Mathematical Problems in Engineering

Fractional Mathematical Modelling and Optimal Control Problems of Differential Equations

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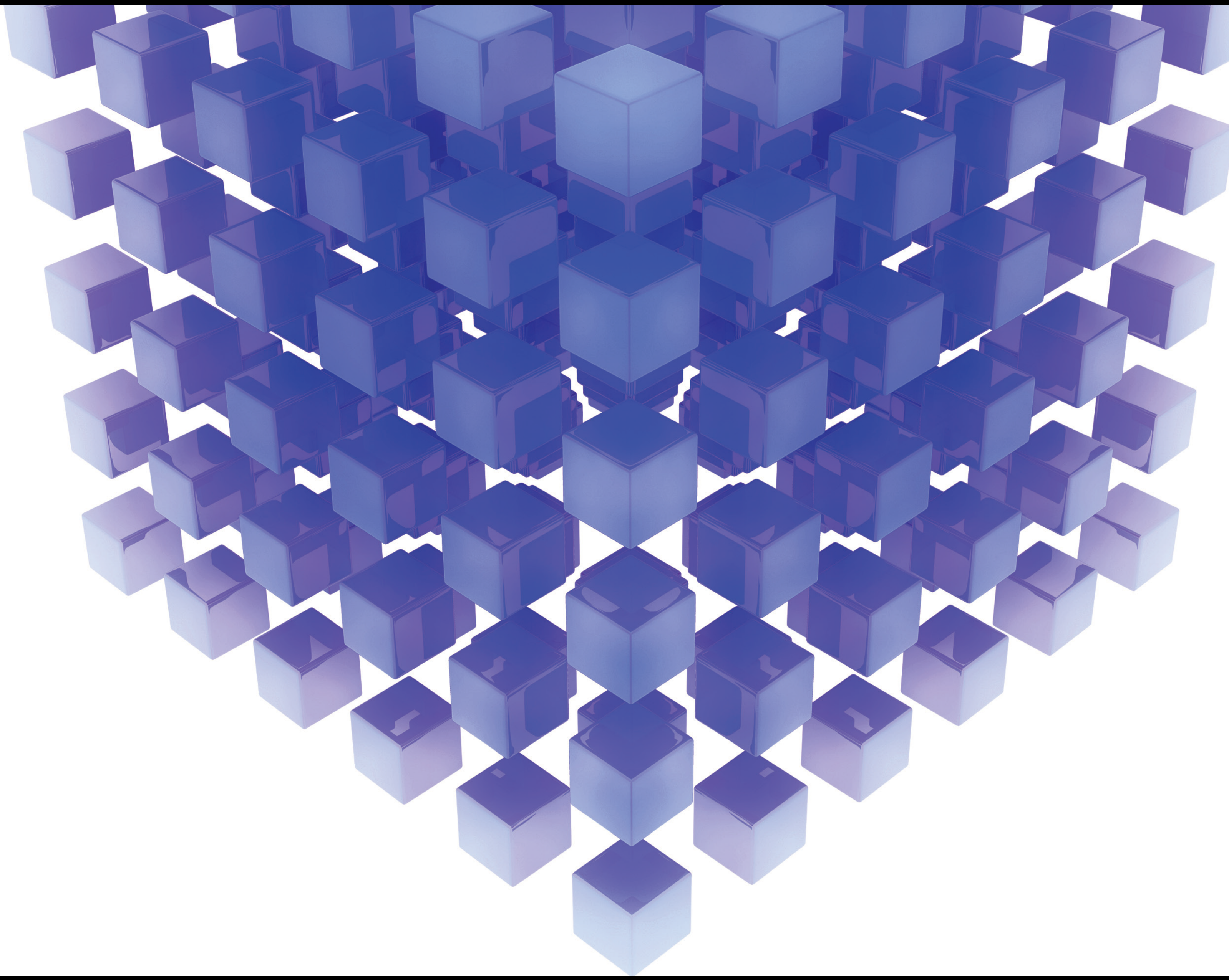
Guest Editors: Lishan Liu, Yong Hong Wu, and Chuanjun Chen



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Research Article

A Numerical Method for Compressible Model of Contamination from Nuclear Waste in Porous Media

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This paper studies and analyzes a model describing the flow of contaminated brines through the porous media under severe thermal conditions caused by the radioactive contaminants. The problem is approximated based on combining the mixed finite element method with the modified method of characteristics. In order to solve the resulting algebraic nonlinear equations efficiently, a two-grid method is presented and discussed in this paper. This approach includes a small nonlinear system on a coarse grid with size H and a linear system on a fine grid with size h. It follows from error estimates that asymptotically optimal accuracy can be obtained as long as the mesh sizes satisfy $H = O(h^{1/3})$.

1. Introduction

A compressible nuclear waste disposal contamination problem in porous media is presented by the following coupled systems of partial differential equations. The physical processes can be concreted to be a high-level waste disposal buried in a salt dome, and next the salt dissolves to generate a brine, radioactive elements decay to generate heat, and finally the radionuclides are transported by the flow.

Fluid:

$$\phi_1 \frac{\partial p}{\partial t} + \nabla \cdot u = -q + R_s'(\hat{c}), \tag{1}$$

$$u = -\frac{k}{\mu(\hat{c})} \nabla p = -a(\hat{c})^{-1} \nabla p, \tag{2}$$

where p and u are the fluid pressure and Darcy velocity, respectively, $\phi_1 = \phi c_w$ and ϕ is the porosity. $q = q(x,t)$ is a production term, $R_s'(\hat{c}) = [c_s \phi K_s f_s / (1 + c_s)](1 - \hat{c})$ is a salt dissolution term, $k(x)$ is the permeability of the rock, and $\mu(\hat{c})$, the viscosity of the fluid, is dependent upon $\hat{c}$, the concentration of the brine in the fluid.

Brine:

$$\phi \frac{\partial \hat{c}}{\partial t} + u \cdot \nabla \hat{c} - \nabla \cdot (E_c \nabla \hat{c}) = g(\hat{c}), \tag{3}$$

where E_c is the diffusion tensor including the molecular diffusion and mechanical diffusion and $E_c = \mathbf{D} + D_m \mathbf{I}$, $\mathbf{D} = (D)_{i,j} = ((\alpha_T |u| \delta_{i,j} + (\alpha_L - \alpha_T) u_i u_j)/|u|)$, and $g(\hat{c}) = -\hat{c}\{[c_s \phi K_s f_s/(1 + c_s)](1 - \hat{c})\} - q_c + R_s$. Here, D_m is molecular diffusion. u_i and u_j are two direction cosines of Darcy velocity. $\mathbf{I}$ is an identity matrix.

Heat:

$$d_1(p) \frac{\partial p}{\partial t} + d_2 \frac{\partial T}{\partial t} + c_p u \cdot \nabla T - \nabla \cdot (\widetilde{E}_H \nabla T) = Q(u, T, \hat{c}, p), \tag{4}$$

where T is the temperature of the fluid, $d_1(p) = -\phi c_w [U_0 + (p/\rho)]$, $d_2 = \phi c_p + (1 - \phi) \rho_R c_{pR}$, $\widetilde{E}_H = \mathbf{D} c_{pw} + K_m' \mathbf{I}$, $K_m' = k_m/\rho_0$, and $Q(u, T, \hat{c}, p) = -\{[\nabla U_0 - c_p \nabla T_0] \cdot u + [U_0 + c_p (T - T_0) + (p/\rho)][-q + R_s']\} - q_L - qH - q_H$.

Radionuclide (component i):

$$\phi K_i \frac{\partial c_i}{\partial t} + u \cdot \nabla c_i - \nabla \cdot (E_c \nabla c_i) + d_3(c_i) \frac{\partial p}{\partial t} = f_i(\hat{c}, c_1, \ldots, c_N), \quad i = 1, \ldots, N, \tag{5}$$

where c_i is the trace concentration of the i th radionuclide, $d_3(c_i) = \phi c_w c_i (K_i - 1)$, and

$$f_i(\widehat{c}, c_1, \ldots, c_N) = c_i \left\{ q - \left[\frac{c_s \phi K_s f_s}{(1+c_s)} \right] (1-\widehat{c}) \right\} - q c_i - q_{ci} + q_{0i} + \sum_{j=1}^{N} k_{ij} \lambda_j K_j \phi c_j - \lambda_i K_i \phi c_i. \tag{6}$$

We assume the following:

(1) Zero Neumann boundary conditions for the equations

(2) The initial conditions are assumed given

(3) The medium Ω is vertically homogeneous and take $\Omega \in \mathbf{R}^2$

(4) The solutions are smooth and Ω periodic

(5) k, ϕ, ϕ_1, d_2, and K_i are bounded below by positive constants, and $R_s'(\widehat{c}), \mu(\widehat{c}), g(\widehat{c}), d_3(c_i), f_i(c_i)$, and $Q(T)$ are twice continuously differentiable with bounded partial derivatives about the variables in parentheses

(6) $\mathbf{D} = 0$

Chou and Li [1], Ewing et al. [2], and Li et al. [3] have presented and studied several numerical methods for system (1)–(5) and its incompressible case. In this paper, we use the mixed finite element method to approximate the fluid problem and treat the brine, heat, and radionuclides by a modified method of characteristic finite element. It is well known that the full discrete approximation scheme is coupled and nonlinear. If simply lagging, the evaluation of the nonlinear items is used to obtain a linear system; it would be inevitable to introduce the constraint conditions about the mesh grid due to the stability requirement. Moreover, it would take an expensive cost to choose the implicit scheme to nonlinear solutions. An efficient method motivated by Xu [4] is considered in this paper. The method is used by Bi et al. [5], Chen et al. [6–10], and Liu et al. [11, 12] for solving some nonlinear problem. We shall relegate all of nonlinear iterations on a coarse grid of size H much coarser than the original fine grid of size h. According to the error estimates in the context, it obtains the asymptotically optimal accuracy to take $H = O(h^{1/3})$.

The remainder of the paper is organized as follows. Notations and approximation assumptions are given in Section 2. A two-grid method is defined and the convergence error estimates are derived in Section 3. In Section 4, we give some conclusions and extensions.

2. Notations and Approximation Results

To analyze the temporal discretization on a time interval $(0, T)$, let M be a positive integer number, $\triangle t = T/M$, $t^n = n\triangle t$ $(0 \le n \le M)$, and $\varphi^n = \varphi(\cdot, t^n)$. Let $L^p(J; W^{j,q}(\Omega))$ denote the usual set of functions with the norm

$$\|\psi\|_{L^p(J;W^{j,q}(\Omega))} = \left(\int_J \|\psi(\cdot,t)\|^p_{W^{j,q}(\Omega)} dt \right)^{1/p}, \tag{7}$$

where if $p = \infty$, the integral is replaced by the essential supremum. Let $l^p(J; W^{j,q}(\Omega))$ denote the time discrete analogue to the space $L^p(J; W^{j,q}(\Omega))$ with the norm

$$\|\psi\|_{l^p(W^{j,q}(\Omega))} = \left(\sum_{n=1}^{N} \|\psi^n\|^p_{W^{j,q}(\Omega)} \Delta t \right)^{1/p}. \tag{8}$$

Let $W = L^2(\Omega)$ and $V = \{v \in H(\text{div}; \Omega); v \cdot \gamma = 0\}$. The weak form is presented as follows:

$$\left(\phi_1 \frac{\partial p}{\partial t}, w \right) + (\nabla \cdot u, w) = (-q + R_s'(\widehat{c}), w), \quad \forall w \in W, \tag{9}$$

$$(a(\widehat{c}) u, v) - (\nabla \cdot v, p) = 0, \quad \forall v \in V, \tag{10}$$

$$\left(\phi \frac{\partial \widehat{c}}{\partial t}, z \right) + (u \cdot \nabla \widehat{c}, z) + (E_c \nabla \widehat{c}, \nabla z) = (g(\widehat{c}), z), \tag{11}$$

$$\left(d_2 \frac{\partial T}{\partial t}, z \right) + c_p (u \cdot \nabla T, z) + (\widetilde{E}_H \nabla T, \nabla z) + \left(d_1(p) \frac{\partial p}{\partial t}, z \right) = (Q(u, T, \widehat{c}, p), z), \tag{12}$$

$$\left(\phi K_i \frac{\partial c_i}{\partial t}, z \right) + (u \cdot \nabla c_i, z) + (E_c \nabla c_i, \nabla z) + \left(d_3(c_i) \frac{\partial p}{\partial t}, z \right) = (f_i(\widehat{c}, c_1, \ldots, c_N), z), \tag{13}$$

for $z \in H^1(\Omega)$ and $i = 1, \ldots, N$.

Assume that $V_h \times W_h$ is the Raviart–Thomas space of index at least k associated with a quasitriangulation of Ω such that the elements have diameters bounded by h_p. Let $M_h = M_{h_c}$ and $R_h = R_{h_T}$ be finite-dimensional subspaces of $W^{1,\infty}(\Omega)$ for the approximation of concentrations and temperature, respectively, and we take M_h and R_h as the piecewise-polynomial space of degree at least l and r, respectively. As in [2, 11, 13], the approximation properties for $V_h \times W_h$ and M_h, R_h are given by

$$\inf_{v_h \in V_h} \|v - v_h\| \le C\|v\|_{k+1} h_p^{k+1}, \tag{14}$$

$$\inf_{v_h \in V_h} \|\nabla \cdot (v - v_h)\| \le C\left(\|v\|_{k+1} + \|\nabla \cdot v\|_{k+1}\right) h_p^{k+1}, \tag{15}$$

for $v \in V \cap H^{k+1}(\Omega)^2$ and $\nabla \cdot v \in H^{k+1}(\Omega)$, and

$$\inf_{w_h \in W_h} \|w - w_h\| \le C\|w\|_{k+1} h_p^{k+1}, \quad w \in H^{k+1}(\Omega), \tag{16}$$

$$\inf_{z_h \in M_h} \|z - z_h\|_{1,q} \le C\|z\|_{l+1,q} h_c^l, \quad z \in W^{l+1,q}(\Omega), 1 \le q \le \infty, \tag{17}$$

$$\inf_{z_h \in R_h} \|z - z_h\|_{1,q} \le C\|z\|_{r+1,q} h_T^r, \quad z \in W^{r+1,q}(\Omega), 1 \le q \le \infty. \tag{18}$$

If the initial solutions $\{p_h^0, u_h^0, \widehat{c}_h^0, \check{T}_h^0, c_{ih}^0\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$, the characteristics-Galerkin and mixed finite element approximation schemes are to find $\{p_h^n, u_h^n, \widehat{c}_h^n, \check{T}_h^n, c_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$ satisfying

$$\left(\phi_1 \frac{p_h^n - p_h^{n-1}}{\Delta t}, w\right) + (\nabla \cdot u_h^n, w) = (-q + R_s'(\widehat{c}_h^n), w), \quad \forall w \in W_h, \tag{19}$$

$$(a(\widehat{c}_h^n) u_h^n, v) - (\nabla \cdot v, p_h^n) = 0, \quad \forall v \in V_h, \tag{20}$$

$$\left(\phi \frac{\widehat{c}_h^n - \overline{\widehat{c}}_h^{n-1}}{\Delta t}, z\right) + (E_c \nabla \widehat{c}_h^n, \nabla z) = (g(\widehat{c}_h^n), z), \quad \forall z \in M_h, \tag{21}$$

$$\begin{aligned}&\left(d_2 \frac{\check{T}_h^n - \overline{\check{T}}_h^{n-1}}{\Delta t}, z\right) + \left(\widetilde{E}_h \nabla \check{T}_h^n, \nabla z\right) + \left(d_1(p_h^n) \frac{p_h^n - p_h^{n-1}}{\Delta t}, z\right) \\ &= \left(Q\left(u_h^n, \check{T}_h^n, \widehat{c}_h^n, p_h^n\right), z\right), \quad \forall z \in R_h,\end{aligned} \tag{22}$$

$$\begin{aligned}&\left(\phi K_i \frac{c_{ih}^n - \overline{c}_{ih}^{n-1}}{\Delta t}, z\right) + (E_c \nabla c_{ih}^n, \nabla z) + \left(d_3(c_{ih}^n) \frac{p_h^n - p_h^{n-1}}{\Delta t}, z\right) \\ &= (f_i(\widehat{c}_h^n, c_{1h}^n, \ldots, c_{Nh}^n), z), \quad \forall z \in M_h,\end{aligned} \tag{23}$$

where $i = 1, \ldots, N$ and

$$\begin{aligned}\overline{\widehat{c}}_h^{n-1} &= \widehat{c}_h^{n-1}\left(\overline{x}^{n-1}\right), \\ \overline{x}^{n-1} &= x - \frac{u_h^n \Delta t}{\phi}, \\ \overline{c}_{ih}^{n-1} &= c_{ih}^{n-1}\left(\overline{x}^{n-1}\right), \\ \overline{x}^{n-1} &= x - \frac{u_h^n \Delta t}{(\phi K_i)}, \\ \overline{\check{T}}_h^{n-1} &= \check{T}_h^{n-1}\left(\overline{x}^{n-1}\right), \\ \overline{x}^{n-1} &= x - \frac{c_p u_h^n \Delta t}{d_2}.\end{aligned} \tag{24}$$

Remark. If $\overline{x}^{n-1}$ is located outside Ω, we can join $\overline{x}^{n-1}$ with $Y \in \partial\Omega$ so that $(\overline{x}^{n-1} - Y)/\|\overline{x}^{n-1} - Y\|$ is the outer-normal direction to the boundary $\partial\Omega$ at Y. Take $x^* \in \Omega$ so that $Y - x^{*n-1} = \overline{x}^{n-1} - Y$, then we define $\overline{x}^{n-1} = x^{*n-1}$.

In order to deduce the error estimates, we employ the elliptic projections by labeling them with tildes.

$$(\nabla \cdot (u - \widetilde{U}), w) + (p - \widetilde{P}, w) = 0, \quad w \in W_h, \tag{25}$$

$$(a(\widehat{c})(u - \widetilde{U}), v) + (\nabla \cdot v, p - \widetilde{P}) = 0, \quad v \in V_h, \tag{26}$$

$$\left(E_c \nabla (\widehat{c} - \widetilde{C}), \nabla z\right) + \lambda (\widehat{c} - t\widetilde{C}, z) = 0, \quad z \in M_h, \tag{27}$$

$$\left(E_H \nabla (T - \widetilde{T}), \nabla z\right) + \lambda (T - \widetilde{T}, z) = 0, \quad z \in R_h, \tag{28}$$

$$\left(E_c \nabla\left(c_i - \widetilde{C}_i\right), \nabla z\right) + \lambda\left(c_i - \widetilde{C}_i, z\right) = 0, \quad z \in M_h, \tag{29}$$

where $\{\widetilde{U}, \widetilde{P}\}$: $J \longrightarrow V_h \times W_h, \widetilde{C}$: $J \longrightarrow M_h, \widetilde{T}$: $J \longrightarrow R_h$ and $\widetilde{C}_i$: $J \longrightarrow M_h$ for $t \in J$ and introduce the following notations:

$$\begin{aligned}\eta &= p - \widetilde{P}, \\ \alpha &= \widetilde{P} - p_h, \\ \varrho &= u - \widetilde{U}, \\ \beta &= \widetilde{U} - u_h, \\ \widehat{\zeta} &= \widehat{c} - \widetilde{C}, \\ \widehat{\chi} &= \widetilde{C} - \widehat{c}_h, \\ \zeta_i &= c_i - \widetilde{C}_i, \\ \chi_i &= \widetilde{C}_i - c_{ih}, \\ \theta &= T - \widetilde{T}, \\ \psi &= \widetilde{T} - \check{T}_h.\end{aligned} \tag{30}$$

Subtracting (19) from (9) and taking $w = d_t \alpha^{n-1}$, we get the error equation about the pressure function as follows:

$$\left(\phi_1 d_t\alpha^{n-1}, d_t\alpha^{n-1}\right) + \left(\nabla\cdot\beta^n, d_t\alpha^{n-1}\right)$$
$$= -\left(\phi_1\frac{\partial\eta^n}{\partial t}, d_t\alpha^{n-1}\right) + \left(\phi_1\left(d_t\widetilde{P}^{n-1} - \frac{\partial\widetilde{P}^n}{\partial t}\right), d_t\alpha^{n-1}\right)$$
$$+ \left(\frac{\partial R_s'}{\partial\widehat{c}}\left(\delta_1^n\right)\left(\widehat{c}^n - \widehat{c}_h^n\right), d_t\alpha^{n-1}\right), \tag{31}$$

where $d_t\alpha^{n-1} = (\alpha^n - \alpha^{n-1})/\Delta t$ and δ_1^n is between $\widehat{c}^n$ and $\widehat{c}_h^n$.

Next, combining (20) from (10) at $t = t^n$ with the test function β^n,

$$\left(a\left(\widehat{c}_h^n\right)\beta^n, \beta^n\right) - \left(\nabla\cdot\beta^n, \alpha^n\right) = -\left(\frac{\partial a}{\partial\widehat{c}}\left(\delta_2^n\right)\widetilde{U}^n\left(\widehat{c}^n - \widehat{c}_h^n\right), \beta^n\right). \tag{32}$$

When $t = t^{n-1}$, we apply the Taylor expansion and obtain that

$$\left(a\left(\widehat{c}_h^{n-1}\right)\beta^{n-1}, \beta^{n-1}\right) - \left(\nabla\cdot\beta^n, \alpha^{n-1}\right)$$
$$= \left(a\left(\widehat{c}_h^{n-1}\right)\beta^{n-1}, \beta^{n-1} - \beta^n\right) - \left(\frac{\partial a}{\partial\widehat{c}}\left(\delta_2^{n-1}\right)\widetilde{U}^{n-1}\left(\widehat{c}^{n-1} - \widehat{c}_h^{n-1}\right), \beta^n\right), \tag{33}$$

where δ_2 is between $\widehat{c}$ and $\widehat{c}_h$.

Combining (32) with (33), we get

$$\begin{aligned}
&\frac{1}{2}d_t\left\{\left(a\left(\widehat{c}_h^{n-1}\right)\beta^{n-1}, \beta^{n-1}\right)\right\} - \left(\nabla\cdot\beta^n, d_t\alpha^{n-1}\right)\\
&= -\frac{1}{2\Delta t}\left(a\left(\widehat{c}_h^n\right)\beta^n, \beta^n\right) + \frac{1}{2\Delta t}\left(a\left(\widehat{c}_h^{n-1}\right)\beta^n, \beta^n\right) - \frac{1}{2\Delta t}\left(a\left(\widehat{c}_h^{n-1}\right)\beta^n, \beta^n\right)\\
&\quad + \frac{1}{\Delta t}\left(a\left(\widehat{c}_h^{n-1}\right)\beta^{n-1}, \beta^n\right) - \frac{1}{2\Delta t}\left(a\left(\widehat{c}_h^{n-1}\right)\beta^{n-1}, \beta^{n-1}\right)\\
&\quad - \left(d_t\left\{\frac{\partial a}{\partial\widehat{c}}\left(\delta_2^{n-1}\right)\widetilde{U}^{n-1}\left(\widehat{c}^{n-1} - \widehat{c}_h^{n-1}\right), \beta^n\right)\right..
\end{aligned} \tag{34}$$

By (31) and (34),

$$\begin{aligned}
&\left(\phi_1 d_t\alpha^{n-1}, d_t\alpha^{n-1}\right) + \frac{1}{2}d_t\left\{\left(a\left(\widehat{c}_h^{n-1}\right)\beta^{n-1}, \beta^{n-1}\right)\right\}\\
&= -\left(\phi_1\frac{\partial\eta^n}{\partial t}, d_t\alpha^{n-1}\right) + \left(\phi_1\left(d_t\widetilde{P}^{n-1} - \frac{\partial\widetilde{P}^n}{\partial t}\right), d_t\alpha^{n-1}\right) + \left(\frac{\partial R_s'}{\partial\widehat{c}}\left(\delta_1^n\right)\left(\widehat{c}^n - \widehat{c}_h^n\right), d_t\alpha^{n-1}\right)\\
&\quad - \frac{1}{2\Delta t}\left(\left[a\left(\widehat{c}_h^n\right) - a\left(\widehat{c}_h^{n-1}\right)\right]\beta^n, \beta^n\right) - \frac{1}{2\Delta t}\left(a\left(\widehat{c}_h^{n-1}\right)\left(\beta^n - \beta^{n-1}\right), \left(\beta^n - \beta^{n-1}\right)\right)\\
&\quad - \left(d_t\left\{\frac{\partial a}{\partial\widehat{c}}\left(\delta_2^{n-1}\right)\widetilde{U}^{n-1}\left(\widehat{c}^{n-1} - \widehat{c}_h^{n-1}\right), \beta^n\right)\right.\\
&\le -\left(\phi_1\frac{\partial\eta^n}{\partial t}, d_t\alpha^{n-1}\right) + \left(\phi_1\left(d_t\widetilde{P}^{n-1} - \frac{\partial\widetilde{P}^n}{\partial t}\right), d_t\alpha^{n-1}\right) + \left(\frac{\partial R_s'}{\partial\widehat{c}}\left(\delta_1^n\right)\left(\widehat{c}^n - \widehat{c}_h^n\right), d_t\alpha^{n-1}\right)\\
&\quad - \frac{1}{2\Delta t}\left(\left[a\left(\widehat{c}_h^n\right) - a\left(\widehat{c}_h^{n-1}\right)\right]\beta^n, \beta^n\right) - \left(d_t\left\{\frac{\partial a}{\partial\widehat{c}}\left(\delta_2^{n-1}\right)\widetilde{U}^{n-1}\left(\widehat{c}^{n-1} - \widehat{c}_h^{n-1}\right), \beta^n\right)\right..
\end{aligned} \tag{35}$$

Using the deduction as [1, 2], we have

$$\begin{aligned}&\left\|d_t\alpha^{n-1}\right\|^2+\frac{1}{2}d_t\left\{\left(a\left(\widehat{c}_h^{n-1}\right)\beta^{n-1},\beta^{n-1}\right)\right\}\\&\leq\left(\left\|\beta^n\right\|_\infty^2+\varepsilon\right)\left\|d_t\widehat{\chi}^{n-1}\right\|^2+C\left(\left\|\widehat{\chi}^{n-1}\right\|^2+\left\|\beta^n\right\|^2+h_p^{2k+2}+h_c^{2l+2}+\Delta t^2\right).\end{aligned}\tag{36}$$

After making the induction hypothesis that $\sup\|\beta^n\|_\infty\longrightarrow 0$, we multiply (36) by Δt and sum over $1\leq n\leq M$ to get

$$\begin{aligned}&\sum_{n=1}^{M}\left\|d_t\alpha^{n-1}\right\|^2\Delta t+\left(a\left(\widehat{c}_h^{M-1}\right)\beta^M,\beta^M\right)\\&\leq\left(\left\|\beta^n\right\|_\infty^2+\varepsilon\right)\sum_{n=1}^{M}\left\|d_t\widehat{\chi}^{n-1}\right\|^2\Delta t\\&\quad+C\left(\sum_{n=1}^{M}\left[\left\|\widehat{\chi}^{n-1}\right\|_1^2+\left\|\beta^n\right\|^2\right]\Delta t+h_p^{2k+2}+h_c^{2l+2}+\Delta t^2\right).\end{aligned}\tag{37}$$

Combining (11) and (21), we get the following equality, in which we choose the test function $z=\widehat{\chi}^n-\widehat{\chi}^{n-1}=d_t\widehat{\chi}^{n-1}\Delta t$ and sum over $1\leq n\leq M$:

$$\begin{aligned}&\sum_{n=1}^{M}\left(\phi d_t\widehat{\chi}^{n-1},d_t\widehat{\chi}^{n-1}\right)\Delta t+\frac{1}{2}\left(E_c\nabla\widehat{\chi}^M,\nabla\widehat{\chi}^M\right)-\frac{1}{2}\left(E_c\nabla\widehat{\chi}^0,\nabla\widehat{\chi}^0\right)\\&\leq\sum_{n=1}^{M}\left(\phi\frac{\partial\widehat{c}^n}{\partial t}+u_h^n\cdot\nabla\widehat{c}^n-\phi\frac{\widehat{c}^n-\overline{\widehat{c}}^{n-1}}{\Delta t},d_t\widehat{\chi}^{n-1}\right)\Delta t+\sum_{n=1}^{M}\left(\phi\frac{\overline{\widehat{\chi}}^{n-1}-\widehat{\chi}^{n-1}}{\Delta t},d_t\widehat{\chi}^{n-1}\right)\Delta t\\&\quad+\sum_{n=1}^{M}\left(\phi d_t\widehat{\zeta}^{n-1},d_t\widehat{\chi}^{n-1}\right)\Delta t+\sum_{n=1}^{M}\left(\phi\frac{\widehat{\zeta}^{n-1}-\overline{\widehat{\zeta}}^{n-1}}{\Delta t},d_t\widehat{\chi}^{n-1}\right)\Delta t+\sum_{n=1}^{M}\lambda\left(\widehat{\zeta}^n,d_t\widehat{\chi}^{n-1}\right)\Delta t\\&\quad+\sum_{n=1}^{M}\left(\left(u^n-u_h^n\right)\cdot\nabla\widehat{c}^n,d_t\widehat{\chi}^{n-1}\right)\Delta t+\sum_{n=1}^{M}\left(\frac{\partial g}{\partial\widehat{c}}\left(\widehat{c}_h^n\right)\left(\widehat{\zeta}^n+\widehat{\chi}^n\right),d_t\widehat{\chi}^{n-1}\right)\Delta t,\end{aligned}\tag{38}$$

where $\overline{\widehat{\chi}}^{n-1}=\widehat{\chi}^{n-1}(\overline{x}^{n-1})$, $\overline{\widehat{\zeta}}^{n-1}=\widehat{\zeta}^{n-1}(\overline{x}^{n-1})$, and $\overline{x}^{n-1}=x-(u_h^n/\phi(x))\Delta t$.

Note that

$$\begin{aligned}&\left|\sum_{n=1}^{M}\left(\phi\frac{\widehat{\zeta}^{n-1}-\overline{\widehat{\zeta}}^{n-1}}{\Delta t},d_t\widehat{\chi}^{n-1}\right)\Delta t\right|\\&=\left|\left[\left(\phi\left(\widehat{\zeta}^{M-1}-\overline{\widehat{\zeta}}^{M-1}\right),\widehat{\chi}^M\right)-\left(\phi\left(\widehat{\zeta}^0-\overline{\widehat{\zeta}}^0\right),\widehat{\chi}^0\right)\right]-\sum_{n=1}^{M-1}\left(\phi d_t\left(\widehat{\zeta}^n-\overline{\widehat{\zeta}}^n\right),\widehat{\chi}^n\right)\right|\\&=C\Delta t\left\{\left\|\widehat{\zeta}^{M-1}\right\|^2+\left\|\widehat{\zeta}^0\right\|^2+\sum_{n=1}^{M-1}\left(\left\|\widehat{\zeta}^{n-1}\right\|^2+\left\|d_t\widehat{\zeta}^n\right\|^2\right)\right\}+\varepsilon\Delta t\left\|\widehat{\chi}^0\right\|_1^2.\end{aligned}\tag{39}$$

The reminder of the right side items in (38) is just as [2], that is,

$$\sum_{n=1}^{M}\left\|d_t\hat{\chi}^{n-1}\right\|^2\Delta t+\left(E_c\nabla\hat{\chi}^M,\nabla\hat{\chi}^M\right)$$
$$\leq C\left(\sum_{n=1}^{M}\left[\left\|\nabla\hat{\chi}^{n-1}\right\|_{\infty}^2\left\|\hat{\chi}^{n-1}\right\|_1^2+\left\|\beta^n\right\|^2\right]\Delta t+h_p^{2k+2}+h_c^{2l+2}+\Delta t^2\right). \tag{40}$$

It follows from the assumption $\sup\|\nabla\hat{\chi}^{n-1}\|_{\infty}\leq C$ and Gronwall lemma that

$$\|\hat{\chi}\|_{L^{\infty}(H^1)}+\left\|d_t\hat{\chi}\right\|_{l^2(L^2)}+\|\beta\|_{L^{\infty}(L^2)}+\left\|d_t\alpha\right\|_{l^2(L^2)}$$
$$\leq C\left(\Delta t+h_c^{l+1}+h_p^{k+1}\right). \tag{41}$$

Then, by the inverse estimate and (40), we know that the induction hypotheses hold if

$$\begin{aligned} h_p^{k+1}&=o\left(h_c\right),\\ h_c^{l+1}&=o\left(h_p\right),\\ \Delta t&=o\left(h_c\right),\\ \Delta t&=o\left(h_p\right). \end{aligned} \tag{42}$$

Finally, from the approximation properties,

$$\left\|\hat{c}-\hat{c}_h\right\|_{L^{\infty}(h^1)}+\left\|d_t\left(\hat{c}-\hat{c}_h\right)\right\|_{l^2(L^2)}+\left\|u-u_h\right\|_{L^{\infty}(L^2)}+\left\|d_t\left(p-p_h\right)\right\|_{l^2(L^2)}$$
$$\leq c\left(\Delta t+h_c^l+h_p^{k+1}\right). \tag{43}$$

Similar to the above analysis, we can obtain the error estimates for the radionuclide equation and heat equation as follows:

$$\sum_{i=1}^{N}\left\|\chi_i\right\|_{L^{\infty}(H^1)}+\sum_{i=1}^{N}\left\|d_t\chi_i\right\|_{l^2(L^2)}\leq C\left(\Delta t+h_c^{l+1}+h_p^{k+1}\right),$$
$$\|\psi\|_{L^{\infty}(H^1)}+\left\|d_t\psi\right\|_{l^2(L^2)}\leq C\left(\Delta t+h_c^{l+1}+h_T^{r+1}+h_p^{k+1}\right), \tag{44}$$

where $h_p^{k+1}=o(h_T), h_c^{l+1}=o(h_T)$, and $\Delta t=o(h_T)$ are satisfied.

Note that the time step is limited to be $o(h)$ due to the theoretical proof.

Theorem 1. *Define* $\{p_h^n,u_h^n,\hat{c}_h^n,\breve{T}_h^n,c_{ih}^n\}\in V_h\times W_h\times M_h\times R_h\times M_h^N$ *for* $n\geq 1$ *by system (19)–(23) and assume that the approximation properties (25)–(29) hold. If*

$$\begin{aligned} h_p^{k+1}&=o\left(h_c\right),\\ h_c^{l+1}&=o\left(h_p\right),\\ \Delta t&=o\left(h_c\right),\\ \Delta t&=o\left(h_p\right),\\ h_p^{k+1}&=o\left(h_T\right),\\ h_c^{l+1}&=o\left(h_T\right),\\ \Delta t&=o\left(h_T\right), \end{aligned} \tag{45}$$

then there exists a positive constant C independent of h and $\triangle t$*, such that*

$$\left\|\hat{c}-\hat{c}_h\right\|_{L^{\infty}(H^1)}+\left\|d_t\left(\hat{c}-\hat{c}_h\right)\right\|_{l^2(L^2)}+\left\|u-u_h\right\|_{L^{\infty}(L^2)}+\left\|d_t\left(p-p_h\right)\right\|_{l^2(L^2)}$$
$$+\sum_{i=1}^{N}\left\|c_i-c_{ih}\right\|_{L^{\infty}(H^1)}+\sum_{i=1}^{N}\left\|d_t\left(c_i-c_{ih}\right)\right\|_{l^2(L^2)}$$
$$+\left\|T-\breve{T}_h\right\|_{L^{\infty}(H^1)}+\left\|d_t\left(T-\breve{T}_h\right)\right\|_{l^2(L^2)}$$
$$\leq C\left(\Delta t+h_c^l+h_T^r+h_p^{k+1}\right). \tag{46}$$

Similarly, we can get the error estimates of fine grid scheme in L^2 norm.

Theorem 2. *Define $\{p_h^n, u_h^n, \hat{c}_h^n, \check{T}_h^n, c_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$ for $n \geq 1$ by system (19)–(23) and assume that the approximation properties (25)–(29) hold. Then, there exists a positive constant C independent of h, H, and Δt, such that*

$$\begin{aligned}
&\|\hat{c} - \hat{c}_h\|_{L^\infty(L^2)} + \|u - u_h\|_{l^2(L^2)} + \|p - p_h\|_{L^\infty(L^2)} \\
&\quad + \sum_{i=1}^{N} \|c_i - c_{ih}\|_{L^\infty(L^2)} + \|T - \check{T}_h\|_{L^\infty(L^2)} \\
&\leq C\left(\Delta t + h_c^{l+1} + h_T^{r+1} + h_p^{k+1}\right).
\end{aligned} \tag{47}$$

3. An Efficient Method

We now use and analyze a two-grid method for iteratively solving the nonlinear problem. The method has two steps.

Stage 1. On the coarse grid $\mathcal{T}_H$ with a mesh size $H \in (0,1)$, solve a small nonlinear system for $\{P_H^n, U_H^n, \hat{C}_H^n, T_H^n, C_{iH}^n\} \in V_H \times W_H \times M_H \times R_H \times M_H^N$ given by (19)–(23).

Stage 2. On the fine grid $\mathcal{T}_h$ with a mesh size $h \in (0,1)\,(h \ll H)$, solve the following linear system for $\{P_h^n, U_h^n, \hat{C}_h^n, T_h^n, C_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$:

$$\begin{aligned}
&\left(\phi_1 \frac{P_h^n - P_h^{n-1}}{\Delta t}, w\right) + (\nabla \cdot U_h^n, w) - \left(\frac{\partial R_s'}{\partial \hat{c}}(\hat{C}_H^n)\hat{C}_h^n, w\right) \\
&\quad = \left(-q + R_s'(\hat{C}_H^n) - \frac{\partial R_s'}{\partial \hat{c}}(\hat{C}_H^n)\hat{C}_H^n, w\right), \quad \forall w \in W_h,
\end{aligned} \tag{48}$$

$$\left(a(\hat{C}_H^n)U_h^n + \frac{\partial a}{\partial \hat{c}}(\hat{C}_H^n)U_H^n\hat{C}_h^n, v\right) - (\nabla \cdot v, P_h^n) = \left(\frac{\partial a}{\partial \hat{c}}(\hat{C}_H^n)U_H^n\hat{C}_H^n, v\right), \quad \forall v \in V_h, \tag{49}$$

$$\begin{aligned}
&\left(\phi \frac{\hat{C}_h^n - \overline{\hat{C}}_h^{n-1}}{\Delta t}, z\right) + \left(U_h^n \cdot \nabla \hat{C}_H^n, z\right) + \left(E_c \nabla \hat{C}_h^n, \nabla z\right) - \left(\frac{\partial g}{\partial \hat{c}}(\hat{C}_H^n)\hat{C}_h^n, z\right) \\
&\quad = \left(U_H^n \cdot \nabla \hat{C}_H^n, z\right) + \left(g(\hat{C}_H^n) - \frac{\partial g}{\partial \hat{c}}(\hat{C}_H^n)\hat{C}_H^n, z\right), \quad \forall z \in M_h,
\end{aligned} \tag{50}$$

$$\begin{aligned}
&\left(d_2 \frac{T_h^n - \overline{T}_h^{n-1}}{\Delta t}, z\right) + (\tilde{E}_H \nabla T_h^n, \nabla z) - \left(\frac{\partial Q^H}{\partial T} T_h^n, z\right) \\
&\quad = -\left(d_1(P_h^n)\frac{P_h^n - P_h^{n-1}}{\Delta t}, z\right) + \left(Q^H, z\right) - \left(\frac{\partial Q^H}{\partial T} T_H^n, z\right), \quad \forall z \in R_h,
\end{aligned} \tag{51}$$

$$\begin{aligned}
&\left(\phi K_i \frac{C_{ih}^n - \overline{C}_{ih}^{n-1}}{\Delta t}, z\right) + (E_c \nabla C_{iH}^n, \nabla z) + \left(\frac{\partial d_3}{\partial c_i}(C_{iH}^n)\frac{P_h^n - P_h^{n-1}}{\Delta t} C_{ih}^n, z\right) \\
&\quad - \left(\frac{\partial f_i^H}{\partial c_1} C_{1h}^n + \cdots + \frac{\partial f_i^H}{\partial c_N} C_{Nh}^n, z\right) = -\left(d_3(C_{iH}^n)\frac{P_h^n - P_h^{n-1}}{\Delta t}, z\right) \\
&\quad + \left(\frac{\partial d_3}{\partial c_i}(C_{iH}^n)\frac{P_h^n - P_h^{n-1}}{\Delta t} C_{iH}^n + f_i^H - \frac{\partial f_i^H}{\partial c_1} C_{1H}^n - \cdots - \frac{\partial f_i^H}{\partial c_N} C_{NH}^n, z\right), \quad \forall z \in M_h,
\end{aligned} \tag{52}$$

where

$$
\begin{aligned}
Q^H &= Q\left(U_h^n, T_H^n, \widehat{C}_h^n, P_h^n\right),\\
f_i^H &= f_i\left(\widehat{C}_h^n, C_{1H}^n, \ldots, C_{NH}^n\right),\\
\overline{\widehat{C}}_h^{n-1} &= \widehat{C}_h^{n-1}\left(\overline{x}^{n-1}\right),\\
\overline{x}^{n-1} &= x - \left(\frac{U_H^n \Delta t}{\phi}\right),\\
\overline{C}_{ih}^{n-1} &= C_{ih}^{n-1}\left(\overline{x}^{n-1}\right),\\
\overline{x}^{n-1} &= x - \left(\frac{U_h^n \Delta t}{(\phi K_i)}\right),\\
\overline{T}_h^{n-1} &= T_h^{n-1}\left(\overline{x}^{n-1}\right),\\
\overline{x}^{n-1} &= x - \left(\frac{c_p U_h^n \Delta t}{d_2}\right),
\end{aligned}
\tag{53}
$$

and the projection on the fine grid is based on the numerical solutions on coarse grid.

The sequential solution processes are defined as follows. Firstly, we apply the Newton iteration to the coupled system on the coarse grid and obtain $\left\{P_H^n, U_H^n, \widehat{C}_H^n, T_H^n, C_{iH}^n\right\}$. Next, combining (48)–(50), we get P_h^n and U_h^n with RT_1 and $\widehat{C}_h^n$ with piecewise linear finite element using a coupled linear system. Finally, from (51) and (52), we can get C_{ih}^n and T_h^n by parallel computation.

In order to analyze the linear scheme on the fine grid, we define

$$
\begin{aligned}
\pi &= \widetilde{P} - P_h,\\
\sigma &= \widetilde{U} - U_h,\\
\widehat{\xi} &= \widetilde{C} - \widehat{C}_h,\\
\xi &= \widetilde{C}_i - C_{ih},\\
\omega &= \widetilde{T} - T_h.
\end{aligned}
\tag{54}
$$

According to Taylor expansion, there exists a positive δ_3 such that

$$
\begin{aligned}
&R_s'(\widehat{c}^n) - R_s'\left(\widehat{C}_H^n\right) - \frac{\partial R_s'}{\partial \widehat{c}}\left(\widehat{C}_H^n\right)\left(\widehat{C}_h^n - \widehat{C}_H^n\right)\\
&= R_s'\left(\widehat{C}_H^n\right) + \frac{\partial R_s'}{\partial \widehat{c}}\left(\widehat{C}_H^n\right)\left(\widehat{c}^n - \widehat{C}_H^n\right) - R_s'\left(\widehat{C}_H^n\right) - \frac{\partial R_s'}{\partial \widehat{c}}\left(\widehat{C}_H^n\right)\left(\widehat{C}_h^n - \widehat{C}_H^n\right)\\
&\quad + \frac{\partial^2 R_s'}{2\,\partial \widehat{c}^2}(\delta_3)\left(\widehat{c}^n - \widehat{C}_H^n\right)^2\\
&= \frac{\partial R_s'}{\partial \widehat{c}}\left(\widehat{C}_H^n\right)\left(\widehat{c}^n - \widehat{C}_h^n\right) + \frac{\partial^2 R_s'}{2\,\partial \widehat{c}^2}(\delta_3)\left(\widehat{c}^n - \widehat{C}_H^n\right)^2.
\end{aligned}
\tag{55}
$$

According to (48), (9), and (25),

$$
\begin{aligned}
&\left(\phi_1 d_t \pi^{n-1}, d_t \pi^{n-1}\right) + \left(\nabla \cdot \sigma^n, d_t \pi^{n-1}\right)\\
&= -\left(\phi_1 \frac{\partial \eta^n}{\partial t}, d_t \pi^{n-1}\right) + \left(\phi_1\left(d_t \widetilde{P}^{n-1} - \frac{\partial \widetilde{P}^n}{\partial t}\right), d_t \pi^{n-1}\right) + \left(\eta^n, d_t \pi^{n-1}\right)\\
&\quad + \left(\frac{\partial R_s'}{\partial \widehat{c}}\left(\widehat{C}_H^n\right)\left(\widehat{c}^n - \widehat{C}_h^n\right) + \frac{\partial^2 R_s'}{2\,\partial \widehat{c}^2}(\delta_1)\left(\widehat{c}^n - \widehat{C}_H^n\right)^2, d_t \pi^{n-1}\right).
\end{aligned}
\tag{56}
$$

Like the deduction of (34), we see that

$$\frac{1}{2}d_t\left\{\left(a\left(\widehat{C}_H^{n-1}\right)\sigma^{n-1},\sigma^{n-1}\right)\right\}-\left(\nabla\cdot\sigma^n,d_t\pi^{n-1}\right)$$
$$=-\frac{1}{2\Delta t}\left(a\left(\widehat{C}_H^n\right)\sigma^n,\sigma^n\right)+\frac{1}{2\Delta t}\left(a\left(\widehat{C}_H^{n-1}\right)\sigma^n,\sigma^n\right)-\frac{1}{2\Delta t}\left(a\left(\widehat{C}_H^{n-1}\right)\sigma^n,\sigma^n\right)$$
$$+\frac{1}{\Delta t}\left(a\left(\widehat{C}_H^{n-1}\right)\sigma^{n-1},\sigma^n\right)-\frac{1}{2\Delta t}\left(a\left(\widehat{C}_H^{n-1}\right)\sigma^{n-1},\sigma^{n-1}\right)-\left(d_t\left\{\frac{\partial a}{\partial\widehat{c}}\left(\widehat{C}_H^{n-1}\right)\widetilde{U}^{n-1}\right.\right.\tag{57}$$
$$\left.\left.\left(\widehat{c}^{n-1}-\widehat{C}_h^{n-1}\right)+\frac{\partial a}{\partial\widehat{c}}\left(\widehat{C}_H^{n-1}\right)\left(\widetilde{U}^{n-1}-U_H^{n-1}\right)\left(\widehat{C}_h^{n-1}-\widehat{C}_H^{n-1}\right)\right\},\sigma^n\right)$$
$$-\frac{1}{2\Delta t}\left(\frac{\partial^2 a}{\partial\widehat{c}^2}(\delta_4)\widetilde{U}^n\left(\widehat{c}^n-\widehat{C}_H^n\right)^2-\frac{\partial^2 a}{\partial\widehat{c}^2}(\delta_4)\widetilde{U}^{n-1}\left(\widehat{c}^{n-1}-\widehat{C}_H^{n-1}\right)^2,\sigma^n\right),$$

where δ_4 is between $\widehat{c}$ and $\widehat{C}_H$. Hence, (56) and (57) can give that

$$\left(\phi_1 d_t\pi^{n-1},d_t\pi^{n-1}\right)+\frac{1}{2}d_t\left\{\left(a\left(\widehat{C}_H^{n-1}\right)\sigma^{n-1},\sigma^{n-1}\right)\right\}$$
$$=-\left(\phi_1\frac{\partial\eta^n}{\partial t},d_t\pi^{n-1}\right)+\left(\phi_1\left(d_t\widetilde{P}^{n-1}-\frac{\partial\widetilde{P}^n}{\partial t}\right),d_t\pi^{n-1}\right)+\left(\eta^n,d_t\pi^{n-1}\right)$$
$$+\left(\frac{\partial R_s'}{\partial\widehat{c}}\left(\widehat{C}_H^n\right)\left(\widehat{c}^n-\widehat{C}_h^n\right)+\frac{\partial^2 R_s'}{2\,\partial\widehat{c}^2}(\delta_3)\left(\widehat{c}^n-\widehat{C}_H^n\right)^2,d_t\pi^{n-1}\right)$$
$$-\frac{1}{2\Delta t}\left(\left[a\left(\widehat{C}_H^n\right)-a\left(\widehat{C}_H^{n-1}\right)\right]\sigma^n,\sigma^n\right)-\frac{1}{2\Delta t}\left(a\left(\widehat{C}_H^{n-1}\right)\left(\sigma^n-\sigma^{n-1}\right),\left(\sigma^n-\sigma^{n-1}\right)\right),\tag{58}$$
$$-\left(d_t\left\{\frac{\partial a}{\partial\widehat{c}}\left(\widehat{C}_H^{n-1}\right)\widetilde{U}^{n-1}\left(\widehat{c}^{n-1}-\widehat{C}_h^{n-1}\right)+\frac{\partial a}{\partial\widehat{c}}\left(\widehat{C}_H^{n-1}\right)\left(\widetilde{U}^{n-1}-U_H^{n-1}\right)\left(\widehat{C}_h^{n-1}-\widehat{C}_H^{n-1}\right)\right\},\sigma^n\right)$$
$$-\frac{1}{2\Delta t}\left(\frac{\partial^2 a}{\partial\widehat{c}^2}(\delta_4)\widetilde{U}^n\left(\widehat{c}^n-\widehat{C}_H^n\right)^2-\frac{\partial^2 a}{\partial\widehat{c}^2}(\delta_4)\widetilde{U}^{n-1}\left(\widehat{c}^{n-1}-\widehat{C}_H^{n-1}\right)^2,\sigma^n\right).$$

The error equation about $\widehat{c}$ shows that

$$\left(\phi\frac{\widehat{\xi}^n-\widehat{\xi}^{n-1}}{\Delta t},z\right)+\left(E_c\nabla\widehat{\xi}^n,\nabla z\right)$$
$$=-\left(\phi\frac{\partial\widehat{c}^n}{\partial t}+u_H^n\cdot\nabla\widehat{c}^n-\phi\frac{\widehat{c}^n-\overline{\widehat{c}}^{n-1}}{\Delta t},z\right)+\left(\phi\frac{\overline{\widehat{\xi}}^{n-1}-\widehat{\xi}^{n-1}}{\Delta t},z\right)$$
$$-\left(\phi\frac{\widehat{\zeta}^n-\overline{\widehat{\zeta}}^{n-1}}{\Delta t},z\right)+\lambda\left(\widehat{\zeta}^n,z\right)-\left(\left(u^n-U_H^n\right)\cdot\left(\nabla\widehat{c}^n-\nabla\widehat{C}_H^n\right)+\left(u^n-U_h^n\right)\cdot\nabla\widehat{C}_H^n,z\right)\tag{59}$$
$$+\left(\frac{\partial g}{\partial\widehat{c}}\left(\widehat{C}_H^n\right)\left(\widehat{\zeta}^n+\widehat{\xi}^n\right),z\right)+\left(\frac{\partial^2 g}{2\partial\widehat{c}^2}(\delta_5)\left(\widehat{c}^n-\widehat{C}_H^n\right)^2,z\right),$$

where $\overline{\widehat{\xi}}^{n-1} = \widehat{\xi}^{n-1}(\overline{X}^{n-1}), \overline{\widehat{\zeta}}^{n-1} = \widehat{\zeta}^{n-1}(\overline{X}^{n-1})$, and $\overline{X}^{n-1} = x - (U_H^n/\phi(x))\Delta t$.

Taking the test function $z = \widehat{\xi}^n - \widehat{\xi}^{n-1} = d_t\widehat{\xi}^{n-1}\Delta t$ and summing over $1 \le n \le M$, we have

$$\begin{aligned}
&\sum_{n=1}^{M}\left(\phi d_t\widehat{\xi}^{n-1}, d_t\widehat{\xi}^{n-1}\right)\Delta t + \frac{1}{2}\left(E_c\nabla\widehat{\xi}^M, \nabla\widehat{\xi}^M\right) - \frac{1}{2}\left(E_c\nabla\widehat{\xi}^0, \nabla\widehat{\xi}^0\right)\\
&\le \sum_{n=1}^{M}\left(\phi\frac{\partial\widehat{c}^n}{\partial t} + u_H^n\cdot\nabla\widehat{c}^n - \phi\,\frac{\widehat{c}^n - \overline{\widehat{c}}^{n-1}}{\Delta t}, d_t\widehat{\xi}^{n-1}\right)\Delta t + \sum_{n=1}^{M}\left(\phi\,\frac{\overline{\widehat{\xi}}^{n-1} - \widehat{\xi}^{n-1}}{\Delta t}, d_t\widehat{\xi}^{n-1}\right)\Delta t\\
&\quad + \sum_{n=1}^{M}\left(\phi\,\frac{\widehat{\zeta}^n - \overline{\widehat{\zeta}}^{n-1}}{\Delta t}, d_t\widehat{\xi}^{n-1}\right)\Delta t + \sum_{n=1}^{M}\lambda\left(\widehat{\zeta}^n, d_t\widehat{\xi}^{n-1}\right)\Delta t\\
&\quad + \sum_{n=1}^{M}\left((u^n - U_H^n)\cdot\left(\nabla\widehat{c}^n - \nabla\widehat{C}_H^n\right) + (u^n - U_h^n)\cdot\nabla\widehat{C}_H^n, d_t\widehat{\xi}^{n-1}\right)\Delta t\\
&\quad + \sum_{n=1}^{M}\left(\frac{\partial g}{\partial\widehat{c}}\left(\widehat{C}_H^n\right)\left(\widehat{\zeta}^n + \widehat{\xi}^n\right) + \frac{\partial^2 g}{2\,\partial\widehat{c}^2}(\delta_4)\left(\widehat{c}^n - \widehat{C}_H^n\right)^2, d_t\widehat{\xi}^{n-1}\right)\Delta t.
\end{aligned} \tag{60}$$

Since

$$\begin{aligned}
\left\|\left(\widehat{c} - \widehat{C}_H\right)^2\right\| &\le \left\|\widehat{c} - \widehat{C}_H\right\|_{L^\infty}\left\|\widehat{c} - \widehat{C}_H\right\|_{L^2}\\
&\le \left(\|\widehat{c} - \widetilde{C}\|_{L^\infty} + \left\|\widetilde{C} - \widehat{C}_H\right\|_{L^\infty}\right)\left\|\left(\widehat{c} - \widehat{C}_H\right)\right\|_{L^2}\\
&\le C\left(|\ln H_c|H_c^{l+1} + |\ln H_c|\left(H_c^{l+1} + H_p^{k+1}\right)\right)\\
&\quad\times\left(H_c^{l+1} + H_p^{k+1}\right)\\
&\le C\left(H_c^{2l+2}|\ln H_c| + H_p^{2k+2}|\ln H_c|\right),
\end{aligned} \tag{61}$$

$$\begin{aligned}
\left\|(u - U_H)\left(\widehat{c} - \widehat{C}_H\right)\right\| &\le \|u - U_H\|\left\|\widehat{c} - \widehat{C}_H\right\|_{L^\infty}\\
&\le C\left(H_c^{l+1} + H_p^{k+1}\right)\\
&\quad\times\left(|\ln H_c|\left(H_c^{l+1} + H_p^{k+1}\right)\right)\\
&\le C\left(H_c^{2l+2}|\ln H_c| + H_p^{2k+2}|\ln H_c|\right),
\end{aligned} \tag{62}$$

$$\begin{aligned}
\left\|(u - U_H)\left(\nabla\widehat{c} - \nabla\widehat{C}_H\right)\right\| &\le \|u - U_H\|\left\|\widehat{c} - \widehat{C}_H\right\|_{1,\infty}\\
&\le C\left(H_c^{l+1} + H_p^{k+1}\right)\\
&\quad\times\left(H_c^{l} + H_c^{-1}\left(H_c^{l+1} + H_p^{k+1}\right)\right)\\
&\le C\left(H_c^{2l+1} + H_p^{2k+2}H_c^{-1}\right).
\end{aligned} \tag{63}$$

It follows from (61)–(63) that

$$\begin{aligned}
&\left\|\widehat{c} - \widehat{C}_h\right\|_{L^\infty(H^1)} + \left\|d_t\left(\widehat{c} - \widehat{C}_h\right)\right\|_{l^2(L^2)} + \|u - U_H\|_{L^\infty(L^2)}\\
&\quad + \|d_t(p - P_h)\|_{l^2(L^2)}\\
&\le C\left(\Delta t + h_c^l + h_p^{k+1} + H_c^{2l+1} + H_p^{2k+2}H_c^{-1}\right).
\end{aligned} \tag{64}$$

At last, we present the error results of the radionuclide equation and heat equation as follows:

$$\begin{aligned}
&\sum_{i=1}^{N}\|c_i - C_{ih}\|_{L^\infty(H^1)} + \sum_{i=1}^{N}\|d_t(c_i - C_{ih})\|_{l^2(L^2)}\\
&\le C\left(\Delta t + h_c^l + h_p^{k+1} + H_c^{2l+2}|\ln H_c| + H_p^{2k+2}|\ln H_c|\right),\\
&\|T - T_h\|_{L^\infty(H^1)} + \|d_t(T - T_h)\|_{l^2(L^2)}\\
&\le C\left(\Delta t + h_c^l + h_T^r + h_p^{k+1} + H_T^{2r+2}|\ln H_T| + H_c^{2l+2}|\ln H|_T + H_p^{2k+2}|\ln H_T|\right).
\end{aligned} \tag{65}$$

If setting $l = 1, r = 1, k = 1$, $h_c = h_T = h_p$, and $H_c = H_T = H_p$, which accords with the assumption (45), we can get the following theorem of fine grid scheme.

Theorem 3. *Define* $\{P_h^n, U_h^n, \widehat{C}_h^n, T_h^n, C_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$ *for* $n \geq 1$ *by system (48)–(52) and assume that the approximation properties (14)–(18) hold. Then, there exists a positive constant* C *independent of* h, H, *and* $\triangle t$, *such that*

$$\begin{aligned}
&\left\|\widehat{c} - \widehat{C}_h\right\|_{L^\infty(H^1)} + \left\|d_t\left(\widehat{c} - \widehat{C}_h\right)\right\|_{l^2(L^2)} + \left\|u - U_h\right\|_{L^\infty(L^2)} + \left\|d_t\left(p - P_h\right)\right\|_{l^2(L^2)} \\
&\quad + \sum_{i=1}^{N}\left\|c_i - C_{ih}\right\|_{L^\infty(H^1)} + \sum_{i=1}^{N}\left\|d_t\left(c_i - C_{ih}\right)\right\|_{l^2(L^2)} \\
&\quad + \left\|T - T_h\right\|_{L^\infty(H^1)} + \left\|d_t\left(T - T_h\right)\right\|_{l^2(L^2)} \\
&\leq C\left(\Delta t + h + H^3\right).
\end{aligned} \tag{66}$$

Theorem 4. *Define* $\{P_h^n, U_h^n, \widehat{C}_h^n, T_h^n, C_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$ *for* $n \geq 1$ *by system (48)–(52) and assume that the approximation properties (14)–(18) hold. Then, there exists a positive constant* C *independent of* h, H, *and* $\triangle t$, *such that*

$$\begin{aligned}
&\left\|\widehat{c} - \widehat{C}_h\right\|_{L^\infty(L^2)} + \left\|u - U_h\right\|_{l^2(L^2)} + \left\|p - P_h\right\|_{L^\infty(L^2)} \\
&\quad + \sum_{i=1}^{N}\left\|c_i - C_{ih}\right\|_{L^\infty(L^2)} + \left\|T - T_h\right\|_{L^\infty(L^2)} \\
&\leq C\left(\Delta t + h^2 + H^3\right).
\end{aligned} \tag{67}$$

4. Conclusions and Extensions

The two-grid method presented in this paper reduces the complexity of problem. It involves a small nonlinear system on a coarse grid of size H and a linear system on a fine grid of size h. It is shown that the coarse space can be extremely coarse and still achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H = O(h^{1/3})$ in H^1 norm. Compared with the implicit scheme, the two-grid method reduces CPU time. Moreover, the method is suitable to make the large-scale computation and long time duration. The future work is to use the discontinuous finite volume element method [14], block-centered finite difference method [15], higher-order finite volume [16], and SAV method [17, 18] to consider this problem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

The Fractional View Analysis of Polytropic Gas, Unsteady Flow System

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Generally, the differential equations of integer order do not properly model various phenomena in different areas of science and engineering as compared to differential equations of fractional order. The fractional-order differential equations provide the useful dynamics of the physical system and thus provide the innovative and effective information about the given physical system. Keeping in view the above properties of fractional calculus, the present article is related to the analytical solution of the time-fractional system of equations which describe the unsteady flow of polytropic gas dynamics. The present method provides the series form solution with easily computable components and a higher rate of convergence towards the targeted problem's exact solution. The present techniques are straightforward and effective for dealing with the solutions of fractional-order problems. The fractional derivatives are expressed in terms of the Caputo operator. The targeted problems' solutions are calculated using the Adomian decomposition method and variational iteration methods along with Shehu transformation. In the current procedures, we first applied the Shehu transform to reduce the problems into a more straightforward form and then implemented the decomposition and variational iteration methods to achieve the problems' comprehensive results. The solution of the nonlinear equations of unsteady flow of a polytropic gas at various fractional orders of the derivative is the core point of the present study. The solution of the proposed fractional model is plotted via two- and three-dimensional graphs. It is investigated that each problem's solution-graphs are best fitted with each other and with the exact solution. The convergence of fractional-order problems can be observed towards the solution of integer-order problems. Less computational time is the major attraction of the suggested methods. The present work will be considered a useful tool to handle the solution of fractional partial differential equations.

1. Introduction

In recent years, nonlinear fractional partial differential equations (FPDEs) have attracted researchers because of their useful applications in science and engineering [1–3]. The analysis of exact solutions to these nonlinear PDEs plays a very significant role in the Soliton theory since much of the information are provided on the description of the physical models, in the transmission of electrical signals, as a standard diffusion-wave equation, the transfer of neutrons by nuclear reactor, the theory of random walks, and so on [4–14].

In recent decades, many researchers have used different approaches to analyze the solutions of nonlinear PDEs, such as Laplace transform [15], Akbari–Ganji's method [16], homotopy analysis method [17], lattice Boltzmann method [18, 19], volume of fluid method [20, 21], Laplace homotopy analysis method [22, 23], Adomian decomposition technique [24–27], the variational iteration technique [28], Adams–Bashforth–Moulton algorithm [29], homotopy perturbation Sumudu transform method [30], the tan$h$ method [31], the sin$h$-cos$h$ method [32], finite difference method [33], the homotopy perturbation method [34], and the Laplace decomposition technique, to handle fractional-order Zakharov–Kuznetsov equations [35].

In the present study, we consider the gas-dynamic equations fractional-order scheme describing the evolution of an ideal gas's two-dimensional unsteady flow. The polytropic gas in astrophysics is given as follows [36]:

$$\psi = k\omega^{1+(1/m)}, \tag{1}$$

where $\psi = (\theta/\phi)$ is the energy density, ϕ is the container volume, θ is the total energy of the gas, m is the polytropic index, and k is a constant. Degenerate adiabatic gas and electron gas are two instances of such gases. In astrophysics and cosmology, the analysis of polytropic gases plays a critical role, and these gases can perform like dark energy [37]. Now consider the gas-dynamic equations scheme, which describes the evolution of unstable flow of a perfect gas with fractional derivatives [36, 38]:

$$\begin{aligned} &D_\eta^\delta \mu + \mu\frac{\partial\mu}{\partial\xi} + \nu\frac{\partial\mu}{\partial\zeta} + \frac{1}{\omega}\frac{\partial\psi}{\partial\xi} = 0,\\ &D_\eta^\delta \nu + \mu\frac{\partial\nu}{\partial\xi} + \nu\frac{\partial\nu}{\partial\zeta} + \frac{1}{\omega}\frac{\partial\psi}{\partial\zeta} = 0,\\ &D_\eta^\delta \omega + \mu\frac{\partial\omega}{\partial\xi} + \nu\frac{\partial\omega}{\partial\zeta} + \omega\left(\frac{\partial\mu}{\partial\xi} + \frac{\partial\nu}{\partial\zeta}\right) = 0,\\ &D_\eta^\delta \psi + \mu\frac{\partial\psi}{\partial\xi} + \nu\frac{\partial\psi}{\partial\zeta} + \tau\psi\left(\frac{\partial\mu}{\partial\xi} + \frac{\partial\nu}{\partial\zeta}\right) = 0, \end{aligned} \tag{2}$$

with initial conditions

$$\begin{aligned} \mu(\xi,\zeta,0) &= \alpha(\xi+\zeta),\\ \nu(\xi,\zeta,0) &= \beta(\xi+\zeta),\\ \omega(\xi,\zeta,0) &= \gamma(\xi+\zeta),\\ \psi(\xi,\zeta,0) &= \Phi(\xi+\zeta), \end{aligned} \tag{3}$$

where $\mu(\xi,\zeta,\eta)$ and $\nu(\xi,\zeta,\eta)$ are the velocity components, $\omega(\xi,\zeta,\eta)$ is the density, $\psi(\xi,\zeta,\eta)$ is the pressure, and τ is the ratio of the specific heat and it represents the adiabatic index. In past decade, the appropriate analytical results of several distinct types of gas-dynamic equations are achieved using many analytical and numerical methods. Various methods have been solved by a gas-dynamic model such as fractional reduced differential transform method [39], Elzaki transform homotopy perturbation method [40], q-homotopy analysis method [36], Adomian decomposition method [41], fractional homotopy analysis transform method [38], and natural decomposition method [42].

The variational iteration transform method (VITM) combines the variational iteration method and the Shehu transform. Many researchers commonly used this technique to solve linear and nonlinear models [43–45]. The method provides a reliable and effective procedure for a broad range of science. VIM does not need discretization, linearization, or perturbation. It provides quick convergence and successive approximations of the exact result [46–48]. Various equations solve the variational iteration method with the help of different transformations, such as Kuramoto–Sivashinsky equations [49] and fourth-order parabolic partial differential equation [45].

The ADM is an efficient and accurate technique that was suggested initially to solve analytically frontier physical models [50]. Since then, ADM has been implemented in nonlinear ODEs and PDEs without using perturbation or linearization procedure. The Shehu decomposition method (SDM) is a mixture of ADM and Shehu transform [51–54].

The motivation and novelty of the current research work are to modify the ADM and VIM along with Shehu transformation to investigate the solution of a nonlinear system of nonlinear FPDEs of unsteady flow of polytropic gas-dynamics equations. Besides the nonlinear system of four equations, the given problem's solution is calculated by an effortless and straightforward procedure. A higher degree of accuracy is achieved with a tiny number of calculations. The fractional-order solutions are achieved with some graphical justifications. The visual representation has confirmed the effectiveness and applicability of the suggested techniques. In the future, the proposed techniques are preferred to solve other nonlinear FPDEs that frequently arise in science and engineering.

2. Preliminaries Concepts

2.1. Definition 1. The Riemann–Liouville fractional integral is given as follows [55, 56]:

$$\begin{aligned} &I_0^\delta h(\eta) = \frac{1}{\Gamma(\delta)}\int_0^\eta (\eta - s)^{\delta-1} h(s)\mathrm{d}s, \quad \delta > 0, \eta > 0,\\ &h(\eta), \quad \delta \end{aligned} \tag{4}$$

2.2. Definition 2. The Caputo's fractional-order derivative of $h(\eta)$ is defined as follows [55, 56]:

$$\begin{aligned} D_\eta^\delta h(\eta) &= I^{n-\delta} f^n, n-1 < \delta < n, \quad n \in \mathbb{N}\\ &\frac{d^n}{\mathrm{d}\eta_n} h(\eta), \quad \delta = n, n \in \mathbb{N}. \end{aligned} \tag{5}$$

2.3. Definition 3. The Shehu transformation is the new and modern transformation which is described for exponential-order functions. In set A, we take a function represented as follows [51, 52, 57, 58]:

$$\begin{aligned} A = \{\nu(\eta)\colon \exists, \rho_1, \rho_2 > 0, |\nu(\eta)| < \mathrm{M}e^{(|\eta|/\rho_i)},\\ \text{if } \eta \in [0,\infty). \end{aligned} \tag{6}$$

The Shehu transform which is given by $S(.)$ for a function $\nu(\eta)$ is defined as

$$\begin{aligned} S\{\nu(\eta)\} = V(s,\mu) = \int_0^\infty e^{(-s\eta/\mu)}\nu(\eta)\mathrm{d}\eta,\\ \eta > 0,\ s > 0. \end{aligned} \tag{7}$$

The Shehu transform of a function $\nu(\eta)$ is $V(s,\mu)$, and then, $\nu(\eta)$ is called the inverse of $V(s,\mu)$, which is given as

$$S^{-1}\{V(s,\mu)\} = \nu(\eta), \text{ for } \eta \geq 0, \quad S^{-1} \text{ is inverse Shehu transformation.} \tag{8}$$

2.4. Definition 4. Shehu transform for nth derivatives is given as follows [51, 52, 57, 58]:

$$S\left\{\nu^{(n)}(\eta)\right\} = \frac{s^n}{u^n}V(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k-1}\nu^{(k)}(0). \tag{9}$$

2.5. Definition 5. Shehu transform for fractional-order derivatives [51, 52, 57, 58]:

$$S\left\{\nu^{(\delta)}(\eta)\right\} = \frac{s^\delta}{u^\delta}V(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\delta-k-1}\nu^{(k)}(0), \quad 0<\beta\leq n. \tag{10}$$

2.6. Definition 6. The Mittag–Leffler function denoted by $E_\delta(z)$ for $\delta>0$ is defined as

$$E_\delta(z) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\delta m+1)}\delta>0, \quad z\in\mathbb{C}. \tag{11}$$

3. The Procedure of VITM

This section describes the VITM solution, the system of FPDEs:

$$\begin{aligned}
&D_\eta^\delta\mu(\xi,\zeta,\eta) + \overline{\mathscr{G}}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu) - \mathscr{G}_1(\xi,\zeta,\eta) = 0,\\
&D_\eta^\delta\nu(\xi,\zeta,\eta) + \overline{\mathscr{G}}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu) - \mathscr{G}_2(\xi,\zeta,\eta) = 0,\\
&\qquad m-1<\delta\leq m,
\end{aligned} \tag{12}$$

with initial conditions

$$\begin{aligned}
\mu(\xi,\zeta,0) &= g_1(\xi,\zeta),\\
\nu(\xi,\zeta,0) &= g_2(\xi,\zeta),
\end{aligned} \tag{13}$$

where $D_\eta^\delta = (\partial^\delta/\partial\eta^\delta)$ is the Caputo fractional derivative of order δ; $\overline{\mathscr{G}}_1$, $\overline{\mathscr{G}}_2$ and $\mathscr{N}_1$, $\mathscr{N}_2$ are linear and nonlinear functions, respectively; and $\mathscr{G}_1,\mathscr{G}_2$ are source operators.

The Shehu transformation is applied to equation (1):

$$\begin{aligned}
&S\left[D_\eta^\delta\mu(\xi,\zeta,\eta)\right] + S\left[\overline{\mathscr{G}}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu) - \mathscr{G}_1(\xi,\zeta,\eta)\right] = 0,\\
&S\left[D_\eta^\delta\nu(\xi,\zeta,\eta)\right] + S\left[\overline{\mathscr{G}}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu) - \mathscr{G}_2(\xi,\zeta,\eta)\right] = 0.
\end{aligned} \tag{14}$$

Applying the differentiation property of Shehu transform, we have

$$\begin{aligned}
&S[\mu(\xi,\zeta,\eta)] - \sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0}\\
&= -S\left[\overline{\mathscr{G}}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu) - \mathscr{G}_1(\xi,\zeta,\eta)\right],\\
&S[\nu(\xi,\zeta,\eta)] - \sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\nu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0}\\
&= -S\left[\overline{\mathscr{G}}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu) - \mathscr{G}_2(\xi,\zeta,\eta)\right].
\end{aligned} \tag{15}$$

The iteration method for equation (15) may be utilized to indicate the major iterative scheme requiring the Lagrange multiplier as

$$\begin{aligned}
&S[\mu_{m+1}(\xi,\zeta,\eta)] = S[\mu_m(\xi,\zeta,\eta)] +\\
&\lambda(s)\left[\frac{s^\delta}{u^\delta}\mu_m(\xi,\zeta,\eta) - \sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0}\right.\\
&\left.-S[\mathscr{G}_1(\xi,\zeta,\eta)] - S\left\{\overline{\mathscr{G}}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu)\right\}\right],\\
&S[\nu_{m+1}(\xi,\zeta,\eta)] = S[\nu_m(\xi,\zeta,\eta)] + \lambda(s)\left[\frac{s^\delta}{u^\delta}\right.\\
&\nu_m(\xi,\zeta,\eta) - \sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\nu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0}\\
&\left.-S[\mathscr{G}_2(\xi,\zeta,\eta)] - S\left\{\overline{\mathscr{G}}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu)\right\}\right].
\end{aligned} \tag{16}$$

A Lagrange multiplier

$$\lambda(s) = -\frac{u^\delta}{s^\delta}, \tag{17}$$

using inverse Shehu transformation S^{-1}, and equation (16) can be written as

$$\begin{aligned}
&\mu_{m+1}(\xi,\zeta,\eta)\\
&= \mu_m(\xi,\zeta,\eta) - S^{-1}\left[\frac{u^\delta}{s^\delta}\left[\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0}\right.\right.\\
&\left.\left.-S[\mathscr{G}_1(\xi,\zeta,\eta)] - S\left\{\overline{\mathscr{G}}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu)\right\}\right]\right],\\
&\nu_{m+1}(\xi,\zeta,\eta)\\
&= \nu_m(\xi,\zeta,\eta) - S^{-1}\left[\frac{u^\delta}{s^\delta}\left[\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\nu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0}\right.\right.\\
&\left.\left.-S[\mathscr{G}_2(\xi,\zeta,\eta)] - S\left\{\overline{\mathscr{G}}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu)\right\}\right]\right].
\end{aligned} \tag{18}$$

The initial value can be found as

$$\mu_0(\xi,\zeta,\eta) = S^{-1}\left[\frac{u^\delta}{s^\delta}\left\{\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0}\right\}\right],$$

$$\nu_0(\xi,\zeta,\eta) = S^{-1}\left[\frac{u^\delta}{s^\delta}\left\{\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\nu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0}\right\}\right]. \tag{19}$$

The converge of this technique is as follows [59, 60].

4. The Procedure of SDM

In this section, we discuss the SDM solution for system of FPDEs:

$$D_\eta^\delta\mu(\xi,\zeta,\eta) + \overline{\mathscr{G}}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu) - \mathscr{G}_1(\xi,\zeta,\eta) = 0,$$
$$D_\eta^\delta\nu(\xi,\zeta,\eta) + \overline{\mathscr{G}}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu) - \mathscr{G}_2(\xi,\zeta,\eta) = 0,$$
$$m-1<\delta\le m, \tag{20}$$

with initial conditions

$$\mu(\xi,\zeta,0) = g_1(\xi,\zeta),$$
$$\nu(\xi,\zeta,0) = g_2(\xi,\zeta), \tag{21}$$

where $D_\eta^\delta = (\partial^\delta/\partial\eta^\delta)$ is the Caputo fractional derivative of order δ; $\overline{\mathscr{G}}_1$, $\overline{\mathscr{G}}_2$ and $\mathscr{N}_1$, $\mathscr{N}_2$ are linear and nonlinear functions, respectively; and $\mathscr{G}_1, \mathscr{G}_2$ are source functions.

Apply Shehu transform to equation (20):

$$S\left[D_\eta^\delta\mu(\xi,\zeta,\eta)\right] + S\left[\overline{\mathscr{G}}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu) - \mathscr{G}_1(\xi,\zeta,\eta)\right] = 0,$$
$$S\left[D_\eta^\delta\nu(\xi,\zeta,\eta)\right] + S\left[\overline{\mathscr{G}}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu) - \mathscr{G}_2(\xi,\zeta,\eta)\right] = 0. \tag{22}$$

Applying the differentiation property of Shehu transform, we have

$$S[\mu(\xi,\zeta,\eta)] = \frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0} + \frac{u^\delta}{s^\delta}S\left[\mathscr{G}_1(\xi,\zeta,\eta)\right] - \frac{u^\delta}{s^\delta}S\left\{\overline{\mathscr{G}}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu)\right\}],$$

$$S[\nu(\xi,\zeta,\eta)] = \frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\nu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0} + \frac{u^\delta}{s^\delta}S\left[\mathscr{G}_2(\xi,\zeta,\eta)\right] - \frac{u^\delta}{s^\delta}S\left\{\overline{\mathscr{G}}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu)\right\}]. \tag{23}$$

SDM solution of infinite series $\mu(\xi,\zeta,\eta)$ and $\nu(\xi,\zeta,\eta)$ is as follows:

$$\mu(\xi,\zeta,\eta) = \sum_{m=0}^{\infty}\mu_m(\xi,\zeta,\eta),$$
$$\nu(\xi,\zeta,\eta) = \sum_{m=0}^{\infty}\nu_m(\xi,\zeta,\eta). \tag{24}$$

Adomian polynomials of nonlinear terms $\mathscr{N}_1$ and $\mathscr{N}_2$ are given as

$$\mathscr{N}_1(\mu,\nu) = \sum_{m=0}^{\infty}\mathscr{A}_m,$$
$$\mathscr{N}_2(\mu,\nu) = \sum_{m=0}^{\infty}\mathscr{B}_m. \tag{25}$$

The nonlinear of Adomian polynomials can be defined as

$$\mathscr{A}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial\lambda^m}\left\{\mathscr{N}_1\left(\sum_{k=0}^{\infty}\lambda^k\mu_k, \sum_{k=0}^{\infty}\lambda^k\nu_k\right)\right\}\right]_{\lambda=0},$$

$$\mathscr{B}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial\lambda^m}\left\{\mathscr{N}_2\left(\sum_{k=0}^{\infty}\lambda^k\mu_k, \sum_{k=0}^{\infty}\lambda^k\nu_k\right)\right\}\right]_{\lambda=0}. \tag{26}$$

Substituting equation (24) and equation (25) into (23) gives

$$S\left[\sum_{m=0}^{\infty}\mu_m(\xi,\zeta,\eta)\right] = \frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0} + \frac{u^\delta}{s^\delta}S\{\mathscr{G}_1(\xi,\zeta,\eta)\} - \frac{u^\delta}{s^\delta}S\left\{\overline{\mathscr{G}}_1\left(\sum_{m=0}^{\infty}\mu_m, \sum_{m=0}^{\infty}\nu_m\right) + \sum_{m=0}^{\infty}\mathscr{A}_m\right\},$$

$$S\left[\sum_{m=0}^{\infty}\nu_m(\xi,\zeta,\eta)\right] = \frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\nu(\xi,\zeta,\eta)}{\partial^k\eta}\Big|_{\eta=0} + \frac{u^\delta}{s^\delta}S\{\mathscr{G}_2(\xi,\zeta,\eta)\} - \frac{u^\delta}{s^\delta}S\left\{\overline{\mathscr{G}}_2\left(\sum_{m=0}^{\infty}\mu_m, \sum_{m=0}^{\infty}\nu_m\right) + \sum_{m=0}^{\infty}\mathscr{B}_m\right\}. \tag{27}$$

Applying the inverse Shehu transformation to equation (20), we get

$$\sum_{m=0}^{\infty}\mu_m(\xi,\zeta,\eta)$$

$$\{\mathcal{G}_1(\xi,\zeta,\eta)\} - \frac{u^\delta}{s^\delta}S\left\{\overline{\mathcal{G}}_1\left(\sum_{m=0}^{\infty}\mu_m,\sum_{m=0}^{\infty}\nu_m\right)+\sum_{m=0}^{\infty}\mathcal{A}_m\right\}\Bigg],$$

$$\sum_{m=0}^{\infty}\nu_m(\xi,\zeta,\eta) = S^{-1}\left[\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\nu(\xi,\zeta,\eta)}{\partial^k\eta}|_{\eta=0}+\frac{u^\delta}{s^\delta}S\right.$$

$$\left.\{\mathcal{G}_2(\xi,\zeta,\eta)\} - \frac{u^\delta}{s^\delta}S\left\{\overline{\mathcal{G}}_2\left(\sum_{m=0}^{\infty}\mu_m,\sum_{m=0}^{\infty}\nu_m\right)+\sum_{m=0}^{\infty}\mathcal{B}_m\right\}\right]. \tag{28}$$

We define the following terms:

$$\begin{aligned}\mu_0(\xi,\zeta,\eta) &= S^{-1}\left[\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\xi,\zeta,\eta)}{\partial^k\eta}|_{\eta=0}\right.\\&\left.+\frac{u^\delta}{s^\delta}S\{\mathcal{G}_1(\xi,\zeta,\eta)\}\right],\\ \nu_0(\xi,\zeta,\eta) &= S^{-1}\left[\frac{u^\delta}{s^\delta}\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\nu(\xi,\zeta,\eta)}{\partial^k\eta}|_{\eta=0}\right.\\&\left.+\frac{u^\delta}{s^\delta}S\{\mathcal{G}_2(\xi,\zeta,\eta)\}\right],\\ \mu_1(\xi,\zeta,\eta) &= -S^{-1}\left[\frac{u^\delta}{s^\delta}S\{\overline{\mathcal{G}}_1(\mu_0,\nu_0)+\mathcal{A}_0\}\right],\\ \nu_1(\xi,\zeta,\eta) &= -S^{-1}\left[\frac{u^\delta}{s^\delta}S\{\overline{\mathcal{G}}_2(\mu_0,\nu_0)+\mathcal{B}_0\}\right],\end{aligned} \tag{29}$$

in general for $m \geq 1$, and we have

$$\begin{aligned}\mu_{m+1}(\xi,\zeta,\eta) &= -S^{-1}\left[\frac{u^\delta}{s^\delta}S\{\overline{\mathcal{G}}_1(\mu_m,\nu_m)+\mathcal{A}_m\}\right],\\ \nu_{m+1}(\xi,\zeta,\eta) &= -S^{-1}\left[\frac{u^\delta}{s^\delta}S\{\overline{\mathcal{G}}_2(\mu_m,\nu_m)+\mathcal{B}_m\}\right].\end{aligned} \tag{30}$$

5. Implementation of the Methods

Example 1. Consider fractional-order system of nonlinear equations of unsteady flow of a polytropic gas [36, 38]:

$$\begin{aligned}D_\eta^\delta\mu + \mu\frac{\partial\mu}{\partial\xi}+\nu\frac{\partial\mu}{\partial\zeta}+\frac{1}{\omega}\frac{\partial\psi}{\partial\xi} &= 0,\\ D_\eta^\delta\nu + \mu\frac{\partial\nu}{\partial\xi}+\nu\frac{\partial\nu}{\partial\zeta}+\frac{1}{\omega}\frac{\partial\psi}{\partial\zeta} &= 0,\\ D_\eta^\delta\omega + \mu\frac{\partial\omega}{\partial\xi}+\nu\frac{\partial\omega}{\partial\zeta}+\omega\left(\frac{\partial\mu}{\partial\xi}+\frac{\partial\nu}{\partial\zeta}\right) &= 0,\\ D_\eta^\delta\psi + \mu\frac{\partial\psi}{\partial\xi}+\nu\frac{\partial\psi}{\partial\zeta}+\tau\psi\left(\frac{\partial\mu}{\partial\xi}+\frac{\partial\nu}{\partial\zeta}\right) &= 0,\end{aligned} \tag{31}$$

with initial conditions

$$\begin{aligned}\mu(\xi,\zeta,0) &= e^{\xi+\zeta},\\ \nu(\xi,\zeta,0) &= -1-e^{\xi+\zeta},\\ \omega(\xi,\zeta,0) &= e^{\xi+\zeta},\\ \psi(\xi,\zeta,0) &= c,\end{aligned} \tag{32}$$

where c is the real constant.

First, SDM is used to solve equation (31).

For this applying Shehu transformation to equation (31),

$$\begin{aligned}S\left\{\frac{\partial^\delta\mu}{\partial\eta^\delta}\right\} &= S\left[-\left\{\mu\frac{\partial\mu}{\partial\xi}+\nu\frac{\partial\mu}{\partial\zeta}+\frac{1}{\omega}\frac{\partial\psi}{\partial\xi}\right\}\right],\\ S\left\{\frac{\partial^\delta\nu}{\partial\eta^\delta}\right\} &= S\left[-\left\{\mu\frac{\partial\nu}{\partial\xi}+\nu\frac{\partial\nu}{\partial\zeta}+\frac{1}{\omega}\frac{\partial\psi}{\partial\zeta}\right\}\right],\\ S\left\{\frac{\partial^\delta\omega}{\partial\eta^\delta}\right\} &= S\left[-\left\{\mu\frac{\partial\omega}{\partial\xi}+\nu\frac{\partial\omega}{\partial\zeta}+\omega\left(\frac{\partial\mu}{\partial\xi}+\frac{\partial\nu}{\partial\zeta}\right),\right\}\right]\\ S\left\{\frac{\partial^\delta\psi}{\partial\eta^\delta}\right\} &= S\left[-\left\{\mu\frac{\partial\psi}{\partial\xi}+\nu\frac{\partial\psi}{\partial\zeta}+\tau\psi\left(\frac{\partial\mu}{\partial\xi}+\frac{\partial\nu}{\partial\zeta}\right),\right\}\right]\end{aligned}$$

$$\begin{aligned}&\frac{s^\delta}{u^\delta}S\{\mu(\xi,\zeta,\eta)\}-\frac{s^{\delta-1}}{u^\delta}\mu(\xi,\zeta,0)\\ &= S\left[-\left\{\mu\frac{\partial\mu}{\partial\xi}+\nu\frac{\partial\mu}{\partial\zeta}+\frac{1}{\omega}\frac{\partial\psi}{\partial\xi}\right\}\right],\\ &\frac{s^\delta}{u^\delta}S\{\nu(\xi,\zeta,\eta)\}-\frac{s^{\delta-1}}{u^\delta}\nu(\xi,\zeta,0)\\ &= S\left[-\left\{\mu\frac{\partial\nu}{\partial\xi}+\nu\frac{\partial\nu}{\partial\zeta}+\frac{1}{\omega}\frac{\partial\psi}{\partial\zeta}\right\}\right],\\ &\frac{s^\delta}{u^\delta}S\{\omega(\xi,\zeta,\eta)\}-\frac{s^{\delta-1}}{u^\delta}\omega(\xi,\zeta,0)\\ &= S\left[-\left\{\mu\frac{\partial\omega}{\partial\xi}+\nu\frac{\partial\omega}{\partial\zeta}+\omega\left(\frac{\partial\mu}{\partial\xi}+\frac{\partial\nu}{\partial\zeta}\right)\right\}\right],\\ &\frac{s^\delta}{u^\delta}S\{\psi(\xi,\zeta,\eta)\}-\frac{s^{\delta-1}}{u^\delta}\psi(\xi,\zeta,0)\\ &= S\left[-\left\{\mu\frac{\partial\psi}{\partial\xi}+\nu\frac{\partial\psi}{\partial\zeta}+\tau\psi\left(\frac{\partial\mu}{\partial\xi}+\frac{\partial\nu}{\partial\zeta}\right)\right\}\right].\end{aligned} \tag{33}$$

The above algorithm is reduced to simplified form:

$$\begin{aligned}
S\{\mu(\xi,\zeta,\eta)\} &= \frac{1}{s}\{\mu(\xi,\zeta,0)\} - \frac{u^{\delta}}{s^{\delta}} S\left[\left\{\mu\frac{\partial\mu}{\partial\xi} + \nu\frac{\partial\mu}{\partial\zeta} + \frac{1}{\omega}\frac{\partial\psi}{\partial\xi}\right\}\right],\\
S\{\nu(\xi,\zeta,\eta)\} &= \frac{1}{s}\{\nu(\xi,\zeta,0)\} - \frac{u^{\delta}}{s^{\delta}} S\left[\left\{\mu\frac{\partial\nu}{\partial\xi} + \nu\frac{\partial\nu}{\partial\zeta} + \frac{1}{\omega}\frac{\partial\psi}{\partial\zeta}\right\}\right],\\
S\{\omega(\xi,\zeta,\eta)\} &= \frac{1}{s}\{\omega(\xi,\zeta,0)\} - \frac{u^{\delta}}{s^{\delta}} S\left[\left\{\mu\frac{\partial\omega}{\partial\xi} + \nu\frac{\partial\omega}{\partial\zeta} + \omega\left(\frac{\partial\mu}{\partial\xi} + \frac{\partial\nu}{\partial\zeta}\right)\right\}\right],\\
S\{\psi(\xi,\zeta,\eta)\} &= \frac{1}{s}\{\psi(\xi,\zeta,0)\} - \frac{u^{\delta}}{s^{\delta}} S\left[\left\{\mu\frac{\partial\psi}{\partial\xi} + \nu\frac{\partial\psi}{\partial\zeta} + \tau\psi\left(\frac{\partial\mu}{\partial\xi} + \frac{\partial\nu}{\partial\zeta}\right)\right\}\right].
\end{aligned} \tag{34}$$

Applying inverse Shehu transformation, we get

$$\begin{aligned}
\mu(\xi,\zeta,\eta) &= \mu(\xi,\zeta,0) - S^{-1}\left[\frac{u^{\delta}}{s^{\delta}} S\left\{\mu\frac{\partial\mu}{\partial\xi} + \nu\frac{\partial\mu}{\partial\zeta} + \frac{1}{\omega}\frac{\partial\psi}{\partial\xi}\right\}\right],\\
\nu(\xi,\zeta,\eta) &= \nu(\xi,\zeta,0) - S^{-1}\left[\frac{u^{\delta}}{s^{\delta}} S\left\{\mu\frac{\partial\nu}{\partial\xi} + \nu\frac{\partial\nu}{\partial\zeta} + \frac{1}{\omega}\frac{\partial\psi}{\partial\zeta}\right\}\right],\\
\omega(\xi,\zeta,\eta) &= \omega(\xi,\zeta,0) - S^{-1}\left[\frac{u^{\delta}}{s^{\delta}} S\left\{\mu\frac{\partial\omega}{\partial\xi} + \nu\frac{\partial\omega}{\partial\zeta} + \omega\left(\frac{\partial\mu}{\partial\xi} + \frac{\partial\nu}{\partial\zeta}\right)\right\}\right],\\
\psi(\xi,\zeta,\eta) &= \psi(\xi,\zeta,0) - S^{-1}\left[\frac{u^{\delta}}{s^{\delta}} S\left\{\mu\frac{\partial\psi}{\partial\xi} + \nu\frac{\partial\psi}{\partial\zeta} + \tau\psi\left(\frac{\partial\mu}{\partial\xi} + \frac{\partial\nu}{\partial\zeta}\right)\right\}\right].
\end{aligned} \tag{35}$$

Equation (35) can be written in an operator form as

$$\begin{aligned}
\mu(\xi,\zeta,\eta) &= \mu(\xi,\zeta,0) - S^{-1}\left[\frac{u^{\delta}}{s^{\delta}} S \left\{A_1(\mu,\mu_\xi) + B_1(\nu,\mu_\zeta) + C_1(\omega,\psi_\xi)\right\}\right],\\
\nu(\xi,\zeta,\eta) &= \nu(\xi,\zeta,0) - S^{-1}\left[\frac{u^{\delta}}{s^{\delta}} S \left\{A_2(\mu,\nu_\xi) + B_2(\nu,\nu_\zeta) + C_2(\omega,\psi_\zeta)\right\}\right],\\
\omega(\xi,\zeta,\eta) &= \omega(\xi,\zeta,0) - S^{-1}\left[\frac{u^{\delta}}{s^{\delta}} S\left\{A_3(\mu,\omega_\xi) + B_3(\nu,\omega_\zeta) + C_3(\omega,\mu_\xi) + D_3(\omega,\nu_\zeta)\right\}\right],\\
\psi(\xi,\zeta,\eta) &= \psi(\xi,\zeta,0) - S^{-1}\left[\frac{u^{\delta}}{s^{\delta}} S\left\{A_4(\mu,\psi_\xi) + B_4(\nu,\psi_\zeta) + \tau C_4(\psi,\mu_\xi) + \tau D_4(\psi,\nu_\zeta)\right\}\right].
\end{aligned} \tag{36}$$

Assume that the unknown functions $\mu(\xi,\zeta,\eta)$, $\nu(\xi,\zeta,\eta)$, $\omega(\xi,\zeta,\eta)$, and $\psi(\xi,\zeta,\eta)$ have infinite series solution as follows:

$$\begin{aligned}
\mu(\xi,\zeta,\eta) &= \sum_{m=0}^{\infty}\mu_m(\xi,\zeta,\eta),\\
\nu(\xi,\zeta,\eta) &= \sum_{m=0}^{\infty}\nu_m(\xi,\zeta,\eta),\\
\omega(\xi,\zeta,\eta) &= \sum_{m=0}^{\infty}\omega_m(\xi,\zeta,\eta),\\
\psi(\xi,\zeta,\eta) &= \sum_{m=0}^{\infty}\psi_m(\xi,\zeta,\eta).
\end{aligned} \tag{37}$$

All forms of nonlinear Adomian polynomials can be defined as

$$
\begin{aligned}
A_1(\mu,\mu_\xi) &= \mu_0\mu_{0\xi} + (\mu_1\mu_{0\xi} + \mu_0\mu_{1\xi}) + \cdots,\\
B_1(\nu,\mu_\zeta) &= \nu_0\mu_{0\xi} + (\nu_1\mu_{0\xi} + \nu_0\mu_{1\xi}) + \cdots,\\
C_1(\omega,\psi_\xi) &= \frac{\psi_{0\xi}}{\omega_0} + \frac{\omega_0\psi_{1\xi} - \omega_1\psi_{0\xi}}{\omega_0} + \cdots,\\
A_2(\mu,\nu_\xi) &= \mu_0\nu_{0\xi} + (\mu_1\nu_{0\xi} + \mu_0\nu_{1\xi}) + \cdots,\\
B_2(\nu,\nu_\zeta) &= \nu_0\nu_{0\xi} + (\nu_1\nu_{0\xi} + \nu_0\nu_{1\xi}) + \cdots,\\
C_2(\omega,\psi_\zeta) &= \frac{\psi_{0\zeta}}{\omega_0} + \frac{\omega_0\psi_{1\zeta} - \omega_1\psi_{0\zeta}}{\omega_0} + \cdots,\\
A_3(\mu,\omega_\xi) &= \mu_0\omega_{0\xi} + (\mu_1\omega_{0\xi} + \mu_0\omega_{1\xi}) + \cdots,\\
B_3(\nu,\omega_\zeta) &= \nu_0\omega_{0\xi} + (\nu_1\omega_{0\xi} + \nu_0\omega_{1\xi}) + \cdots,\\
C_3(\omega,\mu_\xi) &= \omega_0\mu_{0\xi} + (\omega_1\mu_{0\xi} + \omega_0\mu_{1\xi}) + \cdots,\\
D_3(\omega,\nu_\zeta) &= \omega_0\nu_{0\zeta} + (\omega_1\nu_{0\zeta} + \omega_0\nu_{1\zeta}) + \cdots,\\
A_4(\mu,\psi_\xi) &= \mu_0\psi_{0\xi} + (\mu_1\psi_{0\xi} + \mu_0\psi_{1\xi}) + \cdots,\\
C_4(\psi,\mu_\xi) &= \psi_0\mu_{0\xi} + (\psi_1\mu_{0\xi} + \psi_0\mu_{1\xi}) + \cdots,\\
D_4(\psi,\nu_\zeta) &= \psi_0\nu_{0\zeta} + (\psi_1\nu_{0\zeta} + \psi_0\nu_{1\zeta}) + \cdots.
\end{aligned}
\tag{38}
$$

The initial sources are

$$
\begin{aligned}
&\mu_0(\xi,\zeta,\eta) = e^{\xi+\zeta},\\
&\nu_0(\xi,\zeta,\eta) = -1 - e^{\xi+\zeta},\\
&\omega_0(\xi,\zeta,\eta) = e^{\xi+\zeta},\\
&\psi_0(\xi,\zeta,0) = c.\\
&\mu_m(\xi,\zeta,\eta) = -S^{-1}\left[\frac{u^\delta}{s^\delta}S \cdot \left\{A_1(\mu,\mu_\xi) + B_1(\nu,\mu_\zeta) + C_1(\omega,\psi_\xi)\right\}\right],\\
&\nu_m(\xi,\zeta,\eta) = -S^{-1}\left[\frac{u^\delta}{s^\delta}S \cdot \left\{A_2(\mu,\nu_\xi) + B_2(\nu,\nu_\zeta) + C_2(\omega,\psi_\zeta)\right\}\right],\\
&\omega_m(\xi,\zeta,\eta) = -S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{A_3(\mu,\omega_\xi) + B_3(\nu,\omega_\zeta) + C_3(\omega,\mu_\xi) + D_3(\omega,\nu_\zeta)\right\}\right],\\
&\psi_m(\xi,\zeta,\eta) = -S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{A_4(\mu,\psi_\xi) + B_4(\nu,\psi_\zeta) + \tau C_4(\psi,\mu_\xi) + \tau D_4(\psi,\nu_\zeta)\right\}\right].
\end{aligned}
\tag{39}
$$

For $m = 1$,

$$
\begin{aligned}
\mu_1(\xi,\zeta,\eta) &= e^{\xi+\zeta}\frac{\eta^\delta}{\Gamma(\delta+1)},\\
\nu_1(\xi,\zeta,\eta) &= -e^{\xi+\zeta}\frac{\eta^\delta}{\Gamma(\delta+1)},\\
\omega_1(\xi,\zeta,\eta) &= e^{\xi+\zeta}\frac{\eta^\delta}{\Gamma(\delta+1)},\\
\psi_1(\xi,\zeta,0) &= 0.
\end{aligned}
\tag{40}
$$

For $m = 2$,

$$
\begin{aligned}
\mu_2(\xi,\zeta,\eta) &= e^{\xi+\zeta}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)},\\
\nu_2(\xi,\zeta,\eta) &= -e^{\xi+\zeta}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)},\\
\omega_2(\xi,\zeta,\eta) &= e^{\xi+\zeta}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)},\\
\psi_2(\xi,\zeta,0) &= 0.
\end{aligned}
\tag{41}
$$

For $m = 3$,

$$
\begin{aligned}
\mu_3(\xi,\zeta,\eta) &= e^{\xi+\zeta}\frac{\eta^{3\delta}}{\Gamma(3\delta+1)},\\
\nu_3(\xi,\zeta,\eta) &= -e^{\xi+\zeta}\frac{\eta^{3\delta}}{\Gamma(3\delta+1)},\\
\omega_3(\xi,\zeta,\eta) &= e^{\xi+\zeta}\frac{\eta^{3\delta}}{\Gamma(3\delta+1)},\\
\psi_3(\xi,\zeta,0) &= 0.\\
\mu_m(\xi,\zeta,\eta) &= \mu_0(\xi,\zeta,\eta) + \mu_1(\xi,\zeta,\eta) + \cdots,\\
\nu_m(\xi,\zeta,\eta) &= \nu_0(\xi,\zeta,\eta) + \nu_1(\xi,\zeta,\eta) + \cdots,\\
\nu_m(\xi,\zeta,\eta) &= \nu_0(\xi,\zeta,\eta) + \nu_1(\xi,\zeta,\eta) + \cdots,\\
\omega_m(\xi,\zeta,\eta) &= \omega_0(\xi,\zeta,\eta) + \omega_1(\xi,\zeta,\eta) + \cdots,\\
\psi_m(\xi,\zeta,0) &= \psi_0(\xi,\zeta,\eta) + \psi_1(\xi,\zeta,\eta) + \cdots,\\
\mu(\xi,\zeta,\eta) &= e^{\xi+\zeta} + e^{\xi+\zeta}\frac{\eta^\delta}{\Gamma(\delta+1)} + e^{\xi+\zeta}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + e^{\xi+\zeta}\frac{\eta^{3\delta}}{\Gamma(3\delta+1)} + \cdots,\\
\nu(\xi,\zeta,\eta) &= -1 - e^{\xi+\zeta} - e^{\xi+\zeta}\frac{\eta^\delta}{\Gamma(\delta+1)} - e^{\xi+\zeta}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)} - e^{\xi+\zeta}\frac{\eta^{3\delta}}{\Gamma(3\delta+1)} - \cdots,\\
\omega(\xi,\zeta,\eta) &= e^{\xi+\zeta} + e^{\xi+\zeta}\frac{\eta^\delta}{\Gamma(\delta+1)} + e^{\xi+\zeta}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + e^{\xi+\zeta}\frac{\eta^{3\delta}}{\Gamma(3\delta+1)} + \cdots,\\
\psi(\xi,\zeta,0) &= c + 0 + \cdots.
\end{aligned}
\tag{42}
$$

In general, we have

$$\mu_m(\xi,\zeta,\eta) = e^{\xi+\zeta}\frac{\eta^{m\delta}}{\Gamma(m\delta+1)},$$

$$\nu_m(\xi,\zeta,\eta) = -e^{\xi+\zeta}\frac{\eta^{m\delta}}{\Gamma(m\delta+1)},$$

$$\omega_m(\xi,\zeta,\eta) = e^{\xi+\zeta}\frac{\eta^{m\delta}}{\Gamma(m\delta+1)},$$

$$\psi_m(\xi,\zeta,0) = 0, \quad m = 1,2,\ldots. \tag{43}$$

The Approximate Solution by VITM

According to equation (16) and the iteration formulas for system (31), we get

$$\mu_{m+1}(\xi,\zeta,\eta) = \mu_m(\xi,\zeta,\eta) - N^{-}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\cdot\frac{\partial\mu_m}{\partial\eta} + \mu_m\frac{\partial\mu_m}{\partial\xi} + \nu_m\frac{\partial\mu_m}{\partial\zeta} + \frac{1}{\omega_m}\frac{\partial\psi_m}{\partial\xi}\right\}\right],$$

$$\nu_{m+1}(\xi,\zeta,\eta) = \nu_m(\xi,\zeta,\eta) - S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\cdot\frac{\partial\nu_m}{\partial\eta} + \mu_m\frac{\partial\nu_m}{\partial\xi} + \nu_m\frac{\partial\nu_m}{\partial\zeta} + \frac{1}{\omega_m}\frac{\partial\psi_m}{\partial\zeta}\right\}\right],$$

$$\omega_{m+1}(\xi,\zeta,\eta) = \omega_m(\xi,\zeta,\eta) - N^{-}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_m}{\partial\eta}\right] + \mu_m\frac{\partial\omega_m}{\partial\xi} + \nu_m\frac{\partial\omega_m}{\partial\zeta} + \omega_m\left(\frac{\partial\mu_m}{\partial\xi} + \frac{\partial\nu_m}{\partial\zeta}\right)\right\}\right],$$

$$\psi_{m+1}(\xi,\zeta,\eta) = \psi_m(\xi,\zeta,\eta) - S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\frac{\partial\nu_m}{\partial\eta} + \mu_m\frac{\partial\psi_m}{\partial\xi} + \nu_m\frac{\partial\psi_m}{\partial\zeta} + \tau\psi_m\left(\frac{\partial\mu_m}{\partial\xi} + \frac{\partial\nu_m}{\partial\zeta}\right)\right\}\right], \tag{44}$$

where

$$\mu_0(\xi,\zeta,\eta) = e^{\xi+\zeta},$$

$$\nu_0(\xi,\zeta,\eta) = -1 - e^{\xi+\zeta},$$

$$\omega_0(\xi,\zeta,\eta) = e^{\xi+\zeta},$$

$$\psi_0(\xi,\zeta,0) = c. \tag{45}$$

For $m = 0,1,2,\ldots$,

$$\mu_1(\xi,\zeta,\eta) = \mu_0(\xi,\zeta,\eta) - N^{-}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_0}{\partial\eta} + \mu_0\frac{\partial\mu_0}{\partial\xi} + \nu_0\frac{\partial\mu_0}{\partial\zeta} + \frac{1}{\omega_0}\frac{\partial\psi_0}{\partial\xi}\right\}\right],$$

$$\nu_1(\xi,\zeta,\eta) = \nu_0(\xi,\zeta,\eta) - S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\frac{\partial\nu_0}{\partial\eta} + \mu_0\frac{\partial\nu_0}{\partial\xi} + \nu_0\frac{\partial\nu_0}{\partial\zeta} + \frac{1}{\omega_0}\frac{\partial\psi_0}{\partial\zeta}\right\}\right],$$

$$\omega_1(\xi,\zeta,\eta) = \omega_0(\xi,\zeta,\eta) - N^{-}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\cdot\frac{\partial\mu_0}{\partial\eta} + \mu_0\frac{\partial\omega_0}{\partial\xi_0} + \nu_0\frac{\partial\omega_0}{\partial\zeta_0} + \omega_0\left(\frac{\partial\mu_0}{\partial\xi} + \frac{\partial\nu_0}{\partial\zeta}\right)\right\}\right],$$

$$\psi_1(\xi,\zeta,\eta) = \psi_0(\xi,\zeta,\eta) - S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\cdot\frac{\partial\nu_0}{\partial\eta} + \mu_0\frac{\partial\psi_0}{\partial\xi} + \nu_0\frac{\partial\psi_0}{\partial\zeta} + \tau\psi_0\left(\frac{\partial\mu_0}{\partial\xi} + \frac{\partial\nu_0}{\partial\zeta}\right)\right\}\right],$$

$$\mu_1(\xi,\zeta,\eta) = e^{\xi+\zeta}\left\{1 + \frac{\eta^\delta}{\Gamma(\delta+1)}\right\},$$

$$\nu_1(\xi,\zeta,\eta) = -1 - e^{\xi+\zeta}\left\{1 + \frac{\eta^\delta}{\Gamma(\delta+1)}\right\},$$

$$\omega_1(\xi,\zeta,\eta) = e^{\xi+\zeta}\left\{1 + \frac{\eta^\delta}{\Gamma(\delta+1)}\right\},$$

$$\psi_1(\xi,\zeta,0) = c + 0.$$

$$\mu_2(\xi,\zeta,\eta) = \mu_1(\xi,\zeta,\eta) - N^{-}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\frac{\partial\mu_1}{\partial\eta} + \mu_1\frac{\partial\mu_1}{\partial\xi} + \nu_1\frac{\partial\mu_1}{\partial\zeta} + \frac{1}{\omega_1}\frac{\partial\psi_1}{\partial\xi}\right\}\right],$$

$$\nu_2(\xi,\zeta,\eta) = \nu_1(\xi,\zeta,\eta) - S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\frac{\partial\nu_1}{\partial\eta} + \mu_1\frac{\partial\nu_1}{\partial\xi} + \nu_1\frac{\partial\nu_1}{\partial\zeta} + \frac{1}{\omega_1}\frac{\partial\psi_1}{\partial\zeta}\right\}\right],$$

$$\omega_2(\xi,\zeta,\eta) = \omega_1(\xi,\zeta,\eta) - N^{-}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\cdot\frac{\partial\mu_1}{\partial\eta} + \mu_1\frac{\partial\omega_1}{\partial\xi_0} + \nu_1\frac{\partial\omega_1}{\partial\zeta_1} + \omega_1\left(\frac{\partial\mu_1}{\partial\xi} + \frac{\partial\nu_1}{\partial\zeta}\right)\right\}\right],$$

$$\psi_2(\xi,\zeta,\eta) = \psi_1(\xi,\zeta,\eta) - S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{s^\delta}{u^\delta}\cdot\frac{\partial\nu_1}{\partial\eta} + \mu_1\frac{\partial\psi_1}{\partial\xi} + \nu_1\frac{\partial\psi_1}{\partial\zeta} + \tau\psi_1\left(\frac{\partial\mu_1}{\partial\xi} + \frac{\partial\nu_1}{\partial\zeta}\right)\right\}\right],$$

$$\mu_2(\xi,\zeta,\eta) = e^{\xi+\zeta}\left\{1 + \frac{\eta^\delta}{\Gamma(\delta+1)} + \frac{\eta^{2\delta}}{\Gamma(2\delta+1)}\right\},$$

$$\nu_2(\xi,\zeta,\eta) = -1 - e^{\xi+\zeta}\left\{1 + \frac{\eta^{\delta}}{\Gamma(\delta+1)} + \frac{\eta^{2\delta}}{\Gamma(2\delta+1)}\right\},$$
$$\omega_2(\xi,\zeta,\eta) = e^{\xi+\zeta}\left\{1 + \frac{\eta^{\delta}}{\Gamma(\delta+1)} + \frac{\eta^{2\delta}}{\Gamma(2\delta+1)}\right\},$$
$$\psi_2(\xi,\zeta,0) = c + 0.$$
$$\mu_3(\xi,\zeta,\eta) = \mu_2(\xi,\zeta,\eta) - N^{-}\left[\frac{u^{\delta}}{s^{\delta}} S\left\{\frac{s^{\delta}}{u^{\delta}}\frac{\partial \mu_2}{\partial \eta} + \mu_2\frac{\partial \mu_2}{\partial \xi} + \nu_2\frac{\partial \mu_2}{\partial \zeta} + \frac{1}{\omega_2}\frac{\partial \psi_2}{\partial \xi}\right\}\right],$$
$$\nu_3(\xi,\zeta,\eta) = \nu_2(\xi,\zeta,\eta) - S^{-1}\left[\frac{u^{\delta}}{s^{\delta}} S\left\{\frac{s^{\delta}}{u^{\delta}}\frac{\partial \nu_2}{\partial \eta} + \mu_2\frac{\partial \nu_2}{\partial \xi} + \nu_2\frac{\partial \nu_2}{\partial \zeta} + \frac{1}{\omega_2}\frac{\partial \psi_2}{\partial \zeta}\right\}\right],$$
$$\omega_3(\xi,\zeta,\eta) = \omega_2(\xi,\zeta,\eta) - N^{-}\left[\frac{u^{\delta}}{s^{\delta}} S\left\{\frac{s^{\delta}}{u^{\delta}} \cdot \frac{\partial \mu_2}{\partial \eta} + \mu_2\frac{\partial \omega_2}{\partial \xi_0} + \nu_2\frac{\partial \omega_2}{\partial \zeta_2} + \omega_2\left(\frac{\partial \mu_2}{\partial \xi} + \frac{\partial \nu_2}{\partial \zeta}\right)\right\}\right],$$
$$\psi_3(\xi,\zeta,\eta) = \psi_2(\xi,\zeta,\eta) - S^{-1}\left[\frac{u^{\delta}}{s^{\delta}} S\left\{\frac{s^{\delta}}{u^{\delta}} \cdot \frac{\partial \nu_2}{\partial \eta} + \mu_2\frac{\partial \psi_2}{\partial \xi} + \nu_2\frac{\partial \psi_2}{\partial \zeta} + \tau\psi_2\left(\frac{\partial \mu_2}{\partial \xi} + \frac{\partial \nu_2}{\partial \zeta}\right)\right\}\right],$$
$$\mu_3(\xi,\zeta,\eta) = e^{\xi+\zeta}\left\{1 + \frac{\eta^{\delta}}{\Gamma(\delta+1)} + \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + \frac{\eta^{3\delta}}{\Gamma(3\delta+1)}\right\},$$
$$\omega_3(\xi,\zeta,\eta) + \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + \frac{\eta^{3\delta}}{\Gamma(3\delta+1)}\Bigg\},$$
$$\psi_3(\xi,\zeta,0) = c + 0.$$
$$\nu_3(\xi,\zeta,\eta) = -1 - e^{\xi+\zeta}\left\{1 + \frac{\eta^{\delta}}{\Gamma(\delta+1)} + \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + \frac{\eta^{3\delta}}{\Gamma(3\delta+1)}\right\},$$
$$\nu(\xi,\zeta,\eta) = \sum_{m=0}^{\infty}\nu_m(\xi,\zeta) + \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + \frac{\eta^{3\delta}}{\Gamma(3\delta+1)} + \cdots + \frac{\eta^{m\delta}}{\Gamma(m\delta+1)}\Bigg\},$$
$$\omega(\xi,\zeta,\eta) = \sum_{m=0}^{\infty}\omega_m(\xi,\zeta) + \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + \frac{\eta^{3\delta}}{\Gamma(3\delta+1)} + \cdots + \frac{\eta^{m\delta}}{\Gamma(m\delta+1)}\Bigg\},$$
$$\psi(\xi,\zeta,0) = \sum_{m=0}^{\infty}\psi_m(\xi,\zeta)$$
$$\mu(\xi,\zeta,\eta) = \sum_{m=0}^{\infty}\mu_m(\xi,\zeta) + \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + \frac{\eta^{3\delta}}{\Gamma(3\delta+1)} + \cdots + \frac{\eta^{m\delta}}{\Gamma(m\delta+1)}\Bigg\}. \quad (46)$$

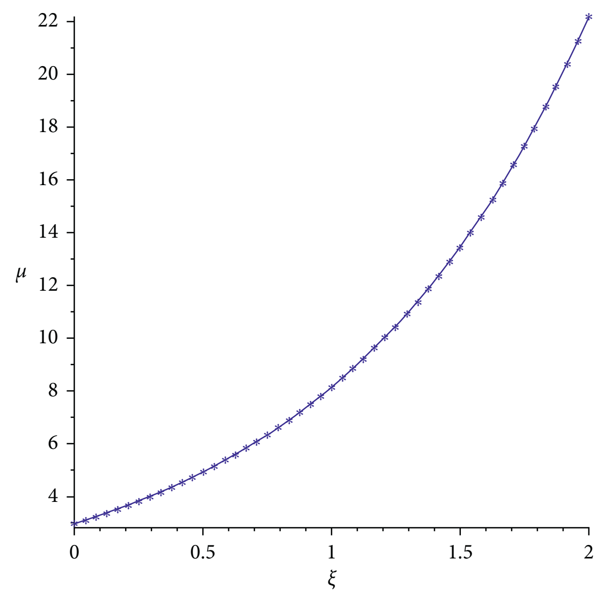

Figure 1: SDM and VITM solution graph of μ at $\zeta = 1$ and $\eta = 0.5$ of Example 1.

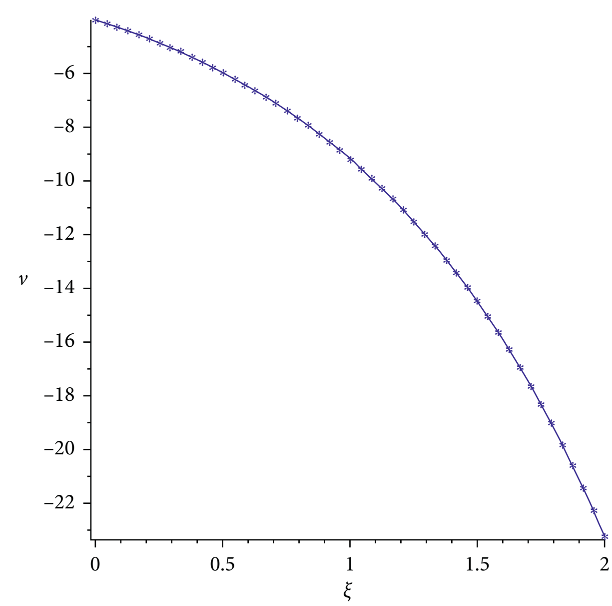

Figure 2: SDM and VITM solution graph of ν at $\zeta = 1$ and $\eta = 0.5$ of Example 1.

The exact solution of equation (31) at $\delta = 1$ is

$$\begin{aligned} \mu(\xi,\zeta,\eta) &= e^{\xi+\zeta+\eta}, \\ \nu(\xi,\zeta,\eta) &= -1 - e^{\xi+\zeta+\eta}, \\ \omega(\xi,\zeta,\eta) &= e^{\xi+\zeta+\eta}, \\ \psi(\xi,\zeta,0) &= c. \end{aligned} \quad (47)$$

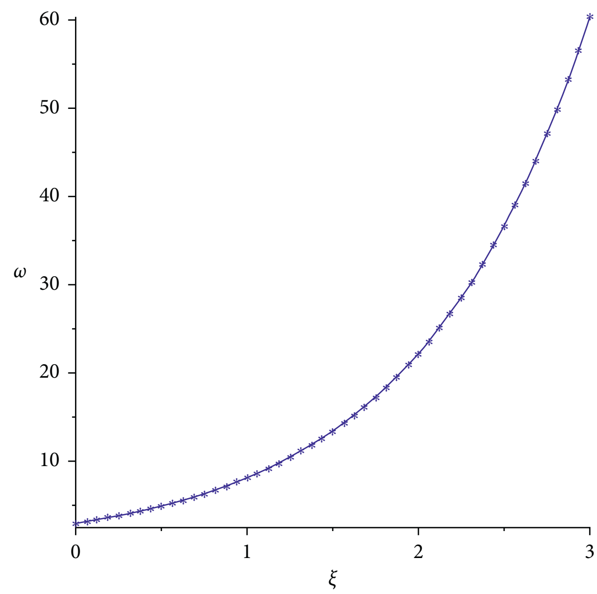

Figure 3: SDM and VITM solution graph of ω at $\zeta = 1$ and $\eta = 0.5$ of Example 1.

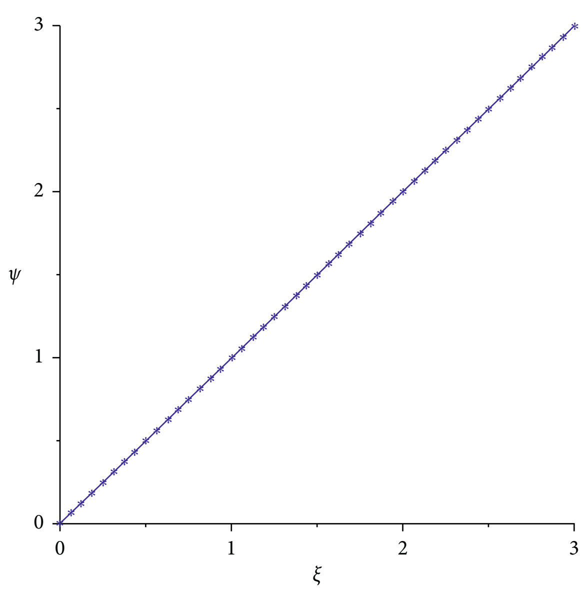

Figure 4: SDM and VITM solution graph of ψ of Example 1.

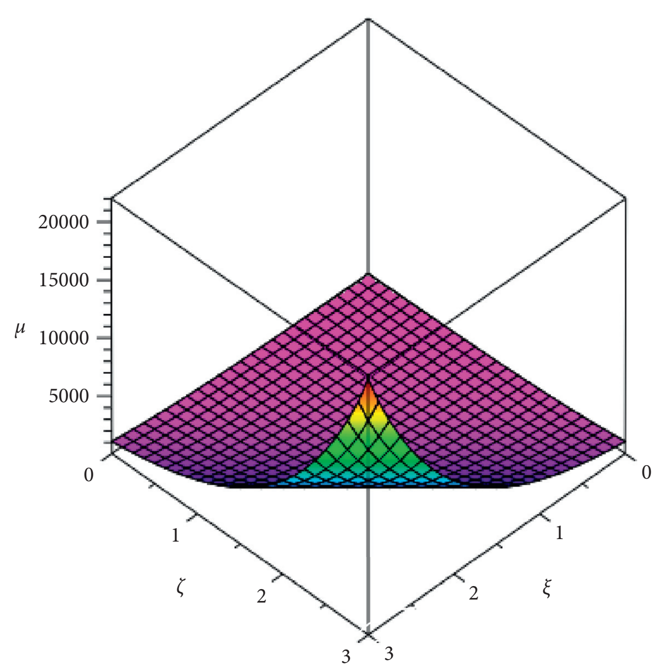

Figure 5: SDM and VITM solution 3d graph of $\mu(\xi, \zeta, \eta)$ of Example 1.

6. Results Discussion

In this section, we discuss the solution-graphs of fractional-order system of nonlinear equations of unsteady flow of a polytropic gas which has been solved by using SDM and VITM. In Figures 1–4, the solutions μ, ν, ω, and ψ obtained by using SDM and VITM are compared by keeping one variable and other constants. The dotted and line subgraphs are, respectively, denoted the SDM and VITM solutions. It is observed that SDM and VITM solution-graphs are identical and within close contact. In similar way, in Figures 5–7, the three-dimensional graphs for variables μ, ν, and ψ are plotted for Example 1. The identical solution-graphs of the suggested methods are attained and confirmed that the results obtained by two different procedures are identical and verified the applicability of the proposed techniques. In Figures 8–10, the SDM and VITM

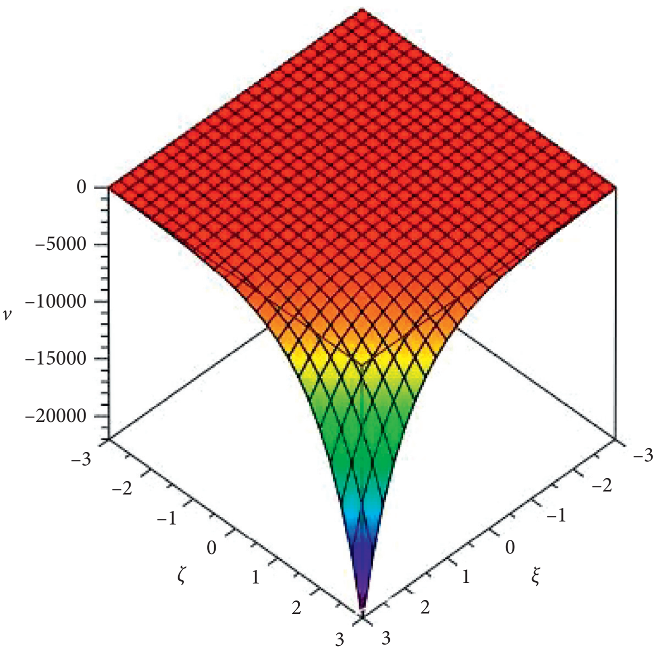

Figure 6: SDM and VITM solution 3d graph of $\nu(\xi, \zeta, \eta)$ of Example 1.

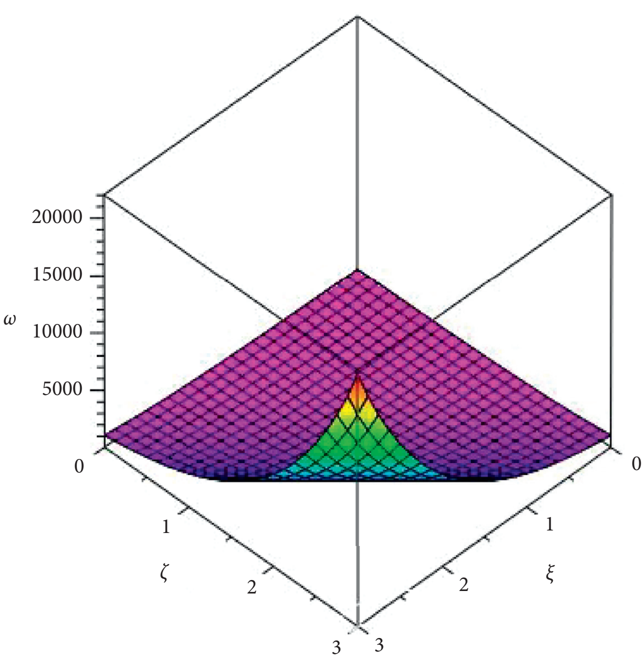

Figure 7: SDM and VITM solution 3d graph of $\omega(\xi, \zeta, \eta)$ of Example 1.

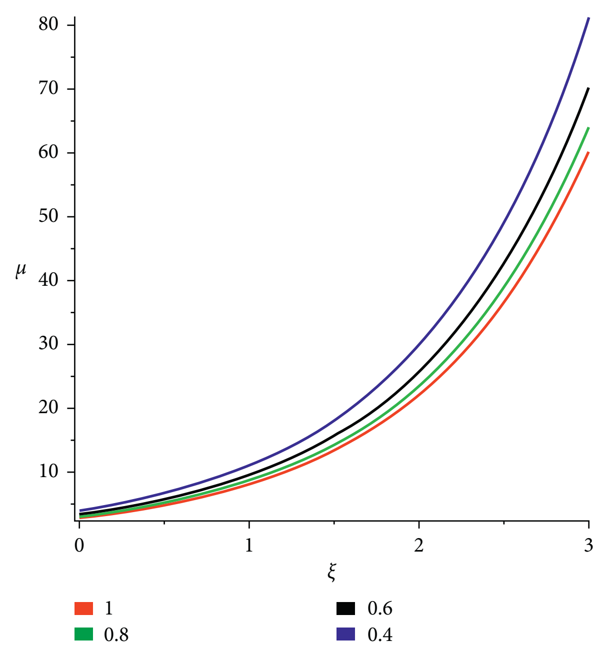

Figure 8: SDM and VITM graph of different value of δ for $\mu(\xi, \zeta, \eta)$ of Example 1.

solutions are plotted in two dimensions at fractional order $\delta = 0.4, 0.6, 0.8, 1$ for Example 1. The convergence phenomenon of the fractional solutions towards integer solution is observed. The three-dimensional graphs of the fractional-order solutions for Example 1 are represented in Figures 11–13 for variables μ, ν, and ω, respectively. In Table 1 and Figure 14, the combined graph for variables μ, ν, and ψ is displayed at $\delta = 1$. The solution comparison of the suggested methods, SDM and VITM, is discussed. The suggested techniques have provided the solutions with the desire degree of accuracy with the consideration of very few terms in its series form solutions.

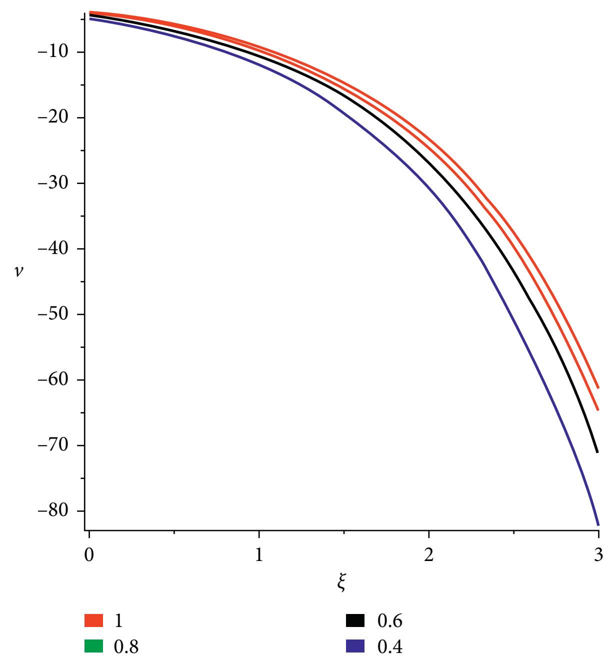

FIGURE 9: SDM and VITM graph of different value of δ for $v(\xi, \zeta, \eta)$ of Example 1.

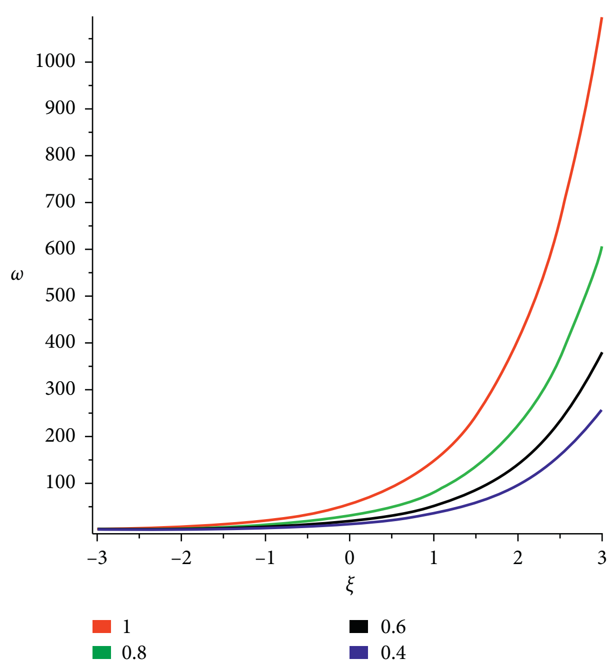

FIGURE 10: SDM and VITM graph of different value of δ for $\omega(\xi, \zeta, \eta)$ of Example 1.

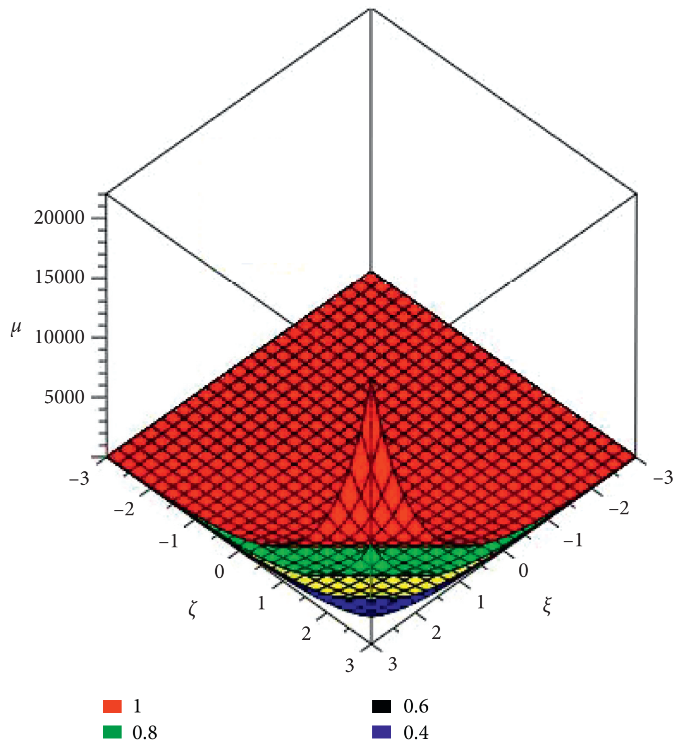

Figure 11: SDM and VITM 3d graph of different value of δ for $\mu(\xi, \zeta, \eta)$ of Example 1.

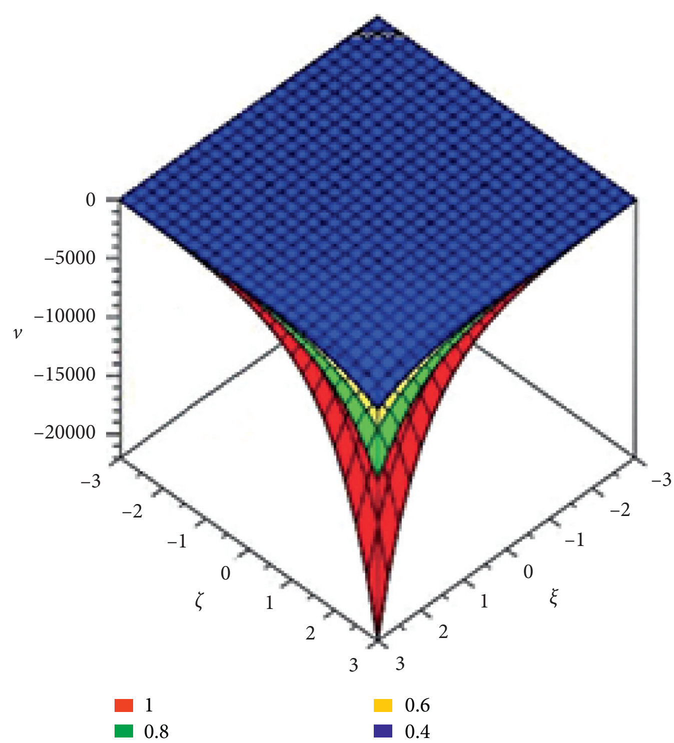

Figure 12: SDM and VITM 3d graph of different value of δ for $\nu(\xi, \zeta, \eta)$ of Example 1.

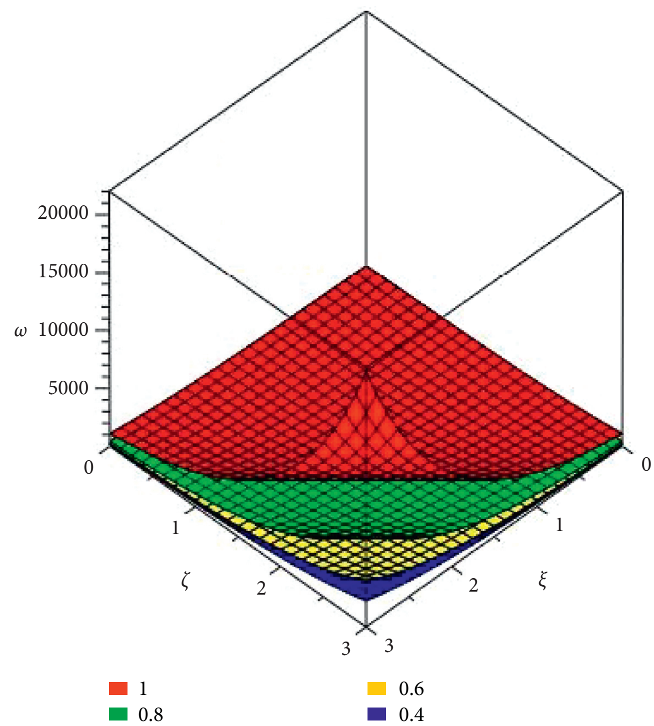

Figure 13: SDM and VITM solution graph of different value of δ for $\omega(\xi, \zeta, \eta)$ of Example 1.

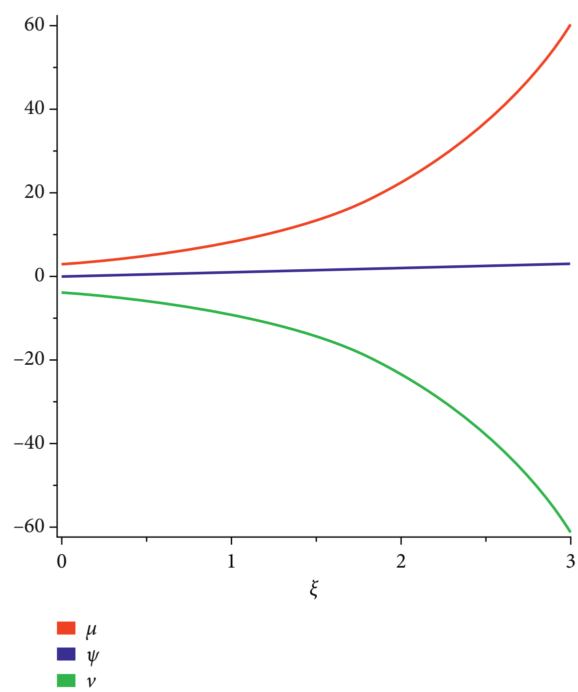

Figure 14: Combine graph of $\mu(\xi, \zeta, \eta)$, $\nu(\xi, \zeta, \eta)$, and $\omega(\xi, \zeta, \eta)$ of Example 1.

Table 1: SDM and VITM solution of Example 1 at $\delta = 1$, $\eta = 0.5$, and $\zeta = 1$.

ξ	AE of μ	AE of ν	AE of ω
0.5	$1.22000E-07$	$1.22000E-07$	$1.22000E-07$
1	$2.01000E-07$	$2.01000E-07$	$2.01000E-07$
1.5	$3.32000E-07$	$3.32000E-07$	$3.32000E-07$
2	$5.50000E-07$	$5.50000E-07$	$5.50000E-07$
2.5	$9.00000E-07$	$9.00000E-07$	$9.00000E-07$
3	$1.49000E-06$	$1.49000E-06$	$1.49000E-06$
3.5	$2.45000E-06$	$2.45000E-06$	$2.45000E-06$
4	$4.06000E-06$	$4.06000E-06$	$4.06000E-06$
4.5	$6.70000E-06$	$6.70000E-06$	$6.70000E-06$
5	$1.11000E-05$	$1.11000E-05$	$1.11000E-05$

7. Conclusion

In this article, the analytical solution of the system of time-fractional partial differential equations of unsteady flow of polytrophic dynamics is investigated by using two different techniques:

(i) The proposed techniques are the mixture of Shehu transformation with Adomian decomposition method and variational iteration method, respectively.

(ii) The obtained solutions of the suggested techniques for both fractional and integer orders are calculated and plotted via two- and three-dimensional graphs.

(iii) A close contact between the actual and the derived results is observed.

(iv) The fractional-order solutions provide various dynamics for a different fractional order of the derivative.

(v) Using analytical solutions, the task can be done rather simple and effective as compared to numerical investigations that need larger calculations.

(vi) After all, the researchers are now able to select the fractional-order problem whose solution is comparatively very close to the experimental results of any physical problem.

(vii) Due to simple and straightforward implementation, the suggested techniques are considered to be preferable to solve other system of FPDEs.

The following abbreviations are used in this article:

Nomenclature

ST: Shehu transform
LT: Laplace transform
FPDEs: Fractional partial differential equations
VITM: Variational iteration transform method
SDM: Shehu decomposition method
ADM: Adomian decomposition method.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

H. K. conceptualized the study, prepared the methodology, wrote the original draft, and did formal analysis. M. A. analyzed using the software and supervised the study. S. I. validated, investigated, and visualized the study and administrated the project. H. K. and M. A. managed resources and reviewed and edited the document.

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Research Article

Convergence Rate Analysis of the Proximal Difference of the Convex Algorithm

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In this paper, we study the convergence rate of the proximal difference of the convex algorithm for the problem with a strong convex function and two convex functions. By making full use of the special structure of the difference of convex decomposition, we prove that the convergence rate of the proximal difference of the convex algorithm is linear, which is measured by the objective function value.

1. Introduction

Difference of convex programming (DCP) is a kind of important optimization problem that the objective function can be written as the difference of convex (DC) functions. The DCP problem has found many applications in assignment and power allocation [1], digital communication system [2], compressed sensing [3], and so on [4–6].

Up to now, one of the classical algorithms for DCP is the DC algorithm (DCA) [7] in which the nonconvex part of the objective function is replaced by a linear approximation. By DCA, only a convex optimization subproblem needs to be solved at each iteration. After that, the DCA has been attracted by a lot of researchers. Le Thi et al. [8] proved the linear convergence rate of DCA by employing the Kurdyka–Lojasiewicz inequality. Assuming that the subproblem of DCA can be easily solved [6], Gotoh et al. [4] proposed the proximal DC algorithm (PDCA) for solving the DCP, in which not only the nonconvex part in the objective function is replaced by the same technique as in DCA but also the convex part is replaced by a quadratic approximal. The PDCA reduces to the classical proximal gradient algorithm for convex programming if the nonconvex part of the objective function is void [9]. To accelerate the PDCA, Wen et al. [10] introduced a new type of proximal algorithm (PDCA_e) with the help of an extrapolation technique. Since the convergence rate of PDCA_e heavily depends on the Kurdyka–Lojasiewicz inequality, PDCA_e converges linearly in general [10].

In this paper, we study the linear convergence rate of PDCA by the structure, which is different from the techniques in [8, 10]. Under conditions that the objection function can be divided into difference of a strong convex function and two convex functions with Lipschitz continuous gradient, we prove the linear convergence rate of PDCA, which is measured by the objective function value.

The remainder of the paper is organized as follows. In Section 2, several useful preliminaries are recalled. In Section 3, more details about the DC optimization problem are given, and the PDCA proposed in [4] is listed for the sake of simplicity. The linear convergence rate of the PDCA is established in Section 4. Final remarks are given in Section 5.

2. Preliminaries

In this section, we recall some useful definitions and properties.

Let $f\colon R^n \longrightarrow [-\infty, +\infty]$ be an extended real function. The domain of f is denoted by

$$\operatorname{dom} f = \{x \in R^n \colon f(x) < +\infty\}. \tag{1}$$

If $f(x)$ never equals $-\infty$ for all $x \in \operatorname{dom} f$ and $\operatorname{dom} f \neq \varnothing$, we say that f is a proper function. If the proper function is lower semicontinuous, then it is called a closed function. A proper closed function $f(x)$ is said to be level bounded if the lower level set of f (i.e., $\{x \in R^n | f t(nxq)h \leq_r x, 7 C\forall; r \in R\}$) is bounded.

Let $f\colon R^n \longrightarrow R \bigcup \{+\infty\}$ be a proper closed function. Then, the limit subdifferential of f at $x \in \operatorname{dom} f$ is defined as follows:

$$\partial f(x) = \left\{ v \in R^n \colon \exists x^t \xrightarrow{f} x, v^t \longrightarrow v \text{ with } \liminf_{y \longrightarrow x^t} \frac{f(y) - f(x^t) - \langle v^t, y - x^t\rangle}{\|y - x^t\|} \geq 0, \quad \forall t \right\}, \tag{2}$$

where $z \xrightarrow{f} x$ denote $z \longrightarrow x$ and $f(z) \longrightarrow f(x)$. Note that $\operatorname{dom} \partial f = \{x \in R^n \colon \partial f(x) \neq \varnothing\}$. It is well known that the limit subdifferential reduces to the classical subdifferential in convex analysis when $f(x)$ is a convex function, that is,

$$\partial f(x) = \{v \in R^n \colon f(u) - f(x) - \langle v, u - x\rangle \geq 0, \quad \forall u \in R^n\}. \tag{3}$$

Furthermore, if f is continuously differentiable, then the limit subdifferential reduces to the gradient of f denoted by ∇f.

3. DC Programming and PDCA

In this section, we begin to consider the DC programming problem:

$$\min_{x \in R^n} \{F(x) \coloneqq f(x) + g(x) - h(x)\}, \tag{4}$$

where $f\colon R^n \longrightarrow R$ is a strong convex function with constant $a > 0$ and $g, h\colon R^n \longrightarrow R$ are convex functions, and their gradients are Lipschitz continuous with constants $L_g > 0$ and $L_h > 0$, respectively. Throughout the paper, we assumed that $F(x)$ is level bounded and $a > 1$. Apparently, (4) is a DC optimization problem and can be solved by the following DCA Algorithm 1.

Although the subproblem (Algorithm 2) is convex, it may not have closed solutions. To solve this drawback, Gotoh et al. proposed the following PDCA.

4. The Convergence Rate of PDCA

In this section, we give the linear convergence rate of PDCA. To continue, the following lemma is useful.

Lemma 1. *Let $f\colon R^n \longrightarrow R$ be a continuous differentiable function with Lipschitz continuous gradient with Lipschitz constant $L > 0$. Then, for any $L' > L$, it holds that*

$$f(x) \leq f(y) + \langle \nabla f(y), x - y\rangle + \frac{L'}{2}\|x - y\|^2, \quad \forall x, y \in R^n. \tag{5}$$

By Lemma 1, we have the following result.

Lemma 2. *Let $\{x_k\}$ be generated in Algorithm 2. Then,*

$$\mu\left(F(x) - F(x_{k+1})\right) \geq \|x - x_{k+1}\|^2 - \|x - x_k\|^2. \tag{6}$$

Proof. Since f is strongly convex with parameter $a > 0$, it holds that

$$f(x) \geq f(x_{k+1}) + \langle \xi_{k+1}, x - x_{k+1}\rangle + \frac{a}{2\mu}\|x - x_{k+1}\|^2, \tag{7}$$

where $\xi_{k+1} \in \partial f(x_{k+1})$.

Since $\nabla h(x)$ is Lipschitz continuous with constant $L_h > 0$, by (5), there exists $0 < \mu \leq 1/L_h$ such that

$$h(x) \leq h(x_x) + \langle \nabla h(x_k), x - x_k\rangle + \frac{1}{2\mu}\|x - x_k\|^2, \tag{8}$$

that is,

$$-h(x) \geq -h(x_k) - \langle \nabla h(x_k), x - x_k\rangle - \frac{1}{2\mu}\|x - x_k\|^2. \tag{9}$$

Since g is a convex function, we have

$$g(x) \geq g(x_k) + \langle \nabla g(x_k), x - x_k\rangle. \tag{10}$$

Summing (7), (9), and (10), we get

$$\begin{aligned} f(x) + g(x) - h(x) \geq{}& f(x_{k+1}) + g(x_k) - h(x_k) + \langle \xi_{k+1}, x - x_{k+1}\rangle + \\ & \langle \nabla g(x_k) - \nabla h(x_k), x - x_k\rangle + \frac{a}{2\mu}\|x - x_{k+1}\|^2 - \frac{1}{2\mu}\|x - x_k\|^2. \end{aligned} \tag{11}$$

On the contrary, since h is a convex function, we have

$$h(x) \geq h(x_k) + \langle \nabla h(x_k), x - x_k\rangle, \tag{12}$$

which is equivalent to the following form:

$$-h(x) \leq -h(x_k) - \langle \nabla h(x_k), x - x_k\rangle. \tag{13}$$

Since $\nabla g(x)$ is Lipschitz continuous with constant $L_g > 0$, by (5), there exists $0 < \mu \leq 1/L_g$ such that

$$g(x) \leq g(x_k) + \langle \nabla g(x_k), x - x_k\rangle + \frac{1}{2\mu}\|x - x_k\|^2. \tag{14}$$

(1) Initial step: choose $\varepsilon > 0$ and $x_0 \in R^n$, and set $k = 0$.
(2) Iterative step: compute the new point by the following formula:
(3) $x_{k+1} = \arg\min_{x\in R^n}\{f(x) + g(x) - h(x_k) - \langle \nabla h(x_k), x - x_k\rangle\}$,
(4) **until** $\|x_{k+1} - x_k\| \le \varepsilon$ is satisfied.

Algorithm 1: DCA for problem (4).

Summing (13) and (14), we have

$$g(x) - h(x) \le g(x_k) - h(x_k) + \langle \nabla g(x_k) - \nabla h(x_k), x - x_k\rangle + \frac{1}{2\mu}\|x - x_k\|^2. \tag{15}$$

Adding $f(x)$ on both sides of (15), we get

$$f(x) + g(x) - h(x) \le f(x) + g(x_k) - h(x_k) + \langle \nabla g(x_k) - \nabla h(x_k), x - x_k\rangle + \frac{1}{2\mu}\|x - x_k\|^2. \tag{16}$$

Taking $x = x_{k+1}$, it follows that

$$f(x_{k+1}) + g(x_{k+1}) - h(x_{k+1}) \le f(x_{k+1}) + g(x_k) - h(x_k) + \langle \nabla g(x_k) - \nabla h(x_k), x_{k+1} - x_k\rangle + \frac{1}{2\mu}\|x_{k+1} - x_k\|^2. \tag{17}$$

By optimality conditions of Algorithm 2, we know that

$$\xi_{k+1} + \nabla g(x_k) - \nabla h(x_k) + \frac{1}{\mu}(x_{k+1} - x_k) = 0, \tag{18}$$

where $\xi_{k+1} \in \partial f(x_{k+1})$, which means that

$$-\frac{1}{\mu}(x_{k+1} - x_k) = \xi_{k+1} + \nabla g(x_k) - \nabla h(x_k). \tag{19}$$

By (11) and (17), it holds that

$$\begin{aligned}
&F(x) - F(x_{k+1})\\
&\ge \langle \xi_{k+1}, x - x_{k+1}\rangle + \langle \nabla g(x_k) - \nabla h(x_k), x - x_{k+1}\rangle\\
&\quad + \frac{a}{2\mu}\|x - x_{k+1}\|^2 - \frac{1}{2\mu}\|x - x_k\|^2 - \frac{1}{2\mu}\|x_k - x_{k+1}\|^2\\
&= -\frac{1}{\mu}\langle x_{k+1} - x_k, x - x_{k+1}\rangle + \frac{a}{2\mu}\|x - x_{k+1}\|^2 - \frac{1}{2\mu}\|x - x_k\|^2 - \frac{1}{2\mu}\|x_k - x_{k+1}\|^2\\
&= \frac{1}{2\mu}\left(\|x_k - x_{k+1}\|^2 + \|x - x_{k+1}\|^2 - \|x - x_k\|^2\right) + \frac{a}{2\mu}\|x - x_{k+1}\|^2 - \frac{1}{2\mu}\|x - x_k\|^2 - \frac{1}{2\mu}\|x_k - x_{k+1}\|^2\\
&= \frac{1}{2\mu}\left((1 + a)\|x - x_{k+1}\|^2 - 2\|x - x_k\|^2\right)\\
&\ge \frac{1}{\mu}\left(\|x - x_{k+1}\|^2 - \|x - x_k\|^2\right),
\end{aligned} \tag{20}$$

where the first equality follows from (19) and the last inequality follows from $a > 1$. The desired result follows.

Now, we are at a position to prove the main theorem as follows. □

Theorem 1. *Let $\{x_k\}$ be generated in Algorithm 2. Then,*

$$F(x_k) - F(x^*) \le \frac{\|x_0 - x^*\|^2}{\mu k}, \tag{21}$$

where x^* is the stationary point of (4).

Proof. By Lemma 2, let $x = x_k$, and we have that

(1) Initial step: choose $0 < \mu < 1/\max\{L_g, L_h\}$, $\varepsilon > 0$, and $x_0 \in R^n$, and set $k = 0$.
(2) Iterative step: compute the new point by the following formula:
(3) $x_{k+1} = \arg\min_{x\in R^n}\{f(x) + g(x_k) - h(x_k) - \langle \nabla g(x_k) - \nabla h(x_k), x - x_k\rangle + 1/2\mu\|x - x_k\|^2\}$,
(4) **until** $\|x_{k+1} - x_k\| \le \varepsilon$ is satisfied.

Algorithm 2:PDCA for problem (4).

$$\mu\left(F(x_k) - F(x_{k+1})\right) \ge \left\|x_{k+1} - x_k\right\|^2 \ge 0. \tag{22}$$

Then, it follows from $\mu > 0$ that $F(x_k) \ge F(x_{k+1})$, which means that the sequence $\{F(x_k)\}$ is nonincreasing. Then, for any $k_0 \in N$, it follows that

$$\sum_{k=0}^{k_0-1} F(x_{k+1}) \ge \sum_{k=0}^{k_0-1} F\left(x_{k_0}\right) = k_0 F\left(x_{k_0}\right). \tag{23}$$

By Lemma 2 again, let $x = x^*$, and we have that

$$\mu\left(F(x^*) - F(x_{k+1})\right) \ge \left\|x_{k+1} - x^*\right\|^2 - \left\|x_k - x^*\right\|^2, \tag{24}$$

which implies that

$$\begin{aligned}\mu\left(k_0 F(x^*) - \sum_{k=0}^{k_0-1} F(x_{k+1})\right) &\ge \left\|x_{k_0} - x^*\right\|^2 - \left\|x_0 - x^*\right\|^2\\ &\ge -\left\|x_0 - x^*\right\|^2.\end{aligned} \tag{25}$$

By (23) and (25), it yields that

$$\mu k_0\left(F(x^*) - F\left(x_{k_0}\right)\right) \ge -\left\|x^* - x^0\right\|^2, \tag{26}$$

and the desired result follows. □

5. Conclusions

In this paper, we give the linear convergence rate of PDCA for the case that the objective function is divided into a strong convex function and two convex functions. Different from the method in [8, 10], which depends heavily on the Kurdyka–Lojasiewicz inequality, we give a simple proof by the special structure of the optimization problem. Actually, there may be some other potential applications about the proposed PDCA. We leave this work in the future. For example, we will study further applications of the PDCA algorithm to some nonconvex problems [11, 12], tensor optimization problems [13, 14], and so on [15–18].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Each author contributed equally to this paper and read and approved the final manuscript.

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Research Article

Using Particle Swarm Optimization Algorithm to Calibrate the Term Structure Model

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One of the advantages of stochastic differential equations (SDE) is that they can follow a variety of different trends so that they can establish complex dynamic systems in the economic and financial fields. Although some estimation methods have been proposed to identify the unknown parameters in virtue of the results in the SDE model to speed up the process, these solutions only focus on using explicit approach to solve SDEs, and therefore they are not reliable to deal with data source merged being large and varied. Thus, this study makes progress in creating a new implicit way to fill in the gaps of accurately calibrating the unknown parameters in the SDE model. Essentially, the primary goal of the article is to generate rigid SDE simulation. Meanwhile, the particle swarm optimization method serves a purpose to search and simultaneously obtain the optimal estimation of the model unknown parameters in the complicated experiment of parameter space in an effective way. Finally, in an interest rate term structure model, it is verified that the method effectively deals with parameter estimation in the SDE model.

1. Introduction

Inevitably, a lot of fluctuation dynamics are observed as a result of both internal and external shocks to the system. Stochastic differential equations (SDEs) seem to be a beneficial way to model these fluctuation phenomena by combining deterministic models with a noise term. As a consequence, SDEs have been extensively used to explain uncertainty of complex systems in the area ranging from the subject of biological sciences to the realm of finance and economics [1]. Particularly, SDEs display as a fundamental explanation for volatility coming from an unexpected direction over the short-term interest rate, as well as asset prices in modern finance and economic theory. As is known, one of the main aims of financial modelling is subjected to the accurate calculation in the model. But it is fraught with difficulties and frustrations to estimate those parameters from large samples of financial data. For this reason, it is noted that the description of parameter estimation has become a more eye-catching way in integral research area [2–4].

When it comes to SDEs, it is more difficult to infer unknown parameters compared to their deterministic counterparts. To be exact, under the same parameters and initial conditions, it is available to generate some unideal simulations with SDEs. Therefore, even though many effective methods are supposed to achieve the goal in deterministic models [2], it occurs to be not obvious in SDEs [5]. As a result, this research has focused on establishing efficient methods for SDEs. A big concern has been the slow convergence of Monte Carlo simulations for SDEs. Henceforth, some efficient methods have been produced to succeed in estimation at length like the Taylor approximation, Runge–Kutta, and SDEs generating a large number of simulations [6]. However, it is expected to be a serious barrier because in most cases the search space for the parameters is complex, and the computational time of the inference is expensive.

Currently, three popular methods to infer the process are discussed with the example of the maximum likelihood estimation (MLE) or simulated maximum likelihood (SML) [7, 8], the method of moments [9, 10], and extended Kalman filter [11]. The other most commonly used method is the Bayesian inference method, which will not be discussed in this work. Among them, the method of moments is quite easy to implement application. Although it has been widely used, the major disadvantage has resulted in the characteristic of frequenting approach and large observations. Therefore, if one has a small sample size, the results often suffer from inaccuracy. Meanwhile the MLE approach is not vulnerable to strict limits on set of data and the results of parameters are evaluated by means of reliable analysis of the probability density function (PDF). In effect, an analytical function of closed-form solution is always misconceived for the MLE; therefore, an SML method is developed to obtain the result of PDF by systematic analysis of numerical simulations. Lately, one study examined by lots of scholars that classical statistical models are mostly not precise for the realization of learning-based process, particularly for nonlinear events. Even though the machine-learning techniques provide a better opportunity to undertake the analysis than classical methods, there is little hope of achieving success because of sample size and time consumption [12].

Two kinds of errors can be found in the research of SML. The primary error would refer to the low-rate convergence with the Monte Carlo simulations. To concur this, there is a major concern in variance reduction methods that have been applied to the reduction of stochastic simulations and the bias in the effect of estimation of the moment. Among the method of variance reduction, the existence of the importance sampler and random number generation methods are induced to illustrate the procedure. The second type of error is pointed out to be the difference discretization method based on numerical results and the original SDE. Attempts to solve this dilemma have resulted in the rapid development of the Euler–Maruyama method, which is applied to form the inference of stochastic process. Simple modelling and Gaussian random variable of the numerical solution are the reason for the popularity. But it fails to be stable and achieve a better convergence rate. Hence, it has been advocated that high-order analytic methods and various forms of implicit methods are employed to maintain the stability and enhance the simulation accuracy. Many scholars have found that the convergence rate of SML is proved to be better than the Markov Chain Monte Carlo (MCMC) approach [13]. Also, it can be analyzed by their numerical solutions that implicit SML methods might accurately generate a vast range of estimations on the purpose of following observing data [14].

The other contribution related to the SML method shows the optimal parameters of the model and raises the efficiency. Several approaches to machine learning, the principle of the genetic algorithm (GA) [14, 15] and the application of particle swarm optimization (PSO) algorithm, have mushroomed with rapid growth in the inference of SML parameters [3, 16]. PSO characterized as an artificial intelligence (AI) method is used to make a process of approximation of the minimization problems, which is a sort of nondifferentiable optimization problem in order to arrive at a solution. A number of comparison studies have been conducted to investigate the efficiency of PSO and GA [17–23]. Also, particle swarm optimization provides an important way in fine-tuning the parameters of finance models and deserved popularity in this field [24–29]. Taguchi's experimental design method has been used to define the user-defined parameters in a comparison study of six algorithms, including the PSO algorithm [30, 31]. It is noted to point out that this method has been turned out to be successfully applied to approximately 700 problems [32]. Empirical studies in recent years have shown that the PSO algorithms achieved high convergence speed for the multiobjective optimization problems [33]. Apart from this, it has been performed to apply to the automatic space exploration on the superscalar computer systems successfully [34, 35]. More notably, it is suggested that PSO has been exercised to greatly improve the implicit SML algorithm through convergence speed and accuracy in consideration of the financial model calibration problems.

To address the issues mentioned above, we design a new research in this study for the application of PSO to infer the unknown parameters in SDE models using the implicit numerical methods for simulating SDEs. The contributions of this work mainly include two parts. The first one is to use implicit methods for simulating stochastic models rather than dominantly using explicit methods in existing research works. This issue is important because the SDE model may have quite a range of stability properties based on the generated parameter samples. The large simulation error may be caused by the stability property of the model rather than the parameter sample. The second contribution is to use efficient PSO algorithm to reduce the computing time. As a heuristic global optimization method, there are still limited research works for applying the PSO algorithm to infer stochastic models. This work will show that the PSO algorithm has greatly improved the efficiency of the implicit SML algorithm with high convergence speed and more accuracy compared to the existing methods.

The remaining sections are organized as follows. Sections 2 and 3 briefly illustrate the research on the interest rate term structure models and method of moments as benchmark for solving models, respectively, while Section 4 will provide our algorithm to generate parameter estimates optimized by PSO for unknown parameter search. Section 5 is presented to be a demonstration of the accuracy and robustness of the proposed algorithm for parameter estimation. Section 6 undertakes an empirical analysis for the application to the US treasury bill data. Finally, in Section 7, conclusions and further research are summarized.

2. Stochastic Model

The standard stochastic differential equation built on the general Brownian motion is analyzed as follows:

$$\mathrm{d}\mathbf{S} = \mu(t, \mathbf{S})\mathrm{d}t + \sigma(t, \mathbf{S})\mathrm{d}\mathbf{B}(t), \tag{1}$$

where $\mathbf{B}(t) = \{(B_t^1, \ldots, B_t^m), t \in [0,T]\}$ is an $\mathscr{A}$-adapted m-dimensional standard Wiener process in the probability space $(\Omega, \mathscr{A}, P)$; $\mu(t, \mathbf{S})$ is the drift term with $\mu(t,s)$: $[0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ being d-dimensional vectors of Borel measurable functions. Also, $\sigma(t, \mathbf{S})$ is the diffusion term with $\sigma(t,s)$ defined on $[0,T] \times \mathbb{R}^d$ which is a $d \times m$-matrix of Borel measurable functions. The Wiener process has an increment of $\Delta\mathbf{B}_n = \mathbf{B}(t_{n+1}) - \mathbf{B}(t_n)$ and follows the Gaussian distribution $\mathbf{N}(0, t_{n+1} - t_n)$.

The Euler–Maruyama method is a computer-based implementation consisting of strong convergence estimation of order 0.5 used in computational finance. However, the above method is the real challenge faced by researchers which is that the result attempts to be stable if the numerical simulation of SDEs with a comparatively large diffusion terms has been solved. A possible solution to this problem is presented to take small step sizes to obtain stable simulations at the art of increasingly computing time consumption. In this study, the implicit Milstein method is utilized to obtain the reliability of results of numerical simulation with the characteristics of precision and stabilization.

The Milstein approach is proposed with a higher-order stochastic process of Taylor expansion technique to achieve a better result in order 1.0 strong convergence as shown below:

$$\begin{aligned} S_{n+1} &= S_n + \mu(t_n, S_n)h + \sigma(t_n, S_n)\Delta B_n \\ &\quad + \frac{1}{2}\sigma(t_n, S_n)\sigma'(t_n, S_n)\left((\Delta B_n)^2 - h\right), \end{aligned} \tag{2}$$

where $\sigma'(t,S)$ is the first derivative of $\sigma(t,S)$. The semi-implicit and fully implicit Milstein methods have a profound impact on the stability and robustness of the system of Milstein scheme only with implicit drift term.

$$\begin{aligned} S_{n+1} &= S_n + \mu(t_{n+1}, S_{n+1})h + \sigma(t_n, S_n)\Delta B_n \\ &\quad + \frac{1}{2}\sigma(t_n, S_n)\sigma'(t_n, S_n)\left((\Delta B_n)^2 - h\right). \end{aligned} \tag{3}$$

The full implicit Milstein method [36] including implicit drift term and the diffusion term is set up by

$$\begin{aligned} S_{n+1} &= S_n + \mu(t_{n+1}, S_{n+1})h + \sigma(t_{n+1}, S_{n+1})\Delta B_n \\ &\quad + \frac{1}{2}\sigma(t_{n+1}, S_{n+1})\sigma'(t_{n+1}, S_{n+1})\left((\Delta B_n)^2 + h\right). \end{aligned} \tag{4}$$

In fact, in light of CIR (Cox, Ingersoll, and Ross) model primarily constructing interest rate term structure, the accuracy of inference method has been detected [3]. The dynamic change of the short term of interest rate is a worthwhile field to finance market because bond prices and mortgage contracts are mostly valued in a way of using the term structure of interest rates, option, and derivative [37, 38]. The model of the CIR has been ready to short interest rate with information [39, 40], which is viewed as a linear stochastic differential equation with mean-reversing [41]. Regarding short term of interest rate acting like a square root controlled by diffusion, the formula is a continuous-time system as below:

$$\mathrm{d}r = \alpha(\beta - r)\mathrm{d}t + \sigma\sqrt{r}\,\mathrm{d}B(t), \tag{5}$$

where $\alpha, \beta, \sigma > 0$, α represents the speed of adjustment (or mean reversion), β is value of the random moving on interest rate of the long term, and σ is a volatility based on constant measure. It has become apparent that the drift is varying along with the volatility change according to the short-term interest rate level.

According to previous literatures, the explicit Milstein method as described in (2) is better than the Euler method. More emphasis is placed on the comparison with the accuracy of the Milstein method in (2) against the semi-implicit Milstein method in (3) for the purpose of parameter inference of the CIR model. On the premise of Milstein method, the application of the CIR is shown as follows:

$$r_{n+1} = r_n + \alpha(\beta - r_n)h + \sigma\sqrt{r_n}\Delta B_n + \frac{\sigma^2}{4}\left(\Delta B_n^2 - h\right), \tag{6}$$

and, with the linear correlation analysis of the interest rate drift term, a semi-implicit algorithm is included as the following formula:

$$r_{n+1} = \frac{1}{1+\alpha h}\left(r_n + \alpha\beta h + \sigma\sqrt{r_n}\,\Delta B_n + \frac{\sigma^2}{4}\left(\Delta B_n^2 - h\right)\right). \tag{7}$$

Figure 1 shows the whole process 5 simulations of the CIR model with result of parameters $\alpha = 0.2$, $\beta = 0.08$, and $\sigma = 0.2$. Given the relatively small level of volatility, the short-term value remains positive.

3. Method of Moments

For the next sections, the analysis of the parameter θ of one-dimensional SDE is employed as follows:

$$\mathrm{d}S = \mu(t, \theta, S)\mathrm{d}t + \sigma(t, \theta, S)\mathrm{d}B(t). \tag{8}$$

If we sample S to get $(N+1)$ observations $S_0, S_1, \ldots, S_N$ on some discrete point-in-time $t_0, t_1, \ldots, t_N$, result θ of the maximum-likelihood (ML) will be estimated based on the maximum likelihood function:

$$G(\theta) = g_0(S_0|\theta)\prod_{n=0}^{N-1} g(S_{n+1}|S_n;\theta). \tag{9}$$

Similarly, minimizing the negative effect of log-likelihood function to estimate θ is shown as follows:

$$-\log G(\theta) = -\log[g_0(S_0|\theta)] - \sum_{n=0}^{N-1}\log[g(S_{n+1}|S_n;\theta)], \tag{10}$$

where $g_0(S_0|\theta)$ is set with the density function of the initial value S_0 and $g(S_{n+1}|S_n t; n\theta)$ presents the corresponding value in the case of the transitional probability density function (PDF) at (t_{n+1}, S_{n+1}). It must be discussed that, according to the Markovian property in equation (8), the transitional PDF correctly satisfies the Fokker–Planck equation. However, considering the failure to reach the

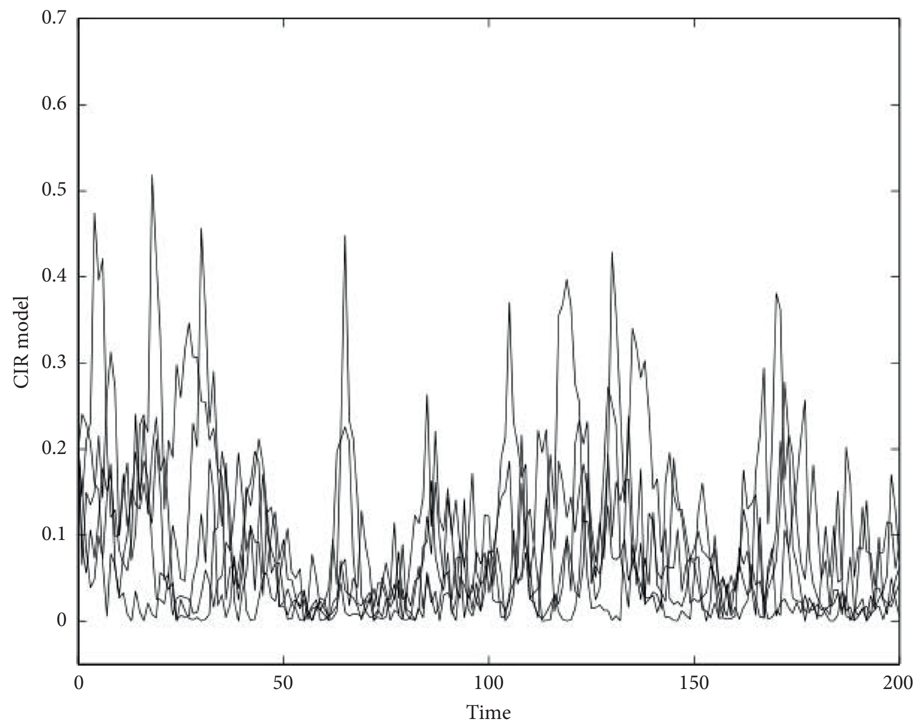

Figure 1: Five simulations of the CIR model.

closed-form solution in the Fokker–Planck equation, the final maximum likelihood estimation will make a difference.

However, it is able to approximate a transitional PDF by the method of SDE in (8). For instance, an application of the Euler–Maruyama method can be used to discretize equation (8) given by

$$S_{n+1} = S_n + \mu(t,\theta,S_n)h + \sigma(t,\theta,S_n)\Delta B_n, \tag{11}$$

where h is time step of the discretization scheme. We can then approximate the transitional PDF of S by normal distribution with mean $S_n + \mu(S_n,\theta)h$ and variance $\sigma^2(S_n,\theta)h$ such that

$$\frac{1}{\sigma(t,\theta,S_n)\sqrt{2\pi h}}\exp\left[-\frac{(S_{n+1}-S_n-\mu(t,\theta,S_n)h)^2}{2\sigma^2(t,\theta,S_n)h}\right]. \tag{12}$$

Alternatively, the simplest form on a discrete maximum likelihood function is called the method of moments, which is achieved by the approximated above-mentioned PDF, not the exact transitional PDF $g(S_{n+1}|S_n t;n\theta)$ in equation (10). In the analysis of this paper, the method is given to be a benchmark in comparison with accuracy and stability of the proposed methods.

As mentioned above, the major focus of the work is placed on the CIR process (5) and to see what happens to the optimal values referring to the parameters α, β, $\overline{\alpha}$, and $\overline{\beta}$ separately as follows [3]:

$$\begin{aligned}
&\overline{\alpha}\left(\overline{\beta}\sum_{n=0}^{N-1}h-\sum_{n=0}^{N-1}S_n h\right)=S_N-S_0,\\
&\overline{\alpha}\left(\overline{\beta}\sum_{n=0}^{N-1}\frac{h}{S_n}-\sum_{n=0}^{N-1}h\right)=\sum_{n=0}^{N-1}\frac{S_{n+1}-S_n}{S_n},
\end{aligned} \tag{13}$$

with the optimal value of $\overline{\sigma}$ being σ such that

$$\overline{\sigma}^2=\frac{1}{N}\sum_{n=0}^{N-1}\frac{\left(S_{n+1}-S_n-\overline{\alpha}\left(\overline{\beta}-S_n\right)h\right)^2}{S_n h}, \tag{14}$$

is satisfied.

The Milstein variant should make a direct contribution to improving the accuracy with the implementation of the discrete maximum likelihood method. However, it appears that it is not easy to express directly the parameter estimation by the transitional PDF [3, 13]. For that reason, the system of the simulation method of maximum likelihood is identified as the transitional PDF method along with several stochastic simulations.

4. Simulated Maximum Likelihood Method

In this section, a simulated implicit numerical scheme is included to illustrate the growth of the efficiency to estimate parameters by PSO algorithm.

In the application of deterministic models, parameter estimation is successfully achieved through fitting numerical simulations to experimental observations. Unfortunately, the method is far from ready to deal with SDE models because a single SDE model can develop. A case is to discuss the simulated maximum likelihood (SML) method for structuring stochastic models [42, 43]. $N + 1$ is given as time sequence observation $\{\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_N\}$ containing a time period $\{t_0, t_1, \ldots, t_N\}$. The joint transitional density function (likelihood function) during the study period is derived as follows:

$$g_0[(t_0, \mathbf{S}_0)|\theta] \prod_{n=1}^{N} g[(t_n, \mathbf{S}_n) \mid (t_{n-1}, \mathbf{S}_{n-1}), \ldots, (t_0, \mathbf{S}_0); \theta], \quad (15)$$

such that parameters $\theta = (\theta_1, \ldots, \theta_n)$ in equation (8) need to be determined. By analyzing the density of state, $g_0[(t_0, \mathbf{S}_0)|\theta]$ is initially presented with

$$g[(t_n, \mathbf{S}_n) | (t_{n-1}, \mathbf{S}_{n-1}), \ldots, (t_0, \mathbf{S}_0); \theta], \quad (16)$$

as the transitional density initializing from $(t_{n-1}, \mathbf{S}_{n-1})$ and running to $(t_n, \mathbf{S}_n)$.

If the framework of financial system is fundamentally prescribed based on the stochastic model (8), we have a stochastic process $\mathbf{S}$ which exhibits the Markov property [44], and the brief description of transitional density function can be shown as follows:

$$g[(t_n, \mathbf{S}_n) \mid (t_{n-1}, \mathbf{S}_{n-1}), \ldots, (t_0, \mathbf{S}_0); \theta] = g[(t_n, \mathbf{S}_n) | (t_{n-1}, \mathbf{S}_{n-1}); \theta]. \quad (17)$$

The equivalence function of the maximum joint transitional density (15) is confirmed to the minimum negative log-likelihood function (10) with the implicit time t in the formula.

Since a closed-form solution obtained from the transitional density (17) is impossible to be evaluated, based on a nonparametric kernel density algorithm it can be used as

$$\overline{g}_M[(t, \mathbf{S}_n) \mid (t_{n-1}, \mathbf{S}_{n-1}); \theta] = \frac{1}{\mathrm{MD}} \sum_{i=1}^{M} K\left(\frac{\mathbf{S}_n - \mathbf{T}_i}{D}\right). \quad (18)$$

In replace for the transitional density, in this case $\mathbf{T}_1, \ldots, \mathbf{T}_M$ are the M realizations of $\mathbf{S}_n$ at a particular time point t_n following the initial condition $(t_{n-1}, \mathbf{S}_{n-1})$, and D is obtained by the kernel bandwidth. We also have that $K(\cdot)$ is devised as a nonnegative kernel function within a probability unit. Regarding a single variable of SDE models linking with the normal Kernel, the specific bandwidth can be represented as $D = 0.9\sigma M^{-0.2}$, with σ as standard deviation of the sample with realization M [42]. In the case of multivariate stochastic models, it can be assumed that random variables are independent or alternatively supported by the existence of the theory of multivariate density estimation [45].

After setting up the objective function, another consideration that has been taken into account is to select a method to improve speed of researching the optimal parameters. Numerical solutions tell us that PSO has better computational saving for computational saving purpose; PSO is studied in presence of numerical solutions for nonlinear and unconstrained problems consisting of continuous design variables. However, the computational saving of integer and combined constrained nonlinear program is much lower.

Eberhart and Kennedy [16] have developed a PSO algorithm based on population stochastic optimization scheme; the discovery of PSO approach is inspired by abundant contents of different social behaviours like bird flocking or fish schooling without the idea of evolution operators like crossover and mutation. The information relevant to this method is that potential solutions are referred to as particles swarm across the problem space of current optimum particles. For the further research, a PSO MATLAB toolbox [46] that was downloaded from the MATLAB File Exchange Central is designed to estimate unknown parameter of SDE models. The software package can be successfully made to settle a number of optimization problems. Fortunately, the optimal solution to unknown parameter θ is identified by SML method referring to the SDEs model (8) by finding the minimum log-likelihood function (10) through the following sampling process and algorithm:

(i) Firstly, the process starts with inputting the system states $\{\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_N\}$ and time points $\{t_0, t_1, \ldots, t_N\}$.

(ii) Secondly, taking $\mathbf{S}_{n-1}$ at time t_{n-1} $(n = 1, \ldots, N)$ as the starting point, several methods are required to realize M realizations $\mathbf{T}_1, \ldots, \mathbf{T}_M$ of $\mathbf{S}_n$ at t_n. The value for random seed is explained specifically by the Gaussian random variables samples.

(iii) Thirdly, the study of the nonparametric density estimation (18) with the normal kernel algorithm and multivariate density function have been generally completed to calculate the transitional density (17).

(iv) Then, it is necessary to repeat the above step for each time point $t_0, \ldots, t_{N-1}$ and describe the log-likelihood function (10).

(v) Finally, it offers a conclusion reading the optimal kinetic rates with the process of a particle swarm optimization algorithm on the account of the minimum $G(\theta)$ in (10).

It should be highlighted that equal increments deriving from the Wiener process should be applied to the numerical simulations method to study diverse values of parameter θ. Also, to minimize variation on estimation, an important research to be summarized is that the condition of the same random seeds or samples mentioned in the second step is applied to estimate numeral values of different candidate parameters.

5. Parameter Estimation of the Interest Rate Models

As mentioned above, we shall now estimate specific parameters from the CIR model. There is no doubt that we can get 20 trajectories from a small step size of ($h = 0.001$) with the help of the semi-implicit Milstein method in a given set of parameters. Great efforts have been made to obtain the estimation of each parameter of the generated trajectories by virtue of the method of moments, the SML method, and the explicit and semi-implicit method of the Milstein scheme. Result can be made by using the PSO algorithm with the population of 40 and generation of 200. The PSO algorithm is implemented 20 times using different random seeds in computation, and thus we achieve 20 estimates of parameters. Hopefully, it is proposed to focus on the mean error and standard deviation (STD) of the estimation.

The above simulation results declare that indeed the method of moments is frequently improper to deal with parameters estimation in model. Because the results of the save parameter have a differential when applying different implementations, for some instances, the parameter relative error will exceed 100%. Worst still, it may have access to a negative coefficients estimation, which happens to be meaningless in finance market, although the noise strength is small. Furthermore, it is important to emphasize that the moment method relies upon observation points Δ correlated with the length and the quantity of various time points of observation. If the length of the data points is large and the chosen observation is big enough, it is proved that the method of moments (see Table 1) can be designed to predict estimates with near acceptable accuracy.

Studying the effect of the noise imprecision of estimation, the volatility of parameter σ is proposed for each SDE model with 3 differing values. From Tables 2 and 3, the explicit Milstein method and the semi-implicit Milstein method were employed for actual parameter estimations on the basis of negligible minor errors and standard deviations with the evidence of small fluctuations in the SDE model ($\sigma = 0.1$). In fact, the final result of the SML method linking with the explicit Milstein scheme is considered to be more accurate and reliable. Reasonably, for modelling nonstiff SDEs, the explicit Milstein method is expected to be more precise than the semi-implicit one. If the actual fluctuations in the SDE models are average instead of extreme ($\sigma = 0.2$), more accurate research results can be attained by the semi-implicit Milstein method compared to the explicit Milstein approach. In terms of application, the stability of the semi-implicit method is slightly prioritized over the precision of the explicit scheme. Particularly, if the noise of interest rate is large enough to estimate ($\sigma = 0.3$), the acceptable results can be achieved by the semi-implicit method. However, if the noise of interest rate tends to be a large extent, the same method becomes less reliable.

Table 1: Parameter estimation results of method of moments.

	Mean	Bias	STD
Exact parameter ($\alpha = 0.2$, $\beta = 0.08$, $\sigma = 0.1$, $\Delta = 5$)			
α	0.0501	0.1499	0.0103
β	−0.0118	0.0918	0.0193
σ	0.1067	0.0933	0.0148
Exact parameter ($\alpha = 0.2$, $\beta = 0.08$, $\sigma = 0.2$, $\Delta = 5$)			
α	0.1013	0.0987	0.0253
β	0.0722	0.0078	0.0271
σ	0.3650	0.1650	0.2647
Exact parameter ($\alpha = 0.2$, $\beta = 0.08$, $\sigma = 0.3$, $\Delta = 5$)			
α	0.0907	0.1093	0.0490
β	0.0746	0.0054	0.0230
σ	0.3319	0.1319	0.2128

Table 2: Parameter estimation results of Milstein method.

	Mean	Bias	STD
Exact parameter ($\alpha = 0.2$, $\beta = 0.08$, $\sigma = 0.1$, $\Delta = 5$)			
α	0.2052	0.0052	6.3E-4
β	0.0814	0.0011	1.1E-4
σ	0.0955	0.0045	3.3E-5
Exact parameter ($\alpha = 0.2$, $\beta = 0.08$, $\sigma = 0.2$, $\Delta = 5$)			
α	0.2184	0.0184	0.0037
β	0.0812	0.0012	0.0005
σ	0.1835	0.0165	0.0006
Exact parameter ($\alpha = 0.2$, $\beta = 0.08$, $\sigma = 0.3$, $\Delta = 5$)			
α	0.3269	0.1269	0.0158
β	0.0496	0.0304	0.0005
σ	0.2434	0.0566	0.0015

Table 3: Parameter estimation results of semi-implicit method.

	Mean	Bias	STD
Exact parameter ($\alpha = 0.2$, $\beta = 0.08$, $\sigma = 0.1$, $\Delta = 5$)			
α	0.2091	0.0091	1.0E-3
β	0.0787	0.0013	1.6E-6
σ	0.0933	0.0067	3.6E-5
Exact parameter ($\alpha = 0.2$, $\beta = 0.08$, $\sigma = 0.2$, $\Delta = 5$)			
α	0.2116	0.0116	0.0028
β	0.0808	0.0008	0.0005
σ	0.1850	0.0150	0.0006
Exact parameter ($\alpha = 0.2$, $\beta = 0.08$, $\sigma = 0.3$, $\Delta = 5$)			
α	0.3023	0.1023	0.0096
β	0.0534	0.0266	0.0006
σ	0.2375	0.0625	0.0007

Since then, a good approach has been extensively explored to simulate the SDE models with less step size, and the use of the fully implicit Milstein method should guarantee the numerical simulations to be more stable.

The robustness property of estimates for the stochastic search methods is a critical problem in the current study. For this study, two research tests of variations are conducted, namely, the change of random seeds for solving the stochastic model, which leads to different outcomes, and changes in the implementation of PSO algorithm that employs varying random samples.

To be specific, the first one is partly caused by the convergence property inherent in the procedure of Monte Carlo simulation under certain conditions. A discussion that presented a problem-solving solution is subject to the increase of stochastic simulations. It is defined to fix the course of the interest rate, and 10 sets of random seeds are needed to apply to simulations the stochastic model. The results from Figures 2(a)–2(c) illustrate that the estimations have progressed to be relatively stable when the amount of simulations exceeded 5000. If the number of simulations was extended to $N = 10000$, the estimates become highly stable. The results are reflection of the application of GA to estimate the calculated rate with constants given in discrete chemical reaction systems preciously [14].

Secondly, the initial model parameters are proposed to accurately estimate the final results in the PSO algorithm. Prior experiments concluded that a significant influence lies between initial parameters and final estimation when the GA was assigned as the stochastic searching method [14, 15]. Various random seeds have been devoted to the PSO algorithm to initialize parameters of the CIR stochastic model. Figures 3(a)–3(c) contain the implication of the final results for parameter estimates. The information on numerical results with the several relevant experiments describes that the PSO method is reliable to estimate independent initial parameters, posing a definite advantage for the PSO algorithm over the GA in the program of model calibration and estimation of parameters in complicated mathematical models. However, given the opportunity cost, it is suffering to pick the optimal estimation from a series of possible candidates according to the designated standard like imparting the robustness in the mathematical model.

6. Application to US Treasury Bill Data

The instantaneous interest rate R of the term structure under the classical single factor model is given as

$$\mathrm{d}R = \alpha(\beta - R)\mathrm{d}t + \sigma R^{\gamma}\mathrm{d}B, \tag{19}$$

where $\mathrm{d}B$ is considered as the standard Wiener process, α is deemed as the speed of adjustment parameter, β is regarded as the mean interest rate, σ is considered as the control of volatility, and γ is considered as the effect of levels, which are devoted to estimate the parameters. According to previous analysis, CIR model has a fine description of a unique and special type of the model in (19) with $\gamma = 0.5$. A plenty of empirical research results have suggested that γ should be in preference to estimate rather than being simply imposed into the model.

Concentrating on parameter estimates under stochastic differential equation (SDE) models (19) is complex but needs to be done. The US three-month Treasury Bill rate (The data are available from the Board of Governors of the Federal Reserve System (US), three-month Treasury Bill: Secondary Market Rate [TB3MS], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/TB3MS) is provided to approximate a reasonable estimate about real short-term interest rates, which is unobservable and instantaneous. For an illustrative purpose, the method of moments is regarded as a benchmark to discuss the approaches. Then the discrete equations that were declared from the single factor model by the Milstein method and semi-Milstein method of SML estimation are suggested as follows:

$$\begin{aligned} R_{n+1} &= R_n + \alpha(\beta - R_n)h + \sigma R_n^{\gamma}\Delta B_n + \frac{1}{2}\sigma^2\gamma R_n^{2\gamma-1}\left(\left(\Delta B_n\right)^2 - h\right), \\ R_{n+1} &= \frac{1}{1+\alpha h}\left(R_n + \alpha\beta h + \sigma R_n^{\gamma}\Delta B_n + \frac{1}{2}\sigma^2\gamma R_n^{2\gamma-1}\left(\left(\Delta B_n\right)^2 - h\right)\right). \end{aligned} \tag{20}$$

The analysis of data is carried out on the US Treasury Bill from the first day of January 1996 to the first day of June 2019 with 282 observations in total on the purpose of parameters estimation of the single-factor model (19). An expression of the results of the estimated parameter from the method of moments and explicit Milstein method as well as semi-implicit Milstein method has been presented at Tables 4–6, respectively. Also, the estimation of standard errors by three methods is explained in the relevant tables. As can be seen, a large number of the estimates are confirmed to be similar without difference. With a large sample size, the method of moments is in line with the result of accuracy of other methods. Although the method of moments is still accurate, the results of the standard errors by these two Milstein methods are much smaller. Detailed analysis is designed for these two Milstein methods with a smaller step size during the numerical simulation, whereas a large number of simulations M are offered for the statistical inference methods. The sign of the improvement to versify the standard error was not much.

Figure 4 gives a specific description of the course of the monthly interest of the US three-month Treasury Bills. As shown in the graph, the results of mean interest rate θ fit precisely with the interest rate data. In accordance with the results, it is reported that the scientific semi-implicit Milstein method is greater and more precise than the others. An outstanding phenomenon supports the evidence that the results of parameter γ of the levels effect by three methods are relatively identical to each other compared to the other parameters estimations. Another important finding broadly supports the analysis that the estimation of standard error of γ is obviously smaller than initial value of $\gamma = 0.5$, reflecting that the results of estimation of γ might be clearly different from 0.5 obtained from the CIR model. These studies further support the idea that the classic single-factor model (19) is better at describing the interest rate than the CIR model. More research confirms that γ should be received from the estimation of financial data instead of simply being imposed to the interest term structure model.

The performance of the levels effect fits better in the interest term structure model, and the appropriate observations were expressed as follows. It is encouraging to estimate parameters of the SML by semi-explicit Milstein method with the given information of $\Delta = 0.01$ and 5000

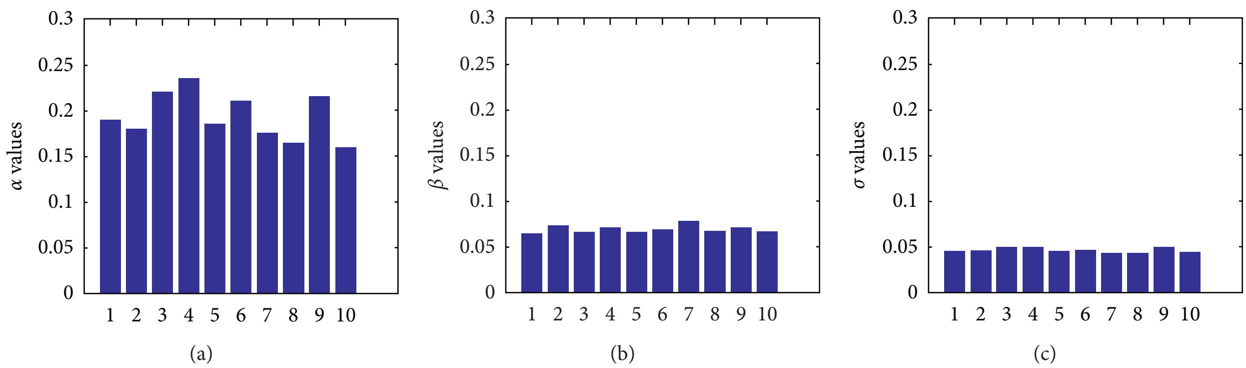

Figure 2: The results of ten sets in the CIR SDE model based on different random seeds: (a) under α parameter estimates; (b) under β parameter estimates; (c) under σ parameter estimates.

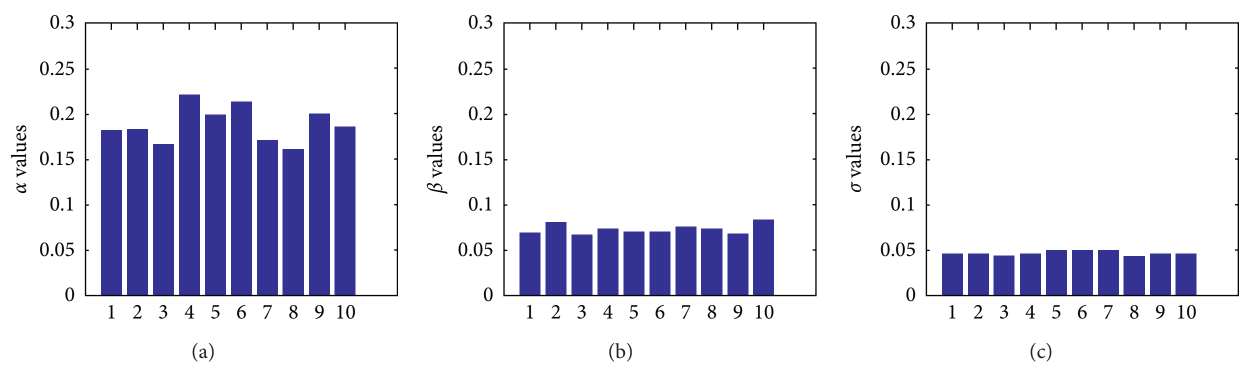

Figure 3: The results of ten sets in PSO algorithm based on different random seeds: (a) under α parameter estimates; (b) under β parameter estimates; (c) under σ parameter estimates.

Table 4: Estimated parameters of method of moments.

Parameters	Moment method $M = 1000, \Delta = 0.01$	Standard error
α	0.0112	0.0094
θ	0.0401	0.0115
σ	0.0155	0.0060
γ	0.6646	0.0923

Table 5: Estimated parameters of Milstein method.

Parameters	Milstein method $M = 1000, \Delta = 0.01$	Standard error
α	0.0107	0.0092
θ	0.0399	0.0106
σ	0.0143	0.0011
γ	0.6680	0.0347

Table 6: Estimated parameters of semi-implicit method.

Method Parameters	Semi-implicit method $M = 1000, \Delta = 0.01$	Standard error
α	0.0105	0.0056
θ	0.0405	0.0058
σ	0.0161	0.0027
γ	0.6683	0.0508

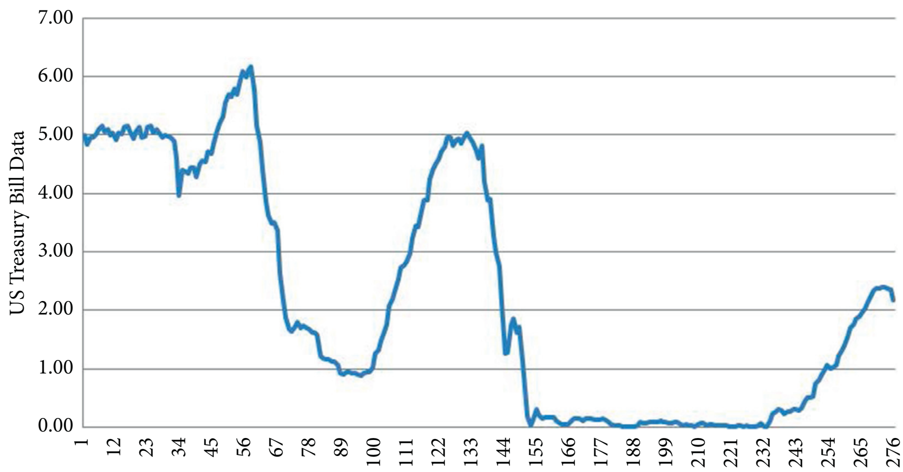

Figure 4: Interest rate series of US three-month Treasury Bills.

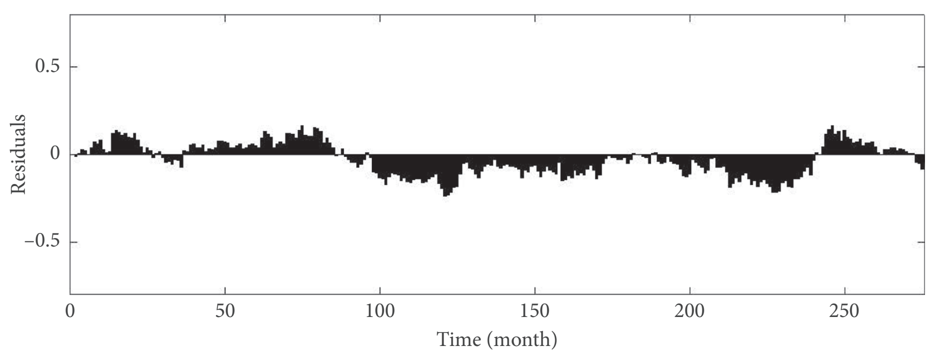

Figure 5: Residual of simulated and real data for 2000 simulations.

simulations in response to be set out to calculate the mean average. The findings from standard deviation range of the actual data were demonstrated in Figure 5, which is acceptable statistically.

7. Conclusions and Future Works

This study provides a new insight into the method of parameters estimation for the SDE models. To undertake a longitudinal analysis of the implicit numerical simulation method in the Monte Carlo simulation along with the PSO method, it can be found that the estimation from the SML by the semi-implicit Milstein method is presented better in efficiency than Euler–Maruyama approach with regard to the convergence and stability. A key discussion on experimental results in moderately stiff SDE model is presented that the semi-implicit Milstein method appears to be more exact than explicit Milstein method. Also, the effects of the PSO method are reliable and almost independent with comparison to the genetic algorithms. For the implementation of the SDE models, especially the interest term structure, actual financial data was explored to estimate the parameters. Therefore, the estimated parameters of the model make a perfect match for actual data.

Compared with the inference of deterministic models, there is more uncertainty in the inference of stochastic models. Generally, it is more difficult for inferring a stochastic model than a deterministic model. Although we have achieved progress in this work for the parameter inference of stochastic model, there are still some limitations of this work. First, the accuracy of the estimated parameters depends on the samples in stochastic simulation. A large number of simulations are needed to ensure the stability of the estimates. It is still an important issue for reducing the computational time for the inference of stochastic models. In addition, it is still a difficult issue to select appropriate estimate from the candidates that all have similar estimation errors. In this work, we use robustness as an additional criterion to select estimates. Other criteria for selecting inference candidates are strongly required. Furthermore, the stochastic differential equations with the generated parameter sample may be highly stiff. In this work, we use the semi-implicit Milstein method to improve the stability property of the explicit Milstein method. It is strongly required to employ numerical methods with better stability properties, such as the fully implicit methods, to simulate stochastic differential equations. All these issues will be interesting topics of our further research.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Jacobian Nonsingularity in Nonlinear Symmetric Conic Programming Problems and Its Application

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This paper considers the nonlinear symmetric conic programming (NSCP) problems. Firstly, a type of strong sufficient optimality condition for NSCP problems in terms of a linear-quadratic term is introduced. Then, a sufficient condition of the nonsingularity of Clarke's generalized Jacobian of the Karush–Kuhn–Tucker (KKT) system is demonstrated. At last, as an application, this property is used to obtain the local convergence properties of a sequential quadratic programming- (SQP-) type method.

1. Introduction

In this paper, we consider the nonlinear symmetric conic programming (NSCP) as follows:

$$\begin{aligned} &\min_{x\in X} f(x),\\ &\text{s.t. } h(x) = 0,\\ &\quad g(x) \in K, \end{aligned} \tag{1}$$

where X and Y are two finite dimensional real vector spaces; $f: X \longrightarrow \mathfrak{R}$, $h: X \longrightarrow \mathfrak{R}^m$, and $g: X \longrightarrow Y$ are twice continuously differentiable functions; and $K \in Y$ is a symmetric cone defined by Euclidean Jordan algebras. In the following part, unless otherwise specified, we denote X, Y, and Z to represent finite dimensional real vector spaces with a scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.

It is well-known that the Karush–Kuhn–Tucker (KKT) conditions of optimization problem (1) are equivalent to the KKT system, which is a nonsmooth system with the metric projector over the symmetric cone. The nonsingularity of Clarke's generalized Jacobian of the KKT system introduced by Pang and Qi [1] is not only one of the most important concepts in perturbation analysis of optimization problems but also plays a vital part in the design of the algorithms and the analysis of the convergence [2–4].

When K in problem (1) is a polyhedral set, Robinson [5] has showed that the strong second-order sufficient condition and the LICQ imply the nonsingularity of Clarke's generalized Jacobian of the KKT system. Interestingly, the converse is also true [2, 6, 7]. Bonnans and Ramírez [8] and Sun [9] demonstrate the equivalent conditions to the nonsingularity of the second-order cone programming problem (SOCP) and the semidefinite programming problem (SDP), respectively.

When K is the class of C^2-cone reducible sets ([3], Definition 3.135), there are lots of most important results about the Aubin property and the robust isolated calmness of the KKT solution mapping, which guarantee the nonsingularity of the KKT system (see [10–13] and the references therein).

For symmetric cone programming problem, Kong, Tunçel, and Xiu [14–16] use a triangular representation of the Jacobian of Löwner operator to characterize the structure of Clarke's generalized Jacobian of metric projection operator onto symmetric cone. They consider the linear symmetric cone programming problem as follows:

$$\begin{aligned} &\min \langle c, x\rangle, \\ &\text{s.t.} \quad A(x) = b,\ x \in K, \end{aligned} \tag{2}$$

and present five equivalent conditions to the nonsingularity of Clarke's generalized Jacobian of KKT system in [15].

In this paper, we focus on the nonsingularity of Clarke's generalized Jacobian of the KKT system in the setting of the nonlinear symmetric cone programming problem (1). In order to present the optimality conditions of NSCP, we need the variational analysis of symmetric cones and some important sets such as tangent cone. We found that, almost at the same time, we [17] and Kong et al. [15] independently obtained the same expressions of the tangent cone and so on by different approaches (see Proposition 2). Importantly, we introduce a linear-quadratic function to establish the second-order optimality conditions. Using the Euclidean Jordan Algebras and the Peirce decomposition of a finite-dimensional vector space, we obtain an upper bound of the linear-quadratic function. Under the constraint nondegeneracy condition, we demonstrate our main result that if a kind of strong second-order sufficient condition holds, any element in Clarke's generalized Jacobian of the KKT system is nonsingular.

In [18], the local convergence for an SQP-type method is ensured by the nonsingularity of Clarke's generalized Jacobian. In this paper, as an application, we give an SQP-type method to solve NSCP (1). The analysis of the local convergence is presented by using the above properties of the nonsingularity, and our proof is a natural extension of the nonlinear programming problem.

The paper is organized as follows. In Section 2, it gives some preliminaries which are used in the paper, including the fundamental notations in Euclidean Jordan algebras. The properties of a linear-quadratic function are developed. In Section 3, we describe the KKT condition and a kind of second-order sufficient condition of NSCP (1) using the linear-quadratic function. Then, we discuss the nonsingularity of Clarke's generalized Jacobian of the equation reformulation of the KKT system. Lastly, the local convergence of a SQP-type method is analyzed by using the nonsingularity in Section 4.

2. Preliminaries

In this section, some preliminaries used in the paper are given firstly. Then, we introduce the decomposition results in Euclidean Jordan algebras, which are vital to this paper.

For a locally Lipschitz continuous function $\Xi\colon \mathcal{O}\subseteq Y \longrightarrow Z$, Clarke's generalized Jacobian of Ξ at y is defined by

$$\partial \Xi(y) := \operatorname{conv}\left\{V\colon V = \lim_{k\longrightarrow\infty} \Xi'\left(y^k\right), y^k \in \mathcal{O}_\Xi, y^k \longrightarrow y\right\}, \tag{3}$$

where $\mathcal{O}_\Xi$ is the set of F-differentiable points in $\mathcal{O}$ and "conv" denotes the convex hull.

The next conclusion of the chain rule is given in [19], which is stronger than its first version in [9].

Lemma 1. *Suppose that $\Psi\colon X \longrightarrow Y$ is a continuously differentiable function and $\Xi\colon \mathcal{O}\subseteq Y \longrightarrow Z$ is a locally Lipschitz continuous function. Denote $y^* := \Psi(x^*) \in \mathcal{O}$ and let $\Psi'(x^*)\colon X \longrightarrow Y$ be onto. Then, the composite function $\Phi(x) := \Xi(\Psi(x))$ is F-differentiable at $x \in \widehat{N}$ if and only if Ξ is F-differentiable at $\Psi(x)$, where $\widehat{N}$ is an open neighborhood of x^* and*

$$\partial_B \Phi(x^*) = \partial_B \Xi(y^*)\Psi'(x^*). \tag{4}$$

The following conclusion of implicit functions can be obtained from [20] (Section 7.1) and [21] (Lemma 2) directly.

Lemma 2. *Let $\Phi\colon X\times Y \longrightarrow X$ be a locally Lipschitz continuous function and $\Phi(\overline{x}, t\overline{y}) = 0$. Suppose that any element in $\Pi_X \partial\Phi(\overline{x}, t\overline{y})$ is nonsingular. Then, there exists a locally Lipschitz continuous function $x(\cdot)\colon \mathcal{O}_Y \longrightarrow X$ satisfying $x(\overline{y}) = \overline{x}$ and*

$$\Phi(x(y), y) = 0, \tag{5}$$

where $\mathcal{O}_Y$ is an open neighborhood of $\overline{y}$. Furthermore, if Φ is (strongly) semismooth, then $x(\cdot)$ is (strongly) semismooth.

In the last part of this section, we provide some properties about the metric projector over a convex set C in Banach space (see [22]).

Lemma 3. *Suppose that C is a convex set in a Banach space Z. Then, for any $y \in Z$ and $V \in\partial\Pi_C(y)$, V is self-adjoint. Furthermore, for any $d \in Z$, $\langle d, V d\rangle \geq 0$ and $\langle V d, d - V d\rangle \geq 0$.*

2.1. Euclidean Jordan Algebras. In this part, we show some useful notations and conclusions on Euclidean Jordan Algebras introduced in [23]. Suppose that $\mathbb{F}$ is the real field $\mathbb{R}$ and $\mathbb{V}$ is a finite-dimensional vector space over $\mathbb{F}$.

For any $x \in \mathbb{V}$, denote

$$\mathscr{L}(x)y := x \cdot y \text{ for every } y \in \mathbb{V}. \tag{6}$$

The pair $\mathbb{A} := (\mathbb{V}, \cdot)$ defined over the real field $\mathbb{R}$ is called a Euclidean Jordan algebra, if, for all $x, y \in \mathbb{V}$:

(i) $x \cdot y = y \cdot x$

(ii) $x \cdot (x^2 \cdot y) = x^2 \cdot (x \cdot y)$, where $x^2 := x \cdot x$

(iii) $\langle x \cdot y, z\rangle_{\mathbb{V}} = \langle y, x \cdot z\rangle_{\mathbb{V}}$

Here are some common concepts used in this paper.

An element $c \in \mathbb{V}$ is called to be the unit element of $\mathbb{A}$ if $x \cdot e = e \cdot x = x$ for all $x \in \mathbb{V}$. We always assume that there exists a unit element $e \in \mathbb{V}$ of $\mathbb{A} = (\mathbb{V}, \cdot)$ in the following paper.

If is called to be idempotent. If two idempotents p and q satisfy $p \cdot q = 0$, they are called orthogonal. And k orthogonal idempotents $\{c_1, c_2, \ldots, c_k\}$ are said to be a complete system if they satisfy

$$\sum_{j=1}^{k} c_j = e. \tag{7}$$

If a nonzero idempotent q cannot be written as the sum of two other nonzero idempotents, we say q is primitive. And Jordan frame is a complete system of orthogonal primitive idempotents.

The following theorem in [23] is very important to show the spectral decomposition.

Theorem 1. *Let $\mathbb{A} = (\mathbb{V}, \cdot)$ be a Euclidean Jordan algebra and $R(\mathbb{A}) = r$. Then, for any $x \in \mathbb{V}$,*

$$x = \sum_{j=1}^{r} \lambda_j(x) c_j = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r, \tag{8}$$

where $\{c_1, c_2, \ldots, c_r\}$ is a Jordan frame and $\lambda_j(x) \in \mathbb{R}$, $j = 1, \ldots, r$, satisfying $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$.

We say the numbers $\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)$ to be eigenvalues of x. Then, x has the spectral decomposition (8) and

$$\operatorname{tr}(x) = \sum_{j=1}^{r} \lambda_j(x). \tag{9}$$

Therefore, another associative inner product can be defined by using $\operatorname{tr}(x)$: $\langle x, y\rangle \coloneqq \operatorname{tr}(x \cdot y)$, $x, y \in \mathbb{V}$. Let $\|\cdot\|$ be the norm on $\mathbb{V}$ induced by this inner product, then

$$\|x\| \coloneqq \sqrt{\langle x, x\rangle} = \sqrt{\sum_{j=1}^{r} \lambda_j^2(x)}, \quad x \in \mathbb{V}. \tag{10}$$

For a scalar-valued function ϕ: $\mathbb{R} \longrightarrow \mathbb{R}$, we define Löwner's operator associated with $\mathbb{A} = (\mathbb{V}, \cdot)$ from [24] as

$$\begin{aligned}\phi_{\mathbb{V}}(x) \coloneqq \sum_{j=1}^{r} \phi\big(\lambda_j(x)\big)c_j &= \phi(\lambda_1(x))c_1 + \phi(\lambda_2(x))c_2 \\ &+ \cdots + \phi\big(\lambda_r(x)\big)c_r,\end{aligned} \tag{11}$$

where $x = \sum_{j=1}^{r} \lambda_j(x) c_j \in \mathbb{V}$.

The metric projection operator on

$$K \coloneqq \left\{y^2 \colon y \in \mathbb{V}\right\} \tag{12}$$

can be described by Löwner's operator using $\phi(t) = \max(0, t)$ as follows:

$$\Pi_K(x) = x_+ = \left(\lambda_1(x)\right)_+ c_1 + \left(\lambda_2(x)\right)_+ c_2 + \cdots + \left(\lambda_r(x)\right)_+ c_r, \tag{13}$$

which is very important in the following research.

It is known from ([23], Theorem III.2.1) that the above cone $K = \{y^2 \colon y \in \mathbb{V}\}$ is a self-dual homogeneous closed convex cone, we call it a symmetric cone, which is the constraint set of problem (1)

For a Jordan frame of $\mathbb{A}$ $\{c_1, c_2, \ldots, c_r\}$, we denote $\mathbb{V}_{ij}$ as follows:

$$\mathbb{V}_{ij} \coloneqq \begin{cases} \mathbb{V}(c_i, 1), & i = j, \\ \mathbb{V}\left(c_i, \dfrac{1}{2}\right) \cap \mathbb{V}\left(c_j, \dfrac{1}{2}\right), & i \neq j. \end{cases} \tag{14}$$

Suppose that $\mathscr{C}_{ij}(x)$ is the orthogonal projection onto $\mathbb{V}_{ij}$. Then, $\mathscr{C}_{ij}(x)$ has the following expression:

$$\mathscr{C}_{jl}(x) = \sum_{i=1}^{d} \langle v_{jl}^{(i)}, \cdot\rangle v_{jl}^{(i)}, \tag{15}$$

where $\left\{v_{jl}^{(i)}\right\}_{i=1}^{d} \in \mathbb{V}$ denotes orthonormal vectors and d satisfies $n = r + d/2r(r-1)$.

It follows from [25] that

$$\begin{aligned} h &= \sum_{j=1}^{r} \mathscr{C}_{jj}(x)h + \sum_{1 \leq j < l \leq r} \mathscr{C}_{jl}(x)h = \sum_{j=1}^{r} \langle c_j, h\rangle c_j \\ &+ \sum_{1 \leq j < l \leq r} 4c_j \cdot (c_l \cdot h), \quad \forall h \in \mathbb{V}. \end{aligned} \tag{16}$$

Actually, all the eigenvectors

$$\left\{c_1, c_2, \ldots, c_r, v_{jl}^{(1)}, v_{jl}^{(2)}, \ldots, v_{jl}^{(d)}, \quad 1 \leq j < l \leq r\right\} \tag{17}$$

form an orthonormal basis of $\mathbb{V}$.

Assume that there exist two integers s and s_1 such that

$$\begin{aligned} &\lambda_1(x) \geq \cdots \geq \lambda_s(x) > 0 = \lambda_{s+1}(x) = \cdots = \lambda_{s_1}(x) > \lambda_{s_1+1}(x) \\ &\quad \geq \cdots \geq \lambda_r(x). \end{aligned} \tag{18}$$

Let us introduce three index sets:

$$\begin{aligned} &\alpha \coloneqq \{1, \ldots, s\}, \\ &\beta \coloneqq \{s+1, \ldots, s_1\}, \text{ and } \gamma \coloneqq \{s_1+1, \ldots, r\}. \end{aligned} \tag{19}$$

For $1 \leq j \leq l$, denote

$$\begin{aligned} h_{jj} &\coloneqq \mathscr{C}_{jj}(x)h = \langle c_j, h\rangle c_j, \\ h_{jl} &\coloneqq \mathscr{C}_{jl}(x)h = \sum_{i=1}^{d} \langle v_{jl}^{(i)}, h\rangle v_{jl}^{(i)}, \\ h_{\alpha\alpha} &= \sum_{j=1}^{s} h_{jj} + \sum_{1 \leq j < l \leq s} h_{jl}, \\ h_{\alpha\beta} &= \sum_{j=1}^{s} \sum_{l=s+1}^{s_1} h_{jl}, \\ h_{\alpha\gamma} &= \sum_{j=1}^{s} \sum_{l=s_1+1}^{r} h_{jl}, \\ h_{\beta\beta} &= \sum_{j=s+1}^{s_1} h_{jj} + \sum_{s+1 \leq j < l \leq s_1} h_{jl}, \\ h_{\beta\gamma} &= \sum_{j=s+1}^{s_1} \sum_{l=s_1+1}^{r} h_{jl}, \\ h_{\gamma\gamma} &= \sum_{j=s_1+1}^{r} h_{jj} + \sum_{s_1+1 \leq j < l \leq r} h_{jl}. \end{aligned} \tag{20}$$

For the merit projector Π_K onto the symmetric cone, Sun and Sun [25] has proved that Π_K is strongly semismooth everywhere. Wang [17] gave a characterization pertaining the structure of the B-subdifferential of Π_K. Recently, Kong, Tunçel, and Xiu presented an exact expression for B-subdifferential and Clarke's generalized Jacobian of Π_K [14, 15]. Here, we rewrite the conclusion obtained by Kong et al. [15] as follows.

Proposition 1. *Let $x = \sum_{j=1}^{r} \lambda_j(x) c_j$ and the indexsets α, β, γ be given by (19). Then, for any $V \in \partial \Pi_K(x)$, there exists $W \in \partial \Pi_{K^{|\beta|}}(0)$ such that*

$$Vh = \sum_{j=1}^{s} h_{jj} + \sum_{j=1}^{s} \sum_{l=j+1}^{s_1} h_{jl} + \sum_{j=1}^{s} \sum_{l=s_1+1}^{r} \frac{\lambda_j(x)}{\lambda_j(x) - \lambda_l(x)} h_{jl} + Wh, \tag{21}$$

where $K^{|\beta|}$ is the symmetric cone in subspace $\mathbb{V}_{\beta\beta} := \oplus_{i \le j, i \in \beta, j \in \beta} \mathbb{V}_{ij}$.

In order to describe the optimality conditions of NSCP (1), we need the expressions of some important sets, such as the tangent cone of K at x_+ $(\mathcal{T}_K(x_+))$, the linearity of $\mathcal{T}_K(x_+)$ $(\mathrm{lin}(\mathcal{T}_K(x_+)))$, critical cone $C(x_+)$, and the affine space of $C(x_+)$, denoted by $\mathrm{aff}(C(x_+))$. As Wang [17] and Kong et al. [15] have obtained the characterizations independently, we only show the topological results in the following proposition.

Proposition 2. *Let x have eigenvalues as in (18). It holds that*

$$\mathcal{T}_K(x_+) = \{z \colon h_{\beta\beta} + h_{\beta\gamma} + h_{\gamma\gamma} \succcurlyeq 0\}, \tag{22}$$

$$\mathrm{lin}(\mathcal{T}_K(x_+)) = \{h \colon h_{\beta\beta} = 0, h_{\beta\gamma} = 0, h_{\gamma\gamma} = 0\}, \tag{23}$$

$$C(x_+) = \{h \colon h_{\beta\beta} \succcurlyeq 0, h_{\beta\gamma} = 0, h_{\gamma\gamma} = 0\}, \tag{24}$$

$$\mathrm{aff}(C(x_+)) = \{h \colon h_{\beta\gamma} = 0, h_{\gamma\gamma} = 0\}. \tag{25}$$

2.2. A Linear-Quadratic Function. Inspired by the works by Bonnans and Shapiro [2] and Sun [9], we define a linear-quadratic function as follows, which is vital for establishing a kind of strong second-order sufficient condition.

Definition 1. For any given $v \in \mathbb{V}$, define a linear-quadratic function $\Upsilon_v \colon \mathbb{V} \times \mathbb{V} \longrightarrow \mathfrak{R}$ by

$$\begin{gathered}\Upsilon_v(z, h) := 4\langle z \cdot h, v^{\dagger} \cdot h \rangle, \\ (z, h) \in \mathbb{V} \times \mathbb{V},\end{gathered} \tag{26}$$

where $v^{\dagger}$ is the Moore–Penrose pseudoinverse of v.

Here, the linear-quadratic function is different from Definition 3.2 in [15]. Even in the especial case, $v \in K$ and $v \cdot z = 0$, the value of the linear-quadratic function (26) is twice the one in [15].

For the linear-quadratic function (16), we have the following conclusions.

Proposition 3. *If $h \in \mathrm{aff}\, C(x_+)$, then*

$$\Upsilon_{x_+}(x - x_+, h) = \sum_{j=1}^{s} \sum_{l=s_1+1}^{r} \frac{\lambda_l}{\lambda_j} \left\| h_{jl} \right\|^2. \tag{27}$$

Proof. As $x_+ = \sum_{j=1}^{s} \lambda_j c_j$, $x - x_+ = \sum_{j=s_1+1}^{r} \lambda_j c_j$, we have

$$\begin{gathered}(x - x_+) \cdot h = \sum_{j=1}^{s} \sum_{l=s_1+1}^{r} \sum_{i=1}^{d} \lambda_l \langle v_{jl}^{(i)}, h \rangle \left(c_l \cdot v_{jl}^{(i)}\right), \frac{n!}{r!(n-r)!}, \\ (x_+)^{\dagger} \cdot h = \sum_{j=1}^{s} \lambda_j^{-1} \langle c_j, h \rangle c_j + \sum_{j=1}^{s} \sum_{l=s_1+1}^{r} \sum_{i=1}^{d} \lambda_j^{-1} \langle v_{jl}^{(i)}, h \rangle \left(c_j \cdot v_{jl}^{(i)}\right),\end{gathered} \tag{28}$$

and (28) comes from the definition of $\Upsilon_{x_+}(x - x_+, h)$. $\square$

Lemma 4. *Let $b \in K$ and $-r \in \mathcal{N}_K(b)$. Then, for any $v \in \partial \Pi_K(b - r)$,*

$$\langle \Delta b, \Delta r \rangle \ge \Upsilon_b(r, \Delta b), \tag{29}$$

where $\Delta b, \Delta r \in \mathbb{V}$ satisfying $\Delta b = v(\Delta b + \Delta r)$.

Proof. Denote $x := b - r$. Then,

$$b = \Pi_K(b - r) = \Pi_K(x) \text{ and } b \cdot r = r \cdot b = 0, \quad b \in K, r \in K. \tag{30}$$

Thus, we assume that x has the following spectral decomposition:

$$x = \sum_{j=1}^{s} \lambda_j(x) c_j + \sum_{j=s_1+1}^{r} \lambda_j(x) c_j \tag{31}$$

satisfying (18), and b and r have the spectral decompositions as follows:

$$\begin{gathered}b = \sum_{j=1}^{s} \lambda_j(x) c_j, \\ r = \sum_{j=s_1+1}^{r} \left(-\lambda_j(x)\right) c_j.\end{gathered} \tag{32}$$

From Proposition 1, we have, for any $V \in \partial \Pi_K(x)$, there exists $W \in \partial \Pi_{K^{|\beta|}}(0)$ such that

$$Vh = \sum_{j=1}^{s} h_{jj} + \sum_{j=1}^{s} \sum_{l=j+1}^{s_1} h_{jl} + \sum_{j=1}^{s} \sum_{l=s_1+1}^{r} \frac{\lambda_j(x)}{\lambda_j(x) - \lambda_l(x)} h_{jl} + Wh. \tag{33}$$

Therefore, we have from $\Delta b = v(\Delta b + \Delta r)$ that

$$\sum_{j=1}^{s}\left[(\Delta b)_{jj}+(\Delta r)_{jj}\right]+W(\Delta b+\Delta r)$$
$$+\sum_{j=1}^{s}\sum_{l=j+1}^{s_1}\left[(\Delta b)_{jl}+(\Delta r)_{jl}\right]+\sum_{j=1}^{s}\sum_{l=s_1+1}^{s}\frac{\lambda_j(x)}{\lambda_j(x)-\lambda_l(x)}$$
$$\cdot\left[(\Delta b)_{jl}+(\Delta r)_{jl}\right]$$
$$=\sum_{j=1}^{s}(\Delta b)_{jj}+\sum_{j=s_1}^{r}(\Delta b)_{jj}+(\Delta b)_{\beta\beta}+(\Delta b)_{\alpha\gamma}+(\Delta b)_{\beta\gamma}+(\Delta b)_{\gamma\gamma}, \tag{34}$$

which implies

$$\begin{gathered}\sum_{j=1}^{s}(\Delta r)_{jj}=0,\\ \sum_{j=s_1+1}^{r}(\Delta b)_{jj}=0,\\ \sum_{j=1}^{s}\sum_{l=j+1}^{s_1}(\Delta r)_{jl}=0,\\ \sum_{l=s_1+1}^{r}\sum_{j=s+1}^{l-1}(\Delta b)_{jl}=0,\end{gathered} \tag{35}$$

$$\sum_{j=1}^{s}\sum_{l=s_1+1}^{r}\frac{\lambda_j(x)}{\lambda_j(x)-\lambda_l(x)}\left[(\Delta r)_{jl}+(\Delta b)_{jl}\right]=\sum_{j=1}^{s}\sum_{l=s_1+1}^{r}(\Delta b)_{jl}, \tag{36}$$

$$W(\Delta b+\Delta r)=(\Delta b)_{\beta\beta}. \tag{37}$$

We can easily check that

$$W(\Delta b+\Delta r)=W\left((\Delta b)_{\beta\beta}+(\Delta r)_{\beta\beta}\right). \tag{38}$$

Then, by the properties of the projection of the metric projector in Lemma 3,

$$\begin{aligned}\left\langle(\Delta b)_{\beta\beta},(\Delta r)_{\beta\beta}\right\rangle&=\langle W(\Delta b+\Delta r),(\Delta b)_{\beta\beta}\\&\quad+(\Delta r)_{\beta\beta}-W(\Delta b+\Delta r)\rangle\\&=\left\langle(\Delta b)_{\beta\beta}+(\Delta r)_{\beta\beta},\left(W-W^2\right)\right.\\&\quad\left.\cdot\left((\Delta b)_{\beta\beta}+(\Delta r)_{\beta\beta}\right)\right\rangle\geq 0.\end{aligned} \tag{39}$$

Hence, by equations (35)–(37), we obtain from $b^\dagger=\sum_{j=1}^{r}\lambda_j^{-1}(x)c_j$ that

$$\begin{aligned}\langle\Delta b,\Delta r\rangle&=\left\langle\sum_{j=1}^{s}(\Delta b)_{jj}+\sum_{j=1}^{s}\sum_{l=j+1}^{r}(\Delta b)_{jl}+\Delta b_{\beta\beta},\sum_{j=s_1+1}^{r}(\Delta r)_{jj}\right.\\&\quad\left.+\sum_{l=s_1+1}^{r}\sum_{j=1}^{l-1}(\Delta r)_{jl}+\Delta r_{\beta\beta}\right\rangle\\&\geq\left\langle\sum_{j=1}^{s}\sum_{l=s_1+1}^{r}(\Delta b)_{jl},\sum_{j=1}^{s}\sum_{l=s_1+1}^{r}(\Delta r)_{jl}\right\rangle\\&=-\sum_{j=1}^{s}\sum_{l=s_1+1}^{r}\frac{\lambda_l(x)}{\lambda_j(x)}\left\|(\Delta b)_{jl}\right\|^2=4\langle r\cdot\Delta b,b^\dagger\cdot\Delta b\rangle\\&=\Upsilon_b(r,\Delta b).\end{aligned} \tag{40}$$

The proof is completed. □

3. Optimality Conditions and Nonsingularity

We consider nonlinear conic problem (1). Let x belong to the feasible set of problem (1). If

$$0\in\operatorname{int}\left\{\begin{pmatrix}h(x)\\ g(x)\end{pmatrix}+\begin{pmatrix}h'(x)\\ g'(x)\end{pmatrix}X-\begin{pmatrix}0\\ K\end{pmatrix}\right\}, \tag{41}$$

we say that Robinson's constraint qualification holds at x. Then, there exists a Lagrange multiplier $(y,z)\in\mathfrak{R}^m\times Y$ satisfying the following KKT conditions:

$$\begin{gathered}\nabla_x L(x,y,z)=0,\\ h(x)=0,\quad -z\in\mathcal{N}_K(g(x)),\end{gathered} \tag{42}$$

where

$$L(x,y,z)=f(x)+\langle y,h(x)\rangle-\langle z,g(x)\rangle \tag{43}$$

is the Lagrangian function of (1). Let $\Lambda(x)\subset\mathfrak{R}^m\times Y$ be the set of all the Lagrangian multipliers.

It is easy to verify the KKT conditions (42) which can be equivalently expressed as

$$\begin{aligned}F(x,y,z)&=\begin{pmatrix}\nabla_x L(x,y,z)\\ h(x)\\ g(x)-\Pi_K(g(x)-z)\end{pmatrix}\\&=\begin{pmatrix}\nabla_x L(x,y,z)\\ h(x)\\ z-\Pi_K(z-g(x))\end{pmatrix}=0.\end{aligned} \tag{44}$$

Step 0: for a given initial point (x^1, y^1, z^1), calculate the value of $h(x^1), g(x^1), \nabla f(x^1), h'(x^1)$ and $g'(x^1)$. Set $k := 1$.
Step 1: if $\nabla_x L(x^k, y^k, z^k) = 0, h(x^k) = 0, g(x^k) \in Q$, stop.
Step 2: calculate $M_k = M(x^k, y^k, z^k)$ and find a KKT point $(\Delta x^k, y^k_{QP}, z^k_{QP})$ of (Sub_k) by solving the KKT system (67).
Step 3: set $x^{k+1} := x^k + \Delta x^k, y^{k+1} := y^k_{QP}, z^{k+1} := z^k_{QP}$.
Step 4: calculate $h(x^{k+1}), g(x^{k+1}), \nabla f(x^{k+1}), h'(x^{k+1})$ and $g'(x^{k+1})$. Set $k := k+1$. Go to step 1.

Algorithm 1: Local SQP method.

For a general constraint $G(x) \in K'$, $x \in X$, where $G: X \longrightarrow Y'$ is a continuously differentiable function and $K' \subseteq Y'$ is a closed convex set, if for a feasible point x,

$$G'(x)X + \operatorname{lin}(\mathcal{T}_{K'}(G(x))) = Y', \tag{45}$$

then the constraint nondegeneracy condition holds at x. So, for conic optimization problem (1), the constraint nondegeneracy condition at x^* has the following form:

$$\begin{pmatrix} h'(x^*) \\ g'(x^*) \end{pmatrix} X + \begin{pmatrix} 0 \\ \operatorname{lin}(\mathcal{T}_K(g(x^*))) \end{pmatrix} = \begin{pmatrix} \mathfrak{R}^m \\ Y \end{pmatrix}. \tag{46}$$

It follows from [2] that, if a locally optimal solution x^* satisfies (46), then $\Lambda(x^*)$ is a singleton.

For a KKT point $(x^*, y^*, z^*) \in X \times \mathfrak{R}^m \times Y$ of problem (1), suppose that $u^* := g(x^*) - z^*$ has the spectral decomposition:

$$u^* = \lambda_1 c_1 + \cdots + \lambda_r c_r, \tag{47}$$

satisfying

$$\lambda_1 \geq \cdots \geq \lambda_s > 0 = \lambda_{s+1} = \cdots = \lambda_{s_1} > \lambda_{s_1+1} \geq \cdots \geq \lambda_r. \tag{48}$$

Then,

$$\begin{aligned} g(x^*) &= \sum_{j=1}^{s} \lambda_j c_j, \\ z^* &= -\sum_{j=s_1+1}^{r} \lambda_j c_j. \end{aligned} \tag{49}$$

According to (23) and (25), we have

$$\operatorname{lin}(\mathcal{T}_K(g(x^*))) = \left\{ w \in Y \colon w_{\beta\beta} = 0, w_{\beta\gamma} = 0, w_{\gamma\gamma} = 0 \right\}. \tag{50}$$

Although the critical cone $\mathcal{C}(x^*)$ of problem (1) has an explicit formula

$$\begin{aligned} \mathcal{C}(x^*) = \{ & d \in X \colon f'(x^*)d \leq 0, \\ & h'(x^*)d = 0, g'(x^*)d \in \mathcal{T}_K(g(x^*)) \}, \end{aligned} \tag{51}$$

$\operatorname{aff}\mathcal{C}(x^*)$ is not easy to describe. Instead, we define the following outer approximation set to $\operatorname{aff}\mathcal{C}(x^*)$ with respect to (y^*, z^*):

$$\begin{aligned} \operatorname{app}(y^*, z^*) = \Big\{ & d \in X \colon h'(x^*)d = 0, [g'(x^*)d]_{\beta\gamma} \\ & = 0, [g'(x^*)d]_{\gamma\gamma} = 0 \Big\}. \end{aligned} \tag{52}$$

It is easy to get that, for any $(y^*, z^*) \in \Lambda(x^*)$,

$$\operatorname{aff}\mathcal{C}(x^*) \subseteq \operatorname{app}(y^*, z^*). \tag{53}$$

We now introduce a kind of strong second-order sufficient condition for problem (1), which is coincided with the strong second-order sufficient condition in [8, 9] when K is a SDP cone and a second-order cone.

Definition 2. Let x^* be a feasible point of (1) such that constraint nondegeneracy condition (46) holds at x^*. We say that the strong second-order sufficient condition holds at x^* if

$$\begin{aligned} \langle d, \nabla^2_{xx} L(x^*, y^*, z^*) d \rangle + \Upsilon_{g(x^*)}(z^*, g'(x^*)d) > 0, \\ \forall d \in \operatorname{app}(y^*, z^*) \setminus \{0\}, \end{aligned} \tag{54}$$

where $\{(y^*, z^*)\} = \Lambda(x^*) \subset \mathfrak{R}^m \times Y$ and $\operatorname{app}(y^*, z^*)$ is defined by (52).

The following theorem establishes the relationship between the strong second-order sufficient condition and the nonsingularity of Clarke's Jacobian of the mapping F defined by (44).

Theorem 2. *Let x^* is a local minimizer of (1). Assume that $(x^*, y^*, z^*) \in X \times \mathfrak{R}^m \times Y$ is a KKT point to (1). If the constraint nondegeneracy condition and the second-order sufficient condition (53) hold at x^*, then any element in $\partial F(x^*, y^*, z^*)$ is nonsingular.*

Proof. Firstly, we assume that the strong second-order sufficient condition (53) holds at x^* with the constraint nondegeneracy condition (46). We shall prove that any $\mathcal{W} \in \partial F(x^*, y^*, z^*)$ is nonsingular. Let $(\Delta x, \Delta y, \Delta z) \in X \times \mathfrak{R}^m \times Y$ satisfying the condition

$$\mathcal{W}(\Delta x, \Delta y, \Delta z) = 0. \tag{55}$$

Suppose that $u = g(x^*) - z^*$ has the spectral decomposition (47) satisfying (48), then we can write $g(x^*)$ and z^* in the form of (49). From Lemma 1, there exists $V \in \partial \Pi_K(g(x^*))$ such that

$$\begin{aligned} & \mathcal{W}(\Delta x, \Delta y, \Delta z) \\ & = \begin{pmatrix} \nabla^2_{xx} L(x^*, y^*, z^*)\Delta x + \nabla h(x^*)\Delta y - \nabla g(x^*)\Delta z, \\ -h'(x^*)\Delta x, \\ -g'(x^*)\Delta x + V(g'(x^*)\Delta x - \Delta z) \end{pmatrix} = 0. \end{aligned} \tag{56}$$

From the third equality in (56), namely,

$$g'(x^*)\Delta x = V(g'(x^*)\Delta x - \Delta z), \tag{57}$$

it has

$$\begin{gathered}\sum_{j=s_1+1}^{r}\langle c_j, g'(x^*)\Delta x\rangle c_j = 0,\\ \sum_{l=s_1+1}^{r}\sum_{j=s+1}^{l-1}\sum_{i=1}^{d}\langle v_{jl}^{(i)}, (g'(x^*)\Delta x)\rangle v_{jl}^{(i)} = 0.\end{gathered} \tag{58}$$

Then, it implies from (52) and (56) that

$$\begin{aligned}&\Delta x \in \operatorname{app}(y^*, z^*),\\ &0 = \langle \Delta x, \nabla_{xx}^2 L(x^*, y^*, z^*)\Delta x + \nabla h(x^*)\Delta y - \nabla g(x^*)\Delta z\rangle\\ &\quad = \langle \Delta x, \nabla_{xx}^2 L(x^*, y^*, z^*)\Delta x\rangle + \langle -\Delta z, g'(x^*)\Delta x\rangle.\end{aligned} \tag{59}$$

Combining with the last equation of (36) and Lemma 4, we obtain

$$\langle \Delta x, \nabla_{xx}^2 L(x^*, y^*, z^*)\Delta x\rangle + \Upsilon_{g(x^*)}(z^*, g'(x^*)\Delta x) \le 0. \tag{60}$$

Comparing with the strong second-order sufficient condition in the strong form (53), we get $\Delta x = 0$. Thus, (36) can be expressed in the following form:

$$\begin{pmatrix}\nabla h(x^*)\Delta y - \nabla g(x^*)\Delta z\\ V(\Delta z)\end{pmatrix} = 0. \tag{61}$$

From $V(\Delta z) = 0$, we have

$$\begin{gathered}(\Delta z)_{\alpha\alpha} = 0,\\ (\Delta z)_{\alpha\beta} = 0,\\ (\Delta z)_{\alpha\gamma} = 0.\end{gathered} \tag{62}$$

Then, it follows from the nondegeneracy (46) that there exists a vector $d \in X$ and $w \in \operatorname{lin}(\mathcal{T}_K(g(x^*)))$ satisfying

$$\begin{gathered}h'(x^*)d = \Delta y,\\ g'(x^*)d + w = -\Delta z.\end{gathered} \tag{63}$$

Therefore, we have

$$\begin{aligned}\langle \Delta y, \Delta y\rangle + \langle \Delta z, \Delta z\rangle &= \langle \Delta y, h'(x^*)d\rangle - \langle \Delta z, g'(x^*)d + w\rangle\\ &= -\langle w, \Delta z\rangle = 0,\end{aligned} \tag{64}$$

where the last equation can be obtained by (50) and (62). Thus, $\Delta y = 0$, $\Delta z = 0$. Together with $\Delta x = 0$, we get that $\mathcal{W}$ is nonsingular. The proof is completed. □

4. Application

In this section, as an application of the nonsingularity in Theorem 2, we will study the sequential quadratic programming- (SQP-) type method to solving problem (1). Let (x^k, y^k, z^k) be the current iteration point. The new iteration points $(x^{k+1}, y^{k+1}, z^{k+1})$ will be generated by solving the following quadratic problem:

(Sub_k)

$$\begin{aligned}&\min_{\Delta x} \nabla f(x^k)^T \Delta x + \frac{1}{2}\Delta x^T M_k \Delta x,\\ &\text{s.t.}\quad \begin{aligned}&h(x^k) + h'(x^k)\Delta x = 0,\\ &g(x^k) + g'(x^k)\Delta x \in K,\end{aligned}\end{aligned} \tag{65}$$

where $M_k := M(x^k, y^k, z^k)$ and $M\colon X \times \mathfrak{R}^m \times Y \longrightarrow X \times X$ is a matrix function satisfying $M(x^*, y^*, z^*) = \nabla_{xx}^2 L(x^*, y^*, z^*)$. This model introduced in [18] to solve the classical nonlinear programming problems is used to solve the nonlinear SDP problems [26, 27] and nonlinear second-order cone programming problems [28, 29], where they may choose the different forms of $M(\cdot,\cdot,\cdot)$.

Theorem 3. *Suppose that $(x^*, y^*, z^*) \in X \times \mathfrak{R}^m \times Y$ is a KKT point to (1) and the second-order sufficient condition (53) with constraint nondegeneracy condition holds at $x^* x^*$. Let the matrix function $M\colon X \times \mathfrak{R}^m \times Y \longrightarrow X \times X$ satisfying $M(x^*, y^*, z^*) = \nabla_{xx}^2 L(x^*, y^*, z^*)$ be semismooth at (x^*, y^*, z^*). Then, for any $(x^k, y^k, z^k) \in \mathcal{U}$, a neighborhood of (x^*, y^*, z^*), (39) has a KKT solution $(\Delta x^k, y_{QP}^k, z_{QP}^k)$ satisfying*

$$\begin{aligned}&\left\|\Delta x^k\right\| + \left\|y_{QP}^k - y^*\right\| + \left\|z_{QP}^k - z^*\right\|\\ &\quad = O\left(\left\|\left(x^k, y^k, z^k\right) - (x^*, y^*, z^*)\right\|\right).\end{aligned} \tag{66}$$

Proof. Let $(\Delta x, y_{QP}, z_{QP})$ be a KKT point of (Sub_k). Then,

$$\begin{aligned}&\nabla f(x^k) + z_k\Delta x + \nabla h(x^k) y_{QP} - \nabla g(x^k) z_{QP} = 0,\\ &h(x^k) + h'(x^k)\Delta x = 0,\\ &g(x^k) + g'(x^k)\Delta x = \Pi_K\big(g(x^k)\\ &\qquad + g'(x^k)\Delta x - z_{QP}\big),\end{aligned} \tag{67}$$

namely,

$$\widehat{F}\left(\Delta x, y_{QP}, z_{QP}, x^k, y^k, z^k\right) = 0, \tag{68}$$

where

$$\widehat{F}(\zeta, \eta, \xi, x, y, z) := \begin{pmatrix}\nabla f(x) + M(x, y, z)\zeta + \nabla h(x)\eta - \nabla g(x)\xi,\\ h(x) + h'(x)\zeta,\\ g(x) + g'(x)\zeta - \Pi_K(g(x) + g'(x)\zeta - \xi)\end{pmatrix}. \tag{69}$$

is a function with the variable $(\zeta, \eta, \xi, x, \lambda, \mu) \in X \times \mathfrak{R}^m \times Y \times X \times \mathfrak{R}^m \times Y$. Let $\tau := (\zeta, \eta, \xi) \in X \times \mathfrak{R}^m \times Y$ and $\nu := (x, \lambda, \mu) \in X \times \mathfrak{R}^m \times Y$. Let $\tau^* = (0, y^*, z^*)$ and $\nu^* = (x^*, y^*, z^*)$. We can easily have the following equation:

$$\widehat{F}(0, y^*, z^*, x^*, y^*, z^*) = \widehat{F}(\tau^*, \nu^*) = 0. \tag{70}$$

By replacing (x^k, M_k) in (65) with , we obtain a new problem expressed as (Sub_*), and $(0, y^*, z^*)$ is an KKT point of (Sub_*).

Therefore, the strong second-order sufficient condition of (Sub_*) at $(0, y^*, z^*)$ under the constraint nondegeneracy condition (46) has the same expression as condition (53). It is concluded from Theorem 2 that any element $W \in \Pi_{X\times\mathfrak{R}^m\times Y}\partial\widehat{F}(\tau^*, z^*)$ is nonsingular. Then, there exists a strong semismooth function $\tau(\cdot)\colon \mathscr{U} \longrightarrow X\times\mathfrak{R}^m\times Y$ satisfying

$$\begin{aligned} \tau(\nu^*) &= \tau^*, \\ \widehat{F}(\tau(\nu), \nu) &= 0, \quad \forall \nu \in \mathscr{U}, \end{aligned} \tag{71}$$

where $\mathscr{U}$ is a neighborhood of ν^*. Denote $(\Delta x^k, y^k_{QP}, z^k_{QP}) \coloneqq \tau(\nu^k)$. Then, $(\Delta x^k, y^k_{QP}, z^k_{QP})$ is a KKT point of (Sub_k) if $\nu^k \in \mathscr{U}$. By the strong semismoothness of $\tau(\cdot)$,

$$\left\|\tau\left(\nu^k\right) - \tau^*\right\| = O\left(\left\|\nu^k - \nu^*\right\|\right). \tag{72}$$

The proof is completed.

Now we present a local SQP method based on solving SQP-type model (Sub_k) in each iteration to solve (1). □

Next, the primal-dual quadratic convergence of Algorithm 1 is demonstrated by using the semismoothness of M and Theorem 2.

Theorem 4. *Suppose that all the hypotheses of Theorem 3 still hold for Algorithm 1. Then, there exists a neighborhood $\mathscr{U}$ of (x^*, y^*, z^*) such that, for any $(x^1, y^1, z^1) \in \mathscr{U}$, the sequence $\{(x^k, y^k, z^k)\}$ generated by Algorithm 1 converges to (x^*, y^*, z^*) quadratically.*

Proof. According to Theorem 3, it is easy to verify that the algorithm is well-defined. Denote

$$\delta_k \coloneqq \left\|\left(x^k, y^k, z^k\right) - (x^*, y^*, z^*)\right\|. \tag{73}$$

We obtain

$$\begin{aligned} \Delta x^k &= O(\delta_k), \\ y^{k+1} - y^* &= O(\delta_k), \\ z^{k+1} - z^* &= O(\delta_k), \end{aligned} \tag{74}$$

where Δx^k is a solution to (65), and $y^{k+1} = y^k_{QP}, z^{k+1} = z^k_{QP}$ is the associated multiplier.

As $M(x^*, y^*, z^*) = \nabla^2_{xx}L(x^*, y^*, z^*)$ and M is semismooth at (x^*, y^*, z^*),

$$M_k - \nabla^2_{xx}L(x^*, y^*, z^*) = O\left(\left\|\left(x^k, y^k, z^k\right) - (x^*, y^*, z^*)\right\|\right). \tag{75}$$

It follows from the Taylor expansion to (67) at (x^*, y^*, z^*), $\nabla_x L(x^*, y^*, z^*) = 0$, $x^{k+1} = x^k + \Delta x^k$, (74), and (75) that

$$\begin{aligned} &\nabla^2_{xx}L(x^*, y^*, z^*)\left(x^{k+1} - x^*\right) + \nabla h(x^*)\left(y^{k+1} - y^*\right) \\ &\quad - \nabla g(x^*)\left(z^{k+1} - z^*\right) = O\left(\delta_k^2\right), \end{aligned} \tag{76}$$

$$h'(x^*)\left(x^{k+1} - x^*\right) = O\left(\delta_k^2\right). \tag{77}$$

Because of the strongly semismoothness of the projection operator $\Pi_K(\cdot)$, we get $V \in \partial\Pi_K(g(x^*) - z^*)$ satisfying

$$\begin{aligned} \Pi_K(g(x^*) - z^*) &= \Pi_K\left(g\left(x^k\right) + g'\left(x^k\right)\Delta x^k - z^k_{QP}\right) \\ &\quad + V\left(g(x^*) - z^* - g\left(x^k\right) - g'\left(x^k\right)\Delta x^k + z^k_{QP}\right) \\ &\quad + O\left(\left\|g(x^*) - z^* - g\left(x^k\right) - g'\left(x^k\right)\Delta x^k + z^k_{QP}\right\|^2\right). \end{aligned} \tag{78}$$

Since

$$\begin{aligned} &g(x^*) - z^* - g\left(x^k\right) - g'\left(x^k\right)\Delta x^k + z^k_{QP} \\ &\quad = g(x^*) - g(x^*) - g'(x^*)\left(x^k - x^*\right) + O\left(\left\|x^k - x^*\right\|^2\right) \\ &\qquad - g'(x^*)\Delta x^k + O\left(x^k - x^*\right)\Delta x^k - z^* + z^{k+1}g'(x^*) \\ &\qquad \cdot\left(x^* - x^{k+1}\right) - z^* + z^{k+1} + O\left(\delta_k^2\right), \end{aligned} \tag{79}$$

it is known that

$$\begin{aligned} &\Pi_K\left(g\left(x^k\right) + g'\left(x^k\right)\Delta x^k - z^k_{QP}\right) = \Pi_K(g(x^*) - z^*) \\ &\quad - V\left(g'(x^*)\left(x^* - x^{k+1}\right) - z^* + z^{k+1}\right) + O\left(\delta_k^2\right), \end{aligned} \tag{80}$$

which, together with $\Pi_K(g(x^*) - z^*) = g(x^*)$ and $\Pi_K(g(x^k) + g'(x^k)\Delta x^k - z^k_{QP}) = g(x^k) + g'(x^k)\Delta x^k$, implies

$$(V - I)g'(x^*)\left(x^{k+1} - x^*\right) - V\left(z^{k+1} - z^*\right) = O\left(\delta_k^2\right). \tag{81}$$

Following from (76), (77), with (81), we obtain

$$\begin{aligned} &\begin{pmatrix} \nabla^2_{xx}L(x^*, y^*, z^*) & \nabla h(x^*) & -\nabla g(x^*) \\ h'(x^*) & 0 & 0 \\ -g'(x^*) + Vg'(x^*) & 0 & -V \end{pmatrix}\begin{pmatrix} x^{k+1} - x^* \\ y^{k+1} - y^* \\ z^{k+1} - z^* \end{pmatrix} \\ &\quad = O\left(\delta_k^2\right). \end{aligned} \tag{82}$$

Combining (46) and (54) with Theorem 2, we show that

$$\left\|\begin{pmatrix} x^{k+1} - x^* \\ y^{k+1} - y^* \\ z^{k+1} - z^* \end{pmatrix}\right\| = O\left(\delta_k^2\right), \tag{83}$$

which means the quadratic convergence of the sequence $\{(x^k, y^k, z^k)\}$. Then, we complete the proof. □

5. Conclusion

In this paper, we discuss the nonlinear symmetric conic programming problems. We show that a kind of strong second-order sufficient condition, together with constraint nondegeneracy condition, implies the nonsingularity of Clarke's generalized Jacobian of the mapping F at the KKT point. In the special cases of NLP, SCOP, and SDP, the converses are also true. Then, we demonstrate the local quadratic convergence of the SQP-type method for solving the nonlinear symmetric conic programming problems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Nonparametric Estimation of Fractional Option Pricing Model

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The establishment of the fractional Black–Scholes option pricing model is under a major condition with the normal distribution for the state price density (SPD) function. However, the fractional Brownian motion is deemed to not be martingale with a long memory effect of the underlying asset, so that the estimation of the state price density (SPD) function is far from simple. This paper proposes a convenient approach to get the fractional option pricing model by changing variables. Further, the option price is transformed as the integral function of the cumulative density function (CDF), so it is not necessary to estimate the distribution function individually by complex approaches. Finally, it encourages to estimate the fractional option pricing model by the way of nonparametric regression and makes empirical analysis with the traded 50 ETF option data in Shanghai Stock Exchange (SSE).

1. Introduction

In the financial market, the memory effect of asset price has been described by the fractional Brownian motion (FBM). The first finding of long memory effects in stock returns was reported by Mandelbrot and Van Ness who also defined the fractional Brownian motion [1]. The memory effect between 0 and 1 is measured by Hurst index (H). Specifically speaking, the asset price has long memory effects if the Hurst index is between 1/2 and 1 whereas the asset price has short memory effects if the Hurst index is between 0 and 1/2. However, there is no memory effect when the Hurst index H is equal to 1/2.

According to the stochastic differential equation driven by the fractional Brownian motion, a large number of literature studies have studied the option pricing models of improving the classical Black–Scholes option pricing model (see Black and Scholes [2]). For instance, the study was reported by Necula [3], Rostek [4], and Hu and Øksendal [5] that fractional Black–Scholes pricing model (FBS) is obtained on the condition that the underlying asset price process obeys the fractional Brownian motion (FBM). Some results reflect the study reported by Ren et al. [6] who found that the option pricing model is linking with the Hurst index between 0.5 and 1. One study done by Wang et al. [7] examined the fractional option pricing formula is carried out when the Hurst index is between 1/3 and 1/2. One study by Chen et al. [8] offers another empirical analysis of the mixed fractional-fractional version of the Black–Scholes model with the Hurst index between 0 and 1.

There are two defects for the existing fractional Black–Scholes option pricing models. Firstly, the existing fractional Black–Scholes option pricing models corroborate the condition of the lognormal distribution of SPD. In practice, it is hard to undertake the estimation of the state price density (SPD) function when the underlying asset process is not a martingale; in addition, the state price density function (SPD) is unknown.

This paper is designed to relax the assumption in the fractional Black–Scholes pricing model (FBS) so that the returns of the underlying asset obey the lognormal distribution, and the option price will be transformed to the integral function of the cumulative density function (CDF). As a result, it is not necessary to estimate the distribution function individually via complex approaches. This idea of variable transformation is inspired by the research found by Ait-Sahalia [9], Xiu [10], and Vogt [11]. The option price can be transformed to a regression equation with the changing variables, which can be estimated by the local polynomial model proposed by Fan and Gijbels [12] and Li and Racine [13].

Nonparametric pricing option has been present among researchers Ait-Sahalia and Lo [14, 15]. In order to overcome model errors, the semiparametric Black–Scholes model (SBS) has been proposed by Ait-Sahalia and Lo [14] with the implied volatility in the Black–Scholes option pricing model. The research done by Dumas et al. [16] carried out the so-called ad hoc Black–Scholes model in that implied volatility is the parabolic function of moneyness. Inspired by Ait-Sahalia and Lo [14], Fan and Mancini [17] proposed the semiparametric Black-Sholes model in that the implied volatility was the nonparametric estimator of moneyness.

The outline of the article is illustrated as follows. In Section 2, the analysis of the fractional Black–Scholes option pricing model and nonparametric fractional option pricing model established when a variable happens to change along with nonparametric fractional option pricing models is by the local polynomial regression. In Section 3, with the use of the traded 50 ETF option prices in Shanghai Stock Exchange (SSE), the experimental work compares the analysis of the effectiveness among classical Black–Scholes (BS) option pricing model, semiparametric Black–Scholes pricing model (SBS), semiparametric fractional Black–Scholes (SFBS) option pricing model, and nonparametric fractional (NF) option pricing model. In Section 4, several conclusions are given about the different option pricing models.

2. Pricing European Option by Changing Variables

Although the fractional Black–Scholes has improved the pricing performance, the application of the model is still under the condition of lognormal distribution and the framework of parametric Black–Scholes. The importance of the study is that it explores a new achievement in an orthogonal way instead of improving the pricing model to a more flexible level. The nonparametric fractional Black–Scholes model is established to improve the pricing performance by relaxing the lognormal distribution of the returns of the underlying asset (or random variable) to be nonparametric.

2.1. Black–Scholes Option Pricing Model by Changing Variables. This section will propose the following changing variables to obtain closed-form expressions of the Black–Scholes option pricing model. Let $P(f_0^{Q_1}, U_1)$ be the European put option price, and S_T is considered as the underlying asset price at time T and K is the strike price. Then, $\tau = T - t$ is regarded as the time to maturity and $f_0^{Q_1}(S_T|\tau)$ means the state price probability density function, while r is the riskless interest rate, and the price of European put option refers to the discounted expressed payoff in the risk-neutral world:

$$\begin{aligned} P\left(f_0^{Q_1}, U_1\right) &= e^{-r\tau} E\left[\max\left(K - S_T, 0\right)\right] \\ &= e^{-r\tau} \int_0^K (K - S) f_0^{Q_1}(S| \end{aligned} \tag{1}$$

The underlying asset price S_t follows the Brownian motion:

$$\mathrm{d}S_t = rS_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \tag{2}$$

where r is the riskless rate, σ is the diffusion coefficient, and B_t is the standard Brownian motion.

According to Ito's lemma, the price process is as follows:

$$\ln\left(\frac{S_T}{S_0}\right) = \mu_1(U_1) + \sigma_1(U_1) Z_1, \tag{3}$$

where $\mu_1(U_1)$ and $\sigma_1(U_1)$ are the known functions of the characteristics of option parameters $U_1 = (S, K, \tau, r, \sigma)$, and $\mu_1(U_1) = (r - 1/2\sigma^2)\tau$ and $\sigma_1(U_1) = \sigma\sqrt{\tau}$. $Z_1 \sim f_0(\cdot|\tau)$, in which $f_0(\cdot|\tau)$ is the unknown state price density function to be nonparametrically estimated by the market data.

From equation (3), Brownian motion is concretely described by the underlying asset as follows:

$$Z_1 = \frac{\left[\ln\left(S_T/S_0\right) - \mu_1(U_1)\right]}{\sigma_1(U_1)}. \tag{4}$$

By changing variables, the option valuation equation (1) becomes

$$\begin{aligned} P\left(f_0^{Q_1}, U_1\right) &= e^{-r\tau} E\left[\max\left(K - S_T, 0\right)\right] = e^{-r\tau} \int_0^K (K - S) f_0^{Q_1}(S| \\ &= e^{-r\tau} \int_{+\infty}^{d_1(U_1)} \left(K - S_0 e^{\mu_1(U_1) + \sigma_1(U_1) U_1}\right) f_0(Z_1|\tau) \mathrm{d}Z_1 = P_{U_1}(f_0, U_1), \end{aligned} \tag{5}$$

where

$$d_1(U_1) = \frac{\left[\ln\left(K/S_0\right) - \mu_1(U_1)\right]}{\sigma_1(U_1)}. \tag{6}$$

The relationship between $f_0^{Q_1}(S_T|\tau)$ and $f_0(Z_1|\tau)$ is as follows:

$$f_0^{Q_1}(S_T|\tau) = \left[S_0 \sigma_1(U_1) e^{\mu_1(U_1) + \sigma_1(U_1) Z_1}\right]^{-1} f_0(Z_1|\tau). \tag{7}$$

The state price density function $f_0(Z_1|\tau)$ is the normal distribution as follows:

$$f_0(Z_1|\tau) = \frac{1}{\sqrt{2\pi}} e^{-Z_1^2/2}. \tag{8}$$

The systematic analysis of option valuation is the Black–Scholes option pricing model [2] as follows:

$$\begin{aligned}
P\left(f_0^{Q_1}, U_1\right) &= e^{-r\tau} \int_0^{d_1(U_1)} \left(K - S_0 e^{\mu_1(U_1)+\sigma_1(U_1)Z_1}\right) f_0(Z_1|\tau) dZ_1 \\
&= e^{-r\tau} K \int_0^{d_1(U_1)} f_0(Z_1|\tau) dZ_1 - e^{-r\tau} \int_0^{d_1(U_1)} S_0 e^{\mu_1(U_1)+\sigma_1(U_1)Z_1} f_0(Z_1|\tau) dZ_1 \\
&= e^{-r\tau} K \int_0^{d_1(U_1)} \frac{1}{\sqrt{2\pi}} e^{-Z_1^2/2} dZ_1 - e^{-r\tau} \int_0^{d_1(U_1)} S_0 e^{\mu_1(U_1)+\sigma_1(U_1)Z_1} \frac{1}{\sqrt{2\pi}} e^{-Z_1^2/2} dZ_1 \\
&= e^{-r\tau} K N(d_1(U_1)) - S_0 e^{\mu_1(U_1)+1/2\sigma_1^2(U_1)} \int_0^{d_1(U_1)} \frac{1}{\sqrt{2\pi}} e^{-1/2[Z_1-\sigma_1(U_1)]^2} dZ_1 \\
&= e^{-r\tau} K N(d_1(U_1)) - S_0 e^{\mu_1(U_1)+1/2\sigma_1^2(U_1)} \int_0^{d_1(U_1)-\sigma_1(U_1)} \frac{1}{\sqrt{2\pi}} e^{-1/2x^2} dx \\
&= e^{-r\tau} K N(d_1(U_1)) - S_0 N(d_1(U_1) - \sigma_1(U_1)) \\
&= e^{-r\tau} K N\left(\frac{\ln(K/S_0) - \left(r - \left(\sigma^2/2\right)\right)\tau}{\sigma\sqrt{\tau}}\right) - S_0 N\left(\frac{\ln(K/S_0) - \left(r - \left(\sigma^2/2\right)\right)\tau}{\sigma\sqrt{\tau}} - \sigma\sqrt{\tau}\right) \\
&= e^{-r\tau} K N\left(-\frac{\ln(S_0/K) + \left(r - \left(\sigma^2/2\right)\right)\tau}{\sigma\sqrt{\tau}}\right) - S_0 N\left(-\frac{\ln(S_0/K) + \left(r + \left(\sigma^2/2\right)\right)\tau}{\sigma\sqrt{\tau}}\right) \\
&= P_{BS}(f_0, Z_1),
\end{aligned} \tag{9}$$

where let $x = Z_1 - \sigma_1(Z_1)$. That is the classical Black–Scholes option pricing model when volatility turns to the history volatility:

$$P_{\mathrm{BS}} = e^{-r\tau} K N(-d_{12}) - S_0 N(-d_{11}), \tag{10}$$

where $d_{11} = \ln(S_0/K) + (r + (\sigma^2/2))\tau/\sigma\sqrt{\tau}$ and $d_{12} = d_{11} - \sigma\sqrt{\tau} = \ln(S_0/K) + (r - (\sigma^2/2))\tau/\sigma\sqrt{\tau}$.

Furthermore, model (10) has a fine description about semiparametric Black–Scholes model (SBS) proposed by Ait-Sahalia and Lo [14] and Fan and Mancini [17] with implied volatility. Fan and Mancini [17] proposed a nonparametric approach to fit the implied volatility function:

$$\sigma_{t,i}^{IV} = G\left(m_{t,i}\right) + \varepsilon_{t,i}, \quad i = 1, 2, \ldots, n, \tag{11}$$

where $m_{t,i} = K/F_{t,\tau}$ is the moneyness and $F_{t,\tau} = (C_t - P_t)e^{-r_{t,\tau}\tau} + K = S_t e^{(r_{t,\tau} - \delta_{t,\tau})\tau}$ means the forward price, the forward price is obtained from the put-call parity $C_t + Ke^{-r_{t,\tau}\tau} = P_t + F_{t,\tau}e^{-r_{t,\tau}\tau}$, P denotes the put price, and C denotes the call price.

However, the random variable Z_1 does not obey the log-normal distribution, which is unknown. By changing variables, option price can be illustrated by the integral function about random variable Z_1 depending on function $d_1(U_1)$. When the state price cumulative density function $F_0(Z_1|\tau)$ is unknown,

$$\begin{aligned}
P\left(f_0^{Q_1}, U_1\right) &= e^{-r\tau} \int_0^K f_0^{Q_1}(S|\tau) dS = e^{-r\tau} \int_0^{d_1(U_1)} \\
&\quad \cdot \left(K - S_0 e^{\mu_1(U_1)+\sigma_1(U_1)Z_1}\right) f_0(Z_1|\tau) dZ_1 \\
&= e^{-r\tau} \int_0^{d_1(U_1)} K\sigma_1(U_1) f_0(Z_1|\tau) dZ_1 \\
&= e^{-r\tau} K\sigma_1(U_1) \int_0^{d_1(U_1)} f_0(Z_1|\tau) dZ_1,
\end{aligned} \tag{12}$$

where $f_0(Z_1|\tau)$ is the cumulative density function (CDF) of random variable Z_1 and $f_0(Z_1|\tau)$ is unknown function.

Because the function $f_0(Z_1|\tau)$ is unknown, and let $\int_0^{d_1(U_1)} f_0(Z_1|\tau) dZ_1 = G(d_1(U_1))$, equation (12) will be the form as follows:

$$\begin{aligned}
P\left(f_0^{Q_1}, U_1\right) &= e^{-r\tau} K\sigma_1(U_1) \int_0^{d_1(U_1)} f_0(Z_1|\tau) dZ_1 \\
&= e^{-r\tau} K\sigma_1(U_1) G(d_1(U_1)).
\end{aligned} \tag{13}$$

It can be found that the option price is the function of one-dimensional variable $d_1(U_1)$ and distribution function $F_0(B|\tau)$.

From equation (13), the nonparametric estimation equation has been established between put option price function $P(f_0^{Q_1}, Z_1)$ and variable $d_1(U_1)$ as follows:

$$Y_i = G(X_i) + \varepsilon_i, \quad i = 1, 2, \ldots, n, \tag{14}$$

where $G(\cdot)$ is the unknown function to be estimated, $Y_i = e^{r\tau}/K\sigma_1(U_1) P(f_0^{Q_1}, Z_1)$, $X_i = \ln(K_i/F_{t,\tau}) + (\sigma^2/2)\tau/\sigma\sqrt{\tau}$, and ε_i features i.i.d with zero mean and common variance σ^2.

2.2. Fractional Option Pricing Model by Changing Variables. The correlational analysis of stock price is set out by a fractional Brownian motion when the stock price process has memory effects. In this section, the fractional option pricing model and nonparametric fractional option pricing model have been established on the condition that the stock price is subject to the fractional Brownian motion by changing variables.

Assume that the underlying asset price S_t follows the fractional Brownian motion:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}B_H(t), \tag{15}$$

where $B_H(t)$ is subject to fractional Brownian motion and H means the Hurst index and H can be estimated by R/S analysis approach.

The fractional Brownian motion $B_H(t)$ can be denoted by the standard Brownian motion $B(t)$ as follows:

$$B_H(t) = C_H\left[\int_{-\infty}^{0}\left((t-s)^{H-1/2} - (-s)^{H-1/2}\right)\mathrm{d}B(s) + \int_0^t (t-s)^{H-1/2}\mathrm{d}B(s)\right]. \tag{16}$$

The increment of the fractional Brownian motion $\Delta B_H(t)$ obeys the standard normal distribution:

$$\Delta B_H(t) \sim N\left(0, (\Delta t)^{2H}\right). \tag{17}$$

The autocovariance function of between $\Delta B_H(t)$ and $\Delta B_H(t+s)$ is as follows:

$$\operatorname{Cov}(\Delta B_H(t), \Delta B_H(t+s)) = \frac{1}{2}(\Delta t)\left(|s+1|^{2H} + |s-1|^{2H} - 2|s|^{2H}\right). \tag{18}$$

From equation (15), the stock price process is as follows:

$$\ln\left(\frac{S_T}{S_0}\right) = \mu_2(U_2) + \sigma_2(U_2)B_H(t), \tag{19}$$

where $U_2 = (S, K, \tau, r, \sigma, H)$ and $\mu_2(U_2) = \mu\tau - 1/2\sigma^2(\tau)^{2H}$, $\sigma_2(U_2) = \sigma(\tau)^H$.

In order to make the variable transformation, let Z_2 be the random variable with memory:

$$Z_2 = \frac{\ln(S_T/S_0) - \mu_2(U_2)}{\sigma_2(U_2)}. \tag{20}$$

Then, equation (1) will be

$$\begin{aligned} P\left(f_0^{Q_2}, U_2\right) &= e^{-r\tau}E[\max(K - S_T, 0)] \\ &= e^{-r\tau}\int_0^K (K-S) f_0^{Q_2}(S|\tau)\mathrm{d}S \\ &= e^{-r(T-\tau)}\int_{+\infty}^{d_2(U_2)}\left(K - S_0 e^{\mu_2(U_2)+\sigma_2(U_2)U_2}\right) \\ &\quad \cdot f_0(Z_2|\tau)\mathrm{d}Z_2 \\ &= P_{U_2}(f_0, U_2), \end{aligned} \tag{21}$$

where

$$d_2(U_2) = \frac{\ln(K/S_0) - \mu_2(U_2)}{\sigma_2(U_2)}. \tag{22}$$

The density function $f_0(Z_2|\tau)$ is given as normal distribution as follows:

$$f_0(Z_2|\tau) = \frac{1}{\sqrt{2\pi}}e^{-Z_2^2/2}. \tag{23}$$

The option valuation is discussed as the fractional Black–Scholes (FBSM) option pricing model (see Necula [3]):

$$\begin{aligned} P\left(f_0^{Q_2}, U_2\right) &= e^{-r\tau}\int_0^{d_2(U_2)}\left(K - S_0 e^{\mu_2(U_2)+\sigma_2(U_2)Z_2}\right) f_0(Z_2)\mathrm{d}Z_2 \\ &= e^{-r\tau}K\int_0^{d_2(U_2)} f_0(Z_2)\mathrm{d}Z_2 - e^{-r\tau}\int_0^{d_2(U_2)} S_0 e^{\mu_2(U_2)+\sigma_2(U_2)Z_2} f_0(Z_2)\mathrm{d}Z_2 \\ &= e^{-r\tau}K\int_0^{d_2(U_2)}\frac{1}{\sqrt{2\pi}}e^{-Z_2^2/2}\mathrm{d}Z_2 - e^{-r\tau}\int_0^{d_2(U_2)} S_0 e^{\mu_2(U_2)+\sigma_2(U_2)Z_2}\frac{1}{\sqrt{2\pi}}e^{-Z_2^2/2}\mathrm{d}Z_2 \\ &= e^{-r\tau}KN(d_2(U_2)) - S_0 e^{\mu_2(U_2)+1/2\sigma_2^2(U_2)}\int_0^{d_2(U_2)}\frac{1}{\sqrt{2\pi}}e^{-1/2[Z_2-\sigma_2(U_2)]^2}\mathrm{d}Z_2 \\ &= e^{-r\tau}KN(d_2(U_2)) - S_0 e^{\mu_2(U_2)+1/2\sigma_2^2(U_2)}\int_0^{d_2(U_2)-\sigma_2(U_2)}\frac{1}{\sqrt{2\pi}}e^{-1/2x^2}\mathrm{d}x \\ &= e^{-r\tau}KN(d_2(U_2)) - S_0 N(d_2(U_2) - \sigma_2(U_2)) \\ &= e^{-r\tau}KN\left(\frac{\ln(K/S_0) - \left(r\tau - \left(\sigma^2/2\right)\tau^{2H}\right)}{\sigma\tau^H}\right) - S_0 N\left(\frac{\ln(K/S_0) - \left(r\tau - \left(\sigma^2/2\right)\tau^{2H}\right)}{\sigma\tau^H} - \sigma\tau^H\right) \\ &= e^{-r\tau}KN\left(-\frac{\ln(S_0/K) + \left(r\tau - \left(\sigma^2/2\right)\tau^{2H}\right)}{\sigma\tau^H}\right) - S_0 N\left(-\frac{\ln(S_0/K) + \left(r\tau + \left(\sigma^2/2\right)\tau^{2H}\right)}{\sigma\tau^H}\right) \\ &= P_{\mathrm{FBS}}(f_0, Z_2), \end{aligned} \tag{24}$$

where let $x = Z_2 - \sigma_2(Z_2)$. Generally, the fractional Black–Scholes option pricing model is given by

$$P_{\text{FBS}} = e^{-r\tau} KN(-d_{22}) - S_0 N(-d_{21}), \tag{25}$$

where $d_{21} = \ln(S_0/K) + (r\tau + (\sigma^2/2)\tau^{2H})/\sigma\tau^H$ and $d_{22} = d_{21} - \sigma\tau^H = \ln(S_0/K) + (r\tau - (\sigma^2/2)\tau^{2H})/\sigma\tau^H$.

However, the state price density function is unknown in practice. What makes it more complicated is that the fractional Brownian motion is neither martingale nor semimartingale. Therefore, the estimation of the density function is difficult to estimate due to the existing memory effects of the underlying asset:

$$\begin{aligned} P\left(f_0^{Q_2}, U_2\right) &= e^{-r\tau}\int_0^K (K-S) f_0^{Q_2}(S|\tau)\mathrm{d}S = e^{-r\tau}\int_0^{d_2(U_2)} \\ &\cdot\left(K - S_0 e^{\mu_2(U_2)+\sigma_2(U_2)Z_2}\right) f_0(Z_2|\tau)\mathrm{d}Z_2 \\ &= e^{-r\tau}\int_0^{d_2(U_2)} K\sigma_2(U_2) f_0(Z_2|\tau)\mathrm{d}Z_2 \\ &= e^{-r\tau} K\sigma_2(U_2)\int_0^{d_2(U_2)} f_0(Z_2|\tau)\mathrm{d}Z_2. \end{aligned} \tag{26}$$

In fact, the density function $f_0(Z_2|\tau)$ is hard to estimate for two reasons: $f_0(Z_2|\tau)$ is unknown and $f_0(Z_2|\tau)$ has memory effects. Therefore, a new idea is put forward not to estimate the function $f_0(Z_2|\tau)$ directly. Let $\int_0^{d_2(U_2)} f_0(Z_2|\tau)\mathrm{d}Z_2 = G(d_2(U_2))$ and the nonparametric regression equation is proposed as follows:

$$\begin{aligned} P\left(f_0^{Q_2}, U_2\right) &= e^{r\tau} K\sigma_2(U_2)\int_0^{d_2(U_2)} f_0(Z_2|\tau)\mathrm{d}Z_2 \\ &= e^{r\tau} K\sigma_2(U_2) G(d_2(U_2)). \end{aligned} \tag{27}$$

According to equation (27), the nonparametric regression equation is given by

$$Y_i = G(X_i) + \varepsilon_i, \quad i = 1, 2, \ldots, n, \tag{28}$$

where $Y_i = e^{r\tau}/K\sigma_2(Z_2)p(f_0^{Q_2}, U_2) = e^{r\tau}/K\sigma\tau^H P(f_0^{Q_2}, U_2)$, $X_i = \mathrm{d}(U_2) = \ln(K_i/S_0) - \mu_2(U_2)/\sigma_2(U_2) = \ln(K_i/F_{t,\tau}) + (\sigma^2/2)\tau^{2H}/\sigma\tau^H$, and ε_i features i.i.d. with zero mean and common variance σ^2.

2.3. Nonparametric Regression Estimation of Option Prices. We can estimate the nonparametric regression model (28) by local polynomial approach in Fan and Gijbels [12]:

$$Y_i = G(X_i) + \varepsilon_i, \quad i = 1, 2, \ldots, n, \tag{29}$$

where $Y = e^{r\tau}/K\sigma\tau^H P(f_0^{Q_2}, U_2)$, $X = \ln(K_i/F_{t,\tau}) + (\sigma^2/2)\tau^{2H}/\sigma\tau^H$, and ε_i features i.i.d with zero mean and common variance σ^2.

We approximate the unknown regression function $G(X)$ locally by a polynomial of order m, and the Taylor expansion of $G(X)$ in the neighborhood of x is given by

$$G(X) = \sum_{k=0}^{m} \frac{m^{(k)}(x)}{k!}(X - x)^k. \tag{30}$$

The nonparametric regression equation (29) will be estimated by a weighted least squares regression problem [12]:

$$\min \sum_{i=0}^{n} \left\{ Y_i - \sum_{k=0}^{m} \beta_k(x)(X - x)^k \right\}^2 K_h\left(\frac{X_i - x}{h}\right), \tag{31}$$

where $K(\cdot)$ is the kernel function, $K(z) = 0.75(1 - z^2)I(|z| < 1)$ (Epanechnikov kernel), h is the bandwidth, and $h = 3.45\sigma n^{-1/5}$ from the experience of cross-validation (CV) approach [15], σ is the std. dev of the regressors, and n is the number of samples.

Generally, the majority of recent studies involve the nonparametric equation by applying a local quadratic polynomial approximation with $m = 2$. It is more convenient to write the weighted least squares problems (31) as matrix notation:

$$\text{minimize}_\beta (Y - X\beta)^T W (Y - X\beta), \tag{32}$$

where

$$\begin{aligned} X &= \begin{pmatrix} 1 & X_1 - x & (X_1 - x)^2 \\ 1 & X_2 - x & (X_2 - x)^2 \\ 3 & & \\ 1 & X_n - x & (X_n - x)^2 \end{pmatrix}_{n\times 3}, \\ Y &= \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_n \end{pmatrix}_{n\times 1}, \\ W &= \mathrm{diag}\{K_h(X - x)\}, \end{aligned} \tag{33}$$

where W is the weight matrix. And the coefficient $\beta_k(x)$ can be denoted by

$$\beta_{(x)} = \begin{pmatrix} \beta_0(x) \\ \beta_1(x) \\ \beta_2(x) \end{pmatrix}_{3\times 1} = \begin{pmatrix} \beta_0(x_1) & \beta_0(x_2) & \ldots & \beta_0(x_n) \\ \beta_1(x_1) & \beta_1(x_2) & \ldots & \beta_1(x_n) \\ \beta_2(x_1) & \beta_2(x_2) & \ldots & \beta_2(x_n) \end{pmatrix}_{3\times n}. \tag{34}$$

The solution vector of (32) is given as

$$\widehat{\beta} = \left(X^T W X\right)^{-1} X^T W Y. \tag{35}$$

3. Empirical Analysis

3.1. Data and Option Contacts. This section will make an empirical analysis by the option market data in China. The analysis is sourcing from the closing prices of European put option on the 50ETF in China from February 9, 2015, to August 21, 2015, and the option contacts contain from March 2015 to September 2015. To retain only liquid options, it is encouraged to discard the options with implied volatility larger than 70% and price smaller than 0.05, ending up with 3529. As a conclusion, the riskless rate is 2.25% in the year of 2015, and the history volatility is 20.59%.

3.2. Empirical Results. The Hurst index of 50 ETF is $H = 0.4526$, which is estimated by R/S analysis approach. Table 1 summarizes the pricing errors of different option

Table 1: Empirical result.

Model	Minimum	Maximum	Mean	Std. dev.	RMSE	MAE
BS	−0.218300	−0.000200	−0.072371	0.040202	0.082785	0.459366
SBS	−0.114000	0.144400	0.006069	0.041827	0.042259	0.233587
SFBS	−0.164617	0.356268	0.006773	0.0.045880	0.046373	0.227103
NF	−0.342859	0.235327	0.001111	0.041441	0.041449	0.146550

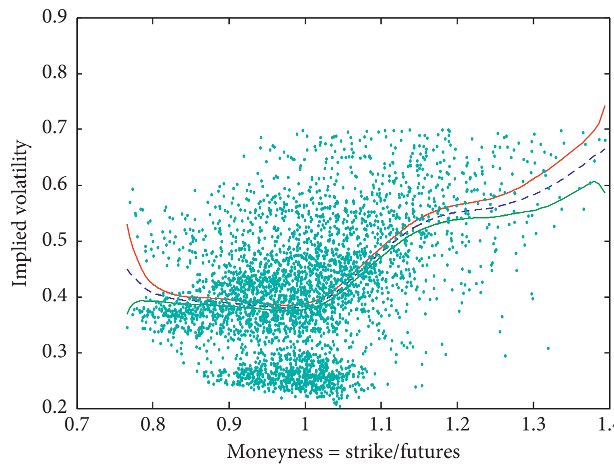

Figure 1: Nonparametric estimation of (11).

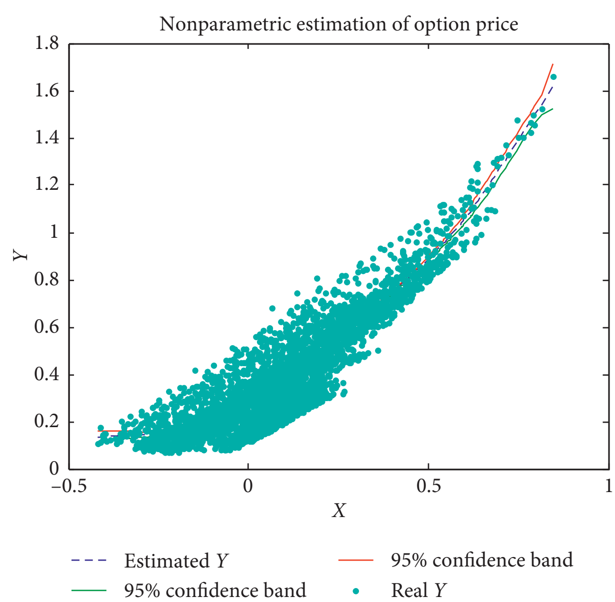

Figure 2: Nonparametric estimation of (28).

pricing models. From Table 1, the result of the MAE and RMSE of NF model is found to be lower than the BS, SBS, and SFBS models. To conclude, the nonparametric fractional option pricing model (NF) is superior to Black–Scholes model (BS), Semiparametric Black–Scholes model (SBS), and semiparametric fractional Black–Scholes pricing model (SFBS).

BS is the classical Black–Scholes option pricing model, and whole SBS is the semiparametric Black–Scholes option pricing model in that implied volatility is the local linear estimator of moneyness; SFBS is the semiparametric fractional Black–Scholes option pricing model, and NF is the nonparametric regression fractional option pricing model. The items are shown as the minimum, maximum, mean, std. dev, RMSE, and MAE of the price error (model price-market price $\text{RMSE} = \sqrt{1/2 \sum_{i=1}^{n} |P_{\text{model}} - P_{\text{market}}|^2}$, $\text{MAE} = 1/2 \sum_{i=1}^{n} |P_{\text{model}} - P_{\text{market}}/P_{\text{marhet}}|$).

Figure 1 presents the expression of the regression of implied volatility smile about moneyness; Figure 2 describes the results of the local quadratic polynomial estimation of equation (28).

Figures 3–6 demonstrate the price error histogram of several models, which is concentrated on zero. From the

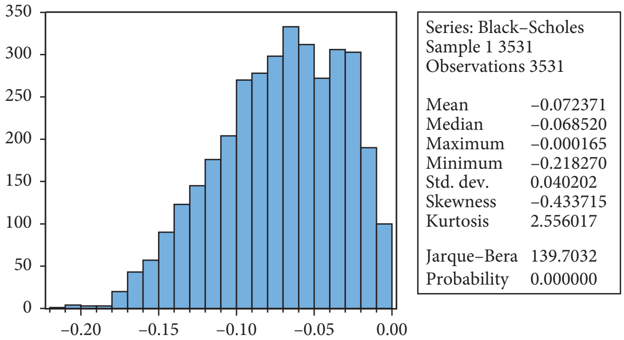

Figure 3: Price error of BS model.

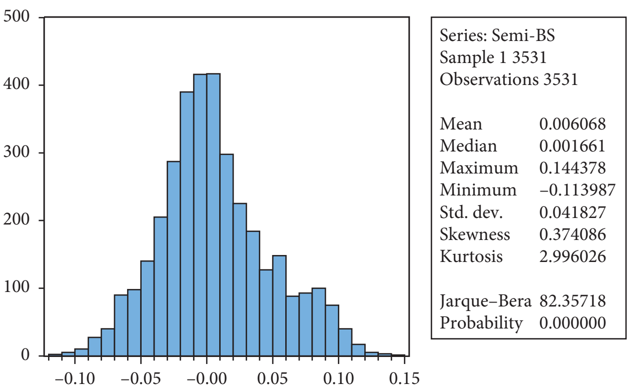

Figure 4: Price error of SBS model.

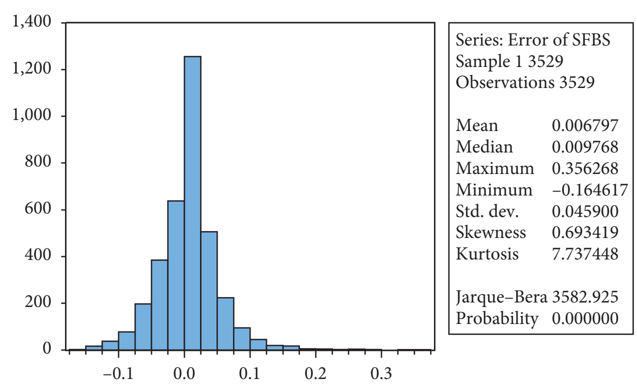

Figure 5: Price error of SFBS model.

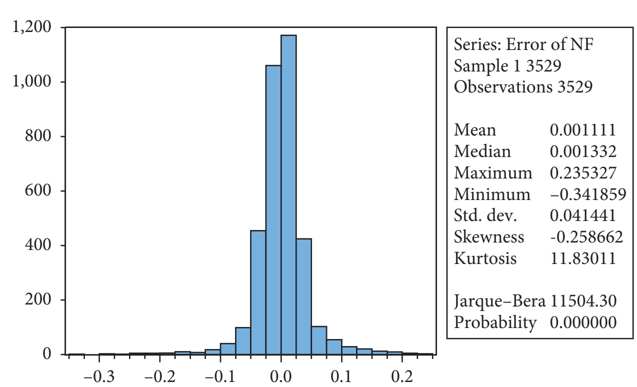

Figure 6: Price error of NF model.

results, it is expected to found that the NF model outperforms the other models.

4. Conclusions

A lot of efforts being spent on proposing the nonparametric fractional option pricing model (NF), which is better than Black–Scholes model (BS), semiparametric Black–Scholes model (SBS), and semiparametric fractional Black–Scholes option pricing model (SFBS). Comparing the pricing error histogram of semiparametric fractional Black–Scholes pricing model (SFBS) to nonparametric fractional option pricing model (NF), the experimental results have revealed that the error of NF is close to zero.

Data Availability

The datasets used and analysed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Refinements of Some Integral Inequalities for (s, m)-Convex Functions

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In this paper, the refinements of integral inequalities for all those types of convex functions are given which can be obtained from (s, m)-convex functions. These inequalities not only provide refinements of bounds for unified integral operators but also for various associated fractional integral operators containing Mittag–Leffler function. At the same time, presented results give generalizations of many known fractional integral inequalities.

1. Introduction

The following fractional integral operator is the well-known Riemann–Liouville fractional integral operator.

Definition 1 (see [1]). Let $f \in L_1[a, b]$. Then, Riemann–Liouville fractional integrals of order μ where $\Re(\mu) > 0$ are defined as follows:

$$
\begin{aligned}
{}^{\mu}I_{a^+} f(x) &= \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)\mathrm{d}t, \quad x > a,\\
{}^{\mu}I_{b^-} f(x) &= \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t)\mathrm{d}t, \quad x < b,
\end{aligned} \tag{1}
$$

where $\Gamma(\cdot)$ is the gamma function.

Next, generalizations of Riemann–Liouville fractional integral operators are given.

Definition 2 (see [2]). Let $f\colon [a, b] \longrightarrow \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b). The left-sided and the right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order μ where $\Re(\mu) > 0$ are defined by

$$
{}_{g}^{\mu}I_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (g(x) - g(t))^{\mu-1} g'(t) f(t)\mathrm{d}t, \quad x > a, \tag{2}
$$

$$
{}_{g}^{\mu}I_{b^-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (g(t) - g(x))^{\mu-1} g'(t) f(t)\mathrm{d}t, \quad x < b, \tag{3}
$$

where $\Gamma(\cdot)$ is the gamma function.

A k-analogue of the above definition is given as follows.

Definition 3 (see [3]). Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order μ where $\Re(\mu)$, $k>0$ are defined by

$$\begin{aligned}
{}_g^{\mu}I_{b^-}^{k} f(x) &= \frac{1}{k\Gamma_k(\mu)}\int_a^x (g(x)-g(t))^{(\mu/k)-1} g'(t) f(t)\mathrm{d}t, \quad x>a,\\
{}_g^{\mu}I_{b^-}^{k} f(x) &= \frac{1}{k\Gamma_k(\mu)}\int_x^b (g(t)-g(x))^{(\mu/k)-1} g'(t) f(t)\mathrm{d}t, \quad x<b,
\end{aligned} \tag{4}$$

where $\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-(t^k/k)}\mathrm{d}t$, $\Re(x)>0$.

The following integral operator is given in [4].

Definition 4. Let $f,g\colon [a,b] \longrightarrow \mathbb{R}$, $0<a<b$, be the functions such that f be positive and $f\in L_1[a,b]$ and g be differentiable and strictly increasing. Also, let ϕ/x be an increasing function on $[a,\infty)$. Then, for $x\in[a,b]$, the left and right integral operators are defined by

$$\begin{aligned}
\left(F_{a^+}^{\phi,g} f\right)(x) &= \int_a^x K_g(x,t;\phi) f(t)\mathrm{d}(g(t)), \quad x>a,\\
\left(F_{b_-}^{\phi,g} f\right)(x) &= \int_x^b K_g(t,x;\phi) f(t)\mathrm{d}(g(t)), \quad x<b,
\end{aligned} \tag{5}$$

where $K_g(x,y;\phi) = ((\phi(g(x)-g(y)))/(g(x)-g(y)))$.

A fractional integral operator containing an extended generalized Mittag–Leffler function in its kernel is defined as follows.

Definition 5 (see [5]). Let $\omega,\mu,\alpha,l,\gamma,c\in\mathbb{C}$, $\Re(\mu),\Re(\alpha),\Re(l)>0$, and $\Re(c)>\Re(\gamma)>0$ with $p\geq 0$, $\delta>0$, and $0<k\leq\delta+\Re(\mu)$. Let $f\in L_1[a,b]$ and $x\in[a,b]$. Then, the generalized fractional integral operators $\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f$ and $\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f$ are defined by

$$\left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(x;p) = \int_a^x (x-t)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-t)^\mu;p) f(t)\mathrm{d}t, \tag{6}$$

$$\left(\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(x;p) = \int_x^b (t-x)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(t-x)^\mu;p) f(t)\mathrm{d}t, \tag{7}$$

where

$$E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{8}$$

is the extended generalized Mittag–Leffler function. For further study of the Mittag–Leffler function, see [6, 7]. $(c)_{nk}$ is the Pochhammer symbol defined by $(c)_{nk} = ((\Gamma(c+nk))/\Gamma(c))$, and β_p is the extended beta function given by

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-(p/(t(1-t)))}\mathrm{d}t, \quad x,y,p\in\mathbb{R}_+. \tag{9}$$

The following identities for the constant function are obtained in [8] (see also [9]):

$$\begin{aligned}
J_{\alpha,a^+}(x;p) &:= \left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} 1\right)(x;p) = (x-a)^\alpha E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(w(x-a)^\mu;p),\\
J_{\beta,b^-}(x;p) &:= \left(\epsilon_{\mu,\beta,l,\omega,b^-}^{\gamma,\delta,k,c} 1\right)(x;p) = (b-x)^\beta E_{\mu,\beta+1,l}^{\gamma,\delta,k,c}(w(b-x)^\mu;p).
\end{aligned} \tag{10}$$

Recently, a unified integral operator is defined as follows.

Definition 6 (see [10]). Let $f,g\colon [a,b]\longrightarrow\mathbb{R}$, $0<a<b$, be the functions such that f be positive and $f\in L_1[a,b]$ and g be differentiable and strictly increasing. Also, let ϕ/x be an increasing function on $[a,\infty)$ and $\alpha,l,\gamma,c\in\mathbb{C}$, $p,\mu,\delta\geq 0$, and $0<k\leq\delta+\mu$. Then, for $x\in[a,b]$, the left and the right integral operators are defined by

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(x,\omega;p) = \int_a^x K_x^y\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right) f(y)\mathrm{d}(g(y)), \tag{11}$$

$$\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f\right)(x,\omega;p) = \int_x^b K_y^x\left(E_{\mu,\beta,l}^{\gamma,\delta,k,c}, g;\phi\right) f(y)\mathrm{d}(g(y)), \tag{12}$$

where the involved kernel is defined by

$$K_x^y\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right) = \frac{\phi(g(x)-g(y))}{g(x)-g(y)} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x)-g(y))^\mu;p). \tag{13}$$

The known fractional integrals studied in [2, 11–22] can be reproduced from the above definition, see [23], Remarks 6 and 7.

The aim of this study is to obtain the bounds of all known fractional integral operators defined in [2, 11–22] in a unified form for strongly (s,m)-convex functions. In the result, we get refinements of many known integral and fractional integral inequalities. Next, we recall definitions of convex, strongly convex, s-convex, m-convex, (s,m)-convex, and strongly (s,m)-convex functions.

Definition 7 (see [24]). A function $f\colon I\longrightarrow\mathbb{R}$ is said to be a convex function if the inequality

$$f(ta+(1-t)b)\leq tf(a)+(1-t)f(b), \tag{14}$$

holds for all $a,b\in I$ and $t\in[0,1]$.

The concept of a strongly convex function is defined as follows.

Definition 8 (see [25]). Let I be a nonempty convex subset of a normed space. A real-valued function f is said to be strongly convex with modulus $\lambda\geq 0$ on I if for each $a,b\in I$ and $t\in[0,1]$, we have

$$f(ta+(1-t)b)\leq tf(a)+(1-t)f(b)-\lambda t(1-t)\|b-a\|^2. \tag{15}$$

A generalization of the convex function defined on the right half of the real line is called the s-convex function, and it is given as follows.

Definition 9 (see [26]). Let $s \in [0,1]$. A function $f\colon [0,\infty) \longrightarrow \mathbb{R}$ is said to be an s-convex function in the second sense if

$$f(ta+(1-t)b) \le t^s f(a) + (1-t)^s f(b), \tag{16}$$

holds for all $a,b \in [0,\infty)$ and $t \in [0,1]$.

The notion of the m-convex function and strongly m-convex function is defined as follows.

Definition 10 (see [27]). A function $f\colon [0,b] \longrightarrow \mathbb{R}$ is said to be an m-convex function, where $m \in [0,1]$ and $b>0$, if for every $x,y \in [0,b]$ and $t \in [0,1]$, we have

$$f(tx+m(1-t)y) \le t f(x) + m(1-t) f(y). \tag{17}$$

Definition 11 (see [28]). A function $f\colon [0,+\infty) \longrightarrow \mathbb{R}$ is said to be a strongly m-convex function with modulus λ if

$$f(ta+m(1-t)b) \le t f(a) + m(1-t) f(b) - \lambda m t(1-t)(b-a)^2, \tag{18}$$

with $a,b \in [0,+\infty)$ and $m \in [0,1]$.

A further generalized convexity is given as follows.

Definition 12 (see [29]). A function $f\colon [0,b] \longrightarrow \mathbb{R}$ is said to be an (s,m)-convex function, where $(s,m) \in [0,1]^2$ and $b>0$, if for every $x,y \in [0,b]$ and $t \in [0,1]$, we have

$$f(tx+m(1-t)y) \le t^s f(x) + m(1-t)^s f(y). \tag{19}$$

The notion of the strongly (s,m)-convex function is defined as follows.

Definition 13 (see [30]). A function $f\colon [0,+\infty) \longrightarrow \mathbb{R}$ is said to be a strongly (s,m)-convex function, with modulus $\lambda \ge 0$, for $(s,m) \in [0,1]^2$, if

$$f(ta+m(1-t)b) \le t^s f(a) + m(1-t)^s f(b) - \lambda t(1-t)|b-a|^2, \tag{20}$$

holds for all $a,b \in [0,+\infty)$ and $t \in [0,1]$.

Using strongly (s,m)-convexity and utilizing fractional operators (6) and (7), some fractional integral inequalities are obtained as in [31]. The following result provides the bound of sum of left and right fractional integrals (6) and (7) for strongly (s,m)-convex functions at an arbitrary point.

Theorem 1 (see [31]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a real-valued function. If f is positive and strongly (s,m)-convex, then for $\alpha,\beta \ge 1$, the following fractional integral inequality holds:*

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-} f\right)(x;p) \le \left(\frac{f(a)+mf(x/m)}{s+1} - \lambda\frac{(x-ma)^2}{6m^2}\right)\\
&(x-a)J_{\alpha-1,a^+}(x;p) + \left(\frac{f(b)+mf(x/m)}{s+1} - \lambda\frac{(mb-x)^2}{6m^2}\right)(b-x)\\
&J_{\beta-1,b^-}(x;p), \quad x \in [a,b].
\end{aligned} \tag{21}$$

The following Hadamard-type inequality holds for generalized fractional integral operators for strongly (s,m)-convex functions.

Theorem 2 (see [31]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$, $a>b$, be a real-valued function. If f is positive, strongly (s,m)-convex and $f((a+mb-x)/m) = f(x)$, then for $\alpha,\beta>0$, the following fractional integral inequality holds:*

$$\begin{aligned}
&\frac{2^s}{1+m}\left(f\left(\frac{a+mb}{2}\right)\left(J_{\alpha+1,a^+}(b;p) + J_{\beta+1,b^-}(a;p)\right) + \frac{\lambda}{4m}\left((b-a)^{\beta+2}J_{\beta+1,b^-}(a;p)\right.\right.\\
&\quad - 2(1+m)(b-a)^{\beta+1}J_{\beta+2,b^-}(a;p) + 2(1+m)^2 J_{\beta+3,b^-}(a;p) + (b-a)^{\alpha+2}\\
&\quad \left.\left.\times J_{\alpha+1,a^+}(b;p) - 2(1+m)(b-a)^{\alpha+1}J_{\alpha+2,a^+}(b;p) + 2(1+m)^2 J_{\alpha+3,a^+}(b;p)\right)\right)\\
&\quad \le \left(\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+} f\right)(b;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-} f\right)(a;p)\right)\\
&\quad \le \left[J_{\beta,b^-}(a;p) + J_{\alpha,a^+}(b;p)\right](b-a)\left(\frac{f(b)+mf(a/m)}{s+1} - \lambda\frac{(mb-a)^2}{6m^2}\right).
\end{aligned} \tag{22}$$

In the following, using the strongly (s,m)-convexity of $|f'|$, a modulus inequality is obtained.

Theorem 3 (see [31]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a real-valued function. If f is differentiable and $|f'|$ is strongly (s,m)-convex, then for $\alpha,\beta \geq 1$, the following fractional integral inequality holds:*

$$\begin{aligned}
&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+} f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-} f\right)(x;p) - \left(J_{\alpha-1,a^+}(x;p) f(a) + J_{\beta-1,b^-}(x;p) f(b)\right)\right| \\
&\leq \left(\frac{|f'(a)| + m|f'(x/m)|}{s+1} - \lambda \frac{(x-ma)^2}{6m^2}\right)(x-a) J_{\alpha-1,a^+}(x;p) \\
&\quad + \left(\frac{|f'(b)| + m|f'(x/m)|}{s+1} - \lambda \frac{(mb-x)^2}{6m^2}\right)(b-x) J_{\beta-1,b^-}(x;p), \quad x \in [a,b].
\end{aligned} \tag{23}$$

In [32], we studied the properties of the kernel given in (13). Here, we are interested in the following property.

P: *let g and ϕ/x be increasing functions. Then, for $x < t < y$, $x, y \in [a,b]$, the kernel $K^y_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)$ satisfies the following inequality:*

$$K^x_t\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi\right) g'(t) \leq K^x_y\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi\right) g'(t). \tag{24}$$

This can be obtained from the following two straightforward inequalities:

$$\begin{aligned}
&\frac{\phi(g(t)-g(x))}{g(t)-g(x)} g'(t) \leq \frac{\phi(g(y)-g(x))}{g(y)-g(x)} g'(t), \\
&E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega(g(t)-g(x))^{\mu}; p\right) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega(g(y)-g(x))^{\mu}; p\right).
\end{aligned} \tag{25}$$

The reverse of inequality (13) holds when g and ϕ/x are decreasing.

The upcoming section contains the results for unified integral operators dealing with the bounds of several fractional integral operators in a compact form by utilizing strongly (s,m)-convex functions. A compact version of the Hadamard inequality is presented, and also a modulus inequality is given for the differentiable function f such that $|f'|$ is a strongly (s,m)-convex function. In the whole paper, we will use

$$I(a,b,g) =: \frac{1}{b-a} \int_a^b g(t) \mathrm{d}t. \tag{26}$$

2. Main Results

The following result provides the upper bound of unified integral operators.

Theorem 4. *Let $f\colon [a,mb] \longrightarrow \mathbb{R}$, $0 \leq a < mb$, be a positive integrable and strongly (s,m)-convex function, $m \neq 0$. Then, for unified integral operators (11) and (12), the following inequality holds:*

$$\begin{aligned}
&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\right)(x,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-} f\right)(x,\omega;p) \leq K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi\right)\Bigg(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a) - \frac{\Gamma(s+1)}{(x-a)^s}\left(mf\left(\frac{x}{m}\right)^s I_{x^-} g(a) - f(a)^s I_{a^+} g(x)\right) \\
&\quad + \frac{\lambda(x-ma)^2}{(x-a)}\left(2I(a,x,I_d g) - (a+x) I(a,x,g)\right)\Bigg) + K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l}, g; \phi\right)\Bigg(f(b)g(b) \\
&\quad - mf\left(\frac{x}{m}\right)g(x) - \frac{\Gamma(s+1)}{(b-x)^s}\left(f(b)^s I_{b^-} g(x) - mf\left(\frac{x}{m}\right)^s I_{x^+} g(b)\right) \\
&\quad + \frac{\lambda(mb-x)^2}{(b-x)}\left(2I(x,b,I_d g) - (x+b) I(x,b,g)\right)\Bigg).
\end{aligned} \tag{27}$$

Proof. By (**P**), the following inequalities hold:

$$K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right)g'(t) \le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right)g'(t), \quad a<t<x, \tag{28}$$

$$K_t^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right)g'(t) \le K_b^x\left(E_{\mu,\beta,l}^{\gamma,\delta,k,c}, g;\phi\right)g'(t), \quad x<t<b. \tag{29}$$

For a strongly (s,m)-convex function, the following inequalities hold for $a<t<x$ and $x<t<b$, respectively:

$$f(t) \le \left(\frac{x-t}{x-a}\right)^s f(a) + m\left(\frac{t-a}{x-a}\right)^s f\left(\frac{x}{m}\right) - \frac{\lambda(x-t)(t-a)(x-ma)^2}{m^2(x-a)^2}, \tag{30}$$

$$f(t) \le \left(\frac{t-x}{b-x}\right)^s f(b) + m\left(\frac{b-t}{b-x}\right)^s f\left(\frac{x}{m}\right) - \frac{\lambda(t-x)(b-t)(mb-x)^2}{m^2(b-x)^2}. \tag{31}$$

From (28) and (30), one can have

$$\begin{aligned}\int_a^x K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right)f(t)\mathrm{d}(g(t)) &\le f(a)K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right)\\ &\times \int_a^x \left(\frac{x-t}{x-a}\right)^s \mathrm{d}(g(t)) + mf\left(\frac{x}{m}\right)K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right)\\ &\cdot \int_a^x \left(\frac{t-a}{x-a}\right)^s \mathrm{d}(g(t))\\ &- K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right)\frac{\lambda(x-ma)^2}{(x-a)^2}\int_a^x (x-t)(t-a)\mathrm{d}(g(t)),\end{aligned} \tag{32}$$

i.e.,

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(x,\omega;p) \le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right) \left(\begin{array}{c} mf\left(\frac{x}{m}\right)g(x) - f(a)g(a) - \frac{\Gamma(s+1)}{(x-a)^s}\left(mf\left(\frac{x}{m}\right)^s I_{x^-}g(a) - f(a)^s I_{a^+}g(x)\right)\\ +\frac{\lambda(x-ma)^2}{(x-a)}\left(2I(a,x,I_d g) - (a+x)I(a,x,g)\right)\end{array}\right). \tag{33}$$

On the other hand, from (29) and (31), one can have

$$\begin{aligned}\int_x^b K_t^x\left(E_{\mu,\beta,l}^{\gamma,\delta,k,c}, g;\phi\right)f(t)\mathrm{d}(g(t)) &\le f(b)K_b^x\left(E_{\mu,\beta,l}^{\gamma,\delta,k,c}, g;\phi\right)\\ &\times \int_x^b \left(\frac{t-x}{b-x}\right)^s \mathrm{d}(g(t)) + mf\left(\frac{x}{m}\right)K_x^b\left(E_{\mu,\beta,l}^{\gamma,\delta,k,c}, g;\phi\right)\int_x^b \left(\frac{b-t}{b-x}\right)^s \mathrm{d}(g(t))\\ &- K_x^b\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi\right)\frac{\lambda(mb-x)^2}{m^2(b-x)^2}\int_x^b (t-x)(b-t)\mathrm{d}(g(t)),\end{aligned} \tag{34}$$

i.e.,

$$\left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f\right)(x,\omega;p)\leq K_b^x\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\left(f(b)g(b)-mf\left(\frac{x}{m}\right)g(x)-\frac{\Gamma(s+1)}{(b-x)^s}\left(f(b)^s I_{b^-}g(x)-mf\left(\frac{x}{m}\right)^s I_{x^+}g(b)\right)\right.$$
$$\left.+\frac{\lambda(mb-x)^2}{(b-x)}\left(2I\left(x,b,I_d g\right)-(x+b)I(x,b,g)\right)\right). \tag{35}$$

By adding (33) and (35), (27) can be obtained. □

Corollary 1. *Setting $p=\omega=0$ in (27), we can obtain the following inequality involving fractional integral operators defined in [4]:*

$$\left(F^{\phi}_{\alpha,a^+}f\right)(x;p)+\left(F^{\phi}_{\beta,b^-}f\right)(x;p)\leq K_g(a,x;\phi)\left(mf\left(\frac{x}{m}\right)g(x)-f(a)g(a)-\frac{\Gamma(s+1)}{(x-a)^s}\left(mf\left(\frac{x}{m}\right)^s I_{x^-}g(a)-f(a)^s I_{a^+}g(x)\right)\right.$$
$$\left.+\frac{\lambda(x-ma)^2}{(x-a)}\left(2I\left(a,x,I_d g\right)-(a+x)I(a,x,g)\right)\right)+K_g(x,b;\phi)\left(f(b)g(b)\right.$$
$$-mf\left(\frac{x}{m}\right)g(x)-\frac{\Gamma(s+1)}{(b-x)^s}\left(f(b)^s I_{b^-}g(x)-mf\left(\frac{x}{m}\right)^s I_{x^+}g(b)\right)$$
$$\left.+\frac{\lambda(mb-x)^2}{(b-x)}\left(2I\left(x,b,I_d g\right)-(x+b)I(x,b,g)\right)\right). \tag{36}$$

Remark 1

(i) If we consider $\lambda=0$ in (27), then Theorem 3.1 in [32] can be obtained, and for $\lambda>0$, we get its refinement

(ii) If we consider $\phi(t)=t^{\alpha}$ and $g(x)=x$ in (27), then Theorem 1 can be obtained

(iii) If we consider $s=m=1$ in the result of (ii), then Corollary 1 in [31] can be obtained

(iv) If we consider $\alpha=\beta$ in the result of (ii), then Corollary 3 in [31] can be obtained

(v) If we consider $f\in L_{\infty}[a,b]$ in the result of (ii), then Corollary 5 in [31] can be obtained

(vi) If we consider $\alpha=\beta$ in the result of (v), then Corollary 7 in [31] can be obtained

(vii) If we consider $s=1$ in the result of (ii), then Corollary 5 in [31] can be obtained

(viii) If we consider $(s,m)=(1,\ 1)$ in (27), then Theorem 2 in [33] is obtained

(ix) If we consider $\alpha=\beta$, $\lambda=0$, and $(s,m)=(1,\ 1)$ in (27), then Theorem 8 in [23] is obtained

(x) If we consider $\lambda=0$ and $p=\omega=0$ in (27), then Theorem 1 in [34] is obtained

(xi) If we consider $\lambda=0$, $\phi(t)=\Gamma(\alpha)t^{\alpha}$, $p=\omega=0$, and $(s,m)=(1,\ 1)$ in (27), then Theorem 1 in [35] is obtained

(xii) If we consider $\alpha=\beta$ in the result of (xi), then Corollary 1 in [35] is obtained

(xiii) If we consider $\lambda=0$, $\phi(t)=t^{\alpha}$, $g(x)=x$, and $m=1$ in (27), then Theorem 2.1 in [36] is obtained

(xvi) If we consider $\alpha=\beta$ in the result of (xiii), then Corollary 2.1 in [36] is obtained

(xv) If we consider $\lambda=0$, $\phi(t)=((\Gamma(\alpha)t^{(\alpha/k)})/(k\Gamma_k(\alpha)))$, $(s,m)=(1,\ 1)$, $g(x)=x$, and $p=\omega=0$ in (27), then Theorem 1 in [37] can be obtained

(xvi) If we consider $\alpha=\beta$ in the result of (xv), then Corollary 1 in [37] can be obtained

(xvii) If we consider $\lambda=0$, $\phi(t)=\Gamma(\alpha)t^{\alpha}$, $g(x)=x$, $p=\omega=0$, and $(s,m)=(1,\ 1)$ in (27), then Theorem 1 in [38] is obtained

(xviii) If we consider $\alpha=\beta$ in the result of (xvii), then Corollary 1 in [38] can be obtained

(xviii) If we consider $\alpha=\beta=1$ and $x=a$ or $x=b$ in the result of (xvii), then Corollary 2 in [38] can be obtained

(xix) If we consider $\alpha=\beta=1$ and $x=((a+b)/2)$ in the result of (xvii), then Corollary 3 in [38] can be obtained

The following lemma is very helpful in the proof of the upcoming theorem, see [31].

Lemma 1. *Let $f\colon [a, mb] \longrightarrow \mathbb{R}$ be a strongly (s, m)-convex function, $0 \leq a < mb$. If f is $f((a+mb-x)/m) = f(x)$, $m \neq 0$, then the following inequality holds:*

$$f\left(\frac{a+mb}{2}\right) \leq \frac{(1+m)f(x)}{2^s} - \frac{\lambda}{4m}(a+mb-x-mx)^2. \quad (37)$$

In the literature, many mathematicians have established many types of Hadamard inequalities, and for their generalizations, see [39–42]. This also motivates us to introduce the more generalized forms of Hadamard-type inequalities. So, by the help of the abovementioned lemma, the following result provides generalized Hadamard inequality for strongly (s, m)-convex functions.

Theorem 5. *Under the assumptions of Theorem 4, in addition to $f(x) = f((a+mb-x)/m)$, the following inequality holds:*

$$\begin{aligned}
f\left(\frac{a+mb}{2}\right)\frac{2^s}{(1+m)}&\left(\begin{array}{c}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}1\right)(b,\omega;p)+\frac{\lambda}{4m}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}(a+mb-x-mx)^2\right)\\ (a,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}1\right)(a,\omega;p)+\frac{\lambda}{4m}\\ \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}(a+mb-x-mx)^2\right)(b,\omega;p)\end{array}\right) \leq \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}f\right)(b,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f\right)(a,\omega;p)\\
&\leq \left(K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)+K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\right)\\
&\cdot\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^s I_{b^-}g(a)\right.\right.\\
&\left.-mf\left(\frac{a}{m}\right)^s I_{a^+}g(b)\right)+\frac{\lambda(mb-a)^2}{(b-a)}\left(2I(a,b,I_d g)\right.\\
&\left.\left.-(a+b)I(a,b,g)\right)\right).
\end{aligned} \quad (38)$$

Proof. By (**P**), the following inequalities hold:

$$K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)g'(x) \leq K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)g'(x), \quad a<x<b, \quad (39)$$

$$K_b^x\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)g'(x) \leq K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)g'(x), \quad a<x<b. \quad (40)$$

A strongly (s, m)-convex function satisfying the following inequalities hold for $a < x < b$:

$$\begin{aligned}
f(x) &\leq \left(\frac{x-a}{b-a}\right)^s f(b) + m\left(\frac{b-x}{b-a}\right)^s f\left(\frac{a}{m}\right)\\
&\quad - \frac{\lambda(b-x)(x-a)(b-ma)^2}{m^2(b-a)^2}.
\end{aligned} \quad (41)$$

From (39) and (41), one can have

$$\begin{aligned}
&\int_a^b K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)f(x)\mathrm{d}(g(x))\\
&\leq mf\left(\frac{a}{m}\right)K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)\int_a^b\left(\frac{b-x}{b-a}\right)^s\mathrm{d}(g(x))\\
&\quad + f(b)K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)\int_a^b\left(\frac{x-a}{b-a}\right)^s\mathrm{d}(g(x))\\
&\quad - K_a^b\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)\\
&\frac{\lambda(b-ma)^2}{m^2(b-a)^2}\int_a^b(x-a)(b-x)\mathrm{d}(g(x)).
\end{aligned} \quad (42)$$

Further, the aforementioned inequality takes the form which involves Riemann–Liouville fractional integrals in the right-hand side, and thus we have upper bound of the unified left-sided integral operator (2) as follows:

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p)\leq K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)\Bigg(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^sI_{b^-}g(a)-mf\left(\frac{a}{m}\right)^sI_{a^+}g(b)\right)$$
$$+\frac{\lambda(mb-a)^2}{(b-a)}\left(2I(a,b,I_dg)-(a+b)I(a,b,g)\right)\Bigg). \tag{43}$$

On the other hand, from (39) and (41), the following inequality holds which involves Riemann–Liouville fractional integrals on the right-hand side and gives the estimate of the integral operator (3):

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p)\leq K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\Bigg(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^sI_{b^-}g(a)-mf\left(\frac{a}{m}\right)^sI_{a^+}g(b)\right)$$
$$+\frac{\lambda(mb-a)^2}{(b-a)}\left(2I(a,b,I_dg)-(a+b)I(a,b,g)\right)\Bigg). \tag{44}$$

By adding (43) and (44), the following inequality can be obtained:

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p)\leq\left(K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)+K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\right)\Bigg(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)$$
$$-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^sI_{b^-}g(a)-mf\left(\frac{a}{m}\right)^sI_{a^+}g(b)\right)+\frac{\lambda(mb-a)^2}{(b-a)}\left(2I(a,b,I_dg)\right.$$
$$\left.-(a+b)I(a,b,g)\right)\Bigg). \tag{45}$$

Multiplying both sides of (37) by $K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x)$ and integrating over $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\int_a^b K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)\mathrm{d}(g(x))$$
$$\leq\left(\frac{1}{2^s}\right)(1+m)\int_a^b K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)f(x)\mathrm{d}(g(x))$$
$$-\frac{\lambda}{4m}\int_a^b K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)(a+mb-x-mx)^2\mathrm{d}(g(x)). \tag{46}$$

From Definition 6, the following inequality is obtained:

$$f\left(\frac{a+mb}{2}\right)\frac{2^s}{(1+m)}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a,\omega;p)$$
$$\leq\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p) \tag{47}$$
$$-\frac{\lambda}{4m}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}(a+mb-x-mx)^2\right)(a,\omega;p).$$

Similarly, multiplying both sides of (37) by $K^x_b(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi)g'(x)$ and integrating over $[a,b]$, we have

$$f\left(\frac{a+mb}{2}\right)\frac{2^s}{(1+m)}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}1\right)(b,\omega;p)$$
$$\leq\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}f\right)(b,\omega;p)$$
$$-\frac{\lambda}{4m}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}(a+mb-x-mx)^2\right)(b,\omega;p). \quad (48)$$

By adding (47) and (48), the following inequality is obtained:

$$f\left(\frac{a+mb}{2}\right)\frac{2^s}{(1+m)}\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}1\right)(b,\omega;p)+\frac{\lambda}{4m}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}(a+mb-x-mx)^2\right)(a,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a,\omega;p)\right.$$
$$\left.+\frac{\lambda}{4m}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}(a+mb-x-mx)^2\right)(b,\omega;p)\right)\leq\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}f\right)(b,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p). \quad (49)$$

Using (45) and (49), inequality (38) can be obtained, which completes the proof. □

Corollary 2. *Setting $p=\omega=0$ in (38), we can obtain the following inequality involving fractional integral operators defined in [4]:*

$$f\left(\frac{a+mb}{2}\right)\frac{2^s}{(1+m)}\left(\begin{array}{c}\left(F^{\phi}_{\beta,a^+}1\right)(b;p)+\frac{\lambda}{4m}\left(F^{\phi}_{\alpha,b^-}(a+mb-x-mx)^2\right)\\(a;p)+\left(F^{\phi}_{\alpha,b^-}1\right)(a;p)+\frac{\lambda}{4m}\left(F^{\phi}_{\beta,a^+}(a+mb-x-mx)^2\right)(b;p)\end{array}\right)$$
$$\leq\left(F^{\phi}_{\beta,a^+}f\right)(b;p)+\left(F^{\phi}_{\alpha,b^-}f\right)(a;p)$$
$$\leq\left(K_g(a,b;\phi)+K_g(a,b;\phi)\right) \quad (50)$$
$$\left(\begin{array}{c}f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^sI_{b^-}g(a)-mf\left(\frac{a}{m}\right)^sI_{a^+}g(b)\right)\\+\frac{\lambda(mb-a)^2}{(b-a)}\left(2I(a,b,I_dg)-(a+b)I(a,b,g)\right)\end{array}\right).$$

Remark 2

(i) If we consider $\phi(t)=t^\alpha$ and $g(x)=x$ in (38), then Theorem 7 in [31] can be obtained

(ii) If we consider $\lambda=0$ in the result of (i), then Theorem 8 in [31] can be obtained

(iii) If we consider $(s,m)=(1,1)$ in (38), then Theorem 3 in [33] is obtained

(iv) If we consider $\lambda=0$ and $(s,m)=(1,1)$ in (38), then Theorem 22 in [23] is obtained

(v) If we consider $\lambda=0$, $\phi(t)=\Gamma(\alpha)t^{\alpha+1}$, $p=\omega=0$, and $(s,m)=(1,1)$ in (38), then Theorem 3 in [35] is obtained

(vi) If we consider $\alpha=\beta$ in the result of (v), then Corollary 3 in [35] is obtained

(vii) If we consider $\lambda=0$, $\phi(t)=t^{\alpha+1}$, $g(x)=x$, and $m=1$ in (38), then Theorem 2.4 in [36] is obtained

(viii) If we consider $\alpha=\beta$ in the result of (vii), then Corollary 2.6 in [36] is obtained

(ix) If we consider $\lambda=0$, $\phi(t)=\Gamma(\alpha)t^{(\alpha/k)+1}$, $(s,m)=(1,1)$, $g(x)=x$, and $p=\omega=0$ in (38), then Theorem 3 in [37] can be obtained

(x) If we consider $\alpha=\beta$ in the result of (ix), then Corollary 6 in [37] can be obtained

(xi) If we consider $\lambda=0$, $\phi(t)=\Gamma(\alpha)t^{\alpha+1}$, $p=\omega=0$, $(s,m)=1$, and $g(x)=x$ in (38), then Theorem 3 in [38] can be obtained

(xii) If we consider $\alpha=\beta$ in the result of (xi), then Corollary 6 in [38] can be obtained

Theorem 6. *Let $f\colon [a, mb] \longrightarrow \mathbb{R}$, $0 \le a < mb$, be a differential function such that $|f'|$ is a strongly (s,m)-convex function, $m \ne 0$. Then, for unified integral operators (11) and (12), the following inequality holds:*

$$\begin{aligned}&\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f * g\right)(x,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-} f * g\right)(x,\omega;p)\right| \\ &\le K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi\right)\begin{pmatrix}\left(m\left|f'\left(\dfrac{x}{m}\right)\right|g(x) - \left|f'(a)\right|g(a) - \dfrac{\Gamma(s+1)}{(x-a)^s}\right) \\ \left(m\left|f'\left(\dfrac{x}{m}\right)\right|^s I_{x^-}g(a) - \left|f'(a)\right|^s I_{a^+}g(x)\right) + \\ \dfrac{\lambda(x-am)^2}{(x-a)}\left(2I(a,x,I_d g) - (a+x)I(a,x,g)\right)\end{pmatrix} \\ &+ K_b^x\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l}, g;\phi\right)\begin{pmatrix}\left(\left|f'(b)\right|g(b) - m\left|f'\left(\dfrac{x}{m}\right)\right|g(x)\right) - \dfrac{\Gamma(s+1)}{(b-x)^s} \\ \left(\left|f'(b)\right|^s I_{b^-}g(x) - m\left|f'\left(\dfrac{x}{m}\right)\right|^s I_{x^+}g(b)\right) + \dfrac{\lambda(mb-x)^2}{(b-x)} \\ \left(2I(x,b,I_d g) - (x+b)I(x,b,g)\right)\end{pmatrix},\end{aligned} \tag{51}$$

where

$$\begin{aligned}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f * g\right)(x,\omega;p) &:= \int_a^x K_x^t\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi\right) f'(t)\,\mathrm{d}(g(t)), \\ \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-} f * g\right)(x,\omega;p) &:= \int_x^b K_t^x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi\right) f'(t)\,\mathrm{d}(g(t)).\end{aligned} \tag{52}$$

Proof. For a strongly (s,m)-convex function $|f'|$, the following inequalities hold for $a < t < x$ and $x < t < b$, respectively:

$$\begin{aligned}\left|f'(t)\right| \le{}& \left(\frac{x-t}{x-a}\right)^s\left|f'(a)\right| + m\left(\frac{t-a}{x-a}\right)^s\left|f'\left(\frac{x}{m}\right)\right| \\ &- \frac{\lambda(x-t)(t-a)(x-ma)^2}{m^2(x-a)^2},\end{aligned} \tag{53}$$

$$\begin{aligned}\left|f'(t)\right| \le{}& \left(\frac{t-x}{b-x}\right)^s\left|f'(b)\right| + m\left(\frac{b-t}{b-x}\right)^s\left|f'\left(\frac{x}{m}\right)\right| \\ &- \frac{\lambda(t-x)(b-t)(mb-x)^2}{m^2(b-x)^2}.\end{aligned} \tag{54}$$

From (28) and (54), the following inequality is obtained:

$$\begin{aligned}&\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}(f * g)\right)(x,\omega;p)\right| \le K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi\right) \\ &\left(\begin{pmatrix} m\left|f'\left(\dfrac{x}{m}\right)\right|g(x) - \left|f'(a)\right|g(a) - \dfrac{\Gamma(s+1)}{(x-a)^s} \\ \left(m\left|f'\left(\dfrac{x}{m}\right)\right|^s I_{x^-}(a) - \left|f'(a)\right|^s I_{a^+}g(x)\right) + \\ \dfrac{\lambda(x-am)^2}{(x-a)}\left(2I(a,x,I_d g) - (a+x)I(a,x,g)\right)\end{pmatrix}\right).\end{aligned} \tag{55}$$

Similarly, from (29) and (55), the following inequality is obtained:

$$\begin{aligned}&\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}(f*g)\right)(x,\omega;p)\right|\leq K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\\&\cdot\left(\left|f'(b)\right|g(b)-m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)-\frac{\Gamma(s+1)}{(b-x)^s}-\left(\left|f'(b)\right|^s I_{b^-}g(x)-m\left|f'\left(\frac{x}{m}\right)\right|^s I_{x^+}g(b)\right)\\&+\frac{\lambda(mb-x)^2}{(b-x)}\left(2I\left(x,b,I_dg\right)-(x+b)I(x,b,g)\right).\end{aligned} \tag{56}$$

By adding (56) and (57), inequality (52) can be achieved. □

Corollary 3. *Setting $p=\omega=0$ in (52), we can obtain the following inequality involving fractional integral operators defined in [4]:*

$$\begin{aligned}&\left|\left(F^{\phi}_{\alpha,a^+}f*g\right)(x,p)+\left(F^{\phi}_{\beta,b^-}f*g\right)(x,p)\right|\leq K_g(a,x;\phi)\\&\begin{pmatrix}\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x)-\left|f'(a)\right|g(a)\right)-\frac{\Gamma(s+1)}{(x-a)^s}\\ \left(m\left|f'\left(\frac{x}{m}\right)\right|^s I_{x^-}g(a)-\left|f'(a)\right|^s I_{a^+}g(x)\right)\\ +\frac{\lambda(x-am)^2}{(x-a)}\left(2I\left(a,x,I_dg\right)-(a+x)I(a,x,g)\right)\end{pmatrix}\\&+K_b(x,b;\phi)\begin{pmatrix}\left(\left|f'(b)\right|g(b)-m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)-\frac{\Gamma(s+1)}{(b-x)^s}\\ \left(\left|f'(b)\right|^s I_{b^-}g(x)-m\left|f'\left(\frac{x}{m}\right)\right|^s I_{x^+}g(b)\right)+\\ \frac{\lambda(mb-x)^2}{(b-x)}\left(2I\left(x,b,I_dg\right)-(x+b)I(x,b,g)\right)\end{pmatrix}.\end{aligned} \tag{57}$$

Remark 3

(i) If we consider $\lambda=0$ in (52), then Theorem 3.4 in [32] can be obtained

(ii) If we consider $\phi(t)=t^{\alpha}$ and $g(x)=x$ in (52), then Theorem 6 in [31] can be obtained

(iii) If we consider $s=m=1$ in the result of (ii), then Corollary 13 in [31] can be obtained

(iv) If we consider $\alpha=\beta$ in the result of (ii), then Corollary 11 in [31] can be obtained

(v) If we consider $(s,m)=(1,1)$ in (52), then Theorem 3 in [33] is obtained

(vi) If we consider $\lambda=0$ and $(s,m)=(1,1)$ in (52), then Theorem 25 in [23] is obtained

(vii) If we consider $\lambda=0$ and $p=\omega=0$ in (52), then Theorem 2 in [34] is obtained

(viii) If we consider $\lambda=0$, $\phi(t)=\Gamma(\alpha)t^{\alpha+1}$, $p=\omega=0$, and $(s,m)=(1,1)$ in (52), then Theorem 2 in [35] is obtained

(ix) If we consider $\alpha=\beta$ in the result of (viii), then Corollary 2 in [35] is obtained

(x) If we consider $\lambda=0$, $\phi(t)=t^{\alpha}$, $g(x)=x$, and $m=1$ in (52), then Theorem 2.3 in [36] is obtained

(xi) If we consider $\alpha=\beta$ in the result of (x), then Corollary 2.5 in [36] is obtained

(xii) If we consider $\lambda=0$, $\phi(t)=\Gamma(\alpha)t^{\alpha/k+1}$, $(s,m)=(1,1)$, $g(x)=x$, and $p=\omega=0$ in (52), then Theorem 2 in [37] can be obtained

(xiii) If we consider $\alpha=\beta$ in the result of (xii), then Corollary 4 in [37] can be obtained

(xiv) If we consider $\alpha=\beta=k=1$ and $x=((a+b)/2)$ in the result of (xii), then Corollary 5 in [37] can be obtained

(xv) If we consider $\lambda=0$, $\phi(t)=\Gamma(\alpha)t^{\alpha+1}$, $g(x)=x$, $p=\omega=0$, and $(s,m)=(1,1)$ in (52), then Theorem 2 in [38] is obtained

(xvi) If we consider $\alpha=\beta$ in the result of (xv), then Corollary 5 in [38] can be obtained

3. Concluding Remarks

In this paper, bounds of a unified integral operator for strongly (s,m)-convex functions are studied. The compact form of these bounds lead to further interesting consequences with respect to fractional integrals of various kinds for convex, (s,m)-convex, m-convex, s-convex, and convex functions. These findings are generalized in nature and give the refinements of many inequalities for unified and fractional integral operators via different types of convex functions.

Data Availability

No data are required for this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Study on the Design of Cantonese Cultural and Creative Products using Analytic Hierarchy Process

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In this paper, we have studied the design of Cantonese cultural and creative products. In the design process of the system, we use the Analytic Hierarchy Process to analyze the needs of users and apply the analysis results to the product design practice, so as to design Cantonese cultural and creative products more in line with the needs of tourists.

1. Introduction

Cantonese Culture, that is, the culture of Cantonese ethnic group of Han nationality, is an important part of Chinese Han civilization and is subordinate to Lingnan Culture [1]. Throughout the history, the population began to move southward in the Qin and Han Dynasties, which brought the impact of the advanced culture of the Central Plains to Lingnan area. Since the Han and Tang Dynasties, Guangdong, led by Guangzhou, has always been an important window for China's foreign trade. Cantonese have formed a typical "marine cultural character" [2]. Therefore, the main characteristics of Cantonese Culture are open-minded and active, daring to be the first in everything, natural and peaceful attitude, focusing on daily life, open and inclusive, pragmatic, and not exclusive.

Based on the understanding of the core of Cantonese Culture, our project team discussed and studied the design of Cantonese cultural and creative products. Usually, cultural and creative products are designed through emotional experience, but lack of rational guidance. In the product design process, it is better to use appropriate mathematical methods, such as Analytic Hierarchy Process (AHP). In 1977, Saaty [3] firstly proposed AHP with the aim of solving problems which can be modeled by a network or hierarchical structure. AHP can be used to solve the problem of comparison of user demand factors. We use it to calculate the weight of factors to carry out the product design.

2. Cantonese Cultural and Creative Product Current Situation Analysis

2.1. Less Outstanding Products and Weak Brand Awareness. The project team investigated the current market of Cantonese cultural and creative products and found that there were few representative products and brands in the market. In the only products, the visual identity is poor and there is lack of systematic design; most enterprises have weak brand awareness and have not formed an influential brand; the lack of clear representative elements makes consumers confused and aesthetic fatigue in regional identification.

2.2. The Application of Graphics Is Hard and the Resonance Is Weak. At present, most of Cantonese cultural and creative products mainly use local representative graphics. For example, the cultural and creative products of Canton tower directly copy the shape of Canton tower, giving people a sense of mechanically copying. The lack of modern design means to analyze, deconstruct, and restructure the characteristic patterns of Cantonese which cannot reflect the added value of cultural and creative products. At the aesthetic level, the expression of product surface decoration is obsolete, less consideration of the aesthetic needs of today's society, and it is difficult to arouse the resonance of consumers.

2.3. Single Product Type and Poor Experience. The great majority Cantonese cultural and creative products on the market are mainly handicraft ornaments, bookmarks, and trinkets. The handicraft thinking of these products is more than the industrial design thinking, and the general practicability is not strong. Nowadays, many Cantonese cultural and creative products still use the product carrier of the traditional handicraft era and do not develop new functions based on the needs of consumers' life. In addition, in the product reflection level, the cultural depth is not fully explored and the communication is not precise enough. Cultural and creative products need more in-depth consideration from the functional level, aesthetic level, and philosophical level to arouse the psychological resonance of consumers.

3. Cantonese Cultural and Creative Product Audience Demands

The consumption groups of Cantonese cultural and creative products are mainly local people and tourists in Guangzhou. According to the data, Guangzhou received 16.2383 million tourists during the National Day golden week in 2019, accounting for about one third of the total number of tourists in the province. Guangzhou is located in the Pearl River Delta, adjacent to Hong Kong and Macao, so the inbound tourists in Guangzhou are mainly composed of foreigners, Hong Kong, Macao, and Taiwan compatriots. Among them, Hong Kong and Macao tourists account for about 60% of the total flow, while Asian tourists account for 30% of the total flow, followed by European and American visitors. Due to the fact that Guangzhou has two sessions of Canton Fair every year, and the geographical advantage of being adjacent to Hong Kong and Macao, the inbound tourists in Guangzhou have their unique characteristics in tourism destination structure. With the heating up of tourism industry, it brings huge business opportunities. However, the low sales revenue of tourist souvenirs in Guangzhou is in contrast with the rapid development of tourism. Cantonese cultural and creative products can be used not only be as tourist souvenirs but also as the name card of the city. Nowadays, the business card of Guangzhou is vague and not good enough to be used.

Western psychological scholars have put forward some different theories of human motivation, which have certain reference value for audience analysis and marketing strategies. Among them, the most popular one is Maslow's "hierarchy of needs" theory [4]. According to the demand hierarchy theory and the market survey completed by the project team in the early stage, we can sort out the audience demand of Cantonese cultural and creative products, so as to enhance the rationality and effectiveness of the later product design practice.

3.1. Appearance Requirements. The lowest level of Maslow's demand level is the basis of the transition to other levels. The external demand for products belongs to the most direct and instinctive response of human beings. It emphasizes the physiological characteristics of users' gaze, feeling, and voice. In the early stage, the project team took frequent visitors to Guangzhou as the main target group, sent out 117 questionnaires, and recovered 117 valid questionnaires. In the preliminary research, among "factors affecting the purchase of Guangzhou souvenirs," 36.84% of the respondents chose "appearance, commemorative significance, and usability," which is a very good illustration that tourists consider whether the products meet the basic demand as the most basic factor to influence the purchase.

3.2. Functional Requirements. Functional requirements are in the middle of Maslow's hierarchy of requirements. The function demand is related to the utility of the product, which emphasizes whether the product can solve the problems in life smoothly for users. Pragmatism is an important feature of western culture. Cultural and creative products with practical functions have the basis of commercial value, and they are also the first choice for tourists in Guangdong Province who combine Chinese and Western culture. In the preliminary investigate and survey, among the question "which features of cultural and creative products do you pay more attention to?" 38.46% of the respondents chose "practicability," accounting for the first place, which proves that consumers are attached great importance by the functional characteristics of Cantonese cultural and creative products.

3.3. Cultural Needs. Cultural needs correspond to Maslow's level of self-demand, and in the pyramid, it refers to a kind of ideological and spiritual transition after functional needs are met. The goal of product design is not only to enable users to use the product but also to meet their deep emotional needs in the multilevel interaction with products in the modern market environment with serious homogenization of functions. In the questionnaire, 74% of the respondents chose "custom" and 63% chose "traditional art" for "which cultural elements do you prefer to buy Guangzhou souvenirs?" Due to the unique composition of Guangzhou tourists, Hong Kong, Macao, and Taiwan compatriots, Asian and Western tourists account for a large proportion. Their aesthetic taste is more influenced by the West and has its uniqueness.

4. Cantonese Cultural and Creative Products Hierarchical Analysis of User Requirements

4.1. Analytic Hierarchy Process. One comprehensively utilized Multicriteria Decision-Making (MCDM) strategy is AHP [3, 5, 6]. Analytic hierarchy process decomposes the decision-making elements into objectives, criteria, schemes, and other levels, on which qualitative and quantitative analysis are carried out. It is a simple, flexible, and practical multicriteria decision-making method for quantitative analysis of qualitative problems. AHP is utilized to measure, order, rank, evaluate decision choices, etc. AHP estimates criteria weights by pairwise comparisons. This method is helpful to determine the relative weight which should have each criterion when we need to make a decision. Therefore, it has been applied in various fields such as environmental management [7], risk assessment [8], and supply chain management [9].

AHP has three advantages: systematization, which regards the object as a whole system and makes decisions according to the thinking steps of decomposition, comparison, judgment, and synthesis; practicability, combining qualitative, and quantitative methods, can deal with problems that cannot be solved by traditional optimization methods; simplicity, it is easy to calculate and clear in results so that decision makers can quickly and directly understand and master.

Now, we give the main steps of the AHP.

Step 1: define the central questions, choices, and judgment criteria

Step 2: using the fundamental scale of Table 1 to create the pairwise comparison matrix

Step 3: determine the criterion weight vector. Normalize the comparison matrix by equation (1); then, calculate the average of each row of the normalized comparison matrix by equation (2) to obtain the weight vector:

$$\overline{a}_{ij} = \frac{a_{ij}}{\sum_{k=1}^{n} a_{ki}}, \quad i, j = 1, 2, \ldots, n, \tag{1}$$

$$W_i = \sum_{j=1}^{n} \frac{\overline{a}_{ij}}{n}, \quad i = 1, 2, \ldots, n. \tag{2}$$

Step 4: using equations (3) and (4) to compute the consistency index (CI) of the comparison matrix:

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n - 1}, \tag{3}$$

$$\lambda_{\max} = \sum_{i=1}^{4} \frac{(AW)_i}{nW_i}. \tag{4}$$

Step 5: calculate the consistency ratio (CR) by equation (5), where the random consistency index (RI) value is determined by Table 2:

$$\mathrm{CR} = \frac{\mathrm{CI}}{\mathrm{RI}}. \tag{5}$$

Step 6: compare the obtained CR with the value considered acceptable for consistency.

4.2. Application of AHP to the Design of Cantonese Cultural and Creative Products. The analytic hierarchy process model is established. First of all, the questionnaire is distributed to the tourists who often travel to and from Guangzhou. Then, it takes the user demand factors of Cantonese cultural and creative products as the goal level, takes the appearance factors, function factors, and culture factors corresponding to Maslow's demand level as the criterion layer, and extracts nine perceptual words from the user demand vocabulary of typical user interviews as the criterion layer, so as to guide the design of product appearance, function, and culture. Figure 1 shows the analytic hierarchy process model of user demand factors of Cantonese cultural and creative products.

Table 1: Scale for pairwise comparison.

Numerical value	Description
1	Equally liked
2	
3	Moderately preferred
4	
5	Strongly preferred
6	
7	Very strongly preferred
8	
9	Extremely preferred

The test group is composed of 117 effective users mentioned above. According to the index system, questionnaire survey is conducted by focus group combined with information method in user survey and decision is made. The importance of the goal layer, criteria layer, and evaluation index layer are scored, respectively. Then, the scoring results are discussed and summarized internally, and the pairwise judgment matrixes (see Tables 3–6) are obtained, and reasonable conclusions are drawn through consistency test. The judgment matrix is constructed and related calculation is carried out, and the relevant data are presented as follows.

The criteria layer judgment matrix is denoted by

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 1/2 \\ 3 & 2 & 1 \end{pmatrix}, \tag{6}$$

then by equation (1), we can get the following normalize the comparison matrix:

$$A_1 = \begin{pmatrix} 0.1667 & 0.1428 & 0.1818 \\ 0.3333 & 0.2857 & 0.2727 \\ 0.5 & 0.5715 & 0.5455 \end{pmatrix}. \tag{7}$$

By equation (2), we get the criterion weight vector:

$$W = (0.1637, 0.2972, 0.5391)^{\mathrm{T}}. \tag{8}$$

Therefore,

$$AW = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 1/2 \\ 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.1637 \\ 0.2972 \\ 0.5391 \end{pmatrix} = \begin{pmatrix} 0.4921 \\ 0.8942 \\ 1.6248 \end{pmatrix}, \tag{9}$$

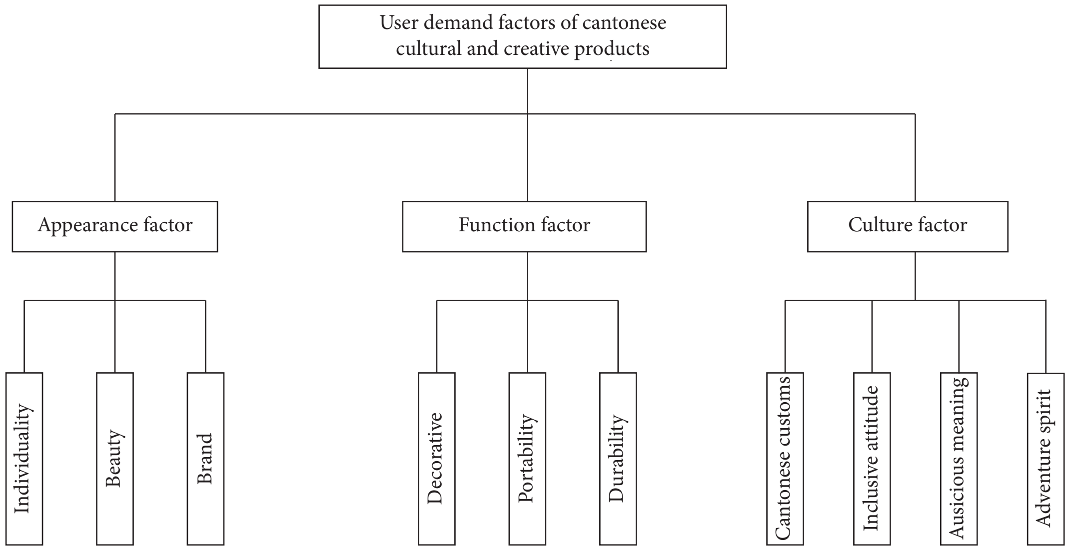

Figure 1: User demand factors of Cantonese cultural and creative product AHP model.

Table 2: Random consistency index (RI).

	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49

Table 3: Criteria layer judgment matrix.

	Appearance	Function	Culture
Appearance	1	1/2	1/3
Function	2	1	1/2
Culture	3	2	1

Table 4: Appearance evaluation index judgment matrix.

	Individuality	Beauty	Brand
Individuality	1	1/4	1/3
Beauty	4	1	3
Brand	3	1/3	1

Table 5: Functional evaluation index judgment matrix.

	Decorative	Portability	Durability
Decorative	1	1/2	1/4
Portability	2	1	1/3
Durability	4	3	1

Table 6: Culture evaluation index judgment matrix.

	Cantonese customs	Inclusive attitude	Auspicious meaning	Adventure spirit
Cantonese customs	1	5	3	7
Inclusive attitude	1/5	1	1/3	2
Auspicious meaning	1/3	3	1	5
Adventure spirit	1/7	1/2	1/5	1

then by equation (4), we obtain

$$\lambda_{\max} = 3.0096. \tag{10}$$

Since the order of A is 3, by equation (3) we obtain

$$\mathrm{CI} = 0.0048. \tag{11}$$

Because $n = 3$, we know that RI = 0.58 from Table 2, then by equation (5), we obtain

$$\mathrm{CR} = 0.0083. \tag{12}$$

Similar with the above procedure, we obtain the results as follows and do not show computational process in detail for simplicity.

$$\begin{aligned} W &= (0.1199, 0.6079, 0.2722)^{\mathrm{T}},\\ \lambda_{\max} &= 3.0742,\\ \mathrm{CI} &= 0.0371,\\ \mathrm{CR} &= 0.0639. \end{aligned} \tag{13}$$

$$\begin{aligned} W &= (0.1372, 0.2394, 0.6234)^{\mathrm{T}},\\ \lambda_{\max} &= 3.0191,\\ \mathrm{CI} &= 0.0095,\\ \mathrm{CR} &= 0.0164. \end{aligned} \tag{14}$$

$$\begin{aligned} W &= (0.5628, 0.1079, 0.2671, 0.0622)^{\mathrm{T}},\\ \lambda_{\max} &= 4.0679,\\ \mathrm{CI} &= 0.0226,\\ \mathrm{CR} &= 0.0252. \end{aligned} \tag{15}$$

Through the analytic hierarchy process, the consistency ratio of each factor layer judgment matrix evaluation index is less than 0.1, which indicates that the test team has passed the consistency test on the user demand hierarchy factors. The order of the importance of the decision-making level is as follows: culture factors > function factors > appearance factors, which shows that culture factors are more important for the design of Cantonese cultural and creative products.

In the order of the importance of functional factors in the evaluation index layer, the order of importance of culture factors is Cantonese customs > auspicious meaning > inclusive attitude > adventure spirit, which shows that Cantonese customs are the most important in culture factors. The order of importance of function factors is durability > portability > decorative, indicating that durability is the most important factor. The order of importance of appearance factors is as follows: beauty > brand > individuality, which indicates that the beauty of products is more important in appearance factors.

Figure 2: Pen holder.

5. Design Practice of Cantonese Cultural and Creative Products

5.1. Appearance Design. Cantonese architecture mainly refers to the traditional architecture in Cantonese area, which refers to the building with traditional style built by using traditional building technology and building materials. Cantonese architecture is closely related to Cantonese culture and style and has strong regional characteristics. The cultural and creative design chooses the "Bahe guild hall" which is very historic and representative in Guangzhou Xiguan as the main design element. "Bahe guild hall" is the guild organization of Cantonese opera artists, formerly known as "Qionghua guild hall" [10]. This guild hall strengthened the unity of people in the opera industry, ensured the normal operation of the troupe, and resumed the troupe business after the lifting of the ban on Cantonese Opera in the Qing Dynasty. "Bahe" have branches all over the world, where there are Chinese, as long as there are Cantonese opera, and there are these guild halls. The long history and profound cultural heritage of Guangzhou's "Bahe," which are respected as their ancestors all over the world, are of great significance. They not only enable the intangible cultural heritage art of Cantonese opera to continue to be inherited but also represent the profound Cantonese opera culture and the indomitable spirit of Cantonese people.

5.2. Function Design. In the early research, we learned that tourists are more interested in some small products and stationery products, so we chose the four treasures of the study as the design objects, including a pen holder, a pen shelf, a paper weight, and an ink slab (see Figures 2–5). In the functional design, people pay more attention to the durability of the product, so in the design of the whole set of products, we strive to achieve a stable and reasonable structure.

5.3. Culture Design. The top half of the pen holder refers to the iconic roof of "Bahe guild hall," while the lower part is added with window decoration elements; the shape of pen

Figure 3: Pen shelf.

Figure 4: Paper weight.

Figure 5: Ink slab.

shelf is transformed from the roof shape of "Bahe guild hall," which is round and beautiful as a whole; the paper weight is decorated with the exterior wall patterns of the guild hall; the ink slab is designed by combining the cloud pattern on the ceiling of the guild hall with the architectural elements.

6. Conclusions

In the research of social science [11] and natural science [12–18], the selection of methods is very important. In this paper, we choose to use an effective method, AHP, to study the design of Cantonese cultural and creative products. Based on the AHP model of user demand, we sort out the user needs of Cantonese cultural and creative products, conduct trade-off screening on the design demand factors of these products, so as to further quantify the needs of tourist groups, determine the most suitable combination of product design factors for users, and apply the conclusions to the design practice. With the vision of tourists, we can design products to meet the needs of tourists so that the cultural and creative products of Cantonese will be loved by more people, and the culture of Cantonese will spread faster and farther.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Positive Solutions for Two-Point Boundary Value Problems for Fourth-Order Differential Equations with Fully Nonlinear Terms

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In this paper, we consider the existence of positive solutions for the fully fourth-order boundary value problem $\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \ 0 \le t \le 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$, where $f\colon [0,1] \times [0,+\infty] \times (-\infty,+\infty) \times (-\infty,0) \times (-\infty,+\infty) \longrightarrow [0,+\infty]$ is continuous. This equation can simulate the deformation of an elastic beam simply supported at both ends in a balanced state. By using the fixed-point index theory and the cone theory, we discuss the existence of positive solutions of the fully fourth-order boundary value problem. We transform the fourth-order differential equation into a second-order differential equation by order reduction method. And then, we examine the spectral radius of linear operators and the equivalent norm on continuous space. After that, we obtain the existence of positive solutions of such BVP.

1. Introduction

In this paper, we study the existence of positive solutions for the fully fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = f\Big(t, u(t), u'(t), u''(t), u'''(t)\Big), & 0 \le t \le 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1}$$

where $f\colon [0,1] \times [0,+\infty] \times (-\infty,+\infty) \times (-\infty,0) \times (-\infty,+\infty) \longrightarrow [0,+\infty]$ is continuous. This boundary value problem can simulate the deformation of an elastic beam, whose one end is fixed and the other end is free in a balanced state. In mechanics, BVP (1) is called a cantilever beam equation. In this equation, each derivative of $u(t)$ has its physical meaning: $u'(t)$ is the slope, $u''(t)$ is the bending moment stiffness, $u'''(t)$ is the shear force stiffness, and $u^{(4)}(t)$ is the load density stiffness. The nonlinear fourth-order differential equation boundary value problem can simulate the deformation of an elastic beam under external force, and different boundary value conditions can show its force under different conditions. Because of its importance in mechanics, many scholars have done a lot of research on the existence of solutions for fourth-order ordinary differential equations using various nonlinear methods [1–11].

As the nonlinear term does not contain the derivative term of the unknown function, equation (1) becomes

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), & 0 \le t \le 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{2}$$

If $f(t,u)$ is superlinear or sublinear growth on u, the authors in [1] used the fixed-point theorem on the cone to obtain the existence of the positive solution of equation (2). In [3], the author used the fixed-point theorem and topological degree theory to study the existence of one or two positive solutions for the fourth-order differential equation boundary value problem:

$$\begin{cases} u^{(4)}(t) - \lambda f(t, u(t)) = 0, & 0 \le t \le 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{3}$$

Under the fixed-point index method on the cone, the authors of [11] discussed the existence of positive solutions for fourth-order boundary value problems with two parameters. Among them, the assumption condition of the nonlinear term f is related to the first eigenvalue of the corresponding linear operator. It is noteworthy that the nonlinear term in the abovementioned boundary value problems does not include the higher-order derivative u'''. When the nonlinear term contains the higher-order derivative of the unknown function, the authors of [8, 9] used the upper and lower solution method to study the existence of solutions for fully fourth-order nonlinear boundary value problems with nonlinear boundary conditions. In [10], the author discussed a fourth-order boundary value problem with fully form:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & 0 \le t \le 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{4}$$

When the nonlinear term satisfies superlinear growth and sublinear growth, the author used the fixed-point index method, combined with the positivity of linear operators and spectral radius, to get the positive solutions for the boundary value problem. But the linear operator in [10] does not involve the first and second derivatives of unknown functions.

In this paper, by using cone theory and the fixed-point index, combined with the spectral radius of linear integral operators, and the application of equivalent norms, we discuss the existence of positive solutions for boundary value problems (1).

2. Preliminaries

In this section, we give some assumptions that are important to our main results:

(i) (H_1): f: $[0,1] \times [0,+\infty] \times (-\infty,+\infty) \times (-\infty,0) \times (-\infty,+\infty) \longrightarrow [0,+\infty]$ is continuous

(ii) (H_2): there exist nonnegative constants $a_1, c_1 \ge 0$ and $r > 0$ such that $a_1 x_1 + c_1 x_3 \le f(t, x_1, x_2 - x_3, x_4)$, $x_1, x_3 \in [0, r]$, $x_2 \in [-r, r]$, and $x_4 \in (-\infty, +\infty)$.

(iii) (H_3): there exist nonnegative constants $a, b, c, d, M \ge 0$ and $0 < q < 1$ such that

$$f(t, x_1, x_2, -x_3, x_4) \le a x_1 + b|x_2| + c x_3 + d|x_4|^q + M, x_1, x_3 \in [0,+\infty], x_2, x_4 \in (-\infty,+\infty). \tag{5}$$

Let $C[0,1]$ denote the Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with norm $\|u\| = \max_{0\le t\le 1}|u(t)|$. Let

$P = \{u \in C[0,1]\ u(t)$is concave on $[0,1]$, $u(0) = u(1) = 0\}$. Then, P is a positive cone on $C[0,1]$.

The functions on cone P have the following properties:

Lemma 1 (see [12]). *Every function $u \in P$ on the cone P is differentiable almost everywhere on $(0,1)$ and satisfies*

$$\begin{aligned} &u(t) \ge t(1-t)\|u\|, \quad t \in [0,1], \\ &|u'(t)| \le \frac{u(t)}{t(1-t)}, \quad \text{a.e. on } [0,1]. \end{aligned} \tag{6}$$

Let $v(t) = -u''(t)$, then the differential equation BVP (1) can be transformed into the following two second-order differential equations:

$$\begin{cases} u''(t) = -v(t), \\ u(0) = u(1) = 0, \end{cases} \tag{7}$$

$$\begin{cases} -v''(t) = f(t, u(t), u'(t), u''(t), u''(t)), \\ v(0) = v(1) = 0. \end{cases} \tag{8}$$

Let $G(t,s) = \begin{cases} s(1-t) & 0 \le s \le t \le 1 \\ t(1-s) & 0 \le t \le s \le 1 \end{cases}$, which is the corresponding Green's function of BVP (7). Thus, BVP (7) can be transformed into an equivalent integral equation:

$$u(t) = \int_0^1 G(t,s) v(s) \mathrm{d}s. \tag{9}$$

By using (9), BVP (8) can be reduced to

$$\begin{cases} -v''(t) = f(t, (Av)(t), (Av)'(t), -v(t), -v'(t)), \\ v(0) = v(1) = 0, \end{cases} \tag{10}$$

where operator A is defined as $(Av)(t) = \int_0^1 G(t,s) v(s) \mathrm{d}s$. Thus, BVP (10) can be reduced to the equivalent integral equation:

$$v(t) = \int_0^1 G(t,s) f(s, (Av)(s), (Av)'(s), -v(s), -v'(s)) \mathrm{d}s. \tag{11}$$

From the expression of Green's function $G(t,s)$, we know that $G(t,s)$ is continuous on $[0,1] \times [0,1]$, and we have

$$t(1-t)s(1-s) \le G(t,s) \le s(1-s) \text{ or } t(1-t), \quad 0 \le t, s \le 1. \tag{12}$$

From the standard proof, we can easily obtain the following statement.

Lemma 2. A: $L[0,1] \longrightarrow C[0,1]$ *is a completely continuous operator.*

For $\forall v \in P$, we define an operator $(Fv)(t) = f(t, (Av)(t), (Av)'(t), -v(t), -v'(t))$. It follows from Lemma 1 and condition (H_3) that

$$\begin{aligned}|(Fv)(t)| &= \left|f\left(t, (Av)(t), (Av)'(t), -v(t), -v'(t)\right)\right| \\ &\leq a(Av)(t) + b\left|(Av)'(t)\right| + cv(t) + d\left|v'(t)\right|^q + M \\ &\leq a\int_0^1 G(t,s)v(s)\mathrm{d}s + b\int_0^1 \left|G_t(t,s)\right|v(s)\mathrm{d}s + cv(t) + d\frac{\|v\|^q}{t^q(1-t)^q} + M.\end{aligned} \tag{13}$$

It shows that F is an operator which is defined on P to $L(0,1)$. It is easy to conclude that F is the continuous bounded operator mapping from P to $L(0,1)$. Therefore, $T = A^\circ F$: $P \longrightarrow P$ is a completely continuous operator. So, BVP (1) is equivalent to the operator equation $u = Tu$, where the operator $T = P \longrightarrow P$ is given by

$$(Tv)(t) = \int_0^1 G(t,s)f\left(s, (Av)(s), (Av)'(s), -v(s), -v'(s)\right)\mathrm{d}s. \tag{14}$$

Let $\tilde{a} = (a,b,c)$ with $a,b,c \geq 0$. We define an operator $T_{\tilde{a}}$ on cone P:

$$\begin{aligned}\left(T_{\tilde{a}}u\right)(t) = a&\int_0^1 G(t,s)\int_0^1 G(s,\tau)u(\tau)\mathrm{d}\tau\,\mathrm{d}s \\ &+ b\int_0^1 G(t,s)\int_0^1 \left|G_s(s,\tau)\right|u(\tau)\mathrm{d}\tau\,\mathrm{d}s \\ &+ c\int_0^1 G(t,s)u(s)\mathrm{d}s, \quad u \in P.\end{aligned} \tag{15}$$

It is easy to see that $T_{\tilde{a}}$: $P \longrightarrow P$ is a linear operator. Let $u_0(t) = t(1-t)$, by (15), we obtain

$$\begin{aligned}\left(T_{\tilde{a}}u_0\right)(t) &\geq at(1-t)\int_0^1 s(1-s)\int_0^1 G(s,\tau)\tau(1-\tau)\,\mathrm{d}\tau\,\mathrm{d}s + bt(1-t) \\ &\cdot \int_0^1 s(1-s)\int_0^1 \left|G_s(s,\tau)\right|\tau(1-\tau)\,\mathrm{d}\tau\,\mathrm{d}s + ct(1-t)\int_0^1 s(1-s)s(1-s)\,\mathrm{d}s = t(1-t) \\ &\cdot \left[a\int_0^1 s(1-s)\int_0^1 G(s,\tau)\tau(1-\tau)\,\mathrm{d}\tau\,\mathrm{d}s + b\int_0^1 s(1-s)\int_0^1 \left|G_s(s,\tau)\right|\tau(1-\tau)\mathrm{d}\tau\,\mathrm{d}s + c\int_0^1 s^2(1-s)^2\,\mathrm{d}s\right].\end{aligned} \tag{16}$$

Then, if scalars $a,b,c \geq 0$, not all equal to zero, there exists a constant $\alpha > 0$ satisfies $T_{\tilde{a}}u_0 \geq \alpha u_0$. Thus, from Krein–Rutman theorem, we know $r(T_{\tilde{a}}) > 0$, and there is $v_{\tilde{a}} \in P/\{\theta\}$ such that $\lambda_{\tilde{a}}T_{\tilde{a}}v_{\tilde{a}} = v_{\tilde{a}}$, where $\lambda_{\tilde{a}} = (r(T_{\tilde{a}}))^{-1}$ is the first eigenvalue of the operator $T_{\tilde{a}}$.

Now, we estimate the range of $\lambda_{\tilde{a}} = (r(T_{\tilde{a}}))^{-1}$. For any $u \in P$, from (15), we get

$$\begin{aligned}\left|\left(T_{\tilde{a}}u\right)(t)\right| \leq a&\int_0^1 s(1-s)\int_0^1 G(s,\tau)u(\tau)\mathrm{d}\tau\,\mathrm{d}s \\ &+ b\int_0^1 s(1-s)\int_0^1 \left|G_s(s,\tau)\right|u(\tau)\mathrm{d}\tau\,\mathrm{d}s \\ &+ c\int_0^1 s(1-s)u(s)\mathrm{d}s, \quad a,b,c \geq 0, \tilde{a} = (a,b,c),\end{aligned} \tag{17}$$

thereby

$$\begin{aligned}\left\|T_{\tilde{a}}u\right\| \leq a&\int_0^1 s(1-s)\int_0^1 G(s,\tau)u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ &+ b\int_0^1 s(1-s)\int_0^1 \left|G_s(s,\tau)\right|u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ &+ c\int_0^1 s(1-s)u(s)\,\mathrm{d}s.\end{aligned} \tag{18}$$

By the expression of Green's function $G(t,s)$ and (15), we infer that

$$\begin{aligned}\left(T_{\tilde{a}}u\right)(t) \geq t(1-t)\Bigg[a&\int_0^1 s(1-s)\int_0^1 G(s,\tau)u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ &+ b\int_0^1 s(1-s)\int_0^1 \left|G_s(s,\tau)\right|u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ &+ c\int_0^1 s(1-s)u(s)\,\mathrm{d}s\Bigg], \quad a,b,c \geq 0, \tilde{a} = (a,b,c).\end{aligned} \tag{19}$$

According to the above two inequalities, we have

$$\left(T_{\tilde{a}}u\right)(t) \geq t(1-t)\left\|T_{\tilde{a}}u\right\|. \tag{20}$$

Recalled that $u_0(t) = t(1-t)$. By (20), we have

$$T_{\tilde{a}}u_0 \geq \left\|T_{\tilde{a}}u_0\right\|u_0. \tag{21}$$

Note the positivity of operator $T_{\tilde{a}}$, we have $T_{\tilde{a}}^2u_0 \geq \|T_{\tilde{a}}u_0\|^2u_0$. Hence, for any $n \in N$, using the recursion method to the above inequality, we get $T_{\tilde{a}}^nu_0 \geq \|T_{\tilde{a}}u_0\|^nu_0$. Therefore, we have

$$\left\|T_{\tilde{a}}^nu_0\right\| \geq \left\|T_{\tilde{a}}u_0\right\|^n\left\|u_0\right\|. \tag{22}$$

Thus,

$$\left\|T_{\tilde{a}}\right\|^{n} \geq \frac{\left\|T_{\tilde{a}}^{n} u_{0}\right\|}{\left\|u_{0}\right\|} \geq \left\|T_{\tilde{a}} u_{0}\right\|^{n}. \tag{23}$$

From this inequality and Gelfand formula on spectral radius, we obtain

$$r\left(T_{\tilde{a}}\right) = \lim_{n \longrightarrow \infty} \sqrt[n]{\left\|T_{\tilde{a}}^{n}\right\|} \geq \left\|T_{\tilde{a}} u_{0}\right\| > 0. \tag{24}$$

From (24), we get

$$\left\|T_{\tilde{a}}\right\| \geq r\left(T_{\tilde{a}}\right) \geq \left\|T_{\tilde{a}} u_{0}\right\|. \tag{25}$$

By $\lambda_{\tilde{a}} = \left(r\left(T_{\tilde{a}}\right)\right)^{-1}$, we can get

$$\frac{1}{\left\|T_{\tilde{a}} u_{0}\right\|} \leq \lambda_{\tilde{a}} \leq \frac{1}{\left\|T_{\tilde{a}}\right\|}. \tag{26}$$

Now, we calculate $\|T_{\tilde{a}}\|$ and $\|T_{\tilde{a}}u_0\|$ as follows. We have known that

$$\begin{aligned} G(s,\tau) &= \begin{cases} s(1-\tau), & 0 \leq s \leq \tau \leq 1, \\ \tau(1-s), & 0 \leq \tau \leq s \leq 1, \end{cases} \\ G_s(s,\tau) &= \begin{cases} 1-\tau, & 0 \leq s \leq \tau \leq 1, \\ -\tau, & 0 \leq \tau < s \leq 1. \end{cases} \end{aligned} \tag{27}$$

We first calculate $\|T_{\tilde{a}}\|$. From (15), we get

$$\begin{aligned} \left\|T_{\tilde{a}}\right\| = \max_{t\in[0,1]} a &\int_0^1 G(t,s) \int_0^1 G(s,\tau) u(\tau)\, d\tau\, ds \\ &+ b \int_0^1 G(t,s) \int_0^1 \left|G_s(s,\tau)\right| u(\tau)\, d\tau\, ds \\ &+ c \int_0^1 G(t,s) u(s)\, ds. \end{aligned} \tag{28}$$

Through integration, we have

$$\left\|T_{\tilde{a}}\right\| = \max_{t\in[0,1]} \left[t(1-t)\left(\frac{1}{12}a + \frac{1}{3}b + \frac{1}{2}c\right)\right] = \frac{1}{48}a + \frac{1}{12}b + \frac{1}{8}c. \tag{29}$$

Next, we calculate $\|T_{\tilde{a}}u_0\|$. Because

$$\begin{aligned} \left\|T_{\tilde{a}} u_0\right\| = \max_{t\in[0,1]} \Bigg[a &\int_0^1 G(t,s) \int_0^1 G(s,\tau) u_0(\tau) d\tau\, ds \\ &+ b \int_0^1 G(t,s) \int_0^1 \left|G_s(s,\tau)\right| u_0(\tau) d\tau\, ds \\ &+ c \int_0^1 G(t,s) u_0(s) ds \Bigg]. \end{aligned} \tag{30}$$

Through integration, we have

$$\begin{aligned} \left\|T_{\tilde{a}} u_0\right\| &= \max_{t\in[0,1]} \left[t(1-t)\left(\frac{1}{60}a + \frac{1}{15}b\right) + \left(\frac{1}{12}t - \frac{1}{6}t^3 + \frac{1}{12}t^4\right)c\right] \\ &= \frac{1}{240}a + \frac{1}{60}b + \frac{5}{192}c. \end{aligned} \tag{31}$$

Combining this with (29), we can obtain

$$\frac{48}{a + 4b + 6c} \leq \lambda_{\tilde{a}} \leq \frac{960}{4a + 16b + 25c}, \quad a, b, c > 0, \tilde{a} = (a, b, c). \tag{32}$$

Now, we consider the special case with $\tilde{b} = (a_1, 0, c_1)$. We make a definition that

$$\begin{aligned} \left(T_{\tilde{b}} u\right)(t) = a_1 &\int_0^1 G(t,s) \int_0^1 G(s,\tau) u(\tau) d\tau\, ds \\ &+ c_1 \int_0^1 G(t,s) u(s) ds, \quad \tilde{b} = (a_1, 0, c_1). \end{aligned} \tag{33}$$

Because of $\lambda_{\tilde{b}} T_{\tilde{b}} v_{\tilde{b}}(t) = v_{\tilde{b}}(t)$, it is equivalent to $\lambda_{\tilde{b}}$ and $v_{\tilde{b}}(t)$ satisfying the differential equation:

$$\begin{cases} u^{(4)}(t) = \lambda_{\tilde{b}}\left[a_1 u(t) - c_1 u''(t)\right], & t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{34}$$

We notice that the function $v_{\tilde{b}}(t)$ is nonnegative. After solving the above differential equations, we may take $v_{\tilde{b}}(t) = \sin \pi t$. Hence, we conclude the value of $\lambda_{\tilde{b}}$ is $\lambda_{\tilde{b}} = \pi^4/a_1 + \pi^2 c_1$.

To prove our main results, we also need the two following lemmas.

Lemma 3 (see [13, 14]). *Let E is a Banach space, P is a cone in E, and $\Omega(P)$ is a bounded open set in P. Assume that A: $\overline{\Omega(P)} \longrightarrow P$ is a completely continuous operator. If there exists $x_0 \in P/\{\theta\}$ such that*

$$x - Ax \neq \mu x_0, \quad \forall x \in \partial\Omega(P), \mu \geq 0. \tag{35}$$

Then, $i(A, \Omega(P), P) = 0$.

Lemma 4 (see [13, 14]). *Let E is a Banach space, P is a cone in E, $\Omega(P)$ is a bounded open set in P, and $\theta \in \Omega(P)$. Assume that A: $\overline{\Omega(P)} \longrightarrow P$ is a completely continuous operator. If*

$$Ax \neq \mu x, \quad \forall x \in \partial\Omega(P), \mu \geq 1, \tag{36}$$

then $i(A, \Omega(P), P) = 1$.

3. Main Results

Theorem 1. *Suppose that conditions (H_1), (H_2), and (H_3) are satisfied, and*

$$\lambda_{\tilde{a}} > 1, \tag{37}$$

$$\lambda_{\tilde{b}} < 1. \tag{38}$$

$\lambda_{\tilde{a}}$ and $\lambda_{\tilde{b}}$ are the first eigenvalues of operator $T_{\tilde{a}}$ and operator $T_{\tilde{b}}$, respectively. Then, BVP (1) has at least one positive solution.

Proof. It follows from (H_1) and (H_2) that, for some $r > 0$, the function f satisfies

$$f(t,x_1,x_2,-x_3,x_4)\ge a_1x_1+c_1x_3,\quad x_1,x_3\in[0,r],\quad x_2\in[-r,r],\quad x_4\in(-\infty,+\infty). \tag{39}$$

It is known that $v_{\widetilde{b}}$ is the positive eigenvector of $T_{\widetilde{b}}$ corresponding to $\lambda_{\widetilde{b}}$, and we have $v_{\widetilde{b}}=\lambda_{\widetilde{b}}T_{\widetilde{b}}v_{\widetilde{b}}$. According to (14), (38), and (39), for any $v\in\partial B_r\cap P$, we have

$$\begin{aligned}|(Av)(t)|&=\int_0^1 G(t,s)|v(s)|\mathrm{d}s\le r\int_0^1 G(t,s)\mathrm{d}s<r,\\ \left|(Av)'(t)\right|&=\int_0^1\left|G_t(t,s)\right|\left|v(s)\right|\mathrm{d}s\le r\int_0^1\left|G_t(t,s)\right|\mathrm{d}s<r.\end{aligned} \tag{40}$$

Thus, we get

$$\begin{aligned}(Tv)(t)&=\int_0^1 G(t,s)f\left(s,(Av)(s),(Av)'(s),-v(s),-v'(s)\right)\mathrm{d}s\\ &\ge\int_0^1 G(t,s)\left(a_1(Av)(s)+c_1v(s)\right)\mathrm{d}s\\ &=a_1\int_0^1 G(t,s)\int_0^1 G(s,\tau)v(\tau)\mathrm{d}\tau\,\mathrm{d}s\\ &\quad+c_1\int_0^1 G(t,s)v(s)\mathrm{d}s=\left(T_{\widetilde{b}}v\right)(t),\quad t\in[0,1].\end{aligned} \tag{41}$$

Let us suppose that T has no fixed point on $\partial B_r\cap P$; otherwise, Theorem 1 is proved.

Now, we prove

$$v-Tv\ne\zeta v_{\widetilde{b}},\quad\forall v\in\partial B_r\cap P,\zeta\ge 0, \tag{42}$$

where $B_r=\{x\in C[0,1]\,|\,\|x\|\le r\}$. If otherwise, there exists $v_0\in\partial B_r\cap P$ and $\zeta_0\ge0$ such that $v_0-Tv_0=\zeta_0v_{\widetilde{b}}$. Then, we have $\zeta_0>0$ and $v_0=Tv_0+\zeta_0v_{\widetilde{b}}\ge\zeta_0v_{\widetilde{b}}$. Let $\zeta^*=\sup\left\{\zeta\,|\,v_0\ge\zeta v_{\widetilde{b}}\right\}$, we can know $\zeta^*\ge\zeta_0>0$ and $v_0\ge\zeta^*v_{\widetilde{b}}$. By $T_{\widetilde{b}}(P)\subset P$, we get $\lambda_{\widetilde{b}}T_{\widetilde{b}}v_0\ge\zeta^*\lambda_{\widetilde{b}}T_{\widetilde{b}}v_{\widetilde{b}}=\zeta^*v_{\widetilde{b}}$. Based on (41), we have

$$v_0=Tv_0+\zeta_0v_{\widetilde{b}}\ge T_{\widetilde{b}}v_0+\zeta_0v_{\widetilde{b}}\ge\left(\frac{\zeta^*}{\lambda_{\widetilde{b}}}+\zeta_0\right)v_{\widetilde{b}}. \tag{43}$$

Then, by the definition of ζ^*, we conclude that $\zeta^*\ge\zeta^*/\lambda_{\widetilde{b}}+\zeta_0$. By (15), we notice that $\lambda_{\widetilde{b}}<1$ and $\zeta_0>0$. So, the above inequality contradicts the definition of ζ^*. Hence, (42) holds. By Lemma 3, we have

$$i(T,B_r\cap P,P)=0. \tag{44}$$

From (H_3), we have

$$\begin{aligned}(Tv)(t)&\le\int_0^1 G(t,s)\left(a\int_0^1 G(s,\tau)v(\tau)\mathrm{d}\tau+b\int_0^1\left|G_s(s,\tau)\right|v(\tau)\mathrm{d}\tau+cv(s)+d\left|v'(s)\right|^q+M\right)\mathrm{d}s\\ &\le a\int_0^1 G(t,s)\int_0^1 G(s,\tau)v(\tau)\mathrm{d}\tau\,\mathrm{d}s+b\int_0^1 G(t,s)\int_0^1\left|G_s(s,\tau)\right|v(\tau)\mathrm{d}\tau\,\mathrm{d}s+c\int_0^1 G(t,s)v(s)\mathrm{d}s+d\int_0^1 G(t,s)\frac{|v(s)|^q}{s^q(1-s)^q}\mathrm{d}s+M\\ &=\left(T_{\widetilde{a}}v\right)(t)+d\int_0^1 G(t,s)\frac{|v(s)|^q}{s^q(1-s)^q}\mathrm{d}s\le\left(T_{\widetilde{a}}v\right)(t)+d\int_0^1 G(t,s)\frac{|v(s)|^q}{s^q(1-s)^q}\mathrm{d}s+M\\ &\le\left(T_{\widetilde{a}}v\right)(t)+d\int_0^1 s(1-s)\frac{|v(s)|^q}{s^q(1-s)^q}\mathrm{d}s+M\le\left(T_{\widetilde{a}}v\right)(t)+d\int_0^1|v(s)|^q\mathrm{d}s+M.\end{aligned} \tag{45}$$

Note that $(r(T_{\widetilde{a}}))^{-1}>1$. Take

$$\varepsilon=\frac{1-r(T_{\widetilde{a}})}{2}. \tag{46}$$

According to Gelfand formula, $r(T_{\widetilde{a}})=\lim_{n\longrightarrow\infty}\sqrt[n]{\|T_{\widetilde{a}}^n\|}$, and there is a natural number N, so that, for $n\ge N$, $\|T_{\widetilde{a}}^n\|\le[r(T_{\widetilde{a}})+\varepsilon]^n$.

For any $v\in C[0,1]$, we define

$$\|v\|^*=\sum_{i=1}^N\left[r(T_{\widetilde{a}})+\varepsilon\right]^{N-i}\left\|T_{\widetilde{a}}^{i-1}v\right\|, \tag{47}$$

where $T_1^0=I$ is an identity operator. It is not difficult to verify that $\|\cdot\|^*$ is a norm in $C[0,1]$, and we have

$$\left\|\int_0^1 v^q(s)\mathrm{d}s\right\|^*\le K\|v\|^{*q},\quad v\in C[0,1], \tag{48}$$

where K is a constant. Take

$$r_1>\max\left\{1,r,4\|M\|^*\varepsilon^{-1},\left(4\,dK\varepsilon^{-1}\right)^{1/1-q}\right\}. \tag{49}$$

Because of $\|v\|^*\ge[r(T_{\widetilde{a}})+\varepsilon]^{N-1}\|v\|$, we can take $r_2>r_1$ so that when $\|v\|\ge r_2$, we have $\|v\|^*>r_1$.

Next, we prove

$$Tv \neq \mu v, \quad \forall v \in \partial B_{r_2} \cap P, \mu \geq 1. \tag{50}$$

If there exists $v_1 \in \partial B_{r_2} \cap P$ and $\mu_0 \geq 1$, such that $Tv_1 = \mu_0 v_1$, by Lemma 1 and (45), we can obtain

$$\begin{aligned}(Tv_1)(t) &= \int_0^1 G(t,s) f(s, (Av_1)(s), (Av_1)'(s), v_1(s), v_1'(s))ds \\ &\leq (T_{\tilde{a}} v_1)(t) + d\int_0^1 v_1^q(s)ds + M.\end{aligned} \tag{51}$$

Hence, we have

$$0 \leq \mu_0 v_1(t) = (Tv_1)(t) \leq (T_{\tilde{a}} v_1)(t) + d\int_0^1 v_1^q(s)ds + M, \quad \forall t \in [0,1]. \tag{52}$$

Furthermore, $T_{\tilde{a}}(P) \subset P$, and we have

$$0 \leqq \left(T_{\tilde{a}}^j (Tv_1)\right)(t) \leq \left(T_{\tilde{a}}^j \left(T_{\tilde{a}} v_1 + d\int_0^1 v_1^q(s)ds + M\right)\right)(t), \quad \forall t \in [0,1], j = 1,2,\ldots,N-1, \tag{53}$$

and we also get

$$\left\|T_{\tilde{a}}^j (Tv_1)\right\| \leq \left\|T_{\tilde{a}}^j \left(T_{\tilde{a}} v_1 + d\int_0^1 v_1^q(s)ds + M\right)\right\|, \quad j = 1,2,\ldots,N-1. \tag{54}$$

Therefore, from (47), we have

$$\begin{aligned}\|Tv_1\|^* &= \sum_{i=1}^N \left[r(T_{\tilde{a}}) + \varepsilon\right]^{N-i} \left\|T_{\tilde{a}}^{i-1}(Tv_1)\right\| \\ &\leq \sum_{i=1}^N \left[r(T_{\tilde{a}}) + \varepsilon\right]^{N-i} \left\|T_{\tilde{a}}^{i-1}\left(T_{\tilde{a}} v_1 + d\int_0^1 v_1^q(s)ds + M\right)\right\| \\ &= \left\|T_{\tilde{a}} v_1 + d\int_0^1 v_1^q(s)ds + M\right\|^*.\end{aligned} \tag{55}$$

When $\|v_1\| = r_2$, we have $\|v_1\|^* > r_1$. Hence, from (48) and (55), it follows that

$$\begin{aligned}\mu_0\|v_1\|^* &= \|Tv_1\|^* \leq \left\|T_{\tilde{a}} v_1 + d\int_0^1 v_1^q(s)ds + M\right\|^* \leq \|T_{\tilde{a}} v_1\|^* + \left\|d\int_0^1 v_1^q(s)ds\right\|^* \\ &\leq \|T_{\tilde{a}} v_1\|^* + dK\|v_1\|^{*q} + \frac{\varepsilon}{4}\|v_1\|^* = \sum_{i=1}^{N-1}\left[r(T_{\tilde{a}}) + \varepsilon\right]^{N-i}\|T_{\tilde{a}}^i v_1\| + dK\|v_1\|^{*q} + \frac{\varepsilon}{4}\|v_1\|^* \\ &\leq \left[r(T_{\tilde{a}}) + \varepsilon\right]\sum_{i=1}^{N-1}\left[r(T_{\tilde{a}}) + \varepsilon\right]^{N-i-1}\|T_{\tilde{a}}^i v_1\| + \|T_{\tilde{a}}^N v_1\| + \frac{\varepsilon}{4}\|v_1\|^{*q} r_1^{1-q} + \frac{\varepsilon}{4}\|v_1\|^* \\ &\leq \left[r(T_{\tilde{a}}) + \varepsilon\right]\sum_{i=1}^{N-1}\left[r(T_{\tilde{a}}) + \varepsilon\right]^{N-i-1}\|T_{\tilde{a}}^i v_1\|\left[r(T_{\tilde{a}}) + \varepsilon\right]^N\|v_1\| + \frac{\varepsilon}{4}\|v_1\|^* + \frac{\varepsilon}{4}\|v_1\|^* \\ &\leq \left[r(T_{\tilde{a}}) + \varepsilon\right]\sum_{i=1}^{N-1}\left[r(T_{\tilde{a}}) + \varepsilon\right]^{N-i}\|T_{\tilde{a}}^{i-1} v_1\| + \frac{\varepsilon}{2}\|v_1\|^* \\ &= \left[r(T_{\tilde{a}}) + \varepsilon\right]\|v_1\|^* + \frac{\varepsilon}{2}\|v_1\|^* \leq \left[r(T_{\tilde{a}}) + \frac{3\varepsilon}{2}\right]\|v_1\|^*.\end{aligned} \tag{56}$$

By $\mu_0 \geq 1$ and (56), we can get $1 \leq r(T_{\tilde{a}}) + 3\varepsilon/2$. But it is contradictory to the definition of $\varepsilon = 1 - r(T_{\tilde{a}})/2$. Hence, (50) holds. According to Lemma 4, we obtain that

$$i\left(T, B_{r_2} \cap P, P\right) = 1. \tag{57}$$

Now, from (44) and (57), it follows that

$$i\left(T, \frac{\left(B_{r_2} \cap P\right)}{\left(\overline{B}_r \cap P\right)}, P\right) = i\left(T, B_{r_2} \cap P, P\right) - i\left(T, B_r \cap P, P\right) = 1. \tag{58}$$

Hence, T has at least one fixed point in $(B_{r_2} \cap P)/(\overline{B}_r \cap P)$. That is to say, BVP (1) has at least one positive solution. The proof of Theorem 1 is completed.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Regional Controllability of Riemann–Liouville Time-Fractional Semilinear Evolution Equations

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In this paper, we discuss the exact regional controllability of fractional evolution equations involving Riemann–Liouville fractional derivative of order $q \in]0,1[$. The result is obtained with the help of the theory of fractional calculus, semigroup theory, and Banach fixed-point theorem under several assumptions on the corresponding linear system and the nonlinear term. Finally, some numerical simulations are given to illustrate the obtained result.

1. Introduction

Time-fractional systems have been proved, with the development of science and technology, to be one of the most effective tools in modeling many phenomena arising in physics, engineering, and real world problems [1–6]. Therefore, the research studies of fractional-order calculus attract lots of attention for these kinds of systems with several fractional derivatives (for more details, see [7–10] and the references therein). Zhou and Jiao [11] introduced a concept of a mild solution based on Laplace transform and probability density functions; several authors presented a tremendous amount of valuable results on controllability and observability, stability analysis, and so on [12–15].

Similar to the integer-order control systems [16–21], the regional controllability problem of fractional systems is a class of control problems presented in many applications in real world. Regional controllability of linear and some nonlinear fractional systems is referred to in literature [22–24] and the references therein. However, regional controllability of Riemann–Liouville fractional semilinear evolution equations with analytic semigroup problem is still open. Then, this paper focuses on the existence of a bounded control steering the system into a bounded desired state defined only in a subregion of the whole evolution domain. Based on Banach fixed-point theorem and some properties of fractional operators, the main result is deduced.

The rest of this paper is organized as follows. In Section 2, some definitions and preparation results are introduced. In Section 3, the regional controllability of the considered system, using theory of analytical semigroup, is established under some conditions. At last, two examples are given to illustrate our given algorithm.

2. Preliminaries and Problem Formulation

In this section, we introduce some basic definitions of fractional operators present in the considered system which will be specified later and some properties which are used further in this paper.

Definition 1 (see [7]). The left sided Riemann–Liouville fractional integral (resp. derivative) of a function y at a point t of order $q \in]0,1[$ can be written as

$$\mathrm{I}_{0^+}^{q} y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\mathrm{d}s, \quad 0 < t \le T, \tag{1}$$

$$^{RL}D_{0^+}^{q} y(t) = \frac{\mathrm{d}}{\mathrm{d}t} I_{0^+}^{1-q} y(t), \quad 0 < t \le T. \tag{2}$$

Let us consider X and Y to be two Banach spaces; we have the following two propositions.

Proposition 1 (see [25]). *Let us consider $f \in L^1_{loc}(0,T;X)$ and T: $[0,T] \longrightarrow \mathscr{L}(X,Y)$ to be strongly continuous.*

Then, the convolution

$$(T * f)(t) := \int_0^t T(t-s) f(s) \mathrm{d}s, \tag{3}$$

exists in Bochner sense and defines a continuous function $T * f$ from $[0,T]$ into Y.

Proposition 2 (see [25]). *(Young's inequality).*

Let us consider $p, r, s \geq 1$ such that $1/r + 1/p = 1 + 1/s$. If $T \in L_p(0,T;\mathscr{L}(X,Y))$ and $f \in L^r(0,T;X)$; then, $T * f \in L^s(0,T;Y)$ and

$$\|T * f\|_{L^s(0,T;Y)} \leq \|f\|_{L^r(0,T;X)} \cdot \|T\|_{L^p(0,T;\mathscr{L}(X,Y))}. \tag{4}$$

Now, we present the considered system. For that, let Ω be a bounded subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let us consider $T > 0$ and denote

$$\begin{aligned} Q &:= \Omega \times]0,T], \\ \Sigma &:= \partial\Omega \times]0,T]. \end{aligned} \tag{5}$$

We consider the following semilinear fractional system involving Riemann–Liouville derivative of order $q \in]0,1[$:

$$\begin{cases} {}^{RL}D^q_{0^+} y(x,t) + Ay(x,t) = Ny(x,t) + \mathscr{B}u(t), & \text{in } Q, \\ y(\xi,t) = 0, & \text{on } \Sigma, \\ \lim\limits_{t \longrightarrow 0^+} I^{1-q}_{0^+} y(x,t) = y_0, & \text{in } \Omega, \end{cases} \tag{6}$$

where $-A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded operator $\{\mathscr{G}(t)\}_{t \geq 0}$ on the Hilbert space $\mathrm{X} = L^2(\Omega)$. Without loss of generality, let $0 \in \rho(A)$ where $\rho(A)$ is the resolvent set of A. Then we define the fractional power A^α for $0 < \alpha < 1$, which is a closed linear operator. Its domain is $D(A^\alpha) = \mathrm{X}^\alpha$, which is a Banach space equipped with the norm $\|.\|_{X^\alpha} = \|A^\alpha(.)\|_X$, N: $L^2(0,T;\mathrm{X}^\alpha) \longrightarrow L^2(0,T;\mathrm{X})$ is a nonlinear operator, and $\mathscr{B}$ is the control operator which is linear (bounded or unbounded) from $\mathbb{R}^p$ into X where p is the number of actuators, u is given in $U := L^2(0,T,\mathbb{R}^p)$, and the initial state y_0 is in X^α.

We use the following definition of mild solution for the previous problem.

Definition 2 (see [26]). For $t \in]0,T]$ and any given $u \in U$, we say that a function $y_u \in C(0,T;X)$ is a mild solution of system (6) if it satisfies the following formula:

$$\begin{aligned} y_u(.,t) = {} & t^{q-1} K_q(t) y_0 \\ & + \int_0^t (t-\tau)^{q-1} K_q(t-\tau) \left(Ny_u(\tau) + Bu(\tau)\right) \mathrm{d}\tau, \end{aligned} \tag{7}$$

where

$$K_q(t) = \alpha \int_0^\infty \theta \phi_q(\theta) \mathscr{G}(t^q \theta) \mathrm{d}\theta, \tag{8}$$

in which ϕ_q is a probability density function defined in $]0,\infty[$.

Moreover A^α and K_q have the following properties.

Proposition 3 (see [27]). *For any $t > 0$, we have*

(i) $\exists M_\alpha > 0$ such that

$$\left\| A^\alpha K_q(t) \right\|_{\mathscr{L}(X,X)} \leq \frac{qM_\alpha}{t^{\alpha q}} \times \frac{\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}. \tag{9}$$

(i) $\exists M > 0$ such that

$$\left\| K_q(t) \right\|_{\mathscr{L}(X^\alpha,X^\alpha)} \leq \frac{M}{\Gamma(q)}. \tag{10}$$

Corollary 1. *Let us consider*

$$H_q(t) = t^{q-1} K_q(t). \tag{11}$$

Then, we have

$$H_q \in L^1(0,T;\mathscr{L}(X,X^\alpha)). \tag{12}$$

Proof. We have $0 < \alpha, q < 1$; then, $q(1-\alpha) > 0$. Therefore, $t^{-\alpha q + q - 1} \in L^1(0,T)$, and by the previous proposition, we have the result.

For the rest of this paper, we denote

$$\beta := \left\| H_q \right\|_{L^1(0,T;\mathscr{L}(X,X^\alpha))} \|\mathscr{B}\|_{\mathscr{L}(X,\mathbb{R}^p)}. \tag{13}$$

3. Regional Controllability

In this section, we formulate and prove conditions for the regional controllability of semilinear Riemann–Liouville fractional control systems. To do this, let ω be a subregion of Ω, and we define the restriction operator in ω by

$$\begin{aligned} \chi_\omega \colon L^2(\Omega) &\longrightarrow L^2(\omega), \\ y &\longrightarrow y_{|_\omega}, \end{aligned} \tag{14}$$

and we denote by χ^*_ω its adjoint.

We have the following definition.

Definition 3. The system (6) is said to be exactly (respectively, approximately) ω-controllable if for all $y_d \in L^2(\omega)$ (respectively, for all $\varepsilon > 0$ and for all $y_d \in L^2(\omega)$), there exists a control $u \in U$ such that $\chi_\omega y_u(T) = y_d$ (respectively, $\|\chi_\omega y_u(T) - y_d\|_{L^2(\omega)} \leq \varepsilon$).

For the rest of this paper, we can write the mild solution as follows:

$$y_u(.,t) = H_q(t) y_0 + \left(H_q * Ny_u\right)(.,t) + \left(H_q \star Bu\right)(.,t), \tag{15}$$

and we define the restriction of the controllability operator in ω by

$$\begin{aligned} H^q_{T\omega}\colon U &\longrightarrow L^2(\omega), \\ u &\longrightarrow \chi_\omega\big(H_q * Bu\big)(.,T). \end{aligned} \tag{16}$$

Consider now the following associate linear system of equation (6):

$$\begin{cases} {}^{RL}D^q_{0^+} y_u(x,t) = A y_u(x,t) + \mathscr{B}u(t), & \text{in } Q, \\ y_u(\xi,t) = 0, & \text{on } \Sigma, \\ \lim_{t\longrightarrow 0^+} I^{1-q}_{0^+} y_u(x,t) = y_0, & \text{in } \Omega, \end{cases} \tag{17}$$

which we assume to be approximately ω–controllable.

Next, we will study the regional controllability of the system (6) in $ImH^q_{T\omega}$ endowed with the norm

$$\|y_d\|_{\mathrm{Im}H^q_{T\omega}} = \left\|H^{q^\dagger}_{T\omega} y_d\right\|_U, \tag{18}$$

where

$$H^{q^\dagger}_{T\omega} \coloneqq H^{q^*}_{T\omega}\big(H^q_{T\omega}H^{q^*}_{T\omega}\big)^{-1}\,(\text{Pseudo}-\text{inverse operator of } H^q_{T\omega}). \tag{19}$$

Now, we shall present the main result; we first define the operator

$$\psi(y_d,u) = H^{q^\dagger}_{T\omega}\big(y_d - \chi_\omega H_q(T)y_0 - \chi_\omega\big(H_q * N y_u\big)(.,T)\big), \tag{20}$$

and we make the following assumptions:

(i) (H_1) For arbitrary $x, y \in L^2(0,T;\mathrm{X}^\alpha)$,

$$\begin{cases} N(0) = 0, \\ \|Nx - Ny\|_{L^2(0,T;X)} \le L_N(\|x\|,\|y\|)\|x-y\|_{L^2(0,T;X^\alpha)}, \end{cases} \tag{21}$$

with $L_N\colon \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying $\lim_{(a,\nu)\longrightarrow(0,0)} L_N(a,\nu) = 0$.

(ii) (H_2) $\|\chi_\omega H_q(.)\|_{\mathscr{L}(X,\mathrm{Im}H^q_{T\omega})} \coloneqq g_q \in L^2(0,T)$.

We obtain the following theorem.

Theorem 1. *If the hypotheses (H_1) and (H_2) are satisfied, then the following assertions hold:*

(1) There exist $\rho > 0$, $\mu = \mu(\rho) > 0$, and $m = m(\rho) > 0$ such that, under the assumption

$$\rho > (1+\beta)\frac{MT}{\Gamma(q)}\|y_0\|_{X^\alpha}, \tag{22}$$

for any state y_d in $B(0,\mu) \subset ImH^q_{T\omega}$, the operator $\Psi(y_d,.)$ has a unique fixed point u^* in $B(0,m)$ that steers system (6) to y_d in ω.

(2) The mapping

$$\begin{aligned} F\colon B(0,\mu) &\longrightarrow U, \\ y_d &\longrightarrow u^* \end{aligned} \tag{23}$$

is a Lipschitz mapping.

Proof

(1) Let us consider

$$\begin{aligned} \mathscr{A}_1 &= \beta\left\|g_q\right\|_{L^2(0,T)} \sup_{a,\nu\le\rho} L_N(a,\nu), \\ \mathscr{A}_2 &= \left\|H_q\right\|_{L^1(0,T;\mathscr{L}(X,X^\alpha))} \sup_{a,\nu\le\rho} L_N(a,\nu) + \frac{MT}{\rho\Gamma(q)}\|y_0\|_{X^\alpha}, \\ \mathscr{A}_3 &= \left\|H_q\right\|_{L^1(0,T;\mathscr{L}(X,X^\alpha))} \sup_{a,\nu\le\rho} L_N(a,\nu). \end{aligned} \tag{24}$$

Using the limit of $L_N(.,.)$ near $(0,0)$, we can see that for $\rho > 0$, there exists $l > 0$ such that

$$L_N(a,\nu) \le l < \frac{1-(1+\beta)MT/\rho\Gamma(q)\|y_0\|_{X^\alpha}}{\left\|H_q\right\|_{L^1(0,T;\mathscr{L}(X,X^\alpha))} + \beta\left\|g_q\right\|_{L^2(0,T)}} \quad \forall a,\nu \le \rho, \tag{25}$$

which gives $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3 < 1$.

Let us consider

$$m = \frac{\rho}{\beta}\left(1 - \left\|H_q\right\|_{L^1(0,T;\mathscr{L}(X,X^\alpha))} \sup_{a\le\rho} L_N(a,0) - \frac{MT}{\rho\Gamma(q)}\|y_0\|_{X^\alpha}\right). \tag{26}$$

We have $m > 0$ and the mapping $f\colon B(0,m) \longrightarrow B(0,\rho)$ such that

$f(u) = y_u$ is a Lipschitz mapping with constant $\beta/1-\mathscr{A}_3$. In fact,

$$\left\|H_q\right\|_{L^1(0,T;\mathscr{L}(X,X^\alpha))} \sup_{a\le\rho} L_N(a,0) + \frac{MT}{\rho\Gamma(q)}\|y_0\|_{X^\alpha} \le A_2 < 1, \tag{27}$$

and hence $m > 0$.

To show the Lipschitz condition of the function f, we use equation (15), and Corollary 1, we have for all $u, \nu \in \mathrm{B}(0,m)$

$$\begin{aligned} &\|y_u - y_\nu\|_{L^2(0,T;X^\alpha)} \\ &= \left\|\big(H_q * N(y_u - y_\nu)\big)(.) + \big(H_q \star B(u-\nu)\big)(.)\right\|_{L^2(0,T;X^\alpha)} \\ &\le \left\|H_q\right\|_{L^1(0,T;\mathscr{L}(X,X^\alpha))}\left\|N(y_u - y_\nu)\right\|_{L^2(0,T;X)} \\ &\quad + \left\|H_q\right\|_{L^1(0,T;\mathscr{L}(X,X^\alpha))}\|\mathscr{B}\|_{\mathscr{L}(X,\mathbb{R}^p)}\|u-\nu\|_U. \end{aligned} \tag{28}$$

Therefore, by hypothesis (H_1), we obtain

$$\|y_u - y_\nu\|_{L^2(0,T;X^\alpha)} \le \mathscr{A}_3\left\|y_u - y_\nu\right\|_{L^2(0,T;X^\alpha)} + \beta\|u-\nu\|_U, \tag{29}$$

and hence f is a Lipschitz mapping with constant $\beta/1-\mathscr{A}_3$.

Now, let $\mu = \rho/\beta(1 - ((1+\beta)MT/\rho\Gamma(q)\|y_0\|_{X^\alpha} + A_1 + A_3)) > 0$ and consider $y_d \in B(0,\mu)$; we show that $\Psi(y_d, .)$ has a unique fixed point in $B(0,m)$.

Let us consider $u, v \in B(0,m)$; we have

$$\begin{aligned}\left\|\Psi(y_d,u) - \Psi(y_d,v)\right\|_U &= \left\|\int_0^T \chi_\omega H_q(T-s)(Ny_u(s) - Ny_v(s))ds\right\|_{\mathrm{Im}\ H^q_{T\omega}} \\ &\le \left\|g_q\right\|_{L^2(0,T)}\left\|Ny_u - Ny_v\right\|_{L^2(0,T;X)} \le \left\|g_q\right\|_{L^2(0,T)} \sup_{a\le\rho} L_N(a,0)\left\|y_u - y_v\right\|_{L^2(0,T;X^\alpha)} \\ &\le \frac{A_1}{\beta}\left\|y_u - y_v\right\|_{L^2(0,T;X^\alpha)}.\end{aligned} \tag{30}$$

Since f is Lipschitz, then

$$\left\|\Psi(y_d,u) - \Psi(y_d,v)\right\|_U \le \frac{\mathscr{A}_1}{1-\mathscr{A}_3}\|u-v\|_U. \tag{31}$$

For $\mathscr{A}_4 := \mathscr{A}_1/1-\mathscr{A}_3$, using inequality (15), we have $\mathscr{A}_4 < 1$, and thus $\Psi(y_d, .)$ is a strict contraction mapping.

Let us show that

$$\Psi(B(0,\mu), B(0,m)) \subset B(0,m). \tag{32}$$

For that, let $u \in B(0,m)$; then, $y_u \in B(0,\rho)$ and

$$\begin{aligned}\left\|\Psi(y_d,u)\right\|_U &= \left\|y_d - \chi_\omega H_q(T)y_0 - \int_0^T \chi_\omega H_q(T-s)Ny_u(s)ds\right\|_{\mathrm{Im}\ H^q_{T\omega}} \\ &\le \left\|y_d\right\|_{\mathrm{Im}\ H^q_{T\omega}} + \frac{MT}{\Gamma(q)}\left\|y_0\right\| + \left\|g_q\right\|_{L^2(0,T)}\left\|Ny_u\right\|_{L^2(0,T;X)} \\ &\le \left\|y_d\right\|_{\mathrm{Im}\ H^q_{T\omega}} + \frac{MT}{\Gamma(q)}\left\|y_0\right\| + \left\|g_q\right\|_{L^2(0,T)}\rho \sup_{a\le\rho} L_N(a,0).\end{aligned} \tag{33}$$

Therefore, for $y_d \in B(0,\mu)$, we have

$$\left\|\Psi(y_d,u)\right\|_U \le \frac{\rho}{\beta}\left(1 - \left\|H_q\right\|_{L^1(0,T;\mathscr{L}(X,X^\alpha))} \sup_{a\le\rho} L_N(a,0) - \frac{MT}{\rho\Gamma(q)}\left\|y_0\right\|_{X^\alpha}\right) = m. \tag{34}$$

Then, $\Psi(B(0,\mu), B(0,m)) \subset B(0,m)$; finally, we deduce from Banach fixed-point theorem that $\Psi(y_d, .)$ admits a unique fixed point $u^* \in B(0,m)$. A direct calculation shows us that u^* obtained is the solution of the regional controllability problem of system (3).

(2) Let us consider z_d and y_d in $B(0,\mu)$; we have

$$\begin{aligned}F(z_d) - F(y_d) = {} & \Psi(z_d, F(z_d)) - \Psi(z_d, F(y_d)) \\ & + \Psi(z_d, F(y_d)) - \Psi(y_d, F(y_d)).\end{aligned} \tag{35}$$

On the other hand,

$$\begin{aligned}&\left\|\Psi(z_d, F(z_d)) - \Psi(z_d, F(y_d))\right\|_U \le \mathscr{A}_4\left\|F(z_d) - F(y_d)\right\|_U, \\ &\left\|\Psi(z_d, F(y_d)) - \Psi(y_d, F(y_d))\right\|_U = \left\|z_d - y_d\right\|_{\mathrm{Im}\ H^q_{T\omega}}.\end{aligned} \tag{36}$$

Then,

$$\left\|F(z_d) - F(y_d)\right\|_U \le \frac{1}{1-A_4}\left\|z_d - y_d\right\|_{\mathrm{Im}\ H^q_{T\omega}}. \tag{37}$$

Therefore, F satisfies the Lipschitz condition. □

Remark 1. In the case where $\mathscr{B}$ is unbounded, we suppose that $\mathscr{B}$ is an admissible control operator for H_q (see [28]), and consequently we can demonstrate the same result with a suitable β.

We give the following proposition.

Proposition 4. *The sequence*

$$\begin{cases} u_0 = 0, \\ u_{n+1} = H^{q\dagger}_{T\omega}\left(y_d - \chi_\omega H_q(T)y_0 - \chi_\omega\left(H_q * Ny_{u_n}\right)(T)\right), \end{cases} \tag{38}$$

converges in $B(0,m) \subset U$ *to* u^*.

Proof. Let us consider $n, k \in \mathbb{N}^*$; we have

$$\|u_{n+k} - u_n\|_U \leq \sum_{l=n}^{n+k-1} \|u_{l+1} - u_l\|_U. \tag{39}$$

From inequality (31), we obtain

$$\begin{aligned} \|u_{l+1} - u_l\|_U &= \|\Psi(y_d, u_l) - \Psi(y_d, u_{l-1})\|_U \\ &\leq \mathscr{A}_4\|u_l - u_{l-1}\|_U \leq \mathscr{A}_4^l\|u_1\|_U. \end{aligned} \tag{40}$$

Since $\mathscr{A}_4^n \underset{n\longrightarrow+\infty}{\longrightarrow} 0$, $\lim_{n\longrightarrow+\infty}\|u_{n+k} - u_n\|_U = 0$.

Then, $(u_n)_n$ is a Cauchy sequence on $B(0,m)$, and we conclude that $(u_n)_n$ converges to u^* in $B(0,m)$.

Passing to the limit in (8), we have $u^* = \Psi(y_d, u^*)$; therefore, $u = u^*$ because $\Psi(y_d, .)$ has a unique fixed point in $B(0,m)$.

Accordingly, we implement the algorithm as follows.

(i) Step 1

q, y_0, the actuator, the subregion ω, y_d and ε small enough.
Choose $r_1 = y_d$.
Calculate $u_1 = H^{q\dagger}_{T\omega}r_1$ and obtain $y_{u_1}(.,T)$.

(ii) Step 2

Repeat
$r_n = r_{n-1} + (y_d - \chi_\omega y_{u_{n-1}}(.,T)), n \geq 2.$
Calculate $u_n = H^{q\dagger}_{T\omega}r_n$ and obtain $y_{u_n}(.,T)$.
Until $\|\chi_\omega y_{u_n}(.,T) - y_d\|_{Im H^q_{T\omega}} < \varepsilon$.

4. Numerical Simulations

In this section, we present two numerical simulations illustrating our theoretical result where the first one is given by using zonal actuator and the second example is given by using a pointwise actuator.

4.1. Zonal Actuator. Let us consider the following one-dimensional fractional system with $q = 0.3$.

$$\begin{cases} {}^{RL}D^{0.3}_{0^+}y_u(x,t) = \dfrac{\partial^2 y_u(x,t)}{\partial x^2} + \displaystyle\sum_{j=1}^{\infty}\left(\langle y_u, \varphi_j\rangle\right)^2\varphi_j(x) + \chi_D u(t) \text{ in }]0,1[\times]0,2[, \\ y_u(\xi,t) = 0 \text{ on } \{0,1\}\times]0,2[, \\ \lim\limits_{t\longrightarrow0^+} I^{0.7}_{0^+}y_u(x,t) = \sin(\pi x) \text{ in }]0,1[, \end{cases} \tag{41}$$

where $\varphi_j = \sqrt{2}\sin(j\pi x)$ and $D = [0.4, 0.6]$.

The subregion where we search the controllability of the considered system is $\omega = [0.2, 0.5]$ and let $y_d(x) = 12x^2(x-1)(x-0.4)$. By using the previous algorithm, we have the following figures.

In Figure 1, we remark that with the given zonal sensor, we obtain successful results which validate the used method and the previous algorithm; indeed, the desired and estimated final states are very close in the subregion $\omega = [0.2, 0.5]$ with the error $\epsilon = 3\times10^{-6}$ which is very small. Figure 2 presents the evolution of control function which has a transfer cost equal to $\|u^*\|^2 = 0.21$.

4.2. Pointwise Actuator. We consider the following system:

$$\begin{cases} {}^{RL}D^{0.5}_{0^+}y_u(x,t) = \dfrac{\partial^2 y_u(x,t)}{\partial x^2} + \displaystyle\sum_{j=1}^{\infty}\left|\langle y_u, \varphi_j\rangle\right|\langle y_u, \varphi_j\rangle\varphi_j(x) + \delta_b u(t) \text{ in }]0,1[\times]0,2[, \\ y_u(\xi,t) = 0 \text{ on } \{0,1\}\times]0,2[, \\ \lim\limits_{t\longrightarrow0^+} I^{0.5}_{0^+}y_u(x,t) = \sin(\pi x) \text{ in }]0,1[. \end{cases} \tag{42}$$

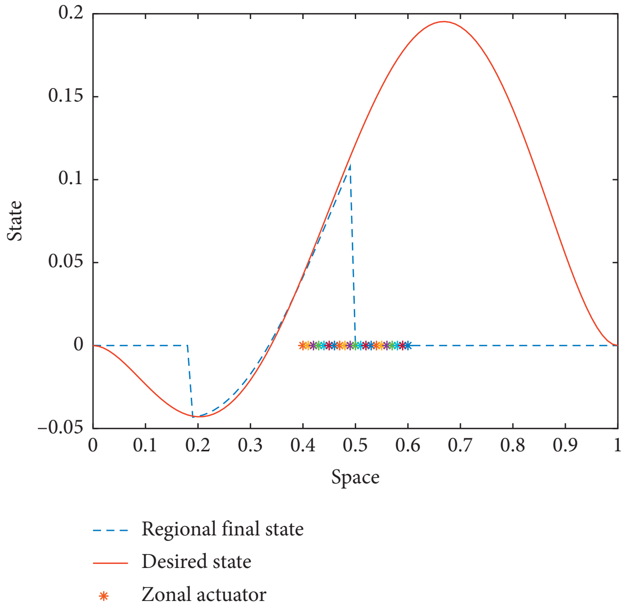

Figure 1: Desired state and estimated final state in ω.

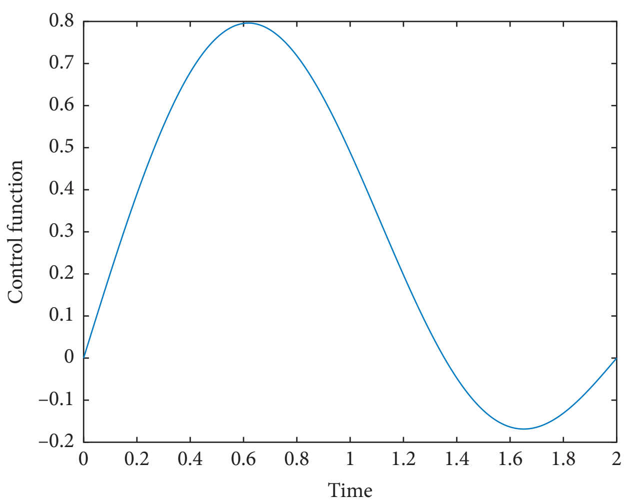

Figure 2: Control input function.

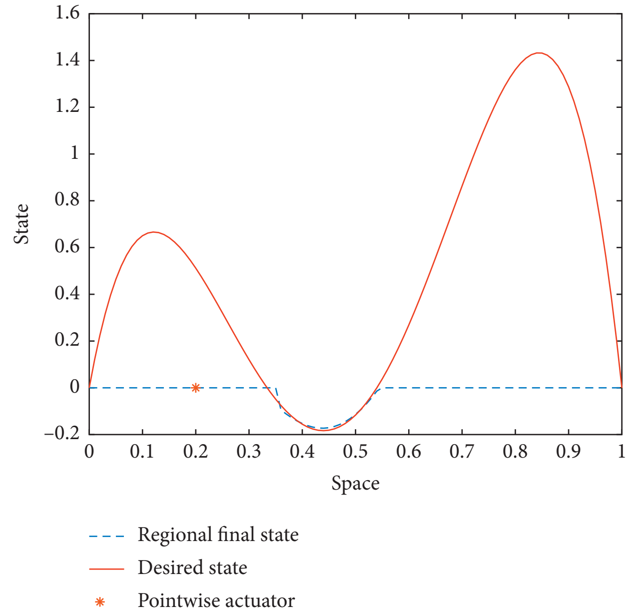

Figure 3: Estimated state and desired state in ω.

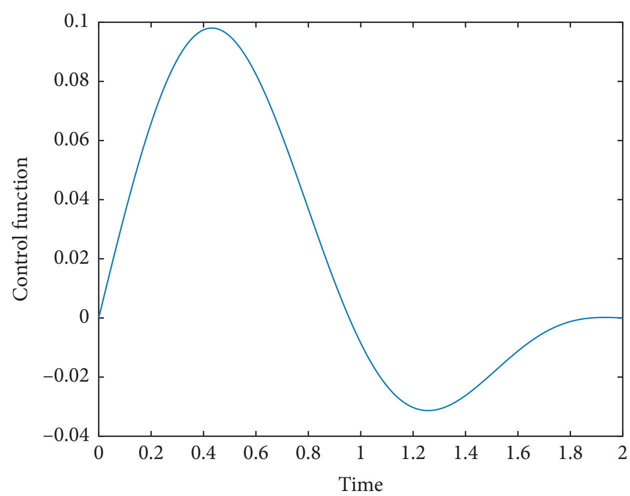

Figure 4: Control function.

By applying the above algorithm for $b = 0.2$, $\omega = [0.35, 0.55]$, and

$$y_d(x) = -52x(x-1)(x-0.33)(x-0.54), \tag{43}$$

we have the following result.

In Figure 3, like the previous part, the given algorithm leads to good results; we remark that the desired state and the reached one are close in $w = [0.35, 0.55]$ using the given pointwise actuator. In this case, the reconstruction error is very small, and it is of order $\varepsilon = 10^{-5}$. Figure 4 shows the evolution of control function depending on the time t with the transfer cost $\|u^*\|^2 = 4 \times 10^{-3}$.

5. Conclusion

In this paper, we have established the regional controllability for a class of Riemann–Liouville fractional semilinear control systems. The idea of applying control theory for this kind of systems is very interesting and constitutes a new issue in the applications. The presented method in this paper covers a large class of this kind of systems. We have also given an algorithm which has been implemented numerically and has very satisfactory results. In addition, the problem of regional controllability remains open for other types of fractional systems that will be the subject of future research.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Existence of Solutions for Fractional Evolution Equations with Infinite Delay and Almost Sectorial Operator

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This paper discusses a class of semilinear fractional evolution equations with infinite delay and almost sectorial operator on infinite interval in Banach space. By using the properties of analytic semigroups and Schauder's fixed-point theorem, this paper obtains the existence of mild solutions of the fractional evolution equation. Moreover, this paper also discusses the existence of mild solution when the analytic semigroup lacks compactness by Kuratowski measures of noncompactness and Darbo–Sadovskii fixed-point theorem.

1. Introduction

Fractional differential models play a very important role in describing many complex phenomena such as chaotic system [1], fluid flow [2, 3], anomalous diffusion [4–7], and so on. Compared with the classical partial differential models such as [8–19], the biggest advantage of models with fractional derivatives is their global property and history memory. Delay is short for time delay, which exists widely in the objective world. In the differential equation model with delay, the function depends not only on the current state but also on the past time state, so it is more suitable to describe the process with time memory. This property of delay is very similar to that of fractional derivatives. So many researchers introduced fractional derivatives into differential equations with delay [20–24]. Evolution equation, which is a general appellation for some partial differential equations with time variable, is mainly used to describe the time-dependent state and process. Common evolution equations include the wave equation, the heat equation, Schrodinger equation, KdV equation, Navier–Stokes equation, and so on. By using the operator semigroup theory, some partial differential evolution equations can be represented to some abstract ordinary differential equations (ODEs) in some special functional spaces. At present, the research on integer-order evolution equations has been relatively perfect [25, 26], but the research on fractional-order evolution equations is still in the preliminary stage. The existence of solutions for fractional evolution equations is also the basis of the following study. The mild solution of integer-order evolution equations is defined by the constant variation method, which cannot be directly extended to fractional-order evolution equations.

Li [20] studied the following fractional evolution equations with almost sectorial operator on finite interval:

$$\left\{ {}^{c}D_{t}^{q}\, x(t) = Ax(t) + f(t, x, x_t), \quad 0 < q < 1, t \in (0, T], x_0 = \phi(t) \in B, \quad t \in (-\infty, 0], \right. \tag{1}$$

where ${}^{c}D_t^q$ is the Caputo fractional derivative operator, the evolution operator A is an almost sectorial operator, and B is a phase space. x_t is the element of B defined by $x_t(\theta) = x(t+\theta)$, $\theta \in (-\infty, 0]$. Here, $x_t(\cdot)$ represents the history of state up to the present time.

Baliki et al. [22] discussed a second-order evolution equation with infinite delay and obtained the existence and attractivity of mild solutions by Schauder's fixed point as follows:

$$\begin{cases} x''(t) - A(t)x(t) = f(t, x_t), & 0 < q < 1, t \in (0, \infty), \\ x_0 = \phi(t) \in B, \ x'(0) = \tilde{x}, \end{cases} \tag{2}$$

where $\{A(t)\}_{0 \le t < \infty}$ is a family of linear closed operators, $x_t(\theta) = x(t+\theta)$, $\theta \in (-\infty, 0]$, and B is a phase space. The existence of mild solutions for fractional evolution equations and evolution equations with infinite delay has been discussed in several papers (see [20, 21]). However, we find that most of the previous papers discuss the fractional evolution equations in the conventional spaces of continuous function on finite or infinite interval and in Banach space on finite interval. To our knowledge, no paper is devoted to the existence of mild solutions with infinite delay and almost sectorial operator on infinite interval on Banach space.

In this paper, we consider the following fractional evolution problem with infinite time delay:

$$\{ {}^{c}D_{0+}^{q} x(t) + Ax(t) = f(t, x_t), \quad t \in (0, +\infty), \ 0 < q < 1, x(t) = \phi(t) \in B, \quad t \in (-\infty, 0], \tag{3}$$

where ${}^{c}D_{0+}^{q}$ is the Caputo fractional derivative operator, the evolution operator A is an almost sectorial operator, f is a given function which will be introduced later, and B is a phase space. For any continuous function x and any $t \ge 0$, x_t is the same as in equation (1) which represents the history of state up to the present time.

The rest of this paper is organized as follows. In Section 2, we recall some definitions, propositions, notations, and lemmas. In Section 3, the main results of this paper are obtained. We consider two cases: the semigroup $Q(t)$ generated by operator A with compactness and without compactness. For the case that $Q(t)$ is compact, we construct a special Banach space B' and obtain the existence of global mild solution by using Schauder's fixed-point theorem. For the case that $Q(t)$ is not compact, we expand the result of Theorem 1.2.4 in Guo et al. [27] from any compact interval to infinite interval (see Lemma 10) and obtain the existence of global mild solution by applying Kuratowski measures of noncompactness theory and Darbo–Sadovskii fixed-point theorem.

2. Preliminaries

In this section, we introduce some notations, definitions, lemmas, and preliminary facts that will be used in the rest of this paper. Let $(E, |\cdot|)$ be a Banach space. Denote $B(E)$ as the space of all bounded linear operators from E to itself with norm $\|\cdot\|_{B(E)}$.

Definition 1 (see [28, 29]). Let $-1 < \gamma < 0$ and $0 < \omega < (\pi/2)$. Denote by $\Theta_\omega^\gamma(E)$ all the linear closed operators A: $D(A) \subset E \longrightarrow E$ which satisfy

(1) $\sigma(A) \subset S_\omega = \{z \in C \setminus \{0\}, \arg|z| \le \omega\} \cup \{0\}$.

(2) For every $\omega < \mu < \pi$, there exists a constant C_μ such that

$$|R(z, A)| \le C_\mu |z|^\gamma \text{ for all } z \in C \backslash S_\mu. \tag{4}$$

A linear operator A will be called an almost sectorial operator on E if $A \in \Theta_\omega^\gamma(E)$.

Define the power of A as

$$A^\beta = \frac{1}{2\pi i} \int_{\Gamma_\theta} z^\beta R(z, A) \mathrm{d}z, \quad \beta > 1 + \gamma, \tag{5}$$

where $\Gamma_\theta = \{R_+ e^{i\theta} \cup R_+ e^{-i\theta}\}$ is an appropriate path oriented counterclockwise and $\omega < \theta < \mu$. Then, the linear power space $X_\beta := D(A^\beta)$ can be defined and X_β is a Banach space with the graph norm $\|x\|_\beta = |A^\beta x|$, $x \in D(A^\beta)$.

Next, let us introduce the semigroup associated with A. If A is an almost sectorial operator, then A generates an analytic semigroup $Q(t)$ of growth order $1 + \gamma$ as follows:

$$Q(t) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{-tz} R(z, A) \mathrm{d}z, \quad t \in S^0_{(\pi/2)-\omega}, \tag{6}$$

where $\Gamma_\theta = \{R_+ e^{i\theta} \cup R_+ e^{-i\theta}\}$ is oriented counterclockwise and $\omega < \theta < \mu < (\pi/2) - \arg|t|$. $S^0_{(\pi/2)-\omega}$ is the open sector $\{z \in C \setminus \{0\}, |\arg z| < (\pi/2) - \omega\}$. Furthermore, $Q(t)$ satisfies the following properties.

Proposition 1 (see [28, 29]). Let $A \in \Theta_\omega^\gamma(E)$ with $-1 < \gamma < 0$ and $0 < \omega < (\pi/2)$. Then, the following properties remain true:

(1) $Q(t)$ is analytic in $S^0_{(\pi/2)-\omega}$ and $(d^n/dt^n)Q(t) = (-A)^n Q(t)$, $t \in S^0_{(\pi/2)-\omega}$.

(2) The functional equation holds: $Q(s+t) = Q(s)Q(t)$ for all $s, t \in S^0_{(\pi/2)-\omega}$.

(3) *There is a constant* $C_0 = C_0(\gamma) > 0$ *such that* $|Q(t)| \le C_0 t^{-\gamma-1}$, $t > 0$.

(4) *The range* $R(Q(t))$ *of* $Q(t)$ $(t \in S^0_{(\pi/2)-\omega})$ *is contained in* $D(A^\infty)$. *Particularly,* $R(Q(t)) \subset D(A^\beta)$ *for all* $\beta \in C$ *with* $\operatorname{Re}\beta > 0$*:*

$$A^\beta Q(t)x = \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\beta e^{-tz} R(z, A)x\mathrm{d}z, \quad t \in S^0_{(\pi/2)-\omega}, x \in E, \tag{7}$$

and there exists a constant $C' = C'(\gamma, \beta) > 0$ *such that for all* $t > 0$*,*

$$\left|A^\beta Q(t)\right| \le C' t^{-\gamma+\operatorname{Re}\beta-1}. \tag{8}$$

(5) *If* $\beta > 1 + \gamma$, *then* $D(A^\beta) \subset \sum_Q = \left\{x \in E, \lim_{t\longrightarrow 0} Q(t)x = x\right\}$.

By Theorem 3.13 in Periago [28], if A is an almost sectorial operator, then for every $\lambda \in C$ with $\operatorname{Re}\lambda > 0$,

$$R(\lambda, -A) = \int_0^{+\infty} e^{-\lambda t} Q(t)\mathrm{d}t. \tag{9}$$

Let X *be the following set:*

$$X := \left\{x\colon R \longrightarrow X_\beta,\ x_{[0,+\infty)} \in C\left([0, +\infty), X_\beta\right),\ \lim_{t\longrightarrow+\infty} e^{-kt}x(t) = 0,\ x_0 \in B\right\}, \tag{10}$$

where $x_{[0,+\infty)}$ *is the restriction of* x *on* $[0, +\infty)$ *and* k *is a constant.*

In this paper, we use an axiomatic definition of the phase space B. $(B, \|\cdot\|_B)$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into E and satisfies the following axioms which are introduced by Hale and Kato in [30].

(A) If $x\colon (-\infty, b] \longrightarrow E$, $b > 0$ is continuous on $[0, b]$ and $x_0 \in B$, then for any $t \in [0, b]$, the following conditions hold:

(i) $x_t \in B$.

(ii) There exists a positive constant H such that $|x| \le H\|x_t\|_B$.

(iii) There exist positive continuous functions $K(\cdot), M(\cdot)$ independent of $x(\cdot)$ such that

$$\|x_t\|_B \le K(t) \sup_{0\le s\le t} \|x(s)\|_\beta + M(t)\|x_0\|_B. \tag{11}$$

(B) For the functions in (A), x_t is a B-value continuous function on $[0, b]$.

(C) The space B is complete.

Definition 2 (see [31, 32]). Let $f \in L^1((0, +\infty), E)$ and $q > 0$; then,

$$I^q_{0+} f(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)\mathrm{d}s \tag{12}$$

is called the Riemann–Liouville fractional integral of order q.

Definition 3 (see [31, 32]). The Caputo fractional derivative of order $q > 0$ of the function $f\colon (0, +\infty) \longrightarrow E$ is given by

$${}^cD^q_{0+} f(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1} f^{(n)}(s)\mathrm{d}s, \tag{13}$$

where n is the smallest integer greater than or equal to q, provided that the right side is well defined on $(0, +\infty)$.

Lemma 1 (see [31, 32]). For all $f, g \in L^q((0, +\infty), E)$, $1 \le q < \infty$,

$$I^q_{0+}(f * g) = (I^q_{0+} f) * g. \tag{14}$$

Next, we will introduce the mild solution of equation (3). Shu et al. [33] define the mild solution of equation (3) as

$$x(t) = S_q(t)\phi(0) + \int_0^t (t-s)^{q-1} P_q(t-s) f(s, x_s)\mathrm{d}s, \tag{15}$$

where $S_q(t)$ and $P_q(t)$ have the following expressions and Γ is an appropriate path in $\rho(-A)$.

$$\begin{aligned} S_q(t) &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{q-1} R(\lambda^q, -A)\mathrm{d}\lambda, \\ P_q(t) &= \frac{t^{1-q}}{2\pi i}\int_\Gamma e^{\lambda t} R(\lambda^q, -A)\mathrm{d}\lambda. \end{aligned} \tag{16}$$

Using the properties of the Mittag-Leffler function (for more details, we refer the readers to [32]),

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i}\int_\Gamma \frac{\lambda^{\alpha-\beta} e^\alpha}{\lambda^\alpha - z}\mathrm{d}\lambda, \tag{17}$$

where Γ is the same path as in (4) (see [32]), the above operators $S_q(t)$ and $P_q(t)$ can be represented as the generalized Mittag-Leffler-type functions:

$$\begin{aligned} S_q(t) &= E_{q,1}(-t^q A) = E_q(-t^q A), \\ P_q(t) &= E_{q,q}(-t^q A). \end{aligned} \tag{18}$$

Moreover, Wang et al. [29] and Zhou et al. [34–36] introduced the function of Wright-type $M_q(z)$:

$$M_q(z) = \sum_{n=1}^{+\infty} \frac{(-z)^{n-1}}{(n-1)!\Gamma(1-nq)}, \quad 0<q<1, z \in C, \tag{19}$$

and obtained another expression of $S_q(t)$, $P_q(t)$:

$$\begin{aligned} S_q(t) &= \int_0^{+\infty} M_q(s) Q(t^q s)\mathrm{d}s, \\ P_q(t) &= \int_0^{+\infty} qs M_q(s) Q(t^q s)\mathrm{d}s. \end{aligned} \tag{20}$$

In fact, these three expressions ((16)–(20)) are equivalent in the case that $t>0$ and $A \in \Theta_\omega^\gamma(E)$. Therefore, in this paper, we use the same expression of $S_q(t)$, $P_q(t)$ as Wang et al. in [29] and Zhou et al. in [34–36]. Then, the global mild solution of problem (3) is given in the following definition.

Definition 4. A function x: $R \longrightarrow X$ is called a global mild solution to the problem (3), if $x(t) \in C(R,X)$ and

$$x(t) = \begin{cases} S_q(t)\phi(0) + \int_0^t (t-s)^{q-1} P_q(t-s) f(s,x_s)\mathrm{d}s, & t \in (0,+\infty), \\ \phi(t), & t \in (-\infty, 0]. \end{cases} \tag{21}$$

Lemma 2 (see [29]). For any fixed $t>0$, $S_q(t)$ and $P_q(t)$ are linear and bounded operators and there exist constants C_s and C_p such that for all $x \in E$,

$$\begin{aligned} \left|S_q(t)x\right| &\le C_s t^{-q(1+\gamma)}|x|, \\ \left|P_q(t)x\right| &\le C_p t^{-q(1+\gamma)}|x|. \end{aligned} \tag{22}$$

Lemma 3 (see [29]). For $t>0$, operators $\{S_q(t)\}$ and $\{P_q(t)\}$ are continuous in the uniform operator topology. Moreover, for every $r>0$, the continuity is uniform on $[r,+\infty)$.

Lemma 4 (see [29]). Let $0<\beta<1-\gamma$; then,

(1) *For $t>0$, the range $R(P_q(t))$ of $P_q(t)$ is contained in $D(A^\beta)$.*

(2) *For all $x \in D(A)$ and $t>0$, $|AS_q(t)x| \le Ct^{-q(1+\gamma)}|Ax|$, where C is a constant depending on γ, q.*

Remark 1. Moreover, for all $x \in D(A^\beta)(0<\beta<1-\gamma)$ and $t>0$,

$$\begin{aligned} \left|A^\beta S_q(t)x\right| &\le C_s t^{-q(1+\gamma)}\left|A^\beta x\right|, \\ \left|A^\beta P_q(t)x\right| &\le C_p t^{-q(1+\gamma)}\left|A^\beta x\right|, \end{aligned} \tag{23}$$

that is,

$$\begin{aligned} \left\|S_q(t)x\right\|_\beta &\le C_s t^{-q(1+\gamma)}\|x\|_\beta, \\ \left\|P_q(t)x\right\|_\beta &\le C_p t^{-q(1+\gamma)}\|x\|_\beta. \end{aligned} \tag{24}$$

Lemma 5 (see [29]). Let $\beta>1+\gamma$; then, $\lim_{t\longrightarrow 0+} S_q(t)x = x$ for all $x \in D(A^\beta)$.

3. Main Results

In this section, our main purpose is to establish sufficient conditions for the existence of global mild solutions to problem (3) in X. Assume that:

(H) f: $[0,+\infty) \times B \longrightarrow X_\beta$, $(1+\gamma<\beta<1-\gamma)$ is continuous and satisfies

$$\|f(t,x)\|_\beta \le p(t)e^{-kt}\|x\|_B, \tag{25}$$

where $p(t)$ is a nonnegative and continuous function on $[0,+\infty)$ and here exists a big enough $k>0$ such that

(i) For any $t \ge 0$,

$$C_p e^{-kt} \int_0^t (t-s)^{-q\gamma-1} p(s)K(s)\mathrm{d}s \le \frac{1}{2}, \tag{26}$$

(ii) $\lim_{t\longrightarrow+\infty} e^{-kt}\int_0^t (t-s)^{-q\gamma-1}p(s)K(s)\mathrm{d}s = 0$,
$\lim_{t\longrightarrow+\infty} e^{-kt}\int_0^t (t-s)^{-q\gamma-1}p(s)M(s)\mathrm{d}s = 0$.

In order to obtain the existence of global mild solution of problem (3), we transform it into a fixed-point problem. For any $\phi(0) \in X_\beta$, define the operator $\widehat{T}$: $X \longrightarrow X$ as

$$\widehat{T}x(t) = \begin{cases} S_q(t)\phi(0) + \int_0^t (t-s)^{q-1} P_q(t-s) f(s,x_s)\mathrm{d}s, & t \in (0,+\infty), \\ \phi(t), & t \in (-\infty, 0]. \end{cases} \tag{27}$$

Let $z(t)$: $R \longrightarrow X$ be the function

$$z(t) = \begin{cases} S_q(t)\phi(0), & t \in (0, +\infty), \\ \phi(t), & t \in (-\infty, 0], \end{cases} \tag{28}$$

and $x(t) = y(t) + z(t)$, $t \in R$. It is easy to know that $x(t)$ satisfies (21) if and only if

$$y(t) = \begin{cases} \int_0^t (t-s)^{q-1} P_q(t-s) f(s, y_s + z_s) \mathrm{d}s, & t \in (0, +\infty), \\ y_0 = 0, & t \in (-\infty, 0]. \end{cases} \tag{29}$$

Define the set $B' := \{y \in X\colon\ y_0 = 0 \in B\}$ endowed with seminorm $\|\cdot\|_b$:

$$\|y\|_b = \|y_0\|_B + \sup_{t \ge 0}\{e^{-kt}\|y(t)\|_\beta\} = \sup_{t \ge 0}\{e^{-kt}\|y(t)\|_\beta\}. \tag{30}$$

Thus, $(B', \|\cdot\|_b)$ is a Banach space. Define the operator T: $B' \longrightarrow B'$ as

$$Ty(t) = \begin{cases} \int_0^t (t-s)^{q-1} P_q(t-s) f(s, y_s + z_s) \mathrm{d}s, & t \in (0, +\infty), \\ y_0 = 0, & t \in (-\infty, 0]. \end{cases} \tag{31}$$

Consequently, the operator $\widehat{T}$: $X \longrightarrow X$ having a fixed point in X is equivalent to the operator T: $B' \longrightarrow B'$ having a fixed point in B'.

Lemma 6. *Assume that condition (H) is valid; then, there exists a constant $r > 0$ such that*

$$C_p e^{-kt} \int_0^t (t-s)^{-q\gamma-1} e^{-ks} p(s)\left[K(s)M + M(s)\|\phi\|_B\right] \mathrm{d}s \le \frac{r}{2}, \tag{32}$$

where M satisfies $\sup_{t>0}\|S_q(t)\phi(0)\|_\beta \le M$. Consider $B_r := \{y \in B', \|y\|_b \le r\}$; then, for any $\phi(0) \in X_\beta$, the operator T: $B_r \longrightarrow B_r$ is continuous.

Proof. By $\phi(0) \in X_\beta$ and Lemma 5 (1), there exists $0 < \delta_1 < T$, and for any $t \in (0, \delta_1]$, such that $\|S_q(t)\phi(0) - \phi(0)\|_\beta < \varepsilon$. for any $t \ge \delta_1$, $\|S_q(t)\phi(0)\|_\beta \le C_s\|\phi(0)\|_\beta \delta_1^{-q(1+\gamma)}$. Therefore, there exists a constant $M > 0$ such that $\sup_{t>0}\|S_q(t)\phi(0)\|_\beta \le M$.

For any $y(t) \in B_r$, $0 < s < t$, note that

$$\begin{aligned} \|y_s + z_s\|_B &\le \|y_s\|_B + \|z_s\|_B, \\ &\le K(s)e^{ks}\|y\|_b + K(s) \sup_{0<\tau\le s}\left\|S_q(\tau)\phi(0)\right\|_\beta \\ &\quad + M(s)\|\phi\|_B, \\ &\le K(s)e^{ks}\|y\|_b + K(s)M + M(s)\|\phi\|_B, \\ &:= \eta(s). \end{aligned} \tag{33}$$

Then, by condition (H) and Remark 1, we have

$$\begin{aligned} e^{-kt}\|Ty(t)\|_\beta &\le e^{-kt}\int_0^t (t-s)^{q-1}\left\|P_q(t-s) f(s, y_s + z_s)\right\|_\beta \mathrm{d}s, \\ &\le C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1} e^{-ks} p(s)\eta(s) \mathrm{d}s, \\ &\le \left(C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1} e^{-ks} p(s) K(s) \mathrm{d}s\right)\|y\|_b, \\ &\quad + C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1} e^{-ks} p(s)\left[K(s)M + M(s)\|\phi\|_B\right] \mathrm{d}s, \\ &\le \frac{r}{2} + \frac{r}{2} = r, \end{aligned} \tag{34}$$

which implies that $\|Ty\|_b \le r$ and T: $B_r \longrightarrow B_r$.

Next, we will prove the continuity of T. Let $\{y^n(t)\}_{n=1}^{\infty} \in B_r$ and $\|y^n - y\|_b \longrightarrow 0$ as $n \longrightarrow \infty$ for any $t \ge 0$. Then, for any $t > 0$, by the continuity of f,

$$\begin{aligned} e^{-kt}\left\|Ty^n(t) - Ty(t)\right\|_\beta &\le C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1} \\ &\quad \left\|f(s, y_s^n) - f(s, y_s)\right\|_\beta \mathrm{d}s \longrightarrow 0 \ (n \longrightarrow \infty), \end{aligned} \tag{35}$$

which implies that $\|Ty^n(t) - Ty(t)\|_b \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, the continuity of T is proved. □

Lemma 7. *Assume that condition (H) is satisfied; then, for any $\phi(0) \in X_\beta$,*

(1) *$\{e^{-kt}Ty(t),\ y \in B'\}$ is equicontinuous on any compact interval of $[0, +\infty)$.*

(2) *For any given $\varepsilon > 0$, there exists a constant $T > 0$ such that $e^{-kt}\|Ty(t)\|_\beta < \varepsilon$ for any $t \ge T$ and $y \in B'$.*

Proof. (1) Without loss of generality, we take $[0,T)\subset[0,+\infty)$ as the compact interval and $0\le t_1<t_2\le T$. Firstly, for $t_1=0$, $t_1<t_2\le T$ and any $y\in B'$, according to the continuity of $p(s)$ and $\eta(s)$, we have

$$\begin{aligned}&\left\|e^{-kt_1}Ty(t_1)-e^{-kt_2}Ty(t_2)\right\|_\beta\le C_pe^{-kt_2}\\&\int_0^{t_2}(t_2-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s\longrightarrow 0\ (t_2\longrightarrow 0).\end{aligned}\tag{36}$$

Next, for $0<t_1<t_2\le T$, by Lemma 2 and Remark 1, we have

$$\begin{aligned}&\left\|e^{-kt_1}Ty(t_1)-e^{-kt_2}Ty(t_2)\right\|_\beta,\\
&\le e^{-kt_2}\int_{t_1}^{t_2}(t_2-s)^{q-1}\left\|P_q(t_2-s)f(s,y_s+z_s)\right\|_\beta\mathrm{d}s,\\
&\quad+\left(e^{-kt_1}-e^{-kt_2}\right)\int_0^{t_1}(t_2-s)^{q-1}\left\|P_q(t_2-s)f(s,y_s+z_s)\right\|_\beta ds,\\
&\quad+e^{-kt_1}\int_0^{t_1}\left[(t_1-s)^{q-1}-(t_2-s)^{q-1}\right]\left\|P_q(t_2-s)f(s,y_s+z_s)\right\|_\beta\mathrm{d}s,\\
&\quad+e^{-kt_1}\int_0^{t_1}(t_1-s)^{q-1}\left\|\left(P_q(t_2-s)-P_q(t_1-s)\right)f(s,y_s+z_s)\right\|_\beta\mathrm{d}s,\\
&\le C_pe^{-kt_2}\int_{t_1}^{t_2}(t_2-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s,\\
&\quad+C_p\left(e^{-kt_1}-e^{-kt_2}\right)\int_0^{t_1}(t_2-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s,\\
&\quad+C_pe^{-kt_1}\int_0^{t_1}\left[(t_1-s)^{q-1}-(t_2-s)^{-q\gamma-1}\right]e^{-ks}p(s)\eta(s)\mathrm{d}s,\\
&\quad+\sup_{s\in[0,t_1-\delta]}\left\|P_q(t_2-s)-P_q(t_1-s)\right\|_{B(E)}e^{-kt_1}\int_0^{t_1-\delta}(t_1-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s,\\
&\quad+e^{-kt_1}\int_{t_1-\delta}^{t_1}(t_1-s)^{q-1}\left\|\left(P_q(t_2-s)-P_q(t_1-s)\right)f(s,y_s+z_s)\right\|_\beta\mathrm{d}s,\\
&\quad\coloneqq I_{11}(t)+I_{12}(t)+I_{13}(t)+I_{14}(t)+I_{15}(t).\end{aligned}\tag{37}$$

For $I_{11}(t)$, $I_{12}(t)$, and $I_{14}(t)$ by the continuity of $p(s)$, $\eta(s)$, e^{-ks}, and $P_q(s)$, we have $I_{11}(t)$, $I_{12}(t)I_{14}(t)\longrightarrow 0$ as $t_2\longrightarrow t_1$, $\delta\longrightarrow 0$. For $I_{13}(t)$ and $I_{15}(t)$, note that

$$I_{1i}(t)\le 2C_pe^{-kt_1}\int_0^{t_1}(t_1-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s,\quad i=3,5.\tag{38}$$

Then, by using Lebesgue's dominated convergence theorem, we have $I_{13}(t),I_{15}(t)\longrightarrow 0$ as $t_2\longrightarrow t_1$, $\delta\longrightarrow 0$. Therefore, for any $0\le t_1<t_2\le T$ and $y\in B'$, $\|Ty(t_1)-Ty(t_2)\|_b\longrightarrow 0$ as $t_2\longrightarrow t_1$, $\delta\longrightarrow 0$.

(2) By condition (H), for big enough $T>0$,

$$e^{-kt}\int_0^t(t-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s<\frac{1}{C_p}\varepsilon.\tag{39}$$

Then, for any $t\ge T$, $y\in B'$, we have

$$e^{-kt}\|Ty(t)\|_\beta\le C_pe^{-kt}\int_0^t(t-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s<\varepsilon.\tag{40}$$

□

3.1. The Case That $Q(t)$ Is Compact. In this section, we assume that $Q(t)$ is compact for $t>0$, i.e., $Q(t)$ is a compact operator for every $t>0$.

Lemma 8. *Let $Z\subseteq B'$ be a bounded set; then, Z is relatively compact in B' if the following conditions hold:*

(1) *The set $\{y(t),y\in Z\}$ is equicontinuous on any compact interval of $[0,+\infty)$ and for any $t\ge 0$, $\{y(t),y\in Z\}$ is relatively compact in X.*
(2) *For any given $\varepsilon>0$, there exists a constant $T=T(\varepsilon)>0$ such that $e^{-kt}\|y(t)\|_\beta<\varepsilon$ for any $t\ge T$ and $y(t)\in Z$.*

Proof. It is sufficient to prove that Z is totally bounded. We consider the compact interval $[0,T]$ of $[0,+\infty)$. Define

$$Z_{[0,T]} := \{y(t)\colon\ y(t) \in Z, \quad t \in [0,T]\}, \tag{41}$$

with norm $\|y\|_{b_1} := \sup_{0\le t\le T}\{e^{-kt}\|y(t)\|_\beta\}$; then, condition (1) combined with Arzelà–Ascoli theorem in Banach space indicates that $Z_{[0,T]}$ is relatively compact. Therefore, for any $\varepsilon>0$, there exist finitely many balls $B_\varepsilon(y^i)$ such that $Z_{[0,T]} \subset \cup_{i=1}^n B_\varepsilon(y^i)$, where $y^i \in B'$.

$$B_\varepsilon(y^i) = \left\{ y(t) \in Z_{[0,T]}, \|y-y^i\|_{b_1} = \sup_{0\le t\le T}\left\{e^{-kt}\|y(t)-y^i(t)\|_\beta\right\} \le \varepsilon \right\}. \tag{42}$$

Hence, for any $y(t) \in Z$, there exists an $i \in \{1,2,\ldots,n\}$ such that $y_{[0,T]} \in B_\varepsilon(y^i)$, i.e., for $t \in [0,T]$,

$$e^{-kt}\|y(t)-y^i(t)\|_\beta \le \varepsilon. \tag{43}$$

Moreover, for $t \in [T,+\infty]$, with conditions (3) and (43),

$$\begin{aligned} &e^{-kt}\|y(t)-y^i(t)\|_\beta, \\ &\le \left\|e^{-kt}y(t)-e^{-kT}y(T)\right\|_\beta + \left\|e^{-kT}y(T)-e^{-kT}y^i(T)\right\|_\beta \\ &\quad + \left\|e^{-kT}y^i(T)-e^{-kt}y^i(t)\right\|_\beta, \\ &\le 5\varepsilon. \end{aligned} \tag{44}$$

Therefore, by (43) and (44), we have $\|y(t)-y^i(t)\|_b \le 5\varepsilon$ for any $t\ge 0$. Then, Z can be covered by balls $B_{5\varepsilon}(y^i) = \{y(t) \in Z, |y-y^i|_b \le 5\varepsilon\}$. Consequently, Z is totally bounded and the process is complete. □

Theorem 1. *Assume that condition* (H) *holds; then, for* $\phi(0) \in X_\beta$, *problem (3) has at least one global mild solution in* B_r.

Proof. We aim to prove this theorem by using Schauder's fixed-point theorem. In view of Lemma 6, $T\colon B_r \longrightarrow B_r$ and T is continuous, so we just need to prove that for any bounded subset $V \subset B_r$, TV is relatively compact in X. Then, it is easy to prove that TV satisfies all conditions in Lemma 8.

Consider Lemma 6; we have proved that $\|Ty\|_b = \sup\{e^{-kt}\|Ty(t)\|_\beta\} \le r$ for any $y \in B_r$ which implies $\{Ty,\ y \in B_r^0\}$ is uniformly bounded. By Lemma 7, $\{Ty, y \in B'\}$ is equicontinuous on any compact interval $[0,T]$ of $[0,+\infty)$ and $e^{-kt}\|Ty(t)\|_\beta < \varepsilon$ for any $t \ge T$ and $y \in B'$. Then, it remains to show that $V(t) = \{(Ty)(t),\ y(t) \in V\}$ is relatively compact in X for any $t \in [0,T]$.

It is easy to know that $V(0) = \{0\}$ is compact in X. Let $t \in [0,T)$ be fixed and for any $\varepsilon \in (0,t)$, $\delta>0$, we define an operator T_ε^δ on V by the formula

$$\begin{aligned} \left(T_\varepsilon^\delta y\right)(t) &= \int_0^{t-\varepsilon}\int_\delta^{+\infty} q\theta(t-s)^{q-1}M_q(\theta)\,Q\left((t-s)^q\theta\right)f(s,y_s+z_s)\mathrm{d}\theta\mathrm{d}s, \\ &= Q(\varepsilon^q\delta)\int_0^{t-\varepsilon}\int_\delta^{+\infty} q\theta(t-s)^{q-1}M_q(\theta)\,Q\left((t-s)^q\theta-\varepsilon^q\delta\right)f(s,y_s+z_s)\mathrm{d}\theta\mathrm{d}s, \end{aligned} \tag{45}$$

where $y \in V$. Under the compactness of $Q(\varepsilon^q\delta)\,(\varepsilon^q\delta>0)$ and the boundedness of

$$\begin{aligned} &\int_0^{t-\varepsilon}\int_\delta^{+\infty} q\theta(t-s)^{q-1}M_q(\theta)Q\left((t-s)^q\theta-\varepsilon^q\delta\right) \\ &f(s,y_s+z_s)\mathrm{d}\theta\mathrm{d}s, \end{aligned} \tag{46}$$

we obtain that the set $V_\varepsilon^\delta(t) = \{(T_\varepsilon^\delta y)(t),\ y \in V\}$ is relatively compact in X for any $\varepsilon \in (0,t)$ and $\delta>0$. Moreover, for any $y \in V$, $t>0$, we have

$$\begin{aligned} &e^{-kt}\left\|(Ty)(t)-\left(T_\varepsilon^\delta y\right)(t)\right\|_\beta, \\ &\le qe^{-kt}\left\|\int_0^t\int_0^\delta \theta(t-s)^{q-1}M_q(\theta)\left((t-s)^q\theta\right)f(s,y_s+z_s)\mathrm{d}\theta\mathrm{d}s\right\|_\beta, \\ &\quad + e^{-kt}\left\|\int_{t-\varepsilon}^t\int_\delta^{+\infty} q\theta(t-s)^{q-1}M_q(\theta)\,Q\left((t-s)^q\theta\right)f(s,y_s+z_s)\mathrm{d}\theta\mathrm{d}s\right\|_\beta, \\ &\le qC_pe^{-kt}\int_0^t (t-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s\int_0^\delta \theta^{-\gamma}M_q(\theta)\mathrm{d}\theta, \\ &\quad + C_pe^{-kt}\int_{t-\varepsilon}^t (t-s)^{-q\gamma-1}e^{-ks}p(s)\eta(s)\mathrm{d}s. \end{aligned} \tag{47}$$

According to $\int_0^{+\infty}\theta^r M_q(\theta)d\theta = (\Gamma(1+r)/\Gamma(1+qr))$ and condition (H), we have

$$\int_0^{\delta}\theta^{-\gamma}M_q(\theta)\mathrm{d}\theta \longrightarrow 0,$$
$$\int_{t-\varepsilon}^{t}(t-s)^{-q\gamma-1}p(s)\eta(s)\mathrm{d}s \longrightarrow 0, \quad \text{as } \varepsilon \longrightarrow 0, \delta \longrightarrow 0, \tag{48}$$

which implies $\|(Ty)(t) - (T_\varepsilon^\delta y)(t)\|_b \longrightarrow 0$ as $\varepsilon \longrightarrow 0, \delta \longrightarrow 0$.

Therefore, the relatively compact set $V_\varepsilon^\delta(t)$ is arbitrarily close to the set $V(t)$. Hence, for any $t \in [0,T]$, the set $V(t)$, $t \in [0,T]$ is also relatively compact in X.

Hence, T: $B_r \longrightarrow B_r$ is a completely continuous operator. So, by Schauder's fixed-point theorem, T has at least one fixed point in B_r which implies that problem (3) has at least one global mild solution in B_r. □

3.2. The Case That $Q(t)$ Is Not Compact. In this section, we assume that $Q(t)$ is not compact. In the following, α and $\alpha_{B'}$ denote the Kuratowski measures of noncompactness of bounded sets in X_β and in B'. For more details about Kuratowski measures of noncompactness, we refer the readers to [27]. Assume that:

(H^*) There exists $m(t) \in L([0,+\infty),[0,+\infty))$ such that $I_{0+}^q m$ exists and for any bounded set $V \subset B$,

$$\alpha(f(t,V)) \le m(t)\, e^{-kt}\sup\nolimits_{-\infty<\tau\le 0}\alpha(V(\tau)), \tag{49}$$

and for any $t \ge 0$,

$$C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}m(s)\mathrm{d}s < 1. \tag{50}$$

Lemma 9 (see [27]). If $V \subset C(J,E)$ is bounded and equicontinuous, then $\alpha(V(t))$ is continuous and

$$\alpha\left(\left\{\int_J y(t)\mathrm{d}t, \quad y \in V\right\}\right) \le \int_J \alpha(V(t))\mathrm{d}t, \tag{51}$$

where J is any compact interval of $[0,+\infty)$.

Lemma 10. *Let V be a bounded set in B'. Suppose that $V(t)$ is equicontinuous on any compact interval $[0,T]$ of $[0,+\infty)$ and for any $t \ge T$, $\varepsilon > 0$, and $y \in V$,*

$$e^{-kt}\|y(t)\|_\beta < \varepsilon. \tag{52}$$

Then, for each $V(t) = \{y(t),\ y \in V\}$,

$$\alpha_{B'}(V) = \sup_{t\ge 0}\left\{e^{-kt}\alpha(V(t))\right\}. \tag{53}$$

Proof. First, we prove that $\alpha_{B'}(V) \ge \sup_{t\ge0}\{e^{-kt}\alpha(V(t))\}$. For the above given $\varepsilon > 0$, $t \ge 0$, there exists a partition $V = \cup_{j=1}^n V_j$ such that

$$\operatorname{diam}\left(V_j\right) < \alpha_{B'}(V) + \varepsilon, \quad \text{for any } j = 1,2,\ldots,n. \tag{54}$$

Then, $V(t) = \cup_{j=1}^n V_j(t)$. For any $u, v \in V_j$, $t \ge 0$,

$$e^{-kt}\|u(t) - v(t)\|_\beta \le \operatorname{diam}\left(V_j\right) < \alpha_{B'}(V) + \varepsilon. \tag{55}$$

Therefore, $\operatorname{diam}(V_j(t)) \le e^{kt}(\alpha_{B'}(V) + \varepsilon)$ which implies

$$\sup_{t\ge0}\left\{e^{-kt}\alpha(V(t))\right\} \le \alpha_{B'}(V), \tag{56}$$

by the arbitrariness of ε.

Next, we show that $\alpha_{B'}(V) \le \sup_{t\ge0}\{e^{-kt}\alpha(V(t))\}$. By the equicontinuity of $V(t)$ on $[0,T]$, there exists a partition $0 = t_0 < t_1 < \cdots < t_m = T$ such that

$$\left\|e^{-kt_i'}y(t_i') - e^{-kt_i''}y(t_i'')\right\|_\beta < \varepsilon, \tag{57}$$

for any $t_i', t_i'' \in [t_i, t_{i+1}]$, $y \in V$, $i = 0,1,\ldots,m-1$. Let $I_i = [t_i, t_{i+1}]$, $i = 0,1,\ldots,m-1$ and $I_m = [t_m, +\infty)$; then, by (51) and (57),

$$\left\|e^{-kt_i'}y(t_i') - e^{-kt_i''}y(t_i'')\right\|_\beta < 2\varepsilon, \quad \text{for any } y \in V,\ t_i', t_i'' \in I_i, \quad i = 0,1,\ldots,m. \tag{58}$$

For each $i \in \{0,1,\ldots,m\}$, there exists a division $V = \cup_{j=1}^n V_j^i$ such that $V(t_i') = \cup_{j=1}^n V_j^i(t_i')$ and

$$\operatorname{diam}\left(V_j^i(t_i')\right) < \alpha(V(t_i')) + 2\varepsilon, \quad j = 1,2,\ldots,n. \tag{59}$$

Let Y be the finite set of all maps $i \longrightarrow \gamma(i)$ of $\{0,1,\ldots,m\}$ into $\{1,2,\ldots,n\}$. For $\gamma \in Y$,

$$Z_\gamma := \left\{y \in V, y(t_i') \in V_{\gamma(i)}^i(t_i'), \quad i = 0,1,\ldots,m\right\}, \tag{60}$$

so $V = \{y(t),\ y \in Z_\gamma, \gamma \in Y\}$. For any $u, v \in Z_\gamma$ and $t \geq 0$, there exists $i \in \{0, 1, \ldots, m\}$ such that $t \in I_i$; then,

$$\begin{aligned}
&e^{-kt}\|u(t) - v(t)\|_\beta,\\
&\leq \left\|e^{-kt}u(t) - e^{-kt_i'}u(t_i')\right\|_\beta + \left\|e^{-kt_i'}u(t_i') - e^{-kt_i'}v(t_i')\right\|_\beta + \left\|e^{-kt}v(t) - e^{-kt_i'}v(t_i')\right\|_\beta,\\
&< \alpha(V(t_i')) + 6\varepsilon.
\end{aligned} \tag{61}$$

Therefore, $\operatorname{diam}(Z_\gamma) \leq \alpha(V(t_i')) + 6\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have

$$\alpha_{B'}(V) \leq \sup_{t\geq 0}\left\{e^{-kt}\alpha(V(t))\right\}. \tag{62}$$

□

Lemma 11 (see [27]). Let D be a bounded, closed, and convex subset of Banach space E. If the operator $T\colon D \longrightarrow D$ is a strict set contraction, then T has a fixed point in D.

Remark 2. A bounded and continuous operator $T\colon D \longrightarrow E$ is called a strict set contraction if there is a constant $0 \leq \lambda < 1$ such that $\alpha(TV) \leq \lambda\alpha(V)$ for any bounded set $V \subset D$.

Theorem 2. *Assume that conditions (H), (H^*) are satisfied; then, for $\phi(0) \in X_\beta$, problem (3) has a global mild solution in B_r.*

Proof. Let V be an arbitrary bounded set in B_r. According to Lemmas 6 and 7, we know that $T\colon B_r \longrightarrow B_r$ is bounded and continuous and $\{Ty(\cdot), y \in V\}$ is equicontinuous on $[0, T]$ and $e^{-kt}\|Ty(t)\|_\beta < \varepsilon$ for any $t \geq T$, $y \in V$, $\varepsilon > 0$. Then, by Lemma 3.6, it follows that

$$\alpha_{B'}(TV) = \sup_{t\geq 0}\left\{e^{-kt}\alpha(TV(t))\right\}. \tag{63}$$

Consider Lemma 9 and condition (H^*); let any $t \geq 0$ be fixed, and for the above $\varepsilon > 0$, we have

$$\begin{aligned}
e^{-kt}\alpha(TV(t)) &= e^{-kt}\alpha\left(\left\{\int_0^t (t-s)^{q-1}P_q(t-s)f(s, y_s + z_s)\mathrm{d}s,\quad y \in V\right\}\right),\\
&\leq e^{-kt}\int_0^t \alpha\left(\left\{(t-s)^{q-1}P_q(t-s)f(s, y_s + z_s),\quad y \in V\right\}\right)\mathrm{d}s,\\
&\leq C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}\alpha(\{f(s, y_s + z_s),\quad y \in V\})\mathrm{d}s,\\
&\leq C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}m(s)e^{-ks}\sup_{0\leq\tau\leq s}\alpha(V(\tau))\mathrm{d}s,\\
&\leq \left(C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}m(s)\mathrm{d}s\right)\alpha_{B'}(V),
\end{aligned} \tag{64}$$

which implies that $\alpha_{B'}(TV) \leq \lambda\alpha_{B'}(V)$ where $\lambda \coloneqq C_p e^{-kt}\int_0^t (t-s)^{-q\gamma-1}m(s)\mathrm{d}s < 1$. Then, T is a strict set contraction.

Consequently, by Lemma 11, T has a fixed point in B_r which implies that problem (3) has a global mild solution in B_r. The proof process is completed. □

4. Conclusions

In this paper, we investigated a class of fractional evolution equations with infinite delay and almost sectorial operator on unbounded domains in Banach space. We considered the case of compact semigroups and noncompact semigroups and obtained sufficient conditions of the existence of global mild solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Some Existence Results for High Order Fractional Impulsive Differential Equation on Infinite Interval

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In this paper, we consider the high order impulsive differential equation on infinite interval $\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),J_{0+}^{\beta}u(t),D_{0+}^{\alpha-1}u(t))=0,\ t\in[0,\infty)\setminus\{t_k\}_{k=1}^{m} \\ \triangle u(t_k)=I_k(u(t_k)),\ t=t_k,k=1,\ldots,m \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,D_{0+}^{\alpha-1}u(\infty)=u_0 \end{cases}$ By applying Schauder fixed points and Altman fixed points, we obtain some new results on the existence of solutions. The nonlinear term of the equation contains fractional integral operator $J^{\beta}u(t)$ and lower order derivative operator $D_{0+}^{\alpha-1}u(t)$. An example is presented to illustrate our results.

1. Introduction

In this paper, we are concerned with the following impulsive differential equation on infinite interval:

$$\begin{cases} D_{0+}^{\alpha}u(t)+f\left(t,u(t),J_{0+}^{\beta}u(t),D_{0+}^{\alpha-1}u(t)\right)=0, & t\in[0,\infty)\setminus\{t_k\}_{k=1}^{m}, \\ \triangle u(t_k)=I_k(u(t_k)), & t=t_k,k=1,\ldots,m, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, & D_{0+}^{\alpha-1}u(\infty)=u_0. \end{cases} \tag{1}$$

where $u_0\in R$, $\alpha,\beta\in(n-1,n]$, $n>2$, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative, $0=t_0<t_1<t_2<\cdots<t_m<\infty$, $\triangle u(t_k)=u(t_k^+)-u(t_k^-)$, $u(t_k^-)=u(t_k)$, $u(t_k^+)=\lim_{h\longrightarrow 0+}u(t_k+h)$ and $u(t_k^-)=\lim_{h\longrightarrow 0-}u(t_k-h)$ represent the right and left limits of $u(t)$ at $t=t_k$, and $D_{0+}^{\alpha-1}u(\infty)=\lim_{t\longrightarrow\infty}D_{0+}^{\alpha-1}u(t)$. Also, $f\in C([0,+\infty)\times R\times R\times R,R)$, $I_k\in C(R,R)$.

During the past decades, fractional differential equations have drawn wide concerns. Compared with integer order differential equations, fractional differential equations have more extensive application range, such as control theory, physics, aerodynamics, polymer rheology, chemistry, biology, and so forth. There are many papers focused on the existence of positive solutions for fractional differential equations (see [1–3]).

Since the last century, the dynamics of populations subject to abrupt changes was described by impulsive differential system. And other phenomena, for instance, harvesting, diseases, and so on, also have been described by using impulsive differential systems. Impulsive differential equations of fractional order play an important role in fractional differential equations theory and applications. Recently, impulsive fractional differential equations have been studied extensively. For example, Wang et al. studied the existence and multiplicity of solutions for impulsive fractional boundary value problem with p-Laplacian in [4], and Liu considered fractional impulsive differential equations using bifurcation techniques in [5]. For more articles related to impulsive fractional differential equations, refer to [6–12].

Recently, in [13], Liu investigated the existence of solutions for higher order impulsive fractional differential equations given by

$$\begin{cases} {}^{c}D_{0+}^{q}x(t) = F(t, x(t)), & t \in (t_i, t_{i+1}], i \in N_0, \\ \triangle x|_{t=t_i} = I(t_i, x(t_i)), & i \in N, \\ x(0) = x_0, \end{cases}$$
$$\begin{cases} D_{0+}^{q}x(t) = G(t, x(t)), & t \in (t_i, t_{i+1}], i \in N_0, \\ \lim\limits_{t\longrightarrow t_i^+} \left(t - t_i^{1-\alpha}\right)x(t) = J(t_i, x(t_i)), & i \in N, \\ \lim\limits_{t\longrightarrow 0^+} t^{1-q}x(t) = x_0, \end{cases} \tag{2}$$

where $q \in (0,1)$, $t \in [0,T]$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} < T$, I, J: $\{t_k\colon k \in N\} \times R \longrightarrow R$ are discrete Carathéodory functions, and F, G: $(0,T) \times R \longrightarrow R$ are strong Carathéodory functions. By using Schauder's fixed-point theorem, Liu established some existence results.

In [10], Liu and Ahmad studied the following problems:

$$\begin{cases} {}^{c}D_{0+}^{\alpha}x(t) = q(t) f\left(t, x(t), {}^{c}D_{0+}^{p}x(t)\right), & t \in (0,\infty), \\ \triangle x(t_k) = I_k(t_k, x(t_k)), & k = 1, 2, \ldots, \\ x(0) = x_0, \end{cases}$$
$$\begin{cases} {}^{c}D_{*}^{\alpha}x(t) = q(t) f\left(t, x(t), {}^{c}D_{*}^{p}x(t)\right), & t \in (0,\infty), \\ \triangle x(t_k) = I_k(t_k, x(t_k)), & k = 1, 2, \ldots, \\ x(0) = x_0, \end{cases} \tag{3}$$

where $x_0 \in R$, $\alpha \in (0,1]$, $0 < p < \alpha$, $0 = t_0 < t_1 < t_2 < \cdots$ with $\lim_{k\longrightarrow\infty} t_k = \infty$, q: $(0,\infty) \longrightarrow R$ satisfies that there exists $l > -\alpha$ such that $|q(t)| \le t^l$ for all $t \in (0,\infty)$, and q may be singular at $t = 0$. And f: $[0,\infty) \times R^2 \longrightarrow R$ is a Carathéodory function, I_k: $(0,\infty) \times R \longrightarrow R$ $(k = 1, 2, \ldots,)$, I_k is a Carathéodory function sequence, and $\triangle x(t_k) = \lim_{t\longrightarrow t_k^+} x(t) - \lim_{t\longrightarrow t_k^-} x(t)$, $k = 1, 2, \ldots,$. By using Schauder's fixed-point theorem, the authors studied the existence of solution. And the authors also considered the uniqueness of solution under some appropriate conditions.

In [9], Zhao and Ge considered the following boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, & t \in (0,\infty), t \ne t_k, k = 1, 2, \ldots, m, \\ u(t_k^+) - u(t_k^-) = -I_k(u(t_k)), & k = 1, 2, \ldots, m, \\ u(0) = 0, \quad D_{0+}^{\alpha}u(\infty) = 0, \end{cases} \tag{4}$$

where α is a real number with $1 < \alpha \le 2$, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative, $t_0 = 0$, $1 < t_1 < t_2 < \cdots < t_m < \infty$, $u(t_k^+) = \lim_{h\longrightarrow 0^+} u(t_k + h)$, $u(t_k^-) = \lim_{h\longrightarrow 0^+} u(t_k - h)$, $D_{0+}^{\alpha-1}u(\infty) = \lim_{t\longrightarrow\infty} D_{0+}^{\alpha-1}u(t)$, $f(t, (1 + t^{\alpha})u)$: $[0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ is continuous, and I_k: $[0,\infty) \longrightarrow [0,\infty)$ $(k = 1, 2, \ldots, m)$ are continuous. Wang and Ge proved that the problem they studied has at least three positive solutions.

Motivated by the aforementioned work, we studied existence of solution of problem (1) by Schauder's fixed-point theorem and Altman's fixed-point theorem. The main features of this paper are as follows. Firstly, the nonlinear term not only involved fractional order derivative but also contained fractional integral. Compared with [9, 10, 13], our nonlinear terms are more general. Many articles contain derivatives for nonlinear terms, but few articles contain both derivatives and integrals. Secondly, we studied the problem on the infinite interval. To the best of our knowledge, there are few articles involving the impulsive fractional order differential equations on the infinite interval. If the nonlinear term contained fractional integral and $t \in [0,\infty)$, it will bring new obstacles to solve the problem. For this purpose, we overcome obstacles by constructing a special cone. Thirdly, our problem is higher order impulsive fractional equation. Compared with [9], we allowed $\alpha \in (n-1, n]$, where $n > 2$. It is obvious that our problem is more general.

This paper is organized as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, we give our main results by fixed-point theorem. In Section 4, one example is presented to illustrate the main results.

2. Preliminaries and Lemmas

Let u: $[0,\infty) \longrightarrow \mathbb{R}$, $J = [0,\infty)$, $J_0 = [0,t_1]$, $J_m = (t_m,\infty)$, $J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m-1$. For $k = 1, 2, \ldots, m$, define the function $u_k(t) = u(t)$. Let $C(J,R)$ be the Banach space of continuous functions from J to $\mathbb{R}$. Let us to introduce the Banach spaces

$$\mathrm{PC}(J,\mathbb{R}) = \left\{u\colon u_k \in C(J_k,\mathbb{R}),\ k = 0, 1, \ldots, m, u \cdot (t_k^+) \text{ and } u(t_k^-) \text{ exist, } u(t_k) = u(t_k^-),\ \lim_{t\longrightarrow\infty} \frac{u(t)}{1 + t^{\alpha-1}} \text{ exsits}\right\}, \tag{5}$$

with the norm

$$\|u\|_{PC} = \sup_{t\in[0,\infty)}\left|\frac{u(t)}{1+t^{\alpha-1}}\right|,$$

$$\begin{aligned}\mathrm{PC}^1(J,\mathbb{R}) = \Big\{& u \in \mathrm{PC}(J,R)\colon D^{\alpha-1}u(t) \in C(J_k,\mathbb{R}), k=0,1,\ldots,m,\ D^{\alpha-1}u(t_k^+) \text{ and } D^{\alpha-1}u(t_k^-) \text{ exist}, D^{\alpha-1}u(t_k^-) \\ &= D^{\alpha-1}u(t_k), \lim_{t\longrightarrow\infty} D^{\alpha-1}u(t) \text{ exists}\Big\},\end{aligned} \tag{6}$$

with the norm

$$\|u\|_{PC^1} = \max\left\{\sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}}, \sup_{t\in J}\left|D^{\alpha-1}u(t)\right|\right\}. \tag{7}$$

Definition 1. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f\colon (0,\infty) \longrightarrow \mathbb{R}$ is given by

$$J_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \tag{8}$$

where the right side is pointwise defined on $(0,\infty)$.

Definition 2. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $f\colon (0,\infty) \longrightarrow \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\mathrm{d}s, \tag{9}$$

where n is the smallest integer greater than or equal to α and the right side is pointwise defined on $(0,\infty)$. In particular, for $\alpha = n$, $D_{0+}^{\alpha} f(t) = f^{(n)}(t)$.

Lemma 1. *Let $\alpha > 0$, and n denotes the smallest integer greater than or equal to α. For all $t \in [a,b]$,*

$$J_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{10}$$

where $c_j \in \mathbb{R}$, $j = 1,2,\ldots,n$.

Lemma 2 (see [2]). *Let $\Omega \subseteq PC^1$. Then, Ω is relatively compact in PC^1 if the following conditions hold:*

(1) Ω is bounded in PC^1

(2) For any $u(t) \in \Omega$, $u(t)/1 + t^{\alpha-1}$ and $D^{\alpha-1}u(t)$ are equicontinuous on any interval J_k

(3) Given $\varepsilon > 0$, there exists a constant $N = N(\varepsilon) > 0$ such that

$$\begin{aligned}&\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon, \\ &\left|D^{\alpha-1}u(t_1) - D^{\alpha-1}u(t_2)\right| < \varepsilon,\end{aligned} \tag{11}$$

for any $t_1, t_2 \geq N$ and $u(t) \in \Omega$.

Theorem 1 (Schauder fixed-point theorem). *If U is a closed bounded convex subset of a Banach space X and $T\colon U \longrightarrow U$ is completely continuous, then T has at least one fixed point in U.*

Theorem 2 (Altman theorem). *Let Ω be an open bounded subset of a Banach space E with $0 \in \Omega$ and $T\colon \overline{\Omega} \longrightarrow E$ be a completely continuous operator. Then, T has a fixed point in $\overline{\Omega}$, provided that*

$$\|\mathrm{Tx} - x\|^2 \geq \|\mathrm{Tx}\|^2 - \|x\|^2, \quad \forall x \in \partial\Omega. \tag{12}$$

Lemma 3. *For a given $y \in C(J,\mathbb{R})$, a function $u \in PC^1(J,\mathbb{R})$ is a solution of the following boundary value problem:*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, & t \in [0,\infty)\setminus\{t_k\}_{k=1}^m, \\ \triangle u(t_k) = I_k(u(t_k)), & t = t_k, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & D_{0+}^{\alpha-1} u(\infty) = u_0, \end{cases} \tag{13}$$

if and only if $u \in PC^1(J,\mathbb{R})$ is a solution of the impulsive fractional integral equation

$$\begin{aligned}u(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\infty} y(s)\mathrm{d}s \\ &+ \frac{t^{\alpha-1}}{\Gamma(\alpha)} u_0 - t^{\alpha-1}\sum_{t<t_i} I_i t_i^{1-\alpha}.\end{aligned} \tag{14}$$

Proof. Assume $u(t)$ satisfies (13). We denote the solution of (13) by $u(t) \triangleq u_k(t)$ in J_k $(k = 0,1,\ldots,m)$.

For $t \in [0,t_1]$, applying Lemma 1, we have

$$\begin{aligned}u_0(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s + C_{01}t^{\alpha-1} + C_{02}t^{\alpha-2} \\ &+ \cdots + C_{0n}t^{\alpha-n}.\end{aligned} \tag{15}$$

From $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we know $C_{0n} = \cdots = C_{03} = C_{02} = 0$. So, we get

$$\begin{aligned}u_0(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s + C_{01}t^{\alpha-1}, \quad t \in [0,t_1], \\ u(t_1^-) &= -\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1} y(s)\mathrm{d}s + C_{01}t_1^{\alpha-1}.\end{aligned} \tag{16}$$

For $t \in (t_1,t_2]$, by applying Lemma 1, we know

$$u_1(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + C_{11}t^{\alpha-1} + C_{12}t^{\alpha-2} + \cdots + C_{1n}t^{\alpha-n}. \tag{17}$$

In view of $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we have $C_{1n} = \cdots = C_{13} = C_{12} = 0$. So, we know

$$\begin{aligned} u_1(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + C_{11}t^{\alpha-1}, \\ u(t_1^+) &= -\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}y(s)ds + C_{11}t_1^{\alpha-1}. \end{aligned} \tag{18}$$

And from impulsive condition of (13), $\triangle u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$. Then,

$$-\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}y(s)ds + C_{11}t_1^{\alpha-1} - \left(-\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}y(s)ds + C_{01}t_1^{\alpha-1}\right) = I_1(u(t_1)). \tag{19}$$

Thus,

$$C_{11} = C_{01} + t_1^{1-\alpha}I_1(u(t_1)). \tag{20}$$

Then,

$$u_1(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + t^{\alpha-1}C_{01} + t^{\alpha-1}t_1^{1-\alpha}I_1(u(t_1)), \quad t \in (t_1, t_2]. \tag{21}$$

For $t \in (t_2, t_3]$, by applying Lemma 1, we obtain

$$u_2(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + C_{21}t^{\alpha-1} + C_{22}t^{\alpha-2} + \cdots + C_{2n}t^{\alpha-n}. \tag{22}$$

In view of $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we have $C_{2n} = \cdots = C_{23} = C_{22} = 0$. So, we know

$$\begin{aligned} u_2(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + C_{21}t^{\alpha-1}, \\ u(t_2^+) &= -\frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1}y(s)ds + C_{21}t_2^{\alpha-1}. \end{aligned} \tag{23}$$

And from impulsive condition, $\triangle u(t_2) = u(t_2^+) - u(t_2^-) = I_2(u(t_2))$. Then,

$$-\frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1}y(s)ds + C_{21}t_2^{\alpha-1} - \left(-\frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1}y(s)ds + C_{11}t_2^{\alpha-1}\right) = I_2(u(t_2)). \tag{24}$$

We get

$$C_{21} = C_{11} + t_2^{1-\alpha}I_2(u(t_2)) = C_{01} + t_1^{1-\alpha}I_1(u(t_1)) + t_2^{1-\alpha}I_2(u(t_2)) = C_{01} + \sum_{i=1}^{2} t_i^{1-\alpha}I_i. \tag{25}$$

Consequently,

$$u_2(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + t^{\alpha-1}C_{01} + t^{\alpha-1}\sum_{i=1}^{2} t_i^{1-\alpha}I_i, \quad t \in (t_2, t_3]. \tag{26}$$

By the recurrent method and Lemma 1, for $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, we can say that

$$u(t)u_k(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + t^{\alpha-1}C_{01} + t^{\alpha-1}\sum_{i=1}^{k} t_i^{1-\alpha}I_i. \tag{27}$$

Thus, for $t \in (t_m, \infty)$, we have

$$u(t) = u_m(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + t^{\alpha-1}C_{01} + t^{\alpha-1}\sum_{i=1}^{m} t_i^{1-\alpha}I_i. \tag{28}$$

From $D_{0+}^{\alpha-1}u(\infty) = u_0$, we get

$$-\int_0^\infty y(s)ds + \Gamma(\alpha)\sum_{i=1}^{m} t_i^{1-\alpha}I_i + \Gamma(\alpha)C_{01} = u_0. \tag{29}$$

So,

$$C_{01} = \frac{1}{\Gamma(\alpha)}u_0 + \frac{1}{\Gamma(\alpha)}\int_0^\infty y(s)ds - \sum_{i=1}^{m} t_i^{1-\alpha}I_i. \tag{30}$$

Therefore, for $t \in [0, \infty)$, we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}u_0 - t^{\alpha-1}\sum_{i=1}^{m} t_i^{1-\alpha}I_i + t^{\alpha-1}\sum_{t_i<t} t_i^{1-\alpha}I_i \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}u_0 - t^{\alpha-1}\sum_{t<t_i} t_i^{1-\alpha}I_i. \end{aligned} \tag{31}$$

Conversely, assume that $u(t)$ satisfies impulsive fractional integral equation (14). Obviously, we get $u(0) = u'(0) = \cdots = u^{n-2}(0) = 0$, and $D_{0+}^{\alpha-1}u(\infty) = u_0$. Using the fact $D_{0+}^{\alpha}t^{\alpha-1} = 0$, we obtain $D_{0+}^{\alpha}u(t) = -y(t)$. Also, we can easily show that $\triangle u(t_k) = I_k(u(t_k))$, $k = 1, 2, \ldots, m$. Then, u is also the solution of problem (13). $\square$

3. Main Results

In this section, we will prove the existence of solution of (1) by using Schauder fixed-point theorem and Altman theorem.

According to Lemma 3, we obtain the following lemma first.

Lemma 4. *$u \in PC^1(J, \mathbb{R})$ is a solution of problem (1) if and only if $u \in PC^1(J, \mathbb{R})$ is a solution of the impulsive fractional integral equation*

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, u(s), J^\beta u(s), D^{\alpha-1}u(s)\right)ds \\ &+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty f\left(s, u(s), J^\beta u(s), D^{\alpha-1}u(s)\right)ds \\ &+\frac{t^{\alpha-1}}{\Gamma(\alpha)}u_0 - t^{\alpha-1}\sum_{t<t_i} I_i t_i^{1-\alpha}, \quad t \in J. \end{aligned} \tag{32}$$

Define an operator T: $PC^1(J, \mathbb{R}) \longrightarrow PC^1(J, \mathbb{R})$ as follows:

$$\begin{aligned} (Tu)(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, u(s), J^\beta u(s), D^{\alpha-1}u(s)\right)ds \\ &+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty f\left(s, u(s), J^\beta u(s), D^{\alpha-1}u(s)\right)ds \\ &+\frac{t^{\alpha-1}}{\Gamma(\alpha)}u_0 - t^{\alpha-1}\sum_{t<t_i} I_i t_i^{1-\alpha}, \quad t \in J. \end{aligned} \tag{33}$$

Then, problem (1) has a solution if and only if the operator T has a fixed point.

Theorem 3. *Assume that following conditions hold:*

$(H1)$ For $f \in C([0, +\infty)) \times R \times R \times R, R)$, there exist nonnegative functions $a(t), b(t), c(t), e(t) \in L^1(J)$ such that

$$\begin{gathered} |f(t, x, y, z)| \le a(t)|x| + b(t)|y| + e(t)|z| + c(t), \\ \int_0^{+\infty}\left(\left(1+t^{\alpha-1}\right)a(t) + b(t)\right)dt < \infty, \\ \int_0^{+\infty} c(t)dt < \infty, \quad \int_0^{+\infty}\frac{(1+t)^{\alpha-1}t^\beta}{\Gamma(\beta+1)}e(t)dt < \infty. \end{gathered} \tag{34}$$

$(H2)$ For $I_k \in C(R, R)$, for all $u \in \mathbb{R}$, there exist some constants $L_k > 0$ such that $|I_k(u)| < L_k$, $k = 1, 2, \ldots, m$.

Then, problem (1) has at least one solution $u(t)$ in $PC^1(J, \mathbb{R})$.

Proof. We will use five steps to prove our conclusion. Firstly, we will show T: $PC^1(J, \mathbb{R}) \longrightarrow PC^1(J, \mathbb{R})$ is continuous. From (33), we know

$$D^{\alpha-1}Tu(t) = -\int_0^t f\left(s, u(s), J^\beta u(s), D^{\alpha-1}u(s)\right)ds + \int_0^\infty f\left(s, u(s), J^\beta u(s), D^{\alpha-1}u(s)\right)ds + u_0 - \Gamma(\alpha)\sum_{t<t_i} I_i t_i^{1-\alpha}. \tag{35}$$

From $(H1)$, we have

$$\begin{aligned} \int_0^\infty \left|f\left(s, u(s), J^\beta u(s), D^{\alpha-1}u(s)\right)\right|ds &\le \int_0^{+\infty}\left[a(s)|u(s)| + b(s)\left|D^{\alpha-1}u(s)\right| + e(s)\left|J^\beta u(s)\right| + c(s)\right]ds \\ &\le \int_0^{+\infty}\left[\left(1+s^{\alpha-1}\right)a(s)\|u\|_{PC} + b(s)\left|D^{\alpha-1}u(s)\right| + \frac{(1+s)^{\alpha-1}s^\beta}{\Gamma(\beta+1)}e(s)\|u\|_{PC} + c(s)\right]ds \\ &\le \|u\|_{PC^1}\int_0^{+\infty}\left[\left(1+s^{\alpha-1}\right)a(s) + b(s) + \frac{(1+s)^{\alpha-1}s^\beta}{\Gamma(\beta+1)}e(s)\right]ds + \int_0^{+\infty} c(s)ds \\ &< \infty. \end{aligned} \tag{36}$$

Let $u_n, u \in PC^1(J, \mathbb{R})$ be such that $u_n \longrightarrow u (n \longrightarrow \infty)$. Then, $\|u_n\|_{PC^1} < \infty$ and $\|u\|_{PC^1} < \infty$. By (36) and the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} &\lim_{n\longrightarrow\infty}\int_0^\infty f\left(s, u_n(s), J^\beta u_n(s), D^{\alpha-1}u_n(s)\right)ds \\ &= \int_0^\infty f\left(s, u(s), J^\beta u(s), D^{\alpha-1}u(s)\right)ds. \end{aligned} \tag{37}$$

By $(H1)$, $(H2)$, and (36), we have

$$\begin{aligned}\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right| &= \left|-\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}} f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\mathrm{d}s + \frac{t^{\alpha-1}}{1+t^{\alpha-1}}\frac{1}{\Gamma(\alpha)}\int_0^{\infty} f\left(s,u(s),J^{\beta}u\right.\right.\\ &\quad \left.\left.\cdot(s),D^{\alpha-1}u(s)\right)\mathrm{d}s + \frac{t^{\alpha-1}}{1+t^{\alpha-1}}\frac{1}{\Gamma(\alpha)}u_0 - \frac{t^{\alpha-1}}{1+t^{\alpha-1}}\sum_{t<t_i} I_i t_i^{1-\alpha}\right| \\ &\le \frac{2}{\Gamma(\alpha)}\int_0^{\infty}\left|f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s + \frac{|u_0|}{\Gamma(\alpha)} + \sum_{t<t_i} L_i t_i^{1-\alpha} < \infty,\end{aligned} \tag{38}$$

$$\begin{aligned}\left|D^{\alpha-1}\mathrm{Tu}(t)\right| &= \left|-\int_0^t f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\mathrm{d}s + \int_0^{\infty} f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right) + u_0 - \Gamma(\alpha)\sum_{t<t_i} I_i t_i^{1-\alpha}\right| \\ &\le \int_0^{\infty}\left|f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s + \int_0^{\infty}\left|f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s + |u_0| + \Gamma(\alpha)\sum_{t<t_i} L_i t_i^{1-\alpha} < \infty.\end{aligned} \tag{39}$$

Hence, according to (37)–(39) and Lebesgue dominated convergence theorem, we can easily get

$$\left\|\mathrm{Tu}_n - \mathrm{Tu}\right\|_{\mathrm{PC}^1} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{40}$$

Therefore, T: $\mathrm{PC}^1(J,\mathbb{R}) \longrightarrow \mathrm{PC}^1(J,\mathbb{R})$ is continuous. Secondly, choose r such that

$$r \ge \frac{2\int_0^{\infty} c(s)\mathrm{d}s + |u_0| + \Gamma(\alpha)\sum_{t<t_i} L_i t_i^{1-\alpha}}{1 - 2\int_0^{\infty}\left(\left(1+s^{\alpha-1}\right)a(s) + b(s) + (1+s)^{\alpha-1}s^{\beta}/\Gamma(\beta+1)e(s)\right)\mathrm{d}s}, \tag{41}$$

and let $B_r = \left\{\|u \in \mathrm{PC}^1 \|u\|_{PC^1} \le r\right\} \subset \mathrm{PC}^1(J,\mathbb{R})$.

For any $u(t) \in B_r$, by (41) and condition $(H1)$, we have

$$\begin{aligned}\left|\frac{\mathrm{Tu}(t)}{1+t^{\alpha-1}}\right| &\le \frac{2}{\Gamma(\alpha)}\int_0^{\infty}\left|f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s + \frac{|u_0|}{\Gamma(\alpha)} + \sum_{t<t_i} L_i t_i^{1-\alpha} \\ &\le \frac{2\|u\|_{\mathrm{PC}^1}}{\Gamma(\alpha)}\int_0^{+\infty}\left[\left(1+s^{\alpha-1}\right)a(s) + b(s)\right]\mathrm{d}s + \frac{2}{\Gamma(\alpha)}\int_0^{+\infty} c(s)\mathrm{d}s \\ &\quad + \frac{2\|u\|_{\mathrm{PC}^1}}{\Gamma(\alpha)}\int_0^{+\infty}\frac{(1+s)^{\alpha-1}s^{\beta}}{\Gamma(\beta+1)}e(s)\mathrm{d}s + \frac{|u_0|}{\Gamma(\alpha)} + \sum_{t<t_i} L_i t_i^{1-\alpha} \\ &\le r, \\ \left|D^{\alpha-1}\mathrm{Tu}(t)\right| &\le 2\int_0^{\infty}\left|f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s + |u_0| + \Gamma(\alpha)\sum_{t<t_i} L_i t_i^{1-\alpha} \\ &\le 2\|u\|_{\mathrm{PC}^1}\int_0^{+\infty}\left[\left(1+s^{\alpha-1}\right)a(s) + b(s) + \frac{(1+s)^{\alpha-1}s^{\beta}}{\Gamma(\beta+1)}e(s)\right]\mathrm{d}s \\ &\quad + 2\int_0^{+\infty} c(s)\mathrm{d}s + |u_0| + \Gamma(\alpha)\sum_{t<t_i} L_i t_i^{1-\alpha} \\ &\le r.\end{aligned} \tag{42}$$

So, $\|\mathrm{Tu}\|_{PC^1} \le r$ and $T\colon B_r \longrightarrow B_r$.

Thirdly, we show that TB_r is uniformly bounded. From (38) and (39), we know

$$\sup_{t\in J}\left|\frac{\mathrm{Tu}(t)}{1+t^{\alpha-1}}\right| < \infty,$$
$$\sup_{t\in J}\left|D^{\alpha-1}\mathrm{Tu}(t)\right| < \infty. \tag{43}$$

So, for $u \in B_r$, it is easy to know that $\|\mathrm{Tu}\|_{PC^1} < \infty$. Hence, TB_r is uniformly bounded.

Fourth, we prove that for any $u(t) \in B_r$, $(\mathrm{Tu}(t)/1+t^{\alpha-1})$ and $D^{\alpha-1}\mathrm{Tu}(t)$ are equicontinuous on any interval J_k.

For any $u(t) \in B_r$, $t_1, t_2 \in J_k$ $(k = 0, 1, 2, \ldots, m)$, $t_1 < t_2$, we have

$$\begin{aligned}
\left|\frac{\mathrm{Tu}(t_2)}{1+t_2^{\alpha-1}} - \frac{\mathrm{Tu}(t_1)}{1+t_1^{\alpha-1}}\right| \le{}& \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\left|f\left(s,u(s),J^\beta u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\left|f\left(s,u(s),J^\beta u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\left|f\left(s,u(s),J^\beta u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\\
&+\frac{|u_0|}{\Gamma(\alpha)}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right| + \frac{\sum_{t<t_2}L_i t_i^{1-\alpha}}{\Gamma(\alpha)}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right| \longrightarrow 0 \text{ if } t_2 \longrightarrow t_1,\\
\left|D^{\alpha-1}\mathrm{Tu}(t_2) - D^{\alpha-1}\mathrm{Tu}(t_1)\right| \le{}& \int_{t_1}^{t_2}\left|f\left(s,u(s),J^\beta u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s + \left|\Gamma(\alpha)\sum_{t_2<t_i} I_i t_i^{1-\alpha} - \Gamma(\alpha)\sum_{t_1<t_i} I_i t_i^{1-\alpha}\right| \longrightarrow 0 \text{ if } t_2 \longrightarrow t_1.
\end{aligned} \tag{44}$$

Therefore, for any $u(t) \in B_r$, $\mathrm{Tu}(t)/1+t^{\alpha-1}$ and $D^{\alpha-1}\mathrm{Tu}(t)$ are equicontinuous on any interval J_k.

Fifth, we need to verify that condition (3) in Lemma 2 is satisfied. It means that we need to verify $\mathrm{Tu}(t)/1+t^{\alpha-1}$ and $D^{\alpha-1}\mathrm{Tu}(t)$ are equiconvergent at $t = J_k$ $(k = 1, 2, \ldots, m, \ldots,)$ and $t = \infty$ for any $u \in B_r$. We have

$$\begin{aligned}
\lim_{t\longrightarrow\infty}\frac{|\mathrm{Tu}(t)|}{1+t^{\alpha-1}} \le{}& \lim_{t\longrightarrow\infty}\left[\frac{2}{\Gamma(\alpha)}\int_0^\infty\left|f\left(s,u(s),J^\beta u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s\frac{t^{\alpha-1}}{1+t^{\alpha-1}} + \frac{|u_0|}{\Gamma(\alpha)}\frac{t^{\alpha-1}}{1+t^{\alpha-1}} + \sum_{t<t_i}L_i t_i^{1-\alpha}\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\right]\\
\le{}& \left(\frac{2\|u\|_{PC^1}}{\Gamma(\alpha)}\int_0^\infty\left(\left(1+s^{\alpha-1}\right)a(s) + b(s) + \frac{(1+s)^{\alpha-1}s^\beta}{\Gamma(\beta+1)}e(s)\right)\mathrm{d}s + \frac{2}{\Gamma(\alpha)}\int_0^\infty c(s)\mathrm{d}s + \frac{u_0}{\Gamma(\alpha)} + \sum_{t<t_i}L_i t_i^{1-\alpha}\right)\\
&\lim_{t\longrightarrow\infty}\frac{t^{\alpha-1}}{1+t^{\alpha-1}} < \infty,\\
\lim_{t\longrightarrow\infty}\left|D^{\alpha-1}\mathrm{Tu}(t)\right| <{}& \lim_{t\longrightarrow\infty}\left[\int_0^\infty\left|f\left(s,u(s),J^\beta u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s + \int_0^\infty\left|f\left(s,u(s),J^\beta u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s + |u_0| + \Gamma(\alpha)\sum_{t<t_i}L_i t_i^{1-\alpha}\right]\\
<{}& \lim_{t\longrightarrow\infty}\left[2\|u\|_{PC^1}\int_0^\infty\left(\left(1+s^{\alpha-1}\right)a(s) + b(s) + \frac{(1+s)^{\alpha-1}s^\beta}{\Gamma(\beta+1)}e(s)\right)\mathrm{d}s + 2\int_0^\infty c(s)\mathrm{d}s + |u_0| + \Gamma(\alpha)\sum_{t<t_i}L_i t_i^{1-\alpha}\right]\\
<{}& \infty.
\end{aligned} \tag{45}$$

Hence, TB_r is equiconvergent at infinity.

Then, we prove that $Tu(t)/1+t^{\alpha-1}$ and $D^{\alpha-1}Tu(t)$ are equiconvergent at $t \longrightarrow t_k^+$ $(k=0,1,2,\ldots,)$. We have

$$\lim_{t\longrightarrow t_k^+}\Bigg|\frac{Tu(t)}{1+t_k^{\alpha-1}}+\frac{1}{\Gamma(\alpha)}\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{1+t_k^{\alpha-1}}f\Big(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\Big)ds-\frac{t_k^{\alpha-1}}{\Gamma(\alpha)\big(1+t_k^{\alpha-1}\big)}$$
$$\int_0^{\infty}f\Big(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\Big)ds$$
$$-\frac{t_k^{\alpha-1}}{\Gamma(\alpha)\big(1+t_k^{\alpha-1}\big)}u_0+\frac{t_k^{\alpha-1}}{1+t_k^{\alpha-1}}\sum_{t_k<t_i}I_i t_i^{1-\alpha}\Bigg|=0,$$
$$\lim_{t\longrightarrow t_k^+}\Bigg|D^{\alpha-1}Tu(t)+\int_0^{t_k}f\Big(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\Big)ds-\int_0^{\infty}f\Big(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\Big)ds-u_0+\Gamma(\alpha)\sum_{t_k<t_i}I_i t_i^{1-\alpha}\Bigg|=0. \tag{46}$$

Therefore, $Tu(t)/1+t^{\alpha-1}$ and $D^{\alpha-1}Tu(t)$ are equiconvergent at $t=J_k$ $(k=1,2,\ldots,m,\ldots,)$ and $t=\infty$ for any $u\in B_r$. By using Lemma 2, we obtain that TB_r is relatively compact, that is, T is a compact operator.

Therefore, Schauder's fixed-point theorem implies that problem (1) has at least one solution in B_r.

Our second result is based on Altman fixed-point theorem. □

Theorem 4. *Assume $(H2)$ and the following condition hold:*

$(H3)$ For $f\in C([0,+\infty)\times R\times R\times R, R)$, there exist nonnegative functions $a(t)$, $b(t)$, $c(t)$ defined on $[0,\infty)$ and constants $p,q,l\geq 0$ such that

$$|f(t,x,y,z)|\leq a(t)+b(t)|x|^p+c(t)|y|^q+e(t)|z|^l,$$
$$\int_0^{+\infty}a(t)dt=a^*<\infty,\quad \int_0^{+\infty}\big(1+t^{\alpha-1}\big)^p b(t)dt=b^*<+\infty,\quad \int_0^{+\infty}c(t)dt=c^*<+\infty, \tag{47}$$
$$\int_0^{+\infty}\left(\frac{(1+t)^{\alpha-1}t^{\beta}}{\Gamma(\beta+1)}\right)^l e(t)dt=e^*<\infty.$$

If $0\leq p,q,l<1$, then problem (1) has at least one solution $u(t)$ in $PC^1(J,\mathbb{R})$.

Proof. Let us choose

$$R\geq\max\left\{12a^*,\big(12b^*\big)^{1/1-p},\big(12c^*\big)^{1/1-q},\big(12e^*\big)^{1/1-l},6|u_0|,6\Gamma(\alpha)\sum_{t<t_i}L_i t_i^{1-\alpha}\right\}, \tag{48}$$

and define $U = \{u \in PC^1 | \|u\|_{PC^1} \le R\}$. According to Theorem 3, we know $T\colon U \longrightarrow U$ is a completely continuous operator. For any $u \in \partial U$, by $(H3)$, we have

$$\begin{aligned}
\frac{\mathrm{Tu}(t)}{1+t^{\alpha-1}} &\le \frac{2}{\Gamma(\alpha)}\int_0^{\infty}\left|f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s + \frac{|u_0|}{\Gamma(\alpha)} + \sum_{t<t_i} L_i t_i^{1-\alpha} \\
&\le \frac{2}{\Gamma(\alpha)}\left(\int_0^{\infty}\left[a(s)+b(s)|u(s)|^p + c(s)\left|D^{\alpha-1}u(s)\right|^q + e(s)\left|J^{\beta}u(s)\right|^l\right]\mathrm{d}s\right) + \frac{|u_0|}{\Gamma(\alpha)} + \sum_{t<t_i} L_i t_i^{1-\alpha} \\
&\le \frac{2}{\Gamma(\alpha)}\left(a^* + \int_0^{\infty} b(s)\left(1+s^{\alpha-1}\right)^p \frac{|u(s)|^p}{\left(1+s^{\alpha-1}\right)^p}\mathrm{d}s + \int_0^{\infty} c(s)\|u\|_{PC^1}^q \mathrm{d}s + \int_0^{+\infty} e(s)\left(\frac{(1+s)^{\alpha-1}s^{\beta}}{\Gamma(\beta+1)}\right)^l \|u\|_{PC^1}^l \mathrm{d}s\right) \\
&\quad + \frac{|u_0|}{\Gamma(\alpha)} + \sum_{t<t_i} L_i t_i^{1-\alpha} \\
&\le \frac{2}{\Gamma(\alpha)}\left(a^* + b^*\|u\|_{PC^1}^p + c^*\|u\|_{PC^1}^q + e^*\|u\|_{PC^1}^l\right) + \frac{|u_0|}{\Gamma(\alpha)} + \sum_{t<t_i} L_i t_i^{1-\alpha} \\
&\le \frac{2}{\Gamma(\alpha)}\left(\frac{R}{12}+\frac{R}{12}+\frac{R}{12}+\frac{R}{12}\right) + \frac{R}{6\Gamma(\alpha)} + \frac{R}{6\Gamma(\alpha)} \\
&< R,
\end{aligned} \tag{49}$$

$$\begin{aligned}
\left|D^{\alpha-1}\mathrm{Tu}(t)\right| &\le 2\int_0^{\infty}\left|f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s + |u_0| + \Gamma(\alpha)\sum_{t<t_i} L_i t_i^{1-\alpha} \\
&\le 2\left(a^* + b^*\|u\|_{PC^1}^p + c^*\|u\|_{PC^1}^q + e^*\|u\|_{PC^1}^l\right) + |u_0| + \Gamma(\alpha)\sum_{t<t_i} L_i t_i^{1-\alpha} \\
&\le 2\left(\frac{R}{12}+\frac{R}{12}+\frac{R}{12}+\frac{R}{12}\right) + \frac{R}{6} + \frac{R}{6} = R.
\end{aligned} \tag{50}$$

Thus, from (49) and (50), we have $\mathrm{TU} \subset U$ and $\|\mathrm{Tu}\|_{PC^1} \le \|u\|_{PC^1}$, $\forall u \in \partial U$. So, by Theorem 2, we know that problem (1) has at least one solution. □

Theorem 5. *Assume that conditions (H2) and (H3) are satisfied. If* $p = q = l = 1$, $(1+\Gamma(\alpha))(b^* + c^*) < \Gamma(\alpha)$, *then problem (1) has at least one solution.*

Proof. Let us take

$$R > \frac{|u_0| + \Gamma(\alpha)\sum_{t<t_i} L_i t_i^{1-\alpha} + 2a^*}{1 - 2\left(b^* + c^* + e^*\right)}, \tag{51}$$

and define $U = \{u \in PC^1 | \|u\|_{PC^1} < R\}$.

For any $u \in \partial U$, we have

$$\begin{aligned}
\frac{\mathrm{Tu}(t)}{1+t^{\alpha-1}} &\le \frac{2}{\Gamma(\alpha)}\int_0^{\infty}\left|f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s+\frac{|u_0|}{\Gamma(\alpha)}+\sum_{t<t_i}L_i t_i^{1-\alpha}\\
&\le \frac{2}{\Gamma(\alpha)}\left(\int_0^{\infty}\left[a(s)+b(s)|u(s)|+c(s)\left|D^{\alpha-1}u(s)\right|+e(s)\left|J^{\beta}u(s)\right|\right]\mathrm{d}s\right)+\frac{|u_0|}{\Gamma(\alpha)}+\sum_{t<t_i}L_i t_i^{1-\alpha}\\
&\le \frac{2}{\Gamma(\alpha)}a^*+\left[\int_0^{\infty}b(s)\left(1+s^{\alpha-1}\right)\frac{|u(s)|}{\left(1+s^{\alpha-1}\right)}\mathrm{d}s+\int_0^{\infty}c(s)\|u\|_{\mathrm{PC}^1}\mathrm{d}s+\int_0^{\infty}e(s)\frac{|u(s)|}{\left(1+s^{\alpha-1}\right)}\frac{(1+s)^{\alpha-1}s^{\beta}}{\Gamma(\beta+1)}\right]+\frac{|u_0|}{\Gamma(\alpha)}+\sum_{t<t_i}L_i t_i^{1-\alpha}\\
&\le \frac{2}{\Gamma(\alpha)}\left(a^*+b^*\|u\|_{\mathrm{PC}^1}+c^*\|u\|_{\mathrm{PC}^1}+e^*\|u\|_{\mathrm{PC}^1}\right)+\frac{|u_0|}{\Gamma(\alpha)}+\sum_{t<t_i}L_i t_i^{1-\alpha}\\
&\le \frac{2}{\Gamma(\alpha)}\left(a^*+b^*R+c^*R+e^*R\right)+\frac{|u_0|}{\Gamma(\alpha)}+\sum_{t<t_i}L_i t_i^{1-\alpha}\\
&< R,
\end{aligned} \tag{52}$$

$$\begin{aligned}
\left|D^{\alpha-1}\mathrm{Tu}(t)\right| &\le 2\int_0^{\infty}\left|f\left(s,u(s),J^{\beta}u(s),D^{\alpha-1}u(s)\right)\right|\mathrm{d}s+|u_0|+\Gamma(\alpha)\sum_{t<t_i}L_i t_i^{1-\alpha}\\
&\le 2\left(a^*+b^*\|u\|_{\mathrm{PC}^1}+c^*\|u\|_{\mathrm{PC}^1}+e^*\|u\|_{\mathrm{PC}^1}\right)+|u_0|+\Gamma(\alpha)\sum_{t<t_i}L_i t_i^{1-\alpha}\\
&\le 2\left(a^*+b^*R+c^*R+e^*R\right)+|u_0|+\Gamma(\alpha)\sum_{t<t_i}L_i t_i^{1-\alpha}\\
&< R.
\end{aligned} \tag{53}$$

Thus, from (52) and (53), we have $\mathrm{TU}\subset U$ and $\|\mathrm{Tu}\|_{\mathrm{PC}^1}\le\|u\|_{\mathrm{PC}^1}$, $\forall u\in\partial U$. So, by Theorem 2, we know that problem (1) has at least one solution. □

Remark 1. If we use other conditions instead of the condition "$p=q=1$", for example, $0\le p<1, q=1$ or $p>1, q=1$ or $0\le q<1, p=1$ or $q>1, p=1$ or $p,q>1$ or $0\le p<1, q>1$ or $0\le q<1, p>1$, and choose proper R, respectively, then we can obtain the same result. The proof is similar to Theorem 4 or Theorem 5, so we omit it.

4. Example

In this section, we give an example to illustrate of our main result.

Example 1. Consider the following impulsive boundary value problem of fractional order:

$$\begin{cases}
D_{0+}^{3/2}u(t)+\dfrac{\ln\left(\left(1+\left|D_{0+}^{1/2}u(t)\right|\right)\right)}{20\left(1+t^2\right)}+\dfrac{\sqrt{\left|u(t)D_{0+}^{1/2}u(t)\right|}}{20e^{\sqrt{t}}}+\dfrac{\left|J^{3/2}u(t)\right|}{20e^{t}}=0, \quad t\in[0,\infty)\setminus\left\{\dfrac{1}{2}\right\},\\
\triangle u\left(\dfrac{1}{2}\right)=I\left(u\left(\dfrac{1}{2}\right)\right), \quad t=\dfrac{1}{2},\\
u(0)=u'(0)=0, \quad D_{0+}^{1/2}u(\infty)=u_0,
\end{cases} \tag{54}$$

where $\alpha = 3/2$, $f(t, x, y, z) = \ln(1 + |y|)/20(1 + t^2) + \sqrt{|xy|}/20e^{\sqrt{t}} + |z|/20e^t$, $k = 1, t_1 = 1/2$.

Let $I(u) = 1/(u + 1/u)$. Then, we have

$$|f(t, x, y)| \leq \frac{1}{40e^{\sqrt{t}}}|x| + \left(\frac{1}{20\left(1 + t^2\right)} + \frac{1}{40e^{\sqrt{t}}}\right)|y| + \frac{1}{20e^t}|z|,$$
$$I(u) = \frac{1}{|u| + 1/|u|} \leq 1. \tag{55}$$

By computing, we know that

$$\int_0^{+\infty}\left[\left(1 + t^{\alpha-1}\right)a(t) + b(t)\right]\mathrm{d}t = \int_0^{+\infty}\left[\left(1 + t^{1/2}\right)\frac{1}{40e^{\sqrt{t}}} + \frac{1}{20\left(1 + t^2\right)} + \frac{1}{40e^{\sqrt{t}}}\right]\mathrm{d}t = \frac{1}{5} + \frac{\pi}{40} \approx 0.2785 < \infty,$$
$$\int_0^{+\infty}\frac{(1 + t)^{\alpha-1}t^{\beta}}{\Gamma(\beta + 1)}e(t)\mathrm{d}t = \int_0^{+\infty}\frac{(1 + t)^{1/2}t^{3/2}}{\Gamma(5/2)}\frac{1}{20e^t} \approx 64.5850 < \infty. \tag{56}$$

Thus, the conditions of Theorem 3 are satisfied, and hence problem (54) has at least one solution.

Remark 2. By theorems in [9, 10, 13], this problem could not be solved.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On Highly Dimensional Elastic and Nonelastic Interaction between Internal Waves in Straight and Varying Cross-Section Channels

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This manuscript studies the computational solutions of the highly dimensional elastic and nonelastic interaction between internal waves through the fractional nonlinear $(4 + 1)$-dimensional Fokas equation. This equation is considered as the extension model of the two-dimensional Davey–Stewartson (DS) and Kadomtsev–Petviashvili (KP) equations to a four spatial dimensions equation with time domain. The modified Khater method is employed along the Atangana–Baleanu (AB) derivative operator to construct many novel explicit wave solutions. These solutions explain more physical and dynamical behavior of that kind of the interaction. Moreover, 2D, 3D, contour, and stream plots are demonstrated to explain the detailed dynamical characteristics of these solutions. The novelty of our paper is shown by comparing our results with those obtained in previous published research papers.

1. Introduction

Internal waves are waves that spread inside a stream, with gradients of intensity [1–3]. The surface gravity waves pass along the broad pressure boundary between air and water, while internal waves migrate inside the ocean over gradients of intensity [4–7]. Perturbations of these gradients of intensity are preserved by momentum, which creates a propagating motion [8–11].

Globally, internal waves play a significant role in the ocean, providing nutrients to surface waters that facilitate the growth of phytoplankton, the foundation of the ocean food chain [12–15]. Created primarily by the tide's interaction with ocean floor and water topography, internal waves may bring the energy from these forces through the entire ocean basins [16, 17]. As internal waves pass through the continental shelf, they interact with the topography, and as the gravity of the surface steepens and splits on the sea, internal waves steep their energy in the shelf and dissipate it [18, 19]. When the internal waves rise, they turn into nonlinear waves of fluid that may assume several forms (e.g., solitons, bores, and boluses), all of which have the potential to bring deep water that has different properties (probably colder, higher in nutrients, lower in oxygen, lower in pH) across the shelf and into shallower waters [20–22].

Depending on the potential of the nonlinear partial differential equation to describe several complicated processes in diverse fields such as physiology, plasma physics, hydrodynamics, fluid mechanics, and optics, numerous precise and computational schemes such as in [23–26] have been developed. Using inspired schemes, computational and technical advances are seen as the basic usefulness of solving these phenomena [27–31]. Such schemes have recently been regarded as simple methods for discovering the different formulas of moving wave solutions to these dynamic phenomena [32–34]. However, in the nonlinear partial differential equation (NLPDE), with an integer instruction, several researchers have struggled to extract and formulate certain complex phenomena [35, 36]. The fractional equation is then deemed an appropriate solution to this issue because it includes a nonlocal property that is not NLPDE-based with an integer [37–40].

In this research, we study the nonlinear fractional $(4+1)$-dimensional Fokas model that is mathematically given by [41–43]

$$4D_t^{\alpha}U_x - U_{xxxy} + U_{xyyy} + 12U_yU_x + 12UU_{xy} - 6U_{wz} = 0, \quad (0<\alpha<1), \tag{1}$$

where U is the function of the elastic and nonelastic interaction between internal waves in straight and varying cross-section channels. Implementation of the following AB-derivative definitions on equation (1) with the following wave transformation $U(x,y,z,w,t) = \psi(\zeta), \zeta = (((\alpha-1)\lambda t^{-\alpha n})/B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/1-\alpha))^n\Gamma(1-\alpha n)) + k_4 w + k_1 x + k_2 y + k_3 z$, where k_i and λ $(i=1,2,3,4)$ are arbitrary constants [37–40], while $B(\alpha)$ is a normalized function, converts the fractional PDE into the next integer order ODE:

$$\alpha_1\psi'' + \alpha_2\psi'''' + \alpha_3\psi'^2 + \alpha_4\psi\psi'' = 0, \tag{2}$$

where $\alpha_1 = (4k_1\lambda - 6k_3k_4), \alpha_2 = (k_1k_2^3 - k_1^3k_2), \alpha_3 = 12k_1k_2$, and $\alpha_4 = -6k_3k_4$.

The remaining parts of our research paper are organized as follows: Section 2 employs the modified Khater system [44–49] to provide the nonlinear fractional Fokas model with novel solitary solutions. Section 3 describes the outcomes and provides the physical description of the sketches seen. This work is concluded in Section 4.

2. Applications

Usage of the modified Khater technique via the concepts of homogeneous equilibrium on equation (2) provides general solutions:

$$\psi(\zeta) = \sum_{i=1}^{m} a_i K^{i\varphi(\zeta)} + \sum_{i=1}^{m} b_i K^{-i\varphi(\zeta)} + a_0 = a_1K^{\varphi(\zeta)} + a_2K^{2\varphi(\zeta)} + a_0 + b_2K^{-2\varphi(\zeta)} + b_1K^{-\varphi(\zeta)}, \tag{3}$$

where $a_i, b_j (i,j = 0,1,2,\ldots,), a_m \neq 0$, or $b_m \neq 0$. Additionally, $\varphi(\zeta)$ is the solution function of $\varphi'(\zeta) = (1/\ln(K))[\delta + \rho K^{f(\zeta)} + \varkappa K^{-f(\zeta)}]$ where δ, ρ, and $\varkappa$ are arbitrary constants. Using equation (3) through its auxiliary equation in the modified Khater technique's framework gives the following families for the above-mentioned arbitrary constants.

Family I:

$$\begin{aligned} &a_1 \longrightarrow 0,\\ &a_2 \longrightarrow 0,\\ &b_1 \longrightarrow \frac{b_2\delta}{\varkappa},\\ &\alpha_1 \longrightarrow -\frac{\alpha_4\left(12a_0\varkappa^2 - b_2\delta^2 - 8b_2\rho\varkappa\right)}{12\varkappa^2},\\ &\alpha_2 \longrightarrow -\frac{\alpha_4 b_2}{12\varkappa^2},\\ &\alpha_3 \longrightarrow \alpha_4. \end{aligned} \tag{4}$$

Family II:

$$\begin{aligned} &a_1 \longrightarrow \frac{a_2\delta}{\rho},\\ &b_1 \longrightarrow 0,\\ &b_2 \longrightarrow 0,\\ &\alpha_1 \longrightarrow -\frac{\alpha_4\left(-a_2\delta^2 + 12a_0\rho^2 - 8a_2\rho\varkappa\right)}{12\rho^2},\\ &\alpha_2 \longrightarrow -\frac{a_2\alpha_4}{12\rho^2},\\ &\alpha_3 \longrightarrow \alpha_4. \end{aligned} \tag{5}$$

Consequently, the explicit solutions of equation (1) are given in the following forms.

In case of $\delta^2 - 4\rho\varkappa < 0, \rho \neq 0$, we obtain

$$\begin{aligned} U(\xi)_{I,1} &= a_0 + \frac{2b_2\rho\left(\delta\sqrt{4\rho\varkappa-\delta^2}\,\tan\left((1/2)\xi\sqrt{4\rho\varkappa-\delta^2}\right) - \delta^2 + 2\rho\varkappa\right)}{\varkappa\left(\delta - \sqrt{4\rho\varkappa-\delta^2}\,\tan\left((1/2)\xi\sqrt{4\rho\varkappa-\delta^2}\right)\right)^2},\\ U(\xi)_{I,2} &= a_0 + \frac{2b_2\rho\left(\delta\sqrt{4\rho\varkappa-\delta^2}\,\cot\left((1/2)\xi\sqrt{4\rho\varkappa-\delta^2}\right) - \delta^2 + 2\rho\varkappa\right)}{\varkappa\left(\delta - \sqrt{4\rho\varkappa-\delta^2}\,\cot\left((1/2)\xi\sqrt{4\rho\varkappa-\delta^2}\right)\right)^2},\\ U(\xi)_{II,1} &= \frac{a_2\left(-\left(\delta^2-4\rho\varkappa\right)\sec^2\left((1/2)\xi\sqrt{4\rho\varkappa-\delta^2}\right) - 4\rho\varkappa\right)}{4\rho^2} + a_0,\\ U(\xi)_{II,2} &= \frac{a_2\left(-\left(\delta^2-4\rho\varkappa\right)\csc^2\left((1/2)\xi\sqrt{4\rho\varkappa-\delta^2}\right) - 4\rho\varkappa\right)}{4\rho^2} + a_0. \end{aligned} \tag{6}$$

In case of $\delta^2 - 4\rho\varkappa > 0, \rho \neq 0$, we obtain

$$U(\xi)_{I,3} = a_0 - \frac{2b_2\rho\left(\delta\sqrt{\delta^2 - 4\rho\varkappa}\tanh\left((1/2)\xi\sqrt{\delta^2 - 4\rho\varkappa}\right) + \delta^2 - 2\rho\varkappa\right)}{\varkappa\left(\sqrt{\delta^2 - 4\rho\varkappa}\tanh\left((1/2)\xi\sqrt{\delta^2 - 4\rho\varkappa}\right) + \delta\right)^2}, \tag{7}$$

$$U(\xi)_{I,4} = a_0 - \frac{2b_2\rho\left(\delta\sqrt{\delta^2 - 4\rho\varkappa}\coth\left((1/2)\xi\sqrt{\delta^2 - 4\rho\varkappa}\right) + \delta^2 - 2\rho\varkappa\right)}{\varkappa\left(\sqrt{\delta^2 - 4\rho\varkappa}\coth\left((1/2)\xi\sqrt{\delta^2 - 4\rho\varkappa}\right) + \delta\right)^2}, \tag{8}$$

$$U(\xi)_{II,3} = \frac{a_2\left(-\left(\delta^2 - 4\rho\varkappa\right)\operatorname{sech}^2\left((1/2)\xi\sqrt{\delta^2 - 4\rho\varkappa}\right) - 4\rho\varkappa\right)}{4\rho^2} + a_0, \tag{9}$$

$$U(\xi)_{II,4} = \frac{a_2\left(\left(\delta^2 - 4\rho\varkappa\right)\operatorname{csch}^2\left((1/2)\xi\sqrt{\delta^2 - 4\rho\varkappa}\right) - 4\rho\varkappa\right)}{4\rho^2} + a_0. \tag{10}$$

In case of $\rho\varkappa > 0, \varkappa \neq 0, \rho \neq 0,$ and $\delta = 0$, we obtain

$$\begin{aligned} U(\xi)_{I,5} &= a_0 + \frac{b_2\rho\cot^2(\xi\sqrt{\rho\varkappa})}{\varkappa}, \\ U(\xi)_{I,6} &= a_0 + \frac{b_2\rho\tan^2(\xi\sqrt{\rho\varkappa})}{\varkappa}, \\ U(\xi)_{II,5} &= \frac{a_2\varkappa\tan^2(\xi\sqrt{\rho\varkappa})}{\rho} + a_0, \\ U(\xi)_{II,6} &= \frac{a_2\varkappa\cot^2(\xi\sqrt{\rho\varkappa})}{\rho} + a_0. \end{aligned} \tag{11}$$

In case of $\rho\varkappa < 0, \varkappa \neq 0, \rho \neq 0,$ and $\delta = 0$, we obtain

$$\begin{aligned} U(\xi)_{I,7} &= a_0 + \frac{b_2\rho\cot^2(\xi\sqrt{\rho}\sqrt{\varkappa})}{\varkappa}, \\ U(\xi)_{I,8} &= a_0 + \frac{b_2\rho\tan^2(\xi\sqrt{\rho}\sqrt{\varkappa})}{\varkappa}, \\ U(\xi)_{II,7} &= \frac{a_2\varkappa\tan^2(\xi\sqrt{\rho}\sqrt{\varkappa})}{\rho} + a_0, \\ U(\xi)_{II,8} &= \frac{a_2\varkappa\cot^2(\xi\sqrt{\rho}\sqrt{\varkappa})}{\rho} + a_0. \end{aligned} \tag{12}$$

In case of $\delta = 0$ and $\varkappa = -\rho$, we obtain

$$U(\xi)_{I,9} = a_0 + b_2\tanh^2(\xi\varkappa), \tag{13}$$

$$U(\xi)_{II,9} = a_2\coth^2(\xi\varkappa) + a_0. \tag{14}$$

In case of $\delta = (\varkappa/2) = \kappa$ and $\rho = 0$, we obtain

$$U(\xi)_{I,10} = a_0 + \frac{b_2 e^{\kappa\xi}}{2\left(e^{\kappa\xi} - 2\right)^2}. \tag{15}$$

In case of $\delta = \rho = \kappa$ and $\varkappa = 0$, we obtain

$$U(\xi)_{II,10} = \frac{1}{4}a_2\operatorname{csch}^2\left(\frac{\kappa\xi}{2}\right) + a_0. \tag{16}$$

In case of $\varkappa = 0, \delta \neq 0,$ and $\rho \neq 0$, we obtain

$$U(\xi)_{II,11} = \frac{2a_2\delta^2 e^{\delta\xi}}{\rho\left(\rho e^{\delta\xi} - 2\right)^2} + a_0. \tag{17}$$

In case of $\delta = \rho = 0$ and $\varkappa \neq 0$, we obtain

$$U(\xi)_{I,11} = a_0 + \frac{b_2}{\xi^2\varkappa^2}. \tag{18}$$

In case of $\delta = \varkappa = 0$ and $\rho \neq 0$, we obtain

$$U(\xi)_{II,12} = \frac{a_2}{\xi^2\rho^2} + a_0. \tag{19}$$

In case of $\delta = 0$ and $\varkappa = \rho$, we obtain

$$\begin{aligned} U(\xi)_{I,12} &= a_0 + b_2\cot^2(C + \xi\varkappa), \\ U(\xi)_{II,13} &= a_2\tan^2(C + \xi\varkappa) + a_0. \end{aligned} \tag{20}$$

In case of $\rho = 0, \delta \neq 0,$ and $\varkappa \neq 0$, we obtain

$$U(\xi)_{I,13} = \frac{a_0\varkappa\left(\varkappa - \delta e^{\delta\xi}\right)^2 + b_2\delta^3 e^{\delta\xi}}{\varkappa\left(\varkappa - \delta e^{\delta\xi}\right)^2}. \tag{21}$$

In case of $\delta^2 - 4\rho\varkappa = 0$, we obtain

$$\begin{aligned} U(\xi)_{I,14} &= a_0 - \frac{b_2\delta^3\xi(\delta\xi + 4)}{4\varkappa^2(\delta\xi + 2)^2}, \\ U(\xi)_{II,14} &= \frac{2a_2\varkappa(\delta\xi + 2)\left(2\rho\varkappa(\delta\xi + 2) - \delta^3\xi\right)}{\delta^4\xi^2\rho} + a_0. \end{aligned} \tag{22}$$

3. Results and Discussion

This section shows our obtained solutions and their novelty. Also, we compare our obtained solutions with those of previously published articles to show the similarity and

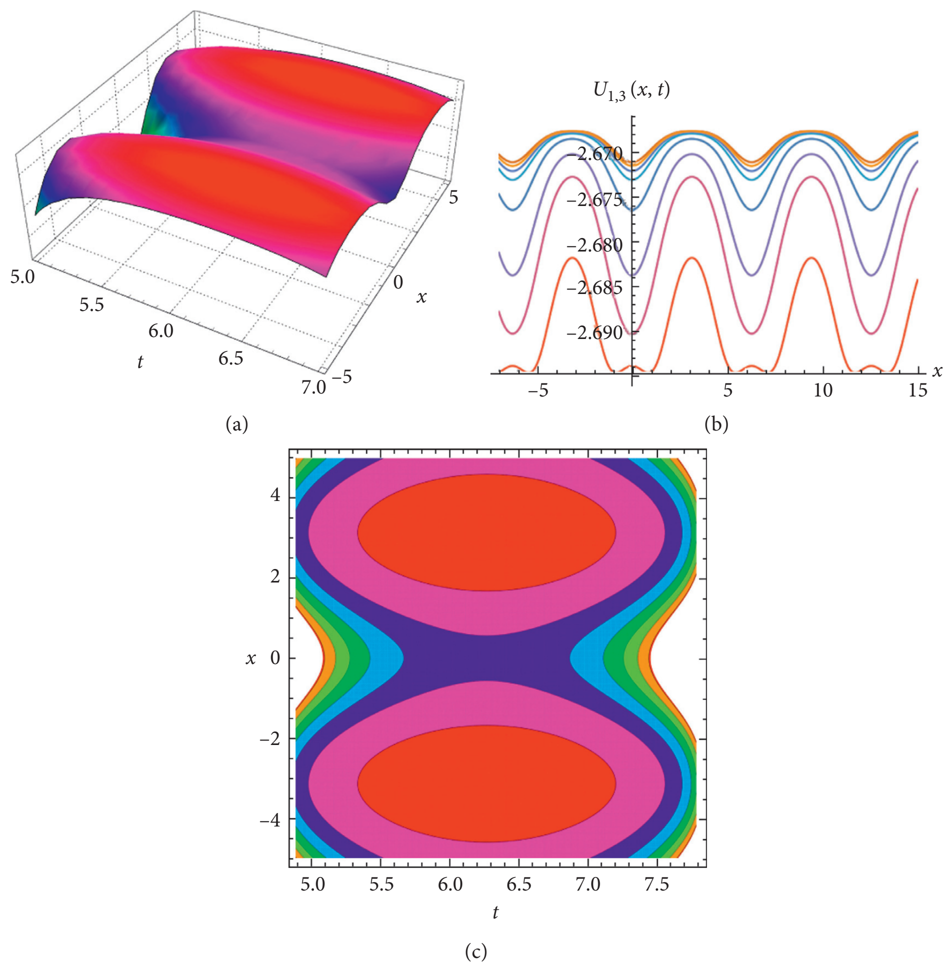


Figure 1: Solitary wave solutions equation (7) in three, two, and contour plots.

difference between our and their solutions. Our discussion is divided into three main parts, which are the used analytical method, obtained solutions, and figure interpretation:

(1) The used computational scheme:

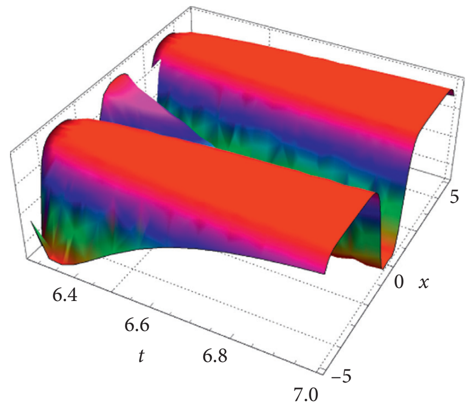
The modified Khater method have been used for the first time for applying to the fractional nonlinear $(4+1)$-dimensional Fokas equation. This modified method is considered as one of the most general analytical schemes in this field; especially, it covers more than twelve recent analytical schemes [50].

(2) The obtained solutions:

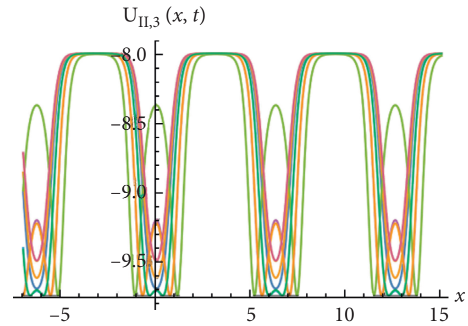
This part gives a comparison between our obtained solutions and those obtained in previously accepted papers. In [41–43] by Wan-Jun Zhang and Tie-Cheng Xia, Ruoxia Yao, Yali Shen, and Zhibin Li, and Wei Li and Yinping Liu, respectively, who applied the Hirota bilinear method, the bilinear form, and Hirota method, receptively to a fractional nonlinear $(4+1)$-dimensional Fokas equation, many distinct types of solutions for these fractional nonlinear models were obtained. All our obtained solutions of the investigated model are new and different from those obtained in [41–43].

(3) The figures interpretation:

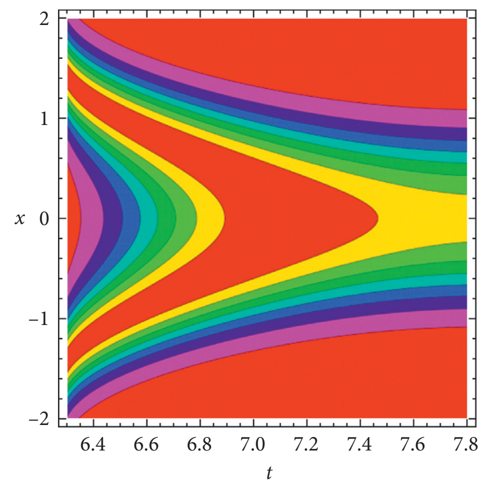
We have represented some of our obtained solutions in three distinct types of figures (3D, 2D, and contour plots) to explain kink, antikink, periodic, and singular shapes to illustrate the perspective view of the solution, the wave propagation pattern of the wave along x-axis, and the overhead view of the solution for the following values of the parameters:



(a)

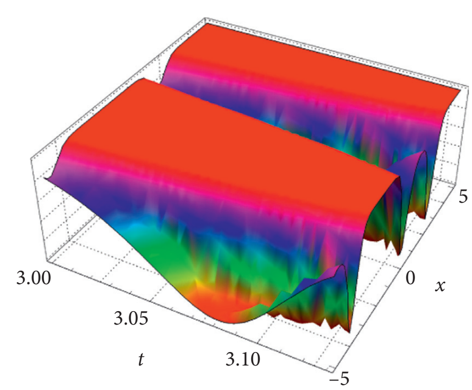


(b)

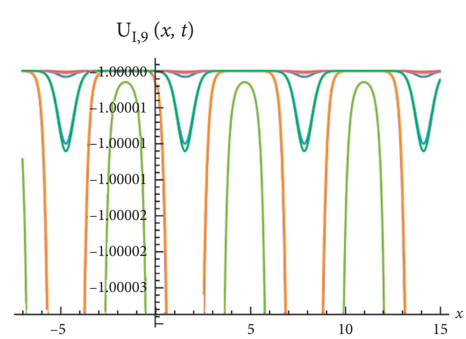


(c)

FIGURE 2: Solitary wave solutions equation (9) in three, two, and contour plots.

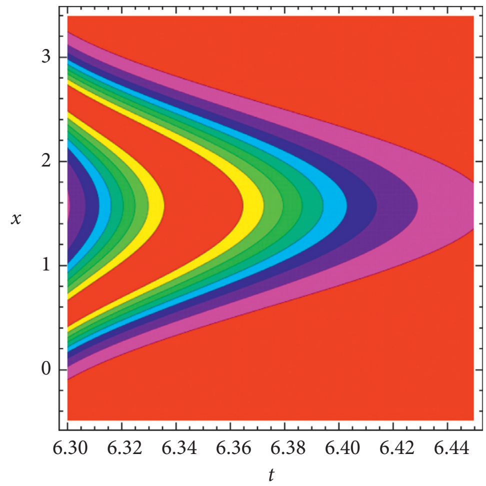


(a)



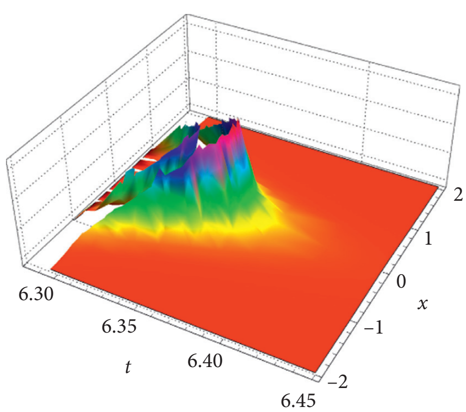
(b)

FIGURE 3: Continued.

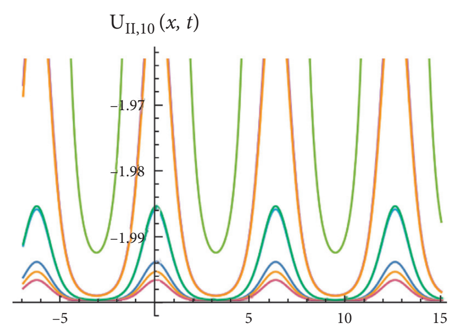


(c)

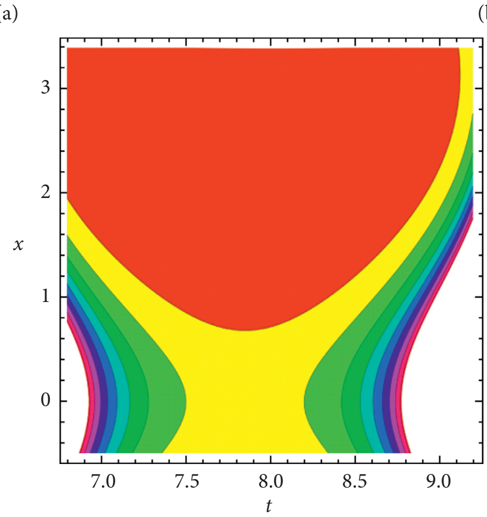
Figure 3: Solitary wave solutions equation (13) in three, two, and contour plots.



(a)



(b)



(c)

Figure 4: Solitary wave solutions equation (16) in three, two, and contour plots.

$$\left[a_0 = -2, b_2 = 1, \delta = 5, k_1 = -1, k_2 = 3, k_3 = 4, k_4 = \frac{33}{2}, \lambda = -6, \rho = 2, w = -\frac{2}{3}, y = 1, z = 2, \varkappa = 3a_2 = 7, \right.$$
$$\& a_0 = 6, \delta = 3, k_1 = 5, k_2 = 3, k_3 = 4, k_4 = \frac{33}{2}, \lambda = 9, \rho = 1, w = -\frac{2}{3}, y = 1, z = 2, \varkappa = 2 \,\& a_0 = -2, b_2 = 1,$$
$$\delta = 0, k_1 = -1, k_2 = 3, k_3 = 4, k_4 = \frac{33}{2}, \lambda = -6, \rho = -3, w = -\frac{2}{3}, y = 1, z = 2, \varkappa = 3 \,\& a_2 = 1,\ a_0 = -2, \delta = 0, \quad (23)$$
$$\left. \kappa = 2, k_1 = -1, k_2 = 3, k_3 = 4, k_4 = \frac{33}{2}, \lambda = -6, w = -\frac{2}{3}, y = 1, z = 2 \right].$$

4. Conclusion

This research paper has successfully investigated the nonlinear fractional nonlinear $(4+1)$-dimensional Fokas model via the modified Khater method that has used the Atangana–Baleanu derivative operator to convert the fractional form of the studied model to a nonlinear ordinary differential equation with an integer order. Many distinct exact traveling and solitary wave solutions have been obtained. These solutions have been illustrated via various sketches (Figures 1–4) that explain more novel properties of the considered fractional models. The accuracy and novelty of our obtained solutions have been explained. The powerfulness and effectiveness of the used techniques are also explained and verified.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

Mostafa M. A. Khater and Qiang Zheng are responsible for the analytical simulation. Haiyong Qin and Raghda A. M. Attia are responsible for the final editing and revision of the whole paper.

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Research Article

A Simulation Study on the Risk Assessment of the Modernization of Traditional Sports Culture Based on a Cellular Automaton Model

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Based on the cellular automaton model, the risk of modernization of traditional sports culture is evaluated and measured by using the methods of documentary, interview, and simulation experiments. The results of simulation experiment show that, in the complex system of the modernization of traditional sports culture, the mutation factors will lead to the sudden decline of the survival suitability of traditional sports culture and the risk of being unable to evolve; in the process of the modernization of traditional sports culture, there are two kinds of risks: self-extinction of dying out and self-extinction of homogenization; risk of dislocation failure lies in the government's support policies; risk of imbalance between fitness and scale allocation lies in modernity of derivative traditional sports culture. *Countermeasures*. The government should measure the environmental carrying capacity of the development of traditional sports culture in the region according to the regional resource situation; on this basis, heterogeneous sports activities are selectively introduced, the vitality of local traditional sports culture is enhanced, and the policy support and intervention time are reasonably controlled, to promote the balanced allocation of the suitability and scale of the derivative traditional sports culture.

1. Introduction

The traditional sports culture is the root and soul of sports development in China. But in the process of modernization, the soil on which the traditional sports culture depends has undergone great changes and is facing the risk that it may bring a survival crisis. The risk sources of traditional sports culture are diverse and complex, and different scholars also reveal and define the risk from different aspects. From the perspective of social transformation, national traditional sports culture faces the risk of lack of modern factors, elimination of living environment, and shrinking of target groups [1]; from the perspective of change, the four levels of material, institution, custom, and thought and value are inevitably facing crisis [2]; in the aspect of inheritance and development, the modernization of traditional sports culture faces many contradictions, such as nationalization and globalization, regionalization, and nationalization [3]. As far as current research is concerned, there is a lack of scientific measurement and intuitive quantitative research for the specific risk factor such as government policy. Moreover, the government policy has more countermeasures to deal with the risk, and there is less research on the risk brought by the policy itself, which provides the space for this research, that is, using cellular automata as the basis for constructing the risk assessment model, and based on explaining the composition and classification of cellular automata, the government policy is chosen as the key control parameter to carry out the simulation experiment of the risk of modernization of traditional sports culture and to measure and identify the risk categories to provide a reference for the formulation of accurate development policy.

2. Simulation Experiment of Traditional Sports Culture Modernization Risk Based on Cellular Automata

2.1. Introduction of Cellular Automata Model

2.1.1. Physical Definition of Cellular Automata. From a physical point of view, cellular automaton is a dynamic system that demonstrates a system that evolves in discrete time dimensions within a cellular space composed of cells with discrete, finite states, according to certain local rules [4].

In cellular automata, space is divided into several cells by a certain regular grid. Each cell in these regular grids is called a cell, and the value of each cell has a fixed limit; that is, it can only be taken in a set of finite discrete states. All cells must follow the same rules in their evolution. A large number of cells constantly update their self-state according to the rules of local evolution. Cellular automata can follow a series of evolutionary rules rather than strictly defined physical equations or functions. The model of cellular automata is similar to the "field" which can produce close action in traditional physics, which can be said to be a discrete model of "field" [5].

The basic idea of cellular automata is to use a large number of simple cells, thick through simple operation rules (that is, determined local evolution rules), in discrete time and space to continue to run, and then simulate the complex and rich various phenomena, providing effective model tools for the study of the overall behavior and complex phenomena of the system, and has been widely used in the research fields of economics, sociology, and other disciplines since its emergence [6].

2.1.2. Composition of Cellular Automata. The cellular automata model consists of four parts: cellular, cellular space, cellular neighbors, and cellular evolution rules. By mathematical definition, cellular automata can be understood as being composed of a cellular space and a transformation function scheduled for that space [7].

(1) Cellular. Cell, or unit, is the most basic unit of composition that can be calculated in cellular automata. Cells are all dispersed on lattice points in discrete 1, 2, or multidimensional Euclidean spaces [8]. Strictly speaking, each cell must have its own state value at a fixed time, thus forming a state set, which can be a binary form of {0, 1} or a discrete set of {s0, s0, s1,. . .,*sn*}integer form.

(2) Cellular Space. Cell space refers to a set of spatial lattice points distributed by cells. In theory, cellular space can be Euclidean space of arbitrary dimension, but the main research is 1-dimensional and 2-dimensional cellular automata (2-dimensional cellular automata model is used in this paper).

Geometric Division of Cellular Space. For 1-dimensional cellular automata, it is obvious that there is only one division of its cellular space, while in 2-dimensional cellular automata, its cellular spatial structure mainly involves three mesh arrangement modes: triangle, square, and regular hexagon. And this topic uses a square grid.

Boundary Conditions in Cellular Space. Even the most advanced computer simulation technology cannot complete the automata evolution experiment of an infinite grid. Data simulation must require the system to be finite and bounded. For the cell on the boundary, we can generally consider adding the boundary condition to the cell space. The general condition mainly has the fixed boundary, that is, adding the "virtual cell" of the fixed state outside the boundary of the cell space so that the boundary looks complete; the periodic boundary, in the 2D grid, mainly the left and right boundary and the upper and lower boundary, needs to be connected separately; there is also a kind of heat insulation boundary, adding "virtual cell" outside the boundary of the cell space, and the state determination of these "virtual cell" needs certain rules, which can give them the state of the boundary cell with which they are adjacent. This topic adopts fixed boundaries.

(3) Cellular Neighbors. The cellular and cellular spaces only define the static part of the cellular automata model. The cellular automata need to be equipped with corresponding evolution rules to carry out dynamic evolution. These rules are all defined in the local range, that is, the state of cellular at the next moment, depending on its state at the current moment, and the state of its neighbor cell. Therefore, the neighbors of the cell must be determined first before given the cell rules. In the 2d cellular automata model, the commonly used neighbor structure is von Neumann and Moore type, and this subject is adopted by von Neumann.

The upper, lower, left, and right adjacent four cells of a cell are the neighbors of the cell. Mathematics is defined as

$$N_{\text{Neumann}} = \left\{ v_i = \left(v_{ix}, v_{iy}\right) : \left|v_{ix} - v_{ox}\right| + \left|v_{iy} - v_{oy}\right| \leq 1, \quad \left(v_{ix}, v_{iy}\right) \in Z^2 \right\}, \tag{1}$$

where $v_i = (v_{ix}, v_{iy})$ denotes the neighbor cell and its coordinates and (v_{ox}, v_{oy}) is the coordinates of the central cell.

(4) Rules of Cellular Evolution. According to the cellular state at present and its neighbor state, the dynamic function of determining the state of the cell at the next time is generally called the state transfer function [9]. For the given cell i, all of its neighbors N_i, the state of cellular i recorded at t time is S_i^t so the state of the entire neighborhood can be recorded as $S_{N_i}^t$. The state transfer function can be recorded as F. As a

result, every cell i has $S_i^{t+1} = F(S_i^t, S_{N_i}^t)$. It can be said that the cellular evolution rule is the key to cellular automata and the dominant factor. The reasonable selection of the evolution rule is related to the success of a cellular automata model.

2.2. Construction of Risk Simulation Model of Traditional Sports Culture Modernization Based on Cellular Automata. Taking the key control parameters of traditional sports culture development, government policy as the research object, the following CA models are established:

$$S^t = f\left(\Omega, S^{t-1}, \phi(N), A, R^t\right). \tag{2}$$

The meaning of the elements in formula (2) is as follows:

(1) Ω mark cell space, $\Omega = \{(i, j), \quad i, j = 1, 2, \ldots, N\}$.

(2) $S^t(i, j)$ and $S^{t-1}(i, j)$ are the strategies for labeling cells at x and y times, which is defined as follows:

$$S^t(i, j) = \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}, \tag{3}$$

where 0 represents no policy involvement in project development, 1 represents the macropolicy guidance of the sports culture represented by a project, and 2 represents the sports culture of a project.

(3) $\phi(N)$ labels the neighborhood of intercellular interaction, and this paper uses the von Neumann model.

(4) $R^t(i, j)$ represents the winning rate of (i, j) cell at t time, also known as the fitness of the cell.

During the development of traditional sports culture, when the government policy begins to intervene in the development of culture, the behavior derived from traditional sports culture is the game behavior of policy and cultural development system; that is, the derivative state of traditional sports culture will have a certain influence after receiving the influence of different policies. In Ω, the formula for calculating the suitability of internal cells R is

$$R^t(i, j) = \left[1 - \left(\sum \frac{S^{t-1}(i, j)}{N}\right)\right]\left(1 - L^{t-1}\sigma\right)\left(1 + E^{t-1}\right). \tag{4}$$

For the convenience of simulation, this study stipulates that, from the beginning of the reform and opening up in 1989, the influence of the policy on the development of traditional sports culture is set a fixed constant:

$$R^t(i, j) = 0.1\left(1 - L^{t-1}\sigma\right)\left(1 + E^{t-1}\right). \tag{5}$$

The relationship between the elements in formulae (4) and (5) is explained as follows: $\sum S^{t-1}(i, j)$ is expressed as the number of ecological traditional sports cultural items under the influence of policy at $t-1$ time, that is, the number of relevant policies selected as 1 or 2 in the cellular space, and this study will be one of the indicators to measure the risk of sports culture modernization.

N indicates the number of all traditional sports cultural items (including original and new ecology) under the influence of policy.

$1 - (\sum(S^{t-1}(i, j))/N)$ is expressed in a certain time and space, the survival of ecological traditional sports, and cultural projects due to the dependence on policy caused by their own growth block.

$1 - L^{t-1}\sigma$ reflects the game competition influence of all kinds of traditional sports culture items under the influence of policy.

L^{t-1} denotes the number of cells in the neighborhood space belonging to the evolution state of national traditional sports culture at $t-1$ time.

σ represents the effect of competition between cells on the cell in the neighborhood space.

E^{t-1} is a remarkable feature and one of the important parameters in the modernization of traditional sports culture.

Generally, we regard the scale change in derivative traditional sports culture is affected by policies such as urban and rural S-shaped curve change; that is, at the beginning, the response to the policy is slow, and then, it enters the rapid development period. With the enhancement of adaptability, the policy stimulus tends to decline. Therefore, the influence of policy parameters conforms to the typical diffusion curve, and the calculation formula used is as follows:

$$E^{t-1} = \frac{1}{\left(1 + a\exp\left(-b\sum E^t(i, j)\right)\right)}. \tag{6}$$

In (6), at time t, the diffusion rate affected by policy is the parameter of the above model. Parameter a determines the position of the curve, and parameter b determines the shape of the curve. The larger the value of b, the steeper the curve is, which means that the response speed of traditional sports cultural projects to policies is very fast, and the derivative scale is larger. Therefore, this study is mainly through the study of parameter b to explore the influence of policy parameters on the development scale of traditional sports culture.

(5) f is the state transition rule function.

Each traditional sports culture item may have a random variation on the derivative route. Because of the policy influence, different projects have a certain contraction strategy, that is, either new or choose to die out.

2.3. Simulation Steps

Step 1: $t = 0$ assigns an initial value to the threshold for the number of experiments T, maximum policy tolerance N, threshold r, a, b, and σ threshold value. At the moment of $t = 1$, at the center of the cell space ($20 * 20$ matrix), there exists a $2 * 2$ matrix; that is, the derivative formed under the influence of the policy of the beginning of the traditional sports culture project.

Step 2: end if $t > T$;

Step 3: E^{t-1} from formula (6);

Step 4: find R^{t-1} and R^t from formulae (4) and (5);

Step 5: calculation $S^t(i, j)$, $\overline{R}^t(i, j)$, $\sum s^t(i, j)$, using rules f;

Step 6: calculation $S^{t+1}(i, j)$, using rules f;

Step 7: $t = t + 1$, step 2.

All the above steps can be programmed by MATLAB to complete the simulation calculation.

2.4. Simulation Diagram. In the simulation process of this study, the cell network of $20 * 20$ is mainly adopted, the number of simulations is specified as $T = 70, N = (1/2)(20 * 20)$, and $r = 0.3$, and then different initial values are given to b and σ; the following series of simulation diagrams are obtained, as shown in Figures 1–8.

(1) $a = 60, \sigma = (1/8), b = 0.2$, and the simulation results are shown in Figures 1 and 2.

(2) $a = 60, \sigma = (1/8), b = 0.8$, and the simulation results are shown in Figures 3 and 4.

(3) $a = 60, \sigma = (1/2), b = 0.2$, and the simulation results are shown in Figures 5 and 6.

(4) $a = 60, \sigma = (1/2), b = 0.8$, and the simulation results are shown in Figures 7 and 8.

3. Interpretation of Result

From the results of the above eight diagrams, we can analyze and obtain four significant risks of the cultural development of the derivative traditional sports culture project.

3.1. Risk 1: Extinction Self-Extinguishing. When $a = 60, \sigma = (1/8)$ (see Figures 1 and 2); when the policy parameter $b = 0.2$, that is, the impact of policy is not strong. At the same time, the average suitability of the policy to the traditional sports culture has changed from the initial intense response to zero. This phenomenon is called extinction self-extinction. The most likely reason for this is that the policy is not in line with the market demand, which leads to the traditional sports culture.

The project's derivative ecology appeared "survival suffocation" and then "revived from the dead" with the strong support of the government. There is a growing gap in development. At this time, the policy measures are not conducive to the health of traditional sports culture development.

As the policy parameters $b = 0.8$, that is, the policy changes, the average, fitness, and development scale of the traditional sports culture evolution ecology gradually appear as local peak, but after the local peak appears, there is no big change, which indicates that the support strategy can only support it to the current high level, but the support fatigue phenomenon. The possible reasons are as follows: first, the traditional sports culture in the evolution of the "ride" behavior, the project because of regional rules, and no one to support innovation, and this situation will lead to the entire project's evolutionary die out. Secondly, maintaining the leading edge of technology, leading innovation projects in traditional sports culture will be taken strict knowledge protection measures by the higher authorities to prevent the spread of innovation technology; the signing of confidentiality agreements for the descendants and the registration of trademarks and patent rights in the form of expression are all-powerful measures to protect the development of local projects, but these measures can often also become resistant to their development.

3.2. Risk 2: Homogenization Self-Extinguishing. As the traditional sports culture system has not died out, when the policy parameters change from 0.2 to 0.8, the scale of development of the traditional sports culture is shrinking at $x = (1/2)$ (Figures 5–8), which shows that when the government's support policy is strengthened, the external economic diffusion effect of traditional sports culture is obvious, and the traditional sports culture system has already manifested competitive advantages such as the scale of participating artists, the rapid spread of influence, and the strong regional brands. Of course, this cannot say which policy is the most reasonable, and only after the market test can we determine the good policy, but this time, often many homogeneous sports in the system have died out or are in a state of extinction, and the emergence of homogenization is self-extinguishing.

3.3. Risk 3: Risk of Dislocation Failure in Government Support Policies. The simulation results show that when the market competition is in a smooth state, the policy support of the government or the competent department does not promote the evolution of the traditional sports culture modernization but only affects the upper limit of the system scale of the traditional sports culture. Suppose the government support policy is the main control parameter. Figures 1–4 show that when the policy is not strong, $b = 2$; the average fitness $\overline{R}^t(i, j)$ is 0.3–0.7 when the evolutionary risk effect of all sports culture modernization in the region spreads slowly, while the traditional sports culture is affected by the policy, the development scale $\sum S^t(i, j)$ fluctuation range is 60–120; when the policy is strong, that is, $b = 0.8$, the fluctuation range of the average fitness $\overline{R}^t(i, j)$ is 0.3–0.9, while the development scale $\sum S^t(i, j)$ fluctuation range of traditional sports culture is 20–120. It can be seen that when the social

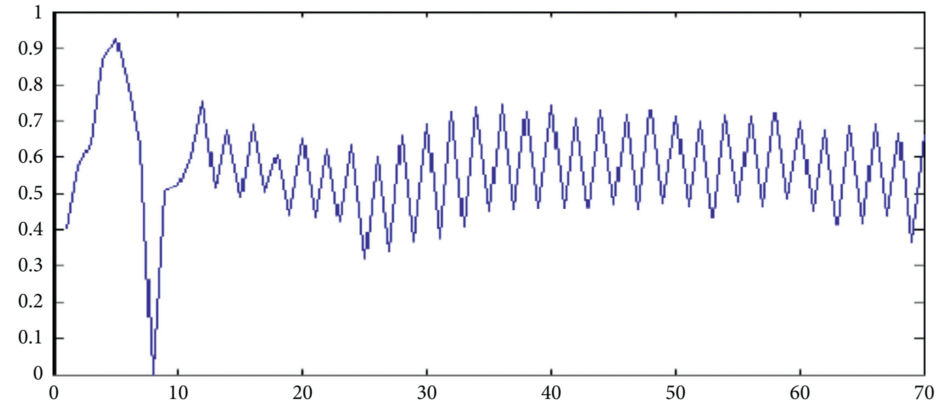

Figure 1: Average fitness $\overline{R}^t(i, j)$ of all sports culture-derived ecology in the region.

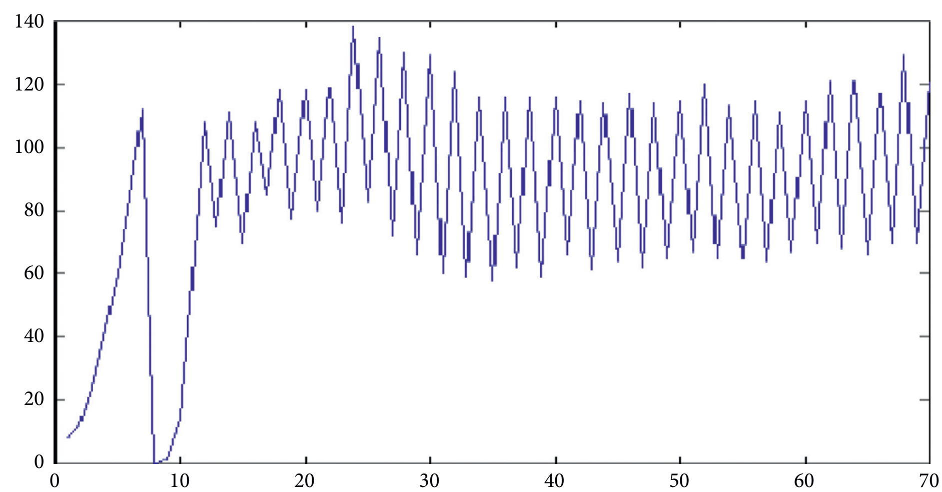

Figure 2: $\sum S^t(i, j)$ of development scale of derivative ecology after being affected by policy.

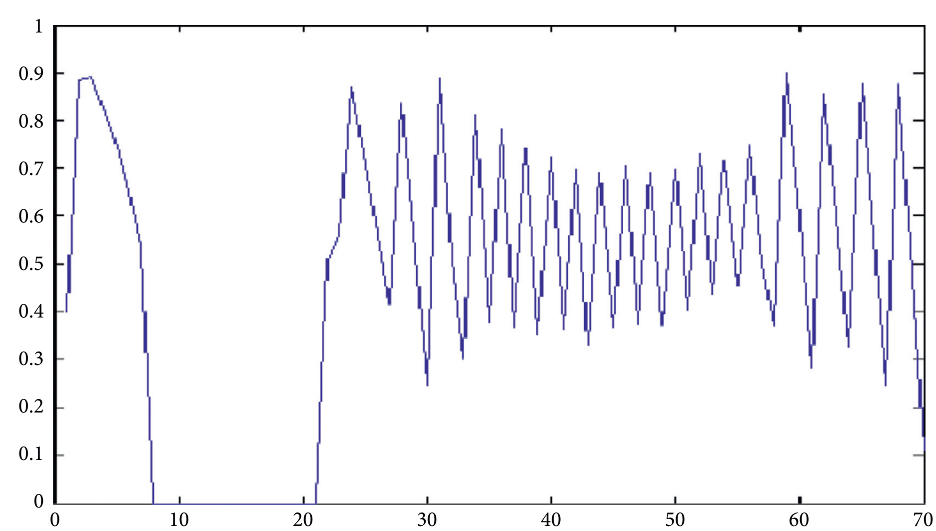

Figure 3: Average fitness $\overline{R}^t(i, j)$ of all sports culture-derived ecology in the region.

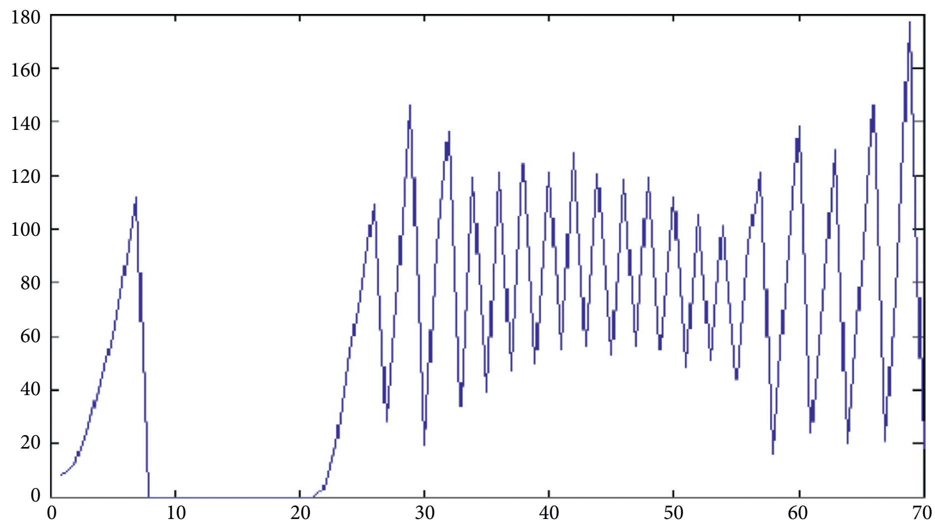

Figure 4: $\sum S^t(i, j)$ of development scale of the derivative ecology after being affected by policy.

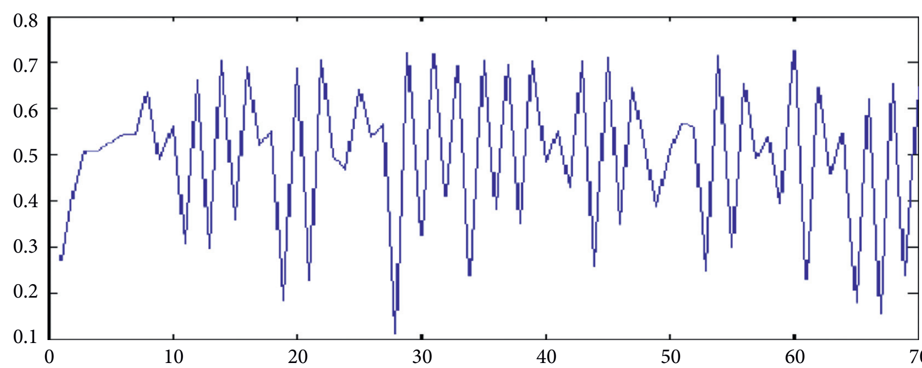

Figure 5: Average fitness $\overline{R}^t(i, j)$ for all sports culture-derived ecology in the region.

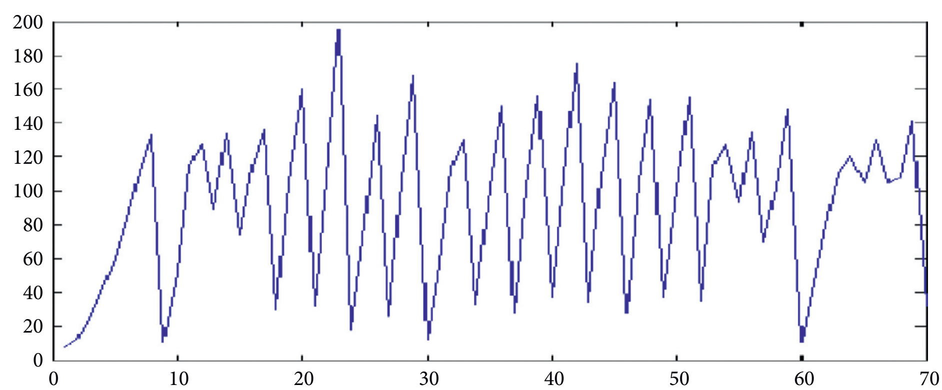

Figure 6: $\sum S^t(i, j)$ of development scale of the derivative ecology after being affected by policy.

environment changes, mainly in the environment of fierce market competition, the support of the government has little influence on the development scale derived from the traditional sports culture, which shows that blindly introducing policies to support will make the policy miss the best time to intervene, which will not only reduce the efficiency of the policy but also affect the self-adjustment of the traditional sports culture.

3.4. Risk 4: Unbalance of Fitness and Scale Allocation of Traditional Sports Culture. When $a = 60$ and $\sigma = (1/2)$, we can see that when the government policy is not strong, $b = 0.2$, the average fitness of the items in the traditional sports culture system is 0.1–0.7, and the scale of the traditional sports culture is 20–160; if the government policy is strong, $b = 0.8$, the average fitness of the items in the traditional sports culture system is 0.2–0.8, and the scale of the

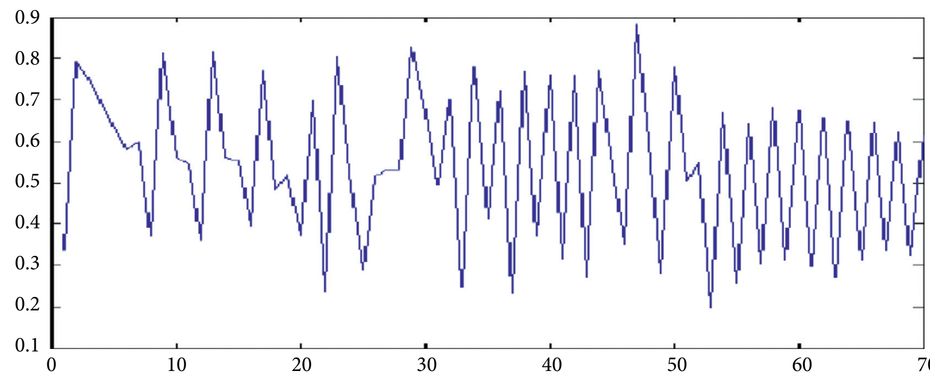

FIGURE 7: Average fitness $\overline{R}^t(i, j)$ for all sports culture-derived ecology in the region.

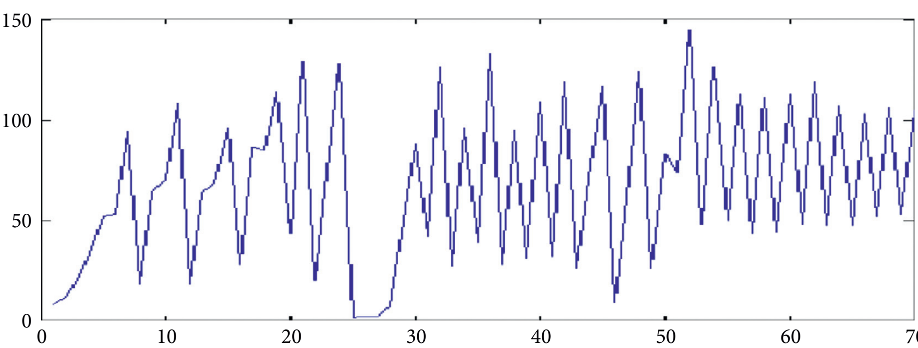

FIGURE 8: $\sum S^t(i, j)$ of development scale of the derivative ecology after being affected by policy.

traditional sports culture is 0–150. This shows that the policy will enhance the suitability of the components in the traditional sports culture project system, but it will inhibit the development of the traditional sports culture scale. Therefore, whether to improve the conformity of the traditional sports culture or to enlarge the scale of its modern development tests the management wisdom of relevant departments. Partially emphasizing conformity or development scale may bring great challenge to traditional sports culture.

4. Risk-Avoiding Strategy for Modernization of Traditional Sports Culture

4.1. Actively Introducing Heterogeneous Sports. To avoid the phenomenon of self-extinguishing, the government should actively introduce as many heterogeneous sports as possible to stimulate the innovation and development of local traditional sports. Of course, in the process of development, the government cannot forcibly change its development structure, but through the external environment to change its internal factors, thus causing its evolution. To complete the modernization and evolution of local traditional sports, the competent authorities should take the search for sports with heterogeneity and stimulation as one of the important tasks.

4.2. Measuring the Environmental Carrying Capacity of the Development of Traditional Sports Culture in the Region. It is suggested to do a good job of statistics and track the scale of sports culture development after the policy release and to predict the environmental carrying capacity of traditional sports development in the region before the new policy release. Evaluate the suitability of the policy, use the CA model, measure the development scale of traditional sports culture, and prevent the risk of self-extinguishing, unbalance, and even failure. Therefore, in the development of the traditional sports culture system, the measurement of environmental carrying capacity is necessary and crucial.

4.3. Policy Support and Timing of Intervention. According to the current situation of inheritance and protection of traditional sports culture in China, government guidance and support are still the main forces to promote its modernization [10]. But the simulation experiment shows that the government policy support strength and the intervention time, should according to the external environment situation, realize the precision support, thus enhancing the policy utility. So when does government support work on traditional sports culture? When the strength of policy support changes from 0.2 to 0.8, the average fitness of sports culture development ecology has been improved to some extent, but the threshold of development scale fluctuation is different. Therefore, when the government formulates the support policy to the traditional sports culture, it must first measure key control parameters (economic and geographical conditions) in its evolution process, so it is possible to determine the timing of policy intervention and better control the influence [11].

4.4. To Promote the Balanced Allocation of Fitness and Scale of Traditional Sports Culture in Derivative State. Because fitness and scale are the important factors that affect the development of ecological traditional sports culture, it should be determined according to the external environment and development stage of traditional sports culture [12]. It can be divided into the following situations: first, when the internal and external environmental resources of a certain region are limited and the competition between projects is fierce, the government should minimize the policy interference and let the different traditional sports in a free competition state; when the competition reaches a certain stage, the government chooses the opportunity to enter, support some projects, and improve the suitability of the project system; second, when the internal and external resources available in a certain region are abundant and the competition between projects is small, the government should strengthen the policy, on the one hand, improve the suitability of traditional sports culture, on the other hand, make full use of external resources to promote its scale development, and achieve the double promotion of suitability and scale; third, when the regional internal and external environmental resources are abundant and the project competition is more intense, the government should make full use of the resources to promote the development of the project scale at this time [13].

5. Conclusion

According to the research needs, this paper selects the government policy as the key control parameter to simulate the four risks in the process of modernization of traditional sports culture, which is of great significance for the protection and inheritance of national culture. In fact, in the era of big data, the "cloud computing" of the modernization development of traditional sports culture in the region is not difficult, and it is expected that soon, the modernization development of traditional sports culture projects can achieve the precise control of various influencing factors and develop towards the direction of human needs and hopes.

Data Availability

Data sharing is not applicable as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Research Article

Coupled Orbit-Attitude Dynamics of Tethered-SPS

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In this paper, a dynamic model for long-term on-orbit operation of the tethered solar power satellite (Tethered-SPS) is established. Because the Tethered-SPS is a super-large and super-flexible structure, the coupling among the orbit, attitude, and structure vibration of the system should be considered in comparison with the traditional spacecraft dynamic models. Based on the absolute nodal coordinate formulation (ANCF), the dynamic equation of the Tethered-SPS is established by Hamiltonian variational principle, and the dual equations of the Hamiltonian system are established by introducing generalized momentum through Legendre transformation. The symplectic Runge–Kutta method is used for numerical simulation, and the validation of the modeling method is verified by a numerical example. The effects of orbital altitude, initial attitude angle, length of solar panel, and orbital eccentricity on the orbit and attitude of the Tethered-SPS are analyzed. The numerical simulation results show that the effect of orbital altitude and length of solar panel on the orbital error of midpoint of beam is small. However, the initial attitude angle has a significant effect on the orbital error of midpoint of solar panel. The effect of the length of solar panel on the attitude angle of the system is not significant, but orbital altitude, orbital eccentricity, and initial attitude angle of the system severely affect the attitude angle of the system. Then, the stability of the system is affected.

1. Introduction

With the use of a large number of mineral energy, the global natural environment has been seriously polluted, which has also caused the destruction of the ecological environment. How to make full use of solar energy, a clean and renewable energy, has attracted worldwide attention [1]. In order to utilize solar energy more effectively, Glaser [2] first proposed the concept of space solar power station (SPS): solar energy is converted into electric energy in space; then the electric energy is transmitted to the ground rectenna receiver through microwave, and finally the electric energy is transmitted to users. NASA and Doe attached great importance to the concept of SPS and put forward the first systematic concept: the 1979 SPS Reference System [3]. Subsequently, the Sun Tower concept [4, 5], the Integrated Symmetrical Concentrator [6], and the SPS-ALPHA [7] were proposed. Thereafter, based on the Sun Tower model, the European Sail Tower Concept was put forward by DLR/ESA [4]. Since the introduction of SPS in the United States, several configurations have been proposed by Japan, such as the tethered solar power satellite (Tethered-SPS) and the NASDA reference system [8]. China also attached great importance to the research of SPS and proposed MR-SPS [9, 10] and SSPS-OMEGA [11].

The above research on SPS is mainly configurational design. However, there are many problems to be solved about SPS research, such as thermal management [9] and structural health monitoring [12, 13]. With the deepening of research, many researchers began to pay attention to the dynamics and control of SPS. Liu et al. [6, 14] simplified the segmented reflector as a particle, the metering structure boom as an Euler-Bernoulli beam, the solar collector, and the microwave transmitter as a rigid body. A coupled dynamic model of the Integrated Symmetric Concentrator system was established by using the floating frame of reference formulation (FFRF). In [15], the coupled orbit-attitude dynamic equations of European Sail Tower were established by using FFRF. The gravitational force and torques were expanded to fourth-order terms through Taylor series. The effects of higher order terms on the orbit and attitude motions were analyzed. Hu et al. [16] established the coupled dynamic model of space flexible beams by FFRF and analyzed the effect of weak damping on the vibration of beams according to six different initial values. Afterwards, considering the effect of aspheric perturbation, Hu and Deng [17] established the coupled dynamic model of flexible spatial beams by using FFRF and analyzed the effect of nonsphere perturbation of the Earth and weak damping of beams on the transverse vibration and the stable fixed point of beams. Considering the effect of solar pressure, gravity gradient, and space thermal radiation, Mu et al. [10] established the complex dynamic equation of the flexible beam by using FFRF. The effects of initial attitude angle, structure size and space environment on the structural vibration and attitude of the beam were analyzed. The above dynamic equations were established by the FFRF, which was the most commonly dynamic modeling method for flexible systems. However, when the flexible body undergoes large deformation and rotation, the modeling accuracy and computational efficiency of the FFRF will be reduced [18]. Then Yang et al. established finite element model of MR-SPS by simplifying the antenna array as thin plates [19].

The Tethered-SPS has many advantages, such as no active attitude control, no active heat control, high robustness and stability, and easy integration and maintenance [20]. So many researchers paid attention to its dynamics and control. In 2003, Fujii et al. [21] established a one-dimensional flexible beam for Tethered-SPS and designed a method to control attitude and structural vibration of the system by adjusting the tethers. In order to control the vibration of solar array panel of the Tethered-SPS, Fujii et al. [22] designed a feedback controller based on task function method by simplifying solar array panel as Euler-Bernoulli beams and verified the feasibility of this control method through ground tests. Senda and Goto [23] established a dynamic model composed of rigid and flexible bodies connected by springs and joints. The attitude control algorithm of the Tethered-SPS was designed and simulated by using geomagnetic force. However, the above studies mainly focused on the control of the attitude motion and structural vibration and did not study the coupled dynamic characteristics of the system. Based on the Hamiltonian variational principle, Hu et al. [24] established a damped beam-spring-mass model of Tethered-SPS. The symplectic dimension reduction method was used to decouple the dynamic equation, and the structure-preserving method was used to solve the dynamic equation. However, [24] was only a one-dimensional dynamic model. Li and Cai [25] simplified the solar array panel as a rigid panel. A dynamic model was established by considering the rotation of solar array panel, the deployment and retrieval of tethers, and the vibration of tethers. The effects of initial attitude errors and the vibration of the tethers on the system were analyzed. However, the deformations of solar array panel were not considered in [25]. The dynamic model of the Tethered-SPS was established by simplifying the bus system as a particle, the tethers as a mass-free springs, and the solar array panel as flexible panels in [26], and the effect of thermal deformation of solar panels on the attitude of the system was studied. Lately, Hu et al. [27] studied the characteristics of the vibration and elastic wave propagation by establishing damping panel-spring-mass model. However, the above models were based on the premise of small deformation and small attitude swing, so it was difficult to study large deformation and large attitude swing.

The absolute node coordinate formulation (ANCF) has the advantages of constant mass matrix in dynamic equation, no Coriolis force and centrifugal force term [28]. Therefore, the method can describe the dynamic characteristics of flexible system more accurately when large angle rotary maneuver and orbital transfer occur. In recent years, more and more researchers began to use ANCF to establish the dynamic models of space flexible system [29–32]. Sun et al. [31] simplified the tether as a flexible beam and the satellites as particles. The dynamic model of the space tethered satellite was established by using the ANCF. Li et al. [32] used the ANCF to establish the dynamic model of large flexible beam and flexible plate in space. A new method for calculating Jacobian matrix of the generalized gravitational force was proposed. The method was successfully applied to attitude control simulation of solar subarrays. The Tethered-SPS is a super-large and super-flexible spacecraft. Its orbit dynamics, attitude dynamics, and structural vibration are quite complex, and even there will be strong coupling between them. At the same time, when the Tethered-SPS is maneuvering and trajectory transferring at large angle, the system will undergo large deformation and rotation. At this time, the ANCF can be used to establish its accurate dynamic model. Wei et al. [33] established the dynamic model of the Tethered-SPS by adopting the ANCF. The effects of the bus system mass, orbital altitude, and tether length on the vibration characteristics of the solar array panel were studied. Xu et al. [30] simplified solar array panels as Euler-Bernoulli beams, tethers as massless springs, and bus system as particle. A simplified model of the Tethered-SPS was established, and the effects of solar pressure on structural vibration and attitude of the system were analyzed. Although [24, 30, 33] established a one-dimensional coupled dynamic model for the Tethered-SPS, they were based on the simplified model of the Tethered-SPS. In this paper, a multi-tethers dynamic model will be established.

Because the dynamic equations of the Tethered-SPS based on ANCF are strongly coupled nonlinear equations, it can only be solved by numerical method. Symplectic algorithm can maintain the stability, energy conservation, and momentum conservation of the system in numerical simulation, which has attracted extensive attention of many researchers [34–37]. At the same time, symplectic method has also been applied in celestial mechanics and aerospace dynamics simulation [37–41]. Aiming at the deployment process of solar receivers in SPS-ALPHA, a damping dynamic model was established by Yin et al. [37]. The classical damping system was separated into undamped system by the separation transformation method, and the numerical simulation was carried out by using the symplectic Runge–Kutta method. Hernandez [38] proposed a symplectic method to solve the N-body problem quickly, and simulated the three-body problem numerically. Li and Zhu [39] established the dynamics model of the tethered satellite by finite element method and used the fourth-order symplectic Runge–Kutta method for numerical simulation. Compared with the classical Runge–Kutta method, the symplectic Runge–Kutta method can maintain long-term numerical stability. Thereafter, they established the dynamic model of flexible electric solar sail and used symplectic method to simulate the dynamic equation [42]. Hubaux et al. [40] established the dynamic model of space debris in the complex space environment. The numerical results showed that the symplectic method can obtain stable and accurate numerical results even with a large step. Li et al. [42] established the orbit-attitude coupled dynamic model of a spacecraft rotating around a small celestial body. The simulation results showed that the coupled effect had a significant impact on the orbit of the system, especially when the higher-order terms were considered. Based on the above background, the symplectic method is used to study the orbit and attitude dynamic response of the Tethered-SPS.

2. Dynamic Model of the Tethered-SPS

In order to accurately reflect the attitude motion of the Tethered-SPS, the dynamic model of orbit-attitude-structure coupling is established. In order to facilitate the study, this paper only considers the motion of the system in the orbit plane. The Tethered-SPS system can be simplified as a dynamic model consisting of Euler-Bernoulli beam (beam AB) and particle (point P). The beam AB and point P are connected by 21 springs [20], and point C is the midpoint of beam AB (see Figure 1). The inertial coordinate system is established, in which the coordinate origin O coincides with Earth's center of mass, and the Ox axis and the Oy axis are in the orbit plane.

On the premise of accuracy and calculation efficiency [30], beam AB is discretized into 20 ANCF elements. So the absolute coordinate vector of any point on the ith beam element can be given by

$$\mathbf{r}(x_e) = [x(x_e), y(x_e)]^T = \mathbf{S}(x_e)\mathbf{e}_i, \tag{1}$$

where $x_e \in [0, l_e]$ is local coordinate along the beam axis, $l_e = l_{AB}/20$ is length of ANCF elements, the shape function $\mathbf{S}(x_e)$ can be obtained in [28], and the node coordinate vector $\mathbf{e}_i$ of the ith beam element is taken as

$$\mathbf{e}_i = \left[e_{i,1}, e_{i,2}, e_{i,3}, e_{i,4}, e_{i+1,1}, e_{i+1,2}, e_{i+1,3}, e_{i+1,4}\right]^{\mathrm{T}}, \tag{2}$$

where

$$\begin{cases} e_{i,1} = x(0), e_{i,2} = y(0), e_{i+1,1} = x(l_e), e_{i+1,2} = y(l_e), \\ e_{i,3} = \dfrac{\partial x(0)}{\partial x_e}, e_{i,4} = \dfrac{\partial y(0)}{\partial x_e}, e_{i+1,3} = \dfrac{\partial x(l_e)}{\partial x_e}, e_{i+1,4} = \dfrac{\partial y(l_e)}{\partial x_e}. \end{cases} \tag{3}$$

The mass matrix of the ith beam element can be written as follows [28]:

$$\mathbf{M}_i = \int_0^{l_e} \frac{m_{AB}}{l_{AB}} \mathbf{S}^T \mathbf{S} \mathrm{d}x_e. \tag{4}$$

Then, the total kinetic energy of the system can be written in an abbreviated form as

$$T = \frac{1}{2}\sum_{i=1}^{20} \dot{e}_i^{\mathrm{T}} \mathbf{M}_i \dot{e}_i + \frac{1}{2} m_P \left(\dot{x}_P^2 + \dot{y}_P^2\right) = \frac{1}{2}\dot{q}^{\mathrm{T}} \mathbf{M} \dot{q}, \tag{5}$$

where $\mathbf{q} = [e_{1,1}, e_{1,2}, e_{1,3}, e_{1,4}, \ldots, e_{21,4}, x_P, y_P]^T$ is the generalized coordinate vector of the system, $\mathbf{M}$ is the mass matrix of the system, and $[x_P, y_P]^{\mathrm{T}}$ is the absolute coordinate vector of point P.

Neglecting the shear deformation and using the assumptions of Euler-Bernoulli beam theory, the elastic potential energy of the ith beam element can be expressed as [28]

$$U_{i,\mathrm{elast}} = \frac{1}{2}\int_0^{l_e} \left(E_{AB} A_{AB} \left(\frac{\partial u_l}{\partial x_e}\right)^2 + E_{AB} I_{AB} \left(\frac{\partial^2 u_t}{\partial x_e^2}\right)^2 \right) \mathrm{d}x_e, \tag{6}$$

where u_l and u_t are, respectively, the axial and transverse displacements. Equation (6)fd6 is calculated according to [32].

Unlike the traditional spring model, tether pressure is not considered [27]. So elastic coefficient of the ith tether can be written as

$$k_i = \frac{\operatorname{sign}(\Delta l_i) + 1}{2} \frac{E_t A_t}{l_i}, \tag{7}$$

where sign( ) is the sign function and l_i is the original length, which can be expressed as

$$l_i = \sqrt{l_{PC}^2 + \left[\frac{l_{AB}}{2} - (i-1) l_e\right]^2}. \tag{8}$$

Then, the elastic potential energy of the ith tether can be expressed as

$$U_{i,\mathrm{tether}} = \frac{1}{2} k_i \left[\sqrt{(x_P - x_{Ni})^2 + (y_P - e_{Ni})^2} - l_i\right]^2, \tag{9}$$

where x_{Ni} and y_{Ni} are the coordinates of the ith node, respectively.

Therefore, the elastic potential energy of the system is expressed as

$$U_{\mathrm{elast}} = \sum_{i=1}^{20} U_{i,\mathrm{elast}} + \sum_{i=1}^{21} U_{i,\mathrm{tether}}. \tag{10}$$

The gravitational potential energy of the system includes the gravitational potential energy of the bus system and beam AB, which can be expressed as [20]

$$U_{\mathrm{grav}} = -\frac{\mu m_P}{\sqrt{x_P^2 + y_P^2}} - \sum_{i=1}^{20} \int_0^{l_e} \frac{\mu m_{AB}}{l_{AB}\sqrt{x^2 + y^2}} \mathrm{d}x_e, \tag{11}$$

where $\mu = 3.986 \times 10^{14}\ \mathrm{m}^3/\mathrm{s}^2$ is gravitational parameter of the Earth; x and y can be obtained by (1)fd1. Equation (11) fd11 is calculated by the Taylor approximation method [32], which has the advantage of high computational efficiency

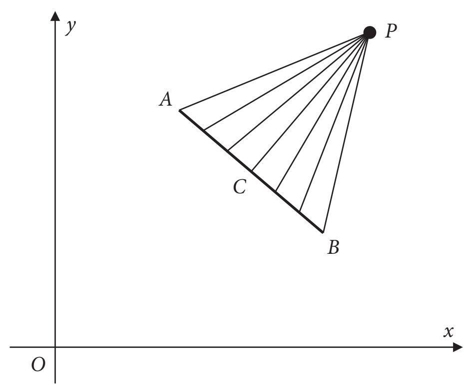

Figure 1: Dynamic model of the Tethered-SPS.

and easy calculation of gravitational potential energy and Jacobian matrix.

According to (5), (10), and (11), the Hamiltonian function of the system can be expressed as [30]

$$H = T + U_{\mathrm{elast}} + U_{\mathrm{grav}}. \tag{12}$$

The generalized momentum can be obtained using Legendre transformations:

$$\mathbf{p} = \frac{\partial H}{\partial \dot{q}} = \mathbf{M}\dot{q}. \tag{13}$$

The Hamiltonian function of the system is rewritten to a dual equation about $\mathbf{q}$ and $\mathbf{p}$; that is,

$$H = \frac{1}{2}\mathbf{p}^{\mathrm{T}}\mathbf{M}\mathbf{p} + U_{\mathrm{elast}} + U_{\mathrm{grav}}. \tag{14}$$

Then Hamiltonian canonical equation can be rewritten in the following form [36]:

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} = \mathbf{M}^{\mathrm{T}}\mathbf{p}, \\ \dot{p} = -\dfrac{\partial H}{\partial \mathbf{q}} = -\dfrac{\partial U_{\mathrm{elast}}}{\partial \mathbf{q}} - \dfrac{\partial U_{\mathrm{grav}}}{\partial \mathbf{q}}. \end{cases} \tag{15}$$

Equation (15)fd15 is a system of first-order ordinary differential equations, which can be solved by Runge–Kutta method [43]. The s-stage Runge–Kutta method formulated can be written as

$$\begin{cases} \mathbf{u}_{n+1} = \mathbf{u}_n + \tau \sum_{j=1}^{s} b_j f\left(t_n + c_j\tau, \mathbf{k}_j\right), \\ \mathbf{k}_i = \mathbf{u}_n + \tau \sum_{j=1}^{s} a_{ij} f\left(t_n + c_i\tau, \mathbf{k}_j\right), \end{cases} \tag{16}$$

where $\sum_{j=1}^{s} a_{ij} = c_i$, $\sum_{i=1}^{s} c_i = 1$, $\sum_{i=1}^{s} b_i = 1$, and $c_j \geq 0$, $i, j = 1, 2, \ldots, s$. Equation (16)fd16 is symplectic, if the coefficients satisfy following conditions:

$$b_i b_j - a_{ij} b_i - a_{ji} b_j = 0, \tag{17}$$

where $i, j = 1, 2, \ldots, s$. The 2-stage symplectic Runge–Kutta parameters are adopted as [37, 44]

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{4} & \dfrac{1}{4} - \dfrac{\sqrt{3}}{6} \\ \dfrac{1}{4} + \dfrac{\sqrt{3}}{6} & \dfrac{1}{4} \end{bmatrix}, \tag{18}$$
$$\mathbf{b} = \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix}.$$

3. Validation of the Proposed Model

A body-fixed coordinate system $P\xi\zeta$ is established as shown in Figure 2. Its origin is located at the center of mass of bus system, the ξ axis is initially parallel to the beam AB, and the ζ axis is perpendicular to the beam AB at the origin. In the following example and numerical simulation analysis, the deflection of midpoint of beam AB is expressed as δ; that is, $\delta = \zeta - l_{PC}$.

The parameters of the system are consistent with the design parameters in [20, 26], as shown in Table 1.

In order to verify the correctness of the model, the Tethered-SPS is simplified to have only two springs connected, and the system parameters are changed to be consistent with [30, 33].

In [33], the tethers are simplified as equidistant constraints; that is, the elastic coefficient of the tethers is considered as infinite. In this model, the elastic coefficient of the tethers is set to a very large number, so that the deformations of the tethers are very small. Figure 3 is the deflection of the midpoint of the beam changing with time. The results of this paper are consistent with [30, 33]. It verifies the correctness of the dynamic model and the numerical algorithm in this paper. It is known that the numerical simulation results of symplectic Runge–Kutta method can maintain the inherent characteristics of the original system. In the following numerical simulation, the symplectic Runge–Kutta method is used.

4. Numerical Simulation and Analysis

This section focuses on the analysis of the orbit and attitude dynamics of the Tethered-SPS. Except for special instructions, the system parameters are shown in Table 1. Point M is the center of mass of the system, and the attitude angle of the system is expressed as shown in Figure 4. Initially, the PC points to the ground, the beam is in an undistorted state, and all the tethers are in a straight state, but not subject to tension and pressure. Because Tethered-SPS transmit electricity to ground rectenna receivers through microwave transmitting antenna, it is necessary to consider the effect of the system parameters on the orbit and attitude of the system. We define the orbital error of the midpoint of the beam as $r_{\mathrm{error}} = r - r_0$, where r represents the orbital radius of the midpoint of the beam and r_0 represents the orbital radius of the midpoint of the beam at the initial time.

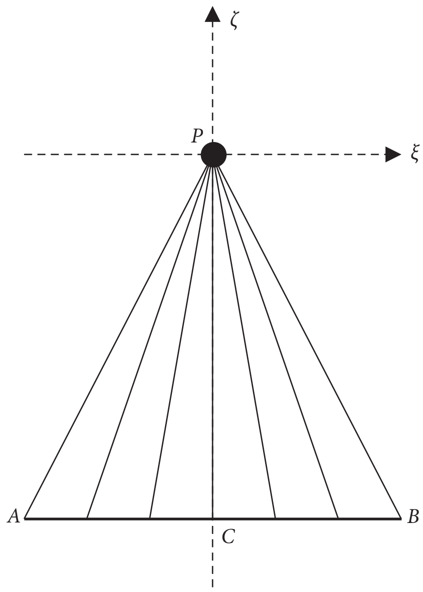

Figure 2: Body-fixed coordinate system of the Tethered-SPS.

Table 1: Parameters of the system.

Parameter	Symbol	Value
Mass of beam AB	m_{AB}	1.9×10^7 kg
Mass of bus system P	m_P	1.0×10^6 kg
Length of beam AB	l_{AB}	2000 m
Cross-sectional area of beam AB	A_{AB}	1.9 m^2
Second moment of section of beam AB	I_{AB}	4.7027×10^{-3} m^4
Elastic modulus of beam AB	E_{AB}	70 GPa
Cross-sectional area of tether	A_t	1.6464×10^{-4} m^2
Modulus of elasticity of tethers	E_t	70 GPa
Original distance from point P to point C	l_{PC}	10 km

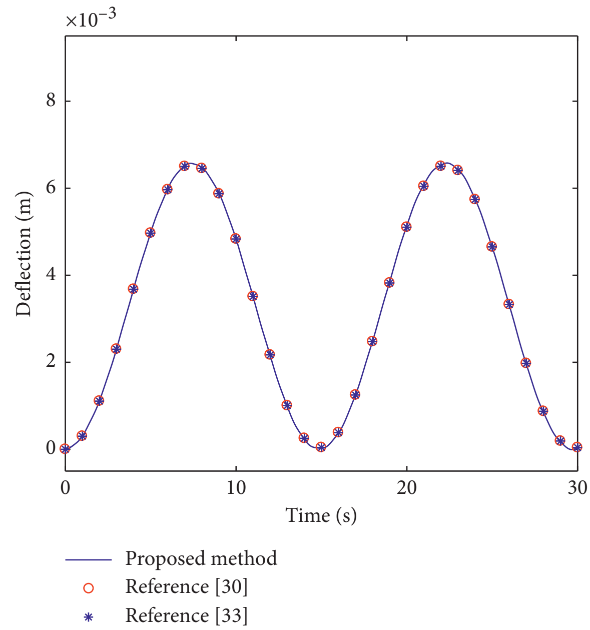

Figure 3: Deflection of midpoint of the beam.

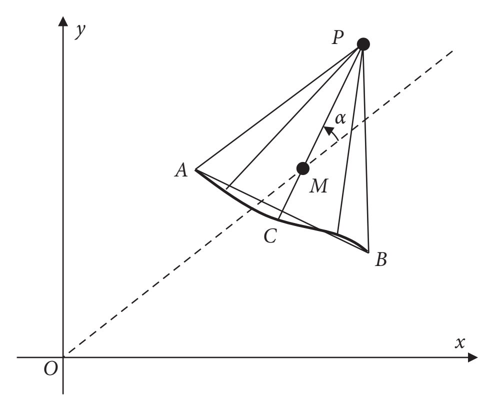

Figure 4: Definition of attitude angles.

4.1. Effect of Orbital Altitude on Orbit and Attitude. The Tethered-SPS can be placed in a geostationary Earth orbit. The advantages are that the solar radiation can be received stably in 99% of the time, and the energy can be transmitted to the ground in real time. The disadvantages are the high launch cost brought by high orbit and the low efficiency of microwave transmission. Therefore, many researchers began to study other orbits [45–49], such as low Earth orbit, Sun-synchronous orbit, and medium orbit. To compare the effects of orbital altitude, the following orbital altitudes are chosen to be 650 km, 6500 km, 10^4 km, 20200 km, and 35786 km, respectively. The initial attitude angle of the system is $\alpha = 0$ rad, and the bus system mass is assumed to be 10^6 kg. For simplicity, Figures 5 and 6 only draw the graphs of orbital altitudes as 650 km, 10^4 km, 20200 km, and 35786 km and record them as Case 1, Case 2, Case 3, and Case 4, respectively.

It can be seen from Figures 5 and 7 that the orbital altitudes will affect the orbital error of midpoint of beam. The orbital error of midpoint of beam decreases with the increase of orbital altitude. When the system is in geostationary Earth orbit, the orbital error of midpoint of beam is too small to be neglected. Although the maximum of orbital error of midpoint of beam will increase as the orbital altitude decreases, the maximum error is not more than 6 m, which is negligible relative to the orbit radius of the system. So the orbital altitude has little effect on the orbital error of midpoint of beam.

It can be seen from Figures 6 and 8 that the attitude of the system will oscillate periodically even at the initial attitude angle $\alpha = 0$ rad. The higher the orbital altitude is, the longer the swing period will be. With the decrease of the orbital altitude of the system, the maximum swing amplitude of the attitude angle of the system will increase. When the Tethered-SPS is in geostationary Earth orbit, the maximum attitude angle of the system is 2×10^{-6} rad. However, the maximum attitude angle of the system is 2.5×10^{-4} rad in low altitude orbit 650 km. The maximum attitude angle in low altitude orbit 650 km is about 100 times that of in geostationary Earth orbit. Therefore, it is necessary to

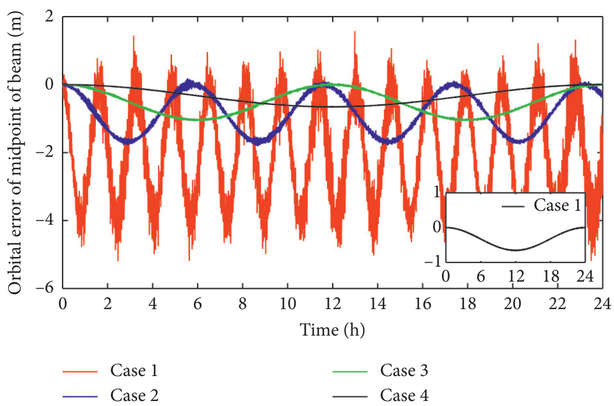

Figure 5: The orbital error of midpoint of beam with time.

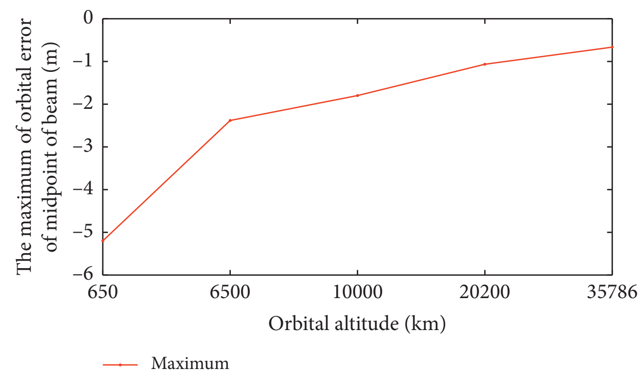

Figure 7: The maximum of orbital error of midpoint of beam with different orbital altitudes.

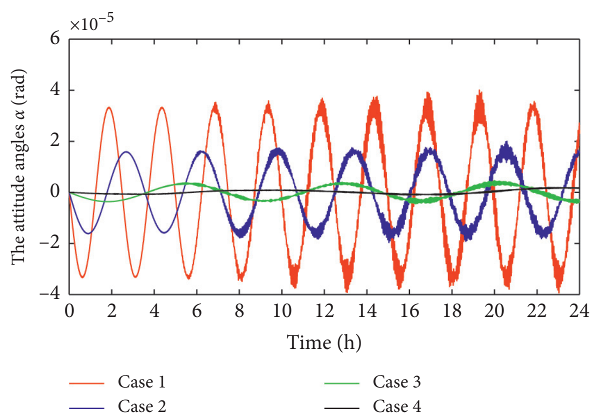

Figure 6: The attitude angle with time.

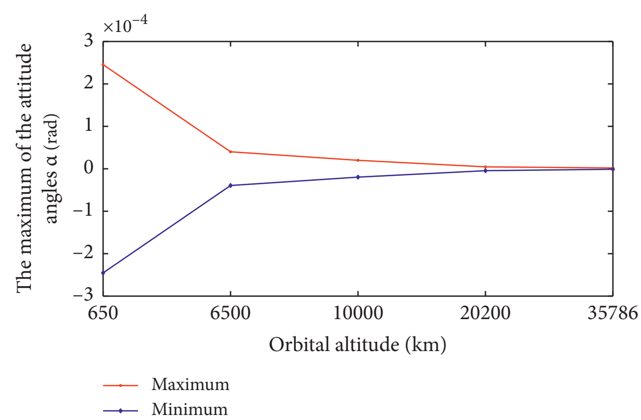

Figure 8: The maximum and minimum of attitude angle with different orbital altitudes.

consider the effect of orbital altitude on the attitude angle of the system.

4.2. Effect of Length of Beam AB on Orbit and Attitude. For the length of solar panel in the Tethered-SPS, the length of solar panel is 2 km in [20]. However, the length of solar panel is 2.6 km in refrence [48] and 1.5 km in [26], respectively. For the convenience of research, the solar panel is simplified as Euler-Bernoulli beam in [22]. In this paper, the solar panel of the Tethered-SPS is simplified to beam AB. In order to investigate the effect of the length of beam AB on the orbit and attitude of the Tethered-SPS, it is assumed that the midpoint of beam AB operates in geostationary Earth orbit. The initial attitude angle of the system is $\alpha = 0$ rad, and the bus system mass is 10^6 kg. The lengths of beam AB are chosen to be 1 km, 1.5 km, 2 km, 2.5 km, and 3 km, respectively. For simplicity, Figures 9 and 10 only draw the graphs of length of beam AB as 1 km, 2 km, 2.5 km, and 3 km and record them as Case 1, Case 2, Case 3, and Case 4, respectively.

It can be seen from Figure 9 that the orbital error of midpoint of beam decreases with the increase of length of beam AB. It shows that the longer the beam AB is, the smaller the orbital error is. It can be seen from Figure 11 that the maximum of orbital error of midpoint of beam is less than 2.5 m, when the length of beam AB is 1 km. This shows that the effect of the length of beam AB on the orbital error of midpoint of beam is not significant.

Even with the initial attitude angle $\alpha = 0$ rad, the attitude of the system will oscillate periodically from Figures 10 and 12. Although the longer the length of beam AB is, the longer the period of swing will be, the difference is not very big. As can be seen from Figure 12, the maximum swing amplitude of the attitude angle of the system is kept around $\alpha = 2 \times 10^{-6}$ rad, and the swing amplitude is relatively small. So the length of the beam AB does not have a great effect on the attitude angle of the system.

4.3. Effect of Eccentricity on Orbit and Attitude. After the assembly of the SPS, the system needs to be pushed to the designed orbit [50]. If the system adopts Horman orbital transfer, multi-pulse orbital transfer, and other orbital transfer modes, the transfer orbit will be elliptical. In this paper, the dynamic response of the Tethered-SPS in elliptical orbit is studied. The initial time of the system is assumed to be at the perigee of geostationary Earth orbit. In order to investigate the effect of orbital eccentricity on orbit and attitude, the initial orbital altitude of the beam midpoint of a Tethered-SPS is assumed to be 35786 km. The initial attitude

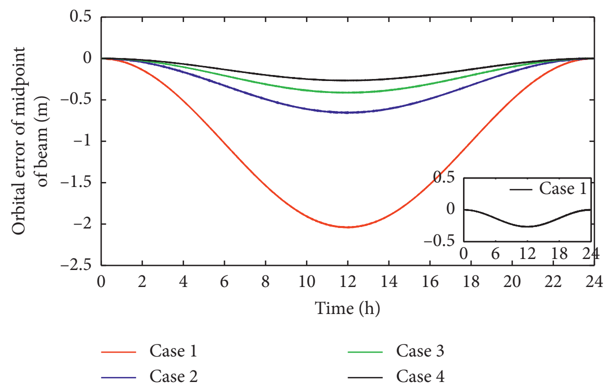

FIGURE 9: The orbital error of midpoint of beam with time.

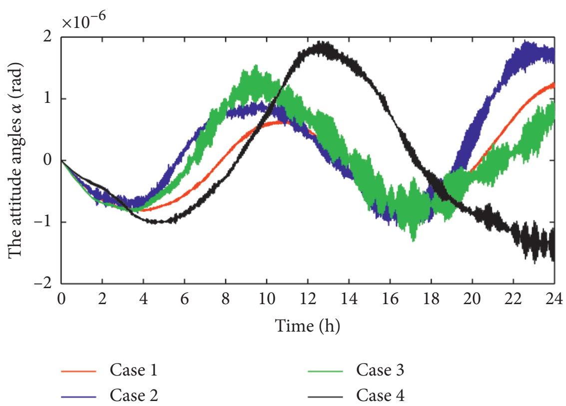

FIGURE 10: The attitude angle with time.

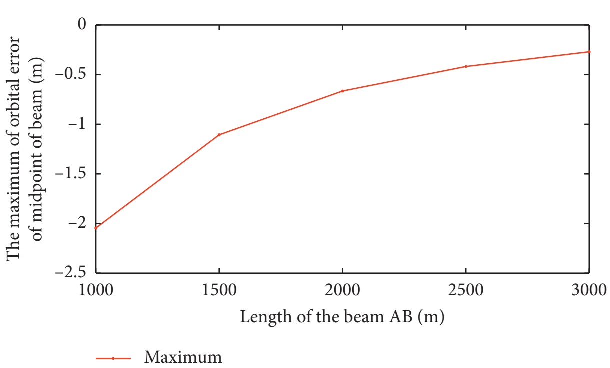

FIGURE 11: The maximum of orbital error of midpoint of beam with different lengths of the beam.

angle of the system is $\alpha = 0$ rad, and the bus system mass is 10^6 kg. To compare the effects of orbital eccentricity, the orbital eccentricity is chosen to be $e = 0$, $e = 0.1$, $e = 0.2$, $e = 0.3$, $e = 0.4$, $e = 0.5$, $e = 0.6$, and $e = 0.7$, $e = 0.8$, respectively. For simplicity, Figure 13 only draws the graphs of orbital eccentricity as $e = 0.1$, $e = 0.3$, $e = 0.5$, and $e = 0.7$ and record them as Case 1, Case 2, Case 3, and Case 4, respectively.

As shown in Figure 13, the attitude angle of the system will oscillate periodically within 24 hours. Figure 14 shows that the maximum of attitude angle of the system increases with the increase of eccentricity. However, when the orbital eccentricity $e \geq 0.5$, the maximum of attitude angle of the system does not increase anymore and remains at 1.6 rad. As is seen in Figures 13 and 14, although the maximum of attitude angle does not increase after orbital eccentricity $e \geq 0.5$, the frequency of swing increases. Therefore, the orbital eccentricity will affect the attitude of the system.

4.4. Effect of Initial Attitude Angle on Orbit and Attitude. When the Tethered-SPS is pushed to the designed orbit, the attitude angle of the system will change. In order to study the effect of initial attitude angle on orbit and attitude, the initial attitude angles of the system are $\alpha = 0$ rad, $\alpha = \pi/108$ rad, $\alpha = \pi/36$ rad, $\alpha = \pi/12$ rad, and $\alpha = \pi/4$ rad, respectively. The initial orbital altitude of the beam midpoint of the Tethered-SPS is 35786 km, and the bus system mass is 10^6 kg. For simplicity, Figures 15 and 16 only draw the graphs of initial attitude angles as $\alpha = 0$ rad, $\alpha = \pi/36$ rad, $\alpha = \pi/12$ rad, and $\alpha = \pi/4$ rad and record them as Case 1, Case 2, Case 3, and Case 4, respectively.

It can be seen from Figures 15 and 17 that the initial attitude angle will affect the orbital error of midpoint of beam but will not change its periodicity. The orbital error of midpoint of beam increases with the increase of initial attitude angle. When the initial attitude angle is less than $\alpha = \pi/36$ rad, the orbital error of midpoint of beam is also small. As shown in Figure 17, when the initial attitude angle is $\alpha = \pi/12$ rad, the maximum of orbital error of midpoint of beam is 200 m. When the initial attitude angle is $\alpha = \pi/4$ rad, the maximum of orbital error of midpoint of beam is 800 m. So, the effect of initial attitude angle on the orbital error of midpoint of beam is relatively large. Therefore, it is necessary to consider the effect of initial attitude angle on the orbital error of the system.

It can be seen from Figures 17 and 18 that the attitude of the system will oscillate periodically for any attitude angle. The larger the initial attitude angle is, the larger the swing amplitude will be. However, attitude angle changes periodically, and the period is basically the same. It shows that different initial attitude angles have little effect on the oscillation period of the attitude angle of the system but have obvious effect on the amplitude of the attitude angle of the system.

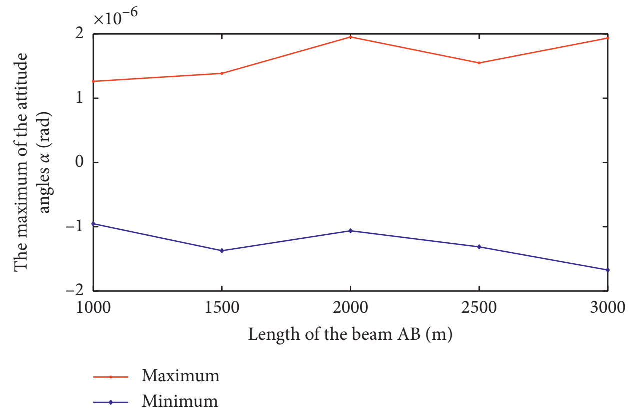

FIGURE 12: The maximum and minimum of attitude angle with different lengths of the beam.

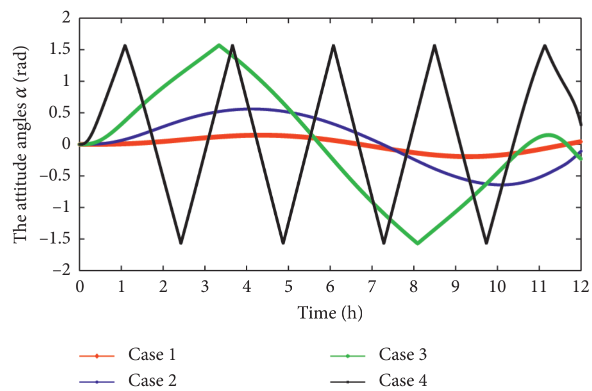

FIGURE 13: The attitude angle with time.

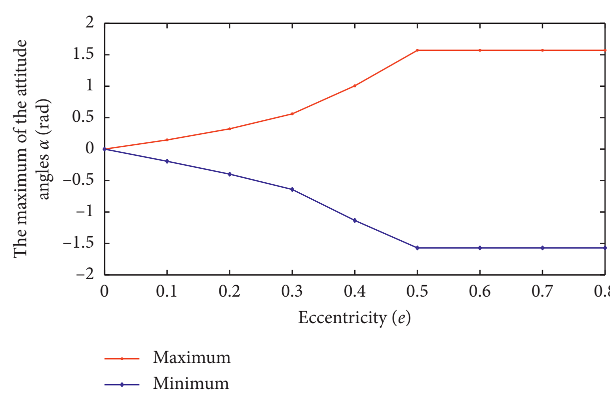

FIGURE 14: The maximum and minimum of attitude angle with different eccentricity.

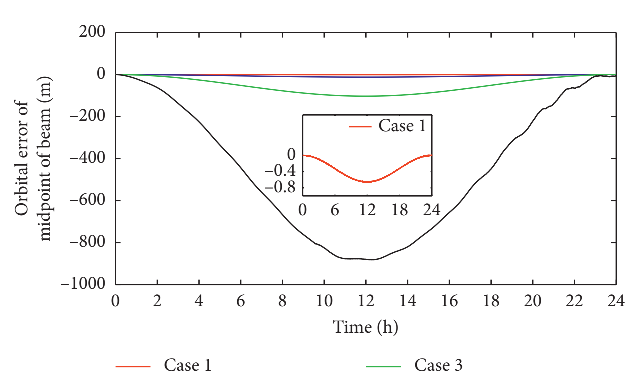

FIGURE 15: The orbital error of midpoint of beam with time.

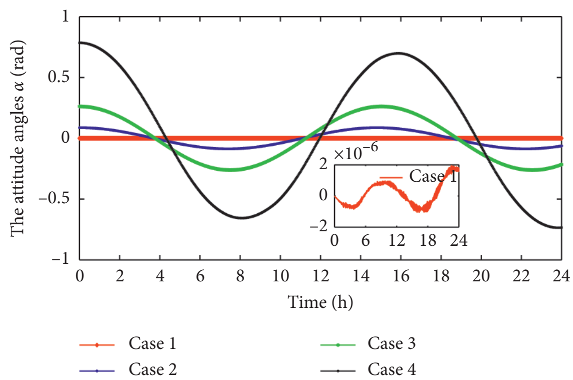

Figure 16: The attitude angle with time.

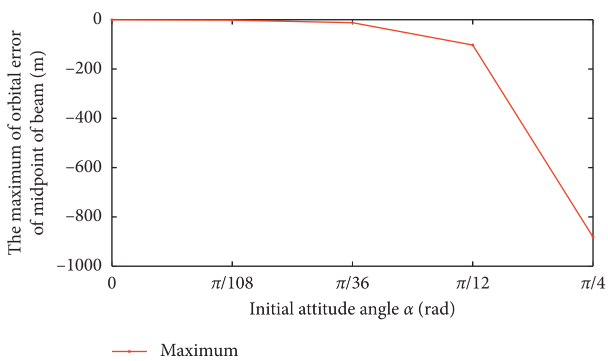

Figure 17: The maximum of orbital error of midpoint of beam with different initial attitude angles.

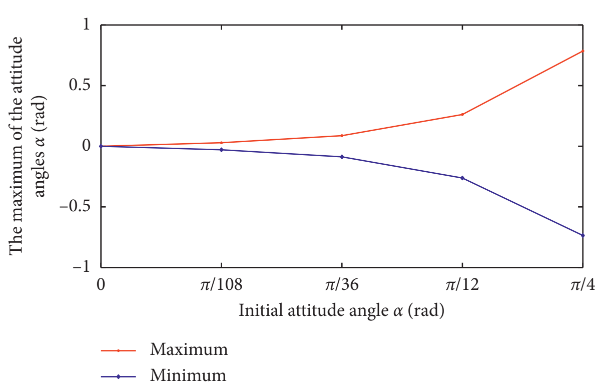

Figure 18: The maximum and minimum of attitude angle with different initial attitude angles.

5. Conclusion

In this paper, the ANCF is used to establish the coupled orbit-attitude-structure dynamic model of Tethered-SPS. The coupled dynamic equation of system is established under Hamiltonian system. The validity of the method is illustrated by an example. The effect of four different parameters on the orbit and attitude of the system is analyzed by using the symplectic Runge–Kutta method. The results are as follows:

(1) The orbital error of midpoint of beam decreases with the increase of orbital altitude. The maximum swing amplitude of system attitude angle increases with the decrease of orbital altitude. It is necessary to consider the effect of orbital altitude on the attitude angle of the system.

(2) As the length of solar panel increases, the orbital error of midpoint of beam decreases. However, the effect of the length of solar panel on the orbital error of midpoint of beam is not significant. Meanwhile, the length of solar panel has little effect on the attitude angle of the system.

(3) With the increase of eccentricity, the maximum of the attitude angle of the system will increase. However, when the orbital eccentricity $e \geq 0.5$, the maximum of attitude angle of the system does not increase anymore and remains at 1.6 rad. Meanwhile, the attitude angle of the system will oscillate periodically, and the frequency of oscillation increases with the increase of eccentricity. Therefore, the orbital eccentricity will affect the attitude of the system.

(4) The initial attitude angle will affect the orbital error of midpoint of beam but will not change its periodicity. The orbital error of midpoint of beam increases with the increase of initial attitude angle. However, different initial attitude angles have little effect on the oscillation period but have obvious effect on the amplitude of the attitude angle of the system.

The modeling theory presented in this paper can be extended to the coupled dynamic model of two-dimensional Tethered-SPS. At the same time, the numerical algorithm in this paper provides a good numerical method for the dynamic analysis of Tethered-SPS with large range of motion. Future works could be devoted to improving the modeling accuracy and computing efficiency.

Data Availability

The data supporting the findings of this article are included within the article and are available upon request to the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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Research Article

Extremal Solutions for a Class of Tempered Fractional Turbulent Flow Equations in a Porous Medium

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In this paper, we are concerned with the existence of the maximum and minimum iterative solutions for a tempered fractional turbulent flow model in a porous medium with nonlocal boundary conditions. By introducing a new growth condition and developing an iterative technique, we establish new results on the existence of the maximum and minimum solutions for the considered equation; at the same time, the iterative sequences for approximating the extremal solutions are performed, and the asymptotic estimates of solutions are also derived.

1. Introduction

Tempered stable laws were introduced to model turbulent velocity fluctuations of physics [1]. Normally, tempered stable laws retain their signature power-law behaviour at infinity and infinite divisibility [2]. By multiplying by an exponential factor for the usual second derivative, one can obtain tempered fractional derivatives and integrals. In [3], an exponential tempering factor was applied to the particle jump density in random walk and stochastic model for turbulence in the inertial range, which is the fractional derivative of Brownian motion exhibiting semi-long range dependence with a power law at moderate time scales.

Tempered stable laws are useful in statistical physics and provide a basic physical model such as turbulent flow for the underlying physical phenomena. Motivated by these physical backgrounds and the sources, in this paper, we focus on the existence of the maximum and minimum iterative solutions for the following tempered fractional turbulent flow equation with nonlocal boundary conditions:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R\mathbb{D}_t^{\beta,\lambda}x(t)\right)\right) = h(t)f(x(t)), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = {}_0^R\mathbb{D}_t^{\beta,\lambda}x(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta,\lambda}x(1) = \int_0^1 e^{-\lambda(1-t)}x(t)\mathrm{d}t, \\ {}_0^R\mathbb{D}_t^{\gamma,\lambda}\left(\varphi_p\left({}_0^R\mathbb{D}_t^{\beta,\lambda}x\right)\right)(1) = \int_0^1 a(t){}_0^R\mathbb{D}^{\gamma,\lambda}\left(\varphi_p\left({}_0^R\mathbb{D}_t^{\beta,\lambda}x(t)\right)\right)\mathrm{d}A(t), \end{cases} \tag{1}$$

where $1<\alpha\leq 2, n-1<\beta\leq n, n\geq 4, 0<\gamma<\alpha-1$, $\lambda>0$ is a constant, φ_p is the p-Laplacian operator defined by $\varphi_p(s)=|s|^{p-2}s$, $p>1$, ${}_0^R\mathbb{D}_t^{\alpha,\lambda}$ is the tempered fractional derivative, $\int_0^1 a(t){}_0^R\mathbb{D}_t^{\gamma,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\beta,\lambda}x(t)))\mathrm{d}A(t)$ denotes a Riemann–Stieltjes integral, A is a function of bounded variation, $f\colon [0,+\infty) \longrightarrow (0,+\infty)$ is continuous, and $h\in [L(0,1),[0,+\infty)]$.

Turbulent flow is a fundamental fluid mechanics problem which can be described by a p-Laplacian equation with a suitable boundary condition; for details, see [4]. Particularly, if the model is of fractional order, then it can describe turbulent flow in a porous medium [5–10]. On the contrary, fractional-order derivative has nonlocal characteristics; based on this property, the fractional differential equation can also interpret many abnormal phenomena that occur in applied science and engineering, such as viscoelastic dynamical phenomena [11–29], advection-dispersion process in anomalous diffusion [30–34], and bioprocesses with genetic attribute [35, 36]. As a powerful tool of modeling the above phenomena, in recent years, the fractional calculus theory has been perfected gradually by many researchers, and various different types of fractional derivatives were studied, such as Riemann–Liouville derivatives [16, 37–62], Hadamard-type derivatives [63–71], Katugampola–Caputo derivatives [72], conformable derivatives [73–76], Caputo–Fabrizio derivatives [77, 78], Hilfer derivatives [79–82], and tempered fractional derivatives [83]. These works also enlarged and enriched the application of the fractional calculus in impulsive theories [84–89], chaotic system [90–93], and resonance phenomena [94–96]. Among them, by using the fixed point theorem of the mixed monotone operator, Zhang et al. [9] established the result of uniqueness of the positive solution for the Riemann–Liouville-type turbulent flow in a porous medium:

$$\begin{cases} \mathscr{D}_{\mathbf{t}}^{\beta}\left(\varphi_p\left(-\mathscr{D}_{\mathbf{t}}^{\alpha} z\right)\right)(t) = -f\left(z(t), \mathscr{D}_{\mathbf{t}}^{\gamma} z(t)\right), \quad t \in (0,1), \\ \mathscr{D}_{\mathbf{t}}^{\alpha} z(0) = \mathscr{D}_{\mathbf{t}}^{\alpha+1} z(0) = \mathscr{D}_{\mathbf{t}}^{\alpha} z(1) = 0, \\ \mathscr{D}_{\mathbf{t}}^{\gamma} z(0) = 0, \mathscr{D}_{\mathbf{t}}^{\gamma} z(1) = \int_0^1 \mathscr{D}_{\mathbf{t}}^{\gamma} z(s) \mathrm{d}A(s), \end{cases} \tag{2}$$

where $0<\gamma\leq 1<\alpha\leq 2<\beta<3, \alpha-\gamma>1$, $\mathscr{D}_{\mathbf{t}}^{\alpha}$, $\mathscr{D}_{\mathbf{t}}^{\beta}$, and $\mathscr{D}_{\mathbf{t}}^{\gamma}$ denote the Riemann–Liouville derivatives, and $\int_0^1 z(s)\mathrm{d}A(s)$ indicates the Riemann–Stieltjes integral, and A is a function of bounded variation; the nonlinear term may be singular at both first variable and second variable. Recently, Zhou et al. [83] investigated a class of tempered fractional differential equations with Riemann–Stieltjes integral boundary conditions; by using the fixed point theorem of the sum-type mixed monotone operator, the existence and uniqueness of positive solutions were established, and iterative sequences for approximating the unique positive solution were also constructed.

However, to the best of our knowledge, there are relatively few results on fractional turbulent flow in a porous medium with nonlocal Riemann–Stieltjes integral boundary conditions, and no work has been reported on the maximal and minimal solutions for the tempered-type fractional turbulent flow equation. Thus, following the previous work, this paper will pay attention to the extremal solutions for the tempered fractional turbulent flow equation in a porous medium with nonlocal Riemann–Stieltjes integral boundary conditions by developing iterative technique, also see [97–100]. Different from [9, 83], in this paper, we will give a new type of growth condition for the nonlinear term to guarantee equation (1) has the extremal solutions. At the same time, the iterative sequences for approximating the extremal solutions are performed, and the asymptotic estimates of solutions are also obtained.

2. Preliminaries and Lemmas

Before starting our work, we firstly recall the definition of the tempered fractional derivative which is an extension of the Riemann–Liouville derivative and integral.

Let $\lambda>0$; the α-order left tempered fractional derivative is defined by

$${}_0^R\mathbb{D}_{\mathbf{t}}^{\alpha,\lambda} x(t) = e^{-\lambda t}\, {}_0^R\mathscr{D}_{\mathbf{t}}^{\alpha}\left(e^{\lambda t} x(t)\right), \tag{3}$$

where ${}_0^R\mathscr{D}_{\mathbf{t}}^{\alpha}$ denotes the standard Riemann–Liouville fractional derivative which can be found in [101].

Let

$$H(t,s) = \begin{cases} \dfrac{\left[\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}t^{\beta-1} - \beta(\beta-1)e^{\lambda s}(t-s)^{\beta-1}\right]e^{-\lambda t}}{(\beta-1)\Gamma(\beta+1)}, & 0\leq s\leq t\leq 1, \\ \dfrac{\left[\beta(1-s)^{\beta-1}(\beta-1+s)\right]e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)} t^{\beta-1}e^{-\lambda t}, & 0\leq t\leq s\leq 1, \end{cases} \tag{4}$$

$$\Delta = \frac{e^{-\lambda}-\delta}{\Gamma(\alpha-\gamma)},$$

$$\delta = \int_0^1 e^{-\lambda s} s^{\alpha-\gamma-1} a(s)\mathrm{d}A(s).$$

The following results have been proven in [83].

Lemma 1. *Given $k \in C[0,1]$; then, the boundary value problem,*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R\mathbb{D}_t^{\beta,\lambda}x(t)\right)\right) = k(t), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = {}_0^R\mathbb{D}_t^{\beta,\lambda}x(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta,\lambda}x(1) = \int_0^1 e^{-\lambda(1-t)}x(t)\mathrm{d}t, \\ {}_0^R\mathbb{D}_t^{\gamma,\lambda}\left(\varphi_p\left({}_0^R\mathbb{D}_t^{\beta,\lambda}x\right)\right)(1) = \int_0^1 a(t){}_0^R\mathbb{D}_t^{\gamma,\lambda}\left(\varphi_p\left({}_0^R\mathbb{D}_t^{\beta,\lambda}x(t)\right)\right)\mathrm{d}A(t), \end{cases} \tag{5}$$

has the unique solution

$$x(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(t,\tau)k(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \tag{6}$$

where $H(t,s)$ is defined by (4) and $G(t,s)$ denotes the Green function as follows:

$$\begin{aligned} G(t,s) &= G_1(t,s) + \frac{t^{\alpha-1}e^{-\lambda t}}{\Delta\Gamma(\alpha-\gamma)}\int_0^1 a(t)G_2(t,s)\mathrm{d}A(t), \\ G_1(t,s) &= \frac{e^{\lambda(s-t)}}{\Gamma(\alpha)}\begin{cases} (1-s)^{\alpha-\gamma-1}t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ (1-s)^{\alpha-2}t^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases} \\ G_2(t,s) &= \frac{e^{\lambda(s-t)}}{\Gamma(\alpha)}\begin{cases} (1-s)^{\alpha-\gamma-1}t^{\alpha-\gamma-1} - (t-s)^{\alpha-\gamma-1}, & 0 \le s \le t \le 1, \\ (1-s)^{\alpha-\gamma-1}t^{\alpha-\gamma-1}, & 0 \le t \le s \le 1. \end{cases} \end{aligned} \tag{7}$$

In order to obtain the positive extremal solutions of tempered fractional turbulent flow equation (1), it is necessary to preserve nonnegativity of the Green function.

(**H**0):

$$\int_0^1 e^{-\lambda s}s^{\alpha-\gamma-1}a(s)\mathrm{d}A(s) < e^{-\lambda}. \tag{8}$$

Lemma 2. *Assume* (**H**0) *holds; then, functions $G(t,s)$ and $H(t,s)$ have the following properties:*

(1) $G(t,s)$ and $H(t,s)$ are nonnegative and continuous for $(t,s) \in [0,1] \times [0,1]$.

(2) For any $t,s \in [0,1]$, $H(t,s)$ satisfies

$$m_1(s)e^{-\lambda t}t^{\beta-1} \le H(t,s) \le M_1(s)e^{-\lambda t}t^{\beta-1}, \tag{9}$$

where

$$\begin{aligned} M_1(s) &= \frac{\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}, \\ m_1(s) &= \frac{\beta s(1-s)^{\beta-1}e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}. \end{aligned} \tag{10}$$

(3)

$$m_2(s)e^{-\lambda t}t^{\alpha-1} \le G(t,s) \le M_2(s)e^{-\lambda t}t^{\alpha-1}, \quad s,t \in [0,1], \tag{11}$$

where

$$\begin{aligned} M_2(s) &= \left[\frac{1}{\Gamma(\alpha)} + \frac{\delta}{\Delta\Gamma(\alpha)\Gamma(\alpha-\gamma)}\right]e^{\lambda s}(1-s)^{\alpha-\gamma-1}, \\ m_2(s) &= \frac{e^{\lambda s}\left[(1-s)^{\alpha-\gamma-1} - (1-s)^{\alpha-1}\right]}{\Gamma(\alpha)}. \end{aligned} \tag{12}$$

Let

$$M_1^* = \frac{\beta^2 e^{\lambda}}{(\beta-1)\Gamma(\beta+1)}. \tag{13}$$

In order to obtain the existence of positive extremal solutions of tempered fractional turbulent flow equation (1), we introduce the following new control conditions.

(**H1**): f: $[0,+\infty) \longrightarrow (0,+\infty)$ is continuous and nondecreasing, and there exists a positive constant $\epsilon > 3/(q-1)$ such that

$$0 < d =: \sup_{s\ge 0}\frac{f(s)}{(s+2)^{\epsilon}} < +\infty. \tag{14}$$

(**H2**):

$$0 < \int_0^1 M_2(\tau)h(\tau)\mathrm{d}\tau < \left(\frac{M_1^*}{3^{\epsilon(q-1)}}\right)^{1/q-1}d^{-1}. \tag{15}$$

Remark 1. Assumption (14) we introduced is a new type of growth condition, which includes a large number of basic functions such as

(1) $f(t) = b_0 + \sum_{i=1}^n b_i(t+2)^{\mu_i}$, where $b_0 > 0, b_i, \mu_i > 3/q-1, i = 1,2,\ldots,n$.

(2) $f(t) = b_0 + [\sum_{i=1}^n b_i(t+2)^{\mu_i}]^{1/\mu}$, where $b_0 > 0, \mu > 0$, $b_i > 0, \mu_i > 0\,(i = 1,2,\ldots,n)$ and

$$\min_{i=1,\ldots,n}\{\mu_i\} > \frac{3\mu}{q-1}. \tag{16}$$

(3) $f(t) = (t+1)^{\mu-1}\ln(1+(1/2+t)) + (2+t)^{\mu-2} + b$, $b > 0, \mu > 2 + (3/q-1)$.

(4) f: $[0,+\infty) \longrightarrow (0,+\infty)$ is continuous and nondecreasing, and there exists a positive constant $\epsilon > 3/q-1$ such that $f(x)/(x+2)^{\epsilon}$ is increasing on x, and

$$\lim_{x\longrightarrow+\infty}\frac{f(x)}{(x+2)^{\epsilon}} = M > 0. \tag{17}$$

(5) $f\colon [0,+\infty) \longrightarrow (0,+\infty)$ is continuous and nondecreasing, and there exists a positive constant $\epsilon > 3/q - 1$ such that $f(x)/(x+2)^{\epsilon}$ is nonincreasing on x.

Proof. For cases (1)–(3), take

$$\epsilon = \max_{i=1,2,\ldots,n}\{\mu_i\},$$
$$\epsilon = \frac{\max_{i=1,2,\ldots,n}\{\mu_i\}}{\mu}, \tag{18}$$
$$\epsilon = \mu - 2,$$

respectively; obviously,

$$0 < d =: \sup_{s\geq 0}\frac{f(s)}{(s+2)^{\epsilon}} < +\infty. \tag{19}$$

For cases (4) and (5), it is clear; we omit the proof. □

Denote $E = C[0,1]$ as all continuous functions equipped the maximum norm

$$\|x\| = \max\{x(t): t \in [0,1]\}. \tag{20}$$

Define a cone P,

$$P = \left\{x \in E\colon \text{there exists a number } 0 < l_x < 1 \text{ such that } 0 \leq x(t) \leq l_x^{-1} e^{-\lambda t} t^{\beta-1}, \quad t \in [0,1]\right\}, \tag{21}$$

and an operator T in E:

$$T(x)(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)f(x(\tau))\mathrm{d}\tau\right)\mathrm{d}s. \tag{22}$$

Then, the fixed point of operator T in E is the solution of tempered fractional turbulent flow equation (1). □

Lemma 3. *Assume that* (**H**0)–(**H**2) *hold. Then,* $T\colon P \longrightarrow P$ *is a continuous, compact operator.*

Proof. It follows from the definition of P that, for any $x \in P$, there exists a number $0 < l_x < 1$ such that

$$0 \leq x(t) \leq l_x^{-1} e^{-\lambda t} t^{\beta-1}, \quad t \in [0,1]. \tag{23}$$

Since T is increasing with respect to x, by (14), (23), and Lemma 2, we have

$$\begin{aligned}
T(x)(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)f(x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
&\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)f(x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
&\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)\frac{f(x(\tau))}{(x(\tau)+2)^{\epsilon}}(x(\tau)+2)^{\epsilon}\mathrm{d}\tau\right)\mathrm{d}s\\
&\leq \varphi_q(d)e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)(x(\tau)+2)^{\epsilon}\mathrm{d}\tau\right)\mathrm{d}s\\
&\leq \varphi_q(d)e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)\left(l_x^{-1}e^{-\lambda\tau}\tau^{\beta-1}+2\right)^{\epsilon}\mathrm{d}\tau\right)\mathrm{d}s\\
&\leq M_1^{*}\varphi_q\left(d\left(l_x^{-1}+2\right)^{\epsilon}\int_0^1 M_2(\tau)h(\tau)d\tau\right)e^{-\lambda t}t^{\beta-1}\\
&\leq \frac{1}{l_x^{*}}e^{-\lambda t}t^{\beta-1},
\end{aligned} \tag{24}$$

where

$$l_x^{*} = \min\left\{\frac{1}{2}, M_1^{*\,-1}\left(d\left(l_x^{-1}+2\right)^{\epsilon}\int_0^1 M_2(\tau)h(\tau)\mathrm{d}\tau\right)^{1-q}\right\}. \tag{25}$$

Thus, it follows from (24) that

$$0 \leq T(x)(t) \leq \frac{1}{l_x^{*}}e^{-\lambda t}t^{\beta-1}, \tag{26}$$

which implies that T is well defined and uniformly bounded, and $T(P) \subset P$.

On the contrary, according to the Arzela–Ascoli theorem and the Lebesgue dominated convergence theorem, it is easy to know that $T\colon P \longrightarrow P$ is completely continuous. □

3. Main Results

Before we begin to state our main result, we first give the following lemma.

Lemma 4. *Suppose* $\epsilon(q-1)>3$ *and* $(\mathbf{H}2)$ *hold; then, the equation*

$$M_1^*\left(d\int_0^1 M_2(\tau)h(\tau)\mathrm{d}\tau\right)^{q-1}(x+3)^{\epsilon(q-1)}(x+1)^{-1}=1, \tag{27}$$

has unique solution δ^* *in* $(0,\infty)$.

Proof. Let

$$\phi(x)=1-M_1^*\left(d\int_0^1 M_2(\tau)h(\tau)\mathrm{d}\tau\right)^{q-1}(x+3)^{\epsilon(q-1)}(x+1)^{-1}. \tag{28}$$

It follows from $\epsilon(q-1)>3$ and $(\mathbf{H}2)$ that

$$\phi(0)=1-M_1^*\left(d\int_0^1 M_2(\tau)h(\tau)\mathrm{d}\tau\right)^{q-1}3^{\epsilon(q-1)}>0, \tag{29}$$

$$\phi(+\infty)=-\infty. \tag{30}$$

On the contrary, $\phi(x)$ is a continuous function in $[0,\infty)$ satisfying

$$\begin{aligned}\phi'(x)&=-\epsilon(q-1)M_1^*\left(d\int_0^1 M_2(\tau)h(\tau)\mathrm{d}\tau\right)^{q-1}(x+3)^{\epsilon(q-1)-1}(x+1)^{-1}+M_1^*\left(d\int_0^1 M_2(\tau)h(\tau)\mathrm{d}\tau\right)^{q-1}(x+3)^{\epsilon(q-1)}(x+1)^{-2}\\&=-M_1^*\left(d\int_0^1 M_2(\tau)h(\tau)\mathrm{d}\tau\right)^{q-1}(x+3)^{\epsilon(q-1)-1}(x+1)^{-1}\left(\epsilon(q-1)-\frac{x+3}{x+1}\right)<0,\quad x\in(0,\infty).\end{aligned} \tag{31}$$

Thus, (29)–(31) imply equation (27) has unique zero point δ^*in $(0,\infty)$. ☐

Theorem 1. *Suppose* $(\mathbf{H}0)$–$(\mathbf{H}2)$ *hold. Then, the following is obtained:*

(i) Existence: equation (1) has a positive minimal solution $\underline{x}$ *and a positive maximal solution* $\overline{x}$.

(ii) Asymptotic estimates: there exist positive numbers $n_i>0$, $i=1,2$, *such that*

$$\begin{aligned}&\frac{\underline{x}(t)}{e^{-\lambda t}t^{\beta-1}}\in[0,n_1],\\&\frac{\overline{x}(t)}{e^{-\lambda t}t^{\beta-1}}\in[0,n_2],t\in(0,1].\end{aligned} \tag{32}$$

(iii) Iterative sequences: for initial values $x^{(0)}(t)=0$ *and* $y^{(0)}(t)=\delta^*+1$, *construct the iterative sequences*

$$\begin{aligned}x^{(n)}(t)&=\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)f\left(x^{(n-1)}(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s,\\y^{(n)}(t)&=\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)f\left(y^{(n-1)}(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s.\end{aligned} \tag{33}$$

Then,

$$\begin{aligned}&\lim_{n\longrightarrow+\infty}x^{(n)}(t)=\underline{x}(t),\\&\lim_{n\longrightarrow+\infty}y^{(n)}(t)=\overline{x}(t),\end{aligned} \tag{34}$$

uniformly, for $t\in[0,1]$, where δ^* is the unique solution of equation (27) in $(0,\infty)$.

Proof. Firstly, let $P_{\delta^*}=\{x\in P: 0\le\|x\|\le\delta^*+1\}$; we shall show $T(P_{\delta^*})\subset P_{\delta^*}$.

For any $x\in P_{\delta^*}$ and for any $t\in(0,1)$, we have

$$0\le x(t)\le\max_{t\in[0,1]}x(t)\le\delta^*+1. \tag{35}$$

Consequently, it follows from $(\mathbf{H}1)$ and Lemma 4 that

$$\begin{aligned}\|T(x)\|&=\max_{t\in[0,1]}\left\{\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)f(x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right\}\\&\le\int_0^1 M_1(s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)f(x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\&\le\int_0^1 M_1(s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)\frac{f(x(\tau))}{(x(\tau)+2)^\epsilon}(x(\tau)+2)^\epsilon\mathrm{d}\tau\right)\mathrm{d}s\\&\le\varphi_q(d)\int_0^1 M_1(s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)(x(\tau)+2)^\epsilon\mathrm{d}\tau\right)\mathrm{d}s\\&\le\varphi_q(d)\int_0^1 M_1(s)\varphi_q\left(\int_0^1 G(s,\tau)h(\tau)(\delta^*+3)^\epsilon\mathrm{d}\tau\right)\mathrm{d}s\\&\le M_1^*\left(d\int_0^1 M_2(\tau)h(\tau)\mathrm{d}\tau\right)^{q-1}(\delta^*+3)^{\epsilon(q-1)}\\&=\delta^*+1,\end{aligned} \tag{36}$$

which implies that $T(P_{\delta^*})\subset P_{\delta^*}$.

Next, take the initial value $x^{(0)}(t)=0$, and let

$$x^{(1)}(t)=T\left(x^{(0)}\right)(t)=T(0)(t),\quad t\in[0,1]. \tag{37}$$

It follows from $x^{(0)}(t)\in P_{\delta^*}$ that $x^{(1)}(t)\in T(P_{\delta^*})\subset P_{\delta^*}$. Denote

$$x^{(n+1)}(t)=Tx^{(n)}(t)=T^{n+1}x^{(0)}(t),\quad n=1,2,\ldots. \tag{38}$$

By $T(P_{\delta^*}) \subset P_{\delta^*}$, we have $x_n \in P_{\delta^*}$ for $n \geq 1$. It follows from the fact of T being a compact operator that $\{x^{(n)}\}$ is a sequentially compact set.

On the contrary, since $x^{(1)}(t) \geq 0 = x^{(0)}(t)$ and T is increasing on x, we have

$$x^{(2)}(t) = \left(Tx^{(1)}\right)(t) \geq \left(Tx^{(0)}\right)(t) = x^{(1)}(t), \quad t \in [0,1]. \tag{39}$$

By induction, one has

$$0 \leq x^{(n)}(t) \leq x^{(n+1)}(t) \leq \delta^* + 1, \quad n = 1, 2, \ldots. \tag{40}$$

Consequently, there exists $\underline{x} \in P_{\delta^*}$ such that $x^{(n)} \longrightarrow \underline{x}$. Noticing that $Tx^{(n)} = x^{(n-1)}$ and letting $n \longrightarrow +\infty$, by the continuity of T, we have $T\underline{x} = \underline{x}$, which implies that $\underline{x}$ is a nonnegative solution of equation (1), and then $\underline{x}$ is a positive solution of equation (1) since $f(0) > 0$.

Now, we take $y^{(0)}(t) = \delta^* + 1$ as the initial value and let

$$y^{(1)}(t) = \left(T\left(y^{(0)}\right)(t)\right), \quad t \in [0,1]. \tag{41}$$

It follows from $y^{(0)}(t) = \delta^* + 1 \in P_{\delta^*}$ that $y^{(1)} \in P_{\delta^*}$. Thus, construct the iterative sequence

$$y^{(n+1)}(t) = Ty^{(n)}(t) = T^{n+1}y^{(0)}(t), \quad n = 1, 2, \ldots. \tag{42}$$

We have

$$y^{(n)}(t) \in P_{\delta*}, \quad n = 0, 1, 2, \ldots, \tag{43}$$

since $T(P_{\delta^*}) \subset P_{\delta^*}$. It follows from Lemma 3 that $\{y^{(n)}\}$ is a sequentially compact set.

Now, since $y^{(1)} \in P_{\delta^*}$ and T is increasing, one has

$$0 \leq y^{(1)}(t) \leq \left\|y^{(1)}\right\| \leq \delta^* + 1 = y^{(0)}(t), \tag{44}$$

and then

$$y^{(2)}(t) = Ty^{(1)}(t) \leq Ty^{(0)(t)} = y^{(1)}(t). \tag{45}$$

It follows from induction that

$$0 \leq y^{(n+1)}(t) \leq y^{(n)}(t) \leq \delta^* + 1, \quad n = 0, 1, 2, \ldots, \tag{46}$$

which implies that there exists $\overline{x} \in P_{\delta^*}$ such that $y^{(n)} \longrightarrow \overline{x}$. Letting $n \longrightarrow +\infty$, from the continuity of T and $Ty^{(n)} = y^{(n-1)}$, we have $T\overline{x} = \overline{x}$, which implies that $\overline{x}$ is another positive solution of equation (1).

Next, we prove that $\underline{x}$ and $\overline{x}$ are the maximum and minimum positive solutions of equation (1). In fact, suppose $\tilde{x}$ is any positive solution of equation (1); then, we have

$$\begin{aligned} &x^{(0)}(t) = 0 \leq \tilde{x}(t) \leq \delta^* + 1 = y^{(0)}(t), \\ &x^{(1)}(t) = Tx^{(0)}(t) \leq T\tilde{x}(t) = \tilde{x}(t) \leq T\left(y^{(0)}\right)(t) = y^{(1)}(t). \end{aligned} \tag{47}$$

Thus, it follows from induction that

$$x^{(n)}(t) \leq \tilde{x}(t) \leq y^{(n)}(t), \quad n = 1, 2, 3, \ldots. \tag{48}$$

Taking the limit, we have

$$\underline{x} \leq \tilde{x} \leq \overline{x}, \tag{49}$$

which implies that $\underline{x}$ and $\overline{x}$ are the maximal and minimal positive solutions of equation (1), respectively.

In the end, since $\underline{x}, \overline{x} \in P_{\delta^*} \subset P$, there exist constants $n_1 >, n_2 > 0$ such that

$$\begin{aligned} &\frac{\underline{x}(t)}{e^{-\lambda t}t^{\beta-1}} \in [0, n_1], \\ &\frac{\overline{x}(t)}{e^{-\lambda t}t^{\beta-1}} \in [0, n_2], \quad t \in (0,1]. \end{aligned} \tag{50}$$

□

4. Example

Since the fractional-order derivative possesses long-memory characteristics, in fluid mechanics, equation (1) can describe a turbulent flow in a porous medium. Here, we give a specific example to illustrate the main results.

Example: consider the following nonlocal tempered fractional turbulent flow equation:

$$\begin{cases} {}_0^R\mathbb{D}_t^{(3/2),1}\left(\varphi_{(3/2)}\left({}_0^R\mathbb{D}_t^{(7/2),1}x(t)\right)\right) = \dfrac{e^{-t}(1-t)^{-(1/4)}}{400}\left((x(t)+1)^3 \ln\left(1 + \dfrac{1}{2+x(t)}\right) + (2+x(t))^2 + 2\right), \\ x(0) = x'(0) = x''(0) = {}_0^R\mathbb{D}_t^{(7/2),1}x(0) = 0, \\ {}_0^R\mathbb{D}_t^{(7/2),1}x(1) = \displaystyle\int_0^1 e^{-(1-t)}x(t)\mathrm{d}t, \\ {}_0^R\mathbb{D}_t^{(1/4),1}\left(\varphi_{(3/2)}\left({}_0^R\mathbb{D}_t^{(7/2),1}x\right)\right)(1) = \displaystyle\int_0^1 {}_0^R\mathbb{D}_t^{(1/4),1}\left(\varphi_{(3/2)}\left({}_0^R\mathbb{D}_t^{(7/2),1}x(t)\right)\right)\mathrm{d}A(t), \end{cases} \tag{51}$$

where

$$A(t)=\begin{cases}0, & t\in\left[0,\dfrac{1}{2}\right),\\ 1, & t\in\left[\dfrac{1}{2},\dfrac{3}{4}\right),\\ \dfrac{1}{2}, & t\in\left[\dfrac{3}{4},1\right].\end{cases}\tag{52}$$

Then, equation (51) has the positive minimal and maximal solutions $\underline{x}$ and $\overline{x}$, and there exist constants $n_1>,n_2>0$ such that

$$\begin{gathered}\frac{\underline{x}(t)}{e^{-t}t^{5/2}}\in[0,n_1],\\ \frac{\overline{x}(t)}{e^{-t}t^{5/2}}\in[0,n_2],\\ t\in(0,1].\end{gathered}\tag{53}$$

Let

$$\begin{aligned}\alpha&=\frac{3}{2},\\ \beta&=\frac{7}{2},\\ \lambda&=1,\\ \gamma&=\frac{1}{4},\\ p&=\frac{3}{2},\\ a(t)&=1,\\ h(t)&=\frac{1}{400}e^{-t}(1-t)^{-1/4},\\ f(x)&=(x+1)^3\ln\left(1+\frac{1}{2+x}\right)+(2+x)^2+2.\end{aligned}\tag{54}$$

Firstly, we have

$$\begin{aligned}\delta&=\int_0^1 e^{-\lambda s}s^{\alpha-\gamma-1}a(s)\mathrm{d}A(s)=\int_0^1 e^{-s}s^{1/4}\mathrm{d}A(s)\\ &=0.2902<e^{-1}=0.3679.\end{aligned}\tag{55}$$

Thus, (**H**0) holds.

Obviously, $f\colon[0,+\infty)\longrightarrow(0,+\infty)$ is continuous and nondecreasing. Take $\epsilon=2>3/q-1=3/2$; then, we have

$$\begin{aligned}0<d=:\sup_{x\geq0}\frac{f(x)}{(x+2)^{\epsilon}}&=\sup_{x\geq0}\left(\left(\frac{x+1}{x+2}\right)^3\ln\left(1+\frac{1}{2+x}\right)^{x+2}\right.\\ &\quad\left.+1+\frac{2}{(x+2)^2}\right)=2<+\infty,\end{aligned}\tag{56}$$

which implies that (**H**1) is satisfied.

Now, we compute M_1^* and Δ:

$$\begin{aligned}M_1^*&=\frac{\beta^2e^{\lambda}}{(\beta-1)\Gamma(\beta+1)}=\frac{3.5^2\times e}{2.5\times\Gamma(4.5)}=0.1550,\\ \Delta&=\frac{e^{-\lambda}-\delta}{\Gamma(\alpha-\gamma)}=\frac{e^{-1}-0.3679}{\Gamma(5/4)}=0.0857.\end{aligned}\tag{57}$$

Thus, we have

$$\begin{aligned}M_2(s)&=\left[\frac{1}{\Gamma(3/2)}+\frac{0.3679}{0.0857\times\Gamma(3/2)\Gamma(5/4)}\right]e^{s}(1-s)^{1/4}\\ &=5.3439e^{s}(1-s)^{1/4}.\end{aligned}\tag{58}$$

Consequently,

$$0<\int_0^1 M_2(\tau)h(\tau)\mathrm{d}\tau=0.013358<\left(\frac{M_1^*}{3^{\epsilon(q-1)}}\right)^{1/q-1}d^{-1}=0.0219.\tag{59}$$

So, condition (**H**3) holds.

Thus, by Theorem 1, equation (51) has a positive minimal solution $\underline{x}$ and a positive maximal solution $\overline{x}$, and there exist constants $n_1>,n_2>0$ such that

$$\begin{gathered}\frac{\underline{x}(t)}{e^{-t}t^{5/2}}\in[0,n_1],\\ \frac{\overline{x}(t)}{e^{-t}t^{5/2}}\in[0,n_2],\\ t\in(0,1].\end{gathered}\tag{60}$$

5. Conclusion

In this work, we establish a new result on the existence of the maximum and minimum solutions for a class of tempered fractional-order differential equations with nonlocal boundary conditions. This type of equation can describe a turbulent flow of a porous medium in fluid mechanics and diffusive interaction. In order to obtain the extremal solutions of the equation, a new type of growth condition is introduced, and the iterative sequences with explicit initial values are constructed which converge uniformly to the maximum and minimum solutions; in addition, the estimations of the upper bounds of the maximum and minimum solutions are also derived.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Research Article

A Gradient Projection Algorithm with a New Stepsize for Nonnegative Sparsity-Constrained Optimization

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Nonnegative sparsity-constrained optimization problem arises in many fields, such as the linear compressing sensing problem and the regularized logistic regression cost function. In this paper, we introduce a new stepsize rule and establish a gradient projection algorithm. We also obtain some convergence results under milder conditions.

1. Introduction

In this paper, we are mainly concerned with the nonnegative sparsity-constrained optimization problem (NN-SCO):

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in S \cap R^n_+, \end{aligned} \tag{1}$$

where $f\colon R^n \longrightarrow R$ is a continuously differential function with a lower bound. $S\colon = \{x \in R^n\colon \|x\|_0 \leq s\}$ is a sparse set, where $s < n$ is a given integer regulating the sparsity level in x and R^n_+ is the nonnegative orthant in R^n. $\|x\|_0$ is the l_0 norm of x, counting the number of nonzero elements in x. Many application problems can be translated into problem (1), such as the widely studied linear compressing sensing problem of $f(x) = (1/2)\|Ax - b\|^2$ with $A \in R^{m\times n}$ being a sensing matrix, $b \in R^m$ is the observation vector, and $\|\cdot\|$ is the Euclidean norm in R^n [1]. Problem (1) has also used to the regularized logistic regression cost function [2].

Recently, a great deal of work has been devoted to algorithms for sparsity-constrained optimization problem. Beck and Eldar [3] established the IHT algorithm which converges to L-stationary under the Lipchitz continuity of the gradient of objective function. Beck and Hallak [4] generalized these results to sparse symmetric sets. Lu [5] designed a nonmonotone algorithm for symmetric set constraint problems. Pan, Xiu, and Zhou [6, 7] established the B-stationary, C-stationary, and α-stationary based on the Bouligand tangent cone and Clarke tangent. Recently, Pan, Zhou, and Xiu [8] established the improved IHT algorithm (IIHT) for problem (1) by using Armijo line search. They proved that any accumulation point converged to an α-stationary point under the restricted strong smoothness of objective function which is weaker than the Lipchitz continuity of the gradient.

Inspired by the above literature studies, in this paper, we establish a gradient projection algorithm with a new stepsize. The new algorithm removes the condition of the restricted strong smoothness of objective function which makes it more applicable. Meanwhile, we prove the convergence of the algorithm.

The rest of this paper is organized as follows. In Section 2, we present some notations, definitions, and lemmas. In Section 3, we give the algorithm of (1) and prove the convergence properties.

2. Preliminaries

2.1. Notations. To make it easier to read, we give some used notations as follows:

$$
\begin{aligned}
&S = \{x \in R^n\colon \|x\|_0 \le s\},\\
&I_1(x) \equiv \{i \in \{1,\dots,n\}\colon x_i \ne 0\},\\
&I_1(xy) = I_1(x) \cup I_1(y),\\
&P_{S\cap R^n_+}(x) \coloneqq \operatorname{argmin}_{y\in S\cap R^n_+}\|x-y\|,\\
&I_0(x) \equiv \{i \in \{1,\dots,n\}\colon x_i = 0\},\\
&|I_1(x)|\colon \text{the cardinality of } I_1(x).
\end{aligned} \tag{2}
$$

2.2. Definitions

Definition 1 (see [8]). Let $x^* \in S\cap R^n_+$ be a given feasible point of (1). We say that x^* is an α-stationary point, if there exists $\alpha > 0$ such that

$$x^* \in P_{S\cap R^n_+}(x^* - \alpha\nabla f(x^*)). \tag{3}$$

Definition 2 (see [9]). A function f is called $2s$-restricted strongly smooth ($2s$-RSS) with parameter $L_{2s} > 0$, and if for any $x, y \in R^n$ satisfying $|I_1(xy)| < 2s$, it holds that

$$f(y) - f(x) - \langle\nabla f(x), y - x\rangle \le \frac{L_{2s}}{2}\|y - x\|^2. \tag{4}$$

Definition 3 (see [9]). A function f is called $2s$-restricted strongly convex ($2s$-RSC) with parameter $l_{2s} > 0$, and if for any $x, y \in R^n$ satisfying $|I_1(xy)| < 2s$, it holds that

$$f(y) - f(x) - \langle\nabla f(x), y - x\rangle \ge \frac{l_{2s}}{2}\|y - x\|^2. \tag{5}$$

If and only if for any $x, y \in R^n$ and $|I_1(xy)| \le 2s$, we have

$$\left\|(\nabla f(x) - \nabla f(y))_{I_1(xy)}\right\| \ge l_{2s}\|x - y\|. \tag{6}$$

In particular, in (5), if $l_{2s} = 0$, the function f is called $2s$-restricted convex ($2s$-RC).

Definition 4 (see [10]). The projected gradient $\nabla_{S\cap R^n_+} f(x)$ of f is defined by

$$
\begin{aligned}
\nabla_{S\cap R^n_+} f(x) &= \left\|P_{T^C_{S\cap R^n_+}}(-\nabla f(x))\right\|,\\
&= \arg\min\left\{\|v + \nabla f(x)\| \;\middle|\; v \in T^C_{S\cap R^n_+}\right\},\\
&= \max\left\{\langle -\nabla f\left(x^k\right), v\rangle, \|v\| = 1\right\}.
\end{aligned} \tag{7}
$$

2.3. Lemmas

Lemma 1 (see [8]). *For $\alpha > 0$, vector $x^* \in S\cap R^n_+$ is an α-stationary point if and only if*

$$\nabla_i f(x^*)\begin{cases} = 0, & i \in I_1(x^*),\\ \ge \alpha M_s(x^*), & i \in I_0(x^*).\end{cases} \tag{8}$$

In particular, when $\|x^*\|_0 = s$, $x^* \ge 0$, $\nabla_i f(x^*) = \begin{cases} 0, & i \in I_1(x^*)\\ \ge -\alpha M_s(x^*), & i \in I_0(x^*)\end{cases}$.

When $\|x^*\|_0 < s$, $x^* \ge 0$, $\nabla_i f(x^*) = \begin{cases} 0, & i \in I_1(x^*)\\ \in R^n_+, & i \in I_0(x^*)\end{cases}$.

Lemma 2 (see [8]). $P_{S\cap R^n_+}(x) = P_S(P_{R^n_+}(x))$.

Lemma 3 (see [8]). *For any $x^* \in S\cap R^n_+$, we have*

$$T^C_{S\cap R^n_+}\left(x^k\right) = \operatorname{span}\left\{e_i, i \in I_1\left(x^k\right)\right\}, \tag{9}$$

where $e_i \in R^n$ is a vector whose i^{th} component is one and others are zeros.

3. Main Results

In this section, we establish a new algorithm which improves the IIHT algorithm for (1) and then we analyze its convergence properties. At first, let us develop the gradient projection algorithm with a new stepsize rule.

Algorithm 1

Step 1. Initialize $x^0 \in S\cap R^n_+$, $0 \le \theta \le 1$, and $\varepsilon > 0$, and set $k \Longleftarrow 0$.

Step 2. Compute $L_k = \sup_{\alpha>0}(\|\nabla f(x^k) - \nabla f(z^k(\alpha,\theta))\|/\|x^k - z^k(\alpha,\theta)\|)$, where

$$
\begin{aligned}
z^k(\alpha,\theta) &= x^k + \theta\left(x^k(\alpha) - x^k\right),\\
x^k(\alpha) &= P_{S\cap R^n_+}\left(x^k - \alpha\nabla f\left(x^k\right)\right).
\end{aligned} \tag{10}
$$

Step 3. Compute $x^{k+1} = P_{S\cap R^n_+}(x^k - \alpha_k\nabla f(x^k))$ where α_k satisfies $0 \le \alpha_k \le (1/3L_k)$.

Step 4. If $\|\nabla_{\Gamma_k} f(x^k)\| \le \varepsilon$, then stop; otherwise, set $k \Longleftarrow k+1$ and go to Step 2.

Next, let us list the following assumptions for convenience:

(1) For any $k > 0$, $L_k < +\infty$

(2) f is bounded below on $S\cap R^n_+$

Lemma 4. *Let the sequence $\{x^k\}$ be generated by Algorithm 1, and set $l_k = 3L_k$. Then, we have*

$$f\left(x^k\right) \le h_{l_k}\left(x^k, x^{k+1}\right), \tag{11}$$

where $h_{l_k}(x^k, x^{k+1}) = f(x^{k+1}) + \langle\nabla f(x^{k+1}), x^k - x^{k+1}\rangle + (l_k/2)\|x^k - x^{k+1}\|^2$.

Proof. Let

$$g(t) = f\left(x^{k+1} + t\left(x^k - x^{k+1}\right)\right). \tag{12}$$

Then,

$$\begin{aligned} g(0) &= f\left(x^{k+1}\right), \\ g(1) &= f\left(x^{k}\right), \\ g'(t) &= \left(x^k - x^{k+1}\right)^T \nabla f\left(x^{k+1} + t\left(x^k - x^{k+1}\right)\right). \end{aligned} \tag{13}$$

Thus,

$$\begin{aligned} f\left(x^k\right) - f\left(x^{k+1}\right) &= g(1) - g(0), \\ &= \int_0^1 g'(t)\mathrm{d}t, \\ &= \int_0^1 \left(x^k - x^{k+1}\right)^T \nabla f\left(x^{k+1} + t\left(x^k - x^{k+1}\right)\right)\mathrm{d}t, \\ &= \int_0^1 \left(x^k - x^{k+1}\right)^T \nabla f\left(x^{k+1}\right)\mathrm{d}t \\ &\quad + \int_0^1 \left(x^k - x^{k+1}\right)^T \left(\nabla f\left(x^{k+1} + t\left(x^k - x^{k+1}\right)\right) - \nabla f\left(x^{k+1}\right)\right)\mathrm{d}t, \\ &= \int_0^1 \left(x^k - x^{k+1}\right)^T \nabla f\left(x^{k+1}\right)\mathrm{d}t \\ &\quad + \int_0^1 \left(x^k - x^{k+1}\right)^T \left(\nabla f\left(x^{k+1} + t\left(x^k - x^{k+1}\right)\right) - \nabla f\left(x^{k+1}\right) + \nabla f\left(x^k\right) - \nabla f\left(x^k\right)\right)\mathrm{d}t, \\ &= \int_0^1 \left(x^k - x^{k+1}\right)^T \nabla f\left(x^{k+1}\right)\mathrm{d}t + \int_0^1 \left(x^k - x^{k+1}\right)^T \left(\nabla f\left(x^k\right) - \nabla f\left(x^{k+1}\right)\right)\mathrm{d}t, \\ &\quad + \int_0^1 \left(x^k - x^{k+1}\right)^T \left(\nabla f\left(x^{k+1} + t\left(x^k - x^{k+1}\right)\right) - \nabla f\left(x^k\right)\right)\mathrm{d}t, \\ &\leq \int_0^1 \left(x^k - x^{k+1}\right)^T \nabla f\left(x^{k+1}\right)\mathrm{d}t + \int_0^1 \left(x^k - x^{k+1}\right)^T \left(\nabla f\left(x^k\right) - \nabla f\left(x^{k+1}\right)\right)\mathrm{d}t, \\ &\quad + \int_0^1 \left\|x^k - x^{k+1}\right\| \left\|\left(\nabla f\left(x^{k+1} + t\left(x^k - x^{k+1}\right)\right) - \nabla f\left(x^k\right)\right)\right\|\mathrm{d}t, \\ &\leq \left(x^k - x^{k+1}\right)^T \nabla f\left(x^{k+1}\right) + L_k\left\|x^k - x^{k+1}\right\|^2 + \left\|x^k - x^{k+1}\right\| \int_0^1 L_k(1-t)\left\|x^k - x^{k+1}\right\|\mathrm{d}t, \\ &\leq \left(x^k - x^{k+1}\right)^T \nabla f\left(x^{k+1}\right) + \frac{3L_k}{2}\left\|x^k - x^{k+1}\right\|^2, \\ &= \left(x^k - x^{k+1}\right)^T \nabla f\left(x^{k+1}\right) + \frac{l_k}{2}\left\|x^k - x^{k+1}\right\|^2. \end{aligned} \tag{14}$$

Then, (11) is tenable. □

Lemma 5. *We suppose* $l \geq l_k$. *For* $x^k \in S \cap R_+^n$ *and* $x^{k+1} = P_{S\cap R_+^n}(x^k - (1/l)\nabla f(x^k))$, *we have*

$$f\left(x^k\right) - f\left(x^{k+1}\right) \geq \sigma\left\|x^k - x^{k+1}\right\|^2, \tag{15}$$

where $\sigma = (l - l_k/2)$.

Proof. Since $x^{k+1} \in P_{S\cap R_+^n}(x^k - (1/l)\nabla f(x^k))$, by the definition of projection, we get

$$x^{k+1} \in \arg\min_{x\in S\cap R_+^n}\left\|x - \left(x^k - \frac{1}{l}\nabla f\left(x^k\right)\right)\right\|^2. \tag{16}$$

Moreover,

$$\begin{aligned} h_l(x, x^k) &= f(x^k) + \langle \nabla f(x^k), x - x^k\rangle + \frac{l}{2}\|x - x^k\|,\\ &= \frac{l}{2}\left\|x - \left(x^k - \frac{l}{2}\nabla f(x^k)\right)\right\|^2 + f(x^k) - \frac{1}{2l}\|\nabla f(x^k)\|^2. \end{aligned} \tag{17}$$

Because $f(x^k) - (1/2l)\|\nabla f(x^k)\|^2$ is a constant independent of x, we can get

$$x^{k+1} \in \arg\min_{x\in P_{S\cap R_+^n}} h_l(x, x^k). \tag{18}$$

Therefore,

$$h_l(x^{k+1}, x^k) \geq h_l(x^k, x^k) = f(x^k). \tag{19}$$

By Lemma 4, we get

$$f(x^{k+1}) \geq h_l(x^{k+1}, x^k). \tag{20}$$

Hence,

$$\begin{aligned} f(x^k) - f(x^{k+1}) &\geq f(x^k) - h_{l_k}(x^{k+1}, x^k),\\ &\geq h_l(x^{k+1}, x^k) - h_{l_k}(x^{k+1}, x^k),\\ &= \frac{l - l_k}{2}\|x^{k+1} - x^k\|^2. \end{aligned} \tag{21}$$

Let $\sigma = (l - l_k/2)$. We get

$$f(x^k) - f(x^{k+1}) \geq \sigma\|x^k - x^{k+1}\|^2. \tag{22}$$

□

Lemma 6. *Let the sequence $\{x^k\}$ be generated by Algorithm 1. Then,*

(1) $f(x^k) - f(x^{k+1}) \geq ((1/\alpha_k) - l_k)/2\|x^k - x^{k+1}\|^2$

(2) $\{f(x^k)\}$ is an increasing sequence, and when $k \longrightarrow \infty$, $\{f(x^k)\}$ converges

(3) $\|x^k - x^{k+1}\| \longrightarrow 0$

(4) for any $k = 0, 1, 2, \ldots$, if $x^k \neq x^{k+1}$, we have $f(x^{k+1}) < f(x^k)$

Proof

(1) Since $0 \leq \alpha_k \leq (1/3L_k)$, we get

$$\frac{1}{\alpha_k} \geq 3L_k = l_k. \tag{23}$$

Setting $l = (1/\alpha_k)$ in (15), formula (1) can be obtained.

(2) We can easily get that $\{f(x^k)\}$ is an increasing sequence by (15). Moreover, by the assumptions (H_2), we can get that $\{f(x^k)\}$ converges.

(3) Let $\mu = ((1/\alpha_k) - l_k/2)$ in (1). We can get

$$f(x^k) - f(x^{k+1}) \geq \mu\|x^k - x^{k+1}\|^2. \tag{24}$$

Summing over both sides of this inequality, we get

$$\begin{aligned} \sum_{k=1}^{\infty}\|x^k - x^{k+1}\|^2 &\leq \sum_{k=1}^{\infty}\frac{2}{\mu}\left(f(x^k) - f(x^{k+1})\right),\\ &= \frac{2}{\mu}\left(f(x^0) - \lim_{k\longrightarrow+\infty} f(x^k)\right). \end{aligned} \tag{25}$$

Since f is bounded below, we get

$$\|x^k - x^{k+1}\| \longrightarrow 0. \tag{26}$$

(4) It easily can be got by (2). □

Lemma 7. *Let the sequence $\{x^k\}$ be generated by Algorithm 1. Suppose that the function f is 2s-RC. We have*

$$\|\nabla f(x^{k+1}) - \nabla f(x^k)\| \leq l_k\|x^{k+1} - x^k\|. \tag{27}$$

Proof. Because the sequence $\{x^k\}$ be generated by Algorithm 1, we get $|I_1(xy)| < 2s$. By Lemma 4 and Lemma 5 in reference [8], we can get

$$\|\nabla f(x^{k+1}) - \nabla f(x^k)\| \leq l_k\|x^{k+1} - x^k\|. \tag{28}$$

□

Theorem 1. *Let the sequence $\{x^k\}$ be generated by Algorithm 1. Then, the following results hold:*

(1) Any accumulation of sequence $\{x^k\}$ is an α-stationary point.

(2) If f is 2s-RC, the projected gradient sequence converges to zero, i.e.,

$$\lim_{k\longrightarrow+\infty}\|\nabla_{\Gamma_k} f(x^k)\| = 0. \tag{29}$$

Proof

(1) Suppose that x^* is an accumulation point of sequence $\{x^k\}$. Then, there exists a subsequence $\{x^{k_n}\}$ converges to x^*.

Because

$$\begin{aligned} \|x^{k_n}\| - \|x^{k_n+1} - x^{k_n}\| \leq \|x^{k_n+1}\| &= \|x^{k_n+1} - x^{k_n} + x^{k_n}\|\\ &\leq \|x^{k_n}\| + \|x^{k_n+1} - x^{k_n}\|, \end{aligned} \tag{30}$$

we get

$$\lim_{n\longrightarrow+\infty} x^{k_n+1} = \lim_{n\longrightarrow+\infty} x^{k_n} = x^*. \tag{31}$$

Moreover,

$$\begin{aligned} x^{k_n+1} &= P_{S\cap R_+^n R^n}\left(x^k - \alpha_{k_n}\nabla f(x^{k_n})\right),\\ &= P_S\left(P_{R_+^n}\left(x^k - \alpha_{k_n}\nabla f(x^{k_n})\right)\right). \end{aligned} \tag{32}$$

We consider the next two cases: □

Case 1. For $i \in I_1(x^*)$, there must exist a sufficiently large index N and a constant $c_0 > 0$ such that

$$\min\left\{x_i^{k_n}, x_i^{k_n+1}\right\} \geq c_0 > 0. \tag{33}$$

By $P_{S\cap R_+^n} = P_S(P_{R_+^n})$ and (33), we can get

$$x_i^{k_n+1} = x_i^k - \alpha_{k_n}\nabla_i f\left(x^{k_n}\right). \tag{34}$$

Since

$$\lim_{n\longrightarrow+\infty} \inf \alpha_{k_n} > 0, \tag{35}$$

without loss of generality, we can suppose $\lim_{k\longrightarrow+\infty}\alpha_{k_n} > c$. Let $n \longrightarrow +\infty$. We get

$$x_i^* = x_i^* - c\nabla_i f(x^*), \tag{36}$$

i.e.,

$$\nabla_i f(x^*) = 0, \quad \forall i \in I_1(x^*). \tag{37}$$

Case 2. For $i \in I_0(x^*)$, we consider two subcases.

Subcase 1. When $\|x^*\|_0 = s$, we get

$$0 = x^* = \lim_{n\longrightarrow+\infty} x_i^{k_n+1} = P_S\left(P_{R_+^n}\left(x^{k_n} - \alpha_{k_n}\nabla f\left(x^{k_n}\right)\right)\right)_i. \tag{38}$$

Due to the property of the projections P_S and $P_{R_+^n}$, we have

$$\max\left\{x_i^{k_n} - \alpha_{k_n}\nabla_i f\left(x^{k_n}\right), 0\right\} \leq M_s(x^*). \tag{39}$$

Thus,

$$x_i^{k_n} - \alpha_{k_n}\nabla_i f\left(x^{k_n}\right) \leq M_s(x^*). \tag{40}$$

Taking limits on both sides, we obtain

$$\nabla_i f(x^*) \geq -\frac{1}{c}M_s(x^*). \tag{41}$$

Subcase 2. When $\|x^*\|_0 < s$, suppose $\nabla_i f(x^*) < 0$, and we have

$$\lim_{n\longrightarrow+\infty}\left(x_i^{k_n} - \alpha_{k_n}\nabla_i f\left(x^{k_n}\right)\right) = -c\nabla_i f(x^*) > 0. \tag{42}$$

For all sufficiently large n, we have

$$\begin{aligned} P_{R_+^n}\left(x_i^{k_n} - \alpha_{k_n}\nabla_i f\left(x^{k_n}\right)\right) &= x_i^{k_n} - \alpha_{k_n}\nabla_i f\left(x^{k_n}\right), \\ &= -c\nabla_i f(x^*) > 0. \end{aligned} \tag{43}$$

Since $\|x^*\|_0 < s$, for all sufficiently large n, we have

$$\begin{aligned} x_i^{k_n+1} &= P_S\left(P_{R_+^n}\left(x^{k_n} - \alpha_{k_n}\nabla f\left(x^{k_n}\right)\right)\right)_i, \\ &= P_{R_+^n}\left(x^{k_n} - \alpha_{k_n}\nabla f\left(x^{k_n}\right)\right)_i > 0. \end{aligned} \tag{44}$$

which contradicts with $i \in I_0(x^*)$. Thus, $\nabla_i f(x^*) \geq 0$.

Summarizing the two cases, we obtain

$$\nabla_i f(x^*)\begin{cases} = 0, & i \in I_1(x^*), \\ \\ \geq -\dfrac{1}{c}M_s(x^*), & i \in I_0(x^*). \end{cases} \tag{45}$$

Thus, x^* is an α-stationary point of (1).

(2) Set $\Gamma^k = I_1(x^k)$. By Lemma 3, we have

$$\begin{aligned} T_{S\cap R_+^n}^C\left(x^k\right) &= R_{\Gamma^k}^n, \\ &= \operatorname{span}\left\{e_i, i \in I_1\left(x^k\right)\right\}. \end{aligned} \tag{46}$$

By Definition 4, we have

$$\begin{aligned} \left\|P_{T_{S\cap R_+^n}^C}\left(-\nabla f\left(x^k\right)\right)\right\| &= \max\left\{\left\langle\left(-\nabla f\left(x^k\right)\right), \nu\right\rangle, \|\nu\| = 1\right\}, \\ &= \left\|\nabla_{\Gamma_k} f\left(x^k\right)\right\|. \end{aligned} \tag{47}$$

Moreover, the maximum value is taken at $\|\nu\| = 1$. For any $\varepsilon > 0$, there exists $\nu^k \in R_{\Gamma^k}^n$ and $\|\nu^k\| = 1$ satisfies

$$\left\|\nabla_{\Gamma_k} f\left(x^k\right)\right\| \leq \langle -\nabla f\left(x^k\right), \nu^k\rangle + \varepsilon. \tag{48}$$

Because $x^{k+1} = P_{S\cap R_+^n}(x^k - \alpha_k\nabla f(x^k))$ and $x_{\Gamma^{k+1}} = y_{\Gamma^{k+1}}$, $x \in P_{S\cap R_+^n}(y)$, we get

$$x_{\Gamma^{k+1}}^{k+1} = \left(x^k - \alpha_k\nabla f\left(x^k\right)\right)_{\Gamma^{k+1}}, \tag{49}$$

i.e.,

$$x_{\Gamma^{k+1}}^{k+1} - \left(x^k - \alpha_k\nabla f\left(x^k\right)\right)_{\Gamma^{k+1}} = 0. \tag{50}$$

Thus, for any $\varpi_{k+1} \in R_{\Gamma^{k+1}}^n$, we get

$$\left\langle\left(x^{k+1} - \left(x^k - \alpha_k\nabla f\left(x^k\right)\right), \varpi_{k+1} - x_{k+1}\right\rangle = 0. \tag{51}$$

Taking $\varpi_{k+1} = x_{k+1} + \nu_{k+1}$, we get

$$\left\langle\left(x^{k+1} - x^k + \alpha_k\nabla f\left(x^k\right), -\nu_{k+1}\right\rangle = 0. \tag{52}$$

By the Cauchy–Schwartz inequality, we get

$$\langle\alpha_k\nabla f\left(x^k\right), -\nu^{k+1}\rangle = \langle x^{k+1} - x^k, \nu^{k+1}\rangle \leq \left\|x^{k+1} - x^k\right\|, \tag{53}$$

i.e.,

$$-\langle\nabla f\left(x^k\right), \nu^{k+1}\rangle \leq \frac{\left\|x^{k+1} - x^k\right\|}{\alpha_k}. \tag{54}$$

By Lemma 7, we get

$$\begin{aligned}-\langle\nabla f(x^{k+1}), v^{k+1}\rangle &= -\langle\nabla f(x^{k+1}) - \nabla f(x^k), v^{k+1}\rangle \\ &\quad - \langle\nabla f(x^k), v^{k+1}\rangle, \\ &\leq l_k\left\|x^{k+1} - x^k\right\| + \frac{\left\|x^{k+1} - x^k\right\|}{\alpha_k}.\end{aligned} \tag{55}$$

Taking limits on both sides and using Lemma 6, we have

$$\lim_{k\longrightarrow+\infty} \sup\langle -\nabla f(x^k), v^{k+1}\rangle \leq 0. \tag{56}$$

By (32), we get

$$\lim_{k\longrightarrow+\infty}\left\|\nabla_{\Gamma_k} f(x^k)\right\| = 0. \tag{57}$$

Theorem 2. *Let the sequence $\{x^k\}$ be generated by Algorithm 1. x^* is an accumulation point of the sequence $\{x^k\}$. Suppose $f(x)$ is $2s$ – RC, then the following results hold:*

(1) If $\|x^*\|_0 < s$, then x^* is a global minimizer of (1)

(2) If $\|x^*\|_0 = s$, then x^* is a local minimizer of (1)

Proof

(1) For $\forall x \in S \cap R^n_+$, we have $|I_1(xx^*)| = |I_1(x) \cup I_1(x^*)| \leq 2s$. Since $f(x)$ is $2s$ – RC, by Definition 3, we have

$$\begin{aligned}f(x) &\geq f(x^*) + \langle\nabla f(x^*), x - x^*\rangle, \\ &= f(x^*) + \sum_{i\in I_1(x^*)} \nabla_i f(x^*)(x_i - x_i^*) \\ &\quad + \sum_{i\in I_0(x^*)} \nabla_i f(x^*)(x_i - x_i^*).\end{aligned} \tag{58}$$

Because x^* is an accumulation point of the sequence $\{x^k\}$. By Theorem 1, x^* is an α-stationary. By Lemma 1, we can get

$$f(x) \geq f(x^*). \tag{59}$$

Thus, x^* is a global minimizer of (1).

(2) If $\|x^*\|_0 = s$, then $I_1(x^*) = I_1(x^k)$.

In fact, for all sufficiently large k, taking $0 < \delta < \min\{x_i^* : i \in I_1(x^*)\}$, we get

$$\left\|x^k - x^*\right\| \leq \delta. \tag{60}$$

For any $i \in I_1(x^*)$, we have

$$x_i^k = x_i^* - \left(x_i^* - x_i^k\right) \geq x_i^* - \left|x_i^* - x_i^k\right| > x_i^* - \delta > 0. \tag{61}$$

Thus,

$$I_1(x^*) \subset I_1(x^k). \tag{62}$$

By $\|x^k\|_0 = s$ and $|I_1(x^*)| = \|x^*\|_0 = s$, we have

$$I_1(x^*) = I_1(x^k). \tag{63}$$

For any $x^k \in S \cap R^n_+$ satisfying $\|x^k - x^*\| \leq \delta$, we have $|I_1(x^k x^*)| = |I_1(x^k) \cup I_1(x^*)| \leq 2s$. Since $f(x)$ is $2s$ – RC, by Definition 3, Theorem 1, and Lemma 1, we have

$$\begin{aligned}f(x^k) &\geq f(x^*) + \langle\nabla f(x^*), x^k - x^*\rangle, \\ &= f(x^*) + \sum_{i\in I_1(x^*)} \nabla_i f(x^*)\left(x_i^k - x_i^*\right) \\ &\quad + \sum_{i\in I_0(x^*)} \nabla_i f(x^*)\left(x_i^k - x_i^*\right), \\ &\geq f(x^*).\end{aligned} \tag{64}$$

Thus, x^* is a local minimizer of (1). □

Theorem 3. *Let the sequence $\{x^k\}$ be generated by Algorithm 1. x^* is a limit of the sequence $\{x^k\}$. Suppose $f(x)$ is $2s$ – RSC with parameter l_{2s} and $\|x^*\|_0 = s$, for all sufficiently large k, and we have*

$$\left\|x^{k+1} - x^*\right\|^2 \leq \left\|x^k - x^*\right\|^2, \quad 0 < \rho < 1, \tag{65}$$

where $\rho = 1 - (-2l_{2s}^2\alpha_k/L_k) + 2l_{2s}^2\alpha_k^2$.

Proof. By Theorem 2, we get $x^k \longrightarrow x^*$. As $f(x)$ is $2s$ – RSC with parameter l_{2s}, for any $x, y \in R^n$ and $|I_1(xy)| \leq 2s$, we have

$$\left\|(\nabla f(x) - \nabla f(y))_{I_1(xy)}\right\| \geq l_{2s}\|x - y\|. \tag{66}$$

Set $\Gamma^k = I_1(x^k)$ and $\Gamma^* = I_1(x^*)$. By Theorem 2, we get $\Gamma^k = \Gamma^*$. For all sufficiently large k, we have

$$\begin{aligned}\left\|\nabla_{\Gamma_*} f(x^*)\right\| &= \lim_{k\longrightarrow+\infty}\left\|\nabla_{\Gamma_k} f(x^k)\right\| = 0, \\ x^{k+1}_{\Gamma^{k+1}} &= \left(x^k - \alpha_k\nabla f(x^k)\right)_{\Gamma^{k+1}}.\end{aligned} \tag{67}$$

For all sufficiently large k, we have

$$\begin{aligned}\left\|x^{k+1} - x^*\right\|^2 &= \left\|\left(x^k_{\Gamma^*} - \alpha_k\nabla_{\Gamma^*} f(x^k)\right) - x^*_{\Gamma^*} + \alpha_k\nabla_{\Gamma^*} f(x^*)\right\|^2 \\ &= \left\|x^k - x^*\right\|^2 - 2\alpha_k\langle x^k - x^*, \nabla f(x^k) - \nabla f(x^*)\rangle \\ &\quad + \alpha_k^2\left\|\left(\nabla f(x^k) - \nabla f(x^*)\right)_{\Gamma^*}\right\|.\end{aligned} \tag{68}$$

Because $\|x^k - x^*\| \geq L_k\|\nabla f(x^k) - \nabla f(x^*)\|$, we get

$$\begin{aligned}\left\|x^{k+1} - x^*\right\|^2 &\leq \left\|x^k - x^*\right\|^2 - \left(\frac{2\alpha_k}{L_k} - \alpha_k^2\right) \\ &\quad \cdot \left\|\left(\nabla f(x^k) - \nabla f(x^*)\right)_{\Gamma^*}\right\|, \\ &\leq \left(1 - \frac{2l_{2s}^2\alpha_k}{L_k} + l_{2s}^2\alpha_k^2\right)\left\|x^k - x^*\right\|^2.\end{aligned} \tag{69}$$

Since $0 \leq \alpha_k \leq (1/3L_k)$ and $\sigma = (l - l_k/2) = (l - L_k/2)$, we have $0 \leq \alpha_k \leq (1/2\sigma + 3L_k)$. Thus,

$$\frac{\beta}{2\sigma + 3L_k} \leq \inf \alpha_k \leq \frac{1}{2\sigma + 3L_k}. \tag{70}$$

Setting $\alpha_* = (\beta/2\sigma + 3L_k)$, we get

$$\alpha_* \leq \alpha_k \leq \frac{1}{3L_k}. \tag{71}$$

Thus,

$$\begin{aligned}
1 - \frac{2l_{2s}^2\alpha_k}{L_k} + l_{2s}^2\alpha_k^2 &= 1 + l_{2s}^2\left(\alpha_k - \frac{1}{L_k}\right)^2 - \frac{l_{2s}^2}{L_k^2},\\
&\leq 1 + l_{2s}^2\left(\alpha_* - \frac{1}{L_k}\right)^2 - \frac{l_{2s}^2}{L_k^2},\\
&= 1 - \frac{2l_{2s}^2\alpha_*}{L_k} + l_{2s}^2\alpha_*^2,\\
&= \rho^2.
\end{aligned} \tag{72}$$

By $l_{2s} \leq L_k$ and $\rho^2 = 1 + l_{2s}^2(\alpha_* - (1/L_k))^2 - (l_{2s}^2/L_k^2)$, we get $\rho > 0$.

From $0 < \beta < 1$ and $\rho^2 = 1 - (2l_{2s}^2\alpha_*/L_k) + l_{2s}^2\alpha_*^2$, we have $\rho < 1$.

Thus,

$$\lim_{k\longrightarrow+\infty} \frac{\left\|x_{k+1} - x_*\right\|}{\left\|x_k - x_*\right\|} \leq \sqrt{\rho}, \tag{73}$$

where $0 < \sqrt{\rho} < 1$. Thus, the sequence $\{x^k\}$ is Q-linear convergence to x^*. □

4. Conclusions

In this paper, we are mainly concerned with the nonnegative sparsity-constrained optimization problem. We introduce a new stepsize rule and propose a new gradient projection algorithm to solve this problem. The new algorithm removes the condition of the restricted strong smoothness of objective function which makes the new algorithm more applicable. Meanwhile, we prove the convergence of the algorithm.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Nontrivial Solutions for a System of Second-Order Discrete Boundary Value Problems

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In this work, we shall study the existence of nontrivial solutions for a system of second-order discrete boundary value problems. Under some conditions concerning the eigenvalues of relevant linear operator, we use the topological degree theory to obtain our main results.

1. Introduction

Nonlinear discrete problems appear in many mathematical models, such as computer science, mechanical engineering, control systems, economics, and fluid mechanics (see [1–4]). Owing to the wide applications, in recent years, there are a large number of researchers paying special attention in this direction (we refer to some results [5–15] and the references therein). For example, in [5], the authors used the Guo–Krasnosel'skii fixed point theorem to study the existence of positive solutions for the following second-order discrete boundary value problem:

$$\begin{cases} \Delta^2 x_{i-1} + f(x_i) = 0, \quad i \in [1, n], \\ x_0 = 0 = x_{n+1}, \end{cases} \tag{1}$$

and the following discrete second-order system:

$$\begin{cases} \Delta^2 x_{i-1} + f(x_i, y_i) = 0, \quad i \in [1, n], \\ \Delta^2 y_{i-1} + g(x_i, y_j) = 0, \quad i \in [1, n], \\ x_0 = x_{n+1} = y_0 = y_{n+1} = 0, \end{cases} \tag{2}$$

where n is a positive integer, $[1, n] = \{1, 2, \ldots, n\}, \Delta$ is the forward difference operator, i.e., $\Delta x_{i-1} = x_i - x_{i-1}$, and $\Delta^2 x_{i-1} = \Delta(\Delta x_{i-1})$.

In [6], the authors used the monotone iterative technique to investigate the existence and uniqueness of positive solutions for the following discrete p-Laplacian fractional boundary value problem:

$$\begin{cases} \Delta_{\nu-1}^{\nu}\left(\phi_p\left(\Delta_{\nu-1}^{\nu} y(t)\right)\right) = f(y(t+\nu-1)), \quad t \in [0, T]_{\mathbb{Z}}, \\ y(\nu-1) = y(\nu+T), \Delta_{\nu-1}^{\nu} y(\nu-1) = \Delta_{\nu-1}^{\nu} y(\nu+T), \end{cases} \tag{3}$$

where $\nu \in (0, 1)$ is a real number, $\Delta_{\nu-1}^{\nu}$ is a discrete fractional operator, and $\phi_p(s) = |s|^{p-2}s$ is the p-Laplacian with $s \in \mathbb{R}, p > 1$.

Coupled systems of discrete problems have also been investigated by many authors; some results can be found in a series of papers [11–15] and the references cited therein (also see some results on differential systems [16–24]). For example, in [11], the authors used the Guo–Krasnosel'skii fixed point theorem to study the following systems of three-point discrete boundary value problems:

$$\begin{cases} \Delta^2 u(n-1) + \lambda a(n) f(u(n), v(n)) = 0, \quad n \in \{1, 2, \ldots, N-1\}, \\ \Delta^2 v(n-1) + \mu b(n) g(u(n), v(n)) = 0, \\ u(0) = \beta u(\eta), u(N) = \alpha u(\eta), v(0) = \beta v(\eta), v(N) = \alpha v(\eta), \end{cases} \tag{4}$$

where $N \geq 4$, $\eta \in \{1, 2, \ldots, N-1\}, \alpha > 0, \beta > 0, \lambda, \mu > 0$. They offered some values for the parameters λ, μ to yield a positive solution for the above system.

In [12], the authors used the fixed point index to study the positive solutions for the following system of first-order discrete fractional boundary value problems:

$$\begin{cases} \Delta_{\nu-1}^{\nu} x(t) = f_1(t+\nu-1, x(t+\nu-1), y(t+\nu-1)), & t \in [0, T]_{\mathbb{Z}}, \\ \Delta_{\nu-1}^{\nu} y(t) = f_2(t+\nu-1, x(t+\nu-1), y(t+\nu-1)), & t \in [0, T]_{\mathbb{Z}}, \\ x(\nu-1) = x(\nu+T), y(\nu-1) = y(\nu+T). \end{cases} \tag{5}$$

By discrete Jensen's inequality, the authors adopted some appropriate nonnegative concave and convex functions to characterize the coupling behavior of the nonlinearities $f_i (i = 1, 2)$.

Motivated by the aforementioned works, in this paper, by means of the topological degree theory, we study the existence of nontrivial solutions for the following system of second-order discrete boundary value problems:

$$\begin{cases} \Delta^2 u(k-1) + f(k, v(k)) = 0, & k \in \{1, 2, \ldots, T\}, \\ \Delta^2 v(k-1) + g(k, u(k)) = 0, \\ u(0) = u(T+1) = v(0) = v(T+1) = 0, \end{cases} \tag{6}$$

where $T > 2$ is a fixed positive integer number, $\triangle u(k) = u(k+1) - u(k), \Delta^2 u(k) = \triangle(\triangle u(k))$, and $f, g\colon \{1, 2, \ldots, T\} \times \mathbb{R} \longrightarrow \mathbb{R} (\mathbb{R} \coloneqq (-\infty, +\infty))$ are continuous and satisfy the following conditions:

(H1) There exist three nonnegative functions $a_i(k), b_i(k) (b_i(k) \not\equiv 0, k \in \mathbb{T}_1)$ and $\beta_i (i = 1, 2)$ on $\mathbb{R}^+$ such that

$$\begin{aligned} f(k, v) &\geq -a_1(k) - b_1(k)\beta_1(v), g(k, u) \\ &\geq -a_2(k) - b_2(k)\beta_2(u), \quad \forall u, v \in \mathbb{R}, t \in \mathbb{T}_1, \end{aligned} \tag{7}$$

where $\mathbb{T}_1 : = \{1, 2, \ldots, T\}$.

(H2) $\lim_{|v| \longrightarrow +\infty} \beta_1(v)/|v| = 0$, $\lim_{|u| \longrightarrow +\infty} \beta_2(u)/|u| = 0$.

(H3) $\liminf_{|v| \longrightarrow +\infty} f(k, v)/|v| > \lambda_1$, $\liminf_{|u| \longrightarrow +\infty} g(k, u)/|u| > \lambda_1$ uniformly on $k \in \mathbb{T}_1$, where $\lambda_1 = 4\sin^2(\pi/(2T+2))$.

(H4) $\limsup_{|v| \longrightarrow 0} |f(k, v)|/|v| < \lambda_1$, $\limsup_{|u| \longrightarrow 0} |g(k, u)|/|u| < \lambda_1$ uniformly on $k \in \mathbb{T}_1$.

Now, we state our main result here.

Theorem 1. *Suppose that (H1)–(H4) hold. Then, (6) has at least one nontrivial solution.*

2. Preliminaries

Let E be the Banach space of real valued functions defined on the discrete interval $\mathbb{T}_2$ with the norm $\|u\| = \max_{k \in \mathbb{T}_2} |u(k)|$, where $\mathbb{T}_2 : = \{0, 1, 2, \ldots, T+1\}$. Define the following sets:

$$\begin{aligned} P &= \{u \in E\colon u(k) \geq 0, \quad \forall k \in \mathbb{T}_2\}, \\ P_0 &= \left\{u \in E\colon \min_{k \in \mathbb{T}_1} u(k) \geq \frac{1}{T}\|u\|\right\}, \end{aligned} \tag{8}$$

and $B_r = \{x \in E\colon \|x\| < r\}$ for $r > 0$. Then, P, P_0 are cones on E, and B_r is an open ball in E.

Lemma 1 (see [11, 15]). *Let $h(k) \in C(\mathbb{T}_1)$. Then, the discrete boundary value problem*

$$\begin{cases} \Delta^2 u(k-1) + h(k) = 0, & k \in \mathbb{T}_1, \\ u(0) = u(T+1) = 0, \end{cases} \tag{9}$$

has a solution with the form

$$u(k) = \sum_{l=1}^{T} G(k, l)h(l), \quad k \in \mathbb{T}_2, \tag{10}$$

where

$$G(k, l) = \frac{1}{T+1} \begin{cases} l(T+1-k), & 1 \leq l \leq k-1 \leq T, \\ k(T+1-k), & 0 \leq k \leq l \leq T. \end{cases} \tag{11}$$

Furthermore, $G(k, l)$ has the following properties (see [13, 15]):

(i) $G(k, l) > 0$ and $G(k, l) = G(l, k)$, for $(k, l) \in \mathbb{T}_1 \times \mathbb{T}_1$.

(ii) $G(l, l)/T \leq G(k, l) \leq G(l, l)$, for $(k, l) \in \mathbb{T}_1 \times \mathbb{T}_1$.

By Lemma 1, system (6) is equivalent to

$$\begin{cases} u(k) = \sum_{l=1}^{T} G(k, l) f(l, v(l)), & k \in \mathbb{T}_2, \\ v(k) = \sum_{l=1}^{T} G(k, l) g(l, u(l)), & k \in \mathbb{T}_2. \end{cases} \tag{12}$$

Then, we can define operators $\mathcal{T}, \mathcal{S}\colon E \longrightarrow E$ by

$$\begin{aligned} (\mathcal{T} v)(k) &= \sum_{l=1}^{T} G(k, l) f(l, v(l)), (\mathcal{S} u)(k) \\ &= \sum_{l=1}^{T} G(k, l) g(l, u(l)), \end{aligned} \tag{13}$$

and operator $\mathcal{A}\colon E \times E \longrightarrow E \times E$ by

$$\mathcal{A}(u, v)(k) = ((\mathcal{T} v), (\mathcal{S} u))(k). \tag{14}$$

Note that $\mathcal{T}, \mathcal{S}, \mathcal{A}$ are completely continuous operators (see [11]), and (u, v) solves (6) if and only if (u, v) is a fixed point of the operator $\mathcal{A}$.

Lemma 2 (see [7, 15]). *Let $\phi(k) = sin(k\pi)/(T+1)$, $k \in \mathbb{T}_2$. Then, $\lambda_1 \sum_{l=1}^{T} G(k, l)\phi(l) = \phi(k)$, $\forall k \in \mathbb{T}_1$.*

Define a linear operator as follows:

$$(Lx)(k) = \sum_{l=1}^{T} G(k, l) x(l), \quad \forall k \in \mathbb{T}_2. \tag{15}$$

Then, we have

$$(L\phi)(k) = \frac{1}{\lambda_1}\phi(k), \tag{16}$$

and we have the following lemma.

Lemma 3. *If $x \in P$, then $Lx \in P_0$.*

This is a direct result by Lemma 1 (ii), so we omit the proof.

Remark 1. $\phi \in P_0$ in Lemma 2.

Lemma 4 (see [25, Theorem A.3.3]). *Let Ω be a bounded open set in a Banach space E and $T\colon \Omega \longrightarrow E$ be a continuous compact operator. If there exists $x_0 \in E\backslash\{0\}$ such that*

$$x - Tx \neq \mu x_0, \quad \forall x \in \partial\Omega, \quad \mu \geq 0, \tag{17}$$

then the topological degree $deg(I - T, \Omega, 0) = 0$.

Lemma 5 (see [25, Lemma 2.5.1]). *Let Ω be a bounded open set in a Banach space E with $0 \in \Omega$ and $T\colon \Omega \longrightarrow E$ be a continuous compact operator. If*

$$Tx \neq \mu x, \quad \forall x \in \partial\Omega, \quad \mu \geq 1, \tag{18}$$

then the topological degree $deg(I - T, \Omega, 0) = 1$.

3. Main Results

In order to obtain the Proof of Theorem 1, we first provide a lemma.

Lemma 6. *There exists a sufficiently large $R > 0$ such that*

$$\deg(I - \mathscr{A}, B_R, 0) = 0. \tag{19}$$

Proof. By (H3), there exist $\varepsilon_1 > 0$ and $X_1 > 0$ such that

$$\begin{aligned} f(k,v) \geq (\lambda_1 + \varepsilon_1)|v|, g(k,u) \geq (\lambda_1 + \varepsilon_1)|u|, \\ \forall k \in \mathbb{T}_1, |u|, |v| > X_1. \end{aligned} \tag{20}$$

Note that when $k \in \mathbb{T}_1, |u|, |v| \leq X_1$, the functions $|f(k,v)|$ and $|g(k,u)|$ are bounded, so we can choose some appropriate positive numbers M_1, M_2 such that

$$\begin{aligned} f(k,v) \geq (\lambda_1 + \varepsilon_1)|v| - M_1, g(k,u) \geq (\lambda_1 + \varepsilon_1)|u| - M_2, \\ \forall k \in \mathbb{T}_1, u, v \in \mathbb{R}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} M_1 &= \max_{k\in\mathbb{T}_1, |u|,|v| \leq X_1} |f(k,v)| + (\lambda_1 + \varepsilon_1)X_1, \\ M_2 &= \max_{k\in\mathbb{T}_1, |u|,|v| \leq X_1} |g(k,u)| + (\lambda_1 + \varepsilon_1)X_1. \end{aligned} \tag{22}$$

From (H2), for any given $\varepsilon, \tilde{\varepsilon} > 0$ with $\varepsilon_1 - \varepsilon\|b_1\| > 0, \varepsilon_1 - \tilde{\varepsilon}\|b_2\| > 0$, there is $X_2 > X_1$ such that

$$\beta_1(v) \leq \varepsilon|v|, \beta_2(u) \leq \tilde{\varepsilon}|u|, \quad \forall |u|, |v| > X_2. \tag{23}$$

Let $\beta_1^* = \max_{|v|\leq X_2}\beta_1(v)$ and $\beta_2^* = \max_{|u|\leq X_2}\beta_2(u)$. Then,

$$\beta_1(v) \leq \varepsilon|v| + \beta_1^*, \beta_2(u) \leq \tilde{\varepsilon}|u| + \beta_2^*, \quad u, v \in \mathbb{R}. \tag{24}$$

Thus, we have

$$\begin{aligned} f(k,v) &\geq (\lambda_1 + \varepsilon_1)|v| - a_1(k) - b_1(k)\beta_1(v) - M_1 \\ &\geq (\lambda_1 + \varepsilon_1)|v| - a_1(k) - b_1(k)[\varepsilon|v| + \beta_1^*] - M_1 \\ &\geq \left(\lambda_1 + \varepsilon_1 - \varepsilon\|b_1\|\right)|v| - a_1(k) - \beta_1^* b_1(k) - M_1, \\ &\qquad \forall k \in \mathbb{T}_1, v \in \mathbb{R}, \end{aligned} \tag{25}$$

$$\begin{aligned} g(k,u) \geq \left(\lambda_1 + \varepsilon_1 - \tilde{\varepsilon}\|b_2\|\right)|u| - a_2(k) - \beta_2^* b_2(k) - M_2, \\ \forall k \in \mathbb{T}_1, u \in \mathbb{R}. \end{aligned} \tag{26}$$

Note that $\varepsilon, \tilde{\varepsilon}$ can be chosen arbitrarily small, so we can let

$$R > \max\{N_1, N_2, N_3, N_4\}, \tag{27}$$

where

$$\begin{aligned} N_1 &= \frac{2\sum_{l=1}^{T} G(l,l)[a_1(l) + \beta_1^* b_1(l) + M_1]}{1 - 2\varepsilon\sum_{l=1}^{T} G(l,l)b_1(l)}, \\ N_2 &= \frac{2\sum_{l=1}^{T} G(l,l)[a_2(l) + \beta_2^* b_2(l) + M_2]}{1 - 2\tilde{\varepsilon}\sum_{l=1}^{T} G(l,l)b_2(l)}, \\ N_3 &= \frac{\left(\lambda_1 T + (1+T)\left(\varepsilon_1 - \varepsilon\|b_1\|\right)\right)\sum_{l=1}^{T} G(l,l)[a_1(l) + a_2(l) + \beta_1^* b_1(l) + \beta_2^* b_2(l) + M_1 + M_2]}{\left(\varepsilon_1 - \varepsilon\|b_1\|\right) - \left(\lambda_1 T + (1+T)\left(\varepsilon_1 - \varepsilon\|b_1\|\right)\right)\left[\varepsilon\sum_{l=1}^{T} G(l,l)b_1(l) + \tilde{\varepsilon}\sum_{l=1}^{T} G(l,l)b_2(l)\right]}, \\ N_4 &= \frac{\left(\lambda_1 T + (1+T)\left(\varepsilon_1 - \varepsilon\|b_2\|\right)\right)\sum_{l=1}^{T} G(l,l)[a_1(l) + a_2(l) + \beta_1^* b_1(l) + \beta_2^* b_2(l) + M_1 + M_2]}{\left(\varepsilon_1 - \varepsilon\|b_2\|\right) - \left(\lambda_1 T + (1+T)\left(\varepsilon_1 - \varepsilon\|b_2\|\right)\right)\left[\varepsilon\sum_{l=1}^{T} G(l,l)b_1(l) + \tilde{\varepsilon}\sum_{l=1}^{T} G(l,l)b_2(l)\right]}. \end{aligned} \tag{28}$$

Now, we prove

$$(u,v)-\mathscr{A}(u,v)\neq\mu(\phi,\phi),\quad \forall u,v\in\partial B_R,\ \mu\geq 0,\tag{29}$$

where $\phi(k)=\sin(k\pi)/(T+1)$, $k\in\mathbb{T}_2$. We argue this claim by indirection. Suppose that there exist $u,v\in\partial B_R,\mu\geq 0$ such that

$$(u,v)-\mathscr{A}(u,v)=\mu(\phi,\phi).\tag{30}$$

$$u(k)=(\mathscr{T}v)(k)+\mu\phi(k)=\sum_{l=1}^{T}G(k,l)f(l,v(l))+\mu\phi(k),\tag{31}$$

$$v(k)=(\mathscr{S}u)(k)+\mu\phi(k)=\sum_{l=1}^{T}G(k,l)g(l,u(l))+\mu\phi(k).\tag{32}$$

$$\begin{aligned}\tilde{v}(k)&=\sum_{l=1}^{T}G(k,l)\left[a_1(l)+b_1(l)\beta_1(v(l))+M_1\right],\\ \tilde{u}(k)&=\sum_{l=1}^{T}G(k,l)\left[a_2(l)+b_2(l)\beta_2(u(l))+M_2\right].\end{aligned}\tag{33}$$

Then by Lemma 3, $\tilde{u},\tilde{v}\in P_0$, and we also have

$$\begin{aligned}u(k)+\tilde{v}(k)&=\sum_{l=1}^{T}G(k,l)\left[f(l,v(l))+a_1(l)\right.\\&\quad\left.+b_1(l)\beta_1(v(l))+M_1\right]+\mu\phi(k),\\ v(k)+\tilde{u}(k)&=\sum_{l=1}^{T}G(k,l)\left[g(l,u(l))+a_2(l)\right.\\&\quad\left.+b_2(l)\beta_2(u(l))+M_2\right]+\mu\phi(k).\end{aligned}\tag{34}$$

Using (24) and (25), we have

$$\begin{aligned}&f(l,v(l))+a_1(l)+b_1(l)\beta_1(v(l))+M_1\in P,\\&g(l,u(l))+a_2(l)+b_2(l)\beta_2(u(l))+M_2\in P.\end{aligned}\tag{35}$$

So, from Lemma 3 and Remark 1, we have

$$v+\tilde{u},u+\tilde{v}\in P_0.\tag{36}$$

Note that $u,v\in\partial B_R$, and using (24), $R>N_1$, and $R>N_2$, we have

$$\begin{aligned}\|\tilde{v}\|&\leq\sum_{l=1}^{T}G(l,l)\left[a_1(l)+b_1(l)\beta_1(v(l))+M_1\right]\\&\leq\sum_{l=1}^{T}G(l,l)\left[a_1(l)+b_1(l)\left(\varepsilon\|v\|+\beta_1^*\right)+M_1\right]<\frac{R}{2},\\ \|\tilde{u}\|&\leq\sum_{l=1}^{T}G(l,l)\left[a_2(l)+b_2(l)\left(\tilde{\varepsilon}\|u\|+\beta_2^*\right)+M_2\right]<\frac{R}{2}.\end{aligned}\tag{37}$$

It is noted that $\|u\|=\|v\|=R,u+\tilde{u}+\tilde{v}\in P_0$, and $v+\tilde{u}+\tilde{v}\in P_0$. Therefore, we get

$$\begin{aligned}u(k)+\tilde{u}(k)+\tilde{v}(k)&\geq\frac{1}{T}\|u+\tilde{u}+\tilde{v}\|\geq\frac{1}{T}\left(\|u\|-\|\tilde{u}+\tilde{v}\|\right)\\&\geq\frac{1}{T}\left[\|u\|-\left(\|\tilde{u}\|+\|\tilde{v}\|\right)\right],\\ v(k)+\tilde{u}(k)+\tilde{v}(k)&\geq\frac{1}{T}\|v+\tilde{u}+\tilde{v}\|\geq\frac{1}{T}\left(\|v\|-\|\tilde{u}+\tilde{v}\|\right)\\&\geq\frac{1}{T}\left[\|v\|-\left(\|\tilde{u}\|+\|\tilde{v}\|\right)\right].\end{aligned}\tag{38}$$

Using $R>N_3$, we have

$$\begin{aligned}&\left(\varepsilon_1-\varepsilon\|b_1\|\right)\sum_{l=1}^{T}G(k,l)\left[v(l)+\tilde{u}(l)+\tilde{v}(l)\right]-\left(\lambda_1+\varepsilon_1-\varepsilon\|b_1\|\right)\sum_{l=1}^{T}G(k,l)\left[\tilde{u}(l)+\tilde{v}(l)\right]\\&\quad\geq\frac{\varepsilon_1-\varepsilon\|b_1\|}{T}\sum_{l=1}^{T}G(k,l)\left[R-\left(\|\tilde{u}\|+\|\tilde{v}\|\right)\right]-\left(\lambda_1+\varepsilon_1-\varepsilon\|b_1\|\right)\sum_{l=1}^{T}G(k,l)\left[\left(\|\tilde{u}\|+\|\tilde{v}\|\right)\right]\geq 0,\end{aligned}\tag{39}$$

and $R>N_4$ implies that

$$\left(\varepsilon_1-\tilde{\varepsilon}\|b_2\|\right)\sum_{l=1}^{T}G(k,l)\left[u(l)+\tilde{u}(l)+\tilde{v}(l)\right]-\left(\lambda_1+\varepsilon_1-\tilde{\varepsilon}\|b_2\|\right)\sum_{l=1}^{T}G(k,l)\left[\tilde{u}(l)+\tilde{v}(l)\right]\geq 0.\tag{40}$$

Consequently, we obtain

$$\begin{aligned}
(\mathcal{T}v)(k)+\tilde{v}(k) &= \sum_{l=1}^{T} G(k,l)\left[f(l,v(l))+a_1(l)+b_1(l)\beta_1(v(l))+M_1\right] \\
&\geq \sum_{l=1}^{T} G(k,l)\left[\left(\lambda_1+\varepsilon_1-\varepsilon\left\|b_1\right\|\right)|v(l)|-a_1(l)-\beta_1^* b_1(l)-M_1+a_1(l)+b_1(l)\beta_1(v(l))+M_1\right] \\
&\geq \sum_{l=1}^{T} G(k,l)\left[\left(\lambda_1+\varepsilon_1-\varepsilon\left\|b_1\right\|\right)|v(l)|-\beta_1^* b_1(l)+b_1(l)\left(\varepsilon|v(l)|+\beta_1^*\right)\right] \geq \left(\lambda_1+\varepsilon_1-\varepsilon\left\|b_1\right\|\right)\sum_{l=1}^{T} G(k,l)|v(l)| \\
&\geq \left(\lambda_1+\varepsilon_1-\varepsilon\left\|b_1\right\|\right)\sum_{l=1}^{T} G(k,l)[v(l)+\tilde{u}(l)+\tilde{v}(l)]-\left(\lambda_1+\varepsilon_1-\varepsilon\left\|b_1\right\|\right)\sum_{l=1}^{T} G(k,l)[\tilde{u}(l)+\tilde{v}(l)] \\
&\geq \lambda_1\sum_{l=1}^{T} G(k,l)[v(l)+\tilde{u}(l)+\tilde{v}(l)] \geq \lambda_1\sum_{l=1}^{T} G(k,l)[v(l)+\tilde{u}(l)], \\
(\mathcal{S}u)(k)+\tilde{u}(k) &= \sum_{l=1}^{T} g(k,l)\left[g(l,v(l))+a_2(l)+b_2(l)\beta_2(u(l))+M_2\right] \\
&\geq \sum_{l=1}^{T} G(k,l)\left[\left(\lambda_1+\varepsilon_1-\tilde{\varepsilon}\left\|b_2\right\|\right)|u(l)|-a_2(l)-\beta_2^* b_2(l)-M_2+a_2(l)+b_2(l)\beta_2(u(l))+M_2\right] \\
&\geq \left(\lambda_1+\varepsilon_1-\tilde{\varepsilon}\left\|b_2\right\|\right)\sum_{l=1}^{T} G(k,l)|u(l)| \\
&\geq \left(\lambda_1+\varepsilon_1-\tilde{\varepsilon}\left\|b_2\right\|\right)\sum_{l=1}^{T} G(k,l)[u(l)+\tilde{u}(l)+\tilde{v}(l)]-\left(\lambda_1+\varepsilon_1-\tilde{\varepsilon}\left\|b_2\right\|\right) \\
&\quad\cdot\sum_{l=1}^{T} G(k,l)[\tilde{u}(l)+\tilde{v}(l)] \geq \lambda_1\sum_{l=1}^{T} G(k,l)[u(l)+\tilde{u}(l)+\tilde{v}(l)] \\
&\geq \lambda_1\sum_{l=1}^{T} G(k,l)[u(l)+\tilde{v}(l)].
\end{aligned} \tag{41}$$

As a result, we get

$$(\mathcal{T}v)(k)+(\mathcal{S}u)(k)+\tilde{u}(k)+\tilde{v}(k)\geq\lambda_1(L(u+v+\tilde{u}+\tilde{v}))(k). \tag{42}$$

In view of (31) and (32), we see

$$\begin{aligned}
u(k)+v(k)+\tilde{u}(k)+\tilde{v}(k) &= (\mathcal{T}v)(k)+(\mathcal{S}u)(k)+\tilde{u}(k) \\
&\quad+\tilde{v}(k)+2\mu\phi(k) \\
&\geq \lambda_1(L(u+v+\tilde{u}+\tilde{v}))(k) \\
&\quad+2\mu\phi(k)\geq 2\mu\phi(k).
\end{aligned} \tag{43}$$

Define $\mu^* = \sup S_\mu := \sup\{\mu>0 : u+v+\tilde{u}+\tilde{v}\geq 2\mu\phi\}$. Then, $S_\mu\neq\varnothing$, $\mu^*\geq\mu$ and $u+v+\tilde{u}+\tilde{v}\geq 2\mu^*\phi$. From $\phi=\lambda_1 L\phi$, we obtain

$$\lambda_1 L(u+v+\tilde{u}+\tilde{v})\geq\lambda_1 L\left(2\mu^*\phi\right)=2\mu^*\lambda_1 L\phi=2\mu^*\phi. \tag{44}$$

Hence,

$$u+v+\tilde{u}+\tilde{v}\geq\lambda_1 L(u+v+\tilde{u}+\tilde{v})+2\mu\phi\geq 2\left(\mu^*+\mu\right)\phi, \tag{45}$$

which contradicts the definition of μ^*. Therefore, (29) holds, and from Lemma 4, we obtain

$$\deg\left(I-\mathcal{A},B_R,0\right)=0. \tag{46}$$

This completes the proof. □

Proof of Theorem 1. From (H4), there exist $\varepsilon_2\in(0,\lambda_1)$ and $r\in(0,R)$ such that

$$\begin{gathered}
|f(k,v)|\leq(\lambda_1-\varepsilon_2)|v|, |g(k,u)|\leq(\lambda_1-\varepsilon_2)|u|, \\
\forall k\in\mathbb{T}_1, u,v\in\mathbb{R}\text{ with }|u|,|v|\leq r.
\end{gathered} \tag{47}$$

This implies that

$$\begin{aligned}|(\mathcal{T}v)(k)| &= \left|\sum_{l=1}^{T} G(k,l) f(l,v(l))\right| \le \sum_{l=1}^{T} G(k,l)|f(l,v(l))| \le (\lambda_1-\varepsilon_2)\sum_{l=1}^{T} G(k,l)|v(l)|,\\ |(\mathcal{S}u)(k)| &= \left|\sum_{l=1}^{T} G(k,l) g(l,u(l))\right| \le \sum_{l=1}^{T} G(k,l)|g(l,u(l))| \le (\lambda_1-\varepsilon_2)\sum_{l=1}^{T} G(k,l)|u(l)|.\end{aligned} \tag{48}$$

Consequently, we have

$$|(\mathcal{T}v)(k)| + |(\mathcal{S}u)(k)| \le (\lambda_1-\varepsilon_2)\sum_{l=1}^{T} G(k,l)[|u(l)| + |v(l)|]. \tag{49}$$

Now, we prove that

$$(u,v) \ne \mu\mathcal{A}(u,v), \tag{50}$$

for all $u, v \in \partial B_r$ and $\mu \in [0,1]$. We argue by contradiction. Suppose that there exist $u, v \in \partial B_r$ and $\mu \in [0,1]$ such that

$$(u,v) = \mu\mathcal{A}(u,v). \tag{51}$$

Therefore,

$$u(k) = \mu(\mathcal{T}v)(k), \text{ and } v(k) = \mu(\mathcal{S}u)(k), \quad k \in \mathbb{T}_1. \tag{52}$$

Hence, we have

$$\begin{aligned}|u(k)| + |v(k)| &\le |(\mathcal{T}v)(k)| + |(\mathcal{S}u)(k)|\\ &\le (\lambda_1-\varepsilon_2)\sum_{l=1}^{T} G(k,l)[|u(l)| + |v(l)|].\end{aligned} \tag{53}$$

From Lemma 1 (i) and Lemma 2, we have

$$\lambda_1 \sum_{k=1}^{T} G(k,l)\phi(k) = \phi(l), \quad \forall l \in \mathbb{T}_1. \tag{54}$$

Multiplying both sides of (53) by $\sin(k\pi)/(T+1)$, then summing from 1 to T, and using (54), we obtain

$$\sum_{k=1}^{T}[|u(k)| + |v(k)|]\frac{\sin(k\pi)}{(T+1)} \le (\lambda_1-\varepsilon_2)\sum_{k=1}^{T}\frac{\sin(k\pi)}{(T+1)}\sum_{l=1}^{T} G(k,l)[|u(l)| + |v(l)|] = \frac{\lambda_1-\varepsilon_2}{\lambda_1}\sum_{l=1}^{T}[|u(l)| + |v(l)|]\frac{\sin(l\pi)}{(T+1)}. \tag{55}$$

This implies that

$$\sum_{k=1}^{T}[|u(k)| + |v(k)|]\frac{\sin(k\pi)}{(T+1)} = 0. \tag{56}$$

Because $\sin(k\pi)/(T+1) \ge 0 (\not\equiv 0)$ for $k \in \mathbb{T}_1$, we have $|u(k)| + |v(k)| \equiv 0$, $k \in \mathbb{T}_1$. This contradicts $u, v \in \partial B_r$. Therefore, (50) holds, and Lemma 5 implies that

$$\deg(I - \mathcal{A}, B_r, 0) = 1. \tag{57}$$

Combining this with Lemma 6, we have

$$\deg\left(I - \mathcal{A}, \frac{B_R}{\overline{B}_r}, 0\right) = \deg(I - \mathcal{A}, B_R, 0) - \deg(I - \mathcal{A}, B_r, 0) = -1. \tag{58}$$

Therefore, the operator $\mathcal{A}$ has at least one fixed point in $B_R/\overline{B}_r$, and (6) has at least one nontrivial solution. This completes the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

This study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Research Article

An SDP Method for Copositivity of Partially Symmetric Tensors

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In this paper, we consider the problem of detecting the copositivity of partially symmetric rectangular tensors. We first propose a semidefinite relaxation algorithm for detecting the copositivity of partially symmetric rectangular tensors. Then, the convergence of the proposed algorithm is given, and it shows that we can always catch the copositivity of given partially symmetric tensors. Several preliminary numerical results confirm our theoretical findings.

1. Introduction

Let $\mathscr{A} = (a_{i_1 i_2,\ldots,i_p j_1 j_2,\ldots,j_q})$ be a real (p,q)-th order $m \times n$-dimensional rectangular tensor, where $a_{i_1 i_2,\ldots,i_p j_1 j_2,\ldots,j_q} \in \mathbb{R}$ for $i_k \in [m]$, $k \in [p]$, $j_l \in [n]$, and $l \in [q]$. If the entries of the tensor are invariant under any permutation of $i_1, i_2, \ldots, i_p$ and $j_1, j_2, \ldots, j_q$, $\mathscr{A}$ is called a partially symmetric tensor. For the sake of simplicity, let $\mathbb{PS}_{p,q}^{m\times n}$ be the set of all partially symmetric rectangular tensors with order (p,q) and dimension $m \times n$. By the relationship between partially symmetric tensors and homogeneous polynomials, we always use the following notation:

$$f(\mathbf{x},\mathbf{y}) = \mathscr{A}\mathbf{x}^p\mathbf{y}^q = \sum_{\substack{i_1,\ldots,i_p \in [m] \\ j_1,\ldots,j_q \in [n]}} a_{i_1 i_2,\ldots,i_p j_1 j_2,\ldots,j_q} x_{i_1} x_{i_2}, \ldots, x_{i_p} y_{j_1} y_{j_2}, \ldots, y_{j_q}. \tag{1}$$

By this notation, we know that $\mathscr{A} = (a_{i_1,\ldots,i_p j_1,\ldots,j_q}) \in \mathbb{PS}_{p,q}^{m\times n}$ is strictly copositive if and only if

$$\mathscr{A}\mathbf{x}^p\mathbf{y}^q \geq (>)0, \quad \text{for all } \mathbf{x} \in \mathbb{R}_+^m, \mathbf{y} \in \mathbb{R}_+^n \text{ with } \|\mathbf{x}\| = 1, \|\mathbf{y}\| = 1. \tag{2}$$

Particularly, if $m = n$ and $\mathbf{x} = \mathbf{y}$, then it reduces to the copositivity of symmetric tensors [1–10].

The copositive tensor has attracted many researches' attention since it plays an important role in polynomial optimization [11], hypergraph theory [1], vacuum stability of a general scalar potential [12], tensor complementarity problem [13, 14], tensor eigenvalue complementarity problem [15, 16], and so on [17–37]. Kannike proved the vacuum stability conditions for more complicated potentials with the help of the copositive tensor [12]. Ling et al. [16] proposed that the tensor generalized eigenvalue complementarity problem is solvable and has one solution at least under assumptions that the related square tensor is strictly copositive. During the process of application, a challenging problem is how to detect the copositivity of tensors numerically.

Recently, several numerical algorithms are proposed to check the copositivity of symmetric tensors. To the best of our knowledge, the first numerical algorithm was proposed by Chen et al. in [2], where the algorithm is based on the representation of the multivariate form in barycentric coordinates with respect to the standard simplex. Then, by a suitable convex subcone of a copositive tensor cone, an updated numerical algorithm for copositivity of tensors was proposed in [1]. It must be pointed out that the methods of [1, 2] can only capture strictly copositive tensors and noncopositive tensors. To overcome this drawback, in [38], Li et al. proposed an SDP relaxation algorithm to test the copositivity of higher-order tensors. Very recently, Nie et al. gave a complete semidefinite relaxation algorithm for detecting the copositivity of a matrix or tensor [39]. If the potential tensor is copositive, the algorithm can get a certificate for the copositivity. Otherwise, the algorithm can get a point that refutes the copositivity. Furthermore, it is showed that the detection can be done by solving a finite number of semidefinite relaxations for all matrices and tensors.

For the copositivity of partially symmetric tensors, Gu et al. gave the first two spectral properties in [40], and some necessary or sufficient conditions for a real partially symmetric rectangular tensor to be copositive are further established. Moreover, an equivalent notion of strictly copositive rectangular tensors is presented [40]. In [41], Wang et al. extended the simplicial partition method for symmetric tensors to check the copositivity of partially symmetric tensors. However, as we discussed above, it can only capture all strictly copositive rectangular tensors or noncopositive rectangular tensors. When the input tensor is copositive but not strictly copositive, the algorithm may not stop in general. To solve this, motivated by the algorithm of [38, 39], we propose a new algorithm to check the copositivity of partially symmetric tensors in this paper.

The remainder of this paper is organized as follows. In Section 2, we recall some preliminaries on polynomials. In Section 3, we formulate the potential problem as a proper polynomial optimization problem which can be efficiently solved by Lasserre-type semidefinite relaxations. Then, a numerical method is proposed to check whether a given partially symmetric tensor is copositive or not, and the convergence of this algorithm is established. Several numerical experiments are listed in Section 4, and final remarks are given in Section 5.

2. Preliminaries

Let $\mathbb{R}[\mathbf{x}]$ be the ring of the polynomial with variables $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Let $\mathbb{R}[\mathbf{x}]_d \subseteq \mathbb{R}[\mathbf{x}]$ denote the vector space of polynomials with degree at most d. The degree of a polynomial f is denoted as $\deg(f)$. Denote $\mathbf{e}$ as the vector of all entries which equals one. A polynomial p is called SOS if there exist $p_1, p_2, \ldots, p_r \in \mathbb{R}[\mathbf{x}]$ such that $p = p_1^2 + p_2^2 +, \ldots, + p_r^2$. Denote by $\sum[\mathbf{x}]$ the set of all SOS polynomials. For $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in N^n$, let $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}, \ldots, x_n^{\alpha_n}$. Then, for any polynomial $f \in \mathbb{R}[\mathbf{x}]$, it can be denoted by $f(\mathbf{x}) = \sum_{\alpha\in N^n} f_\alpha \mathbf{x}^\alpha$, and $\mathrm{vec}(f) := (f_\alpha)_{\alpha\in\mathbb{N}^n}$ denotes its sequence of coefficients in the monomial basis of $\mathbb{R}[\mathbf{x}]$. For matrix A, its transpose is denoted by $A^\top$. For a symmetric matrix $X, X \succcurlyeq 0$ means X is positive semidefinite. More details about polynomial optimization can be seen in [42–45].

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in N^n$, $|\alpha| = \alpha_1 +, \cdots, + \alpha_n$, and denote $N_d^n = \{\alpha \in N^n \mid |\alpha| \le d\}$. For $t \in \mathbb{R}$, $\lceil t \rceil$ denotes the smallest integer that is not smaller than t. If the subset $I \subseteq \mathbb{R}[\mathbf{x}]$ satisfies that $I + I \subseteq I$ and $I \cdot \mathbb{R}[\mathbf{x}] \subseteq \mathbb{R}[\mathbf{x}]$, then I is called ideal. For a polynomial tuple $h = (h_1, h_2, \ldots, h_s)$, the ideal generated by h is defined such that

$$\mathcal{I}(h) = h_1\mathbb{R}[\mathbf{x}] + h_2\mathbb{R}[\mathbf{x}] +, \ldots, + h_s\mathbb{R}[\mathbf{x}]. \tag{3}$$

The k-th truncation ideal generated by h is

$$\mathcal{I}(h)_k = h_1\mathbb{R}[\mathbf{x}]_{k-\deg(h_1)} + h_2\mathbb{R}[\mathbf{x}]_{k-\deg(h_2)} +, \cdots, + h_s\mathbb{R}[\mathbf{x}]_{k-\deg(h_s)}. \tag{4}$$

For complex and real algebraic varieties of polynomial tuple h, define

$$\begin{cases} V_C(h) = \{\mathbf{x} \in C^n \mid h(\mathbf{x}) = 0\}, \\ V_R(h) = V_C(h) \cap \mathbb{R}^n. \end{cases} \tag{5}$$

The quadratic module generated by $g = (g_1, g_2, \ldots, g_t)$ is (denote $g_0 = 1$)

$$Q(g)_k = \sum[\mathbf{x}] + g_1 \sum[\mathbf{x}] +, \ldots, + g_t \sum[\mathbf{x}]. \tag{6}$$

For $\mathbf{y} = (y_\alpha) \in \mathbb{R}^{N_d^n}, \alpha \in N_d^n$, where $\mathbb{R}^{N_d^n}$ is the space of real vectors indexed by $\alpha \in N_d^n$, define

$$\left\langle \sum_{\alpha\in N_d^n} p_\alpha x_1^{\alpha_1} x_2^{\alpha_2}, \ldots, x_n^{\alpha_n}, \mathbf{y} \right\rangle = \sum_{\alpha\in N_d^n} p_\alpha y_\alpha. \tag{7}$$

For a polynomial $q \in \mathbb{R}[\mathbf{x}]_{2k}$, the k-th localizing matrix of q is the symmetric matrix $L_k^q(\mathbf{y})$ satisfying

$$\mathrm{vec}(p_1)^\top \left(L_q^{(k)}(\mathbf{y})\right)\mathrm{vec}(p_2) = \langle q p_1 p_2, \mathbf{y}\rangle, \tag{8}$$

for all $p_1, p_2 \in \mathbb{R}[\mathbf{x}]$ with $\deg(p_1), \deg(p_2) \le k - \lceil \deg(q)/2 \rceil$, where $\mathrm{vec}(p_i)$ denotes the coefficient vector of the polynomial p_i. When $q = 1, L_q^{(k)}(\mathbf{y})$ is the moment matrix $M_k(\mathbf{y}) = L_1^{(k)}(\mathbf{y})$. Let $f = (f_1, f_2, \ldots, f_r)$ be a polynomial tuple; its localizing matrix is defined such that

$$L_f^{(k)}(\mathbf{y}) = \left(L_{f_1}^{(k)}(\mathbf{y}), L_{f_2}^{(k)}(\mathbf{y}), \ldots, L_{f_r}^{(k)}(\mathbf{y})\right). \tag{9}$$

3. The SDP Algorithm for Copositivity of Partially Symmetric Tensors

In this section, we establish an equivalent condition for the copositivity of partially symmetric tensors. Then, the concerned problem can be reformulated as a polynomial optimization problem. To continue, recall that a partially symmetric tensor $\mathcal{A} \in \mathbb{PS}_{p,q}^{m\times n}$ is strictly copositive if and only if

$$\mathcal{A}\mathbf{x}^p\mathbf{z}^q \ge 0\,(>0), \quad \text{for all } \mathbf{x} \in \mathbb{R}_+^m, \mathbf{z} \in \mathbb{R}_+^n \text{ with } \|\mathbf{x}\| = 1, \|\mathbf{z}\| = 1, \tag{10}$$

which is equivalent with the following optimization problem:

$$\begin{aligned} f^* = \min \quad & \mathcal{A}\mathbf{x}^p\mathbf{z}^q \\ \text{s.t.} \quad & \mathbf{e}_1^\top\mathbf{x} = 1, \mathbf{e}_2^\top\mathbf{z} = 1 \\ & \mathbf{x} \in \mathbb{R}_+^m, \mathbf{z} \in \mathbb{R}_+^n. \end{aligned} \tag{11}$$

Clearly, tensor $\mathcal{A}$ is strictly copositive if and only if $f^* \ge 0\,(>0)$. Problem (11) can be solved by classical Lasserre relaxations [46]. Since the objection function is continuous and the feasible region is compact, problem (11) always has a solution. Without loss of generality, assume $(\mathbf{x}^*, \mathbf{z}^*)$ is one of the solutions of (11); then, it satisfies the following KKT-conditions with $\lambda, \mu \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^m$, and $\mathbf{w} \in \mathbb{R}^n$:

$$\begin{cases} p\mathscr{A}\mathbf{x}^{*p-1}\mathbf{z}^{*q} - \lambda\mathbf{e}_1 - \mathbf{v} = \mathbf{0}, \\ q\mathscr{A}\mathbf{x}^{*p}\mathbf{z}^{*q-1} - \mu\mathbf{e}_2 - \mathbf{w} = \mathbf{0}, \\ \mathbf{e}_1^\top\mathbf{x}^* = 1, \mathbf{e}_2^\top\mathbf{z}^* = 1, \\ \mathbf{x}^* \geq \mathbf{0}, \mathbf{z}^* \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}, \mathbf{w} \geq \mathbf{0}, \\ \mathbf{x}^{*\top}\mathbf{v} = 0, \mathbf{z}^{*\top}\mathbf{w} = 0. \end{cases} \tag{12}$$

By (12), we obtain that $\lambda = p\mathscr{A}\mathbf{x}^{*p}\mathbf{z}^{*q}$, $\mu = q\mathscr{A}\mathbf{x}^{*p}\mathbf{z}^{*q}$, and

$$\begin{gathered} p\mathscr{A}\mathbf{x}^{*p-1}\mathbf{z}^q - \lambda\mathbf{e}_1 \geq \mathbf{0},\ q\mathscr{A}\mathbf{x}^{*p}\mathbf{z}^{*q-1} - \mu\mathbf{e}_2 \geq \mathbf{0}, \\ \mathbf{x}^{*\top}\left(p\mathscr{A}\mathbf{x}^{*p-1}\mathbf{z}^{*q}\right) - \lambda\mathbf{x}^{*\top}\mathbf{e}_1 = 0,\ \mathbf{z}^{*\top}\left(q\mathscr{A}\mathbf{x}^{*p}\mathbf{z}^{*q-1}\right) - \mu\mathbf{z}^{*\top}\mathbf{e}_2 = 0. \end{gathered} \tag{13}$$

Combining this with the fact that $\|\mathbf{x}^*\| \leq 1, \|\mathbf{z}^*\| \leq 1$, we consider the following optimization problem:

$$\begin{aligned} \min \quad & \mathscr{A}\mathbf{x}^p\mathbf{z}^q \\ \text{s.t.} \quad & \mathbf{x}^\top\left(\mathscr{A}\mathbf{x}^{p-1}\mathbf{z}^q\right) - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)\mathbf{x}^\top\mathbf{e}_1 = 0, \\ & \mathbf{z}^\top\left(\mathscr{A}\mathbf{x}^p\mathbf{z}^{q-1}\right) - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)\mathbf{z}^\top = 0, \\ & \mathscr{A}\mathbf{x}^{p-1}\mathbf{z}^q - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)\mathbf{e}_1 \geq \mathbf{0}, \\ & \mathscr{A}\mathbf{x}^p\mathbf{z}^{q-1} - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)\mathbf{e}_2 \geq \mathbf{0}, \\ & \mathbf{e}_1^\top\mathbf{x} = 1, \mathbf{e}_2^\top\mathbf{z} = 1, 1 - \|\mathbf{x}\|^2 \geq 0, 1 - \|\mathbf{z}\|^2 \geq 0 \\ & \mathbf{x} \in \mathbb{R}_+^m, \mathbf{z} \in \mathbb{R}_+^n. \end{aligned} \tag{14}$$

It is clear to see that problems (11) and (14) are equivalent in the sense that they have the same optimal solution. To solve (14), we introduce the following notations:

$$\begin{cases} f(\mathbf{x}, \mathbf{z}) = \mathscr{A}\mathbf{x}^p\mathbf{z}^q, \\ g(\mathbf{x}, \mathbf{z}) = \left\{\mathscr{A}\mathbf{x}^{p-1}\mathbf{z}^q - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)\mathbf{e}_1, \mathscr{A}\mathbf{x}^p\mathbf{z}^{q-1} - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)\mathbf{e}_2, 1 - \|\mathbf{x}\|^2, 1 - \|\mathbf{z}\|^2, x_i, z_j\right\}, \\ h(\mathbf{x}, \mathbf{z}) = \left\{\mathbf{e}^\top(\mathbf{x}, \mathbf{0})_{n+m} = 1, \mathbf{e}^\top(\mathbf{0}, \mathbf{z})_{n+m} = 1, x_i\left(\mathscr{A}\mathbf{x}^{p-1}\mathbf{z}^q\right)_i - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)x_i, z_j\left(\mathscr{A}\mathbf{x}^p\mathbf{z}^{q-1}\right)_j - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)z_j\right\}. \end{cases} \tag{15}$$

So, the problem of (14) can be rewritten such that

$$\begin{aligned} f^* = \min \quad & f(\mathbf{x}, \mathbf{z}) \\ \text{s.t.} \quad & g(\mathbf{x}, \mathbf{z}) \geq \mathbf{0}, \\ & h(\mathbf{x}, \mathbf{z}) = \mathbf{0}. \end{aligned} \tag{16}$$

By the Lasserre-type semidefinite relaxations of (16), consider the semidefinite program

$$\begin{aligned} \rho_k = \min \quad & \sum_{\alpha \in N^{n+m}} f_\alpha y_\alpha \\ \text{s.t.} \quad & L_g^{(k)}(\mathbf{y}) \succcurlyeq \mathbf{0},\ L_h^{(k)}(\mathbf{y}) = \mathbf{0}, \\ & y_0 = 1,\ M_k(\mathbf{y}) \succcurlyeq \mathbf{0},\ \mathbf{y} \in \mathbb{R}^{N_{2k}^{n+m}}, \end{aligned} \tag{17}$$

where $k = k_0, k_0 + 1, \ldots$, with $k_0 = \max\{\lceil (p/2) \rceil, \lceil (q/2) \rceil\}$. It is obvious that the feasible set is compact, and the Archimedean condition holds. Thus, the asymptotic convergence of (17) is always guaranteed. Moreover, $\mathscr{A}$ is copositive if $\rho_k \geq 0$ for some k, and ρ_k is a monotonically decreasing sequence, with the decreasing of order k, i.e.,

$$\rho_{k_0} \leq \rho_{k_0+1} \leq, \ldots, \leq \rho_k \leq, \ldots, \leq f^*. \tag{18}$$

Now, we present an algorithm to check the copositivity of a given partially symmetric rectangular tensor (Algorithm 1).

$$\begin{aligned} \rho_k^* = \min \quad & \langle \xi^\top [x, y]_{m+n}, \mathbf{y} \rangle \\ \text{s.t.} \quad & y_0 = 1, L_g^{(k)}(\mathbf{y}) \succcurlyeq \mathbf{0},\ L_h^{(k)}(\mathbf{y}) = \mathbf{0} \\ & M_k(\mathbf{y}) \succcurlyeq \mathbf{0}, L_{\rho_k - f(\mathbf{x}, \mathbf{z})}^{(k)} \succcurlyeq \mathbf{0},\ \mathbf{y} \in \mathbb{R}^{N_{2k}^{n+m}}, \end{aligned} \tag{19}$$

The following theorem shows the convergence of Algorithm 1 for any partially symmetric tensor.

Theorem 1. *Suppose $\mathscr{A} \in \mathbb{PS}_{p,q}^{m\times n}$ is a partially symmetric tensor. Then, the following properties hold:*

(i) For all $k \geq 0$, problem (16) is feasible and achieves its optimal value $\rho_k = f^*$ for all k sufficiently large

(ii) For all $k \geq 0$, problem (19) has an optimizer if it is feasible

(iii) If $\mathscr{A}$ is copositive, Algorithm 1 must stop with $\rho^k \geq 0$ when k is sufficiently large

(iv) If $\mathscr{A}$ is not copositive, Algorithm 1 must stop with $f(x, z) < 0$ for almost all $\xi \in \mathbb{R}^{\mathbb{N}_{p+q}^{n+m}}$ when k is sufficiently large

Proof

(i) Since the feasible set of (11) is compact, it must have a minimizer $(\mathbf{x}^*, \mathbf{z}^*)$. On the contrary, $(\mathbf{x}^*, \mathbf{z}^*)$ is a feasible point for (16), which implies that the semidefinite relaxation (17) is always feasible. Since $L_{1-\|\mathbf{x}\|^2}^{(k)} \succcurlyeq 0$, let $X = \{(\mathbf{x}, 0), (0, \mathbf{z}) \mid \mathbf{x} \in \mathbb{R}^m, \mathbf{z} \in \mathbb{R}^n\} \subseteq \mathbb{R}^{m+n}$; then, it holds that $L_{1-\|\mathbf{x}\|^2}^{(k)} \succcurlyeq 0$. We now show that the feasible set of (17) is compact as follows. First of all, we have

$$1 \geq y_{2e_1} + y_{2e_2} +, \ldots, + y_{2e_{n+m}}. \tag{20}$$

Step 0: given an arbitrary vector $\xi \in \mathbb{R}^{\mathbb{N}^{n+m}_{p+q}}$. Let $k = \max\{\lceil (p/2)\rceil, \lceil (q/2)\rceil\}$.
Step 1: solve the semidefinite relaxation (17). If $\rho_k \geq 0$, then stop, and $\mathscr{A}$ is copositive. If $\rho_k < 0$, go to Step 2.
Step 2: solve the following semidefinite program:
for an optimizer $\mathbf{y}^*$ if it is feasible. If it is infeasible, let $k = k+1$ and go to Step 1.
Step 3: let $(\mathbf{x}^*, \mathbf{z}^*) = ((y^*)_{e_1}, \ldots, (y^*)_{e_m}, (y^*)_{e_{m+1}}, \ldots, (y^*)_{e_{m+n}})$. If $\mathscr{A}\mathbf{x}^{*p}\mathbf{z}^{*q} < 0$, then $\mathscr{A}$ is not copositive and stop. Otherwise, let $k = k+1$ and go to Step 1.

Algorithm 1: An SDP method for copositivity of a partially symmetric tensor $\mathscr{A} \in \mathbb{PS}^{m\times n}_{p,q}$.

Then, $0 \leq y_{2e_i} \leq 1$; since $M_k(\mathbf{y}) \succcurlyeq 0$, $i \in [m+n]$. Furthermore, for all $0 < |\alpha| \leq k-1$, the (α,α)-th diagonal entry of $L^{(k)}_{1-\|\mathbf{x}\|^2}$ is nonnegative, which implies that

$$y_{2\alpha} \geq y_{2e_1+2\alpha} + y_{2e_2+2\alpha} +, \ldots, + y_{2e_{n+m}+2\alpha}. \tag{21}$$

Take $\alpha = e_1, e_2, \ldots, e_{m+n}$ in the following analysis. By the same argument as (21) and repeating $k-1$ times, we can show that $0 \leq y_{2\beta} \leq 1$ for all $|\beta| \leq k$. By the definition of $M_k(\mathbf{y})$, we know that the diagonal entries $M_k(\mathbf{y})$ are precisely $y_{2\beta}$, $|\beta| \leq k$. Since $M_k(\mathbf{y}) \succcurlyeq 0$, all the entries of $M_k(\mathbf{y})$ must be between -1 and 1. So, $\mathbf{y}$ is bounded, and the feasible set of (17) is compact. Hence, the optimal value can always be achieved. In the following, we will show that $\rho_k = f^*$ for all k sufficiently large.

By direct computation, the optimization (16) is equivalent with the following problem:

$$\begin{aligned}
\min \quad & \mathscr{A}\mathbf{x}^p\mathbf{z}^q \\
\text{s.t.} \quad & x_i\left(\mathscr{A}\mathbf{x}^{p-1}\mathbf{z}^q\right)_i - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)x_i = 0, \\
& z_j\left(\mathscr{A}\mathbf{x}^p\mathbf{z}^{q-1}\right)_j - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)z_j = 0, \\
& \mathscr{A}\mathbf{x}^{p-1}\mathbf{z}^q - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)\mathbf{e}_1 \geq \mathbf{0}, \\
& \mathscr{A}\mathbf{x}^p\mathbf{z}^{q-1} - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)\mathbf{e}_2 \geq \mathbf{0}, \\
& \mathbf{e}_1^\top\mathbf{x} = 1, \mathbf{e}_2^\top\mathbf{z} = 1, \\
& \mathbf{x} \in \mathbb{R}^m_+, \mathbf{z} \in \mathbb{R}^n_+.
\end{aligned} \tag{22}$$

For simplicity, denote

$$\begin{cases}
f(\mathbf{x},\mathbf{z}) = \mathscr{A}\mathbf{x}^p\mathbf{z}^q, \\
g(\mathbf{x},\mathbf{z}) = \left\{(\mathbf{x},\mathbf{z}), \mathscr{A}\mathbf{x}^{p-1}\mathbf{z}^q - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)\mathbf{e}_1, \mathscr{A}\mathbf{x}^p\mathbf{z}^{q-1} - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)\mathbf{e}_2\right\} \\
h(\mathbf{x},\mathbf{z}) = \left\{\mathbf{e}^\top(\mathbf{x},\mathbf{0})_{m+n} - 1, \mathbf{e}^\top(\mathbf{0},\mathbf{z})_{m+n} - 1, x_i\left(\mathscr{A}\mathbf{x}^{p-1}\mathbf{z}^q\right)_i - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)x_i, z_j\left(\mathscr{A}\mathbf{x}^p\mathbf{z}^{q-1}\right)_j - (\mathscr{A}\mathbf{x}^p\mathbf{z}^q)z_j\right\}.
\end{cases} \tag{23}$$

Corresponding Lasserre's relaxations for (22) are

$$\begin{aligned}
\rho_k' = \min \sum_{\alpha\in N^n} f_\alpha y_\alpha & \\
\text{s.t.} \quad L_g^{(k)}(\mathbf{y}) \succcurlyeq \mathbf{0},\ L_h^{(k)}(\mathbf{y}) = \mathbf{0}, & \\
y_0 = 1,\ M_k(\mathbf{y}) \succcurlyeq \mathbf{0},\ \mathbf{y} \in \mathbb{R}^{N^{n+m}_{2k}}. &
\end{aligned} \tag{24}$$

For $k = k_0, k_0+1, \ldots,$ where $k_0 = \max\{\lceil (p/2)\rceil, \lceil (q/2)\rceil\}$, any feasible solution of (17) is also a feasible solution of (24), so

$$\rho_k' \leq \rho_k \leq f^*, \quad k = k_0, k_0+1, \ldots. \tag{25}$$

Next, we show that the set of polynomials

$$F = \left\{\left(1-\mathbf{e}^T\mathbf{x}\right)\phi + \sum_{i=1}^n x_i\left(\sum_l s_l^2\right) + \left(1-\mathbf{e}^T\mathbf{z}\right)\psi + \sum_{i=1}^n z_j\left(\sum_t s_t^2\right)\right\}. \tag{26}$$

is Archimedean, i.e., there exists $f \in F$ such that the inequality $f(\mathbf{x}) \geq 0$ defines a compact set in $\mathbb{R}^{m+n}$. Let $f = 2 - \|X\|^2$ and $X = (\mathbf{x},\mathbf{z})_{m+n}$; we have

$$\begin{aligned}
2 - \|\mathbf{x}\|^2 = {} & \left(1-\mathbf{e}^T\mathbf{x}\right)\left(1+\|\mathbf{x}\|^2\right) + \sum_{i=1}^n x_i\left(1-x_i\right)^2 \\
& + \sum_{i\neq j=1}^m x_i^2 x_j + \left(1-\mathbf{e}^T\mathbf{z}\right)\left(1+\|\mathbf{z}\|^2\right) + \sum_{j=1}^n z_j\left(1-z_j\right)^2 \\
& + \sum_{i\neq j=1}^n z_i^2 z_j.
\end{aligned} \tag{27}$$

So, F is Archimedean by Theorem 3.3 of [47]; we know that $\rho_k' = f^*$ when k is sufficiently large. Hence, $\rho_k = f^*$ when all k values are sufficiently large.

(ii) The proof is the same with (i).

(iii) Clearly, $\mathscr{A}$ is copositive if and only if $f^* \geq 0$. By item (i), $\rho_k = f^*$ for all k big enough. Therefore, if $\mathscr{A}$ is copositive, we must have $\rho_k \geq 0$ for all k large enough.

(iv) If $\mathscr{A}$ is not copositive, then $f^* < 0$. By (i), there exists $k_1 \in N$ such that $\rho_k = f^*$ for all $k \geq k_1$. Hence, for all $k \geq k_1$, problem (19) is equivalent with the following problem:

$$\begin{aligned} \widehat{\rho_k} = \min \quad & \langle \xi^\top [\mathbf{x}, \mathbf{y}]_{m+n}, \mathbf{y} \rangle \\ \text{s.t.} \quad & y_0 = 1, L^{(k)}_{1-\mathbf{e}^T\mathbf{x}}(\mathbf{y}) \succcurlyeq \mathbf{0}, L^{(k)}_{1-\mathbf{e}^T\mathbf{y}} \succcurlyeq \mathbf{0}, L^{(k)}_{X}(\mathbf{y}) \succcurlyeq \mathbf{0}, \\ & M_k(\mathbf{y}) \succcurlyeq \mathbf{0}, L^{(k)}_{f^*-f(X)} \succcurlyeq \mathbf{0},\ X = (\mathbf{x}, \mathbf{z})_{m+n},\ \mathbf{y} \in \mathbb{R}^{N^{n+m}_{2k}}. \end{aligned} \tag{28}$$

It is k-th Lasserre's relaxation for the polynomial optimization

$$\begin{aligned} \min \quad & \xi^\top [\mathbf{x}, \mathbf{z}]_{m+n} \\ \text{s.t.} \quad & 1 - \mathbf{e}^\top (\mathbf{x}, \mathbf{0})_{m+n} \geq 0, 1 - \mathbf{e}^\top (\mathbf{0}, \mathbf{z})_{m+n} \geq 0, \mathbf{x} \geq 0, \mathbf{z} \geq 0, f^* - f(X) \geq 0. \end{aligned} \tag{29}$$

The feasible region of (29) is clearly compact. When $\xi \in \mathbb{R}^{\mathbb{N}^{n+m}_{p+q}}$ is arbitrary, (29) has a unique optimizer $X^* = (\mathbf{x}^*, \mathbf{z}^*)$. Hence, for almost all $\xi \in \mathbb{R}^{\mathbb{N}^{n+m}_{p+q}}$, X^* is the unique optimizer. For notation convenience, denote by $\widehat{\mathbf{y}}^k$ the optimizer of (19) with the relaxation order k. Let $X^k = ((\widehat{\mathbf{y}}^k)_{e_1}, \ldots, (\widehat{\mathbf{y}}^k)_{\mathbf{e}_{n+m}})$. By Corollary 3.5 of [48] or Theorem 3.3 of [49], the sequence $\{X^k\}_{k=k_0}^{\infty}$ must converge to X^*. Since $f^* \leq \rho_k^* < 0$, we must have $f(X_k) < 0$ when k is sufficiently large. □

4. Numerical Examples

In this section, we give several numerical examples to show the efficiency of Algorithm 1. Let $S_{\pi(i_1 i_2, \ldots, i_m)}$ denote the set of all permutations of $i_1 i_2, \ldots, i_m$, and let $\rho_k^* = 0$ when $|\rho_k^*| < 1e-5$. All experiments are done in Matlab2014b on a desktop computer with Intel (R) Core (TM)i7-6500 CPU @ 2.50 GHz 2.60 GHz and 16 GB of RAM.

Example 1. Suppose that $\mathscr{A} \in \mathbb{PS}^{2\times 4}_{2,2}$ is given by

$$\begin{cases} a_{1111} = 1, a_{1122} = 1, a_{1133} = 1, a_{1144} = 1, a_{2211} = 1, a_{2222} = 1, a_{2233} = 1, a_{2244} = 1, \\ \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1112)}} a_{i_1 i_2 j_1 j_2} = 2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1134)}} a_{i_1 i_2 j_1 j_2} = 2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1113)}} a_{i_1 i_2 j_1 j_2} = -2, \\ \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1114)}} a_{i_1 i_2 j_1 j_2} = -2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1123)}} a_{i_1 i_2 j_1 j_2} = -2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1124)}} a_{i_1 i_2 j_1 j_2} = -2, \\ \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(2212)}} a_{i_1 i_2 j_1 j_2} = 2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(2234)}} a_{i_1 i_2 j_1 j_2} = 2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(2213)}} a_{i_1 i_2 j_1 j_2} = -2, \\ \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(2214)}} a_{i_1 i_2 j_1 j_2} = -2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(2223)}} a_{i_1 i_2 j_1 j_2} = -2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(2224)}} a_{i_1 i_2 j_1 j_2} = -2, \\ \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1211)}} a_{i_1 i_2 j_1 j_2} = -2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1222)}} a_{i_1 i_2 j_1 j_2} = -2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1212)}} a_{i_1 i_2 j_1 j_2} = -4, \\ \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1244)}} a_{i_1 i_2 j_1 j_2} = -2, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1234)}} a_{i_1 i_2 j_1 j_2} = -4, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1213)}} a_{i_1 i_2 j_1 j_2} = 4, \\ \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1214)}} a_{i_1 i_2 j_1 j_2} = 4, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1223)}} a_{i_1 i_2 j_1 j_2} = 4, \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1224)}} a_{i_1 i_2 j_1 j_2} = 4, \\ \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(1233)}} a_{i_1 i_2 j_1 j_2} = -2. \end{cases} \tag{30}$$

The corresponding polynomial for tensor $\mathscr{A}$ is

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}) = (x_1 - x_2)^2 (y_1 + y_2 - y_3 - y_4)^2, \mathbf{x} \\ = (x_1, x_2), \mathbf{y} = (y_1, y_2, y_3, y_4). \end{aligned} \tag{31}$$

By Algorithm 1, we know that $f^* = 0$ with $\mathbf{x} = (0.5000, 0.5000)$, $\mathbf{y} = (0.2500, 0.2500, 0.2500, 0.2500)$, which implies that rectangular tensor $\mathscr{A}$ is copositive.

Example 2. Suppose that $\mathscr{A} \in \mathbb{PS}^{1\times 2}_{2,2}$ with entries such that

$$\begin{cases} a_{1111} = 1, \\ a_{1122} = 1, \\ \sum\limits_{i_1 i_2 j_1 j_2 \in S_{\pi(11,12)}} a_{i_1 i_2 j_1 j_2} = -2. \end{cases} \tag{32}$$

The corresponding polynomial of $\mathscr{A}$ is

$$f(\mathbf{x},\mathbf{y}) = x_1^2 y_1^2 - 2x_1^2 y_1 y_2 + x_1^2 y_2^2,\ \mathbf{x} = (x_1),\ \mathbf{y} = (y_1, y_2). \tag{33}$$

By Algorithm 1, we obtain that $f^* = 0$ with optimal solution $(\mathbf{x},\mathbf{y}) = (1.0000, 0.7071, 0.7071)$, which implies that $\mathscr{A}$ is copositive but not strictly copositive.

Example 3. Suppose that $\mathscr{A} \in \mathbb{PS}_{2,2}^{2\times 2}$ is given by

$$\begin{cases} a_{1111} = 1, \\ a_{1122} = 1, \\ a_{2211} = 1, \\ a_{2222} = 1, \\ \sum_{i_1 i_2 j_1 j_2 \in S_{\pi(11,12)}} a_{i_1 i_2 j_1 j_2} = -2, \\ \sum_{i_1 i_2 j_1 j_2 \in S_{\pi(22,12)}} a_{i_1 i_2 j_1 j_2} = -2, \\ \sum_{i_1 i_2 j_1 j_2 \in S_{\pi(12,11)}} a_{i_1 i_2 j_1 j_2} = 2, \\ \sum_{i_1 i_2 j_1 j_2 \in S_{\pi(12,22)}} a_{i_1 i_2 j_1 j_2} = 2, \\ \sum_{i_1 i_2 j_1 j_2 \in S_{\pi(12,12)}} a_{i_1 i_2 j_1 j_2} = -4. \end{cases} \tag{34}$$

So, the corresponding polynomial of $\mathscr{A}$ is that

$$\begin{aligned} f(\mathbf{x},\mathbf{y}) = {} & x_1^2 y_1^2 + x_1^2 y_2^2 - 2x_1^2 y_1 y_2 + x_2^2 y_1^2 + x_2^2 y_2^2 \\ & - 2x_2^2 y_1 y_2 + 2x_1 x_2 y_1^2 + 2x_1 x_2 y_2^2 - 4x_1 x_2 y_1 y_2, \end{aligned} \tag{35}$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$. By Algorithm 1, we have $f^* = 0$ with $\mathbf{x}^* = (0.5126, 0.4874)$, $\mathbf{y}^* = (0.5000, .5000)$, which implies that the rectangular tensor is copositive.

Example 4. Suppose that $\mathscr{A} \in \mathbb{PS}_{3,2}^{3\times 2}$ is given by

$$\begin{cases} a_{11122} = 1, \\ a_{22222} = 1, \\ a_{33311} = 1, \\ \sum_{i_1 i_2 i_3 j_1 j_2 \in S_{\pi(123,12)}} a_{i_1 i_2 i_3 j_1 j_2} = -3. \end{cases} \tag{36}$$

The corresponding polynomial of the partially symmetric rectangular tensor $\mathscr{A}$ is

$$f(\mathbf{x},\mathbf{y}) = x_1^3 y_2^2 + x_2^3 y_2^2 + x_3^3 y_1^2 - 4x_1 x_2 x_3 y_1 y_2, \tag{37}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2)$. By Algorithm 1, we know that $f^* = -0.0639$ with $\mathbf{x}^* = (0.7652, 0.4702, 0.7652)$, $\mathbf{y}^* = (0.3572, 0.8724)$, which implies that the rectangular tensor is not copositive.

Example 5. Suppose $\mathscr{A} \in \mathbb{PS}_{2,2}^{2\times 2}$ is a tensor with entries such that

$$\begin{cases} a_{1111} = 1, \\ a_{1122} = -1, \\ a_{2211} = 1, \\ a_{2222} = 1, \\ \sum_{i_1 i_2 j_1 j_2 \in S_{\pi(11,12)}} a_{i_1 i_2 j_1 j_2} = 2, \\ \sum_{i_1 i_2 j_1 j_2 \in S_{\pi(22,12)}} a_{i_1 i_2 j_1 j_2} = 2, \\ \sum_{i_1 i_2 j_1 j_2 \in S_{\pi(12,11)}} a_{i_1 i_2 j_1 j_2} = 2, \\ \sum_{i_1 i_2 j_1 j_2 \in S_{\pi(12,22)}} a_{i_1 i_2 j_1 j_2} = 2, \\ \sum_{i_1 i_2 j_1 j_2 \in S_{\pi(12,12)}} a_{i_1 i_2 j_1 j_2} = -4. \end{cases} \tag{38}$$

The corresponding polynomial of the partially symmetric rectangular tensor $\mathscr{A}$ is

$$\begin{aligned} f(\mathbf{x},\mathbf{y}) = {} & x_1^2 y_1^2 - x_1^2 y_2^2 + 2x_1^2 y_1 y_2 + x_2^2 y_1^2 + x_2^2 y_2^2 \\ & + 2x_2^2 y_1 y_2 + 2x_1 x_2 y_1^2 + 2x_1 x_2 y_2^2 - 4x_1 x_2 y_1 y_2, \end{aligned} \tag{39}$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$. By Algorithm 1, we know that $f^* = 0.3333$ with $\mathbf{x}^* = (0.6666, 0.3334)$, $\mathbf{y}^* = (0.5000, .5000)$, which implies that the rectangular tensor is strictly copositive.

5. Conclusions

In this paper, based on Lasserre's hierarchy of semidefinite relaxations, we propose a new criterion to judge whether a given partially symmetric rectangular tensor is copositive or not. The convergence for the proposed algorithm is established. Furthermore, numerical examples demonstrate that the proposed algorithm is effective when the input rectangular tensor has lower dimension and orders, and it is difficult for the case with higher order or higher dimension. We will continue to study this problem in the future.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Each author contributed equally to this paper and read and approved the final manuscript.

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Research Article

Differential Equation Models for Education Charges of Universities in China and the Applications

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In recent years, with the expansion of university enrollment in China, the cost of running a school is getting higher and higher. Under the circumstance of insufficient state investment in higher education, the education charge paid by students has become an important source of the university income. It has become a hot topic of social concern to formulate a reasonable charging model to enable more students to enter higher education institutions. In this paper, we mainly put forward the basic differential equation models describing the problem of higher education charges in China. Through the qualitative analysis of these two basic models, we draw several conditions for universities to maintain or stabilize their education charges and give some suggestions on macrocontrol of university education charges in China.

1. Introduction to the Present Situation of Higher Education in China

Since the 1990s, under the guidance of the strategy of "rejuvenating the country through science and education and sustainable development (Deng Xiaoping, the chief designer of China, first put forward the important conclusion that science and technology are the first productive force. In 1997, the 15th Congress of the Communist Party of China clearly put forward the strategy of rejuvenating the country through science and education: to fully implement the idea that science and technology are the first productive force, to adhere to education as the basis, and to put science and technology and education in an important position in economic and social development)," the Chinese government has implemented "Project 211 (Project 211 is a construction project of about 100 universities and a number of key disciplines facing the 21st century)," "Project 985 (Project 985 is guided by the key areas and major needs urgently needed by the state or industry and focuses on building a number of platform bases around the national development strategy and the frontier of disciplines)," and "2011 Collaborative Innovation Plan (2011 Collaborative Innovation Plan is called the innovation ability promotion program of colleges and universities)" successively. The "first-class university and first-class discipline construction of higher education" has also been deployed since 2016. China's higher education has achieved a great leap both in quantity and quality due to this series of important measures. Since 1999, China has experienced a huge increase in the enrollment in higher education (see Table 1). The level of the popularization of higher education is also rising.

From the table, we can easily see that China's gross enrollment rate has considerably increased. In 2002, the higher education gross enrollment rate in China reached 15.0%. After 16 years of development, the gross enrollment rate of higher education in China reached an astonishing 48.1% in 2018. Correspondingly, the number of students enrolled in 2018 reached 87.679 million. The Chinese government's financial investment in higher education is also increasing year by year. In 1993, the Chinese government has formulated the Outline of China's Educational

Table 1: Development of higher education in China from 1999 to 2018.

Year	Number of schools	Enrollment in higher education (thousand)	Gross enrollment rate (%)	Number of teaching staff expenses (RMB)	Per public finance budget education expenses	Per public finance public funds (RMB)
1999	1071	1689	10.5	1065	7201.24	2962.37
2000	1041	2335	12.5	1113	7309.58	2921.23
2001	1225	2848	13.3	1214	6818.23	2613.56
2002	1396	3408	15.0	1304	6177.96	2453.47
…	…	…	…	…	…	…
2014	2526	7835	37.5	2336	16102.72	7637.97
2015	2560	8024	40.0	2369	18143.57	8280.08
2016	2596	8153	42.7	2405	18747.65	8067.26
2017	2631	8421	45.7	2443	20298.63	8506.02
2018	2663	8768	48.1	2488	22007.77	9222.23

Remark. The data cited in this paper come from the *China Statistical Yearbook* (1999–2018) and the *China Education Funds Statistical Yearbook* (1999–2018).

Reform and Development in which, for the first time, the Chinese government put forward the target of government spending on education, taking for 4% of GDP. This goal has already been achieved in 2012 (see Table 2), and the proportion has remained above 4% for seven consecutive years (see [1]).

At the same time, we have also realized that although China's higher education has made great achievements in recent decades, there are still many problems or challenges. According to the statistical data, in terms of national financial investment in education, the world's average level is about 7%. Developed countries have reached about 9% and economically underdeveloped countries only reached 4.1% (see [2]). Therefore, there is still a long way to go for China to invest in education funds (see [3]). As far as the current development of education in China is concerned, two issues have aroused wide concern. The first issue to which most Chinese scholars pay attention is how to solve the problem of higher education charges and the problem of tuition payment for the poor students with the premise of a stable educational financial input. The second issue is how to coordinate the relationship among the development of higher education, talent training, and educational equity in order to avoid social contradictions. These two issues are two crucial social problems that China is facing today. Through the years, these problems have been studied from the perspective of educational economics, sociology, and law by various scholars in the field of educational theories (see [2, 4–12]). And, they put forward some corresponding measures on how to calculate the cost of colleges and universities. In order to survive and develop, colleges and universities must pay attention to the dual development of economic and social benefits, improve the efficiency of the use of funds, and further optimize the allocation of educational resources.

2. The Analysis of the Theory of Higher Education Charges in China

At present, the model of raising funds for running universities in China is still mainly based on the state investment and is supplemented by the investment of private and other social forces. This is different from all state investment models implemented in the planned economy in the past. Before the 1990s, higher education institutions are regarded as nonprofit units, so all expenditures are provided by the state finance, while the education department and the finance department do not account for the cost of higher education.

Now, we know that higher education is a special activity process and resource consumption. Higher education cost is the value of educational resources consumed in the process of higher education activities. Because higher education can train technical talents and promote economic development, the government is the beneficiary of higher education. And, because higher education can bring great future benefits to the individuals, the individuals are also beneficiaries of higher education. At the same time, society and universities can also gain their own benefits through the development of higher education. Therefore, from the perspective of market economy theory, it is necessary to implement the cost compensation and allocation system in higher education at present.

According to Martin Throw's theory of the stages of higher education, education in China has changed from elitism to popularity. The process of popularization of higher education is not only the expansion of the educational scale, but also the change of cost sharing. In 1986, Johnston put forward the cost-sharing theory of higher education (see [7]). He believes that the cost of higher education should be reasonably shared by the beneficiaries. Beneficiaries should compensate the cost according to the level of income and the ability to bear it. Therefore, the main body of cost sharing in higher education should include the government, the individuals, the society, and the universities. In the way of sharing, the government shares the cost through financial allocation, the individual by paying tuition and miscellaneous expenses, the society by donating, and the colleges and universities by means of income generation from school-run industries and transformation of scientific research achievements (see [5]). In fact, the tuition charge of higher education is only related to the educational cost shared by the educator. According to the principle of complementarity and affordability of education fees, we believe that the main

Table 2: The proportion of national financial investment in education to GDP in 2008–2018.

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
%	3.48	3.59	3.66	3.93	4.28	4.11	4.10	4.26	4.22	4.14	4.11

Remark. The data cited come from the *China Education Funds Statistical Yearbook* (2008–2018).

influencing factors of higher education fees are the educational costs shared by educators and the unpaid tuition fees of poor students.

In this paper, we aim to establish differential equation models of the higher education cost by analyzing the theory of total cost, input, and output, and then we study the higher educational charges of universities.

3. Differential Equation Models Based on Cost Sharing in Higher Education

3.1. Establishment of Models. According to the theory of sharing in the higher educational cost and the variable relation of difference equation (see [4, 6]), we suppose that the continuous variable N is the number of students in the university; then, we establish the following functional variables which affect the change of the higher education cost:

$R(N)$: educational charge of colleges and universities
$\gamma(N)$: individually shared educational costs
$p(N)$: unpaid tuition of per poor student
$q(N)$: allowance of per unpaid students

From the analysis on the sharing of educational costs in institutions of higher education, we know that the functions $\gamma(N)$, $\gamma(N)$, and $p(N)$ satisfy the following condition: as $\gamma(N)$ increases, $R(N)$ also increases; as $\gamma(N)$ increases, $R(N)$ decreases. So, we can suppose that the relationship between the change rate of $R(N)$ and the function $\gamma(N)$ is a positive linear correlation, and the relationship between the change rate of $R(N)$ and the function $p(N)$ is also linear and negative. At the same time, as the function $p(N)$ increases, $q(N)$ also increases. So, we have

$$q(N) = \alpha p(N), \tag{1}$$

where α is the subsidy coefficient. We can obtain the following differential equation model (I) for the university educational charge:

$$R'(N) = \delta\gamma(N) - \sigma[p(N) + q(N)] = \delta\gamma(N) - \sigma(1+\alpha)p(N). \tag{2}$$

Because the function $p(N)$ of the unpaid charge can increase with the increase in the individually shared educational cost $\gamma(N)$, we have that

$$p(N) = g(\gamma(N)), \tag{3}$$

where $g(\gamma)$ is the monotonically increasing functions. Therefore, we obtain the following differential equation model (II) for the university educational charge:

$$R'(N) = \delta\gamma(N) - \beta g(\gamma(N)), \tag{4}$$

where $\beta = \sigma(1+\alpha)$.

Furthermore, we can obtain the function $\gamma(N)$ by the function $R(N)$. So, we have

$$\gamma(N) = f(R(N)), \tag{5}$$

where $f(R)$ is the monotonically increasing function.

Therefore, we obtain the following differential equation model (III) for the university educational charge:

$$R' = \delta f(R) - \sigma g(f(R)), \tag{6}$$

where $\delta > 0$ and $\sigma > 0$.

3.2. Qualitative Analysis of Educational Charge Model of Universities. From model (I), we can obtain sufficient and necessary conditions for the constant charge of higher education:

$$\delta\gamma(N) = \sigma g(\gamma(N)). \tag{7}$$

This shows that the necessary and sufficient conditions for colleges and universities to keep their educational charge unchanged are as follows: maintaining a balance between the cost of per education shared by individuals and the cost of per unpaid expenses of poor students. In order to achieve this balance, there must exist a positive solution N_0 for the above equation.

Therefore, we can obtain the number N_0 of students enrolled in colleges and universities and individually shared per cost of education $\gamma_0 = \gamma(N_0)$.

At present, China has set an upper limit of 25% for the individual share of education costs. But in fact, the proportion of poor students in colleges and universities has exceeded 30%. This is unbalanced. The objective reason for the imbalance is that the cost of higher education is difficult to calculate. It is easy to make the charge standard exceed the upper limit of the proportion of the education cost shared by individuals. Therefore, balance can be achieved and problems be solved by increasing state finance investment in education, encouraging and attracting social funds to run schools, or by improving the payment ability of poor students by student loans, scholarships, work aids, etc.

Furthermore, by the discriminating method of function extreme, we have the following sufficient condition for keeping the educational charges of colleges and universities invariant:

(1) As $\gamma'(N_0) > 0$, $0 < g'(\gamma(N_0)) < \delta/\sigma$, or $\gamma'(N_0) < 0$, $g^{\sigma}(\gamma(N_0)) > \delta/\sigma$, we have

$$R'(N_0) = 0,\ R''(N_0) > 0. \tag{8}$$

Therefore, there exists the minimum value of educational charges in colleges and universities, i.e., $R(N_0)$. This means that under the conditions at which the per cost $\gamma(N)$ of education shared by individuals increases, the minimum charge $R(N_0)$ of colleges and universities would maintain unchanged as long as the growth rate $g'(\gamma(N_0))$ of the per unpaid expenses for poor students is controlled within a certain limit. But, if the growth rate $g'(\gamma(N_0))$ of the per unpaid expenses of poor students exceeded one degree δ/σ, we can keep the minimum charges $R(N_0)$ of colleges and universities unchanged by reducing the cost $\gamma(N)$ of per education shared by individuals.

(2) As $\gamma'(N_0) < 0$, $0 < g'(\gamma(N_0)) < \delta/\sigma$, or $\gamma'(N_0) > 0$, $g'(\gamma(N_0)) > \delta/\sigma$, we have

$$\begin{aligned} R'(N_0) &= 0, \\ R''(N_0) &< 0. \end{aligned} \tag{9}$$

Therefore, there exists the maximum value of educational charges in colleges and universities, i.e., $R(N_0)$. This means that under the conditions at which the per cost $\gamma(N)$ of education shared by individuals decreases, the maximum charge $R(N_0)$ of colleges and universities would maintain unchanged as long as the growth rate $g'(\gamma(N_0))$ of the per unpaid expenses of poor students is controlled within a certain limit. But, if the growth rate $g'(\gamma(N_0))$ of the per unpaid expenses for poor students exceeded one degree δ/σ, we can keep the maximum charge $R(N_0)$ of colleges and universities unchanged by raising the cost $\gamma(N)$ of per education shared by individuals. However, this situation is equivalent to high tuition charges, and some impoverished students will default more tuition charges or drop out of school because of their inability to pay. Therefore, we should try our best to avoid this problem.

From model (II), we can know that the necessary and sufficient conditions for the constant charge of higher education are as follows:

$$\delta f(R) = \sigma g(f(R)). \tag{10}$$

It is also a balance. Here, we ask that there is a positive solution for the above equation, i.e., $\gamma_0 = f(R_0)$. Because R_0 is the equilibrium point of the differential equation model (II), we can obtain the following stability conclusion for educational charges R_0 by the stability criterion of solutions (see [8]).

(3) As $f'(\gamma_0) > 0$, $0 < g'(\gamma(N_0)) < \delta/\sigma$, we know that R_0 is the equilibrium point of the differential equation model (II). This means that under the conditions at which the per cost $\gamma = f(R)$ of education shared by individuals increases, we can maintain the stability conclusion for educational charges R_0 as long as the growth rate of the per unpaid expenses of poor students is controlled within a certain limit.

(4) As $f'(\gamma_0) < 0$, $g'(\gamma(N_0)) > \delta/\sigma$, we know that R_0 is also the equilibrium point of the differential equation model (II). This means that if the growth rate of the per unpaid expenses of poor students exceeded one degree δ/σ, we can keep the stability conclusion of R_0 by reducing the cost of per education shared by individuals.

4. Conclusion

In the past 30 years, China has experienced a huge increase in the enrollment in higher education. So, we pay attention on how to solve the problem of higher education charges and the problem of tuition payment for the poor students with the premise of a stable educational financial input. In this paper, we mainly put forward the basic differential equation model describing the problem of higher education charges in China. Through the qualitative analysis of these two basic models, we draw several conditions for universities to maintain or stabilize their education charges and give some new conclusions and suggestions on macrocontrol of university education charges.

Data Availability

The data used to support the findings of this study are available from the *China Statistical Yearbook* (1999–2018) and the *China Education Funds Statistical Yearbook* (1999–2018).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Research Article

Iterative Solution for Systems of a Class of Abstract Operator Equations in Banach Spaces and Application

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In this paper, by using the partial order method, the existence and uniqueness of a solution for systems of a class of abstract operator equations in Banach spaces are discussed. The result obtained in this paper improves and unifies many recent results. Two applications to the system of nonlinear differential equations and the systems of nonlinear differential equations in Banach spaces are given, and the unique solution and interactive sequences which converge the unique solution and the error estimation are obtained.

1. Introduction

Guo and Lakshmikantham [1] introduced the definition of the mixed monotone operator and the coupled fixed point, and there are many good results (see [2–23]). Recently, from paper [6], using the monotone iterative techniques, the iterative unique solution of the following nonlinear mixed monotone Fredholm-type integral equations in Banach spaces E is obtained:

$$u(t) = \int_I H(t, s, u(s))\mathrm{d}s, \quad I = [a, b], \tag{1}$$

where $I = [a, b]$ and $H \in C[I \times I \times E, E]$.

In this paper, the following nonlinear abstract operator equations in Banach spaces E are discussed:

$$\begin{cases} u = A(u, v), \\ v = B(v, u), \end{cases} \tag{2}$$

where $A, B\colon D \times D \longrightarrow E$ and D is a partial interval in E which is denoted as the following:

$$D \doteq [u_0, v_0] \equiv \{u \in E \mid u_0 \le u \le v_0\}. \tag{3}$$

For convenience, the following assumptions are made:

(H_1) There exist positive bounded operators $T_i\colon E \longrightarrow E\,(i = 1, 2)$ which satisfy.

$(I + T_1 - T_2)x \ge \theta \Longrightarrow x \in P$, and for any $u_i, v_i \in D\,(i = 1, 2), u_1 \le u_2, v_1 \le v_2$, the following is obtained:

$$\begin{aligned} B(v_2, u_1) - B(v_1, u_2) &\ge -T_1(v_2 - v_1) - T_2(u_2 - u_1), \\ A(v_2, u_1) - A(v_1, u_2) &\ge -T_1(v_2 - v_1) - T_2(u_2 - u_1). \end{aligned} \tag{4}$$

(H_2) $u_0 + T_2(v_0 - u_0) \le A(u_0, v_0), \quad B(v_0, u_0) \le v_0 - T_2(v_0 - u_0)$.

(H_3) There exists a positive bounded operator $L\colon E \longrightarrow E$, and for any $u, v \in D, u \le v$, the following is obtained:

$$-(T_1 + T_2)(v - u) \le B(v, u) - A(u, v) \le L(v - u). \tag{5}$$

(H_4) $LT_2 = T_2L, LT_1 = T_1L, T_1T_2 = T_2T_1$ in which the spectral radius satisfies

$$r(L)+r(T_1)+r(T_2)<\inf\{|\lambda|:\ \lambda\in\sigma(I+T_1-T_2)\},\quad r(T_1-T_2)<1. \tag{6}$$

In this paper, firstly, by using the partial order method, the existence and uniqueness of a solution for systems of a class of abstract operator equations in Banach spaces are discussed. And next, two applications to the system of nonlinear integral equations and the system of nonlinear differential equations in Banach spaces are given, and the unique solution and interactive sequences which converge a unique solution and the error estimation are obtained.

2. The Interactive Solution of Abstract Operator Equations

Let P be a cone in E, i.e., a closed convex subset, such that $\lambda P\subset P$ for any $\lambda\geq 0$ and $P\cap\{-P\}=\{\theta\}$. A partial order $\leq$ in P is defined as $x\leq y\Leftrightarrow y-x\in P$. A cone P is said to be normal if there exists a constant $N>0$ which satisfies $x,y\in E,\theta\leq x\leq y$, implying $\|x\|\leq N\|y\|$, where θ denotes the zero element of E. And, the smallest number N is called as the normal constant of P and denoted as N_P. The cone P is normal iff every ordered interval $[x,y]=\{z\in E:\ x\leq z\leq y\}$ is bounded.

The following theorem is the main results in this section.

Theorem 1. *Let P be a cone in E, $u_0,v_0\in E,u_0\leq v_0$. Suppose that A,B: $D\times D\longrightarrow E$ satisfies conditions $(H_1)-(H_4)$. Then,*

(i) There exists a unique solution of equation (2) (u^*,u^*) in $D\times D$, and for any solutions of equation (2) $(u,u)\in D\times D$, one has $u=u^*$.

(ii) For any initial value $x_0,y_0\in D,x_0\leq y_0$, the following iterative sequences are constructed:

$$\begin{cases} x_n=(I+T_1-T_2)^{-1}[A(x_{n-1},y_{n-1})+T_1x_{n-1}-T_2y_{n-1}],\\ y_n=(I+T_1-T_2)^{-1}[B(y_{n-1},x_{n-1})+T_1y_{n-1}-T_2x_{n-1}],\end{cases} \tag{7}$$

which satisfy $\|x_n-u^*\|\longrightarrow 0,\|y_n-u^*\|\longrightarrow 0\,(n\longrightarrow\infty)$, and for any δ,

$$\frac{r(L)+r(T_1)+r(T_2)}{\inf\{|\lambda|:\ \lambda\in\sigma(I+T_1-t_2)\}}<\delta<1, \tag{8}$$

there exists a natural number n_0 which satisfies as $n\geq n_0$, the following is obtained:

$$\begin{cases}\|x_n-u^*\|\leq 2N_p\delta^n\|v_0-u_0\|,\\ \|y_n-u^*\|\leq 2N_p\delta^n\|v_0-u_0\|.\end{cases} \tag{9}$$

Proof. By $r(T_1-T_2)<1$, it is known that the operator $(I+T_1-T_2)$ is reversible. And, from condition (H_1), $(I+T_1-T_2)^{-1}$ is the positive operator. Let

$$\begin{cases} F(u,v)=(I+T_1-T_2)^{-1}[A(u,v)+T_1u-T_2v],\\ G(v,u)=(I+T_1-T_2)^{-1}[B(v,u)+T_1v-T_2u].\end{cases} \tag{10}$$

Then, equation (7) can be substituted by the following:

$$\begin{cases} x_n=F(x_{n-1},y_{n-1}),\\ y_n=G(y_{n-1},x_{n-1}).\end{cases} \tag{11}$$

By conditions $(H_1)-(H_3)$, it is easy to obtain that operators F and G satisfy the following:

(1) $u_0\leq F(u_0,v_0)\leq G(v_0,u_0)\leq v_0$

(2) F, G: $D\times D\longrightarrow E$ are the mixed monotone operator

(3) $\theta\leq G(v,u)-F(u,v)\leq H(v-u),u_0\leq u\leq v\leq v_0$, where $H=(L+T_1+T_2)(I+T_1-T_2)^{-1}$

Letting $u_n=F(u_{n-1},v_{n-1})$ and $v_n=G(v_{n-1},u_{n-1})$ $(n=1,2,\ldots)$, the following two results are obtained by mathematical induction:

$$u_0\leq u_1\leq\cdots\leq u_n\leq\cdots\leq v_n\leq\cdots\leq v_1\leq v_0, \tag{12}$$

$$\begin{gathered} u_n\leq x_n\leq y_n\leq v_n,\\ \theta\leq v_n-u_n\leq H^n(v_0-u_0),\quad n=1,2,\ldots. \end{gathered} \tag{13}$$

In fact, from (1) and (3), one has

$$\begin{gathered} u_0\leq u_1\leq v_1\leq v_0,\\ u_1\leq x_1\leq y_1\leq v_1,\\ 0\leq v_1-u_1\leq H(v_0-u_0). \end{gathered} \tag{14}$$

Suppose that for $n=k$, one has (12) and (13). Then, as $n=k+1$, by (2) and (3), the following is obtained:

$$\begin{aligned} u_{k+1}&=F(u_k,v_k)\leq x_{k+1}=F(x_k,y_k)\leq G(y_k,x_k)=y_{k+1}\\ &\leq G(v_k,u_k)=v_{k+1},\\ \theta&\leq v_{k+1}-u_{k+1}=G(v_k,u_k)-F(u_k,v_k)\leq H(v_k-u_k)\\ &\leq H^{k+1}(v_k-u_k). \end{aligned} \tag{15}$$

Then, it is known that

$$\begin{gathered} u_k\leq u_{k+1}\leq x_{k+1}\leq y_{k+1}\leq x_{k+1}\leq v_k,\\ \theta\leq v_{k+1}-u_{k+1}\leq H^{k+1}(v_k-u_k). \end{gathered} \tag{16}$$

Then, for any natural number n, (12) and (13) are obtained by mathematical induction.

Next, it is proved that $\{x_n\}$ is Cauchy sequences. From condition (H_4), it is known that

$$(L+T_1+T_2)(I+T_1-T_2)^{-1}=(I+T_1-T_2)^{-1}(L+T_1+T_2), \tag{17}$$

then by ([14], V 3.9), $r(H)\leq[r(L)+r(T_1)+r(T_2)]$ r[{(I+{T_{{1}}}−{T_{{2}}})^{{−1}}}].

Thus, for any δ: $(r(L)+r(T_1)+r(T_2))/(\inf\{|\lambda|:\ \lambda\in\sigma(I+T_1-t_2)\})<\delta<1$, the following is obtained:

$$\lim\|H^n\|^{1/n} = r(H) \leq [r(L) + r(T_1) + r(T_2)]r \cdot \left[(I + T_1 - T_2)^{-1}\right] = \frac{[r(L) + r(T_1) + r(T_2)]}{\inf\{|\lambda|: \lambda \in \sigma(I + T_1 - t_2)\}} < \delta < 1. \tag{18}$$

Then, there exists a natural number n_0 which satisfies

$$\|H^n\| \leq \delta^n, \quad \forall n \geq n_0. \tag{19}$$

And, by (12) and (13), it is obtained that

$$\begin{aligned} &\theta \leq u_n \leq u_{n+p} \leq x_{n+p} \leq y_{n+p} \leq v_{n+p} \leq v_n, \\ &\theta \leq u_n \leq x_n \leq y_n \leq v_n, n, \quad p = 1, 2, \ldots. \end{aligned} \tag{20}$$

So, by (13), it is known that

$$\begin{aligned} &\theta \leq x_{n+p} - u_n \leq v_n - u_n \leq H^n(v_0 - u_0), \\ &\theta \leq x_n - u_n \leq H^n(v_0 - u_0). \end{aligned} \tag{21}$$

Then, by the normality of P and (19), it is known that

$$\left\|x_{n+p} - u_n\right\| \leq N_p \left\|H^n(v_0 - u_0)\right\| \leq N_p \delta^n \left\|v_0 - u_0\right\|, \tag{22}$$

$$\begin{aligned} \left\|x_n - u_n\right\| \leq N_p \left\|H^n(v_0 - u_0)\right\| \leq N_p \delta^n \left\|v_0 - u_0\right\|, \\ n \geq n_0, p = 1, 2, \ldots. \end{aligned} \tag{23}$$

Thus, the following is obtained:

$$\begin{aligned} \left\|x_{n+p} - x_n\right\| \leq \left\|x_{n+p} - u_n\right\| + \left\|x_n - u_n\right\| \leq 2N_p \delta^n \left\|v_0 - u_0\right\|, \\ n \geq n_0, p = 1, 2, \ldots, \end{aligned} \tag{24}$$

i.e., $\{x_n\}$ is Cauchy sequences. So, there exists $u^* \in D$ (D is bounded), such that $\lim_{n \longrightarrow \infty} x_n = u^*$.

And, by $\theta \leq y_n - x_n \leq v_n - u_n \leq H^n(v_0 - u_0)$, the normality of P, and (19), one obtains

$$\left\|y_n - x_n\right\| \leq N_p \delta^n \left\|v_0 - u_0\right\|, \tag{25}$$

therefore

$$\lim_{n \longrightarrow \infty} y_n = u^* = \lim_{n \longrightarrow \infty} x_n, \quad x_n \leq u^* \leq y_n, n = 1, 2, \ldots. \tag{26}$$

Thus, $\|x_n - u_n\| \leq N_p \delta^n \|v_0 - u_0\|$, $\|v_n - x_n\| \leq N_p \delta^n \|v_0 - u_0\|$, and

$$\lim_{n \longrightarrow \infty} u_n = u^* = \lim_{n \longrightarrow \infty} v_n, \tag{27}$$

$$u_n \leq u^* \leq v_n, \quad n = 1, 2, \ldots, \tag{28}$$

so by (2), (3), and (11), it is also obtained that

$$u_n = F(u_{n-1}, v_{n-1}) \leq F(u^*, u^*) \leq G(u^*, u^*) \leq G(v_{n-1}, u_{n-1}) = v_n. \tag{29}$$

Letting $n \longrightarrow \infty$ and by (27), $F(u^*, u^*) = G(u^*, u^*) = u^*$.

Then, by the definition of F and G, one obtains $u^* = A(u^*, u^*), u^* = B(u^*, u^*)$, i.e., (u^*, u^*) is a solution of equation (2).

Lastly, it is proven that the solution is unique. Supposing that $(u, u) \in D \times D$ also satisfies equation (2), then by (11) and mathematical induction, the following is obtained:

$$u_n \leq u \leq v_n, \quad (n = 1, 2, \ldots). \tag{30}$$

Thus, $u = u^*$.

And, letting $p \longrightarrow \infty$ in (24), as $n \geq n_0$, the following is obtained:

$$\left\|x_n - u^*\right\| \leq 2N_p \delta^n \left\|v_0 - u_0\right\|. \tag{31}$$

Similarly, as $n \geq n_0$, the following is obtained:

$$\left\|y_n - u^*\right\| \leq 2N_p \delta^n \left\|v_0 - u_0\right\|. \tag{32}$$

The proof is complete.

Remark 1. In Theorem 1, it is only supposed that operators A and B satisfy the partial condition, and the unique solution and interactive sequences which converge a unique solution are obtained.

3. The Application of Nonlinear Integral Equations

In this section, the following nonlinear integral equations are considered:

$$\begin{cases} u(t) = f_1(t, u(t), v(t)) + \displaystyle\int_0^t g_1(s, u(s), v(s)) \mathrm{d}s, \\ v(t) = f_2(t, v(t), u(t)) + \displaystyle\int_0^t g_2(s, u(s), u(s)) \mathrm{d}s, \end{cases} \tag{33}$$

where $f_i \in [I \times \mathbf{R}_+ \times R_+, R_+]$ (here, the continuity of f_i is not assumed) and $g_i \in C[I \times \mathbf{R}_+ \times \mathbf{R}_+, \mathbf{R}_+]$, $i = 1, 2$, $I = [0, +\infty)$, and E is a real Banach space with norm $\|\cdot\|$.

In this section, the iterative solution of a nonlinear integral equation (33) is discussed. For convenience, the following assumptions are made:

(L_1) For the nonnegative bounded continuous function $a(t), b(t)$, and nonnegative integrable $c(t)$, $d(t)$, one has

$$\begin{aligned} &f_2(t, u, \theta) \leq a(t)u + b(t), \\ &g_2(t, u, \theta) \leq c(t)u + d(t). \end{aligned} \tag{34}$$

(L_2) There exists a constant $M > 0$, for any $u, v \in E, u \leq v$, which satisfies

$$\begin{aligned} &f_i(t, v, u) - f_i(t, u, v) \geq -M(v - u), \\ &g_i(t, v, u) - g_i(t, u, v) \geq 0, \quad (i = 1, 2). \end{aligned} \tag{35}$$

(L_3) For any $u, v \in E, u \leq v$, the following is satisfied:

$$-M(v-u) \le f_2(t,v,u) - f_1(t,u,v) \le c(t)(v-u),$$
$$0 \le g_2(t,v,u) - g_1(t,u,v) \le a(t)(v-u). \tag{36}$$

(L_4) $\max_{t\in I} a(t) < 1$.

In this section, the following main theorem is obtained.

Theorem 2. *Let P be a normal cone in E. Suppose conditions $(L_1)-(L_4)$ hold. Then, there exists a unique solution of equation (2) $(u^*,u^*) \in (E\times E)$, and there are iterative sequences converging to the unique solution, and corresponding error estimates are given.*

Proof. Let $E = C[I,R]$. Then, $P_c = \{x \in C[I,R] \mid x(t) \ge 0, \forall t \in I\}$ is a cone. Thus, by the normal of P, P_c is also normal.

The following operator is considered:

$$A = F_1 + G_1,$$
$$B = F_2 + G_2, \tag{37}$$

where for any $u, v \in P_c, t \in I$,

$$F_1(u,v) = f_1(t,u(t),v(t)),$$
$$G_1(u,v) = \int_0^t g_1(s,u(s),v(s))\mathrm{d}s,$$
$$F_2(v,u) = f_2(t,u(t),u(t)),$$
$$G_1(v,u) = \int_0^t g_2(s,v(s),u(s))\mathrm{d}s. \tag{38}$$

Then, A, B: $P_c \times P_c \longrightarrow E$. It is easy to know that $(u^*,u^*) \in P_c \times P_c$ is a solution of (33) if and only if (u^*,u^*) is a solution of the following integral equations:

$$\begin{cases} u = A(u,v), \\ v = B(v,u). \end{cases} \tag{39}$$

Next, from conditions $(L_1)-(L_4)$, it is obtained that the operators A and B satisfy the whole condition of Theorem 1.

In fact, $\forall u_1,u_2,v_1,v_2 \in P_c$, $u_1 \le u_2, v_1 \le v_2$:

(i) Let

$$Lu = a(t)u + \int_0^t c(s)u(s)\mathrm{d}s,$$
$$h = b(t) + \int_0^t d(s)\mathrm{d}s, \quad t \in I,$$
$$L_1 u = a(t)u,$$
$$L_2 u = \int_0^t c(s)u(s)\mathrm{d}s. \tag{40}$$

Then, $L_1L_2 = L_2L_1$ and $r(L_1) = \max_{t\in I} a(t), r(L_2) = 0$. Thus,

$$r(L) = r(L_1 + L_2) \le r(L_1) + r(L_2) = \max_{t\in I} a(t) < 1. \tag{41}$$

Therefore, for the equation $(I-L)u = h$, there exists a unique solution $v_0 = (I-L)^{-1}h = \sum_{n=0}^{\infty} L^n h \in P$. Then, by (L_1), for any $t \in I$, the following is obtained:

$$\begin{aligned} B(v_0,\theta) &= F_2(v_0,\theta) + G_2(v_0,\theta) = f_2(t,v_0(t),\theta) \\ &\quad + \int_0^t g_2(s,v_0(s),\theta)\mathrm{d}s \\ &\le a(t)v_0 + \int_0^t c(s)v_0(s)\mathrm{d}s + b(t) \\ &\quad + \int_0^t d(s)\mathrm{d}s = Lv_0 + h = v_0. \end{aligned} \tag{42}$$

Obviously, $\theta \le f_1(t,\theta,v_0(t)) + \int_0^t g_1(s,\theta,v_0(s))d = A(\theta,v_0)$.

(ii) By (L_2), the following is obtained:

$$\begin{aligned} &B(v_2,u_1) - B(v_1,u_2) \\ &\quad = f_2(t,v_2(t),u_1(t) - f_2(t,v_1(t),u_2(t)) \\ &\quad + \int_0^t [g_2(s,v_2(s),u_1(s)) - g_2(s,v_1(s),u_2(s))]\mathrm{d}s \\ &\quad \ge f_2(t,v_2(t),u_1(t) - f_2(t,v_1(t),u_2(t)) \ge -M(v_2 - v_1). \end{aligned} \tag{43}$$

Similarly, $A(v_2,u_1) - A(v_1,u_2) \ge -M(v_2 - v_1)$.

(iii) From (L_3) and (L_4), the following is obtained:

$$\begin{aligned} &B(v,u) - A(u,v) \\ &\quad = f_2(t,v(t),u(t)) - f_1(t,v(t),u(t)) + \int_0^t [g_2((s,v(s),u(s)) - g_1(s,v(s),u(s)))]\mathrm{d}s \\ &\quad \ge -M(v-u) + \int_0^t [g_2(s,v(s),u(s)) - g_1(s,v(s),u(s))]\mathrm{d}s \ge -M(v-u), \\ &B(v,u) - A(u,v) \\ &\quad = f_2(t,v(t),u(t)) - f_1(t,v(t),u(t)) + \int_0^t [g_2(s,v(s),u(s)) - g_1(s,v(s),u(s))]\mathrm{d}s \\ &\quad \le a(t)(v-u) + \int_0^t c(s)(v-u)\mathrm{d}s = L(v-u). \end{aligned} \tag{44}$$

Then, by (41), it is known that

$$-M(v-u) \le B(v,u) - A(u,v) \le L(v-u), \quad r(L) < 1. \tag{45}$$

Therefore, from (i), (ii), and (iii), letting $T_1 = M_1 I$, $T_2 = 0$ in Theorem 1, it is easy to know that the condition (H_4) holds.

Finally, for any initial value $x_0, y_0 \in [\theta, v_0]$, $x_0 \le y_0$, by constructing the iterative sequences

$$\begin{cases} x_n(t) = f_1(t, x_{n-1}(t), y_{n-1}(t)) + \int_0^t g_1(s, x_{n-1}(s), y_{n-1}(s))ds, \\ y_n(t) = f_2(t, y_{n-1}(u), x_{n-1}(t)) + \int_0^t g_2(s, y_{n-1}(s), x_{n-1}(s))ds, \end{cases} \tag{46}$$

one has $\|x_n - u^*\| \longrightarrow 0, \|y_n - u^*\| \longrightarrow 0 (n \longrightarrow \infty)$, and for any $\alpha \in (0,1)$, there exists a natural number n_0 which satisfies as $n \ge n_0$, the following is obtained:

$$\begin{cases} \|x_n - u^*\| \le 2N_p \alpha^n \|v_0 - u_0\|, \\ \|y_n - u^*\| \le 2N_p \alpha^n \|v_0 - u_0\|. \end{cases} \tag{47}$$

This completes the proof of Theorem 2.

4. The Application of Nonlinear Differential Equations

In this section, the following nonlinear initial value problems of the differential equation are considered:

$$\begin{cases} u'(t) = f_1(t,u,v) + \int_0^T g_1(s,u,v)ds, & u(0) = u_0, \\ v'(t) = f_2(t,u,v) + \int_0^T g_2(s,u,v)ds, & v(0) = v_0, \end{cases} \tag{48}$$

where $f_i, g_i \in C[I \times \mathbf{R}_+ \times \mathbf{R}_+, \mathbf{R}_+]$, $i = 1,2$, $I = [0,T]$, and E is a real Banach space with norm $\|\cdot\|$.

For convenience, the following assumptions are made:

(C_1) There exists the nonnegative bounded integrable functions $a(t), b(t), c(t), d(t)$ which satisfy

$$\begin{aligned} f_2(t,u,\theta) &\le a(t)u + b(t), \\ g_2(t,u,\theta) &\le c(t)u + d(t). \end{aligned} \tag{49}$$

(C_2) There exists constant $M > 0$, for any $u, v \in E$, $u \le v$, which satisfies

$$\begin{aligned} &f_i(t,v,u) - f_i(t,u,v) \ge -M(v-u), \\ &g_i(t,v,u) - g_i(t,u,v) \ge 0, \quad (i = 1,2). \end{aligned} \tag{50}$$

(C_3) For any $u, v \in E$, $u \le v$, the following is satisfied:

$$\begin{aligned} -M(v-u) \le f_2(t,v,u) - f_1(t,u,v) \le c(t)(v-u), \\ 0 \le g_2(t,v,u) - g_1(t,u,v) \le a(t)(v-u). \end{aligned} \tag{51}$$

(C_4) $\int_0^T a(t)(v-u)dr \int_0^t K(r,s)ds < e^{Mt}, \forall t \in I$.

Then, the following theorem is obtained.

Theorem 3. *Let P be a normal cone in E. Suppose that conditions $(C_1) - (C_4)$ hold. Then, there exists a unique solution of equation (48) (u^*, u^*), and there are iterative sequences converging to the unique solution, and corresponding error estimates are given.*

Proof. Firstly, differential equation (48) is turned into integral equations. For any fixed $\eta \in C^1[J,E]$, the following one-order linear ordinary differential initial value problems in Banach spaces are investigated:

$$\begin{cases} u' = f_1(t,\eta,\eta) - M(u-\eta) + \int_0^T K(t,s) g_1(s,\eta,\eta)ds, & u(0) = u_0, \\ u' = f_2(t,\eta,\eta) - M(u-\eta) + \int_0^T K(t,s) g_2(s,\eta,\eta)ds, & u(0) = u_0. \end{cases} \tag{52}$$

It is easy to know that $(u,u) \in C^1[I,E] \times C^1[I,E]$ is a solution of (52) if and only if (u,u) is a solution of the following integral equations:

$$
\begin{aligned}
u(t) &= e^{-Mt}\left[u_0 + \int_0^T g_1(r,\eta(r),\eta(r))\mathrm{d}r \int_0^t K(s,r)\mathrm{d}s\right] \\
&\quad + e^{-Mt}\int_0^t e^{Ms}\left[f_1(s,\eta(s),\eta(s)) + M\eta(s)\right]\mathrm{d}s, \\
u(t) &= e^{-Mt}\left[u_0 + \int_0^T g_2(r,\eta(r),\eta(r))\mathrm{d}r \int_0^t K(s,r)\mathrm{d}s\right] \\
&\quad + e^{-Mt}\int_0^t e^{Ms}\left[f_2(s,\eta(s),\eta(s)) + M\eta(s)\right]\mathrm{d}s.
\end{aligned}
\tag{53}
$$

Next, the operator $A, B\colon C^1[I,E] \times C^1[I,E] \longrightarrow C^1[I,E]$ is defined as the following:

$$
\begin{aligned}
A(\eta,\eta) &= e^{-Mt}\left[u_0 + \int_0^T g_1(r,\eta(r),\eta(r))\mathrm{d}r \int_0^t K(s,r)\mathrm{d}s\right] \\
&\quad + e^{-Mt}\int_0^t e^{Ms}\left[f_1(s,\eta(s),\eta(s)) + M\eta(s)\right]\mathrm{d}s, \\
B(\eta,\eta) &= e^{-Mt}\left[u_0 + \int_0^T g_2(r,\eta(r),\eta(r))\mathrm{d}r \int_0^t K(s,r)\mathrm{d}s\right] \\
&\quad + e^{-Mt}\int_0^t e^{Ms}\left[f_2(s,\eta(s),\eta(s)) + M\eta_{(s)}\right]\mathrm{d}s.
\end{aligned}
\tag{54}
$$

Obviously, (η,η) is a solution of (48) if and only if

$$
\begin{cases}
\eta = A(\eta,\eta), \\
\eta = B(\eta,\eta).
\end{cases}
\tag{55}
$$

Next, similar to the proof of Theorem 2, it is tested whether the operators A and B satisfy the whole condition of Theorem 1 from conditions $(C_1)-(C_4)$. Therefore, the result of Theorem 3 is obtained from Theorem 1.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The author read and approved the final manuscript.

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