24}
$$

Hence,

$$
w(S) \le \frac{1}{4}\|Q\| + \frac{1}{2}w\left(\widetilde{S_{f,g}}\right), \tag{25}
$$

where $Q = (f(|S|))^2 + (g(|S|))^2$. Thus, inequality (15) obtained in Theorem 2 is better than inequality (12).

Remark 7. For continuous functions f and g in (10), if $\widetilde{S_{f,g}} = 0$, then inequality (15) becomes

$$
w(S) \le \frac{1}{4}\left\|(f(|S|))^2 + (g(|S|))^2\right\|. \tag{26}
$$

In particular, if we take $f(|S|) = |S|^{1/2}$ and $g(|S|) = |S|^{1/2}$, for this choice of f and g if $\widetilde{S_{f,g}} = 0$, then inequality (15) becomes $w(S) \le 1/2\|S\|$, and combined with inequality (3), we get $w(S) = 1/2\|S\|$.

Theorem 8. Let $S \in \mathscr{B}(\mathscr{H})$. *Then, we have*

$$
\begin{aligned}
w^4(S) \le \frac{1}{4}&\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|\|g(|S|)\|\right)^2 \\
&+ \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2,
\end{aligned} \tag{27}
$$

where $P = S^*S + SS^*$ *and* f, g *is nonnegative continuous functions defined on* $[0, \infty)$ *such that* $f(t)g(t) = t$.

Proof. Since $H_\theta = (e^{i\theta}S + e^{-i\theta}S^*)/2$ for all $\theta \in \mathbb{R}$, then we have

$$
\begin{aligned}
{H_\theta}^2 &= \frac{1}{4}\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2} + SS^* + S^*S\right) \\
&= \frac{1}{4}\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2} + P\right),
\end{aligned} \tag{28}
$$

which yields

$$
\begin{aligned}
H_\theta{}^4 &= \frac{1}{16}\left(\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2}\right)^2 + e^{2i\theta}\left(S^2P + PS^2\right)\right.\\
&\quad \left. + e^{-2i\theta}\left(S^{*2}P + PS^{*2}\right) + P^2\right)\\
&= \frac{1}{16}\left(\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2}\right)^2\right.\\
&\quad \left. + 2\left(\mathrm{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right) + P^2\right)\\
&= \frac{1}{16}\left(\left(e^{2i\theta}U|S|U|S| + e^{-2i\theta}|S|U^*|S|U^*\right)^2\right.\\
&\quad \left. + 2\left(\mathrm{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right) + P^2\right)\\
&= \frac{1}{16}\left(\left(e^{2i\theta}Ug(|S|)f(|S|)Ug(|S|)f(|S|)\right.\right.\\
&\quad \left. + e^{-2i\theta}f(|S|)g(|S|)U^*f(|S|)g(|S|)U^*\right)^2\\
&\quad \left. + 2\left(\mathrm{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right) + P^2\right)\\
&= \frac{1}{16}\left(\left(e^{2i\theta}Ug(|S|)\widetilde{S_{f,g}}f(|S|) + e^{-2i\theta}f(|S|)\widetilde{S_{f,g}}^*g(|S|)U^*\right)^2\right.\\
&\quad \left. + 2\left(\mathrm{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right) + P^2\right).
\end{aligned}
\tag{29}
$$

□

Since H_θ is self-adjoint, so $\|H_\theta{}^4\| = \|H_\theta\|^4$. Hence, using the properties of operator norm $\|\cdot\|$ on $\mathscr{B}(\mathscr{H})$, we have

$$
\begin{aligned}
\|H_\theta\|^4 &\le \frac{1}{16}\left(\left(\left\|e^{2i\theta}Ug(|S|)\widetilde{S_{f,g}}f(|S|)\right.\right.\right.\\
&\quad \left.\left. + e^{-2i\theta}f(|S|)\widetilde{S_{f,g}}^*g(|S|)U^*\right\|\right)^2\\
&\quad \left. + 2\left\|\mathrm{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right)\\
&\le \frac{1}{16}\left(\left(\left\|Ug(|S|)\widetilde{S_{f,g}}f(|S|)\right\| + \left\|f(|S|)\widetilde{S_{f,g}}^*g(|S|)U^*\right\|\right)^2\right.\\
&\quad \left. + 2\left\|\mathrm{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right)\\
&\le \frac{1}{16}\left(\left(\|U\|\|g(|S|)\|\left\|\widetilde{S_{f,g}}\right\|\|f(|S|)\|\right.\right.\\
&\quad \left. + \|f(|S|)\|\left\|\widetilde{S_{f,g}}^*\right\|\|g(|S|)\|\|U^*\|\right)^2\\
&\quad \left. + 2\left\|\mathrm{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right)\\
&= \frac{1}{16}\left(4\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|\|g(|S|)\|\right)^2\right.\\
&\quad \left. + 2\left\|\mathrm{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right),
\end{aligned}
\tag{30}
$$

where the equality holds because $\|S\| = \|S^*\|$. Now taking supremum over $\theta \in \mathbb{R}$ in last equality, then applying Lemma 1, yields

$$
\begin{aligned}
w^4(S) &\le \frac{1}{4}\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|\|g(|S|)\|\right)^2\\
&\quad + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|.
\end{aligned}
\tag{31}
$$

For different choices of f and g in (10), we obtain the following inequalities of numerical radius from Theorem 8.

Corollary 9. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0, 1]$*, we have*

$$
\begin{aligned}
w^4(S) &\le \frac{1}{4}\left(\left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|\left\||S|e^{-|S|^\lambda}\right\|\right)^2\\
&\quad + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2,
\end{aligned}
\tag{32}
$$

where $P = S^*S + SS^*$ *and* $\widetilde{S_e} = e^{|S|^\lambda}U|S|e^{-|S|^\lambda}$.

Corollary 10. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0, 1]$*, we have*

$$
\begin{aligned}
w^4(S) &\le \frac{1}{4}\left(\left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|\left\||S|e^{-|S|^\lambda}\right\|\right)^2\\
&\quad + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2,
\end{aligned}
\tag{33}
$$

where $P = S^*S + SS^*$ *and* $\widetilde{S_e} = |S|e^{|S|^{-\lambda}}Ue^{|S|^\lambda}$.

Corollary 11. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0, 1]$*, we have*

$$
w^4(S) \le \frac{1}{4}\left(\|S\|\left\|\widetilde{S}_\lambda\right\|\right)^2 + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2, \tag{34}
$$

where $P = S^*S + SS^*$*. In particular,*

$$
w^4(S) \le \frac{1}{4}\left(\|S\|\|\tilde{S}\|\right)^2 + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2. \tag{35}
$$

Remark 12. It is easy to check that $\|\tilde{S}\| \le \|S^2\|^{1/2}$ (see [9] for details). Using the following inequality

$$
w(YA + AY^*) \le 2\|Y\|w(A), \tag{36}
$$

for all $Y, A \in \mathscr{B}(\mathscr{H})$ (see [14]), Corollary 11 yields

$$
\begin{aligned}
w^4(S) &\le \frac{1}{4}\left(\|S\|\|\tilde{S}\|\right)^2 + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2\\
&\le \frac{1}{4}\left(\|S\|\left\|S^2\right\|^{1/2}\right)^2 + \frac{1}{4}w\left(S^2\right)\|P\| + \frac{1}{16}\|P\|^2\\
&\le \frac{1}{4}\left(\|S\|\left\|S^2\right\|^{1/2}\right)^2 + \frac{1}{4}\left\|S^2\right\|\|P\| + \frac{1}{16}\|P\|^2\\
&= \left(\frac{1}{2}\|S\|\left\|S^2\right\|^{1/2} + \frac{1}{4}\|P\|\right)^2.
\end{aligned}
\tag{37}
$$

We know that $\|S^*S + SS^*\| \le \|S^2\| + \|S\|^2$ (see [17]). Hence,

$$\begin{aligned}\frac{1}{2}\|S\|\left\|S^2\right\|^{1/2} + \frac{1}{4}\|S^*S + SS^*\| &\le \frac{1}{2}\|S\|\left\|S^2\right\|^{1/2} + \frac{1}{4}\left\|S^2\right\| + \frac{1}{4}\|S\|^2 \\ &= \left(\frac{1}{2}\left\|S^2\right\|^{1/2} + \frac{1}{2}\|S\|\right)^2.\end{aligned} \quad (38)$$

Thus, the bound given in Corollary 11 is better than bound (4).

Remark 13. If $\widetilde{S_{f,g}} = 0$ in inequality (27) obtained in Theorem 8 for different choices of f and g in (10), then inequality (27) becomes

$$w^4(S) \le \frac{1}{8} w\left(S^2 P + PS^2\right) + \frac{1}{16}\|P\|^2. \quad (39)$$

If $S^2 = 0$ and $\widetilde{S_{f,g}} = 0$ are equivalent conditions, then inequality (27) becomes

$$w^2(S) = \frac{1}{4}\|P\|. \quad (40)$$

Theorem 14. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, we have*

$$w^3(S) \le \frac{1}{4}\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\| + w\left(S^2S^* + S^*S^2 + SS^*S\right)\right), \quad (41)$$

where f,g is nonnegative continuous functions defined on $[0, \infty)$ such that $f(t)g(t) = t$.

Proof. Since $H_\theta = (e^{i\theta}S + e^{-i\theta}S^*)/2$ for all $\theta \in \mathbb{R}$, then we have

$$H_\theta{}^2 = \frac{1}{4}\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2} + SS^* + S^*S\right), \quad (42)$$

which implies

$$\begin{aligned}H_\theta{}^3 &= \frac{1}{8}\left(\left(e^{i\theta}S + e^{-i\theta}S^*\right)\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2} + SS^* + S^*S\right)\right) \\ &= \frac{1}{8}\left(e^{3i\theta}S^3 + e^{-3i\theta}S^{*3} + 2\,\mathrm{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right).\end{aligned} \quad (43)$$

□

In the last equality, $\mathrm{Re}\,(e^{i\theta}(S^2S^* + S^*S^2 + SS^*S)) = (e^{i\theta}(S^*S^2 + S^{*2}S + S^*SS^*) + e^{-i\theta}(S^2S^* + S^*S^2 + SS^*S))/2$. Hence,

$$\begin{aligned}\|H_\theta\|^3 &\le \frac{1}{8}\left\|e^{3i\theta}Ug(|S|)\left(\widetilde{S_{f,g}}\right)^2 f(|S|)\right\| + \left\|e^{-3i\theta}f(|S|)\left(\widetilde{S_{f,g}}^*\right)^2 g(|S|)U^*\right\| + \left\|2\,\mathrm{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\| \\ &\le \frac{1}{8}\left(\|U\|\|g(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|f(|S|)\| + \|f(|S|)\|\left\|\widetilde{S_{f,g}}^*\right\|^2\|g(|S|)\|\|U^*\| + 2\left\|\mathrm{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right) \\ &= \frac{1}{4}\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\| + \left\|\mathrm{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right).\end{aligned} \quad (44)$$

The first inequality holds because $\|A_1 + A_2\| \le \|A_1\| + \|A_2\|$ where $A, A_1, A_2 \in \mathscr{B}(\mathscr{H})$, the second inequality holds because $\|A_1A_2\| \le \|A_1\|\|A_2\|$ and $\|A^n\| \le \|A\|^n \forall n \in \mathbb{N}$, and the third equality holds because $\|A\| = \|A^*\|$ and $\|U\| = \|U^*\| = 1$. Now taking supremum over $\theta \in \mathbb{R}$ in above equality the using Lemma 1, we obtain

$$w^3(S) \le \frac{1}{4}\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\| + w\left(S^2S^* + S^*S^2 + SS^*S\right)\right), \quad (45)$$

as desired.

Corollary 15. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for $\lambda \in [0, 1]$, we have*

$$w^3(S) \le \frac{1}{4}\left(\left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|^2\left\||S|e^{-|S|^\lambda}\right\| + w\left(S^2S^* + S^*S^2 + SS^*S\right)\right), \quad (46)$$

where $\widetilde{S_e} = e^{|S|^\lambda}U|S|e^{-|S|^\lambda}$.

Corollary 16. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for $\lambda \in [0, 1]$, we have*

$$w^3(S) \le \frac{1}{4}\left(\left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|^2\left\||S|e^{-|S|^\lambda}\right\| + w\left(S^2S^* + S^*S^2 + SS^*S\right)\right), \quad (47)$$

where $\widetilde{S_e} = |S|e^{|S|^{-\lambda}}Ue^{|S|^\lambda}$.

Corollary 17. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for $\lambda \in [0, 1]$, we have*

$$w^3(S) \le \frac{1}{4}\left(\|S\|\left\|\widetilde{S_\lambda}\right\|^2 + w\left(S^2S^* + S^*S^2 + SS^*S\right)\right). \quad (48)$$

In particular,

$$w^3(S) \le \frac{1}{4}\left(\|S\|\|\tilde{S}\|^2 + w\left(S^2S^* + S^*S^2 + SS^*S\right)\right). \quad (49)$$

Remark 18. Yan et al. proved that

$$w^2(S) \le \frac{1}{2}\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|\|g(|S|)\| + \frac{1}{4}\|S^*S + SS^*\|, \quad (50)$$

see [18]. Inequality (41) obtained in Theorem 14 gives better bounds of numerical radius of S for different choices of f and g in (10) when

$$S = \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}. \quad (51)$$

Then, $S = U|S|$ is a polar decomposition of S, where

$$|S| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad (52)$$

and

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (53)$$

is partial isometry.

Bounds (41) and (50) are computed for some choices of f and g in (10) for the given S in Table 1, whereas the numerical radius of S is

$$w(S) = 2.9154. \quad (54)$$

The spectral radius of an operator is defined as

$$r(S) = \sup_{\|x\|=1}\{|\lambda|: \lambda \in \sigma(S)\}, \quad (55)$$

where r denotes the spectral radius. For further inforamtion on spectral radius, see [19]. The following theorem will be used to develop the next inequality of numerical radius.

Theorem 19 (see [17]). *Let $M_1, M_2, N_1, N_2 \in \mathscr{B}(\mathscr{H})$. Then,*

$$\begin{aligned} r(M_1N_1 + M_2N_2) &\le \frac{1}{2}(w(N_1M_1) + w(N_2M_2)) \\ &+ \frac{1}{2}\sqrt{w(N_1M_1) - w(N_2M_2) + 4\|N_1M_2\|\|N_2M_1\|}. \end{aligned} \quad (56)$$

Table 1: Bounds (41) and (50) for different choices of f and g in (10).

(f, g)	Bound (50)	Bound (41)
$(\lvert S\rvert^{1/2}, \lvert S\rvert^{1/2})$	4.2640	3.2655
$(\lvert S\rvert^{1/7}, \lvert S\rvert^{6/7})$	4.0704	3.0724
$\left(e^{\lvert S\rvert^{1/10}}, \lvert S\rvert e^{-\lvert S\rvert^{1/10}}\right)$	4.3582	3.3622
$\left(e^{\lvert S\rvert^{1/10}}, \lvert S\rvert e^{-\lvert S\rvert^{1/10}}\right)$	4.2055	3.2031

Theorem 20. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then,*

$$\begin{aligned} w^3(S) \le \frac{1}{8}\left(w\left(\widetilde{S_{f,g}}^3\right) + \|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\|\right) \\ + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right), \end{aligned} \quad (57)$$

where f,g is nonnegative continuous functions defined on $[0,\infty)$ such that $f(t)g(t) = t$.

Proof. Since $H_\theta = (e^{i\theta}S + e^{-i\theta}S^*)/2$ for all $\theta \in \mathbb{R}$, then we have

$$H_\theta{}^2 = \frac{1}{4}\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2} + SS^* + S^*S\right), \quad (58)$$

which implies

$$\begin{aligned} H_\theta{}^3 &= \frac{1}{8}\left(e^{i\theta}S + e^{-i\theta}S^*\right)\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2} + SS^* + S^*S\right) \\ H_\theta{}^3 &= \frac{1}{8}\left(e^{3i\theta}Ug(|S|)\left(\widetilde{S_{f,g}}\right)^2 f(|S|)\right. \\ &\quad + e^{-3i\theta}f(|S|)\left(\widetilde{S_{f,g}}^*\right)^2 g(|S|)U^* \\ &\quad \left. + 2\,\mathrm{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right). \end{aligned} \quad (59)$$

□

Now by using the properties of operator norm $\|\cdot\|$ on $\mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned} \|H_\theta\|^3 \le \frac{1}{8}\left(\left\|e^{3i\theta}Ug(|S|)\left(\widetilde{S_{f,g}}\right)^2 f(|S|)\right.\right. \\ \left. + e^{-3i\theta}f(|S|)\left(\widetilde{S_{f,g}}^*\right)^2 g(|S|)U^*\right\| \\ \left. + 2\left\|\mathrm{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right) \end{aligned}$$

$$= \frac{1}{8}\left(r\left(e^{3i\theta}Ug(|S|)\left(\widetilde{S_{f,g}}\right)^2 f(|S|) + e^{-3i\theta}f(|S|)\left(\widetilde{S_{f,g}}^{*}\right)^2 g(|S|)U^*\right) + 2\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right) = \frac{1}{8}\left(r(M_1N_1 + M_2N_2) + 2\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right), \tag{60}$$

where $M_1 = e^{3i\theta}Ug(|S|)(\widetilde{S_{f,g}})^2$, $N_1 = f(|S|)$, $M_2 = e^{-3i\theta}f(|S|)(\widetilde{S_{f,g}}^{*})^2$, and $N_2 = g(|S|)U^*$; the first equality holds for Hermitian operator satisfying $r(A) = \|A\|$. Now applying Theorem 19 on last equality with $w(\alpha A) = |\alpha|w(A)$ and $w(A) = w(A^*)$, we obtain

$$\begin{aligned}\|H_\theta\|^3 &\le \frac{1}{8}\left(w\left(\widetilde{S_{f,g}}^{3}\right) + \frac{1}{2}\sqrt{4\left\|(f(|S|))^2\right\|\left\|\left(\widetilde{S_{f,g}}^{*}\right)^2\right\|\left\|(g(|S|))^2\right\|\left\|\widetilde{S_{f,g}}^{2}\right\|} + 2\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right)\\ &\le \frac{1}{8}\left(w\left(\widetilde{S_{f,g}}^{3}\right) + \frac{1}{2}\sqrt{4\|f(|S|)\|^2\left\|\widetilde{S_{f,g}}^{*}\right\|^2\|g(|S|)\|^2\left\|\widetilde{S_{f,g}}\right\|^2} + 2\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right)\\ &\le \frac{1}{8}\left(w\left(\widetilde{S_{f,g}}^{3}\right) + \|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\|\right) + \frac{1}{4}\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|.\end{aligned} \tag{61}$$

Taking supremum over $\theta \in \mathbb{R}$ in last inequality, then applying Lemma 1, we obtain

$$w^3(S) \le \frac{1}{8}\left(w\left(\widetilde{S_{f,g}}^{3}\right) + \|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\|\right) + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right). \tag{62}$$

Corollary 21. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0,1]$ *we have*

$$w^3(S) \le \frac{1}{8}\left(w\left(\widetilde{S_e}^{3}\right) + \left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|^2\left\||S|e^{-|S|^\lambda}\right\|\right) + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right), \tag{63}$$

where $\widetilde{S_e} = e^{|S|^\lambda}U|S|e^{-|S|^\lambda}$.

Corollary 22. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0,1]$ *we have*

$$w^3(S) \le \frac{1}{8}\left(w\left(\widetilde{S_e}^{3}\right) + \left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|^2\left\||S|e^{-|S|^\lambda}\right\|\right) + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right), \tag{64}$$

where $\widetilde{S_e} = |S|e^{|S|^{-\lambda}}Ue^{|S|^\lambda}$.

Corollary 23. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0,1]$ *we have*

$$w^3(S) \le \frac{1}{8}\left(w\left(\widetilde{S_\lambda}^{3}\right) + \|S\|\left\|\widetilde{S_\lambda}\right\|^2\right) + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right). \tag{65}$$

In particular,

$$w^3(S) \le \frac{1}{8}\left(w\left(\tilde{S}^3\right) + \|S\|\|\tilde{S}\|^2\right) + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right). \tag{66}$$

Remark 24. It is easy to observe that inequality (57) obtained in Theorem 20 is better than inequality (41).

Now, we exhibit some examples where numerical radius bounds are computed from inequalities (15), (27), (41), and (57) for some choices of pair f, g in (10) and for a given operator S.

Example 1. Given

$$S = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}. \tag{67}$$

Then, S = U|S| is a polar decomposition of S, where

$$|S| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{68}$$

and

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{69}$$

is partial isometry.

Table 2: Bounds (15), (27), (41), and (57) for different choices of f and g in (10).

(f, g)	Bound (15)	Bound (27)	Bound (41)	Bound (57)
$(\lvert S\rvert^{1/2}, \lvert S\rvert^{1/2})$	2.2704	2.3596	2.4662	2.3811
$(\lvert S\rvert^{1/3}, \lvert S\rvert^{2/3})$	2.3596	2.3264	2.4346	2.3332
$(\lvert S\rvert^{1/4}, \lvert S\rvert^{3/4})$	2.4817	2.3244	2.4327	2.3632
$(\lvert S\rvert^{1/5}, \lvert S\rvert^{4/5})$	2.5826	2.3510	2.4579	2.3462
$\left(e^{\lvert S\rvert^{1/2}}, \lvert S\rvert e^{-\lvert S\rvert^{1/2}}\right)$	8.7223	2.4234	2.5284	2.3854
$\left(e^{\lvert S\rvert^{1/3}}, \lvert S\rvert e^{-\lvert S\rvert^{1/3}}\right)$	5.3389	2.3492	2.4563	2.3459
$\left(e^{\lvert S\rvert^{1/4}}, \lvert S\rvert e^{-\lvert S\rvert^{1/4}}\right)$	4.4307	2.3277	2.4311	2.3314
$\left(e^{\lvert S\rvert^{1/5}}, \lvert S\rvert e^{-\lvert S\rvert^{1/5}}\right)$	4.0333	2.3380	2.4456	2.3394

Table 3: Bounds (15), (27), (41), and (57) for different choices of f and g in (10).

(f, g)	Bound (15)	Bound (27)	Bound (41)	Bound (57)
$(\lvert S\rvert^{1/2}, \lvert S\rvert^{1/2})$	1.5047	1.5065	1.4500	1.3880
$(\lvert S\rvert^{1/3}, \lvert S\rvert^{2/3})$	1.5416	1.5425	1.4901	1.4059
$(\lvert S\rvert^{1/4}, \lvert S\rvert^{3/4})$	1.5754	1.5637	1.5130	1.4432
$(\lvert S\rvert^{1/5}, \lvert S\rvert^{4/5})$	1.6026	1.5769	1.5276	1.4513
$\left(e^{\lvert S\rvert^{1/2}}, \lvert S\rvert e^{-\lvert S\rvert^{1/2}}\right)$	4.7336	1.5272	1.4726	1.4004
$\left(e^{\lvert S\rvert^{1/3}}, \lvert S\rvert e^{-\lvert S\rvert^{1/3}}\right)$	3.6923	1.5335	1.4794	1.4250
$\left(e^{\lvert S\rvert^{1/4}}, \lvert S\rvert e^{-\lvert S\rvert^{1/4}}\right)$	3.3318	1.5585	1.5069	1.4399
$\left(e^{\lvert S\rvert^{1/5}}, \lvert S\rvert e^{-\lvert S\rvert^{1/5}}\right)$	3.1529	1.5729	1.5231	1.4488

Bounds (15), (27), (41), and (57) are computed for some choices of f and g in (10) for the given S in Table 2, whereas the numerical radius of S is

$$w(S) = 2.0565. \tag{70}$$

Example 2. Let

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}. \tag{71}$$

Then, $S = U\lvert S\rvert$ is a polar decomposition of S, where

$$\lvert S\rvert = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{72}$$

and

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{73}$$

Bounds (15), (27), (41), and (57) are computed for some choices of f and g in (10) for the given S in Table 3, whereas the numerical radius of S is

$$w(S) = 1.3662. \tag{74}$$

3. Conclusion

From the results of this paper, we conclude that the inequalities of numerical radius involving generalized Aluthge transform have variety of upper bounds for numerical radius due to the choice of f, g in generalized Aluthge transform (10). The inequalities (15), (27), (41) and (57) obtained in Theorem 2, Theorem 8, Theorem 20, and Theorem 14 are new and generalized upper bounds for numerical radius. These generalized upper bounds can be useful to find better bounds of numerical radius already existing in literature for some choices of f, g in generalized Aluthge transform (10) and certain operators. It is proved that inequality (15) of Theorem 2 generalizes inequality (7) and improves inequality (12) for any choice of f, g in (10). Inequality (27) of Theorem 8 is sharper than inequality (12) for the choice of $f(t) = g(t) = t^{1/2}$ in (10). Inequality (41) of Theorem 14 is better than inequality (57) of Theorem 20. But for inequality (57) of Theorem 20, we can find such matrix and pairs of f, g for which the inequality of Theorem 20 can give better bound of numerical radius available in literature. To support theoretical investigations, some examples are presented where numerical radius and its upper bounds are computed for the pairs f, g in generalized Aluthge transform. Examples 1 and 2 show that there is no comparison between the bounds obtained from the inequalities (15), (27), and (41) of Theorem 2, Theorem 8, and Theorem 20; however, generalized Aluthge transform has choices of the pair f, g in (10) for which better upper bounds can be computed for certain operators.

Data Availability

There is no data required for this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to thank the Deanship of Scientific Research of King Abdulaziz University, Jeddah, Saudi Arabia, for technical and financial support.

References

[1] M. W. Alomari, "On Cauchy-Schwarz type inequalities and applications to numerical radius inequalities," *Ricerche di Matematica*, 2022.

[2] E. Alizadeh and A. Farokhinia, "Some refinements of numerical radius inequalities for Hilbert space operators," *Jordan Journal of Mathematics and Statistics*, vol. 15, no. 1, pp. 55–63, 2022.

[3] G. Muhammad, G. Abbas, and F. Dilawar, "Algebraic properties of ω-Q-fuzzy subgroups," *International Journal of Mathematics and Computer Science*, vol. 15, no. 1, pp. 265–274, 2020.

[4] G. Muhammad, F. Dilawar, D. Alghazzawi, and M. H. Mateen, "A note on complex fuzzy subfield," *Science*, vol. 21, no. 2, pp. 1048–1056, 2021.

[5] F. Kittaneh, "A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix," *Studia Mathematica*, vol. 158, no. 1, pp. 11–17, 2003.

[6] F. Kittaneh, "Numerical radius inequalities for Hilbert space operators," *Studia Mathematica*, vol. 168, no. 1, pp. 73–80, 2005.

[7] P. Bhunia, S. Bag, and K. Paul, "Numerical radius inequalities and its applications in estimation of zeros of polynomials," *Linear Algebra and its Applications*, vol. 573, pp. 166–177, 2019.

[8] A. Aluthge, "On p-hyponormal operators for $0 < p < 1$," *Integral Equations and Operator Theory*, vol. 13, no. 3, pp. 307–315, 1990.

[9] T. Yamazaki, "On upper and lower bounds of the numerical radius and an equality condition," *Studia Mathematica*, vol. 178, no. 1, pp. 83–89, 2007.

[10] P. Bhunia, S. Jana, and K. Paul, "Refined inequalities for the numerical radius of Hilbert space operators," 2021, http://arxiv.org/abs/2106.13949.

[11] K. Okubo, "On weakly unitarily invariant norm and the λ-Aluthge transformation for invertible operator," *Linear Algebra and its Applications*, vol. 419, pp. 48–52, 2006.

[12] A. Abu Omar and F. Kittaneh, "A numerical radius inequality involving the generalized Aluthge transform," *Studia Mathematica*, vol. 216, no. 1, pp. 69–75, 2013.

[13] S. Bag, P. Bhunia, and K. Paul, "Bounds of numerical radius of bounded linear operator using t-Aluthge transform," *Mathematical Inequalities and Applications*, vol. 23, no. 3, pp. 991–1004, 2020.

[14] P. Bhunia and K. Paul, "Proper improvement of well-known numerical radius inequalities and their applications," *Results in Mathematics*, vol. 76, no. 4, 2021.

[15] K. Shebrawi and M. Bakherad, "Generalizations of the Aluthge transform of operators," *Filomat*, vol. 32, no. 18, pp. 6465–6474, 2018.

[16] C. K. Fong and J. A. Holbrook, "Unitarily invariant operator norms," *Canadian Journal of Mathematics*, vol. 35, no. 2, pp. 274–299, 1983.

[17] F. Kittaneh, "Commutator inequalities associated with the polar decomposition," *Proceedings of the American Mathematical Society*, vol. 130, no. 5, pp. 1279–1283, 2002.

[18] T. Yan, J. Hyder, M. Saeed Akram, G. Farid, and K. Nonlaopon, "On numerical radius bounds involving generalized Aluthge transform," *Journal of Function Spaces*, vol. 2022, Article ID 2897323, 8 pages, 2022.

[19] Z. Zhiqiang, M. Saeed Akram, J. Hyder, G. Farid, and T. Yan, "Spectral radius formulas involving generalized Aluthge transform," *Journal of Function Spaces*, vol. 2022, Article ID 3993044, 8 pages, 2022.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 3444854, 10 pages
https://doi.org/10.1155/2022/3444854

Research Article

Properties of Meromorphic Spiral-Like Functions Associated with Symmetric Functions

Daniel Breaz,[1] **Luminița-Ioana Cotîrlă**,[2] **Elangho Umadevi**,[3] **and Kadhavoor R. Karthikeyan**[4]

[1]Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, 510009 Alba Iulia, Romania
[2]Department of Mathematics, Technical University of Cluj-Napoca, Cluj-Napoca, Romania
[3]Department of Mathematics and Statistics, College of Natural and Health Sciences, Zayed University, Abu Dhabi, UAE
[4]Department of Applied Mathematics and Science, National University of Science & Technology, Muscat, P.O. Box 620, Oman

Correspondence should be addressed to Luminița-Ioana Cotîrlă; luminita.cotirla@math.utcluj.ro

Received 2 June 2022; Accepted 13 July 2022; Published 28 August 2022

Academic Editor: Sibel Yalçın

To consolidate or adapt to many studies on meromorphic functions, we define a new subclass of meromorphic functions of complex order involving a differential operator. The defined function class combines the concept of spiral-like functions with other studies pertaining to subclasses of multivalent meromorphic functions. Inclusion relations, integral representation, geometrical interpretation, coefficient estimates and solution to the Fekete-Szegö problem of the defined classes are the highlights of this present study. Further to keep up with the present direction of research, we extend the study using quantum calculus. Applications of our main results are given as corollaries.

1. Introduction

Let $\mathscr{A}$ be the class of function of the form

$$\chi(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \tag{1}$$

which are analytic in the unit disc $\mathbb{E} = \{\xi : |\xi| < 1\}$. Also let $\mathscr{S}$ denote the class of functions $\chi \in \mathscr{A}$ which are univalent in $\mathbb{E}$. The subclasses of $\mathscr{S}$ consisting of functions which map unit disc onto a star-like and convex domain will be symbolized by $\mathscr{S}^*$ and $\mathscr{C}$, respectively. Also let $\mathscr{P}$ denote the class of functions h analytic in the unit disc, given by

$$h(\xi) = 1 + \sum_{n=1}^{\infty} R_n \xi^n, \xi \in \mathbb{E}, R_1 > 0, \tag{2}$$

and satisfies $\text{Re}\,(h(\xi)) > 0$, $\xi \in \mathbb{E}$. For $p \in \mathbb{N} = \{1, 2, \cdots\}$, we let $\mathscr{L}_p$ to denote the class of functions χ of the form

$$\chi(\xi) = \xi^{-p} + \sum_{n=1}^{\infty} d_{n-p} \xi^{n-p}, \tag{3}$$

which are analytic in $\mathbb{E}^* = \{\xi : \xi \in \mathbb{C} \text{ and } 0 < |\xi| < 1\}$. Shi et al. [1] defined the class $\chi(\xi) \in \mathscr{M}\mathscr{S}_p(\sigma, \tau)$ if and only if

$$-e^{i\sigma} \frac{\xi \chi'(\xi)}{\chi(\xi)} \prec \frac{pe^{i\sigma} - \left(2\tau - pe^{-i\sigma}\right)\xi}{1 - \xi} \left(\chi \in \mathscr{L}_p\right), \tag{4}$$

where $|\sigma| < \lambda/2$ and $\tau > p \cos \sigma$. Here, $\prec$ denotes the usual subordination of analytic function. The class $\mathscr{M}\mathscr{S}_p(\sigma, \tau)$ is the meromorphic analogue of the class of p-valent spiral-like functions defined by Uyanik et al. in [2]. Similarly, we let $\mathscr{M}\mathscr{C}_p(\sigma, \tau)$ to denote the class of function in $\mathscr{L}_p$ satisfying the condition

$$-e^{i\sigma} \left(1 + \frac{\xi \chi''(\xi)}{\chi'(\xi)}\right) \prec \frac{pe^{i\sigma} - \left(2\tau - pe^{-i\sigma}\right)\xi}{1 - \xi}. \tag{5}$$

Extending the class of Janowski function ([3]), Aouf [4]

(Equation (4)) (also see [5]) defined the class $h(\xi) \in \mathscr{P}(X, Y, p, \tau)$ if and only if

$$h(\xi) = \frac{p + [pY + (X - Y)(p - \tau)]w(\xi)}{[1 + Yw(\xi)]}, \quad (-1 \le Y < X \le 1, 0 \le \tau < 1), \tag{6}$$

where $w(\xi)$ is the Schwartz function. Motivated by the recent study of Breaz et al. [5] and in view generalizing the superordinate function in (4), Cotîrlă and Karthikeyan in [6] defined and studied the following relation

$$\Delta_\sigma^\tau(\xi) = \frac{\left[\left(1 + Xe^{-2i\sigma}\right)pe^{i\sigma} + \tau(Y - X)\right]h(\xi) + \left[\left(1 - Xe^{-2i\sigma}\right)pe^{i\sigma} - \tau(Y - X)\right]}{[(Y + 1)h(\xi) + (1 - Y)]}, \tag{7}$$

where $-1 \le Y < X \le 1, -\pi/2 < \sigma < \pi/2, \tau > p \cos \sigma$ and $h(\xi) \in \mathscr{P}$.

It is well-known that the function $h(\xi) = 1 + \xi/1 - \xi$ maps the unit disc onto the right half plane. For an admissible choice of the parameter $X = 0.5$, $Y = -0.5$, $p = 1$, $\sigma = \pi/3$, and $\tau = 0.6$, $\Delta_\sigma^\tau(\xi)$ maps unit disc onto a domain which is convex with respect to point 0.5 if $h(\xi) = 1 + \xi/1 - \xi$ (see Figure 1). Similarly, the function $h(\xi) = \xi + \sqrt[3]{1 + \xi^3}$ which is related to the class of functions associated with leaf-like domain (see [7–9]) gets rotated and translated on the impact of $\Delta_\sigma^\tau(\xi)$ (see Figure 2) for a choice of the parameter $X = 0.5$, $Y = -0.5$, $p = 1$, $\sigma = \pi/3$, and $\tau = 0.6$.

Remark 1. The purpose to study $\Delta_\sigma^\tau(\xi)$ was mainly motivated by the study of Karthikeyan et al. [10] and Noor and Malik [11]. Here, we will list some recent studies.

(1) If we let $\sigma = 0$ in (7), then, $\Delta_\sigma^\tau(\xi)$ reduces to

$$\aleph(\xi) = \frac{[(1 + X)p + \tau(Y - X)]h(\xi) + [(1 - X)p - \tau(Y - X)]}{[(Y + 1)h(\xi) + (1 - Y)]}. \tag{8}$$

The function $\aleph(\xi)$ was defined and studied by Breaz et al. in [5].

(2) If we let $X = 1, Y = -1$ and $h(\xi) = (1 + \xi)/(1 - \xi)$ in (7), then, $\Delta_\sigma^\tau(\xi)$ reduces to $2\tau - pe^{-i\sigma} + (2(p \cos \sigma - \tau)/1 - \xi)$ (see the superordinate function in (4)).

It is well-known that if $\chi(\xi)$ given by (1) is in $\mathscr{S}$, then, the ℓ-symmetrical function $\left[\chi(\xi^\ell)\right]^{1/\ell}$, (ℓ is a positive integer) is also in $\mathscr{S}$. Let ℓ be a positive integer and $\varepsilon = \exp(2\pi i/\ell)$. For $\chi \in \mathscr{A}$, let

$$\chi_\ell(\xi) = \frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\frac{\chi(\varepsilon^\nu \xi)}{\varepsilon^\nu}. \tag{9}$$

The function χ is said to be star-like with respect to ℓ-symmetric points if it satisfies the condition

$$\operatorname{Re}\frac{\xi\chi'(\xi)}{\chi_\ell(\xi)} > 0. \tag{10}$$

Here, we will let $\mathscr{S}_\ell^s$ to denote the class of star-like functions with respect to ℓ-symmetric points. The class $\mathscr{S}_\ell^s$ was introduced by Sakaguchi [12] in which he showed that all functions in $\mathscr{S}_\ell^s$ are univalent. Note that $\mathscr{S}_1^s = \mathscr{S}^*$.

A function $\chi \in \mathscr{L}_p$ is said to be ℓ-symmetrical if for each $\xi \in \mathbb{E}$

$$\chi(\varepsilon\xi) = \varepsilon^{-p}\chi(\xi), \tag{11}$$

For $\chi \in \mathscr{L}_p$, Equation (9) can be defined by the following equality

$$\chi_\ell(\xi) = \frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\frac{\chi(\varepsilon^\nu \xi)}{\varepsilon^{-\nu p}}, (\ell = 1, 2, 3, \cdots). \tag{12}$$

Now, we extend the operator defined by Selvaraj and Karthikeyan in [13]. Using Hadamard product (or convolution), we define a operator for functions $\chi \in \mathscr{L}_p$ as follows:

$$I_\mu^m(a_1, a_2, \cdots, a_r, c_1, c_2, \cdots, c_s)\chi = \frac{1}{\xi^p} + \sum_{n=1}^{\infty}\left(\frac{\mu}{n + \mu}\right)^m \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(c_1)_n (c_2)_n \cdots (c_s)_n} d_{n-p}\frac{\xi^{n-p}}{(n)!}, \tag{13}$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \begin{cases} 1 & \text{if } n = 0 \\ x(x + 1)(x + 2) \cdots (x + n - 1) & \text{if } n \in N_0 = \{1, 2, , \cdots\}. \end{cases} \tag{14}$$

For convenience, we shall henceforth denote

$$I_\mu^m(a_1, a_2, \cdots, a_r, c_1, c_2, \cdots, c_s)\chi = I_\mu^m(a_1, c_1)\chi. \tag{15}$$

Note that in [13], $I_\mu^m(a_1, c_1)\chi$ was defined for $\chi \in \mathscr{L}_1$. Here, we skip the discussion on the necessity of using differential or integral operator, refer to [13–17] and reference provided therein for detailed properties of $I_\mu^m(a_1, c_1)\chi$.

Throughout this paper, we assume that $-1 \le Y < X \le 1$, $-\pi/2 < |\sigma| < \pi/2, \tau > p \cos \sigma, \lambda \ge 1, \quad \ell \in \mathbb{N}, \quad \varepsilon = \exp(2\pi i/\ell)$ and

$$\chi_\ell(m, \mu, a_1, c_1; \xi) = \frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\varepsilon^{\nu p}\left[I_\mu^m(a_1, c_1)\chi(\varepsilon^\nu \xi)\right] = \xi^{-p} + \cdots, \tag{16}$$

$$\left(\chi \in \mathscr{L}_p; \ell = 2, 3, \cdots\right). \tag{17}$$

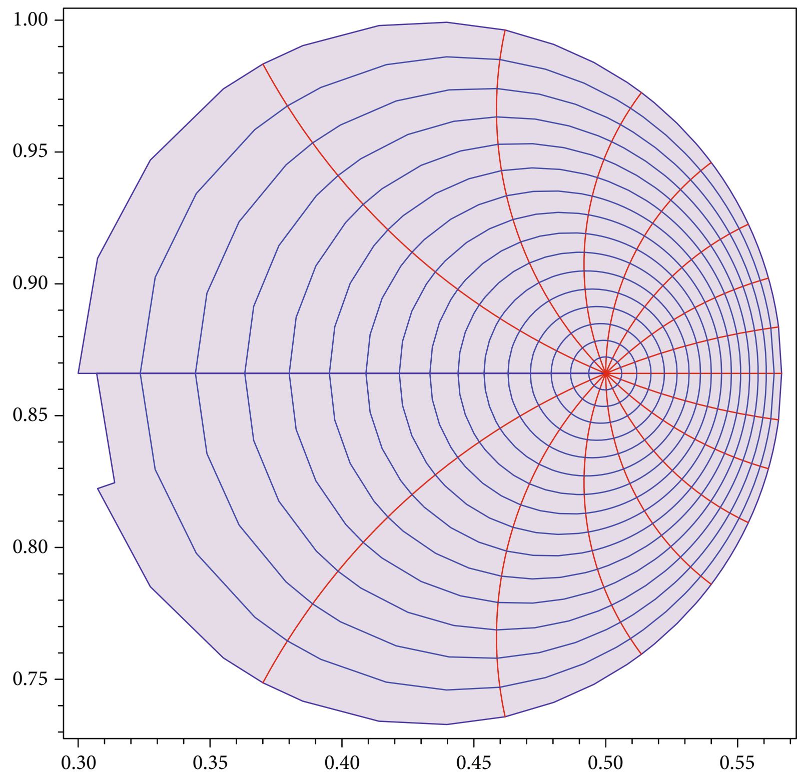


FIGURE 1: The image of the unit disc under the mapping of $\Delta_{\sigma}^{\tau}(\xi)$, if $h(\xi) = 1 + \xi/1 - \xi$.

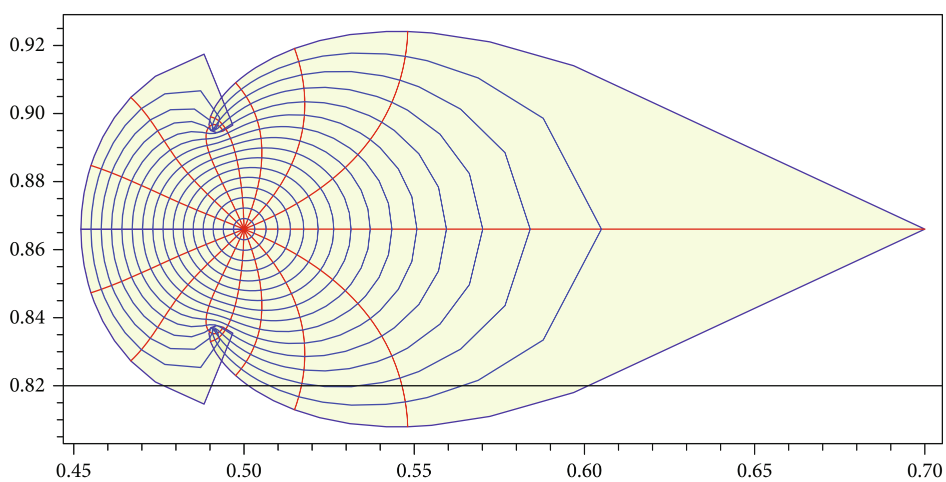


FIGURE 2: The image of the unit disc under the mapping of $\Delta_{\sigma}^{\tau}(\xi)$, if $h(\xi) = \xi + \sqrt[3]{1+\xi^3}$.

1.1. Short Introduction to Quantum Calculus. For $0 < q < 1$, the Jacksons q-derivative operator is defined by (see [18, 19])

$$\mathfrak{D}_q \chi(\xi) := \begin{cases} \chi'(0), & \text{if } \xi = 0, \\ \dfrac{\chi(\xi) - \chi(q\xi)}{(1-q)\xi}, & \text{if } \xi \neq 0. \end{cases} \tag{18}$$

From (18), if χ has the power series expansion (3), we can easily see that $\mathfrak{D}_q \chi(\xi) = [-p]_q \xi^{-p-1} + \sum_{n=1}^{\infty} [n-p]_q d_{n-p} \xi^{n-p-1}$, for $\xi \neq 0$, where the q *-integer number* $[n]_q$ is defined by

$$[n]_q := \frac{1-q^n}{1-q}, \tag{19}$$

and note that $\lim_{q \longrightarrow 1^-} \mathfrak{D}_q \chi(\xi) = \chi'(\xi)$. Throughout this paper, we let denote

$$\left([n]_q\right)_k := [n]_q [n+1]_q [n+2]_q \cdots [n+k-1]_q. \tag{20}$$

The q-Jackson integral is defined by (see [20])

$$I_q[\chi(\xi)] := \int_0^{\xi} \chi(t) d_q t = \xi(1-q) \sum_{n=0}^{\infty} q^n \chi(\xi q^n), \tag{21}$$

provided the q-series converges. Further observe that

$$\mathfrak{D}_q I_q \chi(\xi) = \chi(\xi) \text{ and } I_q \mathfrak{D}_q \chi(\xi) = \chi(\xi) - \chi(0), \tag{22}$$

where the second equality holds if χ is continuous at $\xi = 0$. For details pertaining to the significance of univalent function theory in dual with quantum calculus, refer to [21, 22] (also see [23–26]).

Meromorphic multivalent functions have been extensively studied by various authors, but motivation and references of this study are [1, 13, 27–36].

Definition 2. For $-\pi/2 < \sigma < \pi/2$, $\lambda \geq 1$, $\tau \geq p \cos \sigma$, $b \in \mathbb{C} \setminus \{0\}$ and $I_{\mu}^{m}(a_1, c_1)\chi$ defined as in (13), a function χ belongs to the class $\mathscr{MS}_{\ell}^{m,\lambda}(a_1, c_1; b; h; X, Y)$ if it satisfies

$$e^{i\sigma}\left[p - \frac{1}{b}\left\{\frac{\xi^{(p+1)\lambda-p}\left[I_{\mu}^{m}(a_1, c_1)\chi'(\xi)\right]^{\lambda}}{\chi_{\ell}(m, \mu, a_1, c_1; \xi)} - (-p)^{\lambda}\right\}\right] \prec \Delta_{\sigma}^{\tau}(\xi), \tag{23}$$

where $\prec$ denotes subordination and $h(\zeta)$ is defined as in (2).

Now, we will define a class replacing ordinary derivative with a quantum derivative in $\mathscr{MS}_{\ell}^{m,\lambda}(a_1, c_1; b; h; X, Y)$.

Definition 3. For $-\pi/3 < \sigma < \pi/2$, $0 \leq \eta \leq 1$, $\tau \geq p \cos \sigma$, $b \in \mathbb{C} \setminus \{0\}$ and $I_{\mu}^{m}(a_1, c_1)\chi$ defined as in (13), a function χ belongs to the class $\mathscr{QMS}_{\ell}^{m,\lambda}(a_1, c_1; b; h; X, Y)$ if

$$e^{i\sigma}\left([p]_q - \frac{1}{b}\left\{\frac{\xi^{(p+1)\lambda-p}\left[\mathfrak{D}_q I_{\mu}^{m}(a_1, c_1)\chi(\xi)\right]^{\lambda}}{\chi_{\ell}(m, \mu, a_1, c_1; \xi)} - \left(-[p]_q\right)^{\lambda}\right\}\right) \prec Y_q(\sigma, \tau; \xi), \tag{24}$$

where $Y_q(\sigma, \tau; \xi)$ is the $q-$analogue of $\Delta_{\sigma}^{\tau}(\xi)$, which is defined by

$$Y_q(\sigma, \tau; \xi) = \frac{\left[\left(1 + Xe^{-2i\sigma}\right)[p]_q e^{i\sigma} + \tau(Y - X)\right]h(\xi) + \left[\left(1 - Xe^{-2i\sigma}\right)[p]_q e^{i\sigma} - \tau(Y - X)\right]}{[(Y+1)h(\xi) + (1 - Y)]}. \tag{25}$$

Remark 4. We note that in the definition of $\mathscr{QMS}_{\ell}^{m,\lambda}(a_1, c_1; b; h; X, Y)$, the operator $I_{\mu}^{m}(a_1, c_1)\chi$ and $\chi_{\ell}(m, \mu, a_1, c_1; \xi)$ are the same as used in $\mathscr{MS}_{\ell}^{m,\lambda}(a_1, c_1; b; h; X, Y)$. We have not used the q-analogue operator as it would require the reader to contend with additional set of parameters.

2. Preliminaries and some Supplementary Results

Here, we will discuss the results which would help us to obtain our main results.

We note that everything in classical calculus cannot be generalized to quantum calculus, notably the chain rule needs adaptation. Hence, logarithmic differentiation needs some application of analysis. In [37], Agrawal and Sahoo obtained the following result on logarithmic differentiation. For $\chi \in \mathscr{A}$ and $0 < q < 1$, we have

$$I_q \frac{\mathfrak{D}_q \chi(\xi)}{\chi(\xi)} = \frac{q-1}{\ln q} \log \chi(\xi), \tag{26}$$

where $I_q \chi$ is the Jackson q-integral, defined as in (21). Similarly, we can see that

$$\mathfrak{D}_q\left[\{\chi(\xi)\}^{\lambda-1/\lambda}\right] = \frac{\lambda - 1}{\lambda} \mathfrak{D}_q[\chi(\xi)]\{\chi(\xi)\}^{-1/\lambda}. \tag{27}$$

If ν is an integer, then the following identities follow directly from (16):

$$\chi_{\ell}(m, \mu, a_1, c_1; \varepsilon^{\nu}\xi) = \varepsilon^{-\nu p}\chi_{\ell}(m, \mu, a_1, c_1; \xi). \tag{28}$$

$$\begin{aligned} \chi_{\ell}'(m, \mu, a_1, c_1; \varepsilon^{\nu}\xi) &= \varepsilon^{-\nu p-\nu}\chi_{\ell}'(m, \mu, a_1, c_1; \xi) \\ &= \frac{1}{\ell}\sum_{\nu=0}^{\ell-1} \varepsilon^{\nu+\nu p} I_{\mu}^{m}(a_1, c_1)\chi'(\varepsilon^{\nu}\xi). \end{aligned} \tag{29}$$

Since q-derivative satisfies the linearity condition, (29) holds if the classical derivative is replaced with quantum derivative. That is,

$$\mathfrak{D}_q[\chi_{\ell}(m, \mu, a_1, c_1; \varepsilon^{\nu}\xi)] = \varepsilon^{-\nu p-\nu}\mathfrak{D}_q[\chi_{\ell}(m, \mu, a_1, c_1; \xi)]. \tag{30}$$

We now state the following result which will be used to establish the coefficient inequalities.

Lemma 5 (see [38]). *Let $\vartheta(\xi) = 1 + \sum_{n=1}^{\infty} \vartheta_n \xi^n \in \mathscr{P}$ and also let v be a complex number, then*

$$\left|\vartheta_2 - v\vartheta_1^2\right| \leq 2 \max\{1, |2v - 1|\}, \tag{31}$$

the result is sharp for functions given by

$$\vartheta(\xi) = \frac{1+\xi^2}{1-\xi^2}, \vartheta(\xi) = \frac{1+\xi}{1-\xi}. \tag{32}$$

The Maclaurin series for the function $\Delta_{\sigma}^{\tau}(\xi)$ (see [6]) for the function is given by

$$\Delta_{\sigma}^{\tau}(\xi) = pe^{i\sigma} + \frac{\left[X\left(pe^{-i\sigma} - \tau\right) - Y\left(pe^{i\sigma} - \tau\right)\right]R_1}{2}\xi + \cdots. \tag{33}$$

If we define the function $\vartheta(\xi)$ by

$$\vartheta(\xi) = 1 + \vartheta_1 \xi + \vartheta_2 \xi^2 + \cdots = \frac{1 + w(\xi)}{1 - w(\xi)} \prec \frac{1+\xi}{1-\xi}, (\xi \in \mathbb{E}). \quad (34)$$

We note that $\vartheta(0) = 1$ and $\vartheta \in \mathscr{P}$. Using (34), we have

$$\begin{aligned} w(\xi) &= \frac{\vartheta(\xi) - 1}{\vartheta(\xi) + 1} \\ &= \frac{1}{2}\left[\vartheta_1 \xi + \left(\vartheta_2 - \frac{\vartheta_1^2}{2}\right)\xi^2 + \left(\vartheta_3 - \vartheta_1\vartheta_2 + \frac{\vartheta_1^3}{4}\right)\xi^3 + \cdots\right]. \end{aligned} \quad (35)$$

For some $h(\xi) = 1 + R_1 \xi + R_2 \xi^2 + \cdots$, we have

$$\begin{aligned} & p + b\left\{e^{-i\sigma}\Delta_\sigma^\tau[w(\xi)] - p\right\} \\ &= p + \frac{be^{-i\sigma} R_1 \vartheta_1 \left[X\left(pe^{-i\sigma} - \tau\right) - Y\left(pe^{i\sigma} - \tau\right)\right]}{4}\xi \\ &+ \frac{be^{-i\sigma}\left[X\left(pe^{-i\sigma} - \tau\right) - Y\left(pe^{i\sigma} - \tau\right)\right]R_1}{4} \\ &\cdot \left[\vartheta_2 - \vartheta_1^2\left(\frac{(Y+1)R_1 + 2(1 - (R_2/R_1))}{4}\right)\right]\xi^2 + \cdots. \end{aligned} \quad (36)$$

3. Integral Representations and Closure Properties

We begin with the following.

Theorem 6. *Let $\chi \in \mathscr{MS}_\ell^{m,\lambda}(a_1, c_1; b; h; X, Y)$, then for $\lambda = 1$, we get for $\xi \in \mathbb{E}^*$*

$$\begin{aligned} & \chi_\ell(m, \mu, a_1, c_1; \xi) \\ &= \xi^p \exp\left\{\frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\int_0^{\varepsilon^\nu \xi}\frac{1}{t}\left[b\left\{p - e^{-i\sigma}\Delta_\sigma^\tau[w(t)]\right\} - 2p\right]dt\right\}. \end{aligned} \quad (37)$$

And for $\lambda > 1$, we have for $\xi \in \mathbb{E}^*$

$$\begin{aligned} \chi_\ell(m, \mu, a_1, c_1; \xi) &= \left(\frac{\lambda - 1}{\lambda}\right) \\ &\cdot \left\{\frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\int_0^\xi \left(\frac{\left[b\left\{p - e^{-i\sigma}\Delta_\sigma^\tau[w(\varepsilon^\nu t)]\right\} + (-p)^\lambda\right]}{t^{(p+1)\lambda - p}}\right)^{1/\lambda} dt\right\}^{\lambda - 1/\lambda}, \end{aligned} \quad (38)$$

where $\chi_\ell(m, \mu, a_1, c_1; \xi)$ is defined by equality (16) and $w(\xi)$ is analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(\xi)| < 1$.

Proof. Let $\chi \in \mathscr{MS}_\ell^{m,\lambda}(a_1, c_1; b; h; X, Y)$. In view of (23), we have

$$\frac{\xi^{(p+1)\lambda - p}\left[I_\mu^m(a_1, c_1)\chi'(\xi)\right]^\lambda}{\chi_\ell(m, \mu, a_1, c_1; \xi)} = b\left\{p - e^{-i\sigma}\Delta_\sigma^\tau[w(\xi)]\right\} + (-p)^\lambda, \quad (39)$$

where $w(\xi)$ is analytic in $\mathbb{E}$ and $w(0) = 0$, $|w(\xi)| < 1$. Substituting ξ by $\varepsilon^\nu \xi$ in the equality (39), respectively, $(\nu = 0, 1, 2, \cdots, \ell - 1, \varepsilon^\ell = 1)$, we have

$$\begin{aligned} & \frac{(\varepsilon^\nu \xi)^{(p+1)\lambda - p}\left[I_\mu^m(a_1, c_1)\chi'(\varepsilon^\nu \xi)\right]^\lambda}{\chi_\ell(m, \mu, a_1, c_1; \varepsilon^\nu \xi)} \\ &= b\left\{p - e^{-i\sigma}\Delta_\sigma^\tau[w(\varepsilon^\nu \xi)]\right\} + (-p)^\lambda. \end{aligned} \quad (40)$$

Using (28) in (40), we get

$$\begin{aligned} & \frac{\xi^{(p+1)\lambda - p}\varepsilon^{\nu(p+1)\lambda}\left[I_\mu^m(a_1, c_1)\chi'(\varepsilon^\nu \xi)\right]^\lambda}{\chi_\ell(m, \mu, a_1, c_1; \xi)} \\ &= b\left\{p - e^{-i\sigma}\Delta_\sigma^\tau[w(\varepsilon^\nu \xi)]\right\} + (-p)^\lambda. \end{aligned} \quad (41)$$

Using the equality (29) in (42), we can get

$$\frac{\varepsilon^{\nu + \nu p} I_\mu^m(a_1, c_1)\chi'(\varepsilon^\nu \xi)}{\left[\chi_\ell(m, \mu, a_1, c_1; \xi)\right]^{1/\lambda}} = \left(\frac{\left[b\left\{p - e^{-i\sigma}\Delta_\sigma^\tau[w(\varepsilon^\nu \xi)]\right\} + (-p)^\lambda\right]}{\xi^{(p+1)\lambda - p}}\right)^{1/\lambda}. \quad (42)$$

Let $\nu = 0, 1, 2, \cdots, \ell - 1$ in (42), respectively, and summing them we get

$$\frac{\chi_\ell'(m, \mu, a_1, c_1; \xi)}{\left[\chi_\ell(m, \mu, a_1, c_1; \xi)\right]^{1/\lambda}} = \frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\left(\frac{\left[b\left\{p - e^{-i\sigma}\Delta_\sigma^\tau[w(\varepsilon^\nu \xi)]\right\} + (-p)^\lambda\right]}{\xi^{(p+1)\lambda - p}}\right)^{1/\lambda}. \quad (43)$$

□

Case 1. Let $\lambda = 1$ in (43). We need to integrate from 0 to ξ to find $\chi_\ell(m, \mu, a_1, c_1; \xi)$. But from (43), we notice the presence of the first-order pole at the origin, the difficulty to integrate the above equality is avoided by integrating from ξ_0 to ξ with $\xi_0 \neq 0$, and then, let $\xi_0 \longrightarrow 0$. Therefore, on applying integration, we get

$$\begin{aligned} & \log\left(\frac{\chi_\ell(m, \mu, a_1, c_1; \xi)}{\xi^p}\right) \\ &= \frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\int_0^{\varepsilon^\nu \xi}\frac{1}{t}\left[b\left\{p - e^{-i\sigma}\Delta_\sigma^\tau[w(t)]\right\} - 2p\right]\mathrm{dt}. \end{aligned} \quad (44)$$

Hence, the proof of (37).

Case 2. If $\lambda > 1$, (43) can be rewritten as

$$\left[\{\chi_\ell(m,\mu,a_1,c_1;\xi)\}^{1-1/\lambda}\right]' = \left(1-\left(\frac{1}{\lambda}\right)\right)\frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\left(\frac{\left[b\{p-e^{-i\sigma}\Delta_\sigma^\tau[w(\varepsilon^\nu\xi)]\}+(-p)^\lambda\right]}{\xi^{(p+1)\lambda-p}}\right)^{1/\lambda}. \quad (45)$$

On integrating the above expression we obtain (38). Hence, the proof of Theorem 6.

Theorem 7. *Let $\chi \in \mathcal{QMS}_\ell^{m,\lambda}(a_1,c_1;b;h;X,Y)$, then for $\lambda = 1$, we get*

$$\chi_\ell(m,\mu,a_1,c_1;\xi) = \xi^p \exp\left\{\frac{\ln q}{(q-1)\ell}\sum_{\nu=0}^{\ell-1}\int_0^{\varepsilon^\nu\xi}\frac{1}{t}\left(b\left\{[p]_q - e^{-i\sigma}Y_q(\sigma,\tau;w(t))\right\}-2[p]_q\right)dt\right\}. \quad (46)$$

And for $\lambda > 1$, we have

$$\chi_\ell(m,\mu,a_1,c_1;\xi) = \left(\frac{\lambda-1}{\lambda}\right)\left\{\frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\int_0^\xi\left(\frac{\left[b\left\{[p]_q - e^{-i\sigma}\Delta_\sigma^\tau[w(\varepsilon^\nu t)]\right\}+\left([p]_q\right)^\lambda\right]}{t^{(p+1)\lambda-p}}\right)^{1/\lambda}dt\right\}^{\lambda-1/\lambda}, \quad (47)$$

where $\chi_\ell(m,\mu,a_1,c_1;\xi)$ is defined by equality (16) and $w(\xi)$ is analytic in $\mathbb{E}$ with $w(0)=0$ and $|w(\xi)|<1$.

Proof. In view of (24), (30), and (43), we have

$$\frac{\mathfrak{D}_q[\chi_\ell(m,\mu,a_1,c_1;\xi)]}{[\chi_\ell(m,\mu,a_1,c_1;\xi)]^{1/\lambda}} = \frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\left(\frac{\left(b\left\{[p]_q - e^{-i\sigma}Y_q(\sigma,\tau;w(\varepsilon^\nu\xi))\right\}+\left(-[p]_q\right)^\lambda\right)}{\xi^{(p+1)\lambda-p}}\right)^{1/\lambda}. \quad (48)$$

□

Case 1. Let $\lambda = 1$ in (48). Using the definition of logarithmic differentiation for q-derivative operator (see (26)) in (48), we get $(0<q<1)$

$$\log\left(\frac{\chi_\ell(m,\mu,a_1,c_1;\xi)}{\xi^p}\right) = \frac{\ln q}{(q-1)\ell}\sum_{\nu=0}^{\ell-1}\int_0^{\varepsilon^\nu\xi}\frac{1}{t}\left(b\left\{[p]_q - e^{-i\sigma}Y_q(\sigma,\tau;w(t))\right\}-2[p]_q\right)d_qt, \quad (49)$$

where the integral is q-Jackson integral. Hence, the proof of (37).

Case 2. If $\lambda > 1$, using chain rule (see (27)) for the q-difference operator defined in the previous section, (43) can be rewritten as

$$\mathfrak{D}_q\left[\{\chi_\ell(m,\mu,a_1,c_1;\xi)\}^{1-1/\lambda}\right] = \left(1-\left(\frac{1}{\lambda}\right)\right)\frac{1}{\ell}\sum_{\nu=0}^{\ell-1} \cdot \left(\frac{\left(b\left\{[p]_q - e^{-i\sigma}Y_q(\sigma,\tau;w(\varepsilon^\nu\xi))\right\}+\left(-[p]_q\right)^\lambda\right)}{\xi^{(p+1)\lambda-p}}\right)^{1/\lambda}. \quad (50)$$

On applying q-Jackson integral in the above expression, we obtain (47).

Corollary 8 (see [1, Theorem 1]). *Let $\chi(\xi) \in \mathcal{MS}_p(\sigma,\tau)$, then*

$$\chi(\xi) = \xi^{-p}\exp\left(2(\tau - p\cos\sigma)e^{-i\sigma}\int_0^\xi\frac{w(t)}{t[1-w(t)]}dt\right), (\xi \in \mathbb{E}^*), \quad (51)$$

where $w(\xi)$ is analytic in $\mathbb{E}$ with $w(0)=0$ and $|w(\xi)|<1$.

Proof. Letting $m=2$, $s=1$, $a_1=c_1$, $a_2=1$, $X=1$, $Y=-1$, $\ell = \lambda = b = 1$, and $h(\xi) = (1+\xi)/(1-\xi)$ in Theorem 6, then (43) reduces to the form

$$-e^{i\sigma}\frac{\xi\chi'(\xi)}{\chi(\xi)} = \left(pe^{i\sigma} - \frac{2(\tau - p\cos\sigma)w(\xi)}{1-w(\xi)}\right). \quad (52)$$

Retracing the steps as in Theorem 6, we can establish the assertion of the corollary. □

Setting $m=0$, $r=2$, $s=1$, $a_1=c_1$, and $a_2=1$ in Theorem 6, we get the following

Corollary 9. *Let $\chi \in \mathcal{MS}_\ell^{0,\lambda}(2,1;b;h;X,Y)$, then, for $\lambda=1$, we get for $\xi \in \mathbb{E}^*$*

$$\chi_\ell(\xi) = -p\int_0^\xi u^{-p-1}\exp\left(\frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\int_0^{\varepsilon^\nu u}\frac{1}{t}\left[b\{p - e^{-i\sigma}\Delta_\sigma^\tau[w(t)]\}\right]dt\right)du. \quad (53)$$

And for $\lambda > 1$, we have for $\xi \in \mathbb{E}^*$

$$\chi_\ell(\xi) = -p\left(\frac{\lambda-1}{\lambda}\right)\int_0^\xi u \cdot \left\{\frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\int_0^u\left(\frac{\left[b\{p-e^{-i\sigma}\Delta_\sigma^\tau[w(\varepsilon^\nu t)]\}+(-p)^\lambda\right]}{t^{(p+1)\lambda-p}}\right)^{1/\lambda} dt\right\}^{\lambda-1/\lambda} du, \quad (54)$$

where $\chi_\ell(\xi)$ is defined by equality (12) and $w(\xi)$ is analytic in $\mathbb{E}$ with $w(0)=0$ and $|w(\xi)|<1$.

Letting $\lambda = 1X = 1$, $Y=-1$, $b=1$, and $h(\xi)=(1+\xi)/(1-\xi)$ in Corollary 9, we get the following result.

Corollary 10. (see [1]). *Let $\chi(\xi) \in \mathscr{MC}_p(\sigma,\tau)$, then for $\xi \in \mathbb{E}^*$*

$$\chi(\xi) = -p\int_0^\xi u^{-p-1}\exp \cdot \left(2(\tau - p\cos\sigma)e^{-i\sigma}\int_0^u \frac{w(t)}{t[1-w(t)]}dt\right)du, \quad (55)$$

where $w(\xi)$ is analytic in $\mathbb{E}$ with $w(0)=0$ and $|w(\xi)|<1$.

4. Fekete-Szegö Inequality of $\mathscr{MS}_\ell^{m,\lambda}(a_1, c_1; b; h; X, Y)$ and $\mathscr{QMS}_\ell^{m,\lambda}(a_1, c_1; b; h; X, Y)$

Very few researchers have attempted at finding solution to the Fekete-Szegö problem for class of functions with respect to ℓ-symmetric points, as it is computational tedious. Notable among those works on coefficient inequalities of classes of functions with respect to ℓ-symmetric points were done by Aouf et al. [39].

Throughout this section, we let

$$\Psi_n = \frac{1}{\ell}\sum_{\nu=0}^{\ell-1}\varepsilon^{\nu n}, \ (\ell \in \mathbb{N}; n \ge 1; \varepsilon^\ell = 1),$$
$$\Theta_n = \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{n!(c_1)_n (c_2)_n \cdots (c_s)_n} \text{ and } \Omega_n^m = \left(\frac{\mu}{n+\mu}\right)^m, (n \in \mathbb{N}). \quad (56)$$

Theorem 11. *If $\chi(\xi) \in \mathscr{MS}_\ell^{m,\lambda}(a_1, c_1; b; h; X, Y)$, then, we have for all $\mu \in \mathbb{C}$*

$$\left|d_{2-p} - \mu d_{1-p}^2\right| \le \frac{|b|\left|X\left(pe^{-i\sigma}-\tau\right) - Y\left(pe^{i\sigma}-\tau\right)\right|R_1}{2\left|(-p)^{\lambda-1}\{p\Psi_2 + (2-p)\lambda\}\Theta_2\Omega_2^m\right|} \cdot \max\{1, |2\mathscr{Q}_1 - 1|\}, \quad (57)$$

where $\mathscr{Q}_1$ is given by

$$\mathscr{Q}_1 = \frac{1}{4}\left\{(Y+1)R_1 + 2\left(1-\frac{R_2}{R_1}\right) - \frac{e^{-i\sigma}b\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]\left\{2p^2\Psi_1^2 + 2p(1-p)\Psi_1 + \lambda(\lambda-1)(1-p)^2\right\}R_1}{2(-p)^\lambda[p\Psi_1 + (1-p)\lambda]^2} - \frac{\mu e^{-i\sigma}bR_1\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]\{p\Psi_2+(2-p)\lambda\}\Theta_2\Omega_2^m}{(-p)^{\lambda-1}[p\Psi_1+(1-p)\lambda]^2\Theta_1^2\Omega_1^{2m}}\right\}. \quad (58)$$

The inequality is sharp for each $\mu \in \mathbb{C}$.

Proof. As $\chi \in \mathscr{MS}_\ell^{m,\lambda}(a_1, c_1; b; h; X, Y)$, by (23), we have

$$\frac{\xi^{(p+1)\lambda-p}\left[I_\mu^m(a_1,c_1)\chi'(\xi)\right]^\lambda}{\chi_\ell(m,\mu,a_1,c_1;\xi)} - (-p)^\lambda = -b\left[e^{-i\sigma}\Delta_\sigma^\tau[w(\xi)] - p\right]. \quad (59)$$

Thus, let $\vartheta \in \mathscr{P}$ be of the form $\vartheta(\xi) = 1 + \sum_{\ell=1}^\infty \vartheta_n \xi^n$ and defined by

$$\vartheta(\xi) = \frac{1+w(\xi)}{1-w(\xi)}, \xi \in \Omega. \quad (60)$$

On computation, we have

$$w(\xi) = \frac{1}{2}\vartheta_1\xi + \frac{1}{2}\left(\vartheta_2 - \frac{1}{2}\vartheta_1^2\right)\xi^2 + \frac{1}{2}\left(\vartheta_3 - \vartheta_1\vartheta_2 + \frac{1}{4}\vartheta_1^3\right)\xi^3 + \cdots, \xi \in \Omega. \quad (61)$$

The right hand side of (58)

$$-b\left\{e^{-i\sigma}\Delta_\sigma^\tau[w(\xi)] - p\right\} = -\frac{be^{-i\sigma}R_1\vartheta_1\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]}{4}\xi - \frac{be^{-i\sigma}\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]R_1}{4} \cdot \left[\vartheta_2 - \vartheta_1^2\left(\frac{(Y+1)R_1 + 2(1-(R_2/R_1))}{4}\right)\right]\xi^2 + \cdots. \quad (62)$$

From the left hand side of (58) is given by

$$\frac{\xi^{(p+1)\lambda-p}\left[I_{\mu}^{m}(a_1,c_1)\chi'(\xi)\right]^{\lambda}}{\chi_{\ell}(m,\mu,a_1,c_1;\xi)}-(-p)^{\lambda}$$
$$=(-p)^{\lambda-1}[p\Psi_1+(1-p)\lambda]\Theta_1\Omega_1^m d_{1-p}\xi+\frac{(-p)^{\lambda-1}}{2p}$$
$$\cdot\left[2p\{p\Psi_2+(2-p)\lambda\}\Theta_2\Omega_2^m d_{2-p}-\{2p^2\Psi_1^2+2p(1-p)\Psi_1+\lambda(\lambda-1)(1-p)^2\}d_{1-p}^2\Theta_1^2\Omega_1^{2m}\right]\xi^2+\cdots. \tag{63}$$

From (61) and (62), we obtain

$$d_{1-p}=-\frac{e^{-i\sigma}bR_1\vartheta_1\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]}{4(-p)^{\lambda-1}[p\Psi_1+(1-p)\lambda]\Theta_1\Omega_1^m},$$

$$d_{2-p}=-\frac{e^{-i\sigma}b\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]R_1}{4(-p)^{\lambda-1}\{p\Psi_2+(2-p)\lambda\}\Theta_2\Omega_2^m}\left[\vartheta_2-\frac{1}{4}\left\{(Y+1)R_1+2\left(1-\frac{R_2}{R_1}\right)\right.\right.$$
$$\left.\left.-\frac{e^{-i\sigma}b\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]\{2p^2\Psi_1^2+2p(1-p)\Psi_1+\lambda(\lambda-1)(1-p)^2\}R_1}{2(-p)^{\lambda}[p\Psi_1+(1-p)\lambda]^2}\right\}\vartheta_1^2\right]. \tag{64}$$

Now we consider

$$\left|d_{2-p}-\mu d_{1-p}^2\right|=\left|-\frac{e^{-i\sigma}b\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]R_1}{4(-p)^{\lambda-1}\{p\Psi_2+(2-p)\lambda\}\Theta_2\Omega_2^m}\left[\vartheta_2-\frac{1}{4}\left\{(Y+1)R_1+2\left(1-\frac{R_2}{R_1}\right)\right.\right.\right.$$
$$\left.\left.-\frac{e^{-i\sigma}b\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]\{2p^2\Psi_1^2+2p(1-p)\Psi_1+\lambda(\lambda-1)(1-p)^2\}R_1}{2(-p)^{\lambda}[p\Psi_1+(1-p)\lambda]^2}\right\}\vartheta_1^2\right]$$
$$\left.-\frac{\mu e^{-2i\sigma}b^2R_1^2\vartheta_1^2\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]^2}{16(-p)^{2\lambda-2}[p\Psi_1+(1-p)\lambda]^2\Theta_1^2\Omega_1^{2m}}\right|=\left|-\frac{e^{-i\sigma}b\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]R_1}{4(-p)^{\lambda-1}\{p\Psi_2+(2-p)\lambda\}\Theta_2\Omega_2^m}\right.$$
$$\cdot\left[\vartheta_2-\frac{\vartheta_1^2}{4}\left\{(Y+1)R_1+2\left(1-\frac{R_2}{R_1}\right)-\frac{e^{-i\sigma}b\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]\{2p^2\Psi_1^2+2p(1-p)\Psi_1+\lambda(\lambda-1)(1-p)^2\}R_1}{2(-p)^{\lambda}[p\Psi_1+(1-p)\lambda]^2}\right.\right.$$
$$\left.\left.\left.-\frac{\mu e^{-i\sigma}bR_1\left[X\left(pe^{-i\sigma}-\tau\right)-Y\left(pe^{i\sigma}-\tau\right)\right]\{p\Psi_2+(2-p)\lambda\}\Theta_2\Omega_2^m}{(-p)^{\lambda-1}[p\Psi_1+(1-p)\lambda]^2\Theta_1^2\Omega_1^{2m}}\right\}\right]\right|. \tag{65}$$

On applying Lemma 5, we get the assertion. □

To demonstrate the applications of our results, here, we provide the most simple special case of our result. Note that the following result was obtained [[40], Theorem 6] for functions in $\chi\in\mathscr{A}$.

Corollary 12. *If* $\chi(\xi)\in\mathscr{L}_1$ *satisfies*

$$-\frac{\xi\chi'(\xi)}{\chi(\xi)}\prec h(\xi), \tag{66}$$

and $h(\xi)=1+R_1\xi+R_2\xi^2+\cdots$, *with* $R_1,R_2\in\mathbb{R}$, $R_1>0$, *then for all* $\mu\in\mathbb{C}$ *we have*

$$\left|d_1-\mu d_0^2\right|\leq\frac{R_1}{2}\max\left\{1;\left|\frac{R_2}{R_1}-R_1+2\mu R_1\right|\right\}. \tag{67}$$

The inequality is sharp for the function χ_* *given by*

$$\chi_*(\xi)=\begin{cases}\xi\exp\displaystyle\int_0^{\xi}-\frac{h(t)+1}{t}dt, & \text{if }\left|\dfrac{R_2}{R_1}-R_1+2\mu R_1\right|\geq 1,\\ \xi\exp\displaystyle\int_0^{\xi}-\frac{h(t^2)+1}{t}dt, & \text{if }\left|\dfrac{R_2}{R_1}-R_1+2\mu R_1\right|\leq 1.\end{cases} \tag{68}$$

Proof. In Theorem 11, taking $r=2, s=1$, $a_1=b_1$, $a_2=1$, $X=1$, $Y=-1$, $m=\sigma=\tau=0$, and $\ell=\lambda=p=1$, we get the inequality

$$\left|d_1-\mu d_0^2\right|\leq\begin{cases}\dfrac{R_1}{2}, & \text{if }\left|\dfrac{R_2}{R_1}-R_1+2\mu R_1\right|\leq 1,\\ \dfrac{R_1}{2}\left|\dfrac{R_2}{R_1}-R_1+2\mu R_1\right|, & \text{if }\left|\dfrac{R_2}{R_1}-R_1+2\mu R_1\right|\geq 1.\end{cases} \tag{69}$$

□

Analogous to Theorem 11, we can prove the following.

Theorem 13. *If* $\chi(\xi) \in \mathcal{QMS}_{\ell}^{m,\lambda}(a_1, c_1; b; h; X, Y)$, *then, we have for all* $\mu \in \mathbb{C}$

$$\left| d_{2-p} - \mu d_{1-p}^2 \right| \leq \frac{|b| \left| X\left([p]_q e^{-i\sigma} - \tau\right) - Y\left([p]_q e^{i\sigma} - \tau\right) \right| R_1}{2\left| \left([-p]_q\right)^{\lambda-1} \left\{ [2-p]_q \lambda - [-p]_q \Psi_2 \right\} \Theta_2 \Omega_2^m \right|} \cdot \max\left\{1, |2\mathcal{Q}_2 - 1|\right\}, \tag{70}$$

where $\mathcal{Q}_2$ *is given by*

$$\begin{aligned} \mathcal{Q}_2 = \frac{1}{4}\Bigg\{ &(Y+1)R_1 + 2\left(1 - \frac{R_2}{R_1}\right) \\ &- \frac{e^{-i\sigma} b \left[X\left([p]_q e^{-i\sigma} - \tau\right) - Y\left([p]_q e^{i\sigma} - \tau\right)\right] \left\{2[-p]_q^2 \Psi_1^2 - 2[-p]_q [1-p]_q \Psi_1 + \lambda(\lambda-1)[1-p]_q^2\right\} R_1}{2(-p)^{\lambda} \left\{[1-p]_q \lambda - [-p]_q \Psi_1\right\}^2} \\ &- \frac{\mu e^{-i\sigma} b R_1 \left[X\left([p]_q e^{-i\sigma} - \tau\right) - Y\left([p]_q e^{i\sigma} - \tau\right)\right] \left\{[2-p]_q \lambda - [-p]_q \Psi_2\right\} \Theta_2 \Omega_2^m}{(-p)^{\lambda-1} \left\{[1-p]_q \lambda - [-p]_q \Psi_1\right\}^2 \Theta_1^2 \Omega_1^{2m}} \Bigg\}. \end{aligned} \tag{71}$$

The inequality is sharp.

5. Conclusions

The defined function class $\mathcal{MS}_{\ell}^{m,\lambda}(a_1, c_1; b; h; X, Y)$ though familiar with so called pseudo-star-like functions required lots of adaptation since it involves functions with a removable singularity of order p at the origin. Integral representation and Fekete-Szegö inequalities have been established. Further, we extend the class $\mathcal{MS}_{\ell}^{m,\lambda}(a_1, c_1; b; h; X, Y)$ by replacing the classical derivative with q-derivative. Since all the results involving classical derivative does not get translated to the results involving q-derivative, we used some modified conditions to obtain our main results. We note that these adaptation are essential for future research.

Data Availability

Not applicable.

Conflicts of Interest

Authors declare that they have no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All the authors have read and agreed to the published version of the manuscript.

Acknowledgments

Authors thank the special issue editors and the referee for their helpful comments and suggestions, which helped us to improve the results and its presentation.

References

[1] L. Shi, Z.-G. Wang, and M.-H. Zeng, "Some subclasses of multivalent spirallike meromorphic functions," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.

[2] N. Uyanik, H. Shiraishi, S. Owa, and Y. Polatoglu, "Reciprocal classes of p-valently spirallike and p-valently Robertson functions," *Journal of Inequalities and Applications*, vol. 2011, no. 1, 2011.

[3] W. Janowski, "Some extremal problems for certain families of analytic functions I," *Annales Polonici Mathematici*, vol. 28, no. 3, pp. 297–326, 1973.

[4] M. K. Aouf, "On a class ofp-valent starlike functions of order α," *International Journal of Mathematics and Mathematical Sciences*, vol. 10, no. 4, p. 744, 1987.

[5] D. Breaz, K. R. Karthikeyan, and A. Senguttuvan, "Multivalent prestarlike functions with respect to symmetric points," *Symmetry*, vol. 14, no. 1, p. 20, 2022.

[6] L. I. Cotirla and K. R. Karthikeyan, "Classes of multivalent spirallike functions associated with symmetric regions," *Symmetry*, vol. 14, no. 8, 2022.

[7] K. R. Karthikeyan and G. Murugusundaramoorthy, "Unified solution of initial coefficients and Fekete-Szegö problem for subclasses of analytic functions related to a conic region," *Afrika Matematika*, vol. 33, no. 2, 2022.

[8] B. Khan, H. M. Srivastava, N. Khan, M. Darus, M. Tahir, and Q. Z. Ahmad, "Coefficient estimates for a subclass of analytic functions associated with a certain leaf-like domain," *Mathematics*, vol. 8, no. 8, p. 1334, 2020.

[9] M. H. Priya and R. B. Sharma, "On a class of bounded turning functions subordinate to a leaf-like domain," *Journal of Physics Conference Series*, vol. 1000, article 012056, 2018.

[10] K. R. Karthikeyan, S. Lakshmi, S. Varadharajan, D. Mohankumar, and E. Umadevi, "Starlike functions of complex order with respect to symmetric points defined using higher order derivatives," *Fractal and Fractional*, vol. 6, no. 2, p. 116, 2022.

[11] K. I. Noor and S. N. Malik, "On coefficient inequalities of functions associated with conic domains," *Computers & Mathematcs with Applications*, vol. 62, no. 5, pp. 2209–2217, 2011.

[12] K. Sakaguchi, "On a certain univalent mapping," *Journal of the Mathematical Society of Japan*, vol. 11, no. 1, pp. 72–75, 1959.

[13] C. Selvaraj and K. R. Karthikeyan, "Some inclusion relationships for certain subclasses of meromorphic functions associated with a family of integral operators," *Acta Mathematica Universitatis Comenianae. New Series*, vol. 78, no. 2, pp. 245–254, 2009.

[14] M. K. Aouf, A. O. Mostafa, A. M. Shahin, and S. M. Madian, "Some inclusion properties of p-valent meromorphic functions defined by the Wright generalized hypergeometric function," *Proceedings of the Pakistan Academy of Sciences*, vol. 49, no. 1, pp. 45–52, 2012.

[15] J.-L. Liu and H. M. Srivastava, "A linear operator and associated families of meromorphically multivalent functions," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 2, pp. 566–581, 2001.

[16] J.-L. Liu and H. M. Srivastava, "Classes of meromorphically multivalent functions associated with the generalized hypergeometric function," *Mathematical and Computer Modelling*, vol. 39, no. 1, pp. 21–34, 2004.

[17] H. M. Srivastava, H. M. Hossen, and M. K. Aouf, "A unified presentation of some classes of meromorphically multivalent functions," *Computers & Mathematcs with Applications*, vol. 38, no. 11-12, pp. 63–70, 1999.

[18] M. H. Annaby and Z. S. Mansour, q-*Fractional Calculus and Equations, Lecture Notes in Mathematics*, vol. 2056, Springer, Heidelberg, 2012.

[19] A. Aral, V. Gupta, and R. P. Agarwal, *Applications of* q-*Calculus in Operator Theory*, Springer, New York, 2013.

[20] F. H. Jackson, "On q-definite integrals," *The Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.

[21] H. M. Srivastava, "Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis," *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 44, no. 1, pp. 327–344, 2020.

[22] H. M. Srivastava, "Univalent functions, fractional calculus, and associated generalized hypergeometric functions," in *Univalent Functions, Fractional Calculus, and Their Applications (H. M. Srivastava, S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester)*, pp. 329–354, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[23] K. R. Karthikeyan, G. Murugusundaramoorthy, and T. Bulboaca, "Properties of λ-pseudo-starlike functions of complex order defined by subordination," *Axioms*, vol. 10, no. 2, p. 86, 2021.

[24] H. M. Srivastava, Q. Z. Ahmad, N. Khan, N. Khan, and B. Khan, "Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain," *Mathematics*, vol. 7, no. 2, pp. 181–215, 2019.

[25] H. M. Srivastava, B. Khan, N. Khan, and Q. Z. Ahmad, "Coefficient inequalities for q-starlike functions associated with the Janowski functions," *Hokkaido Mathematical Journal*, vol. 48, no. 2, pp. 407–425, 2019.

[26] H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad, and M. Tahir, "A generalized conic domain and its applications to certain subclasses of analytic functions," *The Rocky Mountain Journal of Mathematics*, vol. 49, pp. 2325–2346, 2019.

[27] M. K. Aouf, "Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function," *Computers & Mathematcs with Applications*, vol. 55, no. 3, pp. 494–509, 2008.

[28] M. K. Aouf, "Certain subclasses of meromorphically p-valent functions with positive or negative coefficients," *Mathematical and Computer Modelling*, vol. 47, no. 9-10, pp. 997–1008, 2008.

[29] D. Breaz, K. R. Karthikeyan, and E. Umadevi, "Subclasses of multivalent meromorphic functions with a pole of order p at the origin," *Mathematics*, vol. 10, no. 4, p. 600, 2022.

[30] H. M. Srivastava, N. Khan, M. Darus, M. T. Rahim, Q. Z. Ahmad, and Y. Zeb, "Properties of spiral-like close-to-convex functions associated with conic domains," *Mathematics*, vol. 7, no. 8, pp. 706–712, 2019.

[31] H. M. Srivastava, N. Raza, E. S. AbuJarad, G. Srivastava, and M. H. AbuJarad, "Fekete-Szegö inequality for classes of (p, q)-starlike and (p, q)-convex functions," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, no. 4, pp. 3563–3584, 2019.

[32] H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, and N. Khan, "Some general classes of q-starlike functions associated with the Janowski functions," *Symmetry*, vol. 11, pp. 1–14, 2019.

[33] H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, and N. Khan, "Some general families of q-starlike functions associated with the Janowski functions," *Univerzitet u Nišu*, vol. 33, pp. 2613–2626, 2019.

[34] H. M. Srivastava, D.-G. Yang, and N.-E. Xu, "Some subclasses of meromorphically multivalent functions associated with a linear operator," *Applied Mathematics and Computation*, vol. 195, no. 1, pp. 11–23, 2008.

[35] Z.-G. Wang, Z.-H. Liu, and R.-G. Xiang, "Some criteria for meromorphic multivalent starlike functions," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 1107–1111, 2011.

[36] Z.-G. Wang, Y. Sun, and Z.-H. Zhang, "Certain classes of meromorphic multivalent functions," *Computers & Mathematcs with Applications*, vol. 58, no. 7, pp. 1408–1417, 2009.

[37] S. Agrawal and S. K. Sahoo, "A generalization of starlike functions of order alpha," *Hokkaido Mathematical Journal*, vol. 46, no. 1, pp. 15–27, 2017.

[38] W. C. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *proceedings of the conference on complex analysis*, pp. 157–169, Tianjin, 1992.

[39] M. K. Aouf, R. M. El-Ashwah, and S. M. El-Deeb, "Fekete-Szegö inequalities for starlike functions with respect tok-symmetric points of complex order," *Journal of Complex Analysis*, vol. 2014, Article ID 131475, 10 pages, 2014.

[40] Z. Tu and L. Xiong, "Unified solution of Fekete-Szegö problem for subclasses of starlike mappings in several complex variables," *Mathematica Slovaca*, vol. 69, no. 4, pp. 843–856, 2019.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 8356125, 10 pages
https://doi.org/10.1155/2022/8356125

Research Article

Some Sharp Results on Coefficient Estimate Problems for Four-Leaf-Type Bounded Turning Functions

Pongsakorn Sunthrayuth[1] **Yousef Jawarneh,**[2] **Muhammad Naeem**[3] **Naveed Iqbal,**[2] **and Jeevan Kafle**[4]

[1]*Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT) Thanyaburi, Pathumthani, Thailand*
[2]*Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia*
[3]*Deanship of Joint First Year Umm Al-Qura University Makkah, P.O. Box 715, Saudi Arabia*
[4]*Central Department of Mathematics, Tribhuvan University, Kritipur, Kathmandu, Nepal*

Correspondence should be addressed to Muhammad Naeem; mfaridoon@uqu.edu.sa and Jeevan Kafle; jeevan.kafle@cdmath.tu.edu.np

Received 31 May 2022; Revised 30 June 2022; Accepted 14 July 2022; Published 2 August 2022

Academic Editor: Sarfraz Nawaz Malik

In this study, we focused on a subclass of bounded turning functions that are linked with a four-leaf-type domain. The primary goal of this study is to explore the limits of the first four initial coefficients, the Fekete-Szegö type inequality, the Zalcman inequality, the Kruskal inequality, and the estimation of the second-order Hankel determinant for functions in this class. All of the obtained findings have been sharp.

1. Introduction and Definitions

Before getting into the key findings, some prior information on function theory fundamentals is required. In this case, the symbols $\mathscr{A}$ and $\mathscr{S}$ indicate the families of normalised holomorphic and univalent functions, respectively. These families are specified in the set-builder form:

$$\mathscr{A} = \left\{ g \in \mathcal{Q}(\mathscr{U}_d) : g(0) = g'(0) - 1 = 0 (z \in \mathscr{U}_d) \right\}, \quad (1)$$

$$\mathscr{S} = \{ g \in \mathscr{A} : g \text{ is univalent in } \mathscr{U}_d \}, \quad (2)$$

where $\mathcal{Q}(\mathscr{U}_d)$ stands for the set of analytic (holomorphic) functions in the disc $\mathscr{U}_d = \{z \in \mathbb{C} \text{and} |z| < 1\}$. Thus, if $g \in \mathscr{A}$, then, it can be stated in the series expansion form by

$$g(z) = z + \sum_{k=2}^{\infty} a_k z^k (z \in \mathscr{U}_d). \quad (3)$$

For the given functions $G_1, G_2 \in \mathcal{Q}(\mathscr{U}_d)$, the function G_1 is subordinated by G_2 (stated mathematically by $G_1 \prec G_2$) if there exists a holomorphic function v in $\mathscr{U}_d$ with the restrictions $v(0) = 0$ and $|v(z)| < 1$ such that $G_1(z) = G_2(v(z))$. Moreover, if G_2 is univalent in $\mathscr{U}_d$, then

$$G_1(z) \prec G_2(z), (z \in \mathscr{U}_d) \Leftrightarrow G_1(0) = G_2(0) \text{ and } G_1(\mathscr{U}_d) \subset G_2(\mathscr{U}_d). \quad (4)$$

Although the function theory was created in 1851, Bieberbach [1] presented the coefficient hypothesis in 1916, and it made the topic a hit as a promising new research field. De-Brages [2] proved this conjecture in 1985. From 1916 to 1985, many of the world's most distinguished scholars sought to prove or disprove this claim. As a result, they investigated a number of subfamilies of the class $\mathscr{S}$ of univalent functions that are associated with various image domains [3–5]. The most fundamental and significant subclasses of the set $\mathcal{S}$ are the families of starlike and convex functions, represented by $\mathcal{S}^*$ and $\mathcal{K}$, respectively. Ma and Minda [6] defined the unified form of the family in 1992 as

$$\mathcal{S}^*(\phi) := \left\{ g \in \mathcal{A} : \frac{zg'(z)}{g(z)} \prec \phi(z)(z \in \mathcal{U}_d) \right\}, \tag{5}$$

where ϕ indicates the analytic function with $\phi'(0) > 0$ and $\mathfrak{Re}\phi > 0$. Also, the region $\phi(\mathcal{U}_d)$ is star-shaped about $\phi(0) = 1$ and is symmetric along the real axis. They examined some interesting aspects of this class. Some significant subfamilies of the collection $\mathcal{A}$ have recently been investigated as unique instances of the class $\mathcal{S}^*(\phi)$. In particular;

(i) The class $\mathcal{S}^*[L, M] \equiv \mathcal{S}^*(1 + Lz/1 + Mz)$, $-1 \le M < L \le 1$, is obtained by selecting $\phi(z) = 1 + Lz/1 + Mz$ and was established in [7]. Moreover, $\mathcal{S}^*(\xi) := \mathcal{S}^*[1 - 2\xi, -1]$ displays the well-known order ξ $(0 \le \xi < 1)$ starlike function class

(ii) The class $\mathcal{S}^*_{\mathcal{L}} := \mathcal{S}^*(\phi(z))$ with $\phi(z) = \sqrt{1+z}$ was designed by the researchers Sokól and Stankiewicz in [8]. Also, they showed that the image of the function $\phi(z) = \sqrt{1+z}$ is bounded by $|w^2 - 1| < 1$.

(iii) The set $\mathcal{S}^*_{\text{car}} := \mathcal{S}^*(\phi(z))$ with $\phi(z) = 1 + 4/3z + 2/3z^2$ has been deduced by Sharma and his coauthors [9] in which they located the image domain of $\phi(z) = 1 + 4/3z + 2/3z^2$, which is bounded by the below cardioid

$$\left(9x^2 + 9y^2 - 18x + 5\right)^2 - 16\left(9x^2 + 9y^2 - 6x + 1\right) = 0. \tag{6}$$

(iv) By selecting $\phi(z) = 1 + \sin z$, we get the class $\mathcal{S}^*(\phi(z)) = \mathcal{S}^*_{\sin}$, which was defined in [10] while $\mathcal{S}^*_e \equiv \mathcal{S}^*(e^z)$ was contributed by the authors [11] and, subsequently, explored some more properties of it in [12]. This class was recently generalized by Srivastava et al. [13] in which the authors determined upper bound of Hankel determinant of order three

(v) The family $\mathcal{S}^*_{\cos} := \mathcal{S}^*(\cos(z))$ and $\mathcal{S}^*_{\cosh} := \mathcal{S}^*(\cosh(z))$ were offered, respectively, by Raza and Bano [14] and Alotaibi et al. [15]. In both the papers, the authors studied some good properties of these families

(vi) By choosing $\phi(z) = 1 + \sinh^{-1}z$, we obtain the recently studied class $\mathcal{S}^*_\rho := \mathcal{S}^*(1 + \sinh^{-1}z)$ created by Al-Sawalha [16]. Barukab and his coauthors [17] studied the sharp Hankel determinant of third-order for the following class in 2021

$$\mathcal{R}_s = \left\{ g \in \mathcal{A} : g'(z) \prec 1 + \sinh^{-1}z, z \in \mathcal{U}_d \right\}. \tag{7}$$

In [18, 19], Pommerenke provided the following Hankel determinant $\mathcal{D}_{q,n}(g)$ containing coefficients of a function $g \in \mathcal{S}$

$$\mathcal{D}_{q,n}(g) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \tag{8}$$

with $q, n \in \mathbb{N} = \{1, 2, \cdots\}$. By varying the parameters q and n, we get the determinants listed below:

$$\mathcal{D}_{2,1}(g) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \tag{9}$$

$$\mathcal{D}_{2,2}(g) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2, \tag{10}$$

$$\mathcal{D}_{3,1}(g) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3\left(a_2a_4 - a_3^2\right) - a_4(a_4 - a_2a_3) + a_5\left(a_3 - a_2^2\right), \tag{11}$$

that referred as first-, second-, and third-order Hankel determinants, respectively. The Hankel determinant for functions belonging to the general family $\mathcal{S}$ has just a few references in the literature. The best established sharp inequality for the function $g \in \mathcal{S}$ is $|\mathcal{D}_{2,n}(g)| \le \lambda\sqrt{n}$, where λ is a constant, and it is because of Hayman [20]. Additionally, it was determined in [21] for the class $\mathcal{S}$ that

$$\left|\mathcal{D}_{2,2}(g)\right| \le \lambda, \text{ for } 1 \le \lambda \le \frac{11}{3}, \tag{12}$$

$$\left|\mathcal{D}_{3,1}(g)\right| \le \mu, \text{ for } \frac{4}{9} \le \mu \le \frac{32 + \sqrt{285}}{15}. \tag{13}$$

Several mathematicians were drawn to the problem of finding the sharp bounds of Hankel determinants in a given family of functions. In this context, Janteng et al. [22, 23] estimated the sharp bounds of $|\mathcal{D}_{2,2}(g)|$, for three basic subfamilies of the set $\mathcal{S}$. These families are $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{R}$ (functions of a bounded turning class), and these bounds are stated as

$$\left|\mathcal{D}_{2,2}(g)\right| \le \begin{cases} \frac{1}{8}, & \text{for } g \in \mathcal{K}, \\ 1, & \text{for } g \in \mathcal{S}^*, \\ \frac{4}{9}, & \text{for } g \in \mathcal{R}. \end{cases} \tag{14}$$

This determinant's exact bound for the unified collection $\mathcal{S}^*(\phi)$ was determined in [24] and subsequently investigated in [25]. In [26–28], this problem was also solved for various families of biunivalent functions.

The formulae provided in (11) make it abundantly evident that the computation of $|\mathcal{D}_{3,1}(g)|$ is much more difficult than determining the bound of $|\mathcal{D}_{2,2}(g)|$. Babalola [29] was the first mathematician who studied third-order Hankel determinant for the $\mathcal{K}$,$\mathcal{S}^*$, and $\mathcal{R}$ families in 2010. Following that, several academics [30–34] used the same method to publish papers regarding $|\mathcal{D}_{3,1}(g)|$ for specific subclasses of univalent functions. However, Zaprawa's work [35] caught the researcher's attention, in which he improved Babalola's results by utilising a revolutionary method to show that

$$|\mathcal{D}_{3,1}(g)| \le \begin{cases} \frac{49}{540}, & \text{for } g \in \mathcal{K}, \\ 1, & \text{for } g \in \mathcal{S}^*, \\ \frac{41}{60}, & \text{for } g \in \mathcal{R}. \end{cases} \quad (15)$$

He also pointed out that these bounds are not sharp. In 2018, Kwon et al. [36] achieved a more acceptable finding for $g \in \mathcal{S}^*$ and demonstrated that $|\mathcal{D}_{3,1}(g)| \le 8/8$, and this limit was further enhanced by Zaprawa and his coauthors [37] in 2021. They got $|\mathcal{D}_{3,1}(g)| \le 5/9$ for $g \in \mathcal{S}^*$. In recent years, Kowalczyk et al. [38] and Lecko et al. [39] got a sharp bound of third Hankel determinant given by

$$|\mathcal{D}_{3,1}(g)| \le \begin{cases} \frac{4}{135}, & \text{for } g \in \mathcal{K}, \\ \frac{1}{9}, & \text{for } g \in \mathcal{S}^*\left(\frac{1}{2}\right), \end{cases} \quad (16)$$

where $\mathcal{S}^*(1/2)$ is the starlike functions family of order 1/2. In [40], the authors obtained the sharp bounds of third Hankel determinant for the subclass of $\mathcal{S}^*_{\sin}$, and Mahmood et al. [41] calculated the third Hankel determinant for starlike functions in q-analogue. For some new literature on sharp third-order Hankel determinant, see [42–45].

In [46], Gandhi introduced a family of bounded turning function connected with a four-leaf function defined by

$$\mathcal{S}^*_{4\mathcal{L}} := \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5, (z \in \mathcal{U}_d) \right\}, \quad (17)$$

and characterized it with some important properties.

Similar to the definition of $\mathcal{S}^*_{4\mathcal{L}}$, we now define a new subfamily of bounded turning functions by the following set builder notation:

$$\mathcal{BT}_{4\mathcal{L}} := \left\{ g \in \mathcal{S} : g'(z) \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5, (z \in \mathcal{U}_d) \right\}. \quad (18)$$

The aim of the current manuscript is to determine the exact bounds of the coefficient inequalities, Fekete-Szegö type problem, Kruskal inequality, and Hankel determinant of order two for functions of bounded turning class linked with four-leaf domain.

2. A Set of Lemmas

We say a function $p \in \mathcal{P}$ if and only if it has the series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n (z \in \mathcal{U}_d), \quad (19)$$

along with the $\Re p(z) \ge 0 (z \in \mathcal{U}_d)$.

Lemma 1. *Let $p \in \mathcal{P}$ be represented by (19). Then*

$$|c_n| \le 2\, n \ge 1. \quad (20)$$

$$|c_{n+k} - \mu c_n c_k| \le 2 \max\{1, |2\mu - 1|\} = \begin{cases} 2 & \text{for } 0 \le \mu \le 1; \\ 2|2\mu - 1| & \text{otherwise.} \end{cases} \quad (21)$$

Also, If $B \in [0, 1]$ with $B(2B - 1) \le D \le B$, we have

$$\left|c_3 - 2Bc_1c_2 + Dc_1^3\right| \le 2. \quad (22)$$

These inequalities (20), (21), and (22) are taken from [47, 48].

Lemma 2. *Let $p \in \mathcal{P}$ and be given by (19). Then, for $x, \delta, \rho \in \bar{\mathcal{U}}_d$, we have*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \quad (23)$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\delta, \quad (24)$$

For the formula c_2, see [48]. The formula c_3 was due to Zlotkiewicz and Libera [49] while the formula for c_4 was proved in [50].

Lemma 3 [51]. *Let α, β, γ, and a satisfy that $a, \alpha \in (0, 1)$ and*

$$\begin{aligned} 8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) \\ + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \le 4a\alpha^2(1-\alpha)^2(1-a). \end{aligned} \quad (25)$$

If $p \in \mathcal{P}$ and be given by (19), then

$$\left|\gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4\right| \le 2. \quad (26)$$

3. Coefficient Inequalities for the Class $\mathcal{BT}_{4\mathcal{L}}$

We begin this section by finding the absolute values of the first four initial coefficients for the function $\mathcal{BT}_{4\mathcal{L}}$.

Theorem 4. *If $g \in \mathcal{BT}_{4\mathcal{L}}$ and has the series representation (3), then*

$$|a_2| \leq \frac{5}{12}, \tag{27}$$

$$|a_3| \leq \frac{5}{18}, \tag{28}$$

$$|a_4| \leq \frac{5}{24}, \tag{29}$$

$$|a_5| \leq \frac{1}{6}. \tag{30}$$

These bounds are best possible.

Proof. Let $g \in \mathcal{BT}_{4\mathcal{L}}$. Then, (18) can be written in the form of Schwarz function as

$$g'(z) = 1 + \frac{5}{6}\omega(z) + \frac{1}{6}(\omega(z))^5, (z \in \mathcal{U}_d). \tag{31}$$

If $p \in \mathcal{P}$, and it may be written in terms of Schwarz function $w(z)$ as

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \tag{32}$$

Equivalently, we have

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}. \tag{33}$$

where

$$\begin{aligned}\omega(z) = {} & \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\right)z^3 \\ & + \left(\frac{1}{2}c_4 - \frac{1}{2}c_1c_3 - \frac{1}{4}c_2^2 - \frac{1}{16}c_1^4 + \frac{3}{8}c_1^2c_2\right)z^4 + \cdots.\end{aligned} \tag{34}$$

From (3), we get

$$g'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots. \tag{35}$$

By simplication and using the series expansion of (34), we get

$$\begin{aligned}1 + \frac{5}{6}\omega(z) + \frac{1}{6}(\omega(z))^5 = {} & 1 + \left(\frac{5}{12}c_1\right)z + \left(-\frac{5}{24}c_1^2 + \frac{5}{12}c_2\right)z^2 \\ & + \left(-\frac{5}{12}c_1c_2 + \frac{5}{12}c_3 + \frac{5}{48}c_1^3\right)z^3 \\ & + \left(\frac{5}{12}c_4 + \frac{5}{16}c_1^2c_2 - \frac{5}{96}c_1^4 - \frac{5}{24}c_2^2 \right.\\ & \left. - \frac{5}{12}c_1c_3\right)z^4 + \cdots.\end{aligned} \tag{36}$$

By comparing (35) and (36), we obtain

$$a_2 = \frac{5}{24}c_1, \tag{37}$$

$$a_3 = \frac{1}{3}\left(-\frac{5}{24}c_1^2 + \frac{5}{12}c_2\right), \tag{38}$$

$$a_4 = \frac{1}{4}\left(-\frac{5}{12}c_1c_2 + \frac{5}{12}c_3 + \frac{5}{48}c_1^3\right), \tag{39}$$

$$a_5 = \frac{1}{5}\left(\frac{5}{12}c_4 + \frac{5}{16}c_1^2c_2 - \frac{5}{96}c_1^4 - \frac{5}{24}c_2^2 - \frac{5}{12}c_1c_3\right). \tag{40}$$

For a_2, implementing (20), in (37), we get

$$|a_2| \leq \frac{5}{12}. \tag{41}$$

For a_3, (38) can be written as

$$a_3 = \frac{5}{36}\left(c_2 - \frac{1}{2}c_1^2\right). \tag{42}$$

Using (21), we get

$$|a_3| \leq \frac{5}{18}. \tag{43}$$

For a_4, we can write (39) as

$$|a_4| = \frac{5}{48}\left|\left(c_3 - 2\left(\frac{1}{2}\right)c_1c_2 + \frac{1}{4}c_1^3\right)\right|. \tag{44}$$

From (22), we have

$$0 \leq B = \frac{1}{2} \leq 1, B = \frac{1}{2} \geq D = \frac{1}{4}, \tag{45}$$

$$B(2B - 1) = 0 \leq D = \frac{1}{4}. \tag{46}$$

Application of triangle inequality plus (22) leads us to

$$|a_4| \leq \frac{5}{24}. \tag{47}$$

For a_5, we may write (40) as

$$|a_5| = \left| -\frac{1}{96}c_1^4 - \frac{1}{24}c_2^2 - \frac{1}{12}c_1c_3 + \frac{1}{16}c_1^2c_2 + \frac{1}{12}c_4 \right|. \quad (48)$$

After simplifying, we have

$$|a_5| = \frac{1}{12}\left| \frac{1}{8}c_1^4 + \frac{1}{2}c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{1}{2}\right)c_1^2c_2 - c_4 \right|. \quad (49)$$

Comparing the right side of (49) with

$$\left| \gamma c_1^4 + ac_2^2 + 2\alpha c_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4 \right|, \quad (50)$$

we get

$$\gamma = \frac{1}{8}, a = \frac{1}{2}, \alpha = \frac{1}{2}, \beta = \frac{1}{2}. \quad (51)$$

It follows that

$$\begin{aligned} &8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) \\ &\quad + \alpha(1-\alpha)(\beta - 2a\alpha)^2 = 0, \end{aligned} \quad (52)$$

$$4a\alpha^2(1-\alpha)^2(1-a) = \frac{1}{16}. \quad (53)$$

From (26), we deduce that

$$|a_5| \le \frac{1}{6}. \quad (54)$$

These bounds are best possible and can be determined by the following extremal functions:

$$g_0(z) = \int_0^z \left(1 + \frac{5}{6}(t) + \frac{1}{6}(t^5)\right) dt = z + \frac{5}{12}z^2 + \frac{1}{36}z^6 + \cdots, \quad (55)$$

$$g_1(z) = \int_0^z \left(1 + \frac{5}{6}(t^2) + \frac{1}{6}(t^{10})\right) dt = z + \frac{5}{18}z^3 + \frac{1}{66}z^{11} + \cdots, \quad (56)$$

$$g_2(z) = \int_0^z \left(1 + \frac{5}{6}(t^3) + \frac{1}{6}(t^{15})\right) dt = z + \frac{5}{24}z^4 + \frac{1}{96}z^{16} + \cdots, \quad (57)$$

$$g_3(z) = \int_0^z \left(1 + \frac{5}{6}(t^4) + \frac{1}{6}(t^{20})\right) dt = z + \frac{1}{6}z^5 + \frac{1}{126}z^{21} + \cdots. \quad (58)$$

□

Theorem 5. *If g is of the form (3) belongs to $\mathscr{BT}_{4\mathscr{L}}$, then*

$$\left|a_3 - \gamma a_2^2\right| \le \max\left\{\frac{5}{18}, \frac{25|\gamma|}{144}\right\}, \quad \textit{for} \quad \gamma \in \mathbb{C}. \quad (59)$$

This inequality is sharp.

Proof. By using (37) and (38), we may have

$$\left|a_3 - \gamma a_2^2\right| = \left|\frac{5}{36}c_2 - \frac{5}{72}c_1^2 - \frac{25}{576}\gamma c_1^2\right|. \quad (60)$$

By rearranging, it yields

$$\left|a_3 - \gamma a_2^2\right| = \frac{5}{36}\left|\left(c_2 - \left(\frac{5\gamma + 8}{16}\right)c_1^2\right)\right|. \quad (61)$$

Application of (21) leads us to

$$\left|a_3 - \gamma a_2^2\right| \le \frac{10}{36} \max\left\{1, \left|\frac{5\gamma + 8}{8} - 1\right|\right\}. \quad (62)$$

After the simplification, we get

$$\left|a_3 - \gamma a_2^2\right| \le \max\left\{\frac{5}{18}, \frac{25|\gamma|}{144}\right\}. \quad (63)$$

This required result is sharp and is determined by

$$g_1(z) = \int_0^z \left(1 + \frac{5}{6}(t^2) + \frac{1}{6}(t^{10})\right) dt = z + \frac{5}{18}z^3 + \frac{1}{66}z^{11} + \cdots. \quad (64)$$

□

Theorem 6. *If g has the form (3) belongs to $\mathscr{BT}_{4\mathscr{L}}$, then*

$$\left|a_2a_3 - a_4\right| \le \frac{5}{24}. \quad (65)$$

This inequality is best possible.

Proof. By employing (37), (38), and (39), we have

$$\left|a_2a_3 - a_4\right| = \frac{5}{48}\left|c_3 - 2\left(\frac{23}{36}\right)c_1c_2 + \frac{7}{18}c_1^3\right|. \quad (66)$$

From (22), we have

$$0 \le B = \frac{23}{36} \le 1, B = \frac{23}{36} \ge D = \frac{7}{18}, \quad (67)$$

$$B(2B - 1) = \frac{115}{648} \le D = \frac{7}{18}. \quad (68)$$

Using (22), we obtain

$$|a_2a_3 - a_4| \le \frac{5}{24}. \tag{69}$$

This inequality is best possible and can be obtained by

$$g_2(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^3\right) + \frac{1}{6}\left(t^{15}\right)\right)dt = z + \frac{5}{24}z^4 + \frac{1}{96}z^{16} + \cdots. \tag{70}$$

□

Theorem 7. *If g belongs to $\mathscr{BT}_{4\mathscr{L}}$, and be of the form (3). Then*

$$|a_5 - a_2a_4| \le \frac{1}{6}. \tag{71}$$

This result is sharp.

Proof. From (37), (39), and (40), we obtain

$$|a_5 - a_2a_4| = \left| -\frac{73}{4608}c_1^4 - \frac{1}{24}c_2^2 - \frac{121}{1152}c_1c_3 + \frac{97}{1152}c_1^2c_2 + \frac{1}{12}c_4 \right|. \tag{72}$$

After simplifying, we have

$$|a_5 - a_2a_4| = \frac{1}{12}\left| \frac{73}{384}c_1^4 + \frac{1}{2}c_2^2 + 2\left(\frac{121}{192}\right)c_1c_3 - \frac{3}{2}\left(\frac{97}{144}\right)c_1^2c_2 - c_4 \right|. \tag{73}$$

Comparing the right side of (73) with

$$\left| \gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4 \right|, \tag{74}$$

we get

$$\gamma = \frac{73}{384}, a = \frac{1}{2}, \alpha = \frac{121}{192}, \beta = \frac{97}{144}. \tag{75}$$

It follows that

$$\begin{aligned} &8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) \\ &\quad + \alpha(1-\alpha)(\beta - 2a\alpha)^2 = 0.00735, \end{aligned} \tag{76}$$

and

$$4a\alpha^2(1-\alpha)^2(1-a) = 0.05431. \tag{77}$$

From (26), we deduce that

$$|a_5 - a_2a_4| \le \frac{1}{6}. \tag{78}$$

The required result is sharp and can be determined by

$$g_3(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^4\right) + \frac{1}{6}\left(t^{20}\right)\right)dt = z + \frac{1}{6}z^5 + \frac{1}{126}z^{21} + \cdots. \tag{79}$$

□

Theorem 8. *If $g \in \mathscr{BT}_{4\mathscr{L}}$, and be of the form (3). Then*

$$|a_5 - a_3^2| \le \frac{1}{6}. \tag{80}$$

This inequality is best possible.

Proof. By using (38) and (40), we have

$$|a_5 - a_3^2| = \left| -\frac{79}{5184}c_1^4 - \frac{79}{1296}c_2^2 - \frac{1}{12}c_1c_3 + \frac{53}{648}c_1^2c_2 + \frac{1}{12}c_4 \right|. \tag{81}$$

After simplifying, we have

$$|a_5 - a_3^2| = \frac{1}{12}\left| \frac{79}{432}c_1^4 + \frac{79}{108}c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{53}{81}\right)c_1^2c_2 - c_4 \right|. \tag{82}$$

Comparing the right side of (82) with

$$\left| \gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4 \right|, \tag{83}$$

we get

$$\gamma = \frac{79}{432}, a = \frac{79}{108}, \alpha = \frac{1}{2}, \beta = \frac{53}{81}. \tag{84}$$

It follows that

$$\begin{aligned} &8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) \\ &\quad + \alpha(1-\alpha)(\beta - 2a\alpha)^2 = 0.00616, \end{aligned} \tag{85}$$

$$4a\alpha^2(1-\alpha)^2(1-a) = 0.04910. \tag{86}$$

From (26), we deduce that

$$|a_5 - a_3^2| \le \frac{1}{6}. \tag{87}$$

This inequality is best possible and can be achieved by

$$g_3(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^4\right) + \frac{1}{6}\left(t^{20}\right)\right)dt = z + \frac{1}{6}z^5 + \frac{1}{126}z^{21} + \cdots. \tag{88}$$

□

4. Kruskal Inequality for the Class $\mathscr{BT}_{4\mathscr{L}}$

In this section, we will give a direct proof of the inequality

$$\left|a_n^p - a_2^{p(n-1)}\right| \le 2^{p(n-1)} - n^p, \tag{89}$$

over the class $\mathscr{BT}_{4\mathscr{L}}$ for the choice of $n=4, p=1$, and for $n=5, p=1$. Krushkal introduced and proved this inequality for the whole class of univalent functions in [52].

Theorem 9. *If g belongs to $\mathscr{BT}_{4\mathscr{L}}$, and be of the form (3). Then*

$$\left|a_4 - a_2^3\right| \le \frac{5}{24}. \tag{90}$$

This result is sharp.

Proof. From (37) and (39), we obtain

$$\left|a_4 - a_2^3\right| = \frac{5}{48}\left|c_3 - 2\left(\frac{1}{2}\right)c_1c_2 + \frac{47}{288}c_1^3\right|. \tag{91}$$

From (22), we have

$$0 \le B = \frac{1}{2} \le 1, B = \frac{1}{2} \ge D = \frac{47}{288}, \tag{92}$$

$$B(2B-1) = 0 \le D = \frac{47}{288}. \tag{93}$$

Using (22), we obtain

$$\left|a_4 - a_2^3\right| \le \frac{5}{24}. \tag{94}$$

This result is sharp and can be obtained by

$$g_2(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^3\right) + \frac{1}{6}\left(t^{15}\right)\right)dt = z + \frac{5}{24}z^4 + \frac{1}{96}z^{16} + \cdots. \tag{95}$$

□

Theorem 10. *If g belongs to $\mathscr{BT}_{4\mathscr{L}}$, and be of the form (3). Then*

$$\left|a_5 - a_2^4\right| \le \frac{1}{6}. \tag{96}$$

This inequality is best possible.

Proof. From (37) and (40), we obtain

$$\left|a_5 - a_2^4\right| = \left|-\frac{4081}{331776}c_1^4 - \frac{1}{24}c_2^2 - \frac{1}{12}c_1c_3 + \frac{1}{16}c_1^2c_2 + \frac{1}{12}c_4\right|. \tag{97}$$

After simplifying, we have

$$\left|a_5 - a_2^4\right| = \frac{1}{12}\left|\frac{4081}{27648}c_1^4 + \frac{1}{2}c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{1}{2}\right)c_1^2c_2 - c_4\right|. \tag{98}$$

Comparing the right side of (98) with

$$\left|\gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4\right|, \tag{99}$$

we get

$$\gamma = \frac{4081}{27648}, a = \frac{1}{2}, \alpha = \frac{1}{2}, \beta = \frac{1}{2}. \tag{100}$$

It follows that

$$\begin{aligned} &8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) \\ &\quad + \alpha(1-\alpha)(\beta - 2a\alpha)^2 = 0.00408, \end{aligned} \tag{101}$$

$$4a\alpha^2(1-\alpha)^2(1-a) = \frac{1}{16}. \tag{102}$$

From (26), we deduce that

$$\left|a_5 - a_2^4\right| \le \frac{1}{6}. \tag{103}$$

This inequality is best possible and can be achieved by

$$g_3(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^4\right) + \frac{1}{6}\left(t^{20}\right)\right)dt = z + \frac{1}{6}z^5 + \frac{1}{126}z^{21} + \cdots. \tag{104}$$

Next, we will calculate the Hankel determinant of order two $|\mathscr{D}_{2,2}(g)|$ for the class $g \in \mathscr{BT}_{4\mathscr{L}}$. □

Theorem 11. *If g belongs to $\mathscr{BT}_{4\mathscr{L}}$, then*

$$\left|\mathscr{D}_{2,2}(g)\right| \le \frac{25}{324}. \tag{105}$$

This inequality is sharp.

Proof. The $\mathscr{D}_{2,2}(g)$ can be written as follows:

$$\mathscr{D}_{2,2}(g) = a_2a_4 - a_3^2. \tag{106}$$

From (37), (38), and (39), we have

$$\mathscr{D}_{2,2}(g) = \frac{25}{1152}c_1c_3 - \frac{25}{10368}c_1^2c_2 + \frac{25}{41472}c_1^4 - \frac{25}{1296}c_2^2. \tag{107}$$

Using (23) and (24) to express c_2 and c_3 in terms of c_1 and, noting that without loss in generality we can write $c_1 = c$, with $0 \le c \le 2$, we obtain

$$\left|\mathscr{D}_{2,2}(g)\right| = \left|-\frac{25}{4608}c^2\left(4-c^2\right)x^2 + \frac{25}{2304}c\left(4-c^2\right)\left(1-|x|^2\right)\delta - \frac{25}{5184}\left(4-c^2\right)^2x^2\right|, \tag{108}$$

with the aid of the triangle inequality and replacing $|\delta| \leq 1, |x| = k$, where $k \leq 1$ and taking $c \in [0, 2]$. So,

$$\left|\mathscr{D}_{2,2}(g)\right| \leq \frac{25}{4608}c^2\left(4-c^2\right)k^2 + \frac{25}{2304}c\left(4-c^2\right)\left(1-k^2\right) + \frac{25}{5184}\left(4-c^2\right)^2k^2 \coloneqq \Xi(c,k). \tag{109}$$

It is not hard to observe that $\Xi'(c,k) \geq 0$ for $[0, 1]$, so we have $\Xi(c,k) \leq \Xi(c,1)$. Putting $k = 1$ gives

$$\left|\mathscr{D}_{2,2}(g)\right| \leq \frac{25}{4608}c^2\left(4-c^2\right) + \frac{25}{5184}\left(4-c^2\right)^2 \coloneqq \Xi(c,1). \tag{110}$$

It is clear that $\Xi'(c,1) < 0$, so $\Xi(c,1)$ is a decreasing function and attains its maximum value at $c = 0$. Thus, we have

$$\left|\mathscr{D}_{2,2}(g)\right| \leq \frac{25}{324}. \tag{111}$$

The required second Hankel determinant is sharp and is obtained by

$$g_1(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^2\right) + \frac{1}{6}\left(t^{10}\right)\right)dt = z + \frac{5}{18}z^3 + \frac{1}{66}z^{11} + \cdots. \tag{112}$$

□

5. Conclusion

In our present investigation, we considered a subclass of bounded turning functions associated with a four-leaf-type domain. We obtained some useful results for such a class, such as the limits of the first four initial coefficients, as well as the Fekete-Szego type inequality, the Zalcman inequality, the Kruskal inequality, and the estimation of the second-order Hankel determinant. All of the obtained results have been proven to be sharp. This work has been used to obtain higher-order Hankel determinants, such as in the investigation of the bounds of fourth-order and fifth-order Hankel determinants. These two determinants have been studied in [45, 53–56], respectively. Also, one can easily use this new methodology to obtain sharp bounds of the third-order Hankel determinant for other subclasses of univalent functions.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work (grant code: 22UQU4310396DSR19).

References

[1] L. Bieberbach, “Über dié koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln,” *Sitzungsberichte Preussische Akademie der Wissenschaften*, vol. 138, pp. 940–955, 1916.

[2] L. De-Brages, “A proof of the Bieberbach conjecture,” *Acta Mathematica*, vol. 154, no. 1-2, pp. 137–152, 1985.

[3] N. Iqbal, A. U. K. Niazi, I. U. Khan, R. Shah, and T. Botmart, “Cauchy problem for non-autonomous fractional evolution equations with nonlocal conditions of order (1, 2),” *AIMS Mathematics*, vol. 7, no. 5, pp. 8891–8913, 2022.

[4] A. U. K. Niazi, N. Iqbal, F. Wannalookkhee, and K. Nonlaopon, “Controllability for fuzzy fractional evolution equations in credibility space,” *Fractal and Fractional*, vol. 5, no. 3, article fractalfract5030112, p. 112, 2021.

[5] M. Naeem, O. F. Azhar, A. M. Zidan, K. Nonlaopon, and R. Shah, “Numerical analysis of fractional-order parabolic equations via Elzaki transform,” *Journal of Function Spaces*, vol. 2021, Article ID 3484482, 10 pages, 2021.

[6] W. C. Ma and D. Minda, “A unified treatment of some special classesof univalent functions,” in *Proceedings of the Conference on Complex Analysis. Tianjin, China, 1992. Conference Proceedings and Lecture Notes in Analysis, vol. I*, Z. Li, F. Ren, L. Yang, and S. Zhang, Eds., pp. 157–169, International Press, Cambridge, Massachusetts, 1994.

[7] W. Janowski, “Extremal problems for a family of functions with positive real part and for some related families,” *Ann. Polon. Math.*, vol. 23, no. 2, pp. 159–177, 1970.

[8] J. Sokół and J. Stankiewicz, “Radius of convexity of some subclasses of strongly starlike functions,” *Zeszyty Naukowe Politechniki Rzeszowskiej*, vol. 19, pp. 101–105, 1996.

[9] K. Sharma, N. K. Jain, and V. Ravichandran, “Starlike functions associated with a cardioid,” *Afrika Matematika*, vol. 27, no. 5-6, article 387, pp. 923–939, 2016.

[10] N. E. Cho, V. Kumar, S. S. Kumar, and V. Ravichandran, “Radius problems for starlike functions associated with the sine function,” *Bulletin of the Iranian Mathematical Society*, vol. 45, no. 1, article 127, pp. 213–232, 2019.

[11] R. Mendiratta, S. Nagpal, and V. Ravichandran, “On a subclass of strongly starlike functions associated with exponential function,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 38, no. 1, article 26, pp. 365–386, 2015.

[12] L. Shi, H. M. Srivastava, M. Arif, S. Hussain, and H. Khan, “An investigation of the third Hankel determinant problem

for certain subfamilies of univalent functions involving the exponential function," *Symmetry*, vol. 11, no. 5, article 598, 2019.

[13] H. M. Srivastava, B. Khan, N. Khan, M. Tahir, S. Ahmad, and N. Khan, "Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with the q-exponential function," *Bulletin des Sciences Mathematiques*, vol. 167, article 102942, 2021.

[14] K. Bano and M. Raza, "Starlike functions associated with cosine Functions," *Bulletin of Iranian Mathematical Society*, vol. 47, no. 5, article 456, pp. 1513–1532, 2021.

[15] A. Alotaibi, M. Arif, M. A. Alghamdi, and S. Hussain, "Starlikness associated with cosine hyperbolic function," *Mathematics*, vol. 8, no. 7, p. 1118, 2020.

[16] M. M. Al-Sawalha, N. Amir, and M. Yar, "Novel analysis of fuzzy fractional Emden-Fowler equations within new iterative transform method," *Journal of Function Spaces*, vol. 2022, Article ID 7731135, 9 pages, 2022.

[17] O. Barukab, M. Arif, M. Abbas, and S. K. Khan, "Sharp bounds of the coefficient results for the family of bounded turning functions associated with petal shaped domain," *Journal of Function Spaces*, vol. 2021, Article ID 5535629, 9 pages, 2021.

[18] C. Pommerenke, "On the coefficients and Hankel determinants of univalent functions," *Journal of the London Mathematical Society*, vol. s1-41, no. 1, pp. 111–122, 1966.

[19] C. Pommerenke, "On the Hankel determinants of univalent functions," *Mathematika*, vol. 14, no. 1, pp. 108–112, 1967.

[20] W. K. Hayman, "On the second Hankel determinant of mean univalent functions," *Proceedings of the London Mathematical Society*, vol. s3-18, no. 1, pp. 77–94, 1968.

[21] M. Obradović and N. Tuneski, "Hankel determinants of second and third order for the class $\mathcal{S}$ of univalent functions," https://arxiv.org/abs/1912.06439.

[22] A. Janteng, S. A. Halim, and M. Darus, "Coefficient inequality for a function whose derivative has a positive real part," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 2, pp. 1–5, 2006.

[23] A. Janteng, S. A. Halim, and M. Darus, "Hankel determinant for starlike and convex functions," *International Journal of Mathematical Analysis*, vol. 1, no. 13, pp. 619–625, 2007.

[24] S. K. Lee, V. Ravichandran, and S. Supramaniam, "Bounds for the second Hankel determinant of certain univalent functions," *Journal of Inequalities and Applications*, vol. 2013, no. 1, Article ID 281, 2013.

[25] A. Ebadian, T. Bulboacă, N. E. Cho, and E. Analouei Adegani, "Coefficient bounds and differential subordinations for analytic functions associated with starlike functions," *Revista de la Real Academia de Ciencias Exactas, Fsicasy Naturales. Series A. Matemáticas*, vol. 114, no. 3, article 128, 2020.

[26] Ş. Altınkaya and S. Yalçın, "Upper bound of second Hankel determinant for Bi-Bazilevič functions," *Mediterranean Journal of Mathematics*, vol. 13, no. 6, article 733, pp. 4081–4090, 2016.

[27] M. Çaglar, E. Deniz, and H. M. Srivastava, "Second Hankel determinant for certain subclasses of bi-univalent functions," *Turkish Journal of Mathematics*, vol. 41, no. 3, pp. 694–706, 2017.

[28] S. Kanas, E. A. Adegani, and A. Zireh, "An unified approach to second Hankel determinant of bi-subordinate functions," *Mediterranean Journal of Mathematics*, vol. 14, no. 6, p. 233, 2017.

[29] K. O. Babalola, "On $H_3(1)$ Hankel determinant for some classes of univalent functions," *Inequality Theory and Applications*, vol. 6, pp. 1–7, 2010.

[30] Ş. Altınkaya and S. Yalçın, "Third Hankel determinant for Bazilevič functions," *Advances in Mathematics*, vol. 5, no. 2, pp. 91–96, 2016.

[31] D. Bansal, S. Maharana, and J. K. Prajapat, "Third order Hankel determinant for certain univalent functions," *Journal of Korean Mathematical Society*, vol. 52, no. 6, pp. 1139–1148, 2015.

[32] D. V. Krishna, B. Venkateswarlu, and T. Ram Reddy, "Third Hankel determinant for bounded turning functions of order alpha," *Society*, vol. 34, no. 2, pp. 121–127, 2015.

[33] M. Raza and S. N. Malik, "Upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli," *Journal Inequalities and Applications*, vol. 2013, no. 1, article 412, 2013.

[34] G. Shanmugam, B. A. Stephen, and K. O. Babalola, "Third Hankel determinant for α-starlike functions," *Gulf Journal of Mathematics*, vol. 2, no. 2, pp. 107–113, 2014.

[35] P. Zaprawa, "Third Hankel determinants for subclasses of univalent functions," *Mediterranean Journal of Mathematics*, vol. 14, no. 1, p. 19, 2017.

[36] O. S. Kwon, A. Lecko, and Y. J. Sim, "The bound of the Hankel determinant of the third kind for starlike functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 2, pp. 767–780, 2019.

[37] P. Zaprawa, M. Obradović, and N. Tuneski, "Third Hankel determinant for univalent starlike functions," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 115, no. 2, article 49, 2021.

[38] B. Kowalczyk, A. Lecko, and Y. J. Sim, "The sharp bound FOR the Hankel determinant of the third kind for convex functions," *Bulletin of the Australian Mathematical Society*, vol. 97, no. 3, article S0004972717001125, pp. 435–445, 2018.

[39] A. Lecko, Y. J. Sim, and B. Śmiarowska, "The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2," *Complex Analysis and Operator Theory*, vol. 13, no. 5, pp. 2231–2238, 2019.

[40] L. Shi, M. Shutaywi, N. Alreshidi, M. Arif, and M. S. Ghufran, "The sharp bounds of the third-order Hankel determinant for certain analytic functions associated with an eight-shaped domain," *Fractal and Fractional*, vol. 6, no. 4, article fractalfract6040223, p. 223, 2022.

[41] S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan, and I. Ali, "Upper bound of the third Hankel determinant for a subclass of q-starlike functions," *Symmetry*, vol. 11, no. 3, article 347, 2019.

[42] M. Arif, M. Raza, H. Tang, S. Hussain, and H. Khan, "Hankel determinant of order three for familiar subsets of analytic functions related with sine function," *Open Mathematics.*, vol. 17, no. 1, pp. 1615–1630, 2019.

[43] L. Shi, I. Ali, M. Arif, N. E. Cho, S. Hussain, and H. Khan, "A study of third Hankel determinant problem for certain subfamilies of analytic functions involving cardioid domain," *Mathematics*, vol. 7, no. 5, article 418, 2019.

[44] M. Raza, H. M. Srivastava, M. Arif, and K. Ahmad, "Coefficient estimates for a certain family of analytic functions involving a q-derivative operator," *The Ramanujan Journal*, vol. 55, no. 1, article 338, pp. 53–71, 2021.

[45] Z. G. Wang, M. Raza, M. Arif, and K. Ahmad, "On the third and fourth Hankel determinants for a subclass of analytic functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 45, no. 1, article 1195, pp. 323–359, 2022.

[46] S. Gandhi, "Radius estimates for three leaf function and convex combination of starlike functions," in *mathematical Analysis 1: Approximation Theory. ICRAPAM*, N. Deo, V. Gupta, and P. Agrawal, Eds., vol. 306, Spinger, Singapore, 2018.

[47] R. J. Libera and E. J. Zlotkiewicz, "Coefficient bounds for the inverse of a function with derivative in *P*," *Proceedings of the American Mathematical Society*, vol. 87, no. 2, pp. 251–257, 1983.

[48] C. Pommerenke, *Univalent Function*, Vanderhoeck & Ruprecht, Göttingen, Germany, 1975.

[49] R. J. Libera and E. J. Złotkiewicz, "Early coefficients of the inverse of a regular convex function," *Proceedings of the American Mathematical Society*, vol. 85, no. 2, pp. 225–230, 1982.

[50] O. S. Kwon, A. Lecko, and Y. J. Sim, "On the fourth coefficient of functions in the Carathéodory class," *Computational Methods and Function Theory*, vol. 18, no. 2, article 229, pp. 307–314, 2018.

[51] V. Ravichandran and S. Verma, "Borne pour le cinquieme coefficient des fonctions etoilees," *Comptes Rendus Mathematique*, vol. 353, no. 6, article S1631073X1500076X, pp. 505–510, 2015.

[52] S. K. Krushkal, *A Short Geometric Proof of the Zalcman and Bieberbach Conjectures*.

[53] H. Tang, M. Arif, M. Haq et al., "Fourth Hankel determinant problem based on certain analytic functions," *Symmetry*, vol. 14, no. 4, article sym14040663, p. 663, 2022.

[54] M. Arif, L. Rani, M. Raza, and P. Zaprawa, "Fourth Hankel determinant for the family of functions with bounded turning," *Bulletin of the Korean Mathematical Society*, vol. 55, no. 6, pp. 1703–1711, 2018.

[55] M. Arif, L. Rani, M. Raza, and P. Zaprawa, "Fourth Hankel determinant for the set of star-like functions," *Mathematical Problems in Engineering*, vol. 2021, Article ID 6674010, 8 pages, 2021.

[56] M. Arif, I. Ullah, M. Raza, and P. Zaprawa, "Investigation of the fifth Hankel determinant for a family of functions with bounded turnings," *Mathematica Slovaca*, vol. 70, no. 2, pp. 319–328, 2020.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 7361558, 8 pages
https://doi.org/10.1155/2022/7361558

Research Article

Fractional Versions of Hermite-Hadamard, Fejér, and Schur Type Inequalities for Strongly Nonconvex Functions

Wenbo Xu,[1] Muhammad Imran,[2] Faisal Yasin,[3] Nazia Jahangir,[2] and Qunli Xia[1]

[1]*School of Astronautics, Beijing Institute of Technology, Beijing 100081, China*
[2]*Department of Mathematics, University of Okara, Okara, Pakistan*
[3]*Department of Mathematics and Statistics, University of the Lahore, Lahore, Pakistan*

Correspondence should be addressed to Qunli Xia; teacher3xia@sina.com

Received 24 May 2022; Revised 15 June 2022; Accepted 28 June 2022; Published 18 July 2022

Academic Editor: Wasim Ul-Haq

In modern world, most of the optimization problems are nonconvex which are neither convex nor concave. The objective of this research is to study a class of nonconvex functions, namely, strongly nonconvex functions. We establish inequalities of Hermite-Hadamard and Fejér type for strongly nonconvex functions in generalized sense. Moreover, we establish some fractional integral inequalities for strongly nonconvex functions in generalized sense in the setting of Riemann-Liouville integral operators.

1. Introduction

The integral and differential operators have remarkable impact on applied sciences, and the interest of researchers is increasing day by day in this research area [1, 2]. Consider a convex function $f : I \subset R \longrightarrow R$ defined on the interval $I \subset \mathbb{R}$ with $a, b \in I$ being constants and $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is Hermite-Hadamard's (see [3, 4]).

The notion of convexity is very old, and it appears in Archimedes treatment of orbit length. Nowadays, convex geometry is a mathematical subject in its own right. There are several modern works on convexity that are for the studies of real analysis, linear algebra, geometry, and functional analysis. The theory of convexity helps us to solve many applied problems. In recent years, the theory of convex analysis gains huge attention of researchers due to its interesting applications in optimizations, geometry, and engineering [5, 6].

The present paper deals with a new class of convex functions and establishes inequalities of Hermite-Hadamard and Fejér. Moreover, we develop some fractional integral inequalities. See [7, 8] for more general inequalities via convexity of functions.

The classical definition of convex functions was given in [3]. Another concept which is used widely in convex analysis is p-convex sets and p-convexity (see [4]). By taking $p = 1$ in the above definition, we get classical notion of convexity. After that, the strongly convex with modulus $\mu > 0$ was introduced in [9]. And in [10], the notion of the strongly p-convex function had been introduced. The notion of generalized convex functions had been introduced in [11, 12].

Motivated by the above researches, [13] introduced the following class of functions.

The function f is strongly nonconvex in generalized sense if

$$f\left(tx^p + (1-t)y^p\right)^{1/p} \leq f(y) + t\eta(f(x), f(y)) - \mu t(1-t)(y^p - x^p)^2 \tag{2}$$

holds for $t \in [0, 1]$.

Definition 1 (see [13, 14]). Consider $f \in L[a, b]$, then the RHS and the LHS Riemann-Liouville fractional integral (RL) of order $\alpha > 0$ with $b > a > 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, x > a,$$
$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-x)^{\alpha-1} f(t) dt, x < b, \tag{3}$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1}. \tag{4}$$

It is to be noted that $J_{\alpha+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

The Riemann integral is reduced to classical integral for $\alpha = 1$ [15–18].

The definition of strong p-convexity was studied in [13]. The aim of this paper is to establish the inequalities of Schur, Fejér, and Hermite-Hadamard type for the strongly nonconvex functions via RL fractional integrals.

2. Inequality of Hermite-Hadamard Type

In order to prove the inequality of Hermite-Hadamard type, the following lemma is very important.

Lemma 2 (see [19]). *Let p be any nonzero real number and α be any positive constant. Further consider an integrable function $w : A \longrightarrow \mathbb{R}$, where $A = [a, b] \subset (0,\infty)$ which is p-symmetric w.r.t. $[a^p + b^p/2]^{1/p}$; then, we have the following:*

(i) If $p > 0$,

$$J_{a^p+}^{\alpha}(wog)(b^p) = J_{b^p-}^{\alpha}(wog)(a^p) = \frac{1}{2}\left[J_{a^p+}^{\alpha}(wog)(b^p) + J_{b^p-}^{\alpha}(wog)(a^p)\right], \tag{5}$$

with $g(x) = x^{1/p}$, $x \in [a^p, b^p]$

(ii) If $p < 0$,

$$J_{b^p+}^{\alpha}(wog)(a^p) = J_{a^p-}^{\alpha}(wog)(a^p) = \frac{1}{2}\left[J_{b^p+}^{\alpha}(wog)(b^p) + J_{a^p-}^{\alpha}(wog)(b^p)\right], \tag{6}$$

with $g(x) = x^{1/p}$, $x \in [b^p, a^p]$

Theorem 3. *Let the strongly generalized p-convex function $f : I \longrightarrow \mathbb{R}$ with magnitude $\mu > 0$ and $\eta(\cdot)$ be bounded above in $f(I) \times f(I)$ y $f \in L[a, b]$. Then, if p is any positive real number, we have*

$$f\left(\frac{a^p + b^p}{2}\right) - M_{\eta} + \frac{\mu(b^p - a^p)^2}{6}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3)}$$
$$\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b^p - a^p)^{\alpha}}\left[J_{a^p+b^p/2+}^{\alpha} fog(b^p) + J_{a^p+b^p/2-}^{\alpha} fog(a^p)\right]$$
$$\leq \frac{f(a)+f(b)}{2} + \frac{\alpha M_{\eta}}{2(\alpha+1)} - \mu\frac{\alpha(\alpha+3)(b^p - a^p)^2}{4(\alpha+1)(\alpha+2)}. \tag{7}$$

Proof. We begin the proof by inserting $x = (ta^p + (1-t)b^p)^{1/p}$ and $y = (tb^p + (1-t)a^p)^{1/p}$

$$f\left[\left(\frac{x^p + y^p}{2}\right)\right]^{1/p} - \frac{M_{\eta}}{2} - \frac{\mu(x^p - y^p)^2}{12} \leq \frac{f(x)+f(y)}{2} + \frac{M_{\eta}}{2} - \frac{\mu(x^p - y^p)^2}{6}. \tag{8}$$

Take $x = [(ta^p + (1-t)b^p)]^{1/p}$ and $y = [(tb^p + (1-t)a^p)]^{1/p}$, then (8) yields

$$f\left[\left(\frac{a^p + b^p}{2}\right)\right]^{1/p} - \frac{M_{\eta}}{2} - \frac{\mu(2t-1)^2(b^p - a^p)^2}{12}$$
$$\leq \frac{1}{2}\left[f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p}\right] + \frac{1}{2}\left[f\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p}\right] + \frac{M_{\eta}}{2} - \frac{\mu(2t-1)^2(b^p - a^p)^2}{6}. \tag{9}$$

Multiplying (9) by $t^{\alpha-1}$ and then integrating w.r.t. t over the interval $[0, 1/2]$,

$$\int_0^{1/2} f\left[\left(\frac{a^p + b^p}{2}\right)\right]^{1/p} t^{\alpha-1} dt - \int_0^{1/2}\frac{M_{\eta}}{2} t^{\alpha-1} dt - \frac{\mu(b^p - a^p)^2}{12}\int_0^{1/2}(2t-1)^2 t^{\alpha-1} dt \leq \frac{1}{2}\int_0^{1/2} t^{\alpha-1} f\left(ta^p + (1-t)b^p\right) dt$$
$$+ \frac{1}{2}\int_0^{1/2} t^{\alpha-1} f\left(tb^p + (1-t)a^p\right) dt + \int_0^{1/2}\frac{M_{\eta}}{2} t^{\alpha-1} dt - \frac{\mu(a^p - b^p)^2}{6}\int_0^{1/2}(2t-1)^2 t^{\alpha-1} dt, \tag{10}$$

$$f\left[\left(\frac{a^p + b^p}{2}\right)\right]^{1/p} - \frac{M_{\eta}}{2} + \frac{\mu(b^p - a^p)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}$$
$$\leq \frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b^p - a^p)^{\alpha}}\left[J_{a^p+b^p/2+}^{\alpha} fog(b^p) + J_{a^p+b^p/2-}^{\alpha} fog(a^p)\right], \tag{11}$$

which is the left side of Theorem 3

Now, to obtain the left-hand side of Theorem 3, we have for $x = [(ta^p + (1-t)b^p)]^{1/p}$,

$$f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p} \le f(b) + t\eta(f(a), f(b)) - \mu t(1-t)\left(b^p - a^p\right)^2, \tag{12}$$

and for $y = [(tb^p + (1-t)a^p)]^{1/p}$,

$$f\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} \le f(a) + t\eta(f(b), f(a)) - \mu t(1-t)\left(b^p - a^p\right)^2. \tag{13}$$

Combining (12) and (13), we have

$$\begin{aligned} f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p} + f\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} &\le f(a) \\ + t\eta(f(a), f(b)) + f(b) + t\eta(f(b), f(a)) \\ - 2\mu t(1-t)\left(b^p - a^p\right)^2. \end{aligned} \tag{14}$$

Multiplying (14) by $2t^{\alpha-1}$ and then integrating w.r.t. t over the interval $[0, 1/2]$, we have

$$\begin{aligned} &2\int_0^{1/2} \left[f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p} t^{\alpha-1} + f\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} t^{\alpha-1} \right] \\ &dt \le 2\int_0^{1/2} (f(a) + f(b)) t^{\alpha-1} dt + 4M_\eta \int_0^{1/2} t^\alpha dt - 4\mu\left(b^p - a^p\right)^2 \\ &\int_0^{1/2} t(1-t) t^{\alpha-1} dt, \end{aligned} \tag{15}$$

$$\begin{aligned} &\frac{\Gamma(\alpha+1)2^{\alpha-1}}{\left(b^p - a^p\right)^\alpha} \left[J^\alpha_{a^p+b^p/2+} fog\left(b^p\right) + J^\alpha_{a^p+b^p/2-} fog\left(a^p\right)\right] \\ &\le \frac{f(a) + f(b)}{2} + \frac{\alpha M_\eta}{2(\alpha+1)} - \frac{\mu\alpha(\alpha+3)\left(b^p - a^p\right)^2}{4(\alpha+1)(\alpha+2)}. \end{aligned} \tag{16}$$

Together (11) and (16) give the required result. □

Remark 4.

(i) Fixing $p = 1$ in Theorem 3 gives Hermite-Hadamard inequality in the sense of the strongly generalized convexity

(ii) Fixing $p = 1$ and $\mu = 0$ in Theorem 3, we obtain [20] (Theorem 2.1)

(iii) Fixing $\eta(x, y) = x - y$ and $\mu = 0$ in Theorem 3 yields [21] (Theorem 2.1)

(iv) Applying both (ii) and (iii) on Theorem 3, we obtain classical fractional version of H-H inequality

Definition 5 (see [22]). Let p be any nonzero real number; then, the function $w : [a, b] \longrightarrow \mathbb{R}$ is p-symmetric w.r.t. $[(a^p + b^p/2)]^{1/p}$ if $w(x) = w[(a^p + b^p - x^p)]^{1/p}$ for all $x \in [a, b]$.

Theorem 6 (inequality of Fejér type). *Suppose that f is a function as in Theorem 3 and an integrable, nonnegative function $w : [a, b] \longrightarrow R$ is symmetric w.r.t. $[(a^p + b^p/2)]^{1/p}$, then*

$$\begin{aligned} &\frac{\Gamma(\alpha)}{2} f\left[\left(\frac{a^p + b^p}{2}\right)\right]^{1/p} \left[J^\alpha_{a^p+} wog\left(b^p\right) + J^\alpha_{b^p-} wog\left(a^p\right)\right] \\ &- \frac{M_\eta \Gamma(\alpha)}{2} \left[J^\alpha_{a^p+} wog\left(b^p\right) + J^\alpha_{b^p-} wog\left(a^p\right)\right] \\ &+ \frac{\mu}{2} \int_{a^p}^{b^p} \left(2x - b^p - a^p\right)^2 \left(b^p - x\right)^{\alpha-1} wog(x) dx \\ &\le \frac{\Gamma(\alpha)}{2} \left[J^\alpha_{a^p+} fwog\left(b^p\right) + J^\alpha_{b^p-} fwog\left(a^p\right)\right] \frac{f(a) + f(b)}{2} \frac{\Gamma(\alpha)}{2} \\ &\left[J^\alpha_{a^p+} wog\left(b^p\right) + J^\alpha_{b^p-} wog\left(a^p\right)\right] + \frac{M_\eta}{\left(b^p - a^p\right)} \int_{a^p}^{b^p} \left(b^p - x\right)^\alpha wog(x) dx \\ &- \mu \int_{a^p}^{b^p} \left(b^p - x\right)^2 \left(x - a^p\right) wog(x) dx. \end{aligned} \tag{17}$$

Proof. Setting $t = 1/2$ in (2),

$$f\left[\left(\frac{x^p + y^p}{2}\right)\right]^{1/p} \le f(y) + \frac{1}{2}\eta(f(x), f(y)) - \frac{\mu}{4}\left(y^p - x^p\right)^2. \tag{18}$$

Substitute $y = [(ta^p + (1-t)b^p)]^{1/p}$ and $x = [(tb^p + (1-t)a^p)]^{1/p}$ in (18),

$$f\left[\left(\frac{x^p + y^p}{2}\right)\right]^{1/p} \le f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p} + \frac{M_\eta}{2} - \frac{\mu}{4}(2t-1)^2\left(b^p - a^p\right)^2. \tag{19}$$

According to the given conditions of w, we have

$$\begin{aligned} w(x) &= w\left[\left(a^p + b^p - x^p\right)\right]^{1/p}, \\ w\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p} &= w\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} \end{aligned} \tag{20}$$

$\forall x, y \in [a, b]$. Multiplying (19) by $2t^{\alpha-1}w[(tb^p + (1-t)a^p)]^{1/p}$ and then integrating w.r.t. t over the interval $[0, 1]$,

$$\begin{aligned} &\int_0^1 2t^{\alpha-1} w\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} \times f\left[\left(\frac{a^p + b^p}{2}\right)\right]^{1/p} \\ &\le \int_0^1 2t^{\alpha-1} w\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} \times f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p} dt \\ &+ \frac{M_\eta}{2} \int_0^1 2t^{\alpha-1} w\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} dt \end{aligned}$$

$$-\frac{\mu}{4}\int_0^1 2t^{\alpha-1}(2t-1)^2\left(b^p-a^p\right)^2 \times w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}dt, \tag{21}$$

$$\begin{aligned} f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\int_{b^p}^{a^p} 2\left(\frac{x-b^p}{a^p-a^p}\right)^{\alpha-1} w\left(x^{1/p}\right)\frac{dx}{\left(a^p-b^p\right)} &\le \int_{b^p}^{a^p} 2f\left(x^{1/p}\right)w\left(x^{1/p}\right)\left(\frac{x-b^p}{a^p-a^p}\right)^{\alpha-1}\frac{dx}{\left(a^p-b^p\right)} \\ &+ \int_{b^p}^{a^p} M_\eta\left(\frac{x-b^p}{a^p-a^p}\right)^{\alpha-1} w\left(x^{1/p}\right)\frac{dx}{\left(a^p-b^p\right)} \\ &- \frac{\mu}{2}\int_{b^p}^{a^p}\left(2x-b^p-a^p\right)^2\left(\frac{x-b^p}{a^p-a^p}\right)^{\alpha-1}\times w\left(x^{1/p}\right)\frac{dx}{\left(a^p-b^p\right)}, \end{aligned} \tag{22}$$

$$\begin{aligned} f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\int_{a^p}^{b^p} 2\left(b^p-x\right)^{\alpha-1}w\left(x^{1/p}\right)\frac{dx}{\left(b^p-a^p\right)^\alpha} &\le \int_{a^p}^{b^p} 2f\left(x^{1/p}\right)w\left(x^{1/p}\right)\left(b^p-x\right)^{\alpha-1}\frac{dx}{\left(b^p-a^p\right)^\alpha} \\ &+ M_\eta\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}w\left(x^{1/p}\right)\frac{dx}{\left(b^p-a^p\right)^\alpha} \\ &- \frac{\mu}{2}\int_{b^p}^{a^p}\left(2x-b^p-a^p\right)^2\left(b^p-x\right)^{\alpha-1}\times w\left(x^{1/p}\right)\frac{dx}{\left(b^p-a^p\right)^\alpha}. \end{aligned} \tag{23}$$

Let $g(x)=x^{1/p}$, then (23) becomes

$$\begin{aligned} & f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\frac{1}{\left(b^p-a^p\right)^\alpha}\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wog(x)dx\frac{2}{\left(b^p-a^p\right)^\alpha} \\ & \int_{a^p}^{b^p} fwog(x)\left(b^p-x\right)^{\alpha-1}dx + \frac{M_\eta}{\left(b^p-a^p\right)^\alpha}\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wog(x)dx \\ & -\frac{\mu}{2\left(b^p-a^p\right)^\alpha}\int_{a^p}^{b^p}\left(2x-b^p-a^p\right)^2\times x\left(b^p-\right)^{\alpha-1}wog(x)dx, \end{aligned} \tag{24}$$

$$\begin{aligned} & \frac{\Gamma(\alpha)}{2}f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\left[J^\alpha_{a^p+}wog\left(b^p\right)+J^\alpha_{b^p-}wog\left(a^p\right)\right] \\ & -\frac{M_\eta}{2}\Gamma(\alpha)\left[J^\alpha_{a^p+}wog\left(b^p\right)+J^\alpha_{b^p-}wog\left(a^p\right)\right] \\ & +\frac{\mu}{2}\int_{a^p}^{b^p}\left(2x-b^p-a^p\right)^2\left(b^p-x\right)^{\alpha-1}wog(x)dx \\ & \le \frac{\Gamma(\alpha)}{2}\left[J^\alpha_{a^p+}fwog\left(b^p\right)+J^\alpha_{b^p-}fwog\left(a^p\right)\right]. \end{aligned} \tag{25}$$

Now, take $x=(ta^p+(1-t)b^p)\forall t\in[0,1]$, then by Def of f,

$$f\left[\left(ta^p+(1-t)b^p\right)\right]^{1/p}\le f(b)+t\eta(f(a),f(b))-\mu t(1-t)\left(b^p-a^p\right)^2. \tag{26}$$

Multiply on both sides of (26) by $2t^{\alpha-1}w[(tb^p+(1-t)a^p)]^{1/p}$ and then integrate w.r.t. t over the interval $[0,1]$,

$$\begin{aligned} & \int_0^1 2f\left[\left(ta^p+(1-t)b^p\right)\right]^{1/p}t^{\alpha-1}w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}dt \\ & \le \int_0^1 2t^{\alpha-1}w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}f(b)dt \\ & \quad + \int_0^1 2t^\alpha M_\eta w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}dt - 2\mu\int_0^1 2t^\alpha(1-t)\left(b^p-a^p\right)^2 \\ & \quad \times w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}dt, \end{aligned} \tag{27}$$

$$\begin{aligned} \int_{b^p}^{a^p} f\left(x^{1/p}\right)w\left(x^{1/p}\right)\left(\frac{x-b^p}{a^p-b^p}\right)^{\alpha-1}\frac{dx}{\left(a^p-b^p\right)} &\le \int_{b^p}^{a^p} f(b)\left(\frac{x-b^p}{a^p-b^p}\right)^{\alpha-1}w\left(x^{1/p}\right)\frac{dx}{\left(a^p-b^p\right)} \\ &+ M_\eta\int_{b^p}^{a^p}\left(\frac{x-b^p}{a^p-b^p}\right)^{\alpha-1}w\left(x^{1/p}\right)\frac{dx}{\left(a^p-b^p\right)} \\ &- \mu\int_{b^p}^{a^p}\left(b^p-x\right)^\alpha\left(x-a^p\right)w\left(x^{1/p}\frac{dx}{\left(b^p-a^p\right)^\alpha}\right). \end{aligned} \tag{28}$$

Take $g(x)=x^{1/p}$ in (28), then we have

$$\begin{aligned} \int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wfog(x)dx &\le \int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wog(x)f(b)dx \\ &+ \frac{M_\eta}{\left(b^p-a^p\right)}\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^\alpha wog(x)dx\right. \\ &- \mu\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^\alpha\left(x-a^p\right)wog(x)dx.\right. \end{aligned} \tag{29}$$

Similarly, we have

$$\begin{aligned} \int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wfog(x)dx &\le \int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wog(x)f(a)dx \\ &+ \frac{M_\eta}{\left(b^p-a^p\right)}\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^\alpha wog(x)dx\right. \\ &- \mu\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^\alpha\left(x-a^p\right)wog(x)dx,\right. \end{aligned} \tag{30}$$

from definition of f by fixing $x=tb^p+(1-t)a^p$. Combining (29) and (30), we obtain

$$\begin{aligned} \int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wfog(x)dx &\le \frac{f(a)+f(b)}{2}\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wog(x)dx \\ &+ \frac{M_\eta}{\left(b^p-a^p\right)}\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^\alpha wog(x)dx-\mu\int_{a^p}^{b^p}\left(b^p-x\right)^\alpha\left(x-a^p\right)wog(x)dx,\right. \end{aligned} \tag{31}$$

$$\begin{aligned}&\frac{\Gamma(\alpha)}{2}\left[J^{\alpha}_{a^p+}fwog(b^p)+J^{\alpha}_{b^p-}fwog(a^p)\right]\\&\leq\frac{f(a)+f(b)}{2}\frac{\Gamma(\alpha)}{2}\left[J^{\alpha}_{a^p+}wog(b^p)+J^{\alpha}_{b^p-}wog(a^p)\right]\\&+\frac{M_\eta}{\left(b^p-a^p\right)}\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^{\alpha}wog(x)dx\right.\\&-\mu\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha}\left(x-a^p\right)wog(x)dx.\end{aligned}\tag{32}$$

Combining (32) and (25) completes the theorem (17). □

3. Fractional Integral Inequalities for Strongly Generalized p-Convex Function

Lemma 7. *Consider a differentiable function $f : I\subset(0,\infty)\longrightarrow R$ on I^o, with $f'\in L[a,b]$, where $a, b\in I$ and $a<b$. If $w:[a,b]\longrightarrow R$ is integrable, then*

$$\begin{aligned}&f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\left[J^{\alpha}_{a^p+b^p/2+}fog(b^p)+J^{\alpha}_{a^p+b^p/2-}fog(a^p)\right]\\&-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}\\&=\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^{\alpha}(fog)'(t)-\left.\int_{a^p+b^p/2}^{b^p}\left(b^p-t\right)^{\alpha}(fog)'(t)\right]dt\\&-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}\end{aligned}\tag{33}$$

holds with $g(x)=x^{1/p}$.

Proof. Let $p>0$, and $x\in[a^p,b^p]$, then for generalized strongly p-convex function, we have

$$\begin{aligned}K&=\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^{\alpha}(fog)'(t)dt\\&-\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\int_{a^p+b^p/2}^{b^p}\left(b^p-t\right)^{\alpha}(fog)'(t)dt-\frac{M_\eta}{2}\\&+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)},\end{aligned}\tag{34}$$

$$K=K_1-K_2-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+!)}{2\Gamma(\alpha+3)},\tag{35}$$

where

$$\begin{aligned}K_1&=\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^{\alpha}(fog)'(t)dt,\\K_2&=\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\int_{a^p+b^p/2}^{b^p}\left.\left(b^p-t\right)^{\alpha}(fog)'(t)\right]dt.\end{aligned}\tag{36}$$

By integration by parts, we have

$$\begin{aligned}K_1&=\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\left(\frac{b^p-a^p}{2}\right)^{\alpha}f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\\&-\frac{\alpha}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^{\alpha-1}fog(t)dt\\&=\frac{1}{2}f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}\Gamma(\alpha)}\\&\times\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^{\alpha-1}fog(t)dt,\end{aligned}\tag{37}$$

$$K_2=-\frac{1}{2}f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}+\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\times J^{\alpha}_{a^p+b^p/2+}fog(b^p).\tag{38}$$

By combining (34), (37), and (38), we have (33). This completes the proof. □

Remark 8. Setting $\mu=0$ and $\eta=x-y$ in Lemma 7 gives us [21] (Lemma 2.1).

Theorem 9. *Let the function f be as in Theorem 3.1. If $|f'|$ is a strongly generalized p-convex function on $[a,b]$ for positive p and α, then*

$$\begin{aligned}&\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^{\alpha}}\left[J^{\alpha}_{a^p+b^p/2+}fog(b^p)+J^{\alpha}_{a^p+b^p/2-}fog(a^p)\right]\right.\\&\left.-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}\right|\leq\frac{b^p-a^p}{2^{1-\alpha}}\left[C_1(\alpha,p)\left|f'(b)\right|\right.\\&\left.+C_2(\alpha,p)\eta\left(\left|f'(b)\right|,\left|f'(a)\right|\right)-C_3(\alpha,p)\mu\left(b^p-a^p\right)^2\right]-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)},\end{aligned}\tag{39}$$

where

$$\begin{aligned}C_1(\alpha,p)&=\int_0^{1/2}\frac{u^{\alpha}}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}du+\int_{1/2}^{1}\frac{(1-u)^{\alpha}}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}du,\\C_2(\alpha,p)&=\int_0^{1/2}\frac{u^{\alpha+1}}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}du+\int_{1/2}^{1}\frac{(1-u)^{\alpha}u}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}du,\\C_3(\alpha,p)&=\int_0^{1/2}\frac{u^{\alpha+1}(1-u)\left(b^p-a^p\right)^2}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}du+\int_{1/2}^{1}\frac{u^{\alpha+1}(1-u)\left(b^p-a^p\right)^2}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}du.\end{aligned}\tag{40}$$

Proof. Theorem (3) gives

$$\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}(b^p-a^p)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog(b^p)+J^\alpha_{a^p+b^p/2-}fog(a^p)\right]\right|$$
$$\le \frac{1}{2^{1-\alpha}(b^p-a^p)^\alpha}\int_{a^p}^{a^p+b^p/2}(t-a^p)^\alpha f'(t^{1/p})\frac{1}{t^{1-1/p}}dt+\frac{1}{2^{1-\alpha}(b^p-a^p)^\alpha}\int_{a^p+b^p/2}^{b^p}(b^p-t)^\alpha f'(t^{1/p})\frac{1}{t^{1-1/p}}dt-\frac{M_\eta}{2}$$
$$+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}\le\frac{b^p-a^p}{2^{1-\alpha}}\int_{a^p}^{a^p+b^p/2}(t-a^p)^\alpha f'(t^{1/p})\frac{1}{t^{1-1/p}}dt$$
$$+\frac{b^p-a^p}{2^{1-\alpha}}\int_{a^p+b^p/2}^{b^p}(b^p-t)^\alpha f'(t^{1/p})\frac{1}{t^{1-1/p}}dt-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}. \tag{41}$$

Setting $t=ua^p+(1-u)b^p$, $dt=(a^p-b^p)du$, we have

$$\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}(b^p-a^p)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog(b^p)+J^\alpha_{a^p+b^p/2-}fog(a^p)\right]-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}\right|$$
$$\le\frac{(b^p-a^p)}{2^{1-\alpha}}\int_0^{1/2}\frac{u^\alpha}{p(ua^p+(1-u)b^p)^{1-1/p}}\left|f'\left(ua^p+(1-u)b^p\right)^{1/p}\right|du$$
$$+\frac{(b^p-a^p)}{2^{1-\alpha}}\int_{1/2}^1\frac{(1-u)^\alpha}{p(ua^p+(1-u)b^p)^{1-1/p}}\left|f'\left(ua^p+(1-u)b^p\right)^{1/p}\right|du-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}. \tag{42}$$

Since $|f'|$ is a strongly generalized p-convex function on $[a,b]$, we have

$$\left|f'\left(ua^p+(1-u)b^p\right)^{1/p}\right|\le|f'(b)|+\mu\eta\left(|f'(a)|,|f'(b)|\right)-\mu(1-u)(b^p-a^p)^2. \tag{43}$$

After combining (48) and (43), we have

$$\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}(b^p-a^p)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog(b^p)+J^\alpha_{a^p+b^p/2-}fog(a^p)\right]-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}\right|$$
$$\le\frac{b^p-a^p}{2^{1-\alpha}}\int_0^{1/2}\frac{u^\alpha}{p(ua^p+(1-u)b^p)^{1-1/p}}\left(|f'(b)|+u\eta\left(|f'(a)|,|f'(b)|\right)-\mu u(1-u)(b^p-a^p)^2\right)du+\frac{b^p-a^p}{2^{1-\alpha}}\int_{\frac{1}{2}}^1\frac{(1-u)^\alpha}{p(ua^p+(1-u)b^p)^{1-1/p}}$$
$$\left(|f'(b)|+u\eta\left(|f'(a)|,|f'(b)|\right)-\mu u(1-u)(b^p-a^p)^2\right)du$$
$$-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)},\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}(b^p-a^p)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog(b^p)+J^\alpha_{a^p+b^p/2-}fog(a^p)\right]-\frac{M_\eta}{2}\right.$$
$$\left.+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}\right|=\frac{b^p-a^p}{2^{1-\alpha}}\int_0^{1/2}\frac{u^\alpha}{p(ua^p+(1-u)b^p)^{1-1/p}}|f'(b)|+\frac{b^p-a^p}{2^{1-\alpha}}\int_{1/2}^1\frac{(1-u)^\alpha}{p(ua^p+(1-u)b^p)^{1-1/p}}|f'(b)|du$$
$$+\frac{b^p-a^p}{2^{1-\alpha}}\int_0^{1/2}\frac{u^{\alpha+1}}{p(ua^p+(1-u)b^p)^{1-1/p}}\eta\left(|f'(a)|,|f'(b|\right)+\frac{b^p-a^p}{2^{1-\alpha}}\int_{1/2}^1\frac{(1-u)^\alpha u}{p(ua^p+(1-u)b^p)^{1-1/p}}\eta\left(|f'(a)|,|f'(b)|\right)du-\frac{M_\eta}{2}$$
$$+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}+\frac{b^p-a^p}{2^{1-\alpha}}\int_0^{1/2}\frac{u^{\alpha+1}(1-u)(b^p-a^p)^2\mu}{p(ua^p+(1-u)b^p)^{1-1/p}}+\frac{b^p-a^p}{2^{1-\alpha}}\int_{1/2}^1\frac{u(1-u)^{\alpha+1}\mu}{p(ua^p+(1-u)b^p)^{1-1/p}},\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\right.$$
$$-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}(b^p-a^p)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog(b^p)+J^\alpha_{a^p+b^p/2-}fog(a^p)\right]-\frac{M_\eta}{2}$$
$$\left.+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}\right|\le\frac{b^p-a^p}{2^{1-\alpha}}\left[C_1(\alpha,p)|f'(b)|+C_2(\alpha,p)\eta\left(|f'(b)|,|f'(a)|\right)-C_3(\alpha,p)\mu(b^p-a^p)^2\right]-\frac{M_\eta}{2}$$
$$+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}. \tag{44}$$

□

Remark 10. If one takes $\eta=x-y$ and $\mu=0$, then we get [21] (Theorem 2.2).

Theorem 11. *Let the function f be as in Theorem 3.1. If $|f'|$ is as in Theorem 9, then*

$$\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}(b^p-a^p)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog(b^p)+J^\alpha_{a^p+b^p/2-}fog(a^p)\right]-M_\eta+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\right|\le\left|(C_5(\alpha,p))^{1-1/q}\left[C_5(\alpha,p)|f'(b)|\right.\right.$$
$$+C_6(\alpha,p)M_\eta-C_7(\alpha,p)\mu(b^p-a^p)^2+(C_8(\alpha,p))^{1-1/q}\left[C_8|f'(b)|\right.$$
$$\left.\left.\left.+C_9M_\eta+C_{10}\mu(b^p-a^p)^2\right]\right]-M_\eta+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\right|, \tag{45}$$

where

$$\begin{aligned}
C_5(\alpha,p)&=\int_0^{1/2}\frac{u^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du,\\
C_6(\alpha,p)&=\int_0^{1/2}\frac{u^{\alpha+1}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du,\\
C_7(\alpha,p)&=\int_0^{1/2}\frac{u^\alpha(1-u)}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du,\\
C_8(\alpha,p)&=\int_{1/2}^{1}\frac{(1-u)^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du,\\
C_9(\alpha,p)&=\int_{1/2}^{1}\frac{u(1-u)^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du,\\
C_{10}(\alpha,p)&=\int_{1/2}^{1}\frac{u(1-u)^{\alpha+1}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du.
\end{aligned}\tag{46}$$

Proof. Let $p>0$:

$$\begin{aligned}
&\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog\left(b^p\right)+J^\alpha_{a^p+b^p/2-}fog\left(a^p\right)\right]\right|\\
&\le\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^\alpha}\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^\alpha f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt\\
&+\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^\alpha}\int_{a^p+b^p/2}^{b^p}\left(b^p-t\right)^\alpha f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt-M_\eta\\
&+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\le\frac{b^p-a^p}{2^{1-\alpha}}\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^\alpha f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt\\
&+\frac{b^p-a^p}{2^{1-\alpha}}\int_{a^p+b^p/2}^{b^p}\left(b^p-t\right)^\alpha f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}.
\end{aligned}\tag{47}$$

Setting $t=ua^p+(1-u)b^p$, $dt=(a^p-b^p)du$, we have

$$\begin{aligned}
&\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog\left(b^p\right)\right.\right.\\
&\left.\left.+J^\alpha_{a^p+b^p/2-}fog\left(a^p\right)\right]-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\right|\\
&\le\frac{\left(b^p-a^p\right)}{2^{1-\alpha}}\int_0^{1/2}\frac{u^\alpha}{p\left[\left(ua^p+(1-u)b^p\right)\right]^{1-1/p}}\left|f'\left[\left(ua^p+(1-u)b^p\right)\right]^{1/p}\right|du\\
&+\frac{\left(b^p-a^p\right)}{2^{1-\alpha}}\times\int_{1/2}^{1}\frac{(1-u)^\alpha}{p\left[\left(ua^p+(1-u)b^p\right)\right]^{1-1/p}}\left|f'\left[\left(ua^p+(1-u)b^p\right)\right]^{1/p}\right|du\\
&-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}.
\end{aligned}\tag{48}$$

Using the inequality of power mean the definition of $|f'|^q$,

$$\begin{aligned}
&\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog\left(b^p\right)+J^\alpha_{a^p+b^p/2-}fog\left(a^p\right)\right]\right.\\
&\left.-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\right|\le\frac{b^p-a^p}{2^{1-\alpha}}\left[\left(\int_0^{1/2}\frac{u^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\left(\int_0^{1/2}\frac{u^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|^q du\right)^{1/q}\right]\\
&+\frac{b^p-a^p}{2^{1-\alpha}}\left[\left(\int_{1/2}^{1}\frac{(1-u)^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\left(\int_{1/2}^{1}\frac{(1-u)^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|^q du\right)^{1/q}\right]\\
&\le\left[\frac{b^p-a^p}{2^{1-\alpha}}\left(\int_0^{1/2}\frac{u^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\left(\int_0^{1/2}\frac{u^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right)\left|f'(b)\right|\right.\\
&\left.+\int_0^{1/2}\frac{u^{\alpha+1}M_\eta}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right]\\
&+\left[\frac{b^p-a^p}{2^{1-\alpha}}\left(\int_0^{1/2}\frac{u^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\int_0^{1/2}\frac{u^{\alpha+1}(1-u)\left(b^p-a^p\right)^2}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right]\\
&+\left[\frac{b^p-a^p}{2^{1-\alpha}}\left(\int_{1/2}^{1}\frac{u(1-u)^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\left(\int_{1/2}^{1}\frac{(1-u)^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right)\left|f'(b)\right|\right.\\
&\left.+\int_0^{1/2}\frac{(1-u)^\alpha uM_\eta}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right]\\
&+\left[\frac{b^p-a^p}{2^{1-\alpha}}\left(\int_{1/2}^{1}\frac{(1-u)^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\int_{1/2}^{1}\frac{(1-u)^{\alpha+1}u\left(b^p-a^p\right)^2}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right]\\
&=\left|\left[\left(C_5(\alpha,p)\right)^{1-1/q}\left[C_5(\alpha,p)\right|\left[\left|f'(b)\right|\right.\right.\right.\\
&+C_6(\alpha,p)M_\eta-C_7(\alpha,p)\mu\left(b^p-a^p\right)^2\Big]\\
&+\Big[C_8(\alpha,p)\Big)^{1-1/q}\Big[C_8(\alpha,p)\left|f'(b)\right|+C_9(\alpha,p)M_\eta
\end{aligned}$$

$$+ C_{10}(\alpha, p)\mu\left(b^p - a^p\right)^2]] - M_\eta + \frac{\mu\left(b^p - a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}|. \quad (49)$$

This completes the proof. □

Remark 12.

(i) Setting $p=1$ and $\mu=0, \eta=xy$ in Theorem 11 gives H-H type inequality for convex functions

(ii) Setting $p=1$ and $\alpha=1, \mu=0, \eta=xy$ in Theorem 11 gives H-H inequality for convex functions

4. Conclusion

In this paper, we established inequalities of Hermite-Hadamard and Fejér type for strongly generalized p-convex functions. We also established some fractional integral inequalities for this class of function in the setting of RL fractional integrals. We also related our results with the existing results and proved that by fixing involved parameters, we get many previous results.

Data Availability

The data required for this research is included within this paper.

Conflicts of Interest

The authors do not have any competing interests.

Authors' Contributions

All authors contributed equally in this paper.

References

[1] I. Abbas Baloch and Y. M. Chu, "Petrović-type inequalities for harmonic -convex functions," *Journal of Function Spaces*, vol. 2020, 7 pages, 2020.

[2] R. S. Ali, S. Mubeen, S. Ali, G. Rahman, J. Younis, and A. Ali, "Generalized Hermite–Hadamard-type integral inequalities forh-Godunova–Levin functions," *Journal of Function Spaces*, vol. 2022, 12 pages, 2022.

[3] J. Hadamard, "Etude sur les properietes des functions entries et en particulier d'une function consideree par Riemann," *Journal De Mathématiques Pures Et Appliquées*, vol. 58, pp. 171–215, 1893.

[4] S. Sezer, Z. Eken, G. Tinaztepe, and G. Adilov, "p–convex functions and some of their properties," *Numerical Functional Analysis and Optimization*, vol. 42, no. 4, pp. 443–459, 2021.

[5] M. B. Khan, H. M. Srivastava, P. O. Mohammed, K. Nonlaopon, and Y. S. Hamed, "Some new Jensen, Schur and Hermite-Hadamard inequalities for log convex fuzzy interval-valued functions," *AIMS Math*, vol. 7, no. 3, pp. 4338–4358, 2022.

[6] J. Nasir, S. Qaisar, S. I. Butt, K. A. Khan, and R. M. Mabela, "Some Simpson's Riemann–Liouville fractional integral inequalities with applications to special functions," *Journal of Function Spaces*, vol. 2022, 12 pages, 2022.

[7] Y. Yue, G. Farid, A. K. Demirel et al., "Hadamard and Fejér–Hadamard inequalities for generalized $ k $-fractional integrals involving further extension of Mittag-Leffler function," *AIMS Mathematics*, vol. 7, no. 1, pp. 681–703, 2021.

[8] L. Chen, W. Nazeer, F. Ali, T. Botmart, and S. Mehfooz, "Some midpoint inequalities forη-convex function via weighted fractional integrals," *Journal of Function Spaces*, vol. 2022, 12 pages, 2022.

[9] N. Merentes and Nikodem, "Remarks on strongly convex functions," *Aequationes Mathematicae*, vol. 80, no. 1-2, pp. 193–199, 2010.

[10] H. Angulo, J. Giménez, A. M. Moros, and K. Nikodem, "On strongly h–convex functions," *Annals of Functional Analysis*, vol. 2, no. 2, pp. 85–91, 2011.

[11] S. Abramovich, "Convexity, subadditivity and generalized Jensen's inequality," *Annals of Functional Analysis*, vol. 4, no. 2, pp. 183–194, 2013.

[12] M. R. Delavar and S. S. Dragomir, "On η- convexity," *Mathematical Inequalities & Applications*, vol. 20, no. 1, pp. 203–216, 2017.

[13] R. Ger and K. Nikodem, "Strongly convex functions of higher order," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 2, pp. 661–665, 2011.

[14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[15] B. Meftah, A. Souahi, and A. Souahi, "Cebysev inequalities for co-ordinated QC-convex and (s, QC)-convex," *Engineering and Applied Science Letters*, vol. 4, no. 1, pp. 14–20, 2021.

[16] M. Iftikhar, A. Qayyum, S. Fahad, and M. Arslan, "A new version of Ostrowski type integral inequalities for different differentiable mapping," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 353–359, 2021.

[17] M. E. Omaba, L. O. Omenyi, and L. O. Omenyi, "Generalized fractional Hadamard type inequalities for (Qs)-class functions of the second kind," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 270–278, 2021.

[18] A. Adu-Sackey, F. T. Oduro, F. T. Oduro, and G. O. Fosu, "Inequalities approach in determination of convergence of recurrence sequences," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 65–72, 2021.

[19] M. U. Awan, M. A. Noor, K. I. Noor, and F. Safdar, "On strongly generalized convex functions," *Univerzitet u Nišu*, vol. 31, no. 18, pp. 5783–5790, 2017.

[20] T. Ali, M. A. Khan, and Y. Khurshidi, "Hermite-Hadamard inequality for fractional integrals via eta-convex functions," *Acta Mathematica Universitatis Comenianae*, vol. 86, no. 1, pp. 153–164, 2017.

[21] M. Kunt and I. Iscan, "Hermite-Hadamard type inequalities for p-convex functions via fractional integrals," *Moroccan Journal of Pure and Applied Analysis*, vol. 3, no. 1, pp. 22–34, 2017.

[22] M. Kunt and I. Iscan, "Hermite-Hadamard-Fejer type inequalities for p-convex functions," *Arab Journal of Mathematical Sciences*, vol. 23, no. 2, pp. 215–230, 2017.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 3950149, 8 pages
https://doi.org/10.1155/2022/3950149

Research Article

Certain Analytic Formulas Linked to Locally Fractional Differential and Integral Operators

Ibtisam Aldawish [1] and Rabha W. Ibrahim [2]

[1] *Department of Mathematics and Statistics, College of Science, IMSIU (Imam Mohammad Ibn Saud Islamic University), Saudi Arabia*
[2] *The Institute of Electrical and Electronics Engineers (IEEE): 94086547, Portland 13765, USA*

Correspondence should be addressed to Rabha W. Ibrahim; rabhaibrahim@yahoo.com

Received 19 May 2022; Accepted 29 June 2022; Published 14 July 2022

Academic Editor: Mohsan Raza

The present investigation is aimed at defining different classes of analytic functions and conformable differential operators in view of the concept of locally fractional differential and integral operators. We present a novel generalized class of analytic functions, which we call it locally fractional analytic functions in the open unit disk. For the suggested class, we look at conditions to get the starlikeness and convexity properties.

1. Introduction

The idea of local fractional calculus (LFC) is an innovative differentiation and integration model for functions affecting on special fractal sets. Experts and scientists have been attracted by this theory. The LFC of a complex variable was established by Yang [1] (see [2] for additional material). The issue of explanation of fractal operators over analytic functions is glowered to be solved since the effort of Viswanathan and Navascues [3]. They presented a technique to define α-fractal operator on $\mathbb{C}^k(I)$, the space of all $k-$real-valued continuous functions defined on a compact interval I. In our work, we extend this idea into a complex domain, which is already compact in the z-plane. Therefore, in our opinion, the current study contributes to the theory of fractal functions and makes it easier for them to find new applications in a variety of domains, such as numerical analysis, functional analysis, and harmonic analysis; for example, in relation to PDEs. We anticipate that the current investigation will open the door to further research on shape-preserving fractal approximation in the different function spaces that are being taken into consideration. The reader is encouraged to see [4] for a fractal approximation that preserves shape in the space of differential functions.

Let $\vartheta \in (0, 1]$ be a fractional number and $\xi = \chi + i\eta$ be a complex number. Then, the fractal complex number ξ^ϑ can be defined by [2]

$$\xi^\vartheta \coloneqq \chi^\vartheta + i^\nu \eta^\vartheta, \quad \chi, \eta \in \mathbb{R}. \tag{1}$$

The local fractional derivative (LFD) at a random point ξ_0 for a complex function $\upsilon(\xi)$ can be formulated as follows:

$$\Delta^\vartheta \upsilon(\xi) = \Gamma(1+\vartheta) \lim_{\xi \longrightarrow \xi_0} \left(\frac{\upsilon(\xi) - \upsilon(\xi_0)}{(\xi - \xi_0)^\vartheta} \right), \quad (\vartheta \in (0, 1], \xi, \xi_0 \in \mathbb{C}, \upsilon : \mathbb{C} \longrightarrow \mathbb{C}). \tag{2}$$

Let $\upsilon(\xi) = (\xi)^{n\vartheta}$, for example; then, the LFD can be calculated as follows:

$$\Delta^\vartheta (\xi)^{n\vartheta} = \left(\frac{\Gamma(1+n\vartheta)}{\Gamma(1+(n-1)\vartheta)} \right) \xi^{(n-1)\vartheta}. \tag{3}$$

Obviously, when $\vartheta = 1$, the LFD reduces to the normal derivative $\xi^n = n\xi^{n-1}$. Moreover, for $n = 1$, the LFD becomes

$$\Delta^{\vartheta}(\xi)^{\vartheta} = \Gamma(1+\vartheta). \tag{4}$$

On Cantor sets, the local fractional differential operator (LFDO) can be used to construct a variety of different transformations and summations (see, e.g., Yang et al. [5–7]).

Let $\mathbb{A}_{\vartheta}, \vartheta \in (0, 1]$ be a class of locally fractional normalized functions in $\Omega := \{\xi \in \mathbb{C} : |\xi| < 1\}$ such that

$$\upsilon\left(\xi^{\vartheta}\right) = \xi^{\vartheta} + \sum_{n=2}^{\infty} v_n \xi^{n\vartheta}, \quad \xi \in \Omega. \tag{5}$$

Clearly, when $\vartheta = 1$, we attain the usual normalized class $\mathbb{A}$:

$$\upsilon(\xi) = \xi + \sum_{n=2}^{\infty} v_n \xi^n, \quad \xi \in \Omega. \tag{6}$$

Two functions $\upsilon, \omega \in \mathbb{A}_{\vartheta}$ are convoluted if they accept

$$\upsilon\left(\xi^{\vartheta}\right) * \omega\left(\xi^{\vartheta}\right) = \xi^{\vartheta} + \sum_{n=2}^{\infty} v_n w_n \xi^{\vartheta n}, \quad \xi \in \Omega, \tag{7}$$

where

$$\omega\left(\xi^{\vartheta}\right) = \xi^{\vartheta} + \sum_{n=2}^{\infty} w_n \xi^{\vartheta n}. \tag{8}$$

Moreover, they are subordinated $\upsilon \prec \omega$ if they satisfy the equality $\upsilon(\xi) = (\omega(\lambda(\xi)))$, where λ is analytic with $|\lambda(\xi)| \le |\xi| < 1$ (see [8]). This definition can be extended to LDC by suggesting $\xi \longrightarrow \xi^{\vartheta}$.

1.1. Convoluted Operators. The local derivative suggests the subsequent functional operator: for $n \ge 1$.

$$\begin{aligned}\frac{\Delta^{\nu}\xi^{\vartheta n}}{\Gamma(1+\vartheta)\xi^{n(\vartheta-1)-\vartheta}} &= \frac{((\Gamma(1+n\vartheta))/(\Gamma(1+(n-1)\vartheta)))\xi^{(n-1)\vartheta}}{\Gamma(1+\nu)\xi^{n(\vartheta-1)-\vartheta}} \\ &= \left(\frac{\Gamma(1+n\vartheta)}{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}\right)\xi^n.\end{aligned} \tag{9}$$

As a result, we define the following function for all $n \ge 1$,

$$\widehat{E}^{\vartheta}(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(1+n\vartheta)}{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}\right)\xi^n. \tag{10}$$

Obviously, $\widehat{\mathbb{E}} \in \mathbb{A}$. By using the convolution operator, we define the the following LFDO: $\mathbb{D}^{\vartheta} : \Omega \longrightarrow \Omega$, such that

$$\mathbb{D}^{\vartheta}\upsilon(\xi) := \left(\widehat{\mathrm{E}}^{\vartheta} * \upsilon\right)(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(1+n\vartheta)}{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}\right)v_n\xi^n. \tag{11}$$

Correspondingly, the fractional locally integral operator (FLIO) is defined as follows:

$$\mathbb{J}^{\vartheta}(\xi) := \left(\mathrm{E}^{\vartheta} * \upsilon\right)(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}{\Gamma(1+n\vartheta)}\right)v_n\xi^n. \tag{12}$$

We proceed to define the $k-$LFDO and $k-$LFIO, as follows:

$$\begin{aligned}\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi) &:= \underbrace{\widehat{\mathrm{E}}^{\vartheta} * \cdots * \widehat{\mathrm{E}}^{\vartheta}}_{k\text{-times}} * \upsilon(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(1+n\vartheta)}{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}\right)^k v_n\xi^n \\ &:= \xi + \sum_{n=2}^{\infty}\left(\widehat{\mathrm{E}}_n\right)^k v_n\xi^n. \\ \left(\mathbb{J}_k^{\vartheta}\upsilon\right)(\xi) &:= \underbrace{\mathrm{E}^{\vartheta} * \cdots * \mathrm{E}^{\vartheta}}_{k\text{-times}} * \upsilon(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}{\Gamma(1+n\vartheta)}\right)^k v_n\xi^n \\ &:= \xi + \sum_{n=2}^{\infty}\left(\mathrm{E}_n\right)^k v_n\xi^n.\end{aligned} \tag{13}$$

It is obvious that we reach the well-known Salagean differential and integral operators, respectively, when $\vartheta = 1$.

Definition 1. Define two formulas of analytic functions $\upsilon \in \mathbb{A}$ as follows:

$$\begin{aligned}\mathcal{S}_{\upsilon}(\xi) &:= \frac{\xi\upsilon'(\xi)}{\upsilon(\xi)}, \quad \xi \in \Omega, \\ \mathcal{K}_{\upsilon}(\xi) &:= 1 + \frac{\xi\upsilon''(\xi)}{\upsilon'(\xi)}, \quad \xi \in \Omega.\end{aligned} \tag{14}$$

Consequently, there are two classes for this investigation, the starlike and convex classes with

$$\begin{aligned}&\mathfrak{R}(\mathcal{S}_{\upsilon}(\xi)) > 0, \\ &\mathfrak{R}(\mathcal{K}_{\upsilon}(\xi)) > 0, \quad \xi \in \Omega,\end{aligned} \tag{15}$$

respectively.

1.2. Locally Fraction Conformable Operator

Definition 2. Let $\wp$ be a nonnegative number, such that $[[\wp]]$ be the integer part of $\wp$. By using the $k-$LFDO, we have the following locally fractional conformable differential operator:

$$\wedge^{\wp}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right] = \wedge^{\wp-[[\wp]]}\left(\wedge^{[[\wp]]}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\right) = \frac{p_1(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left(\wedge^{[[\wp]]}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\right) + \frac{p_0(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\zeta)}\left(\xi\left(\wedge^{[[\wp]]}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\right),\right. \tag{16}$$

where for $q{:=}\wp-[[\wp]] \in [0,1)$,

$$\wedge^{0}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right] = \left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right],$$

$$\begin{aligned}\wedge^{q}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right] &= \frac{p_1(q,\xi)}{p_1(q,\xi)+p_0(q,\xi)}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right] + \frac{p_0(q,\xi)}{p_1(q,\xi)+p_0(q,\xi)}\left(\xi\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]'\right)\\ &= \frac{p_1(q,\xi)}{p_1(q,\xi)+p_0(q,\xi)}\left[\xi+\sum_{n=2}^{\infty}\left(\widehat{\mathrm{E}}_n\right)^k \nu_n\xi^n\right] + \frac{p_0(q,\xi)}{p_1(q,\xi)+p_0(\nu,\xi)}\left(\left[\xi+\sum_{n=2}^{\infty}\left(\widehat{\mathrm{E}}_n\right)^k n\nu_n\xi^n\right]\right)\\ &= \xi+\sum_{n=2}^{\infty}\left(\frac{p_1(q,\xi)+n\,p_0(q,\xi)}{p_1(q,\xi)+p_0(q,\xi)}\right)\left(\widehat{\mathrm{E}}_n\right)^k\nu_n\xi^n\\ &\coloneqq \xi+\sum_{n=2}^{\infty}\theta_n(q,\xi)\left(\widehat{\mathrm{E}}_n\right)^k\nu_n\xi^n\wedge^1\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\\ &= \xi\left(\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\right)\\ &\vdots\end{aligned}$$

$$\wedge^{[[\wp]]}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right] = \wedge^{1}\left(\wedge^{[[\wp]]-1}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\right), \tag{17}$$

where

$$\theta_n(q,\xi) \coloneqq \left(\frac{p_1(q,\xi)+n\,p_0(q,\xi)}{p_1(\nu,\xi)+p_0(q,\xi)}\right), \tag{18}$$

and the functions $p_1, p_0 : [0,1]\times\Omega \longrightarrow \Omega$ are analytic in Ω with

$$\begin{gathered} p_1(q,\xi) \neq -p_0(q,\xi),\\ \lim_{q\longrightarrow 0} p_1(q,\xi) = 1,\\ \lim_{q\longrightarrow 1} p_1(q,\xi) = 0, \quad p_1(q,\xi)\neq 0, \forall\xi\in\Omega, q\in(0,1),\\ \lim_{q\longrightarrow 0} p_0(q,\xi) = 0,\\ \lim_{q\longrightarrow 1} p_0(q,\xi) = 1, \quad p_0(q,\xi)\neq 0, \forall\xi\in\Omega, q\in(0,1). \end{gathered} \tag{19}$$

Remark 3.

(i) For constant coefficients, the operator $\wedge^q[(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)]$ is normalized in Ω. Moreover, if $\wp-[[\wp]]=0, \vartheta=1$, then we realize the Sàlàgean differential operator [9]. If $k=0$, we attain the conformable differential operator in [10], which is based on the same assumptions. Similarly, we can replace the local fractional integral operator using the $k-\mathrm{LFJO}((\mathbb{J}_k^{\vartheta}\upsilon)(\xi))$

(ii) The authors in [11] presented a conformable fractional differential operator by using a combination of fractional integral and differential operators, as follows:

$$D^q\upsilon(\xi) = \xi + \sum_{n=2}^{\infty}\theta_n(q,\xi)\left(\frac{\Gamma(3-q)\Gamma(n+1)}{\Gamma(n+2-q)}\right)^k \nu_n\xi^n, \quad \xi\in\Omega, \tag{20}$$

where

$$\theta_n(q,\xi) = \left(\frac{p_1(q,\xi)+n\,p_0(q,\xi)}{p_1(\nu,\xi)+p_0(q,\xi)}\right) \tag{21}$$

We proceed to discuss the most important geometric properties of $(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)$ and $\wedge^{\wp}[(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)]$, in the next section.

1.3. Lemmas. We request the next preliminaries.

Lemma 4 (see [8]/p135). *Let υ be analytic and ω be univalent in Ω such that $\upsilon(0)=\omega(0)$. Furthermore, let ϕ be analytic in a domain involving $\omega(\Omega)$ and $\omega(\Omega)$. If $\xi\omega'(\xi)\phi(\omega(\xi))$ is starlike, then the subordination*

$$\xi\upsilon'(\xi)\phi(\upsilon(\xi)) \prec \xi\omega'(\xi)\phi(\omega(\xi)) \tag{22}$$

yields

$$\upsilon(\xi)\prec\omega(\xi), \tag{23}$$

and ω is the best dominant.

Lemma 5 (see [12]). *For two analytic functions υ_1 and υ_2 in a complex domain such that $\upsilon_1(0)=\upsilon_2(0)$ and*

$$\upsilon_1(\xi)\prec\upsilon_2(\xi), \tag{24}$$

then

$$\int_0^{2\pi}|\upsilon_1(\xi)|^t d\mu \leq \int_0^{2\pi}|\upsilon_2(\xi)|^t d\mu, \tag{25}$$

where

$$\xi = \nu\exp(i\mu), \quad \nu\in(0,1), t\in\mathbb{R}^+. \tag{26}$$

The next section is about our results of the operator $\wedge^q[(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)]$, and as a special consequence, we connect the operator $[(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)]$.

2. Results

This section deals with the operator $\wedge^q[(\mathbb{D}_k^\vartheta \upsilon)(\xi)] := \wedge^q(\xi)$.

2.1. Starlikeness of $\wedge^q(\xi)$

Theorem 6. *If the following conditions occur:*

(i) Λ is univalent in Ω

(ii) $(\xi\Lambda'(\xi))/(\Lambda(\xi)(\Lambda(\xi)-1))$ is starlike in Ω

(iii) $(\mathscr{K}_{\wedge^q}(\xi)-1)/(\mathscr{S}_{\wedge^q}(\xi)-1) \prec 1+((\xi\Lambda'(\xi))/(\Lambda(\xi)(\Lambda(\xi)-1)))$

Then,

$$\mathscr{S}_{\wedge^q}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega, \tag{27}$$

and Λ is the best dominant. Moreover,

$$\int_0^{2\pi} |\mathscr{S}_{\wedge^q}(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{28}$$

Proof. Put the function H, as follows:

$$H(\xi) := \mathscr{S}_{\wedge^q}(\xi), \quad \xi \in \Omega. \tag{29}$$

A simple calculation yields

$$\mathscr{S}_H(\xi) = \mathscr{K}_{\wedge^q}(\xi) - H(\xi). \tag{30}$$

Consequently, we attain

$$\frac{\mathscr{K}_{\wedge^q}(\xi)-1}{\mathscr{S}_{\wedge^q}(\xi)-1} = \frac{\mathscr{S}_H(\xi)+H(\xi)-1}{H(\xi)-1} = 1 + \frac{\xi H'(\xi)}{H(\xi)(H(\xi)-1)}. \tag{31}$$

Thus, we have

$$\frac{\xi H'(\xi)}{H(\xi)(H(\xi)-1)} \prec \frac{\xi\Lambda'(\xi)}{\Lambda(\xi)(\Lambda(\xi)-1)}, \quad \xi \in \Omega. \tag{32}$$

Based on Lemma 4, we attain the requested result. The second part is a direct application of Lemma 5. □

Theorem 7. *If the following conditions hold:*

(i) Λ is univalent in Ω

(ii) $(\xi\Lambda'(\xi))/(\Lambda(\xi)-1)$ is starlike in Ω

(iii) $\mathscr{S}_{\wedge^q}(\xi)(((\mathscr{K}_{\wedge^q}(\xi)-1)/(\mathscr{S}_{\wedge^q}(\xi)-1)\mathscr{S}_{\wedge^q}(\xi)-1)-1) \prec ((\xi\Lambda'(\xi))/(\Lambda(\xi)-1))$

Then,

$$\mathscr{S}_{\wedge^q}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega, \tag{33}$$

and Λ is the best dominant. Furthermore,

$$\int_0^{2\pi} |\mathscr{S}_{\wedge^q}(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{34}$$

Proof. Formulate the function H as follows:

$$H(\xi) := \mathscr{S}_{\wedge^q}(\xi), \quad \xi \in \Omega. \tag{35}$$

Then, we get the equality

$$\mathscr{S}_H(\xi) + H(\xi) = \mathscr{K}_{\wedge^q}(\xi). \tag{36}$$

Replacing produces the following results:

$$\mathscr{S}_{\wedge^q}(\xi)\left(\frac{\mathscr{K}_{\wedge^q}(\xi)-1}{\mathscr{S}_{\wedge^q}(\xi)-1} - 1\right) = \frac{\xi H'(\xi)}{H(\xi)-1}. \tag{37}$$

Hence,

$$\frac{\xi H'(\xi)}{H(\xi)-1} \prec \frac{\xi\Lambda'(\xi)}{\Lambda(\xi)-1}, \quad \xi \in \Omega. \tag{38}$$

In view of Lemma 4, we have the outcome. Lemma 5 implies the integral inequality. □

Theorem 8. *If the following conditions are fulfilled:*

(i) Λ is univalent in Ω

(ii) $\mathscr{S}_\Lambda$ is starlike in Ω

(iii) $\mathscr{K}_{\wedge^q}(\xi) - \mathscr{S}_{\wedge^q}(\xi) \prec \mathscr{S}_\Lambda(\xi)$

Then,

$$\mathscr{S}_{\wedge_q}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega, \tag{39}$$

and Λ is the best dominant. In addition,

$$\int_0^{2\pi} |\mathscr{S}_{\wedge^q}(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{40}$$

Proof. Present the function H as follows:

$$H(\xi) := \mathscr{S}_{\wedge^q}(\xi), \quad \xi \in \Omega. \tag{41}$$

Therefore, we get

$$\mathscr{S}_H(\xi) + H(\xi) = \mathscr{K}_{\wedge^q}(\xi). \tag{42}$$

Replacing brings that

$$\mathscr{K}_{\wedge^q}(\xi) - \mathscr{S}_{\wedge^q}(\xi) = \mathscr{S}_H(\xi). \tag{43}$$

Hence,

$$\mathscr{S}_H(\xi) \prec \mathscr{S}_\Lambda(\xi), \quad \xi \in \Omega. \tag{44}$$

Hence, Lemma 4 implies $H(\xi) \prec \Lambda(\xi)$. And Lemma 5 yields the last integral inequality

$$\int_0^{2\pi} |H(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{45}$$

□

Theorem 9. *If the following conditions are applied:*

(i) Λ is univalent in Ω

(ii) $\xi\Lambda'(\xi)$ is starlike in Ω

(iii) $\mathcal{S}_{\wedge^q}(\xi)(\mathcal{K}_{\wedge^q}(\xi) - \mathcal{S}_{\wedge^q}(\xi)) \prec \xi\Lambda'(\xi)$

Then,

$$\mathcal{S}_{\wedge^q}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega, \tag{46}$$

and Λ is the best dominant. Also,

$$\int_0^{2\pi} |\mathcal{S}_{\wedge^q}(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{47}$$

Proof. Define the function H as follows:

$$H(\xi) \coloneqq \mathcal{S}_{\wedge^q}(\xi), \quad \xi \in \Omega. \tag{48}$$

Consequently, we have

$$\mathcal{S}_H(\xi) + H(\xi) = \mathcal{K}_{\wedge^q}(\xi). \tag{49}$$

Substituting implies

$$\mathcal{S}_{\wedge^q}(\xi)(\mathcal{K}_{\wedge^q}(\xi) - \mathcal{S}_{\wedge^q}(\xi)) = \xi H'(\xi). \tag{50}$$

Hence,

$$\xi H'(\xi) \prec \xi\Lambda'(\xi), \quad \xi \in \Omega. \tag{51}$$

According to Lemma 4, we have result $H(\xi) \prec \Lambda(\xi)$, while Lemma 5 gives

$$\int_0^{2\pi} |H(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{52}$$

□

Remark 10. Note that, when $q = 0$ in Theorems 6–9, we have the starlikeness of the operator $[(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)]$, as follows:

$$\mathcal{S}_{[(\mathbb{D}_k^{\vartheta}\upsilon)]}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega. \tag{53}$$

Moreover, when $k = 0, q = 0$, we obtain the Ma-Minda class of starlikeness [13], as follows:

$$\mathcal{S}_{\upsilon}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega. \tag{54}$$

The last class is studied widely by many researchers for different types of functions [8] $\Lambda(\xi)$. For example, the inequality $\Lambda(\xi) \coloneqq (1 + i\xi)/(1 - j\xi), -1 \le j < i \le 1$, which indicates that the image of Ω under the description $\mathcal{S}_{\upsilon}(\xi)$ is centered on the x-axis owning diameter of end points $(1 - i)/(1 - j)$ and $(1 + i)/(1 + j)$ (see [14]). Another recent example is that $\Lambda(\xi) = \cos(\xi)$ [15].

We proceed to discover more properties of the locally fractional conformable operator.

Theorem 11. *If the following conditions hold:*

(i) Λ is convex univalent in Ω

(ii) $\mathcal{S}_{\wedge^q}(\xi) \prec \Lambda(\xi), \Lambda(0) = 1$

Then,

$$\wedge^q(\xi) \prec \xi \exp\left(\int_0^{\xi} \frac{\Lambda(\Xi(z))}{z} dz\right), \tag{55}$$

where Ξ fulfills $\Xi(0) = 0$ and $|\Xi(\xi)| < 1$. In addition, the inequality $|\xi| \coloneqq \tau < 1$ yields

$$\exp\left(\int_0^1 \frac{\Lambda(-\tau)}{\tau} d\tau\right) \le \left|\frac{\wedge^q(\xi)}{\xi}\right| \le \exp\left(\int_0^1 \frac{\Lambda(\tau)}{\tau} d\tau\right). \tag{56}$$

Proof. The conditions (i)-(ii) yield

$$\frac{(\wedge^q(\xi))}{\wedge^q(\xi)} - \frac{1}{\xi} = \frac{\Lambda(\Xi(\xi)) - 1}{\xi}. \tag{57}$$

Integration implies that

$$\wedge^q(\xi) \prec \xi \exp\left(\int_0^{\xi} \frac{\Lambda(\Xi(z))}{z} dz\right), \tag{58}$$

which is equivalent to

$$\frac{\wedge^q(\xi)}{\xi} \prec \exp\left(\int_0^{\xi} \frac{\Lambda(\Xi(z))}{z} dz\right). \tag{59}$$

But, we have

$$\Lambda(-\tau|\xi|) \le \Re(\Lambda(\Xi(\xi\tau))) \le \Lambda(\tau|\xi|). \tag{60}$$

Then, we obtain

$$\int_0^1 \frac{\Lambda(-\tau|\xi|)}{\rho} d\tau \le \int_0^1 \frac{\Re(\Lambda(\Xi(\xi\tau)))}{\tau} d\tau \le \int_0^1 \frac{\Lambda(\tau|\xi|)}{\tau} d\tau. \tag{61}$$

The above conclusion leads to

$$\int_0^1 \frac{\Lambda(-\tau|\xi|)}{\rho} d\tau \le \log\left|\frac{\wedge^q(\xi)}{\xi}\right| \le \int_0^1 \frac{\Lambda(\tau|\xi|)}{\tau} d\tau. \tag{62}$$

This leads to

$$\exp\left(\int_0^1 \frac{\Lambda(-\tau)}{\tau}d\tau\right) \le \left|\frac{\wedge^q(\xi)}{\xi}\right| \le \exp\left(\int_0^1 \frac{\Lambda(\tau)}{\tau}d\tau\right). \quad (63)$$

□

Corollary 12. *Let the conditions of Theorem 11 hold for q = 0. Then,*

$$\left[\left(\mathbb{D}_k^\vartheta \upsilon\right)(\xi)\right] \prec \xi \exp\left(\int_0^\xi \frac{\Lambda(\Xi(z))}{z}dz\right), \quad (64)$$

where Ξ achieves $\Xi(0)=0$ and $|\Xi(\xi)|<1$. Moreover, the inequality $|\xi| \coloneqq \tau < 1$ yields

$$\exp\left(\int_0^1 \frac{\Lambda(-\tau)}{\tau}d\tau\right) \le \left|\frac{\left[\left(\mathbb{D}_k^\vartheta \upsilon\right)(\xi)\right]}{\xi}\right| \le \exp\left(\int_0^1 \frac{\Lambda(\tau)}{\tau}d\tau\right). \quad (65)$$

Corollary 13. *Let the conditions of Theorem 11 hold for q = 0 and k = 0. Then,*

$$[\upsilon(\xi)] \prec \xi \exp\left(\int_0^\xi \frac{\Lambda(\Xi(z))}{z}dz\right), \quad (66)$$

where Ξ satisfies $\Xi(0)=0$ and $|\Xi(\xi)|<1$. Moreover, the inequality $|\xi| \coloneqq \tau < 1$ yields

$$\exp\left(\int_0^1 \frac{\Lambda(-\tau)}{\tau}d\tau\right) \le \left|\frac{\upsilon(\xi)}{\xi}\right| \le \exp\left(\int_0^1 \frac{\Lambda(\tau)}{\tau}d\tau\right). \quad (67)$$

2.2. Positive Real Part of $\wedge^q(\xi)$. In this part, we aim to present the sufficient conditions for the operator $\wedge^q$ to be in the class of the real positive part ($\mathscr{P}$.).

Theorem 14. *Let $\ell \in (0,1)$ and $\wp \in [0,\infty)$. If*

$$p_0(\wp - [[\wp]], \xi) = \left(\frac{\ell}{1-\ell}\right) p_1(\wp - [[\wp]], \xi)\,;\, q = \wp - [[\wp]], \quad (68)$$

then

$$\frac{\wedge^{\wp+2}(\xi)}{\wedge^{\wp+1}(\xi)} \in \mathscr{P} \Longrightarrow \frac{\wedge^{\wp+1}(\xi)}{\wedge^{\wp}(\xi)} \in \mathscr{P}. \quad (69)$$

Proof. By the condition $p_0(\wp-[[\wp]],\xi) = (\ell/(1-\ell))p_1(\wp-[[\wp]],\xi)$ and by Definition 2, we have

$$\begin{aligned} \wedge^{\wp}(\xi) &= (1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)]\right), \\ \wedge^{\wp+1}(\xi)] &= \xi\left(\wedge^{[[\wp]]}(\xi)]\right) + \ell\xi^2\left(\wedge^{[[\wp]]}(\xi)]\right), \\ \wedge^{\wp+2}(\xi)] &= \xi\left(\wedge^{[[\wp]]}(\xi)]\right) + (1+2\ell)\wp^2\left(\wedge^{[[\wp]]}(\xi)\right]\right) + \ell\xi^3\left(\wedge^{[[\wp]]}(\xi)]\right). \end{aligned} \quad (70)$$

We realize that

$$\Re\left(\frac{\wedge^{\wp+2}(\xi)}{\wedge^{\wp+1}(\xi)}\right) > 0, \quad (71)$$

whenever the following real is positive:

$$\Re\left\{1 + \frac{(1+\ell)\xi\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi^2\left(\wedge^{[[\wp]]}(\xi)\right)}{\left(\wedge^{[[\wp]]}(\xi)\right]\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)\right)}\right\} > 0. \quad (72)$$

Hence, from the inequality (74) with the idea of convex functional

$$(1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left[\wedge^{[[\wp]]}(\xi)\right], \quad (73)$$

we have

$$\Re\left\{1 + \frac{\xi\left[(1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)\right)\right]}{\left[(1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)\right)\right]}\right\} > 0. \quad (74)$$

Since convexity implies starlikeness, then we realize

$$\Re\left\{\frac{\xi\left[(1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)\right)\right]}{(1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)\right)}\right\} > 0. \quad (75)$$

The inequality (75) holds whenever

$$\Re\left(\frac{\wedge^{\wp+1}(\xi)]}{\wedge^{\wp}(\xi)]}\right) > 0. \quad (76)$$

□

Note that when $k=0$ in Theorem 14, we have [10]—Theorem 3.1.

Next, we deal with the functional

$$\frac{\wedge^{\wp+1}(\xi)}{\xi\left(\wedge^{[[\wp]]}(\xi)\right)'} \quad (77)$$

to be in the class of positive real part $\mathscr{P}$.

Theorem 15. *Let $\ell \in (0,1)$, $\wp \in [0,\infty)$, and*

$$p_1(\wp - [[\wp]], \xi) = \left(\frac{\ell}{1-\ell}\right) p_0(\wp - [[\wp]], \xi). \quad (78)$$

If $\wedge^{[[\wp]]}(\xi) \in \mathscr{C}$ (the class of convex univalent functions in Ω), then

$$\frac{\wedge^{\wp+1}(\xi)]}{\xi\left(\wedge^{[[\wp]]}(\xi)\right]\right)} \in \mathscr{P}(\ell). \quad (79)$$

Proof. Applying the differential operator rule to

$$\wedge^{\wp+1}(\xi)] = \wedge^1\left(\wedge^{\wp}(\xi)\right) \quad (80)$$

yields

$$\begin{aligned}
\wedge^{\wp+1}(\xi)] &= \wedge^{\wp-[[\wp]]}\left(\wedge^{[[\wp]]+1}(\xi)\right) = \wedge^{\wp-[[\wp]]}\left\{\wedge\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&= \wedge^{\wp-[[\wp]]}\left\{\xi\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&= \frac{p_1(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left\{\xi\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&\quad + \frac{p_0(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)} \\
&\quad \times\left\{\xi\left[\left(\wedge^{[[\wp]]}(\xi)\right)+\xi\left(\wedge^{[[\wp]]}(\xi)\right)\right]\right\} \\
&= \frac{p_1(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left\{\xi\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&\quad + \frac{p_0(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left\{\xi\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&\quad + \frac{p_0(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left\{\xi^2\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&= \xi\left[\wedge^{[[\wp]]}(\xi)\right]\right] + \frac{p_0(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left\{\xi^2\left[\wedge^{[[\wp]]}(\xi)\right]\right\}.
\end{aligned} \tag{81}$$

Dividing equation (80) by the formula $\xi(\wedge^{[[\wp]]}(\xi))$ and employing the condition

$$p_1(\wp-[[\wp]],\xi) = \left(\frac{\ell}{1-\ell}\right)p_0(\wp-[[\wp]],\xi), \tag{82}$$

we attain

$$\frac{\wedge^{\wp+1}(\xi)}{\xi\left(\wedge^{[[\wp]]}(\xi)\right)} = 1 + (1-\ell)\frac{\xi\left(\wedge^{[[\wp]]}(\xi)\right)}{\left(\wedge^{[[\wp]]}(\xi)\right)}. \tag{83}$$

The convexity of $\wedge^{[[\wp]]}(\xi), \xi \in \Omega$, brings

$$\Re\left\{1 + \frac{\xi\left(\wedge^{[[\wp]]}(\xi)\right)}{\left(\wedge^{[[\wp]]}(\xi)\right)}\right\} > 0. \tag{84}$$

Hence, it yields that

$$\Re\left\{\frac{\wedge^{\wp+1}(\xi)}{\xi\left(\wedge^{[[\wp]]}(\xi)\right)}\right\} > \ell. \tag{85}$$

□

Note that when $k = 0$ in Theorem 15, we have [10]—Theorem 3.2.

3. Conclusion

By using the concept of locally fractional differential and integral of a complex variable, we illustrated a set of differential and integral operators acting on a class of normalized analytic functions. Moreover, we presented a conformable differential operator linking with the suggested locally fractional differential operator (similarly if we take the locally fractional integral). Different investigations are introduced, including the stralikeness and convexity properties.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally and significantly to writing this article. All authors read and agreed to the published version of the manuscript.

References

[1] X.-J. Yang, *Advanced Local Fractional Calculus and its Applications*, World Science Publisher, 2012.

[2] C. Cattani, H. M. Srivastava, X.-J. Yang, and F. Dynamics, *Walter De Gruyter GmbH*, de Gruyter Open, Berlin/Boston, 2015.

[3] P. Viswanathan and M. A. Navascues, "A fractal operator on some standard spaces of functions," *Proceedings of the Edinburgh Mathematical Society*, vol. 60, no. 3, pp. 771–786, 2017.

[4] P. Viswanathan, A. K. B. Chand, and M. A. Navascues, "Fractal perturbation preserving fundamental shapes: bounds on the scale factors," *Journal of Mathematical Analysis and Applications*, vol. 419, no. 2, pp. 804–817, 2014.

[5] X.-J. Yang, D. Baleanu, and J. A. Tenreiro Machado, "Systems of Navier-Stokes equations on Cantor sets," *Mathematical Problems in Engineering*, vol. 2013, Article ID 769724, 8 pages, 2013.

[6] A.-M. Yang, C. Cattani, C. Zhang, G.-N. Xie, and X.-J. Yang, "Local fractional Fourier series solutions for nonhomogeneous heat equations arising in fractal heat flow with local fractional derivative," *Advances in Mechanical Engineering*, vol. 6, Article ID 514639, 2014.

[7] X.-J. Yang, H. M. Srivastava, J.-H. He, and D. Baleanu, "Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives," *Physics Letters A*, vol. 377, no. 28-30, pp. 1696–1700, 2013.

[8] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, CRC Press, 2000.

[9] G. S. Salagean, "Subclasses of univalent functions," in *Complex Analysis-Fifth Romanian-Finnish Seminar*, pp. 362–372, Springer, Berlin, Heidelberg, 1983.

[10] R. W. Ibrahim and J. M. Jahangiri, "Conformable differential operator generalizes the Briot-Bouquet differential equation in a complex domain," *AIMS Math*, vol. 4, no. 6, pp. 1582–1595, 2019.

[11] R. W. Ibrahim and D. Baleanu, "On a combination of fractional differential and integral operators associated with a class of normalized functions," *AIMS Mathematics*, vol. 6, no. 4, pp. 4211–4226, 2021.

[12] J. E. Littlewood, "On inequalities in the theory of functions," *Proceedings of the London Mathematical Society*, vol. 2, no. 1, pp. 481–519, 1925.

[13] W. Ma, "A unified treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, pp. 157–169, China, 1992.

[14] M. Obradovic and S. Owa, "On certain properties for some classes of starlike functions," *Journal of Mathematical Analysis and Applications*, vol. 145, no. 2, pp. 357–364, 1990.

[15] H. Tang, H. M. Srivastava, S.-H. Li, and G.-T. Deng, "Majorization results for subclasses of starlike functions based on the sine and cosine functions," *Bulletin of the Iranian Mathematical Society*, vol. 46, no. 2, pp. 381–388, 2020.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 4449502, 14 pages
https://doi.org/10.1155/2022/4449502

Research Article

The Study of Solutions of Several Systems of Nonlinear Partial Differential Difference Equations

Hong Yan Xu[1,2] **Meiying Yu**[3] **and Keyu Zhang**[3]

[1]College of Arts and Sciences, Suqian University, Suqian, Jiangsu 223800, China
[2]School of Mathematics and Computer Science, Shangrao Normal University, Shangrao Jiangxi 334001, China
[3]School of Mathematics, Qilu Normal University, Shandong, Jinan 250200, China

Correspondence should be addressed to Meiying Yu; zkyymy@126.com

Received 30 May 2022; Accepted 16 June 2022; Published 12 July 2022

Academic Editor: Sibel Yalçın

Our main aim is to describe the entire solutions of several systems of $\begin{cases}[\alpha_1 f_1(z)]^2+[\alpha_2 f_2(z+c)]^2=1,\\ [\beta_1 f_2(z)]^2+[\beta_2 f_1(z+c)]^2=1,\end{cases}$ $\begin{cases}(\alpha_1\partial f_1/\partial z_1)^{n_1}+[\alpha_2 f_2(z+c)]^{m_1}=1,\\ (\beta_1\partial f_2/\partial z_1)^{n_2}+[\beta_2 f_1(z+c)]^{m_2}=1,\end{cases}$ and $\begin{cases}(\alpha_1\partial f_1/\partial z_1)^2+[\alpha_2 f_2(z+c)]^2=1,\\ (\beta_1\partial f_2/\partial z_1)^2+[\beta_2 f_1(z+c)]^2=1,\end{cases}$ $\begin{cases}(\alpha_1\partial f_1/\partial z_1)^2+[\alpha_2 f_2(z+c)+\alpha_3 f_1(z)]^2=1,\\ (\beta_1\partial f_2/\partial z_1)^2+[\beta_2 f_1(z+c)+\beta_3 f_2(z)]^2=1,\end{cases}$ where $\alpha_j, \beta_j (j=1,2,3)$ are nonzero constants in $\mathbb{C}$ and $m_j, n_j (j=1,2)$ are positive integers. We obtain several theorems on the existence and the forms of solutions for these systems, which are some improvements and supplements of the previous theorems given by Xu and Cao, Gao, and Liu and Yang. Moreover, we give some examples to explain the existence of solutions for such systems.

1. Introduction

As everyone knows, the study of the existence of solutions for Fermat type equations has always been an important and interesting problem. The famous Fermat's Last Theorem has attracted the attention of many mathematical scholars [1, 2]. About 60 years ago or even earlier, Montel [3] and Gross [4] had considered the equation $f^m+g^m=1$ and obtained that the entire solutions of $f^2+g^2=1$ are $f=\cos\zeta(z)$, $g=\sin\zeta(z)$ for the case $m=2$, where $\zeta(z)$ is an entire function, and this equation does not admit any nonconstant entire solution for any positive integer $m>2$.

With the establishment and rapid development of Nevanlinna value distribution theory for meromorphic functions and theirs difference [5–7], Liu [8] in 2009, Liu et al. [9] in 2012, and Liu and Yang [10] in 2013 studied some complex Fermat type difference and Fermat type differential difference equations and obtained some results.

Theorem 1 (see [9], Theorem 1.1). *The transcendental entire solutions with finite order of*

$$f(z)^2+f(z+c)^2=1 \quad (1)$$

must satisfy $f(z)=\sin(Az+B)$, *where* B *is a constant and* $A=(4k+1)\pi/2c$, *with* k *an integer.*

Theorem 2 (see [9], Theorem 1.3). *The transcendental entire solutions with finite order of*

$$f'(z)^2+f(z+c)^2=1 \quad (2)$$

must satisfy $f(z)=\sin(z\pm Bi)$, *where* B *is a constant and* $c=2k\pi$ *or* $c=(2k+1)\pi$, *with* k *an integer.*

After that, Gao [11] in 2016 extended Theorem 2 from complex differential difference equation to the system of complex differential difference equations.

Theorem 3 (see [11], Theorem 1.1). *Suppose that* (f_1, f_2) *is a pair of finite-order transcendental entire solutions for the system of differential difference equations*

$$\begin{cases}[f_{1'}(z)]^2+f_2(z+c)^2=1,\\ [f_{2'}(z)]^2+f_1(z+c)^2=1.\end{cases} \quad (3)$$

Then, (f_1, f_2) *satisfies*

$$(f_1, f_2) = (\sin(z - bi), \sin(z - b_1 i)), \quad (4)$$

or

$$(f_1, f_2) = (\sin(z + bi), \sin(z + b_1 i)), \quad (5)$$

where b, b_1 *are constants and* $c = k\pi$, *where* k *is an integer.*

In recent, Xu and Cao [12, 13] further discussed the solutions for some Fermat type PDDEs and obtained the following:

Theorem 4 (see [13], Theorem 1.4). *Let* $c \in \mathbb{C}^n - \{0\}$. *Then, any nonconstant entire solution with finite order of the equation*

$$f(z)^2 + f(z+c)^2 = 1 \quad (6)$$

has the form of $f(z) = \cos(L(z) + B)$, *where* L *is a linear function of the form* $L(z) = a_1 z_1 + \cdots + a_n z_n$ *on* $\mathbb{C}^n$ *such that* $L(c) = -\pi/2 - 2k\pi, k \in \mathbb{Z}$, *and* B *is a constant on* $\mathbb{C}$.

Theorem 5 (see [13], Theorem 1.1). *Let* $c = (c_1, c_2) \in \mathbb{C}^2$. *Then,*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^n + f(z_1 + c_1, z_2 + c_2)^m = 1 \quad (7)$$

does not have any transcendental entire solution with finite order, where m *and* n *are two distinct positive integers.*

Theorem 6 (see [13], Theorem 1.2). *Let* $c = (c_1, c_2) \in \mathbb{C}^2$. *Then, any transcendental entire solution with finite order of the PDDE*

$$\left(\frac{\partial f}{\partial z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1 \quad (8)$$

has the form of $f(z_1, z_2) = \sin(Az_1 + B)$, *where* A *is a constant on* $\mathbb{C}$ *satisfying* $Ae^{iAc_1} = 1$ *and* B *is a constant on* $\mathbb{C}$; *in the special case whenever* $c_1 = 0$, *we have* $f(z_1, z_2) = \sin(z_1 + B)$.

By analyzing Theorems 3–6, a natural question is as follows: *What will happen about the transcendental entire solutions for the system of the PDDEs of Fermat type*? Although many scholars have paid considerable attention to the complex difference equation with a single variable and the complex Fermat type difference equation in recent years, a series of important and meaningful results (including [7, 14–22]) were obtained, however, to our knowledge, there were not much results about the complex difference equation in several complex variables. Of course, the references involving the results of systems of complex PDDEs are even less.

This manuscript is aimed at studying the solutions of several Fermat type systems involving both difference operator and partial differential. We establish four theorems on the forms of solutions for several systems of Fermat type PDDEs, which are improvement of the previous theorems given by Liu et al., Gao, and Xu and Cao [8, 9, 11, 13]. We mainly employ the Nevanlinna value distribution theory and difference Nevanlinna theory of several complex variables in this article, and the readers can refer to [23, 24]. Now, we start to state our main results below.

Theorem 7. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$, $\alpha_j, \beta_j (j = 1, 2) \in \mathbb{C} - \{0\}$, *and* $m_j, n_j (j = 1, 2) \in \mathbb{N}_+$. *If the Fermat type PDDE system*

$$\begin{cases} \left(\alpha_1 \dfrac{\partial f_1}{\partial z_1}\right)^{n_1} + [\alpha_2 f_2(z+c)]^{m_1} = 1, \\ \left(\beta_1 \dfrac{\partial f_2}{\partial z_1}\right)^{n_2} + [\beta_2 f_1(z+c)]^{m_2} = 1 \end{cases} \quad (9)$$

satisfies one of the conditions

(i) $m_1 m_2 > n_1 n_2$

(ii) $n_j > m_j / m_j - 1$ *and* $m_j \geq 2$, $j = 1, 2$

then system (9) does not exist any pair of finite-order transcendental entire solution.

Remark 8. Here, we say that (f, g) is a pair of finite-order transcendental entire solution for

$$\begin{cases} f^{n_1} + g^{m_1} = 1, \\ f^{n_2} + g^{m_2} = 1, \end{cases} \quad (10)$$

if f, g are transcendental entire functions satisfying the above system and $\rho = \max\{\rho(f), \rho(g)\} < \infty$.

Remark 9. We list an example to demonstrate that the condition $m_j \geq 2$ in Theorem 7 cannot be removed. Let

$$\begin{aligned} f_1 = f_2 = {} & 1 + \frac{1}{4}c_1^2 - \frac{1}{4}z_1^2 + \left(\frac{c_1}{2c_2}z_2 + b + e^{(2\pi i/c_2)z_2}\right)(z_1 - c_1) \\ & - \left[\frac{c_1}{2c_2}(z_2 - c_2) + b + e^{(2\pi i/c_2)z_2}\right]^2, \end{aligned} \quad (11)$$

where $c_1, b \in \mathbb{C}$ and $c_2 \neq 0$. Thus, (f_1, f_2) satisfies the system (9) with $n_1 = n_2 = 2$, $m_1 = m_2 = 1$, and $\alpha_j = \beta_j = 1$, $j = 1, 2$.

Theorem 10. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $\alpha_j, \beta_j (j = 1, 2) \in \mathbb{C} - \{0\}$. *If the system of Fermat type difference equations*

$$\begin{cases} [\alpha_1 f_1(z)]^2 + [\alpha_2 f_2(z+c)]^2 = 1, \\ [\beta_1 f_2(z)]^2 + [\beta_2 f_1(z+c)]^2 = 1 \end{cases} \quad (12)$$

admits a pair of finite-order transcendental entire solution (f_1, f_2), *then* $\alpha_1^2/\beta_2^2 = \beta_1^2/\alpha_2^2 = 1$, *and* (f_1, f_2) *have the* *following forms*

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_0} + e^{-(L(z)+B_0)}}{2\alpha_1}, \frac{\beta_2}{\beta_1}\frac{A_{21}e^{L(z)+B_0} + A_{22}e^{-(L(z)+B_0)}}{2\alpha_1}\right), \tag{13}$$

where $L(z) = a_1z_1 + a_2z_2$, $a_1, a_2, B_0 \in \mathbb{C}$, *and* A_{21}, A_{22}, c *satisfy one of the following cases.*

(i) $L(c) = k\pi i$, $A_{21} = -i$ *and* $A_{22} = i$ *or* $A_{21} = i$ *and* $A_{22} = -i$, *k is a integer*

(ii) $L(c) = (2k \pm 1/2)\pi i$, $A_{21} = -1$ *and* $A_{22} = -1$, *or* $A_{21} = 1$ *and* $A_{22} = 1$.

Remark 11. From Theorem 10, we can conclude that f_1, f_2 have the following relationships

(i) $f_2 = \eta f_1$

(ii) $f_2 = i\eta e^{L(z)+B_1} - e^{-(L(z)+B_1)}/2\alpha_1$, where $\eta = \pm\beta_2/\beta_1$ and $f_1(z) = e^{L(z)+B_1} + e^{-(L(z)+B_1)}/2\alpha_1$.

Now, two examples can verify the existence of solutions for (12).

Example 1. Let c_1, c_2 and $L(z) = a_1z_1 + a_2z_2$ satisfy $L(c) = a_1 c_1 + a_2c_2 = (2k \pm 1/2)\pi i$, and $B_0 \in \mathbb{C}$. Then, the function

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_0} + e^{-L(z)-B_0}}{4}, -\frac{e^{L(z)+B_0} + e^{-L(z)-B_0}}{2}\right) \tag{14}$$

satisfies the system (12) with $\alpha_1 = 2$, $\alpha_2 = 1$, and $\beta_1 = \beta_2 = 1$.

Example 2. Let c_1, c_2 and $L(z) = a_1z_1 + a_2z_2$ satisfy $L(c) = a_1 c_1 + a_2c_2 = k\pi i$, and $B_0 \in \mathbb{C}$. Then, the function

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_0} + e^{-L(z)-B_0}}{2}, \frac{1}{3}\frac{e^{L(z)+B_0} - e^{-L(z)-B_0}}{2i}\right) \tag{15}$$

satisfies the system (12) with $\alpha_1 = 1 = \beta_2$ and $\alpha_2 = 3 = \beta_1$.

Theorem 12. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $\alpha_j, \beta_j (j = 1, 2) \in \mathbb{C} - \{0\}$. *If the system of Fermat type PDDEs*

$$\begin{cases} \left(\alpha_1 \dfrac{\partial f_1}{\partial z_1}\right)^2 + [\alpha_2 f_2(z+c)]^2 = 1, \\ \left(\beta_1 \dfrac{\partial f_2}{\partial z_1}\right)^2 + [\beta_2 f_1(z+c)]^2 = 1 \end{cases} \tag{16}$$

admits a pair of finite-order transcendental entire solution (f_1, f_2), *then* $(\alpha_1\alpha_2)^2 = (\beta_1\beta_2)^2$ *and* (f_1, f_2) *is the form of*

$$(f_1, f_2) = \left(\frac{A_{11}e^{L(z)+B_0} + A_{12}e^{-(L(z)+B_0)}}{2\beta_2}, \eta\frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{2\beta_2}\right), \tag{17}$$

where $L(z) = a_1z_1 + a_2z_2$, B_0 *is a constant in* $\mathbb{C}$, *and* $a_1, c, A_{11}, A_{12}, \eta$ *satisfy* $a_1^2 = -\beta_2^2/\alpha_1^2 = -\alpha_2^2/\beta_1^2$ *and one of the following cases*

(i) $L(c) = 2k\pi i$, *and* $\eta = -1$, $A_{11} = -i$, $A_{12} = i$, *or* $\eta = 1$, $A_{11} = i$, $A_{12} = -i$

(ii) $L(c) = (2k + 1)\pi i$, *and* $\eta = -1$, $A_{11} = i$, $A_{12} = -i$, *or* $\eta = 1$, $A_{11} = -i$, $A_{12} = i$

(iii) $L(c) = (2k + 1/2)\pi i$, *and* $\eta = 1$, $A_{11} = -1$, $A_{12} = -1$, *or* $\eta = -1$, $A_{11} = 1$, $A_{12} = 1$

(iv) $L(c) = (2k - 1/2)\pi i$, *and* $\eta = 1$, $A_{11} = 1$, $A_{12} = 1$, *or* $\eta = -1$, $A_{11} = -1$, $A_{12} = -1$.

Here, two examples can verify the existence of solutions for (16).

Example 3. Let $(a_1, a_2) = (i, \pi)$, $A_{11} = -i$, $A_{12} = i$, $\eta = -1$, and $B_0 \in \mathbb{C}$. That is, $L(z) = iz_1 + \pi z_2$ and

$$(f_1(z), f_2(z)) = \left(-i\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{4}, -i\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{2}\right). \tag{18}$$

Thus, (f_1, f_2) satisfies the system (16) with $(c_1, c_2) = (\pi, i)$, $\alpha_1 = 2$, $\beta_1 = 1$, $\alpha_2 = 1$, and $\beta_2 = 2$.

Example 4. Let $(a_1, a_2) = (1, -\pi i)$, $A_{11} = -1$, $A_{12} = -1$, $\eta = 1$, and $B_0 \in \mathbb{C}$. That is, $L(z) = z_1 - \pi i z_2$ and

$$(f_1(z), f_2(z)) = \left(-\frac{e^{L(z)+B_0} + e^{-(L(z)+B_0)}}{4i}, -\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{2i}\right). \tag{19}$$

Thus, (f_1, f_2) satisfies the system (16) with $(c_1, c_2) = (\pi i, 1/2)$, $\alpha_1 = 2$, $\beta_1 = 1$, $\alpha_2 = i$, and $\beta_2 = 2i$.

Example 5. Let $(a_1, a_2) = (2i, i)$, $A_{11} = i$, $A_{12} = -i$, $\eta = -1$, and $B_0 \in \mathbb{C}$. That is, $L(z) = 2iz_1 + iz_2$ and

$$(f_1(z), f_2(z)) = \left(i\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{4}, -i\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{8}\right). \tag{20}$$

Thus, (f_1, f_2) satisfies the system (16) with $(c_1, c_2) = (\pi, -\pi)$, $\alpha_1 = 1$, $\beta_1 = 2$, $\alpha_2 = 4$, and $\beta_2 = 2$.

Example 6. Let $(a_1, a_2) = (3, 1)$, $A_{11} = 1$, $A_{12} = 1$, $\eta = i$, and $B_0 \in \mathbb{C}$. That is, $L(z) = 3z_1 + z_2$ and

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_0} + e^{-(L(z)+B_0)}}{6}, i \frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{18} \right). \tag{21}$$

Thus, (f_1, f_2) satisfies the system (16) with $(c_1, c_2) = (\pi, -\pi)$, $\alpha_1 = i$, $\alpha_2 = 9$, $\beta_1 = 3i$, and $\beta_2 = 3$.

Theorem 13. *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $\alpha_j, \beta_j (j = 1, 2, 3) \in \mathbb{C} - \{0\}$. Let (f_1, f_2) be a pair of transcendental entire solutions of finite order for the system*

$$\begin{cases} \left(\alpha_1 \dfrac{\partial f_1}{\partial z_1}\right)^2 + [\alpha_2 f_2(z+c) + \alpha_3 f_1(z)]^2 = 1, \\ \left(\beta_1 \dfrac{\partial f_2}{\partial z_1}\right)^2 + [\beta_2 f_1(z+c) + \beta_3 f_2(z)]^2 = 1. \end{cases} \tag{22}$$

Then, (f_1, f_2) is one of the forms

$$(f_1, f_2) = \left(\frac{e^{iL(z)+B_0} - e^{-iL(z)-B_0}}{2ia_1\alpha_1} + e^{\eta z_2} G_1(z_2), \pm \frac{e^{iL(z)+B_0} - e^{-iL(z)-B_0}}{2ia_1\beta_1} + e^{\eta z_2} G_2(z_2) \right), \tag{23}$$

or

$$(f_1, f_2) = \left(\frac{e^{iL(z)+B_0} - e^{-iL(z)-B_0}}{2ia_1\alpha_1} + e^{\eta z_2} G_1(z_2), \pm \frac{e^{iL(z)+B_0} + e^{-iL(z)-B_0}}{2a_1\beta_1} + e^{\eta z_2} G_2(z_2) \right), \tag{24}$$

where $L(z) = a_1 z_1 + a_2 z_2$, $a_1 (\neq 0), a_2, B_0 \in \mathbb{C}$, and $G_1(z_2), G_2(z_2)$ are entire period functions of finite order with period $2c_2$, and $a_1, a_2, \alpha_j, \beta_j, \eta, c_1, c_2$ satisfy $e^{4iL(c)} = 1$ and the following conditions

(C_1) $\eta = 0$ if $\alpha_2\beta_2 = \alpha_3\beta_3$, and $\eta = \log(\alpha_2\beta_2) - \log(\alpha_3\beta_3)/2c_2$ if $\alpha_2\beta_2 \neq \alpha_3\beta_3$

(C_2) $[\beta_1/\alpha_2(a_1 - \alpha_3/\alpha_1)]^2 = [\alpha_1/\beta_2(a_1 - \beta_3/\beta_1)]^2 = 1$, or $[\beta_1/\alpha_2(a_1 - \alpha_3/\alpha_1)]^2 = [\alpha_1/\beta_2(a_1 + \beta_3/\beta_1)]^2 = 1$

$$(f_1(z), f_2(z)) = \left(e^{\log(-1)/2c_2 z_2} G_1(z_2) + D_1, e^{\log(-1)/2c_2 z_2} G_2(z_2) + D_2 \right), \tag{25}$$

where $D_1 = \alpha_2\xi_2 - \beta_3\xi_1/2\alpha_2\beta_2$, $D_2 = \beta_2\xi_1 - \alpha_3\xi_2/2\alpha_2\beta_2$, $\xi_1 = \pm 1$, and $\xi_2 = \pm 1$;

$$(f_1(z), f_2(z)) = \left(b_1 z_1 + \gamma_1 z_2 + G_1(z_2), -\frac{\alpha_3}{\alpha_2} b_1 z_1 + \gamma_2 z_2 + G_2(z_2) \right), \tag{26}$$

where $G_1(z_2), G_2(z_2)$ are stated as in ((23)) and ((24)), and $b_1 (\neq 0), \gamma_1, \gamma_2$ satisfy

$$\gamma_1 = \frac{\alpha_2 b_3 - \beta_3 b_2 - (\alpha_2\beta_2 + \alpha_3\beta_3) b_1 c_1}{2\alpha_2\beta_2 c_2}, \tag{27}$$

$$\gamma_2 = \frac{\beta_2 b_2 - \alpha_3 b_3 + (\alpha_3\beta_2 + \beta_3\alpha_2) b_1 c_1}{2\alpha_2\beta_2 c_2}, \tag{28}$$

$$b_2^2 + (\alpha_1 b_1)^2 = 1, b_3^2 + \left(\beta_1 \frac{\alpha_3}{\alpha_2} b_1\right)^2 = 1. \tag{29}$$

Here, five examples can verify the existence of solutions for (22).

Example 7. Let $B_0 \in \mathbb{C}$, $a_1 = 1$, $a_2 = 1$, and

$$(f_1(z), f_2(z)) = \left(e^{i(z_1+z_2)+B_0} - e^{-i(z_1+z_2)-B_0}/4i + e^{iz_2}, e^{i(z_1+z_2)+B_0} - e^{-i(z_1+z_2)-B_0}/2i - 4e^{iz_2} \right). \tag{30}$$

Thus, (f_1, f_2) satisfies system (22) with $(c_1, c_2) = (\pi, \pi)$, $\alpha_1 = 2$, $\beta_1 = 1$, $\alpha_2 = -1$, $\beta_2 = 4$, $\alpha_3 = 4$, and $\beta_3 = 1$.

Example 8. Let $B_0 \in \mathbb{C}$, $a_1 = i$, $a_2 = 1$, and

$$f_1(z_1, z_2) = e^{i(iz_1+z_2)+B_0} - e^{-i(iz_1+z_2)-B_0}/-2 + e^{\log[-(1+2i)]/2\pi z_2} e^{iz_2},$$
$$f_2(z_1, z_2) = \frac{e^{i(iz_1+z_2)+B_0} + e^{-i(iz_1+z_2)-B_0}}{-2} + \frac{i}{2} e^{\log[-(1+2i)]/2\pi z_2} e^{iz_2}. \tag{31}$$

Thus, (f_1, f_2) satisfies the system (22) with $(c_1, c_2) = (1/2\pi, \pi)$, $\alpha_1 = 1$, $\beta_1 = i$, $\alpha_2 = 2$, $\beta_2 = i - 2$, $\alpha_3 = -i$, and $\beta_3 = 2$.

Example 9. Let $\alpha_1 \in \mathbb{C}$ and $(f_1(z), f_2(z)) = (e^{3\pi i/2z_2} + 1, e^{3\pi i/2z_2})$. Thus, (f_1, f_2) satisfies the system (22) with $(c_1, c_2) = (c_1, 1)$, $c_1 \in \mathbb{C}$, $\alpha_2 = 2$, $\alpha_2 = 1$, $\beta_2 = 1$, and $\beta_3 = -2$.

Example 10. Let (f_1, f_2) be of the forms

$$(f_1(z), f_2(z)) = \left(e^{iz_2} + \frac{i-1}{4\pi} z_2, 2e^{iz_2} + \frac{1-i}{2\pi} z_2 + \frac{1+i}{2} \right). \tag{32}$$

Thus, (f_1, f_2) satisfies the system (22) with $(c_1, c_2) = (c_1, \pi)$, $c_1 \in \mathbb{C}$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\beta_2 = 2i$, $\beta_3 = i$, and $\alpha_1, \beta_1 \in \mathbb{C}$.

Example 11. Let

$$f_1(z_1, z_2) = \frac{1}{2} z_1 + \frac{\sqrt{15} - 4\sqrt{3} - 4}{16} z_2 + e^{2\pi i z_2},$$
$$f_2(z_1, z_2) = -\frac{1}{4} z_1 + \frac{8 - \sqrt{15} + 4\sqrt{3}}{32} z_2 - \frac{1}{2} e^{2\pi i z_2} + \frac{4\sqrt{3} + \sqrt{15}}{32}. \tag{33}$$

Thus, (f_1, f_2) satisfies system (22) with $c = (c_1, c_2) = (1, 1)$, $\alpha_1 = 1$, $\beta_1 = 1$, $\alpha_2 = 2$, $\beta_2 = 2$, $\alpha_3 = 1$, and $\beta_3 = 4$.

2. Proof of Theorem 7

Proof. Let (f_1, f_2) be a pair of finite-order transcendental entire functions satisfying (9). Here, let us consider two cases below. □

Case 1. $m_1 m_2 > n_1 n_2$. Owing to Refs. [23, 24], we have the following facts that

$$m\left(r, \frac{f_j(z)}{f_j(z+c)}\right) = S\left(r, f_j\right), j = 1, 2 \tag{34}$$

hold for all $r > 0$ outside of a possible exceptional set $E_j \subset [1, +\infty)$ of finite logarithmic measure $\int_{E_j} dt/t < \infty$. Due to the above fact, we have

$$\begin{aligned} T\left(r, f_j\right) = m\left(r, f_j\right) &\le m\left(r, \frac{f_j(z)}{f(z+c)}\right) + m\left(r, f_j(z+c)\right) + \log 2 \\ &= m\left(r, f_j(z+c)\right) + S\left(r, f_j\right) \\ &= T\left(r, f_j(z+c)\right) + S\left(r, f_j\right), j = 1, 2, \end{aligned} \tag{35}$$

for all $r \in E =: E_1 \cup E_2$. By the Mokhon'ko theorem ([25], Theorem 3.4) and the Logarithmic Derivative Lemma [26], it yields from (35) that

$$\begin{aligned} m_1 T(r, f_2) &\le m_1 T(r, f_2(z+c)) + S(r, f_2) \\ &= T(r, [\alpha_2 f_2(z+c)]^{m_1}) + S(r, f_2) \\ &= T\left(r, \left(\alpha_1 \frac{\partial f_1}{\partial z_1}\right)^{n_1} - 1\right) + S(r, f_2) \\ &= n_1 T\left(r, \frac{\partial f_1}{\partial z_1}\right) + S(r, f_1) + S(r, f_2) \\ &= n_1 m\left(r, \frac{\partial f_1}{\partial z_1}\right) + S(r, f_1) + S(r, f_2) \\ &\le n_1 \left(m\left(r, \frac{\partial f_1/\partial z_1}{f_1}\right) + m(r, f_1)\right) + S(r, f_1) + S(r, f_2) \\ &= n_1 T(r, f_1) + S(r, f_1) + S(r, f_2), \end{aligned} \tag{36}$$

for all $r \in E$. Similarly, we also get

$$m_2 T(r, f_1) \le n_2 T(r, f_2) + S(r, f_1) + S(r, f_2), r \in E. \tag{37}$$

Thus, we conclude from (36) and (37) that

$$(m_1 m_2 - n_1 n_2) T\left(r, f_j\right) \le S(r, f_1) + S(r, f_2), r \in E. \tag{38}$$

By combining with the condition that $m_1 m_2 > n_1 n_2$ and f_1, f_2 being transcendental functions, we obtain a contradiction.

Case 2. $n_j > m_j/m_j - 1$ and $m_j \ge 2$, $j = 1, 2$. Thus, it is easy to get that $m_j > n_j/n_j - 1$. In view of the Nevanlinna second fundamental theorem, the difference logarithmic derivative lemma in several complex variables [23, 24], we thus obtain from (9) that

$$\begin{aligned} (n_1 - 1) T\left(r, \frac{\partial f_1}{\partial z_1}\right) &\le \bar{N}\left(r, \frac{\partial f_1}{\partial z_1}\right) \\ &+ \sum_{q=1}^{n_1} \boxtimes \bar{N}\left(r, \frac{1}{\partial f_1/\partial z_1 - w_q/\alpha_1}\right) + S\left(r, \frac{\partial f_1}{\partial z_1}\right) \\ &\le \bar{N}\left(r, \frac{1}{(\alpha_1 \partial f_1/\partial z_1)^{n_1} - 1}\right) + S\left(r, \frac{\partial f_1}{\partial z_1}\right) \\ &\le \bar{N}\left(r, \frac{1}{f_2(z+c)}\right) + S(r, f_1) \le T(r, f_2(z+c)) \\ &+ S(r, f_1) + S(r, f_2), \end{aligned} \tag{39}$$

where w_q is a roots of $w^{n_1} - 1 = 0$. Similarly, we also have

$$(n_2 - 1) T\left(r, \frac{\partial f_2}{\partial z_1}\right) \le T(r, f_1(z+c)) + S(r, f_1) + S(r, f_2). \tag{40}$$

In addition, by applying the Mokhon'ko theorem in several complex variables ([25], Theorem 3.4) for (9), we can conclude

$$\begin{aligned} m_1 T(r, f_2(z+c)) &= T(r, [\alpha_2 f_2(z+c)]^{m_1}) + S(r, f_2) \\ &= T\left(r, \left(\alpha_1 \frac{\partial f_1}{\partial z_1}\right)^{n_1} - 1\right) + S(r, f_2) \\ &= n_1 T\left(r, \frac{\partial f_1}{\partial z_1}\right) + S(r, f_1) + S(r, f_2). \end{aligned} \tag{41}$$

Similarly, we also get

$$m_2 T(r, f_1(z+c)) = n_2 T\left(r, \frac{\partial f_2}{\partial z_1}\right) + S(r, f_1) + S(r, f_2). \tag{42}$$

Due to $m_j > n_j/n_j - 1$, it follows from (39)–(42) that

$$\begin{aligned} \left(m_1 - \frac{n_1}{n_1 - 1}\right) T(r, f_2(z+c)) &\le S(r, f_1) + S(r, f_2), \\ \left(m_2 - \frac{n_2}{n_2 - 1}\right) T(r, f_1(z+c)) &\le S(r, f_1) + S(r, f_2), \end{aligned} \tag{43}$$

and this is a contradiction with f_1, f_2 being transcendental functions.

Therefore, Theorem 7 is proved.

3. The Proof of Theorem 10

Let (f_1, f_2) be a pair of finite-order transcendental entire functions satisfying (12). We firstly rewrite the system (12) as

$$\begin{cases} [\alpha_1 f_1 + i\alpha_2 f_2(z+c)][\alpha_1 f_1 - i\alpha_2 f_2(z+c)] = 1, \\ [\beta_1 f_2 + i\beta_2 f_1(z+c)][\beta_1 f_2 - i\beta_2 f_1(z+c)] = 1. \end{cases} \quad (44)$$

By applying the Hadamard factorization theorem (can be found in [27, 28]), then there exist two polynomials p_1, p_2 such that

$$\begin{cases} \alpha_1 f_1 + i\alpha_2 f_2(z+c) = e^{p_1}, \\ \alpha_1 f_1 - i\alpha_2 f_2(z+c) = e^{-p_1}, \\ \beta_1 f_2 + i\beta_2 f_1(z+c) = e^{p_2}, \\ \beta_1 f_2 - i\beta_2 f_1(z+c) = e^{-p_2}. \end{cases} \quad (45)$$

Thus, we have from (45) that

$$\begin{cases} \alpha_1 f_1 = \dfrac{e^{p_1} + e^{-p_1}}{2}, \\ \alpha_2 f_2(z+c) = \dfrac{e^{p_1} - e^{-p_1}}{2i}, \\ \beta_1 f_2 = \dfrac{e^{p_2} + e^{-p_2}}{2}, \\ \beta_2 f_1(z+c) = \dfrac{e^{p_2} - e^{-p_2}}{2i}, \end{cases} \quad (46)$$

which implies

$$\frac{\alpha_1}{\beta_2 i} e^{p_1(z+c)+p_2} + \frac{\alpha_1}{\beta_2} i e^{p_1(z+c)-p_2} - e^{2p_1(z+c)} \equiv 1, \quad (47)$$

$$\frac{\beta_1}{\alpha_2 i} e^{p_2(z+c)+p_1} + \frac{\beta_1}{\alpha_2} i e^{p_2(z+c)-p_1} - e^{2p_2(z+c)} \equiv 1. \quad (48)$$

By applying [29], Lemma 3.1 (can be found in [30]), for (47) and (48), we have that

$$\begin{aligned} &\alpha_1 i e^{p_1(z+c)-p_2} \equiv \beta_2, \text{or} \alpha_1 e^{p_1(z+c)+p_2} \equiv \beta_2 i, \\ &\beta_1 i e^{p_2(z+c)-p_1} \equiv \alpha_2, \text{or} \beta_1 e^{p_2(z+c)+p_1} \equiv \alpha_2 i. \end{aligned} \quad (49)$$

Here, four cases will be discussed below.

Case 1.

$$\begin{cases} \alpha_1 i e^{p_1(z+c)-p_2} \equiv \beta_2, \\ \beta_1 i e^{p_2(z+c)-p_1} \equiv \alpha_2. \end{cases} \quad (50)$$

Thus, we can conclude from (50) that $p_1(z+c) - p_2(z) \equiv C_1$ and $p_2(z+c) - p_1(z) \equiv C_2$; here and below, C_1, C_2 are constants. So, this leads to $p_1(z) = L(z) + B_1, p_2(z) = L(z) + B_2$, where $L(z) = a_1 z_1 + a_2 z_2$ and a_1, a_2, B_1, B_2 are constants. Thus, by virtue of (47)–(50)), it yields that

$$\begin{cases} \alpha_1 i e^{L(c)+B_1-B_2} \equiv \beta_2, \\ \beta_1 i e^{L(c)-B_1+B_2} \equiv \alpha_2, \\ \alpha_1 e^{-L(c)-B_1+B_2} \equiv \beta_2 i, \\ \beta_1 e^{-L(c)+B_1-B_2} \equiv \alpha_2 i, \end{cases} \quad (51)$$

which implies

$$\frac{\alpha_1^2}{\beta_2^2} = \frac{\beta_1^2}{\alpha_2^2} = 1, e^{4L(c)} = 1, e^{B_1-B_2} = \frac{\beta_2}{\alpha_1 i} e^{-L(c)}. \quad (52)$$

In view of (46), let

$$f_1(z) = \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1}, f_2(z) = \frac{e^{L(z)+B_2} + e^{-L(z)-B_2}}{2\beta_1}. \quad (53)$$

If $e^{L(c)} = 1$, i.e., $L(c) = 2k\pi i$, $k \in \mathbb{Z}$, then $e^{B_2-B_1} = \alpha_1/\beta_2 i$. Thus,

$$\begin{aligned} f_2(z) &= \frac{e^{L(z)+B_2} + e^{-L(z)-B_2}}{2\beta_1} = \frac{e^{L(z)+B_1} e^{B_2-B_1} + e^{-L(z)-B_1} e^{B_1-B_2}}{2\beta_1} \\ &= \frac{\beta_2}{\beta_1} \frac{i e^{L(z)+B_1} - i e^{-L(z)-B_1}}{2\alpha_1}. \end{aligned} \quad (54)$$

If $e^{L(c)} = -1$, i.e., $L(c) = (2k+1)\pi i$, $k \in \mathbb{Z}$, then $e^{B_2-B_1} = -\alpha_1/\beta_2 i$. Thus,

$$f_2(z) = \frac{e^{L(z)+B_1} e^{B_2-B_1} + e^{-L(z)-B_1} e^{B_1-B_2}}{2\beta_1} = -\frac{\beta_2}{\beta_1} \frac{i e^{L(z)+B_1} - i e^{-L(z)-B_1}}{2\alpha_1}. \quad (55)$$

If $e^{L(c)} = i$, i.e., $L(c) = (2k + 1/2)\pi i$, $k \in \mathbb{Z}$, then $e^{B_2-B_1} = -\alpha_1/\beta_2$. Thus,

$$f_2(z) = -\frac{\beta_2}{\beta_1} \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1} = -\frac{\beta_2}{\beta_1} f_1(z). \quad (56)$$

If $e^{L(c)} = -i$, i.e., $L(c) = (2k - 1/2)\pi i$, $k \in \mathbb{Z}$, then $e^{B_2-B_1} = \alpha_1/\beta_2$. Thus,

$$f_2(z) = \frac{\beta_2}{\beta_1} \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1} = \frac{\beta_2}{\beta_1} f_1(z). \quad (57)$$

Case 2.

$$\begin{cases} \alpha_1 i e^{p_1(z+c)-p_2} \equiv \beta_2, \\ \beta_1 e^{p_1-p_2(z+c)} \equiv \alpha_2 i. \end{cases} \quad (58)$$

Thus, it yields from (58) that $p_1(z+c) - p_2(z) \equiv C_1$ and $p_1(z) + p_2(z+c) \equiv C_2$. Hence, we obtain that $p_1(z+2c) + p_1(z) \equiv C_1 + C_2$, and this is a contradiction with p_1 is not a constant.

Case 3.

$$\begin{cases} \alpha_1 e^{p_1(z+c)+p_2} \equiv \beta_2 i, \\ \beta_1 i e^{p_2(z+c)-p_1} \equiv \alpha_2. \end{cases} \quad (59)$$

Thus, it yields from (59) that $p_1(z+c)+p_2(z) \equiv C_1$ and $p_2(z+c)-p_1(z) \equiv C_2$. Hence, we obtain that $p_2(z+2c)+p_2(z) \equiv C_1+C_2$, and this is a contradiction with p_2 is not a constant.

Case 4.

$$\begin{cases} \alpha_1 e^{p_1(z+c)+p_2} \equiv \beta_2 i, \\ \beta_1 i e^{p_2(z+c)+p_1} \equiv \alpha_2. \end{cases} \quad (60)$$

Thus, it yields from (60) that $p_2(z)-p_1(z+c) \equiv C_1$ and $p_1(z)-p_2(z+c) \equiv C_2$. Hence, we obtian that $p_1(z) = L(z) + B_1, p_2(z) = -L(z) + B_2$, where $L(z) = a_1 z_1 + a_2 z_2$, a_1, a_2, B_1, B_2 are constants. By virtue of (47),(48), (60), it yields that

$$\begin{cases} \alpha_1 e^{L(c)+B_1+B_2} \equiv \beta_2 i, \\ \beta_1 e^{-L(c)+B_1+B_2} \equiv \alpha_2 i, \\ \alpha_1 e^{-L(c)-B_1-B_2} \equiv \beta_2 i, \\ \beta_1 e^{L(c)-B_1-B_2} \equiv \alpha_2 i, \end{cases} \quad (61)$$

which implies

$$\frac{\alpha_1^2}{\beta_2^2} = \frac{\beta_1^2}{\alpha_2^2} = 1, e^{4L(c)} = 1, e^{B_1+B_2} = \frac{\beta_2}{\alpha_1 i} e^{-L(c)} = \frac{\alpha_2 i}{\beta_1} e^{L(c)}. \quad (62)$$

In view of (46), let

$$f_1(z) = \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1}, f_2(z) = \frac{e^{-L(z)+B_2} + e^{L(z)-B_2}}{2\beta_1}. \quad (63)$$

If $e^{L(c)} = 1$, i.e., $L(c) = 2k\pi i$, $k \in \mathbb{Z}$, then $e^{B_1+B_2} = \beta_2/\alpha_1 i = \alpha_2/\beta_1 i$. Thus,

$$f_2(z) = \frac{e^{L(z)+B_1} e^{-B_2-B_1} + e^{-L(z)-B_1} e^{B_1+B_2}}{2\beta_1} = \frac{\beta_2}{\beta_1} \frac{-ie^{L(z)+B_1} + ie^{-L(z)-B_1}}{2\alpha_1}. \quad (64)$$

If $e^{L(c)} = -1$, i.e., $L(c) = (2k+1)\pi i$, $k \in \mathbb{Z}$, then $e^{B_1+B_2} = -\beta_2/\alpha_1 i = -\alpha_2/\beta_1 i$. Thus,

$$f_2(z) = \frac{\beta_2}{\beta_1} \frac{ie^{L(z)+B_1} - ie^{-L(z)-B_1}}{2\alpha_1}. \quad (65)$$

If $e^{L(c)} = i$, i.e., $L(c) = (2k+1/2)\pi i$, $k \in \mathbb{Z}$, then $e^{B_1+B_2} = \beta_2/\alpha_1 = -\alpha_2/\beta_1$. Thus,

$$f_2(z) = \frac{\beta_2}{\beta_1} \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1} = \frac{\beta_2}{\beta_1} f_1(z). \quad (66)$$

If $e^{L(c)} = -i$, i.e., $L(c) = (2k-1/2)\pi i$, $k \in \mathbb{Z}$, then $e^{B_1-B_2} = -\beta_2/\alpha_1 = \alpha_2/\beta_1$. Thus,

$$f_2(z) = -\frac{\beta_2}{\beta_1} \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1} = -\frac{\beta_2}{\beta_1} f_1(z). \quad (67)$$

Therefore, this completes the proof of Theorem 10.

4. The Proof of Theorem 12

Proof. Let (f_1, f_2) be a pair of finite-order transcendental entire functions satisfying (16). Firstly, (16) may be represented as the following form:

$$\begin{cases} \left[\alpha_1 \frac{\partial f_1}{\partial z_1} + i\alpha_2 f_2(z+c)\right]\left[\alpha_1 \frac{\partial f_1}{\partial z_1} - i\alpha_2 f_2(z+c)\right] = 1, \\ \left[\beta_1 \frac{\partial f_2}{\partial z_1} + i\beta_2 f_1(z+c)\right]\left[\beta_1 \frac{\partial f_2}{\partial z_1} - i\beta_2 f_1(z+c)\right] = 1. \end{cases} \quad (68)$$

By the Hadamard factorization theorem (can be found in [27, 28]), there are two nonconstant polynomials p_1, p_2 satisfying

$$\begin{cases} \alpha_1 \frac{\partial f_1}{\partial z_1} + i\alpha_2 f_2(z+c) = e^{p_1}, \\ \alpha_1 \frac{\partial f_1}{\partial z_1} - i\alpha_2 f_2(z+c) = e^{-p_1}, \\ \beta_1 \frac{\partial f_2}{\partial z_1} + i\beta_2 f_1(z+c) = e^{p_2}, \\ \beta_1 \frac{\partial f_2}{\partial z_1} - i\beta_2 f_1(z+c) = e^{-p_2}. \end{cases} \quad (69)$$

In view of (69), it yields that

$$\begin{cases} \alpha_1 \frac{\partial f_1}{\partial z_1} = \frac{e^{p_1} + e^{-p_1}}{2}, \\ \alpha_2 f_2(z+c) = \frac{e^{p_1} - e^{-p_1}}{2i}, \\ \beta_1 \frac{\partial f_2}{\partial z_1} = \frac{e^{p_2} + e^{-p_2}}{2}, \\ \beta_2 f_1(z+c) = \frac{e^{p_2} - e^{-p_2}}{2i}, \end{cases} \quad (70)$$

which implies

$$\frac{\alpha_1}{\beta_2 i} \frac{\partial p_2}{\partial z_1} e^{p_1(z+c)+p_2} + \frac{\alpha_1}{\beta_2 i} \frac{\partial p_2}{\partial z_1} e^{p_1(z+c)-p_2} - e^{2p_1(z+c)} \equiv 1, \quad (71)$$

$$\frac{\beta_1}{\alpha_2 i} \frac{\partial p_1}{\partial z_1} e^{p_2(z+c)+p_1} + \frac{\beta_1}{\alpha_2 i} \frac{\partial p_1}{\partial z_1} e^{p_2(z+c)-p_1} - e^{2p_2(z+c)} \equiv 1. \quad (72)$$

Obviously, $\partial p_1/\partial z_1 \equiv 0$. Otherwise, $e^{2p_2(z+c)} \equiv 1$. This leads to a contradiction with p_1 is not a constant. Similarly, $\partial p_2/\partial z_1 \equiv 0$. Thus, due to [29], Lemma 3.1 (can be found in [30]), (71), and (72), we obtain that

$$\begin{aligned} &\frac{\alpha_1}{\beta_2 i}\frac{\partial p_2}{\partial z_1}e^{p_1(z+c)+p_2} \equiv 1 \text{or} \frac{\alpha_1}{\beta_2 i}\frac{\partial p_2}{\partial z_1}e^{p_1(z+c)-p_2(z)} \equiv 1, \\ &\frac{\beta_1}{\alpha_2 i}\frac{\partial p_1}{\partial z_1}e^{p_2(z+c)+p_1} \equiv 1 \text{or} \frac{\beta_1}{\alpha_2 i}\frac{\partial p_1}{\partial z_1}e^{p_2(z+c)-p_1} \equiv 1. \end{aligned} \tag{73}$$

□

Hence, four cases will be discussed below.

Case 1.

$$\begin{cases} \dfrac{\alpha_1}{\beta_2 i}\dfrac{\partial p_2}{\partial z_1}e^{p_1(z+c)+p_2} \equiv 1, \\ \dfrac{\beta_1}{\alpha_2 i}\dfrac{\partial p_1}{\partial z_1}e^{p_2(z+c)+p_1} \equiv 1. \end{cases} \tag{74}$$

Thus, it follows from (74) that $p_1(z+c)+p_2 \equiv C_1$ and $p_2(z+c)+p_1 \equiv C_2$. These lead to $p_1(z+2c)-p_1 \equiv C_1 - C_2$ and $p_2(z+c)-p_2 \equiv C_2 - C_1$. Hence, we obtain that $p_1(z) = L(z) + B_1$, $p_2(z) = -L(z) + B_2$, where $L(z) = a_1 z_1 + a_2 z_2$, $a_1(\neq 0)$, a_2, B_1, B_2 are constants. By combining with (71)–(74), we have

$$\begin{cases} \dfrac{\alpha_1 a_1}{\beta_2} i e^{L(c)+B_1+B_2} \equiv 1, \\ \dfrac{\beta_1 a_1}{\alpha_2 i} e^{-L(c)+B_1+B_2} \equiv 1, \\ \dfrac{\alpha_1 a_1}{\beta_2} i e^{-L(c)-B_1-B_2} \equiv 1, \\ \dfrac{\beta_1 a_1}{\alpha_2 i} e^{L(c)-B_1-B_2} \equiv 1, \end{cases} \tag{75}$$

and this leads to

$$a_1^2 = -\frac{\beta_2^2}{\alpha_1^2} = -\frac{\alpha_2^2}{\beta_1^2}, e^{4L(c)} = 1, e^{B_1+B_2} = \frac{\beta_2}{\alpha_1 a_1 i}e^{-L(c)} = \frac{\alpha_2 i}{\beta_1 a_1}e^{L(c)}. \tag{76}$$

Subcase 1. If $e^{L(c)} = 1$, then $L(c) = 2k\pi i$ and $e^{B_1+B_2} = \beta_2/\alpha_1 a_1 i = \alpha_2 i/\beta_1 a_1$. Due to (70), we have that

$$\begin{aligned} f_1 &= \frac{e^{-L(z)+B_2+L(c)} - e^{L(z)-B_2-L(c)}}{2\beta_2 i} = i\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}, \\ f_2 &= \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{e^{L(z)+B_1} - e^{-L(z)-B_1}}{2\alpha_2 i} = \frac{e^{L(z)-B_2}e^{B_1+B_2} - e^{-L(z)+B_2}e^{-B_1-B_2}}{2\alpha_2 i} = \frac{-\beta_2 i/\alpha_1 a_1 e^{L(z)-B_2} - \beta_2 i/-\alpha_1 a_1 e^{-L(z)+B_2}}{2\alpha_2 i} = \frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}. \end{aligned} \tag{77}$$

Subcase 2. If $e^{L(c)} = -1$, then $L(c) = (2k+1)\pi i$, $k \in \mathbb{Z}$ and $e^{B_1+B_2} = -\beta_2/\alpha_1 a_1 i = -\alpha_2 i/\beta_1 a_1$. Due to (70), we have that

$$\begin{aligned} f_1 &= \frac{e^{-L(z)+B_2+L(c)} - e^{L(z)-B_2-L(c)}}{2\beta_2 i} = -i\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}, \\ f_2 &= \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{-e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_2 i} = \frac{-e^{L(z)-B_2}e^{B_1+B_2} + e^{-L(z)+B_2}e^{-B_1-B_2}}{2\alpha_2 i} = \frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}. \end{aligned} \tag{78}$$

Subcase 3. If $e^{L(c)} = i$, then $L(c) = (2k+1/2)\pi i$, $k \in \mathbb{Z}$, and $e^{B_1+B_2} = -\beta_2/\alpha_1 a_1 = -\alpha_2/\beta_1 a_1$. Due to (70), we have that

$$\begin{aligned} f_1 &= \frac{e^{-L(z)+B_2+L(c)} - e^{L(z)-B_2-L(c)}}{2\beta_2 i} = \frac{e^{L(z)-B_2} + e^{-L(z)+B_2}}{2\beta_2}, \\ f_2 &= \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{-e^{L(z)+B_1} - e^{-L(z)-B_1}}{2\alpha_2} = \frac{-e^{L(z)-B_2}e^{B_1+B_2} - e^{-L(z)+B_2}e^{-B_1-B_2}}{2\alpha_2} = -\frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}. \end{aligned} \tag{79}$$

Subcase 4. If $e^{L(c)}=-i$, then $L(c)=(2k-1/2)\pi i$, $k\in\mathbb{Z}$, and $e^{B_1+B_2}\beta_2/\alpha_1 a_1=\alpha_2/\beta_1 a_1$. Due to (70), we have that

$$\begin{aligned} f_1&=\frac{e^{-L(z)+B_2+L(c)}-e^{L(z)-B_2-L(c)}}{2\beta_2 i}=-\frac{e^{L(z)-B_2}+e^{-L(z)+B_2}}{2\beta_2},\\ f_2&=\frac{e^{L(z)+B_1-L(c)}-e^{-L(z)-B_1+L(c)}}{2\alpha_2 i}=\frac{e^{L(z)+B_1}+e^{-L(z)-B_1}}{2\alpha_2}=\frac{e^{L(z)-B_2}e^{B_1+B_2}+e^{-L(z)+B_2}e^{-B_1-B_2}}{2\alpha_2}=-\frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)-B_2}-e^{-L(z)+B_2}}{2\beta_2}. \end{aligned} \tag{80}$$

Case 2.

$$\begin{cases} \dfrac{\alpha_1}{\beta_2 i}\dfrac{\partial p_2}{\partial z_1}e^{p_1(z+c)+p_2}\equiv 1,\\ \dfrac{\beta_1}{\alpha_2 i}\dfrac{\partial p_1}{\partial z_1}e^{p_2(z+c)-p_1}\equiv 1. \end{cases} \tag{81}$$

Thus, it yields from (81) that $p_1(z+c)+p_2\equiv C_1$ and $p_2(z+c)-p_1\equiv C_2$. We have that $p_2(z+2c)+p_2\equiv C_1+C_2$, and this leads to a contradiction with p_2 being not constant.

Case 3.

$$\begin{cases} \dfrac{\alpha_1}{\beta_2 i}\dfrac{\partial p_2}{\partial z_1}e^{p_1(z+c)-p_2}\equiv 1,\\ \dfrac{\beta_1}{\alpha_2 i}\dfrac{\partial p_1}{\partial z_1}e^{p_2(z+c)+p_1}\equiv 1. \end{cases} \tag{82}$$

Since $p_1(z), p_2(z)$ are polynomials, then from (82), it follows that $p_1(z+c)-p_2(z)\equiv C_1$ and $p_2(z+c)+p_1(z)\equiv C_2$. This means $p_1(z+2c)+p_1(z)\equiv C_1+C_2$, and this is a contradiction because $p_1(z)$ is not a constant.

Case 4.

$$\begin{cases} \dfrac{\alpha_1}{\beta_2 i}\dfrac{\partial p_2}{\partial z_1}e^{p_1(z+c)-p_2}\equiv 1,\\ \dfrac{\beta_1}{\alpha_2 i}\dfrac{\partial p_1}{\partial z_1}e^{p_2(z+c)-p_1}\equiv 1. \end{cases} \tag{83}$$

Then, from (83), it yields that $p_1(z+c)-p_2\equiv C_1$ and $p_2(z+c)-p_1\equiv C_2$, and this leads to $p_1(z+2c)-p_1\equiv C_1+C_2$ and $p_2(z+2c)-p_2\equiv C_2+C_1$. Thus, it follows that $p_1(z)=L(z)+B_1, p_2(z)=L(z)+B_2$, where $L(z)=a_1z_1+a_2z_2$, $a_1(\neq 0), a_2, B_1, B_2$ are constants in $\mathbb{C}$. In view of (71), (72), and (83), we have

$$\begin{cases} \dfrac{\alpha_1}{\beta_2 i}a_1e^{L(c)+B_1-B_2}\equiv 1,\\ \dfrac{\beta_1}{\alpha_2 i}a_1e^{L(c)-B_1+B_2}\equiv 1,\\ \dfrac{\alpha_1}{\beta_2 i}a_1e^{-L(c)-B_1+B_2}\equiv 1,\\ \dfrac{\beta_1}{\alpha_2 i}a_1e^{-L(c)+B_1-B_2}\equiv 1, \end{cases} \tag{84}$$

which implies

$$a_1^2=-\frac{\beta_2^2}{\alpha_1^2}=-\frac{\alpha_2^2}{\beta_1^2}, e^{4L(c)}=1, e^{B_1-B_2}=\frac{\beta_2 i}{\alpha_1 a_1}e^{-L(c)}=\frac{\beta_1 a_1}{\alpha_2 i}e^{L(c)}. \tag{85}$$

Subcase 4.1. If $e^{L(c)}=1$, then $L(c)=2k\pi i$, $k\in\mathbb{Z}$, and $e^{B_1-B_2}=\beta_2 i/\alpha_1 a_1=\beta_1 a_1/\alpha_2 i$. By virtue of (70), it follows that

$$\begin{aligned} f_1&=\frac{e^{L(z)+B_2-L(c)}-e^{-L(z)-B_2+L(c)}}{2\beta_2 i}=-i\frac{e^{L(z)+B_2}-e^{-L(z)-B_2}}{2\beta_2},\\ f_2&=\frac{e^{L(z)+B_1-L(c)}-e^{-L(z)-B_1+L(c)}}{2\alpha_2 i}=\frac{e^{L(z)+B_1}-e^{-L(z)-B_1}}{2\alpha_2 i}\\ &=\frac{e^{L(z)+B_2}e^{B_1-B_2}-e^{-L(z)-B_2}e^{-B_1+B_2}}{2\alpha_2 i}\\ &=\frac{\beta_2 i/\alpha_1 a_1 e^{L(z)+B_2}-\beta_2 i/\alpha_1 a_1 e^{-L(z)-B_2}}{2\alpha_2 i}\\ &=-\frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)+B_2}-e^{-L(z)-B_2}}{2\beta_2}. \end{aligned} \tag{86}$$

Subcase 4.2. If $e^{L(c)}=-1$, then $L(c)=(2k+1)\pi i$, $k\in\mathbb{Z}$, and $e^{B_1-B_2}=-\beta_2 i/\alpha_1 a_1=-\beta_1 a_1/\alpha_2 i$. By virtue of (70), it follows that

$$f_1=\frac{e^{L(z)+B_2-L(c)}-e^{-L(z)-B_2+L(c)}}{2\beta_2 i}=i\frac{e^{L(z)+B_2}-e^{-L(z)-B_2}}{2\beta_2},$$

$$f_2 = \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{-e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_2 i} = \frac{-e^{L(z)+B_2}e^{B_1-B_2} + e^{-L(z)-B_2}e^{-B_1+B_2}}{2\alpha_2 i} = -\frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)+B_2} - e^{-L(z)-B_2}}{2\beta_2}. \tag{87}$$

Subcase 4.3. If $e^{L(c)} = i$, then $L(c) = (2k + 1/2)\pi i$ and $e^{B_1-B_2} = \beta_2/\alpha_1 a_1 = \alpha_2/\beta_1 a_1$. By virtue of (70), we have that

$$f_1 = \frac{e^{L(z)+B_2-L(c)} - e^{-L(z)-B_2+L(c)}}{2\beta_2 i} = -\frac{e^{L(z)+B_2} + e^{-L(z)-B_2}}{2\beta_2},$$
$$f_2 = \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = -\frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_2} = -\frac{e^{L(z)+B_2}e^{B_1-B_2} + e^{-L(z)-B_2}e^{-B_1+B_2}}{2\alpha_2} = \frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)+B_2} - e^{-L(z)-B_2}}{2\beta_2}. \tag{88}$$

Subcase 4.4. If $e^{L(c)} = -i$, then $L(c) = (2k - 1/2)\pi i$ and $e^{B_1-B_2} = -\beta_2/\alpha_1 a_1 = -\alpha_2/\beta_1 a_1$. By virtue of (70), we have that

$$f_1 = \frac{e^{L(z)+B_2-L(c)} - e^{-L(z)-B_2+L(c)}}{2\beta_2 i} = \frac{e^{L(z)-B_2} + e^{-L(z)+B_2}}{2\beta_2},$$
$$f_2 = \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_2} = \frac{e^{L(z)+B_2}e^{B_1-B_2} + e^{-L(z)-B_2}e^{-B_1+B_2}}{2\alpha_2} = \frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)+B_2} - e^{-L(z)-B_2}}{2\beta_2}. \tag{89}$$

Hence, the proof of Theorem 12 is completed.

5. The Proof of Theorem 13

Proof. Assume that (f_1, f_2) is a pair of finite-order transcendental entire functions satisfying (22). Thus, let us discuss two following cases.

(i) Suppose that $\partial f_1/\partial z_1$ is transcendental, then $\alpha_2 f_2(z + c) + \alpha_3 f_1$ is transcendental. Noting that α_j, β_j are nonzero constants, we next prove that $\beta_2 f_1(z + c) + \beta_3 f_2$ and $\beta_1 \partial f_2/\partial z_1$ are transcendental

Suppose that $\alpha_2\partial f_2(z + c)/\partial z_1 + \alpha_3\partial f_1/\partial z_1$ is not transcendental. Since $\partial f_1/\partial z_1$ is transcendental, then $\partial f_2(z + c)/\partial z_1$ and $\partial f_2(z)/\partial z_1$ are transcendental. By observing the second equation of (22), we can conclude that $\beta_2 f_1(z + c) + \beta_3 f_2$ is transcendental.

Suppose that $\alpha_2\partial f_2(z + c)/\partial z_1 + \alpha_3\partial f_1/\partial z_1$ is transcendental. If $\partial f_2(z + c)/\partial z_1$ is transcendental, similar to the above argument, $\beta_2 f_1(z + c) + \beta_3 f_2$ and $\partial f_2/\partial z_1$ are transcendental. If $\partial f_2(z + c)/\partial z_1$ is not transcendental, it thus leads to that $\partial f_2/\partial z_1$ is not transcendental. From (22), we thus get that $\beta_2 f_1(z + c) + \beta_3 f_2$ is not transcendental. Thus, it yields that $\beta_2\partial f_1(z + c)/\partial z_1 + \beta_3\partial f_2/\partial z_1$ is not transcendental. This is a contradiction with $\partial f_1(z + c)/\partial z_1$ is transcendental and $\partial f_2/\partial z_1$ is not transcendental.

Hence, if $\partial f_1/\partial z_1$ is transcendental, then $\alpha_2 f_2(z + c) + \alpha_3 f_1$, $\beta_2 f_1(z + c) + \beta_3 f_2$, and $\partial f_2/\partial z_1$ are transcendental. Hence, system (22) can be represented as

$$\begin{cases} \left[\alpha_1\frac{\partial f_1}{\partial z_1} + i[\alpha_2 f_2(z + c) + \alpha_3 f_1]\right]\left[\alpha_1\frac{\partial f_1}{\partial z_1} - i[\alpha_2 f_2(z + c) + \alpha_3 f_1]\right] = 1, \\ \left[\beta_1\frac{\partial f_2}{\partial z_1} + i[\beta_2 f_1(z + c) + \beta_3 f_2]\right]\left[\beta_1\frac{\partial f_2}{\partial z_1} - i[\beta_2 f_1(z + c) + \beta_3 f_2]\right] = 1. \end{cases} \tag{90}$$

Thus, by the Hadamard factorization theorem (can be found in [27, 28]), there are two nonconstant polynomials p, q such that

$$\begin{cases} \alpha_1\frac{\partial f_1}{\partial z_1} + i[\alpha_2 f_2(z + c) + \alpha_3 f_1] = e^{ip}, \\ \alpha_1\frac{\partial f_1}{\partial z_1} - i[\alpha_2 f_2(z + c) + \alpha_3 f_1] = e^{-ip}, \\ \beta_1\frac{\partial f_2}{\partial z_1} + i[\beta_2 f_1(z + c) + \beta_3 f_2] = e^{iq}, \\ \beta_1\frac{\partial f_2}{\partial z_1} - i[\beta_2 f_1(z + c) + \beta_3 f_2] = e^{-iq}. \end{cases} \tag{91}$$

In view of (91), it yields that

$$\begin{cases} \alpha_1\frac{\partial f_1}{\partial z_1} = \frac{e^{ip} + e^{-ip}}{2}, \\ \alpha_2 f_2(z + c) + \alpha_3 f_1 = \frac{e^{ip} - e^{-ip}}{2i}, \\ \beta_1\frac{\partial f_2}{\partial z_1} = \frac{e^{iq} + e^{-iq}}{2}, \\ \beta_2 f_1(z + c) + \beta_3 f_2 = \frac{e^{iq} - e^{-iq}}{2i}, \end{cases} \tag{92}$$

which implies

$$\frac{\beta_1}{\alpha_2}\left(\frac{\partial p}{\partial z_1} - \frac{\alpha_3}{\alpha_1}\right)e^{i(p+q(z+c))} + \frac{\beta_1}{\alpha_2}\left(\frac{\partial p}{\partial z_1} - \frac{\alpha_3}{\alpha_1}\right)e^{i(q(z+c)-p)} - e^{2iq(z+c)} \equiv 1, \tag{93}$$

$$\frac{\alpha_1}{\beta_2}\left(\frac{\partial q}{\partial z_1} - \frac{\beta_3}{\beta_1}\right)e^{i(q+p(z+c))} + \frac{\alpha_1}{\beta_2}\left(\frac{\partial q}{\partial z_1} - \frac{\beta_3}{\beta_1}\right)e^{i(p(z+c)-q)} - e^{2ip(z+c)} \equiv 1. \tag{94}$$

Obviously, $\partial p/\partial z_1 \neq \alpha_3/\alpha_1$. Otherwise, we have that $-e^{2iq(z+c)} \equiv 1$, and this leads to a contradiction since q is not a constant. Similarly, $\partial q/\partial z_1 \neq \beta_3/\beta_1$. Thus, due to [29], Lemma 3.1 (can be found in [30]), and in view of

(93) and (94), we can deduce that

$$\frac{\beta_1}{\alpha_2}\left(\frac{\partial p}{\partial z_1}-\frac{\alpha_3}{\alpha_1}\right)e^{i[q(z+c)-p]}\equiv 1, \text{or} \frac{\beta_1}{\alpha_2}\left(\frac{\partial p}{\partial z_1}-\frac{\alpha_3}{\alpha_1}\right)e^{i[p+q(z+c)]}\equiv 1,$$
$$\frac{\alpha_1}{\beta_2}\left(\frac{\partial q}{\partial z_1}-\frac{\beta_3}{\beta_1}\right)e^{i[p(z+c)-q]}\equiv 1, \text{or} \frac{\alpha_1}{\beta_2}\left(\frac{\partial q}{\partial z_1}-\frac{\beta_3}{\beta_1}\right)e^{i[q+p(z+c)]}\equiv 1. \tag{95}$$

□

Now, let us consider the following four cases.

Case 1.

$$\begin{cases}\frac{\beta_1}{\alpha_2}\left(\frac{\partial p}{\partial z_1}-\frac{\alpha_3}{\alpha_1}\right)e^{i[q(z+c)-p]}\equiv 1,\\ \frac{\alpha_1}{\beta_2}\left(\frac{\partial q}{\partial z_1}-\frac{\beta_3}{\beta_1}\right)e^{i[p(z+c)-q]}\equiv 1.\end{cases} \tag{96}$$

Then, (96) can lead to that $q(z+c)-p\equiv C_1$ and $p(z+c)-q\equiv C_2$. Thus, we obtain that $p(z+2c)-p\equiv C_2+C_1$ and $q(z+2c)-q\equiv C_1+C_2$. Hence, we can conclude that $p(z)=L(z)+B_1, q(z)=L(z)+B_2$, where $L(z)=a_1z_1+a_2z_2$, a_1, a_2, B_1, B_2 are constants. By combining with (93)–(96), we have

$$\begin{cases}\frac{\beta_1}{\alpha_2}\left(a_1-\frac{\alpha_3}{\alpha_1}\right)e^{i(L(c)+B_2-B_1)}\equiv 1,\\ \frac{\alpha_1}{\beta_2}\left(a_1-\frac{\beta_3}{\beta_1}\right)e^{i(L(c)+B_1-B_2)}\equiv 1,\\ \frac{\beta_1}{\alpha_2}\left(a_1-\frac{\alpha_3}{\alpha_1}\right)e^{-i(L(c)+B_2-B_1)}\equiv 1,\\ \frac{\alpha_1}{\beta_2}\left(a_1-\frac{\beta_3}{\beta_1}\right)e^{-i(L(c)+B_1-B_2)}\equiv 1.\end{cases} \tag{97}$$

This means that

$$\left[\frac{\beta_1}{\alpha_2}\left(a_1-\frac{\alpha_3}{\alpha_1}\right)\right]^2=\left[\frac{\alpha_1}{\beta_2}\left(a_1-\frac{\beta_3}{\beta_1}\right)\right]^2=1, e^{4iL(c)}=1, e^{i(B_1-B_2)}=\frac{\beta_1}{\alpha_2}\left(a_1-\frac{\alpha_3}{\alpha_1}\right)e^{iL(c)}. \tag{98}$$

By combining with (92), f_1, f_2 have the following forms:

$$f_1(z)=\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_1(z_2), f_2(z)=\frac{e^{i(L(z)+B_2)}-e^{-i(L(z)+B_2)}}{2ia_1\beta_1}+\varphi_2(z_2), \tag{99}$$

where $\varphi_1(z_2), \varphi_2(z_2)$ are entire functions of finite order in z_2. Substituting the above expressions into (92), we can deduce that

$$\begin{cases}\alpha_2\varphi_2(z_2+c_2)+\alpha_3\varphi_1(z_2)=0,\\ \beta_2\varphi_1(z_2+c_2)+\beta_3\varphi_2(z_2)=0.\end{cases} \tag{100}$$

This leads to

$$\varphi_1(z_2+2c_2)=\frac{\alpha_3\beta_3}{\alpha_2\beta_2}\varphi_1(z_2), \varphi_2(z_2+2c_2)=\frac{\alpha_3\beta_3}{\alpha_2\beta_2}\varphi_2(z_2). \tag{101}$$

Due to (101), we have

$$\varphi_1(z_2)=e^{\eta z_2}G_1(z_2), \varphi_2(z_2)=e^{\eta z_2}G_2(z_2), \tag{102}$$

where $G_1(z_2), G_2(z_2)$ are entire period functions of finite order with period $2c_2$, and in (102), $\eta=0$, if $\alpha_2\beta_2=\alpha_3\beta_3$, and $\eta=\log(\alpha_2\beta_2)-\log(\alpha_3\beta_3)/2c_2$, if $\alpha_2\beta_2\neq\alpha_3\beta_3$. Further, in view of (100) and (102), we have $G_2(z_2)=-\alpha_3/\alpha_2 G_1(z_2)$; if $\alpha_2\beta_2\neq\alpha_3\beta_3$, we have $G_2(z_2)=-\alpha_3/\alpha_2 G_1(z_2)$.

If $e^{iL(c)}=1$, it follows from (97) that $e^{i(B_1-B_2)}=\pm 1$. Thus, it yields that

$$\begin{aligned}f_2(z)&=\frac{e^{i(L(z)+B_2)}-e^{-i(L(z)+B_2)}}{2ia_1\beta_1}+\varphi_2(z_2)\\ &=\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}e^{i(B_2-B_1)}-e^{-i(L(z)+B_1)}e^{i(B_1-B_2)}}{2ia_1\alpha_1}+\varphi_2(z_2)\\ &=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_2(z_2).\end{aligned} \tag{103}$$

If $e^{iL(c)}=-1$, it follows from (97) that $e^{2i(B_1-B_2)}=1$. Thus, similar to the above argument, we obtain that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_2(z_2). \tag{104}$$

If $e^{iL(c)}=i$, it follows from (97) that $e^{2i(B_1-B_2)}=-1$. Thus, we obtain that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}+e^{-i(L(z)-B_1)}}{2a_1\alpha_1}+\varphi_2(z_2). \tag{105}$$

If $e^{iL(c)}=-i$, it follows from (97) that $e^{2i(B_1-B_2)}=-1$. Thus, we obtain that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}+e^{-i(L(z)-B_1)}}{2a_1\alpha_1}+\varphi_2(z_2). \tag{106}$$

Case 2.

$$\begin{cases}\frac{\beta_1}{\alpha_2}\left(\frac{\partial p}{\partial z_1}-\frac{\alpha_3}{\alpha_1}\right)e^{i(q(z+c)-p(z))}\equiv 1,\\ \frac{\alpha_1}{\beta_2}\left(\frac{\partial q}{\partial z_1}-\frac{\beta_3}{\beta_1}\right)e^{i(q(z)+p(z+c))}\equiv 1.\end{cases} \tag{107}$$

We thus get from (107) that $p(z+c)+q(z)\equiv C_2$ and $q(z+c)-p\equiv C_1$. This means $q(z+2c)+q(z)\equiv C_1+C_2$, and this yields a contradiction with q being not a constant.

Case 3.

$$\begin{cases} \dfrac{\beta_1}{\alpha_2}\left(\dfrac{\partial p}{\partial z_1}-\dfrac{\alpha_3}{\alpha_1}\right)e^{i[p+q(z+c)]}\equiv 1, \\ \dfrac{\alpha_1}{\beta_2}\left(\dfrac{\partial q}{\partial z_1}-\dfrac{\beta_3}{\beta_1}\right)e^{i[p(z+c)-q]}\equiv 1. \end{cases} \tag{108}$$

We thus get from (108) that $q(z+c)+p(z)\equiv C_1$ and $p(z+c)-q(z)\equiv C_2$. So, we conclude that $p(z+2c)+p(z)\equiv C_1+C_2$, and this leads to a contradiction with p being not a constant.

Case 4.

$$\begin{cases} \dfrac{\beta_1}{\alpha_2}\left(\dfrac{\partial p}{\partial z_1}-\dfrac{\alpha_3}{\alpha_1}\right)e^{i[p+q(z+c)]}\equiv 1, \\ \dfrac{\alpha_1}{\beta_2}\left(\dfrac{\partial q}{\partial z_1}-\dfrac{\beta_3}{\beta_1}\right)e^{i[q+p(z+c)]}\equiv 1. \end{cases} \tag{109}$$

Then, it follows from (109) that $p+q(z+c)\equiv C_1$ and $q+p(z+c)\equiv C_2$. These yield that $p(z+2c)-p\equiv C_1+C_2$ and $q(z+2c)-q\equiv C_2+C_1$, which leads to $p=L(z)+B_1, q=-L(z)+B_2$, where $L(z)=a_1z_1+a_2z_2$, a_1, a_2, B_1, B_2 are constants. In view of (93), (94), and (109), we have

$$\begin{cases} \dfrac{\beta_1}{\alpha_2}\left(a_1-\dfrac{\alpha_3}{\alpha_1}\right)e^{-i(L(c)-B_1-B_2)}\equiv 1, \\ \dfrac{\alpha_1}{\beta_2}\left(-a_1-\dfrac{\beta_3}{\beta_1}\right)e^{i(L(c)+B_1+B_2)}\equiv 1, \\ \dfrac{\beta_1}{\alpha_2}\left(a_1-\dfrac{\alpha_3}{\alpha_1}\right)e^{i(L(c)-B_1-B_2)}\equiv 1, \\ \dfrac{\alpha_1}{\beta_2}\left(-a_1-\dfrac{\beta_3}{\beta_1}\right)e^{-i(L(c)+B_1+B_2)}\equiv 1. \end{cases} \tag{110}$$

In view of (110), it follows that

$$\left[\frac{\beta_1}{\alpha_2}\left(a_1-\frac{\alpha_3}{\alpha_1}\right)\right]^2=\left[\frac{\alpha_1}{\beta_2}\left(a_1+\frac{\beta_3}{\beta_1}\right)\right]^2=1, e^{4iL(c)}=1, e^{i(B_1+B_2)}=\frac{\beta_1}{\alpha_2}\left(a_1-\frac{\alpha_3}{\alpha_1}\right)e^{iL(c)}. \tag{111}$$

By combining with (92), f_1, f_2 are of the following forms

$$f_1(z)=\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_1(z_2), f_2(z)=\frac{e^{i(-L(z)+B_2)}-e^{i(L(z)-B_2)}}{2ia_1\beta_1}+\varphi_2(z_2), \tag{112}$$

where $\varphi_1(z_2), \varphi_2(z_2)$ are finite-order entire functions in z_2. By using the same argument as in Case 1, we have (102).

If $e^{iL(c)}=1$, it follows from (110) that $e^{2i(B_1+B_2)}=1$. Thus, we can deduce that

$$\begin{aligned} f_2(z)&=\frac{e^{i(-L(z)+B_2)}-e^{i(L(z)-B_2)}}{2ia_1\beta_1}+\varphi_2(z_2) \\ &=\frac{\alpha_1}{\beta_2}\frac{-e^{i(L(z)+B_1)}e^{-i(B_1+B_2)}+e^{-i(L(z)+B_1)}e^{i(B_1+B_2)}}{2ia_1\alpha_1}+\varphi_2(z_2) \\ &=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_2(z_2). \end{aligned} \tag{113}$$

If $e^{iL(c)}=-1$, it follows from (110) that $e^{2i(B_1+B_2)}=1$. We have that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_2(z_2). \tag{114}$$

If $e^{iL(c)}=i$, it follows from (110) that $e^{2i(B_1+B_2)}=-1$. Thus, we obtain that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}+e^{-i(L(z)-B_1)}}{2a_1\alpha_1}+\varphi_2(z_2). \tag{115}$$

If $e^{iL(c)}=-i$, it follows from (110) that $e^{2i(B_1+B_2)}=-1$. Thus, we obtain that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}+e^{-i(L(z)-B_1)}}{2a_1\alpha_1}+\varphi_2(z_2). \tag{116}$$

Therefore, from (102)–(106) and (113)–(116), we can prove the conclusions (23) and (24) of Theorem 13.

(i)Assume that $\partial f_1/\partial z_1=0$. Thus, from (22), it follows that

$$f_1(z)=\phi_1(z_2), \alpha_2f_2(z+c)+\alpha_3f_1(z)\equiv\xi_1, \xi_1=\pm1. \tag{117}$$

This leads to $\partial f_2/\partial z_1=0$. We thus get from (22) that

$$f_2(z)=\phi_2(z_2), \beta_2f_1(z+c)+\beta_3f_2(z)\equiv\xi_2, \xi_2=\pm1. \tag{118}$$

By combining with (117) and (118), it yields

$$\begin{cases} \alpha_2\phi_2(z_2+c_2)+\alpha_3\phi_1(z_2)\equiv\xi_1, \\ \beta_2\phi_1(z_2+c_2)+\beta_3\phi_2(z_2)\equiv\xi_2, \end{cases} \tag{119}$$

which implies that

$$\phi_1(z_2+2c_2)=\frac{\alpha_3\beta_3}{\alpha_2\beta_2}\phi_1(z_2)+\frac{\alpha_2\xi_2-\beta_3\xi_1}{\alpha_2\beta_2}, \phi_2(z_2+2c_2)=\frac{\alpha_3\beta_3}{\alpha_2\beta_2}\phi_2(z_2)+\frac{\beta_2\xi_1-\alpha_3\xi_2}{\alpha_2\beta_2}. \tag{120}$$

If $\alpha_2\beta_2 = \alpha_3\beta_3$, then from (120), it follows that

$$\phi_1(z_2 + 2c_2) = \phi_1(z_2) + \frac{\alpha_2\xi_2 - \beta_3\xi_1}{\alpha_2\beta_2}, \phi_2(z_2 + 2c_2) = \phi_2(z_2) + \frac{\beta_2\xi_1 - \alpha_3\xi_2}{\alpha_2\beta_2}, \tag{121}$$

which implies that

$$\phi_1(z_2) = G_1(z_2) + \gamma_1 z_2, \phi_2(z_2) = G_2(z_2) + \gamma_2 z_2, \tag{122}$$

where $G_1(z_2)$, $G_2(z_2)$ are entire period functions of finite order with period $2c_2$, and

$$\gamma_1 = \frac{\alpha_2\xi_2 - \beta_3\xi_1}{2c_2\alpha_2\beta_2}, \gamma_2 = \frac{\beta_2\xi_1 - \alpha_3\xi_2}{2c_2\alpha_2\beta_2}. \tag{123}$$

If $\alpha_2\beta_2 \neq \alpha_3\beta_3$, then from (120), it follows that

$$\begin{aligned}\phi_1(z_2) &= e^{\log(\alpha_2\beta_2) - \log(\alpha_3\beta_3)/2c_2 z_2} G_1(z_2) + D_1, \phi_2(z_2) \\ &= e^{\log(\alpha_2\beta_2) - \log(\alpha_3\beta_3)/2c_2 z_2} G_2(z_2) + D_2,\end{aligned} \tag{124}$$

where $D_1 = \alpha_2\xi_2 - \beta_3\xi_1/\alpha_2\beta_2 - \alpha_3\beta_3$ and $D_2 = \beta_2\xi_1 - \alpha_3 \xi_2/\alpha_2\beta_2 - \alpha_3\beta_3$. Substituting (124) into (119), it follows that $\alpha_2\beta_2 = -\alpha_3\beta_3$, $G_1(z_2) = \beta_3/\beta_2 i G_2(z_2 - c_2)$ and $G_2(z_2) = \alpha_3/\alpha_2 i G_1(z_2 - c_2)$. Thus, we have

$$\begin{aligned}\phi_1(z_2) &= e^{\log(-1)/2c_2 z_2} G_1(z_2) + \frac{\alpha_2\xi_2 - \beta_3\xi_1}{2\alpha_2\beta_2}, \phi_2(z_2) \\ &= e^{\log(-1)/2c_2 z_2} G_2(z_2) + \frac{\beta_2\xi_1 - \alpha_3\xi_2}{2\alpha_2\beta_2}.\end{aligned} \tag{125}$$

(ii) Suppose that $\partial f_1(z_1, z_2)/\partial z_1 = b_1 (\neq 0)$. Then, it yields in view of (22) that

$$f_1(z) = b_1 z_1 + \psi_1(z_2), \alpha_2 f_2(z + c) + \alpha_3 f_1(z) = b_2, b_2^2 + (\alpha_1 b_1)^2 = 1, \tag{126}$$

where $\psi_1(z_2)$ is a transcendental entire function of finite order in z_2. Equation (126) leads to $\partial f_2/\partial z_1 = -\alpha_3/\alpha_2 b_1$. Thus, due to the second equation in (22), we have

$$f_2(z) = -\frac{\alpha_3}{\alpha_2} b_1 z_1 + \psi_2(z_2), \beta_2 f_1(z + c) + \beta_3 f_2(z) = b_3, b_3^2 + \left(\frac{\beta_1\alpha_3}{\alpha_2} b_1\right)^2 = 1, \tag{127}$$

where $\psi_2(z_2)$ is a transcendental entire function of finite order in z_2. Combining with (126) and (127), we can deduce that $\alpha_2\beta_2 = \alpha_3\beta_3$ and

$$\begin{cases} \alpha_2\psi_2(z_2 + c_2) + \alpha_3\psi_1(z_2) = b_2 + \alpha_3 b_1 c_1, \\ \beta_2\psi_1(z_2 + c_2) + \beta_3\psi_2(z_2) = b_3 - \beta_2 b_1 c_1, \end{cases} \tag{128}$$

This means that

$$\psi_1(z_2) = G_1(z_2) + \gamma_1 z_2, \psi_2(z_2) = G_2(z_2) + \gamma_2 z_2, \tag{129}$$

where $G_1(z_2)$, $G_2(z_2)$ are entire period functions of finite order with period $2c_2$ satisfying

$$\begin{aligned}&G_2(z_2 + c_2) + \frac{\alpha_3}{\alpha_2} G_1(z_2) = \frac{\beta_2 b_2 + \alpha_3 b_3}{2\alpha_2\beta_2}, G_1(z_2 + c_2) + \frac{\beta_3}{\beta_2} G_2(z_2) = \frac{\beta_3 b_2 + \alpha_2 b_3}{2\alpha_2\beta_2}, \\ &\gamma_1 = \frac{\alpha_2 b_3 - \beta_3 b_2 - 2(\alpha_2\beta_2) b_1 c_1}{2\alpha_2\beta_2 c_2}, \gamma_2 = \frac{\beta_2 b_2 - \alpha_3 b_3 + 2\alpha_3\beta_2 b_1 c_1}{2\alpha_2\beta_2 c_2}.\end{aligned} \tag{130}$$

Hence, from (126)–(129), it is easy to get the cases ((26)) and ((27)) of Theorem 13.

Therefore, the proof of Theorem 13 is completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that none of the authors have any competing interests in the manuscript.

Authors' Contributions

Conceptualization was contributed by H. Y. Xu; writing-original draft preparation was contributed by H.Y. Xu and K.Y. Zhang; writing-review and editing was contributed by H. Y. Xu and M.Y. Yu; funding acquisition was contributed by H. Y. Xu and K.Y. Zhang.

Acknowledgments

This work was supported by the National Natural Science Foundation of China 12161074 and the Talent Introduction Research Foundation of Suqian University.

References

[1] R. Taylor and A. Wiles, "Ring-theoretic properties of certain Hecke algebra," *Annals of Mathematics*, vol. 141, no. 3, pp. 553–572, 1995.

[2] A. Wiles, "Modular elliptic curves and Fermats last theorem," *Annals of Mathematics*, vol. 141, no. 3, pp. 443–551, 1995.

[3] P. Montel, *Lecons sur les familles normales de fonctions analytiques et leurs applications*, Gauthier-Villars, Paris, 1927.

[4] F. Gross, "On the equation $f^n + g^n = 1$," *Bulletin of the American Mathematical Society*, vol. 72, no. 1, pp. 86–88, 1966.

[5] Y. M. Chiang and S. J. Feng, "On the Nevanlinna characteristic of f(z+η) and difference equations in the complex plane," *The Ramanujan Journal*, vol. 16, no. 1, pp. 105–129, 2008.

[6] R. G. Halburd and R. J. Korhonen, "Difference analogue of the lemma on the logarithmic derivative with applications to difference equations," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 477–487, 2006.

[7] R. G. Halburd and R. J. Korhonen, “Nevanlinna theory for the difference operator,” *Annales Academiæ Scientiarum Fennicæ Mathematica*, vol. 31, no. 2, pp. 463–478, 2006.

[8] K. Liu, “Meromorphic functions sharing a set with applications to difference equations,” *Journal of Mathematical Analysis and Applications*, vol. 359, no. 1, pp. 384–393, 2009.

[9] K. Liu, T. B. Cao, and H. Z. Cao, “Entire solutions of Fermat type differential-difference equations,” *Archiv der Mathematik*, vol. 99, no. 2, pp. 147–155, 2012.

[10] K. Liu and L. Z. Yang, “On entire solutions of some differential-difference equations,” *Computational Methods and Function Theory*, vol. 13, no. 3, pp. 433–447, 2013.

[11] L. Y. Gao, “Entire solutions of two types of systems of complex differential-difference equations,” *Acta Mathematica Scientia*, vol. 59, pp. 677–685, 2016.

[12] T. B. Cao and L. Xu, “Logarithmic difference lemma in several complex variables and partial difference equations,” *Annali di Matematica Pura ed Applicata*, vol. 199, no. 2, pp. 767–794, 2020.

[13] L. Xu and T. B. Cao, “Solutions of complex Fermat-type partial difference and differential-difference equations,” *Mediterranean Journal of Mathematics*, vol. 15, no. 6, pp. 1–14, 2018.

[14] R. G. Halburd and R. Korhonen, “Finite-order meromorphic solutions and the discrete Painlevé equations,” *Proceedings of the London Mathematical Society*, vol. 94, no. 2, pp. 443–474, 2007.

[15] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, and K. Tohge, “Complex difference equations of Malmquist type,” *Computational Methods and Function Theory*, vol. 1, no. 1, pp. 27–39, 2001.

[16] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1993.

[17] K. Liu and T. B. Cao, “Fractional differential equations 2012,” *Electronic Journal of Differential Equations*, vol. 2013, no. 59, pp. 1-2, 2013.

[18] J. Rieppo, “On a class of complex functional equations,” *Annales Academiæ Scientiarum Fennicæ Mathematica*, vol. 32, no. 1, pp. 151–170, 2007.

[19] H. Y. Xu and Y. Y. Jiang, “Results on entire and meromorphic solutions for several systems of quadratic trinomial functional equations with two complex variables,” *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 116, pp. 1–19, 2021.

[20] H. Y. Xu, S. Y. Liu, and Q. P. Li, “Entire solutions for several systems of nonlinear difference and partial differential-difference equations of Fermat-type,” *Journal of Mathematical Analysis and Applications*, vol. 483, no. 2, article 123641, 2020.

[21] H. Y. Xu, D. W. Meng, S. Y. Liu, and H. Wang, “Entire solutions for several second-order partial differential-difference equations of Fermat type with two complex variables,” *Advances in Difference Equation*, vol. 2021, no. 52, pp. 1–24, 2021.

[22] H. Y. Xu and L. Xu, “Transcendental entire solutions for several quadratic binomial and trinomial PDEs with constant coefficients,” *Analysis and Mathematical Physics*, vol. 12, pp. 1–21, 2022.

[23] T. B. Cao and R. J. Korhonen, “A new version of the second main theorem for meromorphic mappings intersecting hyperplanes in several complex variables,” *Journal of Mathematical Analysis and Applications*, vol. 444, no. 2, pp. 1114–1132, 2016.

[24] R. J. Korhonen, “A difference Picard theorem for meromorphic functions of several variables,” *Computational Methods and Function Theory*, vol. 12, no. 1, pp. 343–361, 2012.

[25] P.-C. Hu and C.-C. Yang, “Malmquist type theorem and factorization of meromorphic solutions of partial differential equations,” *Complex Variables, Theory and Application: An International Journal*, vol. 27, pp. 269–285, 1995.

[26] A. Biancofiore and W. Stoll, “Another proof of the lemma of the logarithmic derivative in several complex variables,” in *Recent Developments in Several Complex Variables*, J. Fornaess, Ed., pp. 29–45, Princeton University Press, Princeton, 1981.

[27] L. I. Ronkin, *Introduction to the Theory of Entire Functions of Several Variables*, American Mathematical Society, Moscow: Nauka 1971 (Russian), 1974.

[28] W. Stoll, *Holomorphic Functions of Finite Order in Several Complex Variables*, American Mathematical Society, Providence, 1974.

[29] P. C. Hu, P. Li, and C. C. Yang, “Unicity of Meromorphic Mappings,” in *Advances in Complex Analysis and Its Applications*, vol. 1, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.

[30] P. C. Hu and C. C. Yang, “Uniqueness of meromorphic functions on $\mathbb{C}^m$,” *Complex Variables and Elliptic Equations*, vol. 30, pp. 235–270, 1996.