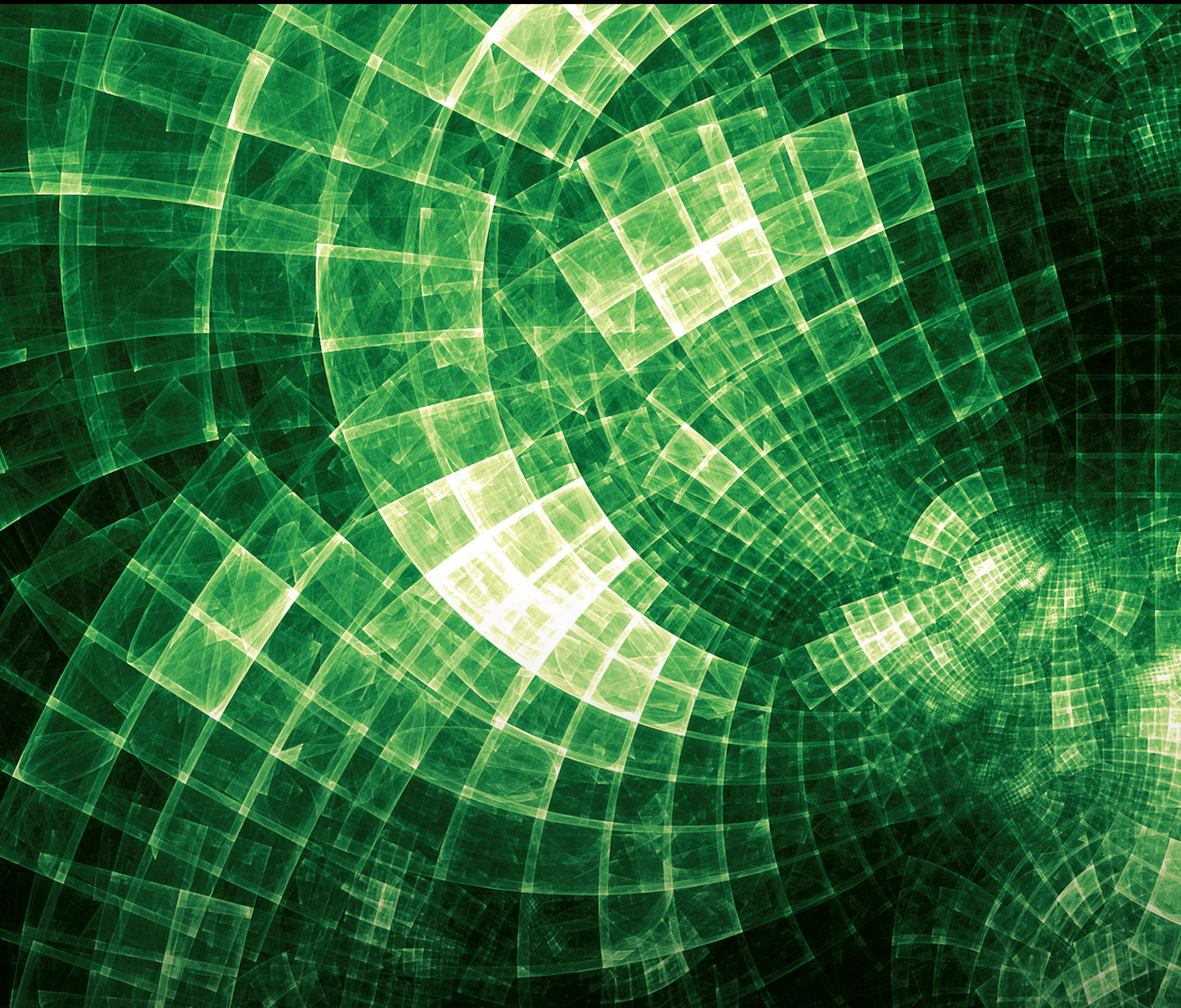
Journal of Mathematics

Recent Trends in Nonlinear and Variational Analysis

Lead Guest Editor: Xiaolong Qin
Guest Editors: Sun Young Cho and Ching-Feng Wen



Recent Trends in Nonlinear and Variational Analysis

Journal of Mathematics

Recent Trends in Nonlinear and Variational Analysis

Lead Guest Editor: Xiaolong Qin
Guest Editors: Sun Young Cho and Ching-Feng Wen

Contents

Contents

Hindawi
Journal of Mathematics
Volume 2021, Article ID 5542994, 6 pages
https://doi.org/10.1155/2021/5542994

Research Article

Condensing Mappings and Best Proximity Point Results

Sarah O. Alshehri,[1,2] Hamed H. Alsulami,[1] and Naseer Shahzad [1]

[1]*King Abdulaziz University, Department of Mathematics, P.O. Box 80203, Jeddah 2159, Saudi Arabia*
[2]*University of Jeddah, College of Science, Department of Mathematics, Jeddah, Saudi Arabia*

Correspondence should be addressed to Naseer Shahzad; nshahzad@kau.edu.sa

Received 22 January 2021; Revised 8 February 2021; Accepted 23 February 2021; Published 21 April 2021

Academic Editor: Xiaolong Qin

Best proximity pair results are proved for noncyclic relatively u-continuous condensing mappings. In addition, best proximity points of upper semicontinuous mappings are obtained which are also fixed points of noncyclic relatively u-continuous condensing mappings. It is shown that relative u-continuity of $\mathfrak{T}$ is a necessary condition that cannot be omitted. Some examples are given to support our results.

1. Introduction

The concept of measure of noncompactness was first introduced by Kuratowski [1]. However, the interest in the concept was revived in 1955 when Darbo [2] proved a generalization of Schauder's fixed point theorem using this concept. Sadovskii [3], in 1967, defined condensing mappings and extended Darbo's theorem. Since then a lot of work has been done using this concept, and several interesting results have appeared, see, for instance, [4–9].

Let (W, Z) be a nonempty pair in a Banach space (that is, both W and Z are nonempty sets). A mapping $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ is called noncyclic provided $\mathfrak{T}(W) \subseteq W$ and $\mathfrak{T}(Z) \subseteq Z$. If there exists $(w, z) \in W \times Z$ which satisfies $w = \mathfrak{T}(w)$, $z = \mathfrak{T}(z)$, and $\|w - z\| = \operatorname{dist}(W, Z)$, then we say that the noncyclic mapping $\mathfrak{T}$ has a best proximity pair. For a multivalued nonself mapping $S\colon W \longrightarrow 2^Z$, a point $w \in W$ is called a fixed point of S if $w \in S(w)$. The necessary condition for the existence of a fixed point for such S is $W \cap Z \neq \varnothing$. If $W \cap Z = \varnothing$, then $\operatorname{dist}(w, S(w)) > 0$ for each $w \in W$. Best proximity point theorems provide sufficient conditions for the existence of at least one solution for the minimization problem, $\min_{w \in W} \operatorname{dist}(w, S(w))$. If $\operatorname{dist}(w, S(w)) = \operatorname{dist}(W, Z)$, the point w is called a best proximity point of S. The existence results of best proximity points for multivalued mappings were obtained in [10–14] and [15]. Best proximity point theorems for relatively nonexpansive and relatively u-continuous were established by Elderd et al. in [16, 17] and by Markin and Shahzad in [18]. In recent years, the topics of best proximity points of single-valued and multivalued mappings have attracted the attention of many researchers, see, for example, the work in [6, 7, 19, 20] and the references cited therein. In this paper, we prove best proximity pair theorems for noncyclic relatively u-continuous condensing mappings. In addition, we obtain best proximity points of upper semicontinuous mappings which are fixed points of noncyclic relatively u-continuous condensing mappings. Also, we give examples to support our results and show by giving an example that relative u-continuity of $\mathfrak{T}$ is a necessary condition that cannot be omitted. Our results extend and complement results of [6, 7, 11].

2. Preliminaries

In this section, we present some notions and known results which will be used in the sequel.

Definition 1. Let K be a bounded set in a metric space X. The Kuratowski noncompactness measure $\alpha(K)$ (or simply, measure of noncompactness) is defined as follows:

$$\alpha(K) = \inf\left\{\eta > 0:\ K \subseteq \bigcup_{l=1}^{m} A_l:\ \operatorname{diam}(A_l) \le \eta, \quad \forall 1 \le l \le m < \infty\right\}. \tag{1}$$

Theorem 1. *Let X be a metric space. Then, for any nonempty bounded pair (C_1, C_2) in X (that is, both C_1 and C_2 are nonempty and bounded sets), the following hold:*

(1) $\alpha(C_1) = 0$ if and only if C_1 is relatively compact

(2) $C_1 \subseteq C_2$ implies $\alpha(C_1) \le \alpha(C_2)$

(3) $\alpha(\overline{C_1}) = \alpha(C_1)$, where $\overline{C_1}$ denotes the closure of C_1

(4) $\alpha(C_1 \cup C_2) = max\{\alpha(C_1), \alpha(C_2)\}$

(5) For a normed space X:

(i) $\alpha(C_1 + x) = \alpha(C_1)$

(ii) $\alpha(C_1 + C_2) \le \alpha(C_1) + \alpha(C_2)$

(iii) $\alpha(\lambda C_1) = |\lambda|\alpha(C_1)$, for any number λ

(iv) $\alpha(\overline{con}(C_1)) = \alpha(con(C_1)) = \alpha(C_1)$, where $con(C_1)$ represents the convex hull of C_1

Theorem 2. *Let $\{F_j\}$ be a decreasing sequence of nonempty closed subsets of a complete metric space X. If $\alpha(F_j) \longrightarrow 0$ as $j \longrightarrow \infty$, then $\cap_{j\in\mathbb{N}} F_j \neq \varnothing$.*

For more details about the measure of noncompactness, see [4].

Definition 2. Let (W, Z) be a nonempty pair in Banach space X and $\mathfrak{T}: W \cup Z \longrightarrow W \cup Z$ a mapping. Then, $\mathfrak{T}$ is said to be noncyclic relatively u-continuous. If $\mathfrak{T}$ is noncyclic and for each $\epsilon > 0$, there is $\gamma > 0$ such that

$$\|\mathfrak{T}(w) - \mathfrak{T}(z)\| < \varepsilon + \operatorname{dist}(W, Z) \text{ whenever } \|w - z\| < \gamma + \operatorname{dist}(W, Z), \tag{2}$$

for each $w \in W$ and $z \in Z$.

Definition 3. Let (W, Z) be a nonempty convex pair in Banach space X. A mapping $\mathfrak{T}: W \cup Z \longrightarrow W \cup Z$ is said to be affine if for each $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ and $x_1, x_2 \in W$ (respectively, $x_1, x_2 \in Z$),

$$\mathfrak{T}(\alpha x_1 + \beta x_2) = \alpha\mathfrak{T}(x_1) + \beta\mathfrak{T}(x_2). \tag{3}$$

Definition 4. Let (W, Z) be a nonempty pair in Banach space X and $S: W \longrightarrow 2^Z$ a multivalued mapping on W, then S is said to be upper semicontinuous if for each closed subset B in Z, $S^{-1}(B) = \{w \in W: S(w) \cap B \neq \varnothing\}$ is closed in W.

Lemma 1. *(see [21]). Let Y be a nonempty, convex, and compact subset of a Banach space X. If $f: Y \longrightarrow 2^Y$ can be written as a finite composition of upper semicontinuous multivalued mappings of nonempty, compact, and convex values, then f has a fixed point.*

Definition 5. Let $\mathfrak{T}: W \cup Z \longrightarrow W \cup Z$ be a noncyclic relatively u-continuous mapping and $S: W \longrightarrow KC(Z)$ be an upper semicontinuous multivalued mapping (here, $KC(Z)$ denotes the collection of all nonempty, convex, and compact subsets of Z), then by the commutativity of $\mathfrak{T}$ and S, we mean that $\mathfrak{T}(S(w)) \subseteq S(\mathfrak{T}(w))$ holds for each $w \in W$.

Given (W, Z), a nonempty pair in Banach space, its proximal pair (W_0, Z_0) is given by

$$\begin{aligned} W_0 &= \left\{w \in W:\ \|w - z^*\| = \operatorname{dist}(W, Z) \text{ for some } z^* \in Z\right\}, \\ Z_0 &= \left\{z \in Z:\ \|w^* - z\| = \operatorname{dist}(W, Z) \text{ for some } w^* \in W\right\}. \end{aligned} \tag{4}$$

Moreover, if (W, Z) is a nonempty, convex, and compact pair in X, then (W_0, Z_0) is also a nonempty, convex, and compact pair.

Definition 6. Let X be a normed space. For a nonempty subset C of X, the metric projection operator $P_C: X \longrightarrow 2^C$ is given by

$$P_C(u) := \{v \in C: \|u - v\| = \operatorname{dist}(u, C)\}. \tag{5}$$

For a nonempty, convex, and compact subset C of a strictly convex Banach space, P_C is a single-valued mapping. Furthermore, for a nonempty, convex, and compact subset C of a Banach space X, P_C is upper semicontinuous with nonempty, convex, and compact values.

Lemma 2. *(see [11]). Let (W, Z) be a nonempty, convex, and compact pair in a strictly convex Banach space X. Let $\mathfrak{T}: W \cup Z \longrightarrow W \cup Z$ be a noncyclic relatively u-continuous and $P: W \cup Z \longrightarrow W \cup Z$ be a mapping given by*

$$P(u) = \begin{cases} P_Z(u), & \text{if } u \in W, \\ P_W(u), & \text{if } u \in Z. \end{cases} \tag{6}$$

Then, $\mathfrak{T}(P(u)) = P(\mathfrak{T}(u))$ for each $u \in W_0 \cup Z_0$.

Theorem 3. *(see [18]). Let (W, Z) be a nonempty, convex, and compact pair in a strictly convex Banach space X. If $\mathfrak{T}: W \cup Z \longrightarrow W \cup Z$ is a noncyclic relatively u-continuous mapping. Then, $\mathfrak{T}$ has best proximity pair.*

In [6], Gabeleh and Markin introduced the class of noncyclic condensing operators.

Recall that a nonempty pair (W, Z) in a Banach space X is called proximinal if $W = W_0$ and $Z = Z_0$.

Definition 7. Let (W, Z) be a nonempty convex pair in a strictly convex Banach space X. A mapping $\mathfrak{T}: W \cup Z \longrightarrow W \cup Z$ is called noncyclic condensing operator provided that, for any nonempty, bounded, closed, convex, proximinal, and $\mathfrak{T}$-invariant pair $(H_1, H_2) \subseteq (W, Z)$ with $\operatorname{dist}(H_1, H_2) = \operatorname{dist}(W, Z)$, there exists $k \in (0, 1)$ such that

$$\alpha(\mathfrak{T}(H_1) \cup \mathfrak{T}(H_2)) \le k\alpha(H_1 \cup H_2). \tag{7}$$

Lemma 3. *(see [11]). Let (W, Z) be a nonempty, convex, and compact pair in a strictly convex Banach space X. If*

$\mathfrak{T}$: $W\cup Z \longrightarrow W\cup Z$ is a noncyclic relatively u-continuous mapping, then T is continuous on W_0 and Z_0.

3. Main Results

Throughout this paper, we will assume that X is a strictly convex Banach space and α is the measure of noncompactness on X.

Remark 1. Let $\mathfrak{T}$: $W \longrightarrow W$ be condensing in the sense of Definition 7 with $k\in(0,1)$. Then, for any bounded subset H of W, $\mathfrak{T}$ satisfies

$$\alpha(\mathfrak{T}(H))\le k\alpha(H). \tag{8}$$

To see this, in (7), set $W=Z$ and $H_1=H_2=H$. Since $H\subseteq\overline{\text{con}}(H)$, then

$$\alpha(\mathfrak{T}(H))\le\alpha(\mathfrak{T}(\overline{\text{con}}(H)))\le k\alpha(\overline{\text{con}}(H))=k\alpha(H). \tag{9}$$

Theorem 4. *Let (W,Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. Suppose $\mathfrak{T}$: $W\cup Z\longrightarrow W\cup Z$ is a noncyclic relatively u-continuous, affine, and condensing mapping. Then, there exists $(u_0,v_0)\in W\times Z$ such that $\mathfrak{T}(u_0)=u_0$, $\mathfrak{T}(v_0)=v_0$ and $\|u_0-v_0\|=\text{dist}(W,Z)$. Moreover, if S: $W\longrightarrow KC(Z)$ is an upper semicontinuous multivalued mapping, $\mathfrak{T}$ and S commute, and for each $x\in W_0$, $S(x)\cap Z_0\neq\varnothing$, there exists $w\in W$ such that $\mathfrak{T}(w)=w$ and $\text{dist}(w,S(w))=\text{dist}(W,Z)$.*

Proof. We follow [6, 11]. Clearly, (W_0,Z_0) is a nonempty, closed, convex, proximinal, and $\mathfrak{T}$-invariant pair. Let $(w_0,z_0)\in W_0\times Z_0$ be such that $\|w_0-z_0\|=\text{dist}(W,Z)$. Suppose $\mathscr{F}$ is a family of nonempty, closed, convex, proximinal, and $\mathfrak{T}$-invariant pairs $(C,D)\subseteq(W,Z)$ such that $(w_0,z_0)\in(C,D)$, then $\mathscr{F}$ is nonempty. Set $(F_1,F_2)=\cap_{(C,D)\in\mathscr{F}}(C,D)$, $G_1=\overline{\text{con}}(\mathfrak{T}(F_1)\cup\{w_0\})$, and $G_2=\overline{\text{con}}(\mathfrak{T}(F_2)\cup\{z_0\})$. So, $(w_0,z_0)\in G_1\times G_2$ and $(G_1,G_2)\subseteq(F_1,F_2)$. Furthermore, $\mathfrak{T}(G_1)\subseteq G_1$ and $\mathfrak{T}(G_2)\subseteq G_2$, that is, $\mathfrak{T}$ is noncyclic on $G_1\cup G_2$. Also, for $x\in G_1$, $x=\sum_{\mathfrak{l}=1}^{m-1}c_{\mathfrak{l}}\mathfrak{T}(w_{\mathfrak{l}})+c_m w_0$, where for all $\mathfrak{l}\in\{1,2,\ldots,m-1\}$ with $c_{\mathfrak{l}}\ge 0$ and $\sum_{\mathfrak{l}=1}^{m}c_{\mathfrak{l}}=1$, $w_{\mathfrak{l}}\in F_1$. Since (F_1,F_2) is proximinal, there is $z_{\mathfrak{l}}\in F_2$ such that $\|w_{\mathfrak{l}}-z_{\mathfrak{l}}\|=\text{dist}(W,Z)$, for each $\mathfrak{l}\in\{1,2,\ldots,m-1\}$. Set $y=\sum_{\mathfrak{l}=1}^{m-1}c_{\mathfrak{l}}\mathfrak{T}(z_{\mathfrak{l}})+c_m z_0$. Then, $y\in G_2$. Moreover,

$$\begin{aligned}\|x-y\| &= \left\|\left(\sum_{\mathfrak{l}=1}^{m-1}c_{\mathfrak{l}}\mathfrak{T}(w_{\mathfrak{l}})+c_m w_0\right)-\left(\sum_{\mathfrak{l}=1}^{m-1}c_{\mathfrak{l}}\mathfrak{T}(z_{\mathfrak{l}})+c_m z_0\right)\right\|\\ &\le \sum_{\mathfrak{l}=1}^{m-1}c_{\mathfrak{l}}\|\mathfrak{T}(w_{\mathfrak{l}})-\mathfrak{T}(z_{\mathfrak{l}})\|+c_m\|w_0-z_0\|\\ &= \text{dist}(W,Z).\end{aligned} \tag{10}$$

So, one can conclude that $(G_1)_0=G_1$. Similarly, $(G_2)_0=G_2$, and hence, $(G_1,G_2)\in\mathscr{F}$, that is, $G_1=F_1$ and $G_2=F_2$. Notice that

$$\begin{aligned}\alpha(G_1\cup G_2) &= \max\{\alpha(G_1),\alpha(G_2)\}=\max\{\alpha(\overline{\text{con}}(\mathfrak{T}(F_1)\cup\{w_0\})),\alpha(\overline{\text{con}}(\mathfrak{T}(F_2)\cup\{z_0\}))\}\\ &= \max\{\alpha(\mathfrak{T}(F_1)),\alpha(\mathfrak{T}(F_2))\}=\alpha(\mathfrak{T}(F_1)\cup\mathfrak{T}(F_2))=\alpha(\mathfrak{T}(G_1)\cup\mathfrak{T}(G_2))\le k\alpha(G_1\cup G_2).\end{aligned} \tag{11}$$

But $k\in(0,1)$, so $\alpha(G_1\cup G_2)=0$. We conclude that (G_1,G_2) is a nonempty, compact, and convex pair with $\text{dist}(G_1,G_2)=\text{dist}(W,Z)$. By Theorem 3, there exists $(u_0,v_0)\in W\times Z$ such that $\mathfrak{T}(u_0)=u_0$, $\mathfrak{T}(v_0)=v_0$ and $\|u_0-v_0\|=\text{dist}(W,Z)$.

Now, let $\text{Fix}(\mathfrak{T})=\{x\in W\cup Z\text{: }\mathfrak{T}(x)=x\}$, $\text{Fix}_W(\mathfrak{T})=\text{Fix}(\mathfrak{T})\cap W_0$, and $\text{Fix}_Z(\mathfrak{T})=\text{Fix}(\mathfrak{T})\cap Z_0$. By the above part, the pair $(\text{Fix}_W(\mathfrak{T}),\text{Fix}_Z(\mathfrak{T}))$ is nonempty. Also, it is a convex pair. Indeed, for α ,$\beta\in[0,1]$, with $\alpha+\beta=1$ and $x,y\in\text{Fix}_W(\mathfrak{T})$ (respectively, $\text{Fix}_Z(\mathfrak{T})$):

$$\mathfrak{T}(\alpha x+\beta y)=\alpha\mathfrak{T}(x)+\beta\mathfrak{T}(y)=\alpha x+\beta y, \tag{12}$$

and by convexity of W_0 (respectively, Z_0), we conclude that $\alpha x+\beta y\in\text{Fix}_W(\mathfrak{T})$ (respectively, $\text{Fix}_Z(\mathfrak{T})$). Furthermore, since $\mathfrak{T}$ is condensing,

$$\begin{aligned}\alpha(\text{Fix}_W(\mathfrak{T})\cup\text{Fix}_Z(\mathfrak{T})) &= \alpha(\mathfrak{T}(\text{Fix}_W(\mathfrak{T}))\cup\mathfrak{T}(\text{Fix}_Z(\mathfrak{T})))\\ &\le k\alpha(\text{Fix}_W(\mathfrak{T})\cup\text{Fix}_Z(\mathfrak{T})),\end{aligned} \tag{13}$$

which implies that the pair $(\text{Fix}_W(\mathfrak{T}),\text{Fix}_Z(\mathfrak{T}))$ is compact.

For $x\in Fix_W(\mathfrak{T})$ and $u\in S(x)$, we have

$$\mathfrak{T}(u)\in\mathfrak{T}(S(x))\subseteq S(\mathfrak{T}(x))=S(x), \tag{14}$$

that is, $S(x)$ is invariant under $\mathfrak{T}$. So, by the invariance of Z_0 under $\mathfrak{T}$, $S(x)\cap Z_0\neq\varnothing$ is invariant under $\mathfrak{T}$. So, in view of Remark 3.1, Darbo's fixed point theorem guarantees the existence of a fixed point for the continuous mapping $\mathfrak{T}$: $S(x)\cap Z_0\longrightarrow S(x)\cap Z_0$. Thus, $S(x)\cap\text{Fix}_Z(\mathfrak{T})\neq\varnothing$, for $x\in\text{Fix}_W(\mathfrak{T})$. Define f: $\text{Fix}_W(\mathfrak{T})\longrightarrow 2^{\text{Fix}_Z(\mathfrak{T})}$ by $f(x)=S(x)\cap\text{Fix}_Z(\mathfrak{T})$, for each $x\in\text{Fix}_W(\mathfrak{T})$. Then, f is an upper semicontinuous multivalued mapping with nonempty, compact, and convex values. Moreover, P_W: $\text{Fix}_Z(\mathfrak{T})\longrightarrow\text{Fix}_W(\mathfrak{T})$ is well-defined. Indeed, for $y\in\text{Fix}_Z(\mathfrak{T})$, there is $x\in W$ such that $\|x-y\|=\text{dist}(W,Z)$. So,

$$y=P_Z(x)\text{ and }x=P_W(y). \tag{15}$$

By relative u-continuity of $\mathfrak{T}$, we conclude that $\|\mathfrak{T}(x)-\mathfrak{T}(y)\|=\text{dist}(W,Z)$. Thus, $\mathfrak{T}(y)=P_Z(\mathfrak{T}(x))$ and $\mathfrak{T}(x)=P_W(\mathfrak{T}(y))$, by (15), $\mathfrak{T}(x)=\mathfrak{T}(P_W(y))=P_W(\mathfrak{T}(y))=P_W(y)$. Then, $P_W(y)\in\text{Fix}_W(\mathfrak{T})$. Consider $P_W{\circ}f$: $\text{Fix}_W(\mathfrak{T})\longrightarrow 2^{\text{Fix}_W(\mathfrak{T})}$, by Lemma 1, there is $w\in\text{Fix}_W(\mathfrak{T})$ such that $w\in(P_W{\circ}f)(w)$, that is, $\mathfrak{T}(w)=w$ and $w\in(P_W(f(w)))$. So, there is $z\in f(w)\subset S(w)\cap Z_0$ such that $w=P_W(z)$. We conclude that $\|z-w\|=\text{dist}(z,W)$. But since

$z \in Z_0$, there is $w^* \in W$ such that $\|w^* - z\| = \text{dist}(W, Z)$. Thus,

$$\begin{aligned}\text{dist}(W,Z) &\leq \text{dist}(w,S(w)) \leq \|w-z\| = \text{dist}(z,W)\\ &\leq \|z-w^*\| = \text{dist}(W,Z).\end{aligned} \tag{16}$$

Hence, $\text{dist}(w,S(w)) = \text{dist}(W,Z)$. □

Example 1. Consider the Hilbert space $X = \ell_2$ over $\mathbb{R}$ with the basis $\{e_n\colon n \in \mathbb{N}\}$ (the canonical basis) and let

$$\begin{aligned}W &= \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta_1 \in [0,4], \zeta_2 = -1\} \text{ and } Z\\ &= \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta_1 \leq 0, \zeta_2 = 1\}.\end{aligned} \tag{17}$$

Then, (W,Z) be a nonempty, convex, and closed pair of X such that W is bounded. Furthermore, $\text{dist}(W,Z) = 2$ and

$$W_0 = \{-e_2\} \text{ and } Z_0 = \{e_2\}. \tag{18}$$

Define the mapping $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ by $\mathfrak{T}(\zeta_1 e_1 + \zeta_2 e_2) = (3\zeta_1/4)e_1 + \zeta_2 e_2$, for each $\zeta_1 e_1 + \zeta_2 e_2 \in W \cup Z$. Then, $\mathfrak{T}$ is a noncyclic relatively u-continuous, affine, and condensing mapping. Now, define $S\colon W \longrightarrow KC(Z)$ by $S(\zeta e_1 - e_2) = \{-\zeta e_1 + e_2\}$; then, S is an upper semicontinuous multivalued mapping, $\mathfrak{T}$ and S commute, and for each $x \in W_0$, $S(x) \cap Z_0 \neq \emptyset$. For $w = -e_2 \in W$, we have $\mathfrak{T}(w) = w$ and $\text{dist}(w,S(w)) = \text{dist}(W,Z)$.

Example 2. Consider the Hilbert space $X = \ell_2$ over $\mathbb{R}$ with the basis $\{e_n\colon n \in \mathbb{N}\}$ and let

$$\begin{aligned}W &= \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta_1 \in [0,4], \zeta_2 \in [1,5]\} \text{ and}\\ Z &= \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta \geq 0, \zeta_2 = 0\}.\end{aligned} \tag{19}$$

Then, (W,Z) be a nonempty, convex, and closed pair of X such that W is bounded with $\text{dist}(W,Z) = 1$ and

$$\begin{aligned}W_0 &= \{\zeta_1 e_1 + e_2\colon \zeta_1 \in [0,4]\},\\ Z_0 &= \{\zeta e_1\colon \zeta \in [0,4]\}.\end{aligned} \tag{20}$$

Define the mapping $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ by $\mathfrak{T}(\zeta_1 e_1 + \zeta_2 e_2) = ((2\zeta_1 + 1)/3)e_1 + \zeta_2 e_2$ for each $\zeta_1 e_1 + \zeta_2 e_2 \in W \cup Z$. Then, $\mathfrak{T}$ is a noncyclic relatively u-continuous, affine, and condensing mapping. Furthermore, for $(u_0, v_0) = (e_2, 0) \in W \times Z$, we have $\mathfrak{T}(u_0) = u_0$, $\mathfrak{T}(v_0) = v_0$, and $\|u_0 - v_0\| = \text{dist}(W,Z)$. Now, let $S\colon W \longrightarrow KC(Z)$ given by $S(\zeta_1 e_1 + \zeta_2 e_2) = \{\gamma e_1\colon \gamma \in [\zeta_1, 4]\}$, then S is an upper semicontinuous multivalued mapping, $\mathfrak{T}$ and S commute, and for each $x \in W_0$, $S(x) \cap Z_0 \neq \emptyset$. For $w = e_1 + e_2 \in W$, we have $\mathfrak{T}(w) = w$ and $\text{dist}(w,S(w)) = \text{dist}(W,Z)$.

Remark 2. The relative u-continuity of $\mathfrak{T}$ is necessary in Theorem 4.

To see this, consider the Hilbert space $X = \ell_2$ over $\mathbb{R}$ with the basis $\{e_n\colon n \in \mathbb{N}\}$ and let $W = \{x \in X\colon \|x\| \leq 1\}$, $Z = \{\zeta e_2\colon \zeta \in [2,3]\}$. Then, (W,Z) is a nonempty, convex, and closed pair in X such that W is bounded. Obviously, $\text{dist}(W,Z) = 1$ and

$$\begin{aligned}W_0 &= \{e_2\},\\ Z_0 &= \{2e_2\}.\end{aligned} \tag{21}$$

Define the mapping $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ by

$$\mathfrak{T}(x) = \left\{\sum_{i=1}^{\infty} \frac{\zeta_i}{2} e_i, \text{for } x \in W, \left(\frac{\zeta_2}{2} + 1\right) e_2 \text{for } x \in Z, \right. \tag{22}$$

for $x = (\zeta_1, \zeta_2, \zeta_3, \ldots) \in W \cup Z$. Then, $\mathfrak{T}$ is a noncyclic, affine, and condensing mapping. Let $S\colon W \longrightarrow KC(Z)$ given by $S((\zeta_1, \zeta_2, \zeta_3, \ldots)) = \{(2 + |\zeta_1|)e_2\}$. Then, S is an upper semicontinuous multivalued mapping, $\mathfrak{T}$ and S commute, and for each $x \in W_0$, $S(x) \cap Z_0 \neq \emptyset$. Here, $w = (0,0,0,\ldots)$ is the only fixed point of $\mathfrak{T}$ in W, but $\text{dist}(w,S(w)) > \text{dist}(W,Z)$. Note that $\|e_2 - 2e_2\| < \text{dist}(W,Z) + \delta$ for all $\delta > 0$ but $\|T(e_2) - T(2e_2)\| > \text{dist}(W,Z) + (1/4)$.

The following corollary follows immediately from Theorem 4.

Corollary 1. *Let (W,Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. Suppose $\mathfrak{T}\colon W \longrightarrow W$ is a continuous, affine, and condensing mapping. If $S\colon W \longrightarrow KC(W)$ is an upper semicontinuous multivalued mapping, $\mathfrak{T}$ and S commute, and then there is $w \in W$ which satisfies $w \in Fix(\mathfrak{T}) \cap Fix(S)$.*

Theorem 5. *Let (W,Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. If $\mathfrak{T}_1, \mathfrak{T}_2\colon W \cup Z \longrightarrow W \cup Z$ are commuting, noncyclic relatively u-continuous, affine, and condensing mappings, then there exists $(u_0, v_0) \in W \times Z$ such that $\mathfrak{T}_1(u_0) = u_0 = \mathfrak{T}_2(u_0)$, $\mathfrak{T}_1(v_0) = v_0 = \mathfrak{T}_2(v_0)$, and $\|u_0 - v_0\| = dist(W,Z)$.*

Proof. Since W_0 is nonempty and by relative u-continuity of $\mathfrak{T}_1$, for $w_0 \in W_0$, there exists $z_0 \in Z$ such that $\|w_0 - z_0\| = \text{dist}(W,Z)$. Consequently, $\|\mathfrak{T}_1(w_0) - \mathfrak{T}_1(z_0)\| = \text{dist}(W,Z)$. That is, W_0 is invariant under $\mathfrak{T}_1$. Thus, Darbo's fixed point theorem guarantees that there is $u \in W_0$ such that $\mathfrak{T}_1(u) = u$. Notice $\mathfrak{T}_1(\text{Fix}_W(\mathfrak{T}_1)) = \text{Fix}_W(\mathfrak{T}_1)$ and so $\alpha(\text{Fix}_W(\mathfrak{T}_1)) = \alpha(\mathfrak{T}_1(\text{Fix}_W(\mathfrak{T}_1))) \leq k\alpha(\text{Fix}_W(\mathfrak{T}_1))$. Thus, $\alpha(\text{Fix}_W(\mathfrak{T}_1)) = 0$, and thus, $Fix_W(\mathfrak{T}_1)$ is compact. Furthermore, $\mathfrak{T}_1(\mathfrak{T}_2(u)) = \mathfrak{T}_2(\mathfrak{T}_1(u)) = \mathfrak{T}_2(u)$. So, $\mathfrak{T}_2\colon \text{Fix}_W(\mathfrak{T}_1) \longrightarrow \text{Fix}_W(\mathfrak{T}_1)$ is a continuous mapping on a compact convex set. By Schauder's fixed point theorem, there is $u_0 \in \text{Fix}_W(\mathfrak{T}_1)$ such that $\mathfrak{T}_2(u_0) = u_0$, that is, $u_0 \in \text{Fix}_W(\mathfrak{T}_1) \cap \text{Fix}_W(\mathfrak{T}_2)$. Let v_0 in Z_0 be the unique closest point to u_0. By relative u-continuity of $\mathfrak{T}_1$ and $\mathfrak{T}_2$, we infer that, since $\|u_0 - v_0\| = \text{dist}(W,Z)$, $\|\mathfrak{T}_1(u_0) - \mathfrak{T}_1(v_0)\| = \text{dist}(W,Z)$ and $\|\mathfrak{T}_2(u_0) - \mathfrak{T}_2(v_0)\| = \text{dist}(W,Z)$. Hence, $\mathfrak{T}_1(u_0) = u_0 = \mathfrak{T}_2(u_0)$, $\mathfrak{T}_1(v_0) = v_0 = \mathfrak{T}_2(v_0)$. □

Lemma 4. *Let (W,Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. Let $\mathscr{C}$ be the collection of the commuting, noncyclic relatively u-continuous, affine, and condensing mappings on $W \cup Z$.*

Then, the mappings in $\mathscr{C}$ have common fixed points $u_0 \in W_0$ and $v_0 \in Z_0$.

Proof. For each $\mathfrak{T} \in \mathscr{C}$, consider $\mathrm{Fix}(\mathfrak{T}), \mathrm{Fix}_W(\mathfrak{T})$, and $\mathrm{Fix}_Z(\mathfrak{T})$ defined previously. Then, $\mathrm{Fix}_W(\mathfrak{T})$ is nonempty, compact, and convex. Let $\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_k$ be a finite subcollection of $\mathscr{C}$. Assume $F = \cap_{1\leq i\leq k}\mathrm{Fix}_W(\mathfrak{T}_i) \neq \varnothing$, $F_1 = F \cap \mathrm{Fix}_W(\mathfrak{T}_{k+1}) = \cap_{1\leq i\leq k+1}\mathrm{Fix}_W(\mathfrak{T}_i)$, and $F_{n+1} = F_n \cap \mathrm{Fix}_W(\mathfrak{T}_{k+n+1}) = \cap_{1\leq i\leq k+n+1}\mathrm{Fix}_W(\mathfrak{T}_i)$, for $n \in \mathbb{N}$. Then, $\{F_n\}$ is a decreasing sequence of compact subsets of X. Furthermore, $F_n \neq \varnothing$ for each $n \in \mathbb{N}$. Indeed, for $w \in F$ and each $m \in \{1, 2, \ldots, k\}$, then $\mathfrak{T}_m(\mathfrak{T}_{k+1}(w)) = \mathfrak{T}_{k+1}(\mathfrak{T}_m(w)) = \mathfrak{T}_{k+1}(w)$, and this implies that $\mathfrak{T}_{k+1}(w) \in F$. Thus, F is invariant under $\mathfrak{T}_{k+1}$. By Schauder's fixed point theorem, we get that $F_1 \neq \varnothing$. Now, for each $n \in \mathbb{N}$ and $m \in \{1, 2, \ldots, k+n\}$, pick $x \in F_n$:

$$\mathfrak{T}_m(\mathfrak{T}_{k+n+1}(x)) = \mathfrak{T}_{k+n+1}(\mathfrak{T}_m(x)) = \mathfrak{T}_{k+n+1}(x), \tag{23}$$

that is, $\mathfrak{T}_{k+n+1}(x) \in F_n$. So, $\mathfrak{T}_{k+n+1}\colon F_n \longrightarrow F_n$ is continuous on F_n, and then there is $y \in F_n$ such that $\mathfrak{T}_{k+n+1}(y) = y$. Therefore, $y \in F_{n+1} \neq \varnothing$. By Theorem 2, $\cap_{n\in\mathbb{N}} F_n \neq \varnothing$. Hence, $\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T}) \neq \varnothing$. Similarly, we can show that $\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T}) \neq \varnothing$. □

Theorem 6. *Let (W, Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. Let $\mathscr{C}$ be the collection of the commuting, noncyclic, relatively u-continuous, affine, and condensing mappings on $W \cup Z$. Then, there is $(u_0, v_0) \in W \times Z$ such that, for each $\mathfrak{T} \in \mathscr{C}$: $\mathfrak{T}(u_0) = u_0$, $\mathfrak{T}(v_0) = v_0$, and $\|u_0 - v_0\| = dist(W, Z)$.*

Proof. Based on the previous lemma, the mappings in $\mathscr{C}$ have a fixed point in common $u_0 \in W$, that is, $\mathfrak{T}(u_0) = u_0$, for each $\mathfrak{T} \in \mathscr{C}$. Let $v_0 \in Z$ be the unique closest point to u_0. By relative u-continuity of $\mathfrak{T}$, since $\|u_0 - v_0\| = \mathrm{dist}(W, Z)$,

$$\begin{aligned}\|u_0 - \mathfrak{T}(v_0)\| &= \|\mathfrak{T}(u_0) - \mathfrak{T}(v_0)\| \\ &= \mathrm{dist}(W, Z), \quad \text{for each } \mathfrak{T} \in \mathscr{C}.\end{aligned} \tag{24}$$

Hence, $\mathfrak{T}(v_0) = v_0$. □

Theorem 7. *Let (W, Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. Let $\mathscr{C}$ be the collection of the commuting, noncyclic relatively u-continuous, affine, and condensing mappings on $W \cup Z$. If $S\colon W \longrightarrow KC(Z)$ is an upper semicontinuous multivalued mapping such that, for each $x \in W_0$: $S(x) \cap Z_0 \neq \varnothing$. If $\mathscr{C}$ commutes with S, then there exists $w \in W$ such that*

$$\mathfrak{T}(w) = w \text{ and } \mathrm{dist}(w, S(w)) = \mathrm{dist}(W, Z). \tag{25}$$

Proof. By Lemma 4, $(\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T}), \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T}))$ is a nonempty compact convex pair. Also, in view to the proof of Theorem 4, for $\mathfrak{T} \in \mathscr{C}$ and for each $x \in \mathrm{Fix}_W(\mathfrak{T})$, we have $S(x)$ and Z_0 are invariant under $\mathfrak{T}$. So, $S(x) \cap (\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T})) \neq \varnothing$.

Define $f\colon \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T}) \longrightarrow 2^{\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T})}$ by $f(x) = S(x) \cap (\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T}))$, for $x \in \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T})$. Then, f is an upper semicontinuous multivalued mapping with nonempty, compact, and convex values. Moreover, $P_W\colon \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T}) \longrightarrow \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T})$ is well-defined. Indeed, for $y \in \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T})$, there exists $x \in W$ such that $\|x - y\| = \mathrm{dist}(W, Z)$. So,

$$y = P_Z(x) \text{ and } x = P_W(y), \tag{26}$$

By relative u-continuity of $\mathfrak{T}$, one can conclude that $\|\mathfrak{T}(x) - \mathfrak{T}(y)\| = \mathrm{dist}(W, Z)$. Thus, $\mathfrak{T}(y) = P_Z(\mathfrak{T}(x))$ and $\mathfrak{T}(x) = P_W(\mathfrak{T}(y))$, and by (26), $\mathfrak{T}(x) = \mathfrak{T}(P_W(y)) = P_W(\mathfrak{T}(y)) = P_W(y)$. Thus, $P_W(y) \in \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T})$. Note that $P_W ∘ f\colon \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T}) \longrightarrow 2^{\mathrm{Fix}_W(\mathfrak{T})}$, and by Lemma 1, there is $w \in \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T})$ such that $w \in (P_W ∘ f)(w)$, that is, for $\mathfrak{T} \in \mathscr{C}$, we have $\mathfrak{T}(w) = w$ and $w \in (P_W(f(w)))$. So, there is $z \in f(w) = S(w) \cap (\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T}))$ such that $w = P_W(z)$. We infer that $\|z - w\| = \mathrm{dist}(z, W)$. But $z \in Z_0$, then there is $w^* \in W$ such that $\|w^* - z\| = \mathrm{dist}(W, Z)$. Then,

$$\begin{aligned}\mathrm{dist}(W, Z) &\leq \mathrm{dist}(w, S(w)) \leq \|w - z\| \leq \|z - w^*\| \\ &= \mathrm{dist}(W, Z).\end{aligned} \tag{27}$$

Hence, $\mathrm{dist}(w, S(w)) = \mathrm{dist}(W, Z)$. □

Example 3. Let $X = \ell_2$ over $\mathbb{R}$ with the basis $\{e_n\colon n \in \mathbb{N}\}$ and let

$$\begin{aligned}W &= \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta_1 \in [-3, -1], \zeta_2 \in [-8, 8]\} \\ &\text{and } Z = \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta_1 \in [0, 3], \zeta_2 \in \mathbb{R}\}.\end{aligned} \tag{28}$$

Then, (W, Z) be a nonempty, convex, and closed pair in X such that W is bounded. Furthermore, $\mathrm{dist}(W, Z) = 1$ and

$$W_0 = \{-e_1 + \zeta_2 e_2\colon \zeta_2 \in [-8, 8]\} \text{ and } Z_0 = \{\zeta_2 e_2\colon \zeta_2 \in [-8, 8]\}. \tag{29}$$

Consider $\mathfrak{T}_1, \mathfrak{T}_2\colon W \cup Z \longrightarrow W \cup Z$ given by

$$\begin{aligned}\mathfrak{T}_1(\zeta_1 e_1 + \zeta_2 e_2) &= \zeta_1 e_1 + \frac{\zeta_2}{2} e_2 \text{ and } \mathfrak{T}_2(\zeta_1 e_1 + \zeta_2 e_2) \\ &= \zeta_1 e_1 + \frac{\zeta_2}{4} e_2,\end{aligned} \tag{30}$$

for each $\zeta_1 e_1 + \zeta_2 e_2 \in W \cup Z$. Then, $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are noncyclic, affine, and condensing mappings. Furthermore, $\mathfrak{T}_1$ and $\mathfrak{T}_2$ commute.

Define $S\colon W \longrightarrow KC(Z)$ by $S(\zeta_1 e_1 + \zeta_2 e_2) = \{\gamma e_1 + \zeta_2 e_2\colon \gamma \in [0, -\zeta_1]\}$, then S is an upper semicontinuous multivalued mapping that commutes with $\mathfrak{T}_1$ and $\mathfrak{T}_2$ and satisfies that, for each $x \in W_0$: $S(x) \cap Z_0 \neq \varnothing$. For $w = -e_1$ and $z = \mathbf{0}$, $\mathfrak{T}_1(w) = \mathfrak{T}_2(w) = w$ and $\mathfrak{T}_1(z) = \mathfrak{T}_2(z) = z$. Furthermore, $\|w - z\| = \mathrm{dist}(W, Z)$ and $\mathrm{dist}(w, S(w)) = \mathrm{dist}(W, Z)$.

4. Conclusion

We have proved some best proximity pair theorems for noncyclic relatively u-continuous and condensing mappings. We have also obtained best proximity points of upper semicontinuous mappings which are fixed points of noncyclic relatively u-continuous condensing mappings.

Moreover, we have given some examples to support our results. It has been shown that relative u-continuity of $\mathfrak{T}$ is a necessary condition that cannot be omitted. We have extended recent results of [6, 11].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] K. Kuratowski, *Introduction to Set Theory and Topology*, Pergamon Press, Oxford, England, 1956.

[2] G. Darbo, "Punti uniti in transformazioni a condomino non compatto," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 24, pp. 84–92, 1955.

[3] B. N. Sadovskii, "On a fixed point principle," *Functional Analysis and Its Applications*, vol. 4, pp. 74–76, 1967.

[4] M. Ayerbe Toledano, T. Dominguez Benavides, and G. Lopez Acedo, "Measures of non-compactness in metric fixed point theory," *Operator Theory: Advances and Applications*, vol. 99, 1997.

[5] L. Cai and J. Liang, "New generalizations of Darbo's fixed point theorem," *Journal of Fixed Point Theory and Applications*, vol. 20158 pages, 2015.

[6] M. Gabeleh and J. Markin, "Optimum solutions for a system of differential equations via measure of noncompactness," *Indagationes Mathematicae*, vol. 29, no. 3, pp. 895–906, 2018.

[7] M. Gabeleh, S. P. Moshokoa, and C. Vetro, "Cyclic (noncyclic) ϕ-condensing operator and its application to a system of differential equations," *Nonlinear Analysis: Modelling and Control*, vol. 24, pp. 985–1000, 2019.

[8] J. Liang, J. H. Liu, M. V. Nguyen, and T. J. Xiao, "Periodic solutions of impulsive differential equations with infinite delay in Banach spaces," *Journal of Nonlinear Functional Analysis*, vol. 201918 pages, 2019.

[9] R. Pietkun, "On some generalizations of the M onch, Sadovskii and Darbo fixed point theorems," *J. Fixed Point Theory Appl.*vol. 20, p. 11, 2018.

[10] A. Abkar and M. Gabeleh, "The existence of best proximity points for multivalued non-self-mappings," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, vol. 107, no. 2, pp. 319–325, 2013.

[11] G. AlNemer, J. Markin, and N. Shahzad, "On best proximity points of upper semicontinuous multivalued mappings," *Journal of Fixed Point Theory and Applications*, vol. 201510 pages, 2015.

[12] M. A. Al-Thagafi and N. Shahzad, "Best proximity pairs and equilibrium pairs for Kakutani multimaps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 3, pp. 1209–1216, 2009.

[13] A. Amini-Harandi, "Best proximity pair and coincidence point theorems for nonexpansive set-valued maps in Hilbert spaces," *Bulletin of the Iranian Mathematical Society*, vol. 37, pp. 229–234, 2011.

[14] M. De la Sen, "Some results on fixed and best proximity points of multivalued cyclic self-mappings with a partial order," *Abstract and Applied Analysis*, vol. 2013, Article ID 968492, 11 pages, 2013.

[15] W. A. Kirk, S. Reich, and P. Veeramani, "Proximinal retracts and best proximity pair theorem," *Numerical Functional Analysis and Optimization*, vol. 24, no. 7-8, pp. 851–862, 2003.

[16] A. A. Elderd, V. S. Raj, and P. Veeramani, "On best proximity pair theorems for relatively u-continuous mapping," *Nonlinear Analysis*, vol. 74, pp. 3870–3875, 2011.

[17] A. A. Eldred, W. A. Kirk, and P. Veeramani, "Proximal normal structure and relatively nonexpansive mappings," *Studia Mathematica*, vol. 171, no. 3, pp. 283–293, 2005.

[18] J. Markin and N. Shahzad, "Best proximity points for relatively u-continuous mappings in Banach and hyperconvex spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 680186, 5 pages, 2013.

[19] E. Naraghirad, "Bregman best proximity points for Bregman asymptotic cyclic contraction mappings in Banach spaces," *Journal of Nonlinear Functional Analysis*, vol. 3, pp. 27–44, 2019.

[20] N. Shahzad, M. A. Alghamdi, and G. AlNemer, "On best proximity points of multivalued nonexpansive mappings," *Dynamic Systems and Applications*, vol. 30, pp. 411–424, 2021.

[21] M. Lassonde, "Fixed points for Kakutani factorizable multifunctions," *Journal of Mathematical Analysis and Applications*, vol. 152, no. 1, pp. 46–60, 1990.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6635026, 7 pages
https://doi.org/10.1155/2021/6635026

Research Article

Strong Convergence Analysis of Iterative Algorithms for Solving Variational Inclusions and Fixed-Point Problems of Pseudocontractive Operators

Zhangsong Yao,[1] Yan-Kuen Wu,[2] and Ching-Feng Wen[3,4]

[1]*School of Information Engineering, Nanjing Xiaozhuang University, Nanjing 211171, China*
[2]*College of International Business and Shaoxing Key Laboratory of Intelligent Monitoring and Prevention of Smart City, Zhejiang Yuexiu University of Foreign Languages, Shaoxin, Zhejiang, China*
[3]*Center for Fundamental Science and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 80708, Taiwan*
[4]*Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung 80708, Taiwan*

Correspondence should be addressed to Yan-Kuen Wu; ykwvnu@gmail.com and Ching-Feng Wen; cfwen@kmu.edu.tw

Received 12 October 2020; Revised 10 November 2020; Accepted 10 March 2021; Published 12 April 2021

Academic Editor: Jen-Chih Yao

Iterative methods for solving variational inclusions and fixed-point problems have been considered and investigated by many scholars. In this paper, we use the Halpern-type method for finding a common solution of variational inclusions and fixed-point problems of pseudocontractive operators. We show that the proposed algorithm has strong convergence under some mild conditions.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed and convex subset of H. Let $f\colon C \longrightarrow H$ and $g\colon H \longrightarrow 2^H$ be two nonlinear operators. Recall that the variational inclusion ([1]) is to solve the following problem of finding $x^{\ddagger} \in 2^H$ verifying

$$0 \in (f + g)x^{\ddagger}. \quad (1)$$

Here, use $(f + g)^{-1}(0)$ to denote the set of solutions of (1).

Special Case 1. Let $\delta_C\colon H \longrightarrow \{0, +\infty\}$ be defined by

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \quad (2)$$

Setting $g = \partial \delta_C$, variational inclusion (1) reduces to find $x^{\ddagger} \in C$ such that

$$\langle f(x^{\ddagger}), x - x^{\ddagger} \rangle \geq 0, \quad \forall x \in C. \quad (3)$$

Problem (3) is the well-known variational inequality which has been studied, extended, and developed in a broad category of jobs (see, e.g., [2–14]).

Special Case 2. Let $\varphi\colon H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and $\partial\varphi$ be the subdifferential of φ. Setting $g = \partial\varphi$, variational inclusion (1) reduces to find $x^{\ddagger} \in H$ such that

$$\langle f(x^{\ddagger}), x - x^{\ddagger} \rangle + \varphi(x) - \varphi(x^{\ddagger}) \geq 0, \quad \forall x \in H. \quad (4)$$

Problem (4) is called the mixed quasi-variational inequality [15] which is a very significant extension of variational inequality (3) involving the nonlinear function φ. It is well known that a large number of practical problems arising in various branches of pure and applied sciences can be formulated as the model of mixed quasi-variational inequality (4).

Problem (1) plays a key role in minimization, convex feasibility problems, machine learning, and others. A popular algorithm for solving problem (1) is the forward-backward algorithm [16] generated by

$$x_{n+1} = (I + \lambda g)^{-1}(I - \lambda f)x_n, \quad n \geq 1, \tag{5}$$

where $I - \lambda f$ is a forward step and $(I + \lambda g)^{-1}$ is a backward step with $\lambda > 0$. This algorithm is a splitting algorithm which solves the difficulty of calculating of the resolvent of $f + g$.

Recently, there has been increasing interest for studying common solution problems relevant to (1) (see for example, [17–27]). Especially, Zhao, Sahu, and Wen [28] presented an iterative algorithm for solving a system of variational inclusions involving accretive operators. Ceng and Wen [29] introduced an implicit hybrid steepest-descent algorithm for solving generalized mixed equilibria with variational inclusions and variational inequalities. Li and Zhao [30] considered an iterate for finding a solution of quasi-variational inclusions and fixed points of nonexpansive mappings.

Motivated by the results in this direction, the main purpose of this paper is to research a common solution problem of variational inclusions and fixed point of pseudocontractions. We suggest a Halpern-type algorithm for solving such problem. We show that the proposed algorithm has strong convergence under some mild conditions.

2. Preliminaries

Let H be a real Hilbert space. Let $g\colon H \longrightarrow 2^H$ be an operator. Write $\mathrm{dom}(g) = \{x \in H\colon g(x) \neq \emptyset\}$. Recall that g is called monotone if $\forall x, y \in \mathrm{dom}(g)$, $u \in g(x)$ and $v \in g(y)$, $\langle x - y, u - v\rangle \geq 0$.

A monotone operator g is maximal monotone if and only if its graph is not strictly contained in the graph of any other monotone operator on H.

For a maximal monotone operator g on H,

(i) Set $g^{-1}0 = \{x \in H\colon 0 \in g(x)\}$

(ii) Denote its resolvent by $J_\lambda^g = (I + \lambda g)^{-1}$ which is single-valued from H into $\mathrm{dom}(g)$

It is known that $g^{-1}0 = \mathrm{Fix}(J_\lambda^g), \forall \lambda > 0$ and J_λ^g is firmly nonexpansive, i.e.,

$$\left\|J_\lambda^g x - J_\lambda^g y\right\|^2 \leq \langle J_\lambda^g x - J_\lambda^g y, x - y\rangle, \tag{6}$$

for all $x, y \in C$.

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that an operator $T\colon C \longrightarrow C$ is said to be

(i) L-Lipschitz if there exists a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{7}$$

If $L = 1$, T is nonexpansive.

(ii) Pseudocontractive if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C. \tag{8}$$

(iii) Inverse-strongly monotone if

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in C, \tag{9}$$

where $\alpha > 0$ is a constant and T is also called α-ism.

Recall that the projection P_C is an orthographic projection from H onto C, which is defined by $\|x - P_C(x)\| = \min_{y\in C}\|x - y\|$. It is known that P_C is nonexpansive.

Lemma 1 (see [23, 31]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T\colon C \longrightarrow C$ be an L-Lipschitz pseudocontractive operator. Then,*

(i) T is demiclosed, i.e., $x_n \rightharpoonup p$ and $T(x_n) \longrightarrow q \Rightarrow T(p) = q$

(ii) For $0 < \zeta < (1/(\sqrt{1 + L^2} + 1))$, $\forall x \in C$ and $y \in \mathrm{Fix}(T)$, we have

$$\|T[(1 - \zeta)x + \zeta T(x)] - y\|^2 \leq \|x - y\|^2 + (1 - \zeta)\|x - T[(1 - \zeta)x + \zeta T(x)]\|^2. \tag{10}$$

Lemma 2 (see [16, 32]). *Let H be a real Hilbert space and let g be a maximal monotone operator on H. Then, we have*

$$\left\|J_s^g(x) - J_t^g(x)\right\|^2 \leq \frac{s - t}{t}\langle J_s^g(x) - J_t^g(x), J_s^g(x) - x\rangle, \tag{11}$$

for all $s, t > 0$ and $x \in H$.

Lemma 3 (see [33]). *Assume that a real number sequence $\{a_n\} \subset [0, \infty)$ satisfies*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n\gamma_n, \tag{12}$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\} \subset (-\infty, +\infty)$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$

(ii) $\limsup_{n\longrightarrow\infty}\delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 4 (see [8]). *Let $\{s_n\} \subset (0, \infty)$ be a sequence. Assume that there exists at least a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ verifying $s_{n_i} \leq s_{n_i+1}$ for all $i \geq 0$. Let $\{\tau(n)\}$ be an integer sequence defined as $\tau(n) = \max\{i \leq n\colon s_{n_i} < s_{n_i+1}\}$. Then $\tau(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ and*

$$\max\{s_{\tau(n)}, s_n\} \le s_{\tau(n)+1}. \tag{13}$$

3. Main Results

Let C be a nonempty closed convex subset of a real Hilbert space H. Let the operator $f\colon C \longrightarrow H$ be an α-ism. Let $g\colon H \longrightarrow 2^H$ be a maximal monotone operator with $\mathrm{dom}(g) \subset C$. Let $T\colon C \longrightarrow C$ be an L–Lipschitz pseudocontractive operator with $L > 1$. Let $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ be two sequences. Let ν and ζ be two constants.

Next, we introduce a Halpern-type algorithm for finding a common solution of variational inclusion (1) and fixed point of pseudocontractive operator T.

Algorithm 1. Let $u \in C$ be a fixed point. Choose $x_0 \in C$. Set $n = 0$.

Step 1. For given x_n, compute y_n by

$$y_n = (1-\nu)x_n + \nu T((1-\zeta)x_n + \zeta T x_n). \tag{14}$$

Step 2. Compute x_{n+1} *by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J^g_{\lambda_n}(I - \lambda_n f) y_n. \tag{15}$$

Step 3. Set $n: = n+1$ and return to Step 1.

Next, we prove the convergence of Algorithm 1.

Theorem 1. *Suppose that* $\Gamma: = \mathrm{Fix}(T) \cap (f+g)^{-1}(0) \neq \emptyset$. *Assume that the following conditions are satisfied:*

$\lim_{n\longrightarrow\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$

$0 < d_1 < \lambda_n < d_2 < 2\alpha$ *and* $0 < \nu < \zeta < (1/(\sqrt{1+L^2}+1))$

Then, the sequence $\{x_n\}$ *generated by Algorithm 1 converges strongly to* $P_\Gamma(u)$.

Proof. Let $x^* \in \mathrm{Fix}(T) \cap (f+g)^{-1}(0)$. Set $u_n = J^g_{\lambda_n}(I - \lambda_n f) y_n$, $\forall n \ge 0$. Since f is α-ism, we have

$$\langle f(y_n) - f(x^*), y_n - x^*\rangle \ge \alpha \|f(y_n) - f(x^*)\|^2. \tag{16}$$

By the nonexpansivity of $J^g_{\lambda_n}$, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \left\|J^g_{\lambda_n}(I-\lambda_n f)y_n - J^g_{\lambda_n}(I-\lambda_n f)x^*\right\|^2 \\
&\le \|y_n - x^* - \lambda_n[f(y_n) - f(x^*)]\|^2 \\
&= \|y_n - x^*\|^2 - 2\lambda_n\langle f(y_n) - f(x^*), y_n - x^*\rangle + \lambda_n^2\|f(y_n) - f(x^*)\|^2 \\
&\le \|y_n - x^*\|^2 - 2\lambda_n\alpha\|f(y_n) - f(x^*)\|^2 + \lambda_n^2\|f(y_n) - f(x^*)\|^2 \\
&= \|y_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|f(y_n) - f(x^*)\|^2 \\
&\le \|y_n - x^*\|^2 - d_1(2\alpha - d_2)\|f(y_n) - f(x^*)\|^2 \ (\text{by condition } (r2)) \\
&\le \|y_n - x^*\|^2.
\end{aligned} \tag{17}$$

Using Lemma 1, we get

$$\|T((1-\zeta)I + \zeta T)x_n - x^*\|^2 \le \|x_n - x^*\|^2 + (1-\zeta)\|x_n - T((1-\zeta)I + \zeta T)x_n\|^2. \tag{18}$$

This together with (14) implies that

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|(1-\nu)x_n + \nu T((1-\zeta)I + \zeta T)x_n - x^*\|^2 \\
&= \|(1-\nu)(x_n - x^*) + \nu(T((1-\zeta)I + \zeta T)x_n - x^*)\|^2 \\
&= (1-\nu)\|x_n - x^*\|^2 + \nu\|T((1-\zeta)I + \zeta T)x_n - x^*\|^2 \\
&\quad - \nu(1-\nu)\|T((1-\zeta)I + \zeta T)x_n - x_n\|^2 \\
&\le \|x_n - x^*\|^2 - \nu(\zeta - \nu)\|T((1-\zeta)I + \zeta T)x_n - x_n\|^2 \\
&\le \|x_n - x^*\|^2.
\end{aligned} \tag{19}$$

According to (15)-(19), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(u - x^*) + (1-\alpha_n)(u_n - x^*)\| \\
&\le \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\
&\le \cdots \\
&\le \max\{\|u - x^*\|, \|x_0 - x^*\|\}.
\end{aligned} \tag{20}$$

Then, the sequence $\{x_n\}$ is bounded. The sequences $\{u_n\}$ and $\{y_n\}$ are also bounded.

Again, by (15)-(19), we deduce

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n(u-x^*)+(1-\alpha_n)(u_n-x^*)\|^2\\ &\le \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|u_n-x^*\|^2 \left(\text{by the convexity of } \|\cdot\|^2\right)\\ &\le \alpha_n\|u-x^*\|^2+\|u_n-x^*\|^2\\ &\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-d_1(2\alpha-d_2)\|f(y_n)-f(x^*)\|^2-\nu(\zeta-\nu)\|T((1-\zeta)I+\zeta T)x_n-x_n\|^2.\end{aligned} \tag{21}$$

It follows that

$$d_1(2\alpha-d_2)\|f(y_n)-f(x^*)\|^2+\nu(\zeta-\nu)\|T((1-\zeta)I+\zeta T)x_n-x_n\|^2 \le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2. \tag{22}$$

Since $J^g_{\lambda_n}$ is firmly nonexpansive, using (6), we have

$$\begin{aligned}\|u_n-x^*\|^2 &= \|J^g_{\lambda_n}(I-\lambda_n f)y_n-J^g_{\lambda_n}(I-\lambda_n f)x^*\|^2\\ &\le \langle (I-\lambda_n f)y_n-(I-\lambda_n f)x^*, u_n-x^*\rangle\\ &= \langle y_n-x^*, u_n-x^*\rangle-\lambda_n\langle u_n-x^*, f(y_n)-f(x^*)\rangle\\ &= \frac{1}{2}\left(\|y_n-x^*\|^2+\|u_n-x^*\|^2-\|y_n-u_n\|^2\right)-\lambda_n\langle y_n-x^*, f(y_n)-f(x^*)\rangle-\lambda_n\langle u_n-y_n, f(y_n)-f(x^*)\rangle\\ &\le \frac{1}{2}\left(\|y_n-x^*\|^2+\|u_n-x^*\|^2-\|y_n-u_n\|^2\right)+\lambda_n\|u_n-y_n\|\|f(y_n)-f(x^*)\|,\end{aligned} \tag{23}$$

which leads to

$$\begin{aligned}\|u_n-x^*\|^2 &\le \|y_n-x^*\|^2-\|y_n-u_n\|^2+2\lambda_n\|u_n-y_n\|\|f(y_n)-f(x^*)\|\\ &\le \|x_n-x^*\|^2-\|y_n-u_n\|^2+2\lambda_n\|u_n-y_n\|\|f(y_n)-f(x^*)\|.\end{aligned} \tag{24}$$

Combining (21) with (24), we obtain

$$\|x_{n+1}-x^*\|^2 \le \alpha_n\|u-x^*\|^2+\|u_n-x^*\|^2 \le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|y_n-u_n\|^2+2\lambda_n\|u_n-y_n\|\|f(y_n)-f(x^*)\|. \tag{25}$$

which results in that

$$\|y_n-u_n\|^2 \le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\lambda_n\|u_n-y_n\|\|f(y_n)-f(x^*)\|. \tag{26}$$

Next, we analyze two cases. (i) $\exists n_0\in\mathbb{N}$ such that $\|x_{n+1}-x^*\|\le\|x_n-x^*\|, \forall n\ge n_0$. (ii) For any $n_0\in\mathbb{N}$, $\exists m\ge n_0$ such that $\|x_m-x^*\|\le\|x_{m+1}-x^*\|$.

In case of (i), $\lim_{n\longrightarrow\infty}\|x_n-x^*\|$ exists. From (22), we deduce

$$\lim_{n\longrightarrow\infty}\|f(y_n)-f(x^*)\|=0 \tag{27}$$

and

$$\lim_{n\longrightarrow\infty}\|T((1-\zeta)I+\zeta T)x_n-x_n\|=0. \tag{28}$$

It follows from (14) that

$$\lim_{n\longrightarrow\infty}\|y_n - x_n\| = \lim_{n\longrightarrow\infty}\nu\|T((1-\zeta)I+\zeta T)x_n - x_n\| = 0. \tag{29}$$

On the basic of (26) and (27), we have

$$\lim_{n\longrightarrow\infty}\|y_n - u_n\| = \lim_{n\longrightarrow\infty}\left\|y_n - J^g_{\lambda_n}(I-\lambda_n f)y_n\right\| = 0. \tag{30}$$

Note that $\|x_{n+1}-x_n\| \le \alpha_n\|u-x_n\| + (1-\alpha_n)\|u_n - x_n\|$. Thanks to (29) and (30), we derive that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\| = 0. \tag{31}$$

However,

$$\begin{aligned}\|x_n - Tx_n\| &\le \|T((1-\zeta)I+\zeta T)x_n - x_n\| + \|Tx_n - T((1-\zeta)I+\zeta T)x_n\|\\ &\le \|T((1-\zeta)I+\zeta T)x_n - x_n\| + \zeta L\|x_n - Tx_n\|,\end{aligned} \tag{32}$$

We have

$$\|x_n - Tx_n\| \le \frac{1}{1-\zeta L}\|T((1-\zeta)I+\zeta T)x_n - x_n\|. \tag{33}$$

This together with (28) implies that

$$\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0. \tag{34}$$

Set $p = P_\Gamma(u)$. Next, we prove that

$$\limsup_{n\longrightarrow\infty}\langle u-p, x_{n+1}-p\rangle \le 0. \tag{35}$$

Since $\{x_{n+1}\}$ is bounded, there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ satisfying

(1) $x_{n_i+1}\rightharpoonup\tilde{x}$ (hence, $x_{n_i}\rightharpoonup\tilde{x}$ by (31))

(2) $\limsup_{n\longrightarrow\infty}\langle u-p, x_{n+1}-p\rangle = \lim_{i\longrightarrow\infty}\langle u-p, x_{n_i+1}-p\rangle$

From (34) and Lemma 1, we obtain $\tilde{x}\in \mathrm{Fix}(T)$. Owing to (29) and (30), we have that $y_{n_i}\rightharpoonup\tilde{x}$ and

$$\lim_{i\longrightarrow\infty}\left\|J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - y_{n_i}\right\| = 0. \tag{36}$$

Since $\lambda_n\in(d_1,d_2)$, without loss of generality, we assume that $\lambda_{n_i}\longrightarrow\lambda^\dagger > 0\,(i\longrightarrow\infty)$. Observe that

$$\begin{aligned}\left\|J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - J^g_{\lambda^\dagger}(I-\lambda^\dagger f)y_{n_i}\right\| &\le \left\|J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - J^g_{\lambda^\dagger}(I-\lambda_{n_i}f)y_{n_i}\right\|\\ &\quad+\left\|J^g_{\lambda^\dagger}(I-\lambda_{n_i}f)y_{n_i} - J^g_{\lambda^\dagger}(I-\lambda^\dagger f)y_{n_i}\right\|\\ &\le \left\|J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - J^g_{\lambda^\dagger}(I-\lambda_{n_i}f)y_{n_i}\right\| + \left|\lambda_{n_i}-\lambda^\dagger\right|\left\|f(y_{n_i})\right\|.\end{aligned} \tag{37}$$

Applying Lemma 2, we obtain

$$\begin{aligned}&\left\|J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - J^g_{\lambda^\dagger}(I-\lambda_{n_i}f)y_{n_i}\right\|^2\\ &\le \frac{\lambda_{n_i}-\lambda^\dagger}{\lambda^\dagger}\langle J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - J^g_{\lambda^\dagger}(I-\lambda_{n_i}f)y_{n_i}, J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - (I-\lambda_{n_i}f)y_{n_i}\rangle\\ &\le \frac{\left|\lambda_{n_i}-\lambda^\dagger\right|}{\lambda^\dagger}\left\|J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - J^g_{\lambda^\dagger}(I-\lambda_{n_i}f)y_{n_i}\right\|\left\|J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - (I-\lambda_{n_i}f)y_{n_i}\right\|.\end{aligned} \tag{38}$$

It follows that

$$\left\|J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - J^g_{\lambda^\dagger}(I-\lambda_{n_i}f)y_{n_i}\right\| \le \frac{\left|\lambda_{n_i}-\lambda^\dagger\right|}{\lambda^\dagger}\left\|J^g_{\lambda_{n_i}}(I-\lambda_{n_i}f)y_{n_i} - (I-\lambda_{n_i}f)y_{n_i}\right\|. \tag{39}$$

Thanks to (37) and (39), we get

$$\left\|J^{g}_{\lambda_{n_i}}\left(I-\lambda_{n_i}f\right)y_{n_i}-J^{g}_{\lambda^\dagger}\left(I-\lambda^\dagger f\right)y_{n_i}\right\|\leq\left|\lambda_{n_i}-\lambda^\dagger\right|\left\|f\left(y_{n_i}\right)\right\|+\frac{\left|\lambda_{n_i}-\lambda^\dagger\right|}{\lambda^\dagger}\left\|J^{g}_{\lambda_{n_i}}\left(I-\lambda_{n_i}f\right)y_{n_i}-\left(I-\lambda_{n_i}f\right)y_{n_i}\right\|. \tag{40}$$

Noting that $\lambda_{n_i}\longrightarrow\lambda^\dagger\ (i\longrightarrow\infty)$, from (36) and (40), we get

$$\lim_{i\longrightarrow\infty}\left\|y_{n_i}-J^{g}_{\lambda^\dagger}\left(I-\lambda^\dagger f\right)y_{n_i}\right\|=0. \tag{41}$$

By Lemma 1, we deduce that $\tilde{x}\in\mathrm{Fix}\left(J^{g}_{\lambda^\dagger}\left(I-\lambda^\dagger f\right)\right)=(f+g)^{-1}(0)$. Therefore, $\tilde{x}\in\Gamma$ and

$$\limsup_{n\longrightarrow\infty}\langle u-p,x_{n+1}-p\rangle=\lim_{i\longrightarrow\infty}\langle u-p,x_{n_i+1}-p\rangle=\langle u-p,\tilde{x}-p\rangle\leq 0. \tag{42}$$

From (15), we have

$$\begin{aligned}\left\|x_{n+1}-p\right\|^2&=\left\|\alpha_n(u-p)+\left(1-\alpha_n\right)\left(u_n-p\right)\right\|^2\\&\leq\left(1-\alpha_n\right)\left\|u_n-p\right\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle\\&\leq\left(1-\alpha_n\right)\left\|x_n-p\right\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle.\end{aligned} \tag{43}$$

Applying Lemma 3 to (43) to deduce $x_n\longrightarrow p$.

In case of (ii), let $s_n=\{\|x_n-x^*\|\}$. So, we have $s_{n_0}\leq s_{n_0+1}$. Define an integer sequence $\{\tau(n)\}$, $\forall n\geq n_0$, by $\tau(n)=\max\{i\in\mathbb{N}|n_0\leq i\leq n,\ s_i\leq s_{i+1}\}$. It is obvious that $\lim_{n\longrightarrow\infty}\tau(n)=\infty$ and $s_{\tau(n)}\leq s_{\tau(n)+1}$ for all $n\geq n_0$. Similarly, we can prove that $\lim_{n\longrightarrow\infty}\|x_{\tau(n)}-Tx_{\tau(n)}\|=0$ and $\lim_{n\longrightarrow\infty}\|J^{g}_{\lambda_{\tau(n)}}(I-\lambda_{\tau(n)}f)x_{\tau(n)}\|=0$. Therefore, all weak cluster points $\omega_w(x_{\tau(n)})\subset\Gamma$. Consequently,

$$\limsup_{n\longrightarrow\infty}\langle u-p,x_{\tau(n)}-p\rangle\leq 0. \tag{44}$$

Note that $s_{\tau(n)}\leq s_{\tau(n)+1}$. From (43), we deduce

$$s^2_{\tau(n)}\leq s^2_{\tau(n)+1}\leq\left(1-\alpha_{\tau(n)}\right)s^2_{\tau(n)}+2\alpha_{\tau(n)}\langle u-p,x_{\tau(n)+1}-p\rangle. \tag{45}$$

It follows that

$$s^2_{\tau(n)}\leq 2\langle u-p,x_{\tau(n)+1}-p\rangle. \tag{46}$$

Combining (44) and (46), we have $\limsup_{n\longrightarrow\infty}s_{\tau(n)}\leq 0$ and hence

$$\lim_{k\longrightarrow\infty}s_{\tau(k)}=0. \tag{47}$$

From (45), we deduce that $\limsup_{n\longrightarrow\infty}s^2_{\tau(n)+1}\leq\limsup_{n\longrightarrow\infty}s^2_{\tau(n)}$. This together with (47) implies that $\lim_{n\longrightarrow\infty}s_{\tau(n)+1}=0$. According to Lemma 4, we get $0\leq s_n\leq\max\{s_{\tau(n)},s_{\tau(n)+1}\}$. Therefore, $s_n\longrightarrow 0$ and $x_n\longrightarrow p$. This completes the proof. □

Remark 1. Since the pseudocontractive operator is nonexpansive, Theorem 1 still holds if T is nonexpansive.

Remark 2. Assumption $(r1)$ imposed on parameter α_n is essential and we do not add any other assumptions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Zhangsong Yao was partially supported by the Grant 19KJD100003. Ching-Feng Wen was partially supported by the Grant of MOST 109-2115-M-037-001.

References

[1] Y.-P. Fang and N.-J. Huang, "H-Monotone operator and resolvent operator technique for variational inclusions," *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 795–803, 2003.
[2] L. C. Ceng, "Composite implicit viscosity extragradient algorithms for systems of variational inequalities with fixed point constraints of asymptotically nonexpansive mappings," *Applied Analysis and Optimization*, vol. 3, pp. 423–437, 2019.
[3] L.-C. Ceng, A. Petrusel, J.-C. Yao, and Y. Yao, "Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces," *Fixed Point Theory*, vol. 19, no. 2, pp. 487–502, 2018.
[4] L. C. Ceng, A. Petrusel, J. C. Yao, and Y. Yao, "Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions," *Fixed Point Theory*, vol. 20, pp. 113–133, 2019.
[5] Q. L. Dong, Y. Peng, and Y. Yao, "Alternated inertial projection methods for the split equality problem," *Journal of Nonlinear and Convex Analysis*, vol. 22, pp. 53–67, 2021.
[6] P. Hartman and G. Stampacchia, "On some non-linear elliptic differential-functional equations," *Acta Mathematica*, vol. 115, pp. 271–310, 1966.
[7] X.-H. Li, Q.-L. Dong, and A. Gibali, "Some new iterative algorithms for finding fixed points of nonexpansive

mappings," *Journal of Optimization Theory and Applications*, vol. 2, pp. 143–154, 2020.

[8] P. E. Maingé and M. L. Gobinddass, "Convergence of one-step projected gradient methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 171, no. 1, pp. 146–168, 2016.

[9] N. Shahzad and H. Zegeye, "Convergence theorems of common solutions for fixed point, variational inequality and equilibrium problems," *Journal of Nonlinear and Variational Analysis*, vol. 3, pp. 189–203, 2019.

[10] F. Wang and H. Pham, "On a new algorithm for solving variational inequality and fixed point problems," *Journal of Nonlinear and Convex Analysis*, vol. 3, pp. 225–233, 2019.

[11] Y. Yao, Y.-C. Liou, and J.-C. Yao, "Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 2, pp. 843–854, 2017.

[12] Y. Yao, M. Postolache, and J. C. Yao, "Iterative algorithms for generalized varational inequalities," *University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics*, vol. 81, pp. 3–16, 2019.

[13] X. Zhao and Y. Yao, "Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems," *Optimization*, vol. 69, no. 9, pp. 1987–2002, 2020.

[14] L. J. Zhu, Y. Yao, and M. Postolache, "Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems," *Politehnica University of Bucharest. Series A*, vol. 83, no. 1, pp. 3–14, 2021.

[15] M. A. Noor, "Generalized set-valued variational inclusions and resolvent equations," *Journal of Mathematical Analysis and Applications*, vol. 228, no. 1, pp. 206–220, 1998.

[16] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, Berlin, Germany, 2011.

[17] P. Cholamjiak, S. Suantai, and P. Sunthrayuth, "An explicit parallel algorithm for solving variational inclusion problem and fixed point problem in Banach spaces," *Banach Journal of Mathematical Analysis*, vol. 14, no. 1, pp. 20–40, 2020.

[18] W. Cholamjiak, S. A. Khan, D. Yambangwai, and K. R. Kazmi, "Strong convergence analysis of common variational inclusion problems involving an inertial parallel monotone hybrid method for a novel application to image restoration," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 114, p. 99, 2020.

[19] C. Izuchukwu, G. N. Ogwo, A. A. Mebawondu, and O. T. Mewomo, "On finite family of monotone variational inclusion problems in reflexive Banach space," *University Politehnica of Bucharest Scientific Bulletin*, vol. 82, pp. 89–104, 2020.

[20] S. Rezapour and S. H. Zakeri, "Hybrid method for equilibrium problems and variational inclusions," *Journal of Inequalities and Applications*, vol. 2020, p. 190, 2020.

[21] M. Wen, C. S. Hu, A. A. Cui, and J. G. Peng, "Algorithms for finding a common element of the set of common fixed points for nonexpansive semigroups, variational inclusions and generalized equilibrium problems," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales-Serie A: Matematicas*, vol. 114, p. 175, 2020.

[22] Y. T. Yang, "A viscosity method for treating variational inequality and variational inclusion problems," *Journal of Nonlinear and Convex Analysis*, vol. 21, pp. 923–931, 2020.

[23] Y. Yao, Y. C. Liou, and J. C. Yao, "Split common fixed point problem for two quasi-pseudo-contractive operators and its algorithm construction," *Fixed Point Theory and Applications*, vol. 2015, p. 127, 2015.

[24] Y. Yao, M. Postolache, and J. C. Yao, "An iterative algorithm for solving the generalized variational inequalities and fixed points problems," *Mathematics*, vol. 7, p. 61, 2019.

[25] C. Zhang, Q. L. Dong, and J. Chen, "Multi-step inertial proximal contraction algorithms for monotone variational inclusion problems," *Carpathian Journal of Mathematics*, vol. 36, pp. 159–177, 2020.

[26] C. Zhang, Z. Zhu, Y. Yao, and Q. Liu, "Homotopy method for solving mathematical programs with bounded box-constrained variational inequalities," *Optimization*, vol. 68, pp. 2293–2312, 2019.

[27] X. Zhao, J. C. Yao, and Y. Yao, "A proximal algorithm for solving split monotone variational inclusions," *University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics*, vol. 82, no. 3, pp. 43–52, 2020.

[28] X. Zhao, D. R. Sahu, D. R. Sahu, and C.-F. Wen, "Iterative methods for system of variational inclusions involving accretive operators and applications," *Fixed Point Theory*, vol. 19, no. 2, pp. 801–822, 2018.

[29] L. C. Ceng and C. F. Wen, "Implicit hybrid steepest-descent methods for generalized mixed equilibria with variational inclusions and variational inequalities," *Journal of Nonlinear and Convex Analysis*, vol. 17, pp. 987–1012, 2016.

[30] D. Li and J. Zhao, "Approximation of solutions of quasi-variational inclusions and fixed points of nonexpansive mappings," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 5, pp. 3002–3009, 2016.

[31] H. Zhou, "Strong convergence of an explicit iterative algorithm for continuous pseudo-contractions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 11, pp. 4039–4046, 2009.

[32] S. Takahashi, W. Takahashi, and M. Toyoda, "Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces," *Journal of Optimization Theory and Applications*, vol. 147, no. 1, pp. 27–41, 2010.

[33] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6682858, 17 pages
https://doi.org/10.1155/2021/6682858

Research Article

Inertial Iterative Schemes for D-Accretive Mappings in Banach Spaces and Curvature Systems

Li Wei[1], **Wenwen Yue**[1], **Yingzi Shang**[1], **and Ravi P. Agarwal**[2]

[1]*School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang 050061, Hebei, China*
[2]*Department of Mathematics, Texas A & M University-Kingsville, Kingsville, TX 78363, USA*

Correspondence should be addressed to Li Wei; diandianba@yahoo.com

Received 30 December 2020; Revised 6 March 2021; Accepted 11 March 2021; Published 31 March 2021

Academic Editor: Xiaolong Qin

We propose and analyze a new iterative scheme with inertial items to approximate a common zero point of two countable d-accretive mappings in the framework of a real uniformly smooth and uniformly convex Banach space. We prove some strong convergence theorems by employing some new techniques compared to the previous corresponding studies. We give some numerical examples to illustrate the effectiveness of the main iterative scheme and present an example of curvature systems to emphasize the importance of the study of d-accretive mappings.

1. Introduction and Preliminaries

In this paper, we assume that E is a real Banach space and E^* is the dual space of E. Suppose that C is a nonempty closed and convex subset of E. The symbols "$\langle x, f\rangle$", "$\rightarrow$" and "$\rightharpoonup$" denote the value of $f \in E^*$ at $x \in E$, the strong convergence, and the weak convergence either in E or E^*, respectively.

The normalized duality mapping J_E: $E \rightarrow 2^{E^*}$ is defined as follows:

$$J_E(x) = \left\{x^* \in E^*: \langle x, x^*\rangle = \|x\|^2 = \left\|x^*\right\|^2\right\}, \quad \forall x \in E. \tag{1}$$

Lemma 1 (see [1]). *Assume that E is real uniformly convex and uniformly smooth Banach space. Then (1) J_E is single-valued and surjective and, for $x \in E$ and $k \in (0, +\infty)$, $J_E(kx) = kJ_E(x)$; (2) $J_E^{-1} = J_{E^*}$ is the normalized duality mapping from E^* to E; (3) both J_E and J_E^{-1} are uniformly continuous on each bounded subset of E or E^*, respectively.*

Definition 1 (see [2]). The Lyapunov functional ϕ: $E \times E \rightarrow R^+$ is defined as follows:

$$\phi(x, y) = \|x\|^2 - 2\langle x, j_E(y)\rangle + \|y\|^2, \quad \forall x, y \in E, j_E(y) \in J_E(y). \tag{2}$$

Similarly, the Lyapunov functional defined on $E^* \times E^*$ can be defined and denoted by $\overline{\phi}$.

Lemma 2 (see [3]). *Let E be a uniformly smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \rightarrow 0$, as $n \rightarrow \infty$, then $x_n - y_n \rightarrow 0$, as $n \longrightarrow \infty$.*

Definition 2 (see [4]). Let $\{C_n\}$ be a sequence of nonempty closed and convex subsets of E; then

(1) $s-\liminf C_n$, which is called strong lower limit of $\{C_n\}$, is defined as the set of all $x \in E$ such that there exists $x_n \in C_n$ for almost all n and it tends to x as $n \rightarrow \infty$ in the norm

(2) $w-\limsup C_n$, which is called weak upper limit of $\{C_n\}$, is defined as the set of all $x \in E$ such that there exists a subsequence $\{C_{n_m}\}$ of $\{C_n\}$ and $x_{n_m} \in C_{n_m}$ for every n_m and it tends to x as $n_m \rightarrow \infty$ in the weak topology

(3) If $s-\liminf C_n = w-\limsup C_n$, then the common value is denoted by $\lim C_n$

Lemma 3 (see [4]). *Let $\{C_n\}$ be a decreasing sequence of closed and convex subsets of E, that is, $C_n \subset C_m$, if $n \geq m$. Then, $\{C_n\}$ converges in E and $\lim C_n = \cap_{n=1}^{\infty} C_n$.*

Definition 3 (see [1, 2]). Suppose that E is a real uniformly smooth and uniformly convex Banach space and C is a nonempty closed and convex subset of E; then, for each $x \in E$, there exists a unique element $v \in C$ such that $\|x-v\| = \inf\{\|x-y\|: y \in C\}$. Such an element v is denoted by $P_C x$ and P_C is called the metric projection of E onto C.

Lemma 4 (see [5]). *Suppose that E is a real uniformly smooth and uniformly convex Banach space and $\{C_n\}$ is a sequence of nonempty closed and convex subsets of E. If $\lim C_n$ exists and is not empty, then $\lim_{n\longrightarrow\infty} P_{C_n} x = P_{\lim C_n} x$, for $\forall x \in E$.*

Definition 4.

(1) A mapping $T: E \longrightarrow E$ is said to be accretive [6] if $\langle Tu_1 - Tu_2, J_E(u_1-u_2)\rangle \geq 0$, $\forall u_i \in E$, $i=1,2$

(2) A mapping $T: E \longrightarrow E$ is said to be d-accretive [7] if $\langle Tu_1 - Tu_2, J_E u_1 - J_E u_2\rangle \geq 0$, $\forall u_i \in E$, $i=1,2$

It is easy to see that accretive mappings and d-accretive mappings are identical in a Hilbert space, while they are different in a non-Hilbert space.

For a nonlinear mapping $A: D(A) \subset E \longrightarrow E$, we use $\mathrm{Fix}(A) = \{x \in D(A): Ax = x\}$ and $A^{-1}0 = \{x \in D(A): Ax = 0\}$ to denote the fixed point set and zero point set of A, respectively.

Lemma 5 (see [8, 9]). *Suppose that E is a real uniformly smooth and uniformly convex Banach space. Let $A: E \longrightarrow E$ be d-accretive mapping such that $R(I+A) = E$. Under the assumption that $A^{-1}0 \neq \emptyset$, one has the following:*

(1) $\forall x \in E^*$, $\forall z \in A^{-1}0$, and $\forall r > 0$,

$$\begin{aligned}&\overline{\phi}\Big(J_E z, \left(J_{E^*} + rAJ_{E^*}\right)^{-1} J_{E^*} x\Big)\\&\quad + \overline{\phi}\Big(\left(J_{E^*} + rAJ_{E^*}\right)^{-1} J_{E^*} x, x\Big) \leq \overline{\phi}(J_E z, x).\end{aligned} \tag{3}$$

(2) If $x_n \in E^*$, $x \in E^*$, $x_n \longrightarrow x$, and $(J_{E^*} + rAJ_{E^*})^{-1} J_{E^*} x_n \longrightarrow x$, as $n \longrightarrow \infty$, then $x = (J_{E^*} + rAJ_{E^*})^{-1} J_{E^*} x$.

Definition 5 (see [10]). Let C be a nonempty closed subset of E and let Q be a mapping of E onto C. Then Q is said to be sunny if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$. A mapping $Q: E \longrightarrow C$ is said to be a retraction if $Q(z) = z$ for every $z \in C$. If E is a smooth and strictly convex Banach space, then a sunny generalized nonexpansive retraction of E onto C is uniquely decided, which is denoted by R_C.

Definition 6 (see [3]). If E is a real uniformly smooth and uniformly convex Banach space and C is a nonempty closed and convex subset of E, then, for each $x \in E$, there exists a unique element $x_0 \in C$ satisfying $\phi(x_0, x) = \inf\{\phi(z,x) : z \in C\}$. In this case, $\forall x \in E$, define $\pi_C: E \longrightarrow C$ by $\pi_C x = x_0$, and π_C is called the generalized projection from E onto C.

Lemma 6 (see [11]). *Suppose that E is a real uniformly convex Banach space and $r \in (0, +\infty)$. Then there exists a continuous and strictly increasing function $g: [0, 2r] \longrightarrow [0, +\infty)$ with $g(0) = 0$ satisfying*

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|), \tag{4}$$

for $\forall \alpha \in [0,1]$, $\forall x, y \in E$ with $\|x\| \leq r$ and $\|y\| \leq r$.

Accretive mappings have been extensively studied until now and some works can be seen in [12–16] and the references therein. However, until 2000, some valuable research work has been done on d-accretive mappings. As we know, in 2000, Alber and Reich [17] presented the following iterative schemes for d-accretive mapping T in a real uniformly smooth and uniformly convex Banach space:

$$x_{n+1} = x_n - \alpha_n T x_n, \tag{5}$$

$$x_{n+1} = x_n - \alpha_n \frac{Tx_n}{\|Tx_n\|}, \quad n \geq 0. \tag{6}$$

They proved that the iterative sequences $\{x_n\}$ generated by (5) and (6) converge weakly to the zero point of T under the assumption that T is uniformly bounded and demicontinuous.

In 2006, Guan [18] presented the following projective method for the d–accretive mapping T in a real uniformly smooth and uniformly convex Banach space E:

$$\begin{cases} x_1 \in D(T), \\ y_n = (I + r_n T)^{-1} x_n, \\ C_n = \{v \in D(T): \phi(v, y_n) \leq \phi(v, x_n)\}, \\ Q_n = \{v \in D(T): \langle x_n - v, J_E x_1 - J_E x_n\rangle \geq 0\}, \\ x_{n+1} = \pi_{C_n \cap Q_n} x_1, \quad n \geq 1. \end{cases} \tag{7}$$

It was shown that the iterative sequences $\{x_n\}$ generated by (7) converge strongly to the zero point of T under the assumptions that (1) $R(I+T) = E$, (2) T is demicontinuous, and (3) J_E is weakly sequentially continuous and satisfies

$$\phi\Big(p, (I + r_n T)^{-1} x\Big) \leq \phi(p, x), \tag{8}$$

for $\forall x \in E$ and $p \in T^{-1}0$. The restrictions are extremely strong, and it is hard for us to find such a d–accretive mapping that is demicontinuous and satisfies (8).

In 2014, Wei et al. [7] presented the following block iterative schemes for approximating common zero points of d–accretive mappings $\{T_i\}_{i=1}^m$ in a Banach space E:

$$\begin{cases} x_1 \in E, \\ y_n = \sum_{i=1}^{m} \omega_{n,i}\left[\alpha_{n,i}x_n + \left(1-\alpha_{n,i}\right)\left(I + r_{n,i}T_i\right)^{-1}x_n\right], \\ x_{n+1} = \sum_{i=1}^{m} \eta_{n,i}\left[\beta_{n,i}x_n + \left(1-\beta_{n,i}\right)\left(I + s_{n,i}T_i\right)^{-1}y_n\right], n \in N, \end{cases} \tag{9}$$

$$\begin{cases} x_1 \in E, \\ u_n = \sum_{i=1}^{m} \omega_{n,i}\left[\alpha_{n,i}x_n + \left(1-\alpha_{n,i}\right)\left(I + r_{n,i}T_i\right)^{-1}x_n\right], \\ v_{n+1} = \sum_{i=1}^{m} \eta_{n,i}\left[\beta_{n,i}x_n + \left(1-\beta_{n,i}\right)\left(I + s_{n,i}T_i\right)^{-1}u_n\right], \\ H_1 = E, \\ H_{n+1} = \{z \in H_n: \phi(u_n, z) \le \phi(x_n, z)\}, \\ x_{n+1} = R_{H_{n+1}}x_1, \quad n \in N. \end{cases} \tag{10}$$

Under mild assumptions, $\{x_n\}$ generated by (9) is proved to be weakly convergent to an element in $\cap_{i=1}^{m} T_i^{-1}0$, while (10) is proved to be strongly convergent to an element in $\cap_{i=1}^{m} T_i^{-1}0$.

In [19], the study on finite d-accretive mappings is extended to that for infinite d-accretive mappings $\{T_i\}_{i=1}^{\infty} \subset E \times E$:

$$\begin{cases} u_1 = v \in E^*, \\ w_{n,i} = \left(I + s_{n,i}J_E T_i J_{E^*}\right)^{-1}u_n, \\ U_1 = E^*, \\ U_{n+1,i} = \left\{z \in E^*: \langle J_{E^*}\left(u_n - w_{n,i}\right), w_{n,i} - z\rangle \ge 0\right\}, \\ U_{n+1} = \left(\bigcap_{i=1}^{\infty} U_{n+1,i}\right) \cap U_n, \\ V_{n+1} = \left\{z \in U_{n+1}: \|v - z\|^2 \le \left\|P_{U_{n+1}}(v) - v\right\|^2 + \tau_{n+1}\right\}, \\ u_{n+1} \in V_{n+1}, \\ \overline{u_n} = J_{E^*}u_n, \quad n \in N. \end{cases} \tag{11}$$

Then, sequence $\{\overline{u_n}\}$ generated by (11) is proved to be strongly convergent to an element in $\cap_{i=1}^{m} T_i^{-1}0$.

A new idea can be seen in (11), where the iterative element $u_n \in V_n$ can be chosen arbitrarily, which is different from the traditional one, for example, (7) in [18]. However, it is found that, for each iterative step n in (11), countable sets $U_{n+1,i}$ should be evaluated for $i \in N$. To simplify it theoretically, the following iterative scheme is designed in [9]:

$$\begin{cases} u_1, e_1 \in E^*, \\ w_n = J_E\left[\alpha_n J_{E^*}u_n + (1-\alpha_n)\sum_{i=1}^{\infty} b_{n,i}J_{E^*}\left(J_{E^*} + s_{n,i}T_i J_{E^*}\right)^{-1}J_{E^*}(u_n + e_n)\right], \\ U_1 = E^* = V_1, \\ U_{n+1} = \left\{z \in U_n: \langle J_{E^*}w_n - \alpha_n J_{E^*}u_n - (1-\alpha_n)J_{E^*}(u_n + e_n), z\rangle \ge \dfrac{\|w_n\|^2 - \alpha_n\|u_n\|^2 - (1-\alpha_n)\|u_n + e_n\|^2}{2}\right\}, \\ V_{n+1} = \left\{z \in U_{n+1}: \|u_1 - z\|^2 \le \left\|P_{U_{n+1}}(u_1) - u_1\right\|^2 + \tau_{n+1}\right\}, \\ u_{n+1} \in V_{n+1}, \\ \overline{u_n} = J_{E^*}u_n, \quad n \in N. \end{cases} \tag{12}$$

Then, $\{\overline{u_n}\}$ generated by (12) is proved to be strongly convergent to an element in $\cap_{i=1}^{m} T_i^{-1}0$.

In 2020, Wei et al. [8] extended the discussion on countable d-accretive mappings $\{T_i\}_{i=1}^{\infty}$ to that for two groups of countable d-accretive mappings $\{T_i\}_{i=1}^{\infty}$ and $\{S_i\}_{i=1}^{\infty}$ and construct two key groups of sets $\{V_n\}$ and $\{Y_n\}$, where the iterative elements $\{y_n\}$ and $\{u_n\}$ can be chosen arbitrarily in $\{V_n\}$ and $\{Y_n\}$, respectively.

$$\begin{cases} u_1 \in E^*, e_1 \in E^*, \varepsilon_1 \in E^*, \\ v_n = J_E\left[\alpha_n J_{E^*} u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J_{E^*}\left(J_{E^*} + r_{n,i} T_i J_{E^*}\right)^{-1} J_{E^*}(u_n + e_n)\right], \\ U_1 = E^* = V_1, \\ U_{n+1} = \left\{ p \in X_n \colon \langle J_{E^*} v_n - \alpha_n J_{E^*} u_n - (1-\alpha_n) J_{E^*}(u_n + e_n), p\rangle \geq \dfrac{\|v_n\|^2 - \alpha_n\|u_n\|^2 - (1-\alpha_n)\|u_n + e_n\|^2}{2} \right\}, \\ V_{n+1} = \left\{ p \in U_{n+1} \colon \|u_1 - p\|^2 \leq \left\|P_{U_{n+1}}(u_1) - u_1\right\|^2 + \lambda_{n+1} \right\}, \\ y_n \in V_{n+1}, \\ z_n = J_E\left[\beta_n J_{E^*} u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_{n,i} J_{E^*}\left(J_{E^*} + s_{n,i} S_i J_{E^*}\right)^{-1} J_{E^*}(y_n + \varepsilon_n)\right], \\ X_{n+1} = \left\{ p \in U_{n+1} \colon \langle J_{E^*} z_n - \beta_n J_{E^*} u_n - (1-\beta_n) J_{E^*}(y_n + \varepsilon_n), p\rangle \geq \dfrac{\|z_n\|^2 - \beta_n\|v_n\|^2 - (1-\beta_n)\|y_n + \varepsilon_n\|^2}{2} \right\}, \\ Y_{n+1} = \left\{ p \in X_{n+1} \colon \|u_1 - p\|^2 \leq \left\|P_{X_{n+1}}(u_1) - u_1\right\|^2 + \delta_{n+1} \right\}, \\ u_{n+1} \in Y_{n+1}, \\ \overline{u_n} = J_{E^*} u_n, \quad n \in N. \end{cases} \tag{13}$$

Then $\{\overline{u_n}\}$ generated by (13) is proved to be strongly convergent to an element in $(\cap_{i=1}^{\infty} T_i^{-1}0) \cap (\cap_{i=1}^{\infty} S_i^{-1}0)$.

Recall that the inertial-type algorithm was first proposed by Polyak [20] as an acceleration process in solving a smooth convex minimization problem. The inertial-type algorithm involves a two-step iterative method where the next iterate is defined by making use of the previous two iterates. For example, in 2015, Lorenz and Pock [21] proposed the following inertial forward-backward algorithm for approximating zero points of $T + S$, where T and S are accretive-type mappings in Hilbert space H:

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily}, \\ v_n = u_n + \theta_n(u_n - u_{n-1}), \\ u_{n+1} = (I + r_n T)^{-1}(v_n - r_n S v_n), \quad n \in N. \end{cases} \tag{14}$$

In (14), the term $\theta_n(u_n - u_{n-1})$ is called the inertial term.

In this paper, motivated by the previous work, some new work is done in the construction of new iterative schemes: (i) the inertial term is inserted for the purpose of possible acceleration; (ii) the combination expressions of T_i or S_i are different from those in (11)–(13). Numerical experiments are conducted, and it is very interesting that the rate of convergence is so quick that only eight steps are enough for some special cases and for different choices of iterative elements. To emphasize the importance of the topic, a kind of curvature systems is studied and is taken as an example of d-accretive mappings.

2. Iterative Schemes and Strong Convergence Theorems

2.1. Iterative Schemes. In this section, we suppose that the following conditions are satisfied:

(A_1) E is a real uniformly convex and uniformly smooth Banach space; $J_E\colon E \longrightarrow E^*$ and $J_{E^*}\colon E^* \longrightarrow E$ are the normalized duality mappings.

(A_2) $T_i, S_i\colon E \longrightarrow E$ are d-accretive mappings such that $R(I + T_i) = E = R(I + S_i)$, for each $i \in N$.

(A_3) $\{r_{n,i}\}$, $\{t_{n,i}\}$, $\{\xi_n\}$ and $\{\tau_n\}$ are real number sequences in $(0, +\infty)$, for $i, n \in N$. $\{\lambda_n\}$ and $\{\vartheta_n\}$ are real number sequences in $(-\infty, +\infty)$. $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real number sequences in $[0, 1]$.

(A_4) $a_0, b_0 \in (0,1)$, $\{a_{n,i}\}_{i=0}^{\infty}$ and $\{b_{n,i}\}_{i=0}^{\infty}$ are real number sequences in $(0,1)$ such that $a_0 + \sum_{i=1}^{\infty} a_{n,i} = b_0 + \sum_{i=1}^{\infty} b_{n,i} = 1$.

(A_5) $\{\varepsilon_n^{(1)}\}$ and $\{\varepsilon_n^{(2)}\}$ are the error sequences in E^*.

(A_6)
$\widetilde{U_n} = J_{E^*}^{-1}\{a_0 J_{E^*} + \sum_{i=1}^{\infty} a_{n,i} J_{E^*}[(J_{E^*} + r_{n,i} T_i J_{E^*})^{-1} J_{E^*}]\}$;
$\widetilde{\widetilde{U_n}} = J_{E^*}^{-1}\{b_0 J_{E^*} + \sum_{i=1}^{\infty} b_{n,i} J_{E^*} \ [(J_{E^*} + t_{n,i} S_i J_{E^*})^{-1} J_{E^*}] \ [(J_{E^*} + t_{n,i-1} S_{i-1} J_{E^*})^{-1} \ J_{E^*}] \cdots [(J_{E^*} + t_{n,1} S_1 J_{E^*})^{-1} J_{E^*}]\}$.

We construct the following iterative scheme:

$$\begin{cases} x_0, x_1, \varepsilon_1^{(1)}, \varepsilon_1^{(2)} \in E^*, \\ u_n = x_n + \lambda_n (x_n - x_{n-1}), \\ z_n = x_n + \vartheta_n (x_n - x_{n-1}), \\ v_n = J_{E^*}^{-1}\left[(1-\alpha_n) J_{E^*} u_n + \alpha_n J_{E^*} \widetilde{U_n}\left(z_n + \varepsilon_n^{(1)}\right)\right], \\ V_1 = W_1 = E^*, \\ V_{n+1} = \left\{p \in V_n : \overline{\phi}(p, v_n) \leq (1-\alpha_n)\overline{\phi}(p, u_n) + \alpha_n \overline{\phi}\left(p, z_n + \varepsilon_n^{(1)}\right)\right\}, \\ W_{n+1} = \left\{p \in V_{n+1} : \|x_1 - p\|^2 \leq \left\|P_{V_{n+1}}(x_1) - x_1\right\|^2 + \tau_{n+1}\right\}, \\ y_n \in W_{n+1} \text{ chosen arbitrarily}, \\ w_n = J_{E^*}^{-1}\left[(1-\beta_n) J_{E^*} u_n + \beta_n J_{E^*} \widetilde{\widetilde{U_n}}\left(y_n + \varepsilon_n^{(2)}\right)\right], \\ U_1 = X_1 = E^*, \\ U_{n+1} = \left\{p \in V_{n+1} : \overline{\phi}(p, w_n) \leq (1-\beta_n)\overline{\phi}(p, u_n) + \beta_n \overline{\phi}\left(p, y_n + \varepsilon_n^{(2)}\right)\right\}, \\ X_{n+1} = \left\{p \in U_{n+1} : \|x_1 - p\|^2 \leq \left\|P_{U_{n+1}}(x_1) - x_1\right\|^2 + \xi_{n+1}\right\}, \\ x_{n+1} \in X_{n+1} \text{ chosen arbitrarily}, \quad n \in N. \end{cases} \tag{15}$$

2.2. Strong Convergence Theorems

Theorem 1. *Consider* $(\cap_{i=1}^{\infty} T_i^{-1} 0) \cap (\cap_{i=1}^{\infty} S_i^{-1} 0) \neq \emptyset$, $\inf_n r_{n,i} > 0$, $\inf_n t_{n,i} > 0$ *for* $i \in N$, $\alpha_n \nrightarrow 0$, $\beta_n \nrightarrow 0$, $\tau_n \longrightarrow 0$, $\lambda_n \nrightarrow \pm\infty$, $\vartheta_n \nrightarrow \pm\infty$, $\xi_n \longrightarrow 0$, $\tau_n \longrightarrow 0$, $\varepsilon_n^{(1)} \longrightarrow 0$, *and* $\varepsilon_n^{(2)} \longrightarrow 0$, *as* $n \longrightarrow \infty$. *Then, the iterative sequence* $\overline{x_n} = J_{E^*} x_n \longrightarrow J_{E^*} \quad P_{\cap_{n=1}^{\infty} U_n}(x_1) = J_{E^*} P_{(\cap_{i=1}^{\infty}(T_i J_{E^*})^{-1} 0) \cap (\cap_{i=1}^{\infty}(S_i J_{E^*})^{-1} 0)} \in (\cap_{i=1}^{\infty} T_i^{-1} 0) \cap (\cap_{i=1}^{\infty} S_i^{-1} 0)$, *as* $n \longrightarrow \infty$.

Proof. The proof is split into eight steps.

Step 1. $\cap_{n=1}^{\infty} U_n \neq \emptyset$.

Since $(\cap_{i=1}^{\infty} T_i^{-1} 0) \cap (\cap_{i=1}^{\infty} S_i^{-1} 0) \neq \emptyset$, there exists $\delta_0 \in E$ such that $T_i \delta_0 = S_i \delta_0 = 0$, $\forall i \in N$. It follows from Lemma 1 that there exists $\eta_0 \in E^*$ such that $J_{E^*} \eta_0 = \delta_0$. Therefore, $(\cap_{i=1}^{\infty}(T_i J_{E^*})^{-1} 0) \cap (\cap_{i=1}^{\infty}(S_i J_{E^*})^{-1} 0) \neq \emptyset$.

Next, we shall use inductive method to prove that $(\cap_{i=1}^{\infty}(T_i J_{E^*})^{-1} 0) \cap (\cap_{i=1}^{\infty}(S_i J_{E^*})^{-1} 0) \subset U_n$, $n \in N$.

$\forall p \in (\cap_{i=1}^{\infty}(T_i J_{E^*})^{-1} 0) \cap (\cap_{i=1}^{\infty}(S_i J_{E^*})^{-1} 0)$. For $n = 1$, it is obvious that $p \in U_1$. Suppose that the result is true for $n = k$. Then, if $n = k+1$, it follows from the definition of the Lyapunov functional, the convexity of $\|\cdot\|^2$, and Lemma 5 that

$$\begin{aligned}
\overline{\phi}(p, v_k) &= \|p\|^2 - 2(1-\alpha_k)\langle p, J_{E^*}u_k\rangle - 2\alpha_k\langle p, J_{E^*}\widetilde{U_k}(z_k + \varepsilon_k^{(1)})\rangle \\
&\quad + \left\|(1-\alpha_k)J_{E^*}u_k + \alpha_k J_{E^*}\widetilde{U_k}(z_k + \varepsilon_k^{(1)})\right\|^2 \\
&\quad \leq \|p\|^2 - 2(1-\alpha_k)\langle p, J_{E^*}u_k\rangle - 2\alpha_k a_0\langle p, J_{E^*}(z_k + \varepsilon_k^{(1)})\rangle - 2\alpha_k\langle p, \sum_{i=1}^{\infty} a_{k,i} J_{E^*}\left(J_{E^*} + r_{k,i}T_i J_{E^*}\right)^{-1} J_{E^*}(z_k + \varepsilon_k^{(1)})\rangle \\
&\quad + (1-\alpha_k)\|u_k\|^2 + \alpha_k a_0\left\|z_k + \varepsilon_k^{(1)}\right\|^2 + \alpha_k\sum_{i=1}^{\infty} a_{k,i}\left\|\left(J_{E^*} + r_{k,i}T_i J_{E^*}\right)^{-1} J_{E^*}(z_k + \varepsilon_k^{(1)})\right\|^2 \\
&= (1-\alpha_k)\overline{\phi}(p, u_k) + \alpha_k a_0\overline{\phi}(p, z_k + \varepsilon_k^{(1)}) + \alpha_k\sum_{i=1}^{\infty} a_{k,i}\overline{\phi}\left(p, \left(J_{E^*} + r_{k,i}T_i J_{E^*}\right)^{-1} J_{E^*}(z_k + \varepsilon_k^{(1)})\right) \\
&\quad \leq (1-\alpha_k)\overline{\phi}(p, u_k) + \alpha_k\overline{\phi}(p, z_k + \varepsilon_k^{(1)}).
\end{aligned} \tag{16}$$

Thus, $p \in V_{k+1}$. Using Lemma 5 repeatedly, similar to the above discussion, one has

$$\begin{aligned}
\overline{\phi}(p, w_k) &\leq (1-\beta_k)\overline{\phi}(p, u_k) + \beta_k\overline{\phi}\left(p, \widetilde{\widetilde{U_k}}(y_k + \varepsilon_k^{(2)})\right) \\
&\leq (1-\beta_k)\overline{\phi}(p, u_k) \\
&\quad + \beta_k b_0\overline{\phi}(p, y_k + \varepsilon_k^{(2)}) + \beta_k\sum_{i=1}^{\infty} b_{k,i}\overline{\phi}(p, y_k + \varepsilon_k^{(2)}) \\
&= (1-\beta_k)\overline{\phi}(p, u_k) + \beta_k\overline{\phi}(p, y_k + \varepsilon_k^{(2)}).
\end{aligned} \tag{17}$$

Then $p \in U_{k+1}$, which implies that $\cap_{n=1}^{\infty} U_n \neq \emptyset$.

Step 2. Both V_n and U_n are closed and convex subsets of E^*, for $n \in N$.

If $n = 1$, the result is obvious. If $n \geq 2$, since

$$\begin{aligned}
&\overline{\phi}(p, v_n) \leq (1-\alpha_n)\overline{\phi}(p, u_n) + \alpha_n\overline{\phi}(p, z_n + \varepsilon_n^{(1)}) \\
&\Leftrightarrow \langle p, (1-\alpha_n)J_{E^*}u_n + \alpha_n J_{E^*}(z_n + \varepsilon_n^{(1)}) - J_{E^*}v_n\rangle \leq \frac{\alpha_n\left\|z_n + \varepsilon_n^{(1)}\right\|^2 + (1-\alpha_n)\|u_n\|^2 - \|v_n\|^2}{2}.
\end{aligned} \tag{18}$$

V_n is a closed and convex subset of E^*, for $n \in N$.

Since

$$\begin{aligned}
&\overline{\phi}(p, w_n) \leq (1-\beta_n)\overline{\phi}(p, u_n) + \beta_n\overline{\phi}(p, y_n + \varepsilon_n^{(2)}) \\
&\Leftrightarrow \langle p, (1-\beta_n)J_{E^*}u_n + \beta_n J_{E^*}(y_n + \varepsilon_n^{(2)}) - J_{E^*}w_n\rangle \leq \frac{\beta_n\left\|y_n + \varepsilon_n^{(2)}\right\|^2 + (1-\beta_n)\|u_n\|^2 - \|w_n\|^2}{2}.
\end{aligned} \tag{19}$$

U_n is a closed and convex subset of E^*, for $n \in N$.

Step 3. $P_{U_n}(x_1) \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, as $n \longrightarrow \infty$.

The result follows from the results of Steps 1 and 2 and Lemmas 3 and 4.

Step 4. $W_n \neq \emptyset$ and $X_n \neq \emptyset$.

Since $\|P_{V_{n+1}}(x_1) - x_1\| = \inf_{q \in V_{n+1}}\|q - x_1\|$; then, for τ_{n+1}, there exists $\theta_{n+1} \in V_{n+1}$ such that

$$\begin{aligned}
\left\|x_1 - \theta_{n+1}\right\|^2 &\leq \left(\inf_{q \in V_{n+1}}\left\|q - x_1\right\|\right)^2 + \tau_{n+1} \\
&= \left\|P_{V_{n+1}}(x_1) - x_1\right\|^2 + \tau_{n+1}.
\end{aligned} \tag{20}$$

Then $W_n \neq \emptyset$. Similarly, $X_n \neq \emptyset$. This ensures that $\{x_n\}$ is well defined.

Step 5. $x_n \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, as $n \longrightarrow \infty$.

Since $x_{n+1} \in X_{n+1}$, $\|x_1 - x_{n+1}\|^2 \le \|P_{U_{n+1}}(x_1) - x_1\|^2 + \xi_{n+1}$. It follows from Step 3 and $\xi_n \longrightarrow 0$ that $\{x_n\}$ is bounded.

Since $x_{n+1} \in X_{n+1} \subset U_{n+1}$ and U_n is convex, for $\forall t \in (0,1)$, $tP_{U_{n+1}}(x_1) + (1-t)x_{n+1} \in U_{n+1}$. Using Lemma 6, one has

$$\begin{aligned}\left\|P_{U_{n+1}}(x_1) - x_1\right\|^2 &\le \left\|tP_{U_{n+1}}(x_1) + (1-t)x_{n+1} - x_1\right\|^2 \\ &\le t\left\|P_{U_{n+1}}(x_1) - x_1\right\|^2 + (1-t)\left\|x_{n+1} - x_1\right\|^2 - t(1-t)g\left(\left\|P_{U_{n+1}}(x_1) - x_{n+1}\right\|\right).\end{aligned} \tag{21}$$

Therefore, $tg(\|P_{U_{n+1}}(x_1) - x_{n+1}\|) \le \|x_{n+1} - x_1\|^2 - \|P_{U_{n+1}}(x_1) - x_1\|^2 \le \xi_{n+1} \longrightarrow 0$, as $n \longrightarrow \infty$. Therefore, $x_{n+1} - P_{U_{n+1}}(x_1) \longrightarrow \infty$, as $n \longrightarrow \infty$. Combining with Step 3, $x_n \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, $n \longrightarrow \infty$.

Step 6. $u_n \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, $z_n \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, $v_n \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, and $y_n \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, as $n \longrightarrow \infty$.

In fact, since $u_n = x_n + \lambda_n(x_n - x_{n-1})$ with $\lambda_n \nrightarrow \pm\infty$, $u_n \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, as $n \longrightarrow \infty$. Similarly, $z_n \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, as $n \longrightarrow \infty$.

Since $x_{n+1} \in X_{n+1} \subset U_{n+1} \subset V_{n+1}$,

$$\begin{aligned}\overline{\phi}(x_{n+1}, v_n) &\le (1-\alpha_n)\overline{\phi}(x_{n+1}, u_n) + \alpha_n\overline{\phi}\left(x_{n+1}, z_n + \varepsilon_n^{(1)}\right) \\ &= \left\|x_{n+1}\right\|^2 + (1-\alpha_n)\left\|u_n\right\|^2 - 2(1-\alpha_n)\langle x_{n+1}, J_{E^*}u_n\rangle + \alpha_n\left\|z_n + \varepsilon_n^{(1)}\right\|^2 \\ &\quad - 2\alpha_n\langle x_{n+1}, J_{E^*}\left(z_n + \varepsilon_n^{(1)}\right)\rangle \\ &= \left\|x_{n+1}\right\|^2 - (1-\alpha_n)\left\|u_n\right\|^2 - \alpha_n\left\|z_n + \varepsilon_n^{(1)}\right\|^2 + 2(1-\alpha_n)\langle u_n - x_{n+1}, J_{E^*}u_n\rangle \\ &\quad + 2\alpha_n\langle z_n + \varepsilon_n^{(1)} - x_{n+1}, J_{E^*}\left(z_n + \varepsilon_n^{(1)}\right)\rangle \\ &\le \left(\left\|x_{n+1}\right\|^2 - \left\|z_n + \varepsilon_n^{(1)}\right\|^2\right) + (1-\alpha_n)\left(\left\|z_n + \varepsilon_n^{(1)}\right\|^2 - \left\|u_n\right\|^2\right) \\ &\quad + 2(1-\alpha_n)\left\|u_n\right\|\left\|x_{n+1} - u_n\right\| + 2\alpha_n\left\|z_n + \varepsilon_n^{(1)}\right\|\left\|z_n + \varepsilon_n^{(1)} - x_{n+1}\right\|.\end{aligned} \tag{22}$$

Since $\varepsilon_n^{(1)} \longrightarrow 0$, it follows from Step 5 and Lemma 2 that $x_{n+1} - v_n \longrightarrow 0$, as $n \longrightarrow \infty$. Therefore, $v_n \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, as $n \longrightarrow \infty$.

Since $x_{n+1} \in X_{n+1} \subset U_{n+1}$,

$$\begin{aligned}&\beta_n\left[\overline{\phi}(x_{n+1}, u_n) - \overline{\phi}\left(x_{n+1}, y_n + \varepsilon_n^{(2)}\right)\right] \\ &\le \overline{\phi}(x_{n+1}, u_n) - \overline{\phi}(x_{n+1}, w_n) \longrightarrow 0.\end{aligned} \tag{23}$$

Since $\beta_n \nrightarrow 0$, there exists a subsequence of $\{n\}$, which is still denoted by $\{n\}$ such that $\overline{\phi}(x_{n+1}, u_n) - \overline{\phi}(x_{n+1}, y_n + \varepsilon_n^{(2)}) \longrightarrow 0$, and then $x_{n+1} - (y_n + \varepsilon_n^{(2)}) \longrightarrow 0$, which ensures that $y_n \longrightarrow P_{\cap_{n=1}^{\infty} U_n}(x_1)$, as $n \longrightarrow \infty$.

Step 7. $P_{\cap_{n=1}^{\infty} U_n}(x_1) \in (\cap_{i=1}^{\infty}(T_iJ_{E^*})^{-1}0) \cap (\cap_{i=1}^{\infty}(S_iJ_{E^*})^{-1}0)$.

$\forall q \in (\cap_{i=1}^{\infty}(T_iJ_{E^*})^{-1}0) \cap (\cap_{i=1}^{\infty}(S_iJ_{E^*})^{-1}0)$, and using Lemma 5, we have

$$\begin{aligned}\overline{\phi}\left(q, \widetilde{U_n}\left(z_n + \varepsilon_n^{(1)}\right)\right) &\le a_0\overline{\phi}\left(q, z_n + \varepsilon_n^{(1)}\right) + \sum_{i=1}^{\infty} a_{n,i}\overline{\phi}\left(q, \left(J_{E^*} + r_{n,i}T_iJ_{E^*}\right)^{-1}J_{E^*}\left(z_n + \varepsilon_n^{(1)}\right)\right) \\ &\le a_0\overline{\phi}\left(q, z_n + \varepsilon_n^{(1)}\right) + \sum_{i=1}^{\infty} a_{n,i}\left[\overline{\phi}\left(q, z_n + \varepsilon_n^{(1)}\right) - \overline{\phi}\left(\left(J_{E^*} + r_{n,i}T_iJ_{E^*}\right)^{-1}J_{E^*}\left(z_n + \varepsilon_n^{(1)}\right), u_n + \varepsilon_n^{(1)}\right)\right] \\ &= \overline{\phi}\left(q, z_n + \varepsilon_n^{(1)}\right) - \sum_{i=1}^{\infty} a_{n,i}\overline{\phi}\left(\left(J_{E^*} + r_{n,i}T_iJ_{E^*}\right)^{-1}J_{E^*}\left(z_n + \varepsilon_n^{(1)}\right), z_n + \varepsilon_n^{(1)}\right).\end{aligned} \tag{24}$$

From iterative scheme (15), we know that

$$\begin{aligned}\overline{\phi}(q,v_n) &\le (1-\alpha_n)\overline{\phi}(q,u_n)+\alpha_n\overline{\phi}\left(q,\widetilde{U_n}\left(z_n+\varepsilon_n^{(1)}\right)\right)\\ &\le (1-\alpha_n)\overline{\phi}(q,u_n)+\alpha_n\overline{\phi}\left(q,z_n+\varepsilon_n^{(1)}\right)-\alpha_n\sum_{i=1}^{\infty}a_{n,i}\overline{\phi}\left(\left(J_{E^*}+r_{n,i}T_iJ_{E^*}\right)^{-1}J_{E^*}\left(u_n+\varepsilon_n^{(1)}\right),z_n+\varepsilon_n^{(1)}\right),\end{aligned}\tag{25}$$

which implies that

$$\begin{aligned}&\alpha_n\sum_{i=1}^{\infty}a_{n,i}\overline{\phi}\left(\left(J_{E^*}+r_{n,i}T_iJ_{E^*}\right)^{-1}J_{E^*}\left(z_n+\varepsilon_n^{(1)}\right),z_n+\varepsilon_n^{(1)}\right)\\ &\le (1-\alpha_n)\overline{\phi}(q,u_n)+\alpha_n\overline{\phi}\left(q,z_n+\varepsilon_n^{(1)}\right)-\overline{\phi}(q,v_n)\\ &\le (1-\alpha_n)\|u_n\|^2+\alpha_n\left\|z_n+\varepsilon_n^{(1)}\right\|^2-\|v_n\|^2-2(1-\alpha_n)\langle q,J_{E^*}u_n\rangle-2\alpha_n\langle q,J_{E^*}\left(z_n+\varepsilon_n^{(1)}\right)\rangle+2\langle q,J_{E^*}v_n\rangle\\ &\le\left(\|u_n\|^2-\|v_n\|^2\right)+\alpha_n\left(\left\|z_n+\varepsilon_n^{(1)}\right\|^2-\|u_n\|^2\right)+2\|q\|\left\|J_{E^*}\left(z_n+\varepsilon_n^{(1)}\right)-J_{E^*}u_n\right\|+2\|q\|\left\|J_{E^*}u_n-J_{E^*}v_n\right\|\longrightarrow 0.\end{aligned}\tag{26}$$

Since $\alpha_n\nrightarrow 0$, there exists a subsequence of $\{n\}$, which is still denoted by $\{n\}$ such that $\overline{\phi}((J_{E^*}+r_{n,i}T_iJ_{E^*})^{-1}J_{E^*}(z_n+\varepsilon_n^{(1)}),z_n+\varepsilon_n^{(1)})\longrightarrow 0$. Then $(J_{E^*}+r_{n,i}T_iJ_{E^*})^{-1}J_{E^*}(u_n+\varepsilon_n^{(1)})\longrightarrow P_{\cap_{n=1}^{\infty}V_n}(x_1)$, as $n\longrightarrow\infty$.

Using Lemma 5 again, we have $P_{\cap_{n=1}^{\infty}U_n}(x_1)=(J_{E^*}+r_{n,i}T_iJ_{E^*})^{-1}J_{E^*}P_{\cap_{n=1}^{\infty}U_n}(x_1)$, $\forall i\in N$. Therefore, $P_{\cap_{n=1}^{\infty}U_n}(x_1)\in\cap_{i=1}^{\infty}(T_iJ_{E^*})^{-1}0$.

Similarly,

$$\begin{aligned}\overline{\phi}\left(q,\widetilde{\widetilde{U_n}}\left(y_n+\varepsilon_n^{(2)}\right)\right)&\le b_0\overline{\phi}\left(q,y_n+\varepsilon_n^{(2)}\right)\\ &\quad+\sum_{i=1}^{\infty}b_{n,i}\overline{\phi}\left(q,\left[\left(J_{E^*}+t_{n,i}S_iJ_{E^*}\right)^{-1}J_{E^*}\right]\left[\left(J_{E^*}+t_{n,i-1}S_{i-1}J_{E^*}\right)^{-1}J_{E^*}\right]\cdots\left[\left(J_{E^*}+t_{n,1}S_1J_{E^*}\right)^{-1}J_{E^*}\right]\left(y_n+\varepsilon_n^{(2)}\right)\right)\\ &\le b_0\overline{\phi}\left(q,y_n+\varepsilon_n^{(2)}\right)\\ &\quad+\sum_{i=1}^{\infty}b_{n,i}\overline{\phi}\left(q,\left[\left(J_{E^*}+t_{n,i-1}S_{i-1}J_{E^*}\right)^{-1}J_{E^*}\right]\ldots\left[\left(J_{E^*}+t_{n,1}S_1J_{E^*}\right)^{-1}J_{E^*}\right]\left(y_n+\varepsilon_n^{(2)}\right)\right)\\ &\quad-\sum_{i=1}^{\infty}b_{n,i}\overline{\phi}\left(\left[\left(J_{E^*}+t_{n,i}S_iJ_{E^*}\right)^{-1}J_{E^*}\right]\cdots\left[\left(J_{E^*}+t_{n,1}S_1J_{E^*}\right)^{-1}J_{E^*}\right]\left(y_n+\varepsilon_n^{(2)}\right),\right.\\ &\quad\left.\cdot\left[\left(J_{E^*}+t_{n,i-1}S_{i-1}J_{E^*}\right)^{-1}J_{E^*}\right]\cdots\left[\left(J_{E^*}+t_{n,1}S_1J_{E^*}\right)^{-1}J_{E^*}\right]\left(y_n+\varepsilon_n^{(2)}\right)\right).\end{aligned}\tag{27}$$

From iterative scheme (15), we have

$$\begin{aligned}\overline{\phi}(q,w_n) &\le (1-\beta_n)\overline{\phi}(q,u_n)+\beta_n\overline{\phi}\Big(q,\widetilde{\overline{U_n}}\big(y_n+\varepsilon_n^{(2)}\big)\Big)\\ &\le (1-\beta_n)\overline{\phi}(q,u_n)+\beta_n b_0\overline{\phi}\big(q,y_n+\varepsilon_n^{(2)}\big)\\ &\quad+\beta_n\left(\sum_{i=1}^{\infty}b_{n,i}\phi\Big(q,\big(J_{E^*}+t_{n,i-1}S_{i-1}J_{E^*}\big)^{-1}J_{E^*}\Big)\big(y_n+\varepsilon_n^{(2)}\big)\right.\\ &\quad-\beta_n\sum_{i=1}^{\infty}b_{n,i}\overline{\phi}\Big(\big(\big(J_{E^*}+t_{n,i}S_iJ_{E^*}\big)^{-1}J_{E^*}\big)\cdots\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big),\\ &\quad\left.\cdot\big(\big(J_{E^*}+t_{n,i-1}S_{i-1}J_{E^*}\big)^{-1}J_{E^*}\big)\cdots\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big)\Big)\right).\end{aligned}\tag{28}$$

Therefore,

$$\begin{aligned}&\beta_n\sum_{i=1}^{\infty}b_{n,i}\Big[\overline{\phi}\Big(\big(\big(J_{E^*}+t_{n,i}S_iJ_{E^*}\big)^{-1}J_{E^*}\big)\cdots\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big),\\ &\qquad\big(\big(J_{E^*}+t_{n,i-1}S_{i-1}J_{E^*}\big)^{-1}J_{E^*}\big)\cdots\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big)\Big]\\ &\le (1-\beta_n)\overline{\phi}(q,u_n)+\beta_n b_0\overline{\phi}\big(q,y_n+\varepsilon_n^{(2)}\big)+\beta_n\sum_{i=1}^{\infty}b_{n,i}\overline{\phi}\big(q,y_n+\varepsilon_n^{(2)}\big)-\overline{\phi}(q,w_n)\\ &\le (1-\beta_n)\|u_n\|^2+\beta_n\big\|y_n+\varepsilon_n^{(2)}\big\|^2-\|w_n\|^2+2\|q\|\big\|y_n+\varepsilon_n^{(2)}-u_n\big\|+2\|q\|\|u_n-w_n\|\longrightarrow 0.\end{aligned}\tag{29}$$

Since $\beta_n\nrightarrow 0$, there exists a subsequence of $\{n\}$, which is still denoted by $\{n\}$ such that

$$\begin{aligned}&\big(\big(J_{E^*}+t_{n,i}S_iJ_{E^*}\big)^{-1}J_{E^*}\big)\cdots\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big)\\ &-\big(\big(J_{E^*}+t_{n,i-1}S_{i-1}J_{E^*}\big)^{-1}J_{E^*}\big)\cdots\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big)\longrightarrow 0.\end{aligned}\tag{30}$$

Repeating the above process,

$$\begin{aligned}&\big(\big(J_{E^*}+t_{n,i-1}S_{i-1}J_{E^*}\big)^{-1}J_{E^*}\big)\cdots\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big)\\ &\big(\big(J_{E^*}+t_{n,i-1}S_{i-2}J_{E^*}\big)^{-1}J_{E^*}\big)\cdots\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big)\longrightarrow 0,\\ &\big(\big(J_{E^*}+t_{n,i-1}S_{i-2}J_{E^*}\big)^{-1}J_{E^*}\big)\cdots\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big)\\ &-\big(\big(J_{E^*}+t_{n,i-1}S_{i-3}J_{E^*}\big)^{-1}J_{E^*}\big)\cdots\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big)\longrightarrow 0,\\ &\vdots\\ &\big(\big(J_{E^*}+t_{n,1}S_1J_{E^*}\big)^{-1}J_{E^*}\big)\big(y_n+\varepsilon_n^{(2)}\big)-\big(y_n+\varepsilon_n^{(2)}\big)\longrightarrow 0, n\longrightarrow\infty.\end{aligned}\tag{31}$$

Repeating (30), (31), and Lemma 5, one has $P_{\cap_{n=1}^{\infty}U_n}(x_1)=(J_{E^*}+t_{n,i}S\quad{}_iJ_{E^*})^{-1}J_{E^*}\quad P_{\cap_{n=1}^{\infty}U_n}(x_1)$, $\forall i\in N$. Therefore, $J_{E^*}P_{\cap_{n=1}^{\infty}U_n}(x_1)\in\quad(\cap_{i=1}^{\infty}T_i^{-1}0)\cap(\cap_{i=1}^{\infty}S_i^{-1}0)$.

Step 8. $\overline{x_n}=J_{E^*}x_n\longrightarrow J_{E^*}P_{\cap_{n=1}^{\infty}U_n}\ (x_1)\in(\cap_{i=1}^{\infty}T_i^{-1}0)\cap(\cap_{i=1}^{\infty}S_i^{-1}0)$, as $n\longrightarrow\infty$.

Using Steps 1 and 7, $\|P_{\cap_{n=1}^{\infty}U_n}(x_1)-x_1\|=\|P_{(\cap_{i=1}^{\infty}(T_iJ_{E^*})^{-1}0)\cap(\cap_{i=1}^{\infty}(S_iJ_{E^*})^{-1}0)}(x_1)-x_1\|$. Since the metric projection is unique, $P_{\cap_{n=1}^{\infty} U_n}(x_1) = P_{(\cap_{i=1}^{\infty}(T_i J_{E*})^{-1}0) \cap (\cap_{i=1}^{\infty}(S_i J_{E*})^{-1}0)}(x_1)$.

This completes the proof. □

Remark 1. If $\lambda_n = \vartheta_n$, then $u_n = z_n$. Two-step inertial iterative scheme (15) reduces to the traditional inertial iterative scheme.

Remark 2. If $\lambda_n = \vartheta_n = 0$, then $u_n = z_n = x_n$; and two-step iterative scheme (15) extends the corresponding work of (13) in [8].

Remark 3. If y_n or x_{n+1} is chosen as $P_{W_{n+1}} x_1$ (or $\pi_{W_{n+1}} x_1$) or $P_{X_{n+1}} x_1$ (or $\pi_{X_{n+1}} x_1$), (15) becomes a projection iterative scheme with inertial items.

Corollary 1. *In Hilbert space H, iterative scheme (15) becomes as follows:*

$$\begin{cases} x_0, x_1, \varepsilon_1^{(1)}, \varepsilon_1^{(2)} \in H, \\ u_n = x_n + \lambda_n (x_n - x_{n-1}), \\ z_n = x_n + \vartheta_n (x_n - x_{n-1}), \\ v_n = (1-\alpha_n) u_n + \alpha_n \overline{U_n}\left(z_n + \varepsilon_n^{(1)}\right), \\ V_1 = W_1 = H, \\ V_{n+1} = \left\{p \in V_n \colon 2(1-\alpha_n)\langle p, u_n\rangle + 2\alpha_n \langle p, z_n + \varepsilon_n^{(1)}\rangle - 2\langle p, v_n\rangle \le (1-\alpha_n)\|u_n\|^2 + \alpha_n \left\|z_n + \varepsilon_n^{(1)}\right\|^2 - \|v_n\|^2\right\}, \\ W_{n+1} = \left\{p \in V_{n+1} \colon \|x_1 - p\|^2 \le \left\|P_{V_{n+1}}(x_1) - x_1\right\|^2 + \tau_{n+1}\right\}, \\ y_n \in W_{n+1} \text{ chosen arbitrarily}, \\ w_n = (1-\beta_n) u_n + \beta_n \overline{\overline{U_n}}\left(y_n + \varepsilon_n^{(2)}\right), \\ U_1 = X_1 = H, \\ U_{n+1} = \left\{p \in V_{n+1} \colon 2(1-\beta_n)\langle p, u_n\rangle + 2\beta_n \langle p, y_n + \varepsilon_n^{(2)}\rangle - 2\langle p, w_n\rangle \le (1-\beta_n)\|u_n\|^2 + \beta_n \left\|y_n + \varepsilon_n^{(2)}\right\|^2 - \|w_n\|^2\right\}, \\ X_{n+1} = \left\{p \in U_{n+1} \colon \|x_1 - p\|^2 \le \left\|P_{U_{n+1}}(x_1) - x_1\right\|^2 + \xi_{n+1}\right\}, \\ x_{n+1} \in X_{n+1} \text{ chosen arbitrarily}, \quad n \in N, \end{cases} \tag{32}$$

where $\overline{U_n} = a_0 I + \sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i} T_i)^{-1}$ *and* $\overline{\overline{U_n}} = b_0 I + \sum_{i=1}^{\infty} b_{n,i}(I + t_{n,i} S_i)^{-1} \quad (I + t_{n,i-1} S_{i-1})^{-1} \cdots (I + t_{n,1} S_1)^{-1}$, $\forall i, n \in N$. *Under the assumptions of Theorem 1, the result of Theorem 1 is still true.*

3. Numerical Experiments

Theorem 2. *Let* $E = (-\infty, +\infty)$, $T_i x = (x/2i)$, *and* $S_i x = (x/i)$, $\forall x \in (-\infty, +\infty)$, $\forall i \in N$. *Let* $a_0 = (1/2) = b_0$, $a_{n,i} = (n + 2/2(n+i+1)(n+i+2))$, $b_{n,i} = (n/2(n+1)^i)$, *and* $t_{n,i} = r_{n,i} = ni$, $\forall i, n \in N$. *Let* $\alpha_1 = \beta_1 = 1$; $\alpha_n = \beta_n = (n - 1/n)$, $(n \ge 2)$. $\lambda_n = \vartheta_n = \tau_n = \xi_n = (1/n)$; $\varepsilon_n^{(1)} = \varepsilon_n^{(2)} = 0$, $\forall n \in N$. *For initial value* $x_0 = 1$, $x_1 = (1/3)$, *the iterative sequence* $\{x_n\}$ *generated by (32) converges strongly to* $0 \in (\cap_{i=1}^{\infty} T_i^{-1}0) \cap (\cap_{i=1}^{\infty} S_i^{-1}0)$ *by the eighth step for two different choices of* $\{y_n\}$ *and* $\{x_n\}$ *in the corresponding sets* W_{n+1} *and* X_{n+1}, *respectively.*

Proof. For the special example, we can easily see that all of the assumptions of Corollary 1 are satisfied; and the iterative scheme (32) can be simplified as follows:

$$\begin{cases} x_0 = 1, \\ x_1 = \dfrac{1}{3}, \\ V_2 = V_1 = (-\infty, +\infty) = W_1, \\ U_1 = (-\infty, +\infty) = X_1, \\ u_n = \dfrac{n+1}{n} x_n - \dfrac{1}{n} x_{n-1}, \\ v_n = \dfrac{n+5}{2n+4} u_n, \\ V_{n+1} = V_n \cap \{p\colon 2(u_n - v_n)p \le (u_n - v_n)(u_n + v_n)\}, \\ W_{n+1} = V_{n+1} \cap \left[x_1 - \sqrt{\left(P_{V_{n+1}} x_1 - x_1\right)^2 + \dfrac{1}{n+1}}, x_1 + \sqrt{\left(P_{V_{n+1}} x_1 - x_1\right)^2 + \dfrac{1}{n+1}} \right], \\ y_n \in W_{n+1}, \\ w_n = \dfrac{1}{n} u_n + \dfrac{(n-1)(n+3)}{2n(n+2)} y_n, \\ \text{set } q_n = \dfrac{(1/n)u_n^2 + (n-1/n) y_n^2 - w_n^2}{2((1/n)u_n + (n-1/n) y_n - w_n)}, \\ U_{n+1} = V_{n+1} \cap \left\{ p\colon 2\left(\dfrac{u_n}{n} + \dfrac{n-1}{n} y_n - w_n\right) p \le \dfrac{u_n^2}{n} + \dfrac{n-1}{n} y_n^2 - w_n^2 \right\}, \\ X_{n+1} = U_{n+1} \cap \left[x_1 - \sqrt{\left(P_{U_{n+1}} x_1 - x_1\right)^2 + \dfrac{1}{n+1}}, x_1 + \sqrt{\left(P_{U_{n+1}} x_1 - x_1\right)^2 + \dfrac{1}{n+1}} \right], \\ x_{n+1} \in X_{n+1}, \quad n \in N. \end{cases} \tag{33}$$

Now, compute step by step and choose two different groups of values of y_n and x_n in W_{n+1} and X_{n+1}, respectively; we can get the two following tables. □

Remark 4. From Tables 1 and 2 derived from the numerical experiments done in Theorem 2, we may find that (1) W_{n+1} is an interval that permits us to choose intermediate iterative element $\{y_n\}$ flexibly; (2) two extreme values of $\{y_n\}$ in W_{n+1}, the largest and the smallest, are chosen, from which we can see that the convergence of the iterative sequence $\{x_n\}$ is not affected.

4. Curvature Systems

To emphasize the importance of d-accretive mappings, the connection among d-accretive mappings, iterative schemes, and nonlinear boundary value problems is set up.

Definition 7 (see [22]). A single-valued mapping $A\colon D(A) = E \longrightarrow E^*$ is hemicontinuous if $A(x + ty) \rightharpoonup Ax$, as $t \longrightarrow 0$, $\forall x, y \in E$.

Definition 8 (see [22, 23]). A multivalued mapping $A\colon D(A) \subset E \longrightarrow 2^{E^*}$ is monotone if $\langle x - y, u - v\rangle \ge 0$, $\forall x, y \in D(A)$, $u \in Ax$, and $v \in Ay$. The monotone operator A is called maximally monotone if $R(J_E + \lambda A) = E^*$, $\forall \lambda > 0$.

Lemma 7 (see [22]). *If $A\colon D(A) = E \longrightarrow E^*$ is everywhere defined, monotone, and hemicontinuous, then it is maximally monotone.*

Example 1. We shall investigate the following curvature systems:

Table 1: y_n is chosen as the largest value in W_{n+1} and x_{n+1} is chosen as the smallest value in X_{n+1}.

Iterative step	Values of u_n, v_n, V_{n+1}, and W_{n+1}	Choice of y_n	Values of w_n, q_n, U_{n+1}, and X_{n+1}	Choice of x_{n+1}
$n = 1$	$u_1 = v_1 = -(1/3)$ $V_2 = V_1 = (-\infty, +\infty)$ $W_2 = [\frac{1}{3} - \frac{\sqrt{2}}{2}, \frac{1}{3} + \frac{\sqrt{2}}{2}]$	$y_1 = (1/3) + (\sqrt{2}/2)$	$w_1 = -(1/3)$ $U_2 = (-\infty, +\infty)$ $X_2 = [\frac{1}{3} - \frac{\sqrt{2}}{2}, \frac{1}{3} + \frac{\sqrt{2}}{2}]$	$x_2 = (1/3) - (\sqrt{2}/2)$
$n = 2$	$u_2 = -0.72732684$ $v_2 = -0.63641098$ $V_3 = [(u_2 + v_2/2), +\infty)$ $= [-0.68186891, +\infty)$ $W_3 = [(1/3) - (\sqrt{3}/3), (1/3) + (\sqrt{3}/3)]$	$y_2 = (1/3) + (\sqrt{3}/3)$	$w_2 = -0.0790748$ $q_2 = 1.9704514$ $U_3 = [(u_2 + v_2/3), q_2]$ $= [-0.68186891, 1.9704514]$ $X_3 = [(1/3) - (\sqrt{3}/3), (1/3) + (\sqrt{3}/3)]$	$x_3 = (1/3) - (\sqrt{3}/3)$
$n = 3$	$u_3 = -0.2007648$ $v_3 = -0.1606118$ $V_4 = [(u_3 + v_3/2), +\infty)$ $= [-0.1806883, +\infty)$ $W_4 = [(1/3) - (\sqrt{4}/4), (1/3) + (\sqrt{4}/4)]$	$x_4 = (1/3) + (\sqrt{4}/4)$	$w_3 = 0.26641173$ $q_3 = 0.91220228$ $U_4 = [(u_3 + v_3/2), q_3]$ $= [-0.1806901, 0.91220228]$ $X_4 = [(1/3) - (\sqrt{4}/4), (1/3) + (\sqrt{4}/4)]$	$x_4 = (1/3) - (\sqrt{4}/4)$
$n = 4$	$u_4 = -0.1473291$ $v_4 = -0.11049682$ $V_5 = [(u_4 + v_4/2), +\infty)$ $= [-0.12891296, +\infty)$ $W_5 = [(1/3) - (\sqrt{5}/5), (1/3) + (\sqrt{5}/5)]$	$y_4 = (1/3) + (\sqrt{5}/5)$	$w_4 = 0.30465688$ $q_4 = 0.75752491$ $U_5 = [(u_4 + v_4/2), q_4]$ $= [-0.12891296, 0.75752491]$ $X_5 = [(1/3) - (\sqrt{5}/5), q_4]$	$x_5 = (1/3) - (\sqrt{5}/5)$
$n = 5$	$u_5 = -0.10332302$ $v_5 = -0.07380214$ $V_6 = [(u_5 + v_5/2), +\infty)$ $= [-0.08856258, +\infty)$ $W_6 = [(1/3) - (\sqrt{6}/6), (1/3) + (\sqrt{6}/6)]$	$y_5 = (1/3) + (\sqrt{6}/6)$	$w_5 = 0.31834414$ $q_5 = 0.67008454$ $U_6 = [(u_5 + v_5/2), q_5]$ $= [-0.08856258, 0.67008454]$ $X_6 = [(1/3) - (\sqrt{6}/6), q_5]$	$x_6 = (1/3) - (\sqrt{6}/6)$
$n = 6$	$u_6 = -0.06842073$ $v_6 = -0.04703925$ $V_7 = [(u_6 + v_6/2), +\infty)$ $= [-0.05772999, +\infty)$ $W_7 = [(1/3) - (\sqrt{7}/7), (1/3) + (\sqrt{7}/7)]$	$x_7 = (1/3) + (\sqrt{7}/7)$	$w_6 = 0.32201738$ $q_6 = 0.61448521$ $U_7 = [(u_6 + v_6/2), q_6]$ $= [-0.05772999, 0.61448521]$ $X_7 = [(1/3) - (\sqrt{7}/7), q_6]$	$x_7 = (1/3) - (\sqrt{7}/7)$
$n = 7$	$u_7 = -0.04030488$ $v_7 = -0.02686992$ $V_8 = [(u_7 + v_7/2), +\infty)$ $= [-0.0335874, +\infty)$ $W_8 = [(1/3) - (\sqrt{8}/8), (1/3) + (\sqrt{8}/8)]$	$y_7 = (1/3) + (\sqrt{8}/8)$	$w_7 = 0.32133083$ $q_7 = 0.5758943$ $U_8 = [(u_7 + v_7/2), q_7]$ $= [-0.0335874, 0.5758943]$ $X_8 = [(1/3) - (\sqrt{8}/8), q_7]$	$y_7 = (1/3) - (\sqrt{8}/8)$
$n = 8$	$u_8 = -0.01716868$ $v_8 = -0.01115964$ $V_9 = [(u_8 + v_8/2), +\infty)$ $= [-0.01416416, +\infty)$ $W_9 = [(1/3) - (\sqrt{9}/9), (1/3) + (\sqrt{9}/9)]$	$y_9 = (1/3) + (\sqrt{9}/9) = (2/3)$	$w_8 = 0.3186872$ $q_8 = 0.5473603$ $U_9 = [(u_8 + v_8/2), q_8]$ $= [-0.01416416, q_8]$ $X_9 = [(1/3) - (\sqrt{9}/9), q_8]$	$x_9 = (1/3) - (\sqrt{9}/9) = 0$

TABLE 2: y_n is chosen as the smallest value in W_{n+1} and x_{n+1} is chosen as the smallest value in X_{n+1}.

Iterative step	Values of u_n, v_n, V_{n+1}, and W_{n+1}	Choice of y_n	Values of w_n, q_n, U_{n+1}, and X_{n+1}	Choice of x_{n+1}
$n = 1$	$u_1 = v_1 = -(1/3)$ $V_2 = V_1 = (-\infty, +\infty)$ $W_2 = [(1/3) - (\sqrt{2}/2), (1/3) + (\sqrt{2}/2)]$	$y_1 = (1/3) - (\sqrt{2}/2)$	$w_1 = -(1/3)$ $U_2 = (-\infty, +\infty)$ $X_2 = [(1/3) - (\sqrt{2}/2), (1/3) + (\sqrt{2}/2)]$	$x_2 = (1/3) - (\sqrt{2}/2)$
$n = 2$	$u_2 = -0.727326847$ $v_2 = -0.636410987$ $V_3 = [(u_2 + v_2/2), +\infty) = [-0.68186892, +\infty)$ $W_3 = [(1/3) - (\sqrt{3}/3), (1/3) + (\sqrt{3}/3)]$	$y_2 = (1/3) - (\sqrt{3}/3)$	$w_2 = -0.43991871$ $q_2 = -1.100971$ $U_3 = [(u_2 + v_2/2), +\infty)$ $X_3 = [(1/3) - (\sqrt{3}/3), (1/3) + (\sqrt{3}/3)]$	$x_3 = (1/3) - (\sqrt{3}/3)$
$n = 3$	$u_3 = -0.2007648$ $v_3 = -0.1606118$ $V_4 = [(u_3 + v_3/2), +\infty) = [-0.1806883, +\infty)$ $W_4 = [(1/3) - (\sqrt{4}/4), (1/3) + (\sqrt{4}/4)]$	$y_3 = (1/3) - (\sqrt{4}/4)$	$w_3 = -0.13358829$ $q_3 = -0.1587173$ $U_4 = [q_3, +\infty]$ $X_4 = [q_3, (1/3) + (\sqrt{4}/4)]$	$x_4 = q_3 = -0.1587173$
$n = 4$	$u_4 = -0.13739232$ $v_4 = -0.10304424$ $V_5 = [(u_4 + v_4/2), +\infty) = [-0.12021834, +\infty)$ $W_5 = [(1/3) - (\sqrt{5}/5), (1/3) + (\sqrt{5}/5)]$	$y_4 = (1/3) - (\sqrt{5}/5)$	$w_4 = -0.0841707$ $q_4 = -0.10342091$ $U_5 = [q_4, +\infty)$ $X_5 = [q_4, (1/3) + (\sqrt{5}/5)]$	$x_5 = q_4 = -0.10342091$
$n = 5$	$u_5 = -0.092361632$ $v_5 = -0.06597259$ $V_6 = [(u_5 + v_5/2), +\infty) = [-0.079167111, +\infty)$ $W_6 = [(1/3) - (\sqrt{6}/6), (1/6) + (\sqrt{6}/6)]$	$y_5 = (1/3) - (\sqrt{6}/6)$	$w_5 = -0.05271917$ $q_5 = -0.06650978$ $U_6 = [q_5, +\infty)$ $X_6 = [q_5, (1/3) + (\sqrt{6}/6)]$	$x_6 = q_5 = -0.06650978$
$n = 6$	$u_6 = -0.06035792$ $v_6 = -0.04149607$ $V_7 = [(u_6 + v_6/2), +\infty) = [-0.050927, +\infty)$ $W_7 = [(1/3) - (\sqrt{7}/7), (1/3) + (\sqrt{7}/7)]$	$y_6 = (1/3) - (\sqrt{7}/7)$	$w_6 = -0.0309805$ $q_6 = -0.04017194$ $U_7 = [q_6, +\infty)$ $X_7 = [q_6, (1/3) + (\sqrt{7}/7)]$	$x_7 = q_6 = -0.04017194$
$n = 7$	$u_7 = -0.03640939$ $v_7 = -0.02427293$ $V_8 = [(u_7 + v_7/2), +\infty) = [-0.03034116, +\infty)$ $W_8 = [(1/3) - (\sqrt{8}/8), (1/3) + (\sqrt{8}/8)]$	$y_7 = (1/3) - (\sqrt{8}/8)$	$w_7 = -0.01482994$ $q_7 = -0.02076458$ $U_8 = [q_7, +\infty)$ $X_8 = [(1/3) - (\sqrt{8}/8), (1/3) + (\sqrt{8}/8)]$	$x_8 = y_7 = -0.02022006$
$n = 8$	$u_8 = -0.01772608$ $v_8 = -0.01152195$ $V_9 = [(u_8 + v_8/2), +\infty) = [-0.01462402, +\infty)$ $W_9 = [(1/3) - (\sqrt{9}/9), (1/3) + (\sqrt{9}/9)]$	$y_8 = (1/3) - (\sqrt{9}/9) = 0$	$w_8 = -0.00221576$ $U_9 = V_9$ $X_9 = [(1/3) - (\sqrt{9}/9), (1/3) + (\sqrt{9}/9)]$	$x_9 = (1/3) - (\sqrt{9}/9) = 0$

$$\begin{cases} -\text{div}\left[\left(1+\left|\nabla u^{(i)}\right|^2\right)^{(s_i/2)}\left|\nabla u^{(i)}\right|^{m_i-1}\nabla u^{(i)}\right]+\varepsilon\left|u^{(i)}\right|^{r_i-2}u^{(i)}+u^{(i)}(x)=h(x), \quad x\in\Omega, \\ -<\nu,\left(1+\left|\nabla u^{(i)}\right|^2\right)^{(s_i/2)}\left|\nabla u^{(i)}\right|^{m_i-1}\nabla u^{(i)}> \ =0, \quad x\in\Gamma,\ i\in N, \end{cases} \tag{34}$$

where $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the norm and inner-product in R^n, respectively. Ω is the bounded conical domain of R^n $(n\geq 1)$ with its boundary $\Gamma\in C^1$, ν is the normal derivative of Γ, $\nabla u^{(i)}=((\partial u^{(i)}/\partial x_1),\ldots,(\partial u^{(i)}/\partial x_n))$, ϵ is a nonnegative constant, and $h(x)$ is a given function. For $i\in N$, $m_i+s_i+1=q_i$, $m_i\geq 0$, and $(2n/n+1)<q_i\leq 2$. If $q_i\geq n$, then suppose that $1\leq r_i<+\infty$; if $q_i<n$, then suppose that $1\leq r_i\leq(nq_i/n-q_i)$. We use $\|\cdot\|_{q_i'}$ and $\|\cdot\|_{1,q_i,\Omega}$ to denote the norm in $L^{q_i\prime}(\Omega)$ and $W^{1,q_i}(\Omega)$, respectively, $\forall i\in N$.

Lemma 8. *For $i\in N$, define B_i: $W^{1,q_i\prime}(\Omega)\longrightarrow(W^{1,q_i\prime}(\Omega))^*$ as follows: $\forall u,v\in W^{1,q_i\prime}(\Omega)$,*

$$\langle v,B_iu\rangle=\int_\Omega\langle\left(1+\left|\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^2\right)^{(s_i/2)}\left|\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^{m_i-1}\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\nabla\left(|v|^{q_i\prime-1}\text{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\rangle \mathrm{d}x. \tag{35}$$

Then, B_i is everywhere defined, hemicontinuous, and monotone, $\forall i\in N$.

Proof. The proof is split into three steps.

Step 1.B_i is everywhere defined.

If $s_i\geq 0$, then

$$\begin{aligned}\left|\langle v,B_iu\rangle\right|&\leq\int_\Omega\left|2\max\left\{1,\left|\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^2\right\}^{(s_i/2)}\right|\left|\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^{m_i}\left|\nabla\left(|v|^{q_i\prime-1}\text{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right|\mathrm{d}x\\ &\leq 2^{(s_i/2)}\left(\int_\Omega\left|\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^{m_iq_i\prime}\mathrm{d}x\right)^{(1/q_i')}\left(\left|\nabla\left(|v|^{q_i\prime-1}\text{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right|^{q_i}\right)^{(1/q_i')}\\ &\quad+2^{(s_i/2)}\left(\int_\Omega\left|\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{qi'}^{2-q_i'}\right)\right|^{q_i\prime}\mathrm{d}x\right)^{(1/q_i')}\left(\left|\nabla\left(|v|^{q_i\prime-1}\text{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right|^{q_i}\right)^{(1/q_i')}\\ &\leq 2^{(s_i/2)}\left\|\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right\|_{q_i'}^{m_i}\left\|\nabla\left(|v|^{q_i\prime-1}\text{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right\|_{1,q_i',\Omega}\\ &\quad+2^{(s_i/2)}\left\|\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right\|_{1,q_i',\Omega}\left\|\nabla\left(|v|^{q_i\prime-1}\text{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right\|_{1,q_i',\Omega}.\end{aligned} \tag{36}$$

If $s_i<0$, then

$$\begin{aligned}\left|\langle v,B_iu\rangle\right|&\leq\int_\Omega\left|\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^{m_i+s_i}\left|\nabla\left(|v|^{q_i\prime-1}\text{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right|\mathrm{d}x\\ &\leq\left\|\nabla\left(|u|^{q_i\prime-1}\text{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right\|_{q_i'}^{\left(q_i/q_i'\right)}\left\|\nabla\left(|v|^{q_i\prime-1}\text{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right\|_{1,q_i',\Omega}.\end{aligned} \tag{37}$$

Thus B_i is everywhere defined.

Step 2. B_i is monotone.

For $\forall u, v \in W^{1,q_i'}(\Omega)$,

$$\begin{aligned}
&\langle u - v, B_i u - B_i v\rangle \\
&= \int_\Omega \langle \left(1 + \left|\nabla\left(|u|^{q_i'-1}\mathrm{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^2\right)^{(s_i/2)} \left|\nabla\left(|u|^{q_i'-1}\mathrm{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^{m_i-1} \nabla\left(|u|^{q_i'-1}\mathrm{sgn}u\|u\|_{q_i'}^{2-q_i'}\right) \\
&\quad - \left(1 + \left|\nabla\left(|v|^{q_i'-1}\mathrm{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right|^2\right)^{(s_i/2)} \left|\nabla\left(|v|^{q_i'-1}\mathrm{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right|^{m_i-1} \nabla\left(|v|^{q_i'-1}\mathrm{sgn}v\|v\|_{q_i'}^{2-q_i'}\right), \nabla\left(|u|^{q_i'-1}\mathrm{sgn}u\|u\|_{q_i'}^{2-q_i'}\right) \\
&\quad - \nabla\left(|v|^{q_i'-1}\mathrm{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\rangle \mathrm{d}x \\
&\geq \int_\Omega \left[\left(1 + \left|\nabla\left(|u|^{q_i'-1}\mathrm{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^2\right)^{(s_i/2)} \left|\nabla\left(|u|^{q_i'-1}\mathrm{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^{m_i} \right. \\
&\quad \left. - \left(1 + \left|\nabla\left(|v|^{q_i'-1}\mathrm{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right|^2\right) \left|\nabla\left(|v|^{q_i'-1}\mathrm{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right|^{m_i}\right] \\
&\quad \times \left(\left|\nabla\left(|u|^{q_i'-1}\mathrm{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right| - \left|\nabla\left(|v|^{q_i'-1}\mathrm{sgn}v\|v\|_{q_i'}^{2-q_i'}\right)\right|\right)\mathrm{d}x.
\end{aligned} \tag{38}$$

From the fact that $h(t) := (1 + t^2)^{(s_i/2)} t^{m_i}$, $\forall t \geq 0$ is monotone; we know that B_i is monotone.

Step 3. B_i is hemicontinuous.

$\forall u, v, w \in W^{1,q_i'}(\Omega)$ and $t \in (0, 1)$; using Lebesgue's dominated convergence theorem, one has

$$\begin{aligned}
0 &\leq \lim_{t\longrightarrow 0} \left|\langle w, B_i(u + tv) - B_i u\rangle\right| \\
&\leq \left|\int_\Omega \lim_{t\longrightarrow 0} < \left(1 + \left|\nabla\left(|u + tv|^{q_i'-1}\mathrm{sgn}u\|u + tv\|_{q_i'}^{2-q_i'}\right)\right|^2\right)^{\frac{s_i}{2}} \left|\nabla\left(|u + tv|^{q_i'-1}\mathrm{sgn}u\|u + tv\|_{q_i'}^{2-q_i'}\right)\right|^{m_i-1} \right. \\
&\quad \times \nabla\left(|u + tv|^{q_i'-1}\mathrm{sgn}(u + tv)\|u + tv\|_{q_i'}^{2-q_i'}\right) - \left(1 + \left|\nabla\left(|u|^{q_i'-1}\mathrm{sgn}u\|u\|_{q_i'}^{2-q_i'}\right)\right|^2\right)^{\frac{s_i}{2}} \left|\nabla\left(|u|^{q_i'-1}\mathrm{sgn}v\|u\|_{q_i'}^{2-q_i'}\right)\right|^{m_i-1} \\
&\quad \left. \cdot \nabla\left(|u|^{q_i'-1}\mathrm{sgn}v\|u\|_{q_i'}^{2-q_i'}\right), \nabla\left(|w|^{q_i'-1}\mathrm{sgn}w\|w\|_{q_i'}^{2-q_i'}\right) > \mathrm{d}x\right| = 0.
\end{aligned} \tag{39}$$

Therefore, B_i is hemicontinuous.

This completes the proof. □

Lemma 9 (see [8]). *For $i \in N$, define C_i: $W^{1,q_i'}(\Omega) \longrightarrow (W^{1,q_i'}(\Omega))^*$ as follows: $\forall u, v \in W^{1,q_i'}(\Omega)$,*

$$\langle v, C_i u\rangle = \int_\Omega uv\mathrm{d}x. \tag{40}$$

Then C_i is maximally monotone, for $\forall i \in N$.

Lemma 10 (see [22]). *For each $i \in N$, there exist the maximal monotone extension of B_i and the maximal monotone extension of C_i, which are denoted by $\overline{B_i}$: $L^{q_i'}(\Omega) \longrightarrow L^{q_i}(\Omega)$ and $\overline{C_i}$: $L^{q_i'}(\Omega) \longrightarrow L^{q_i}(\Omega)$, respectively.*

Lemma 11 (see [24]). *For $i \in N$, if $q_i' \geq 2$, then the normalized duality mapping J_i: $L^{q_i'}(\Omega) \longrightarrow L^{q_i}(\Omega)$ is defined by $J_i u = |u|^{q_i'-1}\mathrm{sgn}u\|u\|_{q_i}^{2-q_i}$, $\forall u \in L^{q_i'}(\Omega)$. Then, J_i^{-1}: $L^{q_i}(\Omega) \longrightarrow L^{q_i'}(\Omega)$ is defined by $J_i^{-1}u = |u|^{q_i'-1}\mathrm{sgn}u$, $\forall u \in L^{q_i}(\Omega)$.*

Based on the above lemmas and imitating Theorems 3.10, 3.11, and 3.12 in [8], one has the following results.

Theorem 3. *For $i \in N$, define T_i: $L^{q_i}(\Omega) \longrightarrow L^{q_i}(\Omega)$ as follows: $\forall u \in L^{q_i}(\Omega)$, $T_i u = \overline{B_i} J_i^{-1} u(x)$. Then T_i is d-accretive and $R(I + \lambda T_i) = L^{q_i}(\Omega)$, $\forall \lambda > 0, i \in N$.*

Theorem 4. *Define the mapping $\overline{S_i}$: $L^{q_i}(\Omega) \longrightarrow L^{q_i}(\Omega)$ by $\overline{S_i} u = \overline{C_i J_i^{-1}} u(x)$, $\forall u(x) \in L^{q_i}(\Omega)$. Then $\overline{S_i}$ is d-accretive and $R(I + \lambda \overline{S_i}) = L^{q_i}(\Omega)$, $\forall \lambda > 0, i \in N$.*

Define the mapping S_i: $L^{q_i}(\Omega) \longrightarrow L^{q_i}(\Omega)$ by $S_i u = \overline{S_i} u(x) - |k|^{q_i - 1} \mathrm{sgn} k$, $\forall u(x) \in L^{q_i}(\Omega)$, where k is a constant. Then S_i is d-accretive and $R(I + \lambda S_i) = L^{q_i}(\Omega)$, $\forall \lambda > 0, i \in N$.

Theorem 5. *If, in (34), $f_i(x) \equiv \varepsilon |k|^{r_i - 1} \mathrm{sgn} k + k$, where k is a constant, then $\{u^{(i)}(x) \equiv k\}$ is the solution of (34). Moreover, $\{u^{(i)}(x) \equiv k\} \subset (\cap_{i=1}^{\infty} T_i^{-1} 0) \cap (\cap_{i=1}^{\infty} S_i^{-1} 0)$.*

Proof. It is obvious that $\{u^{(i)}(x) \equiv k\}$ is the solution of (34). If $u^{(i)}(x) \equiv k$, then $T_i u^{(i)}(x) = \overline{B_i} J_i^{-1} k = B_i J_i^{-1} k = 0$ and $S_i u^{(i)}(x) = \overline{C_i} J_i^{-1} k - |k|^{q_i - 1} \mathrm{sgn} k = C_i J_i^{-1} k - |k|^{q_i - 1} \mathrm{sgn} k$. Since $\langle v, C_i J_i^{-1} k - |k|^{q_i - 1} \mathrm{sgn} k \rangle = \int_{\Omega} (J_i^{-1} k - |k|^{q_i - 1} \mathrm{sgn} k) v dx = 0$, $\{u^{(i)}(x) \equiv k\} \subset (\cap_{i=1}^{\infty} T_i^{-1} 0) \cap (\cap_{i=1}^{\infty} S_i^{-1} 0)$.

This completes the proof. □

Remark 5. From Theorem 5, we can see the relationship between the solution of curvature systems (34) and common zero points of two groups of d-accretive mappings. This will help us to approximate the solution of curvature systems by using iterative schemes introduced in Section 2.

Data Availability

All data generated and analyzed during this study are included within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The first three authors were supported by the Natural Science Foundation of Hebei Province under Grant no. A2019207064, Science and Technology Key Project of Hebei Education Department under Grant no. ZD2019073, and Key Project of Science and Research of Hebei University of Economics and Business under Grant no. 2018ZD06.

References

[1] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitz-type Mappings with Applications*, Springer, Berlin, Germany, 2008.

[2] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and Applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*Dekker, New York, NY, USA, 1996.

[3] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, pp. 938–945, 2003.

[4] U. Mosco, "Convergence of convex sets and of solutions of variational inequalities," *Advances in Mathematics*, vol. 3, no. 4, pp. 510–585, 1969.

[5] M. Tsukada, "Convergence of best approximations in a smooth Banach space," *Journal of Approximation Theory*, vol. 40, no. 4, pp. 301–309, 1984.

[6] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Space*, Noordhoff, Norton Shores, MI, USA, 1976.

[7] L. Wei, Y. X. Liu, and R. P. Agarwal, "Convergence theorems of convex combination methods for treating d-accretive mappings in a Banach space and nonlinear equation," *Journal of Inequalities and Applications*, vol. 482, 2014.

[8] L. Wei, Y. N. Zhang, and R. P. Agarwal, "A new shrinking iterative scheme for d-accretive mappings with applications to capillarity systems," *Journal of Nonlinear Functional Analysis*, vol. 11, 2020.

[9] L. Wei, R. P. Agarwal, and L. L. Duan, "New iterative designs for an infinitely family of d-accretive mappings in a Banach space," *Journal of Nonlinear Functional Analysis*, vol. 4, 2019.

[10] W. Takahashi, "Proximal point algorithms and four resolvents of nonlinear operators of monotone type in Banach spaces," *Taiwanese Journal of Mathematics*, vol. 12, no. 8, pp. 1883–1910, 2008.

[11] H.-K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.

[12] X. Qin, S. Y. Cho, and L. Wang, "Convergence of splitting algorithms for the sum of two accretive operators with applications," *Fixed Point Theory and Applications*, vol. 166, 2014.

[13] X. Qin, S. Y. Cho, and L. Wang, "Strong convergence of an iterative algorithm involving nonlinear mappings of nonexpansive and accretive type," *Optimization*, vol. 67, no. 9, pp. 1377–1388, 2018.

[14] B. A. B. Dehaish, "Weak and strong convergence of algorithms for the sum of two accretive operators with applications," *Journal of Nonlinear and Convex Analysis*, vol. 16, pp. 1321–1336, 2015.

[15] X. Qin and J. C. Yao, "Weak convergence of a Mann-like algorithm for nonexpansive and accretive operators," *Journal of Inequalities and Applications*, vol. 232, 2017.

[16] X. Qin, L. Wang, and J. C. Yao, "Inertial splitting method for maximal monotone mappings," *Journal of Nonlinear and Convex Analysis*, vol. 21, pp. 2325–2333, 2020.

[17] Y. Alber and S. Reich, "Convergence of averaged approximations to null points of a class of nonlinear mapping," *Communications on Applied Nonlinear Analysis*, vol. 7, pp. 1–20, 2000.

[18] W. R. Guan, *Construction of Iterative Algorithm for Equilibrium Points of Nonlinear Systems. Dissertation of Doctoral Degree*, Ordnance Engineering College, Hebei, China, 2007.

[19] L. Wei and R. P. Agarwal, "Relaxed iterative methods for an infinite family of d-accretive mappings in a Banach space and their applications," *Journal of Nonlinear Functional Analysis*, vol. 16, 2018.

[20] B. T. Polyak, "Some methods of speeding up the convergence of iteration methods," *USSR Computational Mathematics and Mathematical Physics*, vol. 4, no. 5, pp. 1–17, 1964.

[21] D. A. Lorena and T. Pock, "An inertial forward-backward algorithm for monotone inclusions," *Journal of Mathematical Imaging and Vision*, vol. 51, pp. 311–325, 2015.

[22] D. Pascali and S. Sburlan, *Nonlinear Mappings of Monotone Type*, Springer, Berlin, Germany, 1978.

[23] L. Wei, R. Chen, Y. N. Zhang, and R. P. Agarwal, "New shrinking iterative methods for infinite families of monotone operators in a banach space, computational experiments and applications," *Journal of Inequalities and Applications*, vol. 67, 2020.

[24] B. D. Calvert and C. P. Gupta, "Nonlinear elliptic boundary value problems in L^p-spaces and sums of ranges of accretive operators," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 2, no. 1, pp. 1–26, 1978.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6642289, 19 pages
https://doi.org/10.1155/2021/6642289

Research Article

Generalized Hadamard Fractional Integral Inequalities for Strongly (s, m)-Convex Functions

Chao Miao,[1] **Ghulam Farid**,[2] **Hafsa Yasmeen**,[2] **and Yanhua Bian**[1]

[1]*Public Education Department, Xuchang Electrical Vocational College, Xuchang 461000, China*
[2]*Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock, Pakistan*

Correspondence should be addressed to Chao Miao; miaoc_1215@126.com and Ghulam Farid; faridphdsms@hotmail.com

Received 20 December 2020; Revised 9 January 2021; Accepted 29 January 2021; Published 25 March 2021

Academic Editor: Xiaolong Qin

This article deals with Hadamard inequalities for strongly (s, m)-convex functions using generalized Riemann–Liouville fractional integrals. Several generalized fractional versions of the Hadamard inequality are presented; we also provide refinements of many known results which have been published in recent years.

1. Introduction

Fractional calculus is related to the integrals and derivatives of any arbitrary real or complex order. Its history starts from the end of the seventeenth century, but now it has many applications in almost every field of mathematics, science, and engineering such as electromagnetic, viscoelasticity, fluid mechanics, and signal processing. Fractional integral and derivative operators are of great importance in fractional calculus. The Riemann–Liouville fractional integrals are playing key role in its development. Sarikaya et al. [1, 2] studied Hadamard inequality through Riemann–Liouville fractional integrals of convex functions. This study has encouraged a number of researchers to work further in the field of mathematical inequalities by using fractional integral operators. As a consequence, Hadamard's inequality is generalized and extended by fractional integral operators in many ways (see [3–9] and the references therein). The following inequality is the well-known Hadamard inequality for convex functions which is stated in [10].

Let $f: I \longrightarrow \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$ and $x, y \in I$ where $x < y$. Then, the following inequality holds:

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(v)\mathrm{d}v \leq \frac{f(x)+f(y)}{2}. \tag{1}$$

For the history of this inequality, we refer the readers to [11, 12]. Use of convex functions in the fields of statistics [13], economics [14], and optimization [15] is of prime importance because they play an important role in development of new concepts and notions. Various scholars extended the research on integral inequalities to fractal sets [16]. In this paper, the Hadamard inequality is studied for generalized Riemann–Liouville fractional integrals of strongly (s, m)-convex functions; also, by using two integral identities, some error bounds of already established fractional inequalities are studied. Bracamonte et al. [17] defined the strongly (s, m)-convex function as follows.

Definition 1. A function $f: [0, +\infty) \longrightarrow \mathbb{R}$ is said to be strongly (s, m)-convex function with modulus $c \geq 0$ in second sense, where $(s, m) \in (0, 1]^2$, if

$$f(xt + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y) - cmt(1-t)|y-x|^2, \tag{2}$$

holds for all $x, y \in [0, +\infty)$ and $t \in [0, 1]$.

The well-known definition of Riemann–Liouville fractional integral is given as follows.

Definition 2 (see [18]) (see also [19]). Let $f \in L[a, b]$. Then, left-sided and right-sided Riemann–Liouville fractional integrals of a function f of order μ where $\Re(\mu)>0$ are given by

$$I_{a^+}^{\mu} f(x)=\frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1} f(t)\mathrm{d}t, \quad x>a, \tag{3}$$

$$I_{b^-}^{\mu} f(x)=\frac{1}{\Gamma(\mu)}\int_x^b (t-x)^{\mu-1} f(t)\mathrm{d}t, \quad x<b, \tag{4}$$

where $\Re(\mu)$ is real part of μ and $\Gamma(\mu)=\int_0^\infty e^{-z}z^{\mu-1}\mathrm{d}z$. The following theorems are the fractional versions of Hadamard inequality by Riemann–Liouville fractional integrals.

Theorem 1 (see [1]). *Let $f\colon [a,b]\longrightarrow \mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L[a,b]$. If f is a convex function on $[a,b]$, then the following fractional integral inequality holds:*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\left[I_{a^+}^{\mu} f(b)+I_{b^-}^{\mu} f(a)\right]\le\frac{f(a)+f(b)}{2}, \tag{5}$$

with $\mu>0$.

Theorem 2 (see [2]). *Under the assumptions of ieorem 1, the following fractional integral inequality holds:*

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\mu-1}\Gamma(\mu+1)}{(b-a)^{\mu}}\left[I_{((a+b)/2)^+}^{\mu} f(b)+I_{((a+b)/2)^-}^{\mu} f(a)\right]\le\frac{f(a)+f(b)}{2}, \tag{6}$$

with $\mu>0$.

By establishing an integral identity, the following error estimation of inequality (6) is proved.

Theorem 3 (see [1]). *Let $f\colon [a,b]\longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following fractional integral inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\left[I_{a^+}^{\mu} f(b)+I_{b^-}^{\mu} f(a)\right]\right|\le\frac{b-a}{2(\mu+1)}\left(1-\frac{1}{2^{\mu}}\right)\left[|f'(a)|+|f'(b)|\right]. \tag{7}$$

A k-analogue of Riemann–Liouville integral is defined as follows.

Definition 3 (see [20]). Let $f\in L[a,b]$. Then, k-fractional Riemann–Liouville integrals of order μ where $\Re(\mu)>0$, $k>0$, are defined as

$${}_kI_{a^+}^{\mu} f(x)=\frac{1}{k\Gamma_k(\mu)}\int_a^x (x-t)^{(\mu/k)-1} f(t)\mathrm{d}t, \quad x>a, \tag{8}$$

$${}_kI_{b^-}^{\mu} f(x)=\frac{1}{k\Gamma_k(\mu)}\int_x^b (t-x)^{(\mu/k)-1} f(t)\mathrm{d}t, \quad x<b, \tag{9}$$

where $\Gamma_k(.)$ is defined by [21]

$$\Gamma_k(\mu)=\int_0^\infty t^{\mu-1}e^{-(t^k/k)}\mathrm{d}t, \quad \Re(\mu)>0. \tag{10}$$

If $k=1$, (8) and (9) coincide with (3) and (4).

Two k-fractional versions of Hadamard inequality for k-fractional Riemann–Liouville integrals are given in the next two theorems.

Theorem 4 (see [22]). *Let $f\colon [a,b]\longrightarrow \mathbb{R}$ be a positive function with $0\le a<b$. If f is a convex function on $[a,b]$, then the following inequality for k-fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma_k(\mu+k)}{2(b-a)^{(\mu/k)}}\left[{}_kI_{a^+}^{\mu} f(b)+{}_kI_{b^-}^{\mu} f(a)\right]\le\frac{f(a)+f(b)}{2}. \tag{11}$$

Theorem 5 (see [23]). *Under the assumption of ieorem 4, the following inequality for k-fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(b-a)^{(\mu/k)}}\left[{}_kI_{((a+b)/2)^+}^{\mu} f(b)+{}_kI_{((a+b)/2)^-}^{\mu} f(a)\right]\le\frac{f(a)+f(b)}{2}. \tag{12}$$

By establishing an integral identity, in the following theorem, the error estimation of Theorem 4 is proved.

Theorem 6 (see [22]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) with $0 \leq a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for k-fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\mu+k)}{2(b-a)^{\mu/k}}\left[{}_kI^{\mu}_{a^+}f(b) + {}_kI^{\mu}_{b^-}f(a)\right]\right| \leq \frac{b-a}{2((\mu/k)+1)}\left(1-\frac{1}{2^{(\mu/k)}}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{13}$$

In the following, we recall the definition of generalized Riemann–Liouville fractional integrals by a monotonically increasing function.

Definition 4 (see [24]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be an integrable function. Also, let ψ be an increasing and positive function on $(a,b]$, having a continuous derivative ψ' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function ψ on $[a,b]$ of order μ where $\Re(\mu) > 0$ are given by*

$$I^{\mu,\psi}_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\mu-1}f(t)\mathrm{d}t, \quad x > a, \tag{14}$$

$$I^{\mu,\psi}_{b^-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{\mu-1}f(t)\mathrm{d}t, \quad x < b. \tag{15}$$

If ψ is identity function, then (14) and (15) coincide with (3) and (4).

The k-analogue of generalized Riemann–Liouville fractional integral is defined as follows.

Definition 5 (see [25]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be an integrable function. Also, let ψ be an increasing and positive function on $(a,b]$, having a continuous derivative ψ' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function ψ on $[a,b]$ of order μ where $\Re(\mu) > 0$, $k > 0$, are defined by*

$${}_kI^{\mu,\psi}_{a^+}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{(\mu/k)-1}f(t)\mathrm{d}t, \quad x > a, \tag{16}$$

$${}_kI^{\mu,\psi}_{b^-}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{(\mu/k)-1}f(t)\mathrm{d}t, \quad x < b. \tag{17}$$

For further study of fractional integrals, see [26, 27]. We will utilize the following well-known hypergeometric, beta, and incomplete beta functions in our results [28].

$$\begin{aligned}
{}_2F_1[a,b;c;z] &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, \quad c > b > 0, |z| < 1,\\
B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\\
B(x,y;z) &= \int_0^z t^{x-1}(1-t)^{y-1}\mathrm{d}t.
\end{aligned} \tag{18}$$

The rest of the paper is organized as follows. In Section 2, we obtain Hadamard inequalities for generalized Riemann-Liouville fractional integrals of strongly (s, m)-convex functions. Many specific cases are given as outcomes of these inequalities; they are related to the results which have been published in different papers. In Section 3, by using two integral identities for generalized fractional integrals, the error bounds of fractional Hadamard inequalities are established for differentiable strongly (s, m)-convex functions. This paper reproduces the results which are explicitly given in [1, 2, 22, 23, 29–37].

2. Main Results

This section is dedicated to the Hadamard inequality for strongly (s, m)-convex functions via generalized Riemann–Liouville fractional integrals. We will give two versions of this inequality. First one is stated and proved in the following theorem.

Theorem 7. *Let $f: [a,b] \longrightarrow \mathbb{R}$, $[a,b] \subset [0,+\infty)$ be a positive function and $f \in L_1[a,b]$. Also, let f be strongly (s,m)-convex function on $[a,b]$ with modulus c, such that $(a/m), (a/m^2), mb \in [a,b]$. Then, for $k > 0$ and $(s,m) \in (0,1]^2$, the following k-fractional integral inequalities hold for operators given in (16) and (17):*

$$f\left(\frac{a+mb}{2}\right) + \frac{cm}{4(\mu+k)(\mu+2k)}\left[\mu(\mu+k)(b-a)^2 + 2k^2\left(\frac{a}{m}-mb\right)^2 + 2k\mu(b-a)\left(\frac{a}{m}-mb\right)\right]$$
$$\leq \frac{\Gamma_k(\mu+k)}{2^s(mb-a)^{\mu/k}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right) + m^{(\mu/k)+1}{}_kI^{\mu,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right)\right] \tag{19}$$
$$\leq \frac{\mu[f(a)+mf(b)]}{2^s(\mu+sk)} + \frac{m\mu B((\mu/k), s+1)\left[f(b)+mf\left(a/m^2\right)\right]}{k2^s} - \frac{cmk\mu\left[(b-a)^2 + m\left(b-\left(a/m^2\right)\right)^2\right]}{2^s(\mu+k)(\mu+2k)},$$

with $\mu > 0$.

Proof. The following inequality holds for strongly (s,m)-convex functions.

$$f\left(\frac{x+my}{2}\right) \leq \frac{f(x)+mf(y)}{2^s} - \frac{cm|y-x|^2}{4}. \tag{20}$$

By setting $x = at + m(1-t)b$, $y = (a/m)(1-t) + tb$, $t \in [0,1]$, in (20), multiplying resulting inequality with $t^{(\mu/k)-1}$, and then integrating with respect to t, we get

$$\frac{k}{\mu}f\left(\frac{a+mb}{2}\right) \leq \frac{1}{2^s}\left[\int_0^1 f(at+m(1-t)b)t^{(\mu/k)-1}\mathrm{d}t + m\int_0^1 f\left(\frac{a}{m}(1-t)+bt\right)t^{(\mu/k)-1}\mathrm{d}t\right]$$
$$-\frac{cm}{4}\left[\frac{k(b-a)^2}{\mu+2k} + \frac{2k^3((a/m)-mb)^2}{\mu(\mu+k)(\mu+2k)} + \frac{2k^2(b-a)((a/m)-mb)}{(\mu+k)(\mu+2k)}\right]. \tag{21}$$

By setting $\psi(u) = at + m(1-t)b$ and $\psi(v) = (a/m)(1-t) + bt$ in (21) and by applying Definition 5, we get the following inequality:

$$f\left(\frac{a+mb}{2}\right) \leq \frac{\Gamma_k(\mu+k)}{2^s(mb-a)^{\mu/k}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right) + m^{(\mu/k)+1}{}_kI^{\mu,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]$$
$$-\frac{cm\mu}{4}\left[\frac{(b-a)^2}{\mu+2k} + \frac{2k^2((a/m)-mb)^2}{\mu(\mu+k)(\mu+2k)} + \frac{2k(b-a)((a/m)-mb)}{(\mu+k)(\mu+2k)}\right]. \tag{22}$$

The above inequality leads to the first inequality of (19). On the other hand, f is strongly (s, m)-convex function with modulus c; for $t \in [0,1]$, we have the following inequality:

$$
\begin{aligned}
& f(ta + m(1-t)b) + mf\left(\frac{a}{m}(1-t) + tb\right) \leq t^{s}[f(a) + mf(b)] \\
& + m(1-t)^{s}\left[f(b) + mf\left(\frac{a}{m^{2}}\right)\right] - cmt(1-t)\left[(b-a)^{2} + m\left(b - \frac{a}{m^{2}}\right)^{2}\right].
\end{aligned} \tag{23}
$$

By integrating (23) over $[0,1]$ after multiplying with $t^{(\mu/k)-1}$, the following inequality holds:

$$
\begin{aligned}
& \int_{0}^{1} t^{(\mu/k)-1} f(ta + m(1-t)b)dt + m\int_{0}^{1} t^{(\mu/k)-1} f\left(\frac{a}{m}(1-t) + tb\right)dt \\
& \leq \frac{[f(a) + mf(b)]k}{\mu + sk} + m\left[f(b) + mf\left(\frac{a}{m^{2}}\right)\right]B\left(\frac{\mu}{k}, s+1\right) \\
& - \frac{cmk^{2}\left[(b-a)^{2} + m\left(b - \left(a/m^{2}\right)\right)^{2}\right]}{(\mu + k)(\mu + 2k)}.
\end{aligned} \tag{24}
$$

Again using substitutions as considered in (21), we get

$$
\begin{aligned}
& \frac{k\Gamma_{k}(\mu)}{(mb - a)^{(\mu/k)}}\left[{}_{k}I^{\mu,\psi}_{\psi^{-1}(a)^{+}}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right) + \mathrm{m}^{(\mu/k)+1}{}_{k}I^{\mu,\psi}_{\psi^{-1}(b)^{-}}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right] \\
& \leq \frac{[f(a) + mf(b)]k}{\mu + sk} + m\left[f(b) + mf\left(\frac{a}{m^{2}}\right)\right]B\left(\frac{\mu}{k}, s+1\right) \\
& - \frac{cmk^{2}\left((b-a)^{2} + m\left(b - \left(a/m^{2}\right)\right)^{2}\right)}{(\mu + k)(\mu + 2k)}.
\end{aligned} \tag{25}
$$

This leads to the second inequality of (19). □

Remark 1. Under the assumption of Theorem 7, the following outcomes are noted.

(i) If $s = 1$, $m = 1$, then the inequality stated in [[32], Theorem 9] is obtained.

(ii) If $c = 0$, $s = 1$, $m = 1$, and ψ is the identity function in (19), then Theorem 4 is obtained.

(iii) If $c = 0$, $s = 1$, $m = 1$, $k = 1$, and ψ is the identity function in (19), then Theorem 1 is obtained.

(iv) If $k = 1$, $s = 1$, $m = 1$, and ψ is the identity function in (19), then refinement of Theorem 1 is obtained.

(v) If $\mu = 1$, $k = 1$, $s = 1$, $m = 1$, $c = 0$, and ψ is the identity function in (19), then Hadamard inequality is obtained.

(vi) If $m = 1$, $s = 1$, and $c = 0$ in (19), then the inequality [[34], Theorem 1] is obtained.

(vii) If $c = 0$, $k = 1$, $m = 1$, and $s = 1$ in (19), then the inequality stated in [[33], Theorem 2.1] is obtained.

(viii) If $s = 1$, $k = 1$, and ψ is the identity function in (19), then the inequality stated in [[31], Theorem 6] is obtained.

(ix) If $k = 1$, $m = 1$, $s = 1$, $\mu = 1$, and ψ is the identity function in (19), then the inequality stated in [[35], Theorem 6] is obtained.

(x) If $\alpha = 1$, $k = 1$, $c = 0$, and ψ is the identity function in (19), then the inequality stated in [[29], Theorem 2.1] is obtained.

Corollary 1. *Under the assumption of Theorem 7 with $c = 0$ in (19), the following inequality holds:*

$$f\left(\frac{a+mb}{2}\right) \leq \frac{\Gamma_k(\mu+k)}{2^s(mb-a)^{(\mu/k)}} {}_kI^{\mu,\psi}_{\psi^{-1}(a)^+}\left[(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right) + \mathrm{m}^{(\mu/\mathrm{k})+1}{}_\mathrm{k}\mathrm{I}^{\mu,\psi}_{\psi^{-1}(\mathrm{b})^-}(\mathrm{f}\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right)\right]$$
$$\leq \frac{\mu[f(a)+mf(b)]}{2^s(\mu+sk)} + \frac{m\mu B((\mu/k), s+1)\left[f(b)+mf\left(a/m^2\right)\right]}{k2^s}. \tag{26}$$

Theorem 8. *Under the assumption of Theorem 7, the following k-fractional integral inequality holds:*

$$f\left(\frac{a+mb}{2}\right) + \frac{cm}{16(\mu+k)(\mu+2k)}\left[\mu(b-a)^2 + \left(\frac{a}{m}-mb\right)^2\left(\mu^2+5k\mu+8k^2\right) + 2\mu(b-a)\left(\frac{a}{m}-mb\right)(\mu+3k)\right]$$
$$\leq \frac{2^{(\mu/k)-s}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right) + \mathrm{m}^{(\mu/\mathrm{k})+1}{}_\mathrm{k}\mathrm{I}^{\mu,\psi}_{\psi^{-1}((\mathrm{a+mb})/2\mathrm{m})^-}(\mathrm{f}\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right)\right]$$
$$\leq \frac{\mu[f(a)+mf(b)]}{2^{2s}(\mu+sk)} + \frac{m_2F_1(-s, (\mu/k), ((\mu+k)/k); (1/2))\left[f(b)+mf\left(a/m^2\right)\right]}{2^s} - \frac{cm(\mu+3k)\mu\left[(b-a)^2 + m\left(b-\left(a/m^2\right)\right)\right]}{2^{s+2}(\mu+k)(\mu+2k)}, \tag{27}$$

with $\mu > 0$.

Proof. By setting $x = (at/2) + m((2-t)/2)b$, $y = (a/m)((2-t)/2) + (bt/2)$, $t \in [0,1]$ in (20), multiplying resulting inequality with $t^{(\mu/k)-1}$, and then integrating with respect to t, we get

$$\frac{k}{\mu}f\left(\frac{a+mb}{2}\right) \leq \frac{1}{2^s}\left[\int_0^1 f\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)t^{(\mu/k)-1}\mathrm{d}t + m\int_0^1 f\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)t^{(\mu/k)-1}\mathrm{d}t\right]$$
$$-\frac{cm}{4}\left[\frac{(b-a)^2k}{4(\mu+2k)} + \frac{k((a/m)-mb)^2\left(\mu^2+5k\mu+8k^2\right)}{4\mu(\mu+k)(\mu+2k)} + \frac{(b-a)((a/m)-mb)(\mu+3k)k}{2(\mu+k)(\mu+2k)}\right]. \tag{28}$$

By setting $\psi(u) = (at/2) + bm((2-t)/2)$ and $\psi(v) = (a/m)((2-t)/2) + (bt/2)$ and by applying Definition 5, we get the following inequality:

$$\frac{k}{\mu} f\left(\frac{a+mb}{2}\right) \leq \frac{2^{\mu/k} k\Gamma_k(\mu)}{2^s (mb-a)^{\mu/k}} \left[{}_k I^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+} (f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right) + \mathrm{m}^{(\mu/k)+1} {}_k I^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-} (f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right] - \frac{cm}{4}\left[\frac{k(b-a)^2}{4(\mu+2k)} + \frac{k((a/m)-mb)^2\left(\mu^2+5k\mu+8k^2\right)}{4\mu(\mu+k)(\mu+2k)} + \frac{k(b-a)((a/m)-mb)(\mu+3k)}{2(\mu+k)(\mu+2k)}\right]. \tag{29}$$

The above inequality leads to the first inequality of (27). On the other hand, f is strongly (s,m)-convex function with modulus c; for $t \in [0,1]$, we have the following inequality:

$$f\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) + mf\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) \leq \left(\frac{t}{2}\right)^s [f(a) + mf(b)] + m\left(\frac{2-t}{2}\right)^s\left[f(b) + mf\left(\frac{a}{m^2}\right)\right] - \frac{cmt(2-t)\left[(b-a)^2 + m\left(b-\left(a/m^2\right)\right)^2\right]}{4}. \tag{30}$$

By integrating (30) over $[0,1]$ after multiplying with $t^{(\mu/k)-1}$, the following inequality holds:

$$\int_0^1 f\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right)t^{(\mu/k)-1}\mathrm{d}t + m\int_0^1 f\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right)t^{(\mu/k)-1}\mathrm{d}t \leq \frac{k[f(a)+mf(b)]}{2^s(sk+\mu)} + \frac{mk\left[f(b) + f\left(a/m^2\right)\right]{}_2F_1(-s,(1+\mu/k),(2+(\mu+k)/k);(1/2))}{\mu} - \frac{cmk(\mu+3k)\left[(b-a)^2 + m\left(b-\left(a/m^2\right)\right)^2\right]}{4(\mu+k)(\mu+2k)}. \tag{31}$$

Again using substitutions as considered in (28), we get

$$\begin{aligned}&\frac{2^{\mu/k}k\Gamma_k(\mu)}{(mb-a)^{\mu/k}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)+\mathrm{m}^{(\mu/k)+1}{}_k\mathrm{I}^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\\&\leq\frac{k[f(a)+mf(b)]}{2^s(sk+\mu)}+\frac{mk\left[f(b)+mf\left(a/m^2\right)\right]{}_2F_1(-s,(\mu/k),((\mu+k)/k);(1/2))}{\mu}\\&-\frac{cmk(\mu+3k)\left[(b-a)^2+m\left(b-\left(a/m^2\right)\right)^2\right]}{4(\mu+k)(\mu+2k)}.\end{aligned}\tag{32}$$

This leads to the second inequality of (27). □

Remark 2. Under the assumption of Theorem 8, the following outcomes are noted.

(i) If $s=1$ and $m=1$ in (27), then the inequality stated in [[32], Theorem 10] is obtained.

(ii) If $s=1$, $m=1$, $k=1$, $c=0$, and ψ is the identity function in (27), then Theorem 2 is obtained.

(iii) If $s=1$, $m=1$, $k=1$, and ψ is the identity function in (27), then refinement of Theorem 2 is obtained.

(iv) If $s=1$, $m=1$, $k=1$, $\mu=1, c=0$, and ψ is the identity function in (27), then Hadamard inequality is obtained.

(v) If $s=1$, $m=1$, $c=0$, and ψ is the identity function in inequality (27), then Theorem 5 is obtained.

(vi) If $s=1$, $m=1$, and $c=0$ in (27), then the inequality stated in [[32], Corollary 5] is obtained.

(vii) If $s=1$, $k=1$, and ψ is the identity function in inequality (27), then the inequality stated in [[31], Theorem 7] is obtained.

Corollary 3. *Under the assumption of Theorem 8 with $c=0$ in (27), the following inequality holds:*

$$\begin{aligned}f\left(\frac{a+mb}{2}\right)&\leq\frac{2^{(\mu/k)-s}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)+\mathrm{m}^{(\mu/k)+1}{}_k\mathrm{I}^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\\&\leq\frac{\mu[f(a)+mf(b)]}{2^{2s}(\mu+sk)}\\&+\frac{m\left[f(b)+mf\left(a/m^2\right)\right]{}_2F_1(-s,(\mu/k),((\mu+k)/k);(1/2))}{2^s}.\end{aligned}\tag{33}$$

3. Error Estimations of Hadamard Inequalities via Strongly (s,m)-Convex Functions

In this section, we will study error estimations of Hadamard inequalities for generalized Riemann–Liouville fractional integrals of strongly (s,m)-convex functions. The estimations obtained here provide refinements of many well-known results. The Mathematica program is used for integration. We recall the well-known Hölder's integral inequality.

Theorem 9 (see [38]). *Let $p>1$ and $(1/p)+(1/q)=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b|f(x)g(x)|\mathrm{d}x\leq\left(\int_a^b|f(x)|^p\mathrm{d}x\right)^{1/p}\left(\int_a^b g(x)^q\mathrm{d}x\right)^{1/q},\tag{34}$$

with equality holding iff $A|f(x)|^p=B|g(x)|^q$ almost everywhere, where A and B are constants.

In order to prove the next result, the following lemma is useful.

Lemma 1 (see [34]). *Let $a<b$ and $f\colon[a,b]\longrightarrow\mathbb{R}$ be a differentiable mapping on (a,b). Also, suppose that $f'\in L[a,b]$, $\psi(x)$ is an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b), and $\alpha\in(0,1)$. Then, for $k>0$, the following identity holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\mu+k)}{2(b-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_k\mathrm{I}^{\mu,\psi}_{\psi^{-1}(\mathrm{b})^-}(\mathrm{f}\circ\psi)\left(\psi^{-1}(\mathrm{a})\right)\right]$$
$$=\frac{b-a}{2}\int_0^1\left[(1-t)^{(\mu/k)}-t^{(\mu/k)}\right]f'(ta+(1-t)b)\mathrm{d}t. \tag{35}$$

Theorem 10. *Let $f\colon [a,b]\longrightarrow\mathbb{R}$, $[a,b]\subset[0,+\infty)$ be a differentiable mapping on (a,b) such that $f'\in L_1[a,b]$. Also, suppose that $|f'|$ is strongly (s,m)-convex on $[a,b]$ with modulus c. Then, for $k>0$ and $(s,m)\in(0,1]^2$, the following k-fractional integral inequality holds for operators given in (16) and (17):*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\mu+k)}{2(b-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_k\mathrm{I}^{\mu,\psi}_{\psi^{-1}(\mathrm{b})^-}(\mathrm{f}\circ\psi)\left(\psi^{-1}(\mathrm{a})\right)\right]\right|$$
$$\leq\frac{b-a}{2}\left[\left|f'(a)\right|\left(2B\left(\frac{1}{2};s+1,\frac{\mu}{k}+1\right)+\frac{1-(1/2)^{s+(\mu/k)}}{s+(\mu/k)+1}-B\left(1+s,1+\frac{\mu}{k}\right)\right)\right.$$
$$+m\left|f'\left(\frac{b}{m}\right)\right|\left(\frac{1-(1/2)^{s+(\mu/k)+1}}{s+(\mu/k)+1}-\frac{{}_2F_1(-s,1+(\mu/k),2+(\mu/k);(1/2))}{2^{1+(\mu/k)}((\mu/k)+1)}\right. \tag{36}$$
$$\left.\left.+B(1+s,1+(\mu/k))-\frac{k\left(k+\mu+2^s(k+ks+\mu)\right){}_2F_1(-s,1+(\mu/k),2+(\mu/k);(1/2))}{2^{s+(\mu/k)+1}(sk+\mu+k)(\mu+k)}\right)\right]$$
$$-\frac{c\left((b/m)-a\right)^2}{((\mu/k)+2)((\mu/k)+3)}\left(1-\frac{(\mu/k)+4}{2^{(\mu/k)+2}}\right),$$

with $\mu>0$ and ${}_2F_1(-s,1+(\mu/k),2+(\mu/k);(1/2))$ being the hypergeometric function.

Proof. By Lemma 1, it follows that

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\mu+k)}{2(b-a)^{\mu/k}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_k\mathrm{I}^{\mu,\psi}_{\psi^{-1}(\mathrm{b})^-}(\mathrm{f}\circ\psi)\left(\psi^{-1}(\mathrm{b})\right)\right]\right|$$
$$\leq\frac{b-a}{2}\int_0^1\left|(1-t)^{\mu/k}-t^{\mu/k}\right|\left|f'(ta+(1-t)b)\right|\mathrm{d}t. \tag{37}$$

Since $|f'|$ is strongly (s,m)-convex function on $[a,b]$, for $t\in[0,1]$, we have

$$\left|f'(ta+(1-t)b)\right|\leq t^s\left|f'(a)\right|+m(1-t)^s\left|f'\left(\frac{b}{m}\right)\right|-cmt(1-t)\left(\frac{b}{m}-a\right)^2. \tag{38}$$

Now using (38) in (37), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\mu+k)}{2(b-a)^{\mu/k}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_kI^{\mu,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}(b)\right)\right]\right|$$
$$\leq\frac{b-a}{2}\int_0^1\left|\left(1-t\right)^{\mu/k}-t^{\mu/k}\right|\left(t^s|f'(a)|+m(1-t)^s\left|f'\left(\frac{b}{m}\right)\right|-cmt(1-t)\left(\frac{b}{m}-a\right)^2\right)dt.$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\mu+k)}{2(b-a)^{\mu/k}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_kI^{\mu,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}(b)\right)\right]\right|$$
$$\begin{aligned}&\leq\frac{b-a}{2}\int_0^{1/2}\left((1-t)^{\mu/k}-t^{\mu/k}\right)\left(t^s|f'(a)|+m(1-t)^s\left|f'\left(\frac{b}{m}\right)\right|-cmt(1-t)\left(\frac{b}{m}-a\right)^2\right)\\&+\int_{1/2}^1\left(t^{\mu/k}-(1-t)^{\mu/k}\right)\left(t^s|f'(a)|+m(1-t)^s\left|f'\left(\frac{b}{m}\right)\right|-cmt(1-t)\left(\frac{b}{m}-a\right)^2\right)dt\\&\leq\frac{b-a}{2}\left[|f'(a)|\left(\int_0^{1/2}t^s\left((1-t)^{\mu/k}-t^{\mu/k}\right)dt+\int_{1/2}^1t^s\left(t^{\mu/k}-(1-t)^{\mu/k}\right)dt\right)\right.\\&+m\left|f'\left(\frac{b}{m}\right)\right|\left(\int_0^{1/2}(1-t)^s\left((1-t)^{\mu/k}-t^{\mu/k}\right)dt+\int_{\frac{1}{2}}^1(1-t)^s\left(t^{\mu/k}-(1-t)^{\mu/k}\right)dt\right)\\&\left.-cm\left(\frac{b}{m}-a\right)^2\left(\int_0^{1/2}t(1-t)\left((1-t)^{\mu/k}-t^{\mu/k}\right)dt+\int_{1/2}^1t(1-t)\left(t^{\mu/k}-(1-t)^{\mu/k}\right)dt\right)\right].\end{aligned}\tag{39}$$

We now evaluate integrals that appear on the right side of the above inequality:

$$\begin{aligned}&\int_0^{1/2}t^s\left((1-t)^{\mu/k}-t^{\mu/k}\right)dt+\int_{1/2}^1t^s\left(t^{\mu/k}-(1-t)^{\mu/k}\right)dt\\&=2B\left(\frac{1}{2};s+1,\frac{\mu}{k}+1\right)+\frac{1-(1/2)^{s+(\mu/k)}}{s+(\mu/k)+1}-B\left(1+s,1+\frac{\mu}{k}\right),\end{aligned}\tag{40}$$

$$\begin{aligned}&\int_0^{1/2}(1-t)^s\left((1-t)^{\mu/k}-t^{\mu/k}\right)dt+\int_{1/2}^1(1-t)^s\left(t^{\mu/k}-(1-t)^{\mu/k}\right)dt\\&=\frac{1-(1/2)^{s+(\mu/k)+1}}{s+(\mu/k)+1}-\frac{(1/2)^{1+(\mu/k)}{}_2F_1(-s,1+(\mu/k),2+(\mu/k);(1/2))}{(\mu/k)+1}+B(1+s,1+(\mu/k))\\&-\frac{(1/2)^{s+(\mu/k)+1}k\left(k+\mu+2^s(k+ks+\mu)_2F_1(-s,1+(\mu/k),2+(\mu/k);(1/2)\right)}{(sk+\mu+k)(\mu+k)},\end{aligned}\tag{41}$$

$$\int_0^{1/2}t(1-t)\left((1-t)^{\mu/k}-t^{\mu/k}\right)dt+\int_{1/2}^1t(1-t)\left(t^{\mu/k}-(1-t)^{\mu/k}\right)dt=\frac{1-\left(((\mu/k)+4)/2^{(\mu/k)+2}\right)}{((\mu/k)+2)((\mu/k)+3)}.\tag{42}$$

Using (40)–(42) in (39), we get (36). □

Remark 3. Under the assumption of Theorem 10, the following outcomes are noted.

(i) If $s=1$ and $m=1$ in (45), then the inequality stated in [[32], Theorem 11] is obtained.

(ii) If $s=1, m=1$, and $c=0$ in (45), then the inequality stated in [[32], Corollary 10] is obtained.

(iii) If $s=1$, $m=1$, $k=1$, and ψ is the identity function in (45), then a refined error estimation of the fractional Hadamard inequality is obtained.

(iv) If $m=1$ and $c=0$ in (45), then the inequality stated in [[34], Theorem 2] is obtained.

(v) If $m=1$, $s=1$, $c=0$, and ψ is the identity function in (45), then Theorem 6 is obtained.

(vi) If $k=1$, $m=1$, $s=1$, $c=0$, and ψ is the identity function in (45), then Theorem 3 is obtained.

(vii) If $k=1$, $s=1$, and ψ is the identity function in (45), then the inequality stated in [[31], Theorem 8] is obtained.

Corollary 5. *Under the assumption of Theorem 10 with $c=0$ in (3.10), the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\mu+k)}{2(b-a)^{\mu/k}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{b})\right)+{}_k\mathrm{I}^{\mu,\psi}_{\psi^{-1}(\mathrm{b})^-}(\mathrm{f}\circ\psi)\left(\psi^{-1}(\mathrm{a})\right)\right]\right| \\
&\leq\frac{b-a}{2}\left[\left|f'(a)\right|\left(2B\left(\frac{1}{2};s+1,\frac{\mu}{k}+1\right)+\frac{1-(1/2)^{s+(\mu/k)}}{s+(\mu/k)+1}-B\left(1+s,1+\frac{\mu}{k}\right)\right)\right. \\
&+m\left|f'\left(\frac{b}{m}\right)\right|\left(\frac{1-(1/2)^{s+(\mu/k)+1}}{s+(\mu/k)+1}-\frac{{}_2F_1(-s,1+(\mu/k),2+(\mu/k);(1/2))}{2^{1+(\mu/k)}((\mu/k)+1)}\right. \\
&\left.\left.+B\left(1+s,1+\frac{\mu}{k}\right)-\frac{k\left(k+\mu+2^s(k+ks+\mu)_2F_1(-s,1+(\mu/k),2+(\mu/k);(1/2)\right)}{2^{s+(\mu/k)+1}(sk+\mu+k)(\mu+k)}\right)\right].
\end{aligned} \tag{43}$$

Corollary 6. *Under the assumption of Theorem 10 with $k=1$, $\mu=1$, $s=1$, $m=1$, and ψ as the identity function in (3.10), the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}-\int_a^b f(\nu)\mathrm{d}\nu\right|\leq\frac{b-a}{8}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]-\frac{c(b-a)^3}{32}. \tag{44}$$

Lemma 2. *Let $f\colon[a,b]\longrightarrow\mathbb{R}$ be a differentiable mapping on (a,b) such that $f'\in L[a,b]$. Then, for $k>0$ and $m\in(0,1]$, the following identity holds for operators given in (16) and (17):*

$$\begin{aligned}
&\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{\mu/k}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)+\mathrm{m}^{(\mu/k)+1}{}_k\mathrm{I}^{\mu,\psi}_{\psi^{-1}((\mathrm{a+mb})/2\mathrm{m})^-}(\mathrm{f}\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right)\right] \\
&-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right] \\
&=\frac{mb-a}{4}\left[\int_0^1 t^{\mu/k}f'\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)\mathrm{d}t-\int_0^1 t^{\mu/k}f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)\mathrm{d}t\right].
\end{aligned} \tag{45}$$

Proof. Let

$$\begin{aligned}
I_1&=\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)\right], \\
I_2&=\frac{m^{(\mu/k)+1}2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right)\right].
\end{aligned} \tag{46}$$

First, we evaluate I_1:

$$\begin{aligned} I_1 &= \frac{2^{(\mu/k)-1}\mu}{k(mb-a)^{(\mu/k)}}\left[\int_{\psi^{-1}((a+mb)/2)}^{\psi^{-1}(mb)} \psi'(u)(mb-\psi(u))^{(\mu/k)-1}(f\circ\psi)(\mathrm{u})\mathrm{du}\right] \\ &= \frac{-2^{(\mu/k)-1}}{(mb-a)^{(\mu/k)}}\left[\int_{\psi^{-1}((a+mb)/2)}^{\psi^{-1}(mb)} \left(d(mb-\psi(u))^{(\mu/k)}\right)(f(\psi(u)))\right]. \end{aligned} \tag{47}$$

Now integrating by parts, we have

$$I_1 = \frac{1}{2}f\left(\frac{a+mb}{2}\right) + \frac{1}{2}\int_{\psi^{-1}((a+mb)/2)}^{\psi^{-1}(mb)} \left(\frac{2(mb-\psi(u))}{mb-a}\right)^{\mu/k}\psi'(u)f'(\psi(u))\mathrm{d}u. \tag{48}$$

Substituting $t = 2(mb-\psi(u))/(mb-a)$, so that $\psi(u) = (at/2) + m((2-t)/2)b$ in (48), we get the following inequality:

$$I_1 = \frac{1}{2}f\left(\frac{a+mb}{2}\right) + \frac{mb-a}{4}\int_0^1 t^{\mu/k}f'\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right)\mathrm{d}t. \tag{49}$$

Now, we evaluate I_2:

$$\begin{aligned} I_2 &= \frac{m^{(\mu/k)+1}2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right)\right] \\ &= \frac{m^{(\mu/k)+1}\mu 2^{(\mu/k)-1}}{k(mb-a)^{(\mu/k)}}\left[\int_{\psi^{-1}(a/m)}^{\psi^{-1}((a+mb)/2m)} \psi'(v)\left(\psi(v)-\frac{a}{m}\right)^{(\mu/k)-1}(f\circ\psi)(\mathrm{v})\mathrm{dv}\right] \\ &= \frac{m^{(\mu/k)+1}2^{(\mu/k)-1}}{(mb-a)^{(\mu/k)}}\left[\int_{\psi^{-1}(a/m)}^{\psi^{-1}((a+mb)/2m)} \left(f(\psi(v))\left(\mathrm{d}\left(\psi(v)-\frac{a}{m}\right)^{(\mu/k)}\right)\right)\right]. \end{aligned} \tag{50}$$

Integrating by parts, we get

$$I_2 = \frac{m}{2}f\left(\frac{a+mb}{2m}\right) - \frac{m}{2}\int_{\psi^{-1}(a/m)}^{\psi^{-1}((a+mb)/2m)} \left(\left(\frac{2m(\psi(v))-(a/m)}{mb-a}\right)^{\mu/k}\right)\psi'(v)\left(f'(\psi(v))\right)\mathrm{d}v. \tag{51}$$

Substituting $s = 2m((\psi(v))-(a/m))/(mb-a)$, so that $\psi(v) = (a/m)((2-t)/2) + (bt/2)$ in (51), we get the following inequality:

$$I_2 = \frac{m}{2}f\left(\frac{a+mb}{2m}\right) - \frac{(mb-a)}{4}\int_0^1 s^{\mu/k}f'\left(\frac{a}{m}\left(\frac{2-s}{2}\right)+\frac{bs}{2}\right)\mathrm{d}s. \tag{52}$$

Adding (49) and (52), (45) is obtained. □

Remark 4. Under the assumption of Lemma 2, the following outcomes are noted.

(i) If $k = 1$ and ψ is the identity function in (45), then the identity stated in [[30], Lemma 2.3] is obtained.

(ii) If $m = 1$ in (45), then the identity stated in [[32], Lemma 2] is obtained.

(iii) If $m = 1$, $k = 1$, and ψ is the identity function in (45), then the identity stated in [[2], Lemma 3] is obtained.

(iv) If $m = 1$, $k = 1$, $\mu = 1$, and ψ is the identity function in (45), then the identity stated in [[2], Corollary 1] is obtained.

(v) If $m = 1$ and ψ is the identity function in (45), then the identity stated in [[23], Lemma 3.1] is obtained.

Theorem 11. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$, $[a,b] \subset [0,+\infty)$ be a differentiable mapping on (a,b) such that $f' \in Ł[a,b]$. Also, suppose that $|f'|^q$ is strongly (s,m)-convex function on $[a,b]$ for $q \geq 1$. Then, for $k > 0$ and $(s,m) \in (0,1]^2$, the following k-fractional integral inequality holds for operators given in (16) and (17):*

$$\begin{aligned}
&\left|\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)+\mathrm{m}^{(\mu/k)+1}{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\\
&\leq\frac{mb-a}{4((\mu/k)+1)}\left(\frac{1}{2((\mu/k)+2)}\right)^{1/q}\left[\left(\left(\frac{2((\mu/k)+1)((\mu/k)+2)|f'(a)|^q}{2^s((\mu/k)+s+1)}+2m|f'(b)|^q\left(\frac{\mu}{k}+2\right)\right)\right.\right.\\
&\times{}_2F_1\left(-s,1+\frac{\mu}{k},2+\frac{\mu}{k};\frac{1}{2}\right)-\left.\frac{cm(b-a)^2((\mu/k)+1)((\mu/k)+4)}{2((\mu/k)+3)}\right)^{1/q}\\
&+\left(2m\left(\frac{\mu}{k}+2\right)\left|f'\left(\frac{a}{m^2}\right)\right|^q{}_2F_1\left(-s,1+\frac{\mu}{k},2+\frac{\mu}{k};\frac{1}{2}\right)\right.\\
&\left.\left.+\frac{2((\mu/k)+1)((\mu/k)+2)|f'(b)|^q}{2^s((\mu/k)+s+1)}-\frac{cm(b-(a/m^2))^2((\mu/k)+1)((\mu/k)+4)}{2((\mu/k)+3)}\right)^{1/q}\right],
\end{aligned}\tag{53}$$

with $\mu > 0$ and $(1/p)+(1/q)=1$.

Proof. We divide the proof in two cases. □

Case 1. Fix $q = 1$. Applying Lemma 2 and strongly (s,m)-convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)+\mathrm{m}^{(\mu/k)+1}{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{4}\left[\left|\int_0^1 t^{(\mu/k)}f'\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)\right|dt+\int_0^1\left|t^{\mu/k}f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)dt\right|\right]\\
&\leq\frac{mb-a}{4}\left[\left(\left(\frac{|f'(a)|+m|f'(b)|}{2^s}\right)\int_0^1 t^{(\mu/k)+s}dt+\frac{m\left(|f'(b)|+m\left|f'\left(a/m^2\right)\right|\right)}{2^s}\int_0^1(2-t)^s t^{\mu/k}dt\right.\right.\\
&\left.-\frac{cm\left((b-a)^2+m\left(b-\left(a/m^2\right)\right)^2\right)}{4}\int_0^1 t^{(\mu/k)+1}(2-t)dt\right]\leq\frac{mb-a}{4}\left[\frac{k\left(|f'(a)|+m|f'(b)|\right)}{2^s(\mu+sk+k)}\right.\\
&\left.+\frac{mk_2F_1(-s,1+(\mu/k),2+(\mu/k),(1/2))\left(|f'(b)|+m\left|f'\left(a/m^2\right)\right|\right)}{k+\mu}-\frac{cm\left((b-a)^2+m\left(b-\left(a/m^2\right)\right)^2\right)((\mu/k)+4)}{4((\mu/k)+2)((\mu/k)+3)}\right].
\end{aligned}\tag{54}$$

Case 2. For $q > 1$. From Lemma 2 and using power mean inequality, we get

$$\begin{aligned}
&\left|\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)\right.\right.\\
&\left.\left.+m^{(\mu/k)+1}{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right)\right]-\frac{1}{2}\left[\mathrm{f}\left(\frac{\mathrm{a+mb}}{2}\right)+\mathrm{mf}\left(\frac{\mathrm{a+mb}}{2\mathrm{m}}\right)\right]\right|\\
&\leq\frac{mb-a}{4}\left(\int_0^1 t^{(\mu/k)}\mathrm{d}t\right)^{1-(1/q)}\left[\left(\int_0^1 t^{(\mu/k)}\left|f'\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)\right|^q\mathrm{d}t\right)^{(1/q)}\right.\\
&\left.+\left(\int_0^1 mt^{(\mu/k)}\left|f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)\right|^q\mathrm{d}t\right)^{(1/q)}\right]\leq\frac{mb-a}{4((\mu/k)+1)^{1/p}}\left[\left(\frac{|f'(a)|^q}{2^s}\int_0^1 t^{s+(\mu/k)}\mathrm{d}t\right.\right.\\
&\left.+\frac{m|f'(b)|^q}{2^s}\int_0^1(2-t)^s t^{(\mu/k)}\mathrm{d}t-\frac{cm(b-a)^2}{4}\int_0^1(2-t)t^{(\mu/k)+1}\mathrm{d}t\right)^{(1/q)}+\left(\frac{mf'\left(a/m^2\right)}{2^s}\right.\\
&\left.\left.\times\int_0^1(2-t)^s t^{(\mu/k)}\mathrm{d}t+\frac{|f'(b)|^q}{2^s}\int_0^1 t^{s+(\mu/k)}\mathrm{d}t-\frac{cm\left(b-\left(a/m^2\right)\right)^2}{4}\int_0^1(2-t)t^{(\mu/k)+1}\mathrm{d}t\right)^{(1/q)}\right]\\
&\leq\frac{mb-a}{4((\mu/k)+1)^{1/p}}\left[\left(\frac{k|f'(a)|^q}{2^s(sk+\mu+k)}+\frac{mk|f'(b)|^q{}_2F_1(-s,1+(\mu/k),2+(\mu/k),(1/2))}{k+\mu}\right.\right.\\
&\left.-\frac{cm((\mu/k)+4)(b-a)^2}{4((\mu/k)+2)((\mu/k)+3)}\right)^{(1/q)}+\left(\frac{mk\left|f'\left(a/m^2\right)\right|^q{}_2F_1(-s,1+(\mu/k),2+(\mu/k),(1/2))}{k+\mu}\right.\\
&\left.\left.+\frac{k|f'(b)|^q}{2^s(sk+\mu+k)}-\frac{cm((\mu/k)+4)\left(b-\left(a/m^2\right)\right)^2}{4((\mu/k)+2)((\mu/k)+3)}\right)^{(1/q)}\right]\\
&\leq\frac{mb-a}{4((\mu/k)+1)^{1/p}}\left(\frac{1}{2((\mu/k)+1)((\mu/k)+2)}\right)^{1/q}\left[\left(\frac{2k|f'(a)|^q((\mu/k)+1)((\mu/k)+2)}{2^s(sk+\mu+k)}\right.\right.\\
&\left.+2m\left(\frac{\mu}{k}+2\right)|f'(b)|^q{}_2F_1\left(-s,1+\frac{\mu}{k},2+\frac{\mu}{k},\frac{1}{2}\right)-\frac{cm((\mu/k)+4)((\mu/k)+1)(b-a)^2}{2((\mu/k)+3)}\right)^{1/q}\\
&+\left(2m\left|f'\left(\frac{a}{m^2}\right)\right|^q\left(\frac{\mu}{k}+2\right){}_2F_1\left(-s,1+\frac{\mu}{k},2+\frac{\mu}{k},\frac{1}{2}\right)\right.\\
&\left.\left.+\frac{2k((\mu/k)+1)((\mu/k)+2)|f'(b)|^q}{2^s(sk+\mu+k)}-\frac{cm((\mu/k)+1)((\mu/k)+4)\left(b-\left(a/m^2\right)\right)^2}{2((\mu/k)+3)}\right)^{1/q}\right].
\end{aligned}\tag{55}$$

Hence, we get (53).

Remark 5. Under the assumption of Theorem 11, the following outcomes are noted.

(i) If $s = 1$ and $m = 1$ in (53), then the inequality stated in [[32], Theorem 12] is obtained.

(ii) If $s = 1$, $k = 1$, and ψ is the identity function in (53), then the inequality stated in [[31], Theorem 9] is obtained.

(iii) If $s = 1$, $k = 1$, $c = 0$, and ψ is the identity function in (53), then the inequality stated in [[30], Theorem 2.4] is obtained.

(iv) If $c = 0$, $s = 1, m = 1$, and ψ is the identity function in (53), then the inequality stated in [[23], Theorem 3.1] is obtained.

(v) If $s = 1$, $m = 1$, $c = 0$, $k = 1$, and ψ is the identity function in (53), then the inequality stated in [[2], Theorem 5] is obtained.

(vi) If $q = 1, s = 1, m = 1, c = 0, k = 1, \mu = 1$, and ψ is the identity function in (53), then the inequality stated in [[36], Theorem 2.2] is obtained.

Corollary 7. *Under the assumption of Theorem 11 with $c = 0$ in (53), the following inequality holds:*

$$\Bigg|\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\Big[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\big(\psi^{-1}(\mathrm{mb})\big) + m^{(\mu/k)+1}{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\Big(\psi^{-1}\Big(\frac{\mathrm{a}}{\mathrm{m}}\Big)\Big)\Big] - \frac{1}{2}\Big[\mathrm{f}\Big(\frac{\mathrm{a+mb}}{2}\Big) + \mathrm{mf}\Big(\frac{\mathrm{a+mb}}{2\mathrm{m}}\Big)\Big]\Bigg|$$
$$\le \frac{mb-a}{4((\mu/k)+1)}\left(\frac{1}{2((\mu/k)+2)}\right)^{1/q}\Bigg[\Bigg(\frac{2((\mu/k)+1)((\mu/k)+2)|f'(a)|^q}{2^s((\mu/k)+s+1)} + 2m|f'(b)|^q((\mu/k)+2) \tag{56}$$
$$\times {}_2F_1\Big(-s,1+\frac{\mu}{k},2+\frac{\mu}{k};\frac{1}{2}\Big)\Bigg)^{1/q} + \Bigg(2m\Big(\frac{\mu}{k}+2\Big)\Big|f'\Big(\frac{a}{m^2}\Big)\Big|^q$$
$$\times {}_2F_1\Big(-s,1+\frac{\mu}{k},2+\frac{\mu}{k};\frac{1}{2}\Big) + \frac{2((\mu/k)+1)((\mu/k)+2)|f'(b)|^q}{2^s((\mu/k)+s+1)}\Bigg)^{1/q}\Bigg].$$

Corollary 8. *Under the assumption of Theorem 11 with $k = 1$, $s = 1$, $m = 1, q = 1$, $\mu = 1$, and ψ as the identity function in (53), the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(v)\mathrm{d}v - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left[|f'(a)| + |f'(b)| - \frac{5c(b-a)^2}{12}\right]. \tag{57}$$

Lemma 3 (see [39]). *Let $p \ge 1$ be a real number. For $(x_1, x_2, \ldots, x_n) \in [0,\infty)$ and $n \ge 2$, the following inequality holds:*

$$\sum_{i=1}^n x_i^p \le \left(\sum_{i=1}^n x_i\right)^p. \tag{58}$$

Theorem 12. *Let $f\colon I \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. Also, suppose that $|f'|^q$ is strongly (s,m)-convex function on $[a,b]$ for $q>1$. Then, for $k>0$ and $(s,m) \in (0,1]^2$, the following k-fractional integral inequality holds for operators given in (16) and (17):*

$$\left|\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)+\mathrm{m}^{(\mu/k)+1}{}_{\mathrm{k}}\mathrm{I}^{\mu,\psi}_{\psi^{-1}((\mathrm{a+mb})/2\mathrm{m})^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{16}\left(\frac{4}{(\mu p/k)+1}\right)^{1/p}\left[\left(\left(\left|f'(a)\right|\left(\frac{4}{2^s(s+1)}\right)^{1/q}\right.\right.\right. \tag{59}$$
$$\left.\left.+\left|f'(b)\right|\left(\frac{4m(-1+2^{s+1})}{2^s(1+s)}\right)^{1/q}\right)^q-\frac{2cm(b-a)^2}{3}\right)^{1/q}+\left(\left(m\left|f'\left(\frac{a}{m^2}\right)\right|\left(\frac{4m(-1+2^{s+1})}{2^s(1+s)}\right)^{1/q}\right.\right.$$
$$\left.\left.\left.+\left(\frac{4}{2^s(s+1)}\right)^{1/q}\left|f'(b)\right|\right)^q-\frac{2cm(b-a)^2}{3}\right)\right)^{1/q}\right],$$

with $\mu>0$ *and* $(1/p)+(1/q)=1$.

Proof. By applying Lemma 2 and using the property of modulus, we get

$$\left|\frac{2^{\mu/k-1}\Gamma_k(\mu+k)}{(mb-a)^{\mu/k}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)\right.\right.$$
$$\left.\left.+m^{(\mu/k)+1}{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right]-\frac{1}{2}\left[f\left(\frac{\mathrm{a+mb}}{2}\right)+\mathrm{mf}\left(\frac{\mathrm{a+mb}}{2\mathrm{m}}\right)\right]\right| \tag{60}$$
$$\leq\frac{mb-a}{4}\left[\int_0^1\left|t^{(\mu/k)}f'\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)\right|\mathrm{d}t+\int_0^1\left|t^{(\mu/k)}f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)\right|\mathrm{d}t\right].$$

Now applying Hölder's inequality for integrals, we get

$$\left|\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)+\mathrm{m}^{(\mu/k)+1}{}_{\mathrm{k}}\mathrm{I}^{\mu,\psi}_{\psi^{-1}((\mathrm{a+mb})/2\mathrm{m})^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right]\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{b-a}{4}\left(\frac{1}{(\mu p/k)+1}\right)^{1/p} \tag{61}$$
$$\left[\left(\int_0^1\left|f'\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)\right|^q\mathrm{d}t\right)^{1/q}+\left(\int_0^1\left|f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)\right|^q\mathrm{d}t\right)^{1/q}\right].$$

Using strongly (s,m)-convexity of $|f'|^q$, we get

$$\begin{aligned}
&\Bigg|\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)+m^{(\mu/k)+1}{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right)\right]\\
&-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\Bigg|\\
&\leq\frac{mb-a}{4}\left(\frac{1}{(\mu p/k)+1}\right)^{1/p}\Bigg[\Bigg(\frac{|f'(a)|^q}{2^s}\int_0^1 t^s\mathrm{d}t+\frac{m|f'(b)|^q}{2^s}\int_0^1(2-t)^s\mathrm{d}t\\
&-\frac{cm(b-a)^2}{4}\int_0^1 t(2-t)\mathrm{d}t\Bigg)^{1/q}+\left(\frac{m\left|f'\left(a/m^2\right)\right|^q}{2^s}\int_0^1(2-t)^s\mathrm{d}t+\frac{|f'(b)|^q}{2^s}\int_0^1 t^s\mathrm{d}t\right.\\
&\left.-\frac{cm\left(b-\left(a/m^2\right)\right)^2}{4}\int_0^1 t(2-t)\mathrm{d}t\right)^{1/q}\Bigg]=\frac{mb-a}{4}\left(\frac{1}{(\mu p/k)+1}\right)^{1/p}\Bigg[\Bigg(\frac{|f'(a)|^q}{2^s(s+1)}\\
&+\frac{m|f'(b)|^q\left(-1+2^{s+1}\right)}{2^s(1+s)}-\frac{cm(b-a)^2}{6}\Bigg)^{1/q}+\Bigg(\frac{m\left|f'\left(a/m^2\right)\right|^q\left(-1+2^{s+1}\right)}{2^s(1+s)}\\
&+\frac{|f'(b)|^q}{2^s(s+1)}-\frac{cm\left(b-\left(a/m^2\right)\right)^2}{6}\Bigg)^{1/q}\Bigg]\leq\frac{mb-a}{16}\left(\frac{4}{(\mu p/k)+1}\right)^{1/p}\Bigg[\Bigg(\frac{4|f'(a)|^q}{2^s(s+1)}\\
&+\frac{4m|f'(b)|^q\left(-1+2^{s+1}\right)}{2^s(1+s)}-\frac{2cm(b-a)^2}{3}\Bigg)^{1/q}+\Bigg(\frac{4m\left|f'\left(a/m^2\right)\right|^q\left(-1+2^{s+1}\right)}{2^s(1+s)}\\
&+\frac{4|f'(b)|^q}{2^s(s+1)}-\frac{2cm\left(b-\left(a/m^2\right)\right)^2}{3}\Bigg)^{1/q}\Bigg]\leq\frac{mb-a}{16}\left(\frac{4}{(\mu p/k)+1}\right)^{1/p}\Bigg[\Bigg(\Bigg(|f'(a)|\left(\frac{4}{2^s(s+1)}\right)^{1/q}\Bigg)\\
&+|f'(b)|\left(\frac{4m\left(-1+2^{s+1}\right)}{2^s(1+s)}\right)^{1/q}\Bigg)^q-\frac{2cm(b-a)^2}{3}\Bigg)^{1/q}+\Bigg(m\left|f'\left(\frac{a}{m^2}\right)\right|\left(\frac{4m\left(-1+2^{s+1}\right)}{2^s(1+s)}\right)^{1/q}\\
&+\left(\frac{4}{2^s(s+1)}\right)^{1/q}|f'(b)|\Bigg)^q-\frac{2cm\left(b-\left(a/m^2\right)\right)^2}{3}\Bigg)^{1/q}\Bigg].
\end{aligned}\tag{62}$$

Here we have used Lemma 3. This completes the proof. □

Remark 6. Under the assumption of Theorem 12, the following outcomes are noted.

(i) If $s=1$ and $m=1$ in (59), then the inequality stated in [[32], Theorem 13] is obtained.

(ii) If $s=1$ and ψ is the identity function in (53), then the inequality stated in [[31], Theorem 10] is obtained.

(iii) If $s=1, k=1, c=0$, and ψ is the identity function in (53), then the inequality stated in [[30], Theorem 2.7] is obtained.

(iv) If $c=0$, $s=1$, $m=1$, and ψ is the identity function in (59), then the inequality stated in [[23], Theorem 3.2] is obtained.

(v) If $k=1$, $s=1$, $m=1$, $c=0$, and ψ is the identity function in inequality (59), then the inequality stated in [[2], Theorem 6] is obtained.

(vi) If $k=1$, $s=1$, $m=1$, $c=0$, $\mu=1$, and ψ is the identity function in inequality (59), then the inequality stated in [[37], Theorem 2.4] is obtained.

Corollary 9. *Under the assumption of Theorem 12 with $c=0$ in (61), the following inequality holds:*

$$\begin{aligned}
&\left|\frac{2^{\mu/k-1}\Gamma_k(\mu+k)}{(mb-a)^{\mu/k}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)+m^{(\mu/k)+1}{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\\
&\leq\frac{mb-a}{16}\left(\frac{4}{(\mu p/k)+1}\right)^{1/p}\left[\left|f'(a)\right|\left(\frac{4}{2^s(s+1)}\right)^{1/q}+\left|f'(b)\right|\left(\frac{4m(-1+2^{s+1})}{2^s(1+s)}\right)^{1/q}\right.\\
&\left.+\left(\left|f'\left(\frac{a}{m^2}\right)\right|\left(\frac{4m(-1+2^{s+1})}{2^s(1+s)}\right)^{1/q}+\left(\frac{4}{2^s(s+1)}\right)^{1/q}\left|f'(b)\right|\right)\right].
\end{aligned}\tag{63}$$

Corollary 10. *Under the assumption of Theorem 12 with $q\longrightarrow 1$ and $p\longrightarrow\infty$ in (59), the following inequality holds:*

$$\begin{aligned}
&\left|\frac{2^{(\mu/k)-1}\Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}}\left[{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2)^+}(f\circ\psi)\left(\psi^{-1}(\mathrm{mb})\right)+m^{(\mu/k)+1}{}_kI^{\mu,\psi}_{\psi^{-1}((a+mb)/2m)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{\mathrm{a}}{\mathrm{m}}\right)\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\\
&\leq\frac{mb-a}{16}\left[\frac{4\left(\left|f'(a)\right|+\left|f'(b)\right|\right)}{2^s(s+1)}+\frac{4m\left(-1+2^{s+1}\right)\left(\left|f'(b)\right|+\left|f'\left(a/m^2\right)\right|\right)}{2^s(1+s)}-\frac{4cm(b-a)^2}{3}\right].
\end{aligned}\tag{64}$$

4. Conclusion

In this article, we studied the Hadamard inequalities and their estimations for generalized Riemann–Liouville fractional integrals of strongly (s,m)-convex functions. These inequalities represent the generalizations and refinements of a number of well-known inequalities stated in [1, 2, 22, 23, 29–37]. The error estimations of Hadamard inequalities for differentiable strongly (s,m)-convex functions are better as compared to those which are obtained for convex functions, strongly convex functions, and strongly m-convex functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

[2] M. Z. Sarikaya and H. Yildirim, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals," *Miskolc Mathematical Notes*, vol. 17, no. 2, pp. 1049–1059, 2017.

[3] F. Chen, "Extensions of the Hermite-Hadamard inequality for convex functions via fractional integrals," *Journal of Mathematical Inequalities*, vol. 10, no. 1, pp. 75–81, 2016.

[4] M. Iqbal, M. I. Bhatti, and K. Nazeer, "Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals," *Bulletin of the Korean Mathematical Society*, vol. 52, no. 3, pp. 707–716, 2015.

[5] M. Jleli, D. O'Regan, and B. Samet, "On Hermite-Hadamard type inequalities via generalized fractional integrals," *Turkish Journal of Mathematics*, vol. 40, no. 6, pp. 1221–1230, 2016.

[6] M. A. Noor, K. I. Noor, M. U. Awan, and S. Khan, "Fractional Hermite-Hadamard inequalities for some new classes of Godunova-Levin functions," *Applied Mathematics & Information Sciences*, vol. 8, no. 6, pp. 2865–2872, 2014.

[7] P. Agarwal, M. Jleli, and M. Tomar, "Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals," *Journal of Inequalities and Applications*, vol. 2017, no. 1, pp. 1–10, 2017.

[8] S. Mehmood and G. Farid, "Fractional integrals inequalities for exponentially m-convex functions," *Open Journal of Mathematical Analysis*, vol. 4, no. 1, pp. 78–85, 2020.

[9] T. U. Khan and M. A. Khan, "Hermite-Hadamard inequality for new generalized conformable fractional operators," *AIMS Mathematics*, vol. 6, no. 1, pp. 23–28, 2020.

[10] J. E. Pecaric, F. Proschan, and Y. L. Tong, "Convex functions, partial orderings, and statistical applications," *Mathematics in Science and Engineering*, vol. 187, 1992.

[11] Ch. Hermite, "Sur deux limites dune integrale definie," *Mathesis*, vol. 3, p. 82, 1883.
[12] J. Hadamard, "Etude sur les proprietes des fonctions entieres et en particulier dune fonction considree par Riemann," *Journal de Mathématiques Pures et Appliquées*, vol. 58, pp. 171–215, 1893.
[13] T. Zhang, "Statistical behavior and consistency of classification methods based on convex risk minimization," *Annals of Statistics*, vol. 32, no. 1, pp. 56–85, 2004.
[14] K. Murota and A. Tamura, "New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities," *Discrete Applied Mathematics*, vol. 131, no. 2, pp. 495–512, 2003.
[15] S. Hamann, "Minimality conditions for convex composite functions and efficiency conditions in vector optimization," *Applied Set-Valued Analysis and Optimization*, vol. 1, no. 3, pp. 221–229, 2019.
[16] W. Sun, "A unified approach to some inequalities for generalized convex functions on fractal sets and their applications," *Journal of Nonlinear Functional Analysis*, vol. 2018, p. 45, 2018.
[17] M. Bracamonte, J. Gimenez, and M. V. Cortez, "Hermite-Hadamard-Fejer type inequalities for strongly (s, m)-convex functions with modulus c, in second sense," *Applied Mathematics & Information Sciences*, vol. 10, no. 6, pp. 2045–2053, 2016.
[18] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives-Theory and Applications Gordon and Breach*, Linghorne, PA, USA, 1993.
[19] R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differentiable Equations of Fractional Order*, Springer, Wien, Austria, 1997.
[20] S. Mubeen and G. M. Habibullah, "k-fractional integrals and applications," *International Journal of Contemporary Mathematical Sciences*, vol. 7, pp. 89–94, 2012.
[21] S. Mubeen and A. Rehman, "A note on k-Gamma function and Pochhammer k-symbol," *Journal of Mathematical Sciences*, vol. 6, no. 2, pp. 93–107, 2014.
[22] G. Farid, A. U. Rehman, and M. Zahra, "On Hadamard-type inequalities for k-fractional integrals," *Nonlinear Functional Analysis and Applications*, vol. 21, no. 3, pp. 463–478, 2016.
[23] G. Farid, A. U. Rehman, and M. Zahra, "On Hadamard inequalities for k-fractional integrals," *Konuralp Journal of Mathematics*, vol. 4, no. 2, pp. 79–86, 2016.
[24] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, New York, NY, USA, 2006.
[25] A. Akkurt, M. E. Yildirim, and H. Yildirim, "On some integral inequalities for (k, h)-Riemann-Liouville fractional integral," *New Trends in Mathematical Science*, vol. 4, no. 2, p. 138, 2016.
[26] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, NY, USA, 1993.
[27] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, "Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities," *IEEE Access*, vol. 6, pp. 64946–64953, 2018.
[28] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics*, Springer Basel AG, Basel. Switzerland, 1988.
[29] G. Farid, A. U. Rehman, B. Tariq, and A. Waheed, "On Hadamard type inequalities for m-convex functions via fractional integrals," *Journal of Inequalities and Special Functions*, vol. 7, no. 4, pp. 150–167, 2016.
[30] G. Farid, A. U. Rehman, and B. Tariq, "On Hadamard-type inequalities for," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 62, no. 2, pp. 141–150, 2017.
[31] Y. C. Kwun, G. Farid, S. B. Akbar, and S. M. Kang, "Riemann-Liouville fractional versions of Hadamard inequality for strongly m-convex functions," 2020.
[32] G. Farid, H. Yasmeen, C. Y. Jung, S. H. Shim, and G. Ha, *Refinements and generalizations of some fractional integral inequalities via strongly convex functions, Mathematical Problems in Engineering*, In Press.
[33] K. Liu, J. Wang, and D. O'Regan, "On the Hermite-Hadamard type inequality for ψ-Riemann-Liouville fractional integrals via convex functions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–10, 2019.
[34] M. U. Awan, S. Talib, Y. M. Chu, M. A. Noor, and K. I. Noor, "Some new refinements of Hermite-Hadamard-type inequalities involving ψ_k-Riemann-Liouville fractional integrals and applications," *Journal of Mathematical Problems in Engineering*, vol. 2020, pp. 1–10, 2020.
[35] N. Merentes and K. Nikodem, "Remarks on strongly convex functions," *Aequationes Mathematicae*, vol. 80, pp. 193–199, 2010.
[36] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.
[37] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004.
[38] D. S. Mitrinović, *Analytic Inequalities*, Springer, New York, NY, USA, 1970.
[39] B. Belaidi, E. Farissi, and Z. Latreuch, "Inequalities between sum of the powers and the exponential of sum of nonnegative sequence," *Archivum Mathematicum*, vol. 47, pp. 257–262, 2012.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 5553126, 18 pages
https://doi.org/10.1155/2021/5553126

Research Article

Existence and Uniqueness of Weak Solutions for Novel Anisotropic Nonlinear Diffusion Equations Related to Image Analysis

Anas Tiarimti Alaoui and Mostafa Jourhmane

TIAD Laboratory, Department of Mathematics, Faculty of Sciences and Technology, Sultan Moulay Slimane University, Beni Mellal 23000, Morocco

Correspondence should be addressed to Anas Tiarimti Alaoui; a.tiarimti@usms.ma

Received 16 January 2021; Revised 16 February 2021; Accepted 17 February 2021; Published 8 March 2021

Academic Editor: Sun Young Cho

This paper establishes the existence and uniqueness of weak solutions for the initial-boundary value problem of anisotropic nonlinear diffusion partial differential equations related to image processing and analysis. An implicit iterative method combined with a variational approach has been applied to construct approximate solutions for this problem. Then, under some a priori estimates and a monotonicity condition, the existence of unique weak solutions for this problem has been proven. This work has been complemented by a consistent and stable approximation scheme showing its great significance as an image restoration technique.

1. Introduction

In the last three decades, nonlinear diffusion equations have inspired numerous research studies in various application ranges. Perona and Malik [1] were the first to introduce such equation in image processing and analysis in the following manner:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot [c(|\nabla u|)\nabla u] = 0, & \text{in } \Omega \times (0,T], \\ \langle c(|\nabla u|)\nabla u, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x;0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is an image domain in $\mathbb{R}^2$ and c is a positive decreasing function defined on $\mathbb{R}_+$.

When it comes to processing a digital image, Perona and Malik chose the above model to preserve meaningful features such as edges while reducing irrelevant information such as noise in the homogeneous area. Nevertheless, this model, known as an isotropic nonlinear diffusion equation, handles an image feature with the same amount of blurring in all its directions. For instance, this process cannot successfully eliminate noises at edges [2]. Accordingly, it might be wise to consider the orientation of essential features by using anisotropic diffusion. Weickert [2] introduced this property by defining an orientation descriptor using the structure tensor, which is convenient to identify features such as corners and T-junctions. Besides, digital images present some structural difficulties; that is, they are discrete in space and image intensity values. Accordingly, it would be of great interest to adapt the diffusion to digital images' structure by considering vertical, horizontal, and diagonal differential operators. Due to these reasons, we modeled and developed anisotropic nonlinear diffusion equations using a novel diffusion tensor.

Various tools can be used to examine the existence of solutions for nonlinear partial differential equations (PDEs), such as variational techniques, monotonicity method, fixed-point theorems, iterative methods, and truncation techniques; for more detailed information, we refer to [3–7] and the references therein. These PDEs have been motivated by various applications such as image restoration and reconstruction (see, for example, [3, 4, 8–11]). Moreover, the image processing of the brain allows the localization of epileptogenic foci for the patient. A noninvasive method has been examined numerically as an inverse problem in [12].

Under some challenging conditions, the existence and uniqueness of weak solutions for the Perona and Malik model have been investigated in the bounded variation space $BV(\Omega)$ [3, 13]. In some other functional frameworks, Wang and Zhou have thoroughly studied in [4] and proved the existence and uniqueness of weak solutions in the Orlicz space $L\log L(\Omega)$ using a new diffusion function $c(s) = ((s + (s+1)\log(s+1))/(s(s+1)))$ for all $s \geq 0$.

In this paper, we suppose that Ω is an open-bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, and T is a positive number. We denote

$$\begin{cases} \partial_{x_1} u := u_{x_1} := \nabla u \cdot \mathbf{e}_1, \\ \partial_{x_2} u := u_{x_2} := \nabla u \cdot \mathbf{e}_2, \\ \partial_{x_{12}} u := u_{x_{12}} := \nabla u \cdot \dfrac{\mathbf{e}_1 + \mathbf{e}_2}{|\mathbf{e}_1 + \mathbf{e}_2|}, \\ \partial_{x_{-12}} u := u_{x_{-12}} := \nabla u \cdot \dfrac{-\mathbf{e}_1 + \mathbf{e}_2}{|-\mathbf{e}_1 + \mathbf{e}_2|}, \end{cases} \tag{2}$$

where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$. We consider the following anisotropic nonlinear parabolic initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot [\mathbf{D}_{\nabla u} \nabla u] = 0, & \text{in } \Omega \times (0, T], \\ \langle \mathbf{D}_{\nabla u} \nabla u, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega \times (0, T], \\ u(x; 0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{3}$$

where $\mathbf{D}_{\nabla u}$, the diffusion tensor, is a real symmetric positive definite matrix of $\mathbb{R}^{2\times 2}$ defined as follows:

$$\mathbf{D}_{\nabla u} = \begin{pmatrix} g\left(\left|u_{x_1}\right|\right) + \dfrac{g\left(\left|u_{x_{12}}\right|\right) + g\left(\left|u_{x_{-12}}\right|\right)}{2} & \dfrac{g\left(\left|u_{x_{12}}\right|\right) - g\left(\left|u_{x_{-12}}\right|\right)}{2} \\ \dfrac{g\left(\left|u_{x_{12}}\right|\right) - g\left(\left|u_{x_{-12}}\right|\right)}{2} & g\left(\left|u_{x_2}\right|\right) + \dfrac{g\left(\left|u_{x_{12}}\right|\right) + g\left(\left|u_{x_{-12}}\right|\right)}{2} \end{pmatrix}, \tag{4}$$

and $g\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a C^1 positive decreasing function. Then, we can define $\phi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ as a C^2 function such that

$$\phi(s) = \int_0^s r g(r) \mathrm{d}r, \quad s \geq 0, \tag{5}$$

satisfying

$$\begin{cases} \phi(0) = \phi'(0) = 0,\ \phi(s) > 0, \phi'(s) > 0, & \text{for } s \in \mathbb{R}_+^*, \\ \phi''(s) \geq 0,\ s\phi''(s) \leq \phi'(s), & \text{for } s \in \mathbb{R}_+, \\ 0 < \lim_{s\longrightarrow\infty} \dfrac{\phi(s)}{s\log(s)} < \infty,\ 0 < \lim_{s\longrightarrow\infty} \dfrac{\phi'(s)}{\log(s)} < \infty, \\ \lim_{s\longrightarrow 0^+} \dfrac{\phi'(s)}{s} > 0,\ \lim_{s\longrightarrow\infty} \dfrac{\phi'(s)}{s} = 0. \end{cases} \tag{6}$$

To construct an adaptive diffusion tensor, the function g is approximated numerically by a cubic Hermite spline [14] that interpolates numeric data specified at $0 = k_0 < k_1 < \cdots < k_m$ with $m \in \mathbb{N}^*$:

$$g(s) = \begin{cases} p_{k_i} P_{1,k_i k_{i+1}}(s) + v_{k_i} P_{2,k_i k_{i+1}}(s) + p_{k_{i+1}} P_{1,k_{i+1} k_i}(s) + v_{k_{i+1}} P_{2,k_{i+1} k_i}(s), & s \in [k_i, k_{i+1}[\\ & i \in \{0, 1, \ldots, m-1\}, \\ p_{k_m} g_{k_m,1}(s) + v_{k_m} g_{k_m,2}(s), & s \in [k_m, \infty[\end{cases} \tag{7}$$

where $p_.$ and $v_.$ are the coefficients used to define the position and the velocity vector at a specific point, k_i are the threshold parameters, $\{P_{j,cd}\}$ is the family of the basis functions composed of polynomials of degree 3 used on the interval $[c, d[$ such that

$$\begin{cases} P_{1,cd}(s) = \dfrac{(s-d)^2 (2s + d - 3c)}{(d-c)^3}, \\ P_{2,cd}(s) = \dfrac{(s-d)^2 (s-c)}{(d-c)^2}. \end{cases} \tag{8}$$

And we may consider

$$\begin{cases} g_{k_m,1}(s) = \dfrac{k_m}{\log(k_m) + 2} \dfrac{2s(\log(s) + 1) - k_m \log(k_m)}{s^2}, \\ g_{k_m,2}(s) = \dfrac{k_m^2}{\log(k_m) + 2} \dfrac{s(\log(s) + 1) - k_m(\log(k_m) + 1)}{s^2}. \end{cases} \tag{9}$$

From the definition of ϕ, we can deduce

$$\phi(s)=\begin{cases} C_i+\sum_{j=0}^{3}\frac{A_{k_ik_{i+1},j}}{j+2}s^{j+2}, & s\in[k_i,k_{i+1}[,\, i\in\{0,1,\ldots,m-1\},\\ A_{k_m,2}s\log(s)+A_{k_m,1}\log(s)+C_m, & s\in[k_m,\infty[,\end{cases} \tag{10}$$

where C_i and C_m are constants determined by the continuity of ϕ at each k_i. In this case, the values of the coefficients $A_{k_ik_{i+1},j}$ are determined experimentally provided that ϕ satisfying the above conditions on $[0,k_m[$. Besides, we may introduce some sufficient conditions on k_m and $A_{k_{m,.}}$ that guarantee the properties of ϕ on $[k_m,\infty[$:

$$\begin{cases} k_m\geq 1,\\ A_{k_m,2}>0,\\ A_{k_m,1}<k_mA_{k_m,2},\\ A_{k_m,1}\geq -\frac{k_m\log(k_m)}{2}A_{k_m,2}. \end{cases} \tag{11}$$

Anisotropic diffusion model (3) allows strong directional smoothing within the areas where $|u_{x_1}|$, $|u_{x_2}|$, $|u_{x_{12}}|$, or $|u_{x_{-12}}|$ is small and prevents blurring boundaries, contours, or corners that separate neighboring areas, where one or a combination of these differential operators has significant value.

Moreover, the matrix $\mathbf{D}_{\nabla u}$ has two eigenvalues $\lambda_{+/-}$:

$$\begin{aligned}\lambda_{+/-}=\frac{1}{2}\Big(&g\left(\left|u_{x_1}\right|\right)+g\left(\left|u_{x_2}\right|\right)+g\left(\left|u_{x_{12}}\right|\right)+g\left(\left|u_{x_{-12}}\right|\right)\\ &\pm\sqrt{\left(g\left(\left|u_{x_1}\right|\right)-g\left(\left|u_{x_2}\right|\right)\right)^2+\left(g\left(\left|u_{x_{12}}\right|\right)-g\left(\left|u_{x_{-12}}\right|\right)\right)^2}\Big),\end{aligned} \tag{12}$$

with $\theta_{+/-}$ are the corresponding eigenvectors. We can then expand the first equation of (3) into

$$\frac{\partial u}{\partial t}=\nabla\cdot\left[\lambda_+\theta_+\theta_+^T\nabla u\right]+\nabla\cdot\left[\lambda_-\theta_-\theta_-^T\nabla u\right]. \tag{13}$$

Accordingly, it is clear from the expression of $\lambda_{+/-}$ that $\lambda_+\geq\lambda_->0$, which means that the diffusion towards θ_+ is privileged over θ_-. In fact, the difference $(\lambda_+-\lambda_-)^2=(g(|u_{x_1}|)-g(|u_{x_2}|))^2+(g(|u_{x_{12}}|)-g(|u_{x_{-12}}|))^2$ indicates the isotropic diffusion for zero value and anisotropic diffusion for positive values.

Henceforth, we will assume that the initial value satisfies

$$u_0\in L^2(\Omega), \tag{14}$$

and we will introduce the following Orlicz space:

$$L\log L^{k_m}(\Omega)=\left\{u\colon\Omega\longrightarrow\mathbb{R}\,\Big|\int_{\Omega\cap\{|u|\geq k_m\}}|u|\log(|u|)\mathrm{d}x<\infty\right\}. \tag{15}$$

Next, we define weak solutions for problem (3) on $Q_T\coloneqq\Omega\times(0,T]$ with $T>0$:

Definition 1. A function $u\colon\overline{Q}_T\longrightarrow\mathbb{R}$ is a weak solution for problem (3) if the following conditions are satisfied:

(i) $u\in C([0,T];L^2(\Omega))\cap L^1(0,T;W^{1,1}(\Omega))$ with $\partial_{x_i}u\in L\log L^{k_m}(\Omega)$ for $i=1,2$.

(ii) For any $\varphi\in C^1(\overline{Q}_T)$ with $\varphi(.,T)=0$, we have

$$-\int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x+\int_0^T\int_\Omega\left[-u\varphi_t+\mathbf{D}_{\nabla u}\nabla u\cdot\nabla\varphi\right]\mathrm{d}x\mathrm{d}t=0. \tag{16}$$

Now, we state our main theorem.

Theorem 1. *Under assumption (14), there exists a unique weak solution for initial-boundary value problem (3).*

Inspired by [4], this paper will investigate the existence and uniqueness of weak solutions for problem (3) according to the following steps:

(i) First, we approximate nonlinear evolution problem (3) by nonlinear elliptic problems using an implicit iterative method (discretization in time-variable only), and then we prove the existence of a unique weak solution for each elliptic problem adopting a variational approach. These solutions constitute approximate solutions for problem (3).

(ii) Next, we show the uniqueness of solutions for initial-boundary value problem (3) using the monotonicity of the vector field $\mathbf{D}_{\nabla}\nabla .: u \in \mathbb{R}^2 \longrightarrow \mathbf{D}_{\nabla u}\nabla u \in \mathbb{R}^2$.

(iii) Finally, passing to limits in some a priori energy estimates and using the monotonicity condition (17), we demonstrate the existence of weak solutions for problem (3).

2. Preliminaries

In this section, we state some useful lemmas that will be used later in the proofs.

Lemma 1. *For all $a \geq 0$ and $b \geq 1$, we have $ab \leq a \exp(a) + b \log(b)$.*

Proof. If $b \leq \exp(a)$, then $ab \leq a \exp(a) \leq a \exp(a) + b \log(b)$.

If $\exp(a) < b$, then $a < \log(b)$, which means $ab < b \log(b) < a \exp(a) + b \log(b)$. □

Lemma 2. *Suppose $\phi: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a C^2 convex function. Then, for all $\xi_0, \xi_1 \in \mathbb{R}^2$, we have*

$$\left(\mathbf{D}_{\xi_1}\xi_1 - \mathbf{D}_{\xi_0}\xi_0\right) \cdot (\xi_1 - \xi_0) \geq 0. \tag{17}$$

Proof. For each $t \in [0,1]$, we put $\xi_t = (1-t)\xi_0 + t\xi_1$. Then, we have

$$\begin{aligned}\mathbf{D}_{\xi_1}\xi_1 - \mathbf{D}_{\xi_0}\xi_0 = {} & \int_0^1 d\left[(\xi_t \cdot \mathbf{e}_1) g\left(\left|\xi_t \cdot \mathbf{e}_1\right|\right)\mathbf{e}_1\right] + \int_0^1 d\left[(\xi_t \cdot \mathbf{e}_2) g\left(\left|\xi_t \cdot \mathbf{e}_2\right|\right)\mathbf{e}_2\right] \\ & + \int_0^1 d\left[(\xi_t \cdot \mathbf{e}_{12}) g\left(\left|\xi_t \cdot \mathbf{e}_{12}\right|\right)\mathbf{e}_{12}\right] + \int_0^1 d\left[(\xi_t \cdot \mathbf{e}_{-12}) g\left(\left|\xi_t \cdot \mathbf{e}_{-12}\right|\right)\mathbf{e}_{-12}\right].\end{aligned} \tag{18}$$

Since $\phi''(s) = g(s) + sg'(s)$, then we obtain

$$\begin{aligned}\mathbf{D}_{\xi_1}\xi_1 - \mathbf{D}_{\xi_0}\xi_0 = {} & \int_0^1 \left[\phi''\left(\left|\xi_t \cdot \mathbf{e}_1\right|\right)((\xi_1 - \xi_0) \cdot \mathbf{e}_1)\mathbf{e}_1\right]\mathrm{d}t + \int_0^1 \left[\phi''\left(\left|\xi_t \cdot \mathbf{e}_2\right|\right)((\xi_1 - \xi_0) \cdot \mathbf{e}_2)\mathbf{e}_2\right]\mathrm{d}t \\ & + \int_0^1 \left[\phi''\left(\left|\xi_t \cdot \mathbf{e}_{12}\right|\right)((\xi_1 - \xi_0) \cdot \mathbf{e}_{12})\mathbf{e}_{12}\right]\mathrm{d}t + \int_0^1 \left[\phi''\left(\left|\xi_t \cdot \mathbf{e}_{-12}\right|\right)((\xi_1 - \xi_0) \cdot \mathbf{e}_{-12})\,\mathbf{e}_{-12}\right]\mathrm{d}t.\end{aligned} \tag{19}$$

We conclude then

$$\begin{aligned}\left(\mathbf{D}_{\xi_1}\xi_1 - \mathbf{D}_{\xi_0}\xi_0\right) \cdot (\xi_1 - \xi_0) = {} & \int_0^1 \left[\phi''\left(\left|\xi_t \cdot \mathbf{e}_1\right|\right)((\xi_1 - \xi_0) \cdot \mathbf{e}_1)^2\right]\mathrm{d}t + \int_0^1 \left[\phi''\left(\left|\xi_t \cdot \mathbf{e}_2\right|\right)((\xi_1 - \xi_0) \cdot \mathbf{e}_2)^2\right]\mathrm{d}t \\ & + \int_0^1 \left[\phi''\left(\left|\xi_t \cdot \mathbf{e}_{12}\right|\right)((\xi_1 - \xi_0) \cdot \mathbf{e}_{12})^2\right]\mathrm{d}t \\ & + \int_0^1 \left[\phi''\left(\left|\xi_t \cdot \mathbf{e}_{-12}\right|\right)((\xi_1 - \xi_0) \cdot \mathbf{e}_{-12})^2\right]\mathrm{d}t \geq 0,\end{aligned} \tag{20}$$

which completes the proof. □

Lemma 3. *Uniform integrability and weak convergence [15].*

Assume $\Omega \subset \mathbb{R}^2$ is bounded, and let $\{u_i\}_{i=1}^{\infty}$ be a sequence of functions in $L^1(\Omega)$ satisfying

$$\sup_i \left\|u_i\right\|_{L^1(\Omega)} < \infty. \tag{21}$$

Suppose also

$$\lim_{l \longrightarrow \infty} \sup_i \int_{\Omega \cap \{|u_i| \geq l\}} \left|u_i\right| \mathrm{d}x = 0. \tag{22}$$

Then, there exist a subsequence $\left\{u_{i_j}\right\}_{j=1}^{\infty}$ and $\tilde{u} \in L^1(\Omega)$ such that

$$u_{i_j} \rightharpoonup \tilde{u}, \quad \text{weakly in } L^1(\Omega). \tag{23}$$

Lemma 4. *Assume $\Omega \subset \mathbb{R}^2$ is bounded, and let $\{u_i\}_{i=1}^{\infty}$ be a sequence of functions in $L^1(\Omega)$ such that*

$$\sup_i \int_{\Omega\cap\{|u_i|\geq k_m\}} |u_i|\log(|u_i|)\mathrm{d}x < \infty. \tag{24}$$

Then, there exist a subsequence $\{u_{i_j}\}_{j=1}^{\infty}$ and $\tilde{u} \in L^1(\Omega)$ such that

$$u_{i_j} \rightharpoonup \tilde{u}, \quad \text{weakly in } L^1(\Omega), \tag{25}$$

with $\tilde{u} \in L\log L^{k_m}(\Omega)$.

Proof. Given $M > 0$, we may find an $l \geq k_m$ such that $Ms \leq s\log(s)$ for all $s \geq l$. Consequently,

$$\begin{aligned}\int_\Omega |u_i|\mathrm{d}x &= \int_{\Omega\cap\{|u_i|<k_m\}} |u_i|\mathrm{d}x + \int_{\Omega\cap\{|u_i|\geq k_m\}} |u_i|\mathrm{d}x \\ &\leq k_m|\Omega| + \frac{1}{M}\int_{\Omega\cap\{|u_i|\geq k_m\}} |u_i|\log(|u_i|)\mathrm{d}x,\end{aligned} \tag{26}$$

which implies that

$$\sup_i \int_\Omega |u_i|\mathrm{d}x < \infty. \tag{27}$$

On the other hand, there exists a positive constant C such that

$$\begin{aligned}\int_{\Omega\cap\{|u_i|\geq l\}} |u_i|\mathrm{d}x &\leq \frac{1}{M}\int_{\Omega\cap\{|u_i|\geq l\}} |u_i|\log(|u_i|)\mathrm{d}x \\ &\leq \frac{1}{M}\int_{\Omega\cap\{|u_i|\geq k_m\}} |u_i|\log(|u_i|)\mathrm{d}x \\ &\leq \frac{C}{M} = \varepsilon,\end{aligned} \tag{28}$$

which is true for all i and arbitrary $\varepsilon > 0$. It follows then that

$$\lim_{l\longrightarrow\infty}\sup_i \int_{\Omega\cap\{|u_i|\geq l\}} |u_i|\mathrm{d}x = 0. \tag{29}$$

Then, from Lemma 3, there exist a subsequence $\{u_{i_j}\}_{j=1}^{\infty}$ of $\{u_i\}_{i=1}^{\infty}$ and a function $\tilde{u} \in L^1(\Omega)$ such that

$$u_{i_j} \rightharpoonup \tilde{u}, \quad \text{weakly in } L^1(\Omega). \tag{30}$$

It remains to prove that $\tilde{u} \in L\log L^{k_m}(\Omega)$.

We know that the function $f(s) = s\log(s)$ for $s \geq 1$ is increasing and convex, and then the function $f(|s|)$ is also convex for all $s \geq 1$. Therefore, we obtain

$$f(|\tilde{u}|) \leq f\left(\left|u_{i_j}\right|\right) + f'(|\tilde{u}|)\left(\tilde{u} - u_{i_j}\right). \tag{31}$$

Integrating the above inequality over $\Omega_N \cap \left\{|u_{i_j}| \geq k_m\right\}$ with $\Omega_N = \Omega\cap\{k_m \leq |\tilde{u}| \leq N\}$, we have

$$\begin{aligned}\int_{\Omega_N} f(|\tilde{u}|)\mathrm{d}x &\leq \int_{\Omega\cap\left\{\left|u_{i_j}\right|\geq k_m\right\}} f\left(\left|u_{i_j}\right|\right)\mathrm{d}x + \int_{\Omega_N\cap\left\{\left|u_{i_j}\right|\geq k_m\right\}} f'(|\tilde{u}|)\left(\tilde{u} - u_{i_j}\right)\mathrm{d}x \\ &\leq \int_{\Omega\cap\left\{\left|u_{i_j}\right|\geq k_m\right\}} f\left(\left|u_{i_j}\right|\right)\mathrm{d}x + \int_{\Omega\cap\left\{\left|u_{i_j}\right|\geq k_m\right\}} f'(|\tilde{u}|)\chi_{\{k_m\leq|\tilde{u}|\leq N\}}\left(\tilde{u} - u_{i_j}\right)\mathrm{d}x.\end{aligned} \tag{32}$$

Since $f'(|\tilde{u}|)\chi_{\{k_m\leq|\tilde{u}|\leq N\}} \in L^\infty(\Omega)$ and passing to limits as $j \longrightarrow \infty$, we get

$$\int_{\Omega_N} f(|\tilde{u}|)\mathrm{d}x \leq \liminf_{j\longrightarrow\infty} \int_{\Omega\cap\left\{\left|u_{i_j}\right|\geq k_m\right\}} f\left(\left|u_{i_j}\right|\right)\mathrm{d}x < \infty. \tag{33}$$

Then, passing to limits as $N \longrightarrow \infty$, we deduce

$$\int_{\Omega\cap\{|\tilde{u}|\geq k_m\}} |\tilde{u}|\log(|\tilde{u}|)\mathrm{d}x < \infty. \tag{34}$$

It follows then $\tilde{u} \in L\log L^{k_m}(\Omega)$. This finishes the proof. □

3. Approximate Solutions

In this section, we will discretize the time-variable interval $[0, T]$ to get approximate solutions for problem (3). We denote $h = (T/N)$ with $N \in \mathbb{N}^*$, and we designate by u_n an approximate solution at time nh. We define gradually from $n = 1, 2, \ldots, N$ the following elliptic problems:

$$\begin{cases}\dfrac{u_n - u_{n-1}}{h} - \nabla\cdot\left[\mathbf{D}_{\nabla u_n}\nabla u_n\right] = 0, & \text{in } \Omega, \\ \langle \mathbf{D}_{\nabla u_n}\nabla u_n, \mathbf{n}\rangle = 0, & \text{on } \partial\Omega.\end{cases} \tag{35}$$

To solve these equations step by step, we only need to prove the existence and uniqueness of weak solutions of the following elliptic problems:

$$\begin{cases}\dfrac{u - u_0}{h} - \nabla\cdot\left[\mathbf{D}_{\nabla u}\nabla u\right] = 0, & \text{in } \Omega, \\ \langle \mathbf{D}_{\nabla u}\nabla u, \mathbf{n}\rangle = 0, & \text{on } \partial\Omega,\end{cases} \tag{36}$$

where $h > 0$ and $u_0 \in L^2(\Omega)$.

Definition 2. A function $u \in L^2(\Omega)\cap W^{1,1}(\Omega)$ with $\partial_{x_i}u \in L\log L^{k_m}(\Omega)$ for $i = 1, 2$ is called a weak solution for problem (36); if for any $\varphi \in C^1(\overline{\Omega})$, we have

$$\int_\Omega \frac{u - u_0}{h}\varphi\mathrm{d}x + \int_\Omega \mathbf{D}_{\nabla u}\nabla u\cdot\nabla\varphi\mathrm{d}x = 0. \tag{37}$$

And when φ is a constant function, we obtain

$$\int_\Omega u\mathrm{d}x = \int_\Omega u_0\mathrm{d}x. \tag{38}$$

In order to prove the existence and uniqueness of weak solutions for problem (36), we consider the variational problem

$$\min\{E(u)|u \in U\}, \tag{39}$$

where

$$U = \left\{u \in L^2(\Omega)\cap W^{1,1}(\Omega)|\partial_{x_i}u \in L\log L^{k_m}(\Omega)\text{ with } i = 1,2, \int_\Omega u\mathrm{d}x = \int_\Omega u_0\mathrm{d}x\right\}, \tag{40}$$

and when $u \in U$, the functional E is defined as

$$E(u) = \int_\Omega\left[\phi\left(\left|u_{x_1}\right|\right) + \phi\left(\left|u_{x_2}\right|\right) + \phi\left(\left|u_{x_{12}}\right|\right) + \phi\left(\left|u_{x_{-12}}\right|\right)\right] + \frac{1}{2h}\int_\Omega (u-u_0)^2\mathrm{d}x. \tag{41}$$

It is easy to prove that (36) is the Euler–Lagrange equations of the functional E [16].

Theorem 2. *Problem (36) has a unique weak solution.*

Proof. Since

$$0 \le \inf_{u\in U} E(u) \le E(0) = \frac{1}{2h}\int_\Omega u_0^2\mathrm{d}x, \tag{42}$$

then we can construct a minimizing sequence $\{u_q\}_{q=1}^\infty$ in U such that $E(u_q) < E(0) + 1$ and

$$\lim_{q\longrightarrow\infty} E\left(u_q\right) = \inf_{u\in U} E(u). \tag{43}$$

Besides,

$$\begin{aligned}\int_\Omega u_q^2\mathrm{d}x &= \int_\Omega \left(u_q - u_0 + u_0\right)^2\mathrm{d}x\\ &\le 2\int_\Omega\left(u_q-u_0\right)^2 + 2\int_\Omega u_0^2\mathrm{d}x\\ &\le 4h\left(E\left(u_q\right) + E(0)\right)\\ &\le 4h(2E(0)+1).\end{aligned} \tag{44}$$

It follows then

$$\sup_q\left\|u_q\right\|_{L^2(\Omega)} < \infty. \tag{45}$$

On the other hand, given $\varepsilon_0 > 0$, we may find $l_0 = k_m$ such that

$$\begin{aligned}\int_{\Omega\cap\left\{\left|\partial_{x_i}u_q\right|\ge k_m\right\}}\left|\partial_{x_i}u_q\right|\log\left(\left|\partial_{x_i}u_q\right|\right)\mathrm{d}x &\le C\int_{\Omega\cap\left\{\left|\partial_{x_i}u_q\right|\ge k_m\right\}}\phi\left(\left|\partial_{x_i}u_q\right|\right)\mathrm{d}x\\ &\le CE\left(u_q\right) < C(E(0)+1)\end{aligned} \tag{46}$$

with $C = (\varepsilon_0 + (1/A_{k_m,2})) > 0$ and $i = 1,2$. It follows then that for $i = 1,2$,

$$\sup_q\int_{\Omega\cap\left\{\left|\partial_{x_i}u_q\right|\ge k_m\right\}}\left|\partial_{x_i}u_q\right|\log\left(\left|\partial_{x_i}u_q\right|\right)\mathrm{d}x < \infty. \tag{47}$$

Therefore, thanks to Lemma 4 and the weak compactness of $L^2(\Omega)$, we can find a subsequence $\left\{u_{q_j}\right\}_{j=1}^\infty$ of $\{u_q\}_{q=1}^\infty$ and a function $u_1 \in L^2(\Omega)\cap W^{1,1}(\Omega)$ such that

$$u_{q_j}\rightharpoonup u_1, \quad \text{weakly in } L^2(\Omega), \tag{48}$$

and for $i = 1,2$,

$$\begin{aligned}&\partial_{x_i}u_{q_j}\rightharpoonup\partial_{x_i}u_1, \quad \text{weakly in } L^1(\Omega),\\ &\partial_{x_i}u_1 \in L\log L^{k_m}(\Omega).\end{aligned} \tag{49}$$

Therefore, we have

$$\begin{aligned}&\int_\Omega u_1\mathrm{d}x = \lim_{j\longrightarrow\infty}\int_\Omega u_{q_j}\mathrm{d}x = \int_\Omega u_0\mathrm{d}x,\\ &\int_\Omega (u_1-u_0)^2\mathrm{d}x \le \liminf_{j\longrightarrow\infty}\int_\Omega\left(u_{q_j}-u_0\right)^2\mathrm{d}x,\end{aligned} \tag{50}$$

and following the reasoning in the proof of Lemma 4, it is easy to show that for any $a \in \{x_1, x_2, x_{12}, x_{-12}\}$ and for a fixed $\epsilon > 0$, there exists $l \ge k_m$ such that

$$\int_{\Omega\cap\{|\partial_a u_1|\geq l\}}|\partial_a u_1|\log(|\partial_a u_1|)\mathrm{d}x\leq \liminf_{j\longrightarrow\infty}\int_{\Omega\cap\left\{\left|\partial_a u_{q_j}\right|\geq l\right\}}\left|\partial_a u_{q_j}\right|\log\left(\left|\partial_a u_{q_j}\right|\right)\mathrm{d}x, \tag{51}$$

$$\int_{\Omega\cap\{|\partial_a u_1|\geq l\}}\phi(|\partial_a u_1|)\mathrm{d}x\leq\left(\varepsilon+A_{k_m,2}\right)\left(\varepsilon+\frac{1}{A_{k_m,2}}\right)\liminf_{j\longrightarrow\infty}\int_{\Omega\cap\left\{\left|\partial_a u_{q_j}\right|\geq l\right\}}\phi\left(\left|\partial_a u_{q_j}\right|\right)\mathrm{d}x. \tag{52}$$

Similarly, since ϕ is increasing and convex in $[0,l]$, then we can prove that

$$\int_{\Omega\cap\{|\partial_a u_1|< l\}}\phi(|\partial_a u_1|)\mathrm{d}x\leq \liminf_{j\longrightarrow\infty}\int_{\Omega\cap\left\{\left|\partial_a u_{q_j}\right|< l\right\}}\phi\left(\left|\partial_a u_{q_j}\right|\right)\mathrm{d}x. \tag{53}$$

Therefore, we obtain from (52) and (53) that

$$\begin{aligned}\int_{\Omega}\phi(|\partial_a u_1|)\mathrm{d}x &= \int_{\Omega\cap\{|\partial_a u_1|< l\}}\phi(|\partial_a u_1|)\mathrm{d}x+\int_{\Omega\cap\{|\partial_a u_1|\geq l\}}\phi(|\partial_a u_1|)\mathrm{d}x\\ &\leq\left(\varepsilon+A_{k_m,2}\right)\left(\varepsilon+\frac{1}{A_{k_m,2}}\right)\liminf_{j\longrightarrow\infty}\int_{\Omega}\phi\left(\left|\partial_a u_{q_j}\right|\right)\mathrm{d}x.\end{aligned} \tag{54}$$

Thus, by letting $\varepsilon\longrightarrow 0$, we get

$$\int_{\Omega}\phi(|\partial_a u_1|)\mathrm{d}x\leq \liminf_{j\longrightarrow\infty}\int_{\Omega}\phi\left(\left|\partial_a u_{q_j}\right|\right)\mathrm{d}x, \tag{55}$$

for any $a\in\{x_1,x_2,x_{12},x_{-12}\}$. It follows then that

$$E(u_1)\leq \liminf_{j\longrightarrow\infty}E\left(u_{q_j}\right)=\inf_{u\in U}E(u), \tag{56}$$

which signifies that $u_1\in U$ is a minimizer of the energy functional $E(u)$, i.e.,

$$E(u_1)=\min_{u\in U}E(u). \tag{57}$$

Furthermore, for all $\varphi\in C^1(\overline{\Omega})$ and $t\in\mathbb{R}$, we have $u_1+t(\varphi-\varphi_\Omega)\in U$ with $\varphi_\Omega=(1/|\Omega|)\int_\Omega\varphi\mathrm{d}x$. Then, $\rho(0)\leq\rho(t)$ where

$$\rho(t)=E(u_1+t(\varphi-\varphi_\Omega)). \tag{58}$$

Hence, we have $\rho'(0)=0$, which means

$$\int_\Omega\frac{u_1-u_0}{h}(\varphi-\varphi_\Omega)\mathrm{d}x+\int_\Omega\mathbf{D}_{\nabla u_1}\nabla u_1\cdot\nabla\varphi\mathrm{d}x=0. \tag{59}$$

Because of (50), we get

$$\int_\Omega\frac{u_1-u_0}{h}\varphi\mathrm{d}x+\int_\Omega\mathbf{D}_{\nabla u_1}\nabla u_1\cdot\nabla\varphi\mathrm{d}x=0. \tag{60}$$

We conclude then that u_1 is a weak solution for problem (36).

Now, assume that there is another weak solution $\widehat{u}$ of (36). Then, for every $\varphi\in C^1(\overline{\Omega})$, we have

$$\int_\Omega\frac{\widehat{u}-u_0}{h}\varphi\mathrm{d}x+\int_\Omega\mathbf{D}_{\nabla\widehat{u}}\nabla\widehat{u}\cdot\nabla\varphi\mathrm{d}x=0, \tag{61}$$

which leads to

$$\int_\Omega\frac{\widehat{u}-u_1}{h}\varphi\mathrm{d}x+\int_\Omega\left[\mathbf{D}_{\nabla\widehat{u}}\nabla\widehat{u}-\mathbf{D}_{\nabla u_1}\nabla u_1\right]\cdot\nabla\varphi\mathrm{d}x=0. \tag{62}$$

Then, if we choose $\varphi=\widehat{u}-u_1$ as a test function in (62), we get

$$\int_\Omega\frac{(\widehat{u}-u_1)^2}{h}\mathrm{d}x+\int_\Omega\left[\mathbf{D}_{\nabla\widehat{u}}\nabla\widehat{u}-\mathbf{D}_{\nabla u_1}\nabla u_1\right]\cdot(\nabla\widehat{u}-\nabla u_1)\mathrm{d}x=0. \tag{63}$$

Thanks to Lemma 2, we deduce that

$$\int_\Omega\frac{(\widehat{u}-u_1)^2}{h}\mathrm{d}x=0. \tag{64}$$

Therefore, $\hat{u} = u_1$ a.e. in Ω.

In conclusion, we have shown that there exists a unique weak solution $u_n \in U$ satisfying (35) for every $n \in \{1, 2, \ldots, N\}$. Consequently, we define an approximate solution u_h for problem (3) as

$$u_h(x,t) = \begin{cases} u_0(x), & t = 0, \\ u_1(x), & t \in (0, h], \\ \ldots\ldots\ldots, & \ldots\ldots, \\ u_j(x), & t \in ((j-1)h, jh], \\ \ldots\ldots\ldots, & \ldots\ldots, \\ u_N(x), & t \in ((N-1)h, T], \end{cases} \tag{65}$$

for every $h = (T/N)$. □

4. Existence and Uniqueness of Weak Solutions

Proof. of Theorem 1. In the beginning, we establish the uniqueness of solutions for problem (3). For this purpose, we suppose there exist two weak solutions u and v for problem (3). Then, we obtain the following:

$$\begin{cases} \dfrac{\partial(u-v)}{\partial t} - \nabla \cdot [\mathbf{D}_{\nabla u}\nabla u - \mathbf{D}_{\nabla v}\nabla v] = 0, & \text{in } Q_T, \\ \langle \mathbf{D}_{\nabla u}\nabla u - \mathbf{D}_{\nabla v}\nabla v, \mathbf{n}\rangle = 0, & \text{on } \partial\Omega \times (0,T], \\ (u-v)(x;0) = 0, & \text{in } \Omega. \end{cases} \tag{66}$$

By multiplying the first equation of the above problem by $(u-v)$ and integrating over Ω and $[0,t]$, we get

$$\frac{1}{2}\int_\Omega (u-v)^2(t)\mathrm{d}x + \int_0^t \int_\Omega [\mathbf{D}_{\nabla u}\nabla u - \mathbf{D}_{\nabla v}\nabla v] \cdot \nabla(u-v)\mathrm{d}x\mathrm{d}\tau = 0, \tag{67}$$

for every $t \in (0,T]$. Since the second term of the above equation is nonnegative (thanks to Lemma 2), it follows then $u = v$ a.e. in Q_T.

Let us now find our weak solution for problem (3). We intend to send h to zero and show that a subsequence of our solutions u_h of the approximate problems (35) converges to a weak solution for problem (3). To this end, we need to find some a priori estimates.

It follows from (35) that for every $\varphi \in C^1(\overline{\Omega})$,

$$\int_\Omega \frac{u_n - u_{n-1}}{h}\varphi \mathrm{d}x + \int_\Omega \mathbf{D}_{\nabla u_n}\nabla u_n \cdot \nabla\varphi \mathrm{d}x = 0. \tag{68}$$

Then, by taking u_n as a test function in (68) and using $u_n u_{n-1} \le ((u_n^2 + u_{n-1}^2)/2)$, we get

$$\frac{1}{2}\int_\Omega u_n^2 \mathrm{d}x + h\int_\Omega \mathbf{D}_{\nabla u_n}\nabla u_n \cdot \nabla u_n \mathrm{d}x \le \frac{1}{2}\int_\Omega u_{n-1}^2 \mathrm{d}x. \tag{69}$$

For each $t \in (0,T]$, we can find $j \in \{1, \ldots, N\}$ such that $t \in ((j-1)h, jh]$. Then, by adding all the inequalities (69) from $n = 1$ to $n = j$, we get

$$\frac{1}{2}\int_\Omega u_j^2 \mathrm{d}x + h\sum_{n=1}^{j}\int_\Omega \mathbf{D}_{\nabla u_n}\nabla u_n \cdot \nabla u_n \mathrm{d}x \le \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x. \tag{70}$$

Then, by definition of u_h, we obtain for $t \in ((j-1)h, jh]$ that

$$\frac{1}{2}\int_\Omega u_h^2(x,t)\mathrm{d}x + \int_0^{jh}\int_\Omega \mathbf{D}_{\nabla u_h}\nabla u_h \cdot \nabla u_h \mathrm{d}x\mathrm{d}\tau \le \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x. \tag{71}$$

Since $\mathbf{D}_{\nabla u_h}$ is a symmetric positive definite matrix, we have also

$$\frac{1}{2}\int_\Omega u_h^2(x,t)\mathrm{d}x + \int_0^{t}\int_\Omega \mathbf{D}_{\nabla uh}\nabla u_h \cdot \nabla u_h \mathrm{d}x\mathrm{d}\tau \le \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x. \tag{72}$$

Therefore, after taking the supremum over $[0,T]$, we deduce that

$$\sup_{0\le t\le T}\int_\Omega u_h^2(x,t)\mathrm{d}x + \int_0^{T}\int_\Omega \mathbf{D}_{\nabla u_h}\nabla u_h \cdot \nabla u_h \mathrm{d}x\mathrm{d}\tau \le 2\int_\Omega u_0^2 \mathrm{d}x. \tag{73}$$

Recalling that $0 \le \phi(s) \le s\phi'(s)$ for all $s \ge 0$, then we can derive the following:

$$\begin{aligned} \mathbf{D}_{\nabla u_h}\nabla u_h \cdot \nabla u_h &= |\nabla u_h \cdot \mathbf{e}_1|\phi'(|\nabla u_h \cdot \mathbf{e}_1|) + |\nabla u_h \cdot \mathbf{e}_2|\phi'(|\nabla u_h \cdot \mathbf{e}_2|) \\ &\quad + |\nabla u_h \cdot \mathbf{e}_{12}|\phi'(|\nabla u_h \cdot \mathbf{e}_{12}|) + |\nabla u_h \cdot \mathbf{e}_{-12}|\phi'(|\nabla u_h \cdot \mathbf{e}_{-12}|) \\ &\ge \phi(|\nabla u_h \cdot \mathbf{e}_1|) + \phi(|\nabla u_h \cdot \mathbf{e}_2|) + \phi(|\nabla u_h \cdot \mathbf{e}_{12}|) + \phi(|\nabla u_h \cdot \mathbf{e}_{-12}|). \end{aligned} \tag{74}$$

Besides, as in (46), for $|\partial_{x_1} u_h|, |\partial_{x_2} u_h| \geq k_m$, we may find a positive constant C such that

$$\begin{aligned}\left|\partial_{x_1} u_h\right|\log\left(\left|\partial_{x_1} u_h\right|\right) + \left|\partial_{x_2} u_h\right|\log\left(\left|\partial_{x_2} u_h\right|\right) &\leq C\left(\phi\left(\left|\nabla u_h \cdot \mathbf{e}_1\right|\right) + \phi\left(\left|\nabla u_h \cdot \mathbf{e}_2\right|\right)\right)\\ &\leq C\mathbf{D}_{\nabla u_h}\nabla u_h \cdot \nabla u_h.\end{aligned} \tag{75}$$

Thus, we conclude

$$\begin{cases}\sup_{0\leq t\leq T}\int_\Omega u_h^2(x,t)\mathrm{d}x < \infty,\\ \int_0^T\int_{\Omega\cap\left\{\left|\partial_{x_1}u_h\right|\geq k_m\right\}}\left|\partial_{x_1} u_h\right|\log\left(\left|\partial_{x_1} u_h\right|\right)\mathrm{d}x\mathrm{d}\tau < \infty,\\ \int_0^T\int_{\Omega\cap\left\{\left|\partial_{x_2}u_h\right|\geq k_m\right\}}\left|\partial_{x_2} u_h\right|\log\left(\left|\partial_{x_2} u_h\right|\right)\mathrm{d}x\mathrm{d}\tau < \infty.\end{cases} \tag{76}$$

By Lemma 4, we can find a subsequence of $\{u_h\}$ (for simplicity, we also denote it by u_h) such that [17]

$$\begin{aligned}u_h \rightharpoonup u, &\quad \text{weakly}_- * \text{in } L^\infty\left(0,T;L^2(\Omega)\right),\\ u_h \rightharpoonup u, &\quad \text{weakly in } L^1\left(0,T;W^{1,1}(\Omega)\right),\end{aligned} \tag{77}$$

with

$$\begin{cases}\sup_{0\leq t\leq T}\int_\Omega u^2(x,t)\mathrm{d}x < \infty,\\ \int_0^T\int_{\Omega\cap\left\{\left|\partial_{x_1}u\right|\geq k_m\right\}}\left|\partial_{x_1} u\right|\log\left(\left|\partial_{x_1} u\right|\right)\mathrm{d}x\mathrm{d}\tau < \infty,\\ \int_0^T\int_{\Omega\cap\left\{\left|\partial_{x_2}u\right|\geq k_m\right\}}\left|\partial_{x_2} u\right|\log\left(\left|\partial_{x_2} u\right|\right)\mathrm{d}x\mathrm{d}\tau < \infty.\end{cases} \tag{78}$$

So, it remains to prove that u is just a weak solution for problem (3). Let us now denote $\xi_h = \mathbf{D}_{\nabla u_h}\nabla u_h$. We will show that ξ_h is bounded in $[L^2(Q_T)]^2$, so we may find a subsequence of ξ_h that converges weakly in $[L^2(Q_T)]^2$ to a particular vector-valued function. Then, we will prove that this vector-valued function is equal almost everywhere to $\mathbf{D}_{\nabla u}\nabla u$ in Q_T through monotonicity condition (17).

From the expression of $\mathbf{D}_{\nabla u_h}$, we can derive the following:

$$\begin{aligned}\left|\xi_h\right| &= \left|\frac{\partial_{x_1}u_h}{\left|\partial_{x_1}u_h\right|}\phi'\left(\left|\partial_{x_1}u_h\right|\right)\mathbf{e}_1 + \frac{\partial_{x_2}u_h}{\left|\partial_{x_2}u_h\right|}\phi'\left(\left|\partial_{x_2}u_h\right|\right)\mathbf{e}_2\right.\\ &\quad\left.+\frac{\partial_{x_{12}}u_h}{\left|\partial_{x_{12}}u_h\right|}\phi'\left(\left|\partial_{x_{12}}u_h\right|\right)\mathbf{e}_{12} + \frac{\partial_{x_{-12}}u_h}{\left|\partial_{x_{-12}}u_h\right|}\phi'\left(\left|\partial_{x_{-12}}u_h\right|\right)\mathbf{e}_{-12}\right|\\ &\leq 4\phi'\left(\left|\partial_{x_1}u_h\right| + \left|\partial_{x_2}u_h\right|\right).\end{aligned} \tag{79}$$

Given $\varepsilon_1, \varepsilon_2 > 0$, we may find $l_1 = l_2 = k_m$ such that

$$\begin{aligned}\phi'(s) &\leq M\ \log(s),\\ s &\leq \varepsilon_2 s\ \log(s),\end{aligned} \tag{80}$$

for all $s \geq k_m$ with $M = (\varepsilon_1 + A_{k_m,2})$. Thus, we can distinguish two cases:

(i) If $|\partial_{x_1}u_h| + |\partial_{x_2}u_h| < k_m$ then $|\xi_h|^2 \leq (4\phi'(k_m))^2$.

(ii) If $|\partial_{x_1}u_h| + |\partial_{x_2}u_h| \geq k_m$ then

$$\begin{aligned}\left|\xi_h\right| &\leq 4M\ \log\left(\left|\partial_{x_1}u_h\right| + \left|\partial_{x_2}u_h\right|\right),\\ \left|\xi_h\right|^2 &\leq (4M)^2(1+\varepsilon_2\log(2))\left(\left|\partial_{x_1} u_h\right|\log\left(\left|\partial_{x_1} u_h\right|\right) + \left|\partial_{x_2} u_h\right|\log\left(\left|\partial_{x_2} u_h\right|\right)\right),\\ \left|\xi_h\right|\exp\left(\frac{\left|\xi_h\right|}{4M}\right) &\leq 4M(1+\varepsilon_2\log(2))\left(\left|\partial_{x_1} u_h\right|\log\left(\left|\partial_{x_1} u_h\right|\right) + \left|\partial_{x_2} u_h\right|\log\left(\left|\partial_{x_2} u_h\right|\right)\right).\end{aligned} \tag{81}$$

Then, $\{\xi_h\}$ is bounded in $[L^2(Q_T)]^2$, which means that we can find a subsequence of $\{\xi_h\}$ (denote it also by $\{\xi_h\}$) and a function $\xi \in [L^2(Q_T)]^2$ such that

$$\xi_h \rightharpoonup \xi, \quad \text{weakly in } \left[L^2(Q_T)\right]^2. \tag{82}$$

Since $s \mapsto s\ \exp(s)\ (s \geq 0)$ is increasing and convex, then as in the proof of Lemma 4, we deduce that

$$\begin{aligned}&\int_0^T\int_{\Omega\cap\left\{\left|\partial_{x_1}u\right|+\left|\partial_{x_2}u\right|\geq k_m\right\}}|\xi|\exp\left(\frac{|\xi|}{4M}\right)\mathrm{d}x\mathrm{d}t \leq\\ &\liminf_{h\longrightarrow 0}\int_0^T\int_{\Omega\cap\left\{\left|\partial_{x_1}u_h\right|+\left|\partial_{x_2}u_h\right|\geq k_m\right\}}\left|\xi_h\right|\exp\left(\frac{\left|\xi_h\right|}{4M}\right)\mathrm{d}x\mathrm{d}t < \infty.\end{aligned} \tag{83}$$

Then, by using Lemma 1, we get

$$\begin{aligned}\int_0^T\int_\Omega|\xi\cdot\nabla u|\mathrm{d}x\mathrm{d}t &\le \int_0^T\int_\Omega|\xi||\nabla u|\mathrm{d}x\mathrm{d}t\\
&\le \int_0^T\int_\Omega|\xi|\left(\left|\partial_{x_1}u\right|+\left|\partial_{x_2}u\right|\right)\mathrm{d}x\mathrm{d}t\\
&\le k_m\int_0^T\int_{\Omega\cap\left\{\left|\partial_{x_1}u\right|+\left|\partial_{x_2}u\right|<k_m\right\}}|\xi|\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^T\int_{\Omega\cap\left\{\left|\partial_{x_1}u\right|+\left|\partial_{x_2}u\right|\ge k_m\right\}}|\xi|\exp\left(\frac{|\xi|}{4M}\right)\mathrm{d}x\mathrm{d}t\\
&\quad+4M(1+\varepsilon_2)\left[\int_0^T\int_{\Omega\cap\left\{\left|\partial_{x_1}u\right|\ge k_m\right\}}\left|\partial_{x_1}u\right|\log\left(\left|\partial_{x_1}u\right|\right)\mathrm{d}x\mathrm{d}t\right.\\
&\quad\left.+\int_0^T\int_{\Omega\cap\left\{\left|\partial_{x_2}u\right|\ge k_m\right\}}\left|\partial_{x_2}u\right|\log\left(\left|\partial_{x_2}u\right|\right)\mathrm{d}x\mathrm{d}t\right]<\infty.\end{aligned}\tag{84}$$

It follows then $\xi\cdot\nabla u\in L^1(Q_T)$. Next, we will show that $\xi=\mathbf{D}_{\nabla u}\nabla u$ a.e. in Q_T.

For each $\varphi\in C^1(Q_T)$ with $\varphi(.,T)=0$, we take $\varphi(x,nh)$ as a test function in (35):

$$\int_\Omega\frac{u_n(x)-u_{n-1}(x)}{h}\varphi(x,nh)\mathrm{d}x+\int_\Omega\mathbf{D}_{\nabla u_n}\nabla u_n\cdot\nabla\varphi(x,nh)\mathrm{d}x=0,\tag{85}$$

with $n\in\{1,2,\ldots,N\}$. By summing n from 1 to N, we obtain

$$\begin{aligned}&-\frac{1}{h}\int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x+\sum_{n=0}^{N-1}\int_\Omega u_n(x)\frac{\varphi(x,nh)-\varphi(x,(n+1)h)}{h}\mathrm{d}x\\
&+\sum_{n=1}^{N}\int_\Omega\mathbf{D}_{\nabla u_n}\nabla u_n\cdot\nabla\varphi(x,nh)\mathrm{d}x=0.\end{aligned}\tag{86}$$

From the definition of u_h (65), we have

$$\begin{aligned}\sum_{n=0}^{N-1}\int_\Omega u_n(x)\frac{\varphi(x,nh)-\varphi(x,(n+1)h)}{h}\mathrm{d}x&=-\sum_{n=0}^{N-1}\int_{nh}^{(n+1)h}\int_\Omega u_h(x,t)\frac{\varphi_t(x,t)}{h}\mathrm{d}x\mathrm{d}t\\
&=-\frac{1}{h}\int_0^T\int_\Omega u_h(x,t)\varphi_t(x,t)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{87}$$

Therefore,

$$\begin{aligned}&-\int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x-\int_0^T\int_\Omega u_h(x,t)\varphi_t(x,t)\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega D_{\nabla u_h}\nabla u_h\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\\
&+\sum_{n=1}^{N}\int_{(n-1)h}^{nh}\int_\Omega\mathbf{D}_{\nabla u_n}\nabla u_n\cdot[\nabla\varphi(x,nh)-\nabla\varphi(x,t)]\mathrm{d}x\mathrm{d}t=0.\end{aligned}\tag{88}$$

Letting h tend to zero, we get

$$\int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x+\int_0^T\int_\Omega u\varphi_t\mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega\xi\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t.\tag{89}$$

On the other hand, we let $v \in L^1(Q_T)$ with

$$\int_0^T \int_{\Omega\cap\{|\partial_{x_i} v| \geq k_m\}} |\partial_{x_i} v| \log(|\partial_{x_i} v|) \mathrm{d}x\mathrm{d}t < \infty, \tag{90}$$

for $i = 1, 2$. We sum up inequalities (69):

$$\frac{1}{2}\int_\Omega u_h^2(T)\mathrm{d}x + \int_0^T \int_\Omega \mathbf{D}_{\nabla u_h} \nabla u_h \cdot \nabla u_h \mathrm{d}x\mathrm{d}t \leq \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x. \tag{91}$$

We have from Lemma 2 that

$$\int_0^T \int_\Omega \left(\mathbf{D}_{\nabla u_h} \nabla u_h - \mathbf{D}_{\nabla v} \nabla v\right) \cdot (\nabla u_h - \nabla v) \mathrm{d}x\mathrm{d}t \geq 0. \tag{92}$$

Then, we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_h^2(T)\mathrm{d}x + \int_0^T \int_\Omega \mathbf{D}_{\nabla uh} \nabla u_h \cdot \nabla v \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \mathbf{D}_{\nabla v} \nabla v \cdot \nabla u_h \mathrm{d}x\mathrm{d}t \\ &- \int_0^T \int_\Omega \mathbf{D}_{\nabla v} \nabla v \cdot \nabla v \mathrm{d}x\mathrm{d}t \leq \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x. \end{aligned} \tag{93}$$

Letting $h \longrightarrow 0$ and noting that

$$\int_\Omega u^2(T)\mathrm{d}x \leq \liminf_{h\longrightarrow 0} \int_\Omega u_h^2(T)\mathrm{d}x, \tag{94}$$

we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega u^2(T)\mathrm{d}x + \int_0^T \int_\Omega \xi \cdot \nabla v \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \mathbf{D}_{\nabla v} \nabla v \cdot \nabla u \mathrm{d}x\mathrm{d}t \\ &- \int_0^T \int_\Omega \mathbf{D}_{\nabla v} \nabla v \cdot \nabla v \mathrm{d}x\mathrm{d}t \leq \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x. \end{aligned} \tag{95}$$

By using $\varphi = u$ in (89), we get

$$\frac{1}{2}\int_\Omega u^2(T)\mathrm{d}x + \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x = \int_0^T \int_\Omega \xi \cdot \nabla u \mathrm{d}x\mathrm{d}t. \tag{96}$$

Combining (95) with (96), we have

$$\int_0^T \int_\Omega \left(\xi - \mathbf{D}_{\nabla v} \nabla v\right) \cdot (\nabla v - \nabla u) \mathrm{d}x\mathrm{d}t \leq -\int_\Omega u^2(T)\mathrm{d}x. \tag{97}$$

Now, setting $v = u + \lambda w$ for any $\lambda > 0$, $w \in W^{1,2}(Q_T)$, we derive from the above inequality that

$$\int_0^T \int_\Omega \left(\xi - \mathbf{D}_{\nabla(u+\lambda w)} \nabla(u + \lambda w)\right) \cdot \nabla w \mathrm{d}x\mathrm{d}t \leq 0. \tag{98}$$

By letting $\lambda \longrightarrow 0$ and using Lebesgue's dominated convergence theorem, we obtain

$$\int_0^T \int_\Omega (\xi - \mathbf{D}_{\nabla u} \nabla u) \cdot \psi \mathrm{d}x\mathrm{d}t = 0, \tag{99}$$

for every $\psi \in [L^2(\Omega)]^2$. It follows then

$$\xi = \mathbf{D}_{\nabla u} \nabla u, \quad \text{a.e. in } Q_T. \tag{100}$$

Therefore, we conclude from (89) that

$$-\int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x + \int_0^T \int_\Omega [-u\varphi_t + \mathbf{D}_{\nabla u} \nabla u \cdot \nabla \varphi] \mathrm{d}x\mathrm{d}t = 0, \tag{101}$$

for any $\varphi \in C^1(\overline{Q}_T)$ with $\varphi(.,T) = 0$. Finally, we need to prove that $u \in C([0,T], L^2(\Omega))$. If we choose $\varphi \in C_0^\infty(Q_T)$ in (89), we obtain

$$\int_0^T \int_\Omega u\varphi_t \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \xi \cdot \nabla \varphi \mathrm{d}x\mathrm{d}t. \tag{102}$$

Since $\xi \in [L^2(Q_T)]^2$, we conclude that $u_t \in L^1(0,T; H^{-1}(\Omega))$ where $H^{-1}(\Omega)$ is the dual space of $W_0^{1,2}(\Omega)$. Since

$$\begin{aligned} u &= \int_0^t u_\tau \mathrm{d}\tau + u_0, \\ u_0 &\in L^2(\Omega) \hookrightarrow H^{-1}(\Omega). \end{aligned} \tag{103}$$

It follows then that $u \in C(0,T; H^{-1}(\Omega))$. Besides, for every $h > 0$, let $v_h(x,t) = u(x,t+h)$ be the weak solution for problem (3) satisfying $v_h(x;0) = u(x,h)$. Then, $w_h(x,t) = u(x,t+h) - u(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial w_h}{\partial t} - \nabla \cdot \left[\mathbf{D}_{\nabla v_h}\nabla v_h - \mathbf{D}_{\nabla u}\nabla u\right] = 0, & \text{in } \Omega \times (0,T], \\ \langle \mathbf{D}_{\nabla v_h}\nabla v_h - \mathbf{D}_{\nabla u}\nabla u, \mathbf{n}\rangle = 0, & \text{on } \partial\Omega \times (0,T], \\ w_h(x;0) = u(x,h) - u_0(x), & \text{in } \Omega. \end{cases} \tag{104}$$

For each $t_0 \in [0,T]$, we may choose w_h as a test function in the first equation for problem (104) over $[0,t_0]$:

$$\frac{1}{2}\int_\Omega w_h^2(x,t_0)\mathrm{d}x + \int_0^{t_0}\int_\Omega \left(\mathbf{D}_{\nabla v_h}\nabla v_h - \mathbf{D}_{\nabla u}\nabla u\right)\cdot(\nabla v_h - \nabla u)\mathrm{d}x\mathrm{d}t \le \frac{1}{2}\int_\Omega w_h^2(x,0)\mathrm{d}x. \tag{105}$$

Because of Lemma 2, we deduce

$$\int_\Omega \left|u(x,t_0+h) - u(x,t_0)\right|^2\mathrm{d}x \le \int_\Omega \left|u(x,h) - u_0(x)\right|^2\mathrm{d}x. \tag{106}$$

Now, in order to prove that $u \in C([0,T],L^2(\Omega))$, we need to prove

$$\limsup_{h\longrightarrow 0^+}\int_\Omega \left|u(x,h) - u_0(x)\right|^2\mathrm{d}x = 0. \tag{107}$$

We suppose that (107) is not true. Then, there exist a positive number δ and a sequence $\{h_i\}$ with $h_i \longrightarrow 0$ as $i \longrightarrow \infty$ such that

$$\lim_{h_i\longrightarrow 0^+}\int_\Omega \left|u(x,h_i) - u_0(x)\right|^2\mathrm{d}x \ge \delta. \tag{108}$$

From estimate (72), we have

$$\int_\Omega \left|u(x,h_i)\right|^2\mathrm{d}x \le \int_\Omega \left|u_0(x)\right|^2\mathrm{d}x. \tag{109}$$

Then, from (108), we get

$$\liminf_{h_i\longrightarrow 0^+}\left(\int_\Omega \left|u_0(x)\right|^2\mathrm{d}x - \int_\Omega u_0(x)u(x,h_i)\mathrm{d}x\right) \ge \frac{\delta}{2}. \tag{110}$$

From (109), we conclude that $\{u(x,h_i)\}$ is a bounded sequence in $L^2(\Omega)$. Then, we may find a subsequence (denote it also by $\{u(x,h_i)\}$) such that there exists a function $\tilde{u}_0 \in L^2(\Omega)$ such that

$$u(x,h_i) \rightharpoonup \tilde{u}_0, \quad \text{weakly in } L^2(\Omega). \tag{111}$$

Since $u \in C(0,T;H^{-1}(\Omega))$, it follows that

$$u(x,h_i) \rightharpoonup u_0, \quad \text{weakly in } H^{-1}(\Omega). \tag{112}$$

Therefore, we must have $\tilde{u}_0 = u_0$, and since $u \in C(0,T;H^{-1}(\Omega))$, it follows that

$$u(x,h_i) \rightharpoonup u_0, \quad \text{weakly in } L^2(\Omega), \tag{113}$$

which is contradictory with (110). Therefore, we conclude that (107) is true and $u \in C([0,T],L^2(\Omega))$. This completes the proof of Theorem 1. □

5. Numerical Implementation and Experimental Results

5.1. Consistent and Stable Symmetric Finite Difference Approximation. In this section, we provide a consistent and stable discretization scheme using symmetric finite difference approximation: at time $t_n = n\delta_t$, $n \ge 0$, and the mesh points $x_i = i\delta$, $y_j = j\delta\,(0 \le i \le N+1 \text{ and } 0 \le j \le M+1)$, and we denote by $u_{i,j}^n$ the finite difference approximation of $u(x_i,y_j;t_n)$. The time-space derivatives are discretized in the following manner:

$$\begin{aligned}
u_{x_1}\left(x_i, y_j; t_n\right) &= \frac{u\left(x_{i+(1/2)}, y_j; t_n\right) - u\left(x_{i-(1/2)}, y_j; t_n\right)}{\delta} + \mathcal{O}\left(\delta^2\right),\\
u_{x_2}\left(x_i, y_j; t_n\right) &= \frac{u\left(x_i, y_{j+(1/2)}; t_n\right) - u\left(x_i, y_{j-(1/2)}; t_n\right)}{\delta} + \mathcal{O}\left(\delta^2\right),\\
u_{x_{12}}\left(x_i, y_j; t_n\right) &= \frac{u\left(x_{i+(1/2)}, y_{j+(1/2)}; t_n\right) - u\left(x_{i-(1/2)}, y_{j-(1/2)}; t_n\right)}{\sqrt{2}\delta} + \mathcal{O}\left(\delta^2\right),\\
u_{x_{-12}}\left(x_i, y_j; t_n\right) &= \frac{u\left(x_{i-(1/2)}, y_{j+(1/2)}; t_n\right) - u\left(x_{i+(1/2)}, y_{j-(1/2)}; t_n\right)}{\sqrt{2}\delta} + \mathcal{O}\left(\delta^2\right),\\
u_t\left(x_i, y_j; t_n\right) &= \frac{u\left(x_i, y_j; t_{n+1}\right) - u\left(x_i, y_j; t_n\right)}{\delta_t} + \mathcal{O}\left(\delta_t\right).
\end{aligned} \tag{114}$$

By assume $\delta = 1$ and denote

$$\begin{cases}
g^n_{\mathbf{N}_{i,j}} = g\left(\left|\Delta_{\mathbf{N}} u^n_{i,j}\right|\right),\\
g^n_{\mathbf{E}_{i,j}} = g\left(\left|\Delta_{\mathbf{E}} u^n_{i,j}\right|\right),\\
g^n_{\mathbf{S}_{i,j}} = g\left(\left|\Delta_{\mathbf{S}} u^n_{i,j}\right|\right),\\
g^n_{\mathbf{W}_{i,j}} = g\left(\left|\Delta_{\mathbf{W}} u^n_{i,j}\right|\right),\\
g^n_{\mathbf{NE}_{i,j}} = g\left(\left|\frac{\Delta_{\mathbf{NE}} u^n_{i,j}}{\sqrt{2}}\right|\right),\\
g^n_{\mathbf{SE}_{i,j}} = g\left(\left|\frac{\Delta_{\mathbf{SE}} u^n_{i,j}}{\sqrt{2}}\right|\right),\\
g^n_{\mathbf{SW}_{i,j}} = g\left(\left|\frac{\Delta_{\mathbf{SW}} u^n_{i,j}}{\sqrt{2}}\right|\right),\\
g^n_{\mathbf{NW}_{i,j}} = g\left(\left|\frac{\Delta_{\mathbf{NW}} u^n_{i,j}}{\sqrt{2}}\right|\right),
\end{cases}, \quad \text{with} \begin{cases}
\Delta_{\mathbf{N}} u^n_{i,j} = u^n_{i,j+1} - u^n_{i,j},\\
\Delta_{\mathbf{E}} u^n_{i,j} = u^n_{i+1,j} - u^n_{i,j},\\
\Delta_{\mathbf{S}} u^n_{i,j} = u^n_{i,j-1} - u^n_{i,j},\\
\Delta_{\mathbf{W}} u^n_{i,j} = u^n_{i-1,j} - u^n_{i,j},\\
\Delta_{\mathbf{NE}} u^n_{i,j} = u^n_{i+1,j+1} - u^n_{i,j},\\
\Delta_{\mathbf{SE}} u^n_{i,j} = u^n_{i+1,j-1} - u^n_{i,j},\\
\Delta_{\mathbf{SW}} u^n_{i,j} = u^n_{i-1,j-1} - u^n_{i,j},\\
\Delta_{\mathbf{NW}} u^n_{i,j} = u^n_{i-1,j+1} - u^n_{i,j}.
\end{cases} \tag{115}$$

Then, we may approximate problem (3) using the above scheme to obtain the following nonlinear diffusion filter:

$$u^{n+1}_{i,j} = u^n_{i,j} + \delta_t \left[g_{\mathbf{N}} \Delta_{\mathbf{N}} u + g_{\mathbf{E}} \Delta_{\mathbf{E}} u + g_{\mathbf{S}} \Delta_{\mathbf{S}} u + g_{\mathbf{W}} \Delta_{\mathbf{W}} u + \frac{g_{\mathbf{NE}} \Delta_{\mathbf{NE}} u + g_{\mathbf{SE}} \Delta_{\mathbf{SE}} u + g_{\mathbf{SW}} \Delta_{\mathbf{SW}} u + g_{\mathbf{NW}} \Delta_{\mathbf{NW}} u}{2} \right]^n_{i,j}, \tag{116}$$

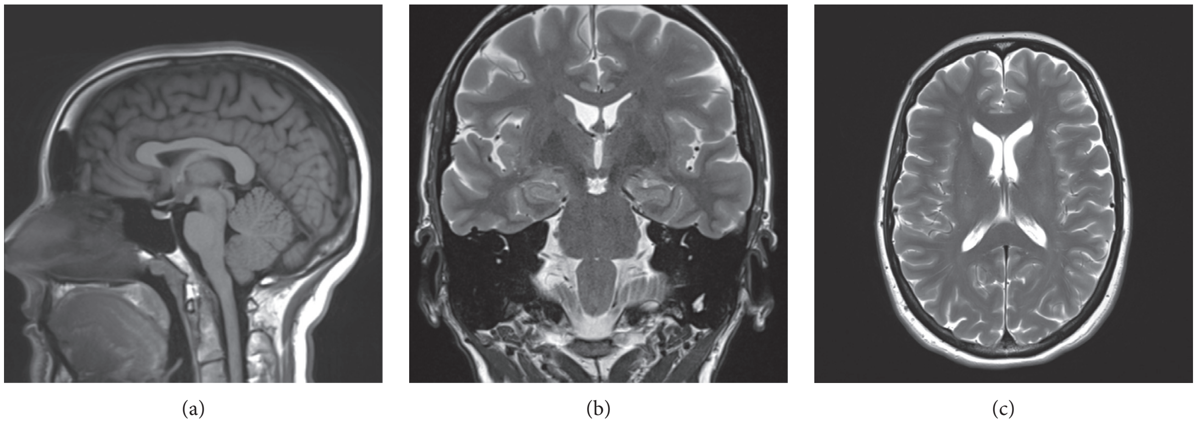


FIGURE 1: Brain MRI scans: Patient30: sagittal T1 of a 30-year-old female patient [20]. Patient50: coronal T2 of a 50-year-old female patient [21]. Patient55: axial T2 of a 55-year-old patient [22]. (a) Patient30, (b) Patient50, and (c) Patient55.

for $1 \le i \le \mathbf{N}$, $1 \le j \le 1 \le i \le \mathbf{M}$, and $n \ge 0$, with the initial condition $u^0_{i,j}$ and the discrete *Neumann* boundary condition:

$$\begin{cases} u^n_{0,j} = u^n_{1,j}, u^n_{\mathbf{N}+1,j} = u^n_{\mathbf{N},j}, & \text{for } 1 \le j \le \mathbf{M}, \\ u^n_{i,0} = u^n_{i,1}, u^n_{i,\mathbf{M}+1} = u^n_{i,\mathbf{M}}, & \text{for } 1 \le i \le \mathbf{N}, \\ u^n_{0,0} = u^n_{1,1}, u^n_{\mathbf{N}+1,0} = u^n_{\mathbf{N},1}, & \\ u^n_{0,\mathbf{M}+1} = u^n_{1,\mathbf{M}}, u^n_{\mathbf{N}+1,\mathbf{M}+1} = u^n_{\mathbf{N},\mathbf{M}}. & \end{cases} \tag{117}$$

A unique sequence $(u^n)_{n\in\mathbb{N}}$ is produced when using filter (116) on a particular initial image u^0 [2]. Besides, due to the continuity of the function g, the sequence u^n depends continuously on u^0 for every finite n. Furthermore, equation (116) satisfies the following maximum-minimum principle, which describes a stability condition for the discrete scheme.

Theorem 3. *Discrete extremum principle [1, 2].*
For an iteration step satisfying

$$0 < \delta_t < \frac{1}{6g(0)}, \tag{118}$$

scheme (116) satisfies

$$\min_{i,j} u^0_{i,j} \le u^n_{i,j} \le \max_{i,j} u^0_{i,j}, \tag{119}$$

for all $1 \le i \le N$, $1 \le j \le M$, *and* $n \in \mathbb{N}$.

5.2. Experimental Results. This section will show the performance of proposed diffusion filter (116) in the image denoising process, under the boundary and initial conditions (117) while respecting the requirements concerning ϕ (Section 1), and δ_t (118). We will use the Peak Signal-to-Noise Ratio (PSNR that is a positive value) [18] and the Structural SIMilarity Index (SSIM that lies in $(0,1)$) [19] to evaluate the quality of the restored images. The best results for the denoising process are equivalent to the higher value of these metrics.

For comparative purposes, we will examine the proposed diffusion function with another one that has the same properties using the same filter (116). Therefore, we will use the following diffusion functions:

(i) The proposed diffusion function ($m = 1$ for instance):

$$g(s) = \begin{cases} p_0 P_{1,0k}(s) + v_0 P_{2,0k}(s) + p_k P_{1,k0}(s) + v_k P_{2,k0}(s), & s \in [0,k[, \\ p_k g_{k,1}(s) + v_k g_{k,2}(s), & s \in [k,\infty[. \end{cases} \tag{120}$$

(ii) The Wang and Zhou diffusion function (WZ) [4]:

$$g(s) = \frac{1}{s+1} + \frac{\log(s+1)}{s}. \tag{121}$$

Additionally, we will consider real test images Figure 1 and evaluate our model's performance on these images, which will be corrupted with different levels of Gaussian white noises with zero mean and variance σ^2.

Table 1 shows the quantitative results on real images, corrupted with various Gaussian noises, filtered by discrete model (116) using proposed diffusion function (120) and the one proposed by WZ (121). These results are obtained using the optimal parameters determined experimentally, as in Table 2 for each diffusion function.

It can be seen from Table 1 and Figure 2 that the proposed model shows remarkable results against the WZ model. From a visual comparison, Figure 2 shows that the restored images using the proposed diffusion function have considerable noise removal and preserve the image essential features better than the restored images by the WZ diffusion function. Besides, compared with the WZ diffusion function,

Table 1: PSNR and SSIM values of the images in Figure 1 affected by different values of Gaussian noise σ^2 and their corresponding iteration number for both functions.

		Noisy		WZ [4]			Proposed		
	σ^2	PSNR	SSIM	PSNR	SSIM	Iter	PSNR	SSIM	Iter
Patient30	0.005	23.4311	0.3426	33.3757	0.9114	30	**33.8333**	**0.9372**	13
	0.010	20.6839	0.2432	31.3367	0.8766	41	**31.5737**	**0.9145**	21
	0.015	19.1190	0.1960	30.1565	0.8521	48	**30.2008**	**0.8980**	24
	0.020	17.9763	0.1660	**29.0905**	0.8298	56	29.0007	**0.8825**	26
	0.100	12.0756	0.0596	**22.6821**	0.6535	117	21.8953	**0.7389**	56
Patient50	0.005	23.6686	0.4424	31.1185	0.8708	24	**31.2934**	**0.8769**	11
	0.010	20.7561	0.3278	29.0893	0.8186	35	**29.1527**	**0.8258**	19
	0.015	19.1313	0.2707	**27.9137**	0.7820	41	27.8981	**0.7920**	26
	0.020	17.9838	0.2334	**27.0011**	0.7508	47	26.9714	**0.7625**	27
	0.100	11.9985	0.0911	**21.4469**	0.5585	93	20.9674	**0.5932**	62
Patient55	0.005	24.0179	0.3867	31.3190	0.9021	26	**31.4668**	**0.9258**	19
	0.010	21.2303	0.2892	29.1310	0.8600	36	**29.0997**	**0.8887**	28
	0.015	19.5990	0.2403	**27.6640**	0.8221	44	27.5635	**0.8561**	35
	0.020	18.4292	0.2096	**26.7717**	0.7938	49	26.5468	**0.8305**	42
	0.100	12.2234	0.0882	**20.6948**	0.5642	98	20.1755	**0.6237**	99

Table 2: The best possible parameters for different diffusion functions.

		WZ [4]	Proposed					
	σ^2	δ_t	δ_t	k	p_0	p_k	v_0	v_k
Patient30	0.005	0.08331	0.14701	4.61411	1.13191	0.66151	−0.00011	−0.10441
	0.010	0.08331	0.14991	5.00191	1.10891	0.45921	−0.00011	−0.04351
	0.015	0.08331	0.15051	5.20221	1.10601	0.45671	−0.00021	−0.03851
	0.020	0.08331	0.14701	5.86411	1.13281	0.46991	−0.00021	−0.03961
	0.100	0.08331	0.15001	5.08081	1.10111	0.45791	−0.00091	−0.03951
Patient50	0.005	0.08331	0.14941	3.18011	1.09241	0.56271	−0.00021	−0.01151
	0.010	0.08321	0.16231	1.89931	1.00171	0.56991	−0.00001	−0.00201
	0.015	0.08331	0.16591	2.06951	0.99011	0.56981	−0.00011	−0.11471
	0.020	0.08331	0.16101	1.79891	1.00001	0.56981	−0.00011	−0.01051
	0.100	0.08331	0.15301	3.87001	0.98701	0.55511	−0.00031	−0.13411
Patient55	0.005	0.08321	0.14681	3.50081	1.12891	0.58971	−0.00021	−0.16571
	0.010	0.08331	0.14701	3.31431	1.12621	0.58281	−0.00131	−0.17311
	0.015	0.08331	0.15141	3.09991	1.09021	0.58881	−0.01401	−0.18991
	0.020	0.08331	0.14881	3.39691	1.08721	0.53891	−0.00011	−0.15801
	0.100	0.08331	0.14991	2.70091	1.08901	0.56861	−0.00021	−0.20951

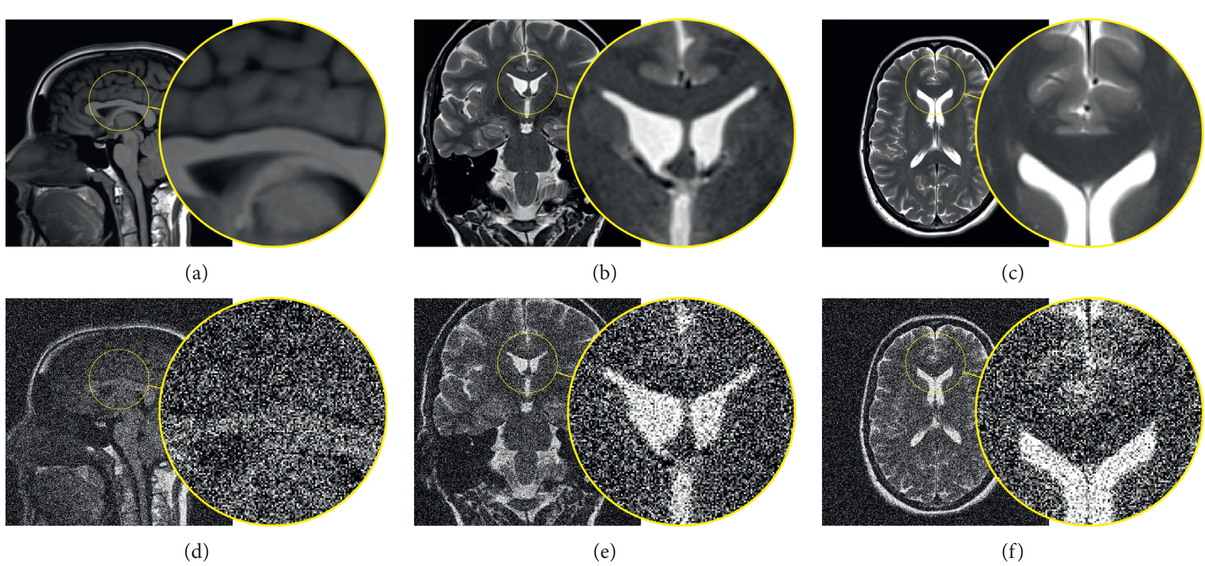


Figure 2: Continued.

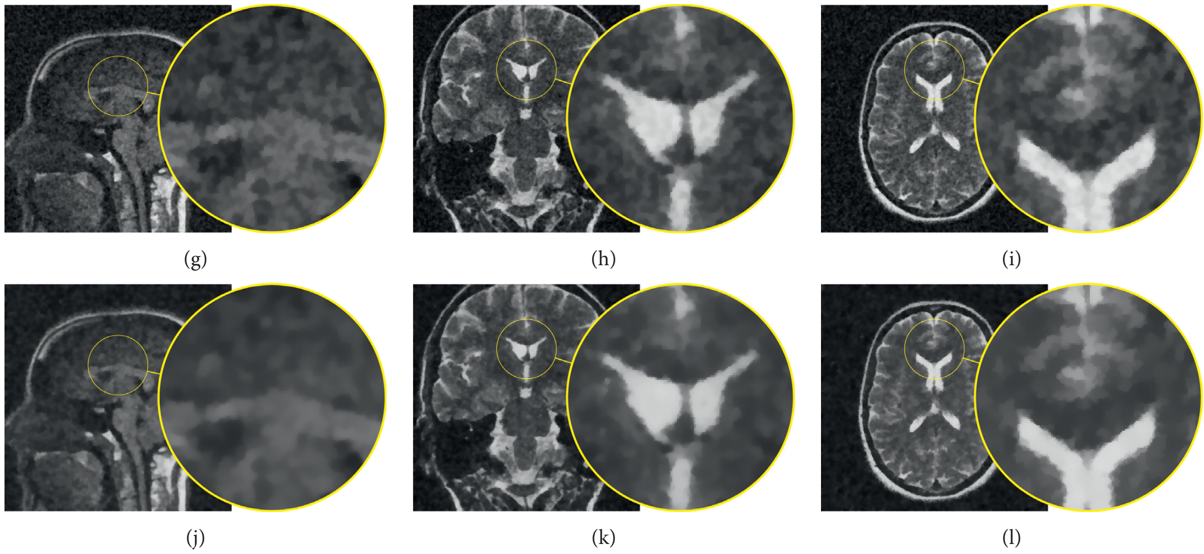


Figure 2: Visual comparison on different real images corrupted by Gaussian noise with $\sigma^2 = 0.1$ and restored images using both functions. (a) Original-Patient30. (b) Original-Patient50. (c) Original-Patient55. (d) Noisy-Patient30. (e) Noisy-Patient50. (f) Noisy-Patient50. (g) WZ-Patient30. (h) WZ-Patient50. (i) WZ-Patient55. (j) Proposed-Patient30. (k) Proposed-Patient50. (l) Proposed-Patient55.

the results from Table 1 prove that the suggested approach has higher values in SSIM, whereas the WZ model shows significant results in PSNR while σ^2-value increases.

6. Conclusion

This paper principally investigates the class of anisotropic diffusion partial differential equations related to image processing and analysis. The existence and uniqueness of weak solutions for this problem have been proven under sufficient conditions satisfied by ϕ. A consistent and stable numerical approximation has been applied, and a discrete nonlinear filter has been tested and revealed its efficiency in the image restoration field.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] P. Perona and J. Malik, "Scale-space and edge detection using anisotropic diffusion," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 12, no. 7, pp. 629–639, 1990.

[2] J. Weickert, *Anisotropic Diffusion in Image Processing*, Springer Vieweg Verlag, Wiesbaden, Germany, 1998.

[3] G. Aubert and P. Kornprobst, *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*, Springer, New York, NY, USA, 2006.

[4] L. Wang and S. Zhou, "Existence and uniqueness of weak solutions for a nonlinear parabolic equation related to image analysis," *Journal of Partial Differential Equations*, vol. 19, no. 2, pp. 97–112, 2006.

[5] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, RI, USA, 2010.

[6] N. Biranvand and A. Salari, "Energy estimate for impulsive fractional advection dispersion equations in anomalous diffusions," *Journal of Nonlinear Functional Analysis*, vol. 2018, pp. 1–17, 2018.

[7] K. P. P. Candito and U. Guarnotta, "Two solutions for a parametric singular p-laplacian problem," *Journal of Nonlinear and Variational Analysis*, vol. 4, pp. 455–468, 2020.

[8] Z. Feng and Z. Yin, "On weak solutions for a class of nonlinear parabolic equations related to image analysis," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2506–2517, 2009.

[9] P. Chen, "Existence and uniqueness of weak solutions for a class of nonlinear parabolic equations," *Electronic Research Announcements in Mathematical Sciences*, vol. 24, pp. 38–52, 2017.

[10] S. Li and P. Li, "Weak solutions for a class of generalised image restoration models," *International Journal of Dynamical Systems and Differential Equations*, vol. 8, no. 3, p. 190, 2018.

[11] T. Humphries, M. Loreto, B. Halter, W. O'Keeffe, and L. Ramírez, "Comparison of regularized and superiorized methods for tomographic image reconstruction," *Journal of Applied and Numerical Optimization*, vol. 2, pp. 77–99, 2020.

[12] M. Jourhmane, *Méthodes numériques de résolution d'un problème d'électro-encéphalographie*, Ph.D. thesis, University of Rennes 1, Rennes, France, 1993.

[13] G. Aubert and L. Vese, "A variational method in image recovery," *SIAM Journal on Numerical Analysis*, vol. 34, no. 5, pp. 1948–1979, 1997.

[14] J. Stoer and R. Bulirsch, *Interpolation*, pp. 37–144, Springer, New York, NY, USA, 2002.

[15] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Taylor & Francis Group, New York, NY, USA, 2015.

[16] A. Tiarimti Alaoui and M. Jourhmane, "Existence and uniqueness of weak solutions for a new class of convex optimization problems related to image analysis," 2020.
[17] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, American Mathematical Society, Providence, RI, USA, 1990.
[18] R. C. Gonzalez and R. E. Woods, *Digital Image Processing*, Prentice-Hall, Upper Saddle River, NJ, USA, 3rd edition, 2006.
[19] Z. Wang, A. C. Bovik, H. R. Sheikh, and E. P. Simoncelli, "Image quality assessment: from error visibility to structural similarity," *IEEE Transactions on Image Processing*, vol. 13, no. 4, pp. 600–612, 2004.
[20] F. Gaillard, "Normal brain mri," 2016.
[21] F. Gaillard, "Normal brain mri (tle protocol)," 2015.
[22] F. Gaillard, "Normal Mri Brain including MR venogram," 2017.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6626660, 8 pages
https://doi.org/10.1155/2021/6626660

Research Article

On New χ-Fixed Point Results for $\lambda - (\Upsilon, \chi)$-Contractions in Complete Metric Spaces with Applications

Babak Mohammadi[1] **Wutiphol Sintunavarat**[2] **and Vahid Parvaneh**[3]

[1]*Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran*
[2]*Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathum Thani 12120, Thailand*
[3]*Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran*

Correspondence should be addressed to Vahid Parvaneh; zam.dalahoo@gmail.com

Received 3 December 2020; Revised 10 January 2021; Accepted 20 February 2021; Published 3 March 2021

Academic Editor: Sun Young Cho

The main aim of this work is to introduce the new concept of $\lambda - (\Upsilon, \chi)$-contraction self-mappings and prove the existence of χ-fixed points for such mappings in metric spaces. Our results generalize and improve some results in existing literature. Moreover, some fixed point results in partial metric spaces can be derived from our χ-fixed points results. Finally, the existence of solutions of nonlinear integral equations is investigated via the theoretical results in this work.

1. Introduction and Preliminaries

One of the most famous metrical fixed point theorem is the Banach contraction principle (BCP) which is the classical tool for solving several nonlinear problems. Based on the noncomplexity and the usefulness of this principle, many mathematicians have improved, extended, and generalized it into several directions. For instance, in [1], on the basis of the probabilistic metric space and the S-metric space, Hu and Gu introduced the concept of the probabilistic metric space, which is called the Menger probabilistic S-metric space. They also proved some fixed point theorems in the framework of Menger probabilistic S-metric spaces. In [2], using the notion of the cyclic representation of a nonempty set with respect to a pair of mappings, Mohanta and Biswas obtained coincidence points and common fixed points of a pair of self-mappings satisfying a type of contraction condition involving comparison functions and (w)-comparison functions in partial metric spaces.

Many researchers attempted to introduce the new idea on generalizations of a metric space and then they investigated fixed point results in new spaces.

In 1994, partial metric spaces were introduced initially by Matthews [3]. One of the important points in this space is the possibility of being nonzero the self-distance.

Definition 1 (see [3]). Let Π be a nonempty set. A mapping $\mathscr{W}\colon \Pi \times \Pi \longrightarrow [0, \infty)$ is called a partial metric if and only if

(p1) $\mathscr{W}(\mathscr{X}, \mathscr{X}) = \mathscr{W}(\mathscr{Y}, \mathscr{Y}) = \mathscr{W}(\mathscr{X}, \mathscr{Y}) \Leftrightarrow \mathscr{X} = \mathscr{Y}$,

(p2) $\mathscr{W}(\mathscr{X}, \mathscr{X}) \leq \mathscr{W}(\mathscr{X}, \mathscr{Y})$,

(p3) $\mathscr{W}(\mathscr{X}, \mathscr{Y}) = \mathscr{W}(\mathscr{Y}, \mathscr{X})$,

(p4) $\mathscr{W}(\mathscr{X}, \mathscr{Y}) \leq \mathscr{W}(\mathscr{X}, \mathscr{Z}) + \mathscr{W}(\mathscr{Z}, \mathscr{Y}) - \mathscr{W}(\mathscr{Z}, \mathscr{Z})$,

for any $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \Pi$. Moreover, the pair $(\Pi, \mathscr{W})$ will be a partial metric space.

Note that any metric space is a partial metric space but the reverse is not true, in general. An example of a partial metric space is the pair $([0,\infty),\mathcal{W})$, where $\mathcal{W}(\mathcal{X},\mathcal{Y})=\max\{\mathcal{X},\mathcal{Y}\}$ for all $\mathcal{X},\mathcal{Y}\in[0,\infty)$. We see that $\mathcal{W}(\mathcal{X},\mathcal{X})$ may not to be zero for some $\mathcal{X}\in\Pi$. For further examples of a partial metric, we refer to [3].

Definition 2 (see [3]). Let $(\Pi,\mathcal{W})$ be a partial metric space.

(1) $\{\mathcal{X}_n\}\subseteq\Pi$ is said to be converging to a point $\mathcal{X}\in\Pi$ if and only if $\lim_{n\longrightarrow\infty}\mathcal{W}(\mathcal{X}_n,\mathcal{X})=\mathcal{W}(\mathcal{X},\mathcal{X})$.

(2) $\{\mathcal{X}_n\}\subseteq\Pi$ is called a Cauchy sequence if and only if $\lim_{n,m\longrightarrow\infty}\mathcal{W}(\mathcal{X}_n,\mathcal{X}_m)$ exists and is finite.

(3) $(\Pi,\mathcal{W})$ is said to be complete if and only if every Cauchy sequence $\{\mathcal{X}_n\}\subseteq\Pi$ converges to some point $\mathcal{X}\in\Pi$ such that $\lim_{n,m\longrightarrow\infty}\mathcal{W}(\mathcal{X}_n,\mathcal{X}_m)=\lim_{n\longrightarrow\infty}\mathcal{W}(\mathcal{X}_n,\mathcal{X})=\mathcal{W}(\mathcal{X},\mathcal{X})$.

Remark 1 (see [3]). If $(\Pi,\mathcal{W})$ is a partial metric space, then the pair $(\Pi,d_{\mathcal{W}})$ is a metric space where $d_{\mathcal{W}}\colon\Pi\times\Pi\longrightarrow[0,\infty)$ is defined by $d_{\mathcal{W}}(\mathcal{X},\mathcal{Y})=2\mathcal{W}(\mathcal{X},\mathcal{Y})-\mathcal{W}(\mathcal{X},\mathcal{X})-\mathcal{W}(\mathcal{Y},\mathcal{Y})$ for all $\mathcal{X},\mathcal{Y}\in\Pi$.

Lemma 1 (see [3]). *Let $(\Pi,\mathcal{W})$ be a partial metric space.*

(i) $\{\mathcal{X}_n\}$ is Cauchy in $(\Pi,\mathcal{W})$ if and only if $\{\mathcal{X}_n\}$ is Cauchy in $(\Pi,d_{\mathcal{W}})$.

(ii) The partial metric space $(\Pi,\mathcal{W})$ is complete if and only if the metric space $(\Pi,d_{\mathcal{W}})$ is complete.

(iii) For each $\{\mathcal{X}_n\}\subseteq\Pi$ and $\mathcal{X}\in\Pi$, $\lim_{n\longrightarrow\infty}d_{\mathcal{W}}(\mathcal{X}_n,\mathcal{X})=0\Leftrightarrow\lim_{n,m\longrightarrow\infty}\mathcal{W}(\mathcal{X}_n,\mathcal{X}_m)=\lim_{n\longrightarrow\infty}\mathcal{W}(\mathcal{X}_n,\mathcal{X})=\mathcal{W}(\mathcal{X},\mathcal{X})$.

According to the published work of Matthews [3], fixed point results in partial metric spaces have been investigated widely by many mathematicians. In 2014, the new concepts of χ-fixed points, χ-Picard mappings, and weakly χ-Picard mappings have been introduced by Jleli et al. [4]. Several χ-fixed point results for mappings satisfying the generalized Banach contractive condition based on the idea of new control function are proved in [4]. Moreover, they also claimed that some fixed point results in partial metric spaces can be derived from these χ-fixed point results in metric spaces. Next, we recall the definitions of χ-fixed points, χ-Picard mappings, and weakly χ-Picard mappings. Before presenting these definitions, some notations are needed.

Let Π be a nonempty set, $\chi\colon\Pi\longrightarrow[0,\infty)$ be a given function, and $\Gamma\colon\Pi\longrightarrow\Pi$ be a mapping.

Throughout this paper, unless otherwise specified, the set of all fixed points of Γ is denoted by $F(\Gamma)=\{\mathcal{X}\in\Pi|\Gamma(\mathcal{X})=\mathcal{X}\}$ and the set of all zeros of χ is denoted by $Z_\chi=\{\mathcal{X}\in\Pi|\chi(\mathcal{X})=0\}$.

Definition 3 (see [4]). Let Π be a nonempty set, $\chi\colon\Pi\longrightarrow[0,\infty)$ be a given function, and $\Gamma\colon\Pi\longrightarrow\Pi$ be a mapping. $\mathcal{Z}\in\Pi$ is called a χ-fixed point of Γ if and only if $\mathcal{Z}$ is a fixed point of Γ such that $\chi(\mathcal{Z})=0$, that is, $\mathcal{Z}\in F(\Gamma)\cap Z_\chi$.

Definition 4 (see [4]). Let Π be a nonempty set and $\chi\colon\Pi\longrightarrow[0,\infty)$ be a given function. A mapping $\Gamma\colon\Pi\longrightarrow\Pi$ is called a χ-Picard mapping if the following conditions hold:

(i) $F(\Gamma)\cap Z_\chi=\{\mathcal{Z}\}$

(ii) $\Gamma^n\mathcal{X}\longrightarrow\mathcal{Z}$ as $n\longrightarrow\infty$ for any $\mathcal{X}\in\Pi$

Definition 5 (see [4]). Let Π be a nonempty set and $\chi\colon\Pi\longrightarrow[0,\infty)$ be a given function. A mapping $\Gamma\colon\Pi\longrightarrow\Pi$ is called a weakly χ-Picard mapping if the following conditions hold:

(i) Γ has at least one χ-fixed point

(ii) The sequence $\{\Gamma^n\mathcal{X}\}$ converges to some χ-fixed point of Γ for any $\mathcal{X}\in\Pi$

A new control function $\Upsilon\colon[0,\infty)^3\longrightarrow[0,\infty)$ has been introduced by Jleli et al. [4] where

(Υ1) $\max\{a,b\}\le\Upsilon(a,b,c)$, for all $a,b,c\in[0,\infty)$

(Υ2) $\Upsilon(0,0,0)=0$

(Υ3) Υ is continuous

Throughout this paper, unless otherwise is specified, the class of all functions satisfying the properties $(\Upsilon1)-(\Upsilon3)$ is denoted by $\overline{\Upsilon}$.

Example 1 (see [4]). Suppose that the mappings $\Upsilon_1,\Upsilon_2,\Upsilon_3\colon[0,\infty)^3\longrightarrow[0,\infty)$ are defined by $\Upsilon_1(a,b,c)=a+b+c,\Upsilon_2(a,b,c)=\max\{a,b\}+c,\Upsilon_3(a,b,c)=a+a^2+b+c$ for all $a,b,c\in[0,\infty)$. Then, $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\overline{\Upsilon}$.

Using the notion of control functions in Υ, Jleli et al. [4] introduced the ideas of (Υ,χ)-contractions and (Υ,χ)-weak contractions and proved existence of χ-fixed point for such mappings as follows.

Definition 6 (see [4]). Let (Π,d) be a metric space, $\chi\colon\Pi\longrightarrow[0,\infty)$ be a given function, and $\Upsilon\in\overline{\Upsilon}$. A mapping $\Gamma\colon\Pi\longrightarrow\Pi$ is called an (Υ,χ)-contraction if and only if there is $k\in(0,\infty)$ such that $\Upsilon(d(\Gamma\mathcal{X},\Gamma\mathcal{Y}),\chi(\Gamma\mathcal{X}),\chi(\Gamma\mathcal{Y}))\le k\Upsilon(d(\mathcal{X},\mathcal{Y}),\chi(\mathcal{X}),\chi(\mathcal{Y}))$, for all $\mathcal{X},\mathcal{Y}\in\Pi$.

Definition 7 (see [4]). Let (Π,d) be a metric space, $\chi\colon\Pi\longrightarrow[0,\infty)$ be a given function, and $\Upsilon\in\overline{\Upsilon}$. A mapping $\Gamma\colon\Pi\longrightarrow\Pi$ is called an (Υ,χ)-weak contraction if and only if there are $k\in(0,\infty)$ and $L\ge0$ such that $\Upsilon(d(\Gamma\mathcal{X},\Gamma\mathcal{Y}),\chi(\Gamma\mathcal{X}),\chi(\Gamma\mathcal{Y}))\le k\Upsilon(d(\mathcal{X},\mathcal{Y}),\chi(\mathcal{X}),\chi(\mathcal{Y}))+L\ (\Upsilon(d(\mathcal{Y},\Gamma\mathcal{X}),\chi(\mathcal{Y}),\chi(\Gamma\mathcal{X}))-\Upsilon(0,\chi(\mathcal{Y}),\chi(\Gamma\mathcal{X})))$ for all $\mathcal{X},\mathcal{Y}\in\Pi$.

Theorem 1 (see [4]). *Let (Π,d) be a complete metric space, $\chi\colon\Pi\longrightarrow[0,\infty)$ be a given function, and $\Upsilon\in\overline{\Upsilon}$. Assume that*

(H1) χ is lower semicontinuous

(H2) $\Gamma\colon\Pi\longrightarrow\Pi$ is an (Υ,χ)-contraction mapping

Then, the following assertions hold:

(i) $F(\Gamma)\subseteq Z_\chi$

(ii) Γ is a χ-Picard mapping

(iii) If $\mathscr{X} \in \Pi$ and $\mathscr{Z}$ is a χ-fixed point of Γ, then $d(\Gamma^n \mathscr{X}, \mathscr{Z}) \leq (k^n/(1-k))\Upsilon (d(\Gamma\mathscr{X}, \mathscr{X}), \chi(\Gamma\mathscr{X}), \chi(\mathscr{X}))$, for all $n \in \mathbb{N}$

Theorem 2 (see [4]). *Let (Π, d) be a complete metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, and $\Upsilon \in \overline{\Upsilon}$. Assume that*

(H1) χ is lower semicontinuous

(H2) $\Gamma\colon \Pi \longrightarrow \Pi$ is an (Υ, χ)-weak contraction mapping

Then, the following assertions hold:

(i) $F(\Gamma) \subseteq Z_\chi$

(ii) Γ is a weakly χ-Picard mapping

(iii) If $\mathscr{X} \in \Pi$ and $\mathscr{Z}$ is a χ-fixed point of Γ, then $d(\Gamma^n \mathscr{X}, \mathscr{Z}) \leq ((k^n)/(1-k))\Upsilon(d(\mathscr{X}, \Gamma\mathscr{X}), \chi(\mathscr{X}), \chi(\Gamma\mathscr{X}))$, for all $n \in \mathbb{N}$

Nowadays, many authors have extended the Banach contractive condition in the BCP into many ways by using various types of the control functions. In 2014, Jleli and Samet [5] presented the new idea of a control function and proved the fixed point results for mappings involving this new control function. Here, we restate the idea of the control function proposed in Jleli and Samet [5] and give the main work in [5] which is the main inspiration in this paper.

Let Λ be the set of all functions $\lambda\colon [0,\infty) \longrightarrow [1,\infty)$ so that

(i) λ is non-decreasing

(ii) For each sequence $\{\iota_n\} \subset [0,\infty)$, $\lim_{n\longrightarrow\infty} \lambda(\iota_n) = 1$ if and only if $\lim_{n\longrightarrow\infty} \iota_n = 0$

(iii) There exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\longrightarrow 0^+}((\lambda(t)-1)/t^r) = l$

Theorem 3 (see [5]). *Let (Π, d) be a complete metric space and $\Gamma\colon \Pi \longrightarrow \Pi$ be a given mapping. Suppose that there exist $\lambda \in \Lambda$ and $k \in (0,1)$ such that for all $\mathscr{X}, \mathscr{Y} \in \Pi$ with $\Gamma\mathscr{X} \neq \Gamma\mathscr{Y}$, one has $\lambda(d(\Gamma\mathscr{X}, \Gamma\mathscr{Y})) \leq [\lambda(d(\mathscr{X}, \mathscr{Y}))]^k$. Then, Γ possesses a unique fixed point.*

Recall that $\chi\colon \Pi \longrightarrow [0,\infty)$ is lower semicontinuous at x_0 if $\liminf_{x\longrightarrow x_0} \chi(x) \geq \chi(x_0)$.

Note that there is no discussion so far on the combination of several ideas of contraction mappings in the literature. The goal of this work is to present the new concept of a λ-(Υ, χ)-contraction self-mappings. The existence results of χ-fixed points for such contraction mappings in metric spaces are provided. The main results of Jleli and Samet [4] and Jleli et al. [5] are particular cases of our main results. Furthermore, we give some fixed point results in partial metric spaces which can be derived from our χ-fixed points results. Finally, we apply the theoretical results in this work to prove the existence of solutions of nonlinear integral equations.

2. Main Results

To present the main result in this paper, we start with the following definition which is larger than the idea of many contraction mappings in the literature.

Definition 8. Let (Π, d) be a metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, $\Upsilon \in \overline{\Upsilon}$, and $\lambda \in \Lambda$. A mapping $\Gamma\colon \Pi \longrightarrow \Pi$ is called a λ-(Υ, χ)-contraction if and only if there exists $k \in (0,1)$ such that

$$\lambda(\Upsilon(d(\Gamma\mathscr{X}, \Gamma\mathscr{Y}), \chi(\Gamma\mathscr{X}), \chi(\Gamma\mathscr{Y}))) \leq [\lambda(\Upsilon(d(\mathscr{X}, \mathscr{Y}), \chi(\mathscr{X}), \chi(\mathscr{Y})))]^k, \tag{1}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$.

Now, we present the main results in this paper.

Theorem 4. *Let (Π, d) be a complete metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, $\Upsilon \in \overline{\Upsilon}$, and $\lambda \in \Lambda$. Assume that*

(i) χ is lower semicontinuous

(ii) $\Gamma\colon \Pi \longrightarrow \Pi$ is an λ-(Υ, χ)-contraction

Then, the following assertions hold:

(i) $F(\Gamma) \subseteq Z_\chi$

(ii) Γ is a χ-Picard mapping

Proof. Suppose that $\mathscr{G} \in \Pi$ is a fixed point of Γ. Appling (1) with $\mathscr{X} = \mathscr{Y} = \mathscr{G}$, we obtain $\lambda(\Upsilon(0, \chi(\mathscr{G}), \chi(\mathscr{G}))) \leq [\lambda(\Upsilon(0, \chi(\mathscr{G}), \chi(\mathscr{G})])]^k$. This implies $\lambda(\Upsilon(0, \chi(\mathscr{G}), \chi(\mathscr{G}))) = 1$ and so $\Upsilon(0, \chi(\mathscr{G}), \chi(\mathscr{G})) = 0$. Then, $\chi(\mathscr{G}) \leq \Upsilon(0, \chi(\mathscr{G}), \chi(\mathscr{G})) = 0$ which implies $\chi(\mathscr{G}) = 0$. Thus, we have proved (i). Now, let $\mathscr{X}$ be an arbitrary point. From (1), we obtain

$$\begin{aligned}
\lambda\left(d\left(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X}\right)\right) &\leq \lambda\left(\Upsilon\left(d\left(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X}\right), \chi\left(\Gamma^n\mathscr{X}\right), \chi\left(\Gamma^{n+1}\mathscr{X}\right)\right)\right),\\
&\leq \lambda\left(\Upsilon\left(d\left(\Gamma^{n-1}\mathscr{X}, \Gamma^n\mathscr{X}\right), \chi\left(\Gamma^{n-1}\mathscr{X}\right), \chi\left(\Gamma^n\mathscr{X}\right)\right)\right)^k\\
&\ \ \vdots\\
&\leq [\lambda(\Upsilon(d(\mathscr{X}, \Gamma\mathscr{X}), \chi(\mathscr{X}), \chi(\Gamma\mathscr{X})))]^{k^n},
\end{aligned} \tag{2}$$

for all $n \in \mathbb{N}$. If $n \longrightarrow \infty$ in the above inequality, we obtain $\lim_{n\longrightarrow\infty} \lambda(d(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X})) = 1$ and so $\lim_{n\longrightarrow\infty} d(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X}) = 0$. Thus, there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\longrightarrow\infty} \frac{\lambda\left(d\left(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X}\right)\right) - 1}{d\left(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X}\right)^r} = l. \tag{3}$$

Similar to the proof of Theorem 2.1 in [5], we deduce $\{\Gamma^n\mathcal{X}\}$ is a Cauchy sequence. Since (Π, d) is complete, there exists $\mathcal{Z} \in \Pi$ such that $\Gamma^n\mathcal{X} \longrightarrow \mathcal{Z}$ as $n \longrightarrow \infty$. From (2), we obtain $1 \leq \lambda(\chi(\Gamma^n\mathcal{X})) \leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{X},\Gamma\mathcal{X}),\chi(\mathcal{X}),\chi(\Gamma\mathcal{X})))]^{k^n}$ for all $n \in \mathbb{N}$. If $n \longrightarrow \infty$ in the above inequality, we obtain $\lim_{n\longrightarrow\infty}\lambda(\chi(\Gamma^n\mathcal{X})) = 1$ and so $\lim_{n\longrightarrow\infty}\chi(\Gamma^n\mathcal{X}) = 0$. Since χ is lower semicontinuous, we obtain $\chi(\mathcal{Z}) = 0$. Again, using (1), we obtain

$$1 \leq \lim_{n\longrightarrow\infty}\lambda\left(\mathrm{d}\left(\Gamma^{n+1}\mathcal{X},\Gamma\mathcal{Z}\right)\right) \leq \lim_{n\longrightarrow\infty}[\lambda(\Upsilon(\mathrm{d}(\Gamma^n\mathcal{X},\mathcal{Z}),\chi(\Gamma^n\mathcal{X}),\chi(\mathcal{Z})))]^k = [\lambda(\Upsilon(0,0,0))]^k = 1. \tag{4}$$

Thus, $\mathrm{d}(\mathcal{Z},\Gamma\mathcal{Z}) = 0$, that is, $\mathcal{Z}$ is a fixed point of Γ. Therefore, $\mathcal{Z}$ is also a χ-fixed point of Γ.

To show the uniqueness of fixed point, let $\mathcal{Z},\mathcal{Z}'$ be two χ-fixed points of Γ. Applying (1) for $\mathcal{X} = \mathcal{Z}, \mathcal{Y} = \mathcal{Z}'$, we get $\lambda(\Upsilon(\mathrm{d}(\mathcal{Z},\mathcal{Z}'),0,0)) \leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{Z},\mathcal{Z}'),0,0))]^k$. This implies $\lambda(\Upsilon(\mathrm{d}(\mathcal{Z},\mathcal{Z}'),0,0)) = 1$ and so $\Upsilon(\mathrm{d}(\mathcal{Z},\mathcal{Z}'),0,0) = 0$. Therefore, $\mathrm{d}(\mathcal{Z},\mathcal{Z}') \leq \Upsilon(\mathrm{d}(\mathcal{Z},\mathcal{Z}'),0,0) = 0$ which gives us $\mathrm{d}(\mathcal{Z},\mathcal{Z}') = 0$. Thus, $\mathcal{Z} = \mathcal{Z}'$. Therefore, we have proved *(ii)*.

Taking $\Upsilon(a,b,c) = a + b + c$ and $\chi \equiv 0$ in the above theorem, we have the following. □

Corollary 1. *Let (Π, d) be a complete metric space and $\lambda \in \Lambda$. Assume that*

(i) There exists $k \in (0,1)$ such that

$$\lambda(\Upsilon(\mathrm{d}(\Gamma\mathcal{X},\Gamma\mathcal{Y}))) \leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{X},\mathcal{Y})))]^k, \tag{5}$$

for all $\mathcal{X},\mathcal{Y} \in \Pi$.

Then, the following assertions hold:

(i) $F(\Gamma) \subseteq Z_\chi$

(ii) Γ is a χ-Picard mapping

Taking $\lambda(t) = \sqrt{t}$ for all $t \geq 0$ in the above corollary, we obtain the BCP.

Next, we present the second idea of the new mappings satisfying the generalized contractive condition which is similar to the first idea and then we prove the existence of a χ-fixed point result for this mapping.

Definition 9. Let (Π, d) be a metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, $\Upsilon \in \overline{\Upsilon}$, and $\lambda \in \Lambda$. A mapping $\Gamma\colon \Pi \longrightarrow \Pi$ is called a λ-(Υ,χ)-weak contraction if and only if there exist $k \in (0,1)$ and $L \geq 0$ such that

$$\lambda(\Upsilon(\mathrm{d}(\Gamma\mathcal{X},\Gamma\mathcal{Y}),\chi(\Gamma\mathcal{X}),\chi(\Gamma\mathcal{Y}))) \leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{X},\mathcal{Y}),\chi(\mathcal{X}),\chi(\mathcal{Y})))]^k [\lambda(\Upsilon(\mathrm{d}(\mathcal{Y},\Gamma\mathcal{X}),\chi(\mathcal{Y}),\chi(\Gamma\mathcal{X})) - \Upsilon(0,\chi(\mathcal{Y}),\chi(\Gamma\mathcal{X})))]^L, \tag{6}$$

for all $\mathcal{X}$, $\mathcal{Y} \in \Pi$.

Theorem 5. *Let (Π, d) be a complete metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, $\Upsilon \in \overline{\Upsilon}$, and $\lambda \in \Lambda$. Assume that*

(i) χ is lower semicontinuous

(ii) $\Gamma\colon \Pi \longrightarrow \Pi$ is a λ-(Υ,χ)-weak contraction

Then, the following assertions hold:

(i) $F(\Gamma) \subseteq Z_\chi$

(ii) Γ is a weakly χ-Picard mapping

Proof. Suppose that $\mathcal{G} \in \Pi$ is a fixed point of Γ. Appling (6) with $\mathcal{X} = \mathcal{Y} = \mathcal{G}$, we obtain

$$\begin{aligned}\lambda(\Upsilon(0,\chi(\mathcal{G}),\chi(\mathcal{G}))) &\leq [\lambda(\Upsilon(0,\chi(\mathcal{G}),\chi(\mathcal{G})]]]^k[\lambda(\Upsilon(0,\chi(\mathcal{G}),\chi(\mathcal{G})) - \Upsilon\Upsilon(0,\chi(\mathcal{G}),\chi(\mathcal{G})))]^L \\ &= [\lambda(\Upsilon(0,\chi(\mathcal{G}),\chi(\mathcal{G})))]^k.\end{aligned} \tag{7}$$

This implies that $\lambda(\Upsilon(0,\chi(\mathcal{G}),\chi(\mathcal{G}))) = 1$ and so $\Upsilon(0,\chi(\mathcal{G}),\chi(\mathcal{G})) = 0$. Then, $\chi(\mathcal{G}) \leq \Upsilon(0,\chi(\mathcal{G}),\chi(\mathcal{G})) = 0$ which implies $\chi(\mathcal{G}) = 0$. Thus, we have proved (i). Now, let $\mathcal{X}$ be an arbitrary point. From (6), we obtain

$$\begin{aligned}\lambda\left(\mathrm{d}\left(\Gamma^n\mathcal{X},\Gamma^{n+1}\mathcal{X}\right)\right) &\leq \lambda\left(\Upsilon\left(\mathrm{d}\left(\Gamma^n\mathcal{X},\Gamma^{n+1}\mathcal{X}\right),\chi\left(\Gamma^n\mathcal{X}\right),\chi\left(\Gamma^{n+1}\mathcal{X}\right)\right)\right), \\ &\leq \left[\lambda\left(\Upsilon\left(\mathrm{d}\left(\Gamma^{n-1}\mathcal{X},\Gamma^n\mathcal{X}\right),\chi\left(\Gamma^{n-1}\mathcal{X}\right),\chi\left(\Gamma^n\mathcal{X}\right)\right)\right)\right]^k \\ &\quad [\lambda((\Upsilon(\mathrm{d}(\Gamma^n\mathcal{X},\Gamma^n\mathcal{X})),\chi(\Gamma^n\mathcal{X}),\chi(\Gamma^n\mathcal{X})) - \Upsilon(0,\chi(\Gamma^n\mathcal{X}),\chi(\Gamma^n\mathcal{X})))]^L \\ &= \left[\lambda\left(\Upsilon\left(d\left(\Gamma^{n-1}\mathcal{X},\Gamma^n\mathcal{X}\right),\chi\left(\Gamma^{n-1}\mathcal{X}\right),\chi\left(\Gamma^n\mathcal{X}\right)\right)\right)\right]^k \\ &\;\;\vdots \\ &\leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{X},\Gamma\mathcal{X}),\chi(\mathcal{X}),\chi(\Gamma\mathcal{X})))]^{k^n}.\end{aligned} \tag{8}$$

Taking the limit as $n \longrightarrow \infty$ in the above inequality, it gives us $\lim_{n\longrightarrow\infty}\lambda(d(\Gamma^n\mathscr{X},\Gamma^{n+1}\mathscr{X}))=1$ and so $\lim_{n\longrightarrow\infty}d(\Gamma^n\mathscr{X},\Gamma^{n+1}\mathscr{X})=0$. Thus, there exist $r\in(0,1)$ and $l\in(0,\infty]$ such that

$$\lim_{n\longrightarrow\infty}\frac{\lambda\left(d\left(\Gamma^n\mathscr{X},\Gamma^{n+1}\mathscr{X}\right)\right)-1}{d\left(\Gamma^n\mathscr{X},\Gamma^{n+1}\mathscr{X}\right)^r}=l. \tag{9}$$

Similar to proof of Theorem 2.1 in [5], we deduce $\{\Gamma^n\mathscr{X}\}$ is a Cauchy sequence. Since (Π,d) is complete, there exists $\mathscr{Z}\in\Pi$ such that $\Gamma^n\mathscr{X}\longrightarrow\mathscr{Z}$ as $n\longrightarrow\infty$. From (8), we obtain. $1\le\lambda(\chi(\Gamma^n\mathscr{X}))\le[\lambda(\Upsilon(d(\mathscr{X},\Gamma\mathscr{X}),\chi(\mathscr{X}),\chi(\Gamma\mathscr{X})))]^{k^n}$.

Taking $n\longrightarrow\infty$, we obtain $\lim_{n\longrightarrow\infty}\lambda(\chi(\Gamma^n\mathscr{X}))=1$ and so $\lim_{n\longrightarrow\infty}\chi(\Gamma^n\mathscr{X})=0$. Since χ is lower semicontinuous, we obtain $\chi(\mathscr{Z})=0$. Again using (11), we obtain

$$\begin{aligned}1&\le\lim_{n\longrightarrow\infty}\lambda\left(d\left(\Gamma^{n+1}\mathscr{X},\Gamma\mathscr{Z}\right)\right)\\&\le\lim_{n\longrightarrow\infty}[\lambda(\Upsilon(d(\Gamma^n\mathscr{X},\mathscr{Z}),\chi(\Gamma^n\mathscr{X}),\chi(\mathscr{Z})))]^k\\&[\lambda(\Upsilon(d(\Gamma^n\mathscr{X},\mathscr{Z}),\chi(\Gamma^n\mathscr{X}),\chi(\mathscr{Z}))-\Upsilon(0,\chi(\Gamma^n\mathscr{X}),\chi(\mathscr{Z})))]^L\\&=[\lambda(\Upsilon(0,0,0))]^k\\&=1.\end{aligned} \tag{10}$$

Therefore, $\lambda(d(\mathscr{Z},\Gamma\mathscr{Z}))=1$ which implies $d(\mathscr{Z},\Gamma\mathscr{Z})=0$, that is, $\mathscr{Z}$ is a fixed point of Γ.

Taking $\Upsilon(a,b,c)=a+b+c$ and $\chi\equiv0$ in the above theorem, we have the following. □

Corollary 2. *Let (Π,d) be a complete metric space and $\lambda\in\Lambda$. Assume that*

(i) There exist $k\in(0,1)$ and $L\ge0$ such that

$$\lambda(\Upsilon(d(\Gamma\mathscr{X},\Gamma\mathscr{Y}),\chi(\Gamma\mathscr{X}),\chi(\Gamma\mathscr{Y})))\le[\lambda(\Upsilon(d(\mathscr{X},\mathscr{Y})))]^k\cdot[\lambda(\Upsilon(d(\mathscr{Y},\Gamma\mathscr{X})))]^L, \tag{11}$$

for all $\mathscr{X},\mathscr{Y}\in\Pi$.

Then, the following assertions hold:

(i) $F(\Gamma)\subseteq Z_\chi$

(ii) Γ is a χ-Picard mapping

Remark 2. Note that the advantage of Corollary 2 is that we can choose the power $L=0$ to obtain Corollary 1. That is, Corollary 2 is more general than Corollary 1. Also, by taking different functions λ, we can obtain many contractive conditions in Theorems 4 and 5.

Next illustrative example is furnished which demonstrates the validity of the hypotheses and degree of utility of Theorem 6 while previous results in the literature are not applicable.

Example 2. Let $\Pi=\{\tau_n|n\in\mathbb{N}\cup\{0\}\}$, where $\tau_n=((n(n+1))/2)$ for all $n\in\mathbb{N}$ and $\tau_0=0$. Obviously, (Π,d) is a complete metric space with the metric $d\colon\Pi\times\Pi\longrightarrow[0,\infty)$ defined by $d(\mathscr{X},\mathscr{Y})=|\mathscr{X}-\mathscr{Y}|$ for all $\mathscr{X},\mathscr{Y}\in\Pi$. Define a mapping $\Gamma\colon\Pi\longrightarrow\Pi$ by $\Gamma\tau_n=\tau_{n-1}$ for all $n\in\mathbb{N}$ and $\Gamma\tau_0=\tau_0=0$. Then, Γ is not a Banach contraction mapping, since

$$\lim_{n\longrightarrow\infty}\frac{d(\Gamma\tau_n,\Gamma\tau_1)}{d(\tau_n,\tau_1)}=\lim_{n\longrightarrow\infty}\frac{\tau_{n-1}-\tau_0}{\tau_n-\tau_1}=\lim_{n\longrightarrow\infty}\frac{((n(n-1))/2)}{((n(n+1))/2)-1}=1. \tag{12}$$

Therefore, the BCP cannot be applied in this example.

Now, we define a function $\lambda\in\Lambda$ by $\lambda(t)=e^{\sqrt{te^t}}$ for all $t\in[0,\infty)$ and a function $\Upsilon\in\overline{\Upsilon}$ by $\Upsilon(a,b,c)=a+b+c$ for all $a,b,c\in[0,\infty)$. Also, we define a function $\chi\colon\Pi\longrightarrow[0,\infty)$ by $\chi(\mathscr{X})=\mathscr{X}$ for all $\mathscr{X}\in\Pi$. We shall show that Γ is a λ-(Υ,χ)-contraction mapping. For any $m,n\in\mathbb{N}$ with $n>m$, we have

$$\begin{aligned}&\frac{d(\Gamma\tau_n,\Gamma\tau_m)+\Gamma\tau_n+\Gamma\tau_m}{d(\tau_n,\tau_m)+\tau_n+\tau_m}e^{d(\Gamma\tau_n,\Gamma\tau_m)+\Gamma\tau_n+\Gamma\tau_m-[d(\tau_n,\tau_m)+\tau_n+\tau_m]}\\&=\frac{((n(n-1))/2)-((m(m-1))/2)+((n(n-1))/2)+((m(m-1))/2)}{((n(n+1))/2)-((m(m+1))/2)+((n(n+1))/2)+((m(m+1))/2)}\\&\cdot e^{((n(n-1))/2)-((m(m-1))/2)+((n(n-1))/2)+((m(m-1))/2)-[((n(n+1))/2)-((m(m+1))/2)+((n(n+1))/2)+((m(m+1))/2)]}\\&=\frac{n(n-1)}{n(n+1)}e^{-2n}\\&\le e^{-2}.\end{aligned} \tag{13}$$

Putting $k = e^{-1}$, the above inequality is equivalent to

$$(\mathrm{d}(\Gamma\tau_n, \Gamma\tau_m) + \Gamma\tau_n + \Gamma\tau_m)e^{\mathrm{d}(\Gamma\tau_n,\Gamma\tau_m)+\Gamma\tau_n+\Gamma\tau_m} \leq k^2(\mathrm{d}(\tau_n, \tau_m) + \tau_n + \tau_m)e^{\mathrm{d}(\tau_n,\tau_m)+\tau_n+\tau_m}, \tag{14}$$

or equivalently

$$e^{\sqrt{(\mathrm{d}(\Gamma\tau_n,\Gamma\tau_m)+\Gamma\tau_n+\Gamma\tau_m)e^{\mathrm{d}(\Gamma\tau_n,\Gamma\tau_m)+\Gamma\tau_n+\Gamma\tau_m}}} \leq e^{k\sqrt{\mathrm{d}(\tau_n,\tau_m)+\tau_n+\tau_m e^{\mathrm{d}(\tau_n,\tau_m)+\tau_n+\tau_m}}}. \tag{15}$$

Therefore, we obtain

$$\lambda(\Upsilon(\mathrm{d}(\Gamma\tau_n, \Gamma\tau_m), \chi(\Gamma\tau_n), \chi(\Gamma\tau_m))) \leq [\lambda(\Upsilon(\mathrm{d}(\tau_n, \tau_m), \chi(\tau_n), \chi(\tau_m)))]^k. \tag{16}$$

Then, all hypotheses of Theorem 7 hold and so Γ has a unique χ-fixed point. Here, $\tau_0 = 0$ is the unique χ-fixed point of Γ.

3. Applications of Theoretical Results

In this section, we give two applications of our main results in the previous section. These applications consist of two parts. The first part is related to the fixed point results in partial metric spaces. The second part shows the application of theoretical results to solve the nonlinear integral equation.

Theorem 6. *Let $(\Pi, \mathscr{W})$ be a complete partial metric space and $\Gamma\colon \Pi \longrightarrow \Pi$ be a mapping such that*

$$\lambda(\mathscr{W}(\Gamma\mathscr{X}, \Gamma\mathscr{Y})) \leq [\lambda(\mathscr{W}(\mathscr{X}, \mathscr{Y}))]^k, \tag{17}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$, where $k \in (0,1)$. Then, Γ has a unique fixed point $\mathscr{Z}$. Moreover, $\chi(\mathscr{Z}) = 0$ implies $\mathscr{W}(\mathscr{Z}, \mathscr{Z}) = 0$.

Proof. Define a metric $d_{\mathscr{W}}\colon \Pi \times \Pi \longrightarrow [0, \infty)$ by

$$\mathrm{d}_{\mathscr{W}}(\mathscr{X}, \mathscr{Y}) = 2\mathscr{W}(\mathscr{X}, \mathscr{Y}) - \mathscr{W}(\mathscr{X}, \mathscr{X}) - \mathscr{W}(\mathscr{Y}, \mathscr{Y}), \tag{18}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$. In addition, we define a new metric $d\colon \Pi \times \Pi \longrightarrow [0, \infty)$ by

$$\mathrm{d}(\mathscr{X}, \mathscr{Y}) = ((d_{\mathscr{W}}(\mathscr{X}, \mathscr{Y}))/2), \tag{19}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$. Also, we set a function $\chi\colon \Pi \longrightarrow [0, \infty)$ and a function $\Upsilon \in \overline{\Upsilon}$ by

$$\chi(\mathscr{X}) = \frac{\mathscr{W}(\mathscr{X}, \mathscr{X})}{2}, \quad \text{for all } \mathscr{X} \in \Pi,$$
$$\Upsilon(a, b, c) = a + b + c, \quad \text{for all } a, b, c \in [0, \infty). \tag{20}$$

Then, from (17), we have

$$\lambda(\mathrm{d}(\Gamma\mathscr{X}, \Gamma\mathscr{Y}) + \mathscr{W}(\Gamma\mathscr{X}, \Gamma\mathscr{X}) + \mathscr{W}(\Gamma\mathscr{Y}, \Gamma\mathscr{Y})) \leq [\lambda(\mathrm{d}(\mathscr{X}, \mathscr{Y}) + \mathscr{W}(\mathscr{X}, \mathscr{X}) + \mathscr{W}(\mathscr{Y}, \mathscr{Y}))]^k, \tag{21}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$. It yields that

$$\lambda(\Upsilon(\mathrm{d}(\Gamma\mathscr{X}, \Gamma\mathscr{Y}), \chi(\Gamma\mathscr{X}), \chi(\Gamma\mathscr{Y}))) \leq [\lambda(\Upsilon(\mathrm{d}(\mathscr{X}, \mathscr{Y}), \chi(\mathscr{X}), \chi(\mathscr{Y})))]^k, \tag{22}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$. By Theorem 7, Γ has a unique χ-fixed point $\mathscr{Z}$. It implies that Γ has a unique fixed point $\mathscr{Z} \in \Pi$. Moreover, $\chi(\mathscr{Z}) = 0$ implies $\mathscr{W}(\mathscr{Z}, \mathscr{Z}) = 0$.

Based on the proof of the above theorem and Theorem 5, we get the following result. □

Theorem 7. *Let $(\Pi, \mathscr{W})$ be a compete partial metric space and $\Gamma\colon \Pi \longrightarrow \Pi$ be a mapping such that*

$$\lambda(\mathscr{W}(\Gamma\mathscr{X}, \Gamma\mathscr{Y})) \leq [\lambda(\mathscr{W}(\mathscr{X}, \mathscr{Y}))]^k [\lambda(\mathscr{W}(\mathscr{Y}, \Gamma\mathscr{X}) - \mathscr{W}(\mathscr{Y}, \mathscr{Y}) - \mathscr{W}(\Gamma\mathscr{X}, \Gamma\mathscr{X}))]^L, \tag{23}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$, where $k \in (0,1)$. Then, Γ has a fixed point $\mathscr{Z}$. Moreover, $\mathscr{W}(\mathscr{Z}, \mathscr{Z}) = 0$.

Next, we will consider the following nonlinear integral equation:

$$\mathcal{X}(t) = \phi(t) + \int_a^b \mathcal{Q}(t, s, \mathcal{X}(s))\mathrm{d}s, \tag{24}$$

where $a, b \in \mathbb{R}$, $\mathcal{X} \in C[a, b]$ (the set of all continuous functions from $[a, b]$ to $\mathbb{R}$), and $\phi\colon [a, b] \longrightarrow \mathbb{R}$ and $\mathcal{Q}\colon [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are two given functions.

Theorem 8. *Consider integral equation (24). Suppose that the following conditions hold:*

(i) $\mathcal{Q}\colon [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and

$$\lambda\left(\int_a^b |\mathcal{Q}(t, s, \alpha)\mathrm{d}s - \mathcal{Q}(t, s, \beta)|\mathrm{d}s\right) \leq \int_a^b \lambda(|\mathcal{Q}(t, s, \alpha) - \mathcal{Q}(t, s, \beta)|)\mathrm{d}s, \tag{25}$$

for all $t, s \in [a, b]$ and for all $\alpha, \beta \in \mathbb{R}$.

(ii) There exist $\lambda \in \Lambda$ and $k \in (0, 1)$ such that

$$\lambda(|\mathcal{Q}(t, s, \alpha) - \mathcal{Q}(t, s, \beta)|) \leq \frac{[\lambda(|\alpha - \beta|)]^k}{b - a}, \tag{26}$$

for all $t, s \in [a, b]$ and for all $\alpha, \beta \in \mathbb{R}$.

Then, integral equation (24) has a unique solution.

Proof. Let $\Pi = C[a, b]$. Define the metric d on Π by $\mathrm{d}(\mathcal{X}, \mathcal{Y}) = \sup_{t\in[a,b]}|\mathcal{X}(t) - \mathcal{Y}(t)|$ for all $\mathcal{X}, \mathcal{Y} \in \Pi$. Then, (Π, d) is a complete metric space. Consider a mapping $\Gamma\colon \Pi \longrightarrow \Pi$ defined by $(\Gamma\mathcal{X})(t) = \phi(t) + \int_a^t \mathcal{Q}(t, s, \mathcal{X}(s))\mathrm{d}s$ for all $\mathcal{X} \in \Pi$. Define the control function $\Upsilon\colon [0, \infty)^3 \longrightarrow [0, \infty)$ by $\Upsilon(a, b, c) = a + b + c$ for all $a, b, c \in [0, \infty)$. Also, define $\chi\colon \Pi \longrightarrow [0, \infty)$ by $\chi(\mathcal{X}) = 0$ for all $\mathcal{X} \in \Pi$. Let $\mathcal{X}, \mathcal{Y} \in \Pi$ and $t \in [a, b]$. Then, we have

$$\begin{aligned}\lambda(|\Gamma\mathcal{X}(t) - \Gamma\mathcal{Y}(t)|) &= \lambda\left(\left|\int_a^t \mathcal{Q}(t, s, \mathcal{X}(s))\mathrm{d}s - \int_a^t \mathcal{Q}(t, s, \mathcal{Y}(s))\mathrm{d}s\right|\right)\\ &\leq \int_a^b \lambda(|\mathcal{Q}(t, s, \mathcal{X}(s)) - \mathcal{Q}(t, s, \mathcal{Y}(s))|)\mathrm{d}s\\ &\leq \int_a^b \frac{[\lambda(|\mathcal{X}(s) - \mathcal{Y}(s)|)]^k}{b - a}\mathrm{d}s\\ &\leq \frac{1}{b - a}\int_a^b [\lambda(d(\mathcal{X}, \mathcal{Y}))]^k \mathrm{d}s\\ &\leq [\lambda(d(\mathcal{X}, \mathcal{Y}))]^k.\end{aligned} \tag{27}$$

Since $\chi(\mathcal{X}) = 0$ for all $\mathcal{X} \in \Pi$, we get

$$\lambda(\mathrm{d}(\Gamma\mathcal{X}, \Gamma\mathcal{Y}) + \chi(\Gamma\mathcal{X}) + \chi(\Gamma\mathcal{Y})) \leq [\lambda(\mathrm{d}(\mathcal{X}, \mathcal{Y}) + \chi(\mathcal{X}) + \chi(\mathcal{Y}))]^k. \tag{28}$$

Therefore,

$$\lambda(\Upsilon(\mathrm{d}(\Gamma\mathcal{X}, \Gamma\mathcal{Y}), \chi(\Gamma\mathcal{X}), \chi(\Gamma\mathcal{Y}))) \leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{X}, \mathcal{Y}), \chi(\mathcal{X}), \chi(\mathcal{Y})))]^k. \tag{29}$$

Thus, $\Gamma\colon \Pi \longrightarrow \Pi$ is a λ-(Υ, χ)-contraction mapping. By Theorem 7, Γ has a unique χ-fixed point $\mathcal{X} \in \Pi$, that is, $(\Gamma\mathcal{X})(t) = \mathcal{X}(t)$ for all $t \in [a, b]$ and $\chi(\mathcal{X}) = 0$ which means that integral equation (24) has a unique solution. □

4. Conclusions

In this paper, we obtained some fixed point results first in a metric space and then in a partial metric space as results. The famous Banach contraction principle is a special case of our results. There are other terms such as $\mathrm{d}(\mathcal{X}, \Gamma\mathcal{Y})$, $\mathrm{d}(\mathcal{Y}, \Gamma\mathcal{Y})$, and $\mathrm{d}(\mathcal{X}, \Gamma\mathcal{X})$ which we can consider in future research. But, certainly, we should also work with other control functions. For more details in this direction, the readers can refer to [6].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This study was supported by Marand Branch, Islamic Azad University, Marand, Iran. This work was supported by Thammasat University Research Unit in Fixed Points and Optimization.

References

[1] P. Hu and F. Gu, "Some fixed point theorems of λ-contractive mappings in Menger PSM-spaces," *Journal of Nonlinear Functional Analysis*, vol. 2020, Article ID 33, 2020.

[2] S. K. Mohanta and P. Biswas, "Cyclic contractions and coincidence points via comparison and (w)-comparison functions on partial metric spaces," *Communications in Optimization Theory*, vol. 2020, Article ID 23, 2020.

[3] S. G. Matthews, "Partial metric topology," *Annals of the New York Academy of Sciences*, vol. 728, pp. 183–197, 1994.

[4] M. Jleli, B. Samet, and C. Vetro, "Fixed point theory in partial metric spaces via φ-fixed point's concept in metric spaces φ-fixed point's concept in metric spaces," *Journal of Inequalities and Applications*, vol. 2014, no. 1, p. 426, 2014.

[5] M. Jleli and B. Samet, "A new generalization of the banach contraction principle," *Journal of Inequalities and Applications*, vol. 2014, Article ID 38, 2014.

[6] N. Hussain, V. Parvaneh, B. Samet, and C. Vetro, "Some fixed point theorems for generalized contractive mappings in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2015, p. 185, 2015.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 5516694, 9 pages
https://doi.org/10.1155/2021/5516694

Research Article

A New Family of Fourth-Order Optimal Iterative Schemes and Remark on Kung and Traub's Conjecture

Chein-Shan Liu [1] **and Tsung-Lin Lee** [2]

[1] *Center of Excellence for Ocean Engineering, Center of Excellence for the Oceans, National Taiwan Ocean University, Keelung 202-24, Taiwan*
[2] *Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan*

Correspondence should be addressed to Tsung-Lin Lee; leetsung@math.nsysu.edu.tw

Received 12 January 2021; Accepted 1 February 2021; Published 17 February 2021

Academic Editor: Xiaolong Qin

Kung and Traub conjectured that a multipoint iterative scheme without memory based on m evaluations of functions has an optimal convergence order $p = 2^{m-1}$. In the paper, we first prove that the two-step fourth-order optimal iterative schemes of the same class have a common feature including a same term in the error equations, resorting on the conjecture of Kung and Traub. Based on the error equations, we derive a constantly weighting algorithm obtained from the combination of two iterative schemes, which converges faster than the departed ones. Then, a new family of fourth-order optimal iterative schemes is developed by using a new weight function technique, which needs three evaluations of functions and whose convergence order is proved to be $p = 2^{3-1} = 4$.

1. Introduction

The most basic problem in engineering and scientific applications is to find the root of a given nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f \in \mathscr{C}(I, \mathbb{R})$ and $I \subset \mathbb{R}$ is an interval we are interested in, and we suppose that $r \in I$ is a simple solution with $f(r) = 0$ and $f'(r) \neq 0$.

The famous Newton method (NM) for iteratively solving equation (1) is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots, \tag{2}$$

which is quadratically convergent. Due to its simplicity and rapid convergence, the Newton method is still the first choice to solve equation (1).

An extension of the NM to a third-order iterative scheme was made by Halley [1]:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)^2 - f(x_n)f''(x_n)}. \tag{3}$$

For the engineering design of the vibrating modes of an elastic system, sometimes we may need to know the eigenvalues of a large-size square matrix, which results in a highly nonlinear and high-order polynomial equation. More often, the function $f(x)$ is itself obtained from other nonlinear ordinary differential equations or partial differential equations. In this situation, it is hard to calculate $f''(x)$ when we apply the Halley method to solve the nonlinear problem.

Kung and Traub conjectured that a multipoint iteration without memory based on m evaluations of functions has an optimal convergence order $p = 2^{m-1}$. It means that the upper bound of the efficiency index $(\text{E.I.}) = p^{(1/m)}$ is $2^{(1-1/m)} < 2$. For $m = 2$, the NM is one of the second-order optimal iterative schemes; however, with $m = 3$, the Halley method is not the optimal one whose $\text{E.I.} = 1.44225$ is low.

The pioneering work of Newton has inspired a lot of studies to solve nonlinear equations, whereby different fourth-order iterative methods were developed for more quickly and stably solving nonlinear equations [2–9]. Many methods to construct the two-step fourth-order optimal schemes were based on the operations of $[f(x_n), f'(x_n), f(y_n)]$ where y_n is obtained from the first Newton step [2, 4–8, 10–14]. Recently, Chicharro et al. [9] proposed a new technique to construct the optimal fourth-order iterative schemes based on the weight function technique.

2. Preliminaries

Before deriving the main results in the next section, we begin with some standard terminologies.

Definition 1. Let the iterative sequence $\{x_n\}$ generated from an iterative scheme converge to a simple root r. If there exists a positive integer p and a real number C such that

$$\lim_{n\longrightarrow\infty} \frac{x_{n+1}-r}{(x_n-r)^p} = C, \tag{4}$$

then p is the order of convergence and C is the asymptotic error constant.

Let $e_n = x_n - r$ be the error in the nth iterate. Then, the relation

$$e_{n+1} = Ce_n^p + \mathcal{O}\left(e_n^{p+1}\right), \tag{5}$$

is called the error equation of an iterative scheme. For example, for the Newton method, the error equation reads as

$$e_{n+1} = c_2 e_n^2 + \mathcal{O}\left(e_n^3\right), \tag{6}$$

where

$$c_n \coloneqq \frac{f^{(n)}(r)}{n!f'(r)}, \quad n = 2, \ldots. \tag{7}$$

Definition 2 (see [10]). An iterative scheme is said to have the optimal order p, if $p = 2^{m-1}$ where m is the number of evaluations of functions (including derivatives).

Definition 3. The efficiency index (E.I.) of an iterative scheme is defined by E.I. $= p^{(1/m)}$.

Definition 4. The conjecture of Kung and Traub asserted that a multipoint iteration without memory based on m evaluations of functions has an optimal order $p = 2^{m-1}$ of convergence [11]. It indicates that the upper bound of the efficiency index is $2^{(1-1/m)} < 2$.

Definition 5. The iterative schemes are of the same class, if they are of the same order p and have the same m evaluations of the same functions.

3. Main Results

We begin with the error equation of the NM:

$$e_{n+1} = c_2 e_n^2 - A_3 e_n^3 - A_4 e_n^4 + \cdots, \tag{8}$$

where

$$A_3 = 2c_2^2 - 2c_3, \tag{9}$$

$$A_4 = 7c_2c_3 - 4c_2^3 - 3c_4. \tag{10}$$

Refer the papers, for instance, [6, 12, 13].

Throughout of the paper, we fix the following notation:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{11}$$

which is the first step of many two-step iterative schemes.

We summarize some fourth-order optimal iterative schemes which were modified from the NM by Chun [14]:

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} - \left(\dfrac{f(x_n)}{f(x_n)-f(y_n)}\right)^2 \dfrac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = \left(2c_2^3 - c_2c_3\right)e_n^4 + \cdots, \end{cases} \tag{12}$$

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} - \dfrac{f^2(x_n)}{f^2(x_n) - 2f(x_n)f(y_n) + 2f^2(y_n)} \dfrac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = \left(3c_2^3 - c_2c_3\right)e_n^4 + \cdots, \end{cases} \tag{13}$$

by Chun [4]:

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} - \left(1 + 2\dfrac{f(y_n)}{f(x_n)} + \dfrac{f^2(y_n)}{f^2(x_n)}\right)\dfrac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = \left(4c_2^3 - c_2c_3\right)e_n^4 + \cdots, \end{cases} \tag{14}$$

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} - \left(1 + 2\dfrac{f(y_n)}{f(x_n)}\right)\dfrac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = \left(5c_2^3 - c_2c_3\right)e_n^4 + \cdots, \end{cases} \tag{15}$$

by King [5]:

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} - \dfrac{f(x_n) + \gamma f(y_n)}{f(x_n) + (\gamma-2)f(y_n)} \dfrac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = \left[(1+2\gamma)c_2^3 - c_2c_3\right]e_n^4 + \cdots, \end{cases} \tag{16}$$

where $\gamma \in \mathbb{R}$, by Chun and Ham [2]:

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} - \dfrac{4f^2(x_n) + 6f(x_n)f(y_n) + 3f^2(y_n)}{4f^2(x_n) - 2f(x_n)f(y_n) - f^2(y_n)} \dfrac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = \left(3c_2^3 - c_2c_3\right)e_n^4 + \cdots, \end{cases} \quad (17)$$

by Kuo et al. [8]:

$$\begin{cases} x_{n+1} = x_n - \dfrac{f^2(x_n) + f^2(y_n)}{f'(x_n)[f(x_n) - f(y_n)]}, \\ e_{n+1} = \left(3c_2^3 - c_2c_3\right)e_n^4 + \cdots, \end{cases} \quad (18)$$

by Ostrowski [15]:

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)[f(x_n) - f(y_n)]}{f'(x_n)[f(x_n) - 2f(y_n)]}, \\ e_{n+1} = \left(c_2^3 - c_2c_3\right)e_n^4 + \cdots, \end{cases} \quad (19)$$

by Maheshwari et al. [16]:

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}\left[\dfrac{f^2(y_n)}{f^2(x_n)} + \dfrac{f(x_n)}{f(x_n) - f(y_n)}\right], \\ e_{n+1} = \left(4c_2^3 - c_2c_3\right)e_n^4 + \cdots, \end{cases} \quad (20)$$

and by Ghanbari [12]:

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} - \left(\dfrac{f(x_n) + 2f(y_n)}{f(x_n) + f(y_n)}\right)^2 \dfrac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = \left(6c_2^3 - c_2c_3\right)e_n^4 + \cdots, \end{cases} \quad (21)$$

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} - \dfrac{f(x_n) + f(y_n)}{f(x_n) - f(y_n)} \dfrac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = \left(3c_2^3 - c_2c_3\right)e_n^4 + \cdots. \end{cases} \quad (22)$$

It is interesting that the iterative schemes (12)-(22) are of the same class because they have same convergence order $p = 4$ and operated with the same evaluations on $[f(x_n), f'(x_n), f(y_n)]$. The efficiency index (E.I.) of the above eleven iterative schemes is the same $\sqrt[3]{4} = 1.5874$, and they are of the optimal fourth-order iterative schemes with three evaluations of $[f(x_n), f'(x_n), f(y_n)]$ in the sense of Kung and Traub, such that $p = 2^{m-1} = 4$. They belong to the same class with the error equations having a common type:

$$e_{n+1} = \left(a_i c_2^3 - c_2c_3\right)e_n^4 + \mathcal{O}\left(e_n^5\right), \quad (23)$$

where a_i are different constants for different optimal fourth-order iterative schemes, which may be zero. Can we raise the order to five by a suitable combination of these iterative schemes? Later, we will reply to this problem.

Theorem 1. *If the conjecture of Kung and Traub is true, then the two-step optimal fourth-order iterative scheme*

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - H[f(x_n), f'(x_n), f(y_n)], \end{cases} \quad (24)$$

which is based on the evaluations of $[f(x_n), f'(x_n), f(y_n)]$, must have the following form of error equation:

$$e_{n+1} = \left(a_0 c_2^3 - c_2c_3\right)e_n^4 + \mathcal{O}\left(e_n^5\right), \quad (25)$$

where a_0 is some constant, which may be zero.

Proof. Suppose that equation (25) is not true, such that we have

$$e_{n+1} = \left(a_0 c_2^3 - b_0 c_2c_3\right)e_n^4 + \mathcal{O}\left(e_n^5\right), \quad (26)$$

where $b_0 \neq 1$.

The weighting factors w_1, w_2, and w_3 are subjected to

$$w_1 + w_2 + w_3 = 1. \quad (27)$$

Then, we consider the weighting average of the error equations in equation (23) with $i = 1, 2$ and equation (26) to be zero in e_n^4:

$$w_1\left(a_1c_2^3 - c_2c_3\right) + w_2\left(a_2c_2^3 - c_2c_3\right) + w_3\left(a_0c_2^3 - b_0c_2c_3\right) = 0, \quad (28)$$

which leads to

$$\begin{aligned} a_1w_1 + a_2w_2 + a_0w_3 &= 0, \\ w_1 + w_2 + b_0w_3 &= 0. \end{aligned} \quad (29)$$

The determinant of the coefficient matrix of the linear equations (27) and (29) is $(b_0 - 1)(a_2 - a_1) \neq 0$ because $b_0 \neq 1$ and $a_1 \neq a_2$. From equations (27) and (29), we have the unique solution of (w_1, w_2, w_3). Thus, we can derive a new iterative scheme by a weighting combination of three optimal fourth-order iterative schemes with the solved factors (w_1, w_2, w_3) whose convergence order is raised to five. This contradicts the conjecture of Kung and Traub, who asserted that the optimal order for the iterative scheme with $m = 3$ is $2^{m-1} = 4$ for a multipoint iteration without memory based on m evaluations of functions.

Obviously, Theorem 1 demonstrates that we cannot raise the convergence order to five by a weighting combination of any three optimal fourth-order convergence iterative schemes. □

Theorem 2. *The following two-step iterative scheme:*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - H(\eta_n)\frac{f(y_n)}{f'(x_n)}, \quad (30)$$

for solving $f(x) = 0$ has fourth-order convergence, where y_n is computed by equation (11), and H is a weight function in terms of

$$\eta_n := \frac{f(y_n)}{f(x_n)}, \quad (31)$$

with

$$\begin{aligned} H(0) &= 1, \\ H'(0) &= 2. \end{aligned} \quad (32)$$

The corresponding error equation is

$$e_{n+1} = \left[\left(5 - \frac{H''(0)}{2}\right)c_2^3 - c_2c_3\right]e_n^4 + \mathcal{O}\left(e_n^5\right). \quad (33)$$

Proof. For the proof of the convergence, we let r be a simple solution of $f(x) = 0$, i.e., $f(r) = 0$ and $f'(r) \neq 0$. We suppose that x_n is sufficiently close to the exact solution r, such that

$$e_n = x_n - r \quad (34)$$

is a small quantity, and it follows that

$$e_{n+1} = e_n + x_{n+1} - x_n. \quad (35)$$

By using the Taylor series, we have

$$f(x_n) = f'(r)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots\right], \quad (36)$$

$$f'(x_n) = f'(r)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots\right]. \quad (37)$$

It immediately leads to

$$\frac{f(x_n)}{f'(x_n)} = \frac{e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots}{1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots + \cdots} = e_n - c_2e_n^2 + A_3e_n^3 + A_4e_n^4 + \cdots. \quad (38)$$

From equations (11) and (38), we have

$$y_n = r + c_2e_n^2 - A_3e_n^3 - A_4e_n^4 + \cdots, \quad (39)$$

$$f(y_n) = f'(r)\left[c_2e_n^2 - A_3e_n^3 - \left(A_4 - c_2^3\right)e_n^4 + \cdots\right]. \quad (40)$$

From equations (40), (37), and (36), it follows that

$$\frac{f(y_n)}{f'(x_n)} = \frac{c_2e_n^2 - A_3e_n^3 - \left(A_4 - c_2^3\right)e_n^4 + \cdots}{1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots} = c_2e_n^2 + \left(2c_3 - 4c_2^2\right)e_n^3 + \left(13c_2^3 - 14c_2c_3 + 3c_4\right)e_n^4 + \cdots, \quad (41)$$

$$\frac{f(y_n)}{f(x_n)} = \frac{c_2e_n^2 - A_3e_n^3 - \left(A_4 - c_2^3\right)e_n^4 + \cdots}{e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots} = c_2e_n + \left(2c_3 - 3c_2^2\right)e_n^2 + \left(3c_4 - 10c_2c_3 + 8c_2^3\right)e_n^3 + \cdots. \quad (42)$$

From equations (31) and (42), we have

$$\eta_n = c_2e_n + \left(2c_3 - 3c_2^2\right)e_n^2 + \left(3c_4 - 10c_2c_3 + 8c_2^3\right)e_n^3 + \cdots. \quad (43)$$

Because the least order of the term $(f(y_n)/f'(x_n))$ as shown in equation (41) is two, we only need to expand $H(\eta_n)$ around zero to the second-order by using equation (43) and

$$H(\eta_n) = H(0) + H'(0)\eta_n + \frac{H''(0)}{2}\eta_n^2 + \cdots = H(0) + c_2H'(0)e_n + \left[\frac{c_2^2}{2}\left[H''(0) - 6H'(0)\right] + 2c_3H'(0)\right]e_n^2 + \cdots. \quad (44)$$

Inserting equations (11), (39), (44), and (41) into equation (30), we have

$$\begin{aligned} e_{n+1} &= c_2e_n^2 - A_3e_n^3 - A_4e_n^4 - \left(H(0) + c_2H'(0)e_n + \left[\frac{c_2^2}{2}\left[H''(0) - 6H'(0)\right] + 2c_3H'(0)\right]e_n^2\right), \\ &\times \left(c_2e_n^2 + \left[2c_3 - 4c_2^2\right]e_n^3 + \left[13c_2^3 - 14c_2c_3 + 3c_4\right]e_n^4\right) + \cdots. \end{aligned} \tag{45}$$

Through some manipulations, we can derive

$$\begin{aligned} e_{n+1} &= \left[c_2 - c_2H(0)\right]e_n^2 - \left[2c_2^2 - 4c_2^2H(0) + H'(0)c_2^2 + 2H(0)c_3 - 2c_3\right]e_n^3, \\ &- \left[7c_2c_3 - 4c_2^3 - 3c_4 + H(0)\left(13c_2^3 - 14c_2c_3 + 3c_4\right) + c_2H'(0)\left(2c_3 - 4c_2^2\right)\right]e_n^4, \\ &- c_2\left[\frac{c_2^2}{2}\left[H''(0) - 6H'(0)\right] + 2c_3H'(0)\right]e_n^4 + \cdots, \end{aligned} \tag{46}$$

which, due to equation (32), can be arranged to that in equation (33). □

Theorem 3 (see [12]). *The following two-step iterative scheme:*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n) + (2+\alpha)f(x_n)f(x_n) + \theta f^2(x_n)}{f^2(x_n) + \alpha f(x_n)f(x_n) + \beta f^2(x_n)}\frac{f(y_n)}{f'(x_n)}, \tag{47}$$

for solving $f(x) = 0$ has fourth-order convergence, where y_n is computed by equation (11). The error equation reads as

$$e_{n+1} = \left[(5 + 2\alpha - \theta + \beta)c_2^3 - c_2c_3\right]e_n^4 + \mathcal{O}\left(e_n^5\right), \tag{48}$$

which is not supplied in [12].

Proof. It is easy to check that the weight function in iterative scheme (47):

$$H(\eta) = \frac{1 + (2+\alpha)\eta + \theta\eta^2}{1 + \alpha\eta + \beta\eta^2}, \tag{49}$$

satisfies equation (32); hence, iterative scheme (47) is a special case of iterative scheme (30).

We can derive

$$H''(\eta) = \frac{1}{A^4(\eta)}\left\{A^2(\eta)\left[A(\eta)B''(\eta) - B(\eta)A''(\eta)\right] - 2\left[A(\eta)B'(\eta) - B(\eta)A'(\eta)\right]A(\eta)A'(\eta)\right\}, \tag{50}$$

where

$$\begin{aligned} A &:= 1 + \alpha\eta + \beta\eta^2, \\ B &:= 1 + (2+\alpha)\eta + \theta\eta^2. \end{aligned} \tag{51}$$

Inserting $A(0) = 1, A'(0) = \alpha, A''(0) = 2\beta, B(0) = 1, B'(0) = 2 + \alpha$, $B''(0) = 2\theta$ into equation (50) by taking $\eta = 0$, we have

$$H''(0) = -2(2\alpha - \theta + \beta). \tag{52}$$

Inserting equation (52) into equation (33), we can derive

$$e_{n+1} = \left[(5 + 2\alpha - \theta + \beta)c_2^3 - c_2c_3\right]e_n^4 + \mathcal{O}\left(e_n^5\right). \tag{53}$$

This ends the proof of this theorem.

Theorem 2 includes those in [9, 17] as special cases. The family developed by Chicharro et al. [9]:

$$x_{n+1} = x_n - G(\eta_n)\frac{f(x_n)}{f'(x_n)}, \tag{54}$$

with $G(0) = G'(0) = 1$ and $G''(0) = 4$ is a special case because we can derive

$$H(\eta_n)\eta_n = G(\eta_n) - 1. \tag{55}$$

Accordingly,

$$\begin{aligned} H(\eta_n) + \eta_n H'(\eta_n) &= G'(\eta_n), \\ 2H'(\eta_n) + \eta_n H''(\eta_n) &= G''(\eta_n), \end{aligned} \tag{56}$$

and $H(0) = 1$ and $H'(0) = 2$ imply $G(0) = G'(0) = 1$ and $G''(0) = 4$. For H, we have only two constraints, but for G, there are three constraints. Hence, iterative scheme (30) is more general than the iterative scheme (54). Moreover, a further differential of the last term in equation (56),

$$3H''(\eta_n) + \eta_n H'''(\eta_n) = G'''(\eta_n), \tag{57}$$

leads to

$$3H''(0) = G'''(0), \tag{58}$$

and hence the error equation of iterative scheme (54) is

$$e_{n+1} = \left[\left(5 - \frac{G'''(0)}{6}\right)c_2^3 - c_2c_3\right]e_n^4 + \mathcal{O}\left(e_n^5\right). \tag{59}$$

In [9], Chicharro et al. derived the error equation as $e_{n+1} = (5c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$ (equation (2) in [9]), which is incorrect to miss the term $-(G'''(0)c_2^3e_n^4/6)$ in the error equation.

The general function of $H(\eta)$ is given by

$$H(\eta) = 1 + 2\eta + \int_0^\eta \int_0^\xi F(z)\mathrm{d}z\mathrm{d}\xi, \tag{60}$$

where $F(z)$ is any integrable function. There are two interesting iterative schemes generated from $F(z) = \cos z$ (COSM) and $F(z) = \sin z$ (SINM):

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - \left(2 + 2\frac{f(y_n)}{f(x_n)} - \cos\frac{f(y_n)}{f(x_n)}\right)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - \left(1 + 3\frac{f(y_n)}{f(x_n)} - \sin\frac{f(y_n)}{f(x_n)}\right)\frac{f(y_n)}{f'(x_n)}. \end{aligned} \tag{61}$$

□

4. Combinations of Iterative Schemes

In this section, we give some methods to combine the iterative schemes as listed in Table 1, which are special cases of the iterative schemes (47) and (30).

From Table 1, we can observe that there exists a cubic term c_2^3 in the error equation for most iterative schemes. Indeed, this term is a dominant factor to enlarge the error, and thus we can combine two iterative schemes by eliminating this term.

Theorem 4. *For the following two-step iterative scheme:*

$$\begin{aligned} x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} &- w_1\frac{f^2(x_n) + (2+\alpha_1)f(x_n)f(y_n) + \theta_1 f^2(y_n)}{f^2(x_n) + \alpha_1 f(x_n)f(y_n) + \beta_1 f^2(y_n)}\frac{f(y_n)}{f'(x_n)} \\ &- w_2\frac{f^2(x_n) + (2+\alpha_2)f(x_n)f(y_n) + \theta_2 f^2(y_n)}{f^2(x_n) + \alpha_2 f(x_n)f(y_n) + \beta_2 f^2(y_n)}\frac{f(y_n)}{f'(x_n)}, \end{aligned} \tag{62}$$

if

$$a_1 := 5 + 2\alpha_1 - \theta_1 + \beta_1 \neq a_2 := 5 + 2\alpha_2 - \theta_2 + \beta_2, \tag{63}$$

$$\begin{aligned} w_1 &= -\frac{a_2}{a_1 - a_2}, \\ w_2 &= \frac{a_1}{a_1 - a_2}, \end{aligned} \tag{64}$$

then the error equation reads as

$$e_{n+1} = -c_2c_3e_n^4 + \mathcal{O}\left(e_n^5\right). \tag{65}$$

Proof. The weighting factors are subjected to

$$w_1 + w_2 = 1. \tag{66}$$

We seek the combination of iterative scheme (47) with two sets of the parameters $(\alpha_1, \beta_1, \theta_1)$ and $(\alpha_2, \beta_2, \theta_2)$ and demand the coefficient preceding $c_2^3e_n^4$ being zero,

$$w_1a_1 + w_2a_2 = w_1(5 + 2\alpha_1 - \theta_1 + \beta_1) + w_2(5 + 2\alpha_2 - \theta_2 + \beta_2) = 0. \tag{67}$$

Solving equations (66) and (67), we can derive equation (64), and the error equation (48) reduces to that in equation (65).

We cannot exhaust all the combinations of the iterative schemes; however, we list the following two: one is the combination of equations (16) and (19), namely, the KOM:

$$x_{n+1} = x_n + \frac{f(x_n)}{2\gamma f'(x_n)} + \frac{1}{2\gamma}\frac{f(x_n) + \gamma f(y_n)}{f(x_n) + (\gamma - 2)f(y_n)}\frac{f(y_n)}{f'(x_n)} - \frac{(1+2\gamma)[f(x_n)f(x_n) - f(y_n)]}{2\gamma f'(x_n)[f(x_n) - 2f(y_n)]}. \tag{68}$$

Table 1: The comparison of different iterative schemes on the error equations.

Algorithm	α	β	θ	Error equation (e_{n+1})
(12)	−2	1	0	$(2c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$
(13)	−2	2	0	$(3c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$
(14)	0	0	1	$(4c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$
(15)	0	0	0	$(5c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$
(16)	$\gamma - 2$	0	0	$((1 + 2\gamma)c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$
(17)	−1/2	−1/4	3/4	$(3c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$
(18)	−1	0	0	$(3c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$
(19)	−2	0	0	$(c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$
(20)	−1	0	−1	$(4c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$
(21)	2	1	4	$(6c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$
(22)	−2	1	−1	$(3c_2^3 - c_2c_3)\, e_n^4 + \mathcal{O}(e_n^5)$

The other one is the combination of equations (12) and (19), namely, the COM:

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)} + \left(\frac{f(x_n)}{f(x_n) - f(y_n)}\right)^2 \frac{f(y_n)}{f'(x_n)} - \frac{2f(x_n)[f(x_n) - f(y_n)]}{f'(x_n)[f(x_n) - 2f(y_n)]}. \quad (69)$$

□

5. Second Family of Optimal Fourth-Order Iterative Schemes

In Theorem 2, we have derived a new family of optimal fourth-order iterative schemes with the assumption that the H-function satisfies $H(0) = 1$ and $H'(0) = 2$. We can relax the conditions to $H(0) = 1$ and derive the following result.

Theorem 5. *Suppose that there are two different functions $H_1(\eta)$ and $H_2(\eta)$ satisfying*

$$\begin{aligned} H_1(0) &= 1, \\ H_2(0) &= 1, \end{aligned} \quad (70)$$

$$H_1'(0) \neq H_2'(0). \quad (71)$$

The following two-step iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - [w_1H_1(\eta_n) + w_2H_2(\eta_n)]\frac{f(y_n)}{f'(x_n)}, \quad (72)$$

for solving $f(x) = 0$ has fourth-order convergence, where y_n is computed by equation (11), and η is defined by equation (31). The corresponding error equation is

$$e_{n+1} = \left[\left(5 - \frac{w_1H_1''(0) + w_2H_2''(0)}{2}\right)c_2^3 - c_2c_3\right]e_n^4 + \mathcal{O}\left(e_n^5\right), \quad (73)$$

where

$$\begin{aligned} w_1 &= \frac{H_2'(0) - 2}{H_2'(0) - H_1'(0)}, \\ w_2 &= \frac{2 - H_1'(0)}{H_2'(0) - H_1'(0)}. \end{aligned} \quad (74)$$

Proof. From equations (46) and (70), it follows that the error equations corresponding to H_1 and H_2 are, respectively,

$$\begin{aligned} e_{n+1} &= [2 - H_1'(0)]c_2^2e_n^3 - A_1e_n^4 + \cdots, \\ e_{n+1} &= [2 - H_2'(0)]c_2^2e_n^3 - A_2e_n^4 + \cdots, \end{aligned} \quad (75)$$

where

$$\begin{aligned} A_1 &:= 9c_2^3 - 7c_2c_3 + c_2H_1'(0)\left(2c_3 - 4c_2^2\right) + c_2\left[\frac{c_2^2}{2}[H_1''(0) - 6H_1'(0)] + 2c_3H_1'(0)\right], \\ A_2 &:= 9c_2^3 - 7c_2c_3 + c_2H_2'(0)\left(2c_3 - 4c_2^2\right) + c_2\left[\frac{c_2^2}{2}[H_2''(0) - 6H_2'(0)] + 2c_3H_2'(0)\right]. \end{aligned} \quad (76)$$

Table 2: The comparison of different methods for the number of iterations.

Functions	$f_1, x_0 = -0.3$	$f_2, x_0 = 0$	$f_3, x_0 = 3$	$f_4, x_0 = 3.5$	$f_5, x_0 = 1$
NM	55	5	7	11	7
KM	49	3	4	7	8
GM	38	3	6	7	11
CM1	12	3	4	5	4
CM2	24	3	4	6	5
OM	56	3	4	5	4
AM	24	3	4	7	5
COSM	45	2	3	6	5
SINM	45	2	3	6	5
KOM	5	3	4	5	6
COM	5	3	4	4	4

We seek a combination of the two iterative schemes corresponding to H_1 and H_2 and ask the coefficient preceding e_n^3 to be zero, such that we have to solve w_1 and w_2 from

$$\begin{aligned} w_1 + w_2 &= 1, \\ w_1\left[2 - H_1'(0)\right] + w_2\left[2 - H_2'(0)\right] &= 0, \end{aligned} \tag{77}$$

whose solutions are given by equation (74). At the same time, the combined error equation is given by

$$\begin{aligned} e_{n+1} &= -w_1 A_1 e_n^4 - w_2 A_2 e_n^4 + \cdots, \\ &= -\left[9c_2^3 - 7c_2c_3 + 2c_2\left(2c_3 - 4c_2^2\right) + \frac{c_2^3}{2}\left[w_1 H_1''(0) + w_2 H_2''(0)\right] + 4c_2c_3 - 6c_2^3\right]e_n^4 + \cdots, \end{aligned} \tag{78}$$

which can be arranged to that in equation (73).

The family in equation (72) includes some optimal fourth-order iterative schemes with two parameters w_1 and w_2, whose error equation again belongs to the type in equation (23). It can be seen that the functions with $H(0) = 1$ are very general, and for this class of iterative schemes, the conjecture of Kung and Traub is also true. □

6. Numerical Experiments

In this section, we give numerical tests of the proposed combined iterative schemes. The test examples are given by

$$\begin{aligned} f_1(x) &= x^3 + 4x^2 - 10, \\ f_2(x) &= x^2 - e^x - 3x + 2, \\ f_3(x) &= (x-1)^3 - 2, \\ f_4(x) &= (x+2)e^x - 1, \\ f_5(x) &= \sin^2 x - x^2 + 1. \end{aligned} \tag{79}$$

The corresponding solutions are, respectively, $r_1 = 1.3652300134$, $r_2 = 0.2575302854$, $r_3 = 2.2599210499$, $r_4 = -0.442854401002$, and $r_5 = 1.4044916482$.

In Table 2, for different functions, we list the number of iterations (NI) obtained by the presently developed algorithms, which are compared to the NM, the CM1 in equation (12), the CM2 in equation (15), the KM in equation (16) with $\gamma = 3$, the OM in equation (19), the AM in equation (20), the GM in equation (21), the KOM in equation (68) with $\gamma = 3$, and the COM in equation (69).

7. Conclusions

Employing a new weight function, the nonlinear equations were solved by using a new family of the fourth-order iterative scheme, which is optimal according to the conjecture of Kung and Traub, and it was proven to be of fourth-order convergence with E.I. = 1.5874. Theorem 1 indicated that if one can develop a fourth-order iterative scheme based on the evaluations of $[f(x_n), f'(x_n), f(y_n)]$ whose coefficient preceding $c_2c_3e_n^4$ is not -1, then the Kung–Traub conjecture would be disproved. We also proposed a combination of two fourth-order iterative schemes of which the dominant term $c_2^3 e_n^4$ in the error equation is eliminated. Upon comparing some examples to other methods, we found that the combined iterative scheme converges faster. The present iterative scheme was competitive to other optimal fourth-order iterative schemes.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] E. Halley, "A new exact and easy method for finding the roots of equations generally and without any previous reduction," *Philosophical Transactions of the Royal Society*, vol. 8, pp. 136–147, 1964.

[2] C. Chun and Y. Ham, "Some fourth-order modifications of Newton's method," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 654–658, 2008.

[3] M. A. Noor, K. I. Noor, and M. Waseem, "Fourth-order iterative methods for solving nonlinear equations," *International Journal of Applied Mathematics and Engineering*, vol. 4, pp. 43–52, 2010.

[4] C. Chun, "Some fourth-order iterative methods for solving nonlinear equations," *Applied Mathematics and Computation*, vol. 195, no. 2, pp. 454–459, 2008.

[5] R. F. King, "A family of fourth order methods for nonlinear equations," *SIAM Journal on Numerical Analysis*, vol. 10, no. 5, pp. 876–879, 1973.

[6] S. Li, "Fourth-order iterative method without calculating the higher derivatives for nonlinear equation," *Journal of Algorithms & Computational Technology*, vol. 13, pp. 1–8, 2019.

[7] C. Chun, "Certain improvements of Chebyshev-Halley methods with accelerated fourth-order convergence," *Applied Mathematics and Computation*, vol. 189, no. 1, pp. 597–601, 2007.

[8] J. Kuo, Y. Li, and X. Wang, "Fourth-order iterative methods free from second derivative," *Applied Mathematics and Computation*, vol. 184, pp. 880–885, 2007.

[9] F. I. Chicharro, A. Cordero, N. Garrido, and J. R. Torregrosa, "Wide stability in a new family of optimal fourth-order iterative methods," *Computational and Mathematical Methods*, vol. 1, Article ID e1023, 2019.

[10] J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, New York, NY, USA, 1964.

[11] H. T. Kung and J. F. Traub, "Optimal order of one-point and multipoint iteration," *Journal of the ACM*, vol. 21, no. 4, pp. 643–651, 1974.

[12] B. Ghanbari, "A new general fourth-order family of methods for finding simple roots of nonlinear equations," *Journal of King Saud University - Science*, vol. 23, no. 4, pp. 395–398, 2011.

[13] C. Chun, M. Y. Lee, B. Neta, and J. Džunić, "On optimal fourth-order iterative methods free from second derivative and their dynamics," *Applied Mathematics and Computation*, vol. 218, no. 11, pp. 6427–6438, 2012.

[14] C. Chun, "Some variants of King's fourth-order family of methods for nonlinear equations," *Applied Mathematics and Computation*, vol. 190, no. 1, pp. 57–62, 2007.

[15] A. M. Ostrowski, *Solutions of Equations and System Equations*, Academic Press, Cambridge, MA, USA, 1960.

[16] A. K. Maheshwari, "A fourth order iterative method for solving nonlinear equations," *Applied Mathematics and Computation*, vol. 211, no. 2, pp. 383–391, 2009.

[17] S. K. Khattri, M. A. Noor, and E. Al-Said, "Unifying fourth-order family of iterative methods," *Applied Mathematics Letters*, vol. 24, no. 8, pp. 1295–1300, 2011.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6636943, 18 pages
https://doi.org/10.1155/2021/6636943

Research Article

Newton-PGSS and Its Improvement Method for Solving Nonlinear Systems with Saddle Point Jacobian Matrices

Yao Xiao, Qingbiao Wu, and Yuanyuan Zhang

Department of Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, China

Correspondence should be addressed to Qingbiao Wu; qbwu@zju.edu.cn

Received 23 November 2020; Revised 11 January 2021; Accepted 21 January 2021; Published 16 February 2021

Academic Editor: Xiaolong Qin

The preconditioned generalized shift-splitting (PGSS) iteration method is unconditionally convergent for solving saddle point problems with nonsymmetric coefficient matrices. By making use of the PGSS iteration as the inner solver for the Newton method, we establish a class of Newton-PGSS method for solving large sparse nonlinear system with nonsymmetric Jacobian matrices about saddle point problems. For the new presented method, we give the local convergence analysis and semilocal convergence analysis under Hölder condition, which is weaker than Lipschitz condition. In order to further raise the efficiency of the algorithm, we improve the method to obtain the modified Newton-PGSS and prove its local convergence. Furthermore, we compare our new methods with the Newton-RHSS method, which is a considerable method for solving large sparse nonlinear system with saddle point nonsymmetric Jacobian matrix, and the numerical results show the efficiency of our new method.

1. Introduction

In this paper, we will explore effective and convenient methods for solving nonlinear nonsymmetric saddle-point problem:

$$F(x) = 0, \tag{1}$$

where $F\colon \mathbb{D} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous differentiable nonlinear function and the function $F = (F_1, \ldots, F_{n+m})^T$ with $F_i = F_i(x), i = 1, 2, \ldots,$ and $x = (x_1, \ldots, x_{n+m})^T$ is defined on an open convex subset of $(n+m)$-dimensional real linear space $\mathbb{R}^{n+m}$. Moreover, the Jacobian matrix $F'(x)$ is large, sparse, and nonsymmetric saddle point with the form

$$F'(x) = \begin{pmatrix} A(x) & B(x) \\ -B^T(x) & 0 \end{pmatrix}, \tag{2}$$

where $A(x) \in \mathbb{R}^{n\times n}$ is a real positive definite matrix and $B(x) \in \mathbb{R}^{n\times m}$ is a full-column rank matrix $(m < n)$. This kind of large sparse nonsymmetric saddle-point nonlinear systems (1) always arises in many scientific and engineering computing areas, such as elastomechanics equations and Stokes equation. Some of them have not been solved analytically, so we can only explore the method to obtain the numerical simulation at our utmost.

In the past, researchers have developed some methods to solve nonlinear function [1–10]. In these methods, the most typical and popular method for solving the nonlinear system (1) is the Newton method. The principle of solving nonlinear equations by the Newton method is very simple. In each step, we expand the nonlinear equation at x_k by Taylor expansion and take its linear part to construct the approximate equation of the nonlinear equation. Then, we calculate the zero point of the approximate equation as the next iteration point, and it is represented as follows:

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k), \quad k = 0, 1, \ldots. \tag{3}$$

The sequence x_k calculated by this iteration will converge to the numerical solution eventually as $k \longrightarrow +\infty$ under certain conditions. We know that an excellent algorithm is not only accurate but also efficient. When the dimension n is large, the cost of each step of the traditional Newton algorithm is very expensive. The reason for this phenomenon is that, at each iterative step, a linear system

$$F'(x_k)s_k = -F(x_k), \quad k \geq 0, s_k = x_{k+1} - x_k, \qquad (4)$$

must be exactly and accurately solved. We hope to give up a little bit of "precision" in exchange for greater "efficiency." This idea led to the development of inexact Newton methods which were first proposed by Dembo et al. [11]. In recent decades, the inexact Newton method has been extensively studied and applied in some fields. The linear equation (4) can be solved efficiently by some methods which will discard some precision, but the calculation amount and time will be greatly reduced. In addition, we know that the traditional Newton method is second-order convergence, and increasing the order of convergence can make the algorithm converge to the exact solution faster. Therefore, we consider improving the Newton method to improve the order of convergence. Next, we introduce the traditional Newton method and the improved Newton method. In the inexact Newton methods, the termination condition of the Newton equation (4) is

$$\left\|F'(x_k)s_k + F(x_k)\right\| \leq \eta_k \left\|F(x_k)\right\|, \quad k \geq 0, \qquad (5)$$

where $s_k = x_{k+1} - x_k$ is obtained by applying some linear iterative methods. The inexact Newton methods usually have the unified form as shown in Algorithm 1 .

Here, $F'(x_k)$ is the Jacobian matrix and $\eta_k \in [0,1)$ is commonly called *forcing term* which is used to control the level of accuracy. The algorithm mentioned above has R-order of convergence two at least. The researchers present the modified Newton iteration to improve convergence order as shown in Algorithm 2.

From what is mentioned above, inexact-modified Newton methods only need to calculate $F'(x_k)^{-1}$ once per m step and have less computation compared with inexact-modified Newton methods. This kind of method has R-order of convergence $m+1$ at least as the outer iteration and the PGSS iteration method as the inner iteration. In this paper, we can establish the modified Newton-PGSS as $m = 2$.

The inexact Newton methods consist of two parts: inner iteration and outer iteration. The outer iteration is the Newton method, which is used to solve nonlinear problems, and each iteration has to solve a linear equation in order to generate the sequence $\{x_k\}$. Linear iterative methods, such as the classical splitting methods or the modern Krylov subspace methods [12, 13], are applied inside the Newton methods to solve the Newton equations approximately. A significant advantage of such inner-outer iterations is that one can reduce the inverse of the Jacobian matrix storage and calculation of each step, so as to improve the operation efficiency. Therefore, this kind of inner-outer iterative methods has been widely studied. Newton–Krylov subspace [3] methods which utilize the Krylov subspace iteration methods as the inner iterations have been effectively and successfully used in many fields, see [14–16].

By introducing the inexact Newton method [1–4, 7, 8], we know that the efficiency of the inner iteration will affect the efficiency of the whole algorithm. Thus, we want to explore the excellent inner iteration to obtain efficient inner-outer iterative methods. In other words, efficient linear

(1) Let the initial guess x_0 be given.
(2) For $k = 0$ until "convergence" do:
$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \quad k = 0, 1, \ldots,$
Find some $\eta_k \in [0,1)$ and s_k that satisfy
$\|F(x_k) + F'(x_k)s_k\| \leq \eta_k \|F(x_k)\|$.
(3) Set $x_{k+1} = x_k + s_k$.

Algorithm 1: Inexact Newton methods.

iteration should be employed to solve the Newton equation (4) with real nonsymmetric saddle-point Jacobian matrix. There are many ways to solve the saddle point linear problem [3, 17–25]. Recently, Cao et al. [26–29] proposed a method which is based on the shift-splitting iteration method presented by Bai et al. [30] to solve the saddle-point problem. This method is more efficient than other algorithms such as the Uzawa-type iteration methods, the successive over-relaxation (SOR-like) iteration methods [31, 32], and the Hermitian and skew-Hermitian splitting (HSS) iteration methods [33–35]. In addition, the PGSS iteration method is convergent unconditionally and the preconditioner generated by it is also very excellent [26]. When applying the PGSS method for solving complex linear system, at each iterative step, it needs to solve single linear subsystem with their coefficient matrices being the $\mathscr{M}_{\mathrm{PGSS}}$ one $(1/2)(\Omega + \mathscr{A})$. Furthermore, in order to increase the efficiency of algorithm, we optimized the outer iteration and then we propose modified Newton-PGSS method to solve the saddle problems. Because there was no Newton method to solve the saddle point system problem, we compare the Newton-PGSS method with the traditional methods, for example, the Newton-RHSS method [31, 36, 37].

The organization of the paper is as follows. In Section 3, we introduce the Newton-PGSS method. In Sections 4 and 5, we offer the convergence properties of this method. We establish local convergence theorem and semilocal convergence properties under some proper hypothesis for the Newton-PGSS method, respectively. We show the modified Newton-PGSS method in Section 6. Numerical examples are presented to confirm the efficiency of our new method in Section 7. Finally, in Section 8, some brief conclusions are given.

2. Preliminaries

First of all, we review the PGSS method [26] for solving large sparse nonsymmetric saddle-point linear system:

$$\mathscr{A}x = b, \quad \mathscr{A} \in \mathbb{R}^{(n+m)\times(n+m)}, x, b \in \mathbb{R}^n, \qquad (6)$$

where $\mathscr{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ is a real nonsymmetric saddle-point matrix.

The PGSS Iteration Method [27]. Given an initial guess $x_0 \in \mathbb{R}^{n+m}$, compute x_{k+1} for $k = 0, 1, 2, \ldots$, using the following iteration scheme until $\{x_k\}$ satisfies the stopping criterion:

(1) Let the initial guess x_0 be given.
(2) For $k = 0$ until "convergence" do:

$$\begin{cases} x_{k,1} = x_k - F'(x_k)^{-1}F(x_k), \\ x_{k,2} = x_{k,1} - F'(x_k)^{-1}F(x_{k,1}), \\ \ldots, \\ x_{k,m-1} = x_{k,m-2} - F'(x_k)^{-1}F(x_{k,m-2}) \\ x_{k+1} = x_{k,m} = x_{k,m-1} - F'(x_k)^{-1}F(x_{k,m-1}). \end{cases}$$

Find some $\eta_k \in [0,1)$ and $s_{k,i}, i = 1,2\ldots m$ that satisfy $\|F(x_{k,i}) + F'(x_k)s_{k,i}\| \leq \eta_{k,i}\|F(x_k)\|$.
(3) Set $x_{k,i} = x_k + s_{k,i}$.

Algorithm 2: Inexact-modified Newton methods.

$$\frac{1}{2}(\Omega + \mathscr{A})x_{k+1} = \frac{1}{2}(\Omega - \mathscr{A})x_k + b, \tag{7}$$

where Ω is a matrix with the form $\begin{pmatrix} \alpha I_1 & 0 \\ 0 & \beta I_2 \end{pmatrix}$, where I_1 is a $n \times n$ identity matrix and I_2 is a $m\times$ m identity matrix and α and β are real numbers greater than 0. We can get x_{k+1} from (7) leading to the PGSS iterative scheme:

$$x_{k+1} = M_{\alpha,\beta}x_k + G_{\alpha,\beta}b, \tag{8}$$

where

$$\begin{aligned} M_{\alpha,\beta} &= (\Omega + \mathscr{A})^{-1}(\Omega - \mathscr{A}), \\ G_{\alpha,\beta} &= (\Omega + \mathscr{A})^{-1}. \end{aligned} \tag{9}$$

Here, $M_{\alpha,\beta}$ is the iteration matrix of the PGSS iteration method.

Theorem 1 (see [27]). *$\mathscr{A} = b \in\in \mathbb{R}^{(n+m)\times(n+m)}$ is a nonsymmetric saddle-point matrix, α is a nonnegative constant, and β is a positive constant. Then, the iteration matrix $M_{\alpha,\beta}$ of PGSS is*

$$M_{\alpha,\beta} = (\Omega + \mathscr{A})^{-1}(\Omega - \mathscr{A}), \tag{10}$$

which satisfies

$$\rho\left(M_{\alpha,\beta}\right) \leq \max_{\lambda_i \in \lambda\left(\Omega^{-(1/2)}\mathscr{A}\Omega^{-(1/2)}\right)} \frac{1-\lambda_i}{1+\lambda_i} \leq 1. \tag{11}$$

3. The Newton-PGSS Method

In this section, we describe an inner-outer iteration method for solving systems of nonlinear equations with complex symmetric Jacobian matrices.

We use Newton methods as outer iteration and apply the PGSS method as the inner solver for the modified Newton method, in other words, the PGSS iteration is employed to solve the following two linear systems:

$$F'(x_k)d_k = -F(x_k), \quad x_k + 1 = x_k + d_k. \tag{12}$$

Then, we get the Newton-PGSS method for solving nonlinear system (1).

The Newton-PGSS Method. Let $F\colon \mathbb{D} \subset \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{n+m}$ be a continuously differentiable function with the complex symmetric Jacobian matrix $F'(x)$ at any $x \in \mathbb{D}$, and let

$$F'(x) = \begin{pmatrix} A(x) & B(x) \\ -B^T(x) & 0 \end{pmatrix}, \tag{13}$$

where $A(x) \in \mathbb{R}^{n\times n}$ is a real-positive definite matrix and $B(x) \in \mathbb{R}^{n\times m}$ is a full column rank matrix $(m < n)$. Given an initial guess $x_0 \in \mathbb{D}$, two positive constants α and β and sequence $\{l_k\}_{k=0}^{\infty}$ of positive integers, compute x_{k+1} for $k = 0,1,2,\ldots,$ until $\{x_k\}$ converges. The algorithm can be concluded as Algorithm 3.

4. Local Convergence of the Newton-PGSS Method

In this section, we prove the local convergence of Newton-PGSS method under the Hölder condition.

Let $F\colon \mathbb{D} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be G-differentiable on an open neighborhood $\mathbb{N}_0 \subset \mathbb{D}$. Suppose $F'(x) = P_{\text{PGSS}}(x) - Q_{\text{PGSS}}(x)$ is modified generalized shift-splitting of the Jacobian matrix $F'(x)$, where $P_{\text{PGSS}}(x) = (1/2)(\Omega(x) + F'(x))$ and $Q_{\text{PGSS}}(x) = (1/2)(\Omega(x) - F'(x))$ and $V(x)$ and $W(x)$ are defined as follows. Suppose $F'(x)$ is continuous and positive definite at a point $x_* \in \mathbb{D}$, at which $F(x_*) = 0$.

Denote with $\mathbb{N}(x_*, r)$ an open ball centered at x_* with radius $r > 0$.

Assumption 1. For all $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$, assume the following conditions hold.

(A1) The bounded condition: there exist positive constants δ and γ such that

$$\begin{aligned} \max\left\{\left\|A(x_*)\right\|, \left\|B(x_*)\right\|\right\} &\leq \delta, \\ \left\|F'(x_*)^{-1}\right\| &\leq \gamma. \end{aligned} \tag{14}$$

(A2) The Hölder condition: there exist nonnegative constants K_w and K_t such that

(1) Given an initial guess x_0, a nonnegative constant α, a positive constant β, and a positive integer sequences $\{l_k\}_{k=0}^{\infty}$.
(2) For $k = 0, 1, \ldots,$ until $\|F(x_k)\| \le \text{tol}\|F(x_0)\|$ do:
(2.1) Set $d_{k,0} = 0$.
(2.2) For $l = 0, 1, \ldots, l_k - 1$, apply algorithm PGSS to the linear system (12):
$(\Omega(x_k) + F'(x_k))d_{k,l+1} = (\Omega(x_k) - F'(x_k))d_{k,l} - F(x_k)$,
and obtain d_{k,l_k} such that
$\|F(x_k) + F'(x_k)d_{k,l_k}\| \le \eta_k\|F(x_k)\|$, for some $\eta_k \in [0, 1)$,
where

$$\Omega(x_k) = \begin{pmatrix} \alpha I_1(x_k) & 0 \\ 0 & \beta I_2(x_k) \end{pmatrix}.$$

I_1 is a $n \times n$ identity matrix and I_2 is a $m \times m$ identity matrix
(2.3) Set
$x_{k+1} = x_k + d_{k,l_k}$.
obtain the following uniform expressions for d_{k,l_k},
$d_{k,l_k} = -\sum_{j=0}^{l_k-1} M_{\alpha,\beta}(\Omega; x_k)^j G_{\alpha,\beta}(\Omega; x_k)F(x_k)$,
where
$M_{\alpha,\beta}(\Omega; x) = (\Omega(x) + F'(x))^{-1}(\Omega(x) - F'(x)))$,
and
$G_{\alpha,\beta}(\Omega; x) = 2(\Omega(x) + F'(x))^{-1}$.
Then, the Newton-PGSS method can be rewritten as
$x_{k+1} = x_k - \sum_{j=0}^{l_k-1} M_{\alpha,\beta}(\Omega; x_k)^j G_{\alpha,\beta}(\Omega; x_k)F(x_k)$, $k = 0, 1, 2, \ldots,$
From the definitions of $M_{\alpha,\beta}(V; x)$ and $G_{\alpha,\beta}(V; x)$, we can obtain
$G_{\alpha,\beta}(\Omega; x)^{-1}(I - M_{\alpha,\beta}(\Omega; x))$
$= G_{\alpha,\beta}(\Omega; x)^{-1} - G_{\alpha,\beta}(\Omega; x)^{-1}M_{\alpha,\beta}(\Omega; x)$
$= (1/2)(\Omega(x) + F'(x)) - (1/2)(\Omega(x) + F'(x))(\Omega(x) + F'(x))^{-1}(\Omega(x) - F'(x))$
$= (1/2)(\Omega(x) + F'(x)) - (1/2)(\Omega(x) - F'(x))$
$= F'(x)$.
Then, the Newton-PGSS method can be equivalently expressed as
$x_{k+1} = x_k - (I - M_{\alpha,\beta}(\Omega; x_k)^{l_k})F'(x_k)^{-1}F(x_k)$, $k = 0, 1, 2, \ldots,$
The Jacobian matrix $F'(x)$ can be rewritten as
$F'(x) = B_{\alpha,\beta}(\Omega; x) - C_{\alpha,\beta}(\Omega; x)$,
with
$B_{\alpha,\beta}(\Omega; x) = G_{\alpha,\beta}(\Omega; x)^{-1}$,
$B_{\alpha,\beta}(\Omega; x)^{-1} = (I - M_{\alpha,\beta}(\Omega; x))F'(x)^{-1}$,
$C_{\alpha,\beta}(\Omega; x) = G_{\alpha,\beta}(V; x)^{-1}M_{\alpha,\beta}(\Omega; x)$.

Algorithm 3: N-PGSS (Newton-PGSS method).

$$\begin{aligned} \|A(x) - A(x_*)\| &\le K_a\|x - x_*\|^p, \\ \|B(x) - B(x_*)\| &\le K_b\|x - x_*\|^p, \end{aligned} \tag{15}$$

with the exponent $p \in (0, 1]$.

Remark 1. We can know the fact that Lipschitz condition is a special case of Hölder condition when $p = 1$, and we can call Hölder condition Lipschitz. Hence, Lipschitz condition is stronger than Hölder condition.

Now, under Assumption 1, we establish the local convergence theorem for the Newton-PGSS, and we can know the properties of function F around the numerical solution x_* and the information about the radius of the neighborhood. The properties and information mentioned above will affect the given method about the local convergence.

Lemma 1. *Under Assumption 1, for all $x, y \in \mathbb{N}(x_*, r)$, if $r \in (0, (1/(\gamma K)))^{(1/p)}$, then $F'(x)^{-1}$ exists. And, the following inequalities hold with $K: = K_a + K_b$ for all $x, y \in \mathbb{N}(x_*, r)$:*

$$\begin{aligned} \|F'(x) - F'(x_*)\| &\le K\|x - x_*\|^p, \\ \|F'(x)^{-1}\| &\le \frac{\gamma}{1 - \gamma K\|x - x_*\|^p}, \\ \|F(x)\| &\le \frac{K}{1 + p}\|x - x_*\|^{1+p} + 2\delta\|x - x_*\|. \end{aligned} \tag{16}$$

Proof

$$\begin{aligned}\left\|F'(x)-F'(x_*)\right\| &\le \begin{pmatrix} A(x)-A(x_*) & B(x)-B(x_*) \\ -\left(B^T(x)-B^T(x_*)\right) & 0\end{pmatrix} \\ &= \left\|\begin{pmatrix} A(x)-A(x_*) & 0 \\ 0 & 0\end{pmatrix}\right\| + \left\|\begin{pmatrix} 0 & B(x)-B(x_*) \\ -\left(B^T(x)-B^T(x_*)\right) & 0\end{pmatrix}\right\| \\ &= \left\|A(x)-A(x_*)\right\| + \left\|B(x)-B(x_*)\right\| \le K_a\left\|x-x_*\right\|^p + K_b\left\|x-x_*\right\|^p \\ &= K\left\|x-x_*\right\|^p.\end{aligned} \tag{17}$$

Since

$$\left\|F'(x)^{-1}\left(F'(x)-F'(x_*)\right)\right\| \le \gamma K\left\|x-x_*\right\|^p < 1, \tag{18}$$

by Banach lemma, $F(x)^{-1}$ exists and inequality

$$\left\|F'(x)^{-1}\right\| \le \frac{\gamma}{1-\gamma K\left\|x-x_*\right\|^p}, \tag{19}$$

holds, and

$$\begin{aligned}\left\|F(x_*)\right\| &= \left\|\begin{pmatrix} A(x_*) & B(x_*) \\ -B^T(x_*) & 0\end{pmatrix}\right\| \\ &\le \left\|\begin{pmatrix} A(x_*) & 0 \\ 0 & 0\end{pmatrix}\right\| + \left\|\begin{pmatrix} 0 & B(x_*) \\ -\left(B^T(x_*)\right) & 0\end{pmatrix}\right\| \\ &\le \left\|A(x_*)\right\| + \left\|B(x_*)\right\| \le 2\delta.\end{aligned} \tag{20}$$

Moreover, since

$$\begin{aligned}F(x) &= F(x) - F(x_*) - F'(x_*)(x-x_*) + F'(x_*)(x-x_*) \\ &= \int_0^1 F'(x_* + t(x-x_*)) - F'(x_*)\mathrm{d}t(x-x_*) \\ &\quad + F'(x_*)(x-x_*),\end{aligned} \tag{21}$$

it holds that

$$\begin{aligned}\|F(x)\| &\le \int_0^1 \left\|F'(x_* + t(x-x_*)) - F'(x_*)\right\|\mathrm{d}t(x-x_*) \\ &\quad + \left\|F'(X_*)\right\|\left\|(x-x_*)\right\| \\ &\le \int_0^1 kt^p\left\|x-x_*\right\|^{p+1}\mathrm{d}t(x-x_*) + 2\delta\left\|x-x_*\right\| \\ &\le \frac{K}{1+p}\left\|x-x_*\right\|^{1+p} + 2\delta\left\|x-x_*\right\|.\end{aligned} \tag{22}$$

This completes the proof of Lemma 1.

Theorem 2. *Under the assumptions of Lemma 1, suppose $r \in (0, r_0)$ and define $r_0 := \min_{1\le j\le 3}\{r_+^{(j)}\}$, where*

$$\begin{aligned} r_+^{(1)} &= \sqrt[q]{\frac{\tau\theta}{\gamma K(1+\theta+\tau\theta)}}, \\ r_+^{(2)} &= \sqrt[q]{\frac{1-2\delta((\tau+1)\theta)^{l_0}}{((4+2p)/(1+p))K\gamma}}, \\ r_+^{(3)} &= \sqrt[q]{\frac{1}{\gamma K}},\end{aligned} \tag{23}$$

with $l_0 = \liminf_{k\longrightarrow\infty} l_k$, and the constant l_0 satisfies

$$l_0 > \lfloor -\frac{\ln 2\ \delta}{\ln((\tau+1)\theta)}\rfloor, \tag{24}$$

where the symbol $\lfloor\cdot\rfloor$ is used to denote the smallest integer no less than the corresponding real number, $\tau \in (0, ((1-\theta)/\theta))$ is a prescribed positive constant, and

$$\begin{aligned}\theta \equiv \theta(\alpha,\beta;x_*) = \left\|M_{\alpha,\beta}(x_*)\right\| &\le \max_{\lambda_i\in\lambda\left(\Omega^{(1/2)}F'(x_*)\Omega^{(1/2)}\right)}\left|\frac{1-\lambda_i}{1+\lambda_i}\right| \\ &\equiv \sigma(\alpha,\beta;x_*) < 1,\end{aligned} \tag{25}$$

where α, β are more than 0.

Then, for any $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0, t \in (0, r)$ and $c > l_0$, it holds that

$$\begin{aligned}&\left\|M_{\alpha,\beta}(V;x)\right\| \le (\tau+1)\theta < 1, \\ &g(t^p;c) = \frac{\gamma}{1-\gamma Kt^p}\left(\frac{3+p}{1+p}Kt^p + 2\beta[(\tau+1)\theta]^c\right) \\ &\qquad \le g\left(r_0^p;l_0\right) < 1.\end{aligned} \tag{26}$$

Proof. Denote

$$\begin{aligned}B_{\alpha,\beta}(\Omega;x) &= G_{\alpha,\beta}(\Omega;x)^{-1}, \\ C_{\alpha,\beta}(\Omega;x) &= G_{\alpha,\beta}(\Omega;x)^{-1}M_{\alpha,\beta}(\Omega;x),\end{aligned} \tag{27}$$

then

$$M_{\alpha,\beta}(V;x) = B_{\alpha,\beta}(\Omega;x)^{-1}C_{\alpha,\beta}(\Omega;x). \tag{28}$$

From the bounded condition, we have

$$\begin{aligned}\left\|B_{\alpha,\beta}(\Omega;x)-B_{\alpha,\beta}(\Omega;x_*)\right\| &\le \frac{1}{2}\left\|F'(x)-F'(x_*)\right\| \\ &\le \frac{K}{2}\left\|x-x_*\right\|^p, \\ \left\|B_{\alpha,\beta}(x_*)^{-1}\right\| = \left\|\left(I-M_{\alpha,\beta}(x_*)\right)F'(x_*)^{-1}\right\| &\le 2\gamma,\end{aligned} \tag{29}$$

and we can get the inequality

$$\left\|B_{\alpha,\beta}(x_*)^{-1}\left(B_{\alpha,\beta}(\Omega;x)-B_{\alpha,\beta}(\Omega;x_*)\right)\right\| \le K\gamma\left\|x-x_*\right\|^p < 1. \tag{30}$$

Hence, by making use of the Banach lemma, we can obtain

$$\left\|B_{\alpha,\beta}(x)^{-1}\right\| \le \frac{2\gamma}{1-\gamma K\left\|x-x_*\right\|^p}. \tag{31}$$

Similarly,

$$\begin{aligned}\left\|C_{\alpha,\beta}(\Omega;x)-C_{\alpha,\beta}(\Omega;x_*)\right\| &\le \frac{1}{2}\left\|F'(x)-F'(x_*)\right\| \le \frac{K}{2}\left\|x-x_*\right\|^p. \\ \left\|C_{\alpha,\beta}(\Omega;x)^{-1}\right\| &\le \frac{2\gamma}{1-\gamma K\left\|x-x_*\right\|^p}.\end{aligned} \tag{32}$$

Then, we have

$$\begin{aligned}&\left\|M_{\alpha,\beta}(\Omega;x)-M_{\alpha,\beta}(\Omega;x_*)\right\| \\ &\quad= \left\|B_{\alpha,\beta}(x)^{-1}C_{\alpha,\beta}(x)-B_{\alpha,\beta}(x_*)^{-1}C_{\alpha,\beta}(x_*)\right\| \\ &\quad= \left\|B_{\alpha,\beta}(x)^{-1}\left(C_{\alpha,\beta}(x)-C_{\alpha,\beta}(x_*)\right)+\left(B_{\alpha,\beta}(x)^{-1}-B_{\alpha,\beta}(x_*)^{-1}\right)C_{\alpha,\beta}(x_*)\right\| \\ &\quad= \left\|B_{\alpha,\beta}(x)^{-1}\left(C_{\alpha,\beta}(x)-C_{\alpha,\beta}(x_*)\right)+B_{\alpha,\beta}(x)^{-1}\left(B_{\alpha,\beta}(\Omega;x_*)-B_{\alpha,\beta}(\Omega;x)\right)B_{\alpha,\beta}(x_*)^{-1}C_{\alpha,\beta}(x_*)\right\| \\ &\quad\le \left\|B_{\alpha,\beta}(\Omega;x)^{-1}\right\|\left(\left\|C_{\alpha,\beta}(\Omega;x)-C_{\alpha,\beta}(\Omega;x_*)\right\|+\left\|B_{\alpha,\beta}(\Omega;x)-B_{\alpha,\beta}(\Omega;x_*)\right\|\left\|M_{\alpha,\beta}(\Omega;x_*)\right\|\right) \\ &\quad\le \frac{-2\gamma}{1-\gamma K\left\|x-x_*\right\|^p}\left(\frac{K}{2}\left\|x-x_*\right\|^p+\frac{K}{2}\left\|x-x_*\right\|^p\theta\right) \\ &\quad= \frac{(1+\theta)\gamma K\left\|x-x_*\right\|^p}{1-\gamma K\left\|x-x_*\right\|^p} \le \tau\theta.\end{aligned} \tag{33}$$

We can use (33); hence,

$$\begin{aligned}\left\|M_{\alpha,\beta}(\Omega;x)\right\| &\le \left\|M_{\alpha,\beta}(\Omega;x)-M_{\alpha,\beta}(\Omega;x_*)\right\| \\ &\quad+\left\|M_{\alpha,\beta}(\Omega;x_*)\right\| \le (1+\tau)\theta < 1.\end{aligned} \tag{34}$$

Now, we turn to estimate the error about the Newton – PGSS iteration $\{x_k\}_0^\infty$ defined above. Clearly, it holds that

$$\begin{aligned}x_{k+1}-x_* &= x_k-x_*-\left(I-M_{\alpha,\beta}(\Omega;x_k)\right)^{l_k}F'(x_k)^{-1}F(x_k) \\ &= -F'(x_k)^{-1}\left(F'(x_k)-F'(x_*)-F'(x_*)(x_k-x_*)\right) \\ &\quad+F'(x_k)^{-1}\left(F(x_k)-F(x_*)\right)(x_k-x_*) \\ &\quad+M_{\alpha,\beta}\left(\Omega;x_k^{l_k}F'(x_k)^{-1}F(x_k)\right),\end{aligned} \tag{35}$$

where

$$\begin{aligned}M_{\alpha,\beta}(\Omega;x_k)^{l_k}F'(x_k)^{-1}F(x_k) = M_{\alpha,\beta}(\Omega;x_k)^{l_k}F'(x_k)^{-1} &\\ \cdot\left(F(x_k)-F(x_*)-F'(x_*)\right. &\\ \left.\cdot(x_k-x_*)\right) &\\ +M_{\alpha,\beta}(\Omega;x_k)^{l_k}F'(x_*) &\\ \cdot(x_k-x_*). &\end{aligned} \tag{36}$$

Hence, we can obtain

$$\begin{aligned}\left\|M_{\alpha,\beta}(\Omega;x_k)^{l_k}F'(x_*)(x_k-x_*)\right\| &\le \left\|M_{\alpha,\beta}\left(\Omega;x_k^{l_k}\right)\right\|\left\|F'(x_k)^{-1}\right\|\left(\left\|F(x_k-F(x_*)-F'(x_*)(x_k-x_*))\right\|+\left\|F'(x_*)(x_k-x_*)\right\|\right)\\
&\le \frac{\gamma((\tau+1)\theta)^{l_k}}{1-\gamma K\left\|x_k-x_*\right\|^p}\left(\frac{K}{1+p}\left\|x_k-x_*\right\|^{1+p}+2\delta\left\|x_k-x_*\right\|\right),\\
\left\|x_{k+1}-x_*\right\| &\le \frac{\gamma}{1-\gamma K\left\|x-x_*\right\|^p}\left(\frac{K}{1+p}\left\|x_k-x_*\right\|^{1+p}+K|x_k-x_*\|^{1+p}\right)\\
&\quad+\frac{\gamma((\tau+1)\theta)^{l_k}}{1-\gamma K\left\|x-x_*\right\|^p}\left(\frac{K}{1+p}\left\|x_k-x_*\right\|^{1+p}+2\delta\left\|x_k-x_*\right\|\right)\\
&=\frac{\gamma}{1-\gamma K\left\|x_k-x_*\right\|^p}\left(\frac{3+p}{1+p}K\left\|x_k-x_*\right\|^p+2\delta[(\tau+1)\theta]^{l_k}\right)\|x_k-x_*\|\\
&=g\left(\left\|x_k-x_*\right\|^p;l_k\right)\left\|x_k-x_*\right\|,\end{aligned}\tag{37}$$

where

$$g(t^p;c)=\frac{\gamma}{1-\gamma Kt^p}\left(\frac{3+p}{1+p}Kt^p+2\beta[(\tau+1)\theta]^c\right).\tag{38}$$

This function is about t increasing and about c decreasing; hence,

$$g\left(\left\|x_k-x_*\right\|^p;l_k\right)\le g\left(r_0^p;l_0\right)<1.\tag{39}$$

In fact, for $k=0$, we have$\|x_0-x_*\|<r<r_0$, as $x^{(0)}\in\mathbb{N}(x_*,r)$. It follows that

$$\left\|x_1-x_*\right\|\le g\left(r_0^p;l_0\right)\left\|x_0-x_*\right\|\le\left\|x_0-x_*\right\|<r_0.\tag{40}$$

Hence, $x_1\in\mathbb{N}(x_*,r)$, and by making use of mathematical methods of induction, suppose $x_m\in\mathbb{N}(x_*,r)$ is valid for some positive integer $k=m$. Then, by making use of the function above again, we can straightforwardly deduce the estimate

$$\left\|x_{m+1}-x_*\right\|\le g\left(r_0^p;l_0\right)\left\|x_m-x_*\right\|,\tag{41}$$

which show that it also holds true for k=m+1 as the following. In addition, we have

$$\left\|x_{m+1}-x_*\right\|\le g\left(r_0^p;\right)\left\|x_m-x_*\right\|\le\left\|x_0-x_*\right\|<r_0,\tag{42}$$

and, hence, $x_{m+1}\in\mathbb{N}(x_*,r_0)$. Now, the conclusion what we are proving above is as follows.

5. Semilocal Convergence of the Newton-PGSS Method

Assumption 2. For all $x\in\mathbb{N}(x_0,r)\subset\mathbb{N}_0$, where $r<(1/2)\sqrt[p]{(1/(L\gamma))}$, assume the following conditions hold.

(A1) The bounded condition: there exist positive constants δ and γ such that

$$\begin{aligned}\max\left\{\left\|A(x_0)\right\|,\left\|B(x_0)\right\|\right\}&\le\beta,\\ \left\|F'(x_0)^{-1}\right\|&\le\gamma,\\ \left\|F(x_0)\right\|&\le\delta.\end{aligned}\tag{43}$$

(A2) The Hölder condition: there exist nonnegative constants L_a and L_b for all x, $y\in\mathbb{N}(x_0,r)\in\mathbb{N}_0$

$$\begin{aligned}\|A(x)-A(y)\|&\le L_a\|x-y\|^p,\\ \|B(x)-B(y)\|&\le L_b\|x-y\|^p,\end{aligned}\tag{44}$$

with the exponent $p\in(0,1]$, and we define $L:=L_a+L_b$.

Lemma 2. *Under Assumption 2, for all x, $y\in\in\mathbb{N}(x_*,r)$, then $F'(x)^{-1}$exists, and we have the following inequations:*

$$\begin{aligned}\left\|F'(x)-F'(y)\right\|&\le L\|x-y\|^p,\\ \left\|F'(x)\right\|&\le L\left\|x-x_0\right\|^p+2\beta,\\ \left\|B_{\alpha,\beta}(\Omega;x)-B_{\alpha,\beta}(\Omega;y)\right\|&=\left\|C_{\alpha,\beta}(\Omega;x)-C_{\alpha,\beta}(\Omega;y)\right\|\\ \left\|B_{\alpha,\beta}(\Omega;x)^{-1}\right\|&\le\frac{2\gamma}{1-L\gamma\|x-y\|^p}.\end{aligned}\tag{45}$$

Proof. The proof is omitted since it is the same as Lemma 1.

Theorem 3. *Under Assumption 2, for all x, $y\in\in\mathbb{N}(x_*,r)$, then $F'(x)^{-1}$exists, and we have the inequations in (45).*

Now, we construct the following sequence of functions:

$$g(t) = \frac{1}{1+p} a t^{1+p} - bt + c,$$
$$h(t) = a t^{p} - 1, \tag{46}$$

with the constants satisfying

$$a = \frac{\gamma L(1+\eta)}{1 + 2^{p}\gamma^{1+p}\delta^{p} L\eta},$$
$$b = 1 - \eta, \tag{47}$$
$$c = 2\gamma\delta,$$

where $\eta = \max_k \{\eta_k\} < 1$ *and* $r = \min(r_1, r_2)$; *let* $t_0 = 0$, *and the sequence* t_k *are generated by the following formula:*

$$t_{k+1} = t_k - \frac{g(t_k)}{h(t_k)}. \tag{48}$$

Some properties of the function $g(t)$ *and* $h(t)$ *and the sequence* t_k *are given by the following lemmas.*

Lemma 3. *Assume that constants satisfy*

$$\delta^{p}\gamma^{p+1} L \leq \frac{1-\eta}{2^{p}\left(1+\eta^{2}\right)}, \tag{49}$$

$$\frac{p}{1+p}\sqrt[p]{\frac{b}{a}} > \frac{c}{b}. \tag{50}$$

Denote $t_* = \sqrt[p]{(b/a)}$, *and then, when* $t \in [0, t_*]$, *the following inequalities hold that*

$$g(t) \geq 0,$$
$$g'(t) < 0,$$
$$g''(t) > 0, \tag{51}$$
$$h(t) < g'(t) < 0.$$

Proof. The proof is omitted since it is straightforward.

Theorem 4. *Under the assumptions of lemma in this section,* $r{:} = \min(r_1, r_2)$ *with*

$$r_1 = \sqrt[p]{\frac{\theta\tau}{\gamma L(1+\tau+\theta\tau)}},$$
$$r_2 = \sqrt[p]{\frac{b}{a}}, \tag{52}$$

satisfying

$$r_1 < r_2. \tag{53}$$

And, define $l_0 = \liminf_{k\longrightarrow\infty} l_k$, *and the constant* l_0 *satisfies*

$$l_0 > \lfloor -\frac{-\ln\eta}{\ln((\tau+1)\theta)} \rfloor, \tag{54}$$

where the symbol $\lfloor\cdot\rfloor$ *is used to denote the smallest integer no less than the corresponding real number,* $\tau \in (0, ((1-\theta)/\theta))$ *a prescribed positive constant, and*

$$\theta \equiv \theta(\alpha, \beta; x_0) = \left\| M_{\alpha,\beta}(x_0) \right\| < 1. \tag{55}$$

Then, the iteration sequence $\{x_k\}_{k=0}^{\infty}$ *generated by the Newton* – *PGSS is well defined and converges to* x_*, *which satisfies* $F(x_*)$.

Proof. Firstly, we construct the sequence

$$t_0 = 0,$$
$$t_{k+1} = t_k - \frac{g(t_k)}{h(t_k)}. \tag{56}$$

We have

$$g\left(\sqrt[p]{\frac{b}{a}}\right) > 0. \tag{57}$$

Furthermore, $g(0) = c > 0$; hence, we have r_* which satisfies $g(r_*) = 0$, where $t_1 = 2\gamma\delta$ because (49) and (50). Hence, we have

$$g(2\gamma\delta) > 0,$$
$$t_1 = 2\gamma\delta < r_* < \sqrt[p]{\frac{b}{a}}. \tag{58}$$

Therefore, we have

$$t_0 < t_1 < r_*. \tag{59}$$

Now, we assume that $t_{k-1} < t_k < r_*$, and by making use of mathematical methods of induction, we have

$$t_{k+1} = t_k - \frac{g(t_k)}{h(t_k)}. \tag{60}$$

Because

$$h(t_k) \leq g'(t_k) < 0,$$
$$g(t_k) < 0, \tag{61}$$

hence

$$t_{k+1} > t_k. \tag{62}$$

Furthermore, $m(t) = t - (g(t)/g'(t)) \Rightarrow m'(t) = ((g(t)g''(t))/g'(t)^2)$; then, $m'(t)$is an increasing function in $(0, \sqrt[p]{(b/a)})$ and $-(1/h(t_k)) \leq -(1/g'(t_k))$; hence, we have $t_{k+1} = t_k - (g(t_k)/h(t_k)) \leq t_k - (g(t_k)/g'(t_k)) \leq r_* - (g(r_*)/g'(r_*)) < r_* < r_2$, and it exists as point $t_* \liminf_{k\longrightarrow\infty} r_k$.

Next, prove the following inference by mathematical induction:

$$\|x_{k+1} - x_k\| \le t_{k+1} - t_k,$$
$$\|F'(x_k)\| \le \frac{1 - \gamma L t_k^p}{\gamma(1+\eta)}(t_{k+1} - t_k), \tag{63}$$

where

$$\|x_1 - x_0\| \le \|F'(x_0)^{-1}F(x_0)\| + \|M_{\alpha,\beta}(\Omega; x_0)^{l_*}F'(x_0)^{-1}F(x_0)\| \le \gamma(1 + \theta^{l_*})\delta,$$
$$t_1 - t_0 = 2\gamma\delta,$$
$$\|F(x_0)\| \le \delta \le \frac{2\delta}{1+\eta} = \frac{1 - \gamma L t_0^p}{\gamma(1+\eta)}(t_1 - t_0),$$
$$\|F(x_k)\| \le \|F(x_k) - F(x_{k-1}) - F'(x_k)(x_k - x_{k-1})\| + \|F(x_k) + F'(x_{k-1})(x_k - x_{k-1})\| \le \frac{L}{1+p}\|x_k - x_{k-1}\|^{p+1} + \eta\|F(x_{k-1})\|. \tag{64}$$

Because

$$\delta^p \gamma^{p+1} L \le \frac{1-\eta}{2^p(1+\eta)} \le \frac{1}{2^p},$$
$$\frac{1}{1 - \gamma L t_k^p} \le \frac{-1}{h(t_k)}, \tag{65}$$
$$t_k > t_1 > 2\gamma\delta,$$

we can derive inequality

$$\frac{(1-\eta)\gamma^2}{1 - \gamma L t_k^p} \le \frac{a}{-h(t_k)}. \tag{66}$$

Hence,

$$\frac{\gamma(1+\eta)}{1 - \gamma L t_k^p}\|F(x_k)\| \le \frac{\gamma(1+\eta)}{1 - \gamma L t_k^p}\left(\frac{2}{1+p}(t_k - t_{k-1})^{1+p} + \eta\frac{1 - \gamma L t_{k-1}^p}{\gamma(1+\eta)}(t_k - t_{k-1})\right) \le \frac{1}{1+p}\frac{a}{-h(t_k)}(t_k - t_{k-1})^{p+1} + \frac{\eta}{-h(t_k)}(t_k - t_{k-1}). \tag{67}$$

And because

$$\frac{a}{1+p}(t_k - t_{k-1})^{p+1} + \eta(t_k - t_{k-1}) \le g(t_k) - g(t_{k-1}) - h(t_{k-1})(t_k - t_{k-1})$$
$$= \frac{a}{1+p}\left(t_k^{p+1} - t_{k-1}^{p+1}\right) - b(t_k - t_{k-1}) - \left(a t_k^p - 1\right)(t_k - t_{k-1}),$$
$$= \frac{1}{1+p}a\left(t_k^{p+1} - -t_{k-1}^{p+1}\right) - a t_{k-1}^p(t_k - t_{k-1}) + \eta(t_k - t_{k-1}), \tag{68}$$

we can give

$$\frac{\gamma(1+\eta)}{1 - \gamma L t^k}\|F(x_k)\| \le \frac{g(t_k) - g(t_{k-1}) - h(t_{k-1})(t_k - t_{k-1})}{-h(t_k)} = \frac{g(t_k)}{-h(t_k)} = (t_{k+1} - t_k). \tag{69}$$

Then,

$$\|x_{k+1} - x_k\| \le \left\|\left(1 - M_{\alpha,\beta}(\Omega; x_k)^{l_0}\right)F'(t_k)^{-1}F(x_k)\right\| \le \left(1 + ((\tau+1)\theta)^{l_*}\right)\frac{\gamma}{1 - \gamma L t^k}\|F(x_k)\|, \tag{70}$$

and we can have inequality

$$\frac{(1+\eta)\gamma}{1 - \gamma L t_k}\|F(x_k)\| \le t_{k+1} - t_k. \tag{71}$$

Since the sequence $\{t_k\}_{k=0}^{\infty}$ converges to t_* and

$$\|x_{k+1} - x_0\| \le \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\| \le t_{k+1} - t_k + t_k - t_{k-1} + \cdots + t_1 - t_0 \le r_*, \tag{72}$$

where $r_* < \sqrt[p]{(b/a)}$, the sequence x_k also converges to x_*. The proof has been completed as above.

6. The Modified Newton-PGSS Method and Its Local Convergence

In this section, we improve Newton-PGSS and introduce the modified Newton-PGSS and prove the local convergence of the modified Newton-PGSS method briefly.

The modified Newton method is a kind of algorithm based on the Newton method. Its principle is to reduce the calculation times of the inverse matrix of Jacobian matrix, making the algorithm more efficient. It only needs to calculate inverse matrix once every two steps. The format of the algorithm is shown below:

$$\begin{aligned} F'(x_k)d_k &= -F(x_k), \\ y_k &= x_k + d_k, \\ F'(x_k)h_k &= -F(y_k), \\ x_{k+1} &= y_k + h_k. \end{aligned} \tag{73}$$

Then, we get the modified Newton-PGSS method for solving nonlinear system (1) (Algorithm 4).

Assumption 3. For all $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$, assume the following conditions hold.

(A1) The bounded condition: there exist positive constants δ and γ such that

$$\begin{aligned} \max\{\|A(x_*)\|, \|B(x_*)\|\} &\leq \delta, \\ \left\|F'(x_*)^{-1}\right\| &\leq \gamma. \end{aligned} \tag{74}$$

(A2) The Hölder condition: there exist nonnegative constants K_w and K_t such that

$$\begin{aligned} \|A(x) - A(x_*)\| &\leq K_a\|x - x_*\|^p, \\ \|B(x) - B(x_*)\| &\leq K_b\|x - x_*\|^p, \end{aligned} \tag{75}$$

with the exponent $p \in (0, 1]$.

Lemma 4. *Under Assumption 3, for all $x, y \in \mathbb{N}(x_*, r)$, if $r \in (0, (1/(\gamma K)))^{(1/p)}$, then $F'(x)^{-1}$ exists. And, the following inequalities hold with $K: = K_a + K_b$ for all x, $y \in\in \mathbb{N}(x_*, r)$:*

$$\begin{aligned} \|F'(x) - F'(x_*)\| &\leq K\|x - x_*\|^p, \\ \left\|F'(x)^{-1}\right\| &\leq \frac{\gamma}{1 - \gamma K\|x - x_*\|^p}, \\ \|F(x)\| &\leq \frac{K}{1 + p}\|x - x_*\|^{1+p} + 2\delta\|y - x_*\|. \end{aligned} \tag{76}$$

Theorem 5. *Under the assumptions of Lemma 4, suppose $r \in (0, r_0)$ and define $r_0: = \min_{1\leq j\leq 3}\{r_+^{(j)}\}$, where*

$$\begin{aligned} r_+^{(1)} &= \sqrt[q]{\frac{\tau\theta}{\gamma K(1 + \theta + \tau\theta)}}, \\ r_+^{(2)} &= \sqrt[q]{\frac{1 - 2\delta((\tau + 1)\theta)^{l_0}}{((4 + 2p)/(1 + p))K\gamma}}, \\ r_+^{(3)} &= \sqrt[q]{\frac{1}{\gamma K}}, \end{aligned} \tag{77}$$

with $l_0 = \liminf_{k\longrightarrow\infty} l_k$, and the constant u satisfies

$$l_0 > \left\lfloor -\frac{\ln 2\,\delta}{\ln((\tau + 1)\theta)}\right\rfloor, \tag{78}$$

where the symbol $\lfloor\cdot\rfloor$ is used to denote the smallest integer no less than the corresponding real number, $\tau \in (0, ((1 - \theta)/\theta))$ is a prescribed positive constant, and

$$\begin{aligned} \theta \equiv \theta(\alpha, \beta; x_*) &= \left\|M_{\alpha,\beta}(x_*)\right\| \\ &\leq \max_{\lambda_i\in\lambda(\Omega^{(1/2)}F'(x_*)\Omega^{(1/2)})}\left|\frac{1 - \lambda_i}{1 + \lambda_i}\right| \equiv \sigma(\alpha, \beta; x_*) < 1, \end{aligned} \tag{79}$$

with α and β are more than 0.

Then, for any $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0, t \in (0, r)$ and $c > u$, it holds that

$$\begin{aligned} &\left\|M_{\alpha,\beta}(V; x)\right\| \leq (\tau + 1)\theta < 1, \\ &g(t^p; c) = \frac{\gamma}{1 - \gamma K t^p}\left(\frac{3 + p}{1 + p}Kt^p + 2\beta[(\tau + 1)\theta]^c\right) \\ &\leq g\left(r_0^p; u\right) < 1. \end{aligned} \tag{80}$$

Proof. It is the same as Theorem 2.

In Theorem 5, we get the fact that $\|x^{(1)} - x_*\| \leq g(r_0^p;)\|x^{(0)} - x_*\|$ which is the modified Newton-PGSS has the similar result as the following.

Theorem 6. *Under the conditions of Theorem 5, we have the fact that, for any $x_0 \in \mathbb{N}(x_*, r)$with corresponding $\{l_k\}_{k=0}^{\infty}, \{m_k\}_{k=0}^{\infty}$ of positive integers, the iteration sequence $\{x_k\}_{k=0}^{\infty}$ which is generated by the modified Newton-PGSS method is well defined and converges to x_*. Furthermore, it has the following properties:*

$$\lim_{t\longrightarrow\infty} \sup\|x_k - x_*\|^{(1/k)} \leq g\left(r_0^p; u\right)^2. \tag{81}$$

Proof. The proof of $\|y_{k+1} - x_*\| \leq g(\|x_k - x_*\|^p; l_k)\|x_k - x_*\|$ is the same as $\|x_{k+1} - x_*\| \leq g(\|x_k - x_*\|^p; l_k)\|x_k - x_*\|$ in

(1) Given an initial guess x_0, a nonnegative constant α, a positive constant β, and two positive integer sequences $\{l_k\}_{k=0}^{\infty}$, $\{m_k\}_{k=0}^{\infty}$.
(2) For $k = 0, 1, \ldots,$ until $\|F(x_k)\| \le \text{tol}\|F(x_0)\|$ do:
(2.1) Set $d_{k,0} = h_{k,0}: = 0$.
(2.2) For $l = 0, 1, \ldots, l_k - 1$, apply algorithm PGSS to the linear system (12):
$(\Omega(x_k) + F'(x_k))\, d_{k,l+(1/2)} = (\alpha\Omega(x_k) - F'(x_k))d_{k,l} - F(x_k)$,
and obtain d_{k,l_k} such that
$\|F(x_k) + F'(x_k)d_{k,l_k}\| \le \eta_k\|F(x_k)\|$, for some $\eta_k \in [0,1)$.
(2.3) Set
$y_k = x_k + d_{k,l_k}$.
(2.4) For $m = 0, 1, \ldots, m_k - 1$, apply algorithm PGSS to the linear system (12):
$(\Omega(x_k) + F'(x_k))h_{k,m+1} = (\alpha\Omega(x_k) - F'(x_k))d_{k,l} - F(y_k)$,
and obtain h_{k,m_k} such that
$\|F(x_k) + F'(x_k)m_{k,h_k}\| \le \widetilde{\eta}_k\|F(y_k)\|$, for some $\widetilde{\eta}_k \in [0,1)$.
(2.5) Set
$x_{k+1} = y_k + h_{k,m_k}$,
Where

$$\Omega(x_k) = \begin{pmatrix} \alpha I_1(x_k) & 0 \\ 0 & \beta I_2(x_k) \end{pmatrix}.$$

I_1 *is a n × n i de ntit y matrix an d* I_2 *is a m × m i de ntit y matrix*

obtain the following uniform expressions for d_{k,l_k} and h_{k,m_k},

$$d_{k,l_k} = -\sum_{j=0}^{l_k-1} M_{\alpha,\beta}(\Omega; x_k)^j G_{\alpha,\beta}(\Omega; x_k) F(x_k),$$

$$h_{k,m_k} = -\sum_{j=0}^{m_k-1} M_{\alpha,\beta}(\Omega; x_k)^j G_{\alpha,\beta}(\Omega; x_k) F(y_k),$$

$M_{\alpha,\beta}(\Omega; x)$ and $G_{\alpha,\beta}(\Omega; x)$ are defined as well as Section 3. Then, the modified Newton-PGSS method can be rewritten as

$$x_{k+1} = x_k - \sum_{j=0}^{l_k-1} M_{\alpha,\beta}(\Omega; x_k)^j G_{\alpha,\beta}(\Omega; x_k) F(x_k)$$
$$- \sum_{j=0}^{m_k-1} M_{\alpha,\beta}(\Omega; x_k)^j G_{\alpha,\beta}(\Omega; x_k) F(y_k),\ k = 0, 1, 2, \ldots,$$

The modified Newton-PGSS method can be equivalently expressed as

$$x_{k+1} = x_k - (I - M_{\alpha,\beta}(\Omega; x_k)^{l_k})F'(x_k)^{-1}F(x_k)$$
$$-(I - M_{\alpha,\beta}(\Omega; x_k)^{m_k})F'(x_k)^{-1}F(y_k),\ k = 0, 1, 2, \ldots,$$

In the following, we analyze the local convergence, and its condition (including assumption) and local convergence theorem are the same as Theorem 2 because their $M_{\alpha,\beta}(\Omega; x)$ and $G_{\alpha,\beta}(\Omega; x)$ are the same. Thus, we only restate them now.

Algorithm 4: MN-PGSS (modified Newton-PGSS method).

Theorem 2. And, from the defination of x_k and in Lemma 4, we can easily get that

$$\begin{aligned} \left\|x_{k+1} - x_*\right\| &= \left\|y_k - x_* - \left(I - M(\alpha,\beta; x_k)^{m_k}\right)F'(x_k)^{-1}F(y_k)\right\| \\ &= \left\|y_k - x_* - F'(x_k)^{-1}F(y_k)\right\| + \left\|M(\alpha,\beta; x_k)^{m_k}F'(x_k)^{-1}F(y_k)\right\|, \end{aligned} \tag{82}$$

where

$$y_k - x_* - F'(x_k)^{-1}F(y_k) = -F'(x_k)^{-1}\left(F\left(y_k - F(x_*) - F'\left(x_{(*)}\right)\right)\right)(y_k - x_*) + F'(x_k)^{-1}\left(F'(x_k) - F'(x_*)\right)(y_k - x_*). \tag{83}$$

By Lemma 4 and similar to the proof of $\|F(x)\|$, we can get it:

$$\begin{aligned} y_k - x_* - F'(x_k)^{-1}F(y_k) &\le \frac{\gamma}{1-\gamma K\|x_k - x_*\|^p}\left(\frac{K}{1+p}\|y_k - x_*\|^p + K\|x_k - x_*\|^p\right)\|y_k - x_*\| \\ \left\|M(\alpha,\beta;x_k)^{m_k}F'(x_k)^{-1}F(y_k)\right\| &= \left\|M(\alpha,\beta;x_k)^{m_k}F'(x_k)^{-1}\left(F(y_k) - F(x_*) - F'(x_*)(y_k - x_*) + F'(x_*)(y_k - x_*)\right)\right\| \\ &\le \frac{\gamma((\tau+1)\theta)^{m_k}}{1-\gamma K\|x_k - x - \ *\|^p}\left(\frac{K}{1+p}\|y_k - x_*\|^{1+p} + 2\delta\|y_k - x_*\|\right). \end{aligned} \tag{84}$$

Combining (84) with (85), we can obtain

$$\begin{aligned} \|x_{k+1} - x_*\| &\le \left(\frac{\gamma K}{1-\gamma K\|x_k - x_*\|^p}\left(\frac{1+((\tau+1)\theta)^{m_k}}{1+p}\|y_k - x_*\|^p + \|x_k - x_*\|^p\right)\right. \\ &\quad \left. + \frac{2\delta\gamma((\tau+1)\theta)^{m_k}}{1-\gamma K\|x_k - x_*\|^p}\right)\|y_k - x_*\| \\ &\le \frac{\gamma g\left(\|x_k - x_*\|^p; l_k\right)}{1-\gamma K\|x_k - x_*\|^p}\left(\frac{2Kg\left(\|x_k - x_*\|^p; l_k\right)^p}{1+p}\|y_k - x_*\|^p + K\|x_k - x_*\|^p\right. \\ &\quad \left. + 2\delta((\tau+1)\theta)^{m_k}\right)\|x_k - x_*\| \\ &\le \frac{\gamma g\left(\|x_k - x_*\|^p; l_k\right)}{1-\gamma K\|x_k - x_*\|^p}\left(\frac{3+p}{1+p}K\|x_k - x_*\|^p + 2\delta((\tau+1)\theta)^{m_k}\right)\|x_k - x_*\| \\ &= g\left(\|x_k - x_*\|^p; l_k\right)g\left(\|x_k - x_*\|^p; m_k\right)\|x_k - x_*\| \\ &\le g\left(r_0^p; u\right)^2\|x_k - x_*\|. \end{aligned} \tag{85}$$

By utilizing mathematical induction, we can get the fact that any $x_0 \in \mathbb{N}(x_*, r)$ and nonnegative integer k, and we have

$$\begin{aligned} \|x_{k+1} - x_*\| &\le g\left(r_0^p; u\right)^2\|x_k - x_*\| \le \cdots \\ &\le g\left(r_0^p; u\right)^{2k}\|x_1 - x_*\| \le g\left(r_0^p; u\right)^{2k}\|x_1 - x_*\| \\ &\le g\left(r_0^p; u\right)^{2(k+1)}\|x_0 - x_*\|. \end{aligned} \tag{86}$$

Because $g(r_0^p; u) < 1$, we can make a conclusion that x_0 converges to x_* as $n \longrightarrow +\infty$ from (86. The proof of theorem is completed.

7. Numerical Example

In this section, we show the efficiency of the modified Newton-PGSS method. Because such problems have not been analyzed before, in this paper, the first step is that we just compare the modified Newton-PGSS method with the

Table 1: Numerical results of inexact Methods for $\nu = 0.1$ and $\eta = 0.4$.

n	Method	Optimal values of $\alpha/(\alpha, \beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
	Newton-PGSS	(9.7, 7.8)	2.2284×10^{-4}	3.89	11	5
16	Newton-MPGSS	(9.7, 7.8)	6.1064×10^{-5}	4.36	7	5
	Newton-RHSS	14.4	2.3976×10^{-4}	7.12	13	10
	Newton – PGSS	(11.3, 9.2)	3.3053×10^{-4}	15.02	12	6
20	Newton-MPGSS	(11.3, 9.2)	1.6762×10^{-4}	14.7	5	6.6
	Newton – RHSS	17	3.3324×10^{-4}	26.92	13	13.6
	Newton – PGSS	(15.5, 9.9)	5.31×10^{-4}	227.52	13	10.3
32	Newton-MPGSS	(11.3, 9.2)	1.4157×10^{-4}	227.8	7	14.4
	Newton – RHSS	20.6	3.5082×10^{-4}	453.84	15	28.2

Table 2: Numerical results of inexact methods for $\nu = 1$ and $\eta = 0.4$.

n	Method	Optimal values of $\alpha/(\alpha, \beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
	Newton-PGSS	(8.8, 7.4)	1.6253×10^{-4}	20.74	15	44.7
16	Newton-MPGSS	(11.3, 9.2)	1.1582×10^{-4}	30.8	8	39
	Newton-RHSS	13.2	1.7534×10^{-4}	82.2	13	124.9
	Newton – PGSS	(7, 7)	2.6262×10^{-4}	195.01	14	85.9
20	Newton-MPGSS	(13, 6)	3.2972×10^{-4}	140.59	7	45.4
	Newton – RHSS	18	2.2695×10^{-4}	253	13	167.5
	Newton – PGSS	(9.4, 10.2)	2.297×10^{-4}	3562	15	153.3
32	Newton-MPGSS	(11.4, 9.8)	8.8723×10^{-5}	3593	8	135.5
	Newton – RHSS	21.7	3.596×10^{-4}	5488.3	13	327.5

Table 3: Numerical results of inexact methods for $\nu = 0.1$ and $\eta = 0.2$.

n	Method	Optimal values of $\alpha/(\alpha, \beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
	Newton-PGSS	(7, 7)	7.6864×10^{-5}	6.57	8	11.5
16	Newton-MPGSS	(7.7, 9.6)	2.5763×10^{-4}	4.66	4	9
	Newton-RHSS	18.6	9.8486×10^{-5}	8.34	9	16.1
	Newton – PGSS	(8.2, 9)	3.3090×10^{-4}	15.43	8	12.4
20	Newton-MPGSS	(12, 6)	2.5400×10^{4}	14.4	4	8.4
	Newton – RHSS	14.6	2.0703×10^{-4}	21.44	8	23.8
	Newton – PGSS	(8.2, 9.5)	2.2875×10^{-4}	394.4	8	29.2
32	Newton-MPGSS	(10.8, 13.5)	1.7649×10^{-4}	346.78	4	21
	Newton – RHSS	20.6	1.9104×10^{-4}	507.3	7	48.5

Table 4: Numerical results of inexact methods for $\nu = 1$ and $\eta = 0.2$.

n	Method	Optimal values of $\alpha/(\alpha, \beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
	Newton-PGSS	(8, 7.6)	2.7300×10^{-4}	41.67	9	75.9
16	Newton-MPGSS	(20, 7)	1.2180×10^{-4}	33.15	5	47.2
	Newton-RHSS	17.8	8.4271×10^{-5}	86.22	8	218.8
	Newton – PGSS	(8, 9)	3.3351×10^{-4}	174	9	112.4
20	Newton-MPGSS	(8, 9.6)	3.5554×10^{-5}	93.2	5	142.3
	Newton – RHSS	18	9.3834×10^{-5}	258	8	289.6
	Newton – PGSS	(8, 9.2)	1.95×10^{-4}	3784	9	304.9
32	Newton-MPGSS	(10.8, 13.5)	1.485×10^{-4}	3679	5	219.2
	Newton – RHSS	21.7	1.4216×10^{-4}	5944	8	564.5

Newton-PGSS method and Newton-RHSS as their inner iterations are splitting methods. And, the second step, we will discuss which is more effective as preconditioner in Newton-GMRES algorithm. The numerical results in Example 1 were computed using MATLAB Version R2011b, on an iMac with a 3.20 GHz Intel Core i5-6500 CPU, and 8.00 GB RAM, with machine accuracy eps $= 2.22 \times 10^{-16}$.

Table 5: Numerical results of inexact methods for $\nu = 0.1$ and $\eta = 0.1$.

n	Method	Optimal values of $\alpha/(\alpha,\beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
	Newton-PGSS	(8.2, 7)	2.7447×10^{-4}	3	7	10.1
16	Newton-MPGSS	(13, 6)	5.8466×10^{-5}	3.85	4	7.3
	Newton-RHSS	14.4	2.0678×10^{-4}	6.35	7	20.6
	Newton – PGSS	(8.2, 7.6)	3.4582×10^{-4}	12.24	7	15.6
20	Newton-MPGSS	(7.6, 11)	1.0698×10^{-4}	11.65	4	17.5
	Newton – RHSS	16.8	2.6135×10^{-4}	23.68	7	28.3
	Newton – PGSS	(8.2, 7.5)	3.2225×10^{-4}	369	7	38.9
32	Newton-MPGSS	(10.8, 13.4)	7.5135×10^{-5}	306	3	31
	Newton – RHSS	20.2	4.4700×10^{-4}	648	6	62.7

Example 1. Consider the Stokes flow problem. Find $\mathbf{u}$ and $\mathbf{w}$ such that

$$\begin{cases} -\nu\triangle\mathbf{u} + \nabla\mathbf{w} = \tilde{f}, & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = \tilde{g}, & \text{in } \Omega, \\ u(t,x,y) = 0, & \text{on } (0,1] \in \partial\Omega, \\ \int_{\Omega}\mathbf{w}(x)\mathrm{d}x = 0, \end{cases} \tag{87}$$

where $\Omega = (0,1)\times(0,1)$, with $\partial\Omega$ is the boundary of Ω, $\mathbf{u}$ is a vector-valued function representing the velocity $\nu > 0$ is the viscosity constant, $\triangle$ is the componentwise Laplace operator, and $\mathbf{w}$ is a scalar function representing the pressure. By discreting the function above with the upwind scheme, we obtain the saddle point problem in which

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} I\otimes T + T\otimes I & 0 \\ 0 & I\otimes T + T\otimes I \end{pmatrix} \in \mathbb{R}^{2p^2\times 2p^2}, \\ \mathbf{B} &= \begin{pmatrix} I\otimes F \\ F\otimes I \end{pmatrix} \in \mathbb{R}^{2p^2\times p^2}, \end{aligned} \tag{88}$$

where

$$\begin{aligned} T &= \frac{\nu}{h^2}\cdot\text{tridiag}(-1,2,-1) \in \mathbb{R}^{p\times p}, \\ F &= \frac{1}{h}\cdot\text{tridiag}(-1,1,0) \in \mathbb{R}^{2p^2\times 1}, \\ \tilde{f} &= \left(e^{u_1 1}, e^{u_1 2}, \ldots, e^{u_2 1}, \ldots, e^{u_p p}, e^{v_1 1},\right. \\ &\quad \left. e^{v_1 2}, \ldots, e^{v_2 1}, \ldots, e^{v_p p}\right)^T \in \mathbb{R}^{p\times p}, \\ \tilde{g} &= (1,1,\ldots,1)^T \in \mathbb{R}^{p^2\times 1}, \end{aligned} \tag{89}$$

with $\otimes$ being the Kronecker product symbol. By applying the centered finite difference scheme on the equidistant discretization grid with the step size $\Delta t = h = (1/(N+1))$, the system of nonlinear equations (1) is obtained with the following form:

$$F(u) = Mu + \Psi(u) = 0, \tag{90}$$

where

$$\begin{aligned} M &= \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & 0 \end{pmatrix}, \\ \Psi(u) &= \left(e^{u_1 1}, e^{u_1 2}, \ldots, e^{u_2 1}, \ldots, e^{u_p p}, e^{v_1 1},\right. \\ &\quad \left. e^{v_1 2}, \ldots, e^{v_2 1}, \ldots, e^{v_p p}, -1, -1, \ldots, -1\right)^T. \end{aligned} \tag{91}$$

Then, the Jacobian matrix is

$$\begin{aligned} F'(u) = M + \text{diag}&\left(\left(e^{-u_{11}}, e^{-u_{12}}, \ldots, e^{-u_{21}},\right.\right. \\ &\left.\left.\ldots, e^{-u_{pp}}, e^{-v_{11}}, e^{-v_{12}}, \ldots, e^{-v_{21}}, \ldots, e^{-v_{pp}}, 1, 1, \ldots, 1\right)\right). \end{aligned} \tag{92}$$

Firstly, we compare the algorithms whose inner iterations are splitting methods, such as Newton RHSS, Newton PGSS, and modified Newton PGSS. The parameters needed in the problem are chosen by using the traversal method for the purpose of comparison: the initial guess $u_0 = 0$, the stopping criterion for the outer iteration is set to be

$$\frac{\|F(u_k)\|_2}{\|F(u_0)\|_2} \le 10^{-5}, \tag{93}$$

and the prescribed tolerance η_k and $\tilde{\eta}_k$ for controlling the accuracy of the iteration are both set to be η, which satisfies inequality

$$\frac{\left\|F'(u_k)d_{k,l_k} + F(u_k)\right\|_2}{\|F(u_k)\|_2} \le \eta. \tag{94}$$

For different inner tolerance $\eta = 0.4$, 0.2, and 0.1 and problem parameters $\nu = 1$ and 0.1, the results about outer IT, inner IT, and CPU are listed in the numerical tables corresponding to the referred inexact Newton methods. Because the linear matrix of the solution is different in each iteration, there is no way to find the optimal parameters in theory. Thus, we get the most efficient algorithm by traversing for the parameters of different algorithms, and then, we tabulate these results. For the selection of a single parameter, we traverse the parameters from 0 with an interval of 1 in the beginning. When the number of steps, time, and error show an earlier increase and later decrease trend, the iteration is stopped to determine the range of parameters.

Table 6: Numerical results of inexact methods for $\nu = 1$ and $\eta = 0.1$.

n	Method	Optimal values of $\alpha/(\alpha,\beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
16	Newton-PGSS	(8.2, 8.2)	2.5370×10^{-4}	40.885	7	96
	Newton-MPGSS	(11, 8.4)	3.9974×10^{-5}	40.9	4	85.4
	Newton-RHSS	16	3.6266×10^{-5}	70.25	6	297.3
20	Newton – PGSS	(8, 10.4)	3.4557×10^{-4}	156	7	145.1
	Newton-MPGSS	(11, 7.3)	7.1642×10^{-6}	134	4	155.2
	Newton – RHSS	17.2	3.9646×10^{-5}	272	6	404.3
32	Newton – PGSS	(10.7, 11.7)	5.1146×10^{-4}	3001	7	251.6
	Newton-MPGSS	(11.4, 7.6)	2.6628×10^{-4}	3241	4	308.8
	Newton – RHSS	21.4	5.8568×10^{-5}	5907	6	790.7

Table 7: Numerical results of preconditioned inexact Newton methods for $\nu = 0.1$ and $\eta = 0.4$.

n	Preconditioner for GMRES	Optimal values of $\alpha/(\alpha,\beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
16	–	–	2.2319×10^{-4}	1.2	15	3.2
	RHSS	27.5	2.6701×10^{-4}	4.3	13	1.1
	PGSS	(2, 1)	2.5914×10^{-4}	0.7	7	1
20	–	–	2.4378×10^{-4}	3.97	21	3.95
	RHSS	23	2.6537×10^{-4}	13.82	14	1.1
	PGSS	(8, 1)	1.6342×10^{-4}	2.58	8	1
32	–	–	5.1384×10^{-4}	16.21	16	9.9
	RHSS	24	4.0483×10^{-4}	180.06	15	1.7
	PGSS	(5, 1)	1.7622×10^{-4}	16.52	8	1

Table 8: Numerical results of preconditioned inexact Newton methods for $\nu = 0.1$ and $\eta = 0.2$.

n	Preconditioner for GMRES	Optimal values of $\alpha/(\alpha,\beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
16	–	–	6.4114×10^{-5}	1.06	9	4.4
	RHSS	14	2.6652×10^{-4}	3.2	9	1
	PGSS	(2, 2)	1.3215×10^{-4}	1.36	7	1
20	–	–	1.2495×10^{-4}	1.06	10	6.3
	RHSS	19	3.3791×10^{-4}	9.63	9	1.6
	PGSS	(3, 2)	3.0298×10^{-4}	1.83	7	1
32	–	–	1.5261×10^{-4}	19.6	10	17.2
	RHSS	22	2.6631×10^{-4}	165.6	10	1.9
	PGSS	(2, 1)	1.3380×10^{-4}	16.69	7	1

Table 9: Numerical results of preconditioned inexact Newton methods for $\nu = 0.1$ and $\eta = 0.1$.

n	Preconditioner for GMRES	Optimal values of $\alpha/(\alpha,\beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
16	–	–	1.0703×10^{-4}	1.37	8	5.3
	RHSS	12	1.9074×10^{-4}	3.33	8	1.1
	PGSS	(3, 2)	2.5352×10^{-4}	1.28	6	1
20	–	–	2.1225×10^{-4}	1.81	7	8.4
	RHSS	16	1.3363×10^{-4}	9.61	8	1.6
	PGSS	(3, 2)	1.1526×10^{-4}	2.25	7	1
32	–	–	5.0033×10^{-4}	14.63	7	20.7
	RHSS	30	4.1546×10^{-4}	138.9	7	2.7
	PGSS	(1, 1)	3.2577×10^{-4}	13.98	6	1

We use this method to narrow the parameter range and get "the best parameters at present" until the result (such as step) does not change. For the selection of two parameters (denoted them as α and β), first, we fix the parameter α and traverse the parameter β by using the single parameter traversal method. Then, we fix the parameter β and traverse the parameter α. We repeat the process until the result does not change. We can get information from Tables 1–6 that Newton-PGSS performs better than Newton-RHSS in the iterative CPU. Moreover, the Newton-MPGSS algorithm is

Table 10: Numerical results of preconditioned inexact Newton methods for $v = 1$ and $\eta = 0.4$.

n	Preconditioner for GMRES	Optimal values of $\alpha/(\alpha, \beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
	–	–	1.5522×10^{-4}	3.14	16	29.3
16	RHSS	18	1.7172×10^{-4}	4.07	13	1
	PGSS	(1, 3)	1.8162×10^{-4}	1.81	11	1
	–	–	2.0082×10^{-4}	9.91	17	50
20	RHSS	18	1.8359×10^{-4}	10.48	13	1
	PGSS	(1, 1)	2.8163×10^{-4}	3.32	9	1
	–	–	4.5286×10^{-4}	294.57	39	79.8
32	RHSS	26	3.3912×10^{-4}	138.79	15	1.07
	PGSS	(1, 3)	3.8558×10^{-4}	29.95	10	1

Table 11: Numerical results of preconditioned inexact Newton methods for $v = 1$ and $\eta = 0.2$.

n	Preconditioner for GMRES	Optimal values of $\alpha/(\alpha, \beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
	–	–	7.5565×10^{-5}	2.78	11	39.5
16	RHSS	16	2.3989×10^{-4}	3.22	9	1
	PGSS	(3, 1)	2.1344×10^{-4}	1.48	7	1
	–	–	2.0648×10^{-4}	9.24	11	71.5
20	RHSS	19	3.4282×10^{-4}	8.92	9	1.2
	PGSS	(1, 2)	1.5814×10^{-4}	3.27	7	1
	–	–	3.9128×10^{-4}	264.06	35	80
32	RHSS	23	5.4007×10^{-4}	119.75	10	1.3
	PGSS	(2, 4)	5.0058×10^{-4}	29.61	7	1

Table 12: Numerical results of preconditioned inexact Newton methods for $v = 1$ and $\eta = 0.1$.

n	Preconditioner for GMRES	Optimal values of $\alpha/(\alpha, \beta)$	Error estimates	CPU time (s)	Outer IT	Inner IT
	–	–	1.2584×10^{-4}	2.23	8	53.9
16	RHSS	21	2.0586×10^{-4}	2.93	7	1.3
	PGSS	(1, 4)	1.3197×10^{-4}	1.09	5	1
	–	–	1.1186×10^{-4}	9.14	10	78.8
20	RHSS	18	2.6957×10^{-4}	8.32	7	1.6
	PGSS	(3, 2)	1.7341×10^{-4}	2.7	5	1
	–	–	3.9128×10^{-4}	265.38	35	80
32	RHSS	23	2.8212×10^{-4}	121.84	8	1.9
	PGSS	(2, 9)	3.2145×10^{-4}	26.51	5	1

much better than Newton-RHSS in the number of generation steps.

As we known, Krylov subspace method is more efficient than the stationary iterative methods in saddle point. Secondly, we will compare the effects of PGSS and RHSS as preconditioners on Newton-GMRES. In Tables 7–12, we can find it that PGSS and RHSS are more efficient as preprocessing operators than without them as using GMRES methods. Furthermore the PGSS is more efficient than RHSS as preconditioners. In the inner iteration, RHSS and PGSS are treated as preprocessing operators, and then the Krylov subspace method is used to solve the problem, which is better than the Krylov subspace method in CPU and step number. Although the effect of PGSS as preconditioner is not much better than that of RHSS when n is small, it can be seen that PGSS has great advantages in both steps and CPU compared with RHSS with the increase of n.

8. Conclusions

The Newton-PGSS method is a considerable method for solving large sparse nonlinear system with nonsymmetric saddle point problems with the nonsymmetric Jacobian matrix. This is the first time to solve this kind of problem, and we utilize the PGSS iteration as the inner solver for the Newton equation. And, we establish a modified Newton-PGSS method for solving large sparse nonlinear system with nonsymmetric saddle point problems with the nonsymmetric Jacobian matrix. We give the local convergence and semilocal convergence analysis of the new method under proper conditions. Finally, the numerical results show that the modified Newton-PGSS outperforms the other splitting method in the sense of CPU time and iterative steps. Furthermore, when we apply the Newton-GMRES method to solve the problems, PGSS will accelerate the algorithm as preconditioner and make it more efficient than RHSS.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant nos. 11771393 and 11632015).

References

[1] Y. Cao, S. Li, and L.-Q. Yao, "A class of generalized shift-splitting preconditioners for nonsymmetric saddle point problems," *Applied Mathematics Letters*, vol. 49, pp. 20–27, 2015.

[2] Y. Cao, "Regularized DPSS preconditioners for non-Hermitian saddle point problems," *Applied Mathematics Letters*, vol. 84, pp. 96–102, 2018.

[3] Z.-Z. Bai, "On spectral clustering of HSS preconditioner for generalized saddle-point matrices," *Linear Algebra and Its Applications*, vol. 555, pp. 285–300, 2018.

[4] K. Wang, J. Di, and D. Liu, "Improved PHSS iterative methods for solving saddle point problems," *Numerical Algorithms*, vol. 71, no. 4, pp. 753–773, 2016.

[5] Y. Dou, A.-L. Yang, and Y.-J. Wu, "Modified parameterized inexact Uzawa method for singular saddle-point problems," *Numerical Algorithms*, vol. 72, no. 2, pp. 325–339, 2016.

[6] D. K. Salkuyeh, "Shifted skew-symmetric/skew-symmetric splitting method and its application to generalized saddle point problems," *Applied Mathematics Letters*, vol. 103, pp. 106–114, 2020.

[7] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, NY, USA, 1970.

[8] D. K. Salkuyeh and M. Masoudi, "A new relaxed HSS preconditioner for saddle point problems," *Numerical Algorithms*, vol. 74, no. 3, pp. 781–795, 2017.

[9] W. C. Rheinboldt, "A unified convergence theory for a class of iterative processes," *SIAM Journal on Numerical Analysis*, vol. 5, no. 1, pp. 42–63, 1968.

[10] A.-L. Yang, "Scaled norm minimization method for computing the parameters of the HSS and the two-parameter HSS preconditioners," *Numerical Linear Algebra with Applications*, vol. 25, no. 4, Article ID e2169, 2018.

[11] R. S. Dembo, S. C. Eisenstat, and T. Steihaug, "Inexact Newton methods," *SIAM Journal on Numerical Analysis*, vol. 19, no. 2, pp. 400–408, 1982.

[12] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, Philadelphia, PA, USA, 2nd edition, 2003.

[13] W. C. Rheinboldt, *Methods of Solving Systems of Nonlinear Equations*, SIAM, Philadelphia, PA, USA, 2nd edition, 1998.

[14] H.-B. An and Z.-Z. Bai, "A globally convergent Newton-GMRES method for large sparse systems of nonlinear equations," *Applied Numerical Mathematics*, vol. 57, no. 3, pp. 235–252, 2007.

[15] H.-B. An, Z.-Y. Mo, and X.-P. Liu, "A choice of forcing terms in inexact Newton method," *Journal of Computational and Applied Mathematics*, vol. 200, no. 1, pp. 47–60, 2007.

[16] D. A. Knoll and D. E. Keyes, "Jacobian-free Newton-Krylov methods: a survey of approaches and applications," *Journal of Computational Physics*, vol. 193, no. 2, pp. 357–397, 2004.

[17] Q.-Q. Shen and Q. Shi, "Generalized shift-splitting preconditioners for nonsingular and singular generalized saddle point problems," *Computers & Mathematics with Applications*, vol. 72, no. 3, pp. 632–641, 2016.

[18] Y. Cao, Z.-Q. Shi, and Q. Shi, "Regularized DPSS preconditioners for generalized saddle point linear systems," *Computers & Mathematics with Applications*, vol. 80, no. 5, pp. 956–972, 2020.

[19] Z.-G. Huang, L.-G. Wang, Z. Xu, and J.-J. Cui, "Improved PPHSS iterative methods for solving nonsingular and singular saddle point problems," *Computers & Mathematics with Applications*, vol. 72, no. 1, pp. 92–109, 2016.

[20] X.-P. Guo, "On semilocal convergence of inexact Newton methods," *Journal of Computational Mathematics*, vol. 25, pp. 231–242, 2007.

[21] S.-W. Zhou, A.-L. Yang, Y. Dou, and Y.-J. Wu, "The modified shift-splitting preconditioners for nonsymmetric saddle-point problems," *Applied Mathematics Letters*, vol. 59, pp. 109–114, 2016.

[22] J. Zhou, S. Zhang, G. Yang, and J. Tan, "A convergence theorem for the inexact Newton methods based on Hlder continuous Fréchet derivative," *Applied Mathematics and Computation*, vol. 197, no. 1, pp. 206–211, 2008.

[23] Q. Zheng and L. Lu, "Extended shift-splitting preconditioners for saddle point problems," *Journal of Computational and Applied Mathematics*, vol. 313, pp. 70–81, 2017.

[24] Y. Xu and L. Yang, "Optimization of a parameterized inexact Uzawa method for saddle point problems," *International Journal of Computer Mathematics*, vol. 95, no. 12, pp. 2539–2548, 2017.

[25] J. H. Bramble, J. E. Pasciak, and A. T. Vassilev, "Analysis of the inexact Uzawa algorithm for saddle point problems," *SIAM Journal on Numerical Analysis*, vol. 34, no. 3, pp. 1072–1092, 1997.

[26] Y. Cao, M.-Q. Jiang, and Y.-L. Zheng, "A splitting preconditioner for saddle point problems," *Numerical Linear Algebra with Applications*, vol. 18, no. 5, pp. 875–895, 2011.

[27] Y. Cao, J. Du, and Q. Niu, "Shift-splitting preconditioners for saddle point problems," *Journal of Computational and Applied Mathematics*, vol. 272, pp. 239–250, 2014.

[28] Y. Cao, S.-X. Miao, and Z.-R. Ren, "On preconditioned generalized shift-splitting iteration methods for saddle point problems," *Computers & Mathematics with Applications*, vol. 74, no. 4, pp. 859–872, 2017.

[29] Y. Cao, Z. Zhen, and L. Yao, "Improved relaxed positive-definite and skew-Hermitian splitting preconditioners for saddle point problems," *Journal of Computational Mathematics*, vol. 37, pp. 95–111, 2018.

[30] Z.-Z. Bai, J.-F. Yin, and Y.-F. Su, "A shift-splitting preconditioner for non-Hermitian positive definite matrices," *Journal of Computational Mathematics*, vol. 24, pp. 539–552, 2006.

[31] G. H. Golub, X. Wu, and J.-Y. Yuan, "SOR-like methods for augmented systems," *Bit Numerical Mathematics*, vol. 41, no. 1, pp. 71–85, 2001.

[32] Z.-Z. Bai, B. N. Parlett, and Z.-Q. Wang, "On generalized successive overrelaxation methods for augmented linear systems," *Numerische Mathematik*, vol. 102, no. 1, pp. 1–38, 2005.

[33] Z.-Z. Bai and G. H. Golub, "Accelerated hermitian and skew-hermitian splitting iteration methods for saddle-point problems," *IMA Journal of Numerical Analysis*, vol. 27, no. 1, pp. 1–23, 2007.

[34] Z.-Z. Bai, "Regularized HSS iteration methods for stabilized saddle-point problems," *IMA Journal of Numerical Analysis*, vol. 39, no. 4, pp. 1888–1923, 2019.

[35] M. Chen, Q. Wu, and R. Lin, "Semilocal convergence analysis for the modified Newton-HSS method under the Hölder condition," *Numerical Algorithms*, vol. 72, no. 3, pp. 667–685, 2016.

[36] F. Xie, R. F. Lin, and Q. B. Wu, "Modified Newton-DSS method for solving a class of systems of nonlinear equations with complex symmetric Jacobian matrices," *Numerical Algorithms*, vol. 19, pp. 1–25, 2019.

[37] M.-H. Chen and Q.-B. Wu, "On modified Newton-DGPMHSS method for solving nonlinear systems with complex symmetric Jacobian matrices," *Computers & Mathematics with Applications*, vol. 76, no. 1, pp. 45–57, 2018.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6624509, 13 pages
https://doi.org/10.1155/2021/6624509

Research Article

New Inertial Relaxed *CQ* Algorithms for Solving Split Feasibility Problems in Hilbert Spaces

Haiying Li,[1] **Yulian Wu**,[1] **and Fenghui Wang**[2]

[1]*College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China*
[2]*Department of Mathematics, Luoyang Normal University, Luoyang 471934, China*

Correspondence should be addressed to Haiying Li; haiyingli2020@163.com

Received 3 January 2021; Revised 24 January 2021; Accepted 28 January 2021; Published 16 February 2021

Academic Editor: Xiaolong Qin

The split feasibility problem (SFP) has received much attention due to its various applications in signal processing and image reconstruction. In this paper, we propose two inertial relaxed *CQ* algorithms for solving the split feasibility problem in real Hilbert spaces according to the previous experience of applying inertial technology to the algorithm. These algorithms involve metric projections onto half-spaces, and we construct new variable step size, which has an exact form and does not need to know a prior information norm of bounded linear operators. Furthermore, we also establish weak and strong convergence of the proposed algorithms under certain mild conditions and present a numerical experiment to illustrate the performance of the proposed algorithms.

1. Introduction

The split feasibility problem in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] in 1994, for modeling inverse problem that arises from the phase retrievals and in medical image reconstruction [2]. The split feasibility problem can also be used to model the intensity-modulated radiation therapy [3].

Let H_1 and H_2 be two real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. C and Q are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and A is a linear bounded operator from H_1 into H_2. The split feasibility problem (SFP) is formulated as follows: find a point $x \in H_1$ satisfying

$$\begin{aligned} &x \in C, \\ &Ax \in Q. \end{aligned} \tag{1}$$

The solution set of the problem (SFP) (1) is denoted by S; that is,

$$S := \{x \in C\colon\ Ax \in Q\}. \tag{2}$$

A very successful method that solves the (SFP) seems to be the CQ algorithm of Byrne [4], which generates $\{x_n\}$ by the iterative procedure: for any initial guess $x_1 \in H$,

$$x_{n+1} = P_C\left(x_n - \gamma A^*\left(I - P_Q\right)Ax_n\right), \quad \forall n \geq 1, \tag{3}$$

where P_C and P_Q are the metric projections onto C and Q, respectively. A^* is the adjoint operator of the linear operator A, and the step size γ is chosen in the open interval $(0, 2/\|A\|^2)$. The step size selection depends on the operator norm (or the largest eigenvalue of A^*A), which also is not a simple work.

The CQ algorithm (3) for solving the problem (SFP) (1) can be obtained from optimization. If we introduce the convex objective function

$$f(x) := \frac{1}{2}\left\|\left(I - P_Q\right)Ax\right\|^2, \quad x \in H_1, \tag{4}$$

then the CQ algorithm (3) comes immediately as a special case of the gradient-projection algorithm (GPA), since the convex objective function f is differentiable and has a Lipschitz gradient given by

$$\nabla f(x) = A^*\left(I - P_Q\right)Ax. \tag{5}$$

To overcome the computational difficulties, many authors have constructed the variable step size that does not require the norm $\|A\|$; see, for example, [5–12]. In particular, Lopez et al. [7] introduced a new choice of the variable step size sequence τ_n as follows:

$$\tau_n := \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad \forall n \geq 1, \tag{6}$$

where $\{\rho_n\}$ is a sequence of positive real numbers, take zero for the lower bound and four for the upper bound. The advantage of the choice (6) of step size is that there is neither prior information about the matrix norm A nor any other conditions on Q and A.

Now let us consider the case when C and Q are level subsets of convex functions, where C and Q are, respectively, given by

$$\begin{aligned} C &= \{x \in H_1: c(x) \leq 0\}, \\ Q &= \{y \in H_2: q(y) \leq 0\}, \end{aligned} \tag{7}$$

where $c: H_1 \longrightarrow (-\infty, +\infty]$ and $q: H_2 \longrightarrow (-\infty, +\infty]$ are two lower semicontinuous convex functions, and ∂c and ∂q are bounded operators. But the associated projections P_C and P_Q do not have closed-form expressions, and the CQ algorithm is that the iterative process cannot be performed. In order to keep it going, Yang [13] made improvements to these two-level subsets; here is how they are defined:

$$\widetilde{C_n} = \{x \in H_1: c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\}, \tag{8}$$

with $\xi_n \in \partial c(x_n)$, and

$$\widetilde{Q_n} = \{y \in H_2: q(Ax_n) + \langle \zeta_n, y - Ax_n \rangle \leq 0\}, \tag{9}$$

with $\zeta_n \in \partial q(Ax_n)$.

It is easy to see that $\widetilde{C_n}$ and $\widetilde{Q_n}$ are both half-spaces, and the projections $P_{\widetilde{C_n}}$ and $P_{\widetilde{Q_n}}$ have closed-form expressions. In what follows, for each $n \geq 1$, define

$$\begin{aligned} f_n(x) &:= \frac{1}{2}\left\|\left(I - P_{\widetilde{Q_n}}\right)Ax\right\|^2, \\ \nabla f_n(x) &= A^*\left(I - P_{\widetilde{Q_n}}\right)Ax. \end{aligned} \tag{10}$$

Since these projections are easy to calculate, the algorithm is very practical.

Afterwards, the inertial technique was developed by Alvarez and Attouch in order to improve the performance of proximal point algorithms [14]. Dang et al. [15] proposed an inertial relaxed CQ algorithm $\{x_n\}$ for solving the problem (SFP) in a real Hilbert space, which is generated as follows: for any $x_0, x_1 \in H$,

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = P_{\overline{C_n}}\left(w_n - \gamma A^T\left(I - P_{\overline{Q_n}}\right)A(w_n)\right), \end{cases} \tag{11}$$

where $0 < \gamma < (2/\|A\|^2)$, and $0 \leq \theta_n \leq \overline{\theta_n}$ with

$$\begin{aligned} \overline{\theta_n} &= \min\left\{\theta, \frac{1}{\max\left\{n^2\|x_n - x_{n-1}\|^2, n^2\|x_n - x_{n-1}\|^2\right\}}\right\}, \quad \forall n \geq 1, \theta \in [0,1), \\ \overline{C_n} &= \{x \in H_1 | c(w_n) + \langle \xi_n, x - w_n \rangle \leq 0\}, \end{aligned} \tag{12}$$

with $\xi_n \in \partial c(w_n)$, and

$$\overline{Q_n} = \{y \in H_2 | q(Aw_n) + \langle \zeta_n, y - Aw_n \rangle \leq 0\}, \tag{13}$$

with $\zeta_n \in \partial q(Aw_n)$. The algorithm $\{x_n\}$ converges weakly to a point of a solution set of the problem (SFP), where step size also depends on the matrix norm $\|A\|$. It is obvious that the calculation of operator norm is more complicated, so Gibali et al. [16] has changed the step size of (11).

$$\begin{aligned} \lambda_n &= \frac{\rho_n f_n(w_n)}{\eta_n^2}, \\ \eta_n &= \max\{1, \|\nabla f_n(w_n)\|\}, \quad 0 \leq \theta_n \leq \overline{\theta_n}, \end{aligned} \tag{14}$$

where

$$\overline{\theta_n} = \begin{cases} \min\left\{\theta, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|^2}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{15}$$

If $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$, then the sequence $\{x_n\}$ generated by (11) with step size λ_n converges weakly to a point of a solution set of the problem (SFP). For recent results on inertial algorithms (see [17–24]).

On the other hand, the CQ algorithm is the gradient-projection method for the variational inequality problem. In [25], Xu gave weak convergence in the setting of Hilbert spaces. Wang and Xu [26] proposed the following algorithm:

$$x_{n+1} = P_C[(1 - \alpha_n)(x_n - \gamma \nabla f(x_n))], \tag{16}$$

where $\gamma \in (0, 2/\|A\|^2)$. Under some conditions, it is proved that the sequence generated by the algorithm (16) strongly converges to the minimum-norm solution of the (SFP).

Motivated and inspired by the work of [7, 27–29], the authors of [30] introduced a self-adaptive CQ-type algorithm for finding a solution of the (SFP) in the setting of infinite-dimensional real Hilbert spaces; the advantage of this algorithm lies in the fact that step sizes are dynamically chosen and do not depend on the operator norm. This algorithm can be formulated as follows:

$$x_{n+1} = P_{C_n}[(1-\beta_n)(x_n - \lambda_n \nabla f_n(x_n))], \tag{17}$$

where $\lambda_n = (\rho_n f_n(x_n)/\|\nabla f_n(x_n)\|^2)$. It is also proved that the sequence generated by the algorithm (17) strongly converges to the minimum-norm solution of the (SFP) under some conditions.

Inspired by the works mentioned above, we propose a new relaxed CQ algorithm to solve the (SFP) in a real Hilbert space by using inertial technology. The new step size proposed in this algorithm is independent of the operator norm in this paper, and we also establish weak convergence theorem of the proposed algorithms under some mild conditions in [31]. We add the inertial term on the basis of the algorithm in [30] to construct a new iterative process, so that the new algorithm strongly converges to a point in the solution set under some conditions.

The remainder of the paper is organized as follows. Some useful definitions and results are collected in Section 2 for the convergence analysis of the proposed algorithm. In Section 3, new inertial algorithms of weak and strong convergence for solving SFP are proposed, followed by the convergence analysis. In Section 4, we provide a numerical experiment to illustrate the performance of the proposed algorithms. Finally, we end the paper with some conclusion.

2. Preliminaries

Let H be a Hilbert space and let C be a nonempty closed convex subset in H. The strong (weak) convergence of a sequence $\{x_n\}$ to x is denoted by $x_n \longrightarrow x (x_n \rightharpoonup x)$, respectively. For any sequence $\{x_n\} \subset H$, $\omega_w(x_n)$ denotes the weak $\omega -$ limit set of $\{x_n\}$; that is,

$$\omega_w(x_n) := \left\{x \in H: x_{n_j} \rightharpoonup x\right\}, \text{for some subsequence} \left\{n_j\right\} \text{ of } \{n\}. \tag{18}$$

Definition 1. An operator $T: C \longrightarrow H$ is called the following:

(i) Nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{19}$$

(ii) Firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \tag{20}$$

(iii) ν-inverse strongly monotone (ν-ism) if there is $\nu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \nu \|Tx - Ty\|^2, \quad \forall x, y \in C. \tag{21}$$

For every element $x \in H$, there exists a unique nearest point in C denoted by $P_C x$, such that

$$\|x - P_C x\| = \min\{\|x - y\| \mid y \in C\}. \tag{22}$$

Then operator P_C is called the metric projection from H onto C.

The projection has the following well-known properties.

Lemma 1 (see [32, 33]). *For all $x, y \in H$ and $z \in C$, we have*

(1) $\langle x - P_C x, z - P_C x\rangle \le 0$
(2) $\|P_C x - P_C y\| \le \|x - y\|$
(3) $\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle$
(4) $\|P_C x - z\|^2 \le \|x - z\|^2 - \|(I - P_C)x\|^2$

Lemma 2. *Let H be a real Hilbert space and $x, y, z \in H$, $t \in R$; then*

(1) $\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x - y\|^2$;
(2) $\|x - y\|^2 = \|y - z\|^2 - \|x - z\|^2 + 2\langle x - y, x - z\rangle$.

Definition 2 (see [34]). Let H be a real Hilbert space and let $f: H \longrightarrow (-\infty, \infty)$ be a convex function. An element $v \in H$ is called the subgradient of f at $\overline{x} \in H$ if

$$\langle v, x - \overline{x}\rangle \le f(x) - f(\overline{x}), \quad \forall x \in H. \tag{23}$$

The collection of all the subgradients of f at $\overline{x}$ is called the subdifferential of the function f at this point, which is denoted by $\partial f(\overline{x})$; that is,

$$\partial f(\overline{x}) = \{v \in H: \langle v, x - \overline{x}\rangle \le f(x) - f(\overline{x}), \forall x \in H\}. \tag{24}$$

Definition 3. Let $f: H \longrightarrow (-\infty, +\infty]$ be a proper function.

(i) f is lower semicontinuous at x if $x_n \longrightarrow x$ implies

$$f(x) \le \liminf_{n\longrightarrow\infty} f(x_n). \tag{25}$$

(ii) f is weakly lower semicontinuous at x if $x_n \rightharpoonup x$ implies

$$f(x) \le \liminf_{n\longrightarrow\infty} f(x_n). \tag{26}$$

(iii) f is lower semicontinuous on H if it is lower semicontinuous at every point $x \in H$; f is weakly lower semicontinuous on H if it is weakly lower semicontinuous at every point $x \in H$.

(iv) f is lower semicontinuous if and only if it is weakly lower semicontinuous.

Lemma 3 (see [34]). *Let $f: H \longrightarrow (-\infty, +\infty]$ be an α-strongly convex function. Then, for all $x, y \in H$,*

$$f(y) \geq f(x) + \langle \xi, y - x \rangle + \frac{\alpha}{2}\|y - x\|^2, \quad \xi \in \partial f(x). \quad (27)$$

Lemma 4 (see [25]). *Let $t > 0$ and $x^* \in H$. Then the following statements are equivalent:*

(1) The point x^* solves the problem (SFP).

(2) The point x^* solves the fixed-point equation

$$x^* = P_C\left(x^* - tA^*\left(I - P_Q\right)Ax^*\right). \quad (28)$$

(3) The point x^* solves the variational inequality problem with respect to the gradient of f; that is, find a point $x \in C$ such that

$$\langle \nabla f(x), y - x \rangle \geq 0, \quad \forall y \in C. \quad (29)$$

Lemma 5 (see [16]). *Let H be a real Hilbert space and let $\{x_n\}$ be a sequence in H such that there exists a nonempty closed and convex subset S of H satisfying the following conditions:*

(i) For all $z \in S$, $\lim_{n\longrightarrow\infty}\|x_n - z\|$ exists

(ii) Any weak cluster point of $\{x_n\}$ belongs to S

Then there exists $x^* \in S$ such that $\{x_n\}$ converges weakly to x^*.

Lemma 6 (see [35]). *Let $\{\phi_n\} \subset [0, \infty)$ and $\{\delta_n\} \subset [0, \infty)$ be two nonnegative real sequences satisfying the following conditions:*

(1) $\phi_{n+1} - \phi_n \leq \theta_n(\phi_n - \phi_{n-1}) + \delta_n$

(2) $\sum_{n=1}^{\infty} \delta_n < \infty$

(3) $\{\theta_n\} \subset [0, \theta]$, where $\theta \in [0, 1)$

Then, $\{\phi_n\}$ is a converging sequence and $\sum_{n=1}^{\infty} [\phi_{n+1} - \phi_n]_+ < \infty$, where $[t]_+ = \max\{t, 0\}$ for any $t \in R$.

Lemma 7 (see [36, 37]). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \beta_n)a_n + \delta_n + \gamma_n, \quad n \geq 1, \quad (30)$$

where $\{\beta_n\}_{n=0}^{\infty}$ is a sequence in $(0, 1)$ and $\{\delta_n\}_{n=0}^{\infty}$ is a real sequence. Assume $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then the following results hold:

(1) If $\delta_n \leq \beta_n M$ for some $M \geq 0$, then $\{a_n\}_{n=0}^{\infty}$ is a bounded sequence

(2) If $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\limsup_{n\longrightarrow\infty} (\delta_n/\beta_n) \leq 0$, then $\lim_{n\longrightarrow\infty} a_n = 0$

Lemma 8 (see [38]). *Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{aligned} s_{n+1} &\leq (1 - \alpha_n)s_n + \alpha_n\delta_n, \quad n \geq 1, \\ s_{n+1} &\leq s_n - \lambda_n + \gamma_n, \quad n \geq 1, \end{aligned} \quad (31)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a sequence of nonnegative real numbers, and $\{\delta_n\}$ and $\{\gamma_n\}$ are two sequences in $\mathfrak{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\lim_{n\longrightarrow\infty}\gamma_n = 0$

(3) $\lim_{k\longrightarrow\infty}\lambda_{n_k} = 0$ implies $\limsup_{k\longrightarrow\infty}\delta_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$

Then $\lim_{n\longrightarrow\infty} s_n = 0$.

3. Convergence Analysis

In this section, we consider the (SFP) in which C is given by

$$C = \{x \in H_1 | c(x) \leq 0\}, \quad (32)$$

where $c: H_1 \longrightarrow (-\infty, +\infty]$ is an α-strongly convex function; the set Q is given by

$$Q = \{y \in H_2 | q(y) \leq 0\}, \quad (33)$$

where $q: H_2 \longrightarrow (-\infty, +\infty]$ is a β-strongly convex function. We assume that the solution set S of the (SFP) is nonempty, and c and q are lower semicontinuous convex functions; furthermore, we also assume that ∂c and ∂q are bounded operators (i.e., bounded on bounded sets).

We agree to build the following sets in our algorithms according to [39]; that is, given the n-th iterative point w_n, we construct C_n as

$$C_n = \left\{x \in H_1 | c(w_n) + \langle \xi_n, x - w_n \rangle + \frac{\alpha}{2}\left\|x - w_n\right\|^2 \leq 0\right\}, \quad (34)$$

where $\xi_n \in \partial c(w_n)$.

$$Q_n = \left\{y \in H_2 | q(Aw_n) + \langle \zeta_n, y - Aw_n \rangle + \frac{\beta}{2}\left\|y - Aw_n\right\|^2 \leq 0\right\}, \quad (35)$$

where $\zeta_n \in \partial q(Aw_n)$.

If $\alpha = 0$ and $\beta = 0$, then C_n and Q_n are reduced to the half-spaces $\overline{C_n}$ and $\overline{Q_n}$, respectively. If $\alpha > 0$ and $\beta > 0$, then C_n and Q_n are nonempty closed ball of radius $(1/\alpha)\sqrt{\|\xi_n\|^2 - 2\alpha c(w_n)}$ centred at $w_n - (1/\alpha)\xi_n$ and $(1/\beta)\sqrt{\|\zeta_n\|^2 - 2\beta q(Aw_n)}$ centred at $Aw_n - (1/\beta)\zeta_n$, respectively.

In addition, for each $n \geq 0$, we define the following functions:

$$\begin{aligned} f_n(x) &= \frac{1}{2}\left\|\left(I - P_{Q_n}\right)Ax\right\|^2, \\ \nabla f_n(x) &= A^*\left(I - P_{Q_n}\right)Ax, \end{aligned} \quad (36)$$

where Q_n is given as in (35), f_n is weakly lower semicontinuous, convex, and differentiable, and its gradient ∇f_n

is Lipschitz continuous. Now we propose new relaxed CQ algorithms for solving the (SFP).

Next, two inertial relaxed CQ algorithms will be introduced. The weak convergence of Algorithm 1 and the strong convergence of Algorithm 2 will be proved under different step sizes.

Algorithm 1. Choose positive sequence $\{\varepsilon_n\}$ satisfying $\sum_{n=0}^{\infty}\varepsilon_n < \infty$.

Let $x_0, x_1 \in C$ be arbitrary. Given x_n, x_{n-1}, update the next iteration via

$$\begin{aligned} w_n &= x_n + \theta_n (x_n - x_{n-1}),\\ x_{n+1} &= P_{C_n}(w_n - \tau_n \nabla f_n(w_n)), \end{aligned} \tag{37}$$

where $0 \le \theta_n < \overline{\theta_n}$, and

$$\overline{\theta_n} = \begin{cases} \min\left\{\theta, \dfrac{\varepsilon_n}{\max\left\{\|x_n - x_{n-1}\|^2, \|x_n - x_{n-1}\|\right\}}\right\}, & \text{if } x_n \ne x_{n-1},\\ \theta, & \text{if } x_n = x_{n-1}, \end{cases} \tag{38}$$

and C_n and Q_n are given as in (34) and (35).

$$\tau_n = \begin{cases} \dfrac{\sigma_n}{\|\nabla f_n(w_n)\|}, & \text{if } \|\nabla f_n(w_n)\| \ne 0,\\ 0, & \text{if } \|\nabla f_n(w_n)\| = 0, \end{cases} \tag{39}$$

where $\sum_{n=1}^{\infty}\sigma_n = \infty$, $\sum_{n=1}^{\infty}\sigma_n^2 < \infty$.

If $x_{n+1} = w_n$, then stop; otherwise, set $n: = n+1$ and go to the next iteration.

By assuming $\overline{\theta_n}$, we know

$$\begin{aligned} &\sum_{n=1}^{\infty}\theta_n \|x_n - x_{n-1}\|^2 < \infty,\\ &\sum_{n=1}^{\infty}\theta_n \|x_n - x_{n-1}\| < \infty, \end{aligned} \tag{40}$$

which means

$$\begin{aligned} &\lim_{n\longrightarrow\infty}\theta_n \|x_n - x_{n-1}\|^2 = 0,\\ &\lim_{n\longrightarrow\infty}\theta_n \|x_n - x_{n-1}\| = 0. \end{aligned} \tag{41}$$

From $\xi_n \in \partial c(w_n)$, applying Lemma 3, we get $C \subseteq C_n$; and a similar way is used to get $Q \subseteq Q_n$.

Now let us show that our proposed algorithm has a very important property: if $x_{n+1} = w_n$ for some $n > 0$, then w_n is a solution of (SFP). Indeed, $x_{n+1} \in C_n$, so that $w_n \in C_n$ as $w_n = x_{n+1}$ by assumption. So we get $c(w_n) \le 0$ from (34), that is, $w_n \in C$. On the other hand, according to the algorithm, we have $w_n = P_{C_n}(w_n - \tau_n A^*(I - P_{Q_n})Aw_n)$, which together with Lemma 4 implies that $Aw_n \in Q_n$. It also implies that $q(Aw_n) \le 0$ from (35); then $Aw_n \in Q$. The conclusion is tenable.

Lemma 9. *Let $\{x_n\}$ and $\{w_n\}$ be the sequences generated by Algorithm 1. Then, for any $z \in S$, it follows that*

$$\|x_{n+1} - z\|^2 \le \|w_n - z\|^2 + \sigma_n^2 - \frac{4\sigma_n f_n(w_n)}{\|\nabla f_n(w_n)\|}. \tag{42}$$

Proof. For $z \in S$, we have $z \in C, Az \in Q$; and we have $z = P_C z = P_{C_n} z, Az = P_Q Az = P_{Q_n} Az$.

It follows from Lemma 1 that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \left\|P_{C_n}(w_n - \tau_n \nabla f_n(w_n)) - z\right\|^2\\ &\le \|(w_n - z) - \tau_n \nabla f_n(w_n)\|^2 - \|(w_n - x_{n+1}) - \tau_n \nabla f_n(w_n)\|^2\\ &= \|w_n - z\|^2 - \|w_n - x_{n+1}\|^2 - 2\tau_n \langle \nabla f_n(w_n), w_n - z\rangle + 2\tau_n \langle \nabla f_n(w_n), w_n - x_{n+1}\rangle, \end{aligned} \tag{43}$$

where

$$\begin{aligned} 2\tau_n \langle \nabla f_n(w_n), w_n - z\rangle &= 2\tau_n \left\langle \left(I - P_{Q_n}\right)Aw_n - \left(I - P_{Q_n}\right)Az, Aw_n - Az\right\rangle\\ &\ge 2\tau_n \left\|\left(I - P_{Q_n}\right)Aw_n\right\|^2\\ &= 4\tau_n f_n(w_n),\\ 2\tau_n \langle \nabla f_n(w_n), w_n - x_{n+1}\rangle &\le 2\tau_n \|\nabla f_n(w_n)\| \cdot \|w_n - x_{n+1}\|\\ &\le \|w_n - x_{n+1}\|^2 + \tau_n^2 \|\nabla f_n(w_n)\|^2. \end{aligned} \tag{44}$$

Hence, we have

$$\|x_{n+1}-z\|^2 \le \|w_n-z\|^2 - \|w_n-x_{n+1}\|^2 - 4\tau_n f_n(w_n) + \|w_n-x_{n+1}\|^2 + \tau_n^2\|\nabla f_n(w_n)\|^2. \quad (45)$$

If $\|\nabla f_n(w_n)\| = 0$, then $\tau_n = 0$, so that

$$\|x_{n+1}-z\|^2 \le \|w_n-z\|^2. \quad (46)$$

If $\|\nabla f_n(w_n)\| \neq 0$, we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \|w_n-z\|^2 + \tau_n^2\|\nabla f_n(w_n)\|^2 - 4\tau_n f_n(w_n)\\ &= \|w_n-z\|^2 + \sigma_n^2 - \frac{4\sigma_n f_n(w_n)}{\|\nabla f_n(w_n)\|}.\end{aligned} \quad (47)$$

The proof is complete.

Theorem 1. *Assume that θ_n satisfies the assumption. Then there exists a subsequent $\{x_{n_j}\}$ of $\{x_n\}$ generated by Algorithm 1 which weakly converges to a solution of (SFP).*

Proof. We first show that, for any $z \in S$, the limit of $\{\|x_n - z\|\}$ exists. By applying Lemma 9, we have

$$\|x_{n+1}-z\|^2 \le \|w_n-z\|^2 + \sigma_n^2 - \frac{4\sigma_n f_n(w_n)}{\|\nabla f_n(w_n)\|}. \quad (48)$$

From the construction of w_n and Lemma 2, we have

$$\begin{aligned}\|w_n-z\|^2 &= \|(1+\theta_n)(x_n-z) - \theta_n(x_{n-1}-z)\|^2\\ &= (1+\theta_n)\|x_n-z\|^2 - \theta_n\|x_{n-1}-z\|^2 + \theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2,\end{aligned} \quad (49)$$

$$\le (1+\theta_n)\|x_n-z\|^2 - \theta_n\|x_{n-1}-z\|^2 + 2\theta_n\|x_n-x_{n-1}\|^2. \quad (50)$$

Combining (48) and (50) immediately, we get

$$\|x_{n+1}-z\|^2 \le (1+\theta_n)\|x_n-z\|^2 - \theta_n\|x_{n-1}-z\|^2 + 2\theta_n\|x_n-x_{n-1}\|^2 + \sigma_n^2. \quad (51)$$

Denote $\phi_n = \|x_n - z\|^2$; from (51), we have

$$\phi_{n+1} - \phi_n \le \theta_n(\phi_n - \phi_{n-1}) + 2\theta_n\|x_n-x_{n-1}\|^2 + \sigma_n^2, \quad (52)$$

where

$$\sum_{n=1}^{\infty}\theta_n\|x_n-x_{n-1}\|^2 < \infty, \qquad \sum_{n=1}^{\infty}\sigma_n^2 < \infty. \quad (53)$$

Using Lemma 6, the limit of ϕ_n exists, and $\sum_{n=1}^{\infty}(\|x_{n+1}-z\|^2 - \|x_n-z\|^2)_+ < \infty$, which implies that $\sum_{n=1}^{\infty}(\|x_{n+1}-z\|^2 - \|x_n-z\|^2) < \infty$, $(\|x_{n+1}-z\|^2 - \|x_n-z\|^2)_+ = \max\{\|x_{n+1}-z\|^2 - \|x_n-z\|^2, 0\}$. This also implies that the sequence $\{x_n\}$ is bounded, so $\{w_n\}$ is bounded.

We next show that $\omega_w(x_n) \subset S$. Since $\{w_n\}$ is bounded, from the Lipschitz continuity of ∇f_n, we get that $\{\|\nabla f_n(w_n)\|\}$ is bounded. From (48) and (50), we get

$$\frac{4\sigma_n f_n(w_n)}{\|\nabla f_n(w_n)\|} \le \|x_n-z\|^2 - \|x_{n+1}-z\|^2 + \theta_n\left(\|x_n-z\|^2 - \|x_{n-1}-z\|^2\right) + 2\theta_n\|x_n-x_{n-1}\|^2 + \sigma_n^2, \quad (54)$$

where $\sum_{n=1}^{\infty}(\|x_{n+1}-z\|^2 - \|x_n-z\|^2) < \infty$, $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\|^2 < \infty$, and $\sum_{n=1}^{\infty}\sigma_n^2 < \infty$, so we have

$$\sum_{n=1}^{\infty}\frac{4\sigma_n f_n(w_n)}{\|\nabla f_n(w_n)\|} < \infty. \quad (55)$$

But $\sum_{n=1}^{\infty}\sigma_n = \infty$, so

$$\liminf_{n\longrightarrow\infty} f_n(w_n) = 0, \qquad \textit{i.e.}, \liminf_{n\longrightarrow\infty}\left\|\left(I - P_{Q_n}\right)Aw_n\right\|^2 = 0. \quad (56)$$

On the other hand, since $\{x_n\}$ is bounded, the set $\omega_w(x_n)$ is nonempty. Let $x^* \in \omega_w(x_n)$; then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. Furthermore,

$$\|w_n - x_n\|^2 = \theta_n^2\|x_n-x_{n-1}\|^2 \le \theta_n\|x_n-x_{n-1}\|^2 \longrightarrow 0. \quad (57)$$

Let $\{w_{n_j}\}$ be a subsequence of the sequence $\{w_n\}$ such that

$$\liminf_{n\longrightarrow\infty}\left\|\left(I-P_{Q_n}\right)Aw_n\right\|^2 = \lim_{j\longrightarrow\infty}\left\|\left(I-P_{Q_{n_j}}\right)Aw_{n_j}\right\|^2 = 0. \quad (58)$$

Since $\{w_{n_j}\}$ is bounded, there exists a subsequence $\{w_{n_{j_m}}\}$ of $\{w_{n_j}\}$, which converges weakly to x^*. Without loss of generality, we can assume that $w_{n_j} \rightharpoonup x^*$, and A is a bounded linear operator, so $Aw_{n_j} \rightharpoonup Ax^*$.

From Lemma 1, we conclude that

$$\langle w_n - \tau_n\nabla f_n(w_n) - x_{n+1}, z - x_{n+1}\rangle \le 0. \quad (59)$$

Since $\tau_n \longrightarrow 0$ and $\{\|\nabla f_n(w_n)\|\}$ is bounded, we have $\tau_n\nabla f_n(w_n) \longrightarrow 0$. Hence, we get

$$\langle x_{n+1} - w_n, x_{n+1} - z\rangle \leq \langle \tau_n \nabla f_n(w_n), z - x_{n+1}\rangle \longrightarrow 0. \tag{60}$$

Since $\sum_{n=1}^{\infty}(\|x_{n+1} - z\|^2 - \|x_n - z\|^2) < \infty$ and $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\|^2 < \infty$, from (50), we obtain

$$\begin{aligned}\|x_{n+1} - w_n\|^2 &= \|w_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\langle x_{n+1} - w_n, x_{n+1} - z\rangle \\ &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \theta_n\left(\|x_n - z\|^2 - \|x_{n-1} - z\|^2\right) + 2\theta_n\|x_n - x_{n-1}\|^2 + 2\langle x_{n+1} - w_n, x_{n+1} - z\rangle \longrightarrow 0.\end{aligned} \tag{61}$$

Thus,

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - w_n\| + \|w_n - x_n\| \longrightarrow 0. \tag{62}$$

Since $P_{Q_{n_j}}Aw_{n_j} \in Q_{n_j}$, by the definition of Q_{n_j},

$$q\left(Aw_{n_j}\right) + \langle \zeta_{n_j}, P_{Q_{n_j}}Aw_{n_j} - Aw_{n_j}\rangle + \frac{\beta}{2}\left\|P_{Q_{n_j}}Aw_{n_j} - Aw_{n_j}\right\|^2 \leq 0, \tag{63}$$

where $\zeta_{n_j} \in \partial q(Aw_{n_j})$. From the boundedness assumption of ∂q and $\lim_{j\longrightarrow\infty}\|(I - P_{Q_{n_j}})Aw_{n_j}\|^2 = 0$, we have

$$q\left(Aw_{n_j}\right) \leq \left\|\zeta_{n_j}\right\| \cdot \left\|\left(I - P_{Q_{n_j}}\right)Aw_{n_j}\right\| - \frac{\beta}{2}\left\|\left(I - P_{Q_{n_j}}\right)Aw_{n_j}\right\|^2 \longrightarrow 0. \tag{64}$$

From the weak lower semicontinuity of the convex function q, it follows that

$$q(Ax^*) \leq \liminf_{j\longrightarrow\infty} q\left(Aw_{n_j}\right) \leq 0, \tag{65}$$

which means that $Ax^* \in Q$.

Furthermore, $x_{n_j+1} \in C_{n_j}$, and, by the definition of C_{n_j},

$$c\left(w_{n_j}\right) + \langle \xi_{n_j}, x_{n_j+1} - w_{n_j}\rangle + \frac{\alpha}{2}\left\|x_{n_j+1} - w_{n_j}\right\|^2 \leq 0, \tag{66}$$

where $\xi_{n_j} \in \partial c(w_{n_j})$. From the boundedness assumption of ∂c and $\|x_{n_j+1} - w_{n_j}\| \longrightarrow 0$, we have

$$c\left(w_{n_j}\right) \leq \left\|\xi_{n_j}\right\| \cdot \left\|w_{n_j} - x_{n_j+1}\right\| - \frac{\alpha}{2}\left\|x_{n_j+1} - w_{n_j}\right\|^2 \longrightarrow 0. \tag{67}$$

From the weak lower semicontinuity of the convex function c, it follows that

$$c(x^*) \leq \liminf_{j\longrightarrow\infty} c\left(w_{n_j}\right) \leq 0, \tag{68}$$

which means that $x^* \in C$. Therefore, $x_{n_j} \rightharpoonup x^* \in S$. The proof is complete.

Algorithm 2. Choose positive sequence $\{\varepsilon_n\}$ satisfying $\sum_{n=0}^{\infty}\varepsilon_n < \infty$.

Let $x_0, x_1 \in C$ be arbitrary. Given x_n, x_{n-1}, update the next iteration via

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= \beta_n u + P_{C_n}[(1 - \beta_n)(w_n - \tau_n \nabla f_n(w_n))], \\ \tau_n &= \frac{\rho_n f_n(w_n)}{\|\nabla f_n(w_n)\|^2}, \end{aligned} \tag{69}$$

where $0 \leq \theta_n < \overline{\theta_n}$, and

$$\overline{\theta_n} = \begin{cases} \min\left\{\theta, \dfrac{\varepsilon_n}{\max\left\{\|x_n - x_{n-1}\|^2, \|x_n - x_{n-1}\|\right\}}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{if } x_n = x_{n-1}, \end{cases} \tag{70}$$

and C_n and Q_n are given as in (34) and (35), $\{\beta_n\} \subset (0,1)$, $\lim_{n\longrightarrow\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, and $\inf_n\rho_n(4 - \rho_n) > 0$.

If $x_{n+1} = w_n$, then stop; otherwise, set $n := n + 1$ and go to the next iteration.

Theorem 2. *Assume that* $\inf_n\rho_n(4 - \rho_n) > 0$ *and* $\varepsilon_n = o(\beta_n)$. *Then the sequence* x_n *generated by Algorithm 2 converges strongly to* $z = P_S u$.

Proof. First, we show that, for any $z \in S$, the sequence $\{x_n\}$ is bounded. From the construction of w_n, we have

$$\begin{aligned}\|w_n - z\| &= \|x_n + \theta_n (x_n - x_{n-1}) - z\| \\ &\leq \|x_n - z\| + \theta_n \|x_n - x_{n-1}\|,\end{aligned} \tag{71}$$

$$\begin{aligned}\|w_n - \tau_n \nabla f_n(w_n) - z\|^2 &= \|w_n - z\|^2 + \tau_n^2 \|\nabla f_n(w_n)\|^2 \\ &\quad - 2\tau_n \langle \nabla f_n(w_n), w_n - z\rangle \\ &\leq \|w_n - z\|^2 + \tau_n^2 \|\nabla f_n(w_n)\|^2 - 4\tau_n f_n(w_n) \\ &= \|w_n - z\|^2 + \frac{\rho_n^2 f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2} - \frac{4\rho_n f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2} \\ &= \|w_n - z\|^2 - \rho_n(4 - \rho_n)\frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2} \\ &\leq \|w_n - z\|^2.\end{aligned} \tag{72}$$

So, combining (71) and (72), we get

$$\begin{aligned}\|x_{n+1} - z\| &= \left\|\beta_n u + P_{C_n}[(1-\beta_n)(w_n - \tau_n \nabla f_n(w_n))] - z\right\| \\ &= \left\|P_{C_n}[(1-\beta_n)(w_n - \tau_n \nabla f_n(w_n))] - (1-\beta_n)z + \beta_n(u - z)\right\| \\ &\leq \left\|P_{C_n}[(1-\beta_n)(w_n - \tau_n \nabla f_n(w_n))] - (1-\beta_n)z\right\| + \beta_n\|u - z\| \\ &\leq \|(1-\beta_n)(w_n - \tau_n \nabla f_n(w_n)) - (1-\beta_n)z\| + \beta_n\|u - z\| \\ &\leq (1-\beta_n)\|w_n - \tau_n \nabla f_n(w_n) - z\| + \beta_n\|u - z\| \\ &\leq (1-\beta_n)\|w_n - z\| + \beta_n\|u - z\| \\ &\leq (1-\beta_n)\left[\|x_n - z\| + \theta_n\|x_n - x_{n-1}\|\right] + \beta_n\|u - z\| \\ &\leq (1-\beta_n)\|x_n - z\| + \beta_n[\sigma_n + \|u - z\|],\end{aligned} \tag{73}$$

where $\sigma_n = (1-\beta_n)(\theta_n/\beta_n)\|x_n - x_{n-1}\|$. According to hypothesis θ_n,

$$\theta_n \leq \frac{\varepsilon_n}{\|x_n - x_{n-1}\|} \Rightarrow \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \leq \frac{\varepsilon_n}{\beta_n} \longrightarrow 0. \tag{74}$$

Note that

$$\lim_{n\longrightarrow\infty} \sigma_n = \lim_{n\longrightarrow\infty} (1-\beta_n)\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| = 0, \tag{75}$$

which implies that the sequence $\{\sigma_n\}$ is bounded. Setting

$$M = \max\left\{\sup_{n\in N} \sigma_n, \|u - z\|\right\}, \tag{76}$$

as well as using Lemma 7, we conclude that the sequence $\{\|x_n - z\|\}$ is bounded. This shows that the sequence $\{x_n\}$ is bounded and so is $\{w_n\}$.

Since $\{\|x_n - z\|\}$ is bounded, assume that there exists a constant M_1 such that $\|x_n - z\| \leq M_1$. Thus,

$$\begin{aligned}\|w_n - z\|^2 &\leq \left(\|x_n - z\| + \theta_n\|x_n - x_{n-1}\|\right)^2 \\ &= \|x_n - z\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - x_{n-1}\| \cdot \|x_n - z\| \\ &\leq \|x_n - z\|^2 + \theta_n\|x_n - x_{n-1}\|^2 + 2M_1 \cdot \theta_n\|x_n - x_{n-1}\|,\end{aligned} \tag{77}$$

and we get

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&=\left\|\beta_n u+P_{C_n}[(1-\beta_n)(w_n-\tau_n\nabla f_n(w_n))]-z\right\|^2\\
&=\left\|P_{C_n}[(1-\beta_n)(w_n-\tau_n\nabla f_n(w_n))]-(1-\beta_n)z+\beta_n(u-z)\right\|^2\\
&\leq\left\|P_{C_n}[(1-\beta_n)(w_n-\tau_n\nabla f_n(w_n))]-(1-\beta_n)z\right\|^2+2\beta_n\langle u-z,x_{n+1}-z\rangle\\
&\leq\left\|(1-\beta_n)(w_n-\tau_n\nabla f_n(w_n))-(1-\beta_n)z\right\|^2+2\beta_n\langle u-z,x_{n+1}-z\rangle\\
&\leq(1-\beta_n)\left[\|w_n-z\|^2-\rho_n(4-\rho_n)\frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2}\right]+2\beta_n\langle u-z,x_{n+1}-z\rangle\\
&\leq(1-\beta_n)\left[\|x_n-z\|^2+\theta_n\|x_n-x_{n-1}\|^2+2M_1\cdot\theta_n\|x_n-x_{n-1}\|-\rho_n(4-\rho_n)\frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2}\right]+2\beta_n\langle u-z,x_{n+1}-z\rangle.
\end{aligned}\tag{78}$$

From (78),

$$\begin{aligned}
\|x_{n+1}-z\|^2&\leq(1-\beta_n)\|x_n-z\|^2+\beta_n\left[\frac{\theta_n}{\beta_n}\|x_n-x_{n-1}\|^2+2M_1\cdot\frac{\theta_n}{\beta_n}\|x_n-x_{n-1}\|+2\langle u-z,x_{n+1}-z\rangle\right],\\
\|x_{n+1}-z\|^2&\leq\|x_n-z\|^2-\rho_n(4-\rho_n)\frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2}+\theta_n\|x_n-x_{n-1}\|^2+2M_1\cdot\theta_n\|x_n-x_{n-1}\|+2\beta_n\langle u-z,x_{n+1}-z\rangle.
\end{aligned}\tag{79}$$

Let

$$\begin{aligned}
s_n&=\|x_n-z\|^2;\\
\delta_n&=\frac{\theta_n}{\beta_n}\|x_n-x_{n-1}\|^2+2M_1\cdot\frac{\theta_n}{\beta_n}\|x_n-x_{n-1}\|+2\langle u-z,x_{n+1}-z\rangle;\\
\eta_n&=\rho_n(4-\rho_n)\frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2};\\
\gamma_n&=\theta_n\|x_n-x_{n-1}\|^2+2M_1\cdot\theta_n\|x_n-x_{n-1}\|+2\beta_n\langle u-z,x_{n+1}-z\rangle.
\end{aligned}\tag{80}$$

Then (78) can reduce to the inequalities

$$\begin{aligned}
s_{n+1}&\leq(1-\beta_n)s_n+\beta_n\delta_n,\quad n\geq 1,\\
s_{n+1}&\leq s_n-\eta_n+\gamma_n.
\end{aligned}\tag{81}$$

Furthermore, we know that

$$\sum_{n=0}^{\infty}\beta_n=\infty,\tag{82}$$

$$\lim_{n\longrightarrow\infty}\gamma_n=\lim_{n\longrightarrow\infty}\left[\theta_n\|x_n-x_{n-1}\|^2+2M_1\cdot\theta_n\|x_n-x_{n-1}\|+2\beta_n\langle u-z,x_{n+1}-z\rangle\right]=0.\tag{83}$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ and suppose that

$$\lim_{k\longrightarrow\infty}\eta_{n_k}=0.\tag{84}$$

Then, we have

$$\lim_{k\longrightarrow\infty}\rho_{n_k}(4-\rho_{n_k})\frac{f_{n_k}^2(w_{n_k})}{\|\nabla f_{n_k}(w_{n_k})\|^2}=0,\tag{85}$$

which implies, by our assumption, that

$$\frac{f_{n_k}^2(w_{n_k})}{\|\nabla f_{n_k}(w_{n_k})\|^2}\longrightarrow 0,\quad \text{as}\,k\longrightarrow\infty.\tag{86}$$

Since $\{\|\nabla f_{n_k}(w_{n_k})\|\}$ is bounded, it follows that $f_{n_k}(w_{n_k})\longrightarrow 0$ as $k\longrightarrow\infty$, so we get $\lim_{k\longrightarrow\infty}\|(I-P_{Q_{n_k}})Aw_{n_k}\|=0$.

We next show that $\omega_w(x_n)\subset S$. Since $\{x_n\}$ is bounded, the set $\omega_w(x_n)$ is nonempty. Let $x^*\in\omega_w(x_n)$; then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$.

$$\|w_n-x_n\|=\|x_n+\theta_n(x_n-x_{n-1})-x_n\|=\theta_n\|x_n-x_{n-1}\|\longrightarrow 0,\tag{87}$$

and then $w_{n_j}\rightharpoonup x^*$, and A is a bounded linear operator, so $Aw_{n_j}\rightharpoonup Ax^*$.

Since $P_{Q_{n_j}}Aw_{n_j}\in Q_{n_j}$, we have

$$q\left(Aw_{n_j}\right)+\langle\zeta_{n_j},P_{Q_{n_j}}Aw_{n_j}-Aw_{n_j}\rangle+\frac{\beta}{2}\left\|P_{Q_{n_j}}Aw_{n_j}-Aw_{n_j}\right\|^2\leq 0,\tag{88}$$

where $\zeta_{n_j} \in \partial q(Aw_{n_j})$, and, by the boundedness of ∂q, we get

$$q\left(Aw_{n_j}\right) \leq \left\|\zeta_{n_j}\right\| \cdot \left\|\left(I - P_{Q_{n_j}}\right)Aw_{n_j}\right\| - \frac{\beta}{2}\left\|\left(I - P_{Q_{n_j}}\right)Aw_{n_j}\right\|^2 \longrightarrow 0, \tag{89}$$

and, using the weak lower semicontinuity of q,

$$q(Ax^*) \leq \liminf_{j\longrightarrow\infty} q\left(Aw_{n_j}\right) \leq 0. \tag{90}$$

Thus, $Ax^* \in Q$.

On the other hand,

$$\begin{aligned}\left\|x_{n+1} - w_n\right\| &\leq (1-\beta_n)\left\|w_n - \tau_n \nabla f_n(w_n) - w_n\right\| + \beta_n\|u - z\| \\ &= (1-\beta_n)\cdot\frac{\rho_n f_n(w_n)}{\left\|\nabla f_n(w_n)\right\|} + \beta_n\|u - z\| \longrightarrow 0.\end{aligned} \tag{91}$$

Since $(x_{n_j+1} - \beta_{n_j}u) \in C_{n_j}$, we have

$$c\left(w_{n_j}\right) + \langle \xi_{n_j}, x_{n_j+1} - \beta_{n_j}u - w_{n_j}\rangle + \frac{\alpha}{2}\left\|x_{n_j+1} - \beta_{n_j}u - w_{n_j}\right\|^2 \leq 0, \tag{92}$$

where $\xi_{n_j} \in \partial c(w_{n_j})$, and, by the boundedness of ∂c, we get

$$\begin{aligned}c\left(w_{n_j}\right) &\leq \left\|\xi_{n_j}\right\| \cdot \left\|w_{n_j} - x_{n_j+1} + \beta_{n_j}u\right\| - \frac{\alpha}{2}\left\|x_{n_j+1} - w_{n_j} - \beta_{n_j}u\right\|^2 \\ &\leq \left\|\xi_{n_j}\right\| \cdot \left[\left\|w_{n_j} - x_{n_j+1}\right\| + \beta_{n_j}\|u\|\right] \\ &\quad - \frac{\alpha}{2}\left[\left\|x_{n_j+1} - w_{n_j}\right\|^2 + \beta_{n_j}^2\|u\|^2 - 2\langle x_{n_j+1} - w_{n_j}, \beta_{n_j}u\rangle\right] \longrightarrow 0,\end{aligned} \tag{93}$$

and, using the weak lower semicontinuity of c,

$$c(x^*) \leq \liminf_{j\longrightarrow\infty} c\left(w_{n_j}\right) \leq 0. \tag{94}$$

Thus, $x^* \in C$; then $x^* \in S$, that is, $\omega_w(x_n) \subset S$.

Next, we have

$$\begin{aligned}\left\|x_{n+1} - x_n\right\| &\leq (1-\beta_n)\left\|w_n - \tau_n\nabla f_n(w_n) - x_n\right\| + \beta_n\left\|u - x_n\right\| \\ &\leq (1-\beta_n)\left[\left\|w_n - x_n\right\| + \tau_n\left\|\nabla f_n(w_n)\right\|\right] + \beta_n\left\|x_n - u\right\| \\ &\leq \left\|w_n - x_n\right\| + \frac{\rho_n f_n(w_n)}{\left\|\nabla f_n(w_n)\right\|} + \beta_n\left\|x_n - u\right\| \longrightarrow 0.\end{aligned} \tag{95}$$

For $z = P_S u$ and $x_{n_k} \rightharpoonup x^* \in S$, using Lemma 1, $\langle u - z, x^* - z\rangle \leq 0$, so

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle u - z, x_n - z\rangle &= \limsup_{k\longrightarrow\infty}\langle u - z, x_{n_k} - z\rangle \\ &= \langle u - z, x^* - z\rangle \leq 0,\end{aligned} \tag{96}$$

and then

$$\begin{aligned}&\limsup_{n\longrightarrow\infty}\langle u - z, x_{n+1} - z\rangle \\ &\quad = \limsup_{n\longrightarrow\infty}\left(\langle u - z, x_{n+1} - x_n\rangle + \langle u - z, x_n - z\rangle\right) \leq 0,\end{aligned} \tag{97}$$

and thus

$$\begin{aligned}\limsup_k \delta_{n_k} = \limsup_k \Bigg[&\frac{\theta_{n_k}}{\beta_{n_k}}\left\|x_{n_k} - x_{n_k-1}\right\|^2 + 2M_1 \\ &\cdot\frac{\theta_{n_k}}{\beta_{n_k}}\left\|x_{n_k} - x_{n_k-1}\right\| + 2\langle u - z, x_{n_k+1} - z\rangle\Bigg] \leq 0.\end{aligned} \tag{98}$$

From (82), (83), (98), and Lemma 8, we conclude that the sequence $\{x_n\}$ converges strongly to $z = P_S u$. The proof is complete.

4. Numerical Experiments

In this section, we present a numerical experiment to illustrate the performance of the proposed algorithms. Our numerical experiments are coded in MATLAB R2007 running on personal computer with 3.50 GHz Intel Core i3 and 4 GB RAM. In what follows, we apply our algorithms to solve the problem of least absolute shrinkage and selection operator, which requires solving a convex optimization problem as

$$\begin{aligned}&\min_{x\in\mathbb{R}^n}, \frac{1}{2}\|Ax - y\|^2, \\ &\text{s.t.,}\quad \|x\|_1 \leq t_0,\end{aligned} \tag{99}$$

where $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$, and $t_0 > 0$ are given elements. In our experiment, we first generate an $m \times n$ matrix A randomly by a standardized normal distribution, and x is a sparse signal with n elements, only K of which is nonzero, which is also generated randomly. The observation y is generated as $y = Ax$. The parameters in this experiment are set with $n = 512, m = 256, \varepsilon = 10^{-4}$, and $t_0 = K$. In this situation, it is readily seen that $C = \{x \in \mathbb{R}^n\colon c(x) \leq 0\}$ with $c(x) = \|x\|_1 - t_0$ and $Q = \{y\}$, which in turn implies that

$$C_n = \left\{x \in \mathbb{R}^n\colon \langle \xi_n, x\rangle \leq \langle \xi_n, w_n\rangle - \left\|w_n\right\|_1 + t_0\right\}, \tag{100}$$

where ξ_n is defined by

$$(\xi_n)_i = \begin{cases} 1, & \text{if } (\xi_n)_i > 0; \\ [-1, 1], & \text{if } (\xi_n)_i = 0; \\ -1, & \text{if } (\xi_n)_i < 0,\end{cases} \tag{101}$$

standing for the subdifferential of $\|\cdot\|_1$. As a half-space, the associated projection onto C_n takes the following form:

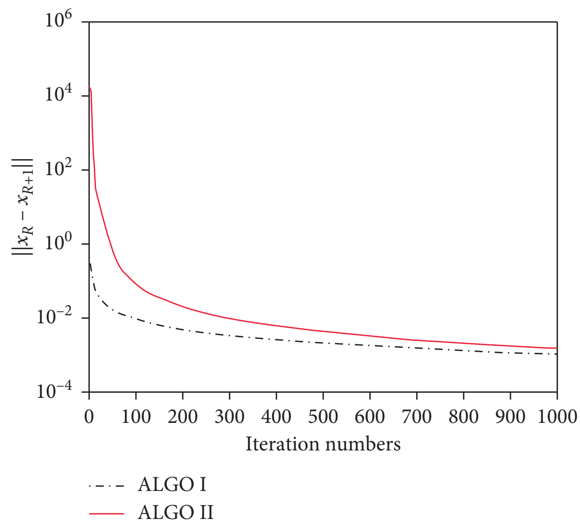


Figure 1: Iterative results with $K = 50$.

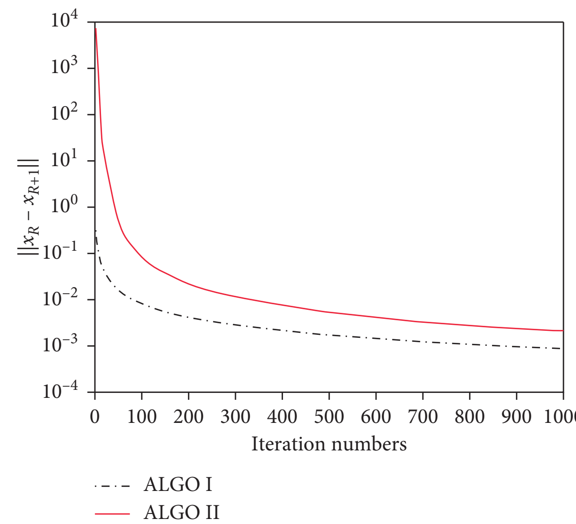


Figure 2: Iterative results with $K = 40$.

$$P_{C_n}(x) = \begin{cases} x + \dfrac{\langle \xi_n, w_n - x\rangle - \|w_n\|_1 + t_0}{\|\xi_n\|^2}\xi_n, & \text{if } x \notin C_n \\ x, & \text{if } x \in C_n. \end{cases} \tag{102}$$

To show the efficiency of our algorithm, we compare it with the algorithm proposed in [40]. The only difference of these two algorithms is that there are no inertial terms in the algorithm proposed in [40]. For the convenience, we denote Algorithm 1 by Algo. I and the algorithm in [40] by Algo. II, respectively. In Algorithm 1, we set

$$\theta_n = \begin{cases} \min\left\{0.8, \dfrac{1}{n^2\|x_n - x_{n-1}\|^2}\right\}, & \text{if } x_n \neq x_{n-1}, \\ 0.8, & \text{if } x_n = x_{n-1}, \end{cases}$$

$$\tau_n = \begin{cases} \dfrac{1}{n\|A^*(Aw_n - y)\|}, & \text{if } \|A^*(Aw_n - y)\| \neq 0, \\ 0, & \text{if } \|A^*(Aw_n - y)\| = 0. \end{cases} \tag{103}$$

In Algo. II, we set $\theta_n \equiv 0$ and τ_n is chosen the same as above. The stopping criterion is that $\|x^{k+1} - x^k\| < \varepsilon$. The initial points are $x_0 = (0, 0, \ldots, 0)^T$ and $x_1 = 100(1, 1, \ldots, 1)^T$. The numerical results of these two algorithms with different choices of the sparsity number K are listed in Figures 1–4. It is easy to see that Algo. I converges faster than Algo. II does, which indicates that our modified algorithm indeed accelerates the convergence of the original algorithm.

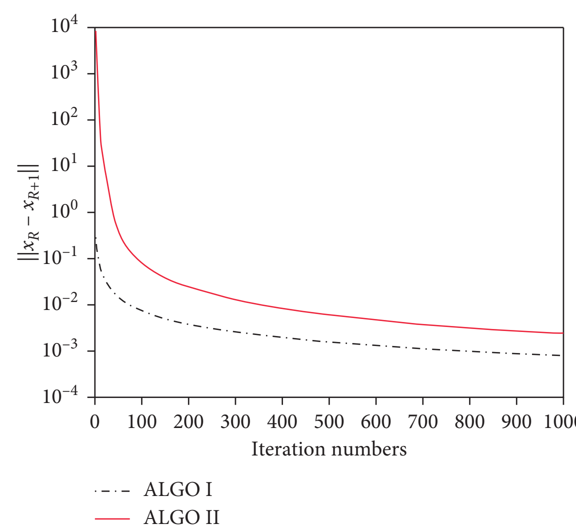


Figure 3: Iterative results with $K = 30$.

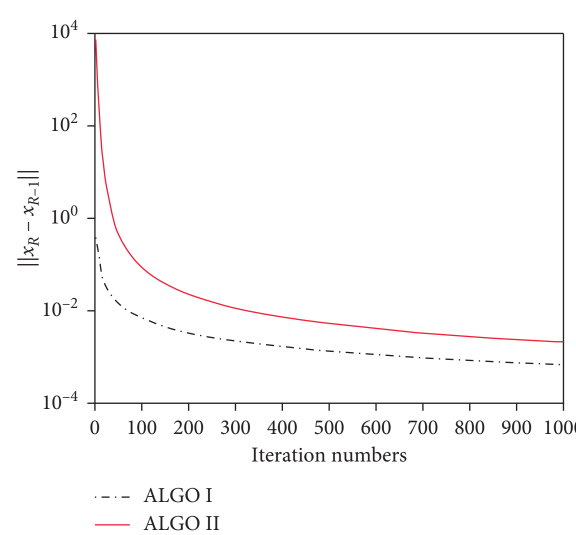


Figure 4: Iterative results with $K = 20$.

5. Conclusions

In this paper, we present two inertial relaxed *CQ* algorithms for solving split feasibility problems in Hilbert spaces by adopting variable step size. These algorithms adopt the new convex subset form, and it is easy to calculate the projections onto these sets. Furthermore, step size selection in the algorithms does not depend on the operator norm. The convergence theorems are established under some mild conditions and a numerical experiment is given to illustrate the performance of the proposed algorithms.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (no. 11771126).

References

[1] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2, pp. 221–239, 1994.
[2] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.
[3] Y. Censor, T. Elfving, N. Kopf, and T. Bortfeld, "The multiple-sets split feasibility problem and its applications for inverse problems," *Inverse Problems*, vol. 21, no. 6, pp. 2071–2084, 2005.
[4] C. Byrne, "Iterative oblique projection onto convex sets and the split feasibility problem," *Inverse Problems*, vol. 18, no. 2, pp. 441–453, 2002.
[5] S. He, H. Tian, and H. K. Xu, "The selective projection method for convex feasibility and split feasibility problems," *Journal of Nonlinear and Convex Analysis*, vol. 19, pp. 1199–1215, 2018.
[6] S. He, Z. Zhao, and B. Luo, "A relaxed self-adaptive *CQ* algorithm for the multiple-sets split feasibility problem," *Optimization*, vol. 64, no. 9, pp. 1907–1918, 2015.
[7] G. L. Acedo, V. Martin-Marquez, F. Wang, and H. Xu, "Solving the split feasibility problem without prior knowledge of matrix norms," *Inverse Problems*, vol. 28, Article ID 085004, 2012.
[8] B. Qu and N. Xiu, "A new halfspace-relaxation projection method for the split feasibility problem," *Linear Algebra and its Applications*, vol. 428, no. 5-6, pp. 1218–1229, 2008.
[9] F. Wang, "A splitting-relaxed projection method for solving the split feasibility problem," *Fixed Point Theory*, vol. 14, pp. 211–218, 2013.
[10] F. Wang, "On the convergence of *CQ* algorithm with variable steps for the split equality problem," *Numerical Algorithms*, vol. 74, no. 3, pp. 927–935, 2017.
[11] F. Wang, "Polyak's gradient method for split feasibility problem constrained by level sets," *Numerical Algorithms*, vol. 77, no. 3, pp. 925–938, 2018.
[12] H.-K. Xu, "A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem," *Inverse Problems*, vol. 22, no. 6, pp. 2021–2034, 2006.
[13] Q. Yang, "The relaxed *CQ* algorithm solving the split feasibility problem," *Inverse Problems*, vol. 20, no. 4, pp. 1261–1266, 2004.
[14] F. Alvarez and H. Attouch, "An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping," *Set-Valued Analysis*, vol. 9, pp. 3–11, 2001.
[15] Y. Dang, J. Sun, J. Sun, and H. Xu, "Inertial accelerated algorithms for solving a split feasibility problem," *Journal of Industrial & Management Optimization*, vol. 13, no. 3, pp. 1383–1394, 2017.
[16] A. Gibali, D. Mai, D. Thi Mai, and N. The Vinh, "A new relaxed *CQ* algorithm for solving split feasibility problems in Hilbert spaces and its applications," *Journal of Industrial & Management Optimization*, vol. 15, no. 2, pp. 963–984, 2019.
[17] Q. L. Dong, Y. J. Cho, and T. M. Rassias, "General inertial Mann algorithms and their convergence analysis for nonexpansive mappings," in *Applications of Nonlinear Analysis*, pp. 175–191, Springer, Berlin, Germany, 2018.
[18] Q. L. Dong, Y. J. Cho, L. L. Zhong, and T. M. Rassias, "Inertial projection and contraction algorithms for variational inequalities," *Journal of Global Optimization*, vol. 70, no. 3, pp. 687–704, 2018.
[19] Q. L. Dong, J. Z. Huang, X. H. Li, Y. J. Cho, and T. M. Rassias, "MiKM: multi-step inertial Krasnosel'skiǐ-Mann algorithm and its applications," *Journal of Global Optimization*, vol. 73, no. 4, pp. 801–824, 2019.
[20] Q. L. Dong, H. B. Yuan, Y. J. Cho, and T. M. Rassias, "Modified inertial Mann algorithm and inertial *CQ*-algorithm for nonexpansive mappings," *Optimization Letters*, vol. 12, no. 1, pp. 87–102, 2018.
[21] Y. Shehu and A. Gibali, "New inertial relaxed method for solving split feasibilities," *Optimization Letters*, 2020.
[22] S. S. Tai, N. Pholasa, and P. Cholamjiak, "The modified inertial relaxed *CQ* algorithm for solving the split feasibility problems," *Journal of Industrial and Management Optimization*, vol. 14, pp. 1595–1615, 2018.
[23] S. S. Tai, N. Pholasa, and P. Cholamjiak, "Relaxed *CQ* algorithms involving the inertial technique for multiple-sets split feasibility problems," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales-Serie A: Matematicas*, vol. 113, pp. 1081–1099, 2019.
[24] S. Y. Cho, "A convergence theorem for generalized mixed equilibrium problems and multivalued asymptotically nonexpansive mappings," *Journal of Nonlinear and Convex Analysis*, vol. 21, pp. 1017–1026, 2020.
[25] H.-K. Xu, "Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces," *Inverse Problems*, vol. 26, no. 10, Article ID 105018, 2010.
[26] F. Wang and H. Xu, "Approximating curve and strong convergence of the CQ algorithm for the split feasibility problem," *Journal of Inequalities and Applications*, vol. 2010, Article ID 102085, 2010.
[27] M. Tian and H. F. Zhang, "The regularized *CQ* algorithm without a priori knowledge of operator norm for solving the split feasibility problem," *Journal of Inequalities and Applications*, vol. 2017, p. 207, 2017.
[28] Y. H. Yao, W. J. Gang, and Y. C. Liou, "Regularized methods for the split feasibility problem," *Abstract and Applied Analysis*, vol. 2012, Article ID 140679, 2012.

[29] Y. Zhou, Z. Haiyun, and P. Wang, "Iterative methods for finding the minimum-norm solution of the standard monotone variational inequality problems with applications in Hilbert spaces," *Journal of Inequalities and Applications*, vol. 2015, 2015.

[30] T. V. Nguyen, C. Prasit, and S. Suthep, "A new CQ algorithm for solving split feasibility problems in Hilbert spaces," *Malaysian Mathematical Sciences Society and Penerbit Universiti Sains Malaysia*, vol. 42, no. 5, 2020.

[31] D. R. Sahu, Y. J. Cho, Q. L. Dong, M. R. Kashyap, and X. H. Li, *Inertial Relaxed CQ algorithms for Solving a Split Feasibility Problem in Hilbert Spaces*, Springer Science. Business Media, Berlin, Germany, 2020.

[32] R. P. Agarwal, D. O. Regan, and D. R. Sahu, "Fixed point theory for Lipschitzian-type mappings with applications," in *Topological Fixed Point Theory and Its Applications*, Springer, New York, NY, USA, 2009.

[33] S. Y. Cho, "A monotone Bregan projection algorithm for fixed point and equilibrium problems in a relative Banach space," *Filomat*, vol. 34, pp. 1487–1497, 2020.

[34] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, NY, USA, 2011.

[35] P. E. Mainge, "Inertial iterative process for fixed points of certain quasi-nonexpansive mapping," *Set Valued Analysis*, vol. 15, pp. 67–79, 2007.

[36] P. E. Mainge, "Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 325, pp. 469–479, 2007.

[37] A. Moudafi and A. Gibali, "regularization of split feasibility problems," *Numerical Algorithms*.vol. 78, no. 3, pp. 1–19, 2017.

[38] S. He and C. Yang, "Solving the variational inequality problem defined on intersection of finite level sets," *Abstract and Applied Analysis*, vol. 2013, Article ID 942315, 2013.

[39] F. Wang and H. Yu, "An inertial relaxed CQ algorithm with an application to the LASSO and elastic net," *Optimization*, vol. 2020, Article ID 1763989, 2020.

[40] P. K. Anh, N. T. Vinh, and V. T. Dung, "A new self-adaptive *CQ* algorithm with an application to the LASSO problem," *Journal of Fixed Point Theory and Applications*, vol. 20, p. 142, 2018.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6651480, 4 pages
https://doi.org/10.1155/2021/6651480

Research Article

A New Algorithm for Privacy-Preserving Horizontally Partitioned Linear Programs

Chengxue Zhang, Debin Kong, Peng Pan, and Mingyuan Zhou

Yantai Nanshan University, Yantai 265713, China

Correspondence should be addressed to Debin Kong; kongdebin@nanshan.edu.cn

Received 13 December 2020; Revised 13 January 2021; Accepted 29 January 2021; Published 11 February 2021

Academic Editor: Sun Young Cho

In a linear programming for horizontally partitioned data, the equality constraint matrix is divided into groups of rows. Each group of the matrix rows and the corresponding right-hand side vector are owned by different entities, and these entities are reluctant to disclose their own groups of rows or right-hand side vectors. To calculate the optimal solution for the linear programming in this case, Mangasarian used a random matrix of full rank with probability 1, but an event with probability 1 is not a certain event, so a random matrix of full rank with probability 1 does not certainly happen. In this way, the solution of the original linear programming is not equal to the solution of the secure linear programming. We used an invertible random matrix for this shortcoming. The invertible random matrix converted the original linear programming problem to a secure linear program problem. This secure linear programming will not reveal any of the privately held data.

1. Introduction

Recently, people have become interested in privacy-preserving classification and data mining [1–10] and have been involved in the field of optimization, especially in linear programming [11–15], where the data to be classified or mined belongs to different entities that are not willing to disclose the data. Mangasarian [13] proposed a random matrix which make the original linear programming problem into a secure linear programming problem. When the random matrix is not full rank [16], especially when the entities collide with each other, the original linear programming problem is not equivalent to the secure linear programming problem. We address this problem by using an invertible matrix multiplied by the two sides of the equality constraints of the linear program. This procedure converts the original linear program to an equivalent secure linear program, and this security linearity does not reveal any private data. This solution vector can be made public and applied by all entities. On the contrary, this algorithm prevents entities from colliding with each other.

Here, we define some symbols. If a vector is not transposed to the row vector by the superscript T, the vector will be a column vector. For a vector $x \in R^n$, the symbol x_j will represent the j^{th} component or j^{th} block of the component. We will define the scalar (inner) product of two vectors x and y in the n-dimensional real space R^n as $x^T y$. The symbol $A \in R^{m\times n}$ will represent a real $m \times n$ matrix. Similarly, A^T will represent the transpose of A and $A_{i.}$ will represent the i row or i block of rows of A and $A_{.j}$ the j^{th} column or the j^{th} block of columns of A. A zero vector in a real space of any dimension will be denoted by 0.

2. Privacy-Preserving Linear Programming for Horizontally Partitioned Data

Consider the following linear programming:

$$\begin{aligned} &\min z = c^T x, \\ &\text{s.t.} \quad \begin{aligned} &Ax = b \\ &x \geq 0. \end{aligned} \end{aligned} \tag{1}$$

Here, $\begin{pmatrix} A & b \end{pmatrix}$ consists of the matrix $A \in R^{m\times n}$ and the right-hand vector $b \in R^m$ and is divided into p horizontal blocks. The number of rows of the p horizontal block is recorded as $m_1, m_2, \ldots, m_p$, where $m_1 + m_2 + \cdots + m_p = m$. An m order identity matrix E is divided into $p$vertical blocks. The number of columns of the p vertical block is recorded as $m_1, m_2, \ldots, m_p$, where $m_1 + m_2 + \cdots + m_p = m$. Each block of rows of $[A \quad b]$ corresponding to the index sets $I_1, I_2, \ldots, I_p$, $\cup_{i=1}^{p} I_i = \{1, 2, \ldots, m\}$, is owned by a distinct entity that is unwilling to make its block of data public or share it with the other entities. We will accomplish this goal by the following transformation.

Each entity $i, i = 1, 2, \ldots, p$, chooses its own private random matrix $B_{.I_i} \in R^{m\times m_i}$, whose corresponding index set is I_i. The value of each element in $B_{.I_i}$ is in the interval $(0, 1)$. The following decompositions can be obtained: $A = \begin{pmatrix} A_{I_1.} \\ A_{I_2.} \\ \vdots \\ A_{I_p.} \end{pmatrix}$ and $b = \begin{pmatrix} b_{I_1} \\ b_{I_2} \\ \vdots \\ b_{I_p} \end{pmatrix}$.

Define

$$B = \begin{pmatrix} B_{.I_1} + \lambda E_{.I_1} & B_{.I_2} + \lambda E_{.I_2} & \cdots & B_{.I_p} + \lambda E_{.I_p} \end{pmatrix}, \quad \lambda \in R, \lambda \geq n. \tag{2}$$

Because the matrix B is an m order strictly diagonally dominant matrix, we can easily conclude that the matrix B is an invertible matrix [17]. Based on this fact, we define the following operation:

$$\begin{aligned} BA &= \begin{pmatrix} B_{.I_1} + \lambda E_{.I_1} & B_{.I_2} + \lambda E_{.I_2} & \cdots & B_{.I_p} + \lambda E_{.I_p} \end{pmatrix} \begin{pmatrix} A_{I_1.} \\ A_{I_2.} \\ \vdots \\ A_{I_p.} \end{pmatrix} \\ &= \left(B_{.I_1} + \lambda E_{.I_1}\right) A_{I_1.} + \left(B_{.I_2} + \lambda E_{.I_2}\right) A_{I_2.} + \cdots + \left(B_{.I_p} + \lambda E_{.I_p}\right) A_{I_p.}, \\ Bb &= \begin{pmatrix} B_{.I_1} + \lambda E_{.I_1} & B_{.I_2} + \lambda E_{.I_2} & \cdots & B_{.I_p} + \lambda E_{.I_p} \end{pmatrix} \begin{pmatrix} b_{I_1} \\ b_{I_2} \\ \vdots \\ b_{I_p} \end{pmatrix} \\ &= \left(B_{.I_1} + \lambda E_{.I_1}\right) b_{I_1} + \left(B_{.I_2} + \lambda E_{.I_2}\right) b_{I_2} + \cdots + \left(B_{.I_p} + \lambda E_{.I_p}\right) b_{I_p}. \end{aligned} \tag{3}$$

According to the above discussion, the original linear programming (1) was converted into the following secure linear programming:

$$\begin{aligned} \min \ & z = c^T x, \\ \text{s.t.,} \ & BAx = Bb \\ & x \geq 0. \end{aligned} \tag{4}$$

The linear programming (1) and the linear programming (4) have the same solution set since the matrix B is invertible. The linear programming (4) is quite safe since only the entity i knows $B_{.I_i}$, $i = 1, 2, \ldots, p$. Other entities cannot compute $A_{I_i.}$ and b_{I_i} from $(B_{.I_i} + \lambda E_{.I_i})A_{I_i.}$ and $(B_{.I_i} + \lambda E_{.I_i})b_{I_i}$ without knowing the random matrix $B_{.I_i}$. We regard the linear programming (4) as a secure linear programming. Whether the linear programming (1) is equivalent to the linear programming (4) or not? Let us discuss next.

Proposition 1. *If the matrix B is an m order invertible matrix; then, the secure linear program (4) is solvable if and only if the linear program (1) is solvable in case the solution sets of the two linear programs are identical.*

Proof. As the matrix B is an m-order invertible matrix, the following relation holds:

$$Ax = b \Leftrightarrow BAx = Bb. \tag{5}$$

Therefore, the feasible regions of the two linear programs are the same. Again according to the objective functions of the linear programming (1) and the linear programming (4), we can conclude that the two linear programs have the same solution set.

The following algorithm can get the best solution of the linear programming (1) without revealing any private data. □

3. Formulation of the Privacy-Preserving Algorithm

As shown in Section 2, the linear program (1) is divided among p entities. We put forward the following algorithm:

Step 1. All entities choose a suitable real number $\lambda, \lambda \geq n$ together.

Step 2. Suppose the matrix $\begin{pmatrix} A_{I_i.} & b_{I_i} \end{pmatrix}$ has m_i rows, where $i = 1, 2, \ldots, p$. A random matrix $B_{.I_i}$ is generated by the entity possessing the matrix $\begin{pmatrix} A_{I_i.} & b_{I_i} \end{pmatrix}$, where $B_{.I_i} \in R^{m\times m_i}$. The value of each element in $B_{.I_i}$ is in the interval (0,1), and $B_{.I_i}$ is not public.

Step 3. The entity that owns the matrix $\begin{pmatrix} A_{I_1.} & b_{I_1} \end{pmatrix}$ is responsible to compute $(B_{.I_1} + \lambda E_{.I_1})A_{I_1.}$ and $(B_{.I_1} + \lambda E_{.I_1})b_{I_1}$, and the result is passed to the entity that owns the matrix $\begin{pmatrix} A_{I_2.} & b_{I_2} \end{pmatrix}$. Then, the entity that owns the matrix $\begin{pmatrix} A_{I_2.} & b_{I_2} \end{pmatrix}$ is responsible to compute $(B_{.I_1} + \lambda E_{.I_1})A_{I_1.} + (B_{.I_2} + \lambda E_{.I_2})A_{I_2.}$ and $(B_{.I_1} + \lambda E_{.I_1}) b_{I_1} + (B_{.I_2} + \lambda E_{.I_2})b_{I_2}$, and the result is passed to the entity that owns the matrix $\begin{pmatrix} A_{I_3.} & b_{I_3} \end{pmatrix}$. And, finally, the entity that owns the matrix $\begin{pmatrix} A_{I_p.} & b_{I_p} \end{pmatrix}$ is responsible to compute the following:

$$\begin{aligned} BA = {} & \left(B_{.I_1} + \lambda E_{.I_1}\right) A_{I_1.} + \left(B_{.I_2} + \lambda E_{.I_2}\right) A_{I_2.} \\ & + \cdots + \left(B_{.I_p} + \lambda E_{.I_p}\right) A_{I_p.}, \end{aligned} \tag{6}$$

$$\begin{aligned} Bb = {} & \left(B_{.I_1} + \lambda E_{.I_1}\right) b_{I_1} + \left(B_{.I_2} + \lambda E_{.I_2}\right) b_{I_2} \\ & + \cdots + \left(B_{.I_p} + \lambda E_{.I_p}\right) b_{I_p}. \end{aligned} \tag{7}$$

Step 4. Utilizing the linear programming (4) to calculate the minimum value and the optimal solution of the objective function, which is the minimum value and the optimal solution of the objective function of the linear programming (1).

Remark 1. Through this algorithm, the solution vector x can be used publicly. However, it does not reveal any entity's data.

4. Numerical Experiments

A linear programming:

$$\begin{aligned} &\min\ z = -3x_1 - 5x_2,\\ &\text{s.t.}\ \begin{cases} x_1 + x_3 = 8,\\ 2x_2 + x_4 = 12,\\ 3x_1 + 4x_2 + x_5 = 36,\\ x_i \geq 0, \quad i = 1, \ldots, 5. \end{cases} \end{aligned} \tag{8}$$

We can find that the optimal solution of (8) is $x^* = (4, 6, 4, 0, 0)^T$.Let $I_1 = \{1, 2\}, I_2 = \{3\}$, $A_{I_1 .} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix}$, $A_{I_2 .} = \begin{pmatrix} 3 & 4 & 0 & 0 & 1 \end{pmatrix}$, $b_{I_1} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$, $b_{I_2} = (36)$, and $\lambda = 5$.

Entity 1 generates a random matrix $B_{.I_1}$ which is not published. Note that

$$B_{.I_1} = \begin{pmatrix} 0.9501 & 0.7621 \\ 0.2311 & 0.7621 \\ 0.6068 & 0.7621 \end{pmatrix}. \tag{9}$$

Entity 1 makes public its matrix product $(B_{.I_1} + \lambda E_{.I_1})A_{I_1 .}$ and $(B_{.I_1} + \lambda E_{.I_1})b_{I_1}$.

Entity 2 generates a random matrix $B_{.I_2}$ which is not published. Note that

$$B_{.I_2} = \begin{pmatrix} 0.7621 \\ 0.7621 \\ 0.7621 \end{pmatrix}. \tag{10}$$

Entity 2 makes public its matrix product $(B_{.I_2} + \lambda E_{.I_2})A_{I_2 .}$ and $(B_{.I_2} + \lambda E_{.I_2})b_{I_2}$.

These products do not reveal any private data, but it can be used to calculate the constraint matrix BA and the right-hand side Bb of the secure linear programming. Next, we derive a linear programming (11) from the linear programming (4), which is equivalent to linear programming (8):

$$\begin{aligned} &\min\ z = -3x_1 - 5x_2,\\ &\text{s.t.}\ \begin{cases} 8.2364x_1 + 4.5726x_2 + 5.9501x_3 + 0.7621x_4 + 0.7621x_5 = 84.1816,\\ 2.5174x_1 + 14.5726x_2 + 0.2311x_3 + 5.7621x_4 + 0.7621x_5 = 98.4296,\\ 17.8931x_1 + 24.5726x_2 + 0.6068x_3 + 0.7621x_4 + 5.7621x_5 = 221.4352,\\ x_i \geq 0 \quad i = 1, \ldots, 5. \end{cases} \end{aligned} \tag{11}$$

The solution of this secure linear programming (11) is the same as that of the linear programming (8). This solution can be made public without revealing any private data.

If we use Mangasarian's study [13] which proposed the algorithm of privacy-preserving horizontally partitioned linear programs, the linear programming (8) needs to be converted into the following linear programming:

$$\begin{aligned} &\min\ z = -3x_1 - 5x_2,\\ &\text{s.t.}\ \begin{cases} 3.2364x_1 + 4.5726x_2 + 0.9501x_3 + 0.7621x_4 + 0.7621x_5 = 44.1816,\\ 2.5174x_1 + 4.5726x_2 + 0.2311x_3 + 0.7621x_4 + 0.7621x_5 = 38.4296,\\ 2.8931x_1 + 4.5726x_2 + 0.6068x_3 + 0.7621x_4 + 0.7621x_5 = 41.4352,\\ x_i \geq 0 \quad i = 1, \ldots, 5. \end{cases} \end{aligned} \tag{12}$$

The optimal solution to secure linear program (12) is $x^{*\prime} = (8, 4, 0, 0, 0)^T$. This is not consistent with the optimal solution for the original linear programming (8). The reason for this error is that the random matrix $\begin{pmatrix} B_{.I_1} & B_{.I_2} \end{pmatrix} = \begin{pmatrix} 0.9501 & 0.7621 & 0.7621 \\ 0.2311 & 0.7621 & 0.7621 \\ 0.6068 & 0.7621 & 0.7621 \end{pmatrix}$ is not a full rank matrix. In this way, the original linear programming (8) is not equivalent to the secure linear program (12).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Aluminum-Copper Strip Material Intelligent Process Control Technology Project Based on Industrial Big Data (Grant no. 2017YFB0306404).

References

[1] L. Sun, W.-S. Mu, B. Qi, and Z.-J. Zhou, "A new privacy-preserving proximal support vector machine for classification of vertically partitioned data," *International Journal of Machine Learning and Cybernetics*, vol. 6, no. 1, p. 109, 2014.

[2] M. A. Xindi, L. I. Hui, M. A. Jianfeng et al., "APPLET: a privacy-preserving framework for location-aware recommender system," *Science China*, vol. 60, no. 9, pp. 5–20, 2017.

[3] D. R. Sahu, J.-C. Yao, M. Verma, and K. K. Shukla, "Convergence rate analysis of proximal gradient methods with applications to composite minimization problems," *Optimization*, vol. 70, no. 1, pp. 1–26, 2020.

[4] E. Luo, Q. Liu, J. H. Abawajy, and G. Wang, "Privacy-preserving multi-hop profile-matching protocol for proximity mobile social networks," *Future Generation Computer Systems*, vol. 68, p. 222, 2017.

[5] S. Y. Cho, "Implicit extragradient-like method for fixed point problems and variational inclusion problems in a Banach space," *Symmetry*, vol. 12, no. 6, p. 998, 2020.

[6] M.-J. Xiao, L.-S. Huang, Y.-L. Luo, and H. Shen, "Privacy preserving id3 algorithm over horizontally partitioned data," in *Proceedings of the Sixth International Conference on Parallel and Distributed Computing Applications and Technologies (PDCAT'05)*, pp. 239–243, Dalian, China, December 2005.

[7] N. T. An, N. M. Nam, and X. Qin, "Solving k-center problems involving sets based on optimization techniques," *Journal of Global Optimization*, vol. 76, no. 1, pp. 189–209, 2020.

[8] S. Y. Cho, "A monotone Bregan projection algorithm for fixed point and equilibrium problems in a reflexive Banach space," *Filomat*, vol. 34, no. 5, pp. 1487–1497, 2020.

[9] K. Chen and L. Liu, "Privacy preserving data classification with rotation perturbation," in *Proceedings of the Fifth International Conference of Data Mining (ICDM'05)*, pp. 589–592, Houston, TX, USA, November 2005.

[10] O. L. Mangasarian, E. W. Wild, and G. M. Fung, "Privacy-preserving classification of vertically partitioned data via random kernels," *ACM Transactions on Knowledge Discovery from Data*, vol. 2, no. 3, pp. 1–16, 2008.

[11] L. V. Nguyen and X. Qin, "The minimal time function associated with a collection of sets," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 26, p. 93, 2020.

[12] O. L. Mangasarian, "Privacy-preserving linear programming," *Optimization Letters*, vol. 5, no. 1, pp. 165–172, 2011.

[13] O. L. Mangasarian, "Privacy-preserving horizontally partitioned linear programs," *Optimization Letters*, vol. 6, no. 3, pp. 431–436, 2012.

[14] S. Y. Cho, "A convergence theorem for generalized mixed equilibrium problems and multivalued asymptotically nonexpansive mappings," *Journal of Nonlinear and Convex Analysis*, vol. 21, pp. 1017–1026, 2020.

[15] W. Li, H. Li, and C. Deng, "Privacy-preserving horizontally partitioned linear programs with inequality constraints," *Optimization Letters*, vol. 7, no. 1, pp. 137–144, 2013.

[16] X. Feng and Z. Zhang, "The rank of a random matrix," *Applied Mathematics and Computation*, vol. 185, no. 1, pp. 689–694, 2007.

[17] Q. Y. N. Chao-Wang and D. Yi-Yi, *Numericalanalysis*, pp. 210-211, Huazhong University of Science and Technology Press, Wuhan, China, 2006.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6659079, 12 pages
https://doi.org/10.1155/2021/6659079

Research Article

A New Iterative Construction for Approximating Solutions of a Split Common Fixed Point Problem

Huimin He [1,2] **Qinwei Fan,** [3] **and Rudong Chen** [4]

[1] School of Mathematics and Statistics, Xidian University, Xi'an 710071, China
[2] School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
[3] School of Science, Xi'an Polytechnic University, Xi'an 710048, China
[4] Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China

Correspondence should be addressed to Huimin He; huiminhe@126.com

Received 23 November 2020; Revised 27 December 2020; Accepted 11 January 2021; Published 30 January 2021

Academic Editor: Sun Young Cho

In this paper, we aim to construct a new strong convergence algorithm for a split common fixed point problem involving the demicontractive operators. It is proved that the vector sequence generated via the Halpern-like algorithm converges to a solution of the split common fixed point problem in norm. The main convergence results presented in this paper extend and improve some corresponding results announced recently. The highlights of this paper shed on the novel algorithm and the new analysis techniques.

1. Introduction

Let H_1 and H_2 be the Hilbert spaces and C and Q be nonempty closed and convex subsets of H_1 and H_2, respectively.

The split feasibility problem (SFP) is known to find

$$x \in C, \quad \text{such that } Ax \in Q, \tag{1}$$

where A: $H_1 \longrightarrow H_2$ is a linear bounded operator.

In [1], the split feasibility problem (SFP) in the finite-dimensional Hilbert spaces was introduced by Censor and Elfving. This problem is equivalent to a number of nonlinear optimization problems and finds numerous real applications, such as signal processing and medical imaging (see, e.g., [2–7]).

For this split problem, simultaneous multiprojections algorithm was employed by Censor and Elfving in the finite-dimensional space R^n to obtain the algorithm as follows:

$$x_{n+1} = A^{-1} P_Q P_{A(C)} A x_n, \tag{2}$$

where both C and Q are convex and closed subsets of R^n, the linear bounded operator A of R^n is an $n \times n$ matrix, and P_Q is the orthogonal projection operator onto the sets Q.

The above algorithm (2) involves the matrix A^{-1} (one always assumes the existence of A^{-1}) at every iterative step. Calculating A^{-1} is very much time-consuming, if the dimensions are large scale, in particular, and thus it does not become popular.

In order to overcome the fault, Byrne [2, 8] proposed the following novel algorithm CQ, which is under the spotlight of recent research

$$x_{n+1} = P_C\left(x_n - \gamma A^*\left(I - P_Q\right)Ax_n\right), \quad n \geq 0, \tag{3}$$

where P_C and P_Q are the orthogonal projection operators onto the sets C and Q, respectively, and $0 < \gamma < (2/\rho)$ with ρ being the spectral radius of the composite mapping A^*A. But, the CQ algorithm's step-size is fixed, and it is related to spectral radius of A^*A. On the other hand, the orthogonal projection onto the subsets C and Q in Hilbert space H_1 is not easily calculated generally except the special cases, such as balls and polyhedrals. With the real applications (intensity-modulated radiation therapy and medical imaging) of the SFP in signal processing, the SFP has obtained much attention. Now, the approximate solutions of the SFP have been studied extensively by scholars and engineers (see, e.g., [9–13]).

In (1), if C and Q are the intersections of fixed point sets of finite many nonlinear operators, the SFP becomes the split common fixed point problem (SCFPP). The SCFPP was studied first by Censor and Segal [14] in 2009, which consists of finding an element $x \in H_1$ with

$$x \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i), \quad s.t.\ Ax \in \bigcap_{j=1}^{n} \mathrm{Fix}(S_j), \tag{4}$$

where $\mathrm{Fix}(T_i)$ denotes the fixed point set of T_i: $H_1 \longrightarrow H_1$ and $\mathrm{Fix}(S_j)$ denotes the fixed point sets of S_i: $H_2 \longrightarrow H_2$, respectively.

In particular, if $m = n = 1$, then

$$x \in \mathrm{Fix}(T), \quad s.t.\ Ax \in \mathrm{Fix}(S), \tag{5}$$

and T: $H_1 \longrightarrow H_1$, S: $H_2 \longrightarrow H_2$, and $\mathrm{Fix}(T)$ denotes the fixed point set of T, and $\mathrm{Fix}(S)$ denotes the fixed point set of S.

The SCFPP becomes a specific case of SFP and closely related to SFP. To solve this problem, the original algorithm for the directed operator was introduced by Censor and Segal [14] in 2009 as follows:

$$x_{n+1} = T\left(x_n - \rho A^* (I - S)Ax_n\right), \quad n \geq 0, \tag{6}$$

where ρ satisfies the constraint condition $0 < \rho < (2/\|A\|^2)$, and the authors got the weak convergence of the sequence $\{x_n\}$ for solving the SCFPP (5) if the SCFPP consists, that is, its solution set is nonempty.

Recently, Cui and Wang [15] studied the following algorithm, and they got the weak convergence of the sequence $\{x_n\}$ for solving the SCFPP (5):

$$x_{n+1} = U_\lambda\left(x_n - \rho_n A^* (I - T)Ax_n\right), \tag{7}$$

where $U_\lambda = (1-\lambda)I + \lambda U$ and ρ_n is given in the following pattern:

$$\rho_n = \begin{cases} \dfrac{(1-\tau)\left\|(I-T)Ax_n\right\|^2}{2\left\|A^*(I-T)Ax_n\right\|^2}, & Ax_n \neq T(Ax_n), \\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

The step-size of this algorithm ρ_n does not depend on the norm of the operator A and searches automatically.

In 2015, Boikanyo [16] extended the main results of Cui and Wang [15] and constructed the Halpern-type algorithm for demicontractive operators that converge to a solution of the SCFPP (5) strongly:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)U_\lambda\left(x_n - \rho_n A^* (I - T)Ax_n\right), \tag{9}$$

where ρ_n is given as (8). In this result, the resolvent $I - \rho_n A^* (I - T)A$ plays an important role. Indeed, the techniques of resolvents is quite popular, and it acts as a bridge between fixed point problems and a number of optimization problems (see, e.g., [17–21] and the references therein).

Motivated by the above results, we propose a novel algorithm on demicontractive operators for approximating a solution of the SCFPP (5):

$$\begin{cases} u_n = x_n - \rho_n A^* (I-T)Ax_n, \\ x_{n+1} = (1-\alpha_n)\{(1-\xi_n)I + \xi_n U[(1-\eta_n)I + \eta_n U]\}u_n + \alpha_n u, \end{cases} \tag{10}$$

where ρ_n is also obtained by (8). Our algorithm is also based on the Halpern iteration. Indeed, it is a core for many algorithms in split problems (see, e.g., [22–26]). We get the strong convergence of the iterative sequence $\{x_n\}$ generated by (10) for solving the SCFPP (5). Our main results are in two folds. First, we construct a novel iterative algorithm to solve the split common fixed point problem for the demicontractive operators. Second, we permit step-size to be selected self-adaptively by the self-adaptive method, which avoids to depend on the norm of the nonlinear operator A. Our results extend and improve some results of Boikanyo [16], Cui and Wang [15], Yao et al. [27], and many others.

2. Preliminaries

In this section, we will present some lemmas, which are useful to prove our main results as follows.

Let H be a Hilbert space, which is endowed with the inner product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$. Then, the following inequalities hold:

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle, \quad \forall u, v \in H, \tag{11}$$

$$\|tu + (1-t)v\|^2 = t\|u\|^2 + (1-t)\|v\|^2 - t(1-t)\|u - v\|^2, \quad \forall t \in R \text{ and } \forall u, v \in H. \tag{12}$$

Definition 1. Let T: $H \longrightarrow H$ be an operator, then $I - T$ called demiclosed at zero, if the following implication holds for any $\{x_n\}$ in H:

$$\left.\begin{array}{ll} x_n & \rightharpoonup x \\ (I-T)x_n & \longrightarrow 0 \end{array}\right\} \Rightarrow x = Tx. \tag{13}$$

Note that the nonexpansive operator is demiclosed at zero [28].

Lemma 1 (see [29]). *Let $\{a_n\}$ be a sequence of real nonnegative numbers with*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \tag{14}$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$

(ii) $\limsup_{n\longrightarrow\infty} (\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2 (see [15]). *Let A: $H_1 \longrightarrow H_2$ be a linear bounded operator and T: $H_2 \longrightarrow H_2$ a τ−demicontractive mapping with $\tau < 1$. If $A^{-1}Fix(T) \neq \emptyset$, then it is as follows:*

(a) $(I-T)A\hat{x}=0 \Leftrightarrow A^*(I-T)A\hat{x}=0$, $\forall \hat{x}\in H_1$.

(b) In addition, for $z\in A^{-1}Fix(T)$,

$$\left\|x-z-\rho A^*(I-T)A\hat{x}\right\|^2+\frac{(1-\tau)^2\|(I-T)A\hat{x}\|^4}{4\left\|A^*(I-T)A\hat{x}\right\|^2}\le\|\hat{x}-z\|^2, \tag{15}$$

where $x\in H_1$, $Ax\neq T(Ax)$ *and*

$$\rho := \frac{(1-\tau)\|(I-T)A\hat{x}\|^2}{2\left\|A^*(I-T)A\hat{x}\right\|^2}. \tag{16}$$

Lemma 3 (see [30]). *Let H be a Hilbert space and let T be an L-Lipschitzian mapping defined on H with the module $L\ge 1$. Set*

$$K := \xi T(\eta T+(1-\eta)I)+(1-\xi)I. \tag{17}$$

If $0<\xi<\eta<(1/1+\sqrt{1+L^2})$, then the following conclusions hold:

(1) K is demiclosed at zero point 0, if T is demiclosed at 0

(2) $\mathrm{Fix}(T)=\mathrm{Fix}(T(\eta T+(1-\eta)I))=\mathrm{Fix}(K)$

(3) If $T: H\longrightarrow H$ is a quasi-pseudo-contractive operator, then the operator K is quasi-non-expansive

Lemma 4 (see [31]). *Let $\{s_k\}$ be a real numbers sequence that does not decrease at infinity in the sense that there exists a subsequence $\left\{s_{k_j}\right\}$ of $\{s_k\}$ such that $\left\{s_{k_j}\right\}<\left\{s_{k_{j+1}}\right\}$ for all $j\ge 0$. Define an integer sequence $\{m_k\}_{k\ge k_0}$ by*

$$m_k=\max\{k_0\le l\le k\colon\ s_l<s_{l+1}\}. \tag{18}$$

Then, $m_k\longrightarrow\infty$ as $k\longrightarrow\infty$ and

$$s_{m_k+1}\ge\max\left\{s_{m_k},s_k\right\}, \tag{19}$$

for all $k\ge k_0$.

3. Some Nonlinear Operators

Definition 2. An operator $T\colon H\longrightarrow H$ is said to be $L-$Lipschitzian if and only if there exists $L>0$ such that

$$\|Tx-Ty\|\le L\|x-y\|, \tag{20}$$

for all $x,y\in C$.

Definition 3. An operator $T\colon H\longrightarrow H$ is said to be non-expansive if and only if

$$\|Tx-Ty\|\le\|x-z\|,\quad \forall x\in H. \tag{21}$$

Definition 4. An operator $T\colon H\longrightarrow H$ is said to be quasi-non-expansive if and only if $\mathrm{Fix}(T)\neq\varnothing$ and

$$\|Tx-z\|\le\|x-z\|,\quad \forall x\in H,\ \forall z\in\mathrm{Fix}(T). \tag{22}$$

Definition 5. An operator $T\colon H\longrightarrow H$ is said to be firmly nonexpansive if and only if

$$\|Tx-Ty\|^2\le\|x-y\|^2-\|(I-T)x-(I-T)y\|^2,\quad \forall x,y\in H. \tag{23}$$

Definition 6. An operator $T\colon H\longrightarrow H$ is said to be firmly quasi-non-expansive if and only if $\mathrm{Fix}(T)\neq\varnothing$ and

$$\|Tx-z\|^2\le\|x-z\|^2-\|(I-T)x\|^2,\quad \forall x\in H,\ \forall z\in\mathrm{Fix}(T). \tag{24}$$

Definition 7. An operator $T\colon H\longrightarrow H$ is said to be pseudocontractive if and only if

$$\langle Tx-Ty,x-y\rangle\le\|x-y\|^2,\quad \forall x,y\in H. \tag{25}$$

Note that T is pseudocontractive if and only if the operator $I-T$ is monotone. There is also an alternative definition for pseudocontractive operators, that is, T is said to be pseudocontractive if and only if

$$\|Tx-Ty\|^2\le\|x-y\|^2+\|(I-T)x-(I-T)y\|^2,\quad \forall x,y\in H. \tag{26}$$

Definition 8. An operator $T\colon H\longrightarrow H$ is said to be quasi-pseudo-contractive if and only if $\mathrm{Fix}(T)\neq\varnothing$ and

$$\left\|Tx-x^*\right\|^2\le\left\|x-x^*\right\|^2+\|Tx-x\|^2,\quad \forall x\in H,\ \forall x^*\in\mathrm{Fix}(T). \tag{27}$$

Definition 9. An operator $T\colon H\longrightarrow H$ is said to be strictly pseudocontractive if and only if there exists $k\in[0,1)$ such that

$$\|Tx-Ty\|^2\le\|x-y\|^2+k\|(I-T)x-(I-T)y\|^2,\quad \forall x,y\in H. \tag{28}$$

Definition 10. A operator $T\colon H\longrightarrow H$ is said to be directed if and only if

$$\langle z-Tx,x-Tx\rangle\le 0,\quad \forall x\in H,\ \forall z\in\mathrm{Fix}(T). \tag{29}$$

Definition 11. An operator $T\colon H\longrightarrow H$ is said to be $\tau-$demicontractive with $\tau<1$ if and only if

$$\|Tx-z\|^2\le\|x-z\|^2+\tau\|x-Tx\|^2,\quad \forall x\in H,\ \forall z\in\mathrm{Fix}(T). \tag{30}$$

It is easy to obtain that (29) is equivalent to

$$\|z - Tx\|^2 + \|x - Tx\|^2 - \|x - z\|^2 \leq 0, \quad \forall x \in H, \ \forall z \in \mathrm{Fix}(T). \tag{31}$$

Remark 1. The classes of k-demicontrative mappings, directed mappings, quasi-non-expansive mappings, and nonexpansive mappings are closely related. By the above definitions, we obtain the following conclusion relations easily (see Figures 1–7).

(1) The nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ is quasi-non-expansive mapping
(2) The quasi-non-expansive mapping is 0–demicontrative mapping
(3) The firmly nonexpansive mapping is nonexpansive mapping
(4) The firmly quasi-non-expansive mapping is quasi-non-expansive mapping
(5) The firmly nonexpansive mapping is firmly quasi-non-expansive mapping
(6) The directed mapping is demicontractive mapping
(7) The demicontractive mapping is quasi-pseudo-contractive mapping
(8) The strictly pseudocontractive mapping is pseudo-contractive mapping
(9) The pseudocontractive mapping is quasi-pseudo-contractive mapping

4. Main Results

In this section, some assumptions are as follows:

(1) H_1 and H_2 are two Hilbert spaces, $A\colon H_1 \longrightarrow H_2$ is a linear bounded operator, and A^* is the adjoint of A
(2) $U\colon H_1 \longrightarrow H_1$ and $T\colon H_2 \longrightarrow H_2$ are two L–Lipschitzian operators with $L \geq 1$, $\mathrm{Fix}(U) \neq \emptyset$, and$\mathrm{Fix}(T) \neq \emptyset$
(3) $U\colon H_1 \longrightarrow H_1$ is a κ-demicontractive operator $(\kappa < 1)$, and $T\colon H_2 \longrightarrow H_2$ is a τ-demicontractive operator $(\tau < 1)$
(4) $I - U$ and $I - T$ are two demiclosed operators at O
(5) The set of solutions of SCFPP (5), denoted by S, is nonempty

The strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ is denoted by $x_n \longrightarrow x$.

Now, we give the new algorithm to find $x^* \in S$.where A is a bounded and linear mapping, A^* is the adjoint of operator A, and ρ_n is obtained as follows:

$$\rho_n = \begin{cases} \dfrac{(1-\tau)\left\|(I-T)Ax_n\right\|^2}{2\left\|A^*(I-T)Ax_n\right\|^2}, & Ax_n \neq T(Ax_n), \\ 0, & \text{otherwise}. \end{cases} \tag{33}$$

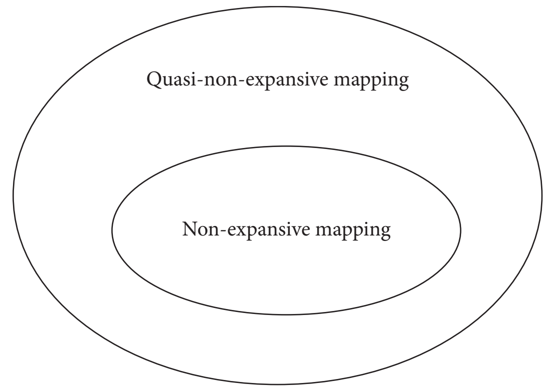


Figure 1: The relations of some nonlinear operators.

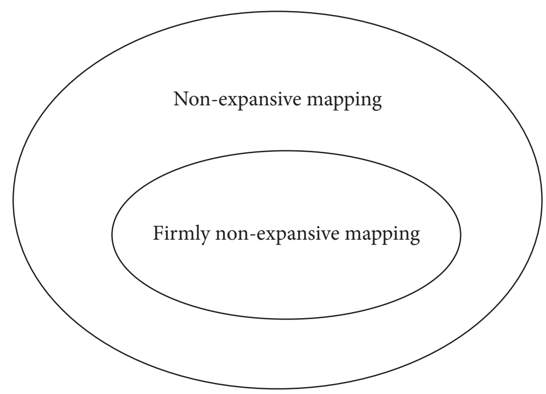


Figure 2: The relations of some nonlinear operators.

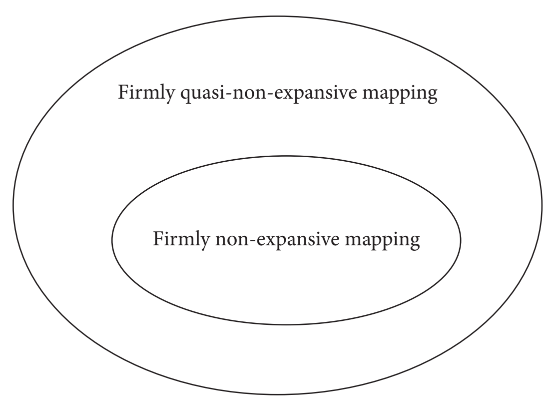


Figure 3: The relations of some nonlinear operators.

Algorithm 1. H_1 is a real Hilbert space, and $\mathrm{Fix}(U) \neq \emptyset$. Take an initial point $x_0 \in H_1$ arbitrarily, and fix $u \in H_1$ and $\{\theta_n\} \subset (0, 1)$. If the n–th iteration x_n is available, then the $(n+1)$–th iteration is constructed via the following formula:

$$\begin{cases} u_n = x_n - \rho_n A^*(I-T)Ax_n, \\ x_{n+1} = \theta_n u + (1-\theta_n)\{(1-\mu_n)I + \mu_n U[(1-\nu_n)I + \nu_n U]\}u_n, \end{cases} \tag{32}$$

Lemma 5. *Assume that H_1 is a Hilbert space, $U\colon H_1 \longrightarrow H_1$ is a κ-demicontractive operator with $\kappa \leq 1$, L–Lipschitzian*

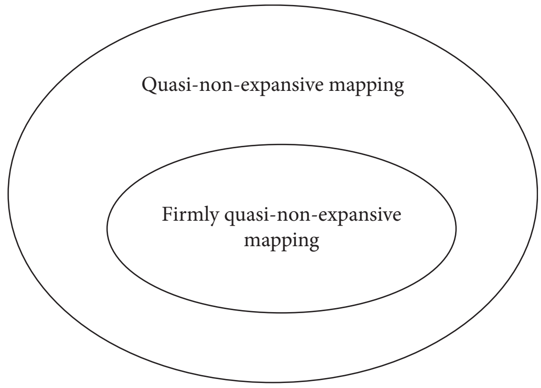


FIGURE 4: The relations of some nonlinear operators.

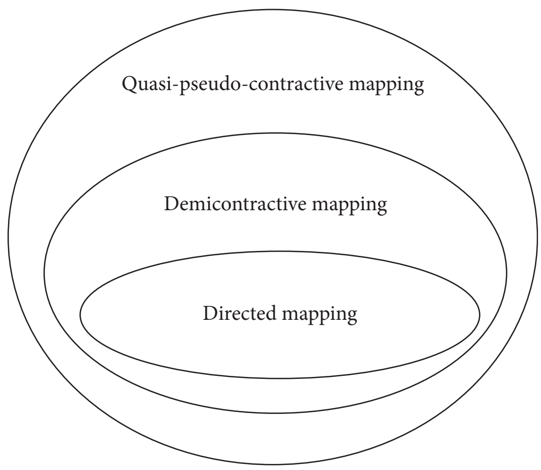


FIGURE 5: The relations of some nonlinear operators.

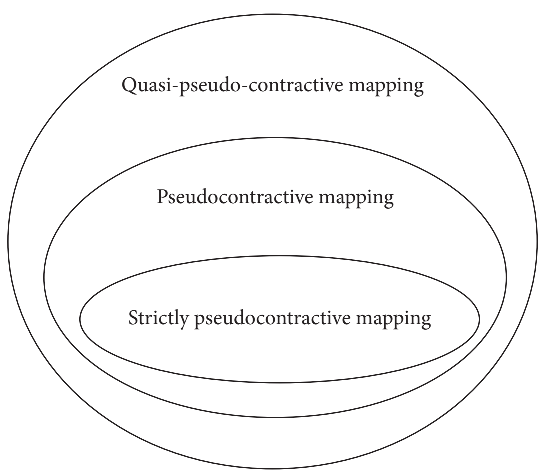


FIGURE 6: The relations of some nonlinear operators.

mappings ($L \geq 1$)*, and* $Fix(U) \neq \emptyset$*. Denote* $U_{\mu,\nu} := (1-\mu)I + \mu U[(1-\nu)I + \nu U]$ *with* $0 < \mu < \nu < (2-\kappa/1+\sqrt{1+L^2(2-\kappa)})$*. Then, for all* $x \in H_1$,

$$\left\| z - U_{\mu,\nu} \right\|^2 \leq \|x - z\|^2 - \mu\nu\left(2 - 2\nu - \kappa - \nu^2 L^2\right)\|Ux - x\|^2, \tag{34}$$

where $z \in Fix(U)$*. Moreover,*

$$\left\| z - U_{\mu,\nu} \right\| \leq \|z - x\|. \tag{35}$$

That is, $U_{\mu,\nu}$ is quasi-non-expansive.

Proof. Since $z \in \mathrm{Fix}(U)$, we get from (30) that

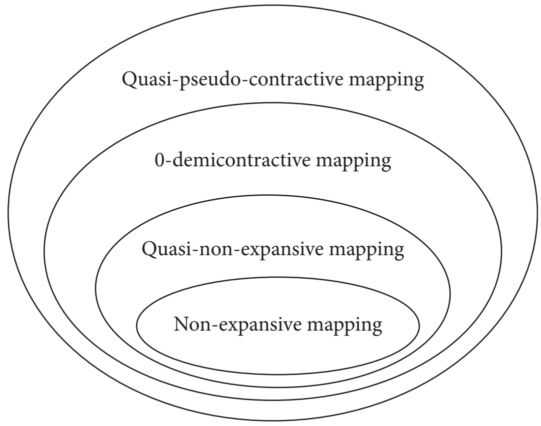


Figure 7: The relations of some nonlinear operators.

$$\begin{aligned}&\|U[(1-\nu)I+\nu U]x-z\|^2\\&\quad\le\|[(1-\nu)I+\nu U]x-z\|^2\\&\qquad+\kappa\|[(1-\nu)I+\nu U]x-U[(1-\nu)I+\nu U]x\|^2\\&\quad\le\|(1-\nu)(x-z)+\nu(Ux-z)\|^2\\&\qquad+\kappa\|[(1-\nu)I+\nu U]x-U[(1-\nu)I+\nu U]x\|^2.\end{aligned}\tag{36}$$

Based on the fact that U is L–Lipschitzian, we get

$$\|Ux-U[(1-\nu)I+\nu U]x\|\le\nu L\|x-Ux\|.\tag{37}$$

Also, from (30) and (12), we can get

$$\begin{aligned}&\|(1-\nu)(x-z)+\nu(Ux-z)\|^2\\&\quad=(1-\nu)\|x-z\|^2+\nu\|Ux-z\|^2-\nu(1-\nu)\|x-Ux\|^2\\&\quad\le(1-\nu)\|x-z\|^2+\nu\left(\|x-z\|^2+\kappa\|Ux-x\|^2\right)\\&\qquad-\nu(1-\nu)\|x-Ux\|^2\\&\quad=\|x-z\|^2+\nu(\nu+\kappa-1)\|Ux-x\|^2.\end{aligned}\tag{38}$$

By (12) and (37), we get

$$\begin{aligned}&\|[(1-\nu)I+\nu U]x-U[(1-\nu)I+\nu U]x\|^2\\&\quad=\|(1-\nu)(x-U[(1-\nu)I+\nu U]x)+\nu(Ux-U[(1-\nu)I+\nu U]x)\|^2\\&\quad=(1-\nu)\|x-U[(1-\nu)I+\nu U]x\|^2+\nu\|Ux-U[(1-\nu)I+\nu U]x\|^2\\&\qquad-\nu(1-\nu)\|x-Ux\|^2\\&\quad\le(1-\nu)\|x-U[(1-\nu)I+\nu U]x\|^2+\nu\nu^2L^2\|Ux-x\|^2\\&\qquad-\nu(1-\nu)\|x-Ux\|^2\\&\quad=(1-\nu)\|x-U[(1-\nu)I+\nu U]x\|^2-\nu\left(1-\nu-\nu^2L^2\right)\|x-Ux\|^2.\end{aligned}\tag{39}$$

Substituting (38) and (39) into (36), we have

$$\begin{aligned}&\|U[(1-\nu)I+\nu U]x-z\|^2\\&\le\|x-z\|^2+\nu(\nu+\kappa-1)\|Ux-x\|^2\\&\quad+(1-\nu)\|x-U[(1-\nu)I+\nu U]x\|^2\\&\quad-\nu\left(1-\nu-\nu^2L^2\right)\|x-Ux\|^2\\&=\|x-z\|^2+(1-\nu)\|x-U[(1-\nu)I+\nu U]x\|^2\\&\quad-\nu\left(2-2\nu-\kappa-\nu^2L^2\right)\|x-Ux\|^2.\end{aligned}\tag{40}$$

Since $\mu<\nu$, combining (12) and (40), we get

$$\begin{aligned}&\|(1-\mu)x+\mu U[(1-\nu)I+\nu U]x-z\|^2\\&=\|(1-\mu)(x-z)+\mu\{U[(1-\nu)I+\nu U]x-z\}\|^2\\&=(1-\mu)\|x-z\|^2+\mu\|U[(1-\nu)I+\nu U]x-z\|^2\\&-\mu(1-\mu)\|U[(1-\nu)I+\nu U]x-x\|^2\\&=(1-\mu)\|x-z\|^2-\mu(1-\mu)\|U[(1-\nu)I+\nu U]x-x\|^2\\&+\mu\left[\|x-z\|^2+(1-\nu)\|x-U[(1-\nu)I+\nu U]x\|^2\right.\\&\left.-\nu\left(2-2\nu-\kappa-\nu^2L^2\right)\|x-Ux\|^2\right]\\&=\|x-z\|^2+\mu(\mu-\nu)\|x-U[(1-\nu)I+\nu U]x\|^2\\&-\nu\left(2-2\nu-\kappa-\nu^2L^2\right)\|x-Ux\|^2\\&\le\|x-z\|^2-\nu\left(2-2\nu-\kappa-\nu^2L^2\right)\|x-Ux\|^2.\end{aligned}\tag{41}$$

Since $\nu<(2-\kappa/1+\sqrt{1+L^2(2-\kappa)})$, we deduce

$$2-2\nu-\kappa-\nu^2L^2>0.\tag{42}$$

Hence,

$$\|(1-\mu)x+\mu U[(1-\nu)I+\nu U]x-z\|^2\le\|x-z\|^2.\tag{43}$$

That is, $U_{\mu,\nu}$ is quasi-non-expansive. □

Theorem 1. *Assume that problem (5) is consistent* $(S\ne\varnothing)$. *Let* $H_1,H_2,A,U,T,\{x_n\}$ *be the same as above. If* $\theta_n\subset(0,1)$ *satisfies* $\lim_{n\longrightarrow\infty}\theta_n=0$ *and* $\sum_{n=0}^{\infty}\theta_n=\infty$, *where a and b are constants and* $\{\mu_n\}$ *and* $\{\nu_n\}$ *satisfies* $0<a<\mu_n<\nu_n<b<(2-\kappa/1+\sqrt{1+L^2(2-\kappa)})$, $\forall n\ge1$, *then the sequence* $\{x_n\}$ *converges to a point* $\overline{x}\in S$ *in norm and* $\overline{x}$ *is the nearest point S to u* $(\overline{x}=tP_Snu)$.

Proof. This proof is split into three parts as follows. □

Step 1. Prove that $\{x_n\}$ is a bounded sequence.

Take $p\in S$. From Theorem 1, we know that U_{μ_n,ν_n} is quasi-non-expansive. From (32), we have

$$\begin{aligned}\left\|x_{n+1}-p\right\|&=\left\|\theta_nu+(1-\theta_n)U_{\mu_n,\nu_n}u_n-p\right\|\\&=\left\|\theta_n(u-p)+(1-\theta_n)\left(U_{\mu_n,\nu_n}u_n-p\right)\right\|\\&\le\theta_n\|u-p\|+(1-\theta_n)\left\|U_{\mu_n,\nu_n}u_n-p\right\|\\&\le\theta_n\|u-p\|+(1-\theta_n)\left\|u_n-p\right\|\\&\le\theta_n\|u-p\|+(1-\theta_n)\left\|x_n-p\right\|.\end{aligned}\tag{44}$$

By induction, we get

$$\left\|x_n-p\right\|\le\max\left\{\|u-p\|,\left\|x_0-p\right\|\right\}.\tag{45}$$

Thus, $\{x_n\}$ is bounded.

Step 2

$$\left\|x_{n+1}-\overline{x}\right\|^2\le(1-\theta_n)\left\|x_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle,\tag{46}$$

where $\overline{x}=P_Su$.

Consider the case $\rho_n\ne0$. From (32), (35), and (11), we get

$$\begin{aligned}\left\|x_{n+1}-\overline{x}\right\|^2&=\left\|\theta_nu+(1-\theta_n)U_{\mu_n,\nu_n}u_n-\overline{x}\right\|\\&=\left\|\theta_n(u-\overline{x})+(1-\theta_n)\left(U_{\mu_n,\nu_n}u_n-\overline{x}\right)\right\|\\&\le(1-\theta_n)^2\left\|U_{\mu_n,\nu_n}u_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle\\&\le(1-\theta_n)\left\|U_{\mu_n,\nu_n}u_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle\\&\le(1-\theta_n)\left\|u_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle\\&\le(1-\theta_n)\left[\left\|x_n-\overline{x}\right\|^2-\frac{(1-\tau)^2}{4}\frac{\left\|(I-T)Ax_n\right\|^4}{\left\|A^*(I-T)Ax_n\right\|^2}\right]\\&\quad+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle\\&\le(1-\theta_n)\left\|x_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle.\end{aligned}\tag{47}$$

Hence,

$$\left\|x_{n+1}-\overline{x}\right\|^2\le(1-\theta_n)\left\|x_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle.\tag{48}$$

Consider the case $\rho_n=0$. From (32) and (11), we get

$$\begin{aligned}\left\|x_{n+1}-\overline{x}\right\|^2&=\left\|\theta_nu+(1-\theta_n)U_{\mu_n,\nu_n}u_n-\overline{x}\right\|\\&=\left\|\theta_n(u-\overline{x})+(1-\theta_n)\left(U_{\mu_n,\nu_n}u_n-\overline{x}\right)\right\|\\&\le(1-\theta_n)^2\left\|U_{\mu_n,\nu_n}u_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle\\&\le(1-\theta_n)\left\|U_{\mu_n,\nu_n}u_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle\\&\le(1-\theta_n)\left\|u_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle\\&\le(1-\theta_n)\left\|x_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x},x_{n+1}-\overline{x}\rangle.\end{aligned}\tag{49}$$

Hence,

$$\|x_{n+1}-\overline{x}\|^2 \le (1-\theta_n)\|x_n-\overline{x}\|^2 + 2\theta_n\langle u-\overline{x}, x_{n+1}-\overline{x}\rangle. \tag{50}$$

Step 3. Prove that $x_n \longrightarrow \overline{x}$ as $n \longrightarrow \infty$.

This step is divided into two cases. Denote $s_n := \|x_n-\overline{x}\|^2$.

Case 1. Assume there exists a positive integer n_0 and the sequence $\{s_n\}$ is decreasing for any $n \ge n_0$. Then, $\{s_n\}$ converges to some point strongly by the monotonic bounded principle.

First, we show that

$$\limsup_{n\longrightarrow\infty}\langle u-\overline{x}, x_n-\overline{x}\rangle \le 0. \tag{51}$$

Using the choice (33) of the step-size ρ_n, (32), (34), (35), and (11), we get

$$\begin{aligned}
\|x_{n+1}-\overline{x}\|^2 &= \left\|\theta_n u+(1-\theta_n)U_{\mu_n,\nu_n}u_n-\overline{x}\right\| \\
&= \left\|\theta_n(u-\overline{x})+(1-\theta_n)\left(U_{\mu_n,\nu_n}u_n-\overline{x}\right)\right\| \\
&\le (1-\theta_n)^2\left\|U_{\mu_n,\nu_n}u_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x}, x_{n+1}-\overline{x}\rangle \\
&\le \left\|U_{\mu_n,\nu_n}u_n-\overline{x}\right\|^2+2\theta_n\langle u-\overline{x}, x_{n+1}-\overline{x}\rangle \\
&\le \|u_n-\overline{x}\|^2-\mu_n\nu_n\left(2-2\nu_n-\kappa-\nu_n^2L^2\right)\|Uu_n-u_n\|^2 \\
&\quad +2\theta_n\langle u-\overline{x}, x_{n+1}-x\rangle \\
&\le \|x_n-\overline{x}\|^2-\frac{\|(I-T)Ax_n\|^4}{\|A^*(I-T)Ax_n\|^2}\frac{(1-\tau)^2}{4} \\
&\quad -\mu_n\nu_n\left(2-2\nu_n-\kappa-\nu_n^2L^2\right)\|Uu_n-u_n\|^2 \\
&\quad +2\theta_n\langle u-\overline{x}, x_{n+1}-\overline{x}\rangle.
\end{aligned} \tag{52}$$

So,

$$\begin{aligned}
&\mu_n\nu_n\left(2-2\nu_n-\kappa-\nu_n^2L^2\right)\|Uu_n-u_n\|^2 \le s_n-s_{n+1}+\theta_n L, \\
&0 \le \frac{(1-\tau)^2\|(I-T)Ax_n\|^4}{4\|A^*(I-T)Ax_n\|^2} \le s_n-s_{n+1}+\theta_n L,
\end{aligned} \tag{53}$$

where L is a nonnegative real constant such that $\sup_{n\in N}\{2\langle f(x_n)-\overline{x}, x_{n+1}-\overline{x}\rangle\} \le L$. Based on the fact that $\{s_n\}$ is convergent, we have

$$\|u_n-Uu_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{54}$$

$$\frac{\|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{55}$$

Moreover,

$$\frac{\|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|} \ge \frac{\|(I-T)Ax_n\|^2}{\|(I-T)Ax_n\|\cdot\|A\|} \ge \frac{\|(I-T)Ax_n\|}{\|A\|}. \tag{56}$$

Hence,

$$\|Ax_n-TAx_n\| \longrightarrow 0. \tag{57}$$

Since

$$\begin{aligned}
\|x_n-u_n\| &= \rho_n\|A^*(I-T)Ax_n\| \\
&= \frac{(1-\tau)\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{58}$$

Since $x_n \rightharpoonup q$, we have $u_n \rightharpoonup q$ due to (58). From (54) and as $I-U$ is demiclosed at zero, we have

$$q \in \mathrm{Fix}(U). \tag{59}$$

From (55) and $I-T$ is demiclosed at zero, we have

$$Aq \in \mathrm{Fix}(T). \tag{60}$$

Thus, $q \in S$ by (59) and (60). Hence, it follows from $\overline{x} = P_S u$ that

$$\begin{aligned}
&\text{limsup\textbackslash limits_}n \longrightarrow \infty\langle u-\overline{x}, x_n-\overline{x}\rangle \\
&\quad = \langle u-\overline{x}, q-\overline{x}\rangle \le 0.
\end{aligned} \tag{61}$$

Secondly, we show that

$$\|x_{n+1}-x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{62}$$

From (32), we have

$$\begin{aligned}
\left\|U_{\mu_n,\nu_n}u_n-u_n\right\| &= \mu_n\|u_n-U[(1-\nu_n)I+\nu_nU]u_n\| \\
&= \mu_n\|u_n-Uu_n+Uu_n-U[(1-\nu_n)I+\nu_nU]u_n\| \\
&\le \mu_n\|u_n-Uu_n\|+\mu_n\|Uu_n-U[(1-\nu_n)I+\nu_nU]u_n\| \\
&\le \mu_n\|u_n-Uu_n\|+\mu_nL\|u_n-[(1-\nu_n)I+\nu_nU]u_n\| \\
&= \mu_n\|u_n-Uu_n\|+\mu_n\nu_nL\|u_n-Uu_n\| \\
&= \mu_n(1+\nu_nL)\|u_n-Uu_n\|.
\end{aligned} \tag{63}$$

From the above equation and (32), (54), and (58), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \theta_n\|u-x_n\| + (1-\theta_n)\|x_n - U_{\mu_n,\nu_n}u_n\|\\ &\le \theta_n\|u-x_n\| + \|x_n-u_n\| + \|u_n - U_{\mu_n,\nu_n}u_n\|\\ &\le \theta_n\|u-x_n\| + \|x_n-u_n\| + \mu_n(1+\nu_nL)\|u_n-Uu_n\|\\ &\le \theta_n\|u-x_n\| + \|x_n-u_n\| + b(1+bL)\|u_n-Uu_n\|.\end{aligned} \tag{64}$$

Combining (54) and 58, we get

$$\|x_{n+1}-x_n\| \longrightarrow 0, \quad \text{as}\, n \longrightarrow \infty. \tag{65}$$

Thirdly, we show that $x_n \longrightarrow \overline{x}$ as $n \longrightarrow \infty$. Together with (51) and (62), we get

$$\limsup_{n\longrightarrow\infty}\langle u-\overline{x}, x_{n+1}-\overline{x}\rangle \le 0. \tag{66}$$

Applying Lemma 2 to (46), which together with the assumption of $\{\theta_n\}$ and (66), we get $x_n \longrightarrow \overline{x}$ as $n \longrightarrow \infty$ easily.

Case 2. Assume that there is no positive integer n_0 and a decreasing sequence $\{s_n\}$ for any $n \ge n_0$. That is, there is a subsequence $\{s_{k_i}\}$ of $\{s_k\}$ such that $s_{k_i} < s_{k_i+1}$ for any $i \in N$.

From Lemma 4, we can define a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \longrightarrow \infty$ as $k \longrightarrow \infty$ and

$$s_{m_k} \le s_{m_k+1}. \tag{67}$$

Firstly, we show

$$\limsup_{n\longrightarrow\infty}\langle u-\overline{x}, x_{m_k}-\overline{x}\rangle \le 0. \tag{68}$$

It follows from (52) and (67) and the boundedness of $\{x_{m_k}\}$ that

$$\begin{aligned}\mu_{m_k}\nu_{m_k}\left(2-2\nu_{m_k}-\kappa-\nu_{m_k}^2L^2\right)\left\|Uu_{m_k}-u_{m_k}\right\|^2 &\le s_{m_k}-s_{m_k+1}+\alpha_{m_k}L\\ &\le \alpha_{m_k}L,\\ 0 \le \frac{(1-\tau)^2}{4}\frac{\left\|(I-T)Ax_{m_k}\right\|^4}{\left\|A^*(I-T)Ax_{m_k}\right\|^2} &\le s_{m_k}-s_{m_k+1}+\alpha_{m_k}L\\ &\le \alpha_{m_k}L.\end{aligned} \tag{69}$$

Thus,

$$\begin{aligned}&\left\|u_{m_k}-Uu_{m_k}\right\| \longrightarrow 0, \quad \text{as}\, n \longrightarrow \infty,\\ &\frac{\left\|(I-T)Ax_{m_k}\right\|^2}{\left\|A^*(I-T)Ax_{m_k}\right\|} \longrightarrow 0, \quad \text{as}\, n \longrightarrow \infty.\end{aligned} \tag{70}$$

Moreover,

$$\frac{1}{\|A\|}\left\|(I-T)Ax_{m_k}\right\| \le \frac{\left\|(I-T)Ax_{m_k}\right\|^2}{\|A\|\cdot\left\|(I-T)Ax_{m_k}\right\|} \le \frac{\left\|(I-T)Ax_{m_k}\right\|^2}{\left\|A^*(I-T)Ax_{m_k}\right\|}. \tag{71}$$

Hence,

$$\left\|Ax_{m_k}-TAx_{m_k}\right\| \longrightarrow 0, \tag{72}$$

due to

$$\begin{aligned}\left\|x_{m_k}-u_{m_k}\right\| &= \rho_{m_k}\left\|A^*(I-T)Ax_{m_k}\right\|\\ &= \frac{(1-\tau)\left\|(I-T)Ax_{m_k}\right\|^2}{2\left\|A^*(I-T)Ax_{m_k}\right\|} \longrightarrow 0, \quad \text{as}\, n \longrightarrow \infty.\end{aligned} \tag{73}$$

Since $x_{m_k} \rightharpoonup q$, then $u_{m_k} \rightharpoonup q$. So, we have $q \in S$ by the similar proofs in Case 1. Hence, it follows from $\overline{x} = P_S u$ that

$$\limsup_{n\longrightarrow\infty}\langle u-\overline{x}, x_{m_k}-\overline{x}\rangle = \langle u-\overline{x}, q-\overline{x}\rangle \le 0. \tag{74}$$

Secondly, we show

$$\left\|x_{m_k+1}-x_{m_k}\right\| \longrightarrow 0, \quad \text{as}\, k \longrightarrow \infty. \tag{75}$$

From (32), we have

$$\begin{aligned}&\left\|U_{\mu_{m_k},\nu_{m_k}}u_{m_k}-u_{m_k}\right\|\\ &= \mu_{m_k}\left\|u_{m_k}-U\left[\left(1-\nu_{m_k}\right)I+\nu_{m_k}U\right]u_{m_k}\right\|\\ &= \mu_{m_k}\left\|u_{m_k}-Uu_{m_k}+Uu_{m_k}-U\left[\left(1-\nu_{m_k}\right)I+\nu_{m_k}U\right]u_{m_k}\right\|\\ &\le \mu_{m_k}\left\|u_{m_k}-Uu_{m_k}\right\| + \mu_{m_k}\left\|Uu_{m_k}-U\left[\left(1-\nu_{m_k}\right)I+\nu_{m_k}U\right]u_{m_k}\right\|\\ &\le \mu_{m_k}\left\|u_{m_k}-Uu_{m_k}\right\| + \mu_{m_k}L\left\|u_{m_k}-\left[\left(1-\nu_{m_k}\right)I+\nu_{m_k}U\right]u_{m_k}\right\|\\ &= \mu_{m_k}\left\|u_{m_k}-Uu_{m_k}\right\| + \mu_{m_k}\nu_{m_k}L\left\|u_{m_k}-Uu_{m_k}\right\|\\ &= \mu_{m_k}\left(1+\nu_{m_k}L\right)\left\|u_{m_k}-Uu_{m_k}\right\|.\end{aligned} \tag{76}$$

By the above equation and (32), we have

$$\begin{aligned}&\left\|x_{m_k+1}-x_{m_k}\right\|\\&\le\alpha_{m_k}\left\|u-x_{m_k}\right\|+\left(1-\alpha_{m_k}\right)\left\|x_{m_k}-U_{\mu_{m_k},\nu_{m_k}}u_{m_k}\right\|\\&\le\alpha_{m_k}\left\|u-x_{m_k}\right\|+\left\|x_{m_k}-u_{m_k}\right\|+\left\|u_{m_k}-U_{\mu_{m_k},\nu_{m_k}}u_{m_k}\right\|\\&\le\alpha_{m_k}\left\|u-x_{m_k}\right\|+\left\|x_{m_k}-u_{m_k}\right\|+\mu_{m_k}\left(1+\nu_{m_k}L\right)\left\|u_{m_k}-Uu_{m_k}\right\|\\&\le\alpha_{m_k}\left\|u-x_{m_k}\right\|+\left\|x_{m_k}-u_{m_k}\right\|+b(1+bL)\left\|u_{m_k}-Uu_{m_k}\right\|.\end{aligned}\tag{77}$$

Combining (54) and the (58), we get

$$\left\|x_{m_k+1}-x_{m_k}\right\|\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.\tag{78}$$

Thirdly, we show that $x_{m_k}\longrightarrow\overline{x}$ as $n\longrightarrow\infty$. Using (68) and (75), we get

$$\limsup_{n\longrightarrow\infty}\langle u-\overline{x},x_{m_k+1}-\overline{x}\rangle\le 0.\tag{79}$$

Based on $s_{m_k}\le s_{m_k+1}$, $\forall k\in N$ and (46), we get

$$\alpha_{m_k}s_{m_k+1}+\left(1-\alpha_{m_k}\right)\left(s_{m_k+1}-s_{m_k}\right)\le 2\alpha_{m_k}\langle u-\overline{x},x_{m_k+1}-\overline{x}\rangle.\tag{80}$$

So,

$$\alpha_{m_k}s_{m_k+1}\le 2\alpha_{m_k}\langle u-\overline{x},x_{m_k+1}-\overline{x}\rangle,\tag{81}$$

that is,

$$s_{m_k+1}\le 2\langle u-\overline{x},x_{m_k+1}-\overline{x}\rangle.\tag{82}$$

Taking the limit $k\longrightarrow\infty$, using (79), we obtain

$$s_{m_k+1}\longrightarrow 0,\quad \text{as } k\longrightarrow\infty.\tag{83}$$

Thus,

$$s_k\longrightarrow 0,\quad \text{as } k\longrightarrow\infty,\tag{84}$$

due to $s_k\le s_{m_k+1}$. The proof is completed.

5. Numerical Example

In the section, we present a numerical experiment to demonstrate the convergence of this algorithm.

Assume $H_1=H_2=(R^3,\|\cdot\|_2)$ and $T,U\colon R^3\longrightarrow R^3$ is defined by

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{3}\begin{pmatrix}x\\y\\z\end{pmatrix},\qquad U\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\a\\b\end{pmatrix}.\tag{85}$$

Let the bounded linear operator A be defined by

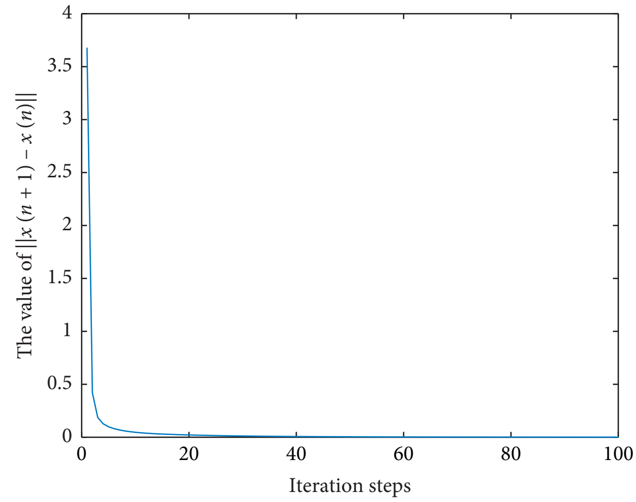


Figure 8: The iterative curves of algorithm (21) under different n.

$$A=\begin{pmatrix}5&-5&-7\\-4&2&-4\\-7&-4&5\end{pmatrix}.\tag{86}$$

Clearly, both U and T are 0−demicontractive mappings. Choose the parameters as follows:

$$\begin{aligned}\theta_n&=\frac{1}{n},\\\mu_n&=\frac{1}{n},\\\nu_n&=\frac{1}{\sqrt{n}},\quad\forall n\ge 1.\end{aligned}\tag{87}$$

ρ_n is chosen in the following way:

$$\rho_n=\begin{cases}\dfrac{(1-\tau)\left\|(I-T)Ax_n\right\|^2}{2\left\|A^*(I-T)Ax_n\right\|^2}, & Ax_n\ne T\left(Ax_n\right),\\[2ex] 0, & \text{otherwise},\end{cases}\tag{88}$$

where A is a bounded and linear mapping and A^* is its adjoint. Then, the iterative algorithm (10) becomes as follows:

$$\begin{cases}u_n=x_n-\rho_nA^*(I-T)Ax_n,\\[1ex] x_{n+1}=\dfrac{1}{n}u+\left(1-\dfrac{1}{n}\right)\left\{\left(1-\dfrac{1}{n}\right)I+\dfrac{1}{n}U\left[\left(1-\dfrac{1}{\sqrt{n}}\right)I+\dfrac{1}{\sqrt{n}}U\right]\right\}u_n,\end{cases}\tag{89}$$

where $u=\begin{pmatrix}1\\-1\\2\end{pmatrix}$ is a fixed point in R^3, and the initial point $x_1=\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}=\begin{pmatrix}1\\-2\\5\end{pmatrix}$ and $x_n=\begin{pmatrix}a_n\\b_n\\c_n\end{pmatrix}$ is generated by the algorithm (10). We plot the numbers of iterations and

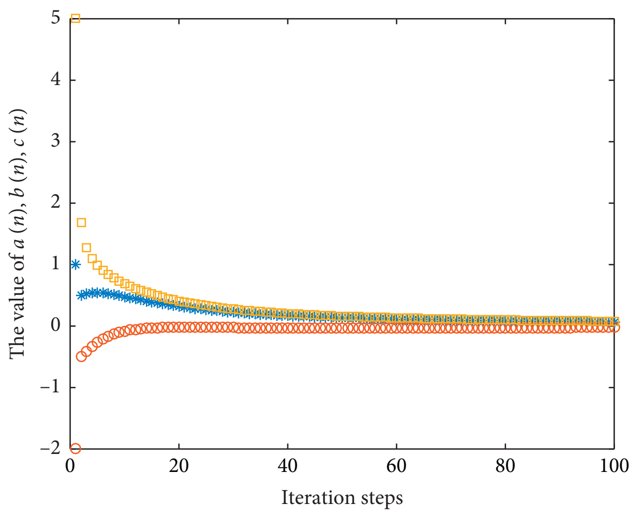


FIGURE 9: The iterative curves of algorithm (21) under different *n*.

$\|x_{n+1} - x_n\|_2$ in the following graphs (Figures 8 and 9), the numbers of iterations and $\{x_n\} = \{a_n, b_n, c_n\}$.

6. Conclusion

In this paper, we proposed a new iteration algorithm (10) and we obtained the strong convergence of the sequence $\{x_n\}$ for split common fixed point problems (5). The main result is an extension of the related results announced in [15, 16, 27]. The research highlights of this paper are novel algorithms and their analysis techniques. The improvement on the extension of the operator, such as the demicontractive mappings, the directed operators, the quasi-non-expansive operators, and quasi-pseudo-contractive operators will be of interest for further research in the future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Fundamental Research Funds for the Central Universities (no. JB180713), the National Science Foundation of China (no. 12031003), the 65th China Postdoctoral Science Foundation (no. 2019 M652837), and the Special Science Research Plan of the Education Bureau of Shaanxi Province of China (no. 18JK0344).

References

[1] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2, pp. 221–239, 1994.

[2] C. Byrne, "Iterative oblique projection onto convex sets and the split feasibility problem," *Inverse Problems*, vol. 18, no. 2, pp. 441–453, 2002.

[3] X. Qin, "Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets," *Computational Optimization and Applications*, vol. 74, no. 3, pp. 821–850, 2019.

[4] Q. An and T. Liu, "Smoothing L_0 regularization for extreme learning machine," *Mathematical Problems in Engineering*, vol. 2020, Article ID 9175106, , 2020.

[5] S. Y. Cho, "Generalized mixed equilibrium and fixed point problems in a Banach space," *Journal of Nonlinear Sciences and Applications*, vol. 09, no. 03, pp. 1083–1092, 2016.

[6] N. T. An, N. M. Nam, and X. Qin, ", "Solving k-center problems involving sets based on optimization techniques," *Journal of Global Optimization*, vol. 76, no. 1, pp. 189–209, 2020.

[7] Q. Fan, L. Niu, and Q. Kang, "Regression and multiclass classification using sparse extreme learning machine via smoothing group L 1/2 regularizer," *IEEE Access*, vol. 8, pp. 191482–191494, 2020.

[8] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

[9] X. Qin, L. Wang, and J. C. Yao, "Inertial splitting method for maximal monotone mappings," *Journal of Nonlinear Convex Analysis*, vol. 21, pp. 2325–2333, 2020.

[10] W. Takahashi, C. F. Wen, and J. C. Yao, "The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space," *Fixed Point Theory*, vol. 19, pp. 407–419, 2018.

[11] W. Takahahsi and J. C. Yao, "The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces," *Journal of Nonlinear Convex Analysis*, vol. 20, pp. 173–195, 2019.

[12] A. Moudafi, "The split common fixed point problem for demicontractive mappings," *Inverse Problem*, vol. 26, p. 55007, 2010.

[13] L. Liu, "A hybrid steepest descent method for solving split feasibility problems involving nonexpansive mappings," *Journal of Nonlinear Convex Analysis*, vol. 20, pp. 471–488, 2019.

[14] Y. Censor and A. Segal, "The split common fixed point problem for directed operators," *Journal of Convex Analysis*, vol. 16, pp. 587–600, 2009.

[15] H. Cui and F. Wang, "Iterative methods for the split common fixed point problem in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2014, p. 78, 2014.

[16] O. A. Boikanyo, "A strongly convergent algorithm for the split common fixed point problem," *Applied Mathematics and Computation*, vol. 265, pp. 844–853, 2015.

[17] S. Y. Cho, "A convergence theorem for generalized mixed equilibrium problems and multivalued asymptotically nonexpansive mappings," *Journal of Nonlinear Convex Analysis*, vol. 21, pp. 1017–1026, 2020.

[18] X. Qin, S. Y. Cho, and L. Wang, "Strong convergence of an iterative algorithm involving nonlinear mappings of nonexpansive and accretive type," *Optimization*, vol. 67, no. 9, pp. 1377–1388, 2018.

[19] W. Takahashi, "The split feasibility problem in Banach spaces," *Journal of Nonlinear Convex Analysis*, vol. 15, pp. 1349–1335, 2014.

[20] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Relaxed extragradient methods for finding minimum-norm solutions of the split feasibility problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 2116–2125, 2012.

[21] S. Y. Cho, "A monotone Bregan projection algorithm for fixed point and equilibrium problems in a reflexive Banach space," *Filomat*, vol. 34, pp. 1487–1497, 2020.

[22] S. M. Alsulami and W. Takahashi, "The split common null point problem for maximal monotone mappings in Hilbert spaces and applications," *Journal of Nonlinear Convex Analysis*, vol. 15, pp. 793–808, 2014.

[23] X. Qin, S. Y. Cho, and L. Wang, "Iterative algorithms with errors for zero points of m-accretive operators," *Fixed Point Theory and Applications*, vol. 2013, p. 148, 2013.

[24] H. He, J. Peng, and Q. Fan, "An iterative viscosity approximation method for the split common fixed-point problem," *Optimization*, vol. 67, 2020.

[25] T. M. Tuye, R. Promka, and P. Sunthrayuth, "Strong convergence of a generalized forwardbackward splitting method in reflexive Banach spaces," *Optimization*, vol. 47, 2020.

[26] X. Qin, S. Y. Cho, and L. Wang, "A regularization method for treating zero points of the sum of two monotone operators," *Fixed Point Theory and Applications*, vol. 2014, p. 75, 2014.

[27] Y. Yao, R. P. Agarwal, M. Postolache, and Y.-C. Liou, "Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem," *Fixed Point Theory and Applications*, vol. 2014, no. 1, p. 183, 2014.

[28] K. Goebel and W. A. Kirk, *Topics in Metric in Metric Fixed Point Theory*, Vol. 28, Cambrige University Press, Cambridge, UK, 1990.

[29] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[30] S. S. Chang, L. Wang, and L. J. Qin, "Split Equality Fixed Point Problem for Quasi-Pseudo-Contractive Mappings with Applications," *Fixed point Theory and Applications*, vol. 2015, p. 208, 2015.

[31] P.-E. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 8, pp. 899–912, 2008.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6687238, 9 pages
https://doi.org/10.1155/2021/6687238

Research Article

Existence and Uniqueness of Fixed Points of Generalized F-Contraction Mappings

A. P. Farajzadeh[1], **M. Delfani**[1], **and Y. H. Wang**[2]

[1]*Department of Mathematics, Razi University, Kermanshah 67149, Iran*
[2]*Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China*

Correspondence should be addressed to Y. H. Wang; yhwang@zjnu.cn

Received 22 November 2020; Revised 6 December 2020; Accepted 12 January 2021; Published 29 January 2021

Academic Editor: Ljubisa Kocinac

The newest generalization of the Banach contraction through the notions of the generalized F-contraction, simulation function, and admissible function is introduced. The existence and uniqueness of fixed points for a self-mapping on complete metric spaces by the new constructed contraction are investigated. The results of this article can be viewed as an improvement of the main results given in the references.

1. Introduction and Preliminaries

In 1922, Banach proved the following famous and fundamental result in fixed-point theory [1]. Let (X, d) be a complete metric space. Let T be a contractive mapping on X; that is, there exists $q \in [0, 1)$ satisfying

$$d(Tx, Ty) \le q.d(x, y), \forall x, y \in X. \tag{1}$$

Then, there exists a unique fixed point $x_0 \in X$ of T. This theorem, which is called the Banach contraction principle that is a forceful tool in nonlinear analysis [9–14] and fixed-point theory, is a fascinating subject, with an enormous number of algorithms and applications in various fields of mathematics, see, e.g., [15–18]. This principle has been generalized in different directions by various researchers. One of them is the following theorem that is presented by Bryant.

Theorem 1 (see [2]). *If f is a mapping of a complete metric space into itself and if, for some positive integer k, f^k is a contraction, then f has a unique fixed point.*

It is obvious that f^k is continuous but there are examples that show it cannot imply the continuity of f and so Theorem 1 is a real extension of the Banach principle.

In 1969, Sehgal [19] proved the following interesting generalization of Theorem 1.

Theorem 2 (see [19]). *Let (X, d) be a complete metric space, $q \in [0, 1)$, and $T: X \longrightarrow X$ be a continuous mapping. If for each $x \in X$ there exists a positive integer $k = k(x)$ such that*

$$d\left(T^{k(x)}x, T^{k(x)}y\right) \le q\, d(x, y), \tag{2}$$

for all $y \in X$, then T has a unique fixed point $u \in X$. Moreover, for any $x \in X$, $u = \lim_{n\longrightarrow\infty} T^n x$.

Several researchers are interested to generalize Banach contraction. Here, we state two of them. Wardowski [8] generalized the Banach contraction as follows.

Definition 1 (see [8]). Let (X, d) be a metric space. The mapping $T: X \longrightarrow X$ is called an F-contraction, if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y)), \tag{3}$$

where $F: (0, \infty) \longrightarrow \mathbb{R}$ is strictly increasing $\lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty$ iff $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and there exists a number $k \in (0, 1)$ such that $\lim_{\alpha\longrightarrow 0^+} \alpha^k F(\alpha) = -\infty$.

Notation. The family of all functions $F\colon (0,+\infty) \longrightarrow \mathbb{R}$ is denoted by $\mathcal{F}$ (see [8]) if F satisfies the following conditions:

(F1) F is strictly increasing

(F2) for every sequence $\{\alpha_n\}$ in $(0,+\infty)$, we have $\lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty$ iff $\lim_{n\longrightarrow\infty}\alpha_n = 0$

(F3) there exists a number $k \in (0,1)$ such that $\lim_{\alpha\longrightarrow 0^+}\alpha^k F(\alpha) = -\infty$

The collection of all functions $F\colon (0,+\infty) \longrightarrow \mathbb{R}$ is denoted by $\mathcal{G}$ ([20]) if F satisfies the following conditions:

(G1) F is strictly increasing

(G2) there exists a sequence $\{\alpha_n\}$ in $(0,+\infty)$ such that $\lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty$, or $\inf F = -\infty$

(G3) F is a continuous mapping

Another way to generalize the Banach contraction is through the following notion.

Definition 2 (see [3, 21]). Let $\zeta\colon [0,\infty)\times[0,\infty) \longrightarrow \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:

(ζ1) $\zeta(0,0) = 0$

(ζ2) $\zeta(t,s) < s - t$ for all $t, s > 0$

(ζ3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\longrightarrow\infty} \zeta(t_n, s_n) < 0 \tag{4}$$

We denote the set of all simulation functions by $\mathcal{Z}$.

Ozturk [4], by using the simulation function and Wardowski [8] idea, extended Theorem 2 as follows.

Theorem 3 (see [4]). *Let (X,d) be a complete metric space and $T\colon X \longrightarrow X$ a mapping which satisfies the condition: If there exist $f \in \mathcal{F}$ and $\tau > 0$ such that for each $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$,*

$$d\left(T^{n(x)}(x), T^{n(x)}(y)\right) > 0 \Rightarrow \zeta\Big(F(d(x,y)), \tau + F\Big(d\left(T^{n(x)}(x), T^{n(x)}(y)\right)\Big)\Big) \geq 0, \tag{5}$$

then T has a unique fixed point $z \in X$ and $T^n(x_0) \longrightarrow z$ for each $x_0 \in X$, as $n \longrightarrow \infty$.

The first aim of this paper is to generalize Theorem 2 by introducing a more general contraction type mapping through the notions of the generalized F-contraction, simulation function, and admissible function. Then, by the new constructed contraction and suitable conditions, the existence and uniqueness of fixed points are investigated.

The following definitions and preliminary results are needed in the next section.

Definition 3 (see [6, 22]). Let $\alpha\colon X\times X \longrightarrow (0,+\infty)$ be a given mapping. The mapping $T\colon X \longrightarrow X$ is said to be an α-admissible, whenever $\alpha(Tx, Ty) \geq 1$ provided $\alpha(x,y) \geq 1$ and $x, y \in X$.

Definition 4 (see [23]). An α-admissible map T is said to have the K-property, while for each sequence $\{x_n\} \subseteq X$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$, the nonnegative integer numbers, there exists a positive integer number k such that $\alpha(Tx_n, Tx_m) \geq 1$, for all $m > n \geq k$.

Lemma 1 (see [5]). *Let $F\colon (0,+\infty) \longrightarrow \mathbb{R}$ be an increasing function and $\{\alpha_n\}$ be a sequence of positive real numbers. Then, the following holds:*

(a) if $\lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty$, then $\lim_{n\longrightarrow\infty}\alpha_n = 0$

(b) if $\inf F = -\infty$ and $\lim_{n\longrightarrow\infty}\alpha_n = 0$, then $\lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty$

Lemma 2 (see [24]). *Let (X,d) be a metric space and $\{x_n\}$ be a sequence in X such that $\lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) > \varepsilon$, $d(x_{m_k}, x_{n_k - 1}) < \varepsilon$, and*

(1) $\lim_{k\longrightarrow\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$

(2) $\lim_{k\longrightarrow\infty} d(x_{m_k - 1}, x_{n_k}) = \varepsilon$

(3) $\lim_{k\longrightarrow\infty} d(x_{m_k}, x_{n_k + 1}) = \varepsilon$

(4) $\lim_{k\longrightarrow\infty} d(x_{m_k - 1}, x_{n_k + 1}) = \varepsilon$

2. Main Results

In this section, the main achievements of this article are presented. The existence and uniqueness of fixed points of the self-mappings on complete metric spaces satisfying the generalized F-contraction (relation (6) of the following theorem) with suitable assumptions are established by the first theorem. The second theorem can be viewed as a generalized version of Suzuki's theorem given in [21]. Of course it ensures existence of fixed points for self-mappings under suitable hypothesis.

Theorem 4 *Let (X,d) be a complete metric space and $\alpha\colon X\times X \longrightarrow (0,+\infty)$ be a symmetric function, where $\alpha(x,y) \geq 1$ and $T\colon X \longrightarrow X$ be a continuous mapping which satisfies the condition: if there exist $F \in \mathcal{F}$, $\tau > 0$, $L \geq 0$, and simulation function ζ such that for all $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$ and $d(T^{n(x)}(x), T^{n(x)}(y)) > 0$,*

$$\zeta\Big(\tau + \alpha(x,y) F\Big(d\Big(T^{n(x)}x, T^{n(x)}y\Big)\Big), F\big(m(x,y) + \mathrm{LN}_1(x,y)\big)\Big) \geq 0, \tag{6}$$

where

$$m(x,y)=\max\left\{d(x,y),d\left(x,T^{n(x)}x\right),d\left(y,T^{n(x)}y\right),\frac{d\left(x,T^{n(x)}y\right)+d\left(y,T^{n(x)}x\right)}{2}\right\},$$
$$N_1(x,y)=\min\left\{d\left(x,T^{n(x)}x\right),d\left(x,T^{n(x)}y\right),d\left(y,T^{n(x)}x\right)\right\}, \tag{7}$$

then T has a unique fixed point.

Proof. We shall built a recursive sequence $\{x_k\}$ as follows: for the chosen arbitrary point $x_0\in X$ with $n_0=n(x_0)$, we set $x_1=T^{n_0}x_0$ and inductively we get $x_{i+1}=T^{n_i}x_i$ with $n_i=n(x_i)$.

We assert that $x_i\neq x_{i+1}$ for all $i\in\mathbb{N}_0$. Suppose, on the contrary, there exists $i_0\in N_0$ such that $x_{i_0}=x_{i_0+1}=T^{n_{i_0}}x_{i_0}$. Then, x_{i_0} turns to be a fixed point of $T^{n_{i_0}}$. On the other hand,

$$Tx_{i_0}=T\left(T^{n_{i_0}}x_{i_0}\right)=T^{n_{i_0}}\left(Tx_{i_0}\right). \tag{8}$$

Thus, Tx_{i_0} form a fixed point of $T^{n_{i_0}}$. If $Tx_{i_0}=x_{i_0}$, then we conclude that T has a fixed point and that terminate the proof. Suppose, on the contrary, that $Tx_{i_0}\neq x_{i_0}$ and hence $d(T^{n_{i_0}}(Tx_{i_0}),T^{n_{i_0}}(x_{i_0}))>0$. Then, by (6), we have

$$\begin{aligned}0&\leq\left(\tau+\alpha\left(x_{i_0},Tx_{i_0}\right)F\left(d\left(T^{n_{i_0}}x_{i_0},T^{n_{i_0}}Tx_{i_0}\right)\right),F\left(m\left(x_{i_0},Tx_{i_0}\right)+\mathrm{LN}_1\left(x_{i_0},Tx_{i_0}\right)\right)\right),\\&\leq F\left(m\left(x_{i_0},Tx_{i_0}\right)+\mathrm{LN}_1\left(x_{i_0},Tx_{i_0}\right)\right)-\left(\tau+\alpha\left(x_{i_0},Tx_{i_0}\right)F\left(d\left(T^{n_{i_0}}x_{i_0},T^{n_{i_0}}Tx_{i_0}\right)\right)\right).\end{aligned} \tag{9}$$

Hence,

$$\begin{aligned}\tau+F\left(d\left(x_{i_0},Tx_{i_0}\right)\right)&=\tau+F\left(d\left(T^{n_{i_0}}x_{i_0},T^{n_{i_0}}Tx_{i_0}\right)\right)\\&\leq\tau+\alpha\left(x_{i_0},Tx_{i_0}\right)F\left(d\left(T^{n_{i_0}}x_{i_0},T^{n_{i_0}}Tx_{i_0}\right)\right)\\&\leq F\left(m\left(x_{i_0},Tx_{i_0}\right)+\mathrm{LN}_1\left(x_{i_0},Tx_{i_0}\right)\right).\end{aligned} \tag{10}$$

However,

$$\begin{aligned}m\left(x_{i_0},Tx_{i_0}\right)&=\max\left\{d\left(x_{i_0},Tx_{i_0}\right),d\left(x_{i_0},T^{n_{i_0}}x_{i_0}\right),d\left(Tx_{i_0},T^{n_{i_0}}Tx_{i_0}\right),\frac{d\left(x_{i_0},T^{n_{i_0}}Tx_{i_0}\right)+d\left(Tx_{i_0},T^{n_{i_0}}x_{i_0}\right)}{2}\right\}=\left\{d\left(x_{i_0},Tx_{i_0}\right)\right\},\\N_1&=\min\left\{d\left(x_{i_0},T^{n_{i_0}}x_{i_0}\right),d\left(x_{i_0},T^{n_{i_0}}Tx_{i_0}\right),d\left(Tx_{i_0},T^{n_{i_0}}x_{i_0}\right)\right\}=0.\end{aligned} \tag{11}$$

Therefore,

$$\tau+F\left(d\left(x_{i_0},Tx_{i_0}\right)\right)\leq F\left(d\left(x_{i_0},Tx_{i_0}\right)\right). \tag{12}$$

So, $\tau\leq 0$, which is a contradiction. Consequently, we deduce that for all $i\in\mathbb{N}_0$, $x_i\neq x_{i+1}$. Then, $d(x_{i+1},x_i)>0$, by (6),

$$\begin{aligned}\tau+F\left(d\left(x_{i+1},x_{i+2}\right)\right)&=\tau+F\left(d\left(T^{n_i}x_i,T^{n_i}x_{i+1}\right)\right.\\&\leq\tau+\alpha\left(x_i,x_{i+1}\right)F\left(d\left(T^{n_i}x_i,T^{n_i}x_{i+1}\right)\right.\\&\leq F\left(m\left(x_i,x_{i+1}\right)+\mathrm{LN}_1\left(x_i,x_{i+1}\right)\right)\\&\leq F\left(m\left(x_i,x_{i+1}\right)+L\,d\left(x_{i+1},x_{i+1}\right)\right)\\&=F\left(m\left(x_i,x_{i+1}\right)\right).\end{aligned} \tag{13}$$

Then,

$$\tau + F(d(x_{i+1}, x_{i+2})) \leq F(m(x_i, x_{i+1})). \quad (14)$$

However,

$$\begin{aligned} m(x_i, x_{i+1}) &= \max\left\{ d(x_i, x_{i+1}), d(x_i, T^{n_i} x_i), d(x_{i+1}, T^{n_i} x_{i+1}), \frac{d(x_i, T^{n_i} x_{i+1}) + d(x_{i+1}, T^{n_i} x_i)}{2} \right\} \\ &= \max\left\{ d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2}), \frac{d(x_i, x_{i+2})}{2} \right\} \\ &\leq \max\left\{ d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2}), \frac{d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})}{2} \right\} \\ &\leq \max\{ d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2}) \}. \end{aligned} \quad (15)$$

If $d(x_{i_0+1}, x_{i_0+2}) \geq d(x_{i_0}, x_{i_0+1})$ for some $i_0 \in \mathbb{N}_0$, then

$$m(x_{i_0}, x_{i_0+1}) \leq d(x_{i_0+1}, x_{i_0+2}), \quad (16)$$

and since F is strictly increasing,

$$F(m(x_{i_0}, x_{i_0+1})) \leq F(d(x_{i_0+1}, x_{i_0+2})), \quad (17)$$

so, it follows from (14) that

$$\tau + F(d(x_{i_0+1}, x_{i_0+2})) \leq F(d(x_{i_0+1}, x_{i_0+2})). \quad (18)$$

So, $\tau \leq 0$, which is a contradiction. Consequently,

$$d(x_{i+1}, x_{i+2}) < d(x_i, x_{i+1}), \quad \forall i \in \mathbb{N}_0. \quad (19)$$

Hence, from (14) and (19), we have

$$\tau + F(d(x_{i+1}, x_{i+2}) \leq F(d(x_i, x_{i+1})) \quad (20)$$

or

$$F(d(x_{i+1}, x_{i+2}) \leq F(d(x_i, x_{i+1})) - \tau. \quad (21)$$

In general, one can get

$$F(d(x_{i+1}, x_{i+2}) \leq F(d(x_0, x_1)) - i\tau. \quad (22)$$

Hence,

$$\lim_{i \longrightarrow \infty} F(d(x_i, x_{i+1})) = -\infty. \quad (23)$$

So, from (F_2), we have

$$\lim_{i \longrightarrow \infty} d(x_i, x_{i+1}) = 0. \quad (24)$$

Therefore, with notice to (F_3), there exists $k \in (0,1)$ such that

$$\lim_{i \longrightarrow \infty} (d(x_i, x_{i+1}))^k F(d(x_i, x_{i+1})) = 0. \quad (25)$$

Now, (22) implies that

$$(d(x_i, x_{i+1}))^k F(d(x_i, x_{i+1})) \leq (d(x_i, x_{i+1}))^k (F(d(x_0, x_1)) - i\tau). \quad (26)$$

Then, it can be easily seen that

$$\lim_{i \longrightarrow \infty} i(d(x_i, x_{i+1}))^k = 0. \quad (27)$$

So, there exists $i_0 \in \mathbb{N}_0$ such that

$$d(x_i, x_{i+1}) \leq \frac{1}{i^{1/k}}, \quad \forall i \geq i_0. \quad (28)$$

Consequently, if $m > n > n_0$, then

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{j=n}^{m} d(x_j, x_{j+1}) \\ &\leq \sum_{j=n}^{m} \frac{1}{j^{(1/k)}} \\ &\leq \sum_{j=n_0}^{\infty} \frac{1}{j^{(1/k)}}. \end{aligned} \quad (29)$$

Since $k \in (0,1)$, the series $\sum_{j=n_0}^{\infty} 1/j^{(1/k)}$ is convergent. Therefore, $\{x_i\}$ is a Cauchy sequence, and since X is complete, there exists $u \in X$ such that $x_i \longrightarrow u$ as $i \longrightarrow \infty$.As a next step, we show that u is a fixed point of $T^{n(u)}$. Indeed, due to the continuity of T, we have

$$d(Tu, u) = \lim_{i \longrightarrow \infty} d(Tx_i, x_i) = \lim_{i \longrightarrow \infty} d(x_{i+1}, x_i) = 0, \quad (30)$$

and so u is a fixed point of T. For proving the uniqueness of the fixed point, let us consider u and v be two distinct fixed points and $n = n(u)$. So, we have $d(u, v) > 0$, and hence, we get that $d(Tu, Tv) > 0$; then, by (6) and $(\zeta 2)$,

$$\begin{aligned} 0 &\leq \zeta(\tau + \alpha(u,v) F(d(Tu, Tv)), F(m(u,v) + \mathrm{LN}_1(u,v))) \\ &\leq F(m(u,v) + LN_1(u,v)) - (\tau + \alpha(u,v) F(d(Tu, Tv))). \end{aligned} \quad (31)$$

Therefore,

$$\tau + \alpha(u,v) F(d(Tu, Tv)) \leq F(m(u,v) + \mathrm{LN}_1(u,v)). \quad (32)$$

Hence, (32) implies that

$$\begin{aligned}\tau + F(d(u,v)) &= \tau + F(d(Tu,Tv))\\ &\le \tau + \alpha(u,v)F(d(Tu,Tv))\\ &\le F\left(m(u,v) + \mathrm{LN}_1(u,v)\right)\\ &\le F\left(m(u,v) + L\,d(u,Tu)\right)\\ &= F(m(u,v)+0)\\ &= F(m(u,v)),\end{aligned} \tag{33}$$

where

$$\begin{aligned}m(u,v) &= \max\left\{d(u,v), d(u,Tu), d(v,Tv), \frac{d(u,Tv)+d(v,Tu)}{2}\right\}\\ &= \max\left\{d(u,v), 0, 0, \frac{d(u,v)+d(v,u)}{2}\right\}\\ &= d(u,v).\end{aligned} \tag{34}$$

So, we have

$$\tau + F(d(u,v)) \le F(d(u,v)), \tag{35}$$

which is a contradiction, as $\tau > 0$. So, $u = v$. □

Corollary 1. *Theorem 3.3 of [7] of Theorem 4 by taking $n(x) = 1$. Because in this case,*

$$\zeta\left(\tau + \alpha(x,y)F(d(Tx,Ty)), F\left(m(x,y) + \mathrm{LN}_1(x,y)\right)\right) \ge 0. \tag{36}$$

Now, by $(\zeta 2)$, we have

$$\begin{aligned}0 &\le \zeta\left(\tau + \alpha(x,y)F(d(Tx,Ty)), F\left(m(x,y) + \mathrm{LN}_1(x,y)\right)\right)\\ &\le F\left(m(x,y) + \mathrm{LN}_1(x,y)\right) - (\tau + \alpha(x,y)F(d(Tx,Ty))).\end{aligned} \tag{37}$$

Therefore,

$$\tau + \alpha(x,y)F(d(Tx,Ty)) \le F\left(m(x,y) + \mathrm{LN}_1(x,y)\right). \tag{38}$$

Corollary 2. *Theorem 3 is contained in Theorem 4 by taking $m(x,y) = d(x,y)$, $\alpha(x,y) = 1$, and $L = 0$. Also, Theorem 4 is reduced to theorem [8] by setting $n(x) = 1$.*

The following example shows that if the mapping T satisfies the condition of Theorem 4, it cannot guarantee in general the continuity of the mapping T.

Example 1. Let $X = \mathbb{R}$ denote the real numbers with the usual metric d. Define function $T\colon X \longrightarrow X$ by

$$Tx = \begin{cases} 1, & x \in \mathbb{Q},\\ 0, & x \in \mathbb{Q}^c.\end{cases} \tag{39}$$

Then, T discontinues at each point of X, and $T^2 = 1$. If α is an arbitrary element of $[0,1)$, then

$$\forall x \in X, \exists n_x = 2; \quad \forall y \in X\colon\ d\left(T^{n_x}x, T^{n_x}y\right) = 0 \le \alpha\, d(x,y). \tag{40}$$

Now, it is obvious that the function $\zeta(t,s) = \alpha s - t$ of condition (6) of Theorem 4 on $[0,\infty)\times[0,\infty)$ is a simulation function and T satisfies following condition:

$$\zeta\left(d\left(T^n x, T^n y\right), d(x,y)\right) \ge 0, \tag{41}$$

but T discontinues at each point of X. Moreover, T satisfies all the assumptions of Theorem 4, when $L = 0$ and the unique fixed point of T is $x = 1$ and Picard's iteration of T; that is, if $y \in X$ is an arbitrary point of X, then $T^n(y)$ is convergent to the fixed point.

Theorem 5. *Let (X,d) be a complete metric space and $\alpha\colon X\times X \longrightarrow (0,+\infty)$ a symmetric function, where $\alpha(x,y) \ge 1$. Assume that $T\colon X \longrightarrow X$ is a mapping in which there exist $F \in \mathcal{G}$, $\tau > 0$, and the simulation function ζ such that for all $x, y \in X$ with $T^{n(x)}x \ne T^{n(x)}y$, where $n(x)$ is a positive integer and $1/2d(x, T^{n(x)}x) \le d(x,y)$ implies*

$$\zeta\left(\tau + \alpha(x,y)F\left(d\left(T^{n(x)}x, T^{n(x)}y\right)\right), F(m(x,y))\right) \ge 0, \tag{42}$$

where $m(x,y)$ is defined as in Theorem 4, satisfying the following conditions:

(i) T is α-admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$,

(iii) if $\{x_n\}$ is a sequence in X such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}_0$, then $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}_0$, and

(iv) T has the K-property,

then T has a fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. The recursive sequence $\{x_k\}$ is inductively constructed as follows: $n_0 = n(x_0)$, and we set $x_1 = T^{n_0}x_0$ and inductively get $x_{i+1} = T^{n_i}x_i$ with $n_i = n(x_i)$.

We assert that $x_i \ne x_{i+1}$ for all $i \in \mathbb{N}_0$. Suppose, on the contrary, that there exists $i_0 \in N_0$ such that $x_{i_0} = x_{i_0+1} = T^{n_{i_0}}x_{i_0}$. Then, x_{i_0} turns to be a fixed point of $T^{n_{i_0}}$. On the other hand,

$$Tx_{i_0} = T\left(T^{n_{i_0}}x_{i_0}\right) = T^{n_{i_0}}\left(Tx_{i_0}\right). \tag{43}$$

Thus, Tx_{i_0} form a fixed point of $T^{n_{i_0}}$. If $Tx_{i_0} = x_{i_0}$, then we conclude that T has a fixed point and that terminate the proof. Suppose, on the contrary, that $Tx_{i_0} \ne x_{i_0}$ and hence $d(T^{n_{i_0}}(Tx_{i_0}), T^{n_{i_0}}(x_{i_0})) > 0$. Then, by (42), we have

$$\begin{aligned}0 &\le \left(\tau + \alpha\left(x_{i_0}, Tx_{i_0}\right)F\left(d\left(T^{n_{i_0}}x_{i_0}, T^{n_{i_0}}Tx_{i_0}\right)\right), F\left(m\left(x_{i_0}, Tx_{i_0}\right)\right)\right),\\ &\le F\left(m\left(x_{i_0}, Tx_{i_0}\right)\right) - \left(\tau + \alpha\left(x_{i_0}, Tx_{i_0}\right)F\left(d\left(T^{n_{i_0}}x_{i_0}, T^{n_{i_0}}Tx_{i_0}\right)\right)\right).\end{aligned} \tag{44}$$

Hence,

$$\begin{aligned}\tau + F\left(d\left(x_{i_0}, Tx_{i_0}\right)\right) &= \tau + F\left(d\left(T^{n_{i_0}}x_{i_0}, T^{n_{i_0}}Tx_{i_0}\right)\right)\\ &\le \tau + \alpha\left(x_{i_0}, Tx_{i_0}\right)F\left(d\left(T^{n_{i_0}}x_{i_0}, T^{n_{i_0}}Tx_{i_0}\right)\right)\\ &\le F\left(m\left(x_{i_0}, Tx_{i_0}\right)\right).\end{aligned} \tag{45}$$

However,

$$\begin{aligned}m\left(x_{i_0}, Tx_{i_0}\right) &= \max\left\{d\left(x_{i_0}, Tx_{i_0}\right), d\left(x_{i_0}, T^{n_{i_0}}x_{i_0}\right), d\left(Tx_{i_0}, T^{n_{i_0}}Tx_{i_0}\right), \frac{d\left(x_{i_0}, T^{n_{i_0}}Tx_{i_0}\right) + d\left(Tx_{i_0}, T^{n_{i_0}}x_{i_0}\right)}{2}\right\}\\ &= \left\{d\left(x_{i_0}, Tx_{i_0}\right)\right\}.\end{aligned} \tag{46}$$

Therefore,

$$\tau + F\left(d\left(x_{i_0}, Tx_{i_0}\right)\right) \le F\left(d\left(x_{i_0}, Tx_{i_0}\right)\right). \tag{47}$$

So, $\tau \le 0$, which is a contradiction. Consequently, we deduce that, for all $i \in \mathbb{N}_0$, $x_i \ne x_{i+1}$. Then, $d(x_{i+1}, x_i) > 0$, and so

$$\frac{1}{2}d\left(x_i, T^{n_i}x_i\right) = \frac{1}{2}d\left(x_i, x_{i+1}\right) \le d\left(x_i, x_{i+1}\right). \tag{48}$$

Now, by (42),

$$\begin{aligned}\tau + F\left(d\left(x_{i+1}, x_{i+2}\right)\right) &= \tau + F\left(d\left(T^{n_i}x_i, T^{n_i}x_{i+1}\right)\right.\\ &\le \tau + \alpha\left(x_i, x_{i+1}\right)F\left(d\left(T^{n_i}x_i, T^{n_i}x_{i+1}\right)\right.\\ &\le F\left(m\left(x_i, x_{i+1}\right)\right).\end{aligned} \tag{49}$$

Hence,

$$\tau + F\left(d\left(x_{i+1}, x_{i+1}\right)\right) \le F\left(m\left(x_i, x_{i+1}\right)\right). \tag{50}$$

However,

$$\begin{aligned}m\left(x_i, x_{i+1}\right) &= \max\left\{d\left(x_i, x_{i+1}\right), d\left(x_i, T^{n_i}x_i\right), d\left(x_{i+1}, T^{n_i}x_{i+1}\right), \frac{d\left(x_i, T^{n_i}x_{i+1}\right) + d\left(x_{i+1}, T^{n_i}x_i\right)}{2}\right\}\\ &= \max\left\{d\left(x_i, x_{i+1}\right), d\left(x_{i+1}, x_{i+2}\right), \frac{d\left(x_i, x_{i+2}\right)}{2}\right\}\\ &\le \max\left\{d\left(x_i, x_{i+1}\right), d\left(x_{i+1}, x_{i+2}\right), \frac{d\left(x_i, x_{i+1}\right) + d\left(x_{i+1}, x_{i+2}\right)}{2}\right\}\\ &\le \max\left\{d\left(x_i, x_{i+1}\right), d\left(x_{i+1}, x_{i+2}\right)\right\}.\end{aligned} \tag{51}$$

If $d(x_{i_0+1}, x_{i_0+2}) \ge d(x_{i_0}, x_{i_0+1})$ for some $i_0 \in \mathbb{N}_0$, then

$$m\left(x_{i_0}, x_{i_0+1}\right) \le d\left(x_{i_0+1}, x_{i_0+2}\right), \tag{52}$$

and since F is strictly increasing,

$$F\left(m\left(x_{i_0}, x_{i_0+1}\right)\right) \le F\left(d\left(x_{i_0+1}, x_{i_0+2}\right)\right), \tag{53}$$

so, it follows from (50) that

$$\tau + F\left(d\left(x_{i_0+1}, x_{i_0+2}\right)\right) \le F\left(d\left(x_{i_0+1}, x_{i_0+2}\right)\right). \tag{54}$$

Hence, $\tau \le 0$, which is a contradiction. Therefore,

$$d\left(x_{i+1}, x_{i+2}\right) < d\left(x_i, x_{i+1}\right), \quad \forall i \in \mathbb{N}_0. \tag{55}$$

Hence, from (50) and (55), we have

$$\tau + F\left(d\left(x_{i+1}, x_{i+2}\right) \le F\left(d\left(x_i, x_{i+1}\right)\right)\right. \tag{56}$$

or

$$F(d)\left(x_{i+1}, x_{i+2}\right) \le F\left(d\left(x_i, x_{i+1}\right)\right) - \tau. \tag{57}$$

Consequently,

$$F\left(d\left(x_{i+1}, x_{i+2}\right) \le F\left(d\left(x_0, x_1\right)\right) - i\tau.\right. \tag{58}$$

Hence,

$$\lim_{i \longrightarrow \infty} F\left(d\left(x_i, x_{i+1}\right)\right) = -\infty. \tag{59}$$

So, from (G_2), we have

$$\lim_{i \longrightarrow \infty} d\left(x_i, x_{i+1}\right) = 0. \tag{60}$$

Now, we claim that $\{x_i\}$ is a Cauchy sequence. If it is not true, then by Lemma 2, there exist $\varepsilon_0 > 0$ and two sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) > \epsilon_0$, $d(x_{m_k}, x_{n_k-1}) < \epsilon_0$, and

$(L1)$ $\lim_{k\longrightarrow\infty} d(x_{n_k}, x_{m_k}) = \varepsilon_0$

$(L2)$ $\lim_{k\longrightarrow\infty} d(x_{n_k}, x_{m_k-1}) = \varepsilon_0$

$(L3)$ $\lim_{k\longrightarrow\infty} d(x_{n_k+1}, x_{m_k}) = \varepsilon_0$

$(L4)$ $\lim_{k\longrightarrow\infty} d(x_{n_k+1}, x_{m_k-1}) = \varepsilon_0$

Therefore, the definition of $m(x, y)$ implies

$$\begin{aligned}\lim_{k\longrightarrow\infty} m\left(x_{n_k}, x_{m_k-1}\right) &= \lim_{k\longrightarrow\infty} \max\left\{ d\left(x_{n_k}, x_{m_k-1}\right), d\left(x_{n_k}, T^{n(x)}x_{n_k}\right), d\left(x_{m_k-1}, T^{n(x)}x_{m_k-1}\right), \frac{d\left(x_{n_k}, T^{n(x)}x_{m_k-1}\right) + d\left(x_{m_k-1}, T^{n(x)}x_{n_k}\right)}{2} \right\} \\ &= \lim_{k\longrightarrow\infty} \max\left\{ d\left(x_{n_k}, x_{m_k-1}\right), d\left(x_{n_k}, x_{n_k+1}\right), d\left(x_{m_k-1}, x_{m_k}\right), \frac{d\left(x_{n_k}, x_{m_k}\right) + d\left(x_{m_k-1}, x_{n_k+1}\right)}{2} \right\} \\ &= \max\left\{ \varepsilon_0, 0, 0, \frac{\varepsilon_0 + \varepsilon_0}{2} \right\} = \varepsilon_0. \end{aligned} \tag{61}$$

So,

$$\lim_{k\longrightarrow\infty} m\left(x_{n_k}, x_{m_k-1}\right) = \varepsilon_0. \tag{62}$$

On the other hand, since $\lim_{k\longrightarrow\infty} d(x_{n_k}, x_{m_k-1}) = \varepsilon_0 > 0$ and $\lim_{k\longrightarrow\infty} d(x_{n_k}, x_{n_k+1}) = 0$, with considering a subsequence if it is needed, one can suppose that there exists $k_1 \in \mathbb{N}$ such that for any $k > k_1$ and $n_k > m_k > k$,

$$d\left(x_{n_k}, x_{n_k+1}\right) \le d\left(x_{n_k}, x_{m_k-1}\right). \tag{63}$$

So, it is obvious that, for all $k > k_1$ and $n_k > m_k > k$,

$$\frac{1}{2} d\left(x_{n_k}, T^{n(x)}x_{n_k}\right) = \frac{1}{2} d\left(x_{n_k}, x_{n_k+1}\right) < d\left(x_{n_k}, x_{m_k-1}\right). \tag{64}$$

Also, using the K-property, there exists $k_2 \in \mathbb{N}$ such that

$$\alpha\left(x_{n_k}, x_{m_k-1}\right) \ge 1, \quad \forall k > k_2. \tag{65}$$

If $k \ge \max\{k_1, k_2\}$, then it follows from (65) that

$$\begin{aligned}\tau + F\left(d\left(T^{n(k)}x_{n_k}, x_{m_k-1}\right)\right) &\le \tau + \alpha\left(x_{n_k}, x_{m_k-1}\right)F \\ &\quad \cdot \left(d\left(T^{n(x)}x_{n_k}, T^{n(x)}x_{m_k-1}\right)\right) \\ &\le F\left(m\left(x_{n_k}, x_{m_k-1}\right)\right). \end{aligned} \tag{66}$$

Letting $n \longrightarrow \infty$, the continuity of F through (L1) and (62) implies

$$\tau + F(\varepsilon_0) \le F(\varepsilon_0), \tag{67}$$

which is contradicted by $\tau > 0$. Consequently, $\{x_i\}$ is a Cauchy sequence in the complete metric space X. Hence, there exists $u \in X$ such that $x_i \longrightarrow u$, as $n \longrightarrow \infty$. To complete the proof, we show that u is a fixed point of T. We first claim, for all $n \ge 0$, that

$$\frac{1}{2} d(x_i, x_{i+1}) \le d(x_i, u), \text{ or } \frac{1}{2} d(x_{i+1}, x_{i+2}) \le d(x_{i+1}, u). \tag{68}$$

In fact, if we assume that, for some $i_0 \ge 0$, both of them are false, then

$$\frac{1}{2} d\left(x_{i_0}, x_{i_0+1}\right) > d\left(x_{i_0}, u\right), \text{ and } \frac{1}{2} d\left(x_{i_0+1}, x_{i_0+2}\right) > d\left(x_{i_0+1}, u\right). \tag{69}$$

Hence, (55) implies

$$\begin{aligned} d\left(x_{i_0}, x_{i_0+1}\right) &\le d\left(x_{i_0}, u\right) + d\left(u, x_{i_0+1}\right) \\ &< \frac{1}{2} d\left(x_{i_0}, x_{i_0+1}\right) + \frac{1}{2} d\left(x_{i_0+1}, x_{i_0+2}\right) \\ &\le \frac{1}{2} d\left(x_{i_0}, x_{i_0+1}\right) + \frac{1}{2} d\left(x_{i_0}, x_{i_0+1}\right) \\ &= d\left(x_{i_0}, x_{i_0+1}\right), \end{aligned} \tag{70}$$

which is a contradiction and the claim is proved.

Now, let us begin with the first part of (68); that is, suppose that

$$\frac{1}{2} d(x_i, x_{i+1}) \le d(x_i, u), \tag{71}$$

and on the contrary, assume that $Tu \ne u$. Without loss of generality, one can imagine that $Tx_i \ne Tu$, for all $i \in \mathbb{N}_0$ (because if $x_{i+1} = Tx_i = Tu$ for infinite values of i, then uniqueness of the limit concludes that $Tu = u$). Then, from (45) and (iii), we get

$$\begin{aligned} \tau + F(d(x_{i+1}, Tu)) &= \tau + F(d(Tx_i, Tu)) \\ &\le \tau + \alpha(x_i, u)F(d(Tx_i, Tu)) \\ &\le F(m(x_i, u)). \end{aligned} \tag{72}$$

And since F is continuous on $(0, +\infty)$, and $d(u, Tu) > 0$, as $i \longrightarrow \infty$, we get

$$\tau + F(d(u, Tu)) \le F\Big(\lim_{i\longrightarrow\infty} (m(x_i, u)). \tag{73}$$

However,

$$m(x_i,u)=\max\left\{d(x_i,u),d(x_i,x_{i+1}),d(u,Tu),\frac{d(x_i,Tu)+d(u,x_{i+1})}{2}\right\}. \tag{74}$$

So, we have

$$\begin{aligned}\lim_{i\longrightarrow\infty}m(x_i,u)&=\max\left\{0,0,d(u,Tu),\frac{d(u,Tu)+0}{2}\right\}\\&=d(u,Tu).\end{aligned} \tag{75}$$

Therefore, if $d(u,Tu)\neq 0$, then from (73), we have

$$\tau+F(d(u,Tu))\leq F(d(u,Tu)), \tag{76}$$

which is contradicted, as $\tau>0$. So, $d(u,Tu)=0$, i.e., $Tu=u$. Finally, if we assume that the second part of (68) is true, i.e.,

$$\frac{1}{2}d(x_{i+1},x_{i+2})\leq d(x_{i+1},u), \tag{77}$$

then by using the same manner, we can prove that $d(u,Tu)=0$, i.e., $Tu=u$.

Suppose that u and v are two fixed points of T. If $u\neq v$, then $d(Tu,Tv)>0$. Furthermore, $\alpha(u,v)\geq 1$, because $u,v\in \mathrm{Fix}(T)$. It is also clear that $1/2d(u,Tu)=0<d(u,v)$. Hence, (45) implies

$$\begin{aligned}\tau+F(d(u,v))&=\tau+F(d(Tu,Tv))\\&\leq\tau+\alpha(u,v)F(d(Tu,Tv))\\&\leq F(m(u,v)),\end{aligned} \tag{78}$$

where

$$\begin{aligned}m(u,v)&=\max\left\{d(u,v),d(u,Tu),d(v,Tv),\frac{d(u,Tv)+d(v,Tu)}{2}\right\}\\&=\max\left\{d(u,v),0,0,\frac{d(u,v)+d(v,u)}{2}\right\}\\&=d(u,v).\end{aligned} \tag{79}$$

So, we get

$$\tau+F(d(u,v))\leq F(d(u,v)), \tag{80}$$

which is a contradicted by $\tau>0$ and so $u=v$. This completes the proof. □

Corollary 3. *If in Theorem 5, we put $n(x)=1$, then*

$$\zeta(\tau+\alpha(x,y)F(d(Tx,Ty)),F(m(x,y)))\geq 0. \tag{81}$$

Now, by $(\zeta 2)$, we have

$$\begin{aligned}0&\leq\zeta(\tau+\alpha(x,y)F(d(Tx,Ty)),F(m(x,y)))\\&\leq F(m(x,y))-(\tau+\alpha(x,y)F(d(Tx,Ty))).\end{aligned} \tag{82}$$

Therefore,

$$\tau+\alpha(x,y)F(d(Tx,Ty))\leq F(m(x,y)). \tag{83}$$

Hence, we get Theorem 3.3 of [7].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The paper was supported by the National Natural Science Foundation of China (no. 11671365).

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] V. W. Bryant, "A remark on a fixed-point theorem for iterated mappings," *The American Mathematical Monthly*, vol. 75, no. 4, pp. 399-400, 1968.

[3] F. Khojasteh, S. Shukla, and S. Radenovic, "A new approach to the study of fixed point theory for simulation functions," *Filomat*, vol. 29, no. 6, pp. 1189–1194, 2015.

[4] A. Öztürk, "A fixed point theorem for mappings with a F-contractive iterate," *Advances in the Theory of Nonlinear Analysis and its Application*, vol. 3, no. 3, pp. 231–235, 2019.

[5] H. Piri and P. Kumam, "Some fixed point theorems concerning F-contraction in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2014, p. 210, 2014.

[6] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for -contractive type mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[7] A. Taheri and A. P. Farajzadeh, "A new generalization of \$\alpha\$-type almost-\$F\$-contractions and \$\alpha\$-type \$F\$-Suzuki contractions in metric spaces and their fixed point theorems," *Carpathian Mathematical Publications*, vol. 11, no. 2, pp. 475–492, 2019.

[8] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2012, p. 94, 2012.

[9] L.-C. Ceng, A. Petrusel, A. Petruşel, J.-C. Yao, and Y. Yao, "Hybrid viscosity extragradient method for systems of variational inequalities, fixed Points of nonexpansive mappings, zero points of accretive operators in Banach spaces," *Fixed Point Theory*, vol. 19, no. 2, pp. 487–502, 2018.

[10] Y. Yao, L. Leng, M. Postolache, and X. Zheng, "Mann-type iteration method for solving the split common fixed point problem," *Journal of Nonlinear and Convex Analysis*, vol. 18, pp. 875–882, 2017.

[11] Y. Yao, M. Postolache, and J. C. Yao, "An iterative algorithm for solving the generalized variational inequalities and fixed points problems," *Mathematics*, vol. 7, p. 61, 2019.

[12] H. Zegeye, N. Shahzad, and Y. Yao, "Minimum-norm solution of variational inequality and fixed point problem in Banach spaces," *Optimization*, vol. 64, no. 2, pp. 453–471, 2015.

[13] N. Shahzad and H. Zegeye, "Convergence theorems of common solutions for fixed point, variational inequality and equilibrium problems," *Journal of Nonlinear and Variational Analysis*, vol. 3, pp. 189–203, 2019.

[14] B. Tan, S. Li, and S. Y. Cho, "Strong convergence of inertial Mann algorithms for solving hierarchical fixed point problems," *Journal of Nonlinear and Variational Analysis*, vol. 4, pp. 337–355, 2020.

[15] L.-C. Ceng, A. Petrusel, A. Petruşel, J.-C. Yao, and Y. Yao, "Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions," *Fixed Point Theory*, vol. 20, no. 1, pp. 113–134, 2019.

[16] Y. Yao, Y. C. Liou, and J. C. Yao, "Split common fixed point problem for two quasi-pseudocontractive operators and its algorithm construction," *Fixed Point Theory and Applications*, vol. 2015, p. 127, 2015.

[17] Y. Yao, Y.-C. Liou, and J.-C. Yao, "Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 2, pp. 843–854, 2017.

[18] Y. Yao, X. Qin, and J. C. Yao, "Projection methods for firmly type nonexpansive operators," *Journal of Nonlinear and Convex Analysis*, vol. 19, pp. 407–415, 2018.

[19] V. M. Sehgal, "A fixed point theorem for mappings with a contractive iterate," *Proceedings of the American Mathematical Society*, vol. 23, no. 3, p. 631, 1969.

[20] NA. Secelean, "Iterated function systems consisting of F-contractions," *Fixed Point Theory and Applications*, vol. 2013, p. 277, 2013.

[21] G. Heidary Joonaghany, A. Farajzadeh, M. Azhini, and F. Khojasteh, "A new common fixed point theorem for Suzuki type contractions via generalized Ψ-simulation functions," *Sahand Communications in Mathematical Analysis*, vol. 16, no. 1, pp. 129–148, 2019.

[22] A. Farajzadeh, C. Noytaptim, and A. Kaewcharoen, "Some fixed point theorems for generalized—geraghty contractive type mappings in partial-metric spaces," *Journal of Informatics and Mathematical Sciences*, vol. 10, no. 3, pp. 455–578, 2018.

[23] H. Alikhani, D. Gopal, M. A. Miandaragh, S. Rezapour, and N. Shahzad, "Some endpoint results for β-generalized weak contractive multifunctions," *Scientific World Journal*, vol. 2013, Article ID 948472, 2013.

[24] B. S. Choudhury, P. Konar, B. E. Rhoades, and N. Metiya, "Fixed point theorems for generalized weakly contractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 6, pp. 2116–2126, 2011.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6674427, 17 pages
https://doi.org/10.1155/2021/6674427

Research Article

A New Study on Halpern and Nonconvex Combination Algorithm for Nonlinear Mappings in Banach Spaces with Applications

Gaobo Li

School of Education, Shandong Women's University, Jinan 250300, China

Correspondence should be addressed to Gaobo Li; gaobolisdwu@163.com

Received 29 November 2020; Accepted 30 December 2020; Published 29 January 2021

Academic Editor: Sun Young Cho

In this paper, we introduce a Halpern algorithm and a nonconvex combination algorithm to approximate a solution of the split common fixed problem of quasi-ϕ-nonexpansive mappings in Banach space. In our algorithms, the norm of linear bounded operator does not need to be known in advance. As the application, we solve a split equilibrium problem in Banach space. Finally, some numerical examples are given to illustrate the main results in this paper and compare the computed results with other ones in the literature. Our results extend and improve some recent ones in the literature.

1. Introduction

Let H_1 be a Hilbert space, and let C be the nonempty closed convex subset of H_1. Let H_2 be a real Hilbert space, and let Q be the nonempty closed convex subset of H_2. Let $A: H_1 \longrightarrow H_2$ be a linear bounded operator. In 1994, Censor and Elfving [1] introduced the split feasibility problem (SFP) as a generalization of convex feasibility problem as follows:

$$\text{find a point } x^* \in C \text{ such that } Ax^* \in Q. \tag{1}$$

Recntly, the SFP and its variants have been investigated by many authors due to its real applications such as medical imaging, radiation therapy, and treatment planning; see, e.g., [2–5]. For solving SFP (1), it needs to get the inverse A^{-1} (assuming the existence of A^{-1}) in algorithm of Censor and Elfving [1]. However, few authors continue to study the algorithm of Censor and Elfving since the difficulty of computing A^{-1}, even if it exists. In fact, another algorithm solving SFP (1) is more popular which is called CQ algorithm given by Byrne [6, 7]. The CQ algorithm of Byrne is a gradient projection method in convex minimization. Since the CQ algorithm does need to compute A^{-1} and only involves the projections P_C and P_Q, it is easy to implement when P_C and P_Q have the closed-form expressions. However, the computations of P_C and P_Q are also difficult if these projections did not have the closed-form expressions which is such that the CQ algorithm of Byrne [6, 7] is not easy to implement in this case. In 2010, Xu [8] investigated the CQ algorithm from the ways of optimization and fixed point, proposed Mann's algorithm, and relaxed CQ algorithm to solve SFP (1). In the relaxed CQ algorithm, the sets C and Q are level sets of convex functions so that the projections involved in the CQ algorithm are onto half-spaces, which makes the algorithm implementable. Also, in 2010, Moudafi [9] proposed an iterative method to solve a split common fixed point problem for quasi-nonexpansive mappings in which the projection is not involved which is such that the algorithm is easy to implement. In 2014, Kraikaew and Saejung [10] combined the Moudafi method and the Halpern algorithm to propose a new iteration in which the projection is not involved for solving the SFP. In the recent years, many algorithms have been given to solve the SFP in Hilbert spaces; see, for instance, [11–15] and the references therein.

However, because of the complexity of properties in Banach space, it is very difficulty to solve SFP (and fixed point problem) in Banach spaces. Until now, only limited works on SFP (and fixed point problem) in Banach spaces have been reported in the literature. For instance, the authors in [16] gave an algorithm to solve SFP in Banach space. In [17], Tang et al. introduced some iterative algorithms to solve a split common fixed point problem for a quasi-strict pseudocontractive mapping and an asymptotically nonexpansive mapping in two Banach spaces and obtained the weak and strong convergence for the proposed algorithms. In [18], Chen et al. proposed a new hybrid projection method for solving split feasibility and fixed point problems involved in Bregman quasi-strictly pseudocontractive mapping in p-uniformly convex and uniformly smooth real Banach spaces. They proved the strong convergence for the proposed algorithm using the Bregman projection method. On the feasible and common fixed point problem, the authors also refer to [19–21].

Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$ and E_2 be a uniformly smooth, strictly convex, and reflexive Banach space. Let S: $E_1 \longrightarrow E_1$ be a closed quasi-ϕ-nonexpansive mapping and A: $E_1 \longrightarrow E_2$ be a linear bounded operator. Very recently, Ma et al. [22] proposed a hybrid projection algorithm to solve the following split feasibility problem and fixed point problem:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q, \tag{2}$$

where $C = \{x \in E_1 : x = Sx\}$ and $Q \subset E_2$ is a nonempty closed convex subset. Precisely, their algorithm to solve (2) is as follows:

$$\begin{cases} x_1 \in E_1, C_1 = E_1, \\ z_n = J^{-1}\left(J_1 x_n + \gamma A^* J_2\left(P_Q - I\right) A x_n\right), \\ y_n = J^{-1}\left[\alpha_n J_1 z_n + (1-\alpha_n) J_1 S z_n\right], \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n), \phi(v, z_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \ge 1, \end{cases} \tag{3}$$

where $\{\alpha\} \subset [\delta, 1)$ with $\delta > 0$, $\gamma \in (0, (1/\|A\|^2 k^2))$, P_Q is the metric projection of E_2 onto Q, and $\Pi_{C_{n+1}}$ is the generalized projection of E_1 in C_{n+1}. The authors proved that the sequence generated by (3) strongly converges to a point which solves (2).

On the contrary, the most algorithms of approximating the fixed points of quasi-ϕ-nonexpansive mappings in Banach spaces are constructed by the hybrid or shrinking projection methods, see [23–25]. However, in 2018, Hieu and Strodiot [26] introduced a new iterative algorithm for solving pseudomonotone equilibrium problem involving the fixed point problem for quasi-ϕ-nonexpansive mapping in Banach space without using the hybrid or shrinking projection methods. More precisely, their algorithm is

$$\begin{cases} y_n = \operatorname{argmin}\left\{\lambda_n f(x_n, y) + \dfrac{1}{2}\phi(y, x_n) : y \in C\right\}, \\ z_n = \operatorname{argmin}\left\{\lambda_n f(y_n, y) + \dfrac{1}{2}\phi(y, x_n) : y \in C\right\}, \\ x_{n+1} = \Pi_C\left(J^{-1}\left(\alpha_n J u + (1-\alpha_n)\left(\beta_n J z_n + (1-\beta_n) J S z_n\right)\right)\right), \end{cases} \tag{4}$$

where $f : C \times C \longrightarrow \mathbb{R}$ is a pseudomonotone bifunction and $S : C \longrightarrow C$ is a quasi-ϕ-nonexpansive mapping. The authors proved that the sequence generated by (4) strongly converges to a common point that solves the pseudomonotone equilibrium problem on f and is a fixed point of S.

In general, there are three kinds of iterations of strong convergence that are used to approximate the fixed point of the nonlinear operator. The iterations are the Halpern iteration, the viscosity iteration, and the hybrid projection iteration. Recently, Hussain et al. [27] proposed a new surprising iteration that strongly converges to a fixed point of a nonexpansive mapping in Hilbert space. More precisely, the iteration is

$$x_1 \in H, \quad x_{n+1} = \alpha_n (1-\mu_n) x_n + (1-\alpha_n) T x_n, \ n \ge 1, \tag{5}$$

where H is a Hilbert space, $T : H \longrightarrow H$ is a nonexpansive mapping, and $\{\alpha_n\}, \{\mu_n\} \subset (0, 1]$ are the control sequences. The authors proved that $\{x_n\}$ generated by (5) strongly converges to a fixed point of T under some certain conditions on $\{\alpha_n\}$ and $\{\mu_n\}$. Later on, Marino et al. [28] extended (5) to strict pseudocontraction.

In this paper, motivated by the work of [22, 26, 27], we introduce some algorithms to solve a split common fixed point problem for two families of quasi-ϕ-nonexpansive mappings in Banach spaces and prove the strong convergence for the proposed algorithms. As the application, we solve a split equilibrium problem in Banach space. Finally, we give a numerical example in infinite dimension Banach space to illustrate the main result of this paper. Our results extend the one of Ma et al. [22] from one quasi-nonexpansive mapping to two quasi-nonexpansive mappings and [27] from Hilbert space to Banach space.

2. Preliminaries

Let E be a Banach space, and let E^* be the dual space of E. For all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^* \rangle$. The duality mapping J on E is defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \left\|x^*\right\|^2\right\}, \quad \forall x \in E. \tag{6}$$

It is known that $J(x)$ is nonempty for all $x \in E$. A Banach space E is said to be smooth if the limit

$$\lim_{n\longrightarrow\infty} \frac{\|x+ty\|-\|x\|}{t} \tag{7}$$

exists for all $x, y \in S(E) = \{z \in E : \|z\| = 1\}$. The space E is smooth if and only if the duality mapping J is single-valued.

A Banach space E is said to be strictly convex if $(\|x+y\|/2) < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$ and uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $(\|x+y\|/2) \le 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \ge \epsilon$. It is known that if E is smooth, strictly convex, and reflexive, then the duality mapping J is single-valued, one-to-one, and onto. Let E be a smooth Banach space. The function ϕ: $E \times E \longrightarrow \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \tag{8}$$

for all $x, y \in E$. From the definition of ϕ, it is easy to see that, for all $x, y, z \in E$, the following hold:

$$\begin{gathered}(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2,\\ \phi\left(x, J^{-1}(\lambda Jy) + (1-\lambda)Jz\right) \le \lambda\phi(x, y) + (1-\lambda)\phi(x, z), \quad \lambda \in (0, 1).\end{gathered} \tag{9}$$

The following is an important property for the function ϕ:

$$\phi(x, y) = \phi(z, y) + \phi(x, z) + 2\langle z - x, Jy - Jz\rangle, \tag{10}$$

for all $x, y, z \in E$.

Lemma 1 (see [29]). *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \longrightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \longrightarrow 0$.*

For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space, the following hold:

$$\phi(x_n, y_n) \longrightarrow 0 \Leftrightarrow \left\|x_n - y_n\right\| \longrightarrow 0 \Leftrightarrow \left\|Jx_n - Jy_n\right\| \longrightarrow 0. \tag{11}$$

Let $\Pi_C : E \longrightarrow C$ be mapping called the generalized projection [30] that assigns to an arbitrary element $x \in E$ the minimum point of the functional $\phi(x, y)$; that is, $\Pi_C x = \operatorname{argmin}_{y\in C}\phi(y, x)$.

Lemma 2 (see [30]). *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E. Then, the following conclusions hold:*

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y)$, $\forall x \in C$, $\forall y \in E$

(b) For $x \in E$, $z = \Pi_C x$ if and only if $\langle z - y, Jx - Jz\rangle \ge 0$, $\forall y \in C$

(c) For $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$

Let E be a strictly convex and reflexive Banach space and C be a nonempty closed and convex subset. The metric projection

$$P_C x = \operatorname{argmin}_{y\in C}\|y - x\|, \quad \forall x \in E. \tag{12}$$

Lemma 3 (see [31]). *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty, closed convex subset of E. Let $x \in E$. Then,*

$$z = P_C x \text{ if and only if } \langle z - y, J(x - z)\rangle \ge 0, \quad \forall y \in C. \tag{13}$$

Let E be a strictly convex, smooth, and reflexive Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*, that is, $J^* = J^{-1}$. Define a mapping V: $E \times E^* \longrightarrow \mathbb{R}$ [32] by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \left\|x^*\right\|^2, \quad \forall (x, x^*) \in E \times E^*. \tag{14}$$

Lemma 4 (see [32]). *Let E be a reflexive, smooth, and strictly convex Banach space. Then,*

$$V(x, x^*) \le V(x, x^* + y^*) - 2\langle J^{-1}x^* - x, y^*\rangle, \tag{15}$$

for all $x \in E$ and $x^*, y^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E$.

Let E be a smooth Banach space. A mapping T: $E \longrightarrow E$ is said to be closed if for any sequence $\{x_n\} \subset E$ with $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$, then $Tx_n = y$. T is said to be quasi-ϕ-nonexpansive mapping if $\mathrm{Fix}(T) \neq \varnothing$ and

$$\phi(p, Tx) \le \phi(p, x), \tag{16}$$

for all $p \in \mathrm{Fix}(T)$ and $x \in E$. For a quasi-ϕ-nonexpansive mapping T, $\mathrm{Fix}(T)$ is convex. If T is closed, then $\mathrm{Fix}(T)$ is closed, see [24].

Lemma 5 (see [33]). *Let $r > 0$. A real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function g: $[0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|), \tag{17}$$

for all $t \in [0, 1]$ and $x, y \in B_r$, where $B_r = \{x \in E : \|x\| \le r\}$.

Lemma 6 (see [33]). *Let $r > 0$. Let E be a 2-uniformly smooth Banach space with the best smoothness constants $k > 0$. Then,*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, Jx\rangle + 2k^2\|y\|^2, \tag{18}$$

for all $x, y \in E$.

Lemma 7 (see [34]). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \quad \forall n \in \mathbb{N}, \tag{19}$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\} \subset \mathbb{R}$ *satisfy the conditions:*

$$\begin{aligned} &\lim_{n\longrightarrow\infty} \gamma_n = 0, \\ &\sum_{n=1}^{\infty} \gamma_n = \infty, \text{ and } \limsup_{n\longrightarrow\infty} \delta_n \le 0. \end{aligned} \tag{20}$$

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 8 (see [35]). *Let* $\{a_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then, there exists a nondecreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \longrightarrow \infty$ *as* $k \longrightarrow \infty$, *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \le a_{m_k+1} \text{ and } a_k \le a_{m_k+1}. \tag{21}$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 9 (see [36]). *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are sequences of nonnegative real numbers such that*

$$a_{n+1} \le a_n + b_n, \quad n \ge 1. \tag{22}$$

If $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\longrightarrow\infty} a_n$ *exists.*

3. Main Results

In this section, let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$ and E_2 be a uniformly smooth, strictly convex, and reflexive Banach space. Define the functions ϕ_1 and ϕ_2 by

$$\begin{aligned} &\phi_1(x,y) = \|x\|_1^2 - 2\langle x, J_1 y\rangle_1 + \|y\|_1^2, \quad \forall x, y \in E_1, \\ &\phi_2(u,v) = \|u\|_2^2 - 2\langle u, J_2 v\rangle_2 + \|v\|_2^2, \quad \forall u, v \in E_2, \end{aligned} \tag{23}$$

where $\langle x, J_1 y\rangle_1$ (resp., $\langle u, J_1 v\rangle_2$) and $\|x\|_1$ (resp., $\|u\|_2$) denote the value of $J_1 y$ at x and norm of x (resp., the value of $J_1 v$ at u and norm of u) in E_1 (resp. E_2), respectively. However, for convenience, we use the same symbols $\langle \cdot,\cdot\rangle$, $\|\cdot\|$, and ϕ in E_1 and E_2 without the confusion.

Let $A: E_1 \longrightarrow E_2$ be a linear bounded operator with adjoint A^*. Let $S: E_1 \longrightarrow E_1$ and $T: E_2 \longrightarrow E_2$ be the quasi-ϕ-nonexpansive mappings. Consider the following split common fixed point problem:

$$\text{find } x \in \text{Fix}(S) \text{ such that } Ax \in \text{Fix}(T). \tag{24}$$

Denote the set of solutions of the above split common fixed point problem by Ω. In this section, assume that S and T are closed and $I - S$ and $I - T$ are demiclosed at zeros in E_1 and E_2. Note that, from the closedness of S and T, it follows that $\text{Fix}(S)$ and $\text{Fix}(T)$ are closed [24], which implies that Ω is closed. The convexity of Ω is from the convexity of $\text{Fix}(S)$. Assume that Ω is nonempty.

Let $x^* = \Pi_\Omega \theta$, where θ is the zero element in E_1. We will prove that sequence $\{x_n\}$ generated by the following algorithm converges strongly to x^*.

Algorithm 1. Take $x_1 \in E_1$, and define a sequence $\{x_n\}$ by

$$\begin{cases} w_n = TAx_n, \\ Q_n = \{w \in E_2 : \phi(w, w_n) \le \phi(w, Ax_n)\}, \\ z_n = J_1^{-1}\left(J_1 x_n - \gamma_n A^* J_2\left(I - P_{Q_n}\right)Ax_n\right), \\ y_n = J_1^{-1}\left(\beta_n J_1 z_n + (1-\beta_n) J_1 S z_n\right), \\ x_{n+1} = J_1^{-1}\left(\alpha_n (1-\tau_n) J_1 x_n + (1-\alpha_n) J_1 y_n\right), \quad n \ge 1, \end{cases} \tag{25}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{\tau_n\} \subset (\tau, 1)$ with $\tau \in (0,1)$ and

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2}{2k^2\left\|A^* J_2\left(I - P_{Q_n}\right)Ax_n\right\|^2}, & \text{if } \left\|\left(P_{Q_n} - I\right)Ax_n\right\| \ne 0, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Lemma 10. *The sequence* $\{x_n\}$ *is well-defined and bounded.*

Proof. Since $\phi(w, w_n) \le \phi(w, Ax_n)$ is equivalent to $2\langle w, J_2 Ax_n - J_2 w_n\rangle \le \|Ax_n\|^2 - \|w_n\|^2$, it follows that Q_n is closed and convex for each $n \ge 1$. For any $p \in \Omega$, it follows that $Ap \in Q_n$ for all $n \ge 1$. Hence, each Q_n is nonempty closed convex, which implies that $\{P_{Q_n} Ax_n\}$ is well-defined. Now, we show that $\|(P_{Q_n} - I)Ax_n\| \ne 0$ implies that $\|A^* J_2 (P_{Q_n} - I)Ax_n\| \ne 0$. Assume that $\|A^* J_2 (P_{Q_n} - I)Ax_n\| = 0$. We have $\langle Ap - P_{Q_n} Ax_n, J_2 (P_{Q_n} - I)Ax_n\rangle \ge 0$ by Lemma 3 and hence

$$\begin{aligned} 0 &= \langle p - x_n, A^* J_2\left(P_{Q_n} - I\right)Ax_n\rangle = \langle Ap - Ax_n, J_2\left(P_{Q_n} - I\right)Ax_n\rangle \\ &= \langle Ap - P_{Q_n} Ax_n, J_2\left(P_{Q_n} - I\right)Ax_n\rangle + \langle P_{Q_n} Ax_n - Ax_n, J_2\left(P_{Q_n} - I\right)Ax_n\rangle \\ &= \langle Ap - P_{Q_n} Ax_n, J_2\left(P_{Q_n} - I\right)Ax_n\rangle + \left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2 \ge \left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2. \end{aligned} \tag{27}$$

It is a contradiction. It follows that $\|(P_{Q_n} - I)Ax_n\| \neq 0$ implies that $\|A^* J_2 (P_{Q_n} - I)Ax_n\| \neq 0$. Hence, $\{z_n\}$ is well-defined. Furthermore, $\{x_n\}$ is well-defined.

Since E_1 is a 2-uniformly convex and 2-uniformly smooth real Banach space, E_1^* is 2-uniformly smooth real Banach space, and $J_1 = (J_1^*)^{-1}$. From (25) and Lemma 6, we have

$$\begin{aligned}\phi(x^*, z_n) &= \|x^*\|^2 - 2\langle x^*, J_1 x_n + \gamma_n A^* J_2 (P_{Q_n} - I)Ax_n\rangle + \left\|J_1 x_n + \gamma_n A^* J_2 (P_{Q_n} - I)Ax_n\right\|^2 \\ &\leq \|x^*\|^2 - 2\langle x^*, J_1 x_n\rangle - 2\gamma_n \langle x^*, A^* J_2 (P_{Q_n} - I)Ax_n\rangle + \|x_n\|^2 \\ &\quad + 2\gamma_n \langle x_n, A^* J_2 (P_{Q_n} - I)Ax_n\rangle + 2\gamma_n^2 k^2 \left\|A^* J_2 (P_{Q_n} - I)Ax_n\right\|^2 \\ &= \phi(x^*, x_n) - 2\gamma_n \langle x^* - x_n, A^* J_2 (P_{Q_n} - I)Ax_n\rangle + 2\gamma_n^2 k^2 \left\|A^* J_2 (P_{Q_n} - I)Ax_n\right\|^2.\end{aligned} \tag{28}$$

Since $Ax^* \in Q_n$, $\langle Ax^* - P_{Q_n} Ax_n, J_2 (P_{Q_n} - I)Ax_n\rangle \geq 0$. Hence, we have

$$\begin{aligned}2\langle x^* - x_n, A^* J_2 (P_{Q_n} - I)Ax_n\rangle &= 2\langle Ax^* - Ax_n, J_2 (P_{Q_n} - I)Ax_n\rangle \\ &= 2\langle Ax^* - P_{Q_n} Ax_n, J_2 (P_{Q_n} - I)Ax_n\rangle + 2\left\|(P_{Q_n} - I)Ax_n\right\|^2 \geq 2\left\|(P_{Q_n} - I)Ax_n\right\|^2.\end{aligned} \tag{29}$$

Combining (28) with (29), we obtain

$$\begin{aligned}\phi(x^*, z_n) &\leq \phi(x^*, x_n) - 2\gamma_n \left\|(P_{Q_n} - I)Ax_n\right\|^2 + 2\kappa^2 \gamma_n^2 \left\|A^* J_2 (P_{Q_n} - I)Ax_n\right\|^2 \\ &= \phi(x^*, x_n) - \frac{\left\|(P_{Q_n} - I)Ax_n\right\|^4}{2k^2 \left\|A^* J_2 (P_{Q_n} - I)Ax_n\right\|^2} \leq \phi(x^*, x_n).\end{aligned} \tag{30}$$

Furthermore, by Lemma 5, (25), and (30) we obtain

$$\begin{aligned}\phi(x^*, y_n) &= \|x^*\|^2 - 2\langle x^*, \beta_n J_1 z_n + (1 - \beta_n) J_1 S z_n\rangle + \|\beta_n J_1 z_n + (1 - \beta_n) J_1 S z_n\|^2 \\ &\leq \|x^*\|^2 - 2\langle x^*, \beta_n J_1 z_n + (1 - \beta_n) J_1 S z_n\rangle + \beta_n \|z_n\|^2 + (1 - \beta_n)\|S z_n\|^2 - \beta_n (1 - \beta_n) g(\|J_1 z_n - J_1 S z_n\|) \\ &= \beta_n \phi(x^*, z_n) + (1 - \beta_n)\phi(x^*, S z_n) - \beta_n (1 - \beta_n) g(\|J_1 z_n - J_1 S z_n\|) \\ &\leq \phi(x^*, x_n) - \frac{\left\|(P_{Q_n} - I)Ax_n\right\|^4}{2k^2 \left\|A^* J_2 (P_{Q_n} - I)Ax_n\right\|^2} - \beta_n (1 - \beta_n) g(\|J_1 z_n - J_1 S z_n\|).\end{aligned} \tag{31}$$

It follows from (25), (31), and Lemma 5 that

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi\left(x^*, J_1^{-1}(\alpha_n(1-\tau_n)J_1x_n + (1-\alpha_n)J_1y_n)\right) = \|x^*\|^2 - 2\alpha_n(1-\tau_n)\langle x^*, J_1x_n\rangle - 2(1-\alpha_n)\langle x^*, J_1y_n\rangle \\
&\quad + \|\alpha_n(1-\tau_n)J_1x_n + (1-\alpha_n)J_1y_n\|^2 \le \|x^*\|^2 - 2\alpha_n(1-\tau_n)\langle x^*, J_1x_n\rangle - 2(1-\alpha_n)\langle x^*, J_1y_n\rangle \\
&\quad + \alpha_n\|(1-\tau_n)J_1x_n\|^2 + (1-\alpha_n)\|J_1y_n\|^2 \le \|x^*\|^2 - 2\alpha_n(1-\tau_n)\langle x^*, J_1x_n\rangle - 2(1-\alpha_n)\langle x^*, J_1y_n\rangle \\
&\quad + \alpha_n(1-\tau_n)\|x_n\|^2 + (1-\alpha_n)\|y_n\|^2 = \alpha_n(1-\tau_n)\phi(x^*, x_n) + (1-\alpha_n)\phi(x^*, y_n) + \alpha_n\tau_n\|x^*\|^2 \\
&\le \alpha_n(1-\tau_n)\phi(x^*, x_n) + (1-\alpha_n)\left(\phi(x^*, x_n) - \frac{\|(P_{Q_n} - I)Ax_n\|^4}{2k^2\|A^*J_2(P_{Q_n} - I)Ax_n\|^2} - \beta_n(1-\beta_n)g(\|J_1z_n - J_1Sz_n\|)\right) \\
&\quad + \alpha_n\tau_n\|x^*\|^2 \\
&= (1-\alpha_n\tau_n)\phi(x^*, x_n) + \alpha_n\tau_n\|x^*\|^2 - (1-\alpha_n)\left(\frac{\|(P_{Q_n} - I)Ax_n\|^4}{2k^2\|A^*J_2(P_{Q_n} - I)Ax_n\|^2} + \beta_n(1-\beta_n)g(\|J_1z_n - J_1Sz_n\|)\right) \\
&\le \max\left\{\phi(x^*, x_n), \|x^*\|^2\right\} \le \cdots \le \max\left\{\phi(x^*, x_1), \|x^*\|^2\right\}, \quad n \ge 1.
\end{aligned} \tag{32}$$

So, $\{\phi(x^*, x_n)\}$ is bounded. □

Lemma 11. *Let $\{x_n\}$ be the sequence generated by Algorithm 1. Then,*

$$\phi(x^*, x_{n+1}) \le (1-\alpha_n\tau_n)\phi(x^*, x_n) + 2\alpha_n\tau_n\langle x^* - x_{n+1}, J_1x^* + (1-\alpha_n)(J_1x_n - J_1y_n)\rangle. \tag{33}$$

Proof. Let $h_n = \alpha_nJ_1x_n + (1-\alpha_n)J_1y_n$. Then, by (31), we have

$$\begin{aligned}
\phi(x^*, J_1^{-1}h_n) &\le \alpha_n\phi(x^*, x_n) + (1-\alpha_n)\phi(x^*, y_n) \\
&\le \alpha_n\phi(x^*, x_n) + (1-\alpha_n)\phi(x^*, x_n) \\
&= \phi(x^*, x_n).
\end{aligned} \tag{34}$$

Note that

$$x_{n+1} = J_1^{-1}((1-\alpha_n\tau_n)h_n + \alpha_n\tau_n(1-\alpha_n)(J_1y_n - J_1x_n)). \tag{35}$$

By (34) and (35) and Lemma 4, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi\left(x^*, J_1^{-1}((1-\alpha_n\tau_n)h_n + \alpha_n\tau_n(1-\alpha_n)(J_1y_n - J_1x_n))\right) \\
&= V(x^*, (1-\alpha_n\tau_n)h_n + \alpha_n\tau_n(1-\alpha_n)(J_1y_n - J_1x_n)) \\
&\le V(x^*, (1-\alpha_n\tau_n)h_n + \alpha_n\tau_n(1-\alpha_n)(J_1y_n - J_1x_n) + \alpha_n\tau_n(J_1x^* - (1-\alpha_n)(J_1y_n - J_1x_n))) \\
&\quad - 2\langle x_{n+1} - x^*, \alpha_n\tau_n(J_1x^* - (1-\alpha_n)(J_1y_n - J_1x_n))\rangle = V(x^*, (1-\alpha_n\tau_n)h_n + \alpha_n\tau_nJ_1x^*) \\
&\quad - 2\langle x_{n+1} - x^*, \alpha_n\tau_n(J_1x^* - (1-\alpha_n)(J_1y_n - J_1x_n))\rangle \le (1-\alpha_n\tau_n)\phi\left(x^*, J_1^{-1}h_n\right) + \alpha_n\tau_n\phi(x^*, x^*) \\
&\quad - 2\langle x_{n+1} - x^*, \alpha_n\tau_n(J_1x^* - (1-\alpha_n)(J_1y_n - J_1x_n))\rangle \\
&\le (1-\alpha_n\tau_n)\phi(x^*, x_n) + 2\alpha_n\tau_n\langle x^* - x_{n+1}, J_1x^* - (1-\alpha_n)(J_1y_n - J_1x_n)\rangle.
\end{aligned} \tag{36}$$

□

Theorem 1 *If the following conditions hold:*

$$\lim_{n\longrightarrow\infty}\alpha_n = 0, \quad \sum_{n=1}^{\infty}\alpha_n = \infty \text{ and } \liminf_{n\longrightarrow\infty}\beta_n(1-\beta_n) > 0, \tag{37}$$

then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the element x^*.

Proof. By (32), we have

$$(1-\alpha_n)\left(\frac{\left\|\left(P_{Q_n}-I\right)Ax_n\right\|^4}{2k^2\left\|A^*J_2\left(P_{Q_n}-I\right)Ax_n\right\|^2}+\beta_n(1-\beta_n)g\left(\left\|J_1z_n-J_1Sz_n\right\|\right)\right)$$
$$\leq(1-\alpha_n\tau_n)\phi(x^*,x_n)-\phi(x^*,x_{n+1})+\alpha_n\tau_n\|x^*\|^2\leq\phi(x^*,x_n)-\phi(x^*,x_{n+1})+\alpha_n\tau_n\|x^*\|^2. \tag{38}$$

Now, we show that $\|x_n-x^*\|\longrightarrow 0$ by the following two cases:

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\phi(x^*,x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\phi(x^*,x_n)\}$ is convergent. By (37) and (38), we have

$$\lim_{n\longrightarrow\infty}\frac{\left\|\left(P_{Q_n}-I\right)Ax_n\right\|^4}{\left\|A^*J_2\left(P_{Q_n}-I\right)Ax_n\right\|^2}=\lim_{n\longrightarrow\infty}g\left(\left\|J_1z_n-J_1Sz_n\right\|\right)=0, \tag{39}$$

which implies that

$$\lim_{n\longrightarrow\infty}\left\|J_1z_n-J_1Sz_n\right\|=0. \tag{40}$$

Since $\left\{\|A^*J_2(P_{Q_n}-I)Ax_n\|\right\}$ is bounded, we have

$$\lim_{n\longrightarrow\infty}\left\|\left(P_{Q_n}-I\right)Ax_n\right\|=0. \tag{41}$$

By (40), we have

$$\left\|J_1y_n-J_1z_n\right\|=(1-\beta_n)\left\|J_1z_n-J_1Sz_n\right\|\longrightarrow 0. \tag{42}$$

Combining (39) with (42), we obtain

$$\left\|J_1y_n-J_1x_n\right\|\leq\left\|J_1y_n-J_1z_n\right\|+\left\|J_1z_n-J_1x_n\right\|=\left\|J_1y_n-J_1z_n\right\|+\gamma_n\left\|A^*J_2\left(P_{Q_n}-I\right)Ax_n\right\|$$
$$=\left\|J_1y_n-J_1z_n\right\|+\frac{\left\|\left(P_{Q_n}-I\right)Ax_n\right\|^2}{2\kappa^2\left\|A^*J_2\left(P_{Q_n}-I\right)Ax_n\right\|}\longrightarrow 0. \tag{43}$$

On the contrary, from (25) and (43), it follows that

$$\left\|J_1z_n-J_1x_{n+1}\right\|\leq\left\|J_1z_n-J_1y_n\right\|+\left\|J_1y_n-J_1x_{n+1}\right\|=\left\|J_1z_n-J_1y_n\right\|+\alpha_n\left\|(1-\tau_n)J_1x_n-J_1y_n\right\|\longrightarrow 0. \tag{44}$$

Since E_1 is a 2-uniformly convex and 2-uniformly smooth real Banach space, J_1 is uniformly norm-to-norm continuous. From (40), (42), and (44), it follows that

$$\lim_{n\longrightarrow\infty}\left\|z_n-Sz_n\right\|=\lim_{n\longrightarrow\infty}\left\|y_n-z_n\right\|=\lim_{n\longrightarrow\infty}\left\|z_n-x_{n+1}\right\|=0. \tag{45}$$

Since $\{z_n\}$ is bounded, there exist a subsequence $\left\{z_{n_k}\right\}$ of $\{z_n\}$ converging weakly to $p\in E_1$ such that

$$\limsup_{n\longrightarrow\infty}\langle x^*-z_n,J_1x^*\rangle=\lim_{k\longrightarrow\infty}\langle x^*-z_{n_k},J_1x^*\rangle$$
$$=\langle x^*-p,J_1x^*\rangle. \tag{46}$$

Now, we show that $p\in\Omega$. First, by (45) and demi-closeness principle at zero of S, we have $p\in\mathrm{Fix}(S)$. On the contrary, since $P_{Q_n}Ax_n\in Q_n$ and $\|P_{Q_n}Ax_n-Ax_n\|\longrightarrow 0$, we have

$$\phi\left(P_{Q_n}Ax_n,w_n\right)\leq\phi\left(P_{Q_n}Ax_n,Ax_n\right)\longrightarrow 0. \tag{47}$$

By Lemma 1, it follows that

$$\left\|P_{Q_n}Ax_n-w_n\right\|=\left\|P_{Q_n}Ax_n-TAx_n\right\|\longrightarrow 0. \tag{48}$$

Hence,

$$\left\|Ax_n-TAx_n\right\|\leq\left\|Ax_n-P_{Q_n}Ax_n\right\|+\left\|P_{Q_n}Ax_n-TAx_n\right\|\longrightarrow 0. \tag{49}$$

Since A is bounded and linear, by (45), we can conclude that $\left\{Ax_{n_k+1}\right\}$ converges weakly to $Ap\in E_2$. By (49) and demi-closedness principle of T, we obtain that $Ap\in\mathrm{Fix}(T)$. Hence, $p\in\Omega$. Therefore, by (45) and Lemma 3,

$$\limsup_{n\longrightarrow\infty}\langle x^*-x_{n+1},J_1x^*\rangle=\limsup_{n\longrightarrow\infty}\langle x^*-z_n,J_1x^*\rangle$$
$$=\langle x^*-p,J_1x^*\rangle\leq 0. \tag{50}$$

Finally, the conclusion $\|x_n-x^*\|\longrightarrow 0$ follows from the hypothesis on $\{\alpha_n\}$, (33), (43), (50), and Lemma 4.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi\left(x^*, x_{n_i}\right) < \phi\left(x^*, x_{n_i+1}\right), \tag{51}$$

for all $i \in \mathbb{N}$.

Then, by Lemma 5, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \longrightarrow \infty$:

$$\begin{aligned}\phi\left(x^*, x_{m_k}\right) &\leq \phi\left(x^*, x_{m_k+1}\right) \text{ and } \phi\left(x^*, x_k\right) \\ &\leq \phi\left(x^*, x_{m_k+1}\right), \quad \forall k \geq 1.\end{aligned} \tag{52}$$

Replacing n with m_k in (38), by (52), we have

$$\begin{aligned}&\left(1-\alpha_{m_k}\right)\left(\frac{\left\|\left(P_{Q_{m_k+1}}-I\right)Ax_{m_k}\right\|^4}{2k^2\left\|A^*J_2\left(P_{Q_{m_k+1}}-I\right)Ax_{m_k}\right\|^2}+\beta_{m_k}\left(1-\beta_{m_k}\right)g\left(\left\|J_1z_{m_k}-J_1Sz_{m_k}\right\|\right)\right) \\ &\quad\leq \phi\left(x^*, x_{m_k}\right)-\phi\left(x^*, x_{m_k+1}\right)+\alpha_{m_k}\tau_{m_k}\left\|x^*\right\|^2 \leq \alpha_{m_k}\tau_{m_k}\left\|x^*\right\|^2.\end{aligned} \tag{53}$$

Then, by a similar process with proving (43)–(50), we can obtain that

$$\lim_{k\longrightarrow\infty}\left\|J_1x_{m_k}-J_1y_{m_k}\right\| = 0 \text{ and } \limsup_{n\longrightarrow\infty}\langle x^*-x_{m_k+1}, J_1x^*\rangle \leq 0. \tag{54}$$

Replacing n with m_k in (33), we have

$$\begin{aligned}\phi\left(x^*, x_{m_k+1}\right) \leq{}& \left(1-\alpha_{m_k}\tau_{m_k}\right)\phi\left(x^*, x_{m_k}\right) \\ &+ 2\alpha_{m_k}\tau_{m_k}\langle x^*-x_{m_k+1}, J_1x^* \\ &+\left(1-\alpha_{m_k}\right)\left(J_1y_{m_k}-J_1x_{m_k}\right)\rangle,\end{aligned} \tag{55}$$

from which we obtain

$$\begin{aligned}\alpha_{m_k}\tau_{m_k}\phi\left(x^*, x_{m_k}\right) &\leq \phi\left(x^*, x_{m_k}\right)-\phi\left(x^*, x_{m_k+1}\right)+2\alpha_{m_k}\tau_{m_k}\langle x^*-x_{m_k+1}, J_1x^*+\left(1-\alpha_{m_k}\right)\left(J_1y_{m_k}-J_1x_{m_k}\right)\rangle \\ &\leq 2\alpha_{m_k}\tau_{m_k}\langle x^*-x_{m_k+1}, J_1x^*+\left(1-\alpha_{m_k}\right)\left(J_1y_{m_k}-J_1x_{m_k}\right)\rangle.\end{aligned} \tag{56}$$

Since $\alpha_{m_k}\tau_{m_k} > 0$, by (54) and (56), we have

$$\begin{aligned}\phi\left(x^*, x_{m_k}\right) \leq{}& 2\langle x^*-x_{m_k+1}, J_1x^* \\ &+\left(1-\alpha_{m_k}\right)\left(J_1y_{m_k}-J_1x_{m_k}\right)\rangle \longrightarrow 0.\end{aligned} \tag{57}$$

Furthermore, by (54), (55), and (57), it follows that

$$\lim_{k\longrightarrow\infty}\phi\left(x^*, x_{m_k+1}\right) = 0. \tag{58}$$

However, $\phi(x^*, x_k) \leq \|x_{m_k+1}-x^*\|$ for all $k \geq 1$. So, we conclude that $\phi(x^*, x_k) \longrightarrow 0$ as $k \longrightarrow \infty$ and hence $\|x_k - x^*\| \longrightarrow 0$ as $k \longrightarrow \infty$ by Lemma 1. The proof is complete. □

Remark 1. If $\|(P_{Q_n}-I)Ax_n\| = 0$ for all $n \geq 1$, then $\gamma_n = 0$ and $z_n = x_n$ for all $n \geq 1$. In this case, $Ax_n = P_{Q_n}Ax_n$ and $\phi(Ax_n, w_n) = \phi(Ax_n, TAx_n) \leq \phi(Ax_n, Ax_n) = 0$, which implies that $Ax_n = TAx_n$ for all $n \geq 1$. The iterative scheme (25) becomes

$$\begin{cases} y_n = J_1^{-1}\left(\beta_nJ_1x_n+(1-\beta_n)J_1Sx_n\right), \\ x_{n+1} = J_1^{-1}\left(\alpha_n(1-\tau_n)J_1x_n+(1-\alpha_n)J_1y_n\right), \quad n \geq 1. \end{cases} \tag{59}$$

By the proof process above, we still can see that $\{x_n\}$ converges strongly to $x^* = P_{\mathrm{Fix}(S)}\theta$. Since A is linear and bounded, $Ax_n \longrightarrow Ax^*$, which implies that $Ax_n \rightharpoonup x^*$. Note that $Ax_n = TAx_n$, for all $n \geq 1$, and $Ax_n - TAx_n \longrightarrow 0$ as $n \longrightarrow \infty$. By the hypothesis that $I - T$ is demi-closedness at zero, we get $Ax^* = TAx^*$. Hence, $x^* \in \Omega$. Hence, without loss generality, we assume that $\gamma_n \neq 0$ for all $n \geq 1$ in the proof process.

Algorithm 2. Take $u = x_1 \in E_1$, and define a sequence $\{x_n\}$ by

$$\begin{cases} w_n = TAx_n, \\ Q_n = \{w \in E_2\colon \phi(w, w_n) \leq \phi(w, Ax_n)\}, \\ z_n = J_1^{-1}\left(J_1x_n-\gamma_nA^*J_2\left(I-P_{Q_n}\right)Ax_n\right), \\ y_n = J_1^{-1}\left(\beta_nJ_1z_n+(1-\beta_n)J_1Sz_n\right), \\ x_{n+1} = J_1^{-1}\left(\alpha_nJ_1u+(1-\alpha_n)J_1y_n\right), \quad n \geq 1, \end{cases} \tag{60}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n}-I\right)Ax_n\right\|^2}{2k^2\left\|A^*J_2\left(I-P_{Q_n}\right)Ax_n\right\|^2}, & \left\|\left(P_{Q_n}-I\right)Ax_n\right\| \neq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{61}$$

Lemma 12. *$\{x_n\}$ is well-defined and bounded.*

Proof. By a similar proof lines of Lemma 10, we can show that $\{x_n\}$ is well-defined. Now, we prove that $\{x_n\}$ is bounded. By (29)–(31), (60), and Lemma 5, for any $\hat{x} \in \Omega$, we have

$$\begin{aligned}\phi(\hat{x}, x_{n+1}) &= \phi\left(\hat{x}, J_1^{-1}(\alpha_n J_1 u + (1-\alpha_n) J_1 y_n)\right) = \|\hat{x}\|^2 - 2\alpha_n \langle \hat{x}, J_1 u\rangle - 2(1-\alpha_n)\langle \hat{x}, J_1 y_n\rangle \\ &\quad + \left\|\alpha_n J_1 u + (1-\alpha_n) J_1 y_n\right\|^2 \leq \|\hat{x}\|^2 - 2\alpha_n \langle \hat{x}, J_1 u\rangle - 2(1-\alpha_n)\langle \hat{x}, J_1 y_n\rangle \\ &\quad + \alpha_n \|u\|^2 + (1-\alpha_n)\left\|y_n\right\|^2 = \alpha_n \phi(\hat{x}, u) + (1-\alpha_n)\phi(\hat{x}, y_n) \leq \alpha_n \phi(\hat{x}, u) + (1-\alpha_n) \\ &\quad \cdot \left(\phi(\hat{x}, x_n) - \frac{\left\|\left(P_{Q_n} - I\right) A x_n\right\|^4}{2k^2 \left\|A^* J_2 \left(P_{Q_n} - I\right) A x_n\right\|^2} - \beta_n (1-\beta_n) g\left(\left\|J_1 z_n - J_1 S z_n\right\|\right) \right) \\ &\leq \alpha_n \phi(\hat{x}, u) + (1-\alpha_n)\phi(\hat{x}, x_n) \leq \alpha_n \phi(\hat{x}, u) + \phi(\hat{x}, x_n), \quad n \geq 1.\end{aligned} \tag{62}$$

By the hypothesis on $\{\alpha_n\}$ and Lemma 9, it follows that the limit of $\{\phi(\hat{x}, x_n)\}$ exists. Hence, $\{x_n\}$ is bounded. □

Theorem 2. *Assume that S and T are closed. If the interior of Ω is nonempty and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions*

$$\sum_{n=1}^{\infty} \alpha_n < \infty \text{ and } \liminf_{n \longrightarrow \infty} \beta_n (1-\beta_n) > 0, \tag{63}$$

then $\{x_n\}$ generated by Algorithm 2 converges strongly to the element $x^* = \lim_{n \longrightarrow \infty} \Pi_\Omega x_n$.

Proof. We first show that $\{x_n\}$ is a Cauchy sequence and hence converges strongly to some point $x^* \in E_1$. Since the interior of Ω is nonempty, there exist $p \in \Omega$ and $r > 0$ such that

$$p + rh \in \Omega, \tag{64}$$

whenever $\|h\| \leq 1$. By (10), we have

$$\begin{aligned}\phi(p, x_n) &= \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - p, J_1 x_n - J_1 x_{n+1}\rangle \\ &= \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - (p+rh), J_1 x_n - J_1 x_{n+1}\rangle + 2r\langle h, J_1 x_n - J_1 x_{n+1}\rangle.\end{aligned} \tag{65}$$

On the contrary, by (10), again we have

$$\begin{aligned}\phi(p+rh, x_n) &= \phi(x_{n+1}, x_n) + \phi(p+rh, x_{n+1}) \\ &\quad + 2\langle x_{n+1} - (p+rh), J_1 x_n - J_1 x_{n+1}\rangle.\end{aligned} \tag{66}$$

Combining (65) with (66), we obtain

$$\begin{aligned}2r\langle h, J_1 x_n - J_1 x_{n+1}\rangle &= \phi(p, x_n) - \left(\phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - (p+rh), J_1 x_n - J_1 x_{n+1}\rangle\right) \\ &= \phi(p, x_n) - \phi(x_{n+1}, x_n) - \phi(p, x_{n+1}) - \phi(p+rh, x_n) + \phi(x_{n+1}, x_n) + \phi(p+rh, x_{n+1}) \\ &= \frac{1}{2r}\left(\phi(p, x_n) - \phi(p, x_{n+1})\right) + \phi(p+rh, x_{n+1}) - \phi(p+rh, x_n).\end{aligned} \tag{67}$$

Since $p + rh \in \Omega$, from (62) and (67), it follows that

$$2r\langle h, J_1x_n - J_1x_{n+1}\rangle \le \phi(p,x_n) - \phi(p,x_{n+1}) + \alpha_n(\phi(p+rh,u) - \phi(p+rh,x_n)) \le \phi(p,x_n) - \phi(p,x_{n+1}) + \alpha_n\phi(p+rh,u). \tag{68}$$

Since h with $\|h\| \le 1$ is arbitrary, we have

$$\|J_1x_n - J_1x_{n+1}\| \le \frac{1}{2r}(\phi(p,x_n) - \phi(p,x_{n+1}) + \alpha_n\phi(p+rh,u)). \tag{69}$$

So, for all $m > n$,

$$\begin{aligned}\|J_1x_n - J_1x_m\| &= \|J_1x_n - J_1x_{n+1} + J_1x_{n+1} - \cdots - J_1x_{m-1} + J_1x_{m-1} + J_1x_m\| \\ &\le \sum_{i=n}^{m-1}\|J_1x_i - J_1x_{i+1}\| \le \frac{1}{2r}\sum_{i=n}^{m-1}(\phi(p,x_i) - \phi(p,x_{i+1}) + \alpha_i\phi(p+rh,u)) \\ &= \frac{1}{2r}\sum_{i=n}^{m-1}(\phi(p,x_i) - \phi(p,x_{i+1})) + \frac{\phi(p+rh,u)}{2r}\sum_{i=n}^{m-1}\alpha_i = \phi(p,x_n) - \phi(p,x_m) + \frac{\phi(p+rh,u)}{2r}\sum_{i=n}^{m-1}\alpha_i.\end{aligned} \tag{70}$$

Since the limit of $\{\phi(p,x_n)\}$ exists and $\sum_{n=1}^{\infty}\alpha_n < \infty$, from (70), we see

$$\lim_{m,n\longrightarrow\infty}\|J_1x_n - J_1x_m\| = 0, \tag{71}$$

which implies that $\{J_1x_n\}$ is a Cauchy sequence in E_1^*. Hence, $\{J_1x_n\}$ converges strongly to some point in E_1^*. Since E_1^* has a Fréchet differentiable norm, then J_1^{-1} is continuous on E_1^*. Hence, x_n converges strongly to some point x^* in E_1.

For any $\hat{x} \in \Omega$, by (62), we have

$$\begin{aligned}&(1-\alpha_n)\left(\frac{\|(P_{Q_n} - I)Ax_n\|^4}{2k^2\|A^*J_2(P_{Q_n} - I)Ax_n\|^2} + \beta_n(1-\beta_n)g(\|J_1z_n - J_1Sz_n\|)\right) \\ &\le \alpha_n\phi(\hat{x},u) + (1-\alpha_n)\phi(\hat{x},x_n) - \phi(\hat{x},x_{n+1}) \le \alpha_n\phi(\hat{x},u) + \phi(\hat{x},x_n) - \phi(\hat{x},x_{n+1}).\end{aligned} \tag{72}$$

Since the limit of $\{\phi(\hat{x},x_n)\}$ exists, by the hypothesis on $\{\alpha_n\}$ and $\{\beta_n\}$, it follows that

$$\lim_{n\longrightarrow\infty}\frac{\|(P_{Q_n} - I)Ax_n\|^4}{\|A^*J_2(P_{Q_n} - I)Ax_n\|^2} = \lim_{n\longrightarrow\infty}g(\|J_1z_n - J_1Sz_n\|) = 0, \tag{73}$$

which implies that

$$\lim_{n\longrightarrow\infty}\|(P_{Q_n} - I)Ax_n\| = 0 \text{ and } \lim_{n\longrightarrow\infty}\|J_1z_n - J_1Sz_n\| = 0, \tag{74}$$

and hence

$$\|z_n - Sz_n\| \longrightarrow 0. \tag{75}$$

On the contrary, by (60) and (73), we have

$$\|J_1z_n - J_1x_n\| = \gamma_n\|A^*J_2(I - P_{Q_n})Ax_n\| = \frac{\|(I - P_{Q_n})Ax_n\|^2}{\|A^*J_2(I - P_{Q_n})Ax_n\|} \longrightarrow 0. \tag{76}$$

It follows that

$$\|z_n - x_n\| \longrightarrow 0. \tag{77}$$

Hence, $\{z_n\}$ converges strongly to $x^* \in E_1$. Since S is closed, by (75), we get $x^* = Sx^*$.

Now, we show that $Ax^* = TAx^*$. From (49), it follows that $\|Ax_n - TAx_n\| \longrightarrow 0$. Since A is linear bounded, $Ax_n \longrightarrow Ax^*$. From the closedness of T, we get $Ax^* = TAx^*$. Therefore, $x^* \in \Omega$. Finally, we show that $x^* = \lim_{n\longrightarrow\infty}\Pi_\Omega x_n$. In fact, since $x^* \in \Omega$, by Lemma 2, we have

$$\phi(x^*, \Pi_\Omega x_n) \le \phi(x^*, x_n) \longrightarrow 0. \tag{78}$$

It follows that $x^* = \lim_{n\longrightarrow\infty}\Pi_\Omega x_n$. The proof is complete.

Let Q be a nonempty closed convex subset of E_2. In Algorithms 1 and 2, if putting $T = I$ and $Q_1 = Q$, we have $w_n = Ax_n$ and $Q_n = Q$ for all $n \geq 1$. Then, we have the following results. □

Corollary 1. *Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$ and E_2 be a uniformly smooth, strictly convex, and reflexive Banach space with a nonempty closed convex subset $Q \subset E_2$. Let A: $E_1 \longrightarrow E_2$ be a linear bounded operator with adjoint A^*. Let S: $E_1 \longrightarrow E_1$ and $Q \subset E_2$ be a nonempty subset. Assume that $I - S$ is demi-closedness at zero and $\Gamma \neq \varnothing$, where $\Gamma = \{x \in E_1: x \in Fix(S),\ Ax \in Q\}$. Let $x_1 \in E_1$ and define a sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J_1^{-1}\left(J_1 x_n - \gamma_n A^* J_2\left(I - P_Q\right)Ax_n\right), \\ y_n = J_1^{-1}\left(\beta_n J_1 z_n + (1-\beta_n)J_1 S z_n\right), \\ x_{n+1} = J_1^{-1}\left(\alpha_n (1-\tau_n) J_1 x_n + (1-\alpha_n) J_1 y_n\right), \quad n \geq 1, \end{cases} \tag{79}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{\tau_n\} \subset (\tau, 1)$ with $\tau \in (0,1)$ and

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2}{2k^2\left\|A^* J_2\left(I - P_{Q_n}\right)Ax_n\right\|^2}, & \left\|\left(P_{Q_n} - I\right)Ax_n\right\| \neq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{80}$$

If the following conditions hold,

$$\lim_{n\longrightarrow\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \liminf_{n\longrightarrow\infty} \beta_n \left(1 - \beta_n\right) > 0, \tag{81}$$

then the sequence $\{x_n\}$ generated by (60) converges strongly to the element $x^* = \Pi_\Gamma \theta$, where θ is the zero element in E_1.

Corollary 2. *Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$ and E_2 be a uniformly smooth, strictly convex, and reflexive Banach space with a nonempty closed convex subset $Q \subset E_2$. Let A: $E_1 \longrightarrow E_2$ be a linear bounded operator with adjoint A^*. Let S: $E_1 \longrightarrow E_1$ and $Q \subset E_2$ be a nonempty subset. Assume that S is closed and the interior of Γ is nonempty, where $\Gamma = \{x \in E_1: x \in Fix(S),\ Ax \in Q\}$. Let $u = x_1 \in E_1$ and define a sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J_1^{-1}\left(J_1 x_n - \gamma_n A^* J_2\left(I - P_Q\right)Ax_n\right), \\ y_n = J_1^{-1}\left(\beta_n J_1 z_n + (1-\beta_n)J_1 S z_n\right), \\ x_{n+1} = J_1^{-1}\left(\alpha_n J_1 u + (1-\alpha_n) J_1 y_n\right),\ n \geq 1, \end{cases} \tag{82}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\gamma_n = \{ \|(P_{Q_n} - I)Ax_n\|^2 / \ 2k^2\|A^* J_2 (I - \ P_{Q_n})Ax_n\|^2, \|(P_{Q_n} - I)Ax_n\| \neq 0, 0, otherwise.$

If the following conditions hold

$$\lim_{n\longrightarrow\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n < \infty \text{ and } \liminf_{n\longrightarrow\infty} \beta_n \left(1 - \beta_n\right) > 0, \tag{83}$$

then the sequence $\{x_n\}$ generated by (82) converges strongly to some element $x^* = lim_{n\longrightarrow\infty}\Pi_\Gamma x_n$.

4. Application

Let E_1 and E_2 be two Banach spaces and f_1: $E_1 \times E_1 \longrightarrow \mathbb{R}$ and f_2: $E_2 \times E_2 \longrightarrow \mathbb{R}$ be the bifunctions. Let A: $E_1 \longrightarrow E_2$ be a linear bounded operator. In this section, we consider a split equilibrium problem: find a point $x^* \in E_1$ such that

$$x^* \in \mathrm{EP}\left(f_1\right) \text{ and } Ax^* \in \mathrm{EP}\left(f_2\right), \tag{84}$$

where $\mathrm{EP}(f_1) = \{x \in E_1: f_1(x,y) \geq 0,\ \forall y \in E_1\}$ and $\mathrm{EP}(f_2) = \{u \in E_2: f_2(u,v) \geq 0,\ \forall v \in E_2\}$. We denote the set of solution of problem (84) by Λ. That is, $\Lambda = \{x \in \mathrm{EP}(f_1): Ax \in \mathrm{EP}(f_2)\}$.

The split equilibrium problem has been studied by many authors in Hilbert space, see [37–41]. However, few results on the split equilibrium problem in Banach space is reported by far.

Lemma 13 (see [24]). *Let E be a strictly convex, reflexive, and uniform smooth Banach space and f: $E \times E \longrightarrow \mathbb{R}$ be a bifunction satisfying the following conditions:*

(A1) $f(x,x) = 0$ for all $x \in E$.

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in E$.

(A3) For all $x, y, z \in E$,

$$\limsup_{t\longrightarrow 0^+} f\left(tz + (1-t)x, y\right) \leq f(x,y). \tag{85}$$

(A4) For all $x \in E$, $f(x,\cdot)$ is convex and lower semicontinuous.

For $r > 0$ and $x \in E$, define a mapping T_r: $E \longrightarrow E$ as follows:

$$\mathrm{T}_r^f x = \left\{ z \in E: f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \text{ for all } y \in E \right\}, \tag{86}$$

for all $x \in E$. Then, the following hold:

(1) T_r^f is single-valued

(2) $Fix(\mathrm{T}_r^f) = EP(f)$

(3) $EP(f)$ is closed and convex

(4) $\phi(q, T_r^f x) + \phi(T_r^f x, x) \leq \phi(q, x)$ *for all* $x \in E$ *and* $q \in EP(f)$, *which shows that* T_r^f *is a quasi-*ϕ*-nonexpansive mapping*

Now, we show that the mapping $I - T_r^f$ *is demi-closedness at zero on a bounded subset of E.*

Lemma 14. *Let E be a strictly convex, reflexive, and uniform smooth Banach space and* $f\colon E \times E \longrightarrow \mathbb{R}$ *be a bifunction satisfying conditions (A1)–(A4). Let* $r > 0$ *and define the mapping* T_r^f *as (86). Assume that* $EP(f) \neq \emptyset$. *Then,* $I - T_r^f$ *is demi-closedness at zero on a bounded set. That is, if* $\{x_n\} \subset E$ *is bounded and weakly converges to* $x \in E$ *and* $\|x_n - T_r^f x_n\| \longrightarrow 0$ *as* $n \longrightarrow \infty$, *then* $x = T_r^f x$.

Proof. et $\{x_n\} \subset E$ be bounded and converges weakly to $x \in E$ and $\|x_n - T_r^f x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. For each $x^* \in \mathrm{EP}(f) = \mathrm{Fix}(T_r^f)$, since T_r^f is quasi-ϕ-nonexpansive, we have

$$\phi\left(x^*, T_r^f x_n\right) \leq \phi(x^*, x_n), \quad n \geq 1, \tag{87}$$

which implies that $\left\{T_r^f x_n\right\}$ is bounded. On the contrary, since J is uniformly norm-to-norm continuous on bounded sets, it follows that

$$\lim_{n\longrightarrow\infty} \left\|JT_r^f x_n - Jx_n\right\| = 0. \tag{88}$$

By (A2), we have

$$\begin{aligned}\frac{1}{r}\langle y - T_r^f x_n, JT_r^f x_n - Jx_n\rangle &\geq -f\left(T_r^f x_n, y\right)\\ &\geq f\left(y, T_r^f x_n\right), \quad \forall y \in E.\end{aligned} \tag{89}$$

Letting $n \longrightarrow 0$ in (89), by (A4) and (88), we obtain

$$f(y, x) \leq 0, \quad \forall y \in E. \tag{90}$$

For $0 < t \leq 1$ and $y \in E$, let $y_t = ty + (1-t)x$. Note that (90) implies that $f(y_t, x) \leq 0$. By (A1), we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, x) \leq tf(y_t, y). \tag{91}$$

Dividing by t, we obtain

$$f(y_t, y) \geq 0, \quad \forall y \in E. \tag{92}$$

Let $t \longrightarrow 0^+$, by (A3), we have

$$f(x, y) \geq 0, \quad \forall y \in E. \tag{93}$$

It follows that $x \in \mathrm{EP}(f)$. That is, $x = T_r^f x$ by Lemma 13. This completes the proof.

Based on the results in Section 3, we give the following conclusion directly. □

Theorem 3. *Let* E_1 *be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant* $k > 0$ *and* E_2 *be a uniformly smooth, strictly convex, and reflexive Banach space. Let* $A\colon E_1 \longrightarrow E_2$ *be a linear bounded operator with adjoint* A^*. *Let* $f_1\colon E_1 \times E_1 \longrightarrow \mathbb{R}$ *and* $f_2\colon E_2 \times E_2 \longrightarrow \mathbb{R}$ *be the bifunctions satisfying conditions (A1)–(A4). Assume that* $\Lambda \neq \emptyset$, *where* $\Lambda = \{x \in E_1\colon x \in EP(f_1), Ax \in EP(f_2)\}$. *Let* $r > 0$. *Take* $x_1 \in E_1$ *and put* $Q_1 = E_2$. *Define a sequence* $\{x_n\}$ *by*

$$\begin{cases} w_n = T_r^{f_2} Ax_n,\\ Q_n = \{w \in Q_n\colon \phi(w, w_n) \leq \phi(w, Ax_n)\},\\ z_n = J^{-1}\left(J_1 x_n + \gamma_n A^* J_2\left(P_{Q_n} - I\right)Ax_n\right),\\ y_n = J^{-1}\left[(1-\beta_n)J_1 z_n + (1-\beta_n)J_1 T_r^{f_1} z_n\right],\\ x_{n+1} = J_1^{-1}\left(\alpha_n(1-\tau_n)J_1 x_n + (1-\alpha_n)J_1 y_n\right), \quad n \geq 1, \end{cases} \tag{94}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{\tau_n\} \subset (\tau, 1)$ *with* $\tau \in (0,1)$ *and*

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2}{2k^2\left\|A^* J_2\left(I - P_{Q_n}\right)Ax_n\right\|^2}, & \left\|\left(P_{Q_n} - I\right)Ax_n\right\| \neq 0,\\ 0, & \text{else}. \end{cases} \tag{95}$$

If the following conditions hold

$$\begin{aligned} &\lim_{n\longrightarrow\infty} \alpha_n = 0,\\ &\sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \liminf_{n\longrightarrow\infty} \beta_n(1-\beta_n) > 0, \end{aligned} \tag{96}$$

then the sequence $\{x_n\}$ *generated by (94) converges strongly to the element* $x^* = \Pi_\Lambda \theta$, *where* θ *is the zero element in* E_1.

Theorem 4. *Let* E_1 *be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant* $k > 0$ *and* E_2 *be a uniformly smooth, strictly convex, and reflexive Banach space. Let* $A\colon E_1 \longrightarrow E_2$ *be a linear bounded operator with adjoint* A^*. *Let* $f_1\colon E_1 \times E_1 \longrightarrow \mathbb{R}$ *and* $f_2\colon E_2 \times E_2 \longrightarrow \mathbb{R}$ *be the bifunctions satisfying conditions (A1)–(A4). Assume that the interior of* Λ *is nonempty, where* $\Lambda = \{x \in E_1\colon x \in EP(f_1), Ax \in EP(f_2)\}$. *Let* $r > 0$. *Take* $u, x_1 \in E_1$ *and put* $Q_1 = E_2$. *Define a sequence* $\{x_n\}$ *by*

$$\begin{cases} w_n = T_r^{f_2} Ax_n,\\ Q_n = \{w \in Q_n\colon \phi(w, w_n) \leq \phi(w, Ax_n)\},\\ z_n = J^{-1}\left(J_1 x_n + \gamma_n A^* J_2\left(P_{Q_n} - I\right)Ax_n\right),\\ y_n = J^{-1}\left[(1-\beta_n)J_1 z_n + (1-\beta_n)J_1 T_r^{f_1} z_n\right],\\ x_{n+1} = J_1^{-1}\left(\alpha_n J_1 u + (1-\alpha_n)J_1 y_n\right), \quad n \geq 1, \end{cases} \tag{97}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *and*

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2}{2k^2\left\|A^* J_2\left(I - P_{Q_n}\right)Ax_n\right\|^2}, & \left\|\left(P_{Q_n} - I\right)Ax_n\right\| \neq 0, \\ 0, & \text{else}. \end{cases} \quad (98)$$

If the following conditions hold

$$\lim_{n\longrightarrow\infty} \alpha_n = 0, \\ \sum_{n=1}^{\infty} \alpha_n < \infty \text{ and } \liminf_{n\longrightarrow\infty} \beta_n \left(1 - \beta_n\right) > 0, \quad (99)$$

then the sequence $\{x_n\}$ *generated by (97) converges strongly to the element* $x^* = lim_{n\longrightarrow\infty}\Pi_\Lambda x_n$.

5. Numerical Examples

In this section, we give the following examples to illustrate the effectiveness of Algorithms 1 and 2. The program is performed by Matlab R2016b running on a PC Desktop with Core(TM) i5CPU M550 3.20 GHz with 4 GB Ram.

We first show the convergence of Algorithm 1 by the following example which has been used by Ma et al. [22]. In [22], the authors compare the computed results using their algorithm (25) with algorithm (100) in Kraikaew and Saejung [10] by the example. Here, we also compare the convergence of our Algorithm 1 with algorithm (25) in [22] and algorithm (100) in [10].

Example 1. Let $E_1 = \mathbb{R}$, $E_2 = \mathbb{R}^2$, $Q = [0, \infty] \times (-\infty, 0)$, $Sx = (x/4)$, for all $x \in E_1$, $Tx = P_Q x$ for all $x \in E_2$, where P_Q is the metric projection from E_2 onto Q, and $A\colon E_1 \longrightarrow E_2$ be a mapping defined by $Ax = (x/2, x/3)$ for all $x \in E_1$. Then, $A^*(u, v) = (u/2) + (v/3)$, for all $(u, v) \in E_2$. It is easy to see that $\Omega = \{x \in E_1\colon x \in \mathrm{Fix}(S), Ax \in \mathrm{Fix}(T)\} = \{0\}$.

Algorithm 3. Let $\{x_n\}$ be the sequence generated by (25) in this paper with $\alpha_n = 1/2n$ and $\beta_n = \tau_n = 6/7$. Then, scheme (25) can be simplified as

$$\begin{cases} x_1 \in E_1, \\ w_n = P_Q\left(\dfrac{x_n}{2}, \dfrac{x_n}{3}\right), \\ Q_n = \left\{w \in E_2\colon \left\|w_n - w\right\| \leq \left\|\left(\dfrac{x_n}{2}, \dfrac{x_n}{3}\right) - w\right\|\right\}, \\ Ax_n = \left(\dfrac{x_n}{2}, \dfrac{x_n}{3}\right), z_n = x_n + \gamma_n A^*\left(P_{Q_n} - I\right)Ax_n, \\ y_n = \dfrac{6}{7}z_n + \dfrac{1}{28}z_n, \\ x_{n+1} = \dfrac{1}{14n}x_n + \dfrac{2n-1}{2n}y_n, \quad n \geq 1, \end{cases} \quad (100)$$

where

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2}{2\left\|A^*\left(I - P_{Q_n}\right)Ax_n\right\|^2}, & \text{if } \left\|\left(P_{Q_n} - I\right)Ax_n\right\| \neq 0, \\ 0, & \text{otherwise}. \end{cases} \quad (101)$$

Algorithm 4. Let $\{x_n\}$ be the sequence generated by algorithm (100) in [10] with $\alpha_n = 1/2n$ and $\gamma = 1$. Then, scheme (100) in [10] can be simplified as

$$x_1 \in E_1, x_{n+1} = \frac{1}{2n}x_1 + \frac{2n-1}{8n}\left(x_n + A^*(T - I)Ax_n\right), \quad n \geq 1. \quad (102)$$

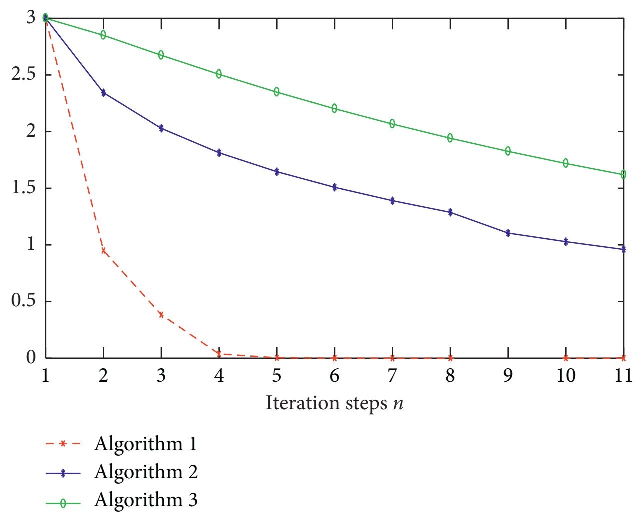


FIGURE 1: Convergence for Algorithms 3–5 with different initial points $x_1 = 3$.

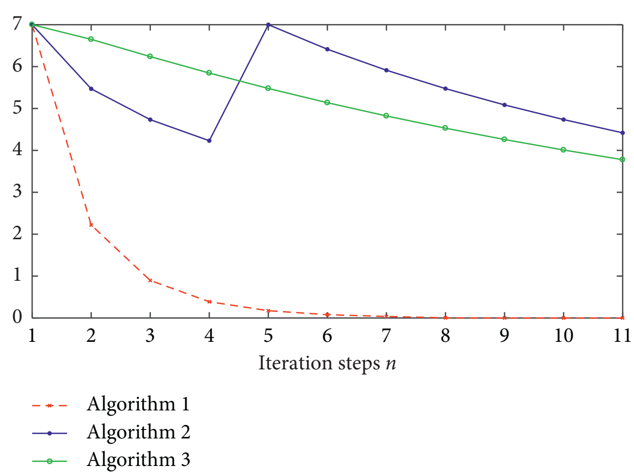


FIGURE 2: Convergence for Algorithms 3–5 with different initial points $x_1 = 7$.

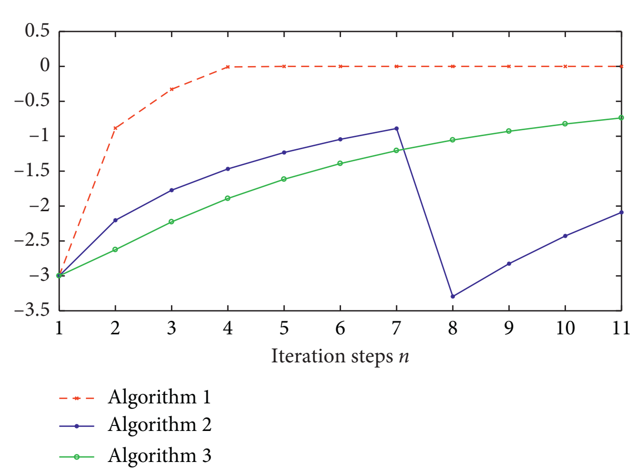


FIGURE 3: Convergence for Algorithms 3–5 with different initial points $x_1 = -3$.

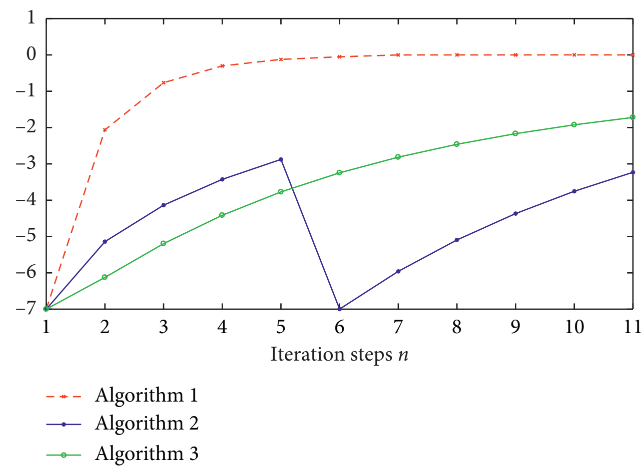


FIGURE 4: Convergence for Algorithms 3–5 with different initial points $x_1 = -7$.

Algorithm 5. Let $\{x_n\}$ be the sequence generated by algorithm (25) in [22] with $\alpha_n = 1/2n$ and $\gamma = 1$. Then, scheme (25) in [22] can be simplified as

$$\begin{cases} x_1 \in E_1, \\ Ax_n = \left(\dfrac{x_n}{2}, \dfrac{x_n}{3}\right), \\ z_n = x_n + A^* (T - I) Ax_n, \\ y_n = \dfrac{2n-1}{2n} z_n + \dfrac{1}{8n} z_n, \\ C_{n+1} = \left\{v_n \colon C_n \colon \left|y_n - v\right| \le \left|x_n - v\right|, \left|z_n - v\right| \le \left|x_n - v\right|\right\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1. \end{cases} \tag{103}$$

We perform schemes (100)–(103) with the different initial points. Figures 1–4 show that the sequence $\{x_n\}$ generated by (100)–(103) converge to 0.

Remark 2. (a) Although Theorem 1 in [22] requires that $\{\alpha_n\}$ in Algorithm 5, i.e., algorithm (25) in [22], takes values in $[\delta, 1)$ with $\delta \in (0, 1)$; here, for comparing the convergence rate of three schemes, we put the same $\alpha_n = 1/2n$. This does not affect the effectiveness of Algorithm 5 since the program stops in finite iterations. (b) Figures 1–4 above show that the convergence rate of Algorithm 3 is faster than that of Algorithms 4 and 5.

Next, we illustrate Theorem 2 by the following example.

Example 2. Let $E_1 = \mathbb{R}^2$ and $E_2 = \mathbb{R}$. Define the mappings $S \colon E_1 \longrightarrow E_1$ by $Sx = ((x_1/2), x_2)$ for all $x = (x_1, x_2) \in E_1$, and $T \colon E_2 \longrightarrow E_2$ by $Tx = x/2$ if $|x| \le 1$ and $Tx = 1$ if $|x| > 1$.

Table 1: Convergence for Algorithm 2 with initial point $x_1 = (3, 6)$.

Iteration steps	x_n
1	(−2, −5)
2	(−1.48705, −5.00000)
3	(−1.13985, −4.99999)
4	(−0.89031, −4.99999)
5	(−0.70347, −4.99999)
⋮	⋮
100	(−0.00085, −4.99999)
261	(−0.00012, −4.99998)
262	(−0.00012, −4.99998)
263	(−0.00011, −4.99998)
264	(−0.00011, −4.99998)
265	(−0.00011, −4.99998)
⋮	⋮
286	(−0.00009, −4.99998)

Table 2: Convergence for Algorithm 2 with initial point $x_1 = (3, 6)$.

Iteration steps	x_n
1	(3, 6)
2	(2.23056, 0.99999)
3	(1.70977, 1.55555)
4	(1.33545, 1.83331)
⋮	⋮
100	(0.00128, 1.99985)
101	(0.00125, 2.00024)
102	(0.00122, 2.00022)
300	(0.00014, 1.99997)
301	(0.00013, 2.00002)
302	(0.00013, 2.00003)
303	(0.00013, 1.99997)
⋮	⋮
349	(0.00010, 2.00002)
350	(0.00010, 2.00000)
351	(0.00009, 2.00003)

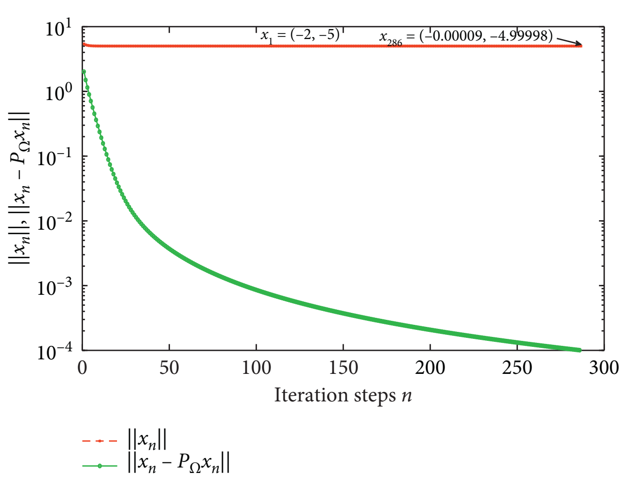


Figure 5: Convergence for Algorithm 2 with different initial points $x_1 = (-2, -2)$.

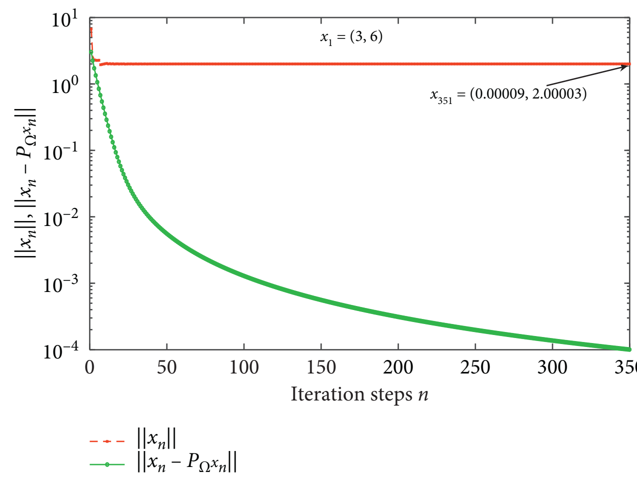


Figure 6: Convergence for Algorithm 2 with different initial points $x_1 = (3, 6)$.

Let $A: E_1 \longrightarrow E_2$ be a mapping defined by $Ax = x_1$ for all $x = (x_1, x_2) \in E_1$. Then, A is linear and bounded and $A^* y = (y, 0)$ for all $y \in E_2$. It is easy to see that $\Omega = \{(0, x_2): x_2 \in \mathbb{R}\}$. All the conditions on S, T, and Ω are satisfied for Theorem 2.

By Algorithm 2, we generate a sequence $\{x_n\}$ with $\alpha_n = 1/n^2$ and $\beta_n = 1/2(1 - e^{(n/2)})$ for all $n \geq 1$. Theorem 2 shows that $\{x_n\}$ will converge to the point $P_\Omega x_n$. We will stop the program when $\|x_n - P_\Omega x_n\| < 10^{-4}$. The computed results of the sequence $\{x_n\}$ are given in Tables 1 and 2. Figures 5 and 6 show the convergence of the sequence $\{x_n\}$.

6. Conclusion

For finding a solution of the split common fixed problem of quasi-ϕ-nonexpansive mappings in Banach space, we introduced a Halpern algorithm and a nonconvex combination algorithm where the norm of the linear bounded operator does not need to be known in advance. The convergence of the algorithms was investigated and some numerical examples were given to illustrate the convergence of the algorithms.

Data Availability

All data for our algorithms are included in this paper.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

[1] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2, pp. 221–239, 1994.

[2] N. T. An, N. M. Nam, and X. Qin, "Solving k-center problems involving sets based on optimization techniques," *Journal of Global Optimization*, vol. 76, no. 1, pp. 189–209, 2020.

[3] Y. Censor, T. Bortfeld, B. Martin, and A. Trofimov, "A unified approach for inversion problems in intensity modulated

radiation therapy," *Physics in Medicine and Biology*, vol. 51, pp. 2356–2365, 2006.

[4] T. H. Cuong, J.-C. Yao, and N. D. Yen, "Qualitative properties of the minimum sum-of-squares clustering problem," *Optimization*, vol. 69, no. 9, pp. 2131–2154, 2020.

[5] L. V. Nguyen and X. Qin, "The minimal time function associated with a collection of sets," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 26, p. 93, 2020.

[6] C. Byrne, "Iterative oblique projection onto convex sets and the split feasibility problem," *Inverse Problems*, vol. 18, no. 2, pp. 441–453, 2002.

[7] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

[8] H.-K. Xu, "Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces," *Inverse Problems*, vol. 26, no. 10, Article ID 105018, 2010.

[9] A. Moudafi, "The split common fixed point problem for demicontractive mapping," *Inverse Problems*, vol. 26, Article ID 055007, 2010.

[10] R. Kraikaew and S. Saejung, "On split common fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 415, no. 2, pp. 513–524, 2014.

[11] A. Moudafi, "A note on the split common fixed-point problem for quasi-nonexpansive operators," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 12, pp. 4083–4087, 2011.

[12] A. Moudafi, "A relaxed alternating CQ-algorithm for convex feasibility problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 79, pp. 117–121, 2013.

[13] X. L. Qin, L. Wang, and J. C. Yao, "Inertial splitting method for maximal monotone mappings," *Journal of Nonlinear Convex Analysis*, vol. 21, pp. 2325–2333, 2020.

[14] W. Takahashi, C. F. Wen, and J. C. Yao, "The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space," *Fixed Point Theory*, vol. 19, pp. 407–419, 2018.

[15] W. Takahahsi and J. C. Yao, "The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces," *Journal of Nonlinear Convex Analysis*, vol. 20, pp. 173–195, 2019.

[16] F. Schopfer, T. Schuster, and A. K. Louis, "An iterative regularization method for the solution of the split feasibility problem in Banach spaces," *Inverse Problems*, vol. 24, Article ID 055008, 20 pages, 2008.

[17] J. F. Tang, S. S. Chang, L. Wang, and X. R. Wang, "On the split common fixed point problem for strict pseudocontractive and asymptotically nonexpansive mappings in Banach spaces," *Journal Inequalities Application*, vol. 305, p. 2015, 2015.

[18] J.-Z. Chen, H.-Y. Hu, and L.-C. Ceng, "Strong convergence of hybrid Bregman projection algorithm for split feasibility and fixed point problems in Banach spaces," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 1, pp. 192–204, 2017.

[19] S. Y. Cho, "Implicit extragradient-like method for fixed point problems and variational inclusion problems in a Banach space," *Symmetry*, vol. 12, no. 6, p. 998, 2020.

[20] Z. Ma and L. Wang, "On the split equality common fixed point problem for asymptotically nonexpansive semigroups in Banach spaces," *Journal of Nonlinear Sciences and Applications*, vol. 09, no. 6, pp. 4003–4015, 2016.

[21] X. Tian, L. Wang, and Z. Ma, "On the split equality common fixed point problem for quasi-nonexpansive multi-valued mappings in Banach spaces," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 10, pp. 5536–5543, 2016.

[22] Z. Ma, L. Wang, and S.-S. Chang, "On the split feasibility problem and fixed point problem of quasi-ϕ-nonexpansive mapping in banach spaces," *Numerical Algorithms*, vol. 80, no. 4, pp. 1203–1218, 2019.

[23] F. Kohasaka and W. Takahashi, "Strong convergence of an iterative sequence for maximal monotone operators in Banach spaces," *Abstract and Applied Analysis*, vol. 3, pp. 239–249, 2004.

[24] X. Qin, Y. J. Cho, and S. M. Kang, "Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 20–30, 2009.

[25] H. Zegeye, "A hybrid iteration scheme for equilibrium problems, variational inequality problems and common fixed point problems in banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 2136–2146, 2010.

[26] D. V. Hieu and J. J. Strodiot, "Strong convergence theorems for equilibrium problems and fixed point problems in Banach spaces," *Journal of Fixed Point Theory and Applications*, vol. 20, no. 20, p. 131, 2018.

[27] N. Hussain, G. Marino, L. Muglia, and B. A. Alamri, "On some Mann's type iterative algorithms," *Fixed Point Theory Application*, vol. 17, 2015.

[28] G. Marino, B. Scardamaglia, and E. Karapinar, "Strong convergence theorem for strict pseudo-contractions in Hilbert spaces," *Journal of Inequalities Application*, vol. 24, p. 134, 2016.

[29] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type Algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.

[30] Y. I. Alber and S. Reich, "An iterative method for solving a class of nonlinear operator equations in Banach spaces," *Panamerican Mathematical Journal*, vol. 4, pp. 39–54, 1994.

[31] H. Y. Zhou and X. L. Qin, *Fixed Points of Nonlinear Operators. Iterative Methods*, De Gruyter, Berlin, Germany, 2020.

[32] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications, in theory and applications of nonlinear operators of accretive and monotone type," in *Lecture Notes in Pure and Applied Mathematics,* Dekker, New York, NY, USA, 1996.

[33] H.-K. Xu, "Inequalities in banach spaces with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.

[34] H.-K. Xu, "Another control condition in an iterative method for nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 65, no. 1, pp. 109–113, 2002.

[35] P.-E. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 7-8, pp. 899–912, 2008.

[36] K. K. Tan and H. K. Xu, "Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process," *Journal of Mathematical Analysis and Applications*, vol. 178, no. 2, pp. 301–308, 1993.

[37] S. Y. Cho and X. Qin, "On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems," *Applied Mathematics and Computation*, vol. 235, pp. 430–438, 2014.

[38] S. Y. Cho, "A convergence theorem for generalized mixed equilibrium problems and multivalued asymptotically nonexpansive mappings," *Journal of nonlinear and convex analysis*, vol. 21, pp. 1017–1026, 2020.

[39] Z. H. He, "The split equilibrium problem and its convergence algorithms," *Journal of Inequalities and Applications*, vol. 2012, p. 162, 2012.

[40] K. R. Kazmi and S. H. Rizvi, "Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem," *Journal of the Egyptian Mathematical Society*, vol. 21, no. 1, pp. 44–51, 2013.

[41] L. V. Nguyen, Q. H. Ansari, and X. L. Qin, "Linear conditioning, weak sharpness and finite convergence for equilibrium problems," *Journal of Global Optimization*, vol. 77, pp. 405–424, 2020.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6624321, 12 pages
https://doi.org/10.1155/2021/6624321

Research Article

Strong Convergence Results of Split Equilibrium Problems and Fixed Point Problems

Li-Jun Zhu,[1] Hsun-Chih Kuo ,[2] and Ching-Feng Wen [3,4]

[1]The Key Laboratory of Intelligent Information and Data Processing of NingXia Province, North Minzu University, Yinchuan 750021, China
[2]Department of Risk Management and Insurance, National Kaohsiung University of Science and Technology, Kaohsiung, Taiwan
[3]Center for Fundamental Science and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 80708, Taiwan
[4]Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung 80708, Taiwan

Correspondence should be addressed to Hsun-Chih Kuo; seankuo@nkust.edu.tw and Ching-Feng Wen; cfwen@kmu.edu.tw

Received 24 November 2020; Revised 22 December 2020; Accepted 13 January 2021; Published 29 January 2021

Academic Editor: George Psihoyios

In this paper, we investigate the split equilibrium problem and fixed point problem in Hilbert spaces. We propose an iterative scheme for solving such problem in which the involved equilibrium bifunctions f and g are pseudomonotone and monotone, respectively, and the operators S and T are all pseudocontractive. We show that the suggested scheme converges strongly to a solution of the considered problem.

1. Introduction

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces. Let C and Q be two nonempty, closed, and convex subsets of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Let $f: C \times C \longrightarrow \mathbb{R}$ be a bifunction. Recall that the equilibrium problem is to find a point $x^* \in C$ such that

$$f(x^*, x) \geq 0, \quad \forall x \in C. \tag{1}$$

Use $\mathrm{SEP}(C, f)$ to denote the solution set of equilibrium problem (1).

Equilibrium problems have been considered broadly in the literature (see e.g. [1–5]). Now, it is known that variational inequalities ([6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]) and fixed point problems ([18, 19, 20, 21, 22, 23]) can be transformed in the form of (1). For every $\sigma > 0$ and $x \in H$, there exists a unique point $z \in C$ such that $f(z, y) + (1/\sigma)\langle z - x, y - x\rangle \geq 0$, $\forall y \in C$ (see [2]). Thus, for solving equilibrium problem (1), an important technique is to use the resolvent of bifunction f ([2]). Another important method for solving equilibrium problem (1) is to use linear search technique [4].

Let $S: C \longrightarrow C$ and $T: Q \longrightarrow Q$ be two operators. Let $\mathrm{Fix}(S)$ and $\mathrm{Fix}(T)$ be the fixed point sets of S and T, respectively. Let $g: Q \times Q \longrightarrow \mathbb{R}$ be a bifunction. Let $A: H_1 \longrightarrow H_2$ be a bounded linear operator. In this paper, we concern the following split problem of finding a point $\tilde{u} \in C$ such that

$$\begin{aligned} &\tilde{u} \in \mathrm{SEP}(C, f) \cap \mathrm{Fix}(S), \\ &A\tilde{u} \in \mathrm{SEP}(Q, g) \cap \mathrm{Fix}(T). \end{aligned} \tag{2}$$

Denote the solution set of (2) by Γ, i.e., $\Gamma = \{x^* \in \mathrm{SEP}(C, f) \cap \mathrm{Fix}(S), Ax^* \in \mathrm{SEP}(Q, g) \cap \mathrm{Fix}(T)\}$.

The split problem has received many concerns (see [13, 24–28]) due to its extensive applications in image recovery and signal processing, control theory, and so on. Note that the split problem (2) includes the following split problems as special cases:

(i) The split equilibrium problem studied in [29, 30] can be formulated to find an element $\tilde{u} \in C$ such that

$$\begin{aligned} &\tilde{u} \in \mathrm{SEP}(C, f), \\ &A\tilde{u} \in \mathrm{SEP}(Q, g). \end{aligned} \tag{3}$$

The solution set of (3) is denoted by Γ_1.

(ii) The split fixed point problem considered in [31, 32, 33, 34] reduces to find a point $\tilde{u} \in C$ such that

$$\begin{aligned} &\tilde{u} \in \mathrm{Fix}(S), \\ &A\tilde{u} \in \mathrm{Fix}(T). \end{aligned} \tag{4}$$

The solution set of (4) is denoted by Γ_2.

Numerical iterative algorithms have been proposed for finding a split problem of the set of solutions of equilibrium problems and the set of fixed points of nonexpansive operators; see, for example, [35–39] and the references therein. Recently, Yao et al. [40] proposed an iterative scheme for solving the split problem (2) and they obtained the weak convergence of the suggested scheme.

In this paper, we continuously study the split problem (2) in which the involved equilibrium bifunctions f and g are pseudomonotone and monotone, respectively, and the operators S and T are all pseudocontractive. We propose an iterative scheme for solving the split problem (2) and strong convergence results are obtained.

2. Preliminaries

Let $\mathscr{H}_1$ be a real Hilbert space with its inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty, convex, and closed subset of $\mathscr{H}_1$. Let P_C: $\mathscr{H}_1 \longrightarrow C$ be the metric projection defined by

$$P_C(x) = \arg\min_{y \in C} \|y - x\|. \tag{5}$$

P_C satisfies: for given $x \in H_1$,

$$\langle x - P_C(x), y - P_C(x) \rangle \leq 0, \quad \forall y \in C. \tag{6}$$

Let f: $C \times C \longrightarrow \mathbb{R}$ be a bifunction. Recall that f is said to be monotone if

$$f\left(u^\dagger, v^\dagger\right) + f\left(v^\dagger, u^\dagger\right) \leq 0, \quad \forall u^\dagger, v^\dagger \in C. \tag{7}$$

f is said to be pseudomonotone if

$$f\left(u^\dagger, v^\dagger\right) \geq 0 \text{ implies } f\left(v^\dagger, u^\dagger\right) \leq 0, \quad \forall u^\dagger, v^\dagger \in C. \tag{8}$$

Let S: $C \longrightarrow C$ be an operator. S is called pseudocontractive if

$$\begin{aligned} \left\|Sx - Sx^\dagger\right\|^2 \leq \left\|x - x^\dagger\right\|^2 + \left\|(I - S)x - (I - S)x^\dagger\right\|^2, \\ \forall x, x^\dagger \in C. \end{aligned} \tag{9}$$

S is called L-Lipschitz if there exists a constant $L \geq 0$ such that

$$\left\|Sx - Sx^\dagger\right\| \leq L\left\|x - x^\dagger\right\|, \quad \forall x, x^\dagger \in C. \tag{10}$$

If $L = 1$, then S is said to be nonexpansive. If $L < 1$, then S is said to be L-contraction.

In the sequel, we use the following symbols. Let $\{x^k\}$ be a sequence in C:

(i) $x^k \rightharpoonup x^\dagger$ means the weak convergence of x^k to $x^\dagger$ as $k \longrightarrow \infty$

(ii) $x^k \longrightarrow x^\dagger$ means the strong convergence of x^k to $x^\dagger$ as $k \longrightarrow \infty$

(iii) $\omega_w(x^k) = \{x^\dagger : \exists \{x^{k_i}\} \subset \{x^k\} \text{ such that } x^{k_i} \rightharpoonup x^\dagger \ (i \longrightarrow \infty)\}$

Recall that f is said to be jointly sequently weakly continuous on $C \times C$, if for two sequences $x^k \in C$ and $y^k \in C$ satisfy $x^k \rightharpoonup u^\dagger$ and $y^k \rightharpoonup v^\dagger$, then we have $f(x^k, y^k) \longrightarrow f(u^\dagger, v^\dagger)$.

Let $\mathscr{H}_2$ be a real Hilbert space with its inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let Q be a nonempty, convex, and closed subset of $\mathscr{H}_2$. Let φ: $Q \longrightarrow (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function. Then, the subdifferential $\partial\varphi$ of φ is defined by

$$\partial\varphi(u) := \left\{v^\dagger \in H_2 : \varphi(u) + \langle v^\dagger, u^\dagger - u \rangle \leq \varphi\left(u^\dagger\right), \forall u^\dagger \in Q\right\}, \tag{11}$$

for each $u \in Q$.

It is well known that

$$u^\dagger = \arg\min_{u \in Q}\{\varphi(u)\} \Longleftrightarrow 0 \in \partial\varphi\left(u^\dagger\right) + N_Q\left(u^\dagger\right), \tag{12}$$

where $N_Q(u^\dagger) = \{\omega \in H_2 : \langle \omega, u - u^\dagger \rangle \leq 0, \forall u \in Q\}$.

The following lemma can be found in [41]. For the completeness, we include the detail of proof.

Lemma 1 (see [41]). *Let S: $C \longrightarrow C$ be an L_1-Lipschitz pseudocontractive operator. Then, for all $\tilde{u} \in C$ and $u^\dagger \in Fix(S)$, we have*

$$\begin{aligned} \left\|u^\dagger - S((1-\eta)\tilde{u} + \eta S\tilde{u})\right\|^2 \leq \left\|\tilde{u} - u^\dagger\right\|^2 + (1-\eta) \\ \cdot \|\tilde{u} - S((1-\eta)\tilde{u} + \eta S\tilde{u})\|^2, \end{aligned} \tag{13}$$

where $0 < \eta < (1/\sqrt{1 + L_1^2} + 1)$.

Proof. Since $u^\dagger \in Fi\tilde{u}(S)$, we have from (9) that

$$\begin{aligned} \left\|S((1-\eta)I + \eta S)\tilde{u} - u^\dagger\right\|^2 \leq \left\|(1-\eta)\left(\tilde{u} - tu^\dagger\right) + \eta\left(S\tilde{u} - u^\dagger\right)\right\|^2 \\ + \|(1-\eta)\tilde{u} + \eta S\tilde{u} - S((1-\eta)\tilde{u} + \eta S\tilde{u})\|^2, \end{aligned} \tag{14}$$

$$\left\|S\tilde{u} - u^\dagger\right\|^2 \leq \left\|\tilde{u} - u^\dagger\right\|^2 + \|S\tilde{u} - \tilde{u}\|^2, \tag{15}$$

for all $\tilde{u} \in C$.

Since S is L_1-Lipschitzian and $\tilde{u} - ((1-\eta)\tilde{u} + \eta S\tilde{u}) = \eta(\tilde{u} - tSn\tilde{u})$, we have

$$\|S\tilde{u} - S((1-\eta)\tilde{u} + \eta S\tilde{u})\| \leq \eta L_1 \|\tilde{u} - S\tilde{u}\|. \tag{16}$$

According to (15), we obtain

$$\begin{aligned}&\left\|(1-\eta)\left(\tilde{u}-tu^{\dagger}\right)+\eta\left(S\tilde{u}-u^{\dagger}\right)\right\|^2\\&\quad=(1-\eta)\left\|\tilde{u}-u^{\dagger}\right\|^2+\eta\left\|S\tilde{u}-u^{\dagger}\right\|^2-\eta(1-\eta)\|\tilde{u}-S\tilde{u}\|^2\\&\quad\le(1-\eta)\left\|\tilde{u}-u^{\dagger}\right\|^2+\eta\left(\left\|\tilde{u}-u^{\dagger}\right\|^2+\|S\tilde{u}-\tilde{u}\|^2\right)\\&\qquad-\eta(1-\eta)\|\tilde{u}-S\tilde{u}\|^2\\&\quad=\left\|\tilde{u}-u^{\dagger}\right\|^2+\eta^2\|S\tilde{u}-\tilde{u}\|^2.\end{aligned}\tag{17}$$

Based on (16), we conclude

$$\begin{aligned}&\|(1-\eta)\tilde{u}+\eta S\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u})\|^2\\&\quad=\|(1-\eta)(\tilde{u}-tSn((1-\eta)\tilde{u}+\eta S\tilde{u}))+\eta(S\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u}))\|^2\\&\quad=(1-\eta)\|\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u})\|^2+\eta\|S\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u})\|^2\\&\qquad-\eta(1-\eta)\|\tilde{u}-S\tilde{u}\|^2\\&\quad\le(1-\eta)\|\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u})\|^2-\eta\left(1-\eta-\eta^2L_1^2\right)\|\tilde{u}-S\tilde{u}\|^2.\end{aligned}\tag{18}$$

By (14), (17), and (18), we obtain

$$\begin{aligned}\left\|S((1-\eta)I+\eta S)\tilde{u}-u^{\dagger}\right\|^2&\le\left\|\tilde{u}-u^{\dagger}\right\|^2+\eta^2\|\tilde{u}-S\tilde{u}\|^2\\&\quad+(1-\eta)\|\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u})\|^2\\&\quad-\eta\left(1-\eta-\eta^2L_1^2\right)\|\tilde{u}-S\tilde{u}\|^2\\&=\left\|\tilde{u}-u^{\dagger}\right\|^2+(1-\eta)\|\tilde{u}-S((1-\eta)I+\eta S)\tilde{u}\|^2\\&\quad-\eta\left(1-2\eta-\eta^2L_1^2\right)\|\tilde{u}-S\tilde{u}\|^2.\end{aligned}\tag{19}$$

Since $\eta<(1/\sqrt{1+L_1^2}+1)$, $1-2\eta-\eta^2L_1^2>0$. Hence, we can deduce the desired result from (19). □

Lemma 2 (see [42]). *Let $S\colon C\longrightarrow C$ be a continuous pseudocontractive operator. Then,*

(i) $\mathrm{Fix}(S)\subset C$ is closed and convex

(ii) S is demiclosedness, i.e., if $x^k\rightharpoonup\tilde{u}$ and $Sx^k\longrightarrow z^{\dagger}$ as $k\longrightarrow\infty$, then $S\tilde{u}=z^{\dagger}$.

Here, we state some conditions on f and g which will be used in the sequel.

Let C and Q be two nonempty, closed, and convex subsets of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Let $f\colon C\times C\longrightarrow\mathbb{R}$ and $g\colon Q\times Q\longrightarrow\mathbb{R}$ be two bifunctions. Assume that

(i) (A1): $f(z^{\dagger},z^{\dagger})=0$ for all $z^{\dagger}\in C$

(ii) (A2): f is pseudomonotone on $\mathrm{SEP}(C,f)$

(iii) (A3): f is jointly sequently weakly continuous on $C\times C$

(iv) (A4): $f(z^{\dagger},\cdot)$ is convex and subdifferentiable on C for all $z^{\dagger}\in C$

(v) (B1): $g(z^{\dagger},z^{\dagger})=0$ for all $z^{\dagger}\in Q$

(vi) (B2): g is monotone on Q

(vii) (B3): $g(u,\cdot)$ is convex and lower semicontinuous on Q for each $u\in Q$

(viii) (B4): for all $u,v,w\in Q$, $\limsup_{\lambda\downarrow 0}g(\lambda w+(1-\lambda)u,v)\le g(u,v)$

Lemma 3 (see [1, 2]). *Assume that g satisfies conditions (B1)–(B4). For $\varsigma>0$ and $u\in H_2$, there exists $w\in Q$ such that*

$$g(w,v)+\frac{1}{\varsigma}\langle v-w,w-u\rangle\ge 0,\quad\forall v\in Q.\tag{20}$$

Let the operator J_{ς}^{g} be defined by

$$J_{\varsigma}^{g}(u)=\left\{w\in Q\colon g(w,v)+\frac{1}{\varsigma}\langle v-w,w-u\rangle\ge 0,\forall v\in Q\right\}.\tag{21}$$

We have the following conclusions:

(i) J_{ς}^{g} is single-valued and firmly nonexpansive, that is, for any $u,v\in H_2$,

$$\left\|J_{\varsigma}^{g}(u)-J_{\varsigma}^{g}(v)\right\|^{2}\leq\langle J_{\varsigma}^{g}(u)-J_{\varsigma}^{g}(v),u-v\rangle. \tag{22}$$

(ii) $\mathrm{SEP}(Q,g)$ is closed and convex and $\mathrm{SEP}(Q,g)=\mathrm{Fix}(J_{\varsigma}^{g})$.

(iii) For $\varsigma_1,\varsigma_2>0$ and $u,v\in H_2$, we have

$$\left\|J_{\varsigma_1}^{g}(u)-J_{\varsigma_2}^{g}(v)\right\|\leq\|u-v\|+\frac{|\varsigma_2-\varsigma_1|}{\varsigma_2}\left\|J_{\varsigma_2}^{g}(v)-v\right\|. \tag{23}$$

Lemma 4 (see [4]). *Assume that f satisfies conditions (A1)-(A4). Let $\{\beta_k\}$ be a sequence satisfying $\beta_k\in[\underline{\beta},\overline{\beta}]\subset(0,1]$. For given $v^k\in C$, let the sequence $\{y^k\}$ be generated by*

$$y^{k}=\arg\min_{u^{\dagger}\in C}\left\{f\left(v^{k},u^{\dagger}\right)+\frac{1}{2\beta_{k}}\left\|v^{k}-u^{\dagger}\right\|^{2}\right\}. \tag{24}$$

Then the boundedness of $\{v^k\}$ implies that $\{y^k\}$ is bounded.

Lemma 5 (see [5]). *Assume that f satisfies conditions (A1)–(A4). For given two points $\overline{u},\overline{v}\in C$ and two sequences $\{a^k\}\subset C$ and $\{b^k\}\subset C$, if $a^k\rightharpoonup\overline{u}$ and $b^k\rightharpoonup\overline{v}$, respectively, then, for any $\varepsilon>0$, there exist $\vartheta>0$ and $N_\varepsilon\in\mathbb{N}$ such that*

$$\partial_2 f\left(b^k,a^k\right)\subset\partial_2 f(\overline{v},t\overline{u})+\frac{\varepsilon}{\vartheta}B, \tag{25}$$

for every $k\geq N_\varepsilon$, where $B:=\{b\in H_1:\ \|b\|\leq1\}$.

Lemma 6 (see [43]). *Let $\{a_n\}\subset(0,\infty)$, $\{b_n\}\subset(0,1)$, and $\{c_n\}$ be three real number sequences. If*

$$a_{n+1}\leq(1-b_n)a_n+c_n, \tag{26}$$

for all $n\geq0$ with $\sum_{n=1}^{\infty}b_n=\infty$ and $\limsup_{n\longrightarrow\infty}(c_n/b_n)\leq0$ or $\sum_{n=1}^{\infty}|c_n|<\infty$, then $\lim_{n\longrightarrow\infty}a_n=0$.

3. Main Results

In this section, in order to solve problem (2), we first present an iterative algorithm and consequently prove its strong convergence.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces. Let C and Q be two nonempty, closed, and convex subsets of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Assume that

(i) $h: C\longrightarrow C$ is a κ-contractive operator

(ii) $S: C\longrightarrow C$ is an L_1-Lipschitz pseudocontractive operator and $T: Q\longrightarrow Q$ is an L_2-Lipschitz pseudocontractive operator with $L_1>1$ and $L_2>1$

(iii) f and g are two bifunctions satisfying conditions (A1)–(A4) and conditions (B1)–(B4), respectively

(iv) $A:\mathscr{H}_1\longrightarrow\mathscr{H}_2$ is a bounded linear operator and A^* is its adjoint

Let $\{\delta_k\}$, $\{\eta_k\}$, $\{\beta_k\}$, $\{\tau_k\}$, $\{\varsigma_k\}$, $\{\zeta_k\}$, $\{\lambda_k\}$, and $\{\mu_k\}$ be real number sequences and α, ϑ, and γ be constants. Next, we introduce our iterative algorithm.

Algorithm 1. Fix an initial point $x^0\in C$. Set $k=0$.

Step 1: assume that x^k is known and compute

$$v^k=(1-\delta_k)x^k+\delta_k S\left[(1-\eta_k)x^k+\eta_k Sx^k\right]. \tag{27}$$

Step 2: compute

$$y^{k}=\arg\min_{y^{\dagger}\in C}\left\{f\left(v^{k},y^{\dagger}\right)+\frac{1}{2\beta_{k}}\left\|v^{k}-y^{\dagger}\right\|^{2}\right\}. \tag{28}$$

If $y^k=v^k$, then set $u^k=v^k$ and go to Step 5. Otherwise, go to Step 3.

Step 3: let $m_k=\min\{1,2,\ldots,k,\ldots\}$ such that

$$f\left(z^{k,m_k},v^k\right)-f\left(z^{k,m_k},y^k\right)\geq\frac{\alpha}{2\beta_k}\left\|v^k-y^k\right\|^2, \tag{29}$$

where

$$z^{k,m_k}=(1-\vartheta^{m_k})v^k+\vartheta^{m_k}y^k. \tag{30}$$

Write $\vartheta_k=\vartheta^{m_k}$ and $z^k=z^{k,m_k}$.

Step 4: compute

$$u^k=P_C\left(v^k-\tau_k\iota_k\nu^k\right), \tag{31}$$

where $\nu^k\in\partial_2 f(z^k,v^k)$ and $\iota_k=(f(z^k,v^k)/\|\nu^k\|^2)$.

Step 5:

For any $v\in Q$, find w^k such that

$$g\left(w^k,v\right)+\frac{1}{\varsigma_k}\langle v-w^k,w^k-Au^k\rangle\geq0. \tag{32}$$

Compute

$$q^k=(1-\zeta_k)w^k+\zeta_k T\left[(1-\lambda_k)w^k+\lambda_k Tw^k\right]. \tag{33}$$

Step 6: compute

$$x^{k+1}=\mu_k h\left(x^k\right)+(1-\mu_k)P_C\left[u^k+\gamma A^*\left(q^k-Au^k\right)\right]. \tag{34}$$

Step 7: set $k:=k+1$ and return to Step 1.

In order to demonstrate the convergence of Algorithm 1, we need some additional assumptions on the iterative parameters. Suppose that the following conditions are satisfied:

(C1): $0<\underline{\delta}<\delta_k<\overline{\delta}<\eta_k<\overline{\eta}<(1/\sqrt{1+L_1^2}+1)\ (\forall k\geq0)$ and $\alpha,\vartheta\in(0,1)$

(C2): $\beta_k\in[\gamma_1,\gamma_2]\subset(0,1]$; $\tau_k\in[\tau_1,\tau_2]\subset(0,2)$ and $0<\varsigma\leq\varsigma_k<+\infty$

(C3): $0<\underline{\zeta}<\zeta_k<\overline{\zeta}<\lambda_k<\overline{\lambda}<(1/\sqrt{1+L_2^2}+1)\ (\forall k\geq0)$ and $\gamma\in(0,1/\|A\|^2)$

(C4): $\lim_{k\longrightarrow+\infty}\mu_k=0$ and $\sum_{k=0}^{+\infty}\mu_k=+\infty$

We have the following remark which can be found in [4].

Remark 1

(1) If $y^k=v^k$, then $y_n\in\mathrm{SEP}(C,f)$

(2) The linesearch rule (29) is well defined

(3) $0 \notin \partial_2 f(z^k, v^k)$

(4) $f(z^k, v^k) > 0$

(5) $\|u^k - p\|^2 \le \|v^k - p\|^2 - \tau_k(2-\tau_k)(\iota_k\|v^k\|)^2$ for all $p \in \mathrm{SEP}(C, f)$

Next, we prove our main result.

Theorem 1. *Suppose that $\Gamma \neq \emptyset$. Then, the sequence $\{x^k\}$ generated by (34) converges strongly to $q^\dagger = P_\Gamma h(q^\dagger)$.*

Proof. Let $x^* \in \Gamma$. We have $x^* \in \mathrm{SEP}(C, f) \cap \mathrm{Fix}(S)$ and $Ax^* \in \mathrm{SEP}(Q, g) \cap \mathrm{Fix}(T)$. By (27) and Lemma 1, we get

$$\begin{aligned}\left\|v^k - x^*\right\|^2 &= \left\|(1-\delta_k)\left(x^k - x^*\right) + \delta_k\left(S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^*\right)\right\|^2\\ &= (1-\delta_k)\left\|x^k - x^*\right\|^2 + \delta_k\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^*\right\|^2\\ &\quad - (1-\delta_k)\delta_k\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2\\ &\le (1-\delta_k)\left\|x^k - x^*\right\|^2 + \delta_k(1-\eta_k)\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2\\ &\quad + \delta_k\left\|x^k - x^*\right\|^2 - (1-\delta_k)\delta_k\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2\\ &= \left\|x^k - x^*\right\|^2 - \delta_k(\eta_k - \delta_k)\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2\\ &\le \left\|x^k - x^*\right\|^2.\end{aligned} \tag{35}$$

From (31) and Remark 1, we have

$$\begin{aligned}\left\|u^k - x^*\right\|^2 &\le \left\|v^k - x^*\right\|^2 - \tau_k(2-\tau_k)\left(\iota_k\left\|v^k\right\|\right)^2\\ &\le \left\|v^k - x^*\right\|^2.\end{aligned} \tag{36}$$

According to (32) and Lemma 3, we have $w^k = J^g_{\varsigma_k} Au^k$ and $Ax^* \in \mathrm{Fix}(J^g_{\varsigma_k})$. Since $J^g_{\varsigma_k}$ is firmly nonexpansive, we deduce

$$\begin{aligned}\left\|w^k - Ax^*\right\|^2 &= \left\|J^g_{\varsigma_k} Au^k - J^g_{\varsigma_k} Ax^*\right\|^2\\ &\le \langle J^g_{\varsigma_k} Au^k - J^g_{\varsigma_k} Ax^*, Au^k - Ax^*\rangle\\ &= \langle w^k - Ax^*, Au^k - Ax^*\rangle\\ &= \frac{1}{2}\left(\left\|w^k - Ax^*\right\|^2 + \left\|Au^k - Ax^*\right\|^2 - \left\|w^k - Au^k\right\|^2\right).\end{aligned} \tag{37}$$

It follows that

$$\left\|w^k - Ax^*\right\|^2 \le \left\|Au^k - Ax^*\right\|^2 - \left\|w^k - Au^k\right\|^2. \tag{38}$$

By virtue of (33) and Lemma 1, we obtain

$$\begin{aligned}\left\|q^k - Ax^*\right\|^2 &= \left\|(1-\zeta_k)\left(w^k - Ax^*\right) + \zeta_k\left(T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - Ax^*\right)\right\|^2\\ &= (1-\zeta_k)\left\|w^k - Ax^*\right\|^2 + \zeta_k\left\|T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - Ax^*\right\|^2\\ &\quad - (1-\zeta_k)\zeta_k\left\|T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - w^k\right\|^2\\ &\le \left\|w^k - Ax^*\right\|^2 + \zeta_k(1-\lambda_k)\left\|T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - w^k\right\|^2\\ &\quad - (1-\zeta_k)\zeta_k\left\|T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - w^k\right\|^2\\ &= \left\|w^k - Ax^*\right\|^2 - (\lambda_k - \zeta_k)\zeta_k\left\|T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - w^k\right\|^2\\ &\le \left\|w^k - Ax^*\right\|^2.\end{aligned} \tag{39}$$

Thanks to (38) and (39), we get

$$\left\|q^k - Ax^*\right\|^2 \le \left\|Au^k - Ax^*\right\|^2 - \left\|w^k - Au^k\right\|^2. \tag{40}$$

Consequently,

$$\begin{aligned}
\langle u^k - x^*, A^*\left(q^k - Au^k\right)\rangle &= \langle Au^k - Ax^*, q^k - Au^k\rangle \\
&= \langle q^k - Ax^*, q^k - Au^k\rangle - \left\|q^k - Au^k\right\|^2 \\
&= \frac{1}{2}\left[\left\|q^k - Ax^*\right\|^2 + \left\|q^k - Au^k\right\|^2 - \left\|Au^k - Ax^*\right\|^2\right] \\
&\quad - \left\|q^k - Au^k\right\|^2 \\
&= \frac{1}{2}\left[\left\|q^k - Ax^*\right\|^2 - \left\|Au^k - Ax^*\right\|^2\right] - \frac{1}{2}\left\|q^k - Au^k\right\|^2 \\
&\le -\frac{1}{2}\left\|w^k - Au^k\right\|^2 - \frac{1}{2}\left\|q^k - Au^k\right\|^2.
\end{aligned} \tag{41}$$

Set $t^k = P_C[u^k + \gamma A^*(q^k - Au^k)]$ for all $k \ge 0$. In view of (35), (36), and (41), using the nonexpansivity of P_C, we have

$$\begin{aligned}
\left\|t^k - x^*\right\|^2 &= \left\|P_C\left[u^k + \gamma A^*\left(q^k - Au^k\right)\right] - P_C[x^*]\right\|^2 \\
&\le \left\|u^k - x^* + \gamma A^*\left(q^k - Au^k\right)\right\|^2 \\
&= \left\|u^k - x^*\right\|^2 + \left\|\gamma A^*\left(q^k - Au^k\right)\right\|^2 + 2\gamma\langle A^*\left(q^k - Au^k\right), u^k - x^*\rangle \\
&\le \left\|u^k - x^*\right\|^2 + \gamma^2\|A\|^2\left\|q^k - Au^k\right\|^2 - \gamma\left\|w^k - Au^k\right\|^2 \\
&\quad - \gamma\left\|q^k - Au^k\right\|^2 \\
&= \left\|u^k - x^*\right\|^2 - \gamma\left(1 - \gamma\|A\|^2\right)\left\|q^k - Au^k\right\|^2 - \gamma\left\|w^k - Au^k\right\|^2 \\
&\le \left\|x^k - x^*\right\|^2 - \tau_k(2 - \tau_k)\left(\iota_k\left\|\nu^k\right\|\right)^2 - \delta_k(\eta_k - \delta_k)\left\|S\left[(1 - \eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2 \\
&\quad - \gamma\left(1 - \gamma\|A\|^2\right)\left\|q^k - Au^k\right\|^2 - \gamma\left\|w^k - Au^k\right\|^2 \\
&\le \left\|x^k - x^*\right\|^2.
\end{aligned} \tag{42}$$

From (34), we get

$$\begin{aligned}
\left\|x^{k+1} - x^*\right\| &= \left\|\mu_k\left(h\left(x^k\right) - x^*\right) + (1 - \mu_k)\left(t^k - x^*\right)\right\| \\
&\le \mu_k\left\|h\left(x^k\right) - h(x^*)\right\| + \mu_k\left\|h(x^*) - x^*\right\| + (1 - \mu_k)\left\|t^k - x^*\right\| \\
&\le \mu_k\kappa\left\|x^k - x^*\right\| + \mu_k\left\|h(x^*) - x^*\right\| + (1 - \mu_k)\left\|x^k - x^*\right\| \\
&= [1 - (1 - \kappa)\mu_k]\left\|x^k - x^*\right\| + \mu_k\left\|h(x^*) - x^*\right\| \\
&\le \max\left\{\left\|x^k - x^*\right\|, \frac{\left\|h(x^*) - x^*\right\|}{(1 - \kappa)}\right\}.
\end{aligned} \tag{43}$$

By induction, we can obtain that $\|x^k - x^*\| \leq \max\{\|x^0 - x^*\|, (\|h(x^*) - x^*\|/(1-\kappa))\}$. Thus, the sequences $\{x^k\}$, $\{u^k\}$, and $\{v^k\}$ are all bounded.

Based on (34), we have

$$\begin{aligned}\left\|x^{k+1}-x^*\right\|^2 &= \left\|\mu_k\left(h\left(x^k\right)-x^*\right)+\left(1-\mu_k\right)\left(t^k-x^*\right)\right\|^2\\ &\leq \left(1-\mu_k\right)^2\left\|t^k-x^*\right\|^2+2\mu_k\langle h\left(x^k\right)-x^*, x^{k+1}-x^*\rangle\\ &\leq \left(1-\mu_k\right)^2\left\|t^k-x^*\right\|^2+2\mu_k\kappa\left\|x^k-x^*\right\|\left\|x^{k+1}-x^*\right\|+2\mu_k\langle h\left(x^*\right)-x^*, x^{k+1}-x^*\rangle\\ &\leq \left(1-\mu_k\right)^2\left\|t^k-x^*\right\|^2+\mu_k\kappa\left\|x^k-x^*\right\|^2+\mu_k\kappa\left\|x^{k+1}-x^*\right\|^2\\ &\quad+2\mu_k\langle h\left(x^*\right)-x^*, x^{k+1}-x^*\rangle.\end{aligned} \tag{44}$$

It follows that

$$\begin{aligned}\left\|x^{k+1}-x^*\right\|^2 &\leq \frac{\left(1-\mu_k\right)^2}{1-\kappa\mu_k}\left\|t^k-x^*\right\|^2+\frac{\kappa\mu_k}{1-\mu_k\kappa}\left\|x^k-x^*\right\|^2+\frac{2\mu_k}{1-\kappa\mu_k}\langle h\left(x^*\right)-x^*, x^{k+1}-x^*\rangle\\ &\leq \frac{\left(1-\mu_k\right)^2}{1-\kappa\mu_k}\left[\left\|x^k-x^*\right\|^2-\tau_k\left(2-\tau_k\right)\left(\iota_k\left\|v^k\right\|\right)^2-\gamma\left(1-\gamma\|A\|^2\right)\left\|q^k-Au^k\right\|^2\right.\\ &\quad\left.-\delta_k\left(\eta_k-\delta_k\right)\left\|S\left[\left(1-\eta_k\right)x^k+\eta_k Sx^k\right]-x^k\right\|^2-\gamma\left\|w^k-Au^k\right\|^2\right]\\ &\quad+\frac{\kappa\mu_k}{1-\mu_k\kappa}\left\|x^k-x^*\right\|^2+\frac{2\mu_k}{1-\kappa\mu_k}\langle h\left(x^*\right)-x^*, x^{k+1}-x^*\rangle\\ &=\left(1-\frac{2(1-\kappa)-\mu_k}{1-\kappa\mu_k}\mu_k\right)\left\|x^k-x^*\right\|^2+\frac{\left(1-\mu_k\right)^2\mu_k}{1-\kappa\mu_k}\left\{-\tau_k\left(2-\tau_k\right)\frac{\left(\iota_k\left\|v^k\right\|\right)^2}{\mu_k}\right.\\ &\quad-\delta_k\left(\eta_k-\delta_k\right)\frac{\left\|S\left[\left(1-\eta_k\right)x^k+\eta_k Sx^k\right]-x^k\right\|^2}{\mu_k}-\gamma\left(1-\gamma\|A\|^2\right)\frac{\left\|q^k-Au^k\right\|^2}{\mu_k}\\ &\quad\left.-\frac{\gamma\left\|w^k-Au^k\right\|^2}{\mu_k}+\frac{2}{\left(1-\mu_k\right)^2}\langle h\left(x^*\right)-x^*, x^{k+1}-x^*\rangle\right\}.\end{aligned} \tag{45}$$

Set $a_k = \|x^k - x^*\|^2$, $b_k = (2(1-\kappa) - \mu_k/1 - \kappa\mu_k)\mu_k$, and

$$\begin{aligned}c_k &= \frac{\left(1-\mu_k\right)^2}{2(1-\kappa)-\mu_k}\left\{-\tau_k\left(2-\tau_k\right)\frac{\left(\iota_k\left\|v^k\right\|\right)^2}{\mu_k}-\gamma\left(1-\gamma\|A\|^2\right)\frac{\left\|q^k-Au^k\right\|^2}{\mu_k}\right.\\ &\quad-\frac{\gamma\left\|w^k-Au^k\right\|^2}{\mu_k}+\frac{2}{\left(1-\mu_k\right)^2}\langle h\left(x^*\right)-x^*, x^{k+1}-x^*\rangle\\ &\quad\left.-\delta_k\left(\eta_k-\delta_k\right)\frac{\left\|S\left\|\left(1-\eta_k\right)x^k+\eta_k Sx^k\right\|-x^k\right\|^2}{\mu_k}\right\}.\end{aligned} \tag{46}$$

for all $k \geq 0$.

Since $\mu_k \longrightarrow 0$ as $k \longrightarrow +\infty$, without loss of generality, we assume that $\mu_k \leq 1-\kappa$ for all $k \geq 0$. From (46), we have

$$
\begin{aligned}
c_k &\leq \frac{2}{2(1-\kappa)-\mu_k}\langle h(x^*)-x^*, x^{k+1}-x^*\rangle \\
&\leq \frac{2}{1-\kappa}\|h(x^*)-x^*\|\|x^{k+1}-x^*\|.
\end{aligned} \tag{47}
$$

So, $\limsup_{k\longrightarrow+\infty} c_k < +\infty$. Next, we show that $\limsup_{k\longrightarrow+\infty} c_k \geq -1$. Assume that $\limsup_{k\longrightarrow+\infty} c_k < -1$. Then, there exists a positive integer number K_0 such that $c_k < -1$ when $k \geq K_0$. We can rewrite (45) as $a_{k+1} \leq (1-b_k)a_k + b_k c_k$. Thus, for all $k \geq K_0$, from (45), we deduce

$$
a_{k+1} \leq (1-b_k)a_k + b_k c_k \leq a_k - b_k, \tag{48}
$$

which leads to $a_{k+1} \leq a_{K_0} - \sum_{i=K_0}^{k} b_k$. Therefore,

$$
\limsup_{k\longrightarrow+\infty} a_{k+1} \leq a_{K_0} - \limsup_{k\longrightarrow+\infty}\sum_{i=K_0}^{k} b_k. \tag{49}
$$

Note that $b_k = (2(1-\kappa)-\mu_k/1-\kappa\mu_k)\mu_k \geq (1-\kappa)\mu_k$. This together with the last inequality implies that $\limsup_{k\longrightarrow+\infty} a_{k+1} \leq -\infty$. It is impossible. Hence, $-1 \leq \limsup_{k\longrightarrow+\infty} c_k < +\infty$. As a result, we can select a subsequence $\{k_i\}$ of $\{k\}$ such that $x^{k_i} \rightharpoonup p^\dagger$ and

$$
\begin{aligned}
\limsup_{k\longrightarrow+\infty} c_k &= \lim_{i\longrightarrow+\infty} c_{k_i} \\
&= \lim_{i\longrightarrow+\infty}\frac{(1-\mu_{k_i})^2}{2(1-\kappa)-\mu_{k_i}}\left\{-\tau_{k_i}(2-\tau_{k_i})\frac{(\iota_{k_i}\|\nu^{k_i}\|)^2}{\mu_{k_i}} - \frac{\gamma\|w^{k_i}-Au^{k_i}\|^2}{\mu_{k_i}}\right. \\
&\quad - \gamma(1-\gamma\|A\|^2)\frac{\|q^{k_i}-Au^{k_i}\|^2}{\mu_{k_i}} + \frac{2}{(1-\mu_{k_i})^2}\langle h(x^*)-x^*, x^{k_i+1}-x^*\rangle \\
&\quad \left. -\delta_{k_i}(\eta_{k_i}-\delta_{k_i})\frac{\|S[(1-\eta_{k_i})x^{k_i}+\eta_{k_i}Sx^{k_i}]-x^{k_i}\|^2}{\mu_{k_i}}\right\}.
\end{aligned} \tag{50}
$$

Since the sequence $\{x^{k_i+1}\}$ is bounded, without loss of generality, we assume that $\lim_{i\longrightarrow+\infty}\langle h(x^*)-x^*, x^{k_i+1}-x^*\rangle$ exists. Consequently, from (50), we obtain

$$
\begin{aligned}
&\lim_{i\longrightarrow+\infty}\left\{\tau_{k_i}(2-\tau_{k_i})\frac{(\iota_{k_i}\|\nu^{k_i}\|)^2}{\mu_{k_i}} + \delta_{k_i}(\eta_{k_i}-\delta_{k_i})\frac{\|S[(1-\eta_{k_i})x^{k_i}+\eta_{k_i}Sx^{k_i}]-x^{k_i}\|^2}{\mu_{k_i}}\right. \\
&\quad \left. + \gamma(1-\gamma\|A\|^2)\frac{\|q^{k_i}-Au^{k_i}\|^2}{\mu_{k_i}} + \frac{\gamma\|w^{k_i}-Au^{k_i}\|^2}{\mu_{k_i}}\right\},
\end{aligned} \tag{51}
$$

exists.

By the assumptions, we have $\liminf_{i\longrightarrow+\infty}\tau_{k_i}(2-\tau_{k_i}) > 0$ and $\liminf_{i\longrightarrow+\infty}\delta_{k_i}(\eta_{k_i}-\delta_{k_i}) > 0$, Therefore, we deduce

$$
\lim_{i\longrightarrow+\infty}\iota_{k_i}\|\nu^{k_i}\| = 0, \tag{52}
$$

$$
\lim_{i\longrightarrow+\infty}\|S[(1-\eta_{k_i})x^{k_i}+\eta_{k_i}Sx^{k_i}]-x^{k_i}\| = 0, \tag{53}
$$

$$
\lim_{i\longrightarrow+\infty}\|q^{k_i}-Au^{k_i}\| = 0, \tag{54}
$$

$$
\lim_{i\longrightarrow+\infty}\|w^{k_i}-Au^{k_i}\| = 0. \tag{55}
$$

By (54) and (55), we get

$$
\lim_{i\longrightarrow+\infty}\|q^{k_i}-w^{k_i}\| = 0. \tag{56}
$$

In addition, from (31), we have

$$\left\|u^k - v^k\right\| = \left\|P_C\left(v^k - \tau_k \iota_k v^k\right) - P_C\left(v^k\right)\right\| \leq \tau_k \iota_k \left\|v^k\right\|. \tag{57}$$

So, we get from (52) that

$$\lim_{i \longrightarrow +\infty} \left\|u^{k_i} - v^{k_i}\right\| = 0. \tag{58}$$

Observe that

$$\begin{aligned}\left\|Sx^k - x^k\right\| &\leq \left\|Sx^k - S\left[(1-\eta_k)x^k + \eta_k Sx^k\right]\right\| \\ &\quad + \left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\| \\ &\leq L\eta_k \left\|Sx^k - x^k\right\| + \left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|.\end{aligned} \tag{59}$$

It follows that

$$\left\|Sx^k - x^k\right\| \leq \frac{1}{1 - L\eta_k} \left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|. \tag{60}$$

This together with (53) implies that

$$\lim_{i \longrightarrow +\infty} \left\|Sx^{k_i} - x^{k_i}\right\| = 0. \tag{61}$$

In addition, by (27) and (53), we have

$$\left\|v^{k_i} - x^{k_i}\right\| \leq \delta_{k_i} \left\|S\left[\left(1 - \eta_{k_i}\right)x^{k_i} + \eta_{k_i} Sx^{k_i}\right] - x^{k_i}\right\| \longrightarrow 0. \tag{62}$$

Since $\{v^k\}$ is bounded, by Lemma 4, $\{y^k\}$ is bounded. Consequently, the sequence $\{z^k\}$ is bounded. Applying Lemma 5, we deduce that $\{v^k\}$ is bounded. According to (52), we derive

$$\lim_{i \longrightarrow +\infty} f\left(z^{k_i}, v^{k_i}\right) = \lim_{i \longrightarrow +\infty} \left(\iota_{k_i} \left\|v^{k_i}\right\|\right) \left\|v^{k_i}\right\| = 0. \tag{63}$$

Since $f(z^{k_i}, \cdot)$ is convex, we have

$$\begin{aligned}0 = f\left(z^{k_i}, z^{k_i}\right) &= f\left(z^{k_i}, \left(1 - \vartheta_{k_i}\right)v^{k_i} + \vartheta_{k_i} y^{k_i}\right) \\ &\leq \left(1 - \vartheta_{k_i}\right) f\left(z^{k_i}, v^{k_i}\right) + \vartheta_{k_i} f\left(z^{k_i}, y^{k_i}\right).\end{aligned} \tag{64}$$

So, we get from (29) that

$$\begin{aligned}f\left(z^{k_i}, v^{k_i}\right) &\geq \vartheta_{k_i}\left[f\left(z^{k_i}, v^{k_i}\right) - f\left(z^{k_i}, y^{k_i}\right)\right] \\ &\geq \frac{\alpha}{2\beta_{k_i}} \vartheta_{k_i} \left\|v^{k_i} - y^{k_i}\right\|^2.\end{aligned} \tag{65}$$

Combining the above inequality with (63), we have

$$\lim_{i \longrightarrow +\infty} \vartheta_{k_i} \left\|v^{k_i} - y^{k_i}\right\|^2 = 0. \tag{66}$$

Note that $x^{k_i} \rightharpoonup p^\dagger \in C$. Then, it follows from (55), (58), and (62) that $u^{k_i} \rightharpoonup p^\dagger$, $v^{k_i} \rightharpoonup p^\dagger$, $Au^{k_i} \rightharpoonup Ap^\dagger$, $Av^{k_i} \rightharpoonup Ap^\dagger$, and $w^{k_i} \rightharpoonup Ap^\dagger \in Q$.

There are two possible cases. □

Case 1. $\limsup_{k \longrightarrow +\infty} \vartheta_{k_i} > 0$. Then, there exist $\overline{\vartheta} > 0$ and a subsequence of $\{\vartheta_{k_i}\}$, still denoted by $\{\vartheta_{k_i}\}$ such that for some $I_0 > 0$, $\vartheta_{k_i} > \overline{\vartheta}$ for all $i \geq I_0$. Consequently, by (66), we deduce

$$\lim_{i \longrightarrow +\infty} \left\|v^{k_i} - y^{k_i}\right\| = 0. \tag{67}$$

Noting that $v^{k_i} \rightharpoonup p^\dagger$, thus $y^{k_i} \rightharpoonup p^\dagger$. According to (28), we obtain

$$0 \in \partial_2 f\left(v^{k_i}, y^{k_i}\right) + \frac{1}{\beta_{k_i}}\left(y^{k_i} - v^{k_i}\right) + N_C\left(y^{k_i}\right), \tag{68}$$

so, there exists $\widehat{v}^{k_i} \in \partial_2 f(v^{k_i}, y^{k_i})$ such that

$$\left\langle \widehat{v}^{k_i}, y - y^{k_i}\right\rangle + \frac{1}{\beta_{k_i}}\left\langle y^{k_i} - v^{k_i}, y - y^{k_i}\right\rangle \geq 0, \quad \forall y \in C. \tag{69}$$

By the subdifferential inequality, we have

$$f\left(v^{k_i}, y\right) - f\left(v^{k_i}, y^{k_i}\right) \geq \left\langle \widehat{v}^{k_i}, y - y^{k_i}\right\rangle, \quad \forall y \in C. \tag{70}$$

Therefore,

$$f\left(v^{k_i}, y\right) - f\left(v^{k_i}, y^{k_i}\right) + \frac{1}{\beta_{k_i}}\left\langle y^{k_i} - v^{k_i}, y - y^{k_i}\right\rangle \geq 0, \quad \forall y \in C. \tag{71}$$

Since

$$\left\langle y^{k_i} - v^{k_i}, y - y^{k_i}\right\rangle \leq \left\|y^{k_i} - v^{k_i}\right\|\left\|y - y^{k_i}\right\|, \tag{72}$$

from (71), we get

$$f\left(v^{k_i}, y\right) - f\left(v^{k_i}, y^{k_i}\right) + \frac{1}{\beta_{k_i}}\left\|y^{k_i} - v^{k_i}\right\|\left\|y - y^{k_i}\right\| \geq 0. \tag{73}$$

Letting $i \longrightarrow +\infty$ in (73), from (A1), (A3), and (67), we obtain

$$f\left(p^\dagger, y\right) \geq f\left(p^\dagger, p^\dagger\right) = 0, \quad \forall y \in C, \tag{74}$$

hence $p^\dagger \in \mathrm{SEP}(C, f)$.

Case 2. $\lim_{i \longrightarrow +\infty} \vartheta_{k_i} = 0$. Since the sequence $\{y^{k_i}\}$ is bounded, without loss of generality, we may assume that $y^{k_i} \rightharpoonup \overline{y}$ as $i \longrightarrow +\infty$. Replacing y by v^{k_i} in (71), we get

$$f\left(v^{k_i}, y^{k_i}\right) \leq -\frac{1}{\beta_{k_i}}\left\|y^{k_i} - v^{k_i}\right\|^2. \tag{75}$$

According to (29), for $m_{k_i} - 1$, we have

$$f\left(z^{k_i, m_{k_i}-1}, v^{k_i}\right) - f\left(z^{k_i, m_{k_i}-1}, y^{k_i}\right) < \frac{\alpha}{2\beta_{k_i}}\left\|y^{k_i} - v^{k_i}\right\|^2. \tag{76}$$

From (75) and (76), we obtain

$$f\left(v^{k_i}, y^{k_i}\right) \leq \frac{2}{\alpha}\left[f\left(z^{k_i, m_{k_i}-1}, y^{k_i}\right) - f\left(z^{k_i, m_{k_i}-1}, v^{k_i}\right)\right]. \tag{77}$$

Letting $i \longrightarrow +\infty$ in (77) and noting that $v^{k_i} \rightharpoonup p^\dagger$, $y^{k_i} \rightharpoonup \overline{y}$ and $z^{k_i, m_{k_i}-1} \rightharpoonup p^\dagger$ as $i \longrightarrow +\infty$, we obtain

$$f\left(p^\dagger, \overline{y}\right) \leq \frac{2}{\alpha} f\left(p^\dagger, \overline{y}\right). \tag{78}$$

Therefore, $f(p^{\dagger}, \overline{y}) = 0$ and $\lim_{i\longrightarrow+\infty}\|y^{k_i} - v^{k_i}\| = 0$. Consequently, by the similar argument as that in Case 1, we get $p^{\dagger} \in \mathrm{SEP}(C, f)$.

At the same time, from (61), $x^{k_i} \rightharpoonup p^{\dagger}$ and Lemma 2, we deduce that $p^{\dagger} \in \mathrm{Fix}(S)$. Therefore, $p^{\dagger} \in \mathrm{Fix}(S) \cap \mathrm{SEP}(C, f)$.

Next, we show that $p^{\dagger} \in \mathrm{Fix}(T) \cap \mathrm{SEP}(Q, g)$. First, by (39), we have

$$\begin{aligned}(\lambda_k - \zeta_k)\zeta_k\left\|T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right] - w^k\right\|^2 &\le \left\|w^k - Ax^*\right\|^2 - \left\|q^k - Ax^*\right\|^2 \\ &\le \left\|w^k - q^k\right\|\left[\left\|w^k - Ax^*\right\| + \left\|q^k - Ax^*\right\|\right].\end{aligned} \tag{79}$$

Since $\liminf_{k\longrightarrow+\infty}(\lambda_k - \zeta_k)\zeta_k > 0$ and $\{w^k\}$ and $\{q^k\}$ are bounded, from (56) and (79), we deduce that

$$\lim_{k\longrightarrow+\infty}\left\|T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right] - w^k\right\| = 0. \tag{80}$$

Observe that

$$\begin{aligned}\left\|Tw^k - w^k\right\| &\le \left\|Tw^k - T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right]\right\| \\ &\quad + \left\|T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right] - w^k\right\| \\ &\le L_2\lambda_k\left\|Tw^k - w^k\right\| + \left\|T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right] - w^k\right\|.\end{aligned} \tag{81}$$

It follows that

$$\left\|Tw^k - w^k\right\| \le \frac{1}{1 - L_2\lambda_k}\left\|T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right] - w^k\right\|. \tag{82}$$

This together with (80) implies that $\lim_{k\longrightarrow+\infty}\|Tw^k - w^k\| = 0$. Combining this with $w^{k_i} \rightharpoonup Ap^{\dagger}$ and the fact that $I - T$ is demiclosed at zero (Lemma 2), it is immediate that $Ap^{\dagger} \in \mathrm{Fix}(T)$.

By Lemma 3, we have

$$\left\|J^g_{\varsigma_k}\left(Au^k\right) - J^g_{\varsigma}\left(Au^k\right)\right\| \le \frac{\varsigma_k - \varsigma}{\varsigma_k}\left\|J^g_{\varsigma_k}\left(Au^k\right) - Au^k\right\|. \tag{83}$$

Hence,

$$\begin{aligned}\left\|J^g_{\varsigma}\left(Au^k\right) - Au^k\right\| &\le \left\|J^g_{\varsigma_k}\left(Au^k\right) - Au^k\right\| + \left\|J^g_{\varsigma_k}\left(Au^k\right) - J^g_{\varsigma}\left(Au^k\right)\right\| \\ &\le 2\left\|J^g_{\varsigma_k}\left(Au^k\right) - Au^k\right\|.\end{aligned} \tag{84}$$

It follows from (55) that $\lim_{k\longrightarrow\infty}\|J^g_{\varsigma}Au^k - Au^k\| = 0$. Since J^g_{ς} is nonexpansive and $Au^{k_i} \rightharpoonup Ap^{\dagger}$, we deduce that $Ap^{\dagger} \in \mathrm{Fix}(J^g_{\varsigma}) = \mathrm{SEP}(Q, g)$ by Lemma 3. So, $p^{\dagger} \in \Gamma$ and $\omega_w(x^k) \subset \Gamma$.

Replacing $x^* = P_C h(q^{\dagger})$ in (45), we have

$$\begin{aligned}\left\|x^{k+1} - P_C h\left(q^{\dagger}\right)\right\|^2 &\le \left(1 - \frac{2(1-\kappa) - \mu_k}{1 - \kappa\mu_k}\mu_k\right)\left\|x^k - P_C h\left(q^{\dagger}\right)\right\|^2 \\ &\quad + \frac{2(1-\kappa) - \mu_k}{1 - \kappa\mu_k}\mu_k \times \frac{2}{2(1-\kappa) - \mu_k} \\ &\quad \cdot \left\langle h\left(P_C h\left(q^{\dagger}\right)\right) - P_C h\left(q^{\dagger}\right), x^{k+1} - P_C h\left(q^{\dagger}\right)\right\rangle.\end{aligned} \tag{85}$$

Noting that $\limsup_{k\longrightarrow+\infty}\langle h(P_C h(q^{\dagger})) - P_C h(q^{\dagger}), x^{k+1} - P_C h(q^{\dagger})\rangle \le 0$, applying Lemma 6 to the last inequality, we deduce that $x^k \longrightarrow P_C h(q^{\dagger})$. This completes the proof. □

Next, we can apply Algorithm 1 and Theorem 1 for solving the split equilibrium problem (3). Setting $S = I$ and $T = I$ in Algorithm 1, we deduce that $v^k = x^k$ and $q^k = w^k$. Consequently, we have the following algorithm and corollary.

Algorithm 2. Fix an initial point $x^0 \in C$. Set $k = 0$.

Step 1: assume that x^k is known and compute

$$y^k = \arg\min_{y^{\dagger}\in C}\left\{f\left(x^k, y^{\dagger}\right) + \frac{1}{2\beta_k}\left\|x^k - y^{\dagger}\right\|^2\right\}. \tag{86}$$

If $y^k = x^k$, then set $u^k = x^k$ and go to Step 4. Otherwise, go to Step 2.

Step 2: let $m_k = \min\{1, 2, \ldots, k, \ldots\}$ such that

$$f\left(z^{k,m_k}, x^k\right) - f\left(z^{k,m_k}, y^k\right) \ge \frac{\alpha}{2\beta_k}\left\|x^k - y^k\right\|^2, \tag{87}$$

where

$$z^{k,m_k} = \left(1 - \vartheta^{m_k}\right)x^k + \vartheta^{m_k}y^k. \tag{88}$$

Write $\vartheta_k = \vartheta^{m_k}$ and $z^k = z^{k,m_k}$.

Step 3: compute

$$u^k = P_C\left(v^k - \tau_k\iota_k v^k\right), \tag{89}$$

where $v^k \in \partial_2 f(z^k, x^k)$ and $\iota_k = (f(z^k, x^k)/\|v^k\|^2)$.

Step 4: for any $v \in Q$, find w^k such that

$$g\left(w^k, v\right) + \frac{1}{\varsigma_k}\langle v - w^k, w^k - Au^k\rangle \ge 0. \tag{90}$$

Step 5: compute

$$x^{k+1} = \mu_k h\left(x^k\right) + (1 - \mu_k)P_C\left[u^k + \gamma A^*\left(w^k - Au^k\right)\right]. \tag{91}$$

Step 6: set $k := k + 1$ and return to Step 1.

Corollary 1. *Assume that $\Gamma_1 \ne \varnothing$. Then, the sequence $\{x^k\}$ generated by (91) strongly converges to a solution $q_1 = P_{\Gamma_1}h(q_1)$.*

Next, we can apply Algorithm 1 and Theorem 1 for solving the split fixed point problem (4). Setting $f = 0$ and $g = 0$ in Algorithm 1, we deduce that $y^k = x^k$ and $w^k = v^k$. Consequently, we have the following algorithm and corollary.

Algorithm 3. Fix an initial point $x^0 \in C$. Define the sequence $\{x^k\}$ iteratively by

$$\begin{cases} v^k = (1-\delta_k)x^k + \delta_k S\left[(1-\eta_k)x^k + \eta_k S x^k\right], \\ q^k = (1-\zeta_k)v^k + \zeta_k T\left[(1-\lambda_k)v^k + \lambda_k T v^k\right], \\ x^{k+1} = \mu_k h\left(x^k\right) + (1-\mu_k) P_C\left[x^k + \gamma A^*\left(q^k - Ax^k\right)\right], \quad k \geq 0. \end{cases} \tag{92}$$

Corollary 2. *Assume that $\Gamma_2 \neq \varnothing$. Then, the sequence $\{x^k\}$ generated by (92) strongly converges to a solution $q_2 = P_{\Gamma_2} h(q_2)$.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgments

Li-Jun Zhu was supported by the National Natural Science Foundation of China (grant no. 11861003) and the Natural Science Foundation of Ningxia Province (grant nos. NZ17015 and NXYLXK2017B09). Ching-Feng Wen was supported by the grant of MOST 109-2115-M-037-001.

References

[1] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, pp. 123–145, 1994.

[2] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, pp. 117–136, 2005.

[3] A. N. Iusem and V. Mohebbi, "Extragradient methods for nonsmooth equilibrium problems in Banach spaces," *Optimization*, vol. 69, no. 11, pp. 2383–2403, 2020.

[4] T. D. Quoc, L. D. Muu, and N. V. Hien, "Extragradient algorithms extended to equilibrium problems," *Optimization*, vol. 57, pp. 749–776, 2008.

[5] P. T. Vuong, J. J. Strodiot, and V. H. Nguyen, "On extragradient-viscosity methods for solving equilibrium and fixed point problems in a hilbert space," *Optimization*, vol. 64, no. 2, pp. 429–451, 2015.

[6] L.-C. Ceng, A. Petrusel, J.-C. Yao, and Y. Yao, "Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in banach spaces," *Fixed Point Theory*, vol. 19, no. 2, pp. 487–502, 2018.

[7] L.-C. Ceng, A. Petrusel, J.-C. Yao, and Y. Yao, "Systems of variational inequalities with hierarchical variational inequality constraints for lipschitzian pseudocontractions," *Fixed Point Theory*, vol. 20, no. 1, pp. 113–133, 2019.

[8] Y. Yao, M. Postolache, and J.-C. Yao, "Iterative algorithms for generalized variational inequalities," *UPB Scientific Bulletin, Series A*, vol. 81, no. 2, pp. 3–16, 2019.

[9] Y. Yao, Y.-C. Liou, and J.-C. Yao, "Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations," *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 2, pp. 843–854, 2017.

[10] Q.-L. Dong, A. Gibali, and D. Jiang, "A modified subgradient extragradient method for solving the variational inequality problem," *Numerical Algorithms*, vol. 79, no. 3, pp. 927–940, 2018.

[11] Y. Shehu, X.-H. Li, and Q.-L. Dong, "An efficient projection-type method for monotone variational inequalities in Hilbert spaces," *Numerical Algorithms*, vol. 84, no. 1, pp. 365–388, 2020.

[12] Y. Yao, M. Postolache, Y.-C. Liou, and Z. Yao, "Construction algorithms for a class of monotone variational inequalities," *Optimization Letters*, vol. 10, no. 7, pp. 1519–1528, 2016.

[13] X. P. Zhao, J.-C. Yao, and Y. Yao, "A proximal algorithm for solving split monotone variational inclusions," *UPB Scientific Bulletin, Series A*, vol. 82, pp. 43–52, 2020.

[14] X. P. Zhao and Y. Yao, "Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems," *Optimization*, vol. 69, pp. 1987–2002, 2020.

[15] A. Gibali and Y. Shehu, "An efficient iterative method for finding common fixed point and variational inequalities in Hilbert spaces," *Optimization*, vol. 68, no. 1, pp. 13–32, 2019.

[16] Q.-L. Dong, Y.-Y. Lu, and J. Yang, "The extragradient algorithm with inertial effects for solving the variational inequality," *Optimization*, vol. 65, no. 12, pp. 2217–2226, 2016.

[17] H. Zegeye, N. Shahzad, and Y. Yao, "Minimum-norm solution of variational inequality and fixed point problem in banach spaces," *Optimization*, vol. 64, no. 2, pp. 453–471, 2015.

[18] S. Y. Cho, X. Qin, J.-C. Yao, and Y. Yao, "Viscosity approximation splitting methods for monotone and nonexpansive operators in hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 19, pp. 251–264, 2018.

[19] Y. Yao, Y.-C. Liou, and M. Postolache, "Self-adaptive algorithms for the split problem of the demicontractive operators," *Optimization*, vol. 67, no. 9, pp. 1309–1319, 2018.

[20] W. Takahashi, N. C. Wong, and J.-C. Yao, "Iterative common solutions for monotone inclusion problems, fixed point problems and equilibrium problems," *Fixed Point Theory and Applications*, vol. 2012181 pages, 2012.

[21] W. Takahashi and J.-C. Yao, "Strong convergence theorems by hybrid methods for two noncommutative nonlinear mappings in Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 41, no. 10, pp. 1149–1171, 2020.

[22] Y. Yao, X. Qin, and J.-C. Yao, "Projection methods for firmly type nonexpansive operators," *Journal of Nonlinear and Convex Analysis*, vol. 19, pp. 407–415, 2018.

[23] S. Y. Cho, "A monotone Bregan projection algorithm for fixed point and equilibrium problems in a reflexive Banach space," *Filomat*, vol. 34, pp. 1487–1497, 2020.

[24] Y. Censor, T. Elfving, N. Kopf, and T. Bortfeld, “The multiple-sets split feasibility problem and its applications for inverse problems,” *Inverse Problems*, vol. 21, no. 6, pp. 2071–2084, 2005.

[25] S.-s. Chang, J.-C. Yao, C.-F. Wen, and L.-c. Zhao, “On the split equality fixed point problem of quasi-pseudo-contractive mappings without a priori knowledge of operator norms with applications,” *Journal of Optimization Theory and Applications*, vol. 185, no. 2, pp. 343–360, 2020.

[26] W. Takahashi, C.-F. Wen, and J.-C. Yao, “Iterative methods for the split common fixed point problem with families of demimetric mappings in banach spaces,” *Journal of Nonlinear and Convex Analysis*, vol. 19, no. 1, pp. 1–18, 2018.

[27] J. Zhao and Y. S. Liang, “On solving general split equality variational inclusion problems in Banach space,” *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 07, pp. 3619–3629, 2017.

[28] J. Zhao, Y. Liang, and Z. Liu, “Strong convergent iterative techniques for 2-generalized hybrid mappings and split equilibrium problems,” *Filomat*, vol. 33, no. 18, pp. 5851–5862, 2019.

[29] J. K. Kim and P. Majee, “Modified krasnoselski-mann iterative method for hierarchical fixed point problem and split mixed equilibrium problem,” *Journal of Inequalities and Applications*, vol. 2020227 pages, 2020.

[30] O. K. Oyewole, O. T. Mewomo, L. O. Jolaoso, and S. H. Khan, “An extragradient algorithm for split generalized equilibrium problem and the set of fixed points of quasi-φ-nonexpansive mappings in Banach spaces,” *Turkish Journal of Mathematics*, vol. 44, no. 4, pp. 1146–1170, 2020.

[31] Y. Censor and A. Segal, “The split common fixed point problem for directed operators,” *Journal of Convex Analysis*, vol. 16, pp. 587–600, 2009.

[32] Z. He and W. S. Du, “Nonlinear algorithms approach to split common solution problems,” *Fixed Point Theory and Applications*, vol. 2012130 pages, 2012.

[33] A. Moudafi, “The split common fixed point problem for demicontractive mappings,” *Inverse Problems*, vol. 26, Article ID 055007, 2010.

[34] Y. Yao, L. Leng, M. Postolache, and X. Zheng, “Mann-type iteration method for solving the split common fixed point problem,” *Journal of Nonlinear and Convex Analysis*, vol. 18, pp. 875–882, 2017.

[35] M. Alansari, K. R. Kazmi, and R. Ali, “Hybrid iterative scheme for solving split equilibrium and hierarchical fixed point problems,” *Optimization Letters*, vol. 14, no. 8, pp. 2379–2394, 2020.

[36] S. A. Harisa, M. A. A. Khan, and F. Mumtaz, “Shrinking cesaro means method for the split equilibrium and fixed point problems in hilbert spaces,” *Advances in Difference Equations*, vol. 2020345 pages, 2020.

[37] W. Inthakon and N. Niyamosot, “The split equilibrium problem and common fixed points of two realtively quasi-nonexpansive mappings in banach spaces,” *Journal of nonlinear and convex analysis*, vol. 20, pp. 685–702, 2019.

[38] L. O. Jolaoso and I. Karahan, “A general alternative regularization method with line search technique for solving split equilibrium and fixed point problems in hilbert spaces,” *Journal of Computational and Applied Mathematics*, vol. 39150 pages, 2020.

[39] N. Petrot, M. Rabbani, M. Khonchaliew, and V. Dadashi, “A new extragradient algorithm for split equilibrium problems and fixed point problems,” *Journal of Inequalities and Applications*, vol. 2019137 pages, 2019.

[40] Y. Yao, H. Li, and M. Postolache, “Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions,” *Optimization*1 page, 2020.

[41] Y. Yao, Y. C. Liou, and J.-C. Yao, “Split common fixed point problem for two quasi-pseudocontractive operators and its algorithm construction,” *Fixed Point Theory and Applications*, vol. 2015127 pages, 2015.

[42] H. Zhou, “Strong convergence of an explicit iterative algorithm for continuous pseudo-contractions in banach spaces,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 11, pp. 4039–4046, 2009.

[43] H.-K. Xu, “Iterative algorithms for nonlinear operators,” *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6617738, 7 pages
https://doi.org/10.1155/2021/6617738

Research Article

Coupled Fixed-Point Theorems in Theta-Cone-Metric Spaces

Sahar Mohamed Ali Abou Bakr

Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt

Correspondence should be addressed to Sahar Mohamed Ali Abou Bakr; saharm_ali@yahoo.com

Received 5 December 2020; Revised 6 January 2021; Accepted 7 January 2021; Published 28 January 2021

Academic Editor: Xiaolong Qin

This paper gives further generalizations of some well-known coupled fixed-point theorems. Specifically, Theorem 3 of the paper is the generalization of the Baskar–Lackshmikantham coupled fixed-point theorem, and Theorem 5 is the generalization of the Sahar Mohamed Ali Abou Bakr fixed-point theorem, where the underlying space is complete θ-cone-metric space.

1. Introduction and Preliminaries

Since 1922, the pioneering fixed-point principle of Banach [1] showed exclusive interest of researchers because it has many applications, including variational linear inequalities and optimization, and applications in differential equations, in the field of approximation theory, and in minimum norm problems.

Since then, several types of contraction mappings have been introduced and many research papers have been written to generalize this Banach contraction principle.

In 1987, Guo and Lakshmikantham [2] introduced one of the most interesting concepts of coupled fixed point.

Definition 1. An element $(x, y) \in E \times E$ is said to be a coupled fixed point of the mapping $T: E \times E \longrightarrow E$ if and only if $T(x, y) = x$ and $T(y, x) = y$.

In 2006, Bhaskar and Lakshmikantham [3] introduced the concept of the mixed monotone property as follows.

Definition 2. Let $(E, \leq)$ be a partially ordered set and T be a mapping from $E \times E$ to E. Then,

(1) T is said to be monotone nondecreasing in x if and only if, for any $y \in E$,

$$\text{if } x_1, x_2 \in E \text{ and } x_1 \leq x_2, \text{ then } T(x_1, y) \leq T(x_2, y), \tag{1}$$

(2) T is said to be monotone nonincreasing in y if and only if, for any $x \in E$,

$$\text{if } y_1, y_2 \in E \text{ and } y_1 \leq y_2, \text{ then } T(x, y_1) \geq T(x, y_2), \tag{2}$$

(3) T is said to have a mixed monotone property if and only if $T(x, y)$ is both monotone nondecreasing in x and monotone nonincreasing in y

Definition 3. An element $(x_0, y_0) \in E \times E$ is said to be a lower-anti-upper coupled point of the mapping $T: E \times E \longrightarrow E$ if and only if

$$x_0 \leq T(x_0, y_0) \text{ and } y_0 \geq T(y_0, x_0). \tag{3}$$

A mapping $T: E \times E \longrightarrow E$ is said to have a lower-upper property if and only if T has at least one lower-anti-upper coupled point.

Definition 4. Let $(E, \leq, \|.\|)$ be a partially ordered normed space. Then,

(1) E is said to be a sequentially lower ordered space if it fulfills the condition: If $\{x_n\}_{n\in N}$ is a nondecreasing

sequence in E such that $\{x_n\}_{n\in N}$ converges strongly to x, then $x_n \leq x$ for all $n \in N$

(2) E is said to be a sequentially upper-ordered space if it fulfills the condition: If $\{y_n\}_{n\in N}$ is a nonincreasing sequence in E such that $\{y_n\}_{n\in N}$ converges strongly to y, then $y_n \geq y$ for all $n \in N$

(3) E is said to be a sequentially lower-upper ordered space if it is both a lower- and upper-ordered space

In 2006, Bhaskar and Lakshmikantham [3] proved the existence of coupled fixed points for mixed monotone mappings with weak contractivity assumption in a partial-ordered Banach space $(E, \|.\|, \leq)$ as follows.

Theorem 1 (see [3]). *Let E be a sequentially both lower- and upper-ordered Banach space and* $T\colon E \times E \longrightarrow E$ *be a mapping with mixed monotone and lower-upper properties. If there is a real number* $0 \leq k < 1$ *such that*

$$\|T(x, y) - T(z, w)\| \leq \frac{k}{2}[\|x - z\| + \|y - w\|],$$
$$\forall x, y, z, w \in E, z \leq x, \text{ and } y \leq w, \tag{4}$$

then T has coupled fixed points in E.

In 2013, Mohamed Ali [4] introduced novel contraction type of mappings and proved the following fixed-point theorem.

Theorem 2 (see [4]). *Let* $(E, \|.\|)$ *be a Banach space and T be a mapping from* $E \times E$ *into E, and we suppose there are three constants* $a, b, c \in [0, 1)$ *and* $a + b + c < 1$ *such that*

$$\|T(x, y) - T(y, z)\| \leq a\|x - y\| + b\|T(x, y) - x\| + c\|T(y, z) - y\|, \quad \forall x, y, z \in E. \tag{5}$$

Then, there is a unique point $x_0 \in E$ such that $T(x_0, x_0) = x_0$.

There are many interesting coupled fixed-point theorems concerning some other type of contraction mappings, see [5–10].

Recently, more advanced approaches for studying coupled fixed points have been presented by the authors in [11–13].

In 2007, Huang and Zhang [14] introduced the concept of cone-metric spaces as follows: First, a subset M of the real Banach space $\mathscr{E}$ is said to be a cone in $\mathscr{E}$ if and only if

(1) M is nonempty closed and $M \neq \{\Theta\}$, where Θ is the zero (neutral) element of $\mathscr{E}$

(2) $\lambda M + \mu M \subset M$ for all nonnegative real numbers λ, μ

(3) $M \cap -M = \{\Theta\}$

If int$M$ is the set of all interior points of M, then a cone M in a normed space $\mathscr{E}$ induces the following ordered relations:

$$u \prec v \Leftrightarrow v - u \in M, \quad u < v \Leftrightarrow (v - u \in M, \text{and}\, u \neq v), \quad u \prec_{\neq} v \Leftrightarrow v - u \in \operatorname{int} M. \tag{6}$$

If E is a nonempty set, the distance $d(x, y)$ between any two elements $x, y \in E$ is defined to be a vector in the cone M, and the space (E, d) is said to be a cone-metric space if and only if d satisfied the three axioms of metric but using the ordered relation $\prec$ induced by M for the triangle inequality instead. They studied the topological characterizations of such a defined space, and then, they applied their concept to have more generalizations of some previous fixed-point theorems for contractive type of mappings.

A mapping $T\colon E \longrightarrow E$ is said to be a contraction if and only if there is a constant $\alpha \in [0, 1)$ such that

$$d(T(x), T(y)) \prec \alpha d(x, y), \quad \forall x, y \in E. \tag{7}$$

In 2019, Mohamed Ali Abou Bakr [15] proved the existence of a unique common fixed point of generalized joint cyclic Banach algebra contractions and Banach algebra Kannan type of mappings on cone quasimetric spaces.

In 2013, Khojasteh et al. [10] introduced the notion of θ-action function, θ: $[0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$, the concept of θ-metric, and then, they studied the topological structures of θ-metric spaces in detail. Their work led to a step-forward generalization of metric spaces.

In 2020, Mohamed Ali Abou Bakr [16] replaced $[0, \infty)$ by a cone M in a normed space and used the ordered relation induced by this cone to introduce the following analogous generalization of θ-action function.

Definition 5. Let $(\mathscr{E}, \prec)$ be an ordered normed space, where $\prec$ is an ordered relation induced by some cone $M \subset \mathscr{E}$ and θ: $M \times M \longrightarrow M$ be a continuous mapping with respect to each variable, and we denote

$$\operatorname{Im}(\theta) = \{t\colon t \in M \text{ such that } \exists u_0, v_0 \in \mathrm{E}, \quad \theta(u_0, v_0) = t\}. \tag{8}$$

Then, θ is said to be an ordered action mapping on $\mathscr{E}$ if and only if it satisfies the following conditions:

(1) $\theta(\Theta, \Theta) = \Theta$ and $\theta(u, v) = \theta(v, u)$ for every $u, v \in M$

(2)

$$\theta(u, v) < \theta(w, t) \text{ if either } \begin{cases} u < w \text{ and } v \prec t; \\ \text{or} \\ u \prec w \text{ and } v < t \end{cases} \tag{9}$$

(3) For every $u \in \operatorname{Im}(\theta)$ and every $\Theta \prec v \prec u$, there is $\Theta \prec w \prec u$ such that $\theta(v, w) = u$

(4) $\theta(u, \Theta) \prec u$ for every $u \in (M/\{\Theta\})$

Because $x - \Theta \in M$ for every $x \in M$, one can write instead $\Theta \prec x$ for every $x \in M$, ($\Theta < x$ for every $x \in (M/\{\Theta\})$).

In addition, Mohamed Ali Abou Bakr [16] gave further replacement, replaced the set of nonnegative real numbers $\mathbb{R}^+$ by a cone M in a normed space, and used θ-ordered actions to introduce the concept of θ-cone-metric space as follows.

Definition 6 (see [16]). Let $(\mathscr{E}, \prec)$ be an ordered normed space, where $\prec$ is the ordered relation induced by some cone $M \subset \mathscr{E}$, and θ be an ordered action on $\mathscr{E}$. If E is a nonempty set, then the function $d_\theta\colon E \times E \longrightarrow M$ is said to be a θ-cone-metric on E if and only if d_θ satisfies the following conditions:

(1) $d_\theta(x, y) = \Theta \Leftrightarrow x = y$
(2) $d_\theta(x, y) = d_\theta(y, x), \forall x, y \in E$
(3) $d_\theta(x, y) \prec \theta(d_\theta(x, z), d_\theta(z, y)), \quad \forall x, y, z \in E$

The double (E, d_θ) is defined to be a θ-cone-metric space.

The author has further given some topological characterizations of this space and then generalized some previous fixed-point theorems in this setting.

Remark 1. If $\theta(u, v) = u + v$, then we have a cone-metric space.

In this paper, we extend and generalize the coupled fixed-point theorem of Baskar–Lackshmikantham (1.5) to a more general one (2.1), where the underlying space (E, d_θ) is a complete θ-cone-metric space. On the other side, if $T\colon E \times E \longrightarrow E$ is a continuous mapping in the second argument and there are three constants $a, b, c \in [0, 1)$ and $a + b + c < 1$ such that

$$d_\theta(T(x, y), T(y, z)) \prec a d_\theta(x, y) + b d_\theta(T(x, y), x) + c d_\theta(T(y, z), y), \quad \forall x, y, z \in E, \tag{10}$$

then we proved that T has a unique fixed point in the sense that there is a unique point $x \in E$ such that $T(x, x) = x$.

We also claim that some results of [6–10, 17] can be proved in the case of θ-cone-metric spaces.

2. Main Results

Let $(E, d_\theta, \leq)$ be a partially ordered θ-cone-metric space. Then, the following relation defines a partial-ordered relation on $E \times E$:

$$(x, y) \ll (z, w) \Leftrightarrow x \leq z, \text{ and } w \leq y. \tag{11}$$

We have the following coupled fixed-point theorem.

Theorem 3. *Let $(E, d_\theta, \leq)$ be a partially ordered, sequentially lower-upper ordered complete θ-cone-metric space and $G\colon E \times E \longrightarrow E$ be a mapping having mixed monotone and lower-upper properties on E. We assume that there exists $r \in [0, 1)$ with*

$$d_\theta(G(x, y), G(z, w)) \prec \frac{r}{2}[d_\theta(x, z) + d_\theta(y, w)], \quad \forall (x, y) \ll (z, w). \tag{12}$$

Then, G has coupled fixed points in E.

Proof. Since G has a lower-upper property, then there exist $x_0, y_0 \in E$ such that

$$x_0 \leq G(x_0, y_0) \text{ and } G(y_0, x_0) \leq y_0. \tag{13}$$

We denote $x_1 = G(x_0, y_0)$ and $y_1 = G(y_0, x_0)$ and then give notations for the elements of the following inductively constructed sequences:

$$\begin{aligned} x_2 &= G(x_1, y_1) \coloneqq G^2(x_0, y_0),\\ y_2 &= G(y_1, x_1) \coloneqq G^2(y_0, x_0),\\ x_3 &= G(x_2, y_2) \coloneqq G^3(x_0, y_0),\\ y_3 &= G(y_2, x_2) \coloneqq G^3(y_0, x_0),\\ &\cdots\\ x_{n+1} &= G(x_n, y_n) \coloneqq G^{n+1}(x_0, y_0),\\ y_{n+1} &= G(y_n, x_n) \coloneqq G^{n+1}(y_0, x_0),\\ &\cdots \end{aligned} \tag{14}$$

Using the mixed monotonicity property of G insures that each step leads to the next step in each of the following:

$$\begin{aligned} &x_0 \leq x_1 = G(x_0, y_0) \leq G(x_1, y_0) \leq G(x_1, y_1) = x_2,\\ &y_2 = G(y_1, x_1) \leq G(y_1, x_0) \leq G(y_0, x_0) = y_1 \leq y_0,\\ &x_1 \leq x_2 = G(x_1, y_1) \leq G(x_2, y_2) = x_3,\\ &y_3 = G(y_2, x_2) \leq G(y_1, x_1) = y_2 \leq y_1\\ &\cdots\\ &x_{n+1} = G(x_n, y_n) \leq G(x_{n-1}, y_{n-1}) = x_n,\\ &y_{n+1} = G(y_n, x_n) \leq G(y_{n-1}, x_{n-1}) = y_n,\\ &\cdots \end{aligned} \tag{15}$$

The mixed monotonicity property, the contractiveness of G, and the inductive process prove the following for every $n \in N$:

$$\begin{aligned} d_\theta\left(G^{n+1}(x_0, y_0), G^n(x_0, y_0)\right) &\prec \left[\frac{r}{2}\right]^n [d_\theta(G(x_0, y_0), x_0) + d_\theta(G(y_0, x_0), y_0)],\\ d_\theta\left(G^{n+1}(y_0, x_0), G^n(y_0, x_0)\right) &\prec \left[\frac{r}{2}\right]^n [d_\theta(G(y_0, x_0), y_0) + d_\theta(G(x_0, y_0), x_0)]. \end{aligned} \tag{16}$$

Consequently, we have

$$\lim_{n\longrightarrow\infty} d_\theta\left(G^{n+1}(x_0, y_0), G^n(x_0, y_0)\right) = \lim_{n\longrightarrow\infty} d_\theta\left(G^{n+1}(y_0, x_0), G^n(y_0, x_0)\right) = \Theta. \tag{17}$$

Hence, we claim that both $\{G^n(x_0, y_0)\}_{n\in N}$ and $\{G^n(y_0, x_0)\}_{n\in N}$ are Cauchy sequences in E. Indeed, if one of them, say $\{G^n(x_0, y_0)\}_{n\in N}$, is not Cauchy, then there exist $v \in \operatorname{Im}(\theta)$, $\Theta < v$ and sequences of natural numbers $\{i_n\}_{n\in N}$ and $\{j_n\}_{n\in N}$ such that, for any $i_n > j_n > n$,

$$v \prec d_\theta\left(G^{i_n}(x_0, y_0), G^{j_n}(x_0, y_0)\right),\quad d_\theta\left(G^{i_n-1}(x_0, y_0), G^{j_n}(x_0, y_0)\right) < v. \tag{18}$$

Since any subsequence of $\{d_\theta(G^{n+1}(x_0, y_0), G^n(x_0, y_0))\}_{n\in N}$ is convergent to Θ, the properties of θ imply the following contradiction:

$$\begin{aligned} v &\prec d_\theta\left(G^{i_n}(x_0, y_0), G^{j_n}(x_0, y_0)\right) \\ &\prec \theta\left(d_\theta\left(G^{i_n-1}(x_0, y_0), G^{j_n}(x_0, y_0)\right), d_\theta\left(G^{i_n}(x_0, y_0), G^{i_n-1}(x_0, y_0)\right)\right) \\ &< \theta\left(v, d_\theta\left(G^{i_n}(x_0, y_0), G^{i_n-1}(x_0, y_0)\right)\right) \\ &< \theta\left(v, \lim_{n\longrightarrow\infty} d_\theta\left(G^{i_n}(x_0, y_0), G^{i_n-1}(x_0, y_0)\right)\right) \\ &< \theta(v, \Theta) < v. \end{aligned} \tag{19}$$

Similarly, the sequence $\{G^n(y_0, x_0)\}_{n\in N}$ is also Cauchy. Since E is a complete θ-cone-metric space, there exist $x, y \in E$ such that

$$\lim_{n\longrightarrow\infty} d_\theta(G^n(x_0, y_0), x) = \Theta,\quad \lim_{n\longrightarrow\infty} d_\theta(G^n(y_0, x_0), y) = \Theta. \tag{20}$$

Now, we are going to show that (x, y) is a coupled fixed point of G. Since the sequence $\{G^n(x_0, y_0) = x_n\}_{n\in N}$ is nondecreasing with $\lim_{n\longrightarrow\infty} G^n(x_0, y_0) = x$, then $G^n(x_0, y_0) \leq x$, and since the sequence $\{G^n(y_0, x_0) = y_n\}_{n\in N}$ is nonincreasing with $\lim_{n\longrightarrow\infty} G^n(y_0, x_0) = y$, then $y \leq G^n(y_0, x_0)$ for every $n \in N$, and accordingly, we have

$$\begin{aligned} d_\theta(G(x, y), x) &\prec \theta\left(d_\theta\left(G(x, y), G^{n+1}(x_0, y_0)\right), d_\theta\left(G^{n+1}(x_0, y_0), x\right)\right), \\ &= \theta\left(d_\theta(G(x, y), G(x_n, y_n)), d_\theta\left(G^{n+1}(x_0, y_0), x\right)\right) \\ &\prec \theta\left(\left[\frac{r}{2}\right]\left[d_\theta(x, x_n) + d_\theta(y, y_n)\right], d_\theta\left(G^{n+1}(x_0, y_0), x\right)\right) \\ &\prec \theta\left(\left[\frac{r}{2}\right]\left[d_\theta(x, G^n(x_0, y_0)) + d_\theta(y, G^n(y_0, x_0))\right], d_\theta\left(G^{n+1}(x_0, y_0), x\right)\right). \end{aligned} \tag{21}$$

Taking the limit as $n \longrightarrow \infty$ with the help of equation (20), we find that

$$d_\theta(G(x, y), x) \prec \theta\left(\left[\frac{r}{2}\right][\Theta + \Theta], \Theta\right) = \theta(\Theta, \Theta) = \Theta. \tag{22}$$

Hence, $d_\theta(G(x, y), x) = \Theta$; therefore, $G(x, y) = x$. Similarly, $G(y, x) = y$. □

If the partial-ordered relation on $E \times E$ is defined as

$$(x, y) \ll_2 (z, w) \Leftrightarrow x \geq z, \text{ and } y \leq w, \tag{23}$$

then the following theorem is similarly proved.

Theorem 4. *Let $(E, d_\theta, \leq)$ be a partially ordered, sequentially lower-upper ordered complete θ-cone-metric space and $G\colon E \times E \longrightarrow E$ be a mapping having mixed monotone property, and we suppose that there are $x_0, y_0 \in E$ such that $T(x_0, y_0) \leq x_0$ and $y_0 \leq T(y_0, x_0)$. If there exists $r \in [0, 1)$ with*

$$d_\theta(G(x, y), G(z, w)) \prec \frac{r}{2}\left[d_\theta(x, z) + d_\theta(y, w)\right],\quad \forall (x, y) \ll_2 (z, w), \tag{24}$$

then G has coupled fixed points in E.

Corollary 1. *Let E be a sequentially both lower- and upper-ordered Banach space and $T\colon E \times E \longrightarrow E$ be a mapping with mixed monotone and lower-upper properties. If there is a real number $0 \leq k < 1$ such that*

$$\|T(x,y)-T(z,w)\| \leq \frac{k}{2}[\|x-z\|+\|y-w\|],$$
$$\forall x,y,z,w \in E, z \leq x, \text{ and } y \leq w, \tag{25}$$

then T has coupled fixed point in E.

Proof. We just notice that any Banach space $(E, \|.\|)$ is a θ-cone-metric space (E, d_θ), where $(\mathcal{E}, \prec) = (\mathbb{R}, |.|)$ is the Banach space of real numbers with the absolute value metric and with the usual ordered relation of real numbers, $\theta(u,v) = u + v$, $\theta: [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$, and the metric d_θ is the metric induced by the norm $\|.\|$ on E, $d_\theta(x,y) = \|x-y\|$. □

Remark 2. Corollary 1 is Baskar–Lackshmikantham coupled fixed-point Theorem 1. This proves that Theorem 3 is a quite good generalization of the Baskar–Lackshmikantham coupled fixed-point theorem.

On the other side, we have the following results:

Lemma 1. *Let (E, d_θ) be a θ-cone-metric space and T be a mapping, $T: E \times E \longrightarrow E$. It is supposed that there are constants $a, b, c \in [0,1)$ and $a + b + c < 1$ such that*

$$d_\theta(T(x,y),T(y,z)) \prec ad_\theta(x,y) + bd_\theta(T(x,y),x) + cd_\theta(T(y,z),y), \quad \forall x,y,z \in E. \tag{26}$$

If x_1 and x_2 are arbitrary elements in E, then the sequence $\{x_n\}_{n=3}^{\infty}$ defined iteratively by

$$x_n = T(x_{n-1}, x_{n-2}), \quad \forall n \in \mathcal{N}, n > 2, \tag{27}$$

which satisfies the following:

$$d_\theta(x_{n+1}, x_n) \prec td_\theta(x_n, x_{n-1}), \quad \forall n > 2, \tag{28}$$

$$d_\theta(x_{n+1}, x_n) \prec t^n d_\theta(x_2, x_1), \quad \forall n > 2, \tag{29}$$

where $t = (a + c/1 - b)$. Moreover, the sequence $\{x_n\}_{n\in\mathcal{N}}$ is a Cauchy sequence.

Proof. Using the contractiveness property of the given mapping gives

$$\begin{aligned} d_\theta(x_{n+1}, x_n) &= (d_\theta(T(x_n, x_{n+1})), T(x_{n-1}, x_{n-2})) \\ &\prec ad_\theta(x_n, x_{n-1}) + bd_\theta(x_{n+1}, x_n) \\ &\quad + cd_\theta(x_n, x_{n-1}). \end{aligned} \tag{30}$$

Hence,

$$d_\theta(x_{n+1}, x_n) \prec \left(\frac{a+c}{1-b}\right) d_\theta(x_n, x_{n-1}), \quad \forall n > 2, \tag{31}$$

and repeating the last step $n-2$ times with the term $d_\theta(x_n, x_{n-1})$ proves the inequalities given in (29). To prove that the sequence (27) is Cauchy, we take the limit of both sides of (29) as $n \longrightarrow \infty$ gives $\lim_{n\longrightarrow\infty} d_\theta(x_{n+1}, x_n) = \Theta$ and suppose that $\{x_n\}_{n\in N}$ is not Cauchy; then, there exist $v \in \operatorname{Im}(\theta)$, $\Theta < v$ and sequences of natural numbers $\{i_n\}_{n\in N}$ and $\{j_n\}_{n\in N}$ such that, for any $i_n > j_n > n$,

$$\begin{aligned} v &\prec d_\theta\left(x_{i_n}, x_{j_n}\right), \\ d_\theta&\left(x_{i_n-1}, x_{j_n}\right) < v. \end{aligned} \tag{32}$$

Since any subsequence of $\{d_\theta(x_{n+1}, x_n)\}_{n\in N}$ is convergent to Θ, the continuity and the properties of θ imply the following contradiction:

$$\begin{aligned} v &\prec d_\theta\left(x_{i_n}, x_{j_n}\right) \prec \theta\left(d_\theta\left(x_{i_n-1}, x_{j_n}\right), d_\theta\left(x_{i_n}, x_{i_n-1}\right)\right) \\ &< \theta\left(v, d_\theta\left(x_{i_n}, x_{i_n-1}\right)\right) \longrightarrow_{n\longrightarrow\infty} \theta(v, \Theta) < v. \end{aligned} \tag{33}$$

□

Theorem 5. *Let (E, d_θ) be a complete θ-cone-metric space and $T: E \times E \longrightarrow E$ be a continuous mapping in the second argument, and we suppose there are three constants $a, b, c \in [0,1)$ and $a + b + c < 1$ such that*

$$d_\theta(T(x,y),T(y,z)) \prec ad_\theta(x,y) + bd_\theta(T(x,y),x) + cd_\theta(T(y,z),y), \quad \forall x,y,z \in E, \tag{34}$$

and then, T has a unique fixed point in the sense that there is a unique point $x_0 \in E$ such that $T(x_0, x_0) = x_0$.

Proof. Since (E, d_θ) is complete, the Cauchy sequence $\{x_n\}_{n=3}^{\infty}$ given in Lemma 1 is converging to some element x_0 in E. We show that x_0 is fixed point of T. Using the properties of θ and the continuity of T, we see that

$$\begin{aligned}
d_\theta(T(x_0,x_0),x_0) &\prec \theta(d_\theta(T(x_0,x_0),x_n),d_\theta(x_n,x_0))\\
&\prec \theta(\theta(d_\theta(T(x_0,x_0),T(x_0,x_{n-1})),d_\theta(T(x_0,x_{n-1}),x_n)),d_\theta(x_n,x_0)),\\
&= \theta(\theta(d_\theta(T(x_0,x_{n-1}),T(x_{n-1},x_{n-2})),d_\theta(T(x_0,x_0),T(x_0,x_{n-1}))),\\
&= \theta(\theta d_\theta T(x_0,x_{n-1}),T(x_{n-1},x_{n-2})),d_\theta(T(x_0,x_0),T(x_0,x_{n-1}))d_\theta(x_n,x_0))\\
&\prec \theta(\theta(ad_\theta(x_0,x_{n-1})+bd_\theta(T(x_0,x_{n-1}),x_0)+cd_\theta(T(x_{n-1},x_{n-2}),x_{n-1}),\\
&\quad d_\theta(T(x_0,x_0),T(x_0,x_{n-1}))),d_\theta(x_n,x_0))\\
&\prec \theta(\theta(ad_\theta(x_0,x_{n-1})+bd_\theta(T(x_0,x_{n-1}),x_0)+cd_\theta(x_n,x_{n-1}),\\
&\quad d_\theta(T(x_0,x_0),T(x_0,x_{n-1}))),d_\theta(x_n,x_0)) \longrightarrow_{n\longrightarrow\infty}\\
&\prec \theta(\theta(a\Theta+bd_\theta(T(x_0,x_0),x_0)+c\Theta,d_\theta(T(x_0,x_0),T(x_0,x_0))),\Theta)\\
&\prec \theta(\theta(a\Theta+bd_\theta(T(x_0,x_0),x_0)+c\Theta,\Theta),\Theta),\\
&= \theta(\theta(bd_\theta(T(x_0,x_0),x_0),\Theta),\Theta)\\
&\prec \theta(bd_\theta(T(x_0,x_0),x_0),\Theta)\\
&\prec bd_\theta(T(x_0,x_0),x_0).
\end{aligned} \tag{35}$$

Since $b<1$, we get $d_\theta(T(x_0,x_0),x_0)=\Theta$; consequently, $T(x_0,x_0)=x_0$. Now, let x and y be two arbitrarily distinct elements in E with $T(x,x)=x$ and $T(y,y)=y$, and we have

$$\begin{aligned}
&d_\theta(T(x,y),x) \prec \theta(d_\theta T(x,x),T(x,y)),d_\theta(T(x,x),x)\\
&\prec \theta(bd_\theta(T(x,x),x)+cd_\theta(T(x,y),x),d_\theta(T(x,x),x))\\
&\prec \theta(b\Theta+cd_\theta(T(x,y),x),\Theta)\\
&\prec cd_\theta(T(x,y),x).
\end{aligned} \tag{36}$$

Thus, $d_\theta(T(x,y),x)=\Theta$, that is, $T(x,y)=x$. Similarly, we get $T(y,x)=y$; therefore, (x,y) is a coupled fixed point of T. On the other hand, we have the following contradiction:

$$\begin{aligned}
d_\theta(x,y) &= d_\theta(T(x,x),T(y,y))\\
&\prec \theta(d_\theta(T(x,x),T(x,y)),d_\theta(T(x,y),T(y,y)))\\
&\prec \theta(cd_\theta(T(x,y),x),ad_\theta(x,y)+bd_\theta(T(x,y),x))\\
&\prec \theta(cd_\theta(x,x),ad_\theta(x,y)+bd_\theta(x,x))\\
&\prec \theta(\Theta,ad_\theta(x,y)+\Theta) \prec ad_\theta(x,y).
\end{aligned} \tag{37}$$

Since $a<1$, we have $d_\theta(x,y)$; consequently, $x=y$. □

We conclude the following.

Corollary 2. *Let $(E,\|.\|)$ be a Banach space and T be a mapping from $E\times E$ into E, and we suppose that there are three constants $a,b,c\in[0,1)$ and $a+b+c<1$ such that*

$$\begin{aligned}
\|T(x,y)-T(y,z)\| \le a\|x-y\| + b\|T(x,y)-x\|\\
+ c\|T(y,z)-y\|, \quad \forall x,y,z\in E.
\end{aligned} \tag{38}$$

Then, there is a unique point $x_0\in E$ such that $T(x_0,x_0)=x_0$.

Proof. It can be proved in a similar way of Corollary 1 with the same notice. □

Remark 3. Corollary 2 is the fixed-point theorem of Mohamed Ali Abou Bakr; accordingly, Theorem 5 is a generalization of fixed-point Theorem 2 in the setting of a complete θ-cone-metric space.

3. Conclusions

This paper gives further generalizations of some well-known coupled fixed-point theorems. Specifically, Theorem 3 generalizes the Baskar–Lackshmikantham coupled fixed-point theorem [3], and Theorem 5 generalizes the Sahar Mohamed Ali Abou Bakr fixed-point theorem [4]; the underlying space (E,d_θ) is a complete θ-cone-metric space, and we claim that some results of [6–10] can be proved in the case of θ-cone-metric spaces.

Data Availability

No data were used to support this study.

Disclosure

This research was performed as part of the employment of Dr. Sahar Mohamed Ali Abou Bakr at Ain Shams University, Faculty of Science, Department of Mathematics, Cairo, Egypt.

Conflicts of Interest

The author has no conflicts of interest.

Authors' Contributions

The sole author contributed 100% to the article. The author read and approved the final manuscript.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] D. Guo and V. Lakshmikantham, "Coupled fixed points of nonlinear operators with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 11, no. 5, pp. 623–632, 1987.

[3] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[4] S. Mohamed Ali, "Fixed point theorems of some type contraction mappings," *Journal of Nonlinear and Convex Analysis*, vol. 14, no. 2, pp. 331–342, 2013.

[5] M. Kir, E. Yolacan, and H. Kiziltunc, "Coupled fixed point theorems in complete metric spaces endowed with a directed graph and application," *Open Math by De Gruyter Open*, vol. 15, Article ID 734744, 2017.

[6] A. Abdullah, M. Saud, and Alsulami, "Coupled coincidence points for monotone operators in partially ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2011, p. 44, 2011.

[7] H. K. Nashine, Z. Kadelburg, and S. Radenović, "Coupled common fixed point theorems for w-compatible mappings in ordered cone metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5422–5432, 2012.

[8] M. Jain, K. Tas, and N. Gupta, "Coupled common fixed point results involving (ϕ, ψ)-contractions in ordered generalized metric spaces with application to integral equations," *Journal of Inequalities and Applications*, vol. 2013, no. 1, p. 372, 2013.

[9] Z. Kadelburg, P. Kumam, S. Radenovi, and W. Sintunavarat, "Common coupled fixed point theorems for Geraghty-type contraction mappings using monotone property," *Fixed Point Theory and Applications*, vol. 2015, p. 27, 2015.

[10] F. Khojasteh, E. Karapinar, and S. Radenovic, "Theta-metric space: a generalization," *Mathematical Problems in Engineering*, vol. 2013, Article ID 504609, , 2013.

[11] F. Gu and W. Shatanawi, "Some new results on common coupled fixed points of two hybrid pairs of mappings in partial metric spaces," *Journal of Nonlinear Function and Analysis*, vol. 2019, p. 13, 2019.

[12] A. Radouane and D. Mentagui, "A solution for the minimax problem via fixed point theory in complete ordered locally convex spaces," *Communications in Optimization Theory*, vol. 2020, p. 11, 2020.

[13] A. Petrusel and G. Petrusel, "Fixed point results for multivalued locally contractive operators," *Applications of Set-Valued Analysis to Optimization*, vol. 2, pp. 175–181, 2020.

[14] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[15] S. Mohamed Ali Abou Bakr, "Common fixed point of generalized cyclic Banach algebra contractions and Banach algebra Kannan types of mappings on cone quasi metric spaces," *Journal of Nonlinear Science and Appllications*, vol. 12, no. 10, pp. 644–655, 2019.

[16] S. Mohamed Ali Abou Bakr, "Theta cone metric spaces and some fixed point theorems," *Journal of Mathematics*, vol. 2020, p. 7, Article ID 8895568, 2020.

[17] S. Mohamed Ali Abou Bakr, "A study on common fixed point of joint $(A; B)$generalized cyclic $\phi - abc$ weak nonexpansive mappings and generalized cyclic $abc; r$ contractions in quasi metric spaces," *Abstract and Applied Analysis*, vol. 2020, Article ID 9427267, 2020.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6629203, 10 pages
https://doi.org/10.1155/2021/6629203

Research Article

Generalized Complementarity Problem with Three Classes of Generalized Variational Inequalities Involving ⊕ Operation

Rais Ahmad,[1] **Iqbal Ahmad**,[2] **Zahoor Ahmad Rather**,[1] **and Yuanheng Wang**[3]

[1] *Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India*
[2] *College of Engineering, Qassim University, P. O. Box 6677, Buraidah 51452, Al-Qassim, Saudi Arabia*
[3] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China*

Correspondence should be addressed to Yuanheng Wang; yhwang@zjnu.cn

Received 11 November 2020; Accepted 8 January 2021; Published 25 January 2021

Academic Editor: Sun Young Cho

In this study, we introduce and study a generalized complementarity problem involving XOR operation and three classes of generalized variational inequalities involving XOR operation. Under certain appropriate conditions, we establish equivalence between them. An iterative algorithm is defined for solving one of the three generalized variational inequalities involving XOR operation. Finally, an existence and convergence result is proved, supported by an example.

1. Introduction

It is well known that the many unrelated free boundary value problems related to mathematical and engineering sciences can be solved by using the techniques of variational inequalities. In a variational inequality formulation, the location of the free boundary becomes an intrinsic part of the solution, and no special devices are needed to locate it. Complementarity theory is an equally important area of operations research and application oriented. The linear as well as nonlinear programs can be distinguished by a family of complementarity problems. The complementarity theory have been elongated for the purpose of studying several classes of problems occurring in fluid flow through porous media, economics, financial mathematics, machine learning, optimization, and transportation equilibrium, for example, [1–5].

The correlations between the variational inequality problem and complementarity problem were recognized by Lions [6] and Mancino and Stampacchia [7]. However, Karamardian [8, 9] showed that both the problems are equivalent if the convex set involved is a convex cone. For more details on variational inequalities and complementarity problems, refer to [6, 10–12].

The exclusive "XOR," sometimes also exclusive disjunction (short: XOR) or antivalence, is a Boolean operation which only outputs true if only exactly one of its both inputs is true (so, if both inputs differ). There are many applications of XOR terminology, that is, it is used in cryptography, gray codes, parity, and CRC checks. Commonly, the ⊕ symbol is used to denote the XOR operation. Some problems related to variational inclusions involving XOR operation were studied by [13–16].

Influenced by the applications of all the above discussed concepts in this study, we introduce and study a generalized complementarity problem involving XOR operation with three classes of generalized variational inequalities involving XOR operation. Some equivalence relations are established between them. An existence and convergence result is proved for one of the three types of generalized variational inequalities involving XOR operation. For illustration, an example is provided.

2. Some Basic Concepts and Formulation of the Problem

Throughout this study, we assume E to be real ordered Banach space with norm $\|\cdot\|$ and E^* be its dual space. Suppose that d is the metric induced by the norm, 2^E (respectively, $CB(E)$) is the family of nonempty (respectively, closed and bounded) subsets of E. The Hausdorff metric $D(.,.)$ on $CB(E)$ is defined as

$$D(A,\mathscr{B}) = \max\left\{\sup_{x\in\mathscr{A}} d(x,\mathscr{B}), \sup_{y\in\mathscr{B}} d(\mathscr{A},y)\right\}, \quad \forall \mathscr{A},\mathscr{B} \in CB(E), \tag{1}$$

where $d(x,\mathscr{B}) = \inf_{y\in\mathscr{B}} d(x,y)$, and $d(\mathscr{A},y) = \inf_{x\in\mathscr{A}} d(x,y)$.

Let C be a pointed closed convex positive cone in E, and $\langle t,x\rangle$ denotes the value of the linear continuous function $t\in E^*$ at x.

The following definitions and concepts are required to achieve the goal of this study, and most of them can be found in [17, 18].

Definition 1. The relation "$\leq$" is called the partial order relation induced by the cone C, that is, $x\leq y$ if and only if $y-x\in C$.

Definition 2. For arbitrary elements $x,y\in E$, if $x\leq y$ (or $y\leq x$) holds, then x and y are said to be comparable to each other (denoted by $x\propto y$).

Definition 3. For arbitrary elements $x,y\in E$, lub$\{x,y\}$ and glb$\{x,y\}$ mean the least upper bound and the greatest upper bound of the set $\{x,y\}$. Suppose lub$\{x,y\}$ and glb$\{x,y\}$ exist, then some binary operations are defined as

(i) $x\vee y = \text{lub}\{x,y\}$
(ii) $x\wedge y = \text{glb}\{x,y\}$
(iii) $x\oplus y = (x-y)\vee(y-x)$
(iv) $x\odot y = (x-y)\wedge(y-x)$

The operations $\vee,\wedge,\oplus$, and $\odot$ are called OR, AND, XOR, and XNOR operations, respectively.

Proposition 1. *Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then, the following relations hold:*

(i) $x\odot x = 0$, $x\odot y = y\odot x$
(ii) if $x\propto 0$, *then* $-x\oplus 0\leq x\leq x\oplus 0$
(iii) $0\leq x\oplus y$, *if* $x\propto y$
(iv) If $x\propto y$, *then* $x\oplus y = 0$ *if and only if* $x = y$
(v) $x\oplus y = y\oplus x$
(vi) $x\oplus x = 0$
(vii) $0\leq x\oplus 0$
(viii) If $x\leq y$ *and* $u\leq v$, *then* $(x+u)\leq(y+v)$
(ix) If $x\propto y$, *then* $(x\oplus 0)\oplus(y\oplus 0)\leq(x\oplus y)\oplus 0 = x\oplus y$, *for all* $x,y,u,v\in E$ *and* $\lambda\in\mathbb{R}$

Proposition 2. *Let C be a cone in E; then, for each $x,y\in E$, the following relations hold:*

(i) $\|0\oplus 0\| = \|0\| = 0$
(ii) $\|x\vee y\|\leq\|x\|\vee\|y\|\leq\|x\|+\|y\|$
(iii) $\|x\oplus y\|\leq\|x-y\|$
(iv) If $x\propto y$, *then* $\|x\oplus y\| = \|x-y\|$

Definition 4. Let $A\colon E\longrightarrow E$ be a single-valued mapping, then

(i) A is said to be a comparison mapping, if $x\propto y$, then $A(x)\propto A(y)$, $x\propto A(x)$, and $y\propto A(y)$, for all $x,y\in E$
(ii) A is said to be a strongly comparison mapping, if A is a comparison mapping and $A(x)\propto A(y)$, if and only if $x\propto y$, for any $x,y\in E$

Definition 5. Let $f\colon E\longrightarrow\mathbb{R}\cup\{+\infty\}$ be a proper functional. A vector $\omega^*\in E^*$ is called subgradient of f at $x\in$ domf, if

$$\langle\omega^*, y-x\rangle\leq f(y)-f(x), \quad \text{for all } y\in E. \tag{2}$$

The set of all subgradients of f at x is denoted by $\partial f(x)$. The mapping $\partial f: E\longrightarrow 2^{E^*}$ defined by

$$\partial f(x) = \{\omega^*\in E^*\colon \langle\omega^*, y-x\rangle\leq f(y)-f(x), \quad \text{for all } y\in E\} \tag{3}$$

is called subdifferential of f.

Definition 6. The resolvent operator $\mathscr{J}_\rho^{\partial f}$ associated with ∂f is given by

$$\mathscr{J}_\rho^{\partial f}(x) = [I+\rho\partial f]^{-1}(x), \quad \text{for all } x\in E, \tag{4}$$

where $\rho>0$ is a constant, and I is the identity operator.

It is well known that the resolvent operator $\mathscr{J}_\rho^{\partial f}$ is single-valued as well as nonexpansive.

Definition 7. A mapping $f\colon C\longrightarrow\mathbb{R}$ is said to be

(i) Positive homogeneous if, for all $\alpha>0$ and $x\in C$, $f(\alpha x) = \alpha f(x)$
(ii) Convex, if $x,y\in C$ and all $\lambda\in[0,1]$

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y). \tag{5}$$

Definition 8. A multivalued mapping $F\colon C\longrightarrow 2^{E^*}\setminus\{\varnothing\}$ is said to be

(i) Upper semicontinuous at $x\in C$ if, for every open set V containing $F(x)$, there exists an open set U containing x such that $F(U)\subseteq V$, where E^* is equipped with ω^* topology
(ii) Upper semicontinuous on C if it is upper semicontinuous at every point of C

(iii) Upper hemicontinuous on C if its restriction to line segments of C is upper semicontinuous

(iv) Monotone if, for every $x, y \in C$

$$\langle t_1 - t_2, y - x\rangle \geq 0, \quad \text{for all}\, t_1 \in F(y), t_2 \in F(x). \tag{6}$$

Definition 9. A multivalued mapping $F: E \longrightarrow 2^E$ is said to be D-Lipschitz continuous, if there exists a constant $\lambda_{D_F} > 0$ such that

$$D(F(x), F(y)) \leq \lambda_{D_F}\|x - y\|, \quad \text{for all}\, x, y \in E. \tag{7}$$

Definition 10. A multivalued mapping $F: E \longrightarrow 2^E$ is said to be relaxed Lipschitz continuous, if there exists a constant $k > 0$ such that

$$\langle w_1 - w_2, x - y\rangle \leq -k\|x - y\|^2, \quad \text{for all}\, w_1 \in F(x), w_2 \in F(y). \tag{8}$$

Let $F: C \longrightarrow 2^{E^*}\setminus\{\varnothing\}$ be a multivalued mapping with nonempty values and $f: C \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper functional. We consider the following generalized complementarity problem involving XOR operation.

Find $\overline{x} \in C$, $\overline{t} \in F(\overline{x})$ such that

$$\begin{gathered}\langle \overline{t}, t\overline{x}\rangle \oplus f(\overline{x}) = 0,\\ \langle \overline{t}, ty\rangle \oplus f(y) \geq 0,\\ \forall y \in C.\end{gathered} \tag{9}$$

We denote by $S_{C\oplus}$ the solution set of generalized complementarity problem involving XOR operation (9).

We mention some special cases of problem (9) as follows.

(i) If we replace $\oplus$ by $+$ and f by $f: C \longrightarrow \mathbb{R}$, then problem (9) reduces to the problem of finding $\overline{x} \in C$ and $\overline{t} \in F(\overline{x})$ such that

$$\begin{gathered}\langle \overline{t}, t\overline{x}\rangle + f(\overline{x}) = 0,\\ \langle \overline{t}, ty\rangle + f(y) \geq 0,\\ \forall y \in C.\end{gathered} \tag{10}$$

Problem (10) is called generalized f complementarity problem, introduced and studied by Huang et al. [19].

(ii) If $f \equiv 0$, then problems (9) as well as (10) reduce to the problem of finding $\overline{x} \in C$ and $\overline{t} \in F(\overline{x})$ such that

$$\begin{gathered}\langle \overline{t}, t\overline{x}\rangle = 0,\\ \langle \overline{t}, ty\rangle \geq 0,\\ \forall y \in C.\end{gathered} \tag{11}$$

Problem (11) can be found in [20, 21].

We remark that for suitable choices of operators involved in the formulation of (9), a number of known complementarity problems can be obtained easily, for example, [17, 22–24].

Simultaneously, we also study the following three types of generalized variational inequalities involving XOR operation.

(1) Find $\overline{x} \in C$ such that

$$\exists \overline{t} \in F(\overline{x}), \quad \forall y \in C\colon \langle \overline{t}, tyn - q\overline{x}\rangle \oplus (f(y) - f(\overline{x})) \geq 0; \tag{12}$$

(2) Find $\overline{x} \in C$ such that

$$\forall y \in C, \quad \exists \overline{t} \in F(\overline{x})\colon \langle \overline{t}, tyn - q\overline{x}\rangle \oplus (f(y) - f(\overline{x})) \geq 0; \tag{13}$$

(3) Find $\overline{x} \in C$ such that

$$\forall y \in C, \quad \forall t \in F(y)\colon \langle t, y - \overline{x}\rangle \oplus (f(y) - f(\overline{x})) \geq 0. \tag{14}$$

We denote the solution set of (12) by $S_{1\oplus}$, (13) by $S_{2\oplus}$, and (14) by $S_{3\oplus}$.

Many known variational inequality problems can be obtained from problems (12)–(14), for example, [25–29] and the references therein.

3. Equivalence Results

We establish the equivalence among problems (9), (12)–(14). First, we establish the equivalence between generalized complementarity problem involving XOR operation (9) and generalized variational inequality problem involving XOR operation (12).

Theorem 1. *Let $F: C \longrightarrow 2^{E^*}\setminus\{\varnothing\}$ be a multivalued mapping with nonempty values and $f: C \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper functional. Then, the following statements are true:*

(i) If $\langle \overline{t}, t\overline{x}\rangle \propto f(\overline{x})$, then $S_{C\oplus} \subseteq S_{1\oplus}$

(ii) If f is positive homogeneous, then $S_{1\oplus} \subseteq S_{C\oplus}$

Proof

(i) Let $\overline{x} \in S_{C\oplus}$, then $\overline{x} \in C$, and there exists $\overline{t} \in F(\overline{x})$ such that

$$\begin{gathered}\langle \overline{t}, t\overline{x}\rangle \oplus f(\overline{x}) = 0,\\ \langle \overline{t}, ty\rangle \oplus f(y) \geq 0.\end{gathered} \tag{15}$$

Since $\langle \overline{t}, t\overline{x}\rangle \propto f(\overline{x})$, by (iv) of Proposition 1, we have

$$\begin{gathered}\langle \overline{t}, t\overline{x}\rangle = f(\overline{x}),\\ \text{Also as}\ \langle \overline{t}, ty\rangle \oplus f(y) \geq 0,\\ \langle \overline{t}, ty\rangle \oplus f(y) \oplus f(y) \geq 0 \oplus f(y),\end{gathered} \tag{16}$$

which implies that

$$\langle \bar{t}, ty\rangle \geq f(y). \tag{17}$$

By using (16) and (17), we have

$$\begin{aligned}&\langle \bar{t}, tyn - q\bar{x}\rangle = \langle \bar{t}\\ &\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq (f(y) - f(\bar{x}))\oplus(f(y) - f(\bar{x})),\end{aligned} \tag{18}$$

that is,

$$\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0, \tag{19}$$

which implies that $\bar{x} \in S_{1\oplus}$. So, we have $S_{C\oplus} \subseteq S_{1\oplus}$.

(ii) Let $\bar{x} \in S_{1\oplus}$, then $\bar{x} \in C$, and there exists $\bar{t} \in F(\bar{x})$ such that

$$\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0, \quad \forall y \in C. \tag{20}$$

Since C is a pointed closed convex positive cone, clearly $y = 2\bar{x} \in C$ and $y = (1/2)\bar{x} \in C$. Putting $y = 2\bar{x}$ in generalized variational inequality involving XOR operation (12) and using positive homogenity of f, we get

$$\begin{aligned}&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0,\\ &\langle \bar{t}, t2n\bar{x}q - h\bar{x}\rangle \oplus (f(2\bar{x}) - f(\bar{x})) \geq 0,\\ &\langle \bar{t}, t\bar{x}\rangle \oplus f(\bar{x}) \geq 0.\end{aligned} \tag{21}$$

Now, putting $y = (1/2)\bar{x}$ in generalized variational inequality involving XOR operation ((12)) and using positive homogenity of f, we get

$$\begin{aligned}&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0,\\ &\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x}))\oplus(f(y) - f(\bar{x})) \geq 0\oplus(f(y) - f(\bar{x})),\end{aligned} \tag{22}$$

which implies that

$$\begin{aligned}&\langle \bar{t}, y - \bar{x}\rangle \geq (f(y) - f(\bar{x})),\\ &\langle \bar{t}, \frac{1}{2}\bar{x} - \bar{x}\rangle \geq \left(f\left(\frac{1}{2}\bar{x}\right) - f(\bar{x})\right),\\ &\langle \bar{t}, -\frac{1}{2}\bar{x}\rangle \geq -\frac{1}{2} f(\bar{x}),\end{aligned} \tag{23}$$

thus,

$$\begin{aligned}&\langle \bar{t}, t\bar{x}\rangle \leq f(\bar{x}),\\ &\langle \bar{t}, t\bar{x}\rangle \oplus f(\bar{x}) \leq f(\bar{x})\oplus f(\bar{x}) = 0,\end{aligned} \tag{24}$$

that is,

$$\langle \bar{t}, t\bar{x}\rangle \oplus f(\bar{x}) \leq 0. \tag{25}$$

Combining (21) and (25), we have

$$\langle \bar{t}, t\bar{x}\rangle \oplus f(\bar{x}) = 0. \tag{26}$$

From generalized variational inequality involving XOR operations (12) and (16), we have

$$\begin{aligned}&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0,\\ &\langle \bar{t}, tyn - q\bar{x}\rangle \oplus ((f(y) - f(\bar{x}))\oplus(f(y) - f(\bar{x}))) \geq 0\oplus(f(y) - f(\bar{x})),\end{aligned} \tag{27}$$

which implies that

$$\begin{aligned}&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus 0 \geq 0\oplus(f(y) - f(\bar{x})),\\ &\langle \bar{t}, tyn - q\bar{x}\rangle \geq (f(y) - f(\bar{x})),\\ &\langle \bar{t}, ty\rangle - \langle \bar{t}\\ &\langle \bar{t}, ty\rangle - f(\bar{x}) \geq f(y) - f(\bar{x}),\\ &\langle \bar{t}, ty\rangle \geq f(y),\\ &\langle \bar{t}, ty\rangle \oplus f(y) \geq f(y)\oplus f(y) = 0,\end{aligned} \tag{28}$$

thus, we have $\langle \bar{t}, ty\rangle \oplus f(y) \geq 0$. So, we have $\bar{x} \in S_{C\oplus}$. That is, $S_{1\oplus} \subseteq S_{C\oplus}$. □

Theorem 2. *The following statements are true.*

(i) $S_{1\oplus} \subseteq S_{2\oplus}$

(ii) If F *is monotone, then* $S_{2\oplus} \subseteq S_{3\oplus}$

(iii) If F *is upper hemicontinuous and* f *is convex, then* $S_{3\oplus} \subseteq S_{2\oplus}$

Proof

(i) Is trivial

(ii) Let $\bar{x} \in S_{2\oplus}$. Then, for all $y \in C$, there exists $\bar{t} \in F(\bar{x})$ such that

$$\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0. \tag{29}$$

Since F is monotone, for every $y \in C, t \in F(y)$, and using the above inequality, we have

$$\begin{aligned}&\langle t - \bar{t}, y - \bar{x}\rangle \geq 0,\\ &\langle t, y - \bar{x}\rangle \geq \langle \bar{t}, tyn - q\bar{x}\rangle,\\ &\langle t, y - \bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq \langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0,\end{aligned} \tag{30}$$

which implies that $\langle t, y - \bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0$. Thus, $\bar{x} \in S_{3\oplus}$.

(iii) Suppose that the conclusion is not true. Then, there exists $\bar{x} \in C$ such that $\bar{x} \in S_{3\oplus}$ and $\bar{x} \notin S_{2\oplus}$. Then, for some $y \in C$ and $t \in F(\bar{x})$, we have

$$\langle t, y - \bar{x}\rangle \oplus (f(y) - f(\bar{x})) < 0. \tag{31}$$

Since F is upper hemicontinuous and f is convex, setting $x_\lambda = \lambda y + (1 - \lambda)\bar{x}$ and taking $\lambda \longrightarrow 0$, we have

$$\begin{aligned}&\langle t_\lambda, y - \bar{x}\rangle \oplus (f(y) - f(\bar{x})) < 0, \quad \forall t_\lambda \in F(x_\lambda),\\ &\langle t_\lambda, y - \bar{x}\rangle \oplus ((f(y) - f(\bar{x}))\oplus(f(y) - f(\bar{x}))) < 0\oplus(f(y) - f(\bar{x})),\end{aligned} \tag{32}$$

which implies that

$$\begin{aligned}&\langle t_\lambda, y-\overline{x}\rangle < (f(y)-f(\overline{x})),\\&\langle t_\lambda, x_\lambda-\overline{x}\rangle < (f(x_\lambda)-f(\overline{x})),\\&\langle t_\lambda, x_\lambda-\overline{x}\rangle \oplus (f(x_\lambda)-f(\overline{x})) < (f(x_\lambda)-f(\overline{x}))\oplus(f(x_\lambda)-f(\overline{x})),\end{aligned} \tag{33}$$

thus,

$$\langle t_\lambda, x_\lambda-\overline{x}\rangle \oplus (f(x_\lambda)-f(\overline{x})) < 0, \tag{34}$$

which contradicts that $\overline{x}\in S_{3\oplus}$. Thus, $\overline{x}\in S_{2\oplus}$, and (iii) is true. □

Remark 1. If we replace ⊕ by + and dropping the concepts related to ⊕ operation, then with slight modification in Theorems 1 and 2, one can obtain some results of Huang et al. [19]. Additionally, for suitable choices of operators in Theorems 1 and 2, one can obtain some results of Farajzadeh and Harandi [30].

4. Existence and Convergence Result

In this section, we first establish the equivalence between the generalized variational inequality problem involving XOR operation (12) and a nonlinear equation. Based on this equivalence, we construct an iterative algorithm for solving generalized variational inequality problem involving XOR operation (12).

Lemma 1. *The generalized variational inequality problem involving XOR operation (12) admits a solution $(\overline{x}, t\overline{t})$, $\overline{x}\in C$ and $\overline{t}\in F(\overline{x})$, if and only if the following relation is satisfied:*

$$\overline{x} = \mathcal{J}_\rho^{\partial f}[\overline{x}+t\rho n\overline{t}], \tag{35}$$

where $\rho>0$ is a constant, $\mathcal{J}_\rho^{\partial f} = [I+\rho\partial f]^{-1}$ is the resolvent operator associated with f, and I is the identity operator.

Proof. From the definition of resolvent operator $\mathcal{J}_\rho^{\partial f}$ associated with f and relation (35), we have

$$\begin{aligned}\overline{x} &= \mathcal{J}_\rho^{\partial f}[\overline{x}\\&=[I+\rho\partial f]^{-1}[\overline{x}\end{aligned} \tag{36}$$

which implies that $\overline{x}+\rho\overline{t}\in \overline{x}+\rho\partial f(\overline{x})$, that is,

$$\overline{t}\in\partial f(\overline{x}). \tag{37}$$

By the definition of subdifferential operator $\partial f(\overline{x})$ and (37), we have

$$(f(y)-f(\overline{x}))\geq\langle\overline{t}, tyn-q\overline{x}\rangle. \tag{38}$$

Using (vi) of Proposition 1, we have

$$\begin{aligned}&\langle\overline{t}, tyn-q\overline{x}\rangle\oplus(f(y)-f(\overline{x}))\geq\langle\overline{t}\\&\langle\overline{t}, tyn-q\overline{x}\rangle\oplus(f(y)-f(\overline{x}))\geq 0.\end{aligned} \tag{39}$$

Thus, the generalized variational inequality problem involving XOR operation (12) is satisfied.

Conversely, suppose that generalized variational inequality problem involving XOR operation (12) is satisfied. That is,

$$\begin{aligned}&\langle\overline{t}, tyn-q\overline{x}\rangle\oplus(f(y)-f(\overline{x}))\geq 0,\\&\langle\overline{t}, tyn-q\overline{x}\rangle\oplus\langle\overline{t}\end{aligned} \tag{40}$$

that is, $(f(y)-f(\overline{x}))\geq\langle\overline{t}, tyn-q\overline{x}\rangle$, which implies that

$$\begin{aligned}\overline{t}&\in\partial f(\overline{x}),\\\rho\overline{t}&\in\rho\partial f(\overline{x}),\\\overline{x}+\rho\overline{t}&\in\overline{x}+\rho\partial f\\\overline{x}+\rho\overline{t}&\in[I+\rho\partial f](\overline{x}),\\\overline{x}&=[I+\rho\partial f]^{-1}[\overline{x}\\\overline{x}&=\mathcal{J}_\rho^{\partial f}[\overline{x}+\rho\overline{t}],\end{aligned} \tag{41}$$

that is, the relation (35) is satisfied.

Based on Lemma 1, we develop the following iterative algorithm for solving the generalized variational inequality problem involving XOR operation (12). □

Iterative Algorithm 1. Let $C\subset E$ be a pointed closed convex positive cone. Suppose that $\overline{t}_n\propto\overline{t}_{n-1}$, for $n=1,2,\ldots$. Let for $\overline{x}_0\in C$, there exists $t_0\in F(\overline{x}_0)$, such that

$$\overline{x}_1=(1-\alpha)\overline{x}_0+\alpha\mathcal{J}_\rho^{\partial f}[\overline{x}_0+\rho\overline{t_0}]. \tag{42}$$

Since $\overline{t}_0\in F(\overline{x}_0)\in CB(E)$, by Nadler [31], there exists $\overline{t}_1\in F(\overline{x}_1)$, using (iv) of Proposition 2, and as $\overline{t}_0\propto\overline{t}_1$, we have

$$\|\overline{t}_0\oplus\overline{t}_1\|=\|\overline{t}_0-\overline{t}_1\|\leq D(F(\overline{x}_0),F(\overline{x}_1)). \tag{43}$$

Continuing this way, compute the sequences $\{\overline{x}_n\}$ and $\{\overline{t}_n\}$ by the following scheme:

$$\overline{x}_{n+1}=(1-\alpha)\overline{x}_n+\alpha\mathcal{J}_\rho^{\partial f}[\overline{x}_n+\rho\overline{t}_n], \tag{44}$$

$$\|\overline{t}_n\oplus\overline{t}_{n-1}\|=\|\overline{t}_n-\overline{t}_{n-1}\|\leq D(F(\overline{x}_n),F(\overline{x}_{n-1})), \tag{45}$$

for $n=1,2,\ldots$, where $\overline{x}_n\in C$, $\overline{t}_n\in F(\overline{x}_n)$ can be chosen arbitrarily, $\alpha\in[0,1]$, $D(.,.)$ is the Hausdorff metric on $CB(E)$, and $\rho>0$ is a constant.

Now, we prove our main result.

Theorem 3. *Let E be a real ordered Banach space and C be a pointed closed convex positive cone in E with partial ordering "$\leq$." Let $f\colon C\longrightarrow\mathbb{R}\cup\{+\infty\}$ be a functional such that the resolvent operator $\mathcal{J}_\rho^{\partial f}$ associated with f is strongly comparison and continuous. Suppose that $F\colon C\longrightarrow 2^{E^*}\setminus\{\varnothing\}$ is a multivalued mapping such that F is the relaxed Lipschitz continuous with constant $k>0$ and D-Lipschitz continuous with constant $\lambda_{D_F}>0$. Let $\overline{x}_n\propto\overline{x}_{n-1}$ and $\overline{t}_n\propto\overline{t}_{n-1}$, where $\overline{t}_n\in F(\overline{x}_n)$ and $\overline{t}_{n-1}\in F(\overline{x}_{n-1})$, $n=1,2,\ldots$, such that for $\rho>0$, the following condition is satisfied:*

$$\left|\rho - \frac{k}{\lambda_{D_F}^2}\right| < \frac{k}{\lambda_{D_F}^2}. \tag{46}$$

Then, the sequences $\{\overline{x}_n\}$ and $\{\overline{t}_n\}$ strongly converge to x^* and t^*, respectively, the solutions of generalized variational inequality problem involving XOR operation (12).

Proof. Since $\overline{x}_{n+1} \propto \overline{x}_n$, for $n = 1, 2, \ldots$, using (iii) of Proposition 1, we evaluate

$$\begin{aligned} 0 &\le \overline{x}_{n+1} \oplus \overline{x}_n \\ &= \left[(1-\alpha)\overline{x}_n + \alpha \mathcal{J}_\rho^{\partial f}\left[\overline{x}_n + \rho \overline{t}_n\right]\right] \oplus \left[(1-\alpha)\overline{x}_{n-1} + \alpha \mathcal{J}_\rho^{\partial f}\left[\overline{x}_{n-1} + \rho \overline{t}_{n-1}\right]\right] \\ &\le (1-\alpha)\left(\overline{x}_n \oplus \overline{x}_{n-1}\right) + \alpha\left[\mathcal{J}_\rho^{\partial f}\left[\overline{x}_n + \rho \overline{t}_n\right] \oplus \mathcal{J}_\rho^{\partial f}\left[\overline{x}_{n-1} + \rho \overline{t}_{n-1}\right]\right]. \end{aligned} \tag{47}$$

From (47), it follows that

$$\begin{aligned} \left\|\overline{x}_{n+1} \oplus \overline{x}_n\right\| &= \left\|(1-\alpha)\left(\overline{x}_n \oplus \overline{x}_{n-1}\right) + \alpha\left[\mathcal{J}_\rho^{\partial f}\left[\overline{x}_n + \rho \overline{t}_n\right] \oplus \mathcal{J}_\rho^{\partial f}\left[\overline{x}_{n-1} + \rho \overline{t}_{n-1}\right]\right]\right\| \\ &\le (1-\alpha)\left\|\overline{x}_n \oplus \overline{x}_{n-1}\right\| + \alpha\left\|\mathcal{J}_\rho^{\partial f}\left[\overline{x}_n + \rho \overline{t}_n\right] \oplus \mathcal{J}_\rho^{\partial f}\left[\overline{x}_{n-1} + \rho \overline{t}_{n-1}\right]\right\|. \end{aligned} \tag{48}$$

As $\overline{x}_n \propto \overline{x}_{n-1}$, $\overline{t}_n \propto \overline{t}_{n-1}$, obviously, $\overline{x}_n + \rho \overline{t}_n \propto \overline{x}_{n-1} + \rho \overline{t}_{n-1}$, for $n = 1, 2, \ldots$. Since the resolvent operator $\mathcal{J}_\rho^{\partial f}$ is strongly comparison, we have

$$\mathcal{J}_\rho^{\partial f}\left[\overline{x}_n + \rho \overline{t}_n\right] \propto \mathcal{J}_\rho^{\partial f}\left[\overline{x}_{n-1} + \rho \overline{t}_{n-1}\right]. \tag{49}$$

Using above facts, (iv) of Proposition 2 and non-expansiveness of $\mathcal{J}_\rho^{\partial f}$, (48) becomes

$$\begin{aligned} \left\|\overline{x}_{n+1} - \overline{x}_n\right\| &\le (1-\alpha)\left\|\overline{x}_n - \overline{x}_{n-1}\right\| + \alpha\left\|\mathcal{J}_\rho^{\partial f}\left[\overline{x}_n + \rho \overline{t}_n\right] - \mathcal{J}_\rho^{\partial f}\left[\overline{x}_{n-1} + \rho \overline{t}_{n-1}\right]\right\| \\ &\le (1-\alpha)\left\|\overline{x}_n - \overline{x}_{n-1}\right\| + \alpha\left\|\left[\overline{x}_n + \rho \overline{t}_n\right] - \left[\overline{x}_{n-1} + \rho \overline{t}_{n-1}\right]\right\| \\ &= (1-\alpha)\left\|\overline{x}_n - \overline{x}_{n-1}\right\| + \alpha\left\|\overline{x}_n - \overline{x}_{n-1} + \rho\left(\overline{t}_n - \overline{t}_{n-1}\right)\right\|. \end{aligned} \tag{50}$$

Since the multivalued mapping F is the relaxed Lipschitz continuous with constant $k > 0$, D-Lipschitz continuous with constant $\lambda_{D_F} > 0$, and using (45) of Iterative Algorithm 1, we have

$$\begin{aligned} \left\|\overline{x}_n - \overline{x}_{n-1} + \rho\left(\overline{t}_n - \overline{t}_{n-1}\right)\right\|^2 &= \left\|\overline{x}_n - \overline{x}_{n-1}\right\|^2 + 2\rho\left\langle \overline{t}_n - \overline{t}_{n-1}, \overline{x}_n - \overline{x}_{n-1}\right\rangle + \rho^2\left\|\overline{t}_n - \overline{t}_{n-1}\right\|^2 \\ &\le \left\|\overline{x}_n - \overline{x}_{n-1}\right\|^2 - 2\rho k\left\|\overline{x}_n - \overline{x}_{n-1}\right\|^2 + \rho^2 \lambda_{D_F}^2\left\|\overline{x}_n - \overline{x}_{n-1}\right\|^2 \\ &= \left(1 - 2\rho k + \rho^2 \lambda_{D_F}^2\right)\left\|\overline{x}_n - \overline{x}_{n-1}\right\|^2, \end{aligned} \tag{51}$$

thus,

$$\begin{aligned} \left\|\overline{x}_n - \overline{x}_{n-1} + \rho\left(\overline{t}_n - \overline{t}_{n-1}\right)\right\| &\le \sqrt{\left(1 - 2\rho k + \rho^2 \lambda_{D_F}^2\right)}\left\|\overline{x}_n - \overline{x}_{n-1}\right\| \\ &= \theta\left\|\overline{x}_n - \overline{x}_{n-1}\right\|, \end{aligned} \tag{52}$$

where $\theta = \sqrt{1 - 2\rho k + \rho^2 \lambda_{D_F}^2}$.

Combining (50) and (52), we have

$$\begin{aligned} \left\|\overline{x}_{n+1} - \overline{x}_n\right\| &\le (1-\alpha)\left\|\overline{x}_n - \overline{x}_{n-1}\right\| + \alpha\theta\left\|\overline{x}_n - \overline{x}_{n-1}\right\| \\ &\le (1 - \alpha + \alpha\theta)\left\|\overline{x}_n - \overline{x}_{n-1}\right\|, \end{aligned} \tag{53}$$

thus, we have

$$\left\|\overline{x}_{n+1} - \overline{x}_n\right\| \le \gamma^n\left\|\overline{x}_1 - \overline{x}_0\right\|, \tag{54}$$

where $\gamma = (1 - \alpha + \alpha\theta)$. Hence, for $m > n > 0$, we have

$$\left\|\overline{x}_n - \overline{x}_m\right\| \le \sum_{i=n}^{m-1}\left\|\overline{x}_{i+1} - \overline{x}_i\right\| \le \left\|\overline{x}_1 - \overline{x}_0\right\| \sum_{i=n}^{m-1} \gamma^i. \tag{55}$$

It is clear from condition (46) that $0 < \gamma < 1$, and consequently, we have $\|\overline{x}_n - \overline{x}_m\| \longrightarrow 0$, as $n \longrightarrow \infty$. Thus, $\{\overline{x}_n\}$ is a Cauchy sequence in E, and as E is complete, $\overline{x}_n \longrightarrow x^* \in E$, as $n \longrightarrow \infty$. From (45) of Iterative Algorithm 1, we have

$$\begin{aligned} \left\|\overline{t}_n \oplus \overline{t}_{n-1}\right\| &= \left\|\overline{t}_n - \overline{t}_{n-1}\right\| \\ &\le D\left(F\left(\overline{x}_n\right), F\left(\overline{x}_{n-1}\right)\right) \\ &\le \lambda_{D_F}\left\|\overline{x}_n - \overline{x}_{n-1}\right\|, \end{aligned} \tag{56}$$

thus, $\{\overline{t}_n\}$ is also a Cauchy sequence in E such that $\overline{t}_n \longrightarrow t^* \in E$, as $n \longrightarrow \infty$. Now, we will show that (x^*, t^*) is a solution of generalized variational inequality problem involving XOR operation (12). As $\overline{x}_n \longrightarrow x^*, \overline{t}_n \longrightarrow t^*$, and resolvent operator $\mathcal{J}_\rho^{\partial f}$ is continuous, we can write

$$\begin{aligned} x^* &= \lim_{n\longrightarrow\infty} \overline{x}_{n+1} \\ &= \lim_{n\longrightarrow\infty}\left[(1-\alpha)\overline{x}_n + \alpha \mathcal{J}_\rho^{\partial f}\left[\overline{x}_n + \rho \overline{t}_n\right]\right] \\ &= (1-\alpha)\lim_{n\longrightarrow\infty}\overline{x}_n + \alpha \mathcal{J}_\rho^{\partial f}\left[\lim_{n\longrightarrow\infty}\overline{x}_n + \rho \lim_{n\longrightarrow\infty}\overline{t}_n\right] \\ &= (1-\alpha)x^* + \alpha \mathcal{J}_\rho^{\partial f}\left[x^* + \rho t^*\right]. \end{aligned} \tag{57}$$

Thus, the relation (35) is satisfied. It remains to show that $t^* \in F(x^*)$. Since $\overline{t}_n \in F(\overline{x}_n)$, we have

$$\begin{aligned} d(t^*, F(x^*)) &\leq \left\|t^* - \overline{t}_n\right\| + d(\overline{t}_n, F(x^*)) \\ &\leq \left\|t^* - \overline{t}_n\right\| + D(F(\overline{x}_n), F(x^*)) \\ &\leq \left\|t^* - \overline{t}_n\right\| + \lambda_{D_F}\left\|\overline{x}_n - x^*\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{58}$$

Hence $d(t^*, F(x^*)) \longrightarrow 0$, $t^* \in F(x^*)$ as $F(x^*) \in CB(E)$. By Lemma 1, $x^* \in C$, $t^* \in F(x^*)$ is a solution of generalized variational inequality problem involving XOR operation (12). This completes the proof. □

Remark 2. Combining Theorems 1 and 3, we assert that the solution $\overline{x} \in C, \overline{t} \in F(\overline{x})$ of generalized variational inequality involving XOR operation (12) is also a solution of generalized complementarity problem involving XOR operation (9).

5. Numerical Example

In this section, we construct a numerical example in support of Theorem 3. Finally, the convergence graphs and the computation tables are provided for the sequences generated by Iterative Algorithm 1.

Example 1. Let $E = E^* = \mathbb{R}$ with the usual inner product and norm. Let $C = \{\overline{x} \in t\mathbb{R}n\colon q0h \leq_{\overline{x}x} \leq 71\}$ be a pointed closed convex positive cone in $\mathbb{R}$. Let $f\colon C \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a functional, $\partial f\colon \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ be the subdifferential of f, $F\colon C \longrightarrow 2^{\mathbb{R}}\setminus\{\varnothing\}$ be a multivalued mapping, and $\mathcal{J}_\rho^{\partial f}$ be the resolvent operator associated with f such that

$$\begin{aligned} f(\overline{x}) &= 2\overline{x}^2 + 1, \\ F(\overline{x}) &= \left\{-\frac{\overline{x}}{7}\right\}, \quad \forall \overline{x} \in C. \end{aligned} \tag{59}$$

Then,

$$\begin{aligned} \partial f(\overline{x}) &= \{4\overline{x}\}, \\ \mathcal{J}_\rho^{\partial f}(\overline{x}) &= \left\{\frac{\overline{x}}{1+4\rho}\right\}, \quad \forall \overline{x} \in C. \end{aligned} \tag{60}$$

One can easily verify that the resolvent operator $\mathcal{J}_\rho^{\partial f}$ is a strongly comparison mapping and continuous.

For $\overline{x}, y \in C$, $w_1 \in F(\overline{x})$, and $w_2 \in F(y)$, we have

$$\begin{aligned} \langle w_1 - w_2, \overline{x} - y\rangle &= \left\langle -\frac{\overline{x}}{7} + \frac{y}{7}, \overline{x} - y\right\rangle \\ &= -\frac{1}{7}\|\overline{x} - ty\|^2 \\ &\leq -\frac{1}{10}\|\overline{x} - ty\|^2, \end{aligned} \tag{61}$$

that is,

$$\langle w_1 - w_2, \overline{x} - y\rangle \leq -\frac{1}{10}\|\overline{x} - ty\|^2. \tag{62}$$

Thus, F is the relaxed Lipschitz continuous with constant $k = (1/10)$.

Also,

$$\begin{aligned} D(F(\overline{x}), F(y)) &= \max\left\{\sup_{\overline{x}\in F(\overline{x})} d(\overline{x}, tFn(y)), \sup_{y\in F(y)} d(F(\overline{x}), y)\right\} \\ &\leq \max\left\{\left\|-\frac{\overline{x}}{7} + \frac{y}{7}\right\|, \left\|-\frac{y}{7} + \frac{\overline{x}}{7}\right\|\right\} \\ &= \frac{1}{7}\max\{\|\overline{x} - ty\|, \|\overline{x} - y\|\} \\ &\leq \frac{1}{7}\|\overline{x} - y\| \\ &\leq \frac{1}{5}\|\overline{x} - y\|, \end{aligned} \tag{63}$$

that is,

$$D(F(\overline{x}), F(y)) \leq \frac{1}{5}\|\overline{x} - ty\|. \tag{64}$$

Thus, F is the D-Lipschitz continuous with constant $\lambda_{D_F} = (1/5)$.

Let us take $\rho = 1$, then for $k = (1/10)$ and $\lambda_{D_F} = (1/5)$, the condition (46)

$$\left|\rho - \frac{k}{\lambda_{D_F}^2}\right| < \frac{k}{\lambda_{D_F}^2}, \tag{65}$$

is satisfied.

Furthermore, for $\rho = 1$ and $\alpha = (1/3)$, we obtain the sequences $\{\overline{x}_n\}$ and $\{\overline{t}_n\}$ generated by the Iterative Algorithm 1 as

$$\begin{aligned} \overline{x}_{n+1} &= (1-\alpha)\overline{x}_n + \alpha \mathcal{J}_\rho^{\partial f}\left[\overline{x}_n + \rho \overline{t}_n\right] \\ &= \frac{2}{3}\overline{x}_n + \frac{1}{15}\left[\overline{x}_n + \overline{t}_n\right], \end{aligned} \tag{66}$$

where $\overline{t}_n \in F(\overline{x}_n)$, and thus, $\overline{t}_n = -(\overline{x}_n/7)$. It is clear that the sequence $\{\overline{x}_n\}$ converges to $x^* = 0$, and consequently, the sequence $\{\overline{t}_n\}$ also converges to $t^* = 0$.

For initial values $\overline{x}_0 = 5, 10$, and 15, we have the following convergence graphs, which ensure that the sequences $\{\overline{x}_n\}$ and $\{\overline{t}_n\}$ converge to 0. Two computation tables are

Table 1: The values of x_n with initial values $x_0 = 5$, $x_0 = 10$, and $x_0 = 15$.

No. of Iteration	For $x_0 = 5$ x_n	For $x_0 = 10$ x_n	For $x_0 = 15$ x_n
$n = 1$	5	10	15
$n = 2$	3.61904761904762	7.23809523809524	10.8571428571429
$n = 3$	2.61950113378685	5.23900226757370	7.85850340136055
$n = 4$	1.89601986826477	3.79203973652953	5.68805960479430
$n = 5$	1.37235723798212	2.74471447596423	4.11707171394635
$n = 6$	0.993325238920389	1.98665047784078	2.97997571676117
$n = 7$	0.718978268170948	1.43795653634190	2.15693480451284
$n = 10$	0.272639416260542	0.545278832521084	0.817918248781626
$n = 14$	0.0748317352528748	0.149663470505750	0.224495205758624
$n = 18$	0.0205391747010088	0.0410783494020177	0.0616175241030265
$n = 21$	0.00778853666217476	0.0155770733243495	0.0233656099865243
$n = 25$	0.00213773093232492	0.00427546186464984	0.00641319279697477
$n = 26$	0.00154731000815899	0.00309462001631798	0.00464193002447697
$n = 27$	0	0	0
$n = 28$	0	0	0

Table 2: The values of t_n with initial values $x_0 = 5$, $x_0 = 10$, and $x_0 = 15$.

No. of Iteration	For $x_0 = 5$ t_n	For $x_0 = 10$ t_n	For $x_0 = 15$ t_n
$n = 1$	−0.714285714285714	−1.42857142857143	−2.14285714285714
$n = 2$	0.102040816326531	0.204081632653061	0.306122448979592
$n = 3$	−0.0145772594752187	−0.0291545189504373	−0.0437317784256560
$n = 4$	0.00208246563931695	0.00416493127863390	0.00624739691795085
$n = 5$	−0.000297495091330993	−0.000594990182661986	−0.000892485273992979
$n = 6$	4.24992987615704e − 05	8.49985975231408e − 05	0.000127497896284711
$n = 7$	−6.07132839451006e − 06	−1.21426567890201e-05	−1.82139851835302e − 05
$n = 10$	−1.23904661112450e − 07	3.54013317464143e − 08	5.31019976196215e − 08
$n = 14$	−5.16054398635777e − 11	1.47444113895936e − 11	2.21166170843905e − 11
$n = 18$	−2.14933110635476e − 14	6.14094601815645e − 15	9.21141902723467e − 15
$n = 21$	6.26627144709842e − 17	−1.79036327059955e − 17	−2.68554490589932e − 17
$n = 25$	2.60985899504307e − 20	5.21971799008614e − 20	7.82957698512922e − 20
$n = 26$	−3.72836999291867e − 21	−7.45673998583735e − 21	−1.11851099787560e − 20
$n = 27$	0	0	0
$n = 28$	0	0	0

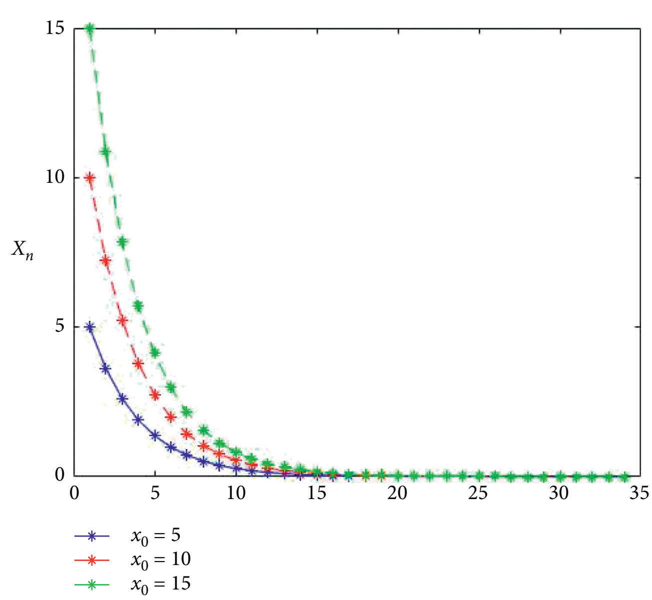


Figure 1: The convergence graph of the sequence $\{\overline{x}_n\}$ with initial values $x_0 = 5$, $x_0 = 10$, and $x_0 = 15$.

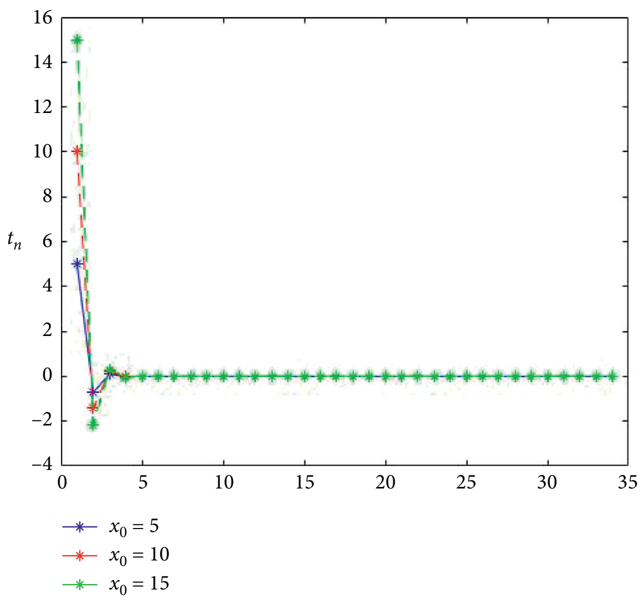


FIGURE 2: The convergence graph of the sequence $\{\bar{t}_n\}$ with initial values $x_0 = 5$, $x_0 = 10$, and $x_0 = 15$.

provided for the iterations (Tables 1 and 2) of the sequences $\{\bar{x}_n\}$ and $\{\bar{t}_n\}$ (Figures 1, and 2).

6. Conclusion

In this study, we introduce and study a generalized complementarity problem involving XOR operation with three classes of generalized variational inequalities involving XOR operation. Some equivalence relations are established between them. Finally, a generalized variational inequality problem involving XOR operation (12) is solved in real ordered Banach spaces. A numerical example is constructed with convergence graphs and computation tables for illustration of our main result.

We remark that our results may be further extended using other tools of functional analysis.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 11671365) and the Natural Science Foundation of Zhejiang Province (Grant no. LY14A010011).

References

[1] N. T. An, N. M. Nam, and X. Qin, "Solving k-center problems involving sets based on optimization techniques," *Journal of Global Optimization*, vol. 76, no. 1, pp. 189–209, 2020.

[2] T. H. Cuong, J. C. Yao, and N. D. Yen, "Qualitative properties of the minimum sum-of-squares clustering problem," *Optimization*, vol. 69, pp. 2131–2154, 2020.

[3] L. V. Nguyen and X. Qin, "The Minimal time function associated with a collection of sets," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 26, p. 93, 2020.

[4] X. Qin and N. T. An, "Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets," *Computational Optimization and Applications*, vol. 74, no. 3, pp. 821–850, 2019.

[5] D. R. Sahu, J. C. Yao, M. Verma, and K. K. Shukla, "Convergence rate analysis of proximal gradient methods with applications to composite minimization problems," *Optimization*, vol. 2020, Article ID 1702040, 2020.

[6] J. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, Germany, 1971.

[7] O. G. Mancino and G. Stampacchia, "Convex programming and variational inequalities," *Journal of Optimization Theory and Applications*, vol. 9, no. 1, pp. 3–23, 1972.

[8] S. Karamardian, "The complementarity problem," *Mathematical Programming*, vol. 2, no. 1, pp. 107–129, 1972.

[9] S. Karamardian, "Generalized complementarity problem," *Journal of Optimization Theory and Applications*, vol. 8, no. 3, pp. 161–168, 1971.

[10] S. Y. Cho, X. Qin, and L. Wang, "Strong convergence of a splitting algorithm for treating monotone operators," *Fixed Point Theory and Applications*, vol. 2014, p. 94, 2014.

[11] X. Qin, S. Y. Cho, and L. Wang, "A regularization method for treating zero points of the sum of two monotone operators," *Fixed Point Theory and Applications*, vol. 2014, p. 75, 2014.

[12] X. Qin, L. Wang, and J. C. Yao, "Inertial splitting method for maximal monotone mappings," *Journal of Nonlinear and Convex Analysis*, vol. 21, pp. 2325–2333, 2020.

[13] I. Ahmad, C. T. Pang, R. Ahmad, and M. Ishtyak, "System of Yosida inclusions involving XOR-operation," *Journal of Nonlinear and Convex Analysis*, vol. 18, pp. 831–845, 2017.

[14] H. G. Li, "A nonlinear inclusion problem involving (α, λ)-NODM set-valued mappings in ordered Hilbert space," *Applied Mathematics Letters*, vol. 25, pp. 1384–1388, 2012.

[15] H. G. Li, D. Qiu, and Y. Zou, "Characterizations of weak-ANODD set-valued mappings with applications to approximate solution of GNMOQV inclusions involving ⊕ operator in ordered Banach spaces," *Fixed Point Theory and Applications*, vol. 2013, p. 241, 2013.

[16] H. G. Li, L. P. Li, and M. M. Jin, "A class of nonlinear mixed ordered inclusion problems for ordered (α_A, λ)-ANODM set valued mappings with strong comparison mapping A," *Fixed Point Theory and Applications*, vol. 2014, p. 79, 2014.

[17] X. P. Ding and F. Q. Xia, "A new class of completely generalized quasi-variational inclusions in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 147, no. 2, pp. 369–383, 2002.

[18] H. G. Li, "Nonlinear inclusion problems for ordered RME set-valued mappings in ordered Hilbert spaces," *Nonlinear Functional Analysis and Applications*, vol. 16, pp. 1–8, 2001.

[19] N.-j. Huang, J. Li, and D. O'Regan, "Generalized complementarity problems in Banach spaces f complementarity

problems in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 12, pp. 3828–3840, 2008.

[20] F. F. Bazán and R. López, "Asymptotic analysis, existence and sensitivity results for a class of multivalued complementarity problems," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 12, pp. 271–293, 2006.

[21] G. Isac, "Complementarity problems," in *Lecture Notes in Mathematics*Springer-Verlag, Berlin, Germany, 1992.

[22] L.-C. Ceng, A. Petruşel, J.-C. Yao, and Y. Yao, "Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces," *Fixed Point Theory*, vol. 19, no. 2, pp. 487–502, 2018.

[23] R. W. Cottle, "Complementarity and variational problems," *Symposia on Mathematica*, vol. 19, pp. 177–208, 1976.

[24] P. T. Harker and J.-S. Pang, "Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications," *Mathematical Programming*, vol. 48, no. 1–3, pp. 161–220, 1990.

[25] Q. H. Ansari, M. Islam, and J.-C. Yao, "Nonsmooth variational inequalities on Hadamard manifolds," *Applicable Analysis*, vol. 99, no. 2, pp. 340–358, 2020.

[26] S. Y. Cho and X. Qin, "On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems," *Applied Mathematics and Computation*, vol. 235, pp. 430–438, 2014.

[27] A. Daniilidis and N. Hadjisavvas, "Coercivity conditions and variational inequalities," *Mathematical Programming*, vol. 86, no. 2, pp. 433–438, 1999.

[28] G. Isac, *Topological Methods in Complementarity Theory*, Kluwer Academic Publishers, Dordrecht, Netherlands, 2000.

[29] I. V. Konnov, "A scalarization approach for vector variational inequalities with applications," *Journal of Global Optimization*, vol. 32, no. 4, pp. 517–527, 2005.

[30] A. P. Farajzadeh and A. Amini-Harandi, "Generalized complementarity problems in Banach spaces," *Albanian Journal of Mathematics*, vol. 3, pp. 35–42, 2009.

[31] S. Nadler, "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6661682, 6 pages
https://doi.org/10.1155/2021/6661682

Research Article

An Inertial Method for Split Common Fixed Point Problems in Hilbert Spaces

Haixia Zhang [1] **and Huanhuan Cui** [2]

[1]Department of Mathematics, Henan Normal University, Xinxiang 453007, China
[2]Department of Mathematics, Luoyang Normal University, Luoyang 471934, China

Correspondence should be addressed to Haixia Zhang; zhx6132004@sina.com

Received 11 October 2020; Revised 8 December 2020; Accepted 16 December 2020; Published 4 January 2021

Academic Editor: Sun Young Cho

In this paper, we consider the split common fixed point problem in Hilbert spaces. By using the inertial technique, we propose a new algorithm for solving the problem. Under some mild conditions, we establish two weak convergence theorems of the proposed algorithm. Moreover, the stepsize in our algorithm is independent of the norm of the given linear mapping, which can further improve the performance of the algorithm.

1. Introduction

In recent years, there has been growing interest in the study of the split common fixed point problem because of its various applications in signal processing and image reconstruction [1–3]. More specifically, the problem consists in finding $\overline{x} \in H_1$ satisfying

$$\begin{aligned} \overline{x} &\in F(U), \\ A\overline{x} &\in F(T), \end{aligned} \tag{1}$$

where $F(U)$ and $F(T)$ stand for the fixed point sets of mappings $U\colon H_1 \longrightarrow H_1$ and $T\colon H_2 \longrightarrow H_2$, respectively, and $A\colon H_1 \longrightarrow H_2$ is a bounded linear mapping. Here, H_1 and H_2 are two Hilbert spaces. In particular, if we let the mappings in (1) be the projections, then it is reduced to the well-known split feasibility problem (SFP): find $\overline{x} \in H_1$ such that

$$\overline{x} \in C, A\overline{x} \in Q, \tag{2}$$

where $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex subsets and $A\colon H_1 \longrightarrow H_2$ is a bounded linear mapping; see, e.g., [1, 4–7].

There are several algorithms for solving the split common fixed point problem. Among them, Censor and Segal [8] introduced an algorithm as

$$x^{k+1} = U\left(x^k - \tau A^* (I - T) A x^k\right), \tag{3}$$

where I stands for the identity mapping, A^* is the adjoint mapping of A, and the stepsize τ is a constant in $(0, 2\|A\|^{-2})$. In particular, when $U = P_C$ and $T = P_Q$, then the above algorithm is reduced to the well-known CQ algorithm for solving the split feasibility problem [4]. Note that this choice of the stepsize requires the exact value or estimation of the norm $\|A\|$. To avoid the calculation of $\|A\|$, Cui and Wang [9] proposed a variable stepsize as

$$\tau_k = \frac{\left\|(I - T) A x^k\right\|^2}{\left\|A^* (I - T) A x^k\right\|^2}. \tag{4}$$

It is readily seen that the above choice of the stepsize does not need any prior knowledge of the linear operator. Recently, Wang [10] introduced a new method for solving (1) as

$$x^{k+1} = x^k - \tau_k\left[(I-U)x^k + A^*(I-T)Ax^k\right], \tag{5}$$

where the stepsize is set as

$$\tau_k = \frac{\left\|(I-U)x^k\right\|^2 + \left\|(I-T)Ax^k\right\|^2}{\left\|(I-U)x^k + A^*(I-T)Ax^k\right\|^2}. \tag{6}$$

Recently, the above algorithms were further extended to the general case; see, e.g., [2, 10–17].

The inertial method was first introduced in [18], and now, it has been successfully applied to solving various optimization problems arising from some applied sciences [19, 20]. In particular, this method was also applied for solving the split feasibility problem [21, 22]. By applying the inertial technique, Dang et al. [21] recently proposed the inertial relaxed CQ algorithm, which is defined as

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} = P_C\left(w^k - \tau A^*(I-P_Q)Aw^k\right), \end{cases} \tag{7}$$

where $0 \le \theta_k < \theta < 1$ and $0 < \tau < (2/\|A\|^2)$. It is clear that the constant stepsize requires the estimation of the norm $\|A\|$. To avoid the estimation of the norm, Gibali et al. [23] modified the above stepsize as

$$\begin{aligned} \tau_k &= \rho_k \frac{\left\|\left(I-P_Q\right)Aw^k\right\|^2}{\eta_k^2}, \\ \eta_k &= \max\left(1, \left\|A^*\left(I-P_Q\right)Aw^k\right\|\right), \end{aligned} \tag{8}$$

with $0 < \rho_k < 4$. It is shown that the inertial relaxed CQ algorithm converges weakly toward a solution of the SFP provided that $\sum_{k=1}^{\infty} \theta_k \|x^k - x^{k-1}\|^2 < \infty$. The main advantage of the inertial method is that it can indeed speed up the convergence of the original algorithm. It is thus natural to extend it to the split common fixed point problem. Recently, Cui et al. [24] proposed a modified algorithm of (3) as

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} = U\left(w^k - \tau_k A^*(I-T)Aw^k\right), \end{cases} \tag{9}$$

where $0 \le \theta_k < \theta < 1$ and τ_k is defined as in (6). It was shown that algorithm (9) converges weakly to a solution of the problem provided that $\sum_{k=1}^{\infty} \theta_k \|x^k - x^{k-1}\|^2 < \infty$.

In this paper, we aim to continue the study of the split common fixed point problem in Hilbert spaces. Motivated by the inertial method, we propose a new algorithm for solving the split common fixed point problem that greatly improves the performance of the original algorithm. Moreover, the stepsize in our algorithm is independent of the norm $\|A\|$. Under some mild conditions, we establish two weak convergence theorems of the proposed algorithm.

2. Preliminary

In the following, we shall assume that problem (1) is consistent, that is, its solution set denoted by $\mathcal{S}$ is nonempty. The notation " $\longrightarrow$ " stands for strong convergence, "$\rightharpoonup$" weak convergence, and $\omega_w\{x_n\}$ the set of weak cluster points of a sequence $\{x_n\}$. Let C be a nonempty closed convex subset. For a mapping T defined on C, we let $F(T) = \{x \in C\colon Tx = x\}$ be its fixed point set and $T' = I - T$ be its complement.

Definition 1. A mapping $T\colon C \longrightarrow H$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{10}$$

T is called quasi-nonexpansive if $F(T) \ne \varnothing$, and

$$\|Tx - y\| \le \|x - y\|, \quad \forall x \in C, y \in F(T). \tag{11}$$

Definition 2. Let $T\colon C \longrightarrow H$ be a mapping with $F(T) \ne \varnothing$. Then, T' is said to be demiclosed at 0 if, for any $\{x^k\}$ in C, there holds the following implication:

$$\left.\begin{aligned} x^k &\rightharpoonup x \\ T'x^k &\longrightarrow 0 \end{aligned}\right] \Rightarrow x \in F(T). \tag{12}$$

It is well known that if T is a nonexpansive mapping, then T' is demiclosed at 0; see [25].

Lemma 1 (see [25]). *If $T\colon C \longrightarrow H$ is quasi-nonexpansive, then*

$$2\langle x - z, T'x\rangle \ge \left\|T'x\right\|^2, \quad \forall z \in F(T), x \in C. \tag{13}$$

Lemma 2 (see [25]). *Assume that $\{x^k\}$ is a sequence in H such that*

(i) For each $z \in C$, the limit of $\{\|x^k - z\|\}$ exists

(ii) Any weak cluster point of $\{x^k\}$ belongs to C

Then, $\{x^k\}$ is weakly convergent to an element in C.

Lemma 3 (see [18]). *Let $\{\phi_k\}$ and $\{\delta_k\}$ be two nonnegative real sequences such that $\sum_{k=0}^{\infty} \delta_k < \infty$ and*

$$\phi_{k+1} - \phi_k \le \theta_k(\phi_k - \phi_{k-1}) + \delta_k, \tag{14}$$

where $0 \le \theta_k \le \theta < 1$. Then, the sequence $\{\phi_k\}$ is convergent.

Lemma 4 (see [25]). *Let $s, t \in \mathbb{R}$ and $x, y \in H$. It then follows that*

$$\|tx + sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - ts\|x - y\|^2. \tag{15}$$

3. The Proposed Algorithm

Algorithm 1. Let x^0, x^1 be arbitrary. Given x^k, x^{k-1}, choose $\theta_k \in [0, 1]$, and set

$$w^k = x^k + \theta_k\left(x^k - x^{k-1}\right). \tag{16}$$

If $\|U'w^k + A^*T'Aw^k\| = 0$, then stop; otherwise, update the next iteration via

$$x^{k+1} = w^k - \tau_k\left[U'w^k + A^*T'Aw^k\right], \tag{17}$$

where

$$\tau_k = \frac{\left\|U'w^k\right\|^2 + \left\|T'Aw^k\right\|^2}{2\left\|U'w^k + A^*T'Aw^k\right\|^2}. \tag{18}$$

Remark 1. In comparison, our stepsize (18) is independent of the norm $\|A\|$ so that the calculation or estimation of $\|A\|$ is avoided.

Remark 2. If $\|U'w^k + A^*T'Aw^k\| = 0$ for some $k \in \mathbb{N}$, then w^k is a solution of the problem. To see this, let $z \in \mathcal{S}$. It then follows from Lemma 1 that $\|U'w^k\|^2 \leq 2\langle U'w^k, w^k - z\rangle$, and

$$\left\|T'Aw^k\right\|^2 \leq 2\langle T'Aw^k, Aw^k - Az\rangle = \langle A^*T'Aw^k, w^k - z\rangle. \tag{19}$$

Combining these inequalities yields

$$\begin{aligned}\left\|U'w^k\right\|^2 + \left\|T'Aw^k\right\|^2 &\leq 2\langle U'w^k + A^*T'Aw^k, w^k - z\rangle \\ &\leq 2\left\|U'w^k + A^*T'Aw^k\right\|\left\|w^k - z\right\|.\end{aligned} \tag{20}$$

This yields $\|U'w^k\| = \|T'Aw^k\| = 0$, which implies $w^k \in \mathcal{S}$.

If we let $\theta_k \equiv 0$ in (16), then we get a new algorithm for problem (1).

Algorithm 2. Let x^0 be arbitrary. Given x^k, if $\|U'x^k + A^*T'Ax^k\| = 0$, then stop; otherwise, update the next iteration via

$$x^{k+1} = x^k - \tau_k\left[U'x^k + A^*T'Ax^k\right], \tag{21}$$

where

$$\tau_k = \frac{\left\|U'x^k\right\|^2 + \left\|T'Ax^k\right\|^2}{2\left\|U'x^k + A^*T'Ax^k\right\|^2}. \tag{22}$$

4. Convergence Analysis

In this section, we shall establish the convergence of the proposed algorithm. By Remark 2, we may assume that Algorithm 1 generates an infinite iterative sequence. To proceed, we first prove the following lemma.

Lemma 5. *Let $\{x^k\}$ and $\{w^k\}$ be the sequences generated by Algorithm 1. Let $\delta_k = (1/(4(1+\|A\|^2)))(\|U'w^k\|^2 + \|T'Aw^k\|^2)$. Then, for any $z \in S$, it follows that*

$$\left\|x^{k+1} - z\right\|^2 \leq \left\|w^k - z\right\|^2 - \delta_k. \tag{23}$$

Proof. Since U is quasi-nonexpansive, we have

$$\begin{aligned}\left\|x^{k+1} - z\right\|^2 &= \left\|w^k - \tau_k\left[U'w^k + A^*T'Aw^k\right] - z\right\|^2 \\ &= \left\|w^k - z\right\|^2 + \tau_k^2\left\|U'w^k + A^*T'Aw^k\right\|^2 \\ &\quad - 2\tau_k\langle U'w^k, w^k - z\rangle - 2\tau_k\langle T'Aw^k, Aw^k - Az\rangle \\ &\leq \left\|w^k - z\right\|^2 + \tau_k^2\left\|U'w^k + A^*T'Aw^k\right\|^2 \\ &\quad - \tau_k\left\|U'w^k\right\|^2 - \tau_k\left\|T'Aw^k\right\|^2.\end{aligned} \tag{24}$$

In view of (18), we have

$$\left\|x^{k+1} - z\right\|^2 \leq \left\|w^k - z\right\|^2 - \frac{\left(\left\|U'w^k\right\|^2 + \left\|T'Aw^k\right\|^2\right)^2}{4\left\|U'w^k + A^*T'Aw^k\right\|^2}. \tag{25}$$

To finish the proof, it suffices to note that

$$\begin{aligned}&\frac{\left(\left\|U'w^k\right\|^2 + \left\|T'Aw^k\right\|^2\right)^2}{\left\|U'w^k + A^*T'Aw^k\right\|^2} \\ &\geq \frac{\left(\left\|U'w^k\right\|^2 + \left\|T'Aw^k\right\|^2\right)^2}{\left(\left\|U'w^k\right\| + \|A\|\left\|T'Aw^k\right\|\right)^2} \\ &\geq \frac{\left(\left\|U'w^k\right\|^2 + \left\|T'Aw^k\right\|^2\right)^2}{\left(1 + \|A\|^2\right)\left(\left\|U'w^k\right\|^2 + \left\|T'Aw^k\right\|^2\right)} \\ &= \frac{1}{1 + \|A\|^2}\left(\left\|U'w^k\right\|^2 + \left\|T'Aw^k\right\|^2\right).\end{aligned} \tag{26}$$

This completes the proof. □

Theorem 1. *Assume that U is quasi-nonexpansive such that U' is demiclosed at 0, and T is quasi-nonexpansive such that T' is demiclosed at 0. If, for each $k \in \mathbb{N}$, $\theta_k \leq \theta < 1$ such that*

(c1) $\sum_{k=1}^{\infty} \theta_k\|x^k - x^{k-1}\|^2 < \infty$,
then the sequence $\{x^k\}$ generated by Algorithm 1 converges weakly to an element in $\mathcal{S}$.

Proof. We first show that the sequence $\{\|x^k - z\|\}$ is convergent for any $z \in \mathcal{S}$. From Lemma 4, we deduce

$$\begin{aligned}\left\|w^k - z\right\|^2 &= \left\|(1+\theta_k)\left(x^k - z\right) - \theta_k\left(x^{k-1} - z\right)\right\|^2 \\ &= (1+\theta_k)\left\|x^k - z\right\|^2 - \theta_k\left\|x^{k-1} - z\right\|^2 \\ &\quad + \theta_k(1+\theta_k)\left\|x^k - x^{k-1}\right\|^2.\end{aligned} \tag{27}$$

By Lemma 5, this yields

$$\begin{aligned}\left\|x^{k+1} - z\right\|^2 &\leq (1+\theta_k)\left\|x^k - z\right\|^2 - \theta_k\left\|x^{k-1} - z\right\|^2 \\ &\quad + 2\theta_k\left\|x^k - x^{k-1}\right\|^2 - \delta_k.\end{aligned} \tag{28}$$

Let $\phi_k\colon = \|x^k - z\|^2$. Then, the above inequality can be rewritten as

$$\phi_{k+1} - \phi_k \le \theta_k (\phi_k - \phi_{k-1}) + 2\theta_k \left\|x^k - x^{k-1}\right\|^2 - \delta_k. \quad (29)$$

By condition (c1), we then apply Lemma 3 to deduce that $\{\phi_k\}$ is convergent, and so is the sequence $\{\|x^k - z\|\}$.

We next show that each weak cluster point of $\{x^k\}$ belongs to $\mathcal{S}$. Since $\{\phi_k\}$ is convergent, this implies that $\phi_k - \phi_{k+1}$ converges to 0 as $n \longrightarrow \infty$. It then follows from (29) that

$$\begin{aligned}\delta_k &\le (\phi_k - \phi_{k+1}) + \theta_k (\phi_k - \phi_{k-1}) + 2\theta_k \left\|x^k - x^{k-1}\right\|^2 \\ &\le |\phi_k - \phi_{k+1}| + |\phi_k - \phi_{k-1}| + 2\theta_k \left\|x^k - x^{k-1}\right\|^2.\end{aligned} \quad (30)$$

Note that $\lim_k \theta_k \|x^k - x^{k-1}\|^2 = 0$ by condition (c1). By passing to the limit in the above inequality, we have δ_k converging to 0 so that

$$\lim_{k\longrightarrow\infty} \left\|U' w^k\right\| = \lim_{k\longrightarrow\infty} \left\|T' A w^k\right\| = 0. \quad (31)$$

Moreover, it is clear that $\{x^k\}$ is bounded; thus, the set $\omega_w(x_n)$ is nonempty. Now, take any $x \in \omega_w(x^k)$, and take a subsequence $\{x^{k_l}\}$ such that it weakly converges to x. On the contrary, we deduce from (c1) that

$$\left\|w^k - x^k\right\|^2 = \theta_k^2 \left\|x^k - x^{k-1}\right\| \le \theta_k \left\|x^k - x^{k-1}\right\| \longrightarrow 0 \quad (32)$$

so that $\{w^{k_l}\}$ also weakly converges to x and $\{Aw^{k_l}\}$ weakly converges to Ax. Since U' and T' are both demiclosed at 0, this together with (31) indicates $x \in F(U)$ and $Ax \in F(T)$; that is, x is an element in $\mathcal{S}$.

Finally, by Lemma 2, the sequence $\{x^k\}$ converges weakly to a solution of problem (1). □

Remark 3. We now construct a sequence satisfying condition (c1). For each $k \in \mathbb{N}$, let

$$\theta_k = \begin{cases} \min\left(0.5, \dfrac{1}{(k+1)^2 \left\|x^k - x^{k-1}\right\|^2}\right), & x^k \ne x^{k-1}, \\ 0.5, & x^k = x^{k-1}. \end{cases} \quad (33)$$

We next study the convergence of Algorithm 1 under another condition. To proceed, we need the following lemma.

Lemma 6. *Let $\{x^k\}$ and $\{w^k\}$ be the sequences generated by Algorithm 1. For any $z \in \mathcal{S}$, let $\phi_k = \|x^k - z\|^2 - \theta_k \|x^{k-1} - z\|^2 + (\theta_k/2)(3 + \theta_k)\|x^k - x^{k-1}\|^2$. If $\{\theta_k\}$ is nondecreasing, then*

$$2(\phi_k - \phi_{k+1}) \ge \left(1 - 4\theta_{k+1} - \theta_{k+1}^2\right)\left\|x^k - x^{k+1}\right\|^2 + \delta_k, \quad (34)$$

where δ_k is defined as in Lemma 5.

Proof. In view of (17) and (18), we get

$$\left\|x^{k+1} - w^k\right\|^2 = \frac{\left(\left\|U' w^k\right\|^2 + \left\|T' A w^k\right\|^2\right)^2}{4\left\|U' w^k + A^* T' A w^k\right\|^2}. \quad (35)$$

It then follows from inequality (25) that

$$\left\|x^{k+1} - z\right\|^2 \le \left\|w^k - z\right\|^2 - \frac{1}{2}\left\|x^{k+1} - w^k\right\|^2 - \frac{1}{2}\delta_k. \quad (36)$$

Moreover, it follows from (27) that

$$\begin{aligned}\left\|x^{k+1} - z\right\|^2 &\le (1 + \theta_k)\left\|x^k - z\right\|^2 - \theta_k \left\|x^{k-1} - z\right\|^2 \\ &\quad + 2\theta_k \left\|x^k - x^{k-1}\right\|^2 - \frac{1}{2}\left\|x^{k+1} - w^k\right\|^2 - \frac{1}{2}\delta_k.\end{aligned} \quad (37)$$

On the contrary, we have

$$\begin{aligned}\left\|w^k - x^{k+1}\right\|^2 &= \left\|x^k - x^{k+1} + \theta_k\left(x^k - x^{k-1}\right)\right\|^2 \\ &= \left\|x^k - x^{k+1}\right\|^2 + \theta_k^2 \left\|x^k - x^{k-1}\right\|^2 \\ &\quad + 2\theta_k \langle x^k - x^{k+1}, x^k - x^{k-1}\rangle \\ &\ge \left\|x^k - x^{k+1}\right\|^2 + \theta_k^2 \left\|x^k - x^{k-1}\right\|^2 \\ &\quad - 2\theta_k \left\|x^k - x^{k+1}\right\| \left\|x^k - x^{k-1}\right\| \\ &\ge \left\|x^k - x^{k+1}\right\|^2 + \theta_k^2 \left\|x^k - x^{k-1}\right\|^2 \\ &\quad - \theta_k\left(\left\|x^k - x^{k+1}\right\|^2 + \left\|x^k - x^{k-1}\right\|^2\right) \\ &= (1 - \theta_k)\left\|x^k - x^{k+1}\right\|^2 - \theta_k (1 - \theta_k)\left\|x^k - x^{k-1}\right\|^2.\end{aligned} \quad (38)$$

Substituting this into (21), we have

$$\begin{aligned}\left\|x^{k+1} - z\right\|^2 &\le (1 + \theta_k)\left\|x^k - z\right\|^2 - \theta_k \left\|x^{k-1} - z\right\|^2 \\ &\quad + \theta_k (1 + \theta_k)\left\|x^k - x^{k-1}\right\|^2 \\ &\quad - \frac{1}{2}(1 - \theta_k)\left\|x^k - x^{k+1}\right\|^2 \\ &\quad + \frac{\theta_k}{2}(1 - \theta_k)\left\|x^k - x^{k-1}\right\|^2 - \frac{1}{2}\delta_k.\end{aligned} \quad (39)$$

Since $\{\theta_k\}$ is nondecreasing, this implies

$$\begin{aligned}\left\|x^{k+1}-z\right\|^2 \le{}& (1+\theta_k)\left\|x^k-z\right\|^2 \\ &-\theta_k\left\|x^{k-1}-z\right\|^2+\frac{\theta_k}{2}(3+\theta_k)\left\|x^k-x^{k-1}\right\|^2 \\ &-\frac{1}{2}(1-\theta_k)\left\|x^k-x^{k+1}\right\|^2-\frac{1}{2}\delta_k \\ \le{}& (1+\theta_{k+1})\left\|x^k-z\right\|^2-\theta_k\left\|x^{k-1}-z\right\|^2 \\ &+\frac{\theta_k}{2}(3+\theta_k)\left\|x^k-x^{k-1}\right\|^2 \\ &-\frac{1}{2}(1-\theta_{k+1})\left\|x^k-x^{k+1}\right\|^2-\frac{1}{2}\delta_k.\end{aligned} \tag{40}$$

From the definition of ϕ_k, we get the desired inequality. □

Theorem 2. *Assume that U is quasi-nonexpansive such that U' is demiclosed at 0, and T is quasi-nonexpansive such that T' is demiclosed at 0. If*

(c2) $\{\theta_k\}$ is nondecreasing and converges to $\theta \in [0, \sqrt{5}-2)$,

then the sequence $\{x^k\}$ generated by Algorithm 1 converges weakly to an element in $\mathscr{I}$.

Proof. We first show that $\{\|x^k-z\|\}$ is convergent for each $z \in \mathscr{I}$. It then follows from Lemma 6 and the range of θ_k that

$$2(\phi_k-\phi_{k+1}) \ge \left(1-4\theta-\theta^2\right)\left\|x^k-x^{k+1}\right\|^2+\delta_k \ge 0 \tag{41}$$

so that $\{\phi_k\}$ is nonincreasing. From the definition of ϕ_k, we get

$$\left\|x^k-z\right\|^2 \le \theta_k\left\|x^{k-1}-z\right\|^2+\phi_k \le \theta\left\|x^{k-1}-z\right\|^2+\phi_1. \tag{42}$$

By induction, we have

$$\left\|x^k-z\right\|^2 \le \left\|x^0-z\right\|^2+\frac{\phi_1}{1-\theta}. \tag{43}$$

Thus, $\{x^k\}$ is bounded. Moreover, from the definition of ϕ_k,

$$\phi_{k+1} \ge -\theta_{k+1}\left\|x^k-z\right\|^2 \ge -\left\|x^k-z\right\|^2 \ge -\left\|x^0-z\right\|^2-\frac{\phi_1}{1-\theta}, \tag{44}$$

which implies that $\{\phi_k\}$ is bounded from below, and thus, it is convergent. Passing to the limit in (41) yields

$$\lim_{k\longrightarrow\infty}\delta_k=\lim_{k\longrightarrow\infty}\left\|x^{k+1}-x^k\right\|=0. \tag{45}$$

On the contrary,

$$\begin{aligned}&\theta_k\left|\left\|x^{k-1}-z\right\|^2-\left\|x^k-z\right\|^2\right| \\ &=\theta_k\left|\left\|x^{k-1}-z\right\|-\left\|x^k-z\right\|\right|\left(\left\|x^{k-1}-z\right\|+\left\|x^k-z\right\|\right) \\ &\le\left\|x^{k-1}-x^k\right\|\left(\left\|x^{k-1}-z\right\|+\left\|x^k-z\right\|\right)\longrightarrow 0,\end{aligned} \tag{46}$$

from which it follows that

$$\lim_{k\longrightarrow\infty}\left\|x^k-z\right\|^2=\frac{1}{1-\theta}\lim_{k\longrightarrow\infty}\phi_k. \tag{47}$$

Here, we used the fact (by the definition of ϕ_k) that

$$\left\|x^k-z\right\|^2=\frac{1}{1-\theta_k}\left(\phi_k+\theta_k\left(\left\|x^{k-1}-z\right\|^2-\left\|x^k-z\right\|^2\right)-\frac{\theta_k(3+\theta_k)}{2}\left\|x^k-x^{k-1}\right\|^2\right). \tag{48}$$

Thus, $\{\|x^k-z\|\}$ is convergent.

We next show that the sequence $\{x^k\}$ converges weakly to a solution of problem (1). By Lemma 2, it suffices to show that each weak cluster point of $\{x^k\}$ belongs to $\mathscr{I}$. Moreover, it is clear that $\{x^k\}$ is bounded; thus, the set $\omega_w(x_n)$ is nonempty. Now, take any $x \in \omega_w(x^k)$. On the contrary, we deduce from (16) and (45) that

$$\left\|w^k-x^k\right\|=\theta_k\left\|x^k-x^{k-1}\right\| \le \left\|x^k-x^{k-1}\right\|\longrightarrow 0. \tag{49}$$

In a similar way, we deduce that $x \in F(U)$ and $Ax \in F(T)$; that is, x is an element in $\mathscr{I}$. Hence, the proof is complete.

If we let $\theta_k \equiv 0$, then it satisfies (c1) and (c2). As a result, we get the following conclusion. □

Corollary 1. *Assume that U is quasi-nonexpansive such that U' is demiclosed at 0, and T is quasi-nonexpansive such that T' is demiclosed at 0. Then, the sequence $\{x^k\}$ generated by Algorithm 2 converges weakly to an element in $\mathscr{I}$.*

5. Concluding Remarks

The main contribution of this paper is to propose a new algorithm for solving the split common fixed point problem in Hilbert spaces. There are two advantages of the proposed algorithm. Compared with the original algorithm for solving the problem, our proposed algorithm is faster in convergence rate. Furthermore, the stepsize in the proposed algorithm is independent of the norm of the given linear mapping, which can further improve its performance.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Natural Science Foundation of China (no. 11701154) and Key Scientific Research Projects of Universities in Henan Province (nos. 19B110010 and 20A110029).

References

[1] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.
[2] G. López, "Solving the split feasibility problem without prior knowledge of matrix norms," *Inverse Problems*, vol. 28, p. 85004, 2012.
[3] H.-K. Xu, "Properties and iterative methods for the Lasso and its variants," *Chinese Annals of Mathematics, Series B*, vol. 35, no. 3, pp. 501–518, 2014.
[4] C. Byrne, "Iterative oblique projection onto convex sets and the split feasibility problem," *Inverse Problems*, vol. 18, no. 2, pp. 441–453, 2002.
[5] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2, pp. 221–239, 1994.
[6] H.-K. Xu, "A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem," *Inverse Problems*, vol. 22, no. 6, pp. 2021–2034, 2006.
[7] H.-K. Xu, "Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces," *Inverse Problems*, vol. 26, no. 10, Article ID 105018, 2010.
[8] Y. Censor and A. Segal, "The split common fixed point problem for directed operators," *Journal of Convex Analysis*, vol. 16, pp. 587–600, 2009.
[9] H. Cui and F. Wang, "Iterative methods for the split common fixed point problem in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 78, pp. 1–8, 2014.
[10] F. Wang, "A new method for split common fixed-point problem without priori knowledge of operator norms," *Journal of Fixed Point Theory and Applications*, vol. 19, no. 4, pp. 2427–2436, 2017.
[11] O. A. Boikanyo, "A strongly convergent algorithm for the split common fixed point problem," *Applied Mathematics and Computation*, vol. 265, pp. 844–853, 2015.
[12] A. Cegielski, "General method for solving the split common fixed point problem," *Journal of Optimization Theory and Applications*, vol. 165, no. 2, pp. 385–404, 2015.
[13] R. Kraikaew and S. Saejung, "On split common fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 415, no. 2, pp. 513–524, 2014.
[14] A. Moudafi, "The split common fixed point problem for demicontractive mappings," *Inverse Problems*, vol. 26, p. 55007, 2010.
[15] A. Moudafi, "A note on the split common fixed-point problem for quasi-nonexpansive operators," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 12, pp. 4083–4087, 2011.
[16] W. Takahahsi and J. C. Yao, "The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 20, pp. 173–195, 2019.
[17] F. Wang, "Strong convergence of two algorithms for the split feasibility problem in Banach spaces," *Optimization*, vol. 67, no. 10, pp. 1649–1660, 2018.
[18] F. Alvarez and H. Attouch, "An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping," *Set-Valued Analysis*, vol. 9, pp. 3–11, 2001.
[19] R. I. Boţ and E. R. Csetnek, "An inertial forward-backward-forward primal-dual splitting algorithm for solving monotone inclusion problems," *Numerical Algorithms*, vol. 71, pp. 519–540, 2016.
[20] D. A. Lorenz and T. Pock, "An inertial forward-backward algorithm for monotone inclusions," *Journal of Mathematical Imaging and Vision*, vol. 51, no. 2, pp. 311–325, 2015.
[21] Y. Dang, J. Sun, and H. Xu, "Inertial accelerated algorithms for solving a split feasibility problem," *Journal of Industrial and Management Optimization*, vol. 13, p. 78, 2016.
[22] X. Qin, L. Wang, and J. C. Yao, "Inertial splitting method for maximal monotone mappings," *Journal of Nonlinear and Convex Analysis*, vol. 21, pp. 2325–2333, 2020.
[23] A. Gibali, D. Mai, D. Thi Mai, and N. The Vinh, "A new relaxed CQ algorithm for solving split feasibility problems in Hilbert spaces and its applications," *Journal of Industrial & Management Optimization*, vol. 15, no. 2, pp. 963–984, 2019.
[24] H. Cui, H. Zhang, and L. Ceng, "An inertial censor-segal algorithm for split common fixed-point problems, to appear in fixed point theory,".
[25] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer-Verlag, Berlin, Germany, 2011.

Hindawi
Journal of Mathematics
Volume 2020, Article ID 6625056, 16 pages
https://doi.org/10.1155/2020/6625056

Research Article

Existence of Solutions for a Periodic Boundary Value Problem with Impulse and Fractional Derivative Dependence

Yaohong Li,[1] **Yongqing Wang**,[2] **Donal O'Regan**,[3] **and Jiafa Xu**[4]

[1]*School of Mathematics and Statistics, Suzhou University, Suzhou 234000, Anhui, China*
[2]*School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China*
[3]*School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, Galway, Ireland*
[4]*School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China*

Correspondence should be addressed to Yongqing Wang; wyqing9801@163.com

Received 10 October 2020; Revised 6 December 2020; Accepted 9 December 2020; Published 29 December 2020

Academic Editor: Xiaolong Qin

In this paper, we present some theorems on impulsive periodic boundary value problems with fractional derivative dependence. In particular, we discuss the existence of solutions of a class of fractional-order impulsive periodic boundary values with nonlinear terms and impulsive terms satisfying certain growth conditions. Three examples are provided to illustrate our results.

1. Introduction

This paper considers the existence of solutions of the following fractional-order impulsive periodic boundary value problem:

$$\begin{cases} {}^cD_t^q u(t) = f(t, u(t), {}^cD_t^\gamma u(t)), \quad t \in J', \\ \Delta u(t_k) = I_k(u(t_k), {}^cD_t^\gamma u(t_k)), \\ \Delta {}^cD_t^\gamma u(t_k) = J_k(u(t_k), {}^{cc}D_t^\gamma u(t_k)), \quad k = 1, 2, \ldots, m, \\ au(0) - bu(1) = 0, aD_t^\gamma u(0) - bD_t^\gamma u(1) = 0, \end{cases} \tag{1}$$

where ${}^cD_t^q$ and ${}^cD_t^\gamma$ represent the common Caputo derivatives of orders q and γ, and $1 < q < 2, 0 < \gamma < 1$, and $J = [0,1], 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1, J' = J\backslash \{t_1, t_2, \ldots, t_m\}$. Here, $f\colon J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $I_k, J_k\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. Now, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right limit and the left limit of $u(t)$ at the impulsive point t_k. Also, $\Delta {}^cD_t^\gamma u(t_k) = {}^cD_t^\gamma u(t_k^+) - {}^cD_t^\gamma u(t_k^-)$, where ${}^cD_t^\gamma u(t_k^+)$ and ${}^cD_t^\gamma u(t_k^-)$ denote the right limit and the left limit of ${}^cD_t^\gamma u(t)$ at the impulsive point t_k. If $u(t_k^-)$ and ${}^cD_t^\gamma u(t_k^-)$ exist, we let $u(t_k) = u(t_k^-)$ and ${}^cD_t^\gamma u(t_k) = {}^cD_t^\gamma u(t_k^-)$, where $k = 1, 2, \ldots, m$. Also, a and b are two real constants with $b > a > 0$.

The theory of fractional differential equation has received a lot of attention because of its wide application in mathematical models (see [1–27] and the references therein). Fractional-order impulsive differential equations are a natural generalization of the case of nonimpulses and are used to describe sudden changes in their states, such as in optimal control, population dynamics, biological systems, financial systems, and mechanical systems with impact. We refer the reader to [28–36] and the references therein. In particular, Bai et al. [37] investigated a mixed boundary value problem of nonlinear impulsive fractional differential equation:

$$\begin{cases} {}^cD_{0^+}^q u(t) = f(t, u(t)), \quad t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) + u(1) = 0, \\ u'(0) + u'(1) = 0, \end{cases} \tag{2}$$

and some sufficient conditions on the existence and uniqueness of solutions for problem (2) are obtained under Lipschitz conditions. In [38], Zhang and Xu studied the following impulsive periodic boundary value problem with the Caputo fractional derivative:

$$\begin{cases} {}^cD_t^q u(t) = f(t, u(t)), & t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \Delta u'(t_k) = J_k(u(t_k)), & k = 1, 2, \ldots, m, \\ au(0) - bu(1) = 0, \\ au'(0) - bu'(1) = 0, \end{cases} \tag{3}$$

using Green's function in [36], and via the symmetry property of Green's function and topological degree theory, the authors obtained the existence of positive solutions for (3) when the growth of f is superlinear and sublinear.

Inspired by the above research studies, in this paper, we consider fractional-order impulsive differential equations with generalized periodic boundary value conditions (1), where the nonlinear term, impulse terms, and periodic boundary conditions all depend on unknown functions and the lower-order fractional derivative of unknown functions. This is obviously more general and more widely applied, but it is also more complex and difficult to solve. Compared with (1), the nonlinear term, pulse term, and periodic boundary conditions of (3) are all independent of fractional derivatives, so it is a special form of (1). In this paper, we first give an equivalent integral form of solutions for problem (1) using some new Green's functions. Next, we present some sufficient conditions for the existence of solutions for problem (1), where the nonlinear and impulse terms satisfy some nonlinear and linear growth conditions, which are different from the conditions in [36–38]. Finally, we present three examples to illustrate our main results.

2. Preliminaries and Lemmas

In this section, we only present some necessary definitions and lemmas about fractional calculus.

Definition 1 (see [39, 40]). The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function $f\colon (0,\infty) \longrightarrow \mathbb{R}$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{4}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2 (see [39, 40]). The Caputo fractional derivative of order $\alpha > 0$ for a continuous and n-order differentiable function $f\colon (0,\infty) \longrightarrow \mathbb{R}$ is defined as

$${}^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{5}$$

where $\Gamma(\cdot)$ is the Euler gamma function and $n = [\alpha] + 1$, where $[\alpha]$ is the smallest integer greater than or equal to α.

Lemma 1 (see [39, 40]). *Let $\alpha > 0$. The differential equation ${}^cD_t^{\alpha} u(t) = 0$ has a unique solution:*

$$u(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, \tag{6}$$

for some $c_i \in \mathbb{R}\,(i = 0, 1, 2, \ldots, n-1)$, where $n = [\alpha] + 1$.

Lemma 2 *Let $y \in C(J)$ and $1 < q < 2$. The unique solution of the following periodic boundary value problem*

$$\begin{cases} {}^cD_t^q u(t) = y(t), & t \in J', \\ \Delta u(t_k) = I_k, \Delta^c D_t^{\gamma} u(t_k) = J_k, & k = 1, 2, \ldots, m, \\ au(0) - bu(1) = 0, aD_t^{\gamma} u(0) - bD_t^{\gamma} u(1) = 0, & 0 < a < b, \end{cases} \tag{7}$$

is expressed by

$$u(t) = \int_0^1 K_1(t,s) y(s) ds + \sum_{i=1}^m K_2(t,t_i) J_i + \sum_{i=1}^m K_3(t,t_i) I_i, \quad t \in J, \tag{8}$$

where

$$K_1(t,s) = \begin{cases} \dfrac{(t-s)^{q-1}}{\Gamma(q)} - \dfrac{b(1-s)^{q-1}}{(b-a)\Gamma(q)} - \dfrac{\Gamma(2-\gamma)(1-s)^{q-\gamma-1}}{\Gamma(q-\gamma)} \left(t - \dfrac{b}{b-a}\right), & 0 \le s \le t \le 1, \\ -\dfrac{b(1-s)^{q-1}}{(b-a)\Gamma(q)} - \dfrac{\Gamma(2-\gamma)(1-s)^{q-\gamma-1}}{\Gamma(q-\gamma)} \left(t - \dfrac{b}{b-a}\right), & 0 \le t \le s \le 1, \end{cases}$$

$$K_2(t,t_i) = \begin{cases} \dfrac{a\Gamma(2-\gamma) t_i}{(b-a) t_i^{1-\gamma}}, & 0 < t_i < t \le 1,\ i = 1, 2, \ldots, m, \\ \dfrac{\Gamma(2-\gamma)}{t_i^{1-\gamma}} \left(\dfrac{bt_i}{b-a} - t\right), & 0 \le t \le t_i < 1,\ i = 1, 2, \ldots, m, \end{cases} \tag{9}$$

$$K_3(t,t_i) = \begin{cases} -\dfrac{a}{a-b}, & 0 < t_i < t \le 1,\ i = 1, 2, \ldots, m, \\ -\dfrac{b}{a-b}, & 0 \le t \le t_i < 1,\ i = 1, 2, \ldots, m. \end{cases}$$

Furthermore,

$$ {}^{c}D_t^{\gamma}u(t) = \int_0^t H_1(t,s)y(s)\mathrm{d}s + \sum_{i=1}^{m} H_2(t,t_i)J_i, \qquad (10) $$

where

$$ H_1(t,s) = \begin{cases} \dfrac{(t-s)^{q-\gamma-1}}{\Gamma(q-\gamma)} - \dfrac{(1-s)^{q-\gamma-1}t^{1-\gamma}}{\Gamma(q-\gamma)}, & 0 \le s \le t \le 1, \\ -\dfrac{(1-s)^{q-\gamma-1}t^{1-\gamma}}{\Gamma(q-\gamma)}, & 0 \le t \le s \le 1, \end{cases} $$

$$ H_2(t,t_i) = \begin{cases} 0, & 0 < t_i < t \le 1,\ i = 1,2,\dots,m, \\ \left(\dfrac{t}{t_i}\right)^{1-\gamma}, & 0 \le t \le t_i < 1,\ i = 1,2,\dots,m. \end{cases} \qquad (11) $$

Proof. Suppose u is a general solution of (7) on each interval $(t_k, t_{k+1}]$ $(k = 0,1,2,\dots,m)$. Then, using Lemma 1, (7) can be transformed into the following equivalent integral equation:

$$ u(t) = I_{0+}^{q}y(t) - c_k - d_k t, \quad t \in (t_k, t_{k+1}], \qquad (12) $$

where $t_0 = 0$ and $t_{m+1} = 1$. Also, we have

$$ {}^{c}D_t^{\gamma}u(t) = I_{0+}^{q-\gamma}y(t) - d_k\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}, \quad t \in (t_k, t_{k+1}]. \qquad (13) $$

From (12) and (13), according to (7), we obtain

$$ ac_0 - bc_m = bd_m - bI_{0+}^{q}y(1), \qquad (14) $$

$$ d_m = \Gamma(2-\gamma)I_{0+}^{q-\gamma}y(1). \qquad (15) $$

Applying the right fractional-order impulsive condition of (7), we obtain

$$ d_{k-1} - d_k = \frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}J_k, \qquad (16) $$

$$ c_{k-1} - c_k = I_k - \frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}J_k. \qquad (17) $$

From (15) and (16), after a recursive calculation, we have

$$ d_0 = d_m + \Gamma(2-\gamma)\sum_{i=1}^{m}\frac{J_i}{t_i^{1-\gamma}} = \Gamma(2-\gamma)I_{0+}^{q-\gamma}y(1) + \Gamma(2-\gamma)\sum_{i=1}^{m}\frac{J_i}{t_i^{1-\gamma}}. \qquad (18) $$

Similar to (18), we see that

$$ d_k = d_0 - \Gamma(2-\gamma)\sum_{i=1}^{k}\frac{J_i}{t_i^{1-\gamma}} = \Gamma(2-\gamma)I_{0+}^{q-\gamma}y(1) + \Gamma(2-\gamma)\sum_{i=k+1}^{m}\frac{J_i}{t_i^{1-\gamma}}. \qquad (19) $$

From (13), (14), and (16), we have

$$ c_0 = \frac{b}{b-a}\left[I_{0+}^{q}y(1) - \Gamma(2-\gamma)I_{0+}^{q-\gamma}y(1) + \sum_{i=1}^{m}\left(I_i - \Gamma(2-\gamma)t_i^{\gamma}J_i\right)\right]. \qquad (20) $$

From (17) and (20), after a recursive calculation, we have

$$ \begin{aligned} c_k = c_0 - \sum_{i=1}^{k}\left[I_i - \Gamma(2-\gamma)t_i^{\gamma}J_i\right] &= \frac{b}{b-a}\left[I_{0+}^{q}y(1) - \Gamma(2-\gamma)I_{0+}^{q-\gamma}y(1)\right] \\ &\quad + \frac{a}{b-a}\sum_{i=1}^{k}\left[I_i - \Gamma(2-\gamma)t_i^{\gamma}J_i\right] + \frac{b}{b-a}\sum_{i=1}^{m}\left[I_i - \Gamma(2-\gamma)t_i^{\gamma}J_i\right]. \end{aligned} \qquad (21) $$

For $t \in J_0 = [t_0, t_1]$, substituting (18) and (20) into (12) and (13), we obtain

$$ \begin{aligned} u(t) &= I_{0+}^{q}y(t) - \frac{b}{b-a}I_{0+}^{q}y(1) - \Gamma(2-\gamma)\left(t - \frac{b}{b-a}\right)I_{0+}^{q-\gamma}y(1) \\ &\quad - \sum_{i=1}^{m}\frac{\Gamma(2-\gamma)}{t_i^{1-\gamma}}\left(t - \frac{bt_i}{b-a}\right)J_i - \frac{b}{b-a}\sum_{i=1}^{m}I_i \\ &\quad - \Gamma(2-\gamma)\left(t - \frac{b}{b-a}\right)\left(\int_0^t + \int_t^1\right)\frac{(1-s)^{q-\gamma-1}}{\Gamma(q-\gamma)}y(s)\mathrm{d}s + \sum_{i=1}^{m}\frac{\Gamma(2-\gamma)}{t_i^{1-\gamma}}\left(\frac{bt_i}{b-a} - t\right)J_i - \frac{b}{b-a}\sum_{i=1}^{m}I_i \end{aligned} $$

$$
\begin{aligned}
&= \int_0^t K_1(t,s)y(s)\mathrm{d}s + \sum_{i=1}^m K_2(t,t_i)J_i + \sum_{i=1}^m K_3(t,t_i)I_i,\\
{}^cD_t^\gamma u(t) &= I_{0+}^{q-\gamma}y(t) - \left[I_{0+}^{q-\gamma}y(1) + \sum_{i=1}^m \frac{J_i}{t_i^{1-\gamma}}\right]t^{1-\gamma}\\
&= \int_0^t \frac{(t-s)^{q-\gamma-1}}{\Gamma(q-\gamma)}y(s)\mathrm{d}s - \left(\int_0^t + \int_t^1\right)\frac{(1-s)^{q-\gamma-1}t^{1-\gamma}}{\Gamma(q-\gamma)}y(s)\mathrm{d}s - \sum_{i=1}^m \left(\frac{t}{t_i}\right)^{1-\gamma}J_i\\
&= \int_0^t H_1(t,s)y(s)\mathrm{d}s + \sum_{i=1}^m H_2(t,t_i)J_i,
\end{aligned}
\tag{22}
$$

where $K_1(t,s), K_2(t,t_i), K_3(t,t_i), H_1(t,s)$, and $H_2(t,t_i)$ are defined by (7) and (9).

For $J_k = [t_k, t_{k+1}], k = 1, 2, \ldots, m$, substituting (20) and (18) into (11) and (12), we have

$$
\begin{aligned}
u(t) &= I_{0+}^q y(t) - \frac{b}{b-a}I_{0+}^q y(1) - \Gamma(2-\gamma)\left(t - \frac{b}{b-a}\right)I_{0+}^{q-\gamma}y(1)\\
&\quad + \frac{a}{b-a}\sum_{i=1}^k \Gamma(2-\gamma)t_k^\gamma J_i - \sum_{i=1}^m \frac{\Gamma(2-\gamma)}{t_i^{1-\gamma}}\left(t - \frac{bt_i}{b-a}\right)J_i - \frac{a}{b-a}\sum_{i=1}^k I_i - \frac{b}{b-a}\sum_{i=1}^m I_i\\
&= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}y(s)\mathrm{d}s - \frac{b}{b-a}\left(\int_0^t + \int_t^1\right)\frac{(1-s)^{q-1}}{\Gamma(q)}y(s)\mathrm{d}s - \Gamma(2-\gamma)\left(t - \frac{b}{b-a}\right)\\
&\quad \cdot \left(\int_0^t + \int_t^1\right)\frac{(1-s)^{q-\gamma-1}}{\Gamma(q-\gamma)}y(s)\mathrm{d}s + \frac{a}{b-a}\sum_{i=1}^k \frac{\Gamma(2-\gamma)t_i}{t_i^{1-\gamma}}J_i + \sum_{i=1}^m \frac{\Gamma(2-\gamma)}{t_i^{1-\gamma}}\left(\frac{bt_i}{b-a} - t\right)J_i\\
&\quad - \frac{a}{b-a}\sum_{i=1}^k I_i - \frac{b}{b-a}\sum_{i=1}^m I_i = \int_0^t K_1(t,s)y(s)\mathrm{d}s + \sum_{i=1}^m K_2(t,t_i)J_i + \sum_{i=1}^m K_3(t,t_i)I_i,\\
{}^cD_t^\gamma u(t) &= I_{0+}^{q-\gamma}y(t) - \left[I_{0+}^{q-\gamma}y(1) + \sum_{i=1}^m \frac{J_i}{t_i^{1-\gamma}}\right]t^{1-\gamma}\\
&= \int_0^t \frac{(t-s)^{q-\gamma-1}}{\Gamma(q-\gamma)}y(s)\mathrm{d}s - \left(\int_0^t + \int_t^1\right)\frac{(1-s)^{q-\gamma-1}t^{1-\gamma}}{\Gamma(q-\gamma)}y(s)\mathrm{d}s - \sum_{i=1}^m \left(\frac{t}{t_i}\right)^{1-\gamma}J_i\\
&= \int_0^1 H_1(t,s)y(s)\mathrm{d}s + \sum_{i=1}^m H_2(t,t_i)J_i,
\end{aligned}
\tag{23}
$$

where $K_1(t,s), K_2(t,t_i), K_3(t,t_i), H_1(t,s)$, and $H_2(t,t_i)$ are defined by (9) and (11). The proof is completed. □

Lemma 3. *Let* $0 < a < b < +\infty$. *Then,* $K_1(t,s) + K_2(t,t_i)$ *and* $K_3(t,t_i)$ *and* $H_1(t,s)$ *and* $H_2(t,t_i)$ *defined as in (9) and (11) are continuous, and the following inequalities hold:*

(i) $|K_1(t,s)| \le ((2b-a)(1-s)^{q-1}/(b-a)\Gamma(q)) + ((2b-a)\Gamma(2-\gamma)(1-s)^{q-\gamma-1}/(b-a)\Gamma(q-\gamma)), |H_1(t,s)| \le (2(1-s)^{q-\gamma-1}/\Gamma(q-\gamma)), t, s \in J$

(ii) $|K_2(t,t_i)| \le b\Gamma(2-\gamma)/b-a$, $|H_2(t,t_i)| \le 1$, $|K_3(t,t_i)| \le b/b-a$, $t, t_i \in J$

.

Proof. Directly observe that

$$\begin{aligned}
\left|K_1(t,s)\right| &\le \frac{(1-s)^{q-1}}{\Gamma(q)} + \frac{b(1-s)^{q-1}}{(b-a)\Gamma(q)} + \frac{\Gamma(2-\gamma)(1-s)^{q-\gamma-1}}{\Gamma(q-\gamma)}\left(1+\frac{b}{b-a}\right)\\
&\le \frac{(2b-a)(1-s)^{q-1}}{(b-a)\Gamma(q)} + \frac{(2b-a)\Gamma(2-\gamma)(1-s)^{q-\gamma-1}}{(b-a)\Gamma(q-\gamma)}, \quad t,s\in J,\\
\left|H_1(t,s)\right| &\le \frac{(1-s)^{q-\gamma-1}}{\Gamma(q-\gamma)} + \frac{(1-s)^{q-\gamma-1}t^{1-\gamma}}{\Gamma(q-\gamma)} \le \frac{\left(1+t^{1-\gamma}\right)(1-s)^{q-\gamma-1}}{\Gamma(q-\gamma)}, \quad \le \frac{2(1-s)^{q-\gamma-1}}{\Gamma(q-\gamma)}, t,s\in J,\\
\left|K_2(t,t_i)\right| &\le \frac{\Gamma(2-\gamma)}{t_i^{1-\gamma}}\frac{bt_i}{b-a} \le \frac{b\Gamma(2-\gamma)}{b-a}t_i^{\gamma} \le \frac{b\Gamma(2-\gamma)}{b-a}, \quad t,t_i\in J,\\
\left|K_3(t,t_i)\right| &\le \frac{b}{a-b}, \quad t,t_i\in J,\\
\left|H_2(t,t_i)\right| &\le \left(\frac{t}{t_i}\right)^{1-\gamma} \le 1, \quad t,t_i\in J.
\end{aligned} \tag{24}$$

Let $E = \{u\colon J \longrightarrow \mathbb{R} | u \in C(J')$, ${}^cD_t^{\gamma}u(t) \in C(J')$, and $u(t_k^-), u(t_k^+), {}^cD_t^{\gamma}u(t_k^-)$, and ${}^cD_t^{\gamma}u(t_k^+)$ exist, where $k = 1, 2, \ldots, m\}$. Note [35] that E is a Banach space equipped with the norm

$$\|u\| = \sup_{t\in J}|u(t)| + \sup_{t\in J}\left|{}^cD_t^{\gamma}u(t)\right|. \tag{25}$$

Lemma 4. *If the function $f(t,u,{}^cD_t^{\gamma}u(t))$ is continuous, then $u \in E$ is a solution of (1) if and only if $u \in E$ is a solution of the following integral equation:*

$$\begin{aligned}
u(t) = &\int_0^1 K_1(t,s) f\left(s,u(s),{}^cD_t^{\gamma}u(s)\right)ds + \sum_{i=1}^m K_2(t,t_i)J_i\left(u(t_i),{}^cD_t^{\gamma}u(t_i)\right)\\
&+ \sum_{i=1}^m K_3(t,t_i)I_i\left(u(t_i),{}^cD_t^{\gamma}u(t_i)\right).
\end{aligned} \tag{26}$$

Proof. Assume that u satisfies (1). From Lemma 2, we see that u satisfies integral equation (26).

Conversely, assume that u satisfies integral equation (26). Applying Definition 2, by a direct fractional derivative computation, it follows that the solution given by (26) and (2) satisfies (1).

Define an operator $T\colon E \longrightarrow E$ as

$$\begin{aligned}
(Tu)(t) = &\int_0^1 K_1(t,s) f\left(s,u(s),{}^cD_t^{\gamma}u(s)\right)\mathrm{d}s + \sum_{i=1}^m K_2(t,t_i)J_i\left(u(t_i),{}^cD_t^{\gamma}u(t_i)\right)\\
&+ \sum_{i=1}^m K_3(t,t_i)I_i\left(u(t_i),{}^cD_t^{\gamma}u(t_i)\right),
\end{aligned} \tag{27}$$

$$\left({}^cD_t^{\gamma}Tu\right)(t) = \int_0^t H_1(t,s) f\left(s,u(s),{}^cD_t^{\gamma}u(s)\right)\mathrm{d}s + \sum_{i=1}^m H_2(t,t_i)J_i\left(u(t_i),{}^cD_t^{\gamma}u(t_i)\right). \tag{28}$$

It is easy to prove that the function u is a solution of (1) if and only if u is a fixed point of the operator T.

For convenience, we list some hypotheses:

(B1) $0 < a < b < +\infty, 1 < q < 2, 0 < \gamma < 1$ with $q - \gamma > 1$

(B2) $f\colon J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $I_k, J_k\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions □

Lemma 5. *Assume that (B1) and (B2) hold. Then, the operator $T\colon E \longrightarrow E$ defined as in (27) is completely continuous.*

Proof. We divide the proof into three steps. Set $\Omega_r = \{u \in E, \|u\| \le r\}$ for some $r > 0$. The steps are as follows:

(i) *Step 1.* T is continuous from the continuity of the functions $K_1, K_2, K_3, H_1, H_2, f, I_k, J_k$.

(ii) *Step 2.* T is uniformly bounded. Now, for $u \in \Omega_r$ we have $|f(t, u, {}^cD_t^\gamma u)| \le m_1, |J_k| \le m_2, |I_k| \le m_3$, where $m_i > 0, i = 1, 2, 3$.

In fact, for each $t \in J_k = [t_k, t_{k+1}], u \in \Omega_r$, $k = 0, 1, 2, \ldots, m$, from Lemma 3, we have

$$\begin{aligned}
|(Tu)(t)| &\le \int_0^1 \left|K_1(t,s) f\left(s, u(s), {}^cD_t^\gamma u(s)\right)\right| \mathrm{d}s + \sum_{i=1}^m \left|K_2(t,t_i) J_i\left(u(t_i), {}^cD_t^\gamma u(t_i)\right)\right| \\
&\quad + \sum_{i=1}^m \left|K_3(t,t_i) I_i\left(u(t_i), {}^cD_t^\gamma u(t_i)\right)\right| \le m_1 \int_0^1 |K_1(t,s)| \mathrm{d}s + m_2 \sum_{i=1}^m |K_2(t,t_i)| + m_3 \sum_{i=1}^m |K_3(t,t_i)|, \\
\left|\left({}^cD_t^\gamma Tu\right)(t)\right| &\le \int_0^1 \left|H_1(t,s) f\left(s, u(s), {}^cD_t^\gamma u(s)\right)\right| \mathrm{d}s + \sum_{i=1}^m \left|H_2(t,t_i) J_i\left(u(t_i), {}^cD_t^\gamma u(t_i)\right)\right| \\
&\le m_1 \int_0^1 |H_1(t,s)| \mathrm{d}s + m_2 \sum_{i=1}^m |H_2(t,t_i)|,
\end{aligned} \tag{29}$$

which and Lemma 4 imply that

$$\begin{aligned}
\|Tu\| &= \sup_{t\in J} |(Tu)(t)| + \sup_{t\in J} \left|\left({}^cD_t^\gamma Tu\right)(t)\right| \\
&\le m_1 \int_0^1 |K_1(t,s)| \mathrm{d}s + m_2 \sum_{i=1}^m |K_2(t,t_i)| + m_3 \sum_{i=1}^m |K_3(t,t_i)| + m_1 \int_0^1 |H_1(t,s)| \mathrm{d}s + m_2 \sum_{i=1}^m |H_2(t,t_i)| \\
&\le m_1 \int_0^1 \left[|K_1(t,s)| + |H_1(t,s)|\right] \mathrm{d}s + m_2 \sum_{i=1}^m \left[|K_2(t,t_i)| + |H_2(t,t_i)|\right] + m_3 \sum_{i=1}^m |K_3(t,t_i)| \\
&\le m_1 \int_0^1 \left[\frac{(2b-a)(1-s)^{q-1}}{(b-a)\Gamma(q)} + \frac{(2b-a)\Gamma(2-\gamma)(1-s)^{q-\gamma-1}}{(b-a)\Gamma(q-\gamma)} + \frac{2(1-s)^{q-\gamma-1}}{\Gamma(q-\gamma)}\right] \mathrm{d}s \\
&\quad + \frac{[b\Gamma(2-\gamma) + b - a] m m_2}{b-a} + \frac{b m m_3}{b-a} = m_1 \left[\frac{(2b-a)}{(b-a)\Gamma(q+1)} + \frac{(2b-a)\Gamma(2-\gamma)}{(b-a)\Gamma(q-\gamma+1)} + \frac{2}{\Gamma(q-\gamma+1)}\right] \\
&\quad + \frac{[b\Gamma(2-\gamma) + b - a] m m_2}{b-a} + \frac{b m m_3}{b-a} = M.
\end{aligned} \tag{30}$$

(iii) *Step 3.* T is equicontinuous. For any $t_1, t_2 \in J_k, k = 0, 1, \ldots, m$, fixed $s \in J$ and for any $\epsilon > 0$, there exists a constant $\delta > 0$ such that for $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
&\left|K_1(t_1, s) - K_1(t_2, s)\right| < \frac{\epsilon}{6m_1},\\
&\left|K_2(t_1, t_i) - K_2(t_2, t_i)\right| < \frac{\epsilon}{6m_2 m},\\
&\left|K_3(t_1, t_i) - K_3(t_2, t_i)\right| < \frac{\epsilon}{6m_3 m},\\
&\left|H_1(t_1, s) - H_1(t_2, s)\right| < \frac{\epsilon}{4m_1},\\
&\left|H_2(t_1, t_i) - H_2(t_2, t_i)\right| < \frac{\epsilon}{4m_2 m}.
\end{aligned} \tag{31}$$

Then,

$$\begin{aligned}
(Tu)(t_1) - (Tu)(t_2) &= \left|\int_0^1 (K_1(t_1, s) - K_1(t_2, s)) f(s, u(s), {}^c D_t^\gamma u(s)) \mathrm{d}s\right.\\
&\quad + \sum_{i=1}^m (K_2(t_1, t_i) - K_2(t_2, t_i)) J_i(u(t_i), {}^c D_t^\gamma u(t_i))\\
&\quad \left. + \sum_{i=1}^m (K_3(t_1, t_i) - K_3(t_2, t_i)) I_i(u(t_i), {}^c D_t^\gamma u(t_i))\right|\\
&\leq m_1 \int_0^1 \left|K_1(t_1, s) - K_1(t_2, s)\right| \mathrm{d}s + m_2 m \left|K_2(t_1, t_i) - K_2(t_2, t_i)\right|\\
&\quad + m_3 m \left|K_3(t_1, t_i) - K_3(t_2, t_i)\right| < \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{2},\\
\left|({}^c D_t^\gamma Tu)(t_1) - ({}^c D_t^\gamma Tu)(t_2)\right| &= \left|\int_0^1 (H_1(t_1, s) - H_1(t_2, s)) f(s, u(s), {}^c D_t^\gamma u(s)) \mathrm{d}s\right.\\
&\quad \left. + \sum_{i=1}^m (H_2(t_1, t_i) - H_2(t_2, t_i)) J_i(u(t_i), {}^c D_t^\gamma u(t_i))\right|\\
&\leq m_1 \int_0^1 \left|H_1(t_1, s) - H_1(t_2, s)\right| \mathrm{d}s + m_2 m \left|H_2(t_1, t_i) - H_2(t_2, t_i)\right| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned} \tag{32}$$

Thus,

$$\left\|(Tu)(t_1) - (Tu)(t_2)\right\| < \varepsilon. \tag{33}$$

which implies that $T(\Omega_r)$ is equicontinuous on any subinterval $J_k, k = 0, 1, \ldots, m$.

From the Arzela–Ascoli theorem, we deduce that $T\colon E \longrightarrow E$ is completely continuous.

Lemma 6 (Schauder fixed-point theorem, see [41, 42]). *Let X be a real Banach space, $C \subset X$ be a nonempty closed bounded and convex subset, and $F\colon C \longrightarrow C$ be compact. Then, T has at least one fixed point in C.*

Lemma 7 (Krasnoselskii fixed point theorem, see [41, 42]). *Let Ω be a closed convex and nonempty subset of a Banach space X. Let Φ and Ψ be the operators such that (i) $\Phi x + \Psi y \in \Omega$*

whenever $x, y \in \Omega$; (ii) $\Phi\colon \Omega \longrightarrow X$ is compact and continuous; and (iii) Ψ is a contraction mapping. Then, there exists an $z \in \Omega$ such that $z = \Phi z + \Psi z$.

Lemma 8 (Banach's fixed point theorem, see [43]). *Let E be a Banach space, $\Omega \subset E$ be closed, and $F\colon \Omega \longrightarrow \Omega$ be a strict contraction, i.e., $|Fx - Fy| \le k|x - y|$ for some $k \in (0,1)$ and all $x, y \in \Omega$. Then, F has a unique fixed point in Ω.*

3. Existence of the Solutions

For convenience, we give the following symbols:

$$\begin{aligned}
A_i &= \int_0^1 \left[\frac{(2b-a)(1-s)^{q-1}}{(b-a)\Gamma(q)} + \frac{(2b-a)\Gamma(2-\gamma)(1-s)^{q-\gamma-1}}{(b-a)\Gamma(q-\gamma)} + \frac{2(1-s)^{q-\gamma-1}}{\Gamma(q-\gamma)} \right] a_i(s)\mathrm{d}s,\\
B_i &= \frac{[b\Gamma(2-\gamma) + b - a]mb_i}{b-a},\\
C_i &= \frac{mbc_i}{b-a}, \quad i = 0,1,2.
\end{aligned} \tag{34}$$

Now, we present our main theorems.

Theorem 1 *Assume that (B1) and (B2) hold, and the following hypotheses are satisfied:*

(C1) There exist three nonnegative functions $a_0, a_1, a_2 \in L(J)$ and two constants $\lambda_1, \lambda_2 \in (0,1)$ such that

$$|f(t,u,v)| \le a_0(t) + a_1(t)|u|^{\lambda_1} + a_2(t)|v|^{\lambda_2}, \quad \forall t \in J,\ u, v \in \mathbb{R}, \tag{35}$$

(C2) There exist eight positive constants $b_1, b_2, c_1, c_2 \ge 0$ and $\mu_1, \mu_2, \nu_1, \nu_2 \in (0,1)$ such that

$$\begin{aligned}
\left|I_i(u,v)\right| &\le b_0 + b_1|u|^{\mu_1} + b_2|v|^{\mu_2},\\
\left|J_i(u,v)\right| &\le c_0 + c_1|u|^{\nu_1} + c_2|v|^{\nu_2}, \quad i = 1,2,\ldots,m,\ \forall u, v \in \mathbb{R}.
\end{aligned} \tag{36}$$

Then, (1) has at least one solution in E.

Proof. Let

$$\begin{aligned}
&R_1 \ge \max\left\{7(A_0 + B_0 + C_0), (7A_1)^{1/1-\lambda_1}, (7A_2)^{1/1-\lambda_2}, (7B_1)^{1/1-\mu_1}, (7B_2)^{1/1-\mu_2}, (7C_1)^{1/1-\nu_1}, (7C_1)^{1/1-\nu_2}\right\},\\
&\Omega_{R_1} = \{u \in E\colon \|u\| \le R_1\}.
\end{aligned} \tag{37}$$

Now, Ω_{R_1} is a closed bounded convex subset of E.

For each $u \in \Omega_{R_1}$, from (C1) and (C2), we have

$$\begin{aligned}
|(Tu)(t)| \le{}& \int_0^1 \left|K_1(t,s) f\left(s, u(s), {}^cD_t^\gamma u(s)\right)\right|\mathrm{d}s + \sum_{i=1}^m \left|K_2(t,t_i) J_i\left(u(t_i), {}^cD_t^\gamma u(t_i)\right)\right|\\
&+ \sum_{i=1}^m \left|K_3(t,t_i) I_i\left(u(t_i), {}^cD_t^\gamma u(t_i)\right)\right| \le \int_0^1 |K_1(t,s)|\left[a_0(s) + a_1(s)|u(s)|^{\lambda_1} + a_2(s)\left|{}^cD_t^\gamma u(s)\right|^{\lambda_2}\right]\mathrm{d}s
\end{aligned}$$

$$+\sum_{i=1}^{m}\left|K_2(t,t_i)\right|\left[b_0+b_1|u(s)|^{\mu_1}+b_2\left|{}^cD_t^{\gamma}u(s)\right|^{\mu_2}\right]+\sum_{i=1}^{m}\left|K_3(t,t_i)\right|\left[c_0+c_1|u(s)|^{\nu_1}+c_2\left|{}^cD_t^{\gamma}u(s)\right|^{\nu_2}\right],$$

$$\begin{aligned}\left|\left({}^cD_t^{\gamma}Tu\right)(t)\right| &\le \int_0^1\left|H_1(t,s)f\left(s,u(s),{}^cD_t^{\gamma}u(s)\right)\right|\mathrm{d}s+\sum_{i=1}^{m}\left|H_2(t,t_i)J_i\left(u(t_i),{}^cD_t^{\gamma}u(t_i)\right)\right| \\ &\le \int_0^1\left|H_1(t,s)\right|\left[a_0(s)+a_1(s)|u(s)|^{\lambda_1}+a_2(s)\left|{}^cD_t^{\gamma}u(s)\right|^{\lambda_2}\right]\mathrm{d}s \\ &\quad+\sum_{i=1}^{m}\left|H_2(t,t_i)\right|\left[b_0+b_1|u(s)|^{\mu_1}+b_2\left|{}^cD_t^{\gamma}u(s)\right|^{\mu_2}\right].\end{aligned} \tag{38}$$

From Lemma 3, we obtain that

$$\begin{aligned}\|Tu\| &= \sup_{t\in J}|u(t)|+\sup_{t\in J}\left|\left({}^cD_t^{\gamma}Tu\right)(t)\right| \\ &\le \int_0^1\left(\left|K_1(t,s)\right|+\left|H_1(t,s)\right|\right)\left(a_0(s)+a_1(s)\|u\|^{\lambda_1}+a_2(s)\|u\|^{\lambda_2}\right)\mathrm{d}s \\ &\quad+\sum_{i=1}^{m}\left(\left|K_2(t,t_i)\right|+\left|H_2(t,t_i)\right|\right)\left(b_0+b_1\|u\|^{\mu_1}+b_2\|u\|^{\mu_2}\right) \\ &\quad+\sum_{i=1}^{m}\left|K_3(t,t_i)\right|\left(c_0+c_1\|u\|^{\nu_1}+c_2\|u\|^{\nu_2}\right) \\ &\le A_0+A_1\left\|R_1\right\|^{\lambda_1}+A_2\left\|R_1\right\|^{\lambda_2}+B_0+B_1\left\|R_1\right\|^{\mu_1}+B_2\left\|R_1\right\|^{\mu_2} \\ &\quad+C_0+C_1\left\|R_1\right\|^{\nu_1}+C_2\left\|R_1\right\|^{\nu_2}\le R_1,\end{aligned} \tag{39}$$

which implies that $T(\Omega_{R_1})\subset\Omega_{R_1}$.

From Lemmas 5 and 6, T has at least one fixed point in Ω_{R_1}, so (1) has at least one solution in E.

Theorem 2. *Assume that (B1) and (B2) hold, and the following hypotheses are satisfied:*

(C3) There exists a nonnegative function $a_0\in L(J)$, such that

$$|f(t,u,v)|\le a_0(t),\quad \forall t\in J, u,v\in\mathbb{R}. \tag{40}$$

(C4) There exist four positive constants $b_1,b_2,c_1,c_2\ge 0$ such that

$$\begin{aligned}\left|I_i(u_1,v_1)-I_i(u_2,v_2)\right|&\le b_1\left|u_1-u_2\right|+b_2\left|v_1-v_2\right|,\quad \forall u_1,u_2,v_1,v_2\in\mathbb{R}, i=1,2,\ldots,m,\\ \left|J_i(u_1,v_1)-J_i(u_2,v_2)\right|&\le c_1\left|u_1-u_2\right|+c_2\left|v_1-v_2\right|,\quad \forall u_1,u_2,v_1,v_2\in\mathbb{R}, i=1,2,\ldots,m.\end{aligned} \tag{41}$$

If $\Lambda=\sum_{i=1}^{2}(B_i+C_i)<1/2$, then (1) has at least one solution in E.

Proof. We first define the operators. For $u\in E$, let

$$
\begin{aligned}
(\Phi u)(t) &= \int_0^1 K_1(t,s) f\left(s, u(s), {}^{c}D_t^{\gamma} u(s)\right) \mathrm{d}s, \\
(\Phi_1 u)(t) &= \int_0^1 H_1(t,s) f\left(s, u(s), {}^{c}D_t^{\gamma} u(s)\right) \mathrm{d}s \\
(\Psi u)(t) &= \sum_{i=1}^{m} K_2(t,t_i) J_i\left(u(t_i), {}^{c}D_t^{\gamma} u(t_i)\right) + \sum_{i=1}^{m} K_3(t,t_i) I_i\left(u(t_i), {}^{c}D_t^{\gamma} u(t_i)\right), \\
(\Psi_1 u)(t) &= \sum_{i=1}^{m} H_2(t,t_i) J_i\left(u(t_i), {}^{c}D_t^{\gamma} u(t_i)\right).
\end{aligned}
\tag{42}
$$

Now,

$$
\begin{aligned}
(Tu)(t) &= (\Phi u)(t) + (\Psi u)(t), \\
\left({}^{c}D_t^{\gamma} Tu\right)(t) &= {}^{c}D_t^{\gamma} (\Phi u)(t) + {}^{c}D_t^{\gamma} (\Psi u)(t) = (\Phi_1 u)(t) + (\Psi_1 u)(t).
\end{aligned}
\tag{43}
$$

Let

$$
\begin{aligned}
M_1 &= \max_{1\le i\le m} \left| I_i(0,0) \right|, \\
M_2 &= \max_{1\le i\le m} \left| J_i(0,0) \right|.
\end{aligned}
\tag{44}
$$

Let

$$
\begin{aligned}
R_2 &\ge \max\left\{ 2A_0, \frac{2\Theta}{1-2\Lambda} \right\}, \\
\Omega_{R_2} &= \left\{ u \in E\colon \|u\| \le R_2 \right\}.
\end{aligned}
\tag{45}
$$

Note that Ω_{R_2} is a nonempty bounded closed convex subset of E.

From Lemma 5, Φ is completely continuous (i.e., condition (ii) of Lemma 7 is satisfied).

For any $u, v \in \Omega_{R_2}$, from hypothesis (C4), we have

$$
\begin{aligned}
\left|(\Psi u)(t) - (\Psi v)(t)\right| &\le \sum_{i=1}^{m} \left|K_2\right| \left|(t,t_i) J_i\left(u(t_i), {}^{c}D_t^{\gamma} u(t_i)\right) - J_i\left(v(t_i), {}^{c}D_t^{\gamma} v(t_i)\right)\right| \\
&\quad + \sum_{i=1}^{m} \left|K_3\right| \left|(t,t_i) I_i\left(u(t_i), {}^{c}D_t^{\gamma} u(t_i)\right) - I_i\left(v(t_i), {}^{c}D_t^{\gamma} v(t_i)\right)\right| \\
&\le \sum_{i=1}^{m} \left|K_2(t,t_i)\right| \left(c_1 |u(s) - v(s)| + c_2 \left|{}^{c}D_t^{\gamma} u(s) - {}^{c}D_t^{\gamma} v(s)\right|\right) \\
&\quad \cdot \sum_{i=1}^{m} \left|K_3(t,t_i)\right| \left(b_1 |u(s) - v(s)| + b_2 \left|{}^{c}D_t^{\gamma} u(s) - {}^{c}D_t^{\gamma} v(s)\right|\right) \\
&\le \sum_{i=1}^{m} \left|K_2(t,t_i)\right| \left(c_1 + c_2\right) \|u - v\| + \sum_{i=1}^{m} \left|K_3(t,t_i)\right| \left(b_1 + b_2\right) \|u - v\|, \\
\left|(\Psi_1 u)(t) - (\Psi_1 v)(t)\right| &\le \sum_{i=1}^{m} \left|H_2\right| \left|(t,t_i) J_i\left(u(t_i), {}^{c}D_t^{\gamma} u(t_i)\right) - J_i\left(v(t_i), {}^{c}D_t^{\gamma} v(t_i)\right)\right| \\
&\le \sum_{i=1}^{m} \left|H_2(t,t_i)\right| \left(c_1 |u(s) - v(s)| + c_2 \left|{}^{c}D_t^{\gamma} u(s) - {}^{c}D_t^{\gamma} v(s)\right|\right) \\
&\le \sum_{i=1}^{m} \left|H_2(t,t_i)\right| \left(c_1 + c_2\right) \|u - v\|.
\end{aligned}
\tag{46}
$$

Therefore,

$$\begin{aligned}\|\Psi u-\Psi v\| &= \max_{t\in J}|\Psi u-\Psi v|+\max_{t\in J}\left|\Psi_1 u-\Psi_1 v\right| \\ &\le \sum_{i=1}^{m}\left[\left|K_2(t,t_i)\right|+\left|H_2(t,t_i)\right|\right](c_1+c_2)\|u-v\| \\ &\quad+\sum_{i=1}^{m}\left|K_3(t,t_i)\right|(b_1+b_2)\|u-v\| \\ &\le (C_1+C_2+B_1+B_2)\|u-v\|=\Lambda\|u-v\|,\end{aligned} \tag{47}$$

and since $\Lambda<1/2$, Ψ is a contraction (so condition (iii) of Lemma 7 is satisfied).

For each $u\in\Omega_{R_2}$, from hypothesis (C3), we have

$$\begin{aligned}|(\Phi u)(t)| &\le \int_0^1\left|K_1(t,s)f(s,u(s),{}^cD_t^\gamma u(s))\right|ds \\ &\le \int_0^1\left|K_1(t,s)\right|a_0(s)ds, \\ \left|(\Phi_1 u)(t)\right| &\le \int_0^1\left|H_1(t,s)f(s,u(s),{}^cD_t^\gamma u(s))\right|ds \\ &\le \int_0^1\left|H_1(t,s)\right|a_0(s)ds.\end{aligned} \tag{48}$$

Consequently,

$$\|\Phi u\|\le\int_0^1\left(|H_1(t,s)|+\left|H_1(t,s)\right|\right)a_0(s)ds\le A_0\le\frac{R_2}{2}. \tag{49}$$

For each $v\in\Omega_{R_2}$, we have

$$\begin{aligned}\|(\Psi v)\| &\le \|(\Psi 0)-(\Psi v)\|+\|(\Psi 0)\| \\ &\le \Lambda\|0-v\|+\sum_{i=1}^{m}\left(|K_2(t,t_i)|+\left|H_2(t,t_i)\right|\right)\max_{t\in J}\left|J_i(0,0)\right| \\ &\quad+\sum_{i=1}^{m}\left|K_3(t,t_i)\right|\max_{t\in J}\left|I_i(0,0)\right|\le\Lambda\|v\|+\Theta,\end{aligned} \tag{50}$$

where

$$\Theta=m\left(\frac{b\Gamma(2-\gamma)+b-a}{b-a}M_2+\frac{b}{b-a}M_1\right). \tag{51}$$

Thus, for any $u,v\in\Omega_{R_2}$, we obtain

$$\|(\Phi u)+(\Psi v)\|\le\|(\Phi u)\|+\|(\Psi v)\|\le A_0+\Lambda\|v\|+\Theta\le R_2, \tag{52}$$

which implies that $\Phi u+\Psi v\in\Omega_{R_2}$ (so condition (i) of Lemma 7 is satisfied).

In view of Lemma 7, there exists a $u\in\Omega_{R_2}$ such that $\Phi u+\Psi u=u$, so (1) has at least one solution in E. □

Theorem 3. *Assume that (B1), (B2), and (C4) hold and the following hypothesis is satisfied:*

(C5) There exist two nonnegative functions $a_1,a_2\in L(J)$ such that

$$\left|f(t,u_1,v_1)-f(t,u_2,v_2)\right|\le a_1(t)\left|u_1-u_2\right|+a_2(t)\left|v_1-v_2\right|,\quad \forall t\in J,u_1,u_2,v_1,v_2\in\mathbb{R}. \tag{53}$$

If $\Pi=(A_1+A_2)+[b\Gamma(2-\gamma)+b-a](c_1+c_2)+b(b_1+b_2)/b-a<1$, then (1) has a unique solution in E.

Proof. Choose

$$R_3\ge\frac{1}{1-\Pi}\{A'M_0+B'\}, \tag{54}$$

where

$$\begin{aligned}M_0 &= \max_{t\in J}|f(t,0,0)|, \\ M_1 &= \max_{1\le i\le m}\left|I_i(0,0)\right|, \\ M_2 &= \max_{1\le i\le m}\left|J_i(0,0)\right|, \\ A' &= \frac{(2b-a)}{(b-a)\Gamma(q+1)}+\frac{(2b-a)\Gamma(2-\gamma)}{(b-a)\Gamma(q-\gamma+1)}+\frac{2}{\Gamma(q-\gamma+1)}, \\ B' &= \frac{b\Gamma(2-\gamma)M_2+bM_1+b-a}{b-a}.\end{aligned} \tag{55}$$

First, we show that $T\Omega_{R_3} \subset \Omega_{R_3}$, where $\Omega_{R_3} = \{u \in E, \|u\| \le R_3\}$. For $u \in \Omega_{R_3}$, from hypotheses (C4) and (C5), we obtain

$$\begin{aligned}
|(Tu)(t)| \le & \int_0^1 |K_1(t,s)| \left[\left| f\left(s,u(s),{}^cD_t^\gamma u(s)\right) - f(t,0,0) \right| + |f(t,0,0)| \right] \mathrm{d}s \\
& + \sum_{i=1}^m |K_2(t,t_i)| \left[\left| J_i\left(u(t_i),{}^cD_t^\gamma u(t_i)\right) - J_i(0,0) \right| + |J_i(0,0)| \right] \\
& + \sum_{i=1}^m |K_3(t,t_i)| \left[\left| I_i\left(u(t_i),{}^cD_t^\gamma u(t_i)\right) - I_i(0,0) \right| + |I_i(0,0)| \right] \\
\le & \int_0^1 |K_1(t,s)| \left[(a_1(t) + a_2(t))\|u\| + M_0 \right] \mathrm{d}s + \sum_{i=1}^m |K_2(t,t_i)| \left[(c_1 + c_2)\|u\| + M_2 \right] \\
& + \sum_{i=1}^m \|K_3(t,t_i)\left[(b_1 + b_2)\|u\| + M_1 \right], \\
\left|\left({}^cD_t^\gamma Tu\right)(t)\right| \le & \int_0^1 |H_1(t,s)| \left[\left| f\left(s,u(s),{}^cD_t^\gamma u(s)\right) - f(t,0,0) \right| + |f(t,0,0)| \right] \mathrm{d}s \\
& + \sum_{i=1}^m |H_2(t,t_i)| \left[\left| J_i\left(u(t_i),{}^cD_t^\gamma u(t_i)\right) - J_i(0,0) \right| + |J_i(0,0)| \right] \\
\le & \int_0^1 |H_1(t,s)| \left[(a_1(t) + a_2(t))\|u\| + M_0 \right] \mathrm{d}s \\
& + \sum_{i=1}^m |H_2(t,t_i)| \left[(c_1 + c_2)\|u\| + M_2 \right].
\end{aligned} \tag{56}$$

Then,

$$\begin{aligned}
\|Tu\| \le & \int_0^1 \left[|K_1(t,s)| + |H_1(t,s)| \right] \left[(a_1(t) + a_2(t))\|u\| + M_0 \right] \mathrm{d}s \\
& + \sum_{i=1}^m \left[|K_2(t,t_i)| + |H_2(t,t_i)| \right] \left[(c_1 + c_2)\|u\| + M_2 \right] \\
& + \sum_{i=1}^m |K_3(t,t_i)| \left[(b_1 + b_2)\|u\| + M_1 \right] \le \Pi\|u\| + A' M_0 + B' \le R_3,
\end{aligned} \tag{57}$$

so $T\Omega_{R_3} \subset \Omega_{R_3}$.

Furthermore, from hypotheses (C4) and (C5), for all $u, v \in \Omega_{R_3}$, we have

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| \le & \int_0^1 |K_1(t,s)| \left| f\left(s,u(s),{}^cD_t^\gamma u(s)\right) - f\left(s,v(s),{}^cD_t^\gamma v(s)\right) \right| \mathrm{d}s \\
& + \sum_{i=1}^m |K_2(t,t_i)| \left| J_i\left(u(t_i),{}^cD_t^\gamma u(t_i)\right) - J_i\left(v(t_i),{}^cD_t^\gamma v(t_i)\right) \right| \\
& + \sum_{i=1}^m |K_3(t,t_i)| \left| I_i\left(u(t_i),{}^cD_t^\gamma u(t_i)\right) - I_i\left(v(t_i),{}^cD_t^\gamma v(t_i)\right) \right|
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 \left|K_1(t,s)\right|\left(a_1(t)+a_2(t)\right)\|u-v\|\mathrm{d}s + \sum_{i=1}^{m}\left|K_2(t,t_i)\right|\left(c_1+c_2\right)\|u-v\| \\
&\quad + \sum_{i=1}^{m}\left|K_3\right\|(t,t_i)\left(b_1+b_2\right)\|u-v\|\right|,
\end{aligned}
$$

$$
\begin{aligned}
\left|\left({}^cD_t^{\gamma}Tu\right)(t)-\left({}^cD_t^{\gamma}Tv\right)(t)\right| &\le \int_0^1 \left|H_1(t,s)\right|\left|f\left(s,u(s),{}^cD_t^{\gamma}u(s)\right)-f\left(s,v(s),{}^cD_t^{\gamma}v(s)\right)\right|\mathrm{d}s \\
&\quad + \sum_{i=1}^{m}\left|H_2(t,t_i)\right|\left|J_i\left(u(t_i),{}^cD_t^{\gamma}u(t_i)\right)-J_i\left(v(t_i),{}^cD_t^{\gamma}v(t_i)\right)\right| \\
&\le \int_0^t \left|H_1(t,s)\right|\left(a_1(t)+a_2(t)\right)\|u-v\|\mathrm{d}s + \sum_{i=1}^{m}\left|H_2(t,t_i)\right|\left(c_1+c_2\right)\|u-v\|.
\end{aligned} \tag{58}
$$

Thus,

$$
\begin{aligned}
\|Tu-Tv\| &\le \int_0^1 \left[\left|K_1(t,s)\right|+\left|H_1(t,s)\right|\right]\left(a_1(t)+a_2(t)\right)\|u-v\|\mathrm{d}s \\
&\quad + \sum_{i=1}^{m}\left[\left|K_2(t,t_i)\right|+\left|H_2(t,t_i)\right|\right]\left(c_1+c_2\right)\|u\| + \sum_{i=1}^{m}\left\|K_3(t,t_i)\left(b_1+b_2\right)\right\|u\| \\
&\le \Pi\|u-v\|,
\end{aligned} \tag{59}
$$

where $\Pi < 1$, so T is a contraction. Lemma 8 guarantees that T has a unique fixed point in Ω_{R_3}, which is the unique solution of (1) in E. This completes the proof. □

4. Examples

In (1), let $q = 1.25, \gamma = 0.15, a = 1, b = 2, t_1 = 0.5$, and $k = 1$ and then, we obtain the following fractional-order impulsive differential equation:

$$
\begin{cases}
{}^cD_t^{1.25}u(t) = f\left(t,u(t),{}^cD_t^{0.15}u(t)\right), & t \in (0,1), t \neq 0.5, \\
\Delta u(0.5) = I_1\left(u(0.5),{}^cD_t^{0.15}u(0.5)\right), \\
\Delta {}^cD_t^{0.15}u(0.5) = J_1\left(u(0.5),{}^cD_t^{0.15}u(0.5)\right), \\
u(0) - 2u(1) = 0, \\
{}^cD_t^{0.15}u(0) - 2{}^cD_t^{0.15}u(1) = 0.
\end{cases} \tag{60}
$$

By a direct observation, note that $0 < a < b < +\infty$, $1 < q < 2, 0 < \gamma < 1$ with $q - \gamma > 1$, so hypothesis (B1) is satisfied.

Example 1. In (60), let

$$
\begin{aligned}
f\left(t,u(t),{}^cD_t^{0.15}u(t)\right) &= \frac{e^t}{50} + \frac{(1-t)^2(u(t))^{0.2}}{100} + \frac{e^{2t}\left({}^cD_t^{0.15}u(t)\right)^{0.3}}{200}, \\
\Delta u(0.5) &= \sin\left(\frac{1+2(u(0.5))^{0.5}+3\left({}^cD_t^{0.15}u(0.5)\right)^{0.4}}{150}\right), \\
\Delta {}^cD_t^{0.25}u(0.5) &= \sin\left(\frac{1+3(u(0.5))^{0.2}+2\left({}^cD_t^{0.15}u(0.5)\right)^{0.1}}{120}\right),
\end{aligned} \tag{61}
$$

so hypothesis (B2) is satisfied. Set ${}^cD_t^{0.15}u(t) = v(t)$, and then, we obtain

$$\begin{aligned}
|f(t,u,v)| &\le \frac{e^t}{50} + \frac{(1-t)^2}{100}|u|^{0.2} + \frac{e^{2t}}{200}|v|^{0.3} = a_0(t) + a_1(t)|u|^{0.2} + a_2(t)|v|^{0.3},\\
|I_1(u,v)| &\le \frac{1}{150} + \frac{1}{75}|u|^{0.5} + \frac{1}{50}|v|^{0.4} = b_0 + b_1|u|^{0.5} + b_2|v|^{0.4},\\
|J_1(u,v)| &\le \frac{1}{120} + \frac{1}{40}|u|^{0.2} + \frac{1}{60}|v|^{0.1} = c_0 + c_1|u|^{0.2} + c_2|v|^{0.1},
\end{aligned} \tag{62}$$

which implies that (C1) and (C2) are satisfied. Thus, all the hypotheses in Theorem 1 are satisfied, so (60) has at least one solution in E.

Example 2. In (60), let

$$\begin{aligned}
f\left(t,u(t),{}^cD_t^{0.15}u(t)\right) &= \frac{(1-s)^2}{50} \times \frac{u(t) + {}^cD_t^{0.15}u(t)}{1 + u(t) + {}^cD_t^{0.15}u(t)},\\
\Delta u(0.5) &= \frac{1 + 2u(0.5) + 3{}^cD_t^{0.15}u(0.5)}{150},\\
\Delta {}^cD_t^{0.15}u(0.5) &= \frac{1 + 3u(0.5) + 2{}^cD_t^{0.15}u(0.5)}{120},
\end{aligned} \tag{63}$$

so hypothesis (B2) is satisfied. Set ${}^cD_t^{0.15}u(t) = v(t)$, and then, we obtain

$$\begin{aligned}
|f(t,u,v)| &\le \frac{(1-s)^2}{50} = a_0(t),\\
\left|I_1(u_1,v_1) - I_1(u_2,v_2)\right| &\le \frac{1}{75}\left|u_1-u_2\right| + \frac{1}{50}\left|v_1-v_2\right|\\
&= b_1\left|u_1-u_2\right| + b_2\left|v_1-v_2\right|,\\
\left|J_1(u_1,v_1) - J_1(u_2,v_2)\right| &\le \frac{1}{40}\left|u_1-u_2\right| + \frac{1}{60}\left|v_1-v_2\right|\\
&= c_1\left|u_1-u_2\right| + c_2\left|v_1-v_2\right|,
\end{aligned} \tag{64}$$

which implies that (C3) and (C4) are satisfied. Also, note that $\Lambda = 0.179707 < 0.5$. Then, all the hypotheses in Theorem 2 are satisfied, so (60) has at least one solution in E.

Example 3. In (60), let

$$\begin{aligned}
f\left(t,u(t),{}^cD_t^{0.15}u(t)\right) &= \frac{e^t}{50} + \frac{(1-s)^2u(t)}{100} + \frac{\sqrt{(1-s)}{}^cD_t^{0.15}u(t)}{200},\\
\Delta u(0.5) &= \frac{1 + 2u(0.5) + 3{}^cD_t^{0.15}u(0.5)}{150},\\
\Delta {}^cD_t^{0.15}u(0.5) &= \frac{1 + 3u(0.5) + 2{}^cD_t^{0.15}u(0.5)}{120},
\end{aligned} \tag{65}$$

so hypothesis (B2) is satisfied. Set ${}^cD_t^{0.15}u(t) = v(t)$, and then, we obtain

$$\begin{aligned}
\left|f(t,u_1,v_1) - f(t,u_2,v_2)\right| &\le \frac{(1-s)^2}{100}\left|u_1-u_2\right| + \frac{\sqrt{1-s}}{200}\left|v_1-v_2\right| = a_1(t)\left|u_1-u_2\right| + a_2(t)\left|v_1-v_2\right|,\\
\left|I_1(u_1,v_1) - I_1(u_2,v_2)\right| &\le \frac{1}{75}\left|u_1-u_2\right| + \frac{1}{50}\left|v_1-v_2\right| = b_1\left|u_1-u_2\right| + b_2\left|v_1-v_2\right|,\\
\left|J_1(u_1,v_1) - J_1(u_2,v_2)\right| &\le \frac{1}{40}\left|u_1-u_2\right| + \frac{1}{60}\left|v_1-v_2\right| = c_1\left|u_1-u_2\right| + c_2\left|v_1-v_2\right|,
\end{aligned} \tag{66}$$

which implies that (C4) and (C5) are satisfied. Note that $\Pi = 0.787135 < 1$. Then, all the hypotheses in Theorem 3 are satisfied, so (60) has a unique solution in E.

5. Conclusion

In this paper, we use fixed-point theorems to study fractional-order impulsive differential equation (1) with generalized periodic boundary value conditions. Very little is known on fractional-order impulsive differential equations with generalized periodic boundary value conditions where nonlinear terms and impulse terms depend on the unknown function and the lower-order fractional derivative of the unknown function. Our main results are obtained under some nonlinear and linear growth conditions corresponding to the relevant linear operators where the symmetry property of a Green's function is not required, so our results generalize and improve works in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration with all authors. All authors read and approved the final manuscript.

Acknowledgments

This work was supported by the University Natural Science Foundation of Anhui Provincial Education Department (Grant nos. KJ2019A0672 and KJ2018A0452), the Foundation of Suzhou University (Grant no. 2016XJGG13), the Natural Science Foundation of Chongqing (Grant no. cstc2020jcyj-msxmX0123), and the Technology Research Foundation of Chongqing Educational Committee (Grant nos. KJQN2019 00539 and KJQN202000528).

References

[1] Y. Wang and H. Wang, "Triple positive solutions for fractional differential equation boundary value problems at resonance," *Applied Mathematics Letters*, vol. 106, Article ID 106376, 2020.

[2] X. Hao, L. Zhang, and L. Liu, "Positive solutions of higher order fractional integral boundary value problem with a parameter," *Nonlinear Analysis: Modelling and Control*, vol. 24, no. 2, pp. 210–223, 2019.

[3] B. Zhu, L. Liu, and Y. Wu, "Existence and uniqueness of global mild solutions for a class of nonlinear fractional reaction-diffusion equations with delay," *Computers & Mathematics with Applications*, vol. 78, no. 6, pp. 1811–1818, 2019.

[4] J. Xu, C. S. Goodrich, and Y. Cui, "Positive solutions for a system of first-order discrete fractional boundary value problems with semipositone nonlinearities," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, no. 2, pp. 1343–1358, 2019.

[5] H. Zhang, Y. Li, and J. Xu, "Positive solutions for a system of fractional integral boundary value problems involving hadamard-type fractional derivatives," *Complexity*, vol. 2019, Article ID 2671539, 11 pages, 2019.

[6] Y. Li, J. Liu, D. O'Regan, and J. Xu, "Nontrivial solutions for a system of fractional q-difference equations involving q-integral boundary conditions," *Mathematics*, vol. 8, no. 5, p. 828, 2020.

[7] X. Zhang, J. Xu, J. Jiang, Y. Wu, and Y. Cui, "The convergence analysis and uniqueness of blow-up solutions for a dirichlet problem of the general k-hessian equations," *Applied Mathematics Letters*, vol. 102, Article ID 106124, 2020.

[8] B. Di and H. Pang, "Existence results for the fractional differential equations with multi-strip integral boundary conditions," *Journal of Applied Mathematics and Computing*, vol. 59, no. 1-2, pp. 1–19, 2019.

[9] Y. Niu and B. Yan, "The existence of positive solutions for the singular two-point boundary value problem," *Topological Methods in Nonlinear Analysis*, vol. 49, no. 2, pp. 665–682, 2017.

[10] B. Yan, "Positive solutions for the singular nonlocal boundary value problems involving nonlinear integral conditions," *Boundary Value Problems*, vol. 38, no. 1, 2014.

[11] S. Meng and Y. Cui, "The extremal solution to conformable fractional differential equations involving integral boundary condition," *Mathematics*, vol. 7, no. 2, p. 186, 2019.

[12] Z. Ali, A. Zada, and K. Shah, "On ulam's stability for a coupled systems of nonlinear implicit fractional differential equations," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 5, pp. 2681–2699, 2019.

[13] H. Zhang, Y. Li, and J. Yang, "New sequential fractional differential equations with mixed-type boundary conditions," *Journal of Function Spaces*, vol. 2020, p. 9, Article ID 6821637, 2020.

[14] H. Zhang, Y. Wang, and J. Xu, "Explicit monotone iterative sequences for positive solutions of a fractional differential system with coupled integral boundary conditions on a half-line," *Advances in Difference Equations*, vol. 2020, p. 396, 2020.

[15] S. Harendra, G. Fahimeh Akhavan, and T. Emran, "Legendre spectral method for the fractional Bratu problem," *Mathematical Methods in the Applied Sciences*, vol. 43, pp. 5941–5952, 2020.

[16] S. Harendra, G. Fahimeh Akhavan, and T. Emran, "Analysis for fractional dynamics of Ebola virus model," *Chaos, Solitons & Fractals*, vol. 138, Article ID 109992, 2020.

[17] S. Harendra and H. M. Srivastava, "Numerical simulation for fractional-order bloch equation arising in nuclear magnetic resonance by using the jacobi polynomials," *Applied Sciences*, vol. 10, p. 285, 2020.

[18] B. Liu and Y. Liu, "Positive solutions of a two-point boundary value problem for singular fractional differential equations in banach space," *Journal of Function Spaces*, vol. 2013, p. 9, Article ID 585639, 2013.

[19] Y. Liu and H. Yu, "Bifurcation of positive solutions for a class of boundary value problems of fractional differential inclusions," *Abstract and Applied Analysis*, vol. 2013, Article ID 942831, 8 pages, 2013.

[20] Y. Liu, "Positive solutions using bifurcation techniques for boundary value problems of fractional differential equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 162418, 79 pages, 2013.

[21] T. Qi, Y. Liu, and Y. Cui, "Existence of solutions for a class of coupled fractional differential systems with nonlocal boundary conditions," *Journal of Function Spaces*, vol. 2017, Article ID 6703860, 2 pages, 2017.

[22] T. Qi, Y. Liu, and Y. Zou, "Existence result for a class of coupled fractional differential systems with integral boundary value conditions," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 07, pp. 4034–4045, 2017.

[23] Y. Wang, Y. Liu, and Y. Cui, "Multiple solutions for a nonlinear fractional boundary value problem via critical point theory," *Journal of Function Spaces*, vol. 2017, Article ID 8548975, 8 pages, 2017.

[24] Y. Wang, Y. Liu, and Y. Cui, "Multiple sign-changing solutions for nonlinear fractional Kirchhoff equations," *Boundary Value Problems*, vol. 2018, p. 193, 2018.

[25] M. Kamenskii, N. Voskovskaya, and M. Zvereva, "On periodic oscillations of some points of a string with a nonlinear boundary condition," *Applied Set-Valued Analysis Optimation*, vol. 2, no. 1, pp. 35–48, 2020.

[26] D. Li, M. Pei, and Z. Yang, "Existence and multiplicity of positive solutions of a third-order periodic boundary value problem with one-parameter," *Journal of Nonlinear Functional Analysis*, vol. 2020, p. 14, 2020.

[27] A. Lachouri, A. Ardjouni, and A. Djoudi, "Positive solutions of a fractional integro-differential equation with integral boundary conditions," *Communications in Optimization Theory*, vol. 2020, p. 1, 2020.

[28] Y. Liu and D. O'Regan, "Controllability of impulsive functional differential systems with nonlocal conditions," *Electronic Journal of Differential Equations*, vol. 194, pp. 1–10, 2013.

[29] Y. Liu, "Bifurcation techniques for a class of boundary value problems of fractional impulsive differential equations," *Journal of Nonlinear Sciences and Applications*, vol. 09, no. 04, pp. 340–353, 2015.

[30] Y. Wang, Y. Liu, and Y. Cui, "Infinitely many solutions for impulsive fractional boundary value problem with p-laplacian," *Boundary Value Problem*, vol. 2018, p. 94, 2018.

[31] X. Xu, Y. Liu, H. Li, and F. E. Alsaadi, "Synchronization of switched boolean networks with impulsive effects," *International Journal of Biomathematics*, vol. 11, no. 06, Article ID 1850080, 2018.

[32] A. Zada, S. Fatima, Z. Ali, J. Xu, and Y. Cui, "Stability results for a coupled system of impulsive fractional differential equations," *Mathematics*, vol. 7, no. 10, p. 927, 2019.

[33] D. Yang, J. Wang, and D. O'Regan, "A class of nonlinear non-instantaneous impulsive differential equations involving parameters and fractional order," *Applied Mathematics and Computation*, vol. 321, pp. 654–671, 2018.

[34] M. Zuo, X. Hao, L. Liu, and Y. Cui, "Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions," *Boundary Value Problems*, vol. 2017, p. 161, 2017.

[35] X. Liu and M. Jia, "Existence of solutions for the integral boundary value problems of fractional order impulsive differential equations," *Mathematical Methods in the Applied Sciences*, vol. 39, no. 3, pp. 475–487, 2016.

[36] K. Zhao and P. Gong, "Positive solutions for impulsive fractional differential equations with generalized periodic boundary value conditions," *Advances in Difference Equations*, vol. 2014, p. 255, 2014.

[37] Z. Bai, X. Dong, and C. Yin, "Existence results for impulsive nonlinear fractional differential equation with mixed boundary conditions," *Boundary Value Problems*, vol. 201663 pages, 2016.

[38] K. Zhang and J. Xu, "Positive solutions for an impulsive boundary value problem with Caputo fractional derivative," *Journal of Nonlinear Sciences and Applications*, vol. 09, no. 06, pp. 4628–4638, 2016.

[39] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA,, 1993.

[40] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies. Elsevier, Amsterdam, Netherlands, 2006.

[41] K. Deimling, *Nonlinear Functional Analysis*, Spring, Berlin, Germany, 1985.

[42] M. A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, New York, NY, USA, 1964.

[43] A. Granas and J. Dugundji, *Fixed Point Theory*, Spring, New York, NY, USA, 2003.

Hindawi
Journal of Mathematics
Volume 2020, Article ID 6689015, 22 pages
https://doi.org/10.1155/2020/6689015

Research Article

Iterative Methods of Weak and Strong Convergence Theorems for the Split Common Solution of the Feasibility Problems, Generalized Equilibrium Problems, and Fixed Point Problems

Shahram Rezapour[1,2] **Yuanheng Wang**[3] **and Seyyed Hasan Zakeri**[1]

[1]*Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran*
[2]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*
[3]*Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China*

Correspondence should be addressed to Yuanheng Wang; yhwang@zjnu.cn

Received 14 October 2020; Revised 11 November 2020; Accepted 18 November 2020; Published 10 December 2020

Academic Editor: Xiaolong Qin

The purpose of this paper is to introduce the extragradient methods for solving split feasibility problems, generalized equilibrium problems, and fixed point problems involved in nonexpansive mappings and pseudocontractive mappings. We establish the results of weak and strong convergence under appropriate conditions. As applications of our three main theorems, when the mappings and their domains take different types of cases, we can obtain nine iterative approximation theorems and corollas on fixed points, variational inequality solutions, and equilibrium points.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces, and let C and Q be two nonempty closed and convex subsets of H_1 and H_2, respectively. Let A: $H_1 \longrightarrow H_2$ be a bounded linear operator with its adjoint A^*. The split feasibility problem (SFP) is to find a point x such that

$$x \in C, \quad Ax \in Q. \tag{1}$$

We denote the solution set of the split feasibility problem (SFP) by

$$\Omega = \{x \in C : Ax \in Q\} = C \cap A^{-1}Q. \tag{2}$$

Problem (1) was first introduced by Censor and Elfving [1] in the finite-dimensional spaces and further has been studied by many researchers (see, for example, [2–6]) and the references therein. To solve the SFP, Byrne [2, 7] first introduced the so-called CQ algorithm as follows:

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = P_C\left(I - \lambda A^*\left(I - P_Q\right)A\right)x_n, \quad \forall n \geq 0, \end{cases} \tag{3}$$

where $0 < \lambda < 2/\rho(A^*A)$, P_C denotes the projection onto C, and $\rho(A^*A)$ is the spectral radius of the self-adjoint operator A^*A. Many authors continue to study the CQ algorithm in its various forms (see, for example, [8–14]). The CQ algorithm can be viewed from two different but equivalent ways: optimization and fixed point [6]. From the view of optimization point, $x^* \in \Omega$ in (2) if and only if x^* is a solution of the following minimization problem with zero optimal value $\min_{x\in C} f(x) \coloneqq (1/2)\|Ax - P_Q Ax\|^2$, where f is a differentiable convex function and has a Lipschitz gradient given by $\nabla f(x) = A^*(I - P_Q)A$, with Lipschitz constant $L = \rho(A^*A)$. Thus, x^* solves the (SFP) if and only if x^* solves the variational inequality problem of finding $x^* \in C$ such that $\langle \nabla f(x^*), y - x^*\rangle \geq 0$ for all $y \in C$.

Xu [6] considered the following Tikhonov regularized problem:

$$\min_{x\in C} f_\alpha(x) := \frac{1}{2}\left\|Ax - P_Q Ax\right\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{4}$$

where $\alpha > 0$ is the regularization parameter. We observe that the gradient

$$\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*\left(I - P_Q\right)A + \alpha I, \tag{5}$$

is $(\alpha + \|A\|^2)$-Lipschitz continuous and α-strongly monotone. The fixed point approach method to solve the SFP is based on the following observations. Let $\lambda > 0$, and assume that $x^* \in \Omega$. Then, $Ax^* \in Q$, which implies that $(I - P_Q)Ax^* = 0$, and thus, $\lambda A^*(I - P_Q)Ax^* = 0$. Hence, we have the fixed point equation $(I - \lambda A^*(I - P_Q)A)x^* = x^*$. Requiring that $x^* \in C$, we consider the fixed point equation

$$P_C(I - \lambda\nabla f)x^* = P_C\left(I - \lambda A^*\left(I - P_Q\right)A\right)x^* = x^*. \tag{6}$$

In [6], it is proved that the solutions of fixed point equation (6) are precisely the solutions of the *SFP*.

Let $A: C \longrightarrow H$ be a nonlinear mapping and F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The generalized equilibrium problem is to find $x^* \in C$ such that $F(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0$, $\forall y \in C$. The set of solutions is denoted by $\mathrm{GEP}(F, A)$. If $A = 0$, then $\mathrm{GEP}(F, A)$ is denoted by $\mathrm{EP}(F)$. If $F(x, y) = 0$ for all $x, y \in C$, then $\mathrm{GEP}(F, A)$ is denoted by $\mathrm{VI}(C, A) = \{x^* \in C\colon \langle Ax^*, y - x^*\rangle \geq 0, \forall y \in C\}$. This is the set of solutions of the variational inequality for A (see, for example, [15–21]). If $C = H$, then $\mathrm{VI}(H, A) = A^{-1}(0)$ where $A^{-1}(0) = \{x \in H\colon Ax = 0\}$.

In 2008, Takahashi and Takahashi [15] have suggested the following iterative method. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle Ax_n, y - y_n\rangle + \dfrac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = a_n x_n + (1 - a_n)T[\beta_n u + (1 - \beta_n)y_n], \quad \forall n \geq 1. \end{cases} \tag{7}$$

Under some appropriate conditions, they proved that the sequence $\{x_n\}$ converges strongly to a point $P_{F(T)\cap \mathrm{GEP}(F,A)}u$.

Motivated and inspired by the above works, we will investigate the weak and strong convergence methods for solving the split feasibility problems, generalized equilibrium problems, and fixed point problems involved in nonexpansive mappings and pseudocontractive mappings. As applications of our three main theorems, when the mappings and their domains take different types of cases, we can obtain nine iterative approximation theorems and corollaries on fixed points, variational inequality solutions, and equilibrium points. So, our results in this paper generalize and improve upon the corresponding modern results of many other authors.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and C be a nonempty, closed, and convex subset of H. Recall that a mapping $A\colon C \longrightarrow H$ is said to be monotone if $\langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in C$ [18, 19]. A mapping A is said to be α-strongly monotone whenever there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^2$ for all $u, v \in C$. A mapping A is said to be α-inverse strongly monotone if there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$ for all $u, v \in C$. Recall that the classical variational inequality problem, which we denote by $\mathrm{VI}(C, A)$, is to find $x \in C$ such that $\langle Ax, y - x\rangle \geq 0$, for all $y \in C$ [16, 17]. It is well known that, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| = \inf_{y\in C}\|x - y\| =: d(x, C)$. It is well known that P_C is a nonexpansive and monotone mapping from H onto C and satisfy the following:

(1) $\langle x - P_C x, z - P_C x\rangle \leq 0$ for all $x \in H, z \in C$

(2) $\|x - z\|^2 \geq \|x - P_C x\|^2 + \|z - P_C x\|^2$ for all $x \in H, z \in C$

(3) The relation $\langle P_C x - P_C z, x - z\rangle \geq \|P_C x - P_C z\|^2$ holds for all $z, x \in H$

Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see from (2) that

$$p \in \mathrm{VI}(C, A) \Leftrightarrow p = P_C(p - \lambda Ap), \quad \forall\lambda > 0. \tag{8}$$

For solving the equilibrium problem, we assume that F satisfies the following conditions:

(i) $(A_1)F(x, x) = 0$ for all $x \in C$

(ii) $(A_2)F$ is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$

(iii) (A_3) for each $x, y, z \in C$, $\lim_{t\longrightarrow 0}F(tz + (1 - t)x, y) \leq F(x, y)$

(iv) (A_4) for each $x \in C$, the function $y \longrightarrow F(x, y)$ is convex and lower semicontinuous

If $F(x, y) = \langle Ax, y - x\rangle$ for every $x, y \in C$, we see that the equilibrium problem is reduced to the variational inequality problem.

Lemma 1 (see [22]). *Let C be a nonempty, closed, and convex subset of H, and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$. For $r > 0$ and $x \in H$, consider the mapping $T_r\colon H \longrightarrow C$ defined by*

$$T_r(x) = \left\{z \in C: F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}. \tag{9}$$

Then, $T_r(x) \neq \varnothing$ for all $x \in H$, T_r is single-valued, $\mathrm{EP}(F)$ is closed and convex, $F(T_r) = \mathrm{EP}(F)$, and T_r is firmly nonexpansive, that is, $\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle$ for all $x, y \in H$.

Lemma 2 (see [23]). *Let C be a nonempty, closed, and convex subset of H, F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1 - A_4)$, ad A_F be a multivalue mapping from H into itself defined by $A_F x = \{z \in C\colon F(z, y) \leq \langle y - x, z\rangle, \forall y \in C\}$ whenever $x \in C$ and $A_F x = \varnothing$ otherwise. Then, A_F is a maximal monotone operator with the domain $T_r(x) = (I + rA_F)^{-1}x$, for all $x \in H$ and $r > 0$.*

Definition 1. Let $T\colon H \longrightarrow H$ be a nonlinear operator.

(1) T is said to be L-Lipschitz whenever there exists $L \geq 0$ such that $\|Tu - Tv\| \leq L\|u - v\|$, $\forall u, v \in H$. If $L = 1$, we call T is nonexpansive, and T is said to be a contraction if $L < 1$.

(2) T is said to be firmly nonexpansive if $2T - I$ is nonexpansive and I is the identity mapping, or equivalently, $\langle Tu - Tv, u - v\rangle \geq \|Tu - Tv\|^2$, $\forall u, v \in H$. Alternatively, T is firmly nonexpansive if and only if T can be expressed as $T = (1/2)(I + S)$, where $S\colon H \longrightarrow H$ is nonexpansive.

(3) T is said to be α-averaged nonexpansive mapping, if there exists a nonexpansive mapping S, such that $T = (1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$. Thus, firmly nonexpansive mappings are $(1/2)$-averaged mapping.

(4) T is said to be pseudocontractive if and only if $\|Tu - Tv\|^2 \leq \|u - v\|^2 + \|(I - T)u- \quad (I - T)v\|^2$, $\forall u, v \in H$.

(5) T is said to be k-strictly pseudocontractive if and only if there exists $0 \leq k < 1$, such that

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + k\|(I - T)u - (I - T)v\|^2, \quad \forall u, v \in H. \tag{10}$$

Remark 1 (see [2]). Let $T\colon C \longrightarrow C$ be a given mapping:

(i) T is nonexpansive if and only if the complement $I - T$ is $(1/2)$-inverse strongly monotone.

(ii) If T is α-inverse strongly monotone, then for $\gamma > 0, \gamma T$ is (α/γ)-inverse strongly monotone.

(iii) T is averaged if and only if the complement $I - T$ is α-inverse strongly monotone for some $\alpha > 1/2$. Indeed, for $\alpha \in (0, 1), T$ is α-averaged if and only if $I - T$ is $(1/2\alpha)$-inverse strongly monotone.

We denote by $F(T)$ the set of fixed points of T. Note that every α-inverse strongly monotone mapping T is Lipschitz and $\|Tu - Tv\| \leq (1/\alpha)\|u - v\|$. Every nonexpansive mapping is a k-strictly pseudocontractive mapping and every k-strictly pseudocontractive mapping is pseudocontractive. Assume that $T\colon C \longrightarrow C$ is a strictly pseudocontractive. If $A = I - T$, we easily find that A is $(1 - k/2)$-inverse strongly monotone and $F(T) = \mathrm{VI}(C, A)$. Note that T is pseudocontractive if and only if $A = I - T$ is monotone, and $F(T) = A^{-1}(0) = \{x \in H\colon Ax = 0\}$. There are a lot works associated with the fixed point algorithms for nonexpansive mappings and pseudocontractive mappings (see, for example, [24–28]).

A set-valued mapping $T\colon H \longrightarrow 2^H$ is called monotone if for all $x, y \in H, f \in Tx$, and $h \in Ty$ imply $\langle x - y, f - h\rangle \geq 0$. A monotone mapping $T\colon H \longrightarrow 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mappings. Also, a monotone mapping $T\colon H \longrightarrow 2^H$ is maximal if and only if, for $(x, f) \in H \times H, \langle x - y, f - h\rangle \geq 0$ for every $(y, h) \in G(T)$ implies $f \in Tx$. Let $A\colon C \longrightarrow H$ be an inverse strongly monotone mapping and let $N_C u$ be the normal cone to C at $u \in C$, i.e., $N_C u = \{v \in H\colon \langle u - w, v\rangle \geq 0, \forall w \in C\}$. Define

$$Tu \coloneqq \begin{cases} Au + N_C u, & u \in C, \\ \varnothing, & u \notin C. \end{cases} \tag{11}$$

It is known that T is maximal monotone and $0 \in Tu$ if and only if $u \in \mathrm{VI}(C, A)$ [29, 30].

Lemma 3 (see [8]). *Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A\colon H_1 \longrightarrow H_2$ be a bounded linear operator and $f\colon H_1 \longrightarrow \mathbb{R}$ be a continuous differentiable function. If $\alpha > 0$ and $\lambda \in (0, (1/\|A\|^2))$, then*

(1) $\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*(I - P_Q)A + \alpha I$ is $(1/\alpha + \|A\|^2)$-inverse strongly monotone mapping

(2) $I - \lambda\nabla f_\alpha$ is $(\lambda(\alpha + \|A\|^2)/2)$-averaged

(3) $P_C(I - \lambda\nabla f_\alpha)$ is ζ-averaged, with $\zeta = (2 + \lambda(\alpha + \|A\|^2)/4)$

(4) $P_C(I - \lambda\nabla f_\alpha)$ is nonexpansive

Lemma 4 (see [31]). *Let H be a real Hilbert space, C be a closed convex subset of H, and $T\colon C \longrightarrow C$ be a continuous pseudocontractive mapping. Then,*

(i) $F(T)$ is a closed convex subset of C

(ii) $(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \longrightarrow x$ and $Tx_n - x_n \longrightarrow 0$; as $n \longrightarrow \infty$, then $x = T(x)$.

Lemma 5 (see [32]). *Let H be a real Hilbert space. Then, for all $x_j \in H$ and $a_j \in [0, 1]$, for $j = 1, 2, 3$ such that $a_1 + a_2 + a_3 = 1$, the following equality holds:*

$$\left\|a_1x_1 + a_2x_2 + a_3x_3\right\|^2 = a_1\left\|x_1\right\|^2 + a_2\left\|x_2\right\|^2 + a_3\left\|x_3\right\|^2 - \sum_{1\leq i,j\leq 3} a_ia_j\left\|x_i - x_j\right\|^2. \tag{12}$$

Lemma 6 (see [33]). *Let C be a nonempty closed and convex subset of a real Hilbert space H and $T\colon C \longrightarrow C$ be a nonexpansive mapping. Then, $I - T$ is demiclosed at zero.*

Lemma 7 (see [34]). *Let $\{x_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real numbers satisfying $x_{n+1} \leq x_n + \gamma_n$. If $\sum_{n=0}^{\infty} \gamma_n$ converges, then $\lim_{n\longrightarrow\infty} x_n$ exists.*

Lemma 8 (see [35]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T: C \longrightarrow C$ be a k-strictly pseudocontraction with a fixed point. Define $S: C \longrightarrow C$ by $Sx = ax + (1-a)Tx$ for each $x \in C$. Then, as $a \in [k, 1)$, S is nonexpansive such that $F(S) = F(T)$.*

Lemma 9 (see [36]). *Let $\{x_n\}$ be a sequence of nonnegative real numbers satisfying $x_{n+1} \leq (1-\beta_n)x_n + \beta_n \gamma_n + \alpha_n$, where $\{\beta_n\} \subset (0,1)$ and $\{\gamma_n\}$ is a sequence such that $\sum_{n=0}^{\infty} \beta_n = \infty$, $\limsup_{n\longrightarrow\infty} \gamma_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n \beta_n| < \infty$, and $\sum_{n=0}^{\infty} \alpha_n < \infty$ where $\alpha_n \geq 0$. Then, $\lim_{n\longrightarrow\infty} x_n = 0$.*

Lemma 10 (see [37]). *Let $\{x_n\}$, $\{\varepsilon_n\}$, and $\{\alpha_n\}$ be the sequences in $[0,\infty)$ such that*

$$x_{n+1} \leq x_n + \varepsilon_n (x_n - x_{n-1}) + \alpha_n, \quad \forall n \geq 0, \tag{13}$$

$\sum_{n=0}^{\infty} \alpha_n < \infty$, and there exists a real number ε with $0 \leq \varepsilon_n \leq \varepsilon < 1$ for all $n \geq 0$. Then, the following holds:

(i) $\sum_{n=0}^{\infty} [x_n - x_{n-1}]_+ < \infty$, where $[t]_+ = \max\{t, 0\}$

(ii) There exists $x^* \in [0,\infty)$ such that $\lim_{n\longrightarrow\infty} x_n = x^*$

Lemma 11 (see [31]). *Let H be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds: $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$.*

3. Weak and Strong Convergence Results

Now, we are ready to state and prove some of our main results in this section.

Theorem 1. *Assume that C and Q are 2 nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A: H_1 \longrightarrow H_2$ be a bounded linear operator, $f: H_1 \longrightarrow \mathbb{R}$ be a continuous differentiable function, F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1)-(A_4)$, M be an α-inverse strongly monotone mapping from C into H_1, $S: C \longrightarrow C$ be a nonexpansive mapping, and $T: C \longrightarrow C$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap \Omega \cap GEP(F, M) \neq \varnothing$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{v_n\}$ be sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ F(v_n, y) + \langle Mx_n, y - v_n\rangle + \dfrac{1}{r_n}\langle y - v_n, v_n - x_n\rangle \geq 0, \\ \quad \forall y \in C, \\ z_n = P_C\left(I - \lambda_n \nabla f_{\alpha_n}\right) v_n, \\ y_n = P_C\left(v_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \\ x_{n+1} = a_n x_0 + b_n x_n + c_n\left((1-\beta_n) S y_n + \beta_n T_n z_n\right), \quad \forall n \geq 0, \end{cases} \tag{14}$$

where $T_n = (1-\gamma_n)I + \gamma_n T$ and $\gamma_n \in (k, 1)$. Suppose the following conditions are satisfied:

(a) $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\lim_{n\longrightarrow\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$

(b) $\{\beta_n\} \subset [\beta_1^*, \beta_2^*]$ for some $\beta_1^*, \beta_2^* \in (0,1)$

(c) $\{\lambda_n\} \subset [e, d]$ for some $e, d \in (0, (1/\|A\|^2))$

(d) $0 < a_n \leq a' < 1, 0 < b \leq b_n \leq b' < 1, 0 < c \leq c_n \leq c' < 1$ and $a_n + b_n + c_n = 1$,

(e) $0 < q_1 \leq r_n \leq q_2 < 2\alpha$

Then, $\{x_n\}$ converges strongly to the point $u = P_\Gamma(x_0)$ provided $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$.

Proof. For any fixed $u \in \Gamma$, we find that $u = P_C(I - \lambda\nabla f)u$ for $\lambda \in (0, (1/\|A\|^2))$ and $Su = u$. We see from Lemma 8 that T_n is nonexpansive and $F(T_n) = F(T)$. It is observed that v_n can be rewritten as $v_n = T_{r_n}(x_n - r_n M x_n)$, $n \geq 0$. From condition (e) and Lemma 1, we have

$$\begin{aligned} \|v_n - u\|^2 &= \left\|T_{r_n}(x_n - r_n M x_n) - u\right\|^2 \\ &= \left\|T_{r_n}(x_n - r_n M x_n) - T_{r_n}(u - r_n M u)\right\|^2 \\ &\leq \left\|(x_n - r_n M x_n) - (u - r_n M u)\right\|^2 \\ &= \left\|x_n - u\right\|^2 + r_n(r_n - 2\alpha)\left\|Mx_n - Mu\right\|^2 \\ &\leq \left\|x_n - u\right\|^2. \end{aligned} \tag{15}$$

From (14), (15), and Lemma 3, it follows that

$$\begin{aligned} \|z_n - u\| &= \left\|P_C\left(I - \lambda_n \nabla f_{\alpha_n}\right) v_n - P_C(I - \lambda_n \nabla f)u\right\| \\ &\leq \left\|P_C\left(I - \lambda_n \nabla f_{\alpha_n}\right) v_n - P_C\left(I - \lambda_n \nabla f_{\alpha_n}\right) u\right\| \\ &\quad + \left\|P_C\left(I - \lambda_n \nabla f_{\alpha_n}\right) u - P_C(I - \lambda_n \nabla f)u\right\| \\ &\leq \|v_n - u\| + \left\|\left(I - \lambda_n \nabla f_{\alpha_n}\right) u - (I - \lambda_n \nabla f)u\right\| \\ &\leq \|v_n - u\| + \lambda_n \alpha_n \|u\| \\ &\leq \|x_n - u\| + \lambda_n \alpha_n \|u\|. \end{aligned} \tag{16}$$

By the property of metric projection, we have

$$\begin{aligned}
\|y_n - u\|^2 &\le \left\|v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - u\right\|^2 - \left\|v_n - \lambda_n \nabla f_{\alpha_n} z_n - y_n\right\|^2 \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), u - y_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n) \\
&\quad - \nabla f_{\alpha_n}(u), u - z_n\rangle \\
&\quad + 2\lambda_n \big(\langle \nabla f_{\alpha_n}(u), u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \big) \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(u), u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \\
&= \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle (\alpha_n I + \nabla f) u, u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \alpha_n \langle u, u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \\
&= \|v_n - u\|^2 - \|v_n - z_n\|^2 - 2\langle v_n - z_n, z_n - y_n\rangle \\
&\quad - \|z_n - y_n\|^2 \\
&\quad + 2\lambda_n \left[\alpha_n \langle u, u - z_n\rangle + \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle\right] \\
&= \|v_n - u\|^2 - \|v_n - z_n\|^2 \\
&\quad + 2\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n\rangle \\
&\quad + 2\lambda_n \alpha_n \langle u, u - z_n\rangle - \|z_n - y_n\|^2.
\end{aligned} \tag{17}$$

Furthermore, by the property of metric projection, we have

$$\begin{aligned}
&\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n\rangle \\
&\quad = \langle v_n - \lambda_n \nabla f_{\alpha_n}(v_n) - z_n, y_n - z_n\rangle \\
&\qquad + \langle \lambda_n \nabla f_{\alpha_n}(v_n) - \lambda_n \nabla f_{\alpha_n}(z_n), y_n - z_n\rangle \\
&\quad \le \langle \lambda_n \nabla f_{\alpha_n}(v_n) - \lambda_n \nabla f_{\alpha_n}(z_n), y_n - z_n\rangle \\
&\quad \le \lambda_n \left\|\nabla f_{\alpha_n}(v_n) - \nabla f_{\alpha_n}(z_n)\right\| \|y_n - z_n\| \\
&\quad \le \lambda_n \left(\alpha_n + \|A\|^2\right) \|v_n - z_n\| \|y_n - z_n\|.
\end{aligned} \tag{18}$$

Hence, we have

$$\begin{aligned}
\|y_n - u\|^2 &\le \|v_n - u\|^2 - \|v_n - z_n\|^2 \\
&\quad + 2\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n\rangle \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n \alpha_n \langle u, u - z_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - z_n\|^2 + 2\lambda_n \left(\alpha_n + \|A\|^2\right) \\
&\quad \cdot \|v_n - z_n\| \|y_n - z_n\| \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n \alpha_n \|u\| \|u - z_n\| \\
&\le \|v_n - u\|^2 - \|v_n - z_n\|^2 + \lambda_n^2 \left(\alpha_n + \|A\|^2\right)^2 \\
&\quad \cdot \|v_n - z_n\|^2 + \|y_n - z_n\|^2 \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n \alpha_n \|u\| \|u - z_n\| \\
&= \|v_n - u\|^2 + \left(\lambda_n^2 \left(\alpha_n + \|A\|^2\right)^2 - 1\right) \|v_n - z_n\|^2 \\
&\quad + 2\lambda_n \alpha_n \|u\| \|u - z_n\| \\
&\le \|v_n - u\|^2 + 2\lambda_n \alpha_n \|u\| \|u - z_n\| \\
&\le \|v_n - u\|^2 + 2\lambda_n \alpha_n \|u\| \left(\|v_n - u\| + \lambda_n \alpha_n \|u\|\right) \\
&\le \|v_n - u\|^2 + 4\lambda_n \alpha_n \|u\| \|v_n - u\| + 4\lambda_n^2 \alpha_n^2 \|u\|^2 \\
&= \left(\|v_n - u\| + 2\lambda_n \alpha_n \|u\|\right)^2.
\end{aligned} \tag{19}$$

So, from (15), we obtain

$$\|y_n - u\|^2 \le \left(\|x_n - u\| + 2\lambda_n \alpha_n \|u\|\right)^2. \tag{20}$$

We find from (14) and (16) and the last inequality that

$$\begin{aligned}
\|x_{n+1} - u\| &= \|a_n x_0 + b_n x_n + c_n ((1 - \beta_n) S y_n + \beta_n T_n z_n) - u\| \\
&\le a_n \|x_0 - u\| + b_n \|x_n - u\| \\
&\quad + c_n \left[(1 - \beta_n) \|S y_n - u\| + \beta_n \|T_n z_n - u\|\right] \\
&\le a_n \|x_0 - u\| + b_n \|x_n - u\| \\
&\quad + c_n \left[(1 - \beta_n) \|y_n - u\| + \beta_n \|z_n - u\|\right] \\
&\le a_n \|x_0 - u\| + b_n \|x_n - u\| \\
&\quad + c_n (1 - \beta_n) \left(\|x_n - u\| + 2\lambda_n \alpha_n \|u\|\right) \\
&\quad + c_n \beta_n \left(\|x_n - u\| + \lambda_n \alpha_n \|u\|\right) \\
&\le a_n \|x_0 - u\| + (1 - a_n) \|x_n - u\| + 2\lambda_n \alpha_n \|u\| \\
&\le \max\left\{\|x_0 - u\|, \|x_n - u\|\right\} + 2\lambda_n \alpha_n \|u\| \\
&\quad \vdots \\
&\le \|x_0 - u\| + 2d \|u\| \sum_{i=0}^{\infty} \alpha_i.
\end{aligned} \tag{21}$$

Consequently, from condition (a), we deduce that $\{x_n\}$ is bounded and so there exist the sequences $\{z_n\}$, $\{v_n\}$, and $\{y_n\}$. Put $t_n = (1-\beta_n)Sy_n + \beta_n T_n z_n$ for all $n \geq 0$. We find from (15), (16), (19), and Lemma 5 that

$$\begin{aligned}
\|t_n - u\|^2 &= \|(1-\beta_n)Sy_n + \beta_n T_n z_n - u\|^2 \\
&\leq (1-\beta_n)\|Sy_n - u\|^2 + \beta_n\|T_n z_n - u\|^2 - \beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2 \\
&\leq (1-\beta_n)\|y_n - u\|^2 + \beta_n\|z_n - u\|^2 - \beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2 \\
&\leq (1-\beta_n)\Big[\|x_n - u\|^2 + \Big(\lambda_n^2\big(\alpha_n + \|A\|^2\big)^2 - 1\Big)\|v_n - z_n\|^2 + r_n(r_n - 2\alpha)\|Mx_n - Mu\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n - u\|\Big] \\
&\quad + \beta_n\big(\|x_n - u\| + \lambda_n\alpha_n\|u\|\big)^2 - \beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2 \\
&\leq \|x_n - u\|^2 + \Big(\lambda_n^2\big(\alpha_n + \|A\|^2\big)^2 - 1\Big)\|v_n - z_n\|^2 \\
&\quad + r_n(r_n - 2\alpha)\|Mx_n - Mu\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n - u\| \\
&\quad + 2\lambda_n\alpha_n\|u\|\|x_n - u\| + \lambda_n^2\alpha_n^2\|u\|^2 - \beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2.
\end{aligned} \tag{22}$$

From (14) and the last inequality, we conclude that

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \|a_n x_0 + b_n x_n + c_n t_n - u\|^2 \\
&\leq a_n\|x_0 - u\|^2 + b_n\|x_n - u\|^2 + c_n\|t_n - u\|^2 - b_n c_n\|x_n - t_n\|^2 \\
&\leq a_n\|x_0 - u\|^2 + b_n\|x_n - u\|^2 - b_n c_n\|x_n - t_n\|^2 \\
&\quad + c_n\Big[\|x_n - u\|^2 + \Big(\lambda_n^2\big(\alpha_n + \|A\|^2\big)^2 - 1\Big)\|v_n - z_n\|^2 + r_n(r_n - 2\alpha)\|Mx_n - Mu\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n - u\| \\
&\quad + 2\lambda_n\alpha_n\|u\|\|x_n - u\| + \lambda_n^2\alpha_n^2\|u\|^2 - \beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2\Big] \\
&\leq a_n\|x_0 - u\|^2 + (1-a_n)\|x_n - u\|^2 - b_n c_n\|x_n - t_n\|^2 \\
&\quad + c_n\Big(\lambda_n^2\big(\alpha_n + \|A\|^2\big)^2 - 1\Big)\|v_n - z_n\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n - u\| \\
&\quad + 2\lambda_n\alpha_n\|u\|\|x_n - u\| + r_n(r_n - 2\alpha)\|Mx_n - Mu\|^2 \\
&\quad + \lambda_n^2\alpha_n^2\|u\|^2 - c_n\beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2 \\
&\leq a_n\|x_0 - u\|^2 + \|x_n - u\|^2 - b_n c_n\|x_n - t_n\|^2 \\
&\quad + c_n\Big(\lambda_n^2\big(\alpha_n + \|A\|^2\big)^2 - 1\Big)\|v_n - z_n\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n - u\| \\
&\quad + 2\lambda_n\alpha_n\|u\|\|x_n - u\| + r_n(r_n - 2\alpha)\|Mx_n - Mu\|^2 \\
&\quad + \lambda_n^2\alpha_n^2\|u\|^2 - c_n\beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2.
\end{aligned} \tag{23}$$

This yields that

$$\begin{aligned} &c\left(1-d^2\left(\alpha_n+\|A\|^2\right)^2\right)\|v_n-z_n\|^2+cb\|x_n-t_n\|^2 \\ &\quad+r_n(2\alpha-r_n)\|Mx_n-Mu\|^2+\beta_1(1-\beta_2)c\|T_nz_n-Sy_n\|^2 \\ &\le c_n\left(1-\lambda_n^2\left(\alpha_n+\|A\|^2\right)^2\right)\|v_n-z_n\|^2+c_nb_n\|x_n-t_n\|^2 \\ &\quad+r_n(2\alpha-r_n)\|Mx_n-Mu\|^2+c_n\beta_n(1-\beta_n)\|T_nz_n-Sy_n\|^2 \\ &\le a_n\|x_0-u\|^2+\|x_n-u\|^2-\|x_{n+1}-u\|^2 \\ &+2\lambda_n\alpha_n\|u\|\left(\|z_n-u\|+\|x_n-u\|+\lambda_n\alpha_n\|u\|\right). \end{aligned} \tag{24}$$

Since $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0$, we have

$$\begin{aligned} &\|x_n-u\|^2-\|x_{n+1}-u\|^2 \\ &\quad\le\left(\|x_n-u\|-\|x_{n+1}-u\|\right) \\ &\quad\cdot\left(\|x_n-u\|+\|x_{n+1}-u\|\right) \\ &\quad\le\|x_{n+1}-x_n\|\left(\|x_n-u\|+\|x_{n+1}-u\|\right)\longrightarrow 0, \\ &\qquad\text{as } n\longrightarrow\infty. \end{aligned} \tag{25}$$

From (97) and the condition (a)–(d), we also obtain

$$\begin{aligned} \lim_{n\longrightarrow\infty}\|T_nz_n-Sy_n\| &= \lim_{n\longrightarrow\infty}\|Mx_n-Mu\| = \lim_{n\longrightarrow\infty}\|x_n-t_n\| \\ &= \lim_{n\longrightarrow\infty}\|v_n-z_n\| = 0. \end{aligned} \tag{26}$$

It is observe that

$$\begin{aligned} \|y_n-z_n\| &= \left\|P_C\left(v_n-\lambda_n\nabla f_{\alpha_n}(z_n)\right)-P_C\left(v_n-\lambda_n\nabla f_{\alpha_n}(v_n)\right)\right\| \\ &\le\left\|v_n-\lambda_n\nabla f_{\alpha_n}(z_n)-\left(v_n-\lambda_n\nabla f_{\alpha_n}(v_n)\right)\right\| \\ &=\lambda_n\left\|\nabla f_{\alpha_n}(z_n)-\nabla f_{\alpha_n}(v_n)\right\| \\ &\le\lambda_n\left(\alpha_n+\|A\|^2\right)\|z_n-v_n\|\longrightarrow 0,\quad \text{as } n\longrightarrow\infty. \end{aligned} \tag{27}$$

Using Lemma 1 and (14), we have

$$\begin{aligned} \|v_n-u\|^2 &= \left\|T_{r_n}(x_n-r_nMx_n)-T_{r_n}(u-r_nMu)\right\|^2 \\ &\le\langle(x_n-r_nMx_n)-(u-r_nMu),v_n-u\rangle \\ &=\frac{1}{2}\|(x_n-r_nMx_n)-(u-r_nMu)\|^2+\frac{1}{2}\|v_n-u\|^2 \\ &\quad-\frac{1}{2}\|(x_n-r_nMx_n)-(u-r_nMu)-(v_n-u)\|^2 \\ &\le\frac{1}{2}\left[\|x_n-u\|^2+\|v_n-u\|^2-\|(x_n-v_n)-2r_n(Mx_n-Mu)\|^2\right] \\ &=\frac{1}{2}\Big[\|x_n-u\|^2+\|v_n-u\|^2-\|x_n-v_n\|^2+2r_n\langle x_n-v_n,Mx_n-Mu\rangle \\ &\quad-r_n^2\|Mx_n-Mu\|^2\Big]. \end{aligned} \tag{28}$$

It follows that

$$\|v_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - v_n\|^2 + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle. \quad (29)$$

From (19) and (29), we find that

$$\begin{aligned}\|t_n - u\|^2 &= \|(1-\beta_n)Sy_n + \beta_n T_n z_n - u\|^2 \\ &\le (1-\beta_n)\|Sy_n - u\|^2 + \beta_n\|T_n z_n - u\|^2 \\ &\le (1-\beta_n)\|y_n - u\|^2 + \beta_n\|z_n - u\|^2 \\ &\le (1-\beta_n)\left[\|x_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\|\right] \\ &\quad + \beta_n\left(\|x_n - u\|^2 - \|x_n - v_n\|^2 + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle\right) \\ &\le \|x_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| \\ &\quad + \beta_n\left(-\|x_n - v_n\|^2 + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle\right).\end{aligned} \quad (30)$$

From (14) and the last inequality, we conclude that

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|a_n x_0 + b_n x_n + c_n t_n - u\|^2 \\ &\le a_n\|x_0 - u\|^2 + b_n\|x_n - u\|^2 + c_n\|t_n - u\|^2 \\ &\le a_n\|x_0 - u\|^2 + b_n\|x_n - u\|^2 \\ &\quad + c_n\left[\|x_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| + \beta_n\left(-\|x_n - v_n\|^2 + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle\right)\right] \\ &\le a_n\|x_0 - u\|^2 + (1 - a_n)\|x_n - u\|^2 - c_n\beta_n\|x_n - v_n\|^2 \\ &\quad + c_n\left[2\lambda_n\alpha_n\|u\|\|u - z_n\| + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle\right).\end{aligned} \quad (31)$$

This yields that

$$\begin{aligned}c_n\beta_n\|x_n - v_n\|^2 &\le a_n\|x_0 - u\|^2 + (1 - a_n)\|x_n - u\|^2 - \|x_{n+1} - u\|^2 \\ &\quad + c_n\left[2\lambda_n\alpha_n\|u\|\|u - z_n\| + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle\right].\end{aligned} \quad (32)$$

It follows from condition (a) and $\lim_{n\longrightarrow\infty}\|x_n - x_{n+1}\| = \lim_{n\longrightarrow\infty}\|Mx_n - Mu\| = 0$ that

$$\lim_{n\longrightarrow\infty}\|x_n - v_n\| = 0. \quad (33)$$

Since $\|x_n - z_n\| \le \|x_n - v_n\| + \|v_n - z_n\|$, $\|T_n z_n - z_n\| \le \|T_n z_n - x_n\| + \|x_n - z_n\|$, $\|T_n z_n - x_n\| \le \|T_n z_n - t_n\| + \|t_n - x_n\|$, $\|T_n z_n - t_n\| = (1-\beta_n)\|T_n z_n - Sy_n\|$, we obtain $\|T_n z_n - t_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Note that $1 - \beta_n > 0$. This implies that

$$\lim_{n\longrightarrow\infty}\|Tz_n - z_n\| = 0. \quad (34)$$

Also, from $\|y_n - x_n\| \le \|y_n - z_n\| + \|z_n - x_n\|$, $\|Sy_n - x_n\| \le \|Sy_n - t_n\| + \|t_n - x_n\|$, $\|Sy_n - t_n\| = \beta_n\|Sy_n - T_n z_n\|$, and $\|Sy_n - y_n\| \le \|Sy_n - x_n\| + \|x_n - y_n\|$, we get

$$\lim_{n\longrightarrow\infty}\|Sy_n - y_n\| = 0. \quad (35)$$

Since $\nabla f = A^*(I - P_Q)A$ is Lipschitz continuous, we obtain $\lim_{n\longrightarrow\infty}\|\nabla f(z_n) - \nabla f(y_n)\| = 0$.

Next, we show that

$$\limsup_{n\longrightarrow\infty}\langle x_0 - u, x_n - u\rangle \le 0, \quad (36)$$

where $u = P_\Gamma(x_0)$. To show it, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle x_0 - u, x_n - u\rangle = \lim_{k\longrightarrow\infty}\langle x_0 - u, x_{n_k} - u\rangle. \quad (37)$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, converges weakly to x^*. Without loss of generality, we assume that $x_{n_k} \longrightarrow x^*$. Since $\|x_n - v_n\| \longrightarrow 0$, $\|x_n - y_n\| \longrightarrow 0$, $\|x_n - z_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain that $y_{n_k} \rightharpoonup x^*$, $v_{n_k} \rightharpoonup x^*$, $z_{n_k} \rightharpoonup x^*$. Since $\{y_{n_k}\} \subset C$ and C is closed and convex, we obtain $x^* \in C$. First, we show that $x^* \in F(T) \cap F(S)$. Then, from (34), (35), Lemma 6, and

Lemma 4, we have that $x^* \in F(T) \cap F(S)$. We now show that $x^* \in \text{GEP}(F, M)$. By $v_n = T_{r_n}(x_n - r_n M x_n)$, we know that

$$F(v_n, y) + \langle M x_n, y - v_n \rangle + \frac{1}{r_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{38}$$

It follows from (A_2) that

$$\langle M x_n, y - v_n \rangle + \frac{1}{r_n} (y - v_n, v_n - x_n) \geq F(y, v_n), \quad \forall y \in C. \tag{39}$$

Hence,

$$\langle M x_{n_j}, y - v_{n_j} \rangle + \langle y - v_{n_j}, \frac{v_{n_j} - x_{n_j}}{r_{n_j}} \rangle \geq F\left(y, v_{n_j}\right), \quad \forall y \in C. \tag{40}$$

For t with $0 < t \leq 1$ and $y \in C$, let $v_t = ty + (1-t)x^*$. Since $y \in C$ and $x^* \in C$, we obtain $v_t \in C$. So, from (74), we have

$$\begin{aligned}
\langle v_t - v_{n_j}, M v_t \rangle &\geq \left(v_t - v_{n_j}, M v_t \right) - \langle v_t - v_{n_j}, M x_{n_j} \rangle \\
&\quad - \left(v_t - v_{n_j}, \frac{v_{n_j} - x_{n_j}}{r_{n_j}} \right) + F\left(v_t, v_{n_j} \right) \\
&= \left(v_t - v_{n_j}, M v_t - M v_{n_j} \right) \\
&\quad + \left(v_t - v_{n_j}, M v_{n_j} - M x_{n_j} \right) \\
&\quad - \left(v_t - v_{n_j}, \frac{v_{n_j} - x_{n_j}}{r_{n_j}} \right) + F\left(v_t, v_{n_j} \right).
\end{aligned} \tag{41}$$

Since $\| v_{n_j} - x_{n_j} \| \longrightarrow 0$, we have $\| M v_{n_j} - M x_{n_j} \| \longrightarrow 0$. Furthermore, from the inverse strongly monotonicity of M, we have $\langle v_t - v_{n_j}, M v_t - M v_{n_j} \rangle \geq 0$. It follows from condition (A_4) and $(v_{n_j} - x_{n_j}/r_{n_j}) \longrightarrow 0$ and $v_{n_j} \rightharpoonup x^*$, we have

$$\langle v_t - x^*, M v_t \rangle \geq F(v_t, x^*), \tag{42}$$

as $j \longrightarrow \infty$. From (A_1) and (A_4), we have

$$\begin{aligned}
0 &= F(v_t, v_t) \\
&\leq tF(v_t, y) + (1-t) F(v_t, x^*) \\
&\leq tF(v_t, y) + (1-t)(v_t - x^*, M v_t) \\
&= tF(v_t, y) + (1-t)t(y - x^*, M v_t),
\end{aligned} \tag{43}$$

and hence,

$$0 \leq F(v_t, y) + (1-t) \langle y - x^*, M v_t \rangle. \tag{44}$$

Letting $t \longrightarrow 0$, we have, for each $y \in C$,

$$F(x^*, y) + \langle y - x^*, M x^* \rangle \geq 0. \tag{45}$$

This implies that $x^* \in \text{GEP}(F, M)$. Next, we show that $x^* \in \Omega$ (1). Let

$$T' p := \begin{cases} \nabla f(p) + N_C p, & p \in C, \\ \varnothing, & p \notin C. \end{cases} \tag{46}$$

Then, T' is maximal monotone and $0 \in T' p$ if and only if $p \in \text{VI}(C, \nabla f)$ [29]. Let $G(T')$ be the graph of T', let $(p, v) \in G(T')$. Then, we have $v \in T'(p) = \nabla f(p) + N_C p$ and hence $v - \nabla f(p) \in N_C p$. Therefore, we have $\langle p - w, v - \nabla f(p) \rangle \geq 0$ for all $w \in C$. By the property of metric projection, from $y_n = P_C(v_n - \lambda_n \nabla f_{\alpha_n} z_n)$ and $p \in C$, we have $\langle p - y_n, y_n - (v_n - \lambda_n \nabla f_{\alpha_n} z_n) \rangle \geq 0$, and hence,

$$\langle p - y_n, \frac{y_n - v_n}{\lambda_n} + \nabla f_{\alpha_n} z_n \rangle \geq 0. \tag{47}$$

From $\langle p - w, v - \nabla f(p) \rangle \geq 0$ for all $w \in C$ and $y_{n_k} \in C$, we have

$$\begin{aligned}
\langle p - y_{n_k}, v \rangle &\geq \langle p - y_{n_k}, \nabla f(p) \rangle \\
&\geq \langle p - y_{n_k}, \nabla f(p) \rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} + \nabla f_{\alpha_n} z_{n_k} \rangle \\
&\geq \langle p - y_{n_k}, \nabla f(p) \rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} + \nabla f\left(z_{n_k}\right) \rangle - \alpha_{n_k} \langle p - y_{n_k}, z_{n_k} \rangle \\
&= \langle p - y_{n_k}, \nabla f(p) - \nabla f\left(y_{n_k}\right) \rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} \rangle - \alpha_{n_k} \langle p - y_{n_k}, z_{n_k} \rangle \\
&\quad + \langle p - y_{n_k}, \nabla f\left(y_{n_k}\right) - \nabla f\left(z_{n_k}\right) \rangle \\
&\geq \langle p - y_{n_k}, \nabla f\left(y_{n_k}\right) - \nabla f\left(z_{n_k}\right) \rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} \rangle - \alpha_{n_k} \langle p - y_{n_k}, z_{n_k} \rangle.
\end{aligned} \tag{48}$$

Thus, we obtain $\langle p - x^*, v\rangle \geq 0$ as $k \longrightarrow \infty$. Since T' is maximal monotone, we have $x^* \in T'^{-1}0$, and hence, $x^* \in VI(C, \nabla f)$. This implies $x^* \in \Omega$. This implies that $x^* \in \Gamma$. Thanks to (37), we arrive at

$$\begin{aligned}\limsup_{n\longrightarrow\infty} x_0 - u, x_n - u\rangle &= \lim_{k\longrightarrow\infty} \langle x_0 - u, x_{n_k} - u\rangle \\ &= \langle x_0 - u, x^* - u\rangle \leq 0.\end{aligned} \tag{49}$$

Next, we show that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. Observe that

$$\begin{aligned}\|t_n - u\| &= \|(1-\beta_n)(Sy_n - u) + \beta_n(T_n z_n - u)\| \\ &\leq (1-\beta_n)\|Sy_n - u\| + \beta_n\|T_n z_n - u\| \\ &\leq (1-\beta_n)\|y_n - u\| + \beta_n\|z_n - u\| \\ &\leq (1-\beta_n)\left(\|x_n - u\| + 2\lambda_n\alpha_n\|u\|\right) \\ &\quad + \beta_n\left(\|x_n - u\| + \lambda_n\alpha_n\|u\|\right) \\ &\leq \|x_n - u\| + 2\lambda_n\alpha_n\|u\|.\end{aligned} \tag{50}$$

With the help of (14), we obtain

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \langle a_n x_0 + b_n x_n + c_n t_n - u, x_{n+1} - u\rangle \\ &\leq a_n\langle x_0 - u, x_{n+1} - u\rangle + \langle b_n(x_n - u) + c_n(t_n - u), x_{n+1} - u\rangle \\ &\leq a_n\langle x_0 - u, x_{n+1} - u\rangle + \left(b_n\|x_n - u\| + c_n\|t_n - u\|\right)\|x_{n+1} - u\| \\ &\leq a_n\langle x_0 - u, x_{n+1} - u\rangle + \left(b_n\|x_n - u\| + c_n\|x_n - u\| + 2\lambda_n\alpha_n\|u\|\right)\|x_{n+1} - u\| \\ &\leq a_n\langle x_0 - u, x_{n+1} - u\rangle + \left((1-a_n)\|x_n - u\| + 2\lambda_n\alpha_n\|u\|\right)\|x_{n+1} - u\| \\ &\leq a_n\langle x_0 - u, x_{n+1} - u\rangle + 2\lambda_n\alpha_n\|u\|\|x_{n+1} - u\| \\ &\quad + \frac{(1-a_n)}{2}\left(\|x_n - u\|^2 + \|x_{n+1} - u\|^2\right),\end{aligned} \tag{51}$$

which implies that

$$\begin{aligned}\|x_{n+1} - u\|^2 &\leq (1-a_n)\|x_n - u\|^2 + 2a_n\langle x_0 - u, x_{n+1} - u\rangle \\ &\quad + 4\lambda_n\alpha_n\|u\|\|x_{n+1} - u\|.\end{aligned} \tag{52}$$

It follows from condition (a) and Lemma 9 that

$$\lim_{n\longrightarrow\infty}\|x_n - u\| = 0. \tag{53}$$

Therefore, from $\|x_n - z_n\| \longrightarrow 0$, $\|x_n - y_n\| \longrightarrow 0$, we can conclude that $\{x_n\}$, $\{z_n\}$, $\{v_n\}$, and $\{y_n\}$ converge strongly to the same point $u = P_\Gamma(x_0)$. The proof is complete. □

In the following, we will discuss the weak convergence of the sequence of the new iteration.

Theorem 2. *Assume that C and Q are 2 nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A: $H_1 \longrightarrow H_2$ be a bounded linear operator and f: $H_1 \longrightarrow \mathbb{R}$ be a continuous differentiable function. Assume that C and Q are 2 nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A_i: $H_1 \longrightarrow H_2$ bounded linear operators, f_i: $H_1 \longrightarrow \mathbb{R}$ be a continuous differentiable function, $i = 1, 2$, and F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$, M be an α-inverse strongly monotone mapping from C into H_1, S: $C \longrightarrow C$ be a nonexpansive mapping, and T: $C \longrightarrow C$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap GEP(F, M) \cap (\cap_{i=1}^2 \Omega_i) \neq \varnothing$. Suppose $\{x_n\}$ and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$
\begin{cases}
x_0 = x \in C, \\
z_n = \gamma_n P_C\big(x_n - \lambda_n \nabla f_{1\alpha_n} x_n\big) + (1-\gamma_n) P_C\big(x_n - \lambda_n \nabla f_{2s_n} x_n\big), \\
F(v_n, y) + \langle M z_n t, nyq - hv_n \rangle + \dfrac{1}{r_n}\langle y - tv_n n, qv_n h - z_n \rangle \geq 0, \quad \forall y \in C, \\
x_{n+1} = a_n x_n + b_n\big((1-\beta_n) z_n + \beta_n T z_n\big) + c_n\big((1-\delta_n) v_n + \delta_n S v_n\big), \quad \forall n \geq 0.
\end{cases}
\tag{54}
$$

Suppose the following conditions are satisfied:

(a) $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\sum_{n=0}^{\infty} s_n < \infty$

(b) $\{\beta_n\} \subset [k, r]$ *for some* $r, k \in (0,1)$, $\{\lambda_n\} \subset [e, d]$ *for some* $e, d \in (0, (1/\|A\|^2))$

(c) $\{\gamma_n\} \subset [t, m]$ *for some* $t, m \in (0,1)$, $\{\delta_n\} \subset [\delta_1^*, \delta_2^*]$ *for some* $\delta_1^*, \delta_2^* \in (0,1)$

(d) $0 < a \leq a_n \leq a' < 1$, $0 < b \leq b_n \leq b' < 1$ *and* $0 < c \leq c_n \leq c' < 1$ *and* $a_n + b_n + c_n = 1$

(e) $0 < q_1 \leq r_n \leq q_2 < 2\alpha$

Then, $\{x_n\}$ *converges weakly to an element* $u \in \Gamma$.

Proof. For any fixed $u \in \Gamma$, we find that $u = P_C(I - \lambda \nabla f)u$ for $\lambda \in (0, (1/\|A\|^2))$ and $Su = u$. Let $y_n = P_C(I - \lambda_n \nabla f_{1\alpha_n}) x_n$, $t_n = P_C(I - \lambda_n \nabla f_{2s_n}) x_n$, and $T_n = (1 - \beta_n) I + \beta_n T$. We see from Lemma 8 that T_n is nonexpansive and $F(T_n) = F(T)$. From (54) and Lemma 3, it follows that

$$
\begin{aligned}
\|y_n - u\| &\leq \big\|P_C\big(I - \lambda_n \nabla f_{1\alpha_n}\big)x_n - P_C\big(I - \lambda_n \nabla f_{1\alpha_n}\big)u\big\| \\
&\quad + \big\|P_C\big(I - \lambda_n \nabla f_{1\alpha_n}\big)u - P_C(I - \lambda_n \nabla f_1)u\big\| \\
&\leq \|x_n - u\| + \big\|\big(I - \lambda_n \nabla f_{1\alpha_n}\big)u - (I - \lambda_n \nabla f_1)u\big\| \\
&\leq \|x_n - u\| + \lambda_n \alpha_n \|u\|.
\end{aligned}
\tag{55}
$$

In a similar way, we have

$$
\begin{aligned}
\|t_n - u\| &\leq \big\|P_C\big(I - \lambda_n \nabla f_{2s_n}\big)x_n - P_C\big(I - \lambda_n \nabla f_{2s_n}\big)u\big\| \\
&\quad + \big\|P_C\big(I - \lambda_n \nabla f_{2s_n}\big)u - P_C(I - \lambda_n \nabla f_2)u\big\| \\
&\leq \|x_n - u\| + \big\|\big(I - \lambda_n \nabla f_{2s_n}\big)u - (I - \lambda_n \nabla f_2)u\big\| \\
&\leq \|x_n - u\| + \lambda_n s_n \|u\|.
\end{aligned}
\tag{56}
$$

This implies that

$$
\begin{aligned}
\|z_n - u\| &\leq \gamma_n \|y_n - u\| + (1-\gamma_n)\|t_n - u\| \\
&\leq \gamma_n\big(\|x_n - u\| + \lambda_n \alpha_n \|u\|\big) + (1-\gamma_n) \\
&\quad \cdot \big(\|x_n - u\| + \lambda_n s_n \|u\|\big) \\
&\leq \|x_n - u\| + \lambda_n \|u\| (\alpha_n + s_n).
\end{aligned}
\tag{57}
$$

Observe that v_n can be rewritten as $v_n = T_{r_n}(z_n - r_n M z_n)$, $n \geq 0$. From (e) and Lemma 1, we have

$$
\begin{aligned}
\|v_n - u\|^2 &= \big\|T_{r_n}(z_n - r_n M z_n) - u\big\|^2 \\
&= \big(T_{r_n}(z_n - r_n M z_n) - T_{r_n}(u - r_n M u)\big)^2 \\
&\leq \|(z_n - r_n M z_n) - (u - r_n M u)\|^2 \\
&= \|z_n - u\|^2 + r_n(r_n - 2\alpha)\|M z_n - M u\|^2 \\
&\leq \|z_n - u\|^2 \\
&\leq \|x_n - u\| + \lambda_n \|u\| (\alpha_n + s_n).
\end{aligned}
\tag{58}
$$

We find from (54) and the last inequality that

$$
\begin{aligned}
\|x_{n+1} - u\| &\leq a_n \|x_n - u\| + b_n \|T_n z_n - u\| \\
&\quad + c_n\big((1-\delta_n)\|v_n - u\| + \delta_n \|S v_n - u\|\big) \\
&\leq a_n \|x_n - u\| + (1 - a_n)\|z_n - u\| \\
&\leq a_n \|x_n - u\| + (1 - a_n)\big(\|x_n - u\| + \lambda_n \|u\| (\alpha_n + s_n)\big) \\
&\leq \|x_n - u\| + \lambda_n \|u\| (\alpha_n + s_n).
\end{aligned}
\tag{59}
$$

Consequently, from condition (a) and Lemma 7, we deduce that, for every $u \in \Gamma$, $\lim_{n \longrightarrow \infty} \|x_n - u\|$ exists and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded. It follows from (55), (56), and Lemma 5 that

$$\begin{aligned}
\|z_n - u\|^2 &\le \gamma_n\|y_n - u\|^2 + (1-\gamma_n)\|t_n - u\|^2 - \gamma_n(1-\gamma_n)\|y_n - t_n\|^2 \\
&\le \gamma_n\left(\|x_n - u\| + \lambda_n\alpha_n\|u\|\right)^2 + (1-\gamma_n)\left(\|x_n - u\| + \lambda_n s_n\|u\|\right)^2 \\
&\quad - \gamma_n(1-\gamma_n)\|y_n - t_n\|^2 \\
&\le \gamma_n\left(2\|x_n - u\|^2 + 2\lambda_n^2\alpha_n^2\|u\|^2\right) + (1-\gamma_n)\left(2\|x_n - u\|^2 + 2\lambda_n^2 s_n^2\|u\|^2\right) \\
&\quad - \gamma_n(1-\gamma_n)\|y_n - t_n\|^2 \\
&\le \|x_n - u\|^2 + 2\lambda_n^2\|u\|^2\left(\alpha_n^2 + s_n^2\right) - \gamma_n(1-\gamma_n)\|y_n - t_n\|^2.
\end{aligned} \tag{60}$$

Let $S_n v_n = (1-\delta_n)v_n + \delta_n S v_n$. We find from (54), (58), and Lemma 5 and the last inequality that

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \|a_n x_n + b_n T_n z_n + c_n((1-\delta_n)v_n + \delta_n S v_n) - u\|^2 \\
&\le a_n\|x_n - u\|^2 + b_n\|T_n z_n - u\|^2 + c_n\Big[(1-\delta_n)\|v_n - u\|^2 + \delta_n\|S v_n - u\|^2 \\
&\quad - (1-\delta_n)\delta_n\|v_n - S v_n\|^2\Big] - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S_n v_n\|^2 \\
&\le a_n\|x_n - u\|^2 + (1-a_n)\|z_n - u\|^2 + r_n(r_n - 2\alpha)\|M z_n - M u\|^2 \\
&\quad - c_n(1-\delta_n)\delta_n\|v_n - S v_n\|^2 - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S_n v_n\|^2 \\
&\le a_n\|x_n - u\|^2 + (1-a_n)\Big[\|x_n - u\|^2 + 2\lambda_n^2\|u\|^2\left(\alpha_n^2 + s_n^2\right) \\
&\quad - \gamma_n(1-\gamma_n)\|y_n - t_n\|^2\Big] - c_n(1-\delta_n)\delta_n\|v_n - S v_n\|^2 \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S_n v_n\|^2 + r_n(r_n - 2\alpha)\|M z_n - M u\|^2 \\
&\le \|x_n - u\|^2 + 2\lambda_n^2\|u\|^2\left(\alpha_n^2 + s_n^2\right) - (1-a_n)\gamma_n(1-\gamma_n)\|y_n - t_n\|^2 \\
&\quad - c_n(1-\delta_n)\delta_n\|v_n - S v_n\|^2 - a_n b_n\|x_n - T_n z_n\|^2 \\
&\quad - a_n c_n\|x_n - S_n v_n\|^2 + r_n(r_n - 2\alpha)\|M z_n - M u\|^2.
\end{aligned} \tag{61}$$

From conditions (b)–(e) and (61), we also obtain

$$\begin{aligned}
&(1-a')t(1-m)\|y_n-t_n\|^2+ab\|x_n-T_nz_n\|^2+c(1-\delta_2)\delta_1\|v_n-Sv_n\|^2\\
&\quad+ac\|x_n-S_nv_n\|^2+r_n(2\alpha-r_n)\|Mz_n-Mu\|^2\\
&\le(1-a_n)\gamma_n(1-\gamma_n)\|y_n-t_n\|^2+c_n(1-\delta_n)\delta_n\|v_n-Sv_n\|^2\\
&\quad+a_nb_n\|x_n-T_nz_n\|^2+a_nc_n\|x_n-S_nv_n\|^2+r_n(2\alpha-r_n)\|Mz_n-Mu\|^2\\
&\le\|x_n-u\|^2-\|x_{n+1}-u\|^2+2\lambda_n^2\|u\|^2\left(\alpha_n^2+s_n^2\right).
\end{aligned}\tag{62}$$

Since $\lim_{n\longrightarrow\infty}\|x_n-u\|$ exists and $\sum_{n=0}^{\infty}(\alpha_n+s_n)<\infty$, we see that

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\|v_n-Sv_n\|&=\lim_{n\longrightarrow\infty}\|x_n-T_nz_n\|=\lim_{n\longrightarrow\infty}\|y_n-t_n\|\\
&=\lim_{n\longrightarrow\infty}\|x_n-S_nv_n\|=\lim_{n\longrightarrow\infty}\|Mz_n-Mu\|=0.
\end{aligned}\tag{63}$$

Since $\|x_{n+1}-x_n\|\le b_n\|x_n-T_nz_n\|+c_n\|x_n-S_nv_n\|$ and $\|z_n-y_n\|\le\|y_n-t_n\|$, $\|z_n-t_n\|\le\|y_n-t_n\|$, it follows that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=\lim_{n\longrightarrow\infty}\|y_n-z_n\|=\lim_{n\longrightarrow\infty}\|t_n-z_n\|=0.\tag{64}$$

Using Lemma 1 and (58), we have

$$\begin{aligned}
\|v_n-u\|^2&=\left\|T_{r_n}(z_n-r_nMz_n)-T_{r_n}(u-r_nMu)\right\|^2\\
&\le\langle(z_n-r_nMz_n)-(u-r_nMu),v_n-u\rangle\\
&=\frac{1}{2}\|(z_n-r_nMz_n)-(u-r_nMu)\|^2+\frac{1}{2}\|v_n-u\|^2\\
&\quad-\frac{1}{2}\|(z_n-r_nMz_n)-(u-r_nMu)-(v_n-u)\|^2\\
&\le\frac{1}{2}\left[\|z_n-u\|^2+\|v_n-u\|^2-\|(z_n-v_n)-2r_n(Mz_n-Mu)\|^2\right]\\
&=\frac{1}{2}\left[\|z_n-u\|^2+\|v_n-u\|^2-\|z_n-v_n\|^2+2r_n\langle z_n-v_n,Mz_n-Mu\rangle-r_n^2\|Mx_n-Mu\|^2\right].
\end{aligned}\tag{65}$$

It follows that

$$\|v_n-u\|^2\le\|z_n-u\|^2-\|z_n-v_n\|^2+2r_n\langle z_n-v_n,Mz_n-Mu\rangle.\tag{66}$$

We find from (54) and (66) that

$$\begin{aligned}
\|x_{n+1}-u\|^2&=\|a_nx_n+b_nT_nz_n+c_n((1-\delta_n)v_n+\delta_nSv_n)-u\|^2\\
&\le a_n\|x_n-u\|^2+b_n\|T_nz_n-u\|^2+c_n\left[(1-\delta_n)\|v_n-u\|^2+\delta_n\|Sv_n-u\|^2\right]\\
&\le a_n\|x_n-u\|^2+(1-a_n)\|z_n-u\|^2-c_n\|z_n-v_n\|^2+2r_n\langle z_n-v_n,Mz_n-Mu\rangle\\
&\le a_n\|x_n-u\|^2+(1-a_n)\left[\|x_n-u\|^2+2\lambda_n^2\|u\|^2\left(\alpha_n^2+s_n^2\right)\right]\\
&\quad-c_n\|z_n-v_n\|^2+2r_n\langle z_n-v_n,Mz_n-Mu\rangle\\
&\le\|x_n-u\|^2+(1-a_n)2\lambda_n^2\|u\|^2\left(\alpha_n^2+s_n^2\right)\\
&\quad-c_n\|z_n-v_n\|^2+2r_n\langle z_n-v_n,Mz_n-Mu\rangle.
\end{aligned}\tag{67}$$

This yields that

$$\|z_n - v_n\|^2 \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + 2\lambda_n^2\|u\|^2\left(\alpha_n^2 + s_n^2\right) + 2r_n\langle z_n - v_n, Mz_n - Mu\rangle. \tag{68}$$

It follows from condition (a), $\lim_{n\longrightarrow\infty}\|Mz_n - Mu\| = 0$, and $\lim_{n\longrightarrow\infty}\|x_n - u\|$ exists that

$$\lim_{n\longrightarrow\infty}\|z_n - v_n\| = 0. \tag{69}$$

Also, from $\|T_nz_n - z_n\| \le \|T_nz_n - x_n\| + \|x_n - z_n\|$, $\|v_n - x_n\| \le \|v_n - S_nv_n\| + \|x_n - S_nv_n\|$, and $\|v_n - S_nv_n\| \le \|v_n - Sv_n\|$, we get

$$\lim_{n\longrightarrow\infty}\|T_nz_n - z_n\| = \lim_{n\longrightarrow\infty}\|v_n - x_n\| = 0. \tag{70}$$

Note that $\|x_n - y_n\| \le \|x_n - z_n\| + \|z_n - y_n\|$, $\|x_n - t_n\| \le \|x_n - z_n\| + \|z_n - t_n\|$, $\beta_n\|Tz_n - z_n\| = \|T_nz_n - z_n\|$. This implies that

$$\lim_{n\longrightarrow\infty}\|x_n - y_n\| = \lim_{n\longrightarrow\infty}\|x_n - t_n\| = \lim_{n\longrightarrow\infty}\|Tz_n - z_n\| = 0. \tag{71}$$

Since $\nabla f = A^*(I - P_Q)A$ is Lipschitz continuous, we obtain $\lim_{n\longrightarrow\infty}\|\nabla f(x_n) - \nabla f(y_n)\| = 0$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that it converges weakly to some x^*. Since $\|x_n - y_n\| \longrightarrow 0$, $\|x_n - z_n\| \longrightarrow 0$, and $\|x_n - v_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain that $y_{n_k} \rightharpoonup x^*$, $z_{n_k} \rightharpoonup x^*$, and $v_{n_k} \rightharpoonup x^*$. Since $\{y_{n_k}\} \subset C$ and C is closed and convex, we obtain $x^* \in C$. First, we show that $x^* \in F(T)\cap F(S)$. Then, from (63), (71), Lemma 6, and Lemma 4, we have that $x^* \in F(T)\cap F(S)$. We now show $x^* \in \mathrm{GEP}(F, M)$. By $v_n = T_{r_n}(z_n - r_nMz_n)$, we know that

$$F(v_n, y) + \langle Mz_n, y - v_n\rangle + \frac{1}{r_n}\langle y - v_n, v_n - z_n\rangle \ge 0, \quad \forall y \in C. \tag{72}$$

It follows from (A_2) that

$$\langle Mz_n, y - v_n\rangle + \frac{1}{r_n}\langle y - v_n, v_n - x_n\rangle \ge F(y, v_n), \quad \forall y \in C. \tag{73}$$

Hence,

$$\langle Mz_{n_j}, y - v_{n_j}\rangle + \langle y - v_{n_j}, \frac{v_{n_j} - z_{n_j}}{r_{n_j}}\rangle \ge F\left(y, v_{n_j}\right), \quad \forall y \in C. \tag{74}$$

For t with $0 < t \le 1$ and $y \in C$, let $v_t = ty + (1 - t)x^*$. Since $y \in C$ and $x^* \in C$, we obtain $v_t \in C$. So, from (74), we have

$$\begin{aligned}\langle v_t - v_{n_j}, Mv_t\rangle &\ge \langle v_t - v_{n_j}, Mv_t\rangle - \langle v_t - v_{n_j}, Mz_{n_j}\rangle \\ &\quad - \langle v_t - v_{n_j}, \frac{v_{n_j} - z_{n_j}}{r_{n_j}}\rangle + F\left(v_t, v_{n_j}\right) \\ &= \langle v_t - v_{n_j}, Mv_t - Mv_{n_j}\rangle \\ &\quad + \langle v_t - v_{n_j}, Mv_{n_j} - Mz_{n_j}\rangle \\ &\quad - \langle v_t - v_{n_j}, \frac{v_{n_j} - z_{n_j}}{r_{n_j}}\rangle + F\left(v_t, v_{n_j}\right).\end{aligned} \tag{75}$$

Since $\|v_{n_j} - z_{n_j}\| \longrightarrow 0$, we have $\|Mv_{n_j} - Mz_{n_j}\| \longrightarrow 0$. Furthermore, from the inverse strongly monotonicity of M, we have $\langle v_t - v_{n_j}, Mv_t - Mv_{n_j}\rangle \ge 0$. It follows from A_4 and $(v_{n_j} - z_{n_j}/r_{n_j}) \longrightarrow 0$ and $v_{n_j} \rightharpoonup x^*$, and we have

$$\langle v_t - v, Mv_t\rangle \ge F(v_t, x^*), \tag{76}$$

as $j \longrightarrow \infty$. From (A_1) and (A_4), we have

$$\begin{aligned}0 &= F(v_t, v_t) \\ &\le tF(v_t, y) + (1 - t)F(v_t, x^*) \\ &\le tF(v_t, y) + (1 - t)\langle v_t - x^*, Mv_t\rangle \\ &= tF(v_t, y) + (1 - t)t\langle y - x^*, Mv_t\rangle,\end{aligned} \tag{77}$$

and hence,

$$0 \le F(v_t, y) + (1 - t)\langle y - x^*, Mv_t\rangle. \tag{78}$$

Letting $t \longrightarrow 0$, we have, for each $y \in C$,

$$F(x^*, y) + \langle y - x^*, Mx^*\rangle \ge 0. \tag{79}$$

This implies that $x^* \in \mathrm{GEP}(F, M)$. Next, we show that $x^* \in \cap_{i=1}^2\Omega_i$ (1). For $i = 1, 2$, let

$$T_i'p := \begin{cases}\nabla f_i(p) + N_Cp, & p \in C, \\ \varnothing, & p \notin C.\end{cases} \tag{80}$$

Then, T_i' is maximal monotone and $0 \in T_i'p$ if and only if $p \in \mathrm{VI}(C, \nabla f_i)$ [29]. Let $G(T_i')$ be the graph of T_i', and $(p, v) \in G(T_i')$. Then, we have $v \in T_i'(p) = \nabla f_i(p) + N_Cp$, and hence, $v - \nabla f_i(p) \in N_Cp$. Therefore, we have $\langle p - w, v - \nabla f_i(p)\rangle \ge 0$ for all $w \in C$. By the property of metric projection, from $y_n = P_C(x_n - \lambda_n\nabla f_{1\alpha_n}x_n)$ and $p \in C$, we have $\langle p - y_n, y_n - (x_n - \lambda_n\nabla f_{1\alpha_n}x_n)\rangle \ge 0$, and hence,

$$\langle p - y_n, \frac{y_n - x_n}{\lambda_n} + \nabla f_{1\alpha_n}x_n\rangle \ge 0. \tag{81}$$

From $\langle p - w, v - \nabla f_1(p)\rangle \ge 0$ for all $w \in C$ and $y_{n_k} \in C$, we have

$$\begin{aligned}
\langle p - y_{n_k}, v\rangle &\geq \langle p - y_{n_k}, \nabla f_1(p)\rangle \\
&\geq \langle p - y_{n_k}, \nabla f_1(p)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_{n_k}} + \nabla f_{1\alpha_n} x_{n_k}\rangle \\
&\geq \langle p - y_{n_k}, \nabla f_1(p)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_{n_k}} + \nabla f_1(x_{n_k})\rangle - \alpha_{n_k}\langle p - y_{n_k}, x_{n_k}\rangle \\
&= \langle p - y_{n_k}, \nabla f_1(p) - \nabla f_1(y_{n_k})\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_{n_k}}\rangle - \alpha_{n_k}\langle p - y_{n_k}, x_{n_k}\rangle \\
&\quad + \langle p - y_{n_k}, \nabla f_1(y_{n_k}) - \nabla f_1(x_{n_k})\rangle \\
&\geq \langle p - y_{n_k}, \nabla f_1(y_{n_k}) - \nabla f_1(x_{n_k})\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_{n_k}}\rangle - \alpha_{n_k}\langle p - y_{n_k}, x_{n_k}\rangle.
\end{aligned} \tag{82}$$

Thus, we obtain $\langle p - x^*, v\rangle \geq 0$ as $k \longrightarrow \infty$. Since T_1' is maximal monotone, we have $x^* \in T_1'^{-1}0$, and hence, $x^* \in \mathrm{VI}(C, \nabla f_1)$. Similarly, we have $x^* \in \mathrm{VI}(C, \nabla f_2)$. This implies $x^* \in \Omega_i$ for $i = 1, 2$. This implies that $x^* \in \Gamma$. Therefore, from $\|x_n - z_n\| \longrightarrow 0$, we can conclude that $\{x_n\}$, $\{z_n\}$, and $\{v_n\}$ converge weakly to a point $u \in \Gamma$. The proof is complete. □

Theorem 3. *Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A: H_1 \longrightarrow H_2$ be a bounded linear operator, $S: C \longrightarrow C$ be a nonexpansive map, and $T: C \longrightarrow C$ be a strictly pseudo-contractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap \Omega \neq \varnothing$. Suppose $\{x_n\}$, $\{v_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases}
x_0 = x \in C, \\
v_n = x_n + \varepsilon_n (x_n - x_{n-1}), \\
z_n = P_C(I - \lambda_n \nabla f_{\alpha_n}) v_n, \\
x_{n+1} = a_n x_n + b_n((1 - \beta_n) z_n + \beta_n T z_n) \\
+ c_n S P_C(v_n - \lambda_n \nabla f_{\alpha_n} z_n), \quad \forall n \geq 0.
\end{cases} \tag{83}$$

Suppose the following conditions are satisfied:

(a) $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$

(b) $\{\beta_n\} \subset [k, r]$ *for some* $r, k \in (0, 1)$

(c) $\{\lambda_n\} \subset [e, d]$ *for some* $e, d \in (0, (1/\|A\|^2))$

(d) $0 < a \leq a_n \leq a' < 1, 0 < b_n \leq b' < 1, 0 < c \leq c_n \leq c' < 1$ *and* $a_n + b_n + c_n = 1$

(e) $\{\varepsilon_n\} \subset [0, \varepsilon]$ *and* $\varepsilon \in [0, 1)$, $\sum_{n=0}^{\infty} \varepsilon_n \|x_n - x_{n-1}\| < \infty$

Then, $\{x_n\}$ *converges weakly to an element* $u \in \Gamma$.

Proof. For any fixed $u \in \Gamma$, we find that $u = P_C(I - \lambda \nabla f)u$ for $\lambda \in (0, (1/\|A\|^2))$ and $Su = u$. Putting $T_n = (1 - \beta_n)I + \beta_n T$, we see from Lemma 8 that T_n is nonexpansive and $F(T_n) = F(T)$. We observe that

$$\begin{aligned}
\|v_n - u\| &= \|x_n + \varepsilon_n (x_n - x_{n-1}) - u\| \\
&\leq \|x_n - u\| + \varepsilon_n \|x_n - x_{n-1}\|.
\end{aligned} \tag{84}$$

From (83) and Lemma 3, it follows that

$$\begin{aligned}
\|z_n - u\| &= \|P_C(I - \lambda_n \nabla f_{\alpha_n}) v_n - P_C(I - \lambda_n \nabla f) u\| \\
&\leq \|P_C(I - \lambda_n \nabla f_{\alpha_n}) v_n - P_C(I - \lambda_n \nabla f_{\alpha_n}) u\| \\
&\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n}) u - P_C(I - \lambda_n \nabla f) u\| \\
&\leq \|v_n - u\| + \|(I - \lambda_n \nabla f_{\alpha_n}) u - (I - \lambda_n \nabla f) u\| \\
&\leq \|v_n - u\| + \lambda_n \alpha_n \|u\|.
\end{aligned} \tag{85}$$

Put $y_n = P_C(v_n - \lambda_n \nabla f_{\alpha_n}(z_n))$ for all $n \geq 0$. Then, by property of metric projection, we have

$$\begin{aligned}
\|y_n - u\|^2 &\leq \left\|v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - u\right\|^2 - \left\|v_n - \lambda_n \nabla f_{\alpha_n} z_n - y_n\right\|^2 \\
&\leq \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), u - y_n \rangle \\
&\leq \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n) - \nabla f_{\alpha_n}(u), u - z_n \rangle \\
&\quad + 2\lambda_n \left(\langle \nabla f_{\alpha_n}(u), u - z_n \rangle + \langle \nabla f_{\alpha_n}(z_n), z_n - y_n \rangle \right) \\
&\leq \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(u), u - z_n \rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n \rangle \\
&\leq \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle (\alpha_n I + \nabla f) u, u - z_n \rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n \rangle \\
&\leq \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \alpha_n \langle u, u - z_n \rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n \rangle \\
&= \|v_n - u\|^2 - \|v_n - z_n\|^2 - 2\langle v_n - z_n, z_n - y_n \rangle - \|z_n - y_n\|^2 \\
&\quad + 2\lambda_n \left[\alpha_n \langle u, u - z_n \rangle + \langle \nabla f_{\alpha_n}(z_n), z_n - y_n \rangle \right] \\
&= \|v_n - u\|^2 - \|v_n - z_n\|^2 + 2\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n \rangle \\
&\quad + 2\lambda_n \alpha_n \langle u, u - z_n \rangle - \|z_n - y_n\|^2.
\end{aligned} \tag{86}$$

Furthermore, by property of metric projection, we have

$$\begin{aligned}
&\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n \rangle \\
&\quad = \langle v_n - \lambda_n \nabla f_{\alpha_n}(v_n) - z_n, y_n - z_n \rangle + \langle \lambda_n \nabla f_{\alpha_n}(v_n) - \lambda_n \nabla f_{\alpha_n}(z_n), y_n - z_n \rangle \\
&\quad \leq \langle \lambda_n \nabla f_{\alpha_n}(v_n) - \lambda_n \nabla f_{\alpha_n}(z_n), y_n - z_n \rangle \\
&\quad \leq \lambda_n \left\| \nabla f_{\alpha_n}(v_n) - \nabla f_{\alpha_n}(z_n) \right\| \|y_n - z_n\| \\
&\quad \leq \lambda_n \left(\alpha_n + \|A\|^2 \right) \|v_n - z_n\| \|y_n - z_n\|.
\end{aligned} \tag{87}$$

Hence, we have

$$\begin{aligned}
\|y_n - u\|^2 &\le \|v_n - u\|^2 - \|v_n - z_n\|^2 + 2\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n\rangle \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n\alpha_n\langle u, u - z_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - z_n\|^2 + 2\lambda_n\left(\alpha_n + \|A\|^2\right)\|v_n - z_n\|\|y_n - z_n\| \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| \\
&\le \|v_n - u\|^2 - \|v_n - z_n\|^2 + \lambda_n^2\left(\alpha_n + \|A\|^2\right)^2\|v_n - z_n\|^2 + \|y_n - z_n\|^2 \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| \\
&= \|v_n - u\|^2 + \left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| \\
&\le \|v_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| \\
&\le \|v_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\left(\|v_n - u\| + \lambda_n\alpha_n\|u\|\right) \\
&\le \|v_n - u\|^2 + 4\lambda_n\alpha_n\|u\|\|v_n - u\| + 4\lambda_n^2\alpha_n^2\|u\|^2 \\
&= \left(\|v_n - u\| + 2\lambda_n\alpha_n\|u\|\right)^2.
\end{aligned} \tag{88}$$

We find from (83), (84), and (85) and the last inequality that

$$\begin{aligned}
\|x_{n+1} - u\| &= \left\|a_n x_n + b_n T_n z_n + c_n S P_C\left(v_n - \lambda_n \nabla f_{\alpha_n}(z_n)\right) - u\right\| \\
&\le a_n\|x_n - u\| + b_n\|T_n z_n - u\| + c_n\|S y_n - u\| \\
&\le a_n\|x_n - u\| + b_n\|z_n - u\| + c_n\|y_n - u\| \\
&\le a_n\|x_n - u\| + b_n\left(\|v_n - u\| + \lambda_n\alpha_n\|u\|\right) + c_n\left(\|v_n - u\| + 2\lambda_n\alpha_n\|u\|\right) \\
&\le a_n\|x_n - u\| + (1 - a_n)\left(\|v_n - u\| + 2\lambda_n\alpha_n\|u\|\right) \\
&\le \|x_n - u\| + \varepsilon_n\|x_n - x_{n-1}\| + 2\lambda_n\alpha_n\|u\|.
\end{aligned} \tag{89}$$

Consequently, from conditions (a) and (e) and Lemma 10, we deduce that, for every $u \in \Gamma$, $\lim_{n\longrightarrow\infty}\|x_n - u\|$ exists and the sequences $\{x_n\}$, $\{z_n\}$, and $\{y_n\}$ are bounded. We find from (83), (84), (85), (88), Lemma 5, and Lemma 11 that

$$
\begin{aligned}
\|x_{n+1}-u\|^2 &= \left\|a_n x_n + b_n T_n z_n + c_n S P_C\left(v_n - \lambda_n \nabla f_{\alpha_n}(z_n)\right) - u\right\|^2 \\
&\le a_n\|x_n-u\|^2 + b_n\|T_n z_n - u\|^2 + c_n\|S y_n - u\|^2 \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le a_n\|x_n-u\|^2 + b_n\|z_n - u\|^2 + c_n\|y_n - u\|^2 \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le a_n\|x_n-u\|^2 + b_n\left(\|v_n-u\| + \lambda_n\alpha_n\|u\|\right)^2 \\
&\quad + c_n\left[\|v_n-u\|^2 + \left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2\right] \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 + 2c_n\lambda_n\alpha_n\|u\|\|z_n - u\| \\
&\le a_n\|x_n-u\|^2 + b_n\left(2\|v_n-u\|^2 + 2\lambda_n^2\alpha_n^2\|u\|^2\right) \\
&\quad + c_n\left[\|v_n-u\|^2 + \left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n-u\|\right] \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le (1+b_n)\|x_n-u\|^2 + 2b_n\lambda_n^2\alpha_n^2\|u\|^2 + 2(2b_n + c_n)\varepsilon_n\langle x_n - x_{n-1}, v_n - u\rangle \\
&\quad + c_n\left[\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2 + \alpha_n\left(\lambda_n^2\|u\|^2 + \|z_n-u\|^2\right)\right] \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le (1+b_n)\|x_n-u\|^2 + 2b_n\lambda_n^2\alpha_n^2\|u\|^2 + 2(2b_n + c_n)\varepsilon_n\|x_n - x_{n-1}\|\|v_n - u\| \\
&\quad + c_n\left[\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2 + \alpha_n\left(\lambda_n^2\right)\|u\|^2 + \left(\|v_n-u\| + \lambda_n\alpha_n\|u\|\right)^2\right] \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le (1+b_n)\|x_n-u\|^2 + 2b_n\lambda_n^2\alpha_n^2\|u\|^2 + 2(2b_n + c_n)\varepsilon_n\|x_n - x_{n-1}\|\|v_n - u\| \\
&\quad + c_n\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2 + \alpha_n\left(\lambda_n^2\|u\|^2 + 2\|v_n-u\|^2 + 2\lambda_n^2\alpha_n^2\|u\|^2\right) \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le (1+b_n+2\alpha_n)\|x_n-u\|^2 + c_n\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|x_n - z_n\|^2 \\
&\quad + \alpha_n\lambda_n^2\|u\|^2\left(1 + 2b_n\alpha_n + 2\alpha_n^2\right) - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\quad + 2(2b_n + c_n + 2\alpha_n)\varepsilon_n\|x_n - x_{n-1}\|\|v_n - u\|.
\end{aligned}
\tag{90}
$$

From conditions (b) and (d), we obtain

$$
\begin{aligned}
&c\left(1 - d^2\left(\alpha_n + \|A\|^2\right)^2\right)\|v_n - z_n\|^2 + ab_n\|x_n - T_n z_n\|^2 + ac\|x_n - S y_n\|^2 \\
&\le c_n\left(1 - \lambda_n^2\left(\alpha_n + \|A\|^2\right)^2\right)\|v_n - z_n\|^2 + a_n b_n\|x_n - T_n z_n\|^2 + a_n c_n\|x_n - S y_n\|^2 \\
&\le (1+b_n+2\alpha_n)\|x_n-u\|^2 - \|x_{n+1}-u\|^2 + \alpha_n\lambda_n^2\|u\|^2\left(1 + 2\alpha_n^2\right) \\
&\quad + 2(2b_n + c_n + 2\alpha_n)\varepsilon_n\|x_n - x_{n-1}\|\|v_n - u\|.
\end{aligned}
\tag{91}
$$

From conditions (a) and (e), we also obtain

$$\lim_{n\longrightarrow\infty}\|x_n - Sy_n\| = \lim_{n\longrightarrow\infty}\|x_n - T_n z_n\| = \lim_{n\longrightarrow\infty}\|v_n - z_n\| = 0. \quad (92)$$

By the definition of $\{v_n\}$ and (e), we have

$$\lim_{n\longrightarrow\infty}\|v_n - x_n\| = \lim_{n\longrightarrow\infty}\varepsilon_n\|x_n - x_{n-1}\| = 0. \quad (93)$$

This implies that

$$\|z_n - x_n\| \le \|z_n - v_n\| + \|v_n - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (94)$$

It is observe that

$$\begin{aligned}\|y_n - z_n\| &= \left\|P_C\left(v_n - \lambda_n\nabla f_{\alpha_n}(z_n)\right) - P_C\left(v_n - \lambda_n\nabla f_{\alpha_n}(v_n)\right)\right\| \\ &\le \left\|v_n - \lambda_n\nabla f_{\alpha_n}(z_n) - \left(v_n - \lambda_n\nabla f_{\alpha_n}(v_n)\right)\right\| \\ &= \lambda_n\left\|\nabla f_{\alpha_n}(z_n) - \nabla f_{\alpha_n}(v_n)\right\| \\ &\le \lambda_n\left(\alpha_n + \|A\|^2\right)\|z_n - v_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \quad (95)$$

Also, from $\|T_n z_n - z_n\| \le \|T_n z_n - x_n\| + \|x_n - z_n\|$, $\|y_n - x_n\| \le \|y_n - z_n\| + \|z_n - x_n\|$, and $\|y_n - v_n\| \le \|y_n - z_n\| + \|z_n - v_n\|$, we get

$$\lim_{n\longrightarrow\infty}\|T_n z_n - z_n\| = \lim_{n\longrightarrow\infty}\|y_n - x_n\| = \lim_{n\longrightarrow\infty}\|y_n - v_n\| = 0. \quad (96)$$

Note that $\|Sy_n - y_n\| \le \|Sy_n - x_n\| + \|x_n - y_n\|$, $\beta_n\|Tz_n - z_n\| = \|T_n z_n - z_n\|$. This implies that

$$\lim_{n\longrightarrow\infty}\|Sy_n - y_n\| = \lim_{n\longrightarrow\infty}\|Tz_n - z_n\| = 0. \quad (97)$$

Since $\nabla f = A^*(I - P_Q)A$ is Lipschitz continuous, we obtain $\lim_{n\longrightarrow\infty}\|\nabla f(z_n) - \nabla f(y_n)\| = 0$.

Since, $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that it converges weakly to some x^*. Since $\|x_n - y_n\| \longrightarrow 0$, $\|x_n - z_n\| \longrightarrow 0$, and $\|x_n - v_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain that $y_{n_k} \rightharpoonup x^*, z_{n_k} \rightharpoonup x^*$, and $v_{n_k} \rightharpoonup x^*$. Since $\{y_{n_k}\} \subset C$ and C is closed and convex, we obtain $x^* \in C$. First, we show that $x^* \in F(T) \cap F(S)$. Then, from (97), Lemma 6, and Lemma 4, we have that $x^* \in F(T) \cap F(S)$. We now show $x^* \in \Omega$ (1). Let

$$T'p := \begin{cases}\nabla f(p) + N_C p, & p \in C, \\ \emptyset, & p \notin C.\end{cases} \quad (98)$$

Then, T' is maximal monotone and $0 \in T'p$ if and only if $p \in \mathrm{VI}(C, \nabla f)$ [29]. Let $G(T')$ be the graph of T', and let $(p, v) \in G(T')$. Then, we have $v \in T'(p) = \nabla f(p) + N_C p$, and hence, $v - \nabla f(p) \in N_C p$. Therefore, we have $\langle p - w, v - \nabla f(p)\rangle \ge 0$ for all $w \in C$. By property of metric projection, from $y_n = P_C(v_n - \lambda_n\nabla f_{\alpha_n} z_n)$ and $p \in C$, we have $\langle p - y_n, y_n - (v_n - \lambda_n\nabla f_{\alpha_n} z_n)\rangle \ge 0$, and hence,

$$\langle p - y_n, \frac{y_n - v_n}{\lambda_n} + \nabla f_{\alpha_n} z_n\rangle \ge 0. \quad (99)$$

From $\langle p - w, v - \nabla f(p)\rangle \ge 0$ for all $w \in C$ and $y_{n_k} \in C$, we have

$$\begin{aligned}\langle p - y_{n_k}, v\rangle &\ge \langle p - y_{n_k}, \nabla f(p)\rangle \\ &\ge \langle p - y_{n_k}, \nabla f(p)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} + \nabla f_{\alpha_{n_k}} z_{n_k}\rangle \\ &\ge \langle p - y_{n_k}, \nabla f(p)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} + \nabla f\left(z_{n_k}\right)\rangle - \alpha_{n_k}\langle p - y_{n_k}, z_{n_k}\rangle \\ &= \langle p - y_{n_k}, \nabla f(p) - \nabla f\left(y_{n_k}\right)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}}\rangle - \alpha_{n_k}\langle p - y_{n_k}, z_{n_k}\rangle \\ &\quad + \langle p - y_{n_k}, \nabla f\left(y_{n_k}\right) - \nabla f\left(z_{n_k}\right)\rangle \\ &\ge \langle p - y_{n_k}, \nabla f\left(y_{n_k}\right) - \nabla f\left(z_{n_k}\right)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}}\rangle - \alpha_{n_k}\langle p - y_{n_k}, z_{n_k}\rangle.\end{aligned} \quad (100)$$

Thus, we obtain $\langle p - x^*, v\rangle \ge 0$ as $k \longrightarrow \infty$. Since T' is maximal monotone, we have $x^* \in T'^{-1}0$, and hence, $x^* \in \mathrm{VI}(C, \nabla f)$. This implies that $x^* \in \Omega$. This implies that $x^* \in \Gamma$. Therefore, from $\|x_n - z_n\| \longrightarrow 0$ and $\|x_n - v_n\| \longrightarrow 0$, we can conclude that $\{x_n\}$, $\{z_n\}$, and $\{v_n\}$ converge weakly to a point $u \in \Gamma$. The proof is complete. □

4. Applications

If, in Theorem 3 and Theorem 1, we assume that $C = H_1$, then we can get the following theorems.

Theorem 4. *Let H_1 and H_2 be real Hilbert spaces, A_i: $H_1 \longrightarrow H_2$ be a bounded linear operator, for $i = 1, 2$, S: $H_1 \longrightarrow H_1$ be a nonexpansive mapping, and T: $H_1 \longrightarrow H_1$ a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap_{i=1}^{2} (\nabla f_i)^{-1} 0 \neq \emptyset$. Suppose $\{x_n\}$ and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \gamma_n\left(x_n - \lambda_n \nabla f_{1\alpha_n} x_n\right) + (1 - \gamma_n)\left(x_n - \lambda_n \nabla f_{2s_n} x_n\right), \\ x_{n+1} = a_n x_n + b_n\left((1 - \beta_n) z_n + \beta_n T z_n\right) \\ + c_n\left((1 - \delta_n) z_n + \delta_n S z_n\right), \quad \forall n \geq 0. \end{cases} \tag{101}$$

If conditions $(a) - (d)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Proof. We have $(\nabla f_i)^{-1} 0 = \mathrm{VI}(H_1, \nabla f_i)$ for $i = 1, 2$ and $P_{H_1} = I$; by Theorem 3, we obtain the desired result. □

Theorem 5. *Let H_1 and H_2 be real Hilbert spaces, A: $H_1 \longrightarrow H_2$ be a bounded linear operator, S: $H_1 \longrightarrow H_1$ be a nonexpansive map, and T: $H_1 \longrightarrow H_1$ be a strictly pseudocontractive mapping with constants k such that $\Gamma = F(T) \cap F(S) \cap (\nabla f)^{-1} 0 \neq \emptyset$. Suppose $\{x_n\}$, $\{v_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ v_n = x_n + \varepsilon_n (x_n - x_{n-1}), \\ z_n = \left(I - \lambda_n \nabla f_{\alpha_n}\right) v_n, \\ x_{n+1} = a_n x_n + b_n\left((1 - \beta_n) z_n + \beta_n T z_n\right) \\ + c_n S\left(v_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \quad \forall n \geq 0. \end{cases} \tag{102}$$

If conditions $(a) - (e)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Proof. We have $(\nabla f)^{-1} 0 = \mathrm{VI}(H_1, \nabla f)$ and $P_{H_1} = I$; by Theorem 3, we obtain the desired result. □

Theorem 6. *Let H_1 and H_2 be real Hilbert spaces, A: $H_1 \longrightarrow H_2$ be a bounded linear operator, S: $H_1 \longrightarrow H_1$ be a nonexpansive map, and T: $H_1 \longrightarrow H_1$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap (\nabla f)^{-1} 0 \neq \emptyset$. Suppose $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \left(I - \lambda_n \nabla f_{\alpha_n}\right) x_n, \\ y_n = x_n - \lambda_n \nabla f_{\alpha_n} z_n, \\ x_{n+1} = a_n x_0 + b_n x_n + c_n\left((1 - \beta_n) S y_n + \beta_n T_n z_n\right), \quad \forall n \geq 0, \end{cases} \tag{103}$$

where $T_n = (1 - \gamma_n) I + \gamma_n T$ and $\gamma_n \subset (k, 1)$. If conditions $(a) - (d)$ are satisfied, then $\{x_n\}$ converges strongly to the point $u = P_\Gamma(x_0)$.

Proof. We have $(\nabla f)^{-1} 0 = \mathrm{VI}(H_1, \nabla f)$ and $P_{H_1} = I$; by Theorem 1, we obtain the desired result. □

Let B: $H \longrightarrow 2^H$ be a maximal monotone mapping. Then, for any $x \in H$ and $r > 0$, consider $J_r^B x = \{y \in H\colon x = y + rBy\}$. Likewise, a J_r^B is called the resolvent of B and is denoted by $J_r^B = (I + rB)^{-1}$.

Theorem 7. *Let H_1 and H_2 be real Hilbert spaces, B_i: $H_1 \longrightarrow 2^{H_1}$ be maximal monotone mappings, for $i = 1, 2$, A_i: $H_1 \longrightarrow H_2$ be bounded linear operators, for $i = 1, 2$, $J_r^{B_i}$ be the resolvents of B_i for each $r > 0$, and T: $H_1 \longrightarrow H_1$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap B_i^{-1} 0 \cap (\nabla f_i)^{-1} 0 \neq \emptyset$. Suppose $\{x_n\}$ and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \gamma_n J_r^{B_1}\left(x_n - \lambda_n \nabla f_{1\alpha_n} x_n\right) + (1 - \gamma_n) J_r^{B_2}\left(x_n - \lambda_n \nabla f_{2s_n} x_n\right), \\ x_{n+1} = a_n x_n + b_n\left((1 - \beta_n) z_n + \beta_n T z_n\right) + c_n z_n, \quad \forall n \geq 0. \end{cases} \tag{104}$$

If conditions $(a) - (d)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Proof. We have $F(J_r^{B_i}) = B_i^{-1} 0$, $(\nabla f_i)^{-1} 0 = \mathrm{VI}(H_1, \nabla f_i)$ for $i = 1, 2$ and $P_{H_1} = I$; by Theorem 3, we obtain the desired result. □

Theorem 8. *Let H_1 and H_2 be real Hilbert spaces, A: $H_1 \longrightarrow H_2$ be a bounded linear operator, B: $H_1 \longrightarrow 2^{H_1}$ be a maximal monotone mapping, J_r^B be the resolvent of B for each $r > 0$, and T: $H_1 \longrightarrow H_1$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap B^{-1} 0 \cap (\nabla f)^{-1} 0 \neq \emptyset$. Suppose $\{x_n\}$, $\{v_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ v_n = x_n + \varepsilon_n (x_n - x_{n-1}), \\ z_n = \left(I - \lambda_n \nabla f_{\alpha_n}\right) v_n, \\ x_{n+1} = a_n x_n + b_n\left((1 - \beta_n) z_n + \beta_n T z_n\right) \\ + c_n J_r^B\left(v_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \quad \forall n \geq 0. \end{cases} \tag{105}$$

If conditions $(a) - (e)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Proof. We have $F(J_r^B) = B^{-1}0$, $(\nabla f)^{-1}0 = \mathrm{VI}(H_1, \nabla f)$, and $P_{H_1} = I$; by Theorem 3, we obtain the desired result. □

Theorem 9. *Let H_1 and H_2 be real Hilbert spaces, A: $H_1 \longrightarrow H_2$ be a bounded linear operator, B: $H_1 \longrightarrow 2^{H_1}$ be a maximal monotone mapping, J_r^B be the resolvent of B for each $r > 0$, and T: $H_1 \longrightarrow H_1$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap B^{-1}0 \cap (\nabla f)^{-1}0 \neq \emptyset$. Suppose $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \left(I - \lambda_n \nabla f_{\alpha_n}\right) x_n, \\ y_n = J_r^B\left(x_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \\ x_{n+1} = a_n x_0 + b_n x_n + c_n\left((1-\beta_n) y_n + \beta_n T_n z_n\right), \quad \forall n \geq 0, \end{cases} \tag{106}$$

where $T_n = (1-\gamma_n)I + \gamma_n T$ and $\gamma_n \subset (k, 1)$. If conditions $(a)-(d)$ are satisfied, then $\{x_n\}$ converges strongly to the point $u = P_\Gamma(x_0)$.

Proof. We have $F(J_r^B) = B^{-1}0$, $(\nabla f)^{-1}0 = \mathrm{VI}(H_1, \nabla f)$, and $P_{H_1} = I$; by Theorem 1, we obtain the desired result. □

If in Theorems 3 and 1 we assume that T is nonexpansive, then we have that T is strictly pseudocontractive with $k = 1$, and hence, we get the following corollaries.

Corollary 1. *Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A_i: $H_1 \longrightarrow H_2$ be bounded linear operators for $i = 1, 2$, S: $C \longrightarrow C$ be a nonexpansive mapping, and T: $C \longrightarrow C$ be a nonexpansive mapping such that $\Gamma = F(T) \cap F(S) \cap_{i=1,2}^{2} \Omega_i \neq \emptyset$. Suppose $\{x_n\}$ and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \gamma_n P_C\left(x_n - \lambda_n \nabla f_{1\alpha_n} x_n\right) \\ + (1-\gamma_n) P_C\left(x_n - \lambda_n \nabla f_{2s_n} x_n\right), \\ x_{n+1} = a_n x_n + b_n\left((1-\beta_n) z_n + \beta_n T z_n\right) \\ + c_n\left((1-\delta_n) z_n + \delta_n S z_n\right), \quad \forall n \geq 0. \end{cases} \tag{107}$$

If conditions $(a)-(d)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Corollary 2. *Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A: $H_1 \longrightarrow H_2$ be a bounded linear operator, S: $C \longrightarrow C$ be a nonexpansive mapping, and T: $C \longrightarrow C$ be a nonexpansive mapping such that $\Gamma = F(T) \cap F(S) \cap \Omega \neq \emptyset$. Suppose that $\{x_n\}$, $\{v_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ v_n = x_n + \varepsilon_n (x_n - x_{n-1}), \\ z_n = P_C\left(I - \lambda_n \nabla f_{\alpha_n}\right) v_n, \\ x_{n+1} = a_n x_n + b_n\left((1-\beta_n) z_n + \beta_n T z_n\right) \\ + c_n S P_C\left(v_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \quad \forall n \geq 0. \end{cases} \tag{108}$$

If conditions $(a)-(e)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Corollary 3. *Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A: $H_1 \longrightarrow H_2$ be a bounded linear operator, S: $C \longrightarrow C$ be a nonexpansive map, and T: $C \longrightarrow C$ be nonexpansive such that $\Gamma = F(T) \cap F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = P_C\left(I - \lambda_n \nabla f_{\alpha_n}\right) x_n, \\ y_n = P_C\left(x_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \\ x_{n+1} = a_n x_0 + b_n x_n + c_n\left((1-\beta_n) S y_n + \beta_n T_n z_n\right), \quad \forall n \geq 0, \end{cases} \tag{109}$$

where $T_n = (1-\gamma_n)I + \gamma_n T$ and $\gamma_n \subset (k, 1)$. If conditions $(a)-(d)$ are satisfied, then $\{x_n\}$ converges strongly to the point $u = P_\Gamma(x_0)$.

Data Availability

All data required for this paper are included within this paper.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

This paper was supported by Azarbaijan Shahid Madani University, Iran, and by the National Natural Science Foundation of China (no. 11671365).

References

[1] Y. Censor and T. Elfving, "A multiprojection algorithm using bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2, pp. 221–239, 1994.
[2] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.
[3] E. Bonacker, A. Gibali, and K. H. Küfer, "Nesterov perturbations and projection methods applied to IMRT," *Jouranl of Nonlinear Variable Analysis*, vol. 4, pp. 63–86, 2020.
[4] X. Qin, L. Wang, and J. C. Yao, "Inertial splitting method for maximal monotone mappings," *Journal of Nonlinear Convex Analysis*, vol. 21, pp. 2325–2333, 2020.
[5] Y. Censor, S. Petra, and C. Schnörr, "Superiorization vs. accelerated convex optimization: the superiorized/regularized

least-squares case," *Journal of Applied and Numerical Optimization*, vol. 2, pp. 15–62, 2020.

[6] H.-K. Xu, "Iterative methods for the split feasibility problem in infinite-dimensional hilbert spaces," *Inverse Problems*, vol. 26, no. 10, p. 105018, 2010.

[7] C. Byrne, "Iterative oblique projection onto convex sets and the split feasibility problem," *Inverse Problems*, vol. 18, no. 2, pp. 441–453, 2002.

[8] X. Qin, A. Petruşel, and J. C. Yao, "CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in hilbert spaces," *Journal of Nonlinear Convex Analysis*, vol. 19, pp. 157–165, 2018.

[9] S. Y. Cho and X. Qin, "On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems," *Applied Mathematics and Computation*, vol. 235, pp. 430–438, 2014.

[10] C. E. Chidume and M. O. Nnakwe, "Iterative algorithms for split variational inequalities and generalized split feasibility problems with applications," *Journal of Nonlinear Variable Analysis*, vol. 3, pp. 127–140, 2019.

[11] X. Qin and J. C. Yao, "A viscosity iterative method for a split feasibility problem," *Journal of Nonlinear Convex Analysis*, vol. 20, pp. 1497–1506, 2019.

[12] W. Takahahsi and J. C. Yao, "The split common fixed point problem for two finite families of nonlinear mappings in hilbert spaces," *Journal of Nonlinear Convex Analysis*, vol. 20, pp. 173–195, 2019.

[13] D. R. Sahu, J. C. Yao, M. Verma, and K. K. Shukla, "Convergence rate analysis of proximal gradient methods with applications to composite minimization problems," *Optimization*, pp. 1–21, 2020.

[14] M. Feng, L. Shi, and R. Chen, "Linear convergence of an iterative algorithm for solving the multiple-sets split equality problem," *Journal of Nonlinear Functional Analysis*, vol. 20199 pages, 2019.

[15] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.

[16] Sh. Rezapour and S. H. Zakeri, "Implicit iterative algorithms for α -inverse strongly accretive operators in banach spaces," *Journal of Nonlinear Convex Analysis*, vol. 20, pp. 1547–1560, 2019.

[17] S. Rezapour and S. H. Zakeri, "Strong convergence theorem for δ-inverse strongly accretive operator in banach spaces," *Applied Set-Valued Analysis and Optimization*, vol. 1, pp. 39–52, 2019.

[18] S. Rezapour and S. H. Zakeri, "Hybrid method for equilibrium problems and variational inclusions," *Journal of Inequalities and Applications*, vol. 2020, no. 1, 190 pages, 2020.

[19] Sh. Rezapour, Y. Yao, and S. H. Zakeri, "Strong convergence theorems for generalized mixed equilibrium problem, variational inequality problems and fixed point problems with pseudocontractive mappings," *Journal of Nonlinear Conventional Analysis*, vol. 21, no. 5, pp. 1097–1108, 2020.

[20] W. Takahashi, C. F. Wen, and J. C. Yao, "The shrinking projection method for a finite family of demi-metric mappings with variational inequality problems in a hilbert space," *Fixed Point Theory*, vol. 19, pp. 407–419, 2018.

[21] S. Rezapour and S. H. Zakeri, "On weak and strong convergence results for generalized equilibrium variational inclusion problems in hilbert spaces," *Advances in Difference Equations*, vol. 2020, p. 462, 2020.

[22] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *Math Studio*, vol. 63, pp. 123–145, 1994.

[23] S. Takahashi, W. Takahashi, and M. Toyoda, "Strong convergence theorems for maximal monotone operators with nonlinear mappings in hilbert spaces," *Journal of Optimization Theory and Applications*, vol. 147, no. 1, pp. 27–41, 2010.

[24] Q. H. Ansari, M. Islam, and J.-C. Yao, "Nonsmooth variational inequalities on hadamard manifolds," *Applicable Analysis*, vol. 99, no. 2, pp. 340–358, 2020.

[25] X. Qin, S. Y. Cho, and L. Wang, "Iterative algorithms with errors for zero points of m-accretive operators," *Fixed Point Theory Applied*, vol. 2013148 pages, 2013.

[26] Y. H. Wang, Sh. Rezapour, and S. H. Zakeri, "Strong convergence theorems for Bregman relatively nonexpansive mappings and continuous monotone mapping," *Journal of Nonlinear Convex Analysis*, vol. 20, pp. 551–564, 2019.

[27] S. Cho, X. Qin, and L. Wang, "Strong convergence of a splitting algorithm for treating monotone operators," *Fixed Point Theory and Applications*, vol. 2014, no. 1, p. 94, 2014.

[28] Sh. Rezapour and S. H. Zakeri, "Hybrid method for equilibrium problems and variational inclusions," *Journal of Inequalities and Applications*, vol. 2020, p. 190, 2020.

[29] R. T. Rockafellar, "On the maximality of sums of nonlinear monotone operators," *Transactions of the American Mathematical Society*, vol. 149, no. 1, p. 75, 1970.

[30] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.

[31] H. Zhou, *Fixed Points of Nonlinear Operators. Iterative Methods*, De Gruyter, Berlin, Germany, 2020.

[32] H. Zegeye and N. Shahzad, "Convergence of mann's type iteration method for generalized asymptotically nonexpansive mappings," *Computers & Mathematics with Applications*, vol. 62, no. 11, pp. 4007–4014, 2011.

[33] F. E. Browder, "Nonlinear equations of evolution and nonlinear accretive operators in banach spaces," *Bulletin of the American Mathematical Society*, vol. 73, no. 6, pp. 867–875, 1967.

[34] K. K. Tan and H. K. Xu, "Approximating fixed points of nonexpansive mappings by the ishikawa iteration process," *Journal of Mathematical Analysis and Applications*, vol. 178, no. 2, pp. 301–308, 1993.

[35] H. Zhou, "Convergence theorems of fixed points for -strict pseudo-contractions in hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 2, pp. 456–462, 2008.

[36] L. S. Liu, "Ishikawa and mann iterative process with errors for nonlinear strongly accretive mappings in banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 1, pp. 114–125, 1995.

[37] F. Alvarez, "Weak convergence of a relaxed and inertial hybrid projection-proximal point Algorithm for maximal monotone operators in hilbert space," *SIAM Journal on Optimization*, vol. 14, no. 3, pp. 773–782, 2004.

Hindawi
Journal of Mathematics
Volume 2020, Article ID 6631047, 6 pages
https://doi.org/10.1155/2020/6631047

Research Article

Characterizations of a Class of Dirichlet-Type Spaces and Related Operators

Xiaosong Liu [1,2] and Songxiao Li [3]

[1] *Department of Mathematics, Shantou University, Shantou 515063, Guangdong, China*
[2] *Department of Mathematics, Jiaying University, Meizhou 514015, Guangdong, China*
[3] *Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu 610054, Sichuan, China*

Correspondence should be addressed to Songxiao Li; jyulsx@163.com

Received 28 October 2020; Revised 17 November 2020; Accepted 23 November 2020; Published 9 December 2020

Academic Editor: Xiaolong Qin

In this paper, some characterizations are given in terms of the boundary value and Poisson extension for the Dirichlet-type space $\mathcal{D}(\mu)$. The multipliers of $\mathcal{D}(\mu)$ and Hankel-type operators from $\mathcal{D}(\mu)$ to $L^2(P_\mu dA)$ are also investigated.

1. Introduction

Let $\mathbb{D}$ be the unit disk of complex plane $\mathbb{C}$. For $0 < p < \infty$, the Hardy space, denoted by H^p, is the space consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta < \infty. \quad (1)$$

Here, $H(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$.

Let $\partial\mathbb{D}$ denote the boundary of $\mathbb{D}$ and dA denote the normalized Lebesgue measure on $\mathbb{D}$. Let μ be a positive Borel measure on $\partial\mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the space $\mathcal{D}(\mu)$, called the Dirichlet-type space, if

$$\int_{\mathbb{D}} \left|f'(z)\right|^2 P_\mu(z) dA(z) < \infty, \quad (2)$$

where

$$P_\mu(z) = \int_0^{2\pi} \frac{1-|z|^2}{\left|e^{it} - z\right|^2} \frac{d\mu(t)}{2\pi}. \quad (3)$$

The space $\mathcal{D}(\mu)$ was introduced by Richter in [1] for studying analytic two isometrics. It was shown in [1] that $\mathcal{D}(\mu) \subset H^2$. The norm on $\mathcal{D}(\mu)$ is defined as follows:

$$\|f\|_{\mathcal{D}(\mu)}^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} \left|f'(z)\right|^2 P_\mu(z) dA(z). \quad (4)$$

The space $\mathcal{D}(\mu)$ is a Hilbert space with

$$\langle f, g\rangle_{\mathcal{D}(\mu)} = \langle f, g\rangle_{H^2} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} P_\mu(z) dA(z), \quad (5)$$

$\mathcal{D}(\mu) = H^2$ when $\mu = 0$. If $d\mu = dm$, then $\mathcal{D}(\mu)$ coincides with the Dirichlet space $\mathcal{D}$. By (Proposition 2.2 in [1]), we have

$$\int_{\partial\mathbb{D}} D_\zeta(f) d\mu(\zeta) = \int_{\mathbb{D}} \left|f'(z)\right|^2 P_\mu(z) dA(z). \quad (6)$$

Here,

$$D_\zeta(f) = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(e^{it}) - f(\zeta)}{e^{it} - \zeta} \right|^2 dt. \quad (7)$$

Let $f \in L^2(\partial\mathbb{D})$. We say that $f \in L^2(\mu)$ if

$$\int_{\partial\mathbb{D}} \int_0^{2\pi} \frac{\left|f\left(e^{i\theta}\right) - f(\zeta)\right|^2}{\left|e^{i\theta} - \zeta\right|^2} d\theta \, d\mu(\zeta) < \infty. \quad (8)$$

The norm of the space $L^2(\mu)$ is given by

$$\|f\|_{L^2(\mu)}^2 = \|f\|_{L^2(\partial\mathbb{D})}^2 + \int_{\partial\mathbb{D}}\int_0^{2\pi} \frac{\left|f\left(e^{i\theta}\right) - f(\zeta)\right|^2}{\left|e^{i\theta} - \zeta\right|^2} \mathrm{d}\theta \mathrm{d}\mu(\zeta). \tag{9}$$

The space $\mathscr{D}(\mu)$ has been investigated by many authors. In [2], Richter and Sundberg studied the cyclic vectors of $\mathscr{D}(\mu)$. Shimorin studied the reproducing kernels and extremal functions of $\mathscr{D}(\mu)$ in [3], see [4–6], for the study of Carleson measure for $\mathscr{D}(\mu)$. The study of composition operators and Toeplitz operators on $\mathscr{D}(\mu)$ can be found in [7, 8], respectively, see [9–11], for more study of the space $\mathscr{D}(\mu)$.

In this paper, we provided some characterizations for the space $\mathscr{D}(\mu)$ by the boundary value and Poisson extension. Moreover, we study the multipliers of $\mathscr{D}(\mu)$ and the Hankel-type operator from $\mathscr{D}(\mu)$ to $L^2(P_\mu dA)$.

In this paper, we always assume that μ is a positive Borel measure on $\partial\mathbb{D}$ and C is a positive constant that may differ from one occurrence to the other. The notation $F \lesssim G$ means that there exists a C such that $F \leq CG$. The notation $F \asymp G$ indicates that $G \lesssim F$ and also $F \lesssim G$.

2. Characterizations of the Space $\mathscr{D}(\mu)$

Let $f \in L^1(\partial\mathbb{D})$. The Poisson extension of f, denoted by $\widehat{f}$, is

$$\widehat{f}(z) = \int_0^{2\pi} f\left(e^{\mathrm{it}}\right) \frac{1 - |z|^2}{\left|e^{\mathrm{it}} - z\right|^2} \frac{\mathrm{d}t}{2\pi}, \quad z \in \mathbb{D}. \tag{10}$$

It is well known that $\widehat{f}$ is a harmonic function on $\mathbb{D}$.

Let $C^1(\mathbb{D})$ denote the space of all functions on $\mathbb{D}$ with continuous partial derivatives. For $f \in C^1(\mathbb{D})$, the gradient of f is defined by

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right). \tag{11}$$

First, we state some lemmas.

Lemma 1 (see [6, 8]). *Let $f \in L^2(\partial\mathbb{D})$. Then,*

$$\int_{\partial\mathbb{D}} D_\zeta(f) \mathrm{d}\mu(\zeta) < \infty, \tag{12}$$

if and only if

$$\int_{\mathbb{D}} |\nabla \widehat{f}(z)|^2 P_\mu(z) \mathrm{d}A(z) < \infty. \tag{13}$$

Remark 1. Let $f \in L^2(\partial\mathbb{D})$ and $F \in C^1(\mathbb{D})$ such that $\lim_{r \longrightarrow 1} F(re^{i\theta}) = f(e^{i\theta})$(a.e.) for $e^{i\theta} \in \partial\mathbb{D}$. Then,

$$\int_{\partial\mathbb{D}} D_\zeta(f) \mathrm{d}\mu(\zeta) \lesssim \|f\|_{L^2(\partial\mathbb{D})}^2 + \int_{\mathbb{D}} |\nabla F(z)|^2 P_\mu(z) \mathrm{d}A(z). \tag{14}$$

For $f \in H^2$, let f_b denote the boundary value of f.

Corollary 1. *Let $f \in H^2$. Then, $f \in \mathscr{D}(\mu)$ if and only if $f_b \in L^2(\mu)$.*

Proof. Since $f \in H^2$, then $f = \widehat{f_b}$. The desired result follows from Lemma 1. □

Lemma 2. *Let $f \in L^2(\partial\mathbb{D})$. Then, the following statements are equivalent:*

(a) $\int_{\partial\mathbb{D}} D_\zeta(f) d\mu(\zeta) < \infty$.
(b) $\int_{\mathbb{D}} |\nabla \widehat{f}(z)|^2 P_\mu(z) dA(z) < \infty$.
(c) $\lim_{r \longrightarrow 1^-} \int_{\mathbb{D}} (\widehat{|f|^2}(z) - |\widehat{f}(z)|^2) d\mu_r(z) < \infty$, *where*

$$\mathrm{d}\mu_r(z) = \int_{\partial\mathbb{D}} \frac{r^2\left(1 - r^2\right)}{|\zeta - rz|} \mathrm{d}\mu(\zeta) \mathrm{d}A(z). \tag{15}$$

Proof. $(a) \Longleftrightarrow (b)$ This implication follows by Lemma 1. □

Proof. $(b) \Longleftrightarrow (c)$ For $z \in \mathbb{D}, r \in (0,1)$, set

$$P_{\mu_r}(z) = \int_{\partial\mathbb{D}} \frac{r^2\left(1 - |z|^2\right)}{|\zeta - rz|^2} \mathrm{d}\mu(\zeta). \tag{16}$$

From [11], we see that $P_{\mu_r}(z)$ is subharmonic with

$$\lim_{r \longrightarrow 1^-} P_{\mu_r}(z) = P_\mu(z). \tag{17}$$

By Green's formula, we obtain

$$\begin{aligned} P_{\mu_r}(z) &= \frac{2}{\pi} \int_{\mathbb{D}} \left(\frac{\partial^2}{\partial w \partial \overline{w}} P_{\mu_r}(w)\right) \log\left|\frac{1 - \overline{w}z}{w - z}\right| \mathrm{d}A(w) \\ &\asymp \int_{\mathbb{D}} \int_{\partial\mathbb{D}} \frac{r^2\left(1 - r^2\right)}{|\zeta - rw|} \mathrm{d}\mu(\zeta) \log\left|\frac{1 - \overline{w}z}{w - z}\right| \mathrm{d}A(w). \end{aligned} \tag{18}$$

According to (17) and (18) and Hardy-Littlewood's identity (see page 238 in [12]), we have

$$\begin{aligned}\int_{\mathbb{D}}|\nabla\widehat{f}(z)|^2P_\mu(z)\mathrm{d}A(z) &= \lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}|\nabla\widehat{f}(z)|^2P_{\mu_r}(z)\mathrm{d}A(z)\\ &\asymp \lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}|\nabla\widehat{f}(z)|^2\left(\int_{\mathbb{D}}\int_{\partial\mathbb{D}}\frac{r^2(1-r^2)}{|\zeta-rw|}\mathrm{d}\mu(\zeta)\log\left|\frac{1-\overline{w}z}{w-z}\right|\mathrm{d}A(w)\right)\mathrm{d}A(z)\\ &= \lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left(\int_{\mathbb{D}}|\nabla\widehat{f}(z)|^2\log\left|\frac{1-\overline{w}z}{w-z}\right|\mathrm{d}A(z)\right)\int_{\partial\mathbb{D}}\frac{r^2(1-r^2)}{|\zeta-rw|}\mathrm{d}\mu(\zeta)\mathrm{d}A(w)\\ &= \lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left(\widehat{|f|^2}(w)-|\widehat{f}(w)|^2\right)\mathrm{d}\mu_r(w).\end{aligned} \tag{19}$$

The proof is complete. □

Theorem 1. *Let $f\in H^2$. Then, the following statements are equivalent:*

(a) $f\in\mathscr{D}(\mu)$.

(b) $\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}|f-\widehat{f}(z)|^2(z)d\mu_r(z)<\infty$.

(c) $\int_{\partial\mathbb{D}}D_\zeta(|f|)d\mu(\zeta)<\infty$ and

$$\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left(\widehat{|f|}^2(z)-|f(z)|^2\right)\mathrm{d}\mu_r(z)<\infty. \tag{20}$$

(d) $\int_{\partial\mathbb{D}}D_\zeta(|f|)d\mu(\zeta)<\infty$ and there exists a harmonic function g such that $|f|\le g$ on $\mathbb{D}$ and

$$\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left(g^2(z)-|f(z)|^2\right)\mathrm{d}\mu_r(z)<\infty. \tag{21}$$

Proof. $(a)\Leftrightarrow(b)$ This implication follows by Lemma 2 and

$$\widehat{|f|^2}(z)-|f(z)|^2=|f-\widehat{f}(z)|^2(z). \tag{22}$$

$(a)\Rightarrow(c)$ If $f\in\mathscr{D}(\mu)$, then $\int_{\partial\mathbb{D}}D_\zeta(|f|)\mathrm{d}\mu(\zeta)<\infty$. Since

$$\begin{aligned}(\widehat{|f|}(z))^2 &= \left(\int_{\partial\mathbb{D}}|f(\zeta)|\frac{1-|z|^2}{|\zeta-z|^2}\frac{|\mathrm{d}\zeta|}{2\pi}\right)^2\\ &\le \int_{\partial\mathbb{D}}|f(\zeta)|^2\frac{1-|z|^2}{|\zeta-z|^2}\frac{|\mathrm{d}\zeta|}{2\pi}\int_{\partial\mathbb{D}}\frac{1-|z|^2}{|\zeta-z|^2}\frac{|\mathrm{d}\zeta|}{2\pi}\\ &= \widehat{|f|^2}(z).\end{aligned} \tag{23}$$

We get (c) from Lemma 2 and Corollary 1.

$(c)\Rightarrow(d)$ Inequality (20) implies

$$\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left((\widehat{|f|(z)})^2-|f(z)|^2\right)\mathrm{d}\mu_r(z)<\infty. \tag{24}$$

Let $g=\widehat{|f|}$. Then, $g^2\le(\widehat{|f|})^2$. Thus,

$$\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left((g(z))^2-|f(z)|^2\right)\mathrm{d}\mu_r(z)<\infty. \tag{25}$$

$(d)\Rightarrow(a)$ By Lemma 2,

$$\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left(\widehat{|f|^2}(z)-(\widehat{|f|}(z))^2\right)\mathrm{d}\mu_r(z)<\infty. \tag{26}$$

Assume that g is a harmonic function such that $|f|\le g$. Note that $\widehat{|f|}$ is the least harmonic function equal to or greater than $|f|$ (see [12]); hence, $\widehat{|f|}\le g$. By Lemmas 1 and 2 and Corollary 1, $f\in\mathscr{D}(\mu)$. The proof is complete. □

3. Multipliers of $\mathscr{D}(\mu)$

Let $I\subset\partial\mathbb{D}$. The Carleson box $S(I)$ is

$$S(I)=\{r\zeta\in\mathbb{D}:\ 1-|I|<r<1;\zeta\in I\}. \tag{27}$$

Assume that ν is a positive Borel measure on $\mathbb{D}$. If $\sup_{I\subset\partial\mathbb{D}}(\nu(S(I))/|I|)<\infty$, then we say that ν is a Carleson measure.

If there exists a constant $C>0$ (see [4, 5])

$$\int_{\mathbb{D}}|f(z)|^2\mathrm{d}\nu(z)\le C\|f\|^2_{\mathscr{D}(\mu)},\quad \text{for all } f\in\mathscr{D}(\mu), \tag{28}$$

then we call that ν is a μ-Carleson measure.

Let $g\in L^\infty(\partial\mathbb{D})$ and $f\in L^2(\mu)$. g is called the pointwise multipliers of $L^2(\mu)$ if $gf\in L^2(\mu)$. We denote the space of all pointwise multipliers of $L^2(\mu)$ by $M(L^2(\mu))$.

Lemma 3. *Let ν be a positive Borel measure on $\mathbb{D}$. Then, ν is a μ-Carleson measure if and only if*

$$\int_{\mathbb{D}}|\widehat{g}(z)|^2\mathrm{d}\nu(z)\lesssim\|g\|^2_{L^2(\mu)}, \tag{29}$$

for all $g\in L^2(\mu)$.

Proof. First, we assume that ν is a μ-Carleson measure. Suppose that $g\in L^2(\mu)$. Without loss of generality, let g be a real-valued function. Suppose that $\widetilde{g}$ is the harmonic conjugate of $\widehat{g}$. Set $f=\widehat{g}+i\widetilde{g}$. Then, $|\nabla\widehat{f}(z)|\asymp|f'(z)|$ by the Cauchy–Riemann equation. From Lemma 2.3 in [7] and Lemma 1, we obtain

$$\begin{aligned}\int_{\mathbb{D}}|\widehat{g}(z)|^2 d\nu(z) &\le \int_{\mathbb{D}}|f(z)|^2 d\nu(z)\\ &\lesssim \|f\|^2_{\mathscr{D}(\mu)}\\ &= \|f\|^2_{H^2} + \int_{\mathbb{D}}\left|f'(z)\right|^2 P_\mu(z) dA(z)\\ &\asymp |f(0)|^2 + \int_{\mathbb{D}}|\nabla\widehat{g}(z)|^2 P_\mu(z) dA(z)\\ &\lesssim \|g\|^2_{L^2(\mu)}.\end{aligned} \tag{30}$$

Conversely, for $f \in \mathscr{D}(\mu)$, by Corollary 1, $f_b \in L^2(\mu)$ and $f = \widehat{f}_b$. Then,

$$\int_{\mathbb{D}}|f(z)|^2 d\nu(z) \lesssim \left\|f_b\right\|^2_{L^2(\mu)} \lesssim \|f\|^2_{\mathscr{D}(\mu)}, \tag{31}$$

which implies that ν is a μ-Carleson measure. □

Theorem 2. *$g \in M(L^2(\mu))$ if and only if $g \in L^\infty(\partial\mathbb{D})$ and $|\nabla\widehat{g}|^2 P_\mu dA$ is a μ-Carleson measure.*

Proof. Assume that $g \in L^\infty(\partial\mathbb{D})$ and $|\nabla\widehat{g}|^2 P_\mu dA$ is a μ-Carleson measure. Let $f \in L^2(\mu)$. By Remark 1, we obtain

$$\begin{aligned}\|fg\|^2_{L^2(\mu)} &\lesssim \|fg\|^2_{L^2(\partial\mathbb{D})} + \int_{\mathbb{D}}|\nabla(\widehat{f}\widehat{g})(z)|^2 P_\mu(z) dA(z)\\ &\le \|fg\|^2_{L^2(\partial\mathbb{D})} + \int_{\mathbb{D}}|\widehat{g}(z)|^2|\nabla\widehat{f}(z)|^2 P_\mu(z) dA(z)\\ &\quad + \int_{\mathbb{D}}|\widehat{f}(z)|^2|\nabla\widehat{g}(z)|^2 P_\mu(z) dA(z).\end{aligned} \tag{32}$$

By Lemma 1 and Corollary 1, we obtain

$$\int_{\mathbb{D}}|\widehat{g}(z)|^2|\nabla\widehat{f}(z)|^2 P_\mu(z) dA(z) \le C\|\widehat{g}\|^2_{L^\infty(\mathbb{D})}\|f\|^2_{L^2(\mu)}. \tag{33}$$

In addition, since $|\nabla\widehat{g}|^2 P_\mu dA$ is a μ-Carleson measure, by Lemma 3, we have

$$\int_{\mathbb{D}}|\widehat{f}(z)|^2|\nabla\widehat{g}(z)|^2 P_\mu(z) dA(z) \le C\|f\|^2_{L^2(\mu)}. \tag{34}$$

Combining (32)–(34), we obtain that $g \in M(L^2(\mu))$.

Conversely, assume that $g \in M(L^2(\mu))$. Then, by Theorem 2.7 in [6], we see that $g \in L^\infty(\partial\mathbb{D})$. For $f \in \mathscr{D}(\mu)$, by the Closed Graph Theorem, Lemma 1, and Corollary 1, we obtain

$$\begin{aligned}\int_{\mathbb{D}}|\nabla(f\widehat{g})(z)|^2 P_\mu(z) dA(z) &\le C\|fg\|^2_{L^2(\mu)} \le C\|f\|^2_{L^2(\mu)}\\ &\le C\|f\|^2_{\mathscr{D}(\mu)}.\end{aligned} \tag{35}$$

Next, we show that $|\nabla\widehat{g}|^2 P_\mu dA$ is a μ-Carleson measure. From the fact that $|\nabla f| \asymp |f'(z)|$, we obtain

$$\begin{aligned}\int_{\mathbb{D}}|\widehat{g}(z)|^2|\nabla f(z)|^2 P_\mu(z) dA(z) &\le C\int_{\mathbb{D}}|\nabla f(z)|^2 P_\mu(z) dA(z)\\ &\asymp C\int_{\mathbb{D}}\left|f'(z)\right|^2 P_\mu(z) dA(z)\\ &\le C\|f\|^2_{\mathscr{D}(\mu)}.\end{aligned} \tag{36}$$

Then, by (35) and (36),

$$\begin{aligned}\int_{\mathbb{D}}|f(z)|^2|\nabla\widehat{g}(z)|^2 P_\mu(z) dA(z) &\le C\int_{\mathbb{D}}\left(|\nabla(f\widehat{g})(z)|^2 + |\widehat{g}(z)|^2|\nabla f(z)|^2\right) P_\mu(z) dA(z)\\ &\le C\|f\|^2_{\mathscr{D}(\mu)},\end{aligned} \tag{37}$$

which implies that $|\nabla\widehat{g}|^2 P_\mu dA$ is a μ-Carleson measure.

By Theorem 2, we obtain the following result. □

Corollary 2. *Let $f \in H^2$. Then, $f \in M(\mathscr{D}(\mu))$ if and only if $f_b \in M(L^2(\mu))$.*

4. Hankel-Type Operators on $\mathscr{D}(\mu)$

Let $\mathscr{P}$ denote the set of all polynomials on $\mathbb{D}$. From [1, 2], we see that $\mathscr{P}$ is dense in $\mathscr{D}(\mu)$. Let

$$Pf(z) = \int_D \frac{f(w)}{(1-\overline{w}z)^2} dA(w). \tag{38}$$

From Theorem 1.10 in [13], we see that $P\colon L^2(\mathbb{D}) \longrightarrow A^2$ is a bounded projection. Here, A^2 is the Bergman space which consists of all $f \in H(\mathbb{D})$ such that $\int_{\mathbb{D}}|f(z)|^2 dA(z) < \infty$. For $f \in A^2$, we define a Hankel-type operator h_f on $\mathscr{P}$ by

$$h_f(g) = \overline{P(f\overline{g})}, \quad g \in \mathscr{P}. \tag{39}$$

Lemma 4 (see Theorem 2.3 in [10]). *Let $\tau, \sigma > -1$. Then, $f \in \mathscr{D}(\mu)$ if and only if*

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z)-f(w)|^2}{|1-\overline{z}w|^{4+\sigma+\tau}} P_\mu(z) dA_\sigma(z) dA_\tau(w) < \infty, \tag{40}$$

where $dA_\sigma(z) = (1-|z|^2)^\sigma dA(z)$.

Lemma 5 (see Theorem 3.4 in [10]). *Let T be the operator defined by*

$$Tg(z) = \int_{\mathbb{D}} \frac{|g(w)|}{|1-\overline{w}z|^2} dA(w), \quad g \in L^2(\mathbb{D}). \tag{41}$$

Then, $T\colon L^2(P_\mu dA) \longrightarrow L^2(P_\mu dA)$ is bounded.

Theorem 3. *Let $g \in L^2(\mathbb{D})$ such that $|g|^2 P_\mu dA$ is a μ-Carleson measure. Then, $|Tg|^2 P_\mu dA$ is a μ-Carleson measure.*

Proof. Suppose that $|g|^2 P_\mu dA$ is a μ-Carleson measure. Then, by Lemma 5,

$$\int_{\mathbb{D}} |T(fg)(z)|^2 P_\mu(z) dA(z) \le C \int_{\mathbb{D}} |f(z)g(z)|^2 P_\mu(z) dA(z) \le C\|f\|^2_{\mathcal{D}(\mu)}, \tag{42}$$

for all $f \in \mathcal{D}(\mu)$. So, it is enough to show that

$$\int_{\mathbb{D}} |f(z)Tg(z) - T(fg)(z)|^2 P_\mu(z) dA(z) \le C\|f\|^2_{\mathcal{D}(\mu)}, \tag{43}$$

for every $f \in \mathcal{D}(\mu)$.

By Hölder's inequality, we have

$$\begin{aligned} |f(z)Tg(z) - T(fg)(z)|^2 &\le \left(\int_{\mathbb{D}} \frac{|f(z) - f(w)|}{|1 - \overline{w}z|^2} |g(w)| dA(w) \right)^2 \\ &\lesssim \int_{\mathbb{D}} |g(w)|^2 dA(w) \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \overline{w}z|^4} dA(w) \\ &= \|g\|^2_{L^2(\mathbb{D})} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \overline{w}z|^4} dA(w). \end{aligned} \tag{44}$$

Consequently, by Lemma 4, we obtain

$$\begin{aligned} &\int_{\mathbb{D}} |f(z)Tg(z) - T(fg)(z)|^2 P_\mu(z) dA(z) \\ &\le \|g\|^2_{L^2(\mathbb{D})} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \overline{w}z|^4} dA(w) P_\mu(z) dA(z) \\ &\le \|g\|^2_{L^2(\mathbb{D})} \|f\|^2_{\mathcal{D}(\mu)}. \end{aligned} \tag{45}$$

The desired result follows. □

Theorem 4. *Let $u \in A^2$. Then, the operator $h_u\colon \mathcal{D}(\mu) \longrightarrow L^2(P_\mu dA)$ is bounded if and only if $|u|^2 P_\mu dA$ is a μ-Carlson measure.*

Proof. Suppose that $|u|^2 P_\mu dA$ is a μ-Carlson measure. Let $g \in \mathcal{D}(\mu)$. Then, $u\overline{g} \in L^2(P_\mu dA)$. By Lemma 4, we get that $h_u(g) \in L^2(P_\mu dA)$ and

$$\begin{aligned} \|h_u(g)\|_{L^2(P_\mu dA)} &\le \|T(u\overline{g})\|_{L^2(P_\mu dA)} \le C\|u\overline{g}\|_{L^2(P_\mu dA)} \\ &\le C\|g\|_{\mathcal{D}(\mu)}. \end{aligned} \tag{46}$$

So, $h_u\colon \mathcal{D}(\mu) \longrightarrow L^2(P_\mu dA)$ is bounded.

Conversely, assume that $h_u\colon \mathcal{D}(\mu) \longrightarrow L^2(P_\mu dA)$ is bounded. We need to prove that

$$\|u\overline{g}\|_{L^2(P_\mu dA)} \le C\|g\|_{\mathcal{D}(\mu)}, \quad \text{for any } g \in \mathcal{D}(\mu). \tag{47}$$

By Hölder's inequality we have

$$\begin{aligned} \left| \int_{\mathbb{D}} \frac{u(w)(\overline{g(z) - g(w)})}{(1 - \overline{w}z)^2} dA(w) \right|^2 &\le \int_{\mathbb{D}} |u(w)|^2 dA(w) \int_{\mathbb{D}} \frac{|g(z) - g(w)|^2}{|1 - \overline{w}z|^4} dA(w) \\ &= \|u\|^2_{A^2} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^2}{|1 - \overline{w}z|^4} dA(w). \end{aligned} \tag{48}$$

Since

$$u(z)\overline{g(z)} - \overline{h_u(g)(z)} = \int_{\mathbb{D}} \frac{u(w)(\overline{g(z) - g(w)})}{(1 - \overline{w}z)^2} dA(w), \tag{49}$$

by Lemma 4 and the fact that $h_u\colon \mathcal{D}(\mu) \longrightarrow L^2(P_\mu dA)$ is bounded, we obtain

$$\begin{aligned} \|u\overline{g}\|^2_{L^2(P_\mu dA)} &\le \|u\|^2_{A^2} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^2}{|1 - \overline{w}z|^4} dA(w) P_\mu(z) dA(z) \\ &\quad + \|h_u\|^2 \|g\|^2_{\mathcal{D}(\mu)} \\ &\lesssim \left(\|u\|^2_{A^2} + \|h_u\|^2 \right) \|g\|^2_{\mathcal{D}(\mu)}. \end{aligned} \tag{50}$$

The proof is complete. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The first author was supported by NNSF of China (nos. 11701222 and 11801347), China Postdoctoral Science Foundation (no. 2018M633090), and Key Projects of Fundamental Research in Universities of Guangdong Province (no. 2018KZDXM034).

References

[1] S. Richter, "A representation theorem for cyclic analytic two-isometries," *Transactions of the American Mathematical Society*, vol. 328, no. 1, pp. 325–349, 1991.
[2] S. Richter and C. Sundberg, "A formula for the local Dirichlet integral," *The Michigan Mathematical Journal*, vol. 38, no. 3, pp. 355–379, 1991.
[3] S. M. Shimorin, "Reproducing kernels and extremal functions in Dirichlet-type spaces," *Journal of Mathematical Sciences*, vol. 107, no. 4, pp. 4108–4124, 2001.
[4] G. R. Chacón, "Carleson measures on Dirichlet-type spaces," *Proceedings of the American Mathematical Society*, vol. 139, no. 5, p. 1605, 2011.
[5] G. R. Chacón, E. Fricain, and M. Shabankhah, "Carleson measures and reproducing kernel thesis in Dirichlet-type spaces," *St. Petersburg Mathematical Journal*, vol. 24, no. 6, pp. 847–861, 2013.
[6] R. Chartrand, "Multipliers and Carleson measures for $ D (\mu) $," *Integral Equations and Operator Theory*, vol. 45, no. 3, pp. 309–318, 2003.
[7] G. R. Chacón, "Closed-range composition operators on Dirichlet-type spaces," *Complex Analysis and Operator Theory*, vol. 7, no. 4, pp. 909–926, 2013.
[8] R. Chartrand, "Toeplitz operator on Dirichlet-type spaces," *The Journal of Operator Theory*, vol. 48, pp. 3–13, 2002.
[9] G. R. Chacón, "Interpolating sequences in harmonically weighted Dirichlet spaces," *Integral Equations and Operator Theory*, vol. 69, no. 1, pp. 73–85, 2011.
[10] X. Liu, G. R. Chacón, and Z. Lou, "Characterizations of the dirichlet-type space," *Complex Analysis and Operator Theory*, vol. 9, no. 6, pp. 1269–1286, 2015.
[11] S. Shimorin, "Complete nevanlinna-pick property of dirichlet-type spaces," *Journal of Functional Analysis*, vol. 191, no. 2, pp. 276–296, 2002.
[12] J. Garnett, *Bounded Analytic Functions*, Academic Press, Cambridge, MA, USA, 1981.
[13] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Springer, New York, NY, USA, 2000.

Hindawi
Journal of Mathematics
Volume 2020, Article ID 8825786, 14 pages
https://doi.org/10.1155/2020/8825786

Research Article

Refinements of Some Integral Inequalities for φ-Convex Functions

Moquddsa Zahra,[1] Yu-Ming Chu ,[2,3] and Ghulam Farid [4]

[1]*Department of Mathematics, University of Wah, Wah Cantt, Pakistan*
[2]*Department of Mathematics, Huzhou University, Huzhou 313000, China*
[3]*Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha 410114, China*
[4]*Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock, Pakistan*

Correspondence should be addressed to Yu-Ming Chu; chuyuming@zjhu.edu.cn

Received 21 September 2020; Revised 23 October 2020; Accepted 26 October 2020; Published 24 November 2020

Academic Editor: Xiaolong Qin

In this paper, we are interested to deal with unified integral operators for strongly φ-convex function. We will present refinements of bounds of these unified integral operators and use them to get associated results for fractional integral operators. Several known results are connected with particular assumptions.

1. Introduction and Preliminaries

Convex functions play an important role in the formation of new definitions of related functions which help to give the generalization of classical results. Therefore, in recent years, many generalizations of convex functions are defined and utilized to study the Hadamard and other well-known inequalities (see [1–9]). In this paper, we deal with the strongly φ-convex functions to study the bounds of unified integral operators. The obtained results are compared with already known results.

First, we give some definitions of functions which are necessary for the findings of this paper.

Definition 1 (see [7]). A function $f\colon I \longrightarrow \mathbb{R}$ is said to be convex on I if

$$f(\varsigma u + (1-\varsigma)v) \leq \varsigma f(u) + (1-\varsigma) f(v), \tag{1}$$

holds for all $u, v \in I$ and $\varsigma \in [0,1]$, where $I \subseteq \mathbb{R}$ is an interval. Reverse of inequality (1) defines f as concave function.

Definition 2 (see [10]). A function $f\colon I \longrightarrow \mathbb{R}$ is said to be strongly convex with modulus $\lambda > 0$ if

$$f(\varsigma u + (1-\varsigma)v) \leq \varsigma f(u) + (1-\varsigma) f(v) - \lambda\varsigma(1-\varsigma)(v-u)^2, \tag{2}$$

holds for all $u, v \in I$ and $\varsigma \in [0,1]$.

Definition 3 (see [3]). A function $f\colon I \longrightarrow \mathbb{R}$ is said to be φ-convex on I if

$$f(\varsigma u + (1-\varsigma)v) \leq f(v) + \varsigma\varphi(f(u), f(v)), \tag{3}$$

holds for all $u, v \in I$ and $\varsigma \in [0,1]$, where φ is a bifunction.

Definition 4 (see [2]). A function $f\colon I \longrightarrow \mathbb{R}$ is said to be strongly φ-convex on I if

$$f(\varsigma u + (1-\varsigma)v) \leq f(v) + \varsigma\varphi(f(u), f(v)) - \lambda\varsigma(1-\varsigma)(v-u)^2, \tag{4}$$

holds for all $u, v \in I$ and $\varsigma \in [0,1]$, $\lambda \geq 0$, where φ is a bifunction.

It is to be noted that for $\varphi(x, y) = x - y$, strongly φ-convex function reduces to strongly convex function. Farid in [11] defined the unified integral operators (5) and (6) and has proved the continuity and the boundedness of these integral operators. The aim of this paper is the study of integral inequalities for strongly φ-convex functions via unified integral operators. Next, we give definition of the unified integral operators.

Definition 5. Let f, g: $[u, v] \longrightarrow \mathbb{R}$ where $0<u<v$ be the function such that f is positive and integrable over $[u, v]$ and g is differentiable and strictly increasing. Also, let Ψ/x be an increasing function on $[u, \infty)$ and $\alpha, \xi, \gamma, \zeta \in \mathbb{C}$, $p, \mu, \delta \geq 0$ and $0<k\leq\delta+\mu$. Then, for $x \in [u, v]$, the left and right integral operators are defined as follows:

$$\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^+} f\right)(x, \eta; p) = \int_u^x J_x^y\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}, g; \Psi\right) g'(y) f(y)\mathrm{d}y, \tag{5}$$

$$\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,v^-} f\right)(x, \eta; p) = \int_x^v J_y^x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}, g; \Psi\right) g'(y) f(y)\mathrm{d}y, \tag{6}$$

where

$$J_x^y\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}, g; \Psi\right) = \frac{\Psi(g(x)-g(y))}{g(x)-g(y)} E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta(g(x)-g(y))^{\mu}; p\right). \tag{7}$$

By choosing specific functions Ψ and g and fixing parameters involved in the Mittag-Leffler function $E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}(\eta(g(x)-g(y))^{\mu}; p)$, various known fractional integrals can be reproduced (see [5], Remarks 6 and 7). In [4], by using unified integral operators, we have obtained integral inequalities for φ-convex functions. In the following, we give these inequalities in the form of Theorems 1–3.

Theorem 1. *Let f: $[u, v] \longrightarrow \mathbb{R}$ be a positive φ-convex function and g: $[u, v] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function. Also, let Ψ/x be an increasing function on $[u, v]$, $\eta, \alpha, \xi, \gamma, \zeta \in \mathbb{C}$, $p, \mu, \nu, \delta \geq 0$, $0<k\leq\delta+\mu$, and $0<k\leq\delta+\nu$. Then, for $x \in [u, v]$, we have*

$$\begin{aligned}
&\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^+} f\right)(x, \eta; p) + \left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi,v^-} f\right)(x, \eta; p) \\
&\quad\leq E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta(g(x)-g(u))^{\mu}; p\right)\Psi(g(x)-g(u)) f(x) \\
&\qquad + J_x^u\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}, g; \Psi\right)\varphi(f(u), f(x))(I(u, x; g)-g(u)) \\
&\qquad + E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi}\left(\eta(g(v)-g(x))^{\nu}; p\right)\Psi(g(v)-g(x)) f(v) \\
&\qquad + J_v^x\left(E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi}, g; \Psi\right)\varphi(f(x), f(v))(I(x, v; g)-g(x)).
\end{aligned} \tag{8}$$

Theorem 2. *Along with the assumptions of Theorem 1, if $f(u+v-x) = f(x)$ and $\varphi(x, y) = x+y$, then the following result holds:*

$$\begin{aligned}
&\frac{1}{2} f\left(\frac{u+v}{2}\right)\left(\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,v^-} 1\right)(u, \eta; p) + \left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^+} 1\right)(v, \eta; p)\right) \\
&\quad\leq \left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,v^-} f\right)(u, \eta; p) + \left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^+} f\right)(v, \eta; p) \\
&\quad\leq 2\Psi(g(v)-g(u)) E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta(g(v)-g(u))^{\mu}; p\right) f(v) \\
&\qquad + 2(f(u)+f(v)) J_v^u\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}, g; \Psi\right)(I(u, v; g)-g(u)).
\end{aligned} \tag{9}$$

Also, the following result holds for the convolution of functions f and g.

Theorem 3. *Let f, g: $[u, v] \longrightarrow \mathbb{R}$ be two differentiable functions such that $|f'|$ is φ-convex and g be strictly increasing for $0<u<v$. Also, Ψ/x be an increasing function on $[u, v]$ and $\alpha, \xi, \gamma, \zeta \in \mathbb{C}$, $p, \mu, \nu, \delta \geq 0$ and $0<k\leq\delta+\mu$ and $0<k\leq\delta+\nu$. Then, for $x \in (u, v)$, we have*

$$\begin{aligned}
&\left|\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^+} f * g\right)(x, \eta; p) + \left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi,v^-} f * g\right)(x, \eta; p)\right| \\
&\quad\leq E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta(g(x)-g(u))^{\mu}; p\right)\Psi(g(x)-g(u))|f'(x)| \\
&\qquad + J_x^u\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}, g; \Psi\right)\varphi\left(|f'(u)|, |f'(x)|\right)(I(u, x; g)-g(u)) \\
&\qquad + E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi}\left(\eta(g(v)-g(x))^{\nu}; p\right)\Psi(g(v)-g(x))|f'(v)| \\
&\qquad + J_v^x\left(E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi}, g; \Psi\right)\varphi\left(|f'(x)|, |f'(v)|\right)(I(x, v; g)-g(x)).
\end{aligned} \tag{10}$$

Although we follow the same method which was adopted to prove the results of [4], here we will get refinements of these results by using strongly φ-convex functions. In Section 2, we give the refinements of bounds of unified integral operators given in Definition 5. In Section 3, we will present refinements of bounds of fractional integral operators.

2. Main Results

Throughout this section, we have adopted the following notations:

$$\begin{aligned} I(u,v;g) &:= \frac{1}{v-u}\int_u^v g(t)\mathrm{d}t, \\ S(\mu,\nu,u^+,v^-) &= \left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^+} f\right)(x,\eta;p) + \left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi,v^-} f\right)(x,\eta;p). \end{aligned} \tag{11}$$

Theorem 4. *If f is positive strongly φ-convex function with modulus $\lambda \geq 0$, along with other assumptions of Theorem 1, then we have*

$$\begin{aligned} S(\mu,\nu,u^+,v^-) \leq{}& E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta(g(x)-g(u))^{\mu};p\right)\Psi(g(x)-g(u))f(x) \\ &+ J^u_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\{\varphi(f(u),f(x))(I(u,x;g)-g(u)) - \lambda(x-u)(2I(u,x;I_d g)-(x+u)I(u,x;g))\} \\ &+ E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi}\left(\eta(g(v)-g(x))^{\nu};p\right)\Psi(g(v)-g(x))f(v) \\ &+ J^x_v\left(E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi},g;\Psi\right)\{\varphi(f(x),f(v))(I(x,v;g)-g(x)) - \lambda(v-x)(2I(x,v;I_d g)-(v+x)I(x,v;g))\}, \end{aligned} \tag{12}$$

where I_d is the identity function.

Proof. For the kernel defined in (7) and the strongly φ-convexity of the function f on $[u,x]$, the following inequalities hold, respectively:

$$J^{\varsigma}_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(\varsigma) \leq J^u_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(\varsigma), \quad \varsigma \in [u,x], x \in (u,v), \tag{13}$$

$$f(\varsigma) \leq f(x) + \frac{x-\varsigma}{x-u}\varphi(f(u),f(x)) - \lambda\left(\frac{x-\varsigma}{x-u}\right)\left(\frac{\varsigma-u}{x-u}\right)(u-x)^2. \tag{14}$$

The aforementioned inequalities are used to obtain the following integral inequality:

$$\begin{aligned} \int_u^x J^{\varsigma}_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(\varsigma)f(\varsigma)\mathrm{d}\varsigma \leq{}& f(x)J^u_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\int_u^x g'(\varsigma)\mathrm{d}\varsigma \\ &+ \frac{\varphi(f(u),f(x))}{x-u}J^u_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\int_u^x (x-\varsigma)g'(\varsigma)\mathrm{d}\varsigma \\ &- \lambda J^u_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\int_u^x (x-\varsigma)(\varsigma-u)g'(\varsigma)\mathrm{d}\varsigma. \end{aligned} \tag{15}$$

In view of Definition 5 and applying integration by parts, from inequality (15), we get the following upper bound of the right-sided unified integral operator:

$$\begin{aligned}\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^+}f\right)(x,\eta;p) \le{}& E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta(g(x)-g(u))^{\mu};p\right)\Psi(g(x)-g(u))f(x)\\ &+ J^u_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\{\varphi(f(u),f(x))(I(u,x;g)-g(u))-\lambda(x-u)(2I(u,x;I_dg)\\ &-(x+u)(I(u,x;g)))\}.\end{aligned}\tag{16}$$

Again for the kernel defined in (7) and the strongly φ-convexity of the function f on $(x,v]$, the following inequalities hold, respectively:

$$J^x_\varsigma\left(E^{\gamma,\delta,k,\zeta}_{v,\alpha,\xi},g;\Psi\right)g'(\varsigma) \le J^x_v\left(E^{\gamma,\delta,k,\zeta}_{v,\alpha,\xi},g;\Psi\right)g'(\varsigma),\tag{17}$$

$$f(\varsigma) \le f(v) + \frac{v-\varsigma}{v-x}\varphi(f(x),f(v)) - \lambda\left(\frac{v-\varsigma}{v-x}\right)\left(\frac{\varsigma-x}{v-x}\right)(x-v)^2.\tag{18}$$

The aforementioned inequalities (17) and (18) are used to obtain the following integral inequality:

$$\begin{aligned}&\int_x^v J^x_\varsigma\left(E^{\gamma,\delta,k,\zeta}_{v,\alpha,\xi},g;\Psi\right)g'(\varsigma)f(\varsigma)\mathrm{d}\varsigma\\ &\le J^v_x\left(E^{\gamma,\delta,k,\zeta}_{v,\alpha,\xi},g;\Psi\right)\left(f(v)\int_x^v g'(\varsigma)\mathrm{d}\varsigma + \frac{\varphi(f(x),f(v))}{v-x}\int_x^v (v-\varsigma)g'(\varsigma)\mathrm{d}\varsigma\right)\\ &-\lambda J^v_x\left(E^{\gamma,\delta,k,\zeta}_{v,\alpha,\xi},g;\Psi\right)\int_x^v (v-\varsigma)(\varsigma-x)g'(\varsigma)\mathrm{d}\varsigma.\end{aligned}\tag{19}$$

In view of Definition 5 and applying integration by parts, from inequality (19), we get the following upper bound of the left-sided unified integral operator:

$$\begin{aligned}\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{v,\alpha,\xi,v^-}f\right)(x,\eta;p) \le{}& E^{\gamma,\delta,k,\zeta}_{v,\alpha,\xi}\left(\eta(g(v)-g(x))^{v};p\right)\Psi(g(v)-g(x))f(v)\\ &+ J^x_v\left(E^{\gamma,\delta,k,\zeta}_{v,\alpha,\xi},g;\Psi\right)\{\varphi(f(x),f(v))(I(x,v;g)-g(x))-\lambda(v-x)(2I(x,v;I_dg)-(v+x)I(x,v;g))\}.\end{aligned}\tag{20}$$

Inequality (12) will be obtained by combining (16) and (20). □

Corollary 1. *By setting $\mu = v$ in (12), we get*

$$
\begin{aligned}
S(\mu,\mu,u^{+},v^{-}) \le\; & E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta(g(x)-g(u))^{\mu};p\right)\Psi(g(x)-g(u))f(x) \\
& + J_{x}^{u}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\right)\left\{\varphi(f(u),f(x))(I(u,x;g)-g(u))-\lambda(x-u)\left(2I(u,x;I_{d}g)-(x+u)(I(u,x;g))\right)\right\} \\
& + E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta(g(v)-g(x))^{\mu};p\right)\Psi(g(v)-g(x))f(v) \\
& + J_{v}^{x}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\right)\left\{\varphi(f(x),f(v))(I(x,v;g)-g(x))-\lambda(v-x)\left(2I(x,v;I_{d}g)-(v+x)I(x,v;g)\right)\right\}.
\end{aligned} \tag{21}
$$

Remark 1. For $\lambda = 0$ in (12), we get inequality (8) of Theorem 1; if $2I(u,x;I_d g) > (x+u)(I(u,x;g))$ and $2I(x,v;I_d g) > (v+x)I(x,v;g)$, then we will get the refinement of (8).

For $\varphi(x,y) = x - y$ in (21), we get the result for strongly convex function.

For $\varphi(x,y) = x - y$ and $\lambda = 0$ in (21), we get the result of Theorem 8 in [5].

We will use the following lemma for our next result.

Lemma 1. *Let f be strongly φ-convex function with modulus $\lambda \ge 0$. If $f(x) = f(u+v-x)$, then*

$$
f\left(\frac{u+v}{2}\right) \le f(x) + \frac{1}{2}\varphi(f(x),f(x)) - \frac{\lambda(u-v)^2}{4}, \tag{22}
$$

holds for $x \in [u,v]$.

Proof. Strongly φ-convexity of f implies

$$
\begin{aligned}
f\left(\frac{u+v}{2}\right) &\le f\left(\frac{x-u}{v-u}u + \frac{v-x}{v-u}v\right) + \frac{1}{2}\varphi\left(f\left(\frac{x-u}{v-u}v + \frac{v-x}{v-u}u\right), f\left(\frac{x-u}{v-u}u + \frac{v-x}{v-u}v\right)\right) - \frac{\lambda(u-v)^2}{4} \\
&= f(u+v-x) + \frac{1}{2}\varphi(f(x), f(u+v-x)) - \frac{\lambda(u-v)^2}{4}.
\end{aligned} \tag{23}
$$

Using the condition $f(x) = f(u+v-x)$ in the above inequality, we get (22). □

Remark 2. For $\lambda = 0$, Lemma 1 reduces to Lemma 1 of [4]. For $\lambda > 0$, we get its refinement.

For $\varphi(x,y) = x - y$ and $\lambda = 0$, Lemma 1 reduces to Lemma 21 of [5].

Theorem 5. *Let $f(u+v-x) = f(x)$ and $\varphi(x,y) = x + y$ in addition with the assumptions of Theorem 4. Then, the following inequality holds:*

$$
\begin{aligned}
& \frac{1}{2}\left(f\left(\frac{u+v}{2}\right) + \frac{\lambda(u-v)^2}{4}\right)\left(\left({}_{g}F_{\mu,\alpha,\xi,v^{-}}^{\Psi,\gamma,\delta,k,\zeta}1\right)(u,\eta;p) + \left({}_{g}F_{\mu,\alpha,\xi,u^{+}}^{\Psi,\gamma,\delta,k,\zeta}1\right)(v,\eta;p)\right) \\
& \le S(\mu,v,u^{+},v^{-}) \le 2\Psi(g(v)-g(u))E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta(g(v)-g(u))^{\mu};p\right)f(v) \\
& \quad + 2(f(u)+f(v))J_{v}^{u}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\right)(I(u,v;g)-g(u)) \\
& \quad - 2\lambda J_{v}^{u}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\right)(v-u)\left(2I(u,v;I_{d}g)-(u+v)I(u,v;g)\right).
\end{aligned} \tag{24}
$$

Proof. For the kernel defined in equation (7) and the strongly φ-convexity of the function f on $[u, v]$, the following inequalities hold, respectively:

$$J_x^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)g'(x) \leq J_v^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)g'(x), \quad x \in (u, v), \tag{25}$$

$$f(x) \leq f(v) + \frac{v-x}{v-u}\varphi(f(u), f(v)) - \lambda(v-x)(x-u). \tag{26}$$

The aforementioned inequalities are used to obtain the following integral inequality:

$$\begin{aligned}\int_u^v J_x^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)f(x)g'(x)\mathrm{d}x \leq{}& f(v)J_v^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\int_u^v g'(x)\mathrm{d}x \\ &+ \frac{\varphi(f(u), f(v))}{v-u}J_v^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\int_u^v (v-x)g'(x)\mathrm{d}x \\ &- \lambda J_v^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\int_u^v (v-x)(x-u)g'(x)\mathrm{d}x.\end{aligned} \tag{27}$$

In view of Definition 5, applying integration by parts, and using $\varphi(x, y) = x + y$, from inequality (27), we get the following upper bound of the left-sided unified integral operator:

$$\begin{aligned}\left({}_gF_{\mu,\alpha,\xi,v^-}^{\Psi,\gamma,\delta,k,\zeta}f\right)(u, \eta; p) \leq{}& E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta(g(v)-g(u))^{\mu}; p\right)\Psi(g(v)-g(u))f(v) \\ &+ J_v^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)(f(u)+f(v))(I(u, v; g) - g(u)) \\ &- \lambda J_v^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)(v-u)(2I(u, v, I_d g) - (v+u)I(u, v, g)).\end{aligned} \tag{28}$$

Also, the following inequality holds:

$$J_v^x\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)g'(x) \leq J_v^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)g'(x), \quad x \in (u, v). \tag{29}$$

The aforementioned inequalities (26) and (29) are used to obtain the following integral inequality:

$$\begin{aligned}&\int_u^v J_v^x\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)g'(x)f(x)\mathrm{d}x \\ &\leq J_v^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\left(f(v)\int_u^v g'(x)\mathrm{d}x + \frac{\varphi(f(u), f(v))}{v-u}\int_u^v g'(x)(v-x)\mathrm{d}x\right) \\ &\quad - \lambda J_v^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\int_u^v (v-x)(x-u)g'(x)\mathrm{d}x.\end{aligned} \tag{30}$$

In view of Definition 5 and applying integration by parts, from inequality (30), we get the following upper bound of the right-sided unified integral operator:

$$\begin{aligned}\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^{+}}f\right)(v,\eta;p) \leq{}& E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta(g(v)-g(u))^{\mu};p\right)\Psi(g(v)-g(u))f(v)\\ &+J^{u}_{v}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)(f(u)+f(v))(I(u,v;g)-g(u))\\ &-\lambda J^{u}_{v}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)(v-u)(2I(u,v,I_{d}g)-(v+u)I(u,v,g)).\end{aligned} \tag{31}$$

Now, using Lemma 1, we can write

$$\begin{aligned}&\int_{u}^{v}f\left(\frac{u+v}{2}\right)J^{u}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(x)\mathrm{d}x\\ &\leq\int_{u}^{v}J^{u}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(x)f(x)\mathrm{d}x+\frac{1}{2}\int_{u}^{v}J^{u}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}g;\Psi\right)g'(x)\varphi(f(x),f(x))\mathrm{d}x\\ &\quad-\frac{\lambda(u-v)^{2}}{4}\int_{u}^{v}J^{u}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(x)\mathrm{d}x.\end{aligned} \tag{32}$$

In view of Definition 5 and $\varphi(x,y)=x+y$, from (32), we get the following upper bound of the left-sided unified integral operator:

$$\begin{aligned}f\left(\frac{u+v}{2}\right)\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,v^{-}}1\right)(u,\eta;p)\leq{}&2\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,v^{-}}f\right)(u,\eta;p)\\ &-\frac{\lambda(u-v)^{2}}{4}\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,v^{-}}1\right)(u,\eta;p).\end{aligned} \tag{33}$$

Also, from Lemma 1, we can write

$$\begin{aligned}&\int_{u}^{v}f\left(\frac{u+v}{2}\right)J^{x}_{v}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(x)\mathrm{d}x\\ &\leq\int_{u}^{v}J^{x}_{v}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(x)f(x)\mathrm{d}x+\frac{1}{2}\int_{u}^{v}J^{x}_{v}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(x)\varphi(f(x),f(x))\mathrm{d}x\\ &\quad-\frac{\lambda(u-v)^{2}}{4}\int_{u}^{v}J^{x}_{v}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(x)\mathrm{d}x.\end{aligned} \tag{34}$$

In view of Definition 5 and $\varphi(x,y)=x+y$, from (34), we get the following upper bound of the right-sided unified integral operator:

$$f\left(\frac{u+v}{2}\right)\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^{+}}1\right)(v,\eta;p)\leq 2\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^{+}}f\right)(v,\eta;p)-\frac{\lambda(u-v)^{2}}{4}\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^{+}}1\right)(v,\eta;p). \tag{35}$$

Inequality (24) will be obtained by using (28), (31), (33), and (35). □

Remark 3. For $\lambda = 0$ in (24), we get (9) of Theorem 2; if $2I(u, v; I_d g) > (u+v)I(u, v; g)$, then we will get refinement of (9).

For $\varphi(x, y) = x - y$ in (24), we get the result for strongly convex function.

For $\varphi(x, y) = x - y$ and $\lambda = 0$ in (24), we get the result of Theorem 22 in [5].

Theorem 6. *If $|f'|$ is strongly φ-convex with modulus $\lambda \geq 0$ along with other assumptions of Theorem 3, then the inequality*

$$\begin{aligned}
&\left|\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^+} f^* g\right)(x, \eta; p) + \left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi,v^-} f^* g\right)(x, \eta; p)\right| \\
&\leq E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta(g(x) - g(u))^{\mu}; p\right)\Psi(g(x) - g(u))\left|f'(x)\right| \\
&\quad + J^u_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}, g; \Psi\right)\left\{\varphi\left(\left|f'(u)\right|, \left|f'(x)\right|\right)(I(u, x; g) - g(u))\right. \\
&\quad \left. - \lambda(x-u)(2I(u, x; I_d g) - (x+u)I(u, x; g))\right\} \\
&\quad + E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi}\left(\eta(g(v) - g(x))^{\nu}; p\right)\Psi(g(v) - g(x))\left|f'(v)\right| \\
&\quad + J^x_v\left(E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi}, g; \Psi\right)\left\{\varphi\left(\left|f'(x)\right|, \left|f'(v)\right|\right)(I(x, v; g) - g(x))\right. \\
&\quad \left. - \lambda(v-x)(2I(x, v; I_d g) - (v+x)I(x, v; g))\right\}
\end{aligned} \tag{36}$$

holds for $x \in (u, v)$, where

$$\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^+} f^* g\right)(x, \eta; p) = \int_u^x J^{\varsigma}_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}, g; \Psi\right)g'(\varsigma)f'(\varsigma)\mathrm{d}\varsigma, \tag{37}$$

$$\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi,v^-} f^* g\right)(x, \eta; p) = \int_x^v J^x_{\varsigma}\left(E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi}, g; \Psi\right)g'(\varsigma)f'(\varsigma)\mathrm{d}\varsigma, \tag{38}$$

and I_d is the identity function.

Proof. Using strongly φ-convexity of $|f'|$ over $[u, x]$ gives

$$\left|f'(\varsigma)\right| \leq \left|f'(x)\right| + \frac{x-\varsigma}{x-u}\varphi\left(\left|f'(u)\right|, \left|f'(x)\right|\right) - \lambda(x-\varsigma)(\varsigma-u), \quad \varsigma \in [u, x]. \tag{39}$$

Using absolute value property, we can write

$$\begin{aligned}
&-\left(\left|f'(x)\right| + \frac{x-\varsigma}{x-u}\varphi\left(\left|f'(u)\right|, \left|f'(x)\right|\right) - \lambda(x-\varsigma)(\varsigma-u)\right) \leq f'(\varsigma) \\
&\leq \left(\left|f'(x)\right| + \frac{x-\varsigma}{x-u}\varphi\left(\left|f'(u)\right|, \left|f'(x)\right|\right) - \lambda(x-\varsigma)(\varsigma-u)\right).
\end{aligned} \tag{40}$$

The aforementioned inequality (13) and second inequality of (40) are used to obtain the following integral inequality:

$$\int_u^x J_x^{\varsigma}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right) g'(\varsigma) f'(\varsigma) \mathrm{d}\varsigma \leq J_x^{u}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\left(\left|f'(x)\right| \int_u^x g'(\varsigma)\mathrm{d}\varsigma + \frac{\varphi\left(\left|f'(u)\right|, \left|f'(x)\right|\right)}{x-u} \int_u^x (x-\varsigma) g'(\varsigma)\mathrm{d}\varsigma - \lambda \int_u^x (x-\varsigma)(\varsigma-u) g'(\varsigma)\mathrm{d}\varsigma\right). \tag{41}$$

In view of (37) and applying integration by parts, from inequality (41), we get the following upper bound:

$$\begin{aligned}
&\left({}_gF_{\mu,\alpha,\xi,u^+}^{\Psi,\gamma,\delta,k,\zeta} f^* g\right)(x,\eta;p) \\
&\leq E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta(g(x)-g(u))^{\mu};p\right)\Psi(g(x)-g(u))\left|f'(x)\right| \\
&\quad + J_x^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\left\{\varphi\left(\left|f'(u)\right|, \left|f'(x)\right|\right)(I(u,x;g)-g(u)) \right.\\
&\quad \left. -\lambda(x-u)\left(2I(u,x;I_d g)-(x+u)I(u,x;g)\right)\right\}.
\end{aligned} \tag{42}$$

Also, inequality (13) and the first inequality of (40) are used to obtain the following integral inequality:

$$\begin{aligned}
&\left({}_gF_{\mu,\alpha,\xi,u^+}^{\Psi,\gamma,\delta,k,\zeta} f^* g\right)(x,\eta;p) \\
&\geq -\left[E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta(g(x)-g(u))^{\mu};p\right)\Psi(g(x)-g(u))\left|f'(x)\right| \right.\\
&\quad + J_x^u\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\left\{\varphi\left(\left|f'(u)\right|, \left|f'(x)\right|\right)(I(u,x;g)-g(u)) \right.\\
&\quad \left.\left. -\lambda(x-u)\left(2I(u,x;I_d g)-(x+u)I(u,x;g)\right)\right\}\right].
\end{aligned} \tag{43}$$

Now, using φ-convexity of $|f'|$ over $(x,v]$, we have

$$\left|f'(\varsigma)\right| \leq \left|f'(v)\right| + \frac{v-\varsigma}{v-x}\varphi\left(\left|f'(x)\right|, \left|f'(v)\right|\right) - \lambda(v-\varsigma)(\varsigma-x), \quad \varsigma \in (x,v]. \tag{44}$$

Inequalities (17), (38), and (44) are used to obtain the following upper bounds:

$$\begin{aligned}
&\left({}_gF_{\nu,\alpha,\xi,v^-}^{\Psi,\gamma,\delta,k,\zeta} f^* g\right)(x,\eta;p) \\
&\leq E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta(g(v)-g(x))^{\nu};p\right)\Psi(g(v)-g(x))\left|f'(v)\right| \\
&\quad + J_v^x\left(E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\left\{\varphi\left(\left|f'(x)\right|, \left|f'(v)\right|\right)(I(x,v;g)-g(x)) \right.\\
&\quad \left. -\lambda(v-x)\left(2I(x,v;I_d g)-(v+x)I(x,v;g)\right)\right\},
\end{aligned} \tag{45}$$

$$\begin{aligned}
&\left({}_gF_{\nu,\alpha,\xi,v^-}^{\Psi,\gamma,\delta,k,\zeta} f^* g\right)(x,\eta;p) \\
&\geq -\left[E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta(g(v)-g(x))^{\nu};p\right)\Psi(g(v)-g(x))\left|f'(v)\right| \right.\\
&\quad + J_v^x\left(E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\left\{\varphi\left(\left|f'(x)\right|, \left|f'(v)\right|\right)(I(x,v;g)-g(x)) \right.\\
&\quad \left.\left. -\lambda(v-x)\left(2I(x,v;I_d g)-(v+x)I(x,v;g)\right)\right\}\right].
\end{aligned} \tag{46}$$

Inequality (36) will be obtained by using (42)–(46). □

Corollary 2. *By setting $\mu = \nu$ in (36), we get the following inequality:*

$$\left|\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,u^+}f^*g\right)(x,\eta;p)+\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,v^-}f^*g\right)(x,\eta;p)\right|$$
$$\le E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}(\eta(g(x)-g(u))^{\mu};p)\Psi(g(x)-g(u))|f'(x)|$$
$$+J^u_x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\{\varphi(|f'(u)|,|f'(x)|)(I(u,x;g)-g(u))$$
$$-\lambda(x-u)(2I(u,x;I_dg)-(x+u)I(u,x;g))\}$$
$$+E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}(\eta(g(v)-g(x))^{\mu};p)\Psi(g(v)-g(x))|f'(v)|$$
$$+J^x_v\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\{\varphi(|f'(x)|,|f'(v)|)(I(x,v;g)-g(x))$$
$$-\lambda(v-x)(2I(x,v;I_dg)-(v+x)I(x,v;g))\}. \tag{47}$$

Remark 4. For $\lambda=0$ in (36), we get inequality (10) of Theorem 3; if $2I(u,x;I_dg)>(x+u)I(u,x;g)$ and $2I(x,v;I_dg)>(v+x)I(x,v;g)$, then we will get the refinement of (10).

For $\varphi(x,y)=x-y$ in (47), we get the result for strongly convex function.

For $\varphi(x,y)=x-y$ and $\lambda=0$ in (47), we get the result of Theorem 25 in [5].

3. Results for Fractional Integral Operators

In this section, we give the bounds of some of the fractional integral operators which will be deduced from the results of Section 2. Throughout this section, we assume that $p=\eta=0$.

Proposition 1. *Under the assumptions of Theorem 4, the following result holds:*

$$\Gamma(\alpha)\left(\left({}^{\alpha}_{g}I_{u^+}f\right)(x)+\left({}^{\alpha}_{g}I_{v^-}f\right)(x)\right)\le(g(x)-g(u))^{\alpha}f(x)+(g(v)-g(x))^{\alpha}f(v)$$
$$+(g(x)-g(u))^{\alpha-1}\{\varphi(f(u),f(x))(I(u,x;g)-g(u))-\lambda(x-u)(2I(u,x;I_dg)-(x+u)I(u,x;g))\}$$
$$+(g(v)-g(x))^{\alpha-1}\{\varphi(f(x),f(v))(I(x,v;g)-g(x))-\lambda(v-x)(2I(x,v;I_dg)-(v+x)I(x,v;g))\}. \tag{48}$$

Proof. For $\Psi(\varsigma)=\varsigma^{\alpha}$, where $\alpha>0$, Theorem 4 gives (48). □

Proposition 2. *Under the assumptions of Theorem 4, the following inequality holds:*

$$\Gamma(\alpha)\left(\left({}_{u^+}I_{\Psi}f\right)(x)+\left({}_{v^-}I_{\Psi}f\right)(x)\right)$$
$$\le\Psi(x-u)f(x)+\frac{\Psi(x-u)}{2}\varphi(f(u),f(x))+\Psi(v-x)f(v)$$
$$+\frac{\Psi(v-x)}{2}\varphi(f(x),f(v))-\Psi(x-u)\frac{\lambda(x-u)^2}{6}-\Psi(v-x)\frac{\lambda(v-x)^2}{6}. \tag{49}$$

Proof. For g as identity function, Theorem 4 gives (49). □

Corollary 3. *For $\Psi(\varsigma)=((\varsigma^{\alpha/k})/(k\Gamma_k(\alpha)))$, (5) and (6) reduce to the fractional integral operators given in [5]. Further, the following bound for $\alpha\ge k$ is also satisfied:*

$$\left({}^{\alpha}_{g}I^k_{u^+}f\right)(x)+\left({}^{\alpha}_{g}I^k_{v^-}f\right)(x)\le\frac{1}{k\Gamma_k(\alpha)}\left[(g(x)-g(u))^{\alpha/k}f(x)+(g(v)-g(x))^{\alpha/k}f(v)\right.$$
$$+(g(x)-g(u))^{(\alpha/k)-1}\{\varphi(f(u),f(x))(I(u,x;g)-g(u))-\lambda(x-u)(2I(u,x;I_dg)$$
$$-(x+u)I(u,x;g))\}+(g(v)-g(x))^{(\alpha/k)-1}$$
$$\left.\cdot\{\varphi(f(x),f(v))(I(x,v;g)-g(x))-\lambda(v-x)(2I(x,v;I_dg)-(v+x)I(x,v;g))\}\right]. \tag{50}$$

Corollary 4. *For $\Psi(\varsigma)=\varsigma^{\alpha}$, where $\alpha\geq 1$, and g as identity function, (5) and (6) give fractional integrals defined in [12]. Further, the following bound is also satisfied:*

$$\begin{aligned}&\Gamma(\alpha)\left(\left({}^{\alpha}I_{u^{+}}f\right)(x)+\left({}^{\alpha}I_{v^{-}}f\right)(x)\right)\\&\leq(x-u)^{\alpha}f(x)+(v-x)^{\alpha}f(v)+\frac{(x-u)^{\alpha}}{2}\varphi(f(u),f(x))\\&+\frac{(v-x)^{\alpha}}{2}\varphi(f(x),f(v))-\lambda\frac{(x-u)^{\alpha+2}}{6}-\lambda\frac{(v-x)^{\alpha+2}}{6}.\end{aligned}\tag{51}$$

Corollary 5. *Using $\Psi(\varsigma)=((\varsigma^{\alpha/k})/(k\Gamma_{k}(\alpha)))$ and g as identity functions, (5) and (6) reduce to the fractional integral operators given in [13]. Further, the following bound is also satisfied:*

$$\begin{aligned}k\Gamma_{k}(\alpha)\left(\left({}^{\alpha}I_{u^{+}}^{k}f\right)(x)+\left({}^{\alpha}I_{v^{-}}^{k}f\right)(x)\right)&\leq(x-u)^{\alpha/k}f(x)+(v-x)^{\alpha/k}f(v)+\frac{(x-u)^{\alpha/k}}{2}\\&\varphi(f(u),f(x))+\frac{(v-x)^{\alpha/k}}{2}\varphi(f(x),f(v))-\lambda\frac{(x-u)^{\alpha/k+2}}{6}\\&-\lambda\frac{(v-x)^{\alpha/k+2}}{6}.\end{aligned}\tag{52}$$

Corollary 6. *For $\Psi(\varsigma)=\varsigma^{\alpha}$, where $\alpha>0$, and $g(x)=x^{\rho}/\rho$, where $\rho>0$, (5) and (6) reduce to the fractional integral operators given in [14]. Further, the following bound is also satisfied:*

$$\begin{aligned}&\left({}^{\rho}I_{u^{+}}^{\alpha}f\right)(x)+\left({}^{\rho}I_{v^{-}}^{\alpha}f\right)(x)\\&\leq\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\Big[\left(x^{\rho}-u^{\rho}\right)^{\alpha}f(x)+\left(v^{\rho}-x^{\rho}\right)^{\alpha}f(v)+\left(x^{\rho}-u^{\rho}\right)^{\alpha-1}(\varphi(f(u),f(x))\\&\cdot\left(\frac{x^{\rho+1}-u^{\rho+1}}{(x-u)(\rho+1)}-u^{\rho}\right)-\lambda\left(\frac{2\left(x^{\rho+2}-u^{\rho+2}\right)}{\rho+2}-\frac{(x+u)\left(x^{\rho+1}-u^{\rho+1}\right)}{\rho+1}\right)\Bigg)\\&+\left(v^{\rho}-x^{\rho}\right)^{\alpha-1}\Bigg(\varphi(f(x),f(v))\left(\frac{v^{\rho+1}-x^{\rho+1}}{(v-x)(\rho+1)}-x^{\rho}\right)\\&-\lambda\left(\frac{2\left(v^{\rho+2}-x^{\rho+2}\right)}{\rho+2}-\frac{(v+x)\left(v^{\rho+1}-x^{\rho+1}\right)}{\rho+1}\right)\Bigg)\Big].\end{aligned}\tag{53}$$

Corollary 7. *For $\Psi(\varsigma)=\varsigma^{\alpha}$, where $\alpha>0$, and $g(x)=((x^{s+1})/(s+1))$, where $s>0$, (5) and (6) give the following fractional integral operators:*

$$\begin{aligned}\left(F_{u^{+}}^{((\varsigma^{\alpha})/(\Gamma(\alpha))),g}f\right)(x)&=\left({}^{s}I_{u^{+}}^{\alpha}f\right)(x)=\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_{u}^{x}\left(x^{s+1}-\varsigma^{s+1}\right)^{\alpha-1}\varsigma^{s}f(\varsigma)\mathrm{d}\varsigma,\\\left(F_{v^{-}}^{((\varsigma^{\alpha})/(\Gamma(\alpha))),g}f\right)(x)&=\left({}^{s}I_{v^{-}}^{\alpha}f\right)(x)=\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{v}\left(\varsigma^{s+1}-x^{s+1}\right)^{\alpha-1}\varsigma^{s}f(\varsigma)\mathrm{d}\varsigma.\end{aligned}\tag{54}$$

Further, the following bound is also satisfied:

$$
\begin{aligned}
(s+1)^{\alpha}\Gamma(\alpha)\left(\left({}^{S}I_{u^{+}}^{\alpha} f\right)(x)+\left({}^{S}I_{v^{-}}^{\alpha} f\right)(x)\right) \leq \left(x^{s+1}-u^{s+1}\right)^{\alpha} f(x) \\
+\left(v^{s+1}-x^{s+1}\right)^{\alpha} f(v)+\left(x^{s+1}-u^{s+1}\right)^{\alpha-1}\left(\varphi\left(f(u), f(x)\left(\frac{x^{s+2}-u^{s+2}}{(x-u)(s+2)}-u^{s+1}\right)\right.\right. \\
\left.\left.-\lambda\left(\frac{2\left(x^{s+3}-u^{s+3}\right)}{s+3}-\frac{(x+u)\left(x^{s+2}-u^{s+2}\right)}{s+2}\right)\right)\right)+\left(v^{s+1}-x^{s+1}\right)^{\alpha-1} \\
\cdot\left(\varphi(f(x), f(v))\left(\frac{v^{s+2}-x^{s+2}}{(v-x)(s+2)}-x^{s+1}\right)\right. \\
\left.-\lambda\left(\frac{2\left(v^{s+3}-x^{s+3}\right)}{s+3}-\frac{(v+x)\left(v^{s+2}-x^{s+2}\right)}{s+2}\right)\right)\Bigg).
\end{aligned}
\tag{55}
$$

Corollary 8. *For* $\Psi(\varsigma)=((\varsigma^{\alpha/k})/k\Gamma_k(\alpha))$ *and* $g(x)=((x^{s+1})/(s+1))$, *where* $s>0$, *(5) and (6) reduce to the fractional integral operators given in [15]. Further, the following bound is also satisfied:*

$$
\begin{aligned}
&\left({}_{k}^{s}I_{u^{+}}^{\alpha} f\right)(x)+\left({}_{k}^{s}I_{v^{-}}^{\alpha} f\right)(x) \\
&\leq \frac{1}{(s+1)^{\alpha/k}k\Gamma_k(\alpha)}\left(x^{s+1}-u^{s+1}\right)^{\alpha/k} f(x)+\left(v^{s+1}-x^{s+1}\right)^{\alpha/k} f(v) \\
&+\left(x^{s+1}-u^{s+1}\right)^{(\alpha/k)-1}\left(\varphi\left(f(u), f(x)\left(\frac{x^{s+2}-u^{s+2}}{(x-u)(s+2)}-u^{s+1}\right)\right)\right. \\
&\left.-\lambda\left(\frac{2\left(x^{s+3}-u^{s+3}\right)}{s+3}-\frac{(x+u)\left(x^{s+2}-u^{s+2}\right)}{s+2}\right)\right)\Bigg)+\left(v^{s+1}-x^{s+1}\right)^{(\alpha/k)-1} \\
&\cdot\left(\varphi(f(x), f(v))\left(\frac{v^{s+2}-x^{s+2}}{(v-x)(s+2)}-x^{s+1}\right)\right. \\
&\left.-\lambda\left(\frac{2\left(v^{s+3}-x^{s+3}\right)}{s+3}-\frac{(v+x)\left(v^{s+2}-x^{s+2}\right)}{s+2}\right)\right)\Bigg).
\end{aligned}
\tag{56}
$$

Corollary 9. *For* $\Psi(\varsigma)=\varsigma^{\alpha}$, *where* $\alpha>0$, $g(x)=((x^{\beta+s})/(\beta+s))$, *where* β *and* $s>0$, *(5) and (6) reduce to the fractional integral operators given in [16]. Further, the following bound is also satisfied:*

$$\left({}^{s}_{\beta}I^{\alpha}_{u^{+}}f\right)(x)+\left({}^{s}_{\beta}I^{\alpha}_{v^{-}}f\right)(x)$$
$$\le\Bigg[\left(x^{\beta+s}-u^{\beta+s}\right)^{\alpha}f(x)+\left(v^{\beta+s}-x^{\beta+s}\right)^{\alpha}f(v)$$
$$+\left(x^{\beta+s}-u^{\beta+s}\right)^{\alpha-1}\Bigg(\varphi(f(u),f(x))\left(\frac{x^{\beta+s}+1-u^{\beta+s+1}}{(x-u)(\beta+s+1)}-u^{\beta+s}\right)$$
$$-\lambda\left(\frac{2\left(x^{\beta+s+2}-u^{\beta+s+2}\right)}{\beta+s+2}-\frac{(x+u)\left(x^{\beta+s+1}-u^{\beta+s+1}\right)}{\beta+s+1}\right)\Bigg) \tag{57}$$
$$+\left(v^{\beta+s}-x^{\beta+s}\right)^{\alpha-1}\Bigg(\varphi(f(x),f(v))\left(\frac{v^{\beta+s+1}-x^{\beta+s+1}}{(v-x)(\beta+s+1)}-x^{\beta+s}\right)$$
$$-\lambda\left(\frac{2\left(v^{\beta+s+2}-x^{\beta+s+2}\right)}{\beta+s+2}-\frac{(v+x)\left(v^{\beta+s+1}-x^{\beta+s+1}\right)}{\beta+s+1}\right)\Bigg)\Bigg].$$

Corollary 10. *Using $\Psi(\varsigma)=\varsigma^{\alpha}$ and $g(x)=((x-u)^{\rho})/\rho$ in (5) and $g(x)=(-(v-x)^{\rho})/\rho$ in (6), where $\rho>0$, fractional integral operators given in [17] are obtained. Further, the following bound is also satisfied:*

$$\left({}^{\rho}I^{\alpha}_{u^{+}}f\right)(x)+\left({}^{\rho}I^{\alpha}_{v^{-}}f\right)(x)\le\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\Bigg[(x-u)^{\rho\alpha}f(x)+\varphi(f(u),f(x))\frac{(x-u)^{\rho\alpha}}{\rho+1}$$
$$-\lambda\frac{\rho}{(\rho+1)(\rho+2)}(x-u)^{\rho\alpha+1}+(v-x)^{\rho\alpha}f(v)+\varphi(f(x),f(v))\frac{(v-x)^{\rho\alpha}}{\rho+1} \tag{58}$$
$$-\lambda\frac{\rho}{(\rho+1)(\rho+2)}(v-x)^{\rho\alpha+1}\Bigg].$$

Corollary 11. *For $\Psi(\varsigma)=((\varsigma^{\alpha/k})/(k\Gamma_{k}(\alpha)))$, where $\alpha>k$, and $g(x)=((x-u)^{\rho})/\rho$ in (5) and $g(x)=(-(v-x)^{\rho})/\rho$ in (6), where $\rho>0$, fractional integral operators given in [18] are obtained. Further, the following bound is also satisfied:*

$$\left({}^{\rho}_{k}I^{\alpha}_{u^{+}}f\right)(x)+\left({}^{\rho}_{k}I^{\alpha}_{v^{-}}f\right)(x)\le\frac{1}{\rho^{\alpha/k}k\Gamma_{k}(\alpha)}\Bigg[(x-u)^{\rho\alpha/k}f(x)+\varphi(f(u),f(x))\frac{(x-u)^{\rho\alpha/k}}{\rho+1}$$
$$-\lambda\frac{\rho}{(\rho+1)(\rho+2)}(x-u)^{\rho(\alpha/k)+2}+(v-x)^{\rho\alpha/k}f(v)+\varphi(f(x),f(v))\frac{(v-x)^{\rho\alpha/k}}{\rho+1} \tag{59}$$
$$-\lambda\frac{\rho}{(\rho+1)(\rho+2)}(v-x)^{\rho(\alpha/k)+2}\Bigg].$$

Remark 5

For $\lambda=0$, all the results of Section 3 reduce to the results of Section 3 in [4]; if $\lambda>0$, then all the results of Section 3 give the refinements of the results of Section 3 in [4].

For $\varphi(x,y)=x-y$ and $\lambda=0$, all the results of Section 3 reduce to the propositions and corollaries of [5].

Further, various bounds can be obtained by applying Theorems 5 and 6 which we leave for the reader.

4. Concluding Remarks

The paper presents bounds of unified integral operators (5) and (6) for strongly φ-convex functions. These bounds are refinements of bounds obtained for unified integral operators for φ-convex functions in [4]. The results for fractional integral operators have been deduced which provide bounds for Riemann–Liouville and other well-known fractional integral operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The research was supported by the National Natural Science Foundation of China (grant nos. 11971142, 11871202, 61673169, 11701176, 11626101, and 11601485).

References

[1] T. Abdeljawad, S. Rashid, Z. Hammouch, and Y.-M. Chu, "Some new local fractional inequalities associated with generalized (s, m)-convex functions and applications," *Advances in Difference Equations*, vol. 2020, p. 406, 2020.

[2] M. Awan, M. Noor, K. Noor, and F. Safdar, "On strongly generalized convex functions," *Filomat*, vol. 31, no. 18, pp. 5783–5790, 2017.

[3] M. E. Gordji, M. R. Delavar, and M. D. L. Sen, "On φ-convex functions," *Journal of Mathematical Inequalities*, vol. 10, no. 1, pp. 173–183, 2016.

[4] Y. C. Kwun, M. Zahra, G. Farid, and S. Zainab, "On a unified integral operator for φ-convex function," *Advances in Difference Equations*, vol. 2020, p. 297, 2020.

[5] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, "Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities," *IEEE Access*, vol. 6, pp. 64946–64953, 2018.

[6] C. Pintea and A. Tofan, "Convex decompositions and the valence of some functions," *Journal of Nonlinear Variable Analysis*, vol. 4, pp. 225–239, 2020.

[7] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, NY, USA, 1973.

[8] S. B. Sun and Y. M. Chu, "Inequalities for the generalized weighted mean values of g-convex functions with applications," *Revista de la real academia de ciencias exactas fisicas y naturales serie A-matematicas*, vol. 114, p. 172, 2020.

[9] W. Sun, "A unified approach to some inequalities for generalized convex functions on fractal sets and their applications," *Journal of Nonlinear Functional Analysis*, vol. 2018, p. 45, 2018.

[10] B. T. Polyak, "Existence theorems and convergence of minimizing sequences in extremum problems with restrictions," *Soviet Mathematics - Doklady*, vol. 7, pp. 72–75, 1966.

[11] G. Farid, "A unified integral operator and further its consequences," *Open Journal of Mathematical Analysis*, vol. 4, no. 1, pp. 1–7, 2020.

[12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Vol. 204, North-Holland Mathematics Studies, Elsevier, New York, NY, USA, 2006.

[13] S. Mubeen and G. M. Habibullah, "k-fractional integrals and applications," *International Journal of Contemporary Mathematical Sciences*, vol. 7, pp. 89–94, 2012.

[14] H. Chen and U. N. Katugampola, "Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals," *Journal of Mathematical Analysis and Applications*, vol. 446, no. 2, pp. 1274–1291, 2017.

[15] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris, and F. Ahmad, "(k, s)-Riemann-Liouville fractional integral and applications," *Hacettepe Journal of Mathematics and Statistics*, vol. 1, no. 45, pp. 77–89, 2016.

[16] T. U. Khan and M. A. Khan, "Generalized conformable fractional operators," *Journal of Computational and Applied Mathematics*, vol. 346, pp. 378–389, 2019.

[17] F. Jarad, E. Ugurlu, T. Abdeljawad, and D. Baleanu, "On a new class of fractional operators," *Advances in Difference Equations*, vol. 2017, p. 247, 2017.

[18] S. Habib, S. Mubeen, and M. N. Naeem, "Chebyshev type integral inequalities for generalized k-fractional conformable integrals," *Journal of Inequalities and Special Functions*, vol. 9, no. 4, pp. 53–65, 2018.

Hindawi
Journal of Mathematics
Volume 2020, Article ID 6615478, 8 pages
https://doi.org/10.1155/2020/6615478

Research Article

Multivalued Problems, Orthogonal Mappings, and Fractional Integro-Differential Equation

R. K. Sharma and Sumit Chandok

School of Mathematics, Thapar Institute of Engineering & Technology, Patiala 147004, Punjab, India

Correspondence should be addressed to Sumit Chandok; sumit.chandok@thapar.edu

Received 9 October 2020; Revised 28 October 2020; Accepted 31 October 2020; Published 21 November 2020

Academic Editor: Sun Young Cho

In this manuscript, we propose some sufficient conditions for the existence of solution for the multivalued orthogonal $\mathcal{F}$-contraction mappings in the framework of orthogonal metric spaces. As a consequence of results, we obtain some interesting results. Also as application of the results obtained, we investigate Ulam's stability of fixed point problem and present a solution for the Caputo-type nonlinear fractional integro-differential equation. An example is also provided to illustrate the usability of the obtained results.

1. Introduction and Preliminaries

The theory of multivalued mappings has an important role in mathematics and allied sciences because of its many applications, for instance, in real and complex analysis as well as in optimal control problems. Over the years, this theory has increased its significance, and hence in the literature, there are many papers focusing on the discussion of abstract and practical problems involving multivalued mappings. As a matter of fact, amongst the various approaches utilized to develop this theory, one of the most interesting approaches is based on methods of fixed point theory.

Acknowledging the work of Nadler [1], Gordji et al. [2], and Wardowski [3–5], the aim of this paper is to introduce the notion of multivalued orthogonal $\mathcal{F}$-contraction mappings in the framework of orthogonal metric space and to establish some sufficient conditions for the existence of fixed points for such class of mappings. Many researchers [6–11] proved the existence of fixed points using the concept of $\mathcal{F}$-contraction introduced by Wardowski [3–5]. In 1974, Reich [12, 13] asked whether we can take into account nonempty closed and bounded set instead of nonempty compact set. Although a lot of fixed point theorists studied this problem, it has not been completely solved. There are some partial affirmative answers to this problem, for instance, Mizoguchi et al. [14] and Olgun et al. [15]. We provide a partial solution to Reich's original problem using multivalued orthogonal $\mathcal{F}$-contraction mappings in the setting of orthogonal metric spaces. Also, as application of the interesting and new results obtained, we investigate Ulam's stability of fixed point problem and present a solution for a Caputo-type nonlinear fractional integro-differential equation.

Recently, Gordji et al. [2] introduced the concept of an orthogonal set (briefly, O-set) and presented some fixed point theorems in orthogonal metric spaces.

Definition 1. Let $\mathcal{X} \neq \emptyset$ and $\perp \subset \mathcal{X} \times \mathcal{X}$ be a binary relation. If $\perp$ satisfies the following condition: there exists $x_0 \in \mathcal{X}$ such that for all $y \in \mathcal{X}$, $y \perp x_0$, or for all $y \in \mathcal{X}$, $x_0 \perp y$, then it is called an orthogonal set (briefly O-set). We denote this O-set by $(\mathcal{X}, \perp)$.

Example 1. Let $\mathcal{X} = \mathbb{Z}$. Define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence, $(\mathcal{X}, \perp)$ is an O-set [2].

Example 2. Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ be a Picard operator, that is, $\mathcal{T}$ has a unique fixed point

$x^* \in \mathcal{X}$ and $\lim_{n \longrightarrow +\infty} \mathcal{T}_n(y) = x^*$ for all $y \in \mathcal{X}$. We define the binary relation $\perp$ on $\mathcal{X}$ by $x \perp y$ if

$$\lim_{n \longrightarrow \infty} d(x, \mathcal{T}_n(y)) = 0. \tag{1}$$

Then, $(\mathcal{X}, \perp)$ is an O-set [2].

Example 3. Let $\mathcal{X}$ be an inner product space with the inner product $\langle .,. \rangle$. Define the binary relation $\perp$ on $\mathcal{X}$ by $x \perp y$ if $\langle x, y \rangle = 0$. It is easy to see that $0 \perp x$ for all $x \in \mathcal{X}$. Hence, $(\mathcal{X}, \perp)$ is an O-set [2].

For more interesting examples for an O-set, see [2].

Definition 2. Let $(\mathcal{X}, \perp)$ be an O-set. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called an orthogonal sequence (briefly, O-sequence) if for all $n, x_n \perp x_{n+1}$, or for all $n, x_{n+1} \perp x_n$.

Definition 3. Let $(\mathcal{X}, \perp, d)$ be an orthogonal metric space ($(\mathcal{X}, \perp)$ is an O-set, and $(\mathcal{X}, d)$ is a metric space). Then $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ is said to be orthogonally continuous (or $\perp$-continuous) at $a \in \mathcal{X}$ if, for each O-sequence $\{a_n\}$ in $\mathcal{X}$ with $a_n \longrightarrow a$, we have $\mathcal{T}(a_n) \longrightarrow \mathcal{T}(a)$. Also, $\mathcal{T}$ is said to be $\perp$-continuous on $\mathcal{X}$ if $\mathcal{T}$ is $\perp$-continuous for each $a \in \mathcal{X}$.

It is easy to see that every continuous mapping is $\perp$-continuous, but the converse is not true [2].

Definition 4. Let $(\mathcal{X}, \perp, d)$ be an orthogonal set with the metric d. Then $\mathcal{X}$ is said to be orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

It is easy to see that every complete metric space is O-complete, but the converse is not true [2].

Definition 5. Let $(\mathcal{X}, \perp)$ be an O-set. A mapping $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ is said to be $\perp$-preserving if $\mathcal{T}(x) \perp \mathcal{T}(y)$, whenever $x \perp y$. Also, $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ is said to be weakly $\perp$-preserving if $\mathcal{T}(x) \perp \mathcal{T}(y)$ or $\mathcal{T}(y) \perp \mathcal{T}(x)$, whenever $x \perp y$.

It is easy to see that every $\perp$-preserving mapping is weakly $\perp$-preserving. But the converse is not true [2].

Definition 6. (see [3, 5]). Let $F: (0, +\infty) \longrightarrow \mathbb{R}$ be a mapping satisfying the following:

(F1) For all $a, b > 0$, $a > b$ implies $F(a) > F(b)$

(F2) For every sequence $\{a_n\}$ in $\mathbb{R}^+$, we have $\lim_{n \longrightarrow +\infty} a_n = 0$ if and only if $\lim_{n \longrightarrow +\infty} F(a_n) = -\infty$

(F3) There exists a number $k \in (0,1)$ such that $\lim_{a \longrightarrow 0^+} a^k F(a) = 0$

If $\lim_{t \longrightarrow 0^+} F(t) = -\infty$, then using (F1), we have $F(t_n) \longrightarrow -\infty \Rightarrow t_n \longrightarrow 0$ [5, 11].

Inspired by the work of Wardowski [3–5], we denote $\mathcal{F}$ be the family of all the functions $F: (0, +\infty) \longrightarrow \mathbb{R}$ satisfying (F1) and (F3)

We denote $\mathcal{F}1$ be the family of all the functions $F: (0, +\infty) \longrightarrow \mathbb{R}$ satisfying (F1), (F3), and (F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$

Here, $\lim_{c \longrightarrow d^-} F(c) = F(d-0) = \lim_{\varepsilon \longrightarrow 0^+} F(d-\varepsilon)$ (left limit at d) and $\lim_{c \longrightarrow d^+} F(c) = F(d+0) = \lim_{\varepsilon \longrightarrow 0^+} F(d+\varepsilon)$ (right limit at d) for all $d \in (0, +\infty)$. From mathematical analysis, the following is true for all $d \in (0, +\infty)$:

$$F(d-0) \leq F(d) \leq F(d+0). \tag{2}$$

Example 4. Let functions F_1, F_2, F_3: $(0, +\infty) \longrightarrow \mathbb{R}$ defined as follows:

(1) $F_1(a) = (-1/\sqrt{a})$, for all $a > 0$.

(2) $F_2(a) = \ln a$, for all $a > 0$.

(3) $F_3(a) = a + \ln a$, for all $a > 0$.

Then $F_1, F_2, F_3 \in \mathcal{F}$.

Let $(\mathcal{X}, d)$ be a metric space and H be a Hausdorff–Pompeiu metric induced by metric d on a set $\mathcal{X}$. Denote $\mathcal{P}(\mathcal{X})$ the family of all nonempty subsets of $\mathcal{X}$, $\mathcal{CB}(\mathcal{X})$ the family of all nonempty, and closed and bounded subsets of $\mathcal{X}$ and $\mathcal{K}(\mathcal{X})$ the family of all nonempty compact subsets of $\mathcal{X}$. H: $\mathcal{CB}(\mathcal{X}) \times \mathcal{CB}(\mathcal{X}) \longrightarrow \mathbb{R}$ defined by, for every $A, B \in \mathcal{CB}(\mathcal{X})$:

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\}, \tag{3}$$

where $d(x, A) = \inf\{d(x, y): y \in A\}$.

2. Multivalued Results

In this section, we establish some results on the existence of fixed point for weak orthogonal multivalued contraction mappings using conditions of Wardowski [3–5].

Now, we define the following orthogonal relation between two nonempty subsets of an orthogonal set.

Definition 7. Let A and B be two nonempty subsets of an orthogonal set $(\mathcal{X}, \perp)$. The set A is orthogonal to set B is denoted by $\perp_1$ and defined as follows: $A \perp_1 B$, if for every $a \in A$ and $b \in B$, $a \perp b$.

It is easy to observe the following results.

Lemma 1. *Let $(\mathcal{X}, \perp, d)$ be an orthogonal metric space, $x \in \mathcal{X}$ and $A \in \mathcal{K}(\mathcal{X})$. Then there exists $a \in A$ such that $d(x, A) = d(x, a)$.*

Lemma 2. *Let $(\mathcal{X}, \perp, d)$ be an orthogonal metric space, and $A, B \in \mathcal{K}(\mathcal{X})$, $a \in A$. Then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Now, we are ready to present our first result.

Theorem 1. *Let $(\mathcal{X}, \perp, d)$ be an O-complete orthogonal metric space and $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{K}(\mathcal{X})$ be a multivalued mapping on $\mathcal{X}$. Assume that the following conditions are satisfied:*

(i) There exists $x_0 \in \mathcal{X}$ such that $\{x_0\} \perp_1 \mathcal{T} x_0$ or $\mathcal{T} x_0 \perp_1 \{x_0\}$

(ii) For all $x, y \in \mathcal{X}$, $x \perp y$ implies $Tx \perp_1 Ty$

(iii) *If $\{x_n\}$ is an orthogonal sequence in $\mathcal{X}$ such that $x_n \longrightarrow x^* \in \mathcal{X}$, then $x_n \perp x^*$ or $x^* \perp x_n$ for all $n \in \mathbb{N}$*

(iv) *If $F \in \mathcal{F}$, there exists $\tau > 0$ such that for all $x, y \in \mathcal{X}$ with $x \perp y$ satisfying the following:*

$$H(\mathcal{T}x, \mathcal{T}y) > 0,\ \tau + F(H(\mathcal{T}x, \mathcal{T}y)) \leq F(d(x, y)). \tag{4}$$

Then $\mathcal{T}$ has at least a fixed point.

Proof. By assumption (i), there exists $x_1 \in \mathcal{T}x_0$ such that $x_0 \perp x_1$ or $x_1 \perp x_0$. By assumption (ii), we get $\mathcal{T}x_0 \perp_1 \mathcal{T}x_1$; that is, there exists $x_2 \in \mathcal{T}x_1$ such that $x_1 \perp x_2$ or $x_2 \perp x_1$. If $x_1 \in \mathcal{T}x_1$, then x_1 is a fixed point of $\mathcal{T}$. Suppose that $x_1 \notin \mathcal{T}x_1$. Since $\mathcal{T}x_1$ is compact, $d(x_1, \mathcal{T}x_1) > 0$. As $d(x_1, \mathcal{T}x_1) \leq H(\mathcal{T}x_0, \mathcal{T}x_1)$, using (F1), we have $F(d(x_1, \mathcal{T}x_1)) \leq F(H(\mathcal{T}x_0, \mathcal{T}x_1))$. Therefore, using (iv), we get

$$F(d(x_1, \mathcal{T}x_1)) \leq F(H(\mathcal{T}x_0, \mathcal{T}x_1)) \leq F(d(x_0, x_1)) - \tau. \tag{5}$$

Continuing this process inductively, we can construct an orthogonal sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_{n+1} \in \mathcal{T}x_n$, for all $n \in \mathbb{N} \cup \{0\}$. Thus we have $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$ for all $n \in \mathbb{N} \cup \{0\}$.

If $x_k \in \mathcal{T}x_k$ for some $k \in \mathbb{N} \cup \{0\}$, then x_k is a fixed point of $\mathcal{T}$.

So, we may assume that $x_n \notin \mathcal{T}x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathcal{T}x_n$ is closed, we have $d(x_n, \mathcal{T}x_n) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Also $d(x_n, \mathcal{T}x_n) \leq H(\mathcal{T}x_{n-1}, \mathcal{T}x_n)$. So using (F1), we have $F(d(x_n, \mathcal{T}x_n)) \leq F(H(\mathcal{T}x_{n-1}, \mathcal{T}x_n))$. Further from (iv) and for every $n \geq 1$, we have

$$F(d(x_n, \mathcal{T}x_n)) \leq F(H(\mathcal{T}x_{n-1}, \mathcal{T}x_n)) \leq F(d(x_{n-1}, x_n)) - \tau. \tag{6}$$

Hence from the strictly increasing property of F, we get $H(\mathcal{T}x_n, \mathcal{T}x_{n-1}) < d(x_n, x_{n-1})$. We know that $x_{n+1} \in \mathcal{T}x_n$, $d(x_n, x_{n+1}) = d(x_n, \mathcal{T}x_n) \leq H(\mathcal{T}x_{n-1}, \mathcal{T}x_n) < d(x_{n-1}, x_n)$. Therefore, the sequence $\{d(x_{n+1}, x_n)\}$ is strictly decreasing sequence. Suppose that $t_n = d(x_{n+1}, x_n) \longrightarrow t$, for some $t \geq 0$.

Furthermore, for all $n \geq n_0$, we have

$$\begin{aligned} \tau + F(d(x_{n+1}, x_n)) &\leq \tau + F(H(\mathcal{T}x_n, \mathcal{T}x_{n-1})) \\ &\leq F(d(x_n, x_{n-1})). \end{aligned} \tag{7}$$

Taking $n \longrightarrow +\infty$ in (7), we get $\tau + F(t+0) \leq F(t+0)$, which is contradiction, and hence $t_n = d(x_{n+1}, x_n) \longrightarrow 0$. By (F3), there exists $k \in (0,1)$ such that

$$\lim_{n \longrightarrow +\infty} t_n^k F(t_n) = 0. \tag{8}$$

Using (6), we get

$$F(t_n) \leq F(t_{n-1}) - \tau \leq F(t_{n-2}) - 2\tau \leq \ldots \leq F(t_0) - n\tau. \tag{9}$$

From (9), the following holds for all $n \in \mathbb{N}$:

$$t_n^k F(t_n) - t_n^k F(t_0) \leq -t_n^k n\tau \leq 0. \tag{10}$$

Letting $n \longrightarrow \infty$ in (10), we get $\lim_{n \longrightarrow +\infty} n t_n^k = 0$. Hence there exists $n_1 \in \mathbb{N}$ such that $n t_n^k \leq 1$ for all $n \geq n_1$. So, we have all for all $n \geq n_1$:

$$t_n \leq \frac{1}{n^{(1/k)}}. \tag{11}$$

Now, we have to show that $\{x_n\}$ is a Cauchy orthogonal sequence. Consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangle inequality and (11), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m), \\ &= t_n + t_{n+1} + \cdots + t_{m-1} = \sum_{i=n}^{m-1} t_i \leq \sum_{i=n}^{+\infty} t_i \leq \sum_{i=n}^{+\infty} \frac{1}{i^{(1/k)}}. \end{aligned} \tag{12}$$

By the convergence of series, $\sum_{i=n}^{+\infty} (1/i^{1/k})$, passing to limit $n \longrightarrow +\infty$, we get $d(x_n, x_m) \longrightarrow 0$.

This shows that $\{x_n\}$ is a Cauchy orthogonal sequence. Since $\mathcal{X}$ is O-complete, there exists $x^* \in \mathcal{X}$ such that $\lim_{n \longrightarrow +\infty} x_n = x^*$.

Now, we claim that $x^* \in \mathcal{T}x^*$. Assume the contrary that $x^* \notin \mathcal{T}x^*$. Hence there exists $n_1 \in \mathbb{N}$ such that $x^* \notin \{x_n\}_{n \geq n_1}$, $H(\mathcal{T}x_n, \mathcal{T}x^*) > 0$. Therefore, further by our assumption, $x_n \perp x^*$ or $x^* \perp x_n$, and using (iv), we get

$$F(d(x_{n+1}, \mathcal{T}x^*)) \leq \tau + F(H(\mathcal{T}x_n, \mathcal{T}x^*)) \leq F(d(x_n, x^*)). \tag{13}$$

Now using strict increasing property of F and $\tau > 0$, we get $d(x_{n+1}, \mathcal{T}x^*) < d(x_n, x^*)$. Taking $n \longrightarrow +\infty$, we get $x^* \in \overline{\mathcal{T}x^*} = \mathcal{T}x^*$. Hence, the result is obtained.

Here it should be noted that in Theorem 1, $\mathcal{T}x$ is compact for all $x \in \mathcal{X}$. Now, we have the following result in which we give a partial answer to Reich's problem for a closed and bounded set. □

Theorem 2. *Let $(\mathcal{X}, \perp, d)$ be an O-complete orthogonal metric space and $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{CB}(\mathcal{X})$ be a multivalued mapping on $\mathcal{X}$. Assume that the following conditions are satisfied:*

(i) *There exists $x_0 \in \mathcal{X}$ such that $\{x_0\} \perp_1 \mathcal{T}x_0$ or $\mathcal{T}x_0 \perp_1 \{x_0\}$*

(ii) *For all $x, y \in \mathcal{X}$, $x \perp y$ implies $\mathcal{T}x \perp_1 \mathcal{T}y$*

(iii) *If $\{x_n\}$ is an orthogonal sequence in $\mathcal{X}$ such that $x_n \longrightarrow x^* \in \mathcal{X}$, then $x_n \perp x^*$ or $x^* \perp x_n$ for all $n \in \mathbb{N}$.*

(iv) *If $F \in \mathcal{F}1$, there exists $\tau > 0$ such that for all $x, y \in \mathcal{X}$ with $x \perp y$ satisfying the following:*

$$H(\mathcal{T}x, \mathcal{T}y) > 0, \tau + F(H(\mathcal{T}x, \mathcal{T}y)) \leq F(d(x, y)). \tag{14}$$

Then $\mathcal{T}$ has at least a fixed point.

Proof. Let $x_0 \in \mathcal{X}$. Since $\mathcal{T}x$ is nonempty for all $x \in \mathcal{X}$, by assumption (i), we can choose $x_1 \in \mathcal{T}x_0$ such that $x_0 \perp x_1$ or $x_1 \perp x_0$. If $x_1 \in \mathcal{T}x_1$, then x_1 is a fixed point of $\mathcal{T}$. Let

$x_1 \notin \mathcal{T}x_1$. Then $d(x_1, \mathcal{T}x_1) > 0$ since $\mathcal{T}x_1$ is closed. Since $d(x_1, \mathcal{T}x_1) \leq H(\mathcal{T}x_0, \mathcal{T}x_1)$, then from (F1), we get

$$F(d(x_1, \mathcal{T}x_1)) \leq F(H(\mathcal{T}x_0, \mathcal{T}x_1)). \tag{15}$$

Using (iv), we get

$$F(d(x_1, \mathcal{T}x_1)) \leq F(H(\mathcal{T}x_0, \mathcal{T}x_1)) \leq F(d(x_0, x_1)) - \tau. \tag{16}$$

From (F4), we get $F(d(x_1, \mathcal{T}x_1)) = \inf_{y \in \mathcal{T}x_1} F(d(x_1, y))$. So from (16), we have

$$\begin{aligned} F(d(x_1,\mathcal{T}x_1)) &= \inf_{y\in\mathcal{T}x_1} F(d(x_1,y)) \leq F(H(\mathcal{T}x_0,\mathcal{T}x_1)), \\ &\leq F(d(x_0,x_1)) - \tau, \\ &< F(d(x_0,x_1)) - \frac{\tau}{2}. \end{aligned} \tag{17}$$

By assumption (ii), we get $\mathcal{T}x_0 \perp_1 \mathcal{T}x_1$. Continuing this process, we construct an orthogonal sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_{n+1} \in \mathcal{T}x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Thus we have $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$ for all $n \in \mathbb{N} \cup \{0\}$.

If $x_k \in \mathcal{T}x_k$ for some $k \in \mathbb{N} \cup \{0\}$, then x_k is a fixed point of $\mathcal{T}$, and so the proof is completed.

So, we may assume that $x_n \notin \mathcal{T}x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathcal{T}x_n$ is closed, we have $d(x_n, \mathcal{T}x_n) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Also $d(x_n, \mathcal{T}x_n) \leq H(\mathcal{T}x_{n-1}, \mathcal{T}x_n)$, and from (F1), we get $F(d(x_n, \mathcal{T}x_n)) \leq F(H(\mathcal{T}x_{n-1}, \mathcal{T}x_n))$.

Furthermore, using (iv), we have

$$\begin{aligned} F(d(x_n, \mathcal{T}x_n)) &\leq F(H(\mathcal{T}x_n, \mathcal{T}x_{n+1})) \\ &\leq F(d(x_n, x_{n+1})) - \tau < F(d(x_n, x_{n+1})) - \frac{\tau}{2}. \end{aligned} \tag{18}$$

Since $F(d(x_n, \mathcal{T}x_n)) = \inf_{y \in \mathcal{T}x_n} F(d(x_n, y))$. Therefore, using (18), we get

$$\begin{aligned} F(d(x_n,\mathcal{T}x_n)) &= \inf_{y\in\mathcal{T}x_n} F(d(x_n,y)) \leq F(H((\mathcal{T}x_{n-1},\mathcal{T}x_n)) \\ &< F(d(x_{n-1},x_n)) - \frac{\tau}{2}. \end{aligned} \tag{19}$$

So from (19), we can get a sequence $\{x_n\}$ in $\mathcal{X}$ such that there exists $x_{n+1} \in \mathcal{T}x_n$ and $F(d(x_n, x_{n+1})) < F(d(x_{n-1}, x_n))$ for all $n \in \mathbb{N}$. Now, proceeding on the same lines of Theorem 1, we get the result. □

3. Consequences

In this section, we give some interesting consequences of the results proved in the previous section.

The following result is an immediate consequence of Theorem 1.

Corollary 1. *Let $(\mathcal{X}, \perp, d)$ be an O-complete orthogonal metric space and $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{K}(\mathcal{X})$. Assume that the following conditions are satisfied:*

(i) There exists $x_0 \in \mathcal{X}$ such that $\{x_0\} \perp_1 \mathcal{T}x_0$ or $\mathcal{T}x_0 \perp_1 \{x_0\}$

(ii) For all $x, y \in \mathcal{X}$, $x \perp y$ implies $\mathcal{T}x \perp_1 \mathcal{T}y$

(iii) If $\{x_n\}$ is an orthogonal sequence in $\mathcal{X}$ such that $x_n \longrightarrow x^* \in \mathcal{X}$, then $x_n \perp x^*$ or $x^* \perp x_n$ for all $n \in \mathbb{N}$.

(iv) There exists some $\tau_i > 0$, $i = 1, 2, 3$ such that for all $x, y \in \mathcal{X}$ with $x \perp y$, $H(\mathcal{T}x, \mathcal{T}y) > 0$, either of the following contractive conditions hold:

$$\begin{aligned} &\tau_1 + H(\mathcal{T}x, \mathcal{T}y) \leq d(x, y); \\ &\tau_2 - \frac{1}{H(\mathcal{T}x, \mathcal{T}y)} \leq -\frac{1}{d(x, y)}; \\ &\tau_3 + \frac{1}{1 - e^{H(\mathcal{T}x, \mathcal{T}y)}} \leq \frac{1}{1 - e^{d(x, y)}}. \end{aligned} \tag{20}$$

Then $\mathcal{T}$ has at least a fixed point in each of these cases.

Proof. As each functions $F_1(r) = r$, $F_2(r) = (-1/r)$, and $F_3(r) = (1/1 - e^r)$, where $r = d(x, y) > 0$, is strictly increasing on $(0, +\infty)$, so the proof immediately follows from Theorem 1.

As a consequence of Theorem 1, we have the following result for single-valued mapping by replacing condition (iii) with $\mathcal{T}$ is $\perp$-continuous. □

Corollary 2. *Let $(\mathcal{X}, \perp, d)$ be an O-complete orthogonal metric space and $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$. Assume that the following conditions are satisfied:*

(i) There exists some $\tau > 0$, such that for all $x, y \in \mathcal{X}$ with $x \perp y$, $d(\mathcal{T}x, \mathcal{T}y) > 0$:

$$\tau + F(d(\mathcal{T}x, \mathcal{T}y)) \leq F(d(x, y)), \tag{21}$$

where $F \in \mathcal{F}$.

(ii) There exists $x_0 \in \mathcal{X}$ such that $x_0 \perp \mathcal{T}x_0$ or $\mathcal{T}x_0 \perp x_0$.

(iii) For all $x, y \in \mathcal{X}$, $x \perp y$ implies $\mathcal{T}x \perp \mathcal{T}y$

(iv) $\mathcal{T}$ is $\perp$-continuous

Then, $\mathcal{T}$ has a fixed point.

Proof. Here, we can choose $\mathcal{T}$ as a multivalued mapping by considering $\mathcal{T}x$ is a singleton set for every $x \in \mathcal{X}$. Arguing on the same lines of Theorem 1, we consider $\{x_n\}$ is a Cauchy orthogonal sequence and $\lim_{n\longrightarrow\infty} x_n = x^*$. As $\mathcal{T}$ is $\perp$-continuous, we have

$$d(x^*, Tx^*) = \lim_{n\longrightarrow+\infty} d(\mathcal{T}x_n, \mathcal{T}x^*) = 0, \tag{22}$$

i.e., x^* is a fixed point of $\mathcal{T}$.

As a consequence of Corollary 2, we have the following result by taking $F(r) = \ln r$, $r > 0$. □

Corollary 3. *Let $(\mathcal{X}, \perp, d)$ be an O-complete orthogonal metric space and $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$. Assume that the following conditions are satisfied:*

(i) There exists some $\tau > 0$, such that for all $x, y \in \mathcal{X}$ with $x \perp y$, $d(\mathcal{T}x, \mathcal{T}y) > 0$:

$$d(\mathcal{T}x, \mathcal{T}y) \leq e^{-\tau} d(x, y), \tag{23}$$

where $F \in \mathcal{F}$.

(ii) There exists $x_0 \in \mathcal{X}$ such that $x_0 \perp \mathcal{T}x_0$ or $\mathcal{T}x_0 \perp x_0$.

(iii) For all $x, y \in \mathcal{X}$, $x \perp y$ implies $\mathcal{T}x \perp \mathcal{T}y$.

(iv) $\mathcal{T}$ is $\perp$-continuous.

Then $\mathcal{T}$ has a fixed point.

4. Illustration

In this section, we illustrate an example which shows that $\mathcal{T}$ is a multivalued orthogonal mapping and satisfies the condition (iv) of Theorem 1, but it is not multivalued orthogonal contraction ($H(\mathcal{T}x, \mathcal{T}y) \leq k\,d(x, y)$, for $k \in [0, 1)$ with $x \perp y$).

Example 5. Let $\mathcal{X} = \{S_n = (n(n+1)/2)\colon n \in \mathbb{N}\}$ and $d\colon \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ be a mapping defined by $d(x, y) = |x - y|$ for all $x, y \in \mathcal{X}$.

Define a relation $\perp$ on $\mathcal{X}$ by $x \perp y$ if and only if $xy \in \{x, y\} \subseteq \mathcal{X} = \{S_n\}$.

Thus $(\mathcal{X}, \perp, d)$ is an O-complete orthogonal metric space. Now, we define a mapping $\mathcal{T}\colon \mathcal{X} \longrightarrow \mathcal{K}(\mathcal{X})$ by

$$\mathcal{T}x = \begin{cases} \{x_1\}, & x = x_1, \\ \{x_1, \ldots, x_{n-1}\}, & x = x_n, n \geq 1. \end{cases} \tag{24}$$

We claim that $\mathcal{T}$ is a multivalued orthogonal mapping satisfying condition (iv) of Theorem 1 with respect to $F(\alpha) = \alpha + \ln(\alpha)$, $\alpha > 0$ and $\tau = 1$. To see this, we have the following cases.

First, we observe that for all $m, n \in \mathbb{N}$, $H(\mathcal{T}x, \mathcal{T}y) > 0$ if and only if $m > 2$ and $n = 1$ or $m > n > 1$.

Case 1. For $m > 2$ and $n = 1$, we have

$$\begin{aligned} &\frac{H(\mathcal{T}x_m, \mathcal{T}x_1)}{d(x_m, x_1)} e^{H(\mathcal{T}x_m, \mathcal{T}x_1) - d(x_m, x_1)} \\ &= \frac{x_{m-1} - x_1}{x_m - x_1} e^{x_{m-1} - x_1} = \frac{m^2 - m - 2}{m^2 + m - 2} e^{-m} < e^{-m} < e^{-1}. \end{aligned} \tag{25}$$

Case 2. For $m > n > 1$, we get

$$\begin{aligned} &\frac{H(\mathcal{T}x_m, \mathcal{T}x_n)}{d(x_m, x_n)} e^{H(\mathcal{T}x_m, \mathcal{T}x_n) - d(x_m, x_n)} \\ &= \frac{x_{m-1} - x_{n-1}}{x_m - x_n} e^{x_{m-1} - x_{n-1} - x_m + x_n} \\ &= \frac{m + n - 1}{m + n + 1} e^{n-m} < e^{n-m} \leq e^{-1}. \end{aligned} \tag{26}$$

This shows that $\mathcal{T}$ satisfies (iv) of Theorem 1. Hence, $\mathcal{T}$ has a fixed point.

On the contrary, $\mathcal{T}$ is not multivalued orthogonal contraction ($H(\mathcal{T}x, \mathcal{T}y) \leq k\,d(x, y)$, $k \in [0, 1)$), as

$$\lim_{n \longrightarrow +\infty} \frac{H(\mathcal{T}x_n, \mathcal{T}x_1)}{d(x_n, x_1)} = \lim_{n \longrightarrow +\infty} \frac{x_{n-1} - 1}{x_n - 1} = 1. \tag{27}$$

5. Applications

In this section, we present the Ulam stability and solve a nonlinear fractional differential-type equation using Corollary 3.

5.1. Ulam Stability. The Ulam [16, 17] stability has attracted attention of several authors in fixed point theory [18]. On orthogonal metric space $(\mathcal{X}, \perp, d)$, $\mathcal{T}\colon \mathcal{X} \longrightarrow \mathcal{X}$, we investigate the fixed point equation:

$$\mathcal{T}v = v, \tag{28}$$

and the inequality (for $\varepsilon > 0$):

$$d(\mathcal{T}x, x) \leq \varepsilon. \tag{29}$$

Equation (28) is called the Ulam stable if it satisfies the following condition:

(A) There is a constant $\delta > 0$, for each $\varepsilon > 0$, and for every solution x^* of the inequality (29), there is a solution $v^* \in X$ for equation (28) such that

$$d(v^*, x^*) \leq \delta\varepsilon. \tag{30}$$

Theorem 3. *Under the hypothesis of Corollary 3, the fixed point equation (28) is Ulam stable.*

Proof. On account of Corollary 3, we guarantee a unique $v^* \in X$ such that $v^* = \mathcal{T}v^*$, that is, $v^* \in \mathcal{X}$ forms a solution of (28). Let $\varepsilon > 0$ and $x^* \in \mathcal{X}$ be an ε-solution, that is,

$$d(\mathcal{T}x^*, x^*) \leq \varepsilon. \tag{31}$$

We have

$$\begin{aligned} d(v^*, x^*) &= d(\mathcal{T}v^*, x^*) \\ &\leq d(\mathcal{T}v^*, \mathcal{T}x^*) + d(\mathcal{T}x^*, x^*) \\ &\leq e^{-\tau} d(v^*, x^*) + \varepsilon. \end{aligned} \tag{32}$$

Hence, $d(v^*, x^*) \leq (1/1 - e^{-\tau})\varepsilon = k\varepsilon$, where $k = (1/1 - e^{-\tau}) > 0$. Therefore, equation (28) is Ulam stable. □

5.2. Application to Nonlinear Fractional Integro-Differential Equation. Here, we give a solution for a Caputo-type nonlinear fractional integro-differential equation. For more details on fractional calculus, see [19–25] and references cited therein.

The Caputo derivative of a continuous mapping $g\colon [0, \infty) \longrightarrow \mathbb{R}$ (order $\delta > 0$) is given by

$$ {}^{C}D^{\delta}g(t) := \frac{1}{\Gamma(n-\delta)}\int_0^t \frac{g^{(n)}(s)\mathrm{d}s}{(t-s)^{\delta-n+1}}, \quad n-1 \le \delta < n, n = [\delta]+1, \tag{33} $$

where Γ represents the gamma function and $[\delta]$ refers to the integer part of the positive real number δ.

In this section, we examine the nonlinear fractional integro-differential equation of the Caputo type:

$$ \begin{cases} {}^{C}D^{\delta}u(t) = \mathcal{G}(t,u(t)), & t \in I = [0,1], 1 < \delta \le 2, \\ u(0) = 0, u(1) = \int_0^{\theta} u(s)\mathrm{d}s, \end{cases} \tag{34} $$

where $u \in (C[0,1],\mathbb{R})$, $\theta \in (0,1)$, and $\mathcal{G}: I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function (for more details, see [20]).

We consider $\mathcal{X} = \{u: u \in (C[0,1],\mathbb{R})\}$ with supremum norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$. So $(X,\|.\|)$ is a Banach space.

The space $\mathcal{X}: = C([0,1],\mathbb{R})$ endowed with the metric $d: \mathcal{X}\times\mathcal{X} \longrightarrow [0,\infty)$ defined as $d(u,v) = \|u-v\| = \sup_{t\in[0,1]}(t)|u(t)-v(t)|$ and define an orthogonal relation $u\perp v$ if and only if $uv \ge 0$, for all $u,v \in \mathcal{X}$. Then $(\mathcal{X},\perp,d)$ is an orthogonal metric space.

Clearly, a solution of equation (34) is a fixed point of the integral equation:

$$ \begin{aligned} \mathcal{T}u(t) = {} & \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\mathcal{G}((s,u(s)))\mathrm{d}s, \\ & -\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\mathcal{G}(s,u(s))\mathrm{d}s, \\ & +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^s (s-m)^{\delta-1}\mathcal{G}(s,u(m))\mathrm{d}m\right)\mathrm{d}s. \end{aligned} \tag{35} $$

Theorem 4. *Assume that $\mathcal{G}: I\times\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying*

$$ |\mathcal{G}(s,u(s)) - \mathcal{G}(s,v(s))| \le \frac{\Gamma(\delta+1)}{5}e^{-\tau}|u(s)-v(s)|, \tag{36} $$

for each $s \in [0,1]$, for some $\tau > 0$ and for all $u,v \in C([0,1],\mathbb{R})$. Then the fractional differential equation (34) with given boundary conditions has a solution.

Proof. The space $\mathcal{X}: = C([0,1],\mathbb{R})$ endowed with the metric $d: \mathcal{X}\times\mathcal{X} \longrightarrow [0,\infty)$ defined as $d(u,v) = \sup_{t\in[0,1]}|u(t)-v(t)|$, for all $u,v \in \mathcal{X}$. Define an orthogonal relation $u\perp v$ if and only if $uv \ge 0$, for all $u,v \in \mathcal{X}$. Then $(\mathcal{X},\perp,d)$ is an orthogonal metric space. Define $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ as in (35). So $\mathcal{T}$ is $\perp$-continuous. First, we show that $\mathcal{T}$ is $\perp$-preserving, let $u(t)\perp v(t)$ for all $t \in [0,1]$. Now, from (35). we have

$$ \begin{aligned} \mathcal{T}u(t) = {} & \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\mathcal{G}(s,u(s))\mathrm{d}s, \\ & -\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\mathcal{G}(s,u(s))\mathrm{d}s, \\ & +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^s (s-m)^{\delta-1}\mathcal{G}(s,u(m))\mathrm{d}m\right)\mathrm{d}s > 0, \end{aligned} \tag{37} $$

which implies that $\mathcal{T}u\perp\mathcal{T}v$.

Now, we have to show that $\mathcal{T}$ satisfies (i) of Corollary 2 for $F(r) = \ln r$, $r > 0$. For all $t \in [0,1]$, $u(t)\perp v(t)$, we have

$$ \begin{aligned} |\mathcal{T}u(t) - \mathcal{T}v(t)| = {} & |\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\mathcal{G}(s,u(s))\mathrm{d}s - \frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\mathcal{G}(s,u(s))\mathrm{d}s \\ & +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^s (s-m)^{\delta-1}\mathcal{G}(s,u(m))\mathrm{d}m\right)\mathrm{d}s, \\ & -\left(\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\mathcal{G}(s,v(s))\mathrm{d}s - \frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\mathcal{G}(s,v(s))\mathrm{d}s \right. \\ & \left. +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^s (s-m)^{\delta-1}\mathcal{G}(s,v(m))\mathrm{d}m\right)\mathrm{d}s\right)|, \end{aligned} $$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}|\mathcal{G}(s,u(s))-\mathcal{G}(s,v(s))|\mathrm{d}s-\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}|\mathcal{G}(s,u(s))-\mathcal{G}(s,v(s))|\mathrm{d}s\\
&\quad+\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^\theta\left(\int_0^s (s-m)^{\delta-1}|\mathcal{G}(s,u(m))-\mathcal{G}(s,v(m))|\mathrm{d}m\right)\mathrm{d}s,\\
&\leq \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\left[\frac{\Gamma(\delta+1)}{5}e^{-\tau}\sup_{s\in[0,1]}|u(s)-v(s)|\right]\mathrm{d}s\\
&\quad-\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\left[\frac{\Gamma(\delta+1)}{5}e^{-\tau}\sup_{s\in[0,1]}|u(s)-v(s)|\right]\mathrm{d}s\\
&\quad+\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^\theta\left(\int_0^s (s-m)^{\delta-1}\left[\frac{\Gamma(\delta+1)}{5}e^{-\tau}\sup_{s\in[0,1]}|u(s)-v(s)|\right]\mathrm{d}m\right)\mathrm{d}s,\\
&\leq \left[\frac{\Gamma(\delta+1)}{5}e^{-\tau}\sup_{s\in[0,1]}|u(s)-v(s)|\right]\times\sup_{t\in[0,1]}\left(\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\mathrm{d}s-\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\mathrm{d}s\right.\\
&\quad\left.+\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^\theta\left(\int_0^s (s-m)^{\delta-1}\mathrm{d}m\right)\mathrm{d}s\right)\leq e^{-\tau}\sup_{s\in[0,1]}|u(s)-v(s)|=e^{-\tau}d(u,v),
\end{aligned} \tag{38}$$

for all $u, v \in \mathcal{X}$. Therefore, the condition (i) of Corollary 2 holds. Accordingly, all axioms of Corollary 2 are verified, and $\mathcal{T}$ has a fixed point. The Caputo-type nonlinear fractional differential equation (34) possesses a solution is yielded.

6. Conclusions

In this manuscript, we prove some existence results for the multivalued orthogonal mappings using the conditions (F1) and (F2) of Wardowski's and obtain the stability of a fixed point problem and a solution for the Caputo-type nonlinear fractional differential equation.

Now, we have an open question, whether we can obtain Theorems 1 and 2 with condition (F1) only of Wardowski in the setting of orthogonal metric space?

Data Availability

No data are used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no known conflicts of financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors are grateful to the AISTDF-DST, India CRD/2018/00017.

References

[1] S. Nadler, "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[2] M. E. Gordji, M. Rameani, M. De La Sen, and Y. J. Cho, "On orthogonal sets and Banach fixed point theorem," *Fixed Point Theory and Applications*, vol. 18, pp. 569–578, 2017.

[3] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2012, no. 1, 2012.

[4] D. Wardowski and N. V. Dung, "Fixed points of F-weak contractions on complete metric space," *Demonstratio Mathematica*, vol. 47, pp. 146–155, 2014.

[5] D. Wardowski, "Solving existence problems via F-contractions," *Proceedings of the American Mathematical Society*, vol. 146, no. 4, pp. 1585–1598, 2018.

[6] I. Altun, G. Minak, and H. Dag, "Multivalued F-contractions on complete metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 16, no. 4, pp. 659–666, 2015.

[7] P. S. Kumari, O. Alqahtani, and E. Karapinar, "Some fixed-point theorems in b-dislocated metric space and applications," *Symmetry*, vol. 10, no. 12, pp. 1–24, 2018.

[8] A. Lukács, S. Kajanto, and S. Kajántó, "Fixed point theorems for various types of F-contractions in complete b-metric spaces," *Fixed Point Theory*, vol. 19, no. 1, pp. 321–334, 2018.

[9] H. Nashine and Z. Kadelburg, "Existence of solutions of cantilever beam problem via (α-β-FG)-contractions in b-metric-like spaces," *Filomat*, vol. 31, no. 11, pp. 3057–3074, 2017.

[10] K. Sawangsup, W. Sintunavarat, and Y. J. Cho, "Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 22, p. 10, 2020.

[11] M. Turinici, "Wardowski implicit contractions in metric spaces," 2013, https://arxiv.org/abs/1211.3164.
[12] S. Reich, "Some fixed point problems," *Atti Della Accademia Nazionale Dei Lincei, Classe Di Scienze Fisiche, Matematiche e Naturali*, vol. 57, pp. 194–198, 1974.
[13] S. Reich, "Some problems and results in fixed point theory," in *Topological Methods in Nonlinear Functional Analysis (Toronto, Ont. 1982)*, pp. 179–187, American Mathematical Society, Providence, RI, USA, 1983.
[14] N. Mizoguchi and W. Takahashi, "Fixed point theorems for multivalued mappings on complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 141, pp. 475–488, 1989.
[15] M. Olgun, G. Minak, and I. Altun, "A new approach to Mizoguchi-Takahashi type fixed point theorems," *Journal of Nonlinear and Convex Analysis*, vol. 17, pp. 579–587, 2016.
[16] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences*, vol. 27, no. 4, pp. 222–224, 1941.
[17] S. S. Ulam, *Problems in Modern Mathematics*, John Wiley Sons, New York, NY, USA, 1964.
[18] W. Sintunavarat, "Generalized Ulam-Hyers stability, well-posedness and limit shadowing of fixed point problems for $\alpha - \beta$-contraction mapping in metric spaces," *Scientific World Journal*, vol. 2014, Article ID 569174, 7 pages, 2014.
[19] B. Ahmad, S. K. Ntouyas, A. Alsaedi, and M. Alghanmi, "Multi-term fractional differential equations and inclusions with generalized nonlocal fractional integro-differential boundary conditions," *Journal of Nonlinear Functional Analysis*, vol. 2018, p. 36, 2018.
[20] D. Baleanu, S. Rezapour, and H. Mohammadi, "Some existence results on nonlinear fractional differential equations," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering*, vol. 371, no. 1990, Article ID 20120144, 2013.
[21] D. Baleanu, A. Jajarmi, and M. Hajipour, "A new formulation of the fractional optimal control problems involving Mittag-Leffler nonsingular kernel," *Journal of Optimization Theory and Applications*, vol. 175, no. 3, pp. 718–737, 2017.
[22] D. Baleanu, A. Jajarmi, J. H. Asad, and T. Blaszczyk, "The motion of a bead sliding on a wire in fractional sense," *Acta Physica Polonica A*, vol. 131, no. 6, pp. 1561–1564, 2017.
[23] A. Cernea, "On the mild solutions of a class of second-order integro-differential inclusions," *Journal of Nonlinear and Variational Analysis*, vol. 2, pp. 25–33, 2018.
[24] A. Lachouri, A. Ardjouni, and A. Djoudi, "Positive solutions of a fractional integro-differential equation with integral boundary conditions," *Communications in Optimization Theory*, vol. 2020, p. 1, 2020.
[25] B. Samet and H. Aydi, "On some inequalities involving Liouville-Caputo fractional derivatives and applications to special means of real numbers," *Mathematics*, vol. 6, no. 10, p. 193, 2018.