Journal of Function Spaces

Developments in Functions and Operators of Complex Variables

Lead Guest Editor: Sarfraz Nawaz Malik
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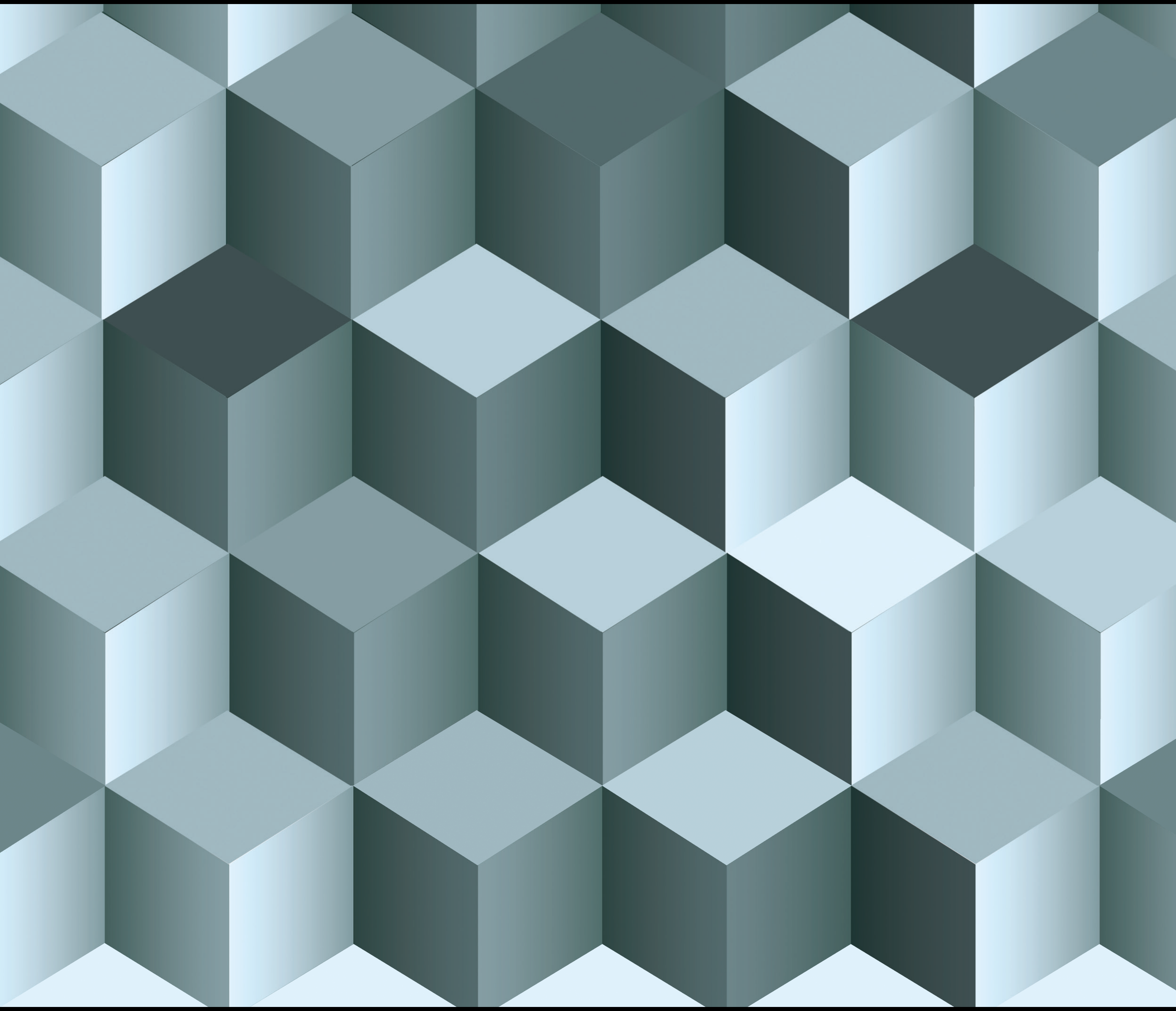

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Research Article

The Extended Laguerre Polynomials $\{A_{q,n}^{(\alpha)}(x)\}$ Involving ${}_qF_q$, $q > 2$

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In this paper, for the proposed extended Laguerre polynomials $\left\{A_{q,n}^{(\alpha)}(x)\right\}$, the generalized hypergeometric function of the type ${}_qF_q$, $q > 2$ and extension of the Laguerre polynomial are introduced. Similar to those related to the Laguerre polynomials, the generating function, recurrence relations, and Rodrigue's formula are determined. Some corollaries are also discussed at the end.

1. Introduction and Applications

Due to its wide applications, the study of orthogonal polynomials has been a popular research topic for many years. Many of these polynomials are generated by hypergeometric functions. Indeed, the orthogonal polynomials have numerous properties of interest, e.g., recurrence relations and differential equations. Based on their Rodrigues formulae, generating functions and solutions of integral equations with orthogonal polynomials as kernels have been extensively investigated.

Generalizations and extensions of orthogonal polynomials are in the another familiar direction of research. One of the polynomial set which has been extended is a set of Laguerre polynomials. Laguerre polynomials are well-known to form an orthogonal set with respect to the weight function $z^\alpha e^{-z}$ on the interval $(0, \infty)$.

A set of Laguerre polynomials is generated by well-known confluent hypergeometric function ${}_1F_1$. It can be also generated by hypergeometric function ${}_0F_1$. Another direction is the study of Laguerre polynomials based on more than one variable which are often used in physical and statistical model. One, too, combinatorial polynomial images, moments, orthogonality relation, and a combinatorial understanding Ikyrana coefficients Al-Salam and Chihara q Laguerre polynomial, can study various aspects. Orthogonal polynomials, namely, Hermite polynomials and Legendre polynomials can also be studied through the finite series involving Laguerre polynomials.

Laguerre polynomials are used to solve noncentral Chi-square distribution. Laguerre polynomials are the orthogonal polynomial satisfied the recurrence relations. Various specializations are studied with application to classical orthogonal polynomials. Kinetic theory of particles based on Laguerre polynomial macroscopic hydrodynamic quantities and kinetic coefficients of different medium is used to set.

There are a large number of generalizations and extensions of Laguerre polynomials, e.g., Shively's polynomials. Many of these generalizations are based on its Rodrigues formulae in addition to hypergeometric functions. Recently, an interesting integral representation of generalized hypergeometric functions has been determined. It is now natural to point to a generalization of Laguerre polynomials based on such a discovery. This idea has motivated the current work. Also, it will explore deeper investigation and extensions of results which we proved in our early studies and research.

In this work, we discuss the features of Extending Laguerre polynomial involving ${}_qF_q$, $q > 2$. Extending Laguerre polynomial set has been a popular research issue well considered for years. There have a number of directions to do so. One direction is to follow the definition of Laguerre polynomials based on the confluent hypergeometric function, explicitly

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1(-n\,;1+\alpha\,;x). \tag{1}$$

Shively [1] extended the Laguerre polynomials as

$$R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n}\,{}_1F_1(-n\,;a+n\,;x). \tag{2}$$

He used a factor $a+n$ instead of $1+\alpha$ in Laguerre polynomials. In his study, he found a large number of its properties including the result that a finite sum of Laguerre polynomials is Shively's polynomials. Habibullah [2] proved the Rodrigues formula for Shively's polynomials in the following form

$$R_n(a+1,x) = \frac{e^x x^{-\alpha-n}}{n!} D^n\left(x^{\alpha+2n}e^{-x}\right), \tag{3}$$

similar to the Rodrigues formula

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} D^n(x^{\alpha+n}e^{-x}), \tag{4}$$

for the Laguerre polynomials.

Researchers have also often based their generalization on extension of Rodrigues formula and subsequently determined properties of extended polynomials. Chatterjea [3] developed an extension of the Laguerre polynomial by strengthening the Rodrigues formula. Chatterjea and Das [4] restructured their definition and the resultant study by considering another version of the Laguerre polynomials.

Chen and Srivastava [5] found a stronger Rodrigues formula to develop a generalization of the Laguerre polynomial.

The forms generalized Rodrigues formulae by Chak [6] show that robust following of this method of defining extensions of the Laguerre polynomial. Since comprehensive literature is available on special functions, we follow Shively's tradition to introduce the definition of the extended Laguerre polynomials set based on special functions similar to that contained the original definition.

Dattoli et al. [7] used an exponential generating functions approach involving Hermite polynomials and Bessel functions introduced new families. He, too, studied their respective recurrence relations and showed that they fulfill different differential equations. Trickovic and Stankovic [8] of the Jacobi and Laguerre polynomial orthogonality of rational functions that have proved equally. Trickovic and Stankovic [8] have proved the orthogonality of the Jacobi and the Laguerre polynomials.

Khan and Shukla [9] have introduced a novel method to give operator representations of certain polynomials. They gave binomial and trinomial operators representations of certain polynomials. Grinshpan [10] has shown that all solutions to the equations of a family of integral equations fulfill modulus inequality. Duenas et al. [11] a derivative of a Dirac delta by adding a perturbation of a Laguerre-Hahn functional gain catalog.

Kim et al. [12] have studied some interesting identities and also studied Bernoulli and Euler's numbers in connection with the properties of Laguerre polynomials. They derived identities by using the orthogonality of Laguerre polynomials w.r.t the relevant inner product. Marinkovic et al. [13] have demonstrated the theory of deformed Laguerre derivative defined by iterated deformed Laguerre operator. Nowak et al. [14] convolution type Laguerre function expansions in order to prove the standard estimates has developed a technique. Khan and Habibullah [15] have introduced $A_{2,n}(x) = {}_2F_2(-n/2, (-n+1/2)\,; 1/2, 1\,; x^2)$.

Khan and Kalim [16] have introduced

$$A_{3,m}^{(\alpha)}(y) = \frac{(1+\alpha)_m}{m!}\,{}_3F_3\left(\frac{-m}{3}, \frac{-m+1}{3}, \frac{-m+2}{3}\,; \frac{1+\alpha}{3}, \frac{2+\alpha}{3}, \frac{3+\alpha}{3}\,; y^3\right). \tag{5}$$

Doha et al. [17] modified generalized Laguerre expansion coefficients of the derivatives of a function in terms of its original expansion coefficients, and an explicit expression for the derivatives of modified generalized Laguerre polynomials of any degree and for any order as a linear combination of modified generalized Laguerre polynomials themselves is also deduced.

Dattoli et al. [18] applied operational techniques to introduce suitable families of special functions. Andrews et al. [19], Trickovic and Stankovic [20], Radulescu [21], and Doha and Youssri [22] have done a lot of work for properties of Laguerre polynomials. Akbary et al. [23] can be referred for other applications of Laguerre polynomials. Li [24], Aksoy et al. [25], Wang [26], and Krasikov and Zarkh [27] have studied problems of permutation of polynomials, bijections that can induce polynomials with integer coefficients is modulo m.

We organize our manuscript as: we present the properties and applications of extended polynomials in Section 2. We give the extended Laguerre polynomials in Section 3. We discuss the generating functions in Section 4. We present the recurrence relations in Section 5. We give the differential equations in Section 6. We discuss the Rodrigues formula in Section 7. We give the special properties in Section 8. We present some other generating functions in Section 9. We give the expansion of the polynomials in Section 10. We present the conclusion in the last section.

2. Extended Polynomial Properties and Application Elementary Results

Das [28] has modified the work of Al-Salam [29]. Carlitz [30] has given a generating function and an explicit polynomial expression for the polynomial $Y_n^c(x\,;k)$, a variant of Laguerre polynomials. Srivastava [31] has derived the several bilinear generating functions by using generalized hypergeometric functions. Explicitly, we can mention [Erdélyi p, 190] [32].

$$D^m\left[x^{\alpha+m}L_n^{(\alpha+m)}(x)\right] = \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)} x^\alpha L_n^{(\alpha)}(x), D = \frac{d}{dx}. \tag{6}$$

One generalization of Laguerre polynomials is $R_n(a,x)$ as Shively defined it by [Rainville p, 298] [33].

$$R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n}\,{}_1F_1(-n\,;a+n\,;x), \tag{7}$$

he used these results as an integral equation involving Shively's polynomials. Karande and Thakare [34] have derived the generating functions, bilinear generating functions, and recurrence relations by using the biorthogonal set of Konhausar. Panda [35] has studied a new generalization based on several known polynomials systems belonging to the families of the classical Jacobi, Hermite, and Laguerre polynomials. Parashar [36] has introduced a new set of Laguerre polynomials $L_n^{(\alpha,h)}(x)$ related to the Laguerre polynomials $L_n^{(\alpha)}(x)$. Sharma and Chongdar [37] have proved an extension of bilateral generating functions of the modified Laguerre polynomials.

Lemma 1. *If $j \in \mathbb{Z}^+$ and n is any nonnegative integer, then*

$$\left(\frac{-n}{q}\right)_j \left(\frac{-n+1}{q}\right)_j \cdots \left(\frac{-n+q-1}{q}\right)_j = (-1)^{qj}\frac{n!}{q^{qj}(n-qj)!}. \tag{8}$$

Proof.

$$\begin{aligned}
&\left(\frac{-n}{q}\right)_j \left(\frac{-n+1}{q}\right)_j \cdots \left(\frac{-n+q-1}{q}\right)_j = \left(\frac{-n}{q}\right)\left(\frac{-n}{q}+1\right)\left(\frac{-n}{q}+2\right)\cdots\left(\frac{-n}{q}+j-1\right),\\
&\left(\frac{-n+1}{q}\right)\left(\frac{-n+1}{q}+1\right)\left(\frac{-n+1}{q}+2\right)\cdots\left(\frac{-n+1}{q}+j-1\right),\\
&\cdots\cdots\cdots\\
&\left(\frac{-n+q-1}{q}\right)\left(\frac{-n+q-1}{q}+1\right)\left(\frac{-n+q-1}{q}+2\right)\cdots\left(\frac{-n+q-1}{q}+j-1\right) = \left(\frac{-n}{q}\right)\left(\frac{-n+q}{q}\right)\left(\frac{-n+2q}{q}\right)\cdots\left(\frac{-n+qj-q}{q}\right),\\
&\left(\frac{-n+1}{q}\right)\left(\frac{-n+q+1}{q}\right)\left(\frac{-n+2q+1}{q}\right)\cdots\left(\frac{-n+qj-q+1}{q}\right),\\
&\cdots\cdots\cdots\\
&\left(\frac{-n+q-1}{q}\right)\left(\frac{-n+2q-1}{q}\right)\left(\frac{-n+3q-1}{q}\right)\cdots\left(\frac{-n+qj-1}{q}\right) = (-1)^{qj}\frac{n!}{qj^{qj}(n-qj)!}.
\end{aligned} \tag{9}$$

Lemma 2. *If $k \in \mathbb{Z}^+$ and n is any nonnegative integer, thus*

$$(\alpha)_{kn} = k^{kn}\left(\frac{\alpha}{k}\right)_n \left(\frac{\alpha+1}{k}\right)_n \cdots \left(\frac{\alpha+k-1}{k}\right)_n, \tag{10}$$

Rainville [33] (p 22).

Lemma 3. *If $k \in \mathbb{Z}^+$ and n is any nonnegative integer, thus*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k), \tag{11}$$

Rainville [33] (p 57).

Lemma 4. *If $k \in \mathbb{Z}^+$ and n is any nonnegative integer, then*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k), \tag{12}$$

Rainville [33] (p 56).

3. The Extended Laguerre Polynomials $A_{q,n}^{(\alpha)}(x)$

We define the extended Laguerre polynomial set $\left\{A_{q,n}^{(\alpha)}(x)\right\}$ by

$$A_{q,n}^{(\alpha)}(x) = \frac{e^x(q+\alpha)_n}{n!}\,{}_qF_q\left(\begin{matrix}\frac{-n}{q}, \frac{-n+1}{q}, \cdots, \frac{-n+q-1}{q}\,; \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}\end{matrix}\,; x^q\right), \tag{13}$$

where $\alpha \in \mathbb{R}$, $n, q \in \mathbb{Z}^+$.

Theorem 5. *If $\left\{A_{q,n}^{(\alpha)}(x)\right\}$ are the extended Laguerre polynomials, then*

$$A_{q,n}^{(\alpha)}(x) = e^x(q+\alpha)_n \sum_{j=0}^{[n/q]} \frac{(-1)^{qj}}{(n-qj)!(q+\alpha)_{qj}}\frac{(x)^{qj}}{(qj)!}, \alpha \in \mathbb{R}, n \in \mathbb{Z}^+. \tag{14}$$

Proof. Consider

$$A_{q,n}^{(\alpha)}(x) = \frac{e^x (q+\alpha)_n}{n!} {}_qF_q\left(\begin{matrix} \frac{-n}{q}, \frac{-n+1}{q}, \cdots, \frac{-n+q-1}{q}; \\ \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q} \end{matrix} ; x^q\right)$$
$$= \frac{e^x (q+\alpha)_n}{n!} \times \sum_{j=0}^{\left[\frac{n}{q}\right]} \left\{ \frac{(-n/q)_j (-n+1/q)_j \cdots (-n+q-1/q)_j}{(q+\alpha/q)_j (q+1+\alpha/q)_j \cdots (2q+\alpha-1/q)_j} \right\} \frac{(x)^{qj}}{(qj)!}. \tag{15}$$

By using Lemma 1

$$A_{q,n}^{(\alpha)}(x) = \frac{e^x (q+\alpha)_n}{n!} \times \sum_{j=0}^{\left[\frac{n}{q}\right]} \left[\frac{(-1)^{qj} n!}{q^{qj}(n-qj)!(q+\alpha/q)_j (q+1+\alpha/q)_j \cdots (2q+\alpha-1/q)_j} \right] \frac{(x)^{qj}}{(qj)!}. \tag{16}$$

Then from Lemma 2, we have

$$A_{q,n}^{(\alpha)}(x) = e^x (q+\alpha)_n \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(-1)^{qj}}{(n-qj)!(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!}. \tag{17}$$

□

4. Generating Functions

The following theorem formulates a generating function for the extended Laguerre polynomials $A_{q,n}^{(\alpha)}(x)$.

Theorem 6. *If $n, j \in \mathbb{Z}^+$, then*

$$\sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(-1)^{qj} e^x t^n}{(n-qj)!(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!} = e^{x+t} {}_0F_q\left(--; \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \left(\frac{-xt}{q}\right)^q\right). \tag{18}$$

Proof. By using Lemma 3, we acquire

$$\sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(-1)^{qj} e^x t^n}{(n-qj)!(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{qj} e^x t^{n+qj}}{n!(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!}$$
$$= e^x \left[\sum_{n=0}^{\infty} \frac{t^n}{n!}\right] \left[\sum_{j=0}^{\infty} \frac{(-1)^{qk} t^{qj}}{(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!}\right]$$
$$= e^{x+t} \sum_{j=0}^{\infty} \frac{(-xt)^{qj}}{(q+\alpha)_{qj}(qj)!}. \tag{19}$$

By using Lemma 2, we acquire

$$\sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(-1)^{qj} e^x t^n}{(n-qj)!(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!}$$
$$= e^{x+t} \sum_{j=0}^{\infty} \frac{(-xt)^{qj}}{q^{qj}(q+\alpha/q)_j (q+1+\alpha/q)_j \cdots (2q+\alpha-1/q)_j (qj)!}$$
$$= e^{x+t} {}_0F_q\left(--; \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \left(\frac{-xt}{q}\right)^q\right). \tag{20}$$

□

Corollary 7. *If $\alpha \in \mathbb{R}$ and $n, q, j \in \mathbb{Z}^+$, then*

$$\sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(x) t^n}{(q+\alpha)_n} = e^{x+t} {}_0F_q\left(\begin{matrix} --; \\ \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \end{matrix} \left(\frac{-xt}{q}\right)^q\right). \tag{21}$$

Proof. From Equation (14), we acquire

$$\sum_{n=0}^{\infty} \left[\frac{A_{q,n}^{(\alpha)}(x)}{(q+\alpha)_n}\right] t^n = \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\left[\frac{n}{q}\right]} \left[\frac{(-1)^{qj}}{(n-qj)!(q+\alpha)_{qj}}\right] \frac{(x)^{qj}}{(qj)!}\right] t^n. \tag{22}$$

A use of Theorem (18), therefore, shows that the extended Laguerre polynomials have the generating function given by

$$\sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(x) t^n}{(q+\alpha)_n} = e^{x+t} {}_0F_q\left(\begin{matrix} --; \\ \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \end{matrix} \left(\frac{-xt}{q}\right)^q\right). \tag{23}$$

□

Theorem 8. *If $c \in \mathbb{Z}^+$, then*

$$\sum_{n=0}^{\infty}\frac{(c)_n A_{q,n}^{(\alpha)}(x)t^n}{(q+\alpha)_n}=\frac{e^x}{(1-t)^c}{}_qF_q\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{-xt}{1-t}\right)^q\right). \tag{24}$$

Proof. From Equation (22), we note that

$$\sum_{n=0}^{\infty}(c)_n\left[\frac{A_{q,n}^{(\alpha)}(x)}{(q+\alpha)_n}\right]t^n=\sum_{n=0}^{\infty}(c)_n e^x\left[\sum_{j=0}^{\left[\frac{n}{q}\right]}\left[\frac{(-1)^{qj}}{(n-qj)!(q+\alpha)_{qj}}\right]\frac{(x)^{qj}}{(qj)!}\right]t^n. \tag{25}$$

By using Lemma 3, we acquire

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(c)_n A_{q,n}^{(\alpha)}(x)t^n}{(q+\alpha)_n}&=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(c)_{n+qj}e^x t^{n+qj}}{n!}\frac{(-1)^{qj}(x)^{qj}}{(q+\alpha)_{qj}(qj)!}\\&=\sum_{j=0}^{\infty}\left[\sum_{n=0}^{\infty}\frac{(c+qj)_n t^n}{n!}\right]\left[\frac{(c)_{qj}}{(q+\alpha)_{qj}}\right]\frac{e^x(-xt)^{qj}}{(qj)!}.\end{aligned} \tag{26}$$

Since $(c)_{n+qj}=(c+qj)_n(c)_{qj}$ and $(1-t)^{-m}=\sum_{n=0}^{\infty}(m)_n t^n/n!$, it thus implies that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(c)_n A_{q,n}^{(\alpha)}(x)t^n}{(q+\alpha)_n}&=\sum_{j=0}^{\infty}\left[\frac{(c)_{qj}}{\left[(1-t)^{c+qj}\right](q+\alpha)_{qj}}\right]\frac{e^x(-xt)^{qj}}{(qj)!}\\&=\frac{e^x}{(1-t)^c}\sum_{k=0}^{\infty}\left[\frac{(c)_{qj}}{(q+\alpha)_{qj}}\right]\frac{1}{(qj)!}\left(\frac{-xt}{1-t}\right)^{qj}.\end{aligned} \tag{27}$$

By using Lemma 2, we consequently obtain the required result

$$\sum_{n=0}^{\infty}\frac{(c)_n A_{q,n}^{(\alpha)}(x)t^n}{(q+\alpha)_n}=\frac{e^x}{(1-t)^c}{}_qF_q\cdot\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{-xt}{1-t}\right)^q\right). \tag{28}$$

□

Corollary 9. *If $\alpha\in\mathbb{R}$ and $n,m,j\in\mathbb{Z}^+$, then*

$$\sum_{n=0}^{\infty}A_{q,n}^{(\alpha)}(x)t^n=\frac{1}{(1-t)^{q+\alpha}}\exp\left(\frac{x-2xt}{1-t}\right). \tag{29}$$

Proof. Put $c=q+\alpha$ in Equation (24), we obtain our desired result. □

5. Recurrence Relations

We describe the recurrence relations for the extended Laguerre polynomials $A_{q,n}^{(\alpha)}(x)$.

Theorem 10. *If $\alpha\in\mathbb{R}$ and $n,j\in\mathbb{Z}^+$, then*

$$xDA_{q,n}^{(\alpha)}(x)=(n+x)A_{q,n}^{(\alpha)}(x)-(q+\alpha+n-1)A_{q,n-1}^{(\alpha)}(x),\ D=\frac{d}{dx}. \tag{30}$$

Proof. From Equation (18)

$$\sum_{n=0}^{\infty}\frac{A_{q,n}^{(\alpha)}(x)t^n}{(q+\alpha)_n}=e^{x+t}{}_0F_q\left(\begin{matrix}--; \\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{-xt}{q}\right)^q\right). \tag{31}$$

Let $\sigma_{q,n}(x)=A_{q,n}^{(\alpha)}(x)/(q+\alpha)_n$.

Suppose that

$${}_0F_q\left(\begin{matrix}--; \\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{-xt}{q}\right)^q\right)=\psi\left(\frac{x^q t^q}{q}\right). \tag{32}$$

$$\text{Then } F=e^{x+t}\psi\left(\frac{x^q t^q}{q}\right)=\sum_{n=0}^{\infty}\sigma_{q,n}(x)t^n, \tag{33}$$

provide that the series is uniformly convergent. By taking partial derivatives,

$$\frac{\partial F}{\partial x}=e^{x+t}\psi+x^{q-1}t^q e^{x+t}\psi', \tag{34}$$

$$\frac{\partial F}{\partial t}=e^{x+t}\psi+x^q t^{q-1}e^{x+t}\psi', \tag{35}$$

$$x\frac{\partial F}{\partial x}-t\frac{\partial F}{\partial t}=xF-tF. \tag{36}$$

Now, since $F=\sum_{n=0}^{\infty}\sigma_{q,n}(x)t^n$, therefore,

$$\frac{\partial F}{\partial x}=\sum_{n=0}^{\infty}\sigma'_{q,n}(x)t^n \quad\text{and}\quad t\frac{\partial F}{\partial t}=\sum_{n=0}^{\infty}n\sigma_{q,n}(x)t^n. \tag{37}$$

Equation (36) then yields

$$\begin{aligned}x\sum_{n=0}^{\infty}\sigma'_{q,n}(x)t^n-\sum_{n=0}^{\infty}n\sigma_{q,n}(x)t^n&=x\sum_{n=0}^{\infty}\sigma_{q,n}(x)t^n-\sum_{n=0}^{\infty}\sigma_{q,n}(x)t^{n+1}\\&=x\sum_{n=0}^{\infty}\sigma_{q,n}(x)t^n-\sum_{n=1}^{\infty}\sigma_{q,n-1}(x)t^n.\end{aligned} \tag{38}$$

We get $\sigma'_{2,0}(x)=0$, and for $n>1$,

$$x\sigma'_{q,n}(x)-n\sigma_{q,n}(x)=x\sigma_{q,n}(x)-\sigma_{q,n-1}(x). \tag{39}$$

This implies that

$$xDA_{q,n}^{(\alpha)}(x)=(n+x)A_{q,n}^{(\alpha)}(x)-(q+\alpha+n-1)A_{q,n-1}^{(\alpha)}(x). \tag{40}$$

□

Theorem 11. *If $\alpha\in\mathbb{R}$ and $n\geq 2$ then*

$$DA_{q,n}^{(\alpha)}(x)=DA_{q,n-1}^{(\alpha)}(x)+A_{q,n}^{(\alpha)}(x)-2A_{q,n-1}^{(\alpha)}(x). \tag{41}$$

Proof. From Equation (29), we get the following

$$(1-t)^{-q-\alpha}\exp\left[x\left(\frac{1-2t}{1-t}\right)\right]=\sum_{n=0}^{\infty}A_{q,n}^{(\alpha)}(x)t^n. \tag{42}$$

$$\text{Let } F=A(t)\exp\left[x\left(\frac{1-2t}{1-t}\right)\right]=\sum_{n=0}^{\infty}y_{q,n}(x)t^n, \tag{43}$$

$$\frac{\partial F}{\partial x}=\left(\frac{1-2t}{1-t}\right)A(t)\exp\left[x\left(\frac{1-2t}{1-t}\right)\right], \tag{44}$$

$$(1-t)\frac{\partial F}{\partial x}=(1-2t)A(t)\exp\left[x\left(\frac{1-2t}{1-t}\right)\right]. \tag{45}$$

By using Equation (42), we obtain

$$(1-t)\frac{\partial F}{\partial x}=(1-2t)F. \tag{46}$$

$$\text{Since } F=\sum_{n=0}^{\infty}y_{q,n}(x)t^n, \text{ therefore we have } \frac{\partial F}{\partial x}=\sum_{n=0}^{\infty}y'_{q,n}(x)t^n. \tag{47}$$

Equation (46) can be expressed as

$$\sum_{n=0}^{\infty}y'_{q,n}(x)t^n-\sum_{n=0}^{\infty}y'_{q,n}(x)t^{n+1}=\sum_{n=0}^{\infty}y_{q,n}(x)t^n-2\sum_{n=0}^{\infty}y_{q,n}(x)t^{n+1}, \tag{48}$$

$$\sum_{n=0}^{\infty}y'_{q,n}(x)t^n-\sum_{n=1}^{\infty}y'_{q,n-1}(x)t^n=\sum_{n=0}^{\infty}y_{q,n}(x)t^n-2\sum_{n=1}^{\infty}y_{q,n-1}(x)t^n. \tag{49}$$

We reach $y'_{q,0}(x)=0$, $y'_{q,1}(x)=0$ and for $n>2$,

$$DA_{q,n}^{(\alpha)}(x)=DA_{q,n-1}^{(\alpha)}(x)+A_{q,n}^{(\alpha)}(x)-2A_{q,n-1}^{(\alpha)}(x). \tag{50}$$

□

Theorem 12. *If $\alpha \in \mathbb{R}$ and $n \geq q$, then*

$$DA_{q,n}^{(\alpha)}(x) = A_{q,n}^{(\alpha)}(x) - \sum_{j=0}^{n-1} A_{q,j}^{(\alpha)}(x). \quad (51)$$

Proof. Equation (46) can be written as

$$\frac{\partial F}{\partial x} = \left[1 - \frac{t}{1-t}\right] F. \quad (52)$$

By using Equation (42), we obtain

$$\frac{\partial F}{\partial x} = \left[1 - \frac{t}{1-t}\right] \sum_{n=0}^{\infty} y_{q,n}(x) t^n. \quad (53)$$

By using Equation (47), we obtain.

$$\begin{aligned} \sum_{n=0}^{\infty} y'_{q,n}(x) t^n &= \sum_{n=0}^{\infty} y_{q,n}(x) t^n - \left[\sum_{n=0}^{\infty} t^{n+1}\right] \left[\sum_{n=0}^{\infty} y_{q,n}(x) t^n\right] \\ &= \sum_{n=0}^{\infty} y_{q,n}(x) t^n - \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} y_{q,j}(x) t^j t^{n+1}. \end{aligned} \quad (54)$$

Since $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k, n-k)$, (Rainville [33], (p 56)).

$$\begin{aligned} \sum_{n=0}^{\infty} y'_{q,n}(x) t^n &= \sum_{n=0}^{\infty} y_{q,n}(x) t^n - \sum_{n=0}^{\infty} \sum_{j=0}^{n} y_{q,j}(x) t^{n+1} \\ &= \sum_{n=0}^{\infty} y_{q,n}(x) t^n - \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} y_{q,j}(x) t^n. \end{aligned} \quad (55)$$

It follows that $y'_{q,0}(x) = 0, y'_{q,1}(x) = 0$ and for $n > q, y'_{q,n}(x) = y_{q,n}(x) - \sum_{j=0}^{n-1} y_{q,j}(x)$, and $DA_{q,n}^{(\alpha)}(x) = A_{q,n}^{(\alpha)}(x) - \sum_{j=0}^{n-1} A_{q,j}^{(\alpha)}(x)$. □

Theorem 13. *If $\alpha \in \mathbb{R}$ and $n \geq q + 1$, then*

$$nA_{q,n}^{(\alpha)}(x) = (3x - q - \alpha) A_{q,n-1}^{(\alpha)}(x) - (q + \alpha + n - 2) A_{q,n-2}^{(\alpha)}(x). \quad (56)$$

□

6. Differential Equation

Since the Extended Laguerre polynomial is a constant multiple of hypergeometric functions ${}_qF_q$, we may obtain the differential equation.

Proof. We can have the following equation after eliminating the derivatives from Equations (30) and (41).

$$\begin{aligned} 0 &= nA_{q,n}^{(\alpha)}(x) - xDA_{q,n-1}^{(\alpha)}(x) \\ &\quad + (2x - q - \alpha - n + 1) A_{q,n-1}^{(\alpha)}(x) nA_{q,n}^{(\alpha)}(x) \\ &= xDA_{q,n-1}^{(\alpha)}(x) - (2x - q - \alpha - n + 1) A_{q,n-1}^{(\alpha)}(x). \end{aligned} \quad (57)$$

Now, by using Equation (30), we finally have

$$\begin{aligned} nA_{q,n}^{(\alpha)}(x) &= (n - 1 + x) A_{q,n-1}^{(\alpha)}(x) \\ &\quad - (q + \alpha + n - 2) A_{q,n-2}^{(\alpha)}(x) \\ &\quad + (2x - q - \alpha - n + 1) A_{q,n-1}^{(\alpha)}(x), \end{aligned} \quad (58)$$

$$nA_{q,n}^{(\alpha)}(x) = (3x - q - \alpha) A_{q,n-1}^{(\alpha)}(x) - (q + \alpha + n - 2) A_{q,n-2}^{(\alpha)}(x). \quad (59)$$

□

Theorem 14. *If $\alpha \in \mathbb{R}$ and $n, q, j \in \mathbb{Z}^+$, then*

$$A_{q,n-1}^{(1+\alpha)}(x) + A_{q,n}^{(\alpha)}(x) = A_{q,n}^{(1+\alpha)}(x).. \quad (60)$$

Proof. From Equation (14), we obtain

$$A_{q,n-1}^{(1+\alpha)}(x) = e^x (q + 1 + \alpha)_{n-1} \sum_{j=0}^{\left[\frac{n-1}{q}\right]} \frac{(-1)^{qj}}{(n - 1 - qj)!(q + 1 + \alpha)_{qj}} \frac{x^{qj}}{(qj)!}, \quad (61)$$

so that $A_{q,n}^{(\alpha)}(x) = e^x (q + \alpha)_n \sum_{j=0}^{[n/q]} ((-1)^{qj}/(n - qj)!(q + \alpha)_{qj}) (x^{qj}/(qj)!)$.

By adding the above equations, we get

Theorem 15. *If $\alpha \in \mathbb{R}$ and $n \geq q$, then*

$$\begin{aligned} &xD^2 A_{q,n}^{(\alpha)}(x) + (q + \alpha - 3x) DA_{q,n}^{(\alpha)}(x) \\ &\quad + (2x + n - q - \alpha) A_{q,n}^{(\alpha)}(x) = 0. \end{aligned} \quad (63)$$

Proof. By taking partial derivatives of Equation (30), we have

$$xD^2A_{q2,n}^{(\alpha)}(x) + DA_{q,n}^{(\alpha)}(x) = (n+x)DA_{q,n}^{(\alpha)}(x) + A_{q,n}^{(\alpha)}(x) - (q+\alpha+n-1)DA_{q,n-1}^{(\alpha)}(x). \tag{64}$$

By using Equation (41), we have

$$xD^2A_{q,n}^{(\alpha)}(x) + DA_{q,n}^{(\alpha)}(x) = (n+x)DA_{q,n}^{(\alpha)}(x) + A_{q,n}^{(\alpha)}(x) - (q+\alpha+n-1)\left[DA_{q,n}^{(\alpha)}(x) - A_{q,n}^{(\alpha)}(x) + 2A_{q,n-1}^{(\alpha)}(x)\right], \tag{65}$$

or

$$xD^2A_{q,n}^{(\alpha)}(x) + (q+\alpha-x)DA_{q,n}^{(\alpha)}(x) = (q+\alpha+n)A_{q,n}^{(\alpha)}(x) - 2(q+\alpha+n-1)A_{q,n-1}^{(\alpha)}(x). \tag{66}$$

By using Equation (30), we have

$$\begin{aligned}
A_{q,n-1}^{(1+a)}(x) + A_{q,n}^{(a)}(x) &= e^x(q+1+\alpha)_{n-1}\sum_{j=0}^{\left[\frac{n-1}{q}\right]}\frac{(-1)^{qj}}{(n-1-qj)!(q+1+\alpha)_{qj}}\frac{x^{qj}}{(qj)!} + e^x(q+\alpha)_n\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{(-1)^{qj}}{(n-qj)!(q+\alpha)_{qj}}\frac{x^{qj}}{(qj)!}\\
&= e^x\left[\sum_{j=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n-1)!(-1)^{qj}}{(n-1-qj)!(q+\alpha+qj)!}\frac{x^{qj}}{(qj)!} + \sum_{k=0}^{\left[\frac{n}{q}\right]}\frac{(q+\alpha+n-1)!(-1)^{qj}}{(n-qj)!(q+\alpha+qj-1)!}\frac{x^{qj}}{(qj)!}\right]\\
&= e^x\left[\sum_{k=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n-1)!(-1)^{qj}}{(n-1-qj)!(q+\alpha+qj)!}\frac{x^{qj}}{(qj)!} + \sum_{j=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n-1)!(-1)^{qj}}{(n-qj)!(q+\alpha+qj-1)!}\frac{x^{qj}}{(qj)!} + \frac{x^{qn}}{(qn)!}\right]\\
&= e^x\left[\sum_{k=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n-1)!x^{qj}(-1)^{qj}}{(qj)!}\left\{\frac{1}{(n-1-qj)!(q+\alpha+qj)!} + \frac{1}{(n-qj)!(q+\alpha+qj-1)!}\right\} + \frac{x^{qn}}{(qn)!}\right]\\
&= e^x\left[\sum_{k=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n-1)!(-1)^{qj}}{(n-qj)!(q+\alpha+qj)!}\{q+\alpha+n\}\frac{x^{qj}}{(qj)!} + \frac{x^{qn}}{(qn)!}\right]\\
&= e^x\left[\sum_{j=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n)!(-1)^{qj}}{(n-qj)!(q+\alpha+qj)!}\frac{x^{qj}}{(qj)!} + \frac{x^{qn}}{(qn)!}\right] = e^x(q+1+\alpha)_n\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{(-1)^{qj}}{(n-qj)!(q+1+\alpha)_{qj}}\frac{x^{qj}}{(qj)!} = A_{q,n}^{(1+\alpha)}(x).
\end{aligned} \tag{62}$$

$$xD^2A_{q,n}^{(\alpha)}(x) + (q+\alpha-x)DA_{q,n}^{(\alpha)}(x) = (q+\alpha+n)A_{q,n}^{(\alpha)}(x) + 2xDA_{q,n}^{(\alpha)}(x) - 2(n+x)DA_{q,n}^{(\alpha)}(x), \tag{67}$$

or

$$xD^2A_{q,n}^{(\alpha)}(x) + (q+\alpha-3x)DA_{q,n}^{(\alpha)}(x) + (2x+n-q-\alpha)A_{q,n}^{(\alpha)}(x) = 0. \tag{68}$$

□

7. Rodrigues Formula

The Rodrigues formula for the Laguerre polynomials is presented as

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}D^n\left(x^{\alpha+n}e^{-x}\right), \tag{69}$$

but we intend to extend this Rodrigues formula.

Theorem 16. *If $\alpha \in \mathbb{R}$ and $n, j \in \mathbb{Z}^+$, then*

$$A_{q,n}^{(\alpha)}(x) = \frac{x^{-(q-1)-\alpha} e^{2x}}{n!} D^n \left(x^{(q-1)+\alpha+n} e^{-x} \right). \tag{70}$$

Proof. Consider the extended Laguerre polynomials involving ${}_qF_q, q > 2$

$$A_{q,n}^{(\alpha)}(x) = \frac{e^x (q+\alpha)_n}{n!} {}_qF_q \left(\begin{array}{c} \frac{-n}{q}, \frac{-n+1}{q}, \cdots, \frac{-n+q-1}{q}; \\ \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \end{array} x^q \right). \tag{71}$$

By Theorem (14), we have

$$\begin{aligned} A_{q,n}^{(\alpha)}(x) &= \frac{e^x}{n!} \sum_{j=0}^{\left[\frac{n}{q}\right]} \left[\frac{n!}{(n-qj)!(qj)!} \right] \frac{(q+\alpha)_n x^{qj}}{(q+\alpha)_{qj}} \\ &= \frac{e^x x^{-(q-1)-\alpha}}{n!} \sum_{j=0}^{\left[\frac{n}{q}\right]} \left[\frac{(-1)^{qj} n!}{(n-qj)!(qj)!} \right] \frac{(q+m)_n x^{qj+\alpha+(q-1)}}{(q+m)_{qj}}. \end{aligned} \tag{72}$$

Since $D^{n-qj}(x^{n+\alpha+(q-1)}) = (q+\alpha)_n x^{qj+\alpha+(q-1)} / (q+\alpha)_{qj}$, therefore, we write it as

$$\begin{aligned} A_{q,n}^{(\alpha)}(x) &= \frac{x^{-(q-1)-\alpha} e^{2x}}{n!} \sum_{j=0}^{\left[\frac{n}{q}\right]} \left[\frac{n!}{(n-qj)!(qj)!} \right] \left[(-1)^{qj} e^{-x} \right] \\ &\cdot \left[D^{n-qj} \left(x^{n+\alpha+(q-1)} \right) \right] = \frac{x^{-(q-1)-\alpha} e^{2x}}{n!} \sum_{j=0}^{\left[\frac{n}{q}\right]^n} C_{qj} D^{n-qj} \\ &\cdot \left(x^{n+\alpha+(q-1)} \right) D^{qj} (e^{-x}). \end{aligned} \tag{73}$$

Lastly, we use the Leibnitz formula for the nth derivative to obtain the following

$$A_{q,n}^{(\alpha)}(x) = \frac{x^{-(q-1)-\alpha} e^{2x}}{n!} D^n \left(x^{(q-1)+\alpha+n} e^{-x} \right). \tag{74}$$

□

8. Special Properties

In this section, we determine the special features of the extended Laguerre polynomials $A_{q,n}^{(\alpha)}(x)$.

Theorem 17. *If $\alpha, \beta \in \mathbb{R}$ and $n, j, q \in \mathbb{Z}^+$, then*

$$A_{q,n}^{(\alpha)}(x) = \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(\alpha-\beta)_{qj} A_{q,n-qj}^{(\beta)}(x)}{(qj)!}. \tag{75}$$

Proof. From Equation (29)

$$\sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x) t^n = \frac{1}{(1-t)^{q+\alpha}} \exp\left(x \left(\frac{1-2t}{1-t} \right) \right). \tag{76}$$

Also, consider

$$\begin{aligned} \frac{1}{(1-t)^{q+\alpha}} \exp\left(x \left(\frac{1-2t}{1-t} \right) \right) &= (1-t)^{-(\alpha-\beta)} (1-t)^{-q-\beta} \exp \\ &\cdot \left(x \left(\frac{1-2t}{1-t} \right) \right), \end{aligned} \tag{77}$$

$$\begin{aligned} \sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x) t^n &= (1-t)^{-(\alpha-\beta)} \sum_{n=0}^{\infty} A_{q,n}^{(\beta)}(x) t^n \\ &= \sum_{n=0}^{\infty} \frac{(\alpha-\beta)_{qn} t^{qn}}{(qn)!} \sum_{n=0}^{\infty} A_{q,n}^{(\beta)}(x) t^n \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha-\beta)_{qj} t^{qj} A_{q,n}^{(\beta)}(x) t^n}{(qj)!}. \end{aligned} \tag{78}$$

By utilizing Lemma 4, we acquire

$$\begin{aligned} \sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x) t^n &= \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(\alpha-\beta)_{qj} t^{qj} A_{q,n-qj}^{(\beta)}(x) t^{n-qj}}{(qj)!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(\alpha-\beta)_{qj} A_{q,n-qj}^{(\beta)}(x) t^n}{(qj)!}. \end{aligned} \tag{79}$$

On comparing the coefficients of t^n, we acquire

$$A_{q,n}^{(\alpha)}(x) = \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(\alpha-\beta)_{qj} A_{q,n-qj}^{(\beta)}(x)}{(qj)!}. \tag{80}$$

□

Theorem 18. *If $\alpha \in \mathbb{R}$ and $n, j \in \mathbb{Z}^+$, then*

$$A_{q,n}^{(\alpha+\beta+q)}(x+y) = \sum_{j=0}^{\left[\frac{n}{q}\right]} A_{q,n-qj}^{(\beta)}(y) A_{q,qj}^{(\alpha)}(x). \tag{81}$$

Proof. Consider

$$(1-t)^{-q-\alpha}\exp\left(x\left(\frac{1-2t}{1-t}\right)\right)(1-t)^{-q-\beta}\exp\left(y\left(\frac{1-2t}{1-t}\right)\right) = (1-t)^{-q-(\alpha+\beta+q)}\exp\left\{(x+y)\left(\frac{1-2t}{1-t}\right)\right\}. \tag{82}$$

By using Equation (75), we acquire

$$\sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x)t^n \sum_{n=0}^{\infty} A_{q,n}^{(\beta)}(y)t^n = \sum_{n=0}^{\infty} A_{q,n}^{(\alpha+\beta+q)}(x+y)t^n, \tag{83}$$

$$\sum_{n=0}^{\infty} A_{q,n}^{(\alpha+\beta+q)}(x+y)t^n = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} A_{q,n}^{(\beta)}(y)t^n A_{q,qj}^{(\alpha)}(x)t^{qj}. \tag{84}$$

By using Lemma 4, we acquire

$$\sum_{n=0}^{\infty} A_{q,n}^{(\alpha+\beta+q)}(x+y)t^n = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]} A_{q,n-qj}^{(\beta)}(y) A_{q,qj}^{(\alpha)}(x)t^n. \tag{85}$$

On comparing the coefficients of t^n, we acquire

$$A_{q,n}^{(\alpha+\beta+q)}(x+y) = \sum_{j=0}^{\left[\frac{n}{q}\right]} A_{q,n-qj}^{(\beta)}(y) A_{q,qj}^{(\alpha)}(x). \tag{86}$$

□

Theorem 19. *If $\alpha \in \mathbb{R}$ and $n, j \in \mathbb{Z}^+$, then*

$$A_{q,n}^{(\alpha)}(xy) = \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(q+\alpha)_n A_{q,qj}^{(\alpha)}(x^q)y^{qj}}{(q+\alpha)_{qj}} \frac{(1-y)^{n-qj}}{(n-qj)!}. \tag{87}$$

Proof. Consider

$$e^{x+t}{}_0F_q\left(\begin{matrix} --; \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \dots, \frac{2q+\alpha-1}{q}; \end{matrix} \left(\frac{-xyt}{q}\right)^q\right) = e^{(1-y)t}e^{x+yt}{}_0F_q\left(\begin{matrix} --; \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \dots, \frac{2q+\alpha-1}{q}; \end{matrix} \left(\frac{-xyt}{q}\right)^q\right). \tag{88}$$

By using Equation (21), we get

$$\sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(xy)t^n}{(q+\alpha)_n} = \sum_{n=0}^{\infty} \frac{(1-y)^n t^n}{n!} \sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(x)y^n t^n}{(q+\alpha)_n} = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{A_{q,qj}^{(\alpha)}(x)y^{qj}t^{qj}}{(q+\alpha)_{qj}} \frac{(1-y)^n t^n}{n!}. \tag{89}$$

By using Lemma 4, we acquire

$$\sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(xy)t^n}{(q+\alpha)_n} = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{A_{q,qj}^{(\alpha)}(x)y^{qj}t^{qj}}{(q+\alpha)_{qj}} \frac{(1-y)^{n-qj}t^{n-qj}}{(n-qj)!} = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{A_{q,qj}^{(\alpha)}(x)y^{qj}}{(q+\alpha)_{qj}} \frac{(1-y)^{n-qj}t^n}{(n-qj)!}. \tag{90}$$

On comparing the coefficients of t^n, we get

$$A_{q,n}^{(\alpha)}(xy) = \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(q+\alpha)_n A_{q,qj}^{(\alpha)}(x^q)y^{qj}}{(q+\alpha)_{qj}} \frac{(1-y)^{n-qj}}{(n-qj)!}. \tag{91}$$

□

Theorem 20. *If $\alpha \in \mathbb{R}$ and $n, j, q \in \mathbb{Z}^+$, then*

$$\sum_{n=0}^{\infty} \frac{(n+qj)! A_{q,n+qj}^{(\alpha)}(x)t^n}{(qj)!n!} = (1-t)^{-q-\alpha-qj}\exp\left(\frac{-xt}{1-t}\right) A_{q,j}^{(\alpha)}\left(\frac{x}{1-t}\right). \tag{92}$$

Proof. Consider the series

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n+qj)! A_{q,n+qj}^{(\alpha)}(x)t^n y^{qj}}{(qj)!n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{n! A_{q,n}^{(\alpha)}(x)t^{n-qj}y^{qj}}{(qj)!(n-qj)!} = \sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x) \sum_{j=0}^{\left[\frac{n}{q}\right]} {}^{n}C_{qj} t^{n-qj}y^{qj} = \sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x)(t+y)^n = (1-t-y)^{-q-\alpha}\exp\left(\frac{x(1-2y-2t)}{1-y-t}\right). \tag{93}$$

Since $(1-t-y)^{-q-\alpha} = (1-t)^{-q-\alpha}(1-y/1-t)^{-q-\alpha}$

$$\exp\left(\frac{x(1-2y-2t)}{1-t}\right) = \exp(x)\exp\left(\frac{-x(y+t)}{(1-t-y)}\right) = \exp(x)\exp\left(\frac{-xt}{1-t}\right)\exp \cdot \left(\frac{(-x/1-t)(y/1-t)}{(1-y/1-t)}\right). \tag{94}$$

Therefore, Equation (93) becomes

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+qj)!A_{q,n+qj}^{(\alpha)}(x^q)t^n y^{qj}}{(qj)!n!} = (1-t)^{-q-\alpha} \cdot \left(1-\frac{y}{1-t}\right)^{-q-\alpha}\exp(x)\exp\left(\frac{-xt}{1-t}\right)\exp \cdot \left(\frac{(-x/1-t)(y/1-t)}{(1-y/1-t)}\right). \tag{95}$$

By using Equation (29), we get

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+qj)!A_{q,n+qj}^{(\alpha)}(x^q)t^n y^{qj}}{(qj)!n!} = (1-t)^{-q-\alpha}\exp \cdot \left(\frac{-xt}{1-t}\right)\sum_{j=0}^{\infty}A_{q,j}^{(\alpha)}\left(\frac{x}{1-t}\right)\left(\frac{y}{1-t}\right)^{qj}. \tag{96}$$

On comparing the coefficients of y^{qj}, we get

$$\sum_{n=0}^{\infty}\frac{(n+qj)!A_{q,n+qj}^{(\alpha)}(x^q)t^n}{(qj)!n!} = (1-t)^{-q-\alpha-qj}\exp \cdot \left(\frac{-xt}{1-t}\right)A_{q,j}^{(\alpha)}\left(\frac{x}{1-t}\right). \tag{97}$$

□

9. Other Generating Functions

In this section, we study some other generating functions.

Theorem 21. *If $\alpha \in \mathbb{R}$ and $n, j, q \in \mathbb{Z}^+$, then*

$$\sum_{j=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n} = (1-t)^{-q-\alpha}\exp\left(\frac{-xt}{1-t}\right)\exp\left(\frac{x-yt}{1-t}\right){}_0F_q \cdot \left(--;\frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q}; \cdot \left(\frac{xyt}{q(1-t)}\right)^q\right). \tag{98}$$

Proof. Consider the series

$$\sum_{n=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n} = \sum_{n=0}^{\infty}\sum_{j=0}^{[n/q]}\frac{n!y^{qj}A_{q,n}^{(\alpha)}(x)(-1)^{qj}t^n}{(qj)!(n-qj)!(q+\alpha)_{qj}}. \tag{99}$$

By using Lemma 3, we get

$$\sum_{n=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n} = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(n+qj)!y^{qj}A_{q,n+qj}^{(\alpha)}(x)(-1)^{qj}t^{n+qj}}{(qj)!n!(q+\alpha)_{qj}}, \tag{100}$$

$$= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+qj)!A_{q,n+qj}^{(\alpha)}(x)t^n}{(qj)!n!}\frac{(-yt)^{qj}}{(q+\alpha)_{qj}}. \tag{101}$$

By using Theorem (92), we get

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n} &= \sum_{j=0}^{\infty}(1-t)^{-q-\alpha-qj}\exp\left(\frac{-xt}{1-t}\right)A_{q,j}^{(\alpha)} \cdot \left(\frac{x}{1-t}\right)\frac{(-yt)^{qj}}{(q+\alpha)_{qj}} = (1-t)^{-q-\alpha}\exp \\ &\cdot \left(\frac{-xt}{1-t}\right)\sum_{j=0}^{\infty}(1-t)^{-qj}A_{q,j}^{(\alpha)} \cdot \left(\frac{x}{1-t}\right)\frac{(-yt)^{qj}}{(q+\alpha)_{qj}} = (1-t)^{-q-\alpha}\exp \\ &\cdot \left(\frac{-xt}{1-t}\right)\times\sum_{j=0}^{\infty}A_{q,j}^{(\alpha)} \cdot \left(\frac{x}{1-t}\right)\frac{(-yt/1-t)^{qj}}{q^{qj}(q+\alpha/q)_j(q+1+\alpha/q)_j\cdots(2q+\alpha-1/q)_j} \\ &= (1-t)^{-q-\alpha}\exp\left(\frac{-xt}{1-t}\right)\sum_{j=0}^{\infty}A_{q,j}^{(\alpha)} \cdot \left(\frac{x}{1-t}\right)\frac{(-yt/q(1-t))^{qj}}{(q+\alpha/q)_j(q+1+\alpha/q)_j\cdots(2q+\alpha-1/q)_j}.\end{aligned} \tag{102}$$

By using Equation (21), we get

$$\sum_{j=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n}=(1-t)^{-q-\alpha}\exp\left(\frac{-xt}{1-t}\right)\exp\left(\frac{x-yt}{1-t}\right){}_0F_q\cdot\left(--;\frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\cdot\left(\frac{xyt}{q(1-t)}\right)^q\right).\tag{103}$$

□

Theorem 22. *If $|t|<1$, $\alpha\in\mathbb{R}$ and $c,n\in\mathbb{Z}^+$, then*

$$(1-t)^{-1-\alpha}\exp\left(\frac{x}{1-t}\right)\left(1-\frac{yt}{1-t}\right)^{-c}{}_qF_q\cdot\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{(-x/(1-t))(yt/1-t)}{1-yt/1-t}\right)^q\right)=\sum_{n=0}^{\infty}{}_{2q}F_q\left(\begin{matrix}\frac{-n}{q},\frac{-n+1}{q},\cdots,\frac{-n+q-1}{q},\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};(qy)^q\end{matrix}\right)A_{q,n}^{(\alpha)}(x)t^n.\tag{104}$$

Proof. Consider the series

$$\sum_{j=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n}=(1-t)^{-q-\alpha}\exp\left(\frac{-(x+y)t}{1-t}\right)\times{}_0F_q\cdot\left(--;\frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots\frac{2q+\alpha-1}{q};\cdot\left(\frac{xyt}{q(1-t)}\right)^q\right).\tag{105}$$

Applying Equation (92), we get

$$\frac{e^x}{(1-t)^c}{}_qF_q\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{-xt}{1-t}\right)^q\right)=\sum_{j=0}^{\infty}\frac{(c)_{qj}A_{q,qj}^{(\alpha)}(x)t^{qj}}{(q+\alpha)_{qj}}.\tag{106}$$

Replacing x by $x(1-t)^{-1}$ and t by $yt(1-t)^{-1}$ yields

$$\exp\left(\frac{x}{1-t}\right)\left(1-\frac{yt}{1-t}\right)^{-c}{}_qF_q\cdot\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{(-x/(1-t))(yt/1-t)}{1-yt/1-t}\right)^q\right)=\sum_{j=0}^{\infty}\frac{(c)_{qj}A_{q,qj}^{(\alpha)}(x/1-t)(yt/1-t)^{qj}}{(q+\alpha)_{qj}},\tag{107}$$

multiplying both sides by $(1-t)^{-q-1}\exp(-xt/1-t)$

$$(1-t)^{-q-\alpha}\exp\left(\frac{x}{1-t}\right)\exp\left(\frac{-xt}{1-t}\right)\left(1-\frac{yt}{1-t}\right)^{-c}\times{}_qF_q\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{(-x/(1-t))(yt/1-t)}{1-yt/1-t}\right)^q\right)=(1-t)^{-q-\alpha}\exp\left(\frac{-xt}{1-t}\right)^2\sum_{j=0}^{\infty}\frac{(c)_{qj}A_{q,qj}^{(\alpha)}(x/1-t)(yt/1-t)^{qj}}{(q+\alpha)_{qj}}=\sum_{j=0}^{\infty}\frac{(c)_{qj}A_{q,qj}^{(\alpha)}(x/1-t)(1-t)^{-q-\alpha-qj}\exp(-xt/1-t)^q y^{qj}t^{qj}}{(q+\alpha)_{qj}}.\tag{108}$$

By using Lemma 4, we acquire

$$(1-t)^{-1-\alpha}\exp\left(\frac{x}{1-t}\right)\left(1-\frac{yt}{1-t}\right)^{-c}{}_qF_q\cdot\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{(-x/(1-t))(yt/1-t)}{1-yt/1-t}\right)^q\right)=\sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{(c)_{qj}n!A_{q,qj}^{(\alpha)}(x)t^{n-qj}y^{qj}t^{qj}}{(qj)!(n-qj)!(q+\alpha)_{qj}}=\sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{(c)_{qj}(-n)_{qj}A_{q,qj}^{(\alpha)}(x)t^n y^{qj}}{(qj)!(q+\alpha)_{qj}}.\tag{109}$$

By using Lemma 1 and 2, we get our required result. □

10. Expansion of Polynomials

Since $A_{q,n}^{(\alpha)}(x)$ forms an orthogonal set, the classical technique for expanding a polynomial. As usual, we prefer to treat the problem by obtaining first the expansion of x^{qn} and then using generating function techniques.

Theorem 23. *If $\alpha \in \mathbb{R}$ and $n, j \in \mathbb{Z}^+$, then*

$$x^{qn} = e^{-x} \sum_{j=0}^{[n/q]} \frac{n!(q+\alpha)_n A_{q,qj}^{(\alpha)}(x)}{(n-qj)!(q+\alpha)_{qj}}. \tag{110}$$

Proof. Equation (21) then yields

$$\begin{aligned} {}_0F_q\left(--; \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots \frac{2q+\alpha-1}{q}; \left(\frac{-xt}{q}\right)^q\right) \\ = e^{-x-t} \sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(x)t^n}{(q+\alpha)_n}, \end{aligned} \tag{111}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-xt/q)^{qn}}{(q+\alpha/q)_n (q+1+\alpha/q)_n \cdots (2q+\alpha-1/q)_n (qn)!} \\ = e^{-x} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(x)t^n}{(q+\alpha)_n}, \end{aligned} \tag{112}$$

$$\sum_{n=0}^{\infty} \frac{(-xt)^{qn}}{(q+\alpha)_{qn}(qn)!} = e^{-x} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^n t^n A_{q,qj}^{(\alpha)}(x) t^{qj}}{n!(q+\alpha)_{qj}}. \tag{113}$$

By using Lemma 4, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n t^n}{(q+\alpha)_n n!} = e^{-x} \sum_{n=0}^{\infty} \sum_{j=0}^{[n/q]} \frac{(-1)^n A_{q,qj}^{(\alpha)}(x) t^n}{(n-qj)!(q+\alpha)_{qj}}. \tag{114}$$

By equating the coefficient of t^n, we get

$$x^{qn} = e^{-x} \sum_{j=0}^{[n/q]} \frac{n!(q+\alpha)_n A_{q,qj}^{(\alpha)}(x)}{(n-qj)!(q+\alpha)_{qj}}. \tag{115}$$

□

11. Conclusion

Finally, in conclusion, we compromised the extended Laguerre polynomials $\left\{A_{q,n}^{(\alpha)}(x)\right\}$ based on the ${}_qF_q$, $q > 2$. We obtained generating functions, recurrence relations, and Rodrigue's formula for these extended Laguerre polynomials. In future work, we can extend it and can get more results. We will apply Laplace transformation, and Elzaki transformation and the same more transformations can apply on the results of extended Laguerre polynomials.

Data Availability

No data were used to support this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] R. L. Shively, *On pseudo-Laguerre polynomials*, 1953.

[2] G. M. Habibullah, "An integral equation involving Shively's polynomials," *Journal of Natural Sciences and Mathematics*, vol. 10, pp. 209–214, 1970.

[3] S. K. Chatterjea, "On a generalization of Laguerre polynomials," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 34, pp. 180–190, 1964.

[4] S. K. Chatterjea and M. K. Das, "Some polynomials related to the generalized Laguerre polynomials," *Publications De L' Institute Mathematique*, vol. 20, pp. 99–104, 1965.

[5] K. Y. Chen and H. M. Srivastava, "A limit relationship between Laguerre and Hermite polynomials," *Integral Transforms and Special Functions*, vol. 16, no. 1, pp. 75–80, 2005.

[6] A. M. Chak, "A class of polynomials and a generalization of Stirling numbers," *Duke Mathematical Journal*, vol. 23, no. 1, pp. 45–56, 1956.

[7] G. Dattoli, H. M. Srivastava, and D. Sacchetti, "The Hermite polynomials and the Bessel functions from a general point of view," *IJMMS*, vol. 2003, no. 57, pp. 3633–3642, 2003.

[8] S. B. Trickovic and M. S. Stankovic, "On the orthogonality of classical orthogonal polynomials," *Integral Transforms and Special Functions*, vol. 14, no. 2, pp. 129–138, 2003.

[9] M. A. Khan and A. K. Shukla, "On binomial and trinomial operator representations of certain polynomials," *Indian Journal of Pure and Applied Mathematics*, vol. 25, pp. 103–110, 2009.

[10] A. Z. Grinshpan, "A family of integral equations with restricted solutions," *Integral Transforms and Special Functions*, vol. 20, no. 10, pp. 723–735, 2009.

[11] H. Duenas, F. Marcellan, and E. Prianes, "Perturbations of Laguerre–Hahn functional: modification by the derivative of a Dirac delta," *Integral Transforms and Special Functions*, vol. 20, no. 1, pp. 59–77, 2009.

[12] D. S. Kim, T. Kim, and D. V. Dolgy, "*Some identities on Laguerre polynomials in connection with Bernoulli and Euler numbers*," *Society*, vol. 2012, pp. 1–10, 2012.

[13] Analytical and operational approach, "Functions induced by iterated deformed Laguerre derivative: analytical and operational approach," *Abstract and Applied Analysis*, vol. 2012, Article ID 190726, 17 pages, 2012.

[14] A. Nowak, Z. Tomasz, and T. Z. Szareka, "Calderón–Zygmund operators related to Laguerre function expansions of convolution type," *Journal of Mathematical Analysis and Applications*, vol. 388, no. 2, pp. 801–816, 2012.

[15] A. Khan and G. M. Habibullah, "Extended Laguerre polynomials," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 22, pp. 1089–1094, 2012.

[16] A. Khan and M. Kalim, "The extended Laguerre polynomials $A_{3,m}^{(\alpha)}(x)$ involving ${}_3F_3$," *Pakistan Journal of Statistics*, vol. 38, no. 1, pp. 89–98, 2022.

[17] E. H. Doha, H. M. Ahmed, and S. I. El-Soubhy, "Explicit formulae for the coefficients of integrated expansions of Laguerre and Hermite polynomials and their integrals," *Integral Transforms and Special Functions*, vol. 20, no. 7, pp. 491–503, 2009.

[18] G. Dattoli, M. R. Martinelli, and P. E. Ricci, "On new families of integral transforms for the solution of partial differential equations," *Integral Transforms and Special Functions*, vol. 16, no. 8, pp. 661–667, 2005.

[19] G. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, 2004.

[20] S. B. Trickovic and M. S. Stankovic, "A new approach to the orthogonality of the Laguerre and Hermite polynomials," *Integral Transforms and Special Functions*, vol. 17, no. 9, pp. 661–672, 2006.

[21] V. Radulescu, "Rodrigues-type for Hermite and Laguerre polynomials," *An. St. Uni. Ovidius Constanta.*, vol. 2, pp. 109–116, 2008.

[22] E. H. Doha and Y. H. Youssri, "On the connection coefficients and recurrence relations arising from expansions in series of modified generalized Laguerre polynomials: applications on a semi-infinite domain," *Nonlinear Engineering*, vol. 8, no. 1, pp. 318–327, 2019.

[23] A. Akbary, D. Ghioca, and Q. Wang, "On permutation polynomials of prescribed shape," *Finite Fields and Their Applications*, vol. 15, no. 2, pp. 195–206, 2009.

[24] S. Li, "Permutation polynomials modulo *m*," *Finite Fields and Their Applications*, vol. 11, pp. 321–325, 2005.

[25] E. Aksoy, A. Cesmelioglu, W. Meidl, and A. Topuzoglu, "On the Carlitz rank of permutation polynomials," *Finite Fields and Their Applications*, vol. 15, no. 4, pp. 428–440, 2009.

[26] Q. Wang, "On inverse permutation polynomials," *Finite Fields and Their Applications*, vol. 15, no. 2, pp. 207–213, 2009.

[27] I. Krasikov and A. Zarkh, "Equioscillatory property of the Laguerre polynomials," *Journal of Approximation Theory*, vol. 162, no. 11, pp. 2021–2047, 2010.

[28] M. K. Das, "Operational representation for the Laguerre polynomials," *Acta Mathematicav Academiae Scientiarum Hungaricae*, vol. 18, no. 3-4, pp. 335–338, 1967.

[29] W. A. Al-Salam, "Operational representations for the Laguerre and other polynomials," *Duke Mathematical Journal*, vol. 31, no. 1, pp. 127–142, 1964.

[30] L. Carlitz, "A note on certain biorthogonal polynomials," *Pacific Journal of Mathematics*, vol. 24, no. 3, pp. 425–430, 1968.

[31] H. M. Srivastava, "An extension of the Hille-Hardy formula," *Mathematics of Computation*, vol. 23, no. 106, pp. 305–311, 1969.

[32] E. Erdélyi, *Higher Transcendental Functions, vol. 2*, McGraw Hill Book Co, New York, NY, USA, 1953.

[33] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, NY, USA, 1965.

[34] B. K. Karande and N. K. Thakare, "Some results for Konhauser-bi-orthogonal polynomials and dual series equations," *Indian Journal of Pure and Applied Mathematics*, vol. 7, pp. 635–646, 1976.

[35] R. Panda, "On a new class of polynomials," *Glasgow Mathematical Journal*, vol. 18, no. 1, pp. 105–108, 1977.

[36] B. P. Parashar, "The difference operators and extended Laguerre polynomials," *Bulletin of the Australian Mathematical Society*, vol. 28, no. 1, pp. 111–119, 1983.

[37] R. Sharma and A. K. Chongdar, "An extension of bilateral generating functions of modified Laguerre polynomials," *Indian Academy of Sciences*, vol. 101, no. 1, pp. 43–47, 1991.

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Research Article

Infinite Product Representation for the Szegö Kernel for an Annulus

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The Szegö kernel has many applications to problems in conformal mapping and satisfies the Kerzman-Stein integral equation. The Szegö kernel for an annulus can be expressed as a bilateral series and has a unique zero. In this paper, we show how to represent the Szegö kernel for an annulus as a basic bilateral series (also known as q-bilateral series). This leads to an infinite product representation through the application of Ramanujan's sum. The infinite product clearly exhibits the unique zero of the Szegö kernel for an annulus. Its connection with the basic gamma function and modified Jacobi theta function is also presented. The results are extended to the Szegö kernel for general annulus and weighted Szegö kernel. Numerical comparisons on computing the Szegö kernel for an annulus based on the Kerzman-Stein integral equation, the bilateral series, and the infinite product are also presented.

1. Introduction

The Ahlfors map is a branching n-to-one map from an n-connected region onto the unit disk. It is intimately tied to the Szegö kernel of an n-connected region [1]. The boundary values of the Szegö kernel satisfy the Kerzman-Stein integral equation, which is a Fredholm integral equation of the second kind for a region with a smooth boundary [2]. The boundary values of the Alhfors map are completely determined from the boundary values of the Szegö kernel [1–3]. For an annulus region Ω, the Szegö kernel can be expressed as a bilateral series from which the zero can be determined analytically [4]. The Kerzman-Stein integral equation has been solved using the Adomian decomposition method in [5] to give another bilateral series form for the Szegö kernel for Ω that converges faster. There are various special functions in the form of bilateral and basic bilateral series [6–8]. For example, the bilateral basic hypergeometric series contain, as special cases, many interesting identities related to infinite products, theta functions, and Ramanujan's identities. It is therefore natural to ask if the bilateral series for the Szegö kernel for Ω can be summed as special functions or an infinite product that exhibits clearly its zero.

In this paper, we show how to express the bilateral series for the Szegö kernel for Ω as a basic bilateral series (also known as q-bilateral series). Ramanujan's sum is then applied to obtain the infinite product representation for the Szegö kernel for Ω. The product clearly exhibits the zero of the Szegö kernel for Ω, and its connection with the q-gamma function and the modified Jacobi theta function is shown. Using the symmetry of Ramanujan's sum, we show how to easily transform the bilateral series for the Szegö kernel for Ω in [4] to the bilateral series in [5].

The plan of the paper is as follows: After the presentation of some preliminaries in Section 2, we derive the basic bilateral series and infinite product representations for the Szegö kernel for Ω in Section 3. We then derive a closed form of the Szegö for Ω in terms of q-gamma function and the modified Jacobi theta function. In Section 4, we show how to extend the representations in Section 3 to the general annulus using the transformation formula for the Szegö kernel under conformal mappings. Similar q-analysis for the weighted Szegö kernel for Ω is presented in Section 5. In Section 6, we give numerical comparisons for computing the Szegö kernel for Ω using bilateral series, infinite product, and integral equation formulations.

2. Preliminaries

Let $\Omega = \{z : \rho < |z| < 1\}$ be an annulus with $0 < \rho < 1$ and a point $a \in \Omega$. The boundary Γ of Ω consists of two smooth Jordan curves with the outer curve Γ_0 oriented counterclockwise and the inner curve Γ_1 oriented clockwise. The positive direction of the contour $\Gamma = \Gamma_0 \cup \Gamma_1$ is usually that for which the region is on the left as one traces the boundary.

Let $\{\varphi_n(z)\}_{n=1}^{\infty}$ be an orthonormal basis for the Hardy spaces $H^2(\Gamma)$. Since the Szegö kernel $S(z,a)$ is the reproducing kernel for $H^2(\Gamma)$, it can be written as [4]

$$S(z,a) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(a)}, a \in \Omega, \tag{1}$$

with absolute and uniform convergence on compact subsets of Ω. An orthogonal basis for $H^2(\Gamma)$ is $\{z^n\}_{n=-\infty}^{\infty}$. Thus

$$\|z^n\|^2 = \int_{\Gamma} |z|^{2n} |dz| = 2\pi\left(1 + \rho^{2n+1}\right), \tag{2}$$

where $|dz|$ is the arc length measure. Therefore, an orthonormal basis for $H^2(\Gamma)$ is [3, 4]

$$\left\{ \frac{z^n}{\sqrt{2\pi(1 + \rho^{2n+1})}} \right\}_{n=-\infty}^{\infty}. \tag{3}$$

Using (1) and (3), the series representation for the Szegö kernel for Ω is given by [4]

$$S(z,a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(z\bar{a})^n}{1 + \rho^{2n+1}}, a \in \Omega, z \in \Omega \cup \Gamma. \tag{4}$$

Series (4) is a bilateral series. It has a zero at $z = -\rho/\bar{a}$ [4].

Another bilateral series representation for the Szegö kernel for Ω is given by [5] (in an equivalent form)

$$S(z,a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \rho^n}{\rho^{2n} - z\bar{a}}, z \in \Omega \cup \Gamma, a \in \Omega, \tag{5}$$

which is initially obtained by solving the Kerzman-Stein integral equation using the Adomian decomposition method. It is also shown in [5] how to derive (5) directly from (4) using geometric series. It is illustrated in [5] that series (5) converges faster than (4).

More generally, if Ω_1 is any doubly connected region with the smooth boundary Γ_1, and $f(z)$ is a biholomorphic map of Ω_1 onto Ω, then the Szegö kernel for Ω_1 can be obtained via the transformation formula as [1]

$$\begin{aligned} S_1(z,a) &= \sqrt{f'(z)}S(f(z), f(a))\overline{\sqrt{f'(a)}} \\ &= \frac{\sqrt{f'(z)}\overline{\sqrt{f'(a)}}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\left(f(z)\overline{f(a)}\right)^n}{1 + \rho^{2n+1}}, a \in \Omega_1, z \in \Omega_1 \cup \Gamma_1, \end{aligned} \tag{6}$$

where ρ is unknown but can be computed.

The Szegö kernel $S_1(z,a)$ can also be computed without using conformal mapping. The boundary values of the Szegö kernel $S_1(z,a)$ on Γ_1 satisfy the Kerzman-Stein integral equation [2, 4],

$$S_1(z,a) + \int_{\Gamma} A(z,w) S_1(w,a) |dw| = g(z), z \in \Gamma_1, \tag{7}$$

where

$$\begin{aligned} A(z,w) &= \begin{cases} \frac{1}{2\pi}\left(\frac{T(w)}{z - w} - \frac{\overline{T(z)}}{\bar{z} - \bar{w}}\right), & z \neq w \in \Gamma_1, \\ 0, & z = w \in \Gamma_1, \end{cases} \\ g(z) &= -\frac{1}{2\pi i} \frac{\overline{T(z)}}{\bar{z} - \bar{a}}, z \in \Gamma_1, \\ T(z) &= \frac{z'(t)}{|z'(t)|}, z \in \Gamma_1, \end{aligned} \tag{8}$$

and $z(t)$ is a parametrization of Γ_1. The function $A(z,w)$ is known as the Kerzman-Stein kernel, and it is continuous on the boundary of Ω_1 [9, 10]. In fact, the integral equation (7) is also valid for an n-connected region.

Since bilateral series and basic bilateral series will be used throughout this paper, we recall some facts about q-series notations and results.

Let $0 < q < 1$ and $\alpha \in \mathbb{C}$. The q-shifted factorial is defined as [7]

$$(q^{\alpha}; q)_n = \begin{cases} 1, & n = 0, \\ (1 - q^{\alpha})(1 - q^{\alpha+1}) \cdots (1 - q^{\alpha+n-1}), & n = 1, 2, \dots, \\ \frac{1}{(1 - q^{\alpha-1})(1 - q^{\alpha-2}) \cdots (1 - q^{\alpha-n})}, & n = -1, -2, \dots. \end{cases} \tag{9}$$

This notation yields the shifted factorial as a special case through

$$\lim_{q \to 1} \frac{(q^{\alpha}; q)_n}{(q; q)_n} = \alpha(\alpha + 1) \cdots (\alpha + n - 1), n = 1, 2, \dots. \tag{10}$$

If α is written in place of q^{α}, then (9) becomes

$$(\alpha;q)_n = \begin{cases} 1, & n=0, \\ (1-\alpha)(1-\alpha q)\cdots\left(1-\alpha q^{n-1}\right), & n=1,2,\ldots, \\ \dfrac{1}{(1-\alpha q^{-1})(1-\alpha q^{-2})\cdots(1-\alpha q^{-n})}, & n=-1,-2,\ldots. \end{cases} \quad (11)$$

It can be shown that [7]

$$\frac{1-\alpha}{1-\alpha q^n} = \frac{(\alpha;q)_n}{(\alpha q;q)_n}, n = 0,\pm1,\pm2,\ldots. \quad (12)$$

If $n \longrightarrow \infty$, it is standard to write

$$(\alpha;q)_\infty = \prod_{n=0}^{\infty}(1-\alpha q^n), \quad (13)$$

which is absolutely convergent for all finite values of α, real or complex, when $|q| < 1$ [6]. This yields

$$(\alpha;q)_n = \frac{(\alpha;q)_\infty}{(\alpha q^n;q)_\infty}. \quad (14)$$

Observe that $(\alpha;q)_\infty$ would have zero as a factor if $\alpha = 1$. It would be zero also if $\alpha = q^{-1}, q^{-2}, q^{-3}, \ldots$, but these are all outside the circle $|z| = 1$ since $|q| < 1$ [8].

The bilateral basic hypergeometric series in base q with one numerator and one denominator parameters is defined by [6–8]

$${}_1\psi_1(\alpha;\beta;q;z) = \sum_{n=-\infty}^{\infty}\frac{(\alpha;q)_n}{(\beta;q)_n}z^n. \quad (15)$$

The series is convergent for $|q| < 1$ and $|\beta/\alpha| < |z| < 1$.

The classical Ramanujan's ${}_1\psi_1$ summation is given by [7, 8]

$${}_1\psi_1(\alpha;\beta;q;z) = \frac{(\alpha z;q)_\infty(q/\alpha z;q)_\infty(\beta/\alpha;q)_\infty(q;q)_\infty}{(z;q)_\infty(\beta/\alpha z;q)_\infty(q/\alpha;q)_\infty(\beta;q)_\infty}, |\beta/\alpha| < |z| < 1. \quad (16)$$

The special case $\beta = \alpha q$ of Ramanujan's ${}_1\psi_1$ summation yields [8]

$$\sum_{n=-\infty}^{\infty}\frac{z^n}{1-\alpha q^n} = \frac{(\alpha z;q)_\infty((q/\alpha z);q)_\infty(q;q)_\infty^2}{(z;q)_\infty((q/z);q)_\infty(\alpha;q)_\infty((q/\alpha);q)_\infty}, \quad (17)$$

also known as Cauchy's formula. Due to symmetry in α and z on the right-hand side of (17), it implies [8]

$$\sum_{n=-\infty}^{\infty}\frac{z^n}{1-\alpha q^n} = \sum_{n=-\infty}^{\infty}\frac{\alpha^n}{1-zq^n}. \quad (18)$$

The q-gamma function is defined as [7]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, 0<q<1, x = \mathbb{C} - \{0,-1,-2,\ldots\}. \quad (19)$$

Another important special function that is used in this paper is the modified Jacobi theta function defined by [7]

$$\theta(x;q) = (x;q)_\infty(q/x;q)_\infty, \quad (20)$$

where $x \neq 0$ and $|q| < 1$. For a more detailed discussion on q-series and historical perspectives, see, for example, [6–8] and the references therein.

3. Szegö Kernel for an Annulus and Basic Bilateral Series

In this section, we express the bilateral series (4) as a basic bilateral series and derive the infinite product representation of the Szegö kernel for Ω. It is given in the following theorem.

Theorem 1. *Let Ω be the annulus $\{z : \rho < |z| < 1\}$ bounded by Γ. For $a \in \Omega$, $z \in \Omega \cup \Gamma$, the Szegö kernel for Ω can be represented by*

$$S(z,a) = \frac{1}{2\pi(1+\rho)}{}_1\psi_1\left(-\rho;-\rho^3;\rho^2;\bar{a}z\right), \quad (21)$$

$$= \frac{1}{2\pi}\prod_{n=0}^{\infty}\frac{\left(1+\bar{a}z\rho^{2n+1}\right)\left(\bar{a}z+\rho^{2n+1}\right)\left(1-\rho^{2n+2}\right)^2}{(1-\bar{a}z\rho^{2n})(\bar{a}z-\rho^{2n+2})(1+\rho^{2n+1})^2}. \quad (22)$$

The zero of $S(z,a)$ in Ω is the zero of the factor $\bar{a}z+\rho$, that is, $z = -\rho/\bar{a}$.

Proof. From (4), we have

$$S(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1+\rho^{2n+1}} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1-(-\rho)\rho^{2n}}. \quad (23)$$

Letting $\alpha = -\rho$ and $q = \rho^2$ yields

$$S(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1-\alpha q^n}, \quad (24)$$

$$= \frac{1}{2\pi(1-\alpha)}\sum_{n=-\infty}^{\infty}\frac{1-\alpha}{1-\alpha q^n}(\bar{a}z)^n. \quad (25)$$

Applying (12) and (15) gives

$$\begin{aligned} S(z,a) &= \frac{1}{2\pi(1-\alpha)}\sum_{n=-\infty}^{\infty}\frac{(\alpha,q)_n}{(\alpha q,q)_n}(\bar{a}z)^n \\ &= \frac{1}{2\pi(1-\alpha)}{}_1\psi_1(\alpha;\alpha q;q;\bar{a}z). \end{aligned} \quad (26)$$

Note that the ${}_1\psi_1$ series above is convergent because $|q| = \rho^2 < 1$ and $|\beta/\alpha| = |\alpha q/\alpha| = |q| = \rho^2 < |\bar{a}z| < 1$. Substituting $\alpha = -\rho$ and $q = \rho^2$ into (26) gives (21).

Applying Ramanujan's sum (16) to (26), gives

$$S(z, a) = \frac{1}{2\pi(1-\alpha)} \frac{(\alpha\bar{a}z\,;q)_\infty (q/\alpha\bar{a}z\,;q)_\infty (q\,;q)_\infty^2}{(\bar{a}z\,;q)_\infty (q/\bar{a}z\,;q)_\infty (q/\alpha\,;q)_\infty (\alpha q\,;q)_\infty}. \tag{27}$$

But from (14), with $n = 1$, we have

$$(1-\alpha)(\alpha q\,;q)_\infty = (\alpha\,;q)_\infty. \tag{28}$$

Thus, (27) becomes

$$S(z, a) = \frac{1}{2\pi} \frac{(\alpha\bar{a}z\,;q)_\infty (q/\alpha\bar{a}z\,;q)_\infty (q\,;q)_\infty^2}{(\bar{a}z\,;q)_\infty (q/\bar{a}z\,;q)_\infty (q/\alpha\,;q)_\infty (\alpha\,;q)_\infty}, \tag{29}$$

$$= \frac{1}{2\pi} \prod_{n=0}^{\infty} \frac{(1-\alpha\bar{a}zq^n)\left(1-q^{n+1}/\alpha\bar{a}z\right)\left(1-q^{n+1}\right)^2}{(1-\bar{a}zq^n)(1-q^{n+1}/\bar{a}z)(1-q^{n+1}/\alpha)(1-\alpha q^n)}. \tag{30}$$

Substituting $\alpha = -\rho$ and $q = \rho^2$ into (30) gives (22).

The infinite product (22) would have poles if

$$1 - \bar{a}z\rho^{2n} = 0 \text{ or } \bar{a}z - \rho^{2n+2} = 0, \tag{31}$$

which implies

$$z = \frac{1}{\bar{a}\rho^{2n}} \text{ or } z = \frac{\rho^{2n+2}}{\bar{a}}. \tag{32}$$

But

$$\frac{1}{|a\rho^{2n}|} > 1, \left|\frac{\rho^{2n+2}}{\bar{a}}\right| < \rho^{2n+1} < \rho. \tag{33}$$

Therefore, the poles are all outside Ω.

The infinite product (22) would have zeros if

$$1 + \bar{a}z\rho^{2n+1} = 0 \text{ or } \bar{a}z + \rho^{2n+1} = 0, \tag{34}$$

which implies

$$z = -\frac{1}{\bar{a}\rho^{2n+1}} \text{ or } z = -\frac{\rho^{2n+1}}{\bar{a}}. \tag{35}$$

For the first case

$$\frac{1}{|a\rho^{2n+1}|} > \frac{1}{\rho^{2n+1}} > 1, \tag{36}$$

which is outside Ω. For the second case, observe that

$$\rho^{2n+1} < \left|\frac{\rho^{2n+1}}{\bar{a}}\right| = \frac{\rho^{2n+1}}{|a|} < \rho^{2n}, \tag{37}$$

which clearly has a zero inside Ω when $n = 0$. Thus, the infinite product (22) for $S(z, a)$ has only one zero inside Ω at $z = -\rho/\bar{a}$. This completes the proof. □

We note that the series representation (21) for $S(z, a)$ is valid only for $\rho \le |z| \le 1$, while the infinite product representation (22) for $S(z, a)$ is meaningful for all $z \in \mathbb{C}$ except for the infinitely many poles at $z = 0, \rho^{-2n}/\bar{a}, \rho^{2n+2}/\bar{a}$.

We next show that the Szegö kernel for Ω can also be expressed in terms of the basic gamma function and modified Jacobi theta function. By applying (20) to (29) and substituting $\alpha = -\rho$ and $q = \rho^2$, we have

$$\begin{aligned} S(z, a) &= \frac{1}{2\pi} \frac{\theta(\alpha\bar{a}z\,;q)_\infty (q\,;q)_\infty^2}{\theta(\bar{a}z\,;q)_\infty (q/\alpha\,;q)_\infty (\alpha\,;q)_\infty} \\ &= \frac{1}{2\pi} \frac{\theta\left(-\rho\bar{a}z\,;\rho^2\right)_\infty \left(\rho^2\,;\rho^2\right)_\infty^2}{\theta\left(\bar{a}z\,;\rho^2\right)_\infty \left(-\rho\,;\rho^2\right)_\infty^2}. \end{aligned} \tag{38}$$

Applying (19) with $q = \rho^2$, observe that

$$\frac{\left(\rho^2\,;\rho^2\right)_\infty}{\left(-\rho\,;\rho^2\right)_\infty} = \frac{\left(\rho^2\,;\rho^2\right)_\infty}{\left(\rho^{2x}\,;\rho^2\right)_\infty} = \frac{\Gamma_{\rho^2}(x)}{\left(1-\rho^2\right)^{1-x}}, \tag{39}$$

where x satisfies $\rho^{2x} = -\rho$. This equation may be written as

$$e^{(2x-1)\ln\rho} = e^{i\pi}, \tag{40}$$

which yields a solution

$$x = \frac{1}{2} + \frac{i\pi}{2\ln\rho}. \tag{41}$$

Thus, (38) becomes

$$S(z, a) = \frac{\left[\Gamma_{\rho^2}(\lambda)\right]^2}{2\pi\left(1-\rho^2\right)^{2(1-\lambda)}} \frac{\theta\left(-\rho\bar{a}z\,;\rho^2\right)_\infty}{\theta\left(\bar{a}z\,;\rho^2\right)_\infty}, \lambda = \frac{1}{2} + \frac{i\pi}{2\ln\rho}. \tag{42}$$

This can be regarded as a closed-form expression for the Szegö kernel for Ω.

In the following, we show how to easily transform series (4) to series (5) using (18). Letting $\alpha = -\rho$ and $q = \rho^2$, (4) becomes

$$S(z, a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(\bar{a}z)^n}{1-\alpha q^n} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\alpha^n}{1-(\bar{a}z)q^n}, \tag{43}$$

where in the last step we have used (18). By replacing $\alpha = -\rho$ and $q = \rho^2$, we get

$$S(z, a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n\rho^n}{1-(\bar{a}z)\rho^{2n}}. \tag{44}$$

Letting $n = -m$ yields

$$S(z,a) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\frac{(-1)^{-m}\rho^{-m}}{1-(\bar{a}z)\rho^{-2m}} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\frac{(-1)^{m}\rho^{m}}{\rho^{2m}-\bar{a}z}, \quad (45)$$

which is the same as (5).

4. Szegö Kernel for General Annulus

Consider the general annulus $\Omega_2 = \{z : r_2 < |z - z_0| < r_1\}$ with boundary denoted by Γ_2. The region Ω_2 reduces to Ω if $z_0 = 0$, $r_2 = \rho$, and $r_1 = 1$.

Theorem 2. *Let $z_0 \in \mathbb{C}$, $z \in \Omega_2 \cup \Gamma_2$, and $a \in \Omega_2$. The Szegö kernel for Ω_2 can be represented by the bilateral series as*

$$S_2(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}-\bar{z}_0)^n}{r_1^{2n+1}+r_2^{2n+1}}(z-z_0)^n, \quad (46)$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n r_1^{n+1} r_2^n}{r_2^{2n}r_1^2 - r_1^{2n}(z-z_0)(\bar{a}-\bar{z}_0)}. \quad (47)$$

The zero of $S_2(z,a)$ in Ω_2 is $z = z_0 - r_1 r_2/\bar{a} - \bar{z}_0$.

Proof. Observe that the function $f(z) = (z - z_0)/r_1$ maps Ω_2 onto Ω with $\rho = r_2/r_1$.

Applying the transformation formula (6) yields

$$\begin{aligned} S_2(z,a) &= \sqrt{f'(z)}S(f(z),f(a))\overline{\sqrt{f'(a)}} \\ &= \frac{1}{\sqrt{r_1}}S\left(\frac{z-z_0}{r_1},\frac{a-z_0}{r_1}\right)\frac{1}{\sqrt{r_1}} \\ &= \frac{1}{r_1}S\left(\frac{z-z_0}{r_1},\frac{a-z_0}{r_1}\right). \end{aligned} \quad (48)$$

Applying (4) to (48) with z and a replaced by $(z - z_0)/r_1$ and $(a - z_0)/r_1$, respectively, gives

$$S_2(z,a) = \frac{1}{2\pi r_1}\sum_{n=-\infty}^{\infty}\frac{\left((z-z_0)(a\bar{z}_0)/r_1^2\right)^n}{1+(r_2/r_1)^{2n+1}}, \quad (49)$$

which simplifies to (46).

Applying (5) to (48) instead of z and a replaced by $(z - z_0)/r_1$ and $(a - z_0)/r_1$, respectively, gives

$$S_2(z,a) = \frac{1}{2\pi r_1}\sum_{n=-\infty}^{\infty}\frac{(-1)^n (r_2/r_1)^n}{(r_2/r_1)^{2n} - (z-z_0)(a\bar{z}_0)/r_1^2}, \quad (50)$$

which simplifies to (47).

Using the fact that $S(z,a)$ has a zero at $z = -\rho/\bar{a}$ for Ω, the zero of $S_2(z,a)$ for Ω_2 is $(z - z_0)/r_1 = -\rho/((a\bar{z}_0)/r_1)$ which implies $z = z_0 - (\rho r_1^2/(\bar{a} - \bar{z}_0)) = z_0 - (r_1 r_2/((\bar{a} - \bar{z}_0)))$. This completes the proof.

Similarly, the infinite product representation of $S_2(z,a)$ for Ω_2 can be obtained by applying (22) to (48) with z and a replaced by $(z - z_0)/r_1$ and $(a - z_0)/r_1$, respectively. □

5. The Weighted Szegö Kernel for an Annulus and Basic Bilateral Series

The weighted Szegö kernel is defined in [11] as

$$\widehat{K}_q^t(z,w) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{w}z)^n}{1+tq^{2n}}, \, t > 0, q < |z|, |w| < 1. \quad (51)$$

To adopt the notations used in this paper, we change q to ρ, w to a, and $\widehat{K}_q^t(z,w)$ to $S_\rho^t(z,a)$ in (51), which gives

$$S_\rho^t(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1+t\rho^{2n}}, \, t > 0, \rho < |z|, |a| < 1. \quad (52)$$

Note that $S_\rho^\rho(z,a)$ is exactly the kernel $S(z,a)$ for Ω discussed in Section 1. The zeros of the kernel $S_t^\rho(z,a)$ are not discussed in [11] but have expressed interest on the effect of the weight on the location of its zeros. In the following theorem, we express the weighted Szegö kernel $S_\rho^t(z,a)$ as a basic bilateral series and derive its associated infinite product representation as well as its zeros.

Theorem 3. *Let Ω be the annulus $\{z : \rho < |z| < 1\}$ bounded by Γ. For $a \in \Omega$, $z \in \Omega \cup \Gamma$, and $t > 0$, the weighted Szegö kernel $S_\rho^t(z,a)$ for Ω can be represented by*

$$S_t^\rho(z,a) = \frac{1}{2\pi(1+t)_1}\psi_1\left(-t;-t\rho^2;\rho^2;\bar{a}z\right), \quad (53)$$

$$= \frac{1}{2\pi}\prod_{n=0}^{\infty}\frac{\left(1+t\bar{a}z\rho^{2n}\right)\left(\bar{a}z+\rho^{2n+2}/t\right)\left(1-\rho^{2n+2}\right)^2}{(1-\bar{a}z\rho^{2n})(\bar{a}z-\rho^{2n+2})(1+\rho^{2n+2}/t)(1+t\rho^{2n})}. \quad (54)$$

The kernel $S_\rho^t(z,a)$ has a zero in Ω only if t takes the form $t = \rho^{\pm(2m+1)}$, $m = 0, 1, 2, \cdots$. In both cases, the zero is $z = -\rho/\bar{a}$.

Proof. Observe that

$$S_\rho^t(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1-(-t)\rho^{2n}}. \quad (55)$$

Letting $\alpha = -t$ and $q = \rho^2$, the above equation becomes

$$S_\rho^t(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1-\alpha q^n}, \quad (56)$$

which is exactly the same form as (24). Applying the result (26) with $\alpha = -t$, the above equation becomes

$$S_\rho^t(z,a) = \frac{1}{2\pi(1+t)_1}\psi_1(-t;-tq;q;\bar{a}z). \quad (57)$$

Series (57) is convergent because $|q| = \rho^2 < 1$ and $|\beta/\alpha| = |-tq/(-t)| = |q| < \rho^2 < |\bar{a}z| < 1$. Substituting $q = \rho^2$ gives (41).

Applying the result (29) with $\alpha = -t$ to (57) yields

$$S_\rho^t(z,a) = \frac{1}{2\pi} \frac{(-t\bar{a}z\,;q)_\infty (q/(-t)\bar{a}z\,;q)_\infty (q\,;q)_\infty^2}{(\bar{a}z\,;q)_\infty (q/\bar{a}z\,;q)_\infty (q/(-t)\,;q)_\infty (-t\,;q)_\infty}. \tag{58}$$

Replacing $q = \rho^2$ and applying (13) give (54).

In the proof of Theorem 1, we have shown that the factors $(1-\bar{a}z\rho^{2n})(\bar{a}z-\rho^{2n+2})$ have no zeros in Ω. The factors $(1+\rho^{2n+2}/t)(1+t\rho^{2n})$ would have zeros if

$$\rho^{2n+2}/t = -1 \text{ or } t\rho^{2n} = -1. \tag{59}$$

Since $t>0$, we conclude that the kernel $S_\rho^t(z,a)$ has no poles in Ω for any $t>0$. The factors $(1+t\bar{a}z\rho^{2n})(\bar{a}z+\rho^{2n+2}/t)$ would have zeros if

$$1 + t\bar{a}z\rho^{2n} = 0 \text{ or } \bar{a}z + \rho^{2n+2}/t = 0, \tag{60}$$

which implies

$$z = -\frac{1}{t\bar{a}\rho^{2n}} \text{ or } z = -\frac{\rho^{2n+2}}{t\bar{a}}. \tag{61}$$

For the first case, observe that

$$\frac{1}{t\rho^{2n}} < \frac{1}{|t\bar{a}\rho^{2n}|} < \frac{1}{t\rho^{2n+1}}. \tag{62}$$

To have a zero in Ω, we must have the condition

$$\rho \le \frac{1}{t\rho^{2n}} < \frac{1}{|t\bar{a}\rho^{2n}|} < \frac{1}{t\rho^{2n+1}} \le 1, \tag{63}$$

which means

$$t \le \frac{1}{\rho^{2n+1}} \text{ and } t \ge \frac{1}{\rho^{2n+1}}. \tag{64}$$

Hence, we must have $t = \rho^{-(2n+1)}$. In this case, the zero of $S_\rho^t(z,a)$ in Ω is $z = -\rho/\bar{a}$.

For the second case, observe that

$$\frac{\rho^{2n+2}}{t} < \frac{\rho^{2n+2}}{|t\bar{a}|} < \frac{\rho^{2n+1}}{t}. \tag{65}$$

To have a zero in Ω, we must have the condition

$$\rho \le \frac{\rho^{2n+2}}{t} < \frac{\rho^{2n+2}}{|t\bar{a}|} < \frac{\rho^{2n+1}}{t} \le 1, \tag{66}$$

which means

$$t \le \rho^{2n+1} \text{ and } t \ge \rho^{2n+1}. \tag{67}$$

Hence, we must have $t = \rho^{2n+1}$. In this case, the zero of $S_\rho^t(z,a)$ in Ω is also $z = -\rho/\bar{a}$. This completes the proof. □

The weighted Szegö kernel can also be expressed in terms of the basic gamma function and the modified Jacobi theta function. By applying (20) to (58) with $q = \rho^2$, we have

$$S_\rho^t(z,a) = \frac{1}{2\pi} \frac{\theta(-t\bar{a}z\,;\rho^2)_\infty (\rho^2\,;\rho^2)_\infty^2}{\theta(\bar{a}z\,;\rho^2)_\infty (\rho^2/(-t)\,;\rho^2)_\infty (-t\,;\rho^2)_\infty}. \tag{68}$$

Observe that

$$\frac{(\rho^2\,;\rho^2)_\infty}{(-t\,;\rho^2)_\infty} = \frac{(\rho^2\,;\rho^2)_\infty}{(\rho^{2x}\,;\rho^2)_\infty} = \frac{\Gamma_{\rho^2}(x)}{(1-\rho^2)^{1-x}}, \tag{69}$$

where x satisfies $\rho^{2x} = -t$. This equation may be written as

$$2x \ln\rho = \ln(-t) = \ln|-t| + i\arg(-t) = \ln t + i\pi, \tag{70}$$

which yields a solution

$$x = \frac{\ln t + i\pi}{2\ln\rho}. \tag{71}$$

Observe also that

$$\frac{(\rho^2\,;\rho^2)_\infty}{(-\rho^2/t\,;\rho^2)_\infty} = \frac{(\rho^2\,;\rho^2)_\infty}{(\rho^{2y}\,;\rho^2)_\infty} = \frac{\Gamma_{\rho^2}(y)}{(1-\rho^2)^{1-y}}, \tag{72}$$

where y satisfies $\rho^{2y} = -\rho^2/t$. This equation may be written as

$$(2y-2)\ln\rho = \ln\left(-\frac{1}{t}\right) = \ln\left|-\frac{1}{t}\right| + i\arg\left(-\frac{1}{t}\right) = -\ln t + i\pi, \tag{73}$$

which yields a solution

$$y = 1 + \frac{-\ln t + i\pi}{2\ln\rho}. \tag{74}$$

Thus, (68) becomes

$$\begin{aligned} S_\rho^t(z,a) &= \frac{\Gamma_{\rho^2}(\mu)\Gamma_{\rho^2}(\nu)\theta(-t\bar{a}z\,;\rho^2)_\infty}{2\pi(1-\rho^2)^{2-\mu-\nu}\theta(\bar{a}z\,;\rho^2)_\infty}, \mu \\ &= \frac{\ln t + i\pi}{2\ln\rho}, \nu \\ &= 1 + \frac{-\ln t + i\pi}{2\ln\rho}. \end{aligned} \tag{75}$$

This can be regarded as a closed-form expression for the weighted Szegö kernel for an annulus Ω. Observe that (75) reduces to (42) when $t = \rho$.

Table 1: Error norms between S_{10} and $\tilde{S}_n$, S_{50} and $\tilde{S}_n$, and S_{100} and $\tilde{S}_n$.

n	$\|S_{10}-\tilde{S}_n\|_\infty$	$\|S_{50}-\tilde{S}_n\|_\infty$	$\|S_{100}-\tilde{S}_n\|_\infty$
16	2.4536 (-02)	2.97754 (-03)	2.97758 (-03)
32	2.75019 (-02)	1.15906 (-05)	1.16299 (-05)
64	2.75136 (-02)	3.91113 (-08)	1.88349 (-10)
128	2.75136 (-02)	3.92996 (-08)	2.28878 (-15)

Table 2: Error norms between S^*_{10} and $\tilde{S}_n$ and S^*_{50} and $\tilde{S}_n$.

n	$\|S^*_{10}-\tilde{S}_n\|_\infty$	$\|S^*_{50}-\tilde{S}_n\|_\infty$
16	2.94797 (-03)	2.97758 (-03)
32	1.78995 (-02)	1.16299 (-05)
64	1.77628 (-04)	1.88351 (-10)
128	1.77628 (-04)	1.81497 (-15)

6. Numerical Computation of the Szegö Kernel for an Annulus

In this section, we compare the speed of convergence of the three formulas for computing the Szegö kernel for Ω based on the two bilateral series (4) and (5) and the infinite product (22).

To approximate (4) numerically, we calculate

$$S(z,a)\approx S_{10}(z,a)=\frac{1}{2\pi}\sum_{k=-10}^{10}\frac{(z\bar{a})^k}{1+\rho^{2k+1}}, \tag{76}$$

and S_{50} and S_{100}.

To approximate (5) numerically, we calculate

$$S(z,a)\approx S^*_{10}(z,a)=\frac{1}{2\pi}\sum_{k=-10}^{10}\frac{(-1)^k\rho^k}{\rho^{2k}-z\bar{a}}, \tag{77}$$

and S^*_{50}.

To approximate (22) numerically, we compute

$$S(z,a)\approx S^{**}_{15}(z,a)=\frac{1}{2\pi}\prod_{k=0}^{15}\frac{\left(1+\bar{a}z\rho^{2k+1}\right)\left(z\bar{a}+\rho^{2k+1}\right)\left(1-\rho^{2k+2}\right)^2}{\left(1-z\bar{a}\rho^{2k}\right)\left(z\bar{a}-\rho^{2k+2}\right)\left(1+\rho^{2k+1}\right)^2}, \tag{78}$$

and S^{**}_{20} and S^{**}_{25}.

The approximations are then compared with the numerical solution of the Kerzman-Stein Equation (7). To solve (7), we used the Nyström method [5] with the trapezoidal rule with n selected nodes on each boundary component Γ_0 and Γ_1. The approximate solution is represented by $\tilde{S}_n$ where n is the number of nodes. All the computations were done using MATHEMATICA 12.3. Four numerical examples are given for different values of a and ρ. The results for the error norms are presented for each example.

Table 3: Error norms between S^{**}_{15} and $\tilde{S}_n$, S^{**}_{20} and $\tilde{S}_n$, and S^{**}_{25} and $\tilde{S}_n$.

n	$\|S^{**}_{15}-\tilde{S}_n\|_\infty$	$\|S^{**}_{20}-\tilde{S}_n\|_\infty$	$\|S^{**}_{25}-\tilde{S}_n\|_\infty$
16	2.97758 (-03)	2.97758 (-03)	2.97758 (-03)
32	1.16296 (-05)	1.16299 (-05)	1.16299 (-05)
64	1.44308 (-10)	1.88038 (-10)	1.8835 (-10)
128	3.1999 (-10)	3.1275 (-13)	1.82618 (-15)

Table 4: Error norms between S_{10} and $\tilde{S}_n$, S_{50} and $\tilde{S}_n$, and S_{100} and $\tilde{S}_n$.

n	$\|S_{10}-\tilde{S}_n\|_\infty$	$\|S_{50}-\tilde{S}_n\|_\infty$	$\|S_{100}-\tilde{S}_n\|_\infty$
16	1.29695 (-02)	1.46732 (-03)	1.46732 (-03)
32	1.56432 (-02)	7.88666 (-06)	7.88666 (-06)
64	1.5646 (-02)	3.26124 (-08)	2.2539 (-10)
128	1.5646 (-02)	3.26942 (-08)	2.85127 (-15)

Table 5: Error norms between S^*_{10} and $\tilde{S}_n$ and S^*_{50} and $\tilde{S}_n$.

n	$\|S^*_{10}-\tilde{S}_n\|_\infty$	$\|S^*_{50}-\tilde{S}_n\|_\infty$
16	1.46686 (-03)	1.46732 (-03)
32	8.4009 (-06)	7.88666 (-06)
64	1.02367 (-06)	2.2539 (-10)
128	1.02367 (-06)	1.25883 (-15)

Table 6: Error norms between S^{**}_{5} and $\tilde{S}_n$, S^{**}_{10} and $\tilde{S}_n$, and S^{**}_{15} and $\tilde{S}_n$.

n	$\|S^{**}_{5}-\tilde{S}_n\|_\infty$	$\|S^{**}_{10}-\tilde{S}_n\|_\infty$	$\|S^{**}_{15}-\tilde{S}_n\|_\infty$
16	1.4675 (-03)	1.46732 (-03)	1.46732 (-03)
32	7.70793 (-06)	7.88666 (-06)	7.88666 (-06)
64	3.72977 (-07)	2.2434 (-10)	2.2539 (-10)
128	3.73107 (-07)	2.2023 (-12)	1.41308 (-15)

Table 7: Error norms between S_{10} and $\tilde{S}_n$, S_{50} and $\tilde{S}_n$, and S_{100} and $\tilde{S}_n$.

n	$\|S_{10}-\tilde{S}_n\|_\infty$	$\|S_{50}-\tilde{S}_n\|_\infty$	$\|S_{100}-\tilde{S}_n\|_\infty$
16	6.45804 (-02)	8.28061 (-03)	8.28061 (-03)
32	6.82534 (-02)	2.2673 (-04)	2.2673 (-04)
64	6.83565 (-02)	9.0045 (-06)	1.79491 (-07)
128	6.83565 (-02)	9.08614 (-06)	1.29631 (-10)

We consider an annulus Ω bounded by

$$\begin{aligned}\Gamma_0 &: z_0(t)=e^{it},\\ \Gamma_1 &: z_1(t)=\rho e^{-it},\end{aligned} \tag{79}$$

with $0\le t\le 2\pi$.

Table 8: Error norms between S_{10}^{*} and $\tilde{S}_n$ and S_{50}^{*} and $\tilde{S}_n$.

n	$\|S_{10}^{*} - \tilde{S}_n\|_\infty$	$\|S_{50}^{*} - \tilde{S}_n\|_\infty$
16	8.28737 (-03)	8.28061 (-03)
32	2.33562 (-04)	2.2673 (-04)
64	1.79806 (-05)	1.79491 (-07)
128	1.78806 (-05)	1.1287 (-15)

Table 9: Error norms between S_5^{**} and $\tilde{S}_n$, S_{10}^{**} and $\tilde{S}_n$, and S_{15}^{**} and $\tilde{S}_n$.

n	$\|S_5^{**} - \tilde{S}_n\|_\infty$	$\|S_{10}^{**} - \tilde{S}_n\|_\infty$	$\|S_{15}^{**} - \tilde{S}_n\|_\infty$
16	8.27577 (-03)	8.28061 (-03)	8.28061 (-03)
32	2.2189 (-04)	2.26729 (-04)	2.2673 (-04)
64	1.13437 (-05)	1.78984 (-07)	1.79491 (-07)
128	1.14253 (-05)	1.19798 (-09)	7.90864 (-14)

Table 10: Error norms between S_{10} and $\tilde{S}_n$, S_{50} and $\tilde{S}_n$, and S_{100} and $\tilde{S}_n$.

n	$\|S_{10} - \tilde{S}_n\|_\infty$	$\|S_{50} - \tilde{S}_n\|_\infty$	$\|S_{100} - \tilde{S}_n\|_\infty$
16	3.15879 (-03)	2.61429 (-04)	2.61429 (-04)
32	3.22447 (-03)	2.08805 (-07)	2.08805 (-07)
64	3.28124 (-03)	5.91022 (-11)	1.33153 (-13)
128	3.28124 (-03)	5.91168 (-11)	1.33233 (-15)

Table 11: Error norms between S_{10}^{*} and $\tilde{S}_n$ and S_{50}^{*} and $\tilde{S}_n$.

n	$\|S_{10}^{*} - \tilde{S}_n\|_\infty$	$\|S_{50}^{*} - \tilde{S}_n\|_\infty$
16	2.61429 (-04)	2.61429 (-04)
32	2.0879 (-07)	2.08805 (-07)
64	1.68217 (-11)	1.33183 (-13)
128	1.67281 (-11)	1.16606 (-15)

Table 12: Error norms between S_5^{**} and $\tilde{S}_n$, S_{10}^{**} and $\tilde{S}_n$, and S_{15}^{**} and $\tilde{S}_n$.

n	$\|S_5^{**} - \tilde{S}_n\|_\infty$	$\|S_{10}^{**} - \tilde{S}_n\|_\infty$	$\|S_{15}^{**} - \tilde{S}_n\|_\infty$
16	2.61429 (-04)	2.61429 (-04)	2.61429 (-04)
32	2.08805 (-07)	2.08805 (-07)	2.08805 (-07)
64	6.46416 (-13)	1.3313 (-13)	1.33121 (-13)
128	6.77069 (-13)	1.49882 (-15)	1.55654 (-15)

Example 1. We consider an annulus Ω with $a = 0.7i$ and $\rho = 0.5$. The results for the error norms are presented in Tables 1–3.

Example 2. We consider an annulus Ω with $a = -0.4 - 0.6i$ and $\rho = 0.3$. The results for the error norms are presented in Tables 4–6.

Example 3. We consider an annulus Ω with $a = -0.8$ and $\rho = 0.4$. The results for the error norms are presented in Tables 7–9.

Example 4. We consider an annulus Ω with $a = -0.4 - 0.5i$ and $\rho = 0.1$. The results for the error norms are presented in Tables 10–12.

The numerical results presented in Tables 1–12 show that computations using the infinite product formula (22) converge faster than the bilateral series formulas (4) and (5).

7. Conclusion

This paper has shown that the bilateral series for the Szegö kernel for Ω is a disguised bilateral basic hypergeometric series ${}_1\psi_1$. Ramanujan's sum for ${}_1\psi_1$ is then applied to obtain the infinite product representation for the Szegö kernel for Ω. The product clearly exhibits the zero of the Szegö kernel for an Ω. The Szegö kernel can also be expressed as a closed form in terms of the q-gamma function and the modified Jacobi theta function. Similar q-analysis has also been conducted for the Szegó kernel for general Ω and for the weighted Szegö kernel for Ω. The numerical comparisons have shown that the infinite product method converges faster than the bilateral series methods for computing the Szegö kernel for Ω.

For future work, it is natural to devote further investigation on the infinite product representation for the Szegö kernel for doubly connected regions via the transformation formula (6) and Theorem 1. This however requires knowledge of conformal mapping of doubly connected regions to annulus [12–15]. For some ideas on numerical methods for computing the zero of the Szegö kernel for doubly connected regions, see [16]. Alternatively, perhaps some computational intelligence algorithms can also be considered to compute the zero, like the monarch butterfly optimization (MBO) [17], earthworm optimization algorithm (EWA) [18], elephant herding optimization (EHO) [19], moth search (MS) algorithm [20], slime mould algorithm (SMA) [21], and Harris hawks optimization (HHO) [22].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] S. R. Bell, *The Cauchy Transform, Potential Theory, and Conformal Mapping*, CRC Press, Boca Raton, 1992.

[2] S. R. Bell, "Numerical computation of the Ahlfors map of a multiply connected planar domain," *Journal of Mathematical Analysis and Applications*, vol. 120, no. 1, pp. 211–217, 1986.

[3] T. J. Tegtmeyer, *The Ahlfors Map and Szegö Kernel in Multiply Connected Domains, [Ph.D. thesis]*, Purdue University, 1998.

[4] T. J. Tegtmeyer and A. D. Thomas, "The Ahlfors map and Szegő kernel for an annulus," *Rocky Mountain Journal of Mathematics*, vol. 29, no. 2, pp. 709–723, 1999.

[5] N. H. A. A. Wahid, A. H. M. Murid, and M. I. Muminov, "Convergence of the series for the Szegö kernel for an annulus region," *AIP Conference Proceedings*, vol. 1974, article 030025, p. 8, 2018.

[6] J. S. Lucy, *Generalized Hypergeometric Functions*, Cambridge University Press, 1996.

[7] G. Gasper and M. Rahman, *Basic Hypergeometric Series, Encyclopedia of Mathematics and Applications, vol. 96*, Cambridge University Press, 2nd edition, 2004.

[8] P. J. Warren, *An Introduction to q-Analysis*, American Mathematical Society, 2020.

[9] N. Kerzman and E. M. Stein, "The Cauchy kernel, the Szegö kernel, and the Riemann mapping function," *Mathematische Annalen*, vol. 236, no. 1, pp. 85–93, 1978.

[10] N. Kerzman and M. R. Trummer, "Numerical conformal mapping via the Szego kernel," *Journal of Computational and Applied Mathematics*, vol. 14, no. 1-2, pp. 111–123, 1986.

[11] S. Mccullough and L. Shen, "On the Szegö kernel of an annulus," *Proceedings of the American Mathematical Society*, vol. 121, no. 4, pp. 1111–1121, 1994.

[12] G. T. Symm, "Conformal mapping of doubly-connected domains," *Numerische Mathematik*, vol. 13, no. 5, pp. 448–457, 1969.

[13] K. Amano, "A charge simulation method for the numerical conformal mapping of interior, exterior and doubly-connected domains," *Journal of Computational and Applied Mathematics*, vol. 53, no. 3, pp. 353–370, 1994.

[14] M. M. S. Nasser, "A boundary integral equation for conformal mapping of bounded multiply connected regions," *Computational Methods and Function Theory*, vol. 9, no. 1, pp. 127–143, 2009.

[15] A. W. K. Sangawi, A. H. M. Murid, and M. M. S. Nasser, "Annulus with circular slit map of bounded multiply connected regions via integral equation method," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 35, no. 4, pp. 945–959, 2012.

[16] A. H. M. Murid, N. H. A. A. Wahid, and M. I. Muminov, "Methods and comparisons for computing the zeros of the Ahlfors map for doubly connected regions," *AIP Conference Proceedings*, vol. 2423, article 020026, 2021.

[17] G.-G. Wang, S. Deb, and Z. Cui, "Monarch butterfly optimization," *Neural Computing and Applications*, vol. 31, no. 7, pp. 1995–2014, 2019.

[18] G.-G. Wang, S. Deb, and L. D. S. Coelho, "Earthworm optimisation algorithm: a bio-inspired metaheuristic algorithm for global optimisation problems," *International Journal of Bio-Inspired Computation*, vol. 12, no. 1, pp. 1–22, 2018.

[19] J. Li, H. Lei, A. H. Alavi, and G.-G. Wang, "Elephant herding optimization: variants, hybrids, and applications," *Journal of Mathematics*, vol. 8, Article ID 1415, 2020.

[20] G.-G. Wang, "Moth search algorithm: a bio-inspired metaheuristic algorithm for global optimization problems," *Memetic Computing*, vol. 10, no. 2, pp. 151–164, 2018.

[21] S. Li, H. Chen, M. Wang, A. A. Heidari, and S. Mirjalili, "Slime mould algorithm: a new method for stochastic optimization," *Future Generation Computer Systems*, vol. 111, pp. 300–323, 2020.

[22] H. M. Alabool, D. Alarabiat, L. Abualigah, and A. A. Heidari, "Harris hawks optimization: a comprehensive review of recent variants and applications," *Neural Computing and Applications*, vol. 33, no. 15, pp. 8939–8980, 2021.

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Research Article

Mobius Group Generated by Two Elements of Order 2, 4, and Reduced Quadratic Irrational Numbers

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The construction of circuits for the evolution of orbits and reduced quadratic irrational numbers under the action of Mobius groups have many applications like in construction of substitution box (s-box), strong-substitution box (s.s-box), image processing, data encryption, in interest for security experts, and other fields of sciences. In this paper, we investigate the behavior of reduced quadratic irrational numbers (RQINs) in the coset diagrams of the set $Q''(\sqrt{m}) = \{\eta/s : \eta \in Q^*(\sqrt{m}), s = 1, 2\}$ under the action of group $H = <x', y' : x'^2 = y'^4 = 1>$, where m is square free integer and $Q^*(\sqrt{m}) = \{(a' + \sqrt{m})/c', (a', (a'^2 - m)/c'c') = 1, c' \neq 0\}$. We discuss the type and reduced cardinality of the orbit $Q''(\sqrt{p})$. By using the notion of congruence, we give the general form of reduced numbers (RNs) in particular orbits under certain conditions on prime p. Further, we classify that for a reduced number r whether $-r, \bar{r}, -\bar{r}$ lying in orbit or not. AMS Mathematics subject classification (2010): 05C25, 20G401.

1. Introduction and Preliminaries

Groups are very helpful algebraic structures, carrying other algebraic structures on them. In abstract algebra, almost all typical structures are illustration of groups. The significance of groups was derived from their action on special structures or spaces. Cryptography is the technique of converting secret knowledge into information and a type of data pretending to reach its terminus without leaking data safely. Modern cryptography is classified into several branches. Although there are two main research fields such that symmetric and public key cryptography, the public and private keys are used in public key cryptography. The same keys are used at both ends to encrypt and decrypt data/information in symmetric key cryptography. It is well known that the substitution box is a standout in symmetric key cryptography. Shahzad et al. investigated the efficient technique for the construction of an S-box by using action of a $PSL(2, Z)$. For constructing an S box, the vertices of the coset diagram are considered in a special way. In this way, the generated S box is highly safe and also closely meeting the optimal values of the standard S-box. In [1–6], the construction of substitution boxes based on coset graphs under the action of modular group $PSL(2, Z)$ has been discussed. In this piece of work, we investigate the structure of coset graphs under the action of modular group H. This work will be more helpful for construction of strong substitution boxes.

The H-circuits of the set upon which the groups act are the equivalence classes of group action. Group H can be written in the form of relations and generators as $<x', y' : x'^2 = y'^4 = 1>$.

Assume that m is nonsquare integer, then $Q(\sqrt{m}) = \{s + t\sqrt{m} : s, t \in Q\}$. In 1878, Cayley was the first who introduced the technique of analysis of the groups through graphs. To investigate the action of infinite groups generated by finite elements on the infinite field by the coset diagram was first introduced by Higman in 1978. A number $\beta = s + t\sqrt{m} \in Q(\sqrt{m})$ is said to be ambiguous number (AN) if β and $\bar{\beta}$ have opposite signs. If $\beta = s + t\sqrt{m}$ is not ambiguous, then it is either totally positive or negative. The real quadratic irrational (RQI) numbers of the form $(a' + \sqrt{m})/c'$, where $(a', (a'^2 - m)/c', c') = 1$ and c' is nonzero integer, make the set represented as $Q^*(\sqrt{m})$. A RQI number $\beta = (a' + \sqrt{m})/c'$ is known as RQIN if $\beta > 0$ and $-1 < \bar{\beta} < 0$. In this paper, we will denote the reduced number by r. If there are k reduced numbers, then they are denoted by $r_1, r_2, r_3, \cdots r_k$. For $\beta \in Q''(\sqrt{m})$, in the orbit of $(\beta)^H$, the count of RQINs is called the reduced length (RL), which is denoted by $|(\beta)^H|_{\text{red}}$. These numbers are very less in $Q''(\sqrt{m})$ and play a significant part in the circuit of an orbit. A circuit made of vertices of a square and edges existing in H-orbits of $Q''(\sqrt{m})$, under the Mobius group H in coset diagram. If $((p_1)_0, (q_1)_1, (r_1)_2, (p_2)_0, (q_2)_1, (r_2)_2 \cdots (p_k)_0, (q_k)_1, (r_k)_2)$ is the type of a circuit, then it makes an element of group $h = (x'y')^{p_1}, (x'y'^2)^{q_1}, (x'y'^3)^{r_1}, (x'y')^{p_2}, (x'y'^2)^{q_2}, (x'y'^3)^{r_2}, \cdots (x'y')^{p_k}, (x'y'^2)^{q_k}, (x'y'^3)^{r_k}$ of H. This h fixes some element exists in this circuit.

In [7, 8], Mushtaq and Aslam presented that there are only finite number of ambiguous numbers (ANS); in the coset diagram for the orbit of $(\beta)^H$, the ambiguous numbers form unique closed path. A cost diagram is introduced in [7, 8] to investigate the action of an infinite group H on the projective line over real quadratic field (RQF). Malik and Zafar [9] investigated the properties of RQI numbers under the action of H. Zafar and Malik [10, 11] investigated the type and ambiguous lengths of the orbit of $Q''(\sqrt{p})$. Farkhanda and Qamar discussed the real quadratic irrational and action of $M = <x', y' : x'^2 = y'^6 = 1>$. Razaq et al. [12, 13] investigated the circuits of length 4 in $\mathrm{PSL}(2, Z)$, group theoretic construction of highly nonlinear substitution box, and its applications in image encryption. Ali and Malik [14, 15] discussed the classification of $\mathrm{PSL}(2, Z)$-circuits and investigated the RQIN and types of G-circuits with length four and six. Chen et al. [16] investigate reduced numbers which play an important role in the study of modular group action on the $PSL(2, Z)$-subset. For more studies of group action on various field, we recommend reading of [17, 18]. The application of group theory and group action is obvious to encryption, physics, and mechanics to construct models and their structures [5, 19–21]. Mateen et al. [22–27] investigated the structure of power digraphs associated with the congruence $x^n \equiv y(\mathrm{mod}\ m)$, the partitioning of a set into two or more disjoint subsets of equal sums, and the symmetry of complete graphs and, moreover, investigated the importance of power digraphs in computer science. Alolaiyan et al. [28] discussed the homomorphic copies in coset graphs for the modular group.

The major contributions of this paper are given below.

(1) This paper presents a graphical study of the action of a Mobius group H on the real quadratic field (RQF)

(2) We discuss the classification of H-circuits and find the numbers that play vital role in the structure of H-circuits

(3) We investigate the RQINs and the types of H-circuits with different length

(4) We give the number of reduced numbers and their general form in different orbits for different values of p under a certain condition on p by using the concept of congruences

Theorem 1. [29]. *If* $<b_1, b_2, b_2, \cdots, b_k>$ *is symmetric continued fraction (CF) and* $<b_1, b_2, b_2, \cdots, b_k> = (R + \sqrt{M})/s$, *then* $M = R^2 + S^2$.

Theorem 2. [9]. *The set* $\mathbb{Q}''(\sqrt{m}) = \{\eta/s : \eta \in \mathbb{Q}^*(\sqrt{m}), s = 1, 2\}$ *is unchanged under the action of H.*

Theorem 3. [10]. *Let* $m \equiv 1(\mathrm{mod}\ 8)$. *Then,* $\mathbb{Q}''(\sqrt{m})$ *splits into four H-subsets. In particular,* $(\sqrt{m}/1)^H$, $(\sqrt{m}/-1)^H$, $((1 + \sqrt{m})/2)^H$, *and* $((1 + \sqrt{m})/4)^H$ *are at least four H-orbits of* $\mathbb{Q}''(\sqrt{m})$.

Theorem 4. [9]. *Let* $m \equiv 3(\mathrm{mod}\ 8)$. *Then,* $\mathbb{Q}''(\sqrt{m})$ *splits into three H-subsets. In particular,* $(\sqrt{m}/1)^H$, $(\sqrt{m}/-1)^H$, *and* $((1 + \sqrt{m})/2)^H$ *are at least three H-orbits of* $\mathbb{Q}''(\sqrt{m})$.

Lemma 5. *Every RQIN in* $\mathbb{Q}''(\sqrt{m})$ *is ambiguous number.*

Theorem 6. [29]. *If* $<b_1, b_2, b_2, \cdots, b_k>$ *is symmetric continued fraction and if* $<b_1, b_2, b_2, \cdots, b_k> = (R + \sqrt{M})/S$, *then* $M = R^2 + S^2$.

Theorem 7. [9]. *The set* $\mathbb{Q}''(\sqrt{m}) = \{\eta/s : \eta \in \mathbb{Q}^*(\sqrt{m}), s = 1, 2\}$ *is unchanged under the action of H.*

Theorem 8. [10]. *Let* $m \equiv 1(\mathrm{mod}\ 8)$. *Then,* $\mathbb{Q}''(\sqrt{m})$ *splits into four H-subsets. In particular,* $(\sqrt{m}/1)^H$, $(\sqrt{m}/-1)^H$, $((1 + \sqrt{m})/2)^H$, *and* $((1 + \sqrt{m})/4)^H$ *are at least four H-orbits of* $\mathbb{Q}''(\sqrt{m})$.

Theorem 9. [9]. *Let* $m \equiv 3(\mathrm{mod}\ 8)$. *Then,* $\mathbb{Q}''(\sqrt{m})$ *splits into three H-subsets. In particular,* $(\sqrt{m}/1)^H$, $(\sqrt{m}/-1)^H$, *and* $((1 + \sqrt{m})/2)^H$ *are at least three H-orbits of* $\mathbb{Q}''(\sqrt{m})$.

Lemma 10. *Every RQIN in* $\mathbb{Q}''(\sqrt{m})$ *is an ambiguous number.*

Lemma 11. [14]. $\beta = (a' + \sqrt{m'})/c'$ *is an ambiguous number if and only if* $c' < 0$ *and* $b' = (a'^2 - m)/c' > 0$ *or* $b' = (a'^2 - m)/c' < 0$, *and* $c' > 0$.

Remark 12. [9]. Let $\beta(a', b', c') \in Q^*(\sqrt{n'})$ and $m \in \mathbb{N}$. Then,

(1) $(x'y')^m(\beta) = (\beta) + m = (y'^3 x')^{-m}(\beta)$.

(2) $(y'x')^m(\beta) = (\beta)/(1 - 2m(\beta)) = (x'y'^3)^{-m}(\beta)$.

(3) $h^m(\beta) = (\beta_1) \in (\beta)^H$.

Remark 13. It should be noted here that for a reduced number $\beta = (a' + \sqrt{m'})/c'$, we have $a' > 0, c' > 0$, and $b' < 0$.

2. Properties of Reduced Quadratic Irrational Numbers in $Q''(\sqrt{m})$

This section is devoted to study the behavior of reduced numbers.

Lemma 14. *If* $r \in Q''(\sqrt{m})$ *is an RQIN, then* $x'(r)$ *is an ambiguous number but not RQIN.*

Proof. Let $r = (a' + \sqrt{m})/c'$ be a reduced quadratic irrational number such that $b' < 0, a' > 0,$ and $c' > 0$, where $b' = (a'^2 - m)/c'$. Then, by using the Mobius transformation $x'(r) = -1/2r$, we have $x'(r) = (-a' + \sqrt{m})/2b' = (a_1 + \sqrt{m})/b_1$, where $a_1 < 0$ and $b_1 < 0$. Since $b' < 0$ by using Remark 12, $x'(r)$ is not RQIN. □

Theorem 15. . *Let* $p \equiv 1$ *or* $5 (\bmod 8)$ *such that* $p - 1 = s^2$. *Then, the circuit of a reduced number* $r \in ((\lfloor\sqrt{p}\rfloor + \sqrt{p})/2)^H$ *has the type* $(2\sqrt{p-1})_2, (\sqrt{p-1})_0)$. *Moreover,* $\bar{r}$, $-r$, *and* $-\bar{r}$ *each exists on the turning points of the circuit and not reduced.*

Proof. $r \in ((\lfloor\sqrt{p}\rfloor + \sqrt{p})/2)^H$, $r = ((\lfloor\sqrt{p}\rfloor + \sqrt{p})/2)$, $(y'^3 x')^{\sqrt{p-1}-1}(r) = -\bar{r}$, where $(\sqrt{p-1} - 1)$ is the number of squares inside the circuit. $x'(y'^2 x')(-\bar{r}) = -r$ which shows that one circuit is lying between the inside and outside boundary of the circuit. $x'(y'x')^{\sqrt{p-1}-2}(-r) = -\bar{r}$, where $\sqrt{p-1} - 1$ is the number of squares inside the circuit. $y'^2(\bar{r}) = r$ which implies that one of the squares is lying between the inside and outside boundary of the circuit. □

Theorem 16. *For* $p \equiv 5$ *or* $1 (\bmod 2^3)$ *such that* $-1 + p = s^2$ *and* $r = (\lfloor\sqrt{p}\rfloor + \sqrt{p})/2$ *be a reduced number, then* $\bar{r}$, $-r$, *and* $-\bar{r}$ *map onto the nonreduced number under the action of* x'.

Proof. Let $r = \sqrt{p} + (\lfloor\sqrt{p}\rfloor - 1)/2$ and $-r = (\lfloor\sqrt{p}\rfloor + \sqrt{p})/-2$. By using linear fractional transformation $x' : \beta = -1/\beta$ and Table 1, where $\beta = (a + \sqrt{m})/c$, $x'(-r) = (-(\lfloor\sqrt{p}\rfloor - 1) + \sqrt{p})/c_1 = (a_1 + \sqrt{m})/c_1$ where $a_1 = -(\lfloor\sqrt{p}\rfloor - 1) < 0$. By using Remark 12, it is not a reduced number. Similarly, $x'(-r)$ is not a reduced number. $\bar{r} = (-(\lfloor\sqrt{p}\rfloor - 1) + \sqrt{p})/-2$ and $\beta = x'(-r) = (-(\lfloor\sqrt{p}\rfloor - 1) + \sqrt{p})/c'$; $c' = 2b > 0$ where $b > 0$. By Remark 12, hence β is not a RQIN. □

Theorem 17. *Let* $r = (a' + \sqrt{m})/c' \in Q''(\sqrt{m})$ *be a RQIN moved to* $1/2((a' + \sqrt{m})/c') \in Q''(\sqrt{m})$ *under a Mobius transformation* x'. *Then,*

$$\left(\frac{c' + \sqrt{m}}{a'}\right)^H \cap \left(\frac{a' + \sqrt{m}}{c'}\right)^H = \Phi. \quad (1)$$

Proof. Suppose $(a' + \sqrt{m})/c' \in Q''(\sqrt{n})$ be RQIN under Mobius transformation x' moved to half of their conjugate, i.e., $x'((a' + \sqrt{m})/c') = 1/2(a' - \sqrt{m}/c')$ by using Table 1, as $m = (a')^2 + (c')^2$. By using Theorem 1, $(a' + \sqrt{m})/c'$ and $-1/(2[(a' + \sqrt{m})/c'])$ have symmetric periodic part, since, in the form of continued fraction, every RQIN has unique description. In similar fashion, $(c' + \sqrt{m})/a'$ and $-1/2[(a' - \sqrt{m})/c']$ with symmetric periodic parts are identical. By Lemma 5 $(a' + \sqrt{m})/c'$ and $(c' + \sqrt{m})/a'$ are not identical. Hence, we conclude that

$$\left(\frac{c' + \sqrt{m}}{a'}\right)^H \cap \left(\frac{a' + \sqrt{m}}{c'}\right)^H = \Phi. \quad (2)$$

□

Lemma 18. *Let* $\beta = (a' + \sqrt{m})/c' \in Q''(\sqrt{m})$ *which moves to half of their conjugate under the linear fractional transformation* x'. *Then,* $Q''(\sqrt{m})$ *has at least 2 distinct circuits*

$$\left(\frac{c' + \sqrt{m}}{a'}\right)^H \text{ and } \left(\frac{a' + \sqrt{m}}{c'}\right)^H. \quad (3)$$

Example 1 reflects Lemma 18.

Example 1. Suppose $m = 13$ and $(2 + \sqrt{m})/3 \in Q''(\sqrt{13})$ be reduced quadratic irrational transformed to half of their conjugate under the x' transformation and $(1_1, 2_0, 1_1, 1_2, 1_1, 2_2, 1_1, 1_2)$ is the type of $((2 + \sqrt{m})/3)$. $(3 + \sqrt{m})/2 \in Q''(\sqrt{13})$ be reduced quadratic irrational transformed to half of their conjugate under the x' transformation and $(2_1, 1_o, 3_2, 1_0, 1_1, 1_2, 3_0)$ is the type of $((3 + \sqrt{m})/2)$. It is easy to see that $((2 + \sqrt{m})/3)$ and $((3 + \sqrt{m})/2)$ are not equivalent, so that $((2 + \sqrt{m})/3) \bigcap ((3 + \sqrt{m})/2) = \phi$ as shown in figures below.

Figures 1 and 2 reflect Lemma 18.

Table 1: Under the action of the group H. The images of elements of $Q^*(\sqrt{m})$ [11].

	$\beta = \frac{a' + \sqrt{m}}{c'}$	a'	b'	c'
$x'(\beta)$	$\frac{-1}{2\beta}$	$-a'$	$\frac{-c'}{2}$	$2b'$
$y'(\beta)$	$\frac{-1}{2(\beta+1)}$	$-a'-c'$	$\frac{-c'}{2}$	$2\left(2a'+b'+c'\right)$
$y'^2(\beta)$	$\frac{-(\beta+1)}{(2\beta+1)}$	$-3a'-2a'-c'$	$2a'+b'+c'$	$4a'+4b'+c'$
$y'^3(\beta)$	$\frac{-(2\beta+1)}{2\beta}$	$-a'+2b'$	$\frac{4a'+4b'+c'}{2}$	$2\left(2a'+b'+c'\right)$
$x'y'(\beta)$	$\beta+1$	$a'+c'$	$2a'+b'+c'$	c
$x'y'^3(\beta)$	$\frac{\beta}{2\beta+1}$	$a'+2b'$	b'	$4a'+4b'+c'$
$y'x'(\beta)$	$\frac{\beta}{1-2\beta}$	$a'-2b'$	b'	$-4a'+4b'+c'$
$y'^2x'(\beta)$	$\frac{1-2\beta}{2(-1+\beta)}$	$3a'-2a'-c'$	$\frac{-4a'+4b'+c'}{2}$	$2\left(-2a'+b'+c'\right)$
$y'^3x'(\beta)$	$\beta-1$	$a'-c'$	$-2a'+b'+c'$	c'

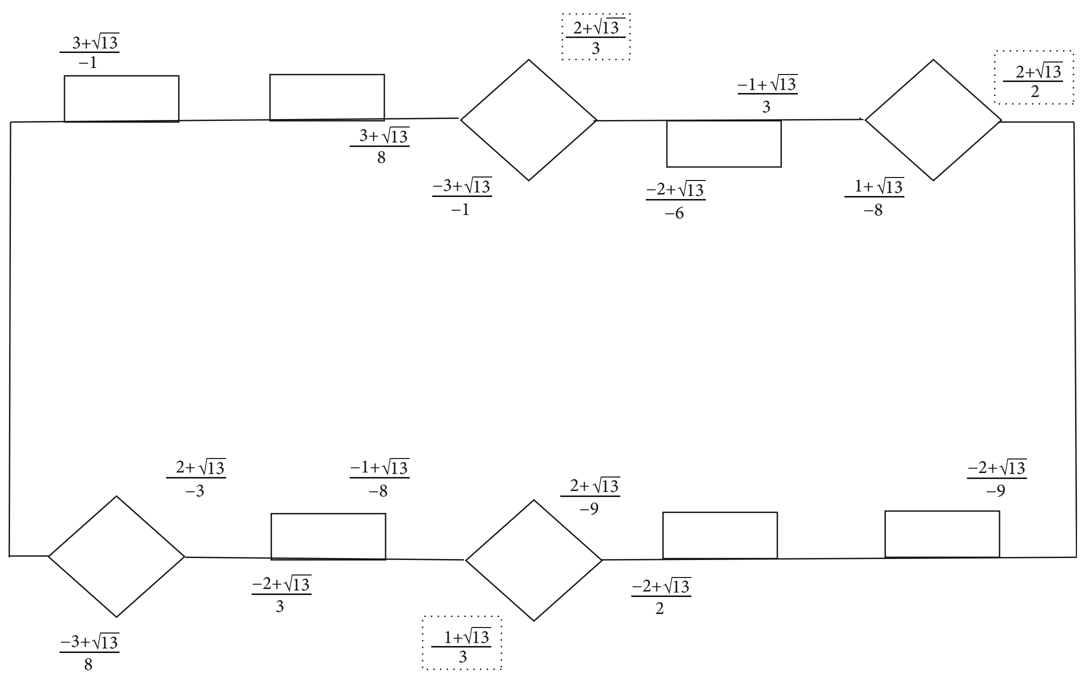

Figure 1: Closed path of $((2+\sqrt{13})/3)^H$.

3. Reduced Length of the H-Circuits of $Q''(\sqrt{P})$

The circuit generates an element of the form $g = (x'y'^{j_k+1})^{m_k} \cdots (x'y'^{j_2+1})^{m_2}(x'y'^{j_1+1})^{m_1}$ of H and fixes some vertex of a square on the closed orbit, and thus, the reduced length of closed orbit is the count of RNs in this closed circuit.

Example 2. The circuit of the type $(1_2, 1_0, 1_2, 1_1, 2_2, 4_0, 2_2, 1_1, 1_2, 1_0, 1_1, 8_0)$ represents that the circuit generates an element $k = (x'y')^8(x'y'^2)(x'y')(x'y'^3)(x'y'^2)(x'y'^3)^2(x'y'^2)(x'y'^3)^2(x'y'^2)(x'y'^3)(x'y')(x'y'^3)$ of H and fixes the vertex $r_1 = 4+\sqrt{19}$. Suppose $r_1 = 4+\sqrt{19}.\cdots\cdots\cdots(1)$, $(x'y'^3)r_1 = \beta_1$, $(x'y')\beta_1 = (3+\sqrt{19})/5 = r_2 \cdots\cdots\cdots..(2)$, $(x'y'^3)r_2 = (-1+\sqrt{19})/9 = \beta_2$, $(x'y'^2)\beta_2 = (2+\sqrt{19})/10 = \beta_3$, $(x'y'^3)^2\beta_3 = (-4+\sqrt{19})/2 = \beta_4$, $(x'y')^4\beta_4 = (4+\sqrt{19})/2 = r_3 \cdots\cdots\cdots..(3)$, $(x'y'^3)^2 r_3 = (-2+\sqrt{19})/10 = \beta_5$,

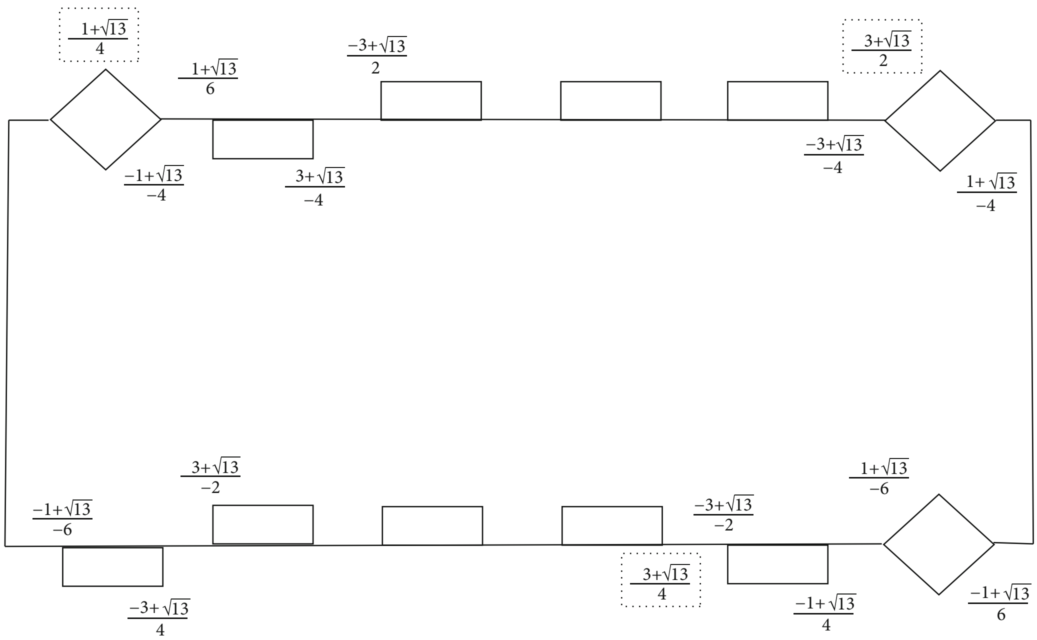

Figure 2: Closed path of $((3+\sqrt{13})/2)^H$.

$(x'y'^2\beta_5 = (1+\sqrt{19})/9 = \beta_6$, $(x'y'^3)\beta_6 = (2+\sqrt{19})/5 = r_4$ $\cdots..(4)$, $(x'y'^2)r_4 = -4+\sqrt{19} = \beta_7$, $(x'y'^8)\beta_7 = 4+\sqrt{19} = r_1$. Equations (1), (2), (3), and (4) follow that r_1, r_2, r_3, and r_4 are only reduced numbers in the orbit. Thus, the reduced length of this orbit is 4.

Now, we investigate the reduced cardinalities of H-orbits.

Theorem 19. *Let $p \equiv 1$ or $5 \pmod 8$ such that $p-1=s^2$ and then the circuit of the reduced number $(((\lfloor\sqrt{p}\rfloor-1)+\sqrt{p})/2)^H$ has the type $((\sqrt{p-1}-1)_2, 1_1, (\sqrt{p-1}-1)_0, 1_1)$, and $|(((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2)^H|_{red}=1$*

Proof. In order to prove that it is enough to find $k\in H$ in such a manner $k(((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2) = (((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2)$. The proof was followed by the following four steps: $(y'^3x')^{\sqrt{p-1}-1}(r) = (y'^3x')^{\sqrt{p-1}-1}(((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2) = ((-(\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/-2) = -\bar{r}$, $x'(y'^2x')(-\bar{r}) = x'(y'^2x')((-(\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/-2) = (((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/-2) = -r$, $x'(y'x')^{\sqrt{p-1}-2}y'(-r) = x'(y'x')^{\sqrt{p-1}-2}y'(((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/-2) = ((-(\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/-2) = \bar{r}$, and $y'^2(\bar{r}) = y'^2((-(\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/-2) = (((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2) = r$. Thus, we obtain $y'^2x'(y'x')^{\sqrt{p-1}-2}y'x'(y'^2x')(y'^3x')^{\sqrt{p-1}-1}(((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2) = (((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2)$. Hence, the circuit of the reduced number $(((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2)^H$ has the type $((\sqrt{p-1}-1)_2, 1_1, (\sqrt{p-1}-1)_0, 1_1)$. Now, we have to prove that $|(((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2)^H|_{red}=1$. Let $r = (((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2)$ be reduced number. Now, by using Theorem 15 and Theorem 16, the numbers $\bar{r}, -\bar{r}$, and $-r$ are on the turning point of the circuit and are not reduced numbers; furthermore, when we will apply linear fractional transformation x' on $\bar{r}, -\bar{r}$, and $-r$, then in result, we get no reduced number. So, $(((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2)$ is only reduced number in $(((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2)^H$. Hence, $|(((\sqrt{\lfloor\sqrt{p}\rfloor}-1)+\sqrt{p})/2)^H|_{red}1$. □

Example 3. Take a prime number $p=17$ such that $17-1=4^2$ and $17\equiv 1 \pmod 8$. It is observed from the coset diagram given below that the reduced number $(((\sqrt{\lfloor 17\rfloor}-1)+\sqrt{17})/2)$ is fixed by the word $(x'y'^2)(y'x')^3(x'y'^2)(y'^3x')^3(((\sqrt{\lfloor 17\rfloor}-1)+\sqrt{17})/2) = (((\sqrt{\lfloor 17\rfloor}-1)+\sqrt{17})/2)$; this shows that type of the circuit $(((\sqrt{\lfloor 17\rfloor}-1)+\sqrt{17})/2)^H$ is $(3_2, 1_1, 3_0, 1_1)$, and it can be seen from the coset diagram given below; $(((\sqrt{\lfloor 17\rfloor}-1)+\sqrt{17})/2)$ is only reduced number in $(((\sqrt{\lfloor 17\rfloor}-1)+\sqrt{17})/2)^H$, and hence, $|(((\sqrt{\lfloor 17\rfloor}-1)+\sqrt{17})/2)^H|_{red}=1$

Figure 3 depicted Theorem 19.

Example 4. Take a prime number $p=101$ such that $p-1=10^2$ and $p\equiv 5 \pmod 8$. It is observed from the coset diagram given below that the reduced number $((\sqrt{\lfloor 101\rfloor}-1)+\sqrt{101}/2)$ is fixed by the word $(x'y'^2)(y'x')^{10}(x'y'^2)(y'^3x')^{10}((\sqrt{\lfloor 101\rfloor}-1)+\sqrt{101}/2) = ((\sqrt{\lfloor 101\rfloor}-1)+\sqrt{101}/2)$; this shows that type of $((\sqrt{\lfloor 101\rfloor}-1)+\sqrt{101}/2)^H$

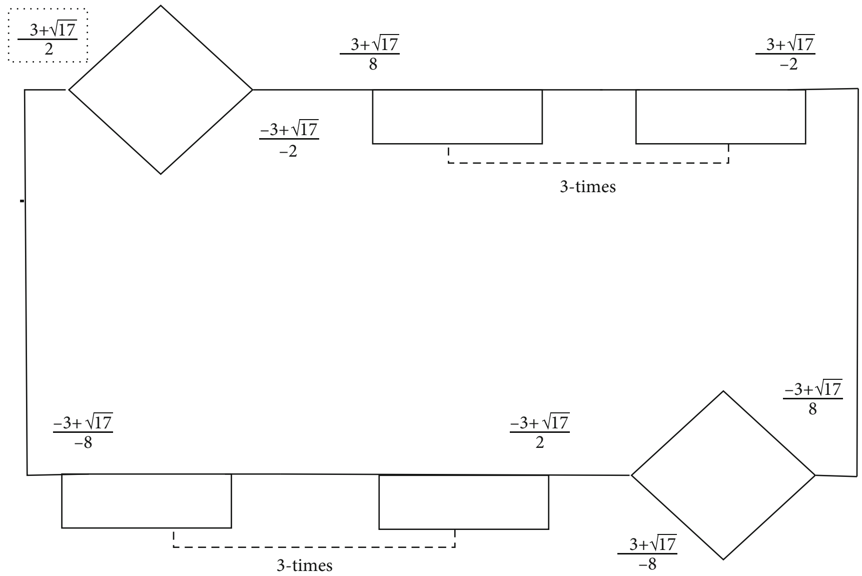

Figure 3: Closed path of $(((\lfloor\sqrt{17}\rfloor - 1) + \sqrt{17})/2)^H$.

of the circuit $((\sqrt{\lfloor 101\rfloor} - 1) + \sqrt{101}/2)^H$ is $(9_2, 1_1, 9_0, 1_1)$, and it can be seen from the coset diagram given below that $((\sqrt{\lfloor 101\rfloor} - 1) + \sqrt{101}/2)$ is only reduced number in $((\sqrt{\lfloor 101\rfloor} - 1) + \sqrt{101}/2)^H$, and hence, $|((\sqrt{\lfloor 101\rfloor} - 1) + \sqrt{101}/2)^H|_{\text{red}} = 1$

Figure 4 reflects Example 4.

Lemma 20. *For $p \equiv 5$ or $3 (\bmod 2^3)$ such that $-2 + p = s^2$, then the orbits of reduced numbers $(s + \sqrt{p})^H$ and $((s + \sqrt{p})/4)^H$ have the type $(((s-1)/2)_2, 1_1, (s-1)_0, 1_1, ((s-1)/2)_2, 2s_o)$ and $|(s + \sqrt{p})^H|_{red} = 2 = |((s + \sqrt{p})/4)^H|_{red}$.*

Proof. To show that it is enough to discover $k \in H$ in such a manner $k(r_1) = r_1$, where $r_1 = s + \sqrt{p}$......(1) using Remark 12(1) and (3), we obtain $(x'y')^{-2s}(r_1) = -2s + r_1 = -r + \sqrt{p} = -\bar{(r_1)}$. Now, $(x'y'^3)^{(s-1)/2}(r_1) = ((s-1)(s^2-p) + s + \sqrt{p})/((s-1)[2s^2 + (s^2-p)(s-1)] + 1) = \alpha$ and $(x'y'^3)^{-(s-1)/2}(-\bar{(r_1)} = (-[(s-1)(s^2-p) + s] + \sqrt{p})/((s-1)[2s + (s^2-p)(s-1)] + 1) = -\bar{(\alpha)}$. In Table 1, $(x'y'^2(\alpha) = ((s-1)(s^2-p+1) + \sqrt{p})/(-(s^2-p)) = \beta$ and $(x'y'^2)^{-1}(-\bar{(\alpha)})\ ((-(s-1)(s^2-p+1) + \sqrt{p})/(-(s^2-p))) = -\bar{\beta} = (r_2) \cdots ..(2)$. Finally, $(x'y')^{-(s-1)}(r_2) = (x'y')^{-(s-1)}(-\bar{\beta}) = ((s-1)(s^2-p+1) + \sqrt{p})/(-(s^2-p)) = \beta$. $(x'y')^{2s}(x'y'^3)^{(s-1)/2}(x'y'^2)(x'y')^{s-1}(x'y'^2)(x'y'^3)^{(s-1)/2}(r_1) = r_1$. Hence, $(((s-1)/2)_2, 1_1, (s-1)_0, 1_1, ((s-1)/2)_2, 2s_o)$ be the type of circuit of reduce number $(s + \sqrt{p})^H$. Similarly, the type of $((s + \sqrt{p})/4)^H$ is same as first one and from equations (1) and (2); hence, $|(s + \sqrt{p})^H|_{\text{red}} = 2 = |((s + \sqrt{p})/4)^H|_{\text{red}}$. □

Example 5. Take a prime number $p = 83$ such that $p - 2 = 9^2$ and $p \equiv 3 (\bmod 8)$. It is observed from the coset diagram given below that the reduced number $(9 + \sqrt{83})$ is fixed by the word $(x'y')^{18}(x'y'^3)^4(x'y'^2)(x'y')^8(x'y'^2)(x'y'^3)^4)(9 + \sqrt{83}) = (9 + \sqrt{83})$; this shows that type of the circuit $(9 + \sqrt{83})^H$ is $(4_2, 1_1, 8_0, 1_1, 4_2, 18_0)$, and it can be seen from the coset diagram given below; $(9 + \sqrt{83})$ and $(8 + \sqrt{83})/2$ are only reduced number in $(9 + \sqrt{83})^H$, and hence, $|(9 + \sqrt{83})^H|_{\text{red}} = 2$.

Figure 5 reflects Lemma 20.

Lemma 21. *If $4|p - 3$ and $1 + p = s^2$, then $(\sqrt{p} + \lfloor\sqrt{p}\rfloor)^H$ and $(\sqrt{p}/-1)^H$ circuits have the type (s_o, s_1). Moreover $|(\lfloor\sqrt{p}\rfloor + \sqrt{p})^H|_{red} = 1$ and $|(\sqrt{p}/-1)^H|_{red} = 0$.*

Proof. Similar proof as of Lemma 20. □

Remark 22. (i) It is not necessary that every circuit contains reduced number. As we can see in the figure given below, the circuit of $(\sqrt{p}/-1)$ contains no reduced number.

Figure 6 reflects Remark 22(i).

3.1. Detection of Reduced Numbers. In the orbits of $(s + \sqrt{p})^H$ and $((s + \sqrt{p})/4)^H$ of $Q''(\sqrt{p})$, where $p \equiv 3$ or 5 (mod 8) such that $p - 2 = s^2$, then

(i) $(s + \sqrt{p})$ and $(((s-1) + \sqrt{p})/2)$ are only reduced numbers in the circuit of $(s + \sqrt{p})^H$

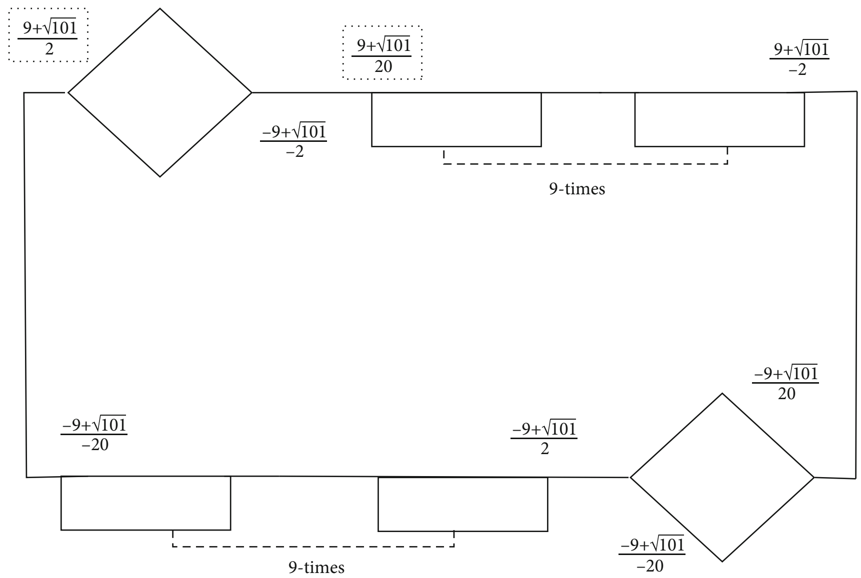

Figure 4: Closed path of $((\sqrt{\lfloor 101\rfloor}-1)+\sqrt{101}/2)^H$.

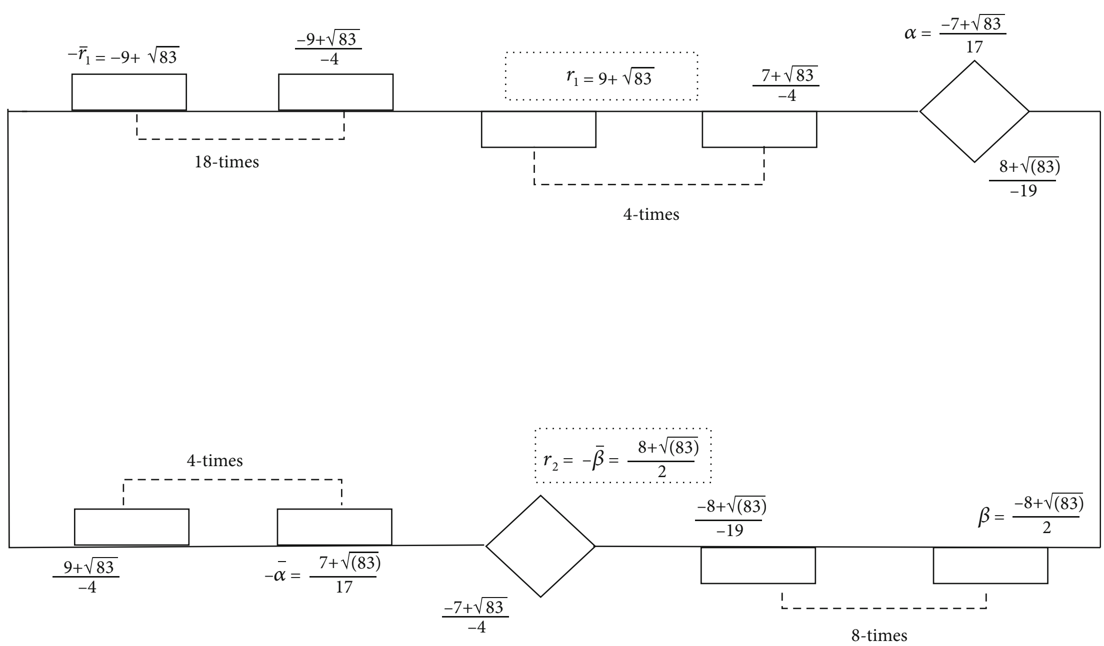

Figure 5: Closed path of $(9+\sqrt{83})^H$.

(ii) $(((s-2)+\sqrt{p})/4)$ and $((s+\sqrt{p})/4)$ are only reduced numbers in the circuit of $((r+\sqrt{p})/4)^H$

$$(s+\sqrt{p})^H \cap \left(\frac{s+\sqrt{p}}{4}\right)^H = \phi. \quad (4)$$

Remark 23. For $p \equiv 3(\text{mod } 2^3)$ such that $-2+p=s^2$.

(i) If a reduced number $r \in (\sqrt{p}+s)^H$, then its negative conjugate $-\bar{r} \in (\sqrt{p}+s)^H$

(ii) If a reduced number $r \in ((\sqrt{p}+s)/4)^H$ then, its negative conjugate $-\bar{r} \in ((s+\sqrt{p})/4)^H$

Lemma 24. *If $p \equiv 7(\text{mod } 2^3)$ and $2+p=s^2$ then, the circuit $((s-1)+\sqrt{p})^H$ of the reduced number has the type $(1_1, (((s-1)/2)-1)_o, 1_2, (s-1)_o, 1_2, (((s-1)/2)-1)_o, 1_1, 2(s-1)_o)$, and hence, $|(s-1)+\sqrt{p})^H|_{red}=4$*

Proof. To illustrate this, it is sufficient to find $k \in H$ such that $k(r_1)=r_1 \cdots .(i)$, where $r_1=(s-1)+\sqrt{p}$ by Remark 12,

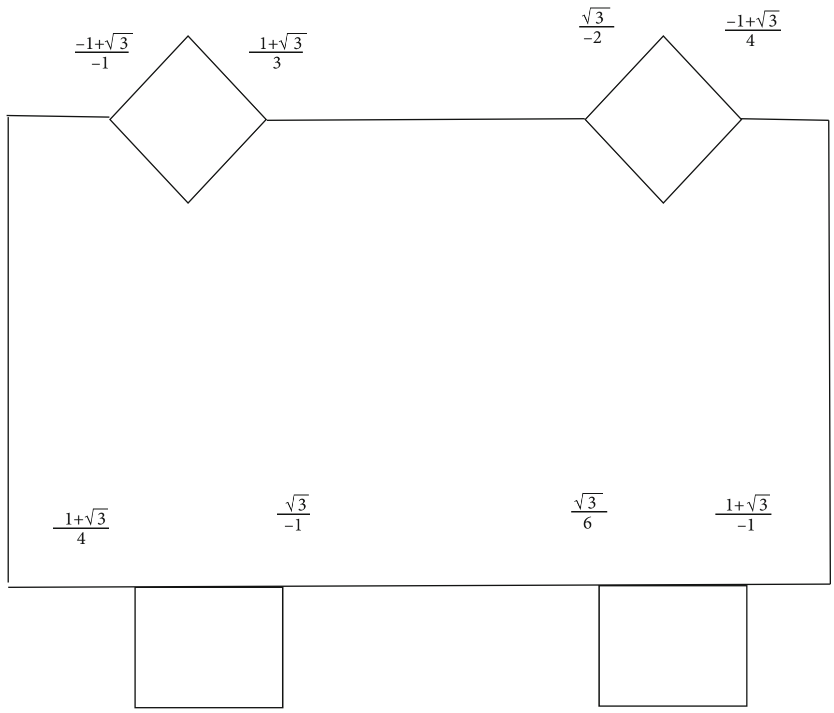

Figure 6: Closed path of $(\sqrt{3}/-1)^H$.

then $(x'y')^{-2(s-1)}(r_1) = (x'y')^{-2(s-1)}((s-1)+\sqrt{p}) = -(s-1)+\sqrt{p}) = -\bar{r}_1$; now, by using Table 1. $(x'y'^2)(r_1) = (2(s^2-p)-s+\sqrt{p})/(2(s^2-p)) = \beta_1$ and $(x'y'^2)^{-1}(-\bar{r}_1) = (-2(s^2-p)+s+\sqrt{p})/(2(s^2-p)) = (-\bar{\beta}_1) = r_2 \cdots\cdots\cdots (ii)$, again by (1.1), we have $(x'y')^{(s-3)/2}(r_2) = (x'y')^{(s-3)/2}(-\bar{\beta}_1) = ((s^2-p)(s-1)-s+\sqrt{p})/(2(s^2-p)) = r_3 \cdots\cdots\cdots .(iii)$, and $(x'y')^{-((s-3)/2)}(r_2) = (x'y')^{-((s-3)/2)}(-\bar{\beta}_1) = (-(s^2-p)(s-1)+s+\sqrt{p})/(2(s^2+p)) = -\bar{r}_3$, by Table 1, $(x'y'^3)(r_3) = (s^3+s(b-3)+1+\sqrt{p})/(s^2-p) = \beta_2$ and $(x'y'^{-1})(-\bar{r}_3) = (s^3+s(p+3)-1+\sqrt{p})/(s^2-p) = -\bar{\beta}_2 = r_4 \cdots .(iv)$; finally, $(x'y')^{s-1}(r_4) = ((2s-1)(s^2-p-1)-s+\sqrt{p})/(s^2-p) = \beta_3$, and $\beta_3 = r_4$. Thus, $(x'y')^{2(s-1)}(x'y'^2)(x'y')^{(((s-1)/2)-1)}(x'y'^3)(x'y')^{s-1}(x'y'^3)(x'y')^{(((s-1)/2)-1)}(x'y'^2)(r_1) = r_1$, and from equations (i), (ii), (iii), and (iv), we get (r_1), r_2, r_3, and r_4 which are only 4 reduced numbers in the circuit of $((s-1)+\sqrt{p})^H$.

Hence, $|((s-1)+\sqrt{p})^H|_{red}$ =4. □

Example 6. Take a prime number $p = 79$ such that $p+2 = 9^2$. It is observed from the coset diagram given below that the reduced number $(8+\sqrt{79})$ is fixed by the word $(x'y')^{16}(x'y'^2)(x'y')^3(x'y'^3)(x'y')^8(x'y'^3)(x'y')^3(x'y'^2)(8+\sqrt{79}) = (8+\sqrt{79})$; this shows that type of $(8+\sqrt{79})^H$ of the circuit is $(1_1, 3_0, 1_2, 8_o, 1_2, 3_0, 1_1, 16_o)$, and also, as can be seen from the coset diagram given below, $(8+\sqrt{79})$, $(7+\sqrt{79})/4$, $(8+\sqrt{79})/2$, and $(5+\sqrt{79})/4$ are only reduced number in $(8+\sqrt{79})^H$ and hence $|(8+\sqrt{79})^H|_{red} = 4$.

Figure 7 reflects Lemma 24.

Lemma 25. *If* $p \equiv 7(\text{mod } 2^3)$ *and* $2+p = s^2$ *then, the circuit* $(((s-1)+\sqrt{p})/(2s-3))^H$ *of the reduced number has the type* $(1_2, ((s-1)/2-1)_o, 1_1, 2(s-1)_o 1_1, ((s-1)/2-1)_o, 1_2, (s-1)_o,)$, *and hence,* $|(((s-1)+\sqrt{p})/2s-3)^H|_{red} = 2$.

Proof. Similar proof as of Lemma 24. □

Example 7. Take $p = 167$ such that $p+2 = 13^2$. It is observed from the coset diagram given below that the reduced number $(12+\sqrt{167})/23$ is fixed by the word $(x'y')^{12}(x'y'^3)(x'y')^5(x'y'^2)(x'y')^{24}(x'y'^2)(x'y')^5(x'y'^3)(12+\sqrt{167})/23 = (12+\sqrt{167})/23$; this shows that type of the circuit $((12+\sqrt{167})/23)^H$ is $(12_o, 1_2, 5_0, 1_1, 24_o, 1_1, 5_0, 1_2)$, and it can be seen from the coset diagram given below, $(12+\sqrt{167})/23$ and $(11+\sqrt{167})/23$ and are only reduced numbers in $((12+\sqrt{167})/23)^H$, and hence, $|((12+\sqrt{167})/23)^H|_{red} = 2$.

Figure 8 reflects Lemma 25.

3.2. Detection of Reduced Numbers. In the circuits of $((s-1)+\sqrt{p})^H$ and $(((s-1)+\sqrt{p})/((2s-3)))^H$ of $Q''(\sqrt{p})$ where $8|p-7$ and $2+p = s^2$, then,

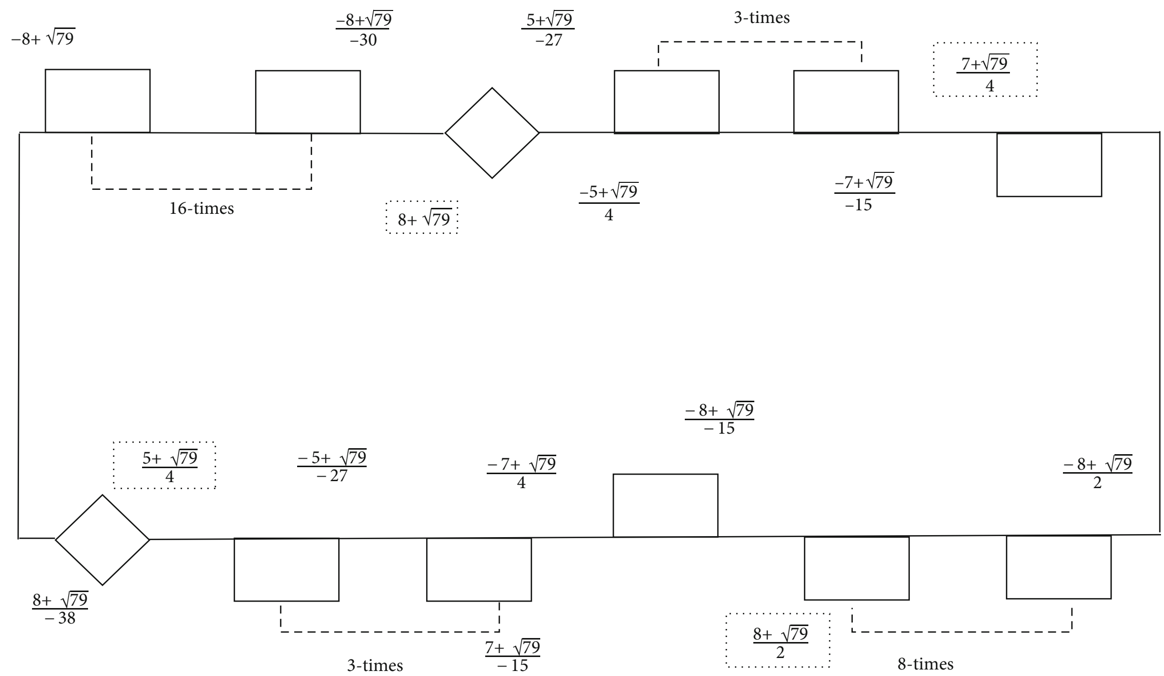

FIGURE 7: Orbit of $((8+\sqrt{79})/1)^H$ 520703).

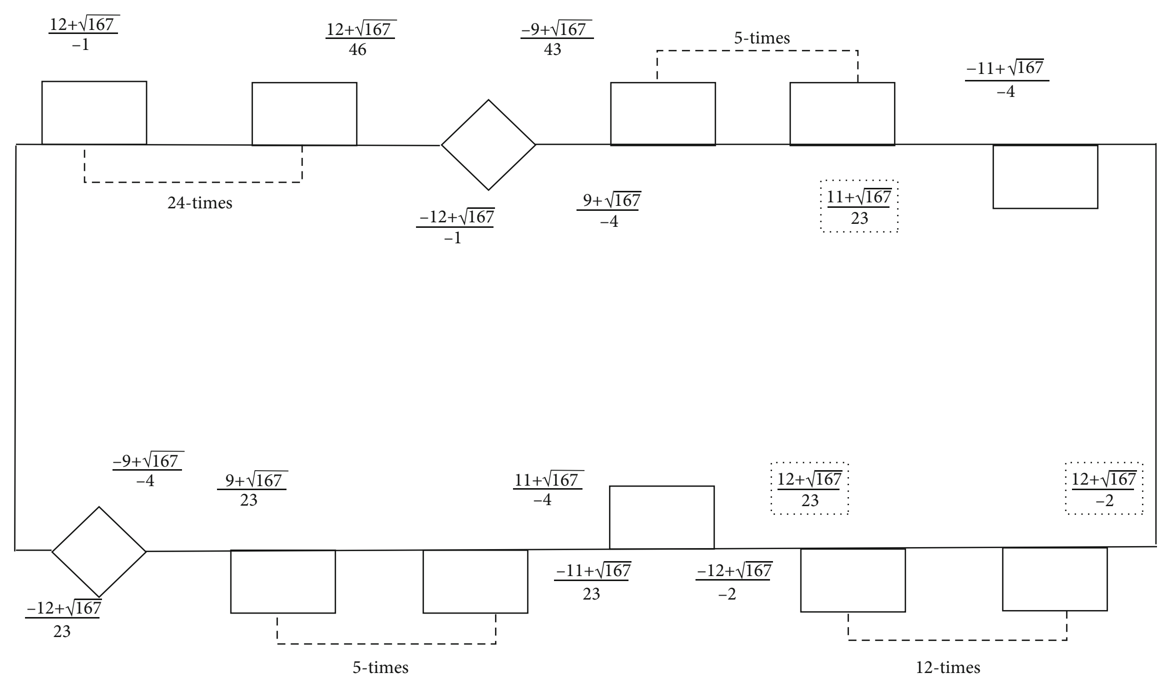

FIGURE 8: Closed path of $((12+\sqrt{167})/23)^H$.

(i) $((s-1)+\sqrt{p})$, $(((s-2)+\sqrt{p})/4)$, $(((s-1)+\sqrt{p})/2)$, and $(((s-4)+\sqrt{p})/4)$ are only reduced numbers in the circuit of $((s-1)+\sqrt{p})^H$

(ii) $(((s-2)+\sqrt{p})/((2s-3)))$ and $(((s-1)+\sqrt{p})/((2s-3)))$ are only reduced numbers in the circuit of $(((s-2)+\sqrt{p})/((2s-3)))^H$

Remark 26. For $p \equiv 7 (\text{mod } 2^3)$ such that $2+p=s^2$.

(1) If $r \in ((s-1)+\sqrt{p})^H$ then $-\bar{r} \in ((s-1)+\sqrt{p})^H$

(2) If $r \in (((s-1)+\sqrt{p})/((2s-3)))^H$, then $-\bar{r} \in (((s-1)+\sqrt{p})/((2s-3)))^H$

(3) $((s-1)+\sqrt{p})^H \bigcap (((s-1)+\sqrt{p})/((2s-3)))^H = \phi$.

4. Conclusion

The idea of types of H-circuits in H-orbits of RQF by Mobius group, which is given in this paper, is new and original. We have presented type of H-circuits with different length in H-orbits $(\beta)^H$, where β is RQIN and H be Mobius group. We have investigated properties of RQINs and classified H-orbits of different length. Furthermore, we proposed reduced length and general form of reduced numbers in different orbits. This work can be extended for the Mobius group $M = <x', y' : x'^2 = y'^6 = 1>$ and $G = <x', y' : x'^2 = y'^3 = 1>$ as well as examined the M-circuits in M-orbits and the G-circuits in G-orbits. Moreover, the reduced length and general form of reduced numbers for different orbits can be discussed.

Data Availability

No real data were used to support this study. The data used in this study are hypothetical, and anyone can use them by citing this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References

[1] I. Shahzad, Q. Mushtaq, and A. Razaq, "Construction of new S-box using action of quotient of the modular group for multimedia security," *Security and Communication Networks*, vol. 2019, Article ID 2847801, 13 pages, 2019.

[2] A. Razaq, H. A. Al-Olayan, A. Ullah, A. Riaz, and A. Waheed, "A novel technique for the construction of safe substitution boxes based on cyclic and symmetric groups," *Security and Communication Networks*, vol. 2018, Article ID 4987021, 9 pages, 2018.

[3] A. Razaq, H. Alolaiyan, M. Aehmad et al., "A novel method for generation of strong substitution-boxes based on coset graphs and symmetric groups," *IEEE Access*, vol. 8, pp. 75473–75490, 2020.

[4] A. Razaq, A. Ullah, H. Alolaiyan, and A. Yousaf, "A novel group theoretic and graphical approach for designing cryptographically strong nonlinear components of block ciphers," *Wireless Personal Communications*, vol. 116, no. 4, pp. 1–26, 2020.

[5] M. A. Yousaf, H. Alolaiyan, M. Ahmad, M. Dilbar, and A. Razaq, "Comparison of pre and post-action of a finite abelian group over certain nonlinear schemes," *IEEE Access*, vol. 8, pp. 39781–39792, 2020.

[6] W. Gao, B. Idrees, S. Zafar, and T. Rashid, "Construction of nonlinear component of block cipher by action of modular group PSL(2, Z) on projective line PL(GF(28))," *IEEE Access*, vol. 8, pp. 136736–136749, 2020.

[7] Q. Mushtaq and M. Aslam, "Modular group acting on real quadratic fields," *Bulletin of the Australian Mathematical Society*, vol. 37, no. 2, pp. 303–309, 1988.

[8] Q. Mushtaqr and M. Aslam, "Group generated by two elements of orders 2 and 4 acting on real quadratic fields," *Acta Mathematics Sinica*, vol. 9, no. 1, pp. 48–54, 1993.

[9] M. A. Malik and M. A. Zafar, "Certain $H-$ subsets of $Q(\sqrt{m}) \setminus Q$ under the action of $<x', y' : x'^2 = y'^4 = 1>$," *PUJM*, vol. 64, 2012.

[10] M. A. Zafar and M. A. Malik, "Malik certain type of the orbits of real quadratic fields by Hecke groups," *Mathematical Sciences Letters*, vol. 5, no. 1, pp. 93–97, 2016.

[11] M. A. Malik and M. A. Zafar, "On orbits of $Q(\sqrt{m}) \setminus Q$ under the action of Hecke group $H(\sqrt{2})$," *Middle-East Journal of Scientific Research*, vol. 15, no. 12, pp. 1641–1650, 2013.

[12] A. Razaq, Q. Mushtaq, and A. Yousaf, "The number of circuits of length 4 in PSL $(2, Z)$-space," *Communications in Algebra*, vol. 46, no. 12, pp. 5136–5145, 2018.

[13] A. Razaq, S. Akhter, A. Yousaf, U. Shuaib, and M. Ahmad, "A group theoretic construction of highly nonlinear substitution box and its applications in image encryption," *Multimedia Tools and Applications*, vol. 81, no. 3, pp. 4163–4184, 2022.

[14] S. Ali and M. A. Malik, "Classification of PSL $(2, Z)$-circuits having length six," *Indian Journal of Science and Technology*, vol. 11, no. 42, 2018.

[15] M. A. Malik and S. Ali, "Reduced quadratic irrational numbers and types of G-circuits with length four by modular group," *Indian Journal of Science and Technology*, vol. 11, no. 30, 2018.

[16] T. Chen, M. N. Bari, M. A. Malik, H. M. Afzal Siddiqui, and J. B. Liu, "Icosahedral group and classification of PSL(2, Z)-orbits of real quadratic fields," *Journal of Mathematics*, vol. 2020, Article ID 9568254, 10 pages, 2020.

[17] Q. Mushtaq and S. Anis, "Coset diagram for the action of picard group on $Q(i, \sqrt{3})$," *Algebra Colloquium*, vol. 23, no. 1, pp. 33–44, 2016.

[18] A. Yousaf, H. Alolaiyan, A. Razaq, and M. Younis, "Evolution of ambiguous numbers under the actions of a Bianchi group," *Journal of Taibah University for Science*, vol. 14, no. 1, pp. 615–620, 2020.

[19] A. Razaq, A. Yousaf, U. Shuaib, N. Siddiqui, A. Ullah, and A. Waheed, "A novel construction of substitution box involving coset diagram and a bijective map," *Security and Communication Networks*, vol. 2017, Article ID 5101934, 16 pages, 2017.

[20] A. Azizi, A. Zekhnini, and M. Taous, "On some metabelian 2 -group whose abelianization is of type $(2, 2, 2)$ andapplications," *Journal of Taibah University for Science*, vol. 9, no. 3, pp. 346–350, 2015.

[21] M. Marin, S. Vlase, R. Ellahi, and M. M. Bhatti, "On the partition of energies for the backward in time problem of thermoelastic materials with a dipolar structure," *Symmetry*, vol. 11, no. 7, p. 863, 2019.

[22] M. H. Mateen and M. K. Mahmood, "Power digraphs associated with the congruence $x^n \equiv y(mod m)$," *Punjab University Journal of Mathematics*, vol. 51, pp. 93–102, 2019.

[23] M. H. Mateen, M. K. Mahmood, D. Alghazzawi, and J.-B. Liu, "Enumeration of components and cycles of digraph over the

map $x^p \equiv y(modm)$," *AIMS Mathematics*, vol. 6, no. 5, pp. 4581–4596, 2021.

[24] M. H. Mateen, M. K. Mahmood, D. A. Kattan, and S. Ali, "A novel approach to find partitions of Z_m with equal sum subsets via complete graphs," *AIMS Mathematics*, vol. 6, no. 9, pp. 9998–10024, 2021.

[25] M. Haris Mateen and M. Khalid Mahmood, "A new approach for the enumeration of components of digraphs over quadratic maps," *Journal of prime research in mathematics*, vol. 16, no. 2, pp. 56–66, 2020.

[26] M. Haris Mateen, M. Khalid Mahmoud, and S. Ali, "Importance of power digraphs in computer science," in *2019 International Conference on Innovative Computing (ICIC)*IEEE.

[27] M. Haris Mateen, M. Khalid Mahmood, S. Ali, and M. D. A. Alam, "On symmetry of complete graphs over quadratic and cubic residues," *Journal of Chemistry*, vol. 2021, Article ID 4473637, 9 pages, 2021.

[28] H. Alolaiyan, A. Razaq, A. Yousaf, and R. Zahra, "A comprehensive overview on the formation of homomorphic copies in coset graphs for the modular group," *Journal of Mathematics*, vol. 2021, Article ID 3905425, 11 pages, 2021.

[29] S. G. Telang, M. G. Nadkarni, and J. S. Dani, *Number Theory*, Tata McGraw-Hill Publishing Company Limited, 1996.

[30] A. Adler and J. E. Coury, *The Theory of Numbers*, Jones and Bartlett Publishers, Inc., 1995.

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Research Article

Novel Algorithms for Solving a System of Absolute Value Variational Inequalities

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The goal of this paper is to study a new system of a class of variational inequalities termed as absolute value variational inequalities. Absolute value variational inequalities present a rational, pragmatic, and novel framework for investigating a wide range of equilibrium problems that arise in a variety of disciplines. We first develop a system of absolute value auxiliary variational inequalities to calculate the approximate solution of the system of absolute variational inequalities, and then by employing the projection technique, we prove the existence of solutions of the system of absolute value auxiliary variational inequalities. By utilizing an auxiliary principle and the existence result, we propose several new iterative algorithms for solving the system of absolute value auxiliary variational inequalities in the frame of four different operators. Furthermore, the convergence of the proposed algorithms is investigated in a thorough manner. The efficiency and supremacy of the proposed schemes is exhibited through some special cases of the system of absolute value variational inequalities and an illustrative example. The results presented in this paper are more general and rehash a number of some previously published findings in this field.

1. Introduction

The theory of variational inequalities, which was presented in the 1960s, exhibits an exceptional evolution as a fascinating and stimulating branch of applied mathematics that assumes a significant role in economics, finance, industry, transportation, optimization, and network analysis. Stampacchia [1] was the first to demonstrate the existence and uniqueness of variational inequality solutions. Variational inequalities have been utilized to examine problems that occurred in a variety of basic and applied sciences since their origin (see [2–6]). These significant applications prompted researchers to develop and broaden variational inequalities and associated optimization problems in various formations employing advanced and innovative methodologies, which include auxiliary principal technique, Wiener-Hopf equations, projection methods, and dynamical systems (see [7–10] and the references therein). It is noted that the operator must be Lipschitz continuous and strongly monotone for projection schemes to converge which is a very difficult set of requirements to verify. This fact led researchers to modify the projection method or to establish new ones. Extragradient-type methods address this difficulty as their convergence requires only the existence of solution and the Lipschitz continuity of the monotone operator. Various modified projection and extragradient-type algorithms have been proposed for finding the solution of variational inequalities. We would like to point out that the projection technique is not appropriate for some variational inequality classes that include nonlinear functions which fail to be differentiable. These factors prompted us to employ the auxiliary principle technique, presented by Glowinski et al. [11]. They employed this method to investigate the existence of a mixed variational inequality solution. Adopting the fixed point approach, this strategy finds the auxiliary variational inequality and proves that the solution obtained from the auxiliary problem is the same as the solution of the underlying problem.

The system of variational inequalities is a natural and useful generalization of variational inequalities because it can be used to describe a variety of equilibrium problems, including traffic equilibrium, spatial equilibrium, the Nash equilibrium, and general equilibrium problems (see [12–16]). The emergence of this approach can be noticed as the simultaneous acquisition of two distinct figurations of research; that is, it validates the qualitative features of the solution of major types of problems, while also empowering us to build useful and effective new problem-solving strategies. Various iterative algorithms have been suggested to solve the different systems of variational inequalities. Agarwal et al. [17] considered a system of generalized nonlinear mixed quasi-variational inclusions and proved its associated sensitivity analysis. However, Pang [18] has shown that several equilibrium-type problems other than the Nash equilibrium problem may also be stated as a variational inequality problem that is equivalent to a system of variational inequalities. Thus, the variational inequality theory gives a natural, comprehensive, ordered, and effective framework for analyzing many linear and nonlinear problems.

In recent years, another remarkable extension of the variational inequalities known as absolute value variational inequalities is introduced and studied by Batool et al. [9]. They have shown that the absolute value variational inequalities can be transformed into a system of absolute value equations if the underlying domain is the entire space. The system of absolute value equations was proposed and analyzed by Mangasarian [19]. In fact, the system of absolute value equations has become an appealing direction for researchers as various mathematical and engineering problems including linear programs, quadratic programs, and bimatrix games can be reduced into an absolute system of equations (see [20–22] and the references therein). Inspired by the significant boost in this field, in this investigation, the approaches for constructing a novel system of absolute value variational inequalities in connection with the fixed point formulation were proposed with the help of the projection method. By equivalency, several new projection algorithms have been developed that are useful for solving the system of absolute value variational inequalities. Moreover, we examine the convergence of these algorithms under suitable constraints. Various special cases are also considered. A test example illustrates the graphical view of our proposed results. The suggested methods associate a variety of iterative algorithms in this direction. The findings in this study are more invigorating and can be viewed as an improvement and extension of the previously known results.

2. Results and Discussion

Let $\mathscr{H}$ be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K_1 and K_2 be two closed and convex sets in $\mathscr{H}$. For given operators $T_1, T_2, B_1, B_2 : \mathscr{H} \longrightarrow \mathscr{H}$, consider the problem of finding $y \in K_1$ and $x \in K_2$ such that

$$\begin{cases} \langle T_1x + B_1|x| - f_1, v - y\rangle \geq 0, & \forall v \in K_1, \\ \langle T_2y + B_2|y| - f_2, v - x\rangle \geq 0, & \forall v \in K_2, \end{cases} \tag{1}$$

where f_1 and f_2 are the continuous functionals defined on $\mathscr{H}$ and $|.|$ contains the absolute values of components of $x, y \in \mathscr{H}$. The system (Equation (1)) is called a system of absolute value variational inequalities with four operators.

We will now discuss some special cases of the system of absolute value variational inequalities (Equation (1)).

(1) If $B_1 = B_2 = B$, then system (Equation (1)) reduces to find $y \in K_1$ and $x \in K_2$ such that

$$\begin{cases} \langle T_1x + B|x| - f_1, v - y\rangle \geq 0, & \forall v \in K_1, \\ \langle T_2y + B|y| - f_2, v - x\rangle \geq 0, & \forall v \in K_2, \end{cases} \tag{2}$$

which is called a system of absolute value variational inequalities with three operators.

(2) If $K_1 = K_2 = K$, then system (Equation (2)) reduces to find $x, y \in K$ such that

$$\begin{cases} \langle T_1x + B|x| - f_1, v - y\rangle \geq 0, & \forall v \in K, \\ \langle T_2y + B|y| - f_2, v - x\rangle \geq 0, & \forall v \in K, \end{cases} \tag{3}$$

which is a system of absolute value variational inequalities.

(3) If $B_1|x| = B_2|y| = 0, \forall x, y \in \mathscr{H}$, then system (Equation (1)) is equivalent to find $y \in K_1$ and $x \in K_2$ such that

$$\begin{cases} \langle T_1x - f_1, v - y\rangle \geq 0, & \forall v \in K_1, \\ \langle T_2y - f_2, v - x\rangle \geq 0, & \forall v \in K_2, \end{cases} \tag{4}$$

which is called the system of variational inequalities.

(4) If $T_1 = T_2 = T, \forall x, y \in \mathscr{H}$, then system (Equation (1)) is equivalent to find $y \in K_1$ and $x \in K_2$ such that

$$\begin{cases} \langle Tx + B_1|x| - f_1, v - y\rangle \geq 0, & \forall v \in K_1, \\ \langle Ty + B_2|y| - f_2, v - x\rangle \geq 0, & \forall v \in K_2, \end{cases} \tag{5}$$

which is a system of absolute value variational inequalities with three operators.

(5) If $T_1 = T_2 = T$ and $K_1 = K_2 = K$, then system (Equation (2)) reduces to find $x \in K$ such that

$$\langle Tx + B|x| - f, y - x\rangle \geq 0, \quad \forall y \in K \tag{6}$$

is called an absolute value variational inequality.

(6) If $B|x| = 0, \forall x \in \mathscr{H}$, then Equation (4) collapses to find $x \in K$ such that

$$\langle Tx - f, y - x\rangle \geq 0, \quad \forall y \in K, \tag{7}$$

which are well-known classical variational inequalities, introduced by Lions and Stampacchia [23, 24] and have been studied extensively in many directions. Variational inequalities are useful to formulate various equilibrium problems.

(7) If $K^* = \{x \in K : \langle x, y\rangle \geq 0, y \in K\}$ is the polar cone of the closed and convex cone K in $\mathscr{H}$, then Equation (4) is equivalent to find $x \in K$ such that

$$x \in K, Tx + B|x| - f \in K^*, \langle Tx - B|x|, x\rangle = 0, \tag{8}$$

which is an absolute value complementarity problem. The absolute value complementarity problem was introduced and studied by Noor et al. [25].

(8) If $B|x| = 0, \forall x \in \mathscr{H}$, then Equation (6) reduces to find $x \in K$ such that

$$x \in K, Tx - f \in K^*, \langle Tx - f, x\rangle = 0, \tag{9}$$

which is called a complementarity problem. The complementarity problem was introduced and studied by Lemke [5] and has also been investigated by Cottle and Dantzig [26].

(9) If $T_1 = T_2 = T$, $B_1 = B_2 = B$, and $K_1 = K_2 = \mathscr{H}$, then Equation (1) is equivalent to find $x \in \mathscr{H}$ such that

$$Tx + B|x| = f, \tag{10}$$

which is known as the system of absolute value equations and is addressed in Reference [27]. The system of absolute value equations is widely applied in various branches of engineering and mathematics. Hence, the proper choice of operators and spaces may generate several known and new types of absolute value variational inequalities and its variant forms.

In order to obtain the main results of this paper, some basic definitions and results are needed which are essential for the further analysis.

Definition 1. $T : \mathscr{H} \longrightarrow \mathscr{H}$ is said to be strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in \mathscr{H}. \tag{11}$$

Definition 2. An operator $T : \mathscr{H} \longrightarrow \mathscr{H}$ is said to be Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|Tx - Ty\| \leq \beta \|x - y\|, \quad \forall x, y \in \mathscr{H}. \tag{12}$$

Definition 3. An operator $T : \mathscr{H} \longrightarrow \mathscr{H}$ is said to be monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in \mathscr{H}. \tag{13}$$

Definition 4. An operator $T : \mathscr{H} \longrightarrow \mathscr{H}$ is said to be pseudomonotone if

$$\langle Tx, y - x\rangle \geq 0 \tag{14}$$

implies

$$\langle Ty, y - x\rangle \geq 0, \quad \forall x, y \in \mathscr{H}. \tag{15}$$

We now consider the well-known projection lemma which is due to Reference [4]. The projection lemma transforms the variational inequalities into a fixed point problem.

Lemma 5 (see [4]). *Let K be a closed and convex set in $\mathscr{H}$. Then, for a given $z \in \mathscr{H}, x \in K$ satisfies*

$$\langle x - z, y - x\rangle \geq 0, \quad \forall y \in K, \tag{16}$$

if and only if

$$x = P_K[z], \tag{17}$$

where P_K is the projection of $\mathscr{H}$ onto a closed and convex set K in $\mathscr{H}$.

It is notable that the projection operator P_K is a nonexpansive operator, that is

$$\|P_K[x] - P_K[y]\| \leq \|x - y\|, \quad \forall x, y \in \mathscr{H}. \tag{18}$$

The above lemma is important to obtain the main results of this paper.

Lemma 6 (see [28]). *If $\{\delta_n\}_{n=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality*

$$\delta_{n+1} \leq (1 - \lambda_n)\delta_n + \sigma_n, \quad \forall n \geq 0, \tag{19}$$

with $0 \leq \lambda_n \leq 1$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, and $\sigma_n = 0(\lambda_n)$, then $\lim_{n \longrightarrow \infty} \delta_n = 0$.

Since the projection-type techniques could not be used to suggest iterative algorithms for mixed variational inequalities, Glowinski et al. [11] suggested a new technique for solving the variational inequalities. It is called auxiliary principle technique which proved to be useful as it does not depend on the projection. Also, it is worth mentioning that unified descent algorithms for variational inequalities can be suggested by using an auxiliary principle technique.

Hence, Equation (1) can easily be written in an equivalent form by using the auxiliary principle technique, that is to find $y \in K_1$ and $x \in K_2$ such that

$$\begin{cases} \langle \gamma_1 T_1 x + \gamma_1 B_1 |x| - \gamma_1 f_1 + y - x, v - y \rangle \geq 0, & \forall v \in K_1, \\ \langle \gamma_2 T_2 y + \gamma_2 B_2 |y| - \gamma_2 f_2 + x - y, v - x \rangle \geq 0, & \forall v \in K_2, \end{cases} \tag{20}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are the constants.

We use this equivalent system to suggest some new iterative algorithms for solving the system of absolute value variational inequalities and its alternative systems.

2.1. Main Results. In this section, we establish the equivalence between system of absolute value Equation (20) and the fixed point problems. We use this equivalent formulation to suggest some iterative algorithms for solving the system of absolute value equations. The convergence analysis of the proposed methods is also demonstrated.

Lemma 7. *The system of absolute value variational inequalities (Equation (20)) has a solution $y \in K_1$ and $x \in K_2$ if and only if $y \in K_1$ and $x \in K_2$ satisfy the relations:*

$$y = P_{K_1}[x - \gamma_1 T_1 x - \gamma_1 B_1 |x| + \gamma_1 f_1], \tag{21}$$

$$x = P_{K_2}[y - \gamma_2 T_2 y - \gamma_2 B_2 |y| + \gamma_2 f_2], \tag{22}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are constants.

It is clear from Lemma 7 that the system (Equation (20)) is equivalent to the fixed point problems (Equations (21) and (22)). This equivalent formulation is very important from theoretical as well as from the numerical point of view (see [29]). We propose and analyze some iterative schemes by using the composition (Equations (21) and (22)).

Equations (21) and (22) can be rewritten in the following equivalent forms:

$$y = (1 - \eta_n) y + \eta_n P_{K_1}[x - \gamma_1 T_1 x - \gamma_1 B_1 |x| + \gamma_1 f_1], \tag{23}$$

$$x = (1 - \zeta_n) x + \zeta_n P_{K_2}[y - \gamma_2 T_2 y - \gamma_2 B_2 |y| + \gamma_2 f_2], \tag{24}$$

where $0 \leq \zeta_n, \eta_n \leq 1$ for all $n \geq 0$.

We use this equivalent formulation to suggest the following iterative algorithms for solving the system of absolute value variational inequalities (Equation (20)) and its related formations.

Algorithm 1. For given $y_0 \in K_2$ and $x_0 \in K_1$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$y_{n+1} = (1 - \eta_n) y_n + \eta_n P_{K_1}[x_n - \gamma_1 T_1 x_n - \gamma_1 B_1 |x_n| + \gamma_1 f]_1, \tag{25}$$

$$x_{n+1} = (1 - \zeta_n) x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T_2 y_n - \gamma_2 B_2 |y_n| + \gamma_2 f_2], \tag{26}$$

where $0 \leq \zeta_n, \eta_n \leq 1$ for all $n \geq 0$.

Algorithm 1 is known as a parallel algorithm which can be considered as the Jacobi method for solving the system of absolute value equations. It is proved that parallel algorithms outperform the sequential schemes.

We now discuss some of the special cases of Algorithm 1.

(1) If $B_1 = B_2 = B$, then Algorithm 1 reduces to the following parallel algorithm to find the solution of the system (Equation (2))

Algorithm 2. For given $y_0 \in K_2$ and $x_0 \in K_1$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$\begin{aligned} y_{n+1} &= (1 - \eta_n) y_n + \eta_n P_{K_1}[x_n - \gamma_1 T_1 x_n - \gamma_1 B |x_n| + \gamma_1 f_1], \\ x_{n+1} &= (1 - \zeta_n) x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T_2 y_n - \gamma_2 B |y_n| + \gamma_2 f_2], \end{aligned} \tag{27}$$

where $0 \leq \zeta_n, \eta_n \leq 1$ for all $n \geq 0$.

(2) If $K_1 = K_2 = K$, then Algorithm 2 reduces to the following projection algorithm to solve the system of absolute value variational inequalities (Equation (3))

Algorithm 3. For given $x_0, y_0 \in K$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$\begin{aligned} y_{n+1} &= (1 - \eta_n) y_n + \eta_n P_{K_1}[x_n - \gamma_1 T_1 x_n - \gamma_1 B |x_n| + \gamma_1 f_1], \\ x_{n+1} &= (1 - \zeta_n) x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T_2 y_n - \gamma_2 B |y_n| + \gamma_2 f_2], \end{aligned} \tag{28}$$

where $0 \leq \zeta_n, \eta_n \leq 1$ for all $n \geq 0$.

(3) If $B_1 |x| = B_2 |y| = 0, \forall x, y \in \mathscr{H}$, then Algorithm 2 reduces to the following projection algorithm to solve the system of variational inequalities (Equation (4))

Algorithm 4. For given $y_0 \in K_2$ and $x_0 \in K_1$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$\begin{aligned} y_{n+1} &= (1-\eta_n)y_n + \eta_n P_{K_1}[x_n - \gamma_1 T_1 x_n + \gamma_1 f_1], \\ x_{n+1} &= (1-\zeta_n)x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T_2 y_n + \gamma_2 f_2], \end{aligned} \tag{29}$$

where $0 \le \zeta_n, \eta_n \le 1$ for all $n \ge 0$.

(4) If $T_1 = T_2 = T$, then Algorithm 1 reduces to the following algorithm

Algorithm 5. For given $y_0 \in K_2$ and $x_0 \in K_1$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$\begin{aligned} y_{n+1} &= (1-\eta_n)y_n + \eta_n P_{K_1}[x_n - \gamma_1 T x_n - \gamma_1 B_1|x_n| + \gamma_1 f_1], \\ x_{n+1} &= (1-\zeta_n)x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T y_n - \gamma_2 B_2|y_n| + \gamma_2 f_2], \end{aligned} \tag{30}$$

where $0 \le \zeta_n, \eta_n \le 1$ for all $n \ge 0$.

(5) If $T_1 = T_2 = T$ and $B_1 = B_2 = B$, then Algorithm 2 reduces to the following parallel algorithm

Algorithm 6. For given $y_0 \in K_2$ and $x_0 \in K_1$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$\begin{aligned} y_{n+1} &= (1-\eta_n)y_n + \eta_n P_{K_1}[x_n - \gamma_1 T x_n - \gamma_1 B|x_n| + \gamma_1 f_1], \\ x_{n+1} &= (1-\zeta_n)x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T y_n - \gamma_2 B|y_n| + \gamma_2 f_2], \end{aligned} \tag{31}$$

where $0 \le \zeta_n, \eta_n \le 1$ for all $n \ge 0$.

Several new and known iterative schemes can be suggested for solving absolute value variational inequalities and the associated problems by making proper and appropriate choice for operators and spaces.

We now examine the convergence analysis of Algorithm 1 which is the key motivation of the next result.

Theorem 8. *Let the operators $T_1, T_2 : \mathscr{H} \longrightarrow \mathscr{H}$ be strongly monotone with constants $\alpha_{T_1} > 0, \alpha_{T_2} > 0$ and Lipschitz continuous with constants $\beta_{T_1} > 0, \beta_{T_2} > 0$ and the operators $B_1, B_2 : \mathscr{H} \longrightarrow \mathscr{H}$ be Lipschitz continuous with constants $\beta_{B_1} > 0, \beta_{B_2} > 0$, respectively. If the following conditions*

(1) $\phi_{T_1} = \sqrt{1 - 2\gamma_1 \alpha_{T_1} + \gamma_1^2 \beta_{T_1}^2} < 1$

(2) $\phi_{T_2} = \sqrt{1 - 2\gamma_2 \alpha_{T_2} + \gamma_2^2 \beta_{T_2}^2} < 1$

(3) $0 \le \zeta_n, \eta_n \le 1 \forall n \ge 0$,

$$\begin{aligned} \zeta_n - \eta_n\left(\phi_{T_1} + \phi_{B_1}\right) &\ge 0, \\ \eta_n - \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right) &\ge 0, \end{aligned} \tag{32}$$

such that

$$\begin{aligned} \sum_{n=0}^{\infty}\left(\zeta_n - \eta_n\left(\phi_{T_1} + \phi_{B_1}\right)\right) &= \infty, \\ \sum_{n=0}^{\infty}\left(\eta_n - \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right)\right) &= \infty, \end{aligned} \tag{33}$$

where

$$\begin{aligned} \phi_{T_1} &= \sqrt{1 - 2\gamma_1 \alpha_{T_1} + \gamma_1^2 \beta_{T_1}^2}, \\ \phi_{T_2} &= \sqrt{1 - 2\gamma_2 \alpha_{T_2} + \gamma_2^2 \beta_{T_2}^2}, \\ \phi_{B_1} &= \gamma_2 \beta_{B_2}, \\ \phi_{B_2} &= \gamma_1 \beta_{B_1}, \end{aligned} \tag{34}$$

hold, then sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 1 converge to x and y, respectively.

Proof. Let $x, y \in \mathscr{H}$ such that $y \in K_1$ and $x \in K_2$ be a solution of the system (Equation (20)). Then, from Equation (23) and Equation (26), we have the following:

$$\begin{aligned} \|x_{n+1} - x\| &= \big\|(1-\zeta_n)x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T_2 y_n - \gamma_2 B_2|y_n| + \gamma_2 f_2]\} - (1-\zeta_n) \\ &\quad \cdot x - \zeta_n P_{K_2}[y - \gamma_2 T_2 y - \gamma_2 B_2|y| + \gamma_2 f_2]\| \le (1-\zeta_n)\|x_n - x\| \\ &\quad + \zeta_n \|P_{K_2}[y_n - \gamma_2 T_2 y_n - \gamma_2 B_2|y_n| + \gamma_2 f_2] - P_{K_2}[y - \gamma_2 T_2 y - \gamma_2 B_2|y| \\ &\quad + \gamma_2 f_2]\| \le (1-\zeta_n)\|x_n - x\| + \zeta_n\|(y_n - y) - \gamma_2(T_2 y_n - T_2 y)\| \\ &\quad + \zeta_n \gamma_2 \|B_2|y_n| - B_2|y|\|. \end{aligned} \tag{35}$$

□

Since the operator T_2 is strongly monotone and Lipschitz continuous with constants $\alpha_{T_2} > 0$ and $\beta_{T_2} > 0$, respectively, then we have the following:

$$\begin{aligned} \|(y_n - y) - \gamma_2(T_2 y_n - T_2 y)\| &\le \|y_n - y\|^2 \\ &\quad - 2\gamma_2\langle T_2 y_n - T_2 y, y_n - y\rangle + \|T_2 y_n - T_2 y\|^2 \\ &\le \sqrt{1 - 2\gamma_2 \alpha_{T_2} + \gamma_2^2 \beta_{T_2}^2}\|y_n - y\|^2. \end{aligned} \tag{36}$$

Also, using the Lipschitz continuity of the operator B_2 with constant $\beta_{B_2} > 0$, we have the following:

$$\|B_2|y_n| - B_2|y|\| \le \beta_{B_2}\|y_n - y\|. \tag{37}$$

Combining Equations (35), (36), and (37), we obtain the following:

$$\begin{aligned}\|x_{n+1}-x\| &\le (1-\zeta_n)\|x_n-x\| \\ &+\zeta_n\left(\sqrt{1-2\gamma_2\alpha_{T_2}+\gamma_2^2\beta_{T_2}^2}+\gamma_2\beta_{B_2}\right)\|y_n-y\| \\ &=(1-\zeta_n)\|x_n-x\|+\zeta_n\left(\phi_{T_2}+\phi_{B_2}\right)\|y_n-y\|.\end{aligned} \tag{38}$$

In a similar way, from Equation (23) and Equation (25), we have the following:

$$\begin{aligned}\|y_{n+1}-y\| &= \big\|(1-\eta_n)y_n+\eta_n P_{K_1}[x_n-\gamma_1 T_1 x_n-\gamma_1 B_1|x_n| \\ &+\gamma_1 f_1]-(1-\eta_n)y-\eta_n P_{K_1}[x-\gamma_1 T_1 x-\gamma_1 B_1|x| \\ &+\gamma_1 f_1]\| \le (1-\eta_n)\|y_n-y\|+\eta_n\|P_{K_1}[x_n-\gamma_1 T_1 x_n \\ &-\gamma_1 B_1|x_n|+\gamma_1 f_1]-P_{K_1}[x-\gamma_1 T_1 x-\gamma_1 B_1|x| \\ &+\gamma_1 f_1]\| \le (1-\eta_n)\|y_n-y\|+\eta_n\|(x_n-x) \\ &-\gamma_1(T_1 x_n-T_1 x)\|+\eta_n\gamma_1\|B_1|x_n|-B_1|x|\| \\ &\le (1-\eta_n)\|y_n-y\|+\eta_n\sqrt{1-2\gamma_1\alpha_{T_1}+\gamma_1^2\beta_{T_1}^2}\|x_n \\ &-x\|+\eta_n\gamma_1\beta_{B_1}\|x_n-x\|=(1-\eta_n)\|y_n-y\|+\eta_n \\ &\cdot\left(\sqrt{1-2\gamma_1\alpha_{T_1}+\gamma_1^2\beta_{T_1}^2}+\gamma_1\beta_{B_1}\right)\|x_n-x\| \\ &=(1-\eta_n)\|y_n-y\|+\eta_n\left(\phi_{T_1}+\phi_{B_1}\right)\|x_n-x\|,\end{aligned} \tag{39}$$

where we have used the strong monotonicity of T_1 with constant $\alpha_{T_1}>0$ and Lipschitz continuity of the operators T_1 and B_1 with constants $\beta_{T_1}>0$ and $\beta_{B_1}>0$, respectively.

Adding Equations (38) and (39), we have the following:

$$\begin{aligned}&\|x_{n+1}-x\|+\|y_{n+1}-y\| \\ &\le\left(1-\zeta_n+\eta_n\left(\phi_{T_1}+\phi_{B_1}\right)\right)\|x_n-x\|+\left(1-\eta_n+\zeta_n\left(\phi_{T_2}+\phi_{B_2}\right)\right) \\ &\quad\cdot\|y_n-y\|\le\max\left\{\left(1-\zeta_n+\eta_n\left(\phi_{T_1}+\phi_{B_1}\right)\right),\left(1-\eta_n+\zeta_n\left(\phi_{T_2}+\phi_{B_2}\right)\right)\right\} \\ &\quad\cdot(\|x_n-x\|+\|y_n-y\|)=\max(\xi_1,\xi_2)(\|x_n-x\|+\|y_n-y\|) \\ &=\phi(\|x_n-x\|+\|y_n-y\|),\end{aligned} \tag{40}$$

where

$$\begin{aligned}\phi&=\max(\xi_1,\xi_2),\\ \xi_1&=1-\zeta_n+\eta_n\left(\phi_{T_1}+\phi_{B_1}\right),\\ \xi_2&=1-\eta_n+\zeta_n\left(\phi_{T_2}+\phi_{B_2}\right).\end{aligned} \tag{41}$$

From assumption (iii), it follows that $\phi<1$. Hence, using Lemma 6, we obtain from Equation (44) the following:

$$\lim_{n\longrightarrow\infty}[\|x_n-x\|+\|y_n-y\|]=0. \tag{42}$$

This further implies that

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_n-x\|=0,\\ \lim_{n\longrightarrow\infty}\|y_n-y\|=0,\end{aligned} \tag{43}$$

which is the required result.

We now propose and examine some new iterative schemes for solving system of absolute value variational inequalities, by employing a useful substitution.

It can easily be shown, by using Lemma 5, that $x,y\in\mathscr{H}$ such that $y\in K_1$ and $x\in K_2$ is a solution of the system of absolute value variational inequalities (Equation (1)), if and only if $x,y\in\mathscr{H}:y\in K_1,x\in K_2$ satisfies the following:

$$y=P_{K_1}[z], \tag{44}$$

$$x=P_{K_2}[w], \tag{45}$$

$$z=x-\gamma_1 T_1 x-\gamma_1 B_1|x|+\gamma_1 f_1, \tag{46}$$

$$w=y-\gamma_2 T_2 y-\gamma_2 B_2|y|+\gamma_2 f_2. \tag{47}$$

By using this alternative formation, we can propose and examine the following iterative schemes to solve the system (Equation (1)).

Algorithm 7. For given $y_0\in K_1$ and $x_0\in K_2$, find x_{n+1} and y_{n+1} by the iterative schemes:

$$y_{n+1}=(1-\eta_n)y_n+\eta_n P_{K_1}[z_n], \tag{48}$$

$$x_{n+1}=(1-\zeta_n)x_n+\zeta_n P_{K_2}[w_n], \tag{49}$$

$$z_n=x_n-\gamma_1 T_1 x_n-\gamma_1 B_1|x_n|+\gamma_1 f_1, \tag{50}$$

$$w_n=y_n-\gamma_2 T_2 y_n-\gamma_2 B_2|y_n|+\gamma_2 f_2, \tag{51}$$

where $0\le\zeta_n,\eta_n\le 1$ for all $n\ge 0$.

By choosing the useful operators and proper spaces, one can have various new as well as known iterative schemes for solving the system of absolute value variational inequalities and its variant forms. Now, we examine the convergence analysis of Algorithm 7 by employing the approach of Theorem 8.

Theorem 9. *Let the operators $T_1,T_2:\mathscr{H}\longrightarrow\mathscr{H}$ be strongly monotone with constants $\alpha_{T_1}>0,\alpha_{T_2}>0$ and Lipschitz continuous with constants $\beta_{T_1}>0,\beta_{T_2}>0$ and the operators $B_1,B_2:\mathscr{H}\longrightarrow\mathscr{H}$ be Lipschitz continuous with constants $\beta_{B_1}>0,\beta_{B_2}>0$, respectively. If the following conditions*

(i) $\phi_{T_1}=\sqrt{1-2\gamma_1\alpha_{T_1}+\gamma_1^2\beta_{T_1}^2}<1$

(ii) $\phi_{T_2}=\sqrt{1-2\gamma_2\alpha_{T_2}+\gamma_2^2\beta_{T_2}^2}<1$

(iii) $0\le\zeta_n,\eta_n\le 1\forall n\ge 0$

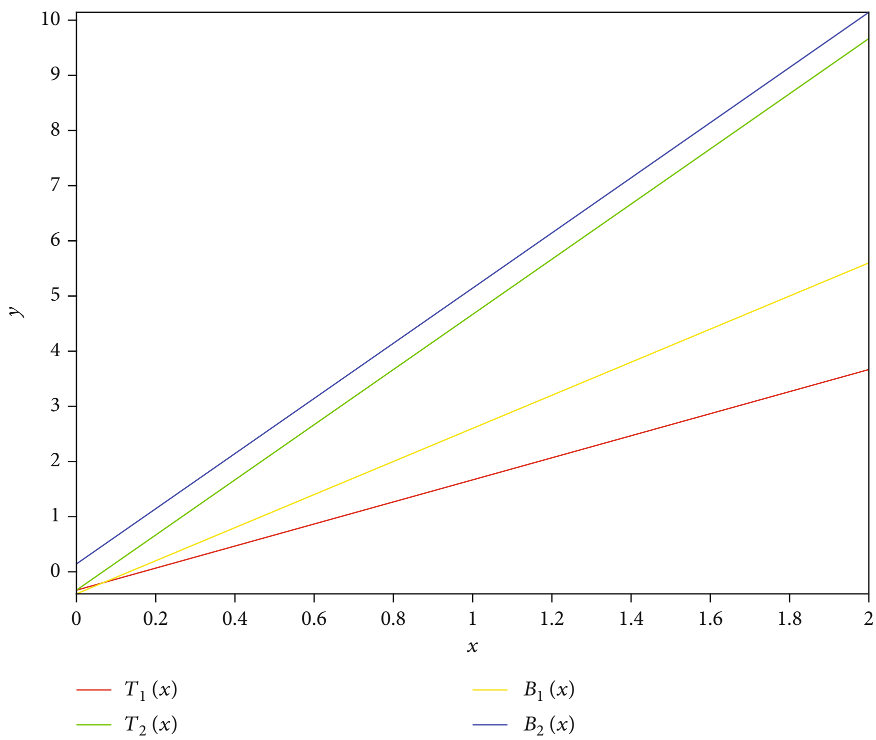

Figure 1: Graphical behaviour of $T_1(x)$, $T_2(x)$, $B_1(x)$, and $B_2(x)$.

$$\begin{aligned} &\zeta_n - \eta_n\left(\phi_{T_1} + \phi_{B_1}\right) \geq 0, \\ &\eta_n - \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right) \geq 0, \end{aligned} \tag{52}$$

such that

$$\begin{aligned} &\sum_{n=0}^{\infty}\left(\zeta_n - \eta_n\left(\phi_{T_1} + \phi_{B_1}\right)\right) = \infty, \\ &\sum_{n=0}^{\infty}\left(\eta_n - \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right)\right) = \infty, \end{aligned} \tag{53}$$

where

$$\begin{aligned} \phi_{T_1} &= \sqrt{1 - 2\gamma_1\alpha_{T_1} + \gamma_1^2\beta_{T_1}^2}, \\ \phi_{T_2} &= \sqrt{1 - 2\gamma_2\alpha_{T_2} + \gamma_2^2\beta_{T_2}^2}, \\ \phi_{B_1} &= \gamma_2\beta_{B_2}, \\ \phi_{B_2} &= \gamma_1\beta_{B_1}, \end{aligned} \tag{54}$$

hold, then sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 7 converge to x and y, respectively.

Proof. Let $x, y \in \mathscr{H}$ such that $y \in K_1$ and $x \in K_2$ be a solution of the system (Equation (20)). Then, from Equation (45) and Equation (49), we have the following:

$$\begin{aligned} \|x_{n+1} - x\| &= \left\|(1-\zeta_n)x_n + \zeta_n P_{K_2}[w_n] - (1-\zeta_n)x - \zeta_n P_{K_2}[w]\right\| \\ &\leq (1-\zeta_n)\|x_n - x\| + \zeta_n\left\|P_{K_2}[w_n] - P_{K_2}[w]\right. \\ &\leq (1-\zeta_n)\|x_n - x\| + \zeta_n\|w_n - w\|. \end{aligned} \tag{55}$$

□

In a similar way, from Equation (44) and Equation (48), we have the following:

$$\begin{aligned} \|y_{n+1} - y\| &= \left\|(1-\eta_n)y_n + \eta_n P_{K_1}[z_n] - (1-\eta_n)y - \eta_n P_{K_1}[z]\right\| \\ &\leq (1-\eta_n)\|y_n - y\| + \eta_n\left\|P_{K_1}[z_n] - P_{K_1}[z]\right. \\ &\leq (1-\eta_n)\|y_n - y\| + \eta_n\|z_n - z\|. \end{aligned} \tag{56}$$

From Equations (36), (37), (47), and (51), we have the following:

$$\begin{aligned} \|w_n - w\| &= \|(y_n - y) - \gamma_2(T_2 y_n - T_2 y)\| + \|B_2|y_n| - B_2|y|\| \\ &\leq \left(\phi_{T_2} + \phi_{B_2}\right)\|y_n - y\|. \end{aligned} \tag{57}$$

Also, from Equations (39), (46), and (50), we have the following:

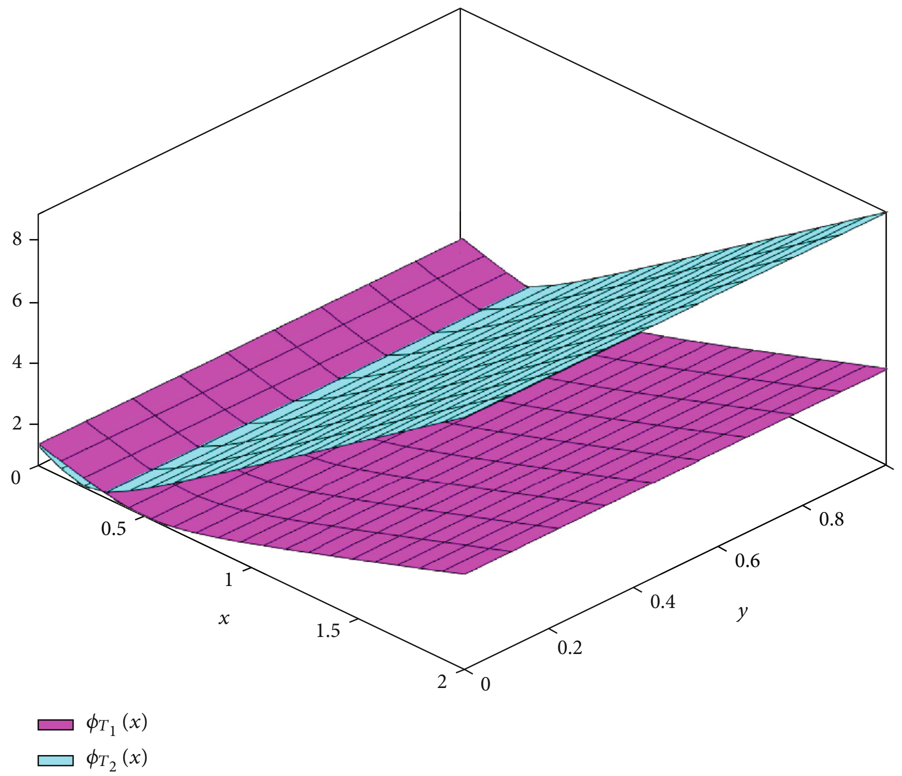

Figure 2: Behaviour of Theorem 8 via a three-dimensional plot satisfies Example 10.

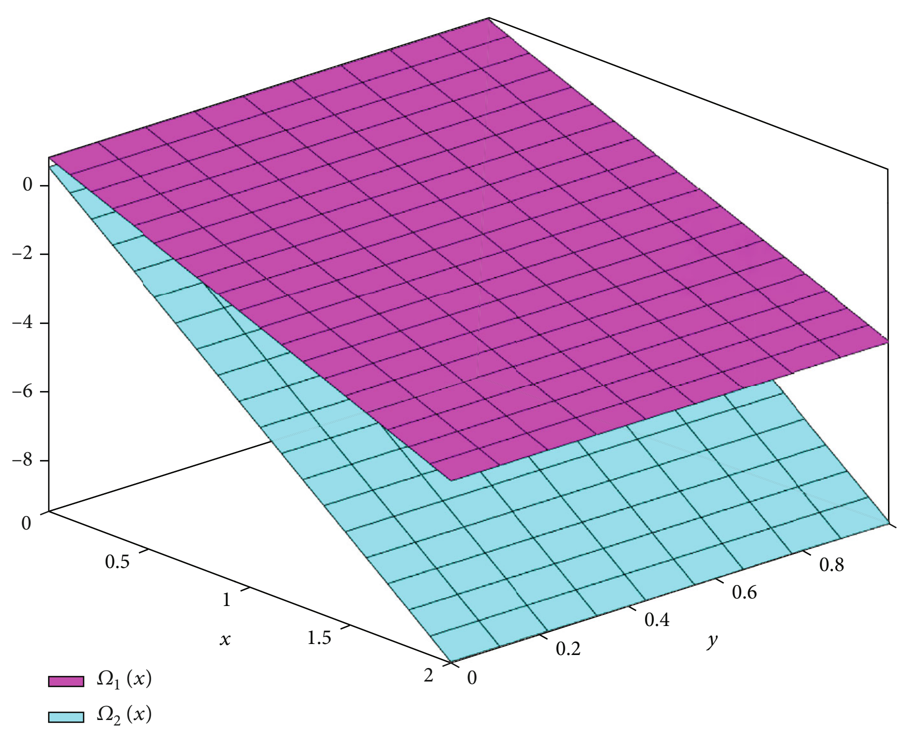

Figure 3: Behaviour of Theorem 9 via a three-dimensional plot satisfies Example 10 approach. The equivalence between the system of absolute value variational inequalities and the system of fixed point problems has been established. Further, we suggested several innovative iterative algorithms for solving our considered system of variational inequalities and the convergence of the proposed schemes has been analyzed under some suitable conditions. Finally, to demonstrate the existence and convergence results, a numerical example was given. We also discussed some special cases of the system of absolute value variational inequalities. The concept and technique of this paper may encourage researchers to analyze the innovative and unique applications of the system of absolute value variational inequalities and its associated optimization problems. In futuristic research, we extend this study to exponential absolute value variational inequalities and their variant forms.

$$\|z_n - z\| = \|(x_n - x) - \gamma_1(T_1x_n - T_1x)\| + \|B_1|x_n| - B_1|x|\| \\ \leq \left(\phi_{T_1} + \phi_{B_1}\right)\|x_n - x\|. \tag{58}$$

Combining Equations (55), (56), (57), and (58), we have the following:

$$\|x_{n+1} - x\| \leq (1-\zeta_n)\|x_n - x\| + \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right)\|y_n - y\|, \tag{59}$$

$$\|y_{n+1} - y\| \leq (1-\eta_n)\|y_n - y\| + \eta_n\left(\phi_{T_1} + \phi_{B_1}\right)\|x_n - x\|. \tag{60}$$

Addition of Equations (59) and (60) implies

$$\begin{aligned}\|x_{n+1} - x\| + \|y_{n+1} - y\| &\leq \left(1 - \zeta_n + \eta_n\left(\phi_{T_1} + \phi_{B_1}\right)\right. \\ &\cdot \|x_n - x\| + \left(1 - \eta_n + \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right)\right)\|y_n - y\| \\ &\leq \max\left\{\left(1 - \zeta_n + \eta_n\left(\phi_{T_1} + \phi_{B_1}\right)\right), \left(1 - \eta_n + \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right)\right)\right\} \\ &\cdot (\|x_n - x\| + \|y_n - y\|) = \max(\xi_1, \xi_2)(\|x_n - x\| + \|y_n - y\|) \\ &= \phi(\|x_n - x\| + \|y_n - y\|),\end{aligned} \tag{61}$$

where

$$\begin{aligned}\phi &= \max(\xi_1, \xi_2), \\ \xi_1 &= 1 - \zeta_n + \eta_n\left(\phi_{T_1} + \phi_{B_1}\right), \\ \xi_2 &= 1 - \eta_n + \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right).\end{aligned} \tag{62}$$

From assumption (iii), it follows that $\phi < 1$. Hence, using Lemma 6, we obtain from Equation (23) the following:

$$\lim_{n\longrightarrow\infty}[\|x_n - x\| + \|y_n - y\|] = 0. \tag{63}$$

This further implies that

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_n - x\| &= 0, \\ \lim_{n\longrightarrow\infty}\|y_n - y\| &= 0,\end{aligned} \tag{64}$$

which is the required result.

Example 10. $\mathscr{H} = \mathbb{R}, K_1 = (-\infty,0]$ and $K_2 = [0,\infty)$.. Let T_1, $T_2 : \mathscr{H} \longrightarrow \mathscr{H}$ be the single-valued mappings defined by the following:

$$T_1(x) = \frac{2x-1}{3}, T_2(x) = \frac{5x-2}{6}, \quad \forall x \in \mathscr{H}. \tag{65}$$

Also, the mappings $B_1, B_2 : \mathscr{H} \longrightarrow \mathscr{H}$ are defined by the following:

$$B_1(x) = \frac{3x-2}{5}, B_2(x) = \frac{5x+1}{7}, \quad \forall x \in \mathscr{H}. \tag{66}$$

Then, it can easily be verified that for each $i = 1, 2$, T_i is strongly monotone and Lipschitz continuous with $\alpha_{T_1} = 2/3 = \beta_{T_1}$ and $\alpha_{T_2} = 5/6 = \beta_{T_2}$, and B_i is strongly monotone and Lipschitz continuous with $\alpha_{B_1} = 3/5 = \beta_{B_1}$ and $\alpha_{B_2} = 5/7 = \beta_{B_2}$.

Then, for $\gamma_1, \gamma_2 = 1$,

$$\begin{aligned}\phi_{T_1} &= \sqrt{1 - 2\gamma_1\alpha_{T_1} + \gamma_1^2\beta_{T_1}^2} = \sqrt{1 - 2\left(\frac{2}{3}\right) + \frac{2^2}{3}} = \frac{2}{3} < 1, \\ \phi_{T_2} &= \sqrt{1 - 2\gamma_2\alpha_{T_2} + \gamma_2^2\beta_{T_2}^2} = \sqrt{1 - 2\left(\frac{5}{6}\right) + \frac{5^2}{6}} = \frac{1}{6} < 1.\end{aligned} \tag{67}$$

Also, for each $n = 1, 2$,

$$\begin{aligned}0 &< \frac{1}{2} = \zeta_n, \\ \eta_n &= \frac{1}{2} < 1,\end{aligned} \tag{68}$$

we have the following:

$$\begin{aligned}\Omega_1 &= \zeta_n - \eta_n\left(\phi_{T_1} + \phi_{B_1}\right) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3} + \frac{3}{5}\right) = \frac{1}{30} > 0, \\ \Omega_2 &= \eta_n - \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{6} + \frac{5}{7}\right) = \frac{5}{84} > 0,\end{aligned} \tag{69}$$

where $\phi_{B_1} = \gamma_1\beta_{B_1} = 3/5$ and $\phi_{B_2} = \gamma_2\beta_{B_2} = 5/7$. Clearly, we see that all the assumptions of Theorem 8 and Theorem 9 are satisfied. Hence, by using Algorithm 1 and Algorithm 7, the conclusions of Theorem 8 and Theorem 9 follow.

Figure 1 is the graphical representation of the operators defined in Example 10. Figure 2 depicts the behaviour of Theorem 8 satisfying Example 10. Similarly, Figure 3 interprets the behaviour of Theorem 9 via a three-dimensional plot satisfying Example 10.

3. Conclusion

In this paper, we have introduced a new system of variational inequalities, called the system of absolute value variational inequalities. To determine the approximate solution of the system of absolute value variational inequalities, we first built a system of absolute value auxiliary variational inequalities. We demonstrate the existence of a solution to the system of absolute value auxiliary variational inequalities using the projection.

Data Availability

The (graphs and an example) data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] G. Stampacchia, "Formes bilineaires coercivites sur les ensembles convexes," *Comptes Rendus Hebdomadaires Des Seances De L Academie Des Sciences*, vol. 258, p. 4413ˆa4416, 1964.

[2] P. T. Harker and J. S. Pang, "Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications," *Mathematical Programming*, vol. 48, no. 1-3, pp. 161–220, 1990.

[3] S. Karamardian, "Generalized complementarity problem," *Journal of Optimization Theory and Applications*, vol. 8, no. 3, pp. 161–168, 1971.

[4] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, London, England, 1980.

[5] C. E. Lemke, "Bimatrix equilibrium points and mathematical programming," *Management Sciences*, vol. 11, no. 7, pp. 681–689, 1965.

[6] A. Nagurney, *Network Economics: A Variational Inequality Approach*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[7] M. A. Noor and Z. Huang, "Wiener-Hopf equation technique for variational inequalities and nonexpansive mappings," *Applied Mathematics and Computation*, vol. 191, no. 2, pp. 504–510, 2007.

[8] A. Bnouhachem and M. A. Noor, "Three-step projection method for general variational inequalities," *International Journal of Modern Physics B*, vol. 26, no. 13, article 1250066, 2012.

[9] S. Batool, M. A. Noor, and K. I. Noor, "Absolute value variational inequalities and dynamical systems," *International Journal of Analysis and Applications*, vol. 18, no. 3, pp. 337–355, 2020.

[10] M. A. Noor, K. I. Noor, and A. G. Khan, "Three step iterative algorithms for solving a class of quasi variational inequalities," *Afrika Matematika*, vol. 26, no. 7-8, pp. 1519–1530, 2015.

[11] R. Glowinski, J. L. Lions, and R. Tremolieres, "Numerical analysis of variational inequalities," in *Studies in Mathematics and its Applications*, vol. 8, p. 776, Oxford: North-Holland Publishing Company, North Holland, Amsterdam, 1981.

[12] L. C. Ceng, S. A. Al-Mezel, and Q. H. Ansari, "Implicit and explicit iterative methods for systems of variational inequalities and zeros of accretive operators," *Abstract and Applied Analysis*, vol. 2013, 19 pages, 2013.

[13] M. Zhang, "Iterative algorithms for a system of generalized variational inequalities in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2012, no. 1, Article ID 232, 2012.

[14] H. Yang, L. Zhou, and Q. Li, "A parallel projection method for a system of nonlinear variational inequalities," *Applied Mathematics and Computation*, vol. 217, no. 5, pp. 1971–1975, 2010.

[15] Q. Liu and Y. Yang, "Global exponential system of projection neural networks for system of generalized variational inequalities and related nonlinear minimax problems," *Neurocomputing*, vol. 73, no. 10-12, pp. 2069–2076, 2010.

[16] L. C. Ceng and Q. Yuan, "Systems of variational inequalities with nonlinear operators," *Mathematics*, vol. 7, no. 4, p. 338, 2019.

[17] R. P. Agarwal, N. J. Huang, and M. Y. Tan, "Sensitivity analysis for a new system of generalized nonlinear mixed quasi-variational inclusions," *Applied Mathematics Letters*, vol. 17, no. 3, pp. 345–352, 2004.

[18] J. S. Pang, "Asymmetric variational inequality problems over product sets: applications and iterative methods," *Mathematical Programming*, vol. 31, no. 2, pp. 206–219, 1985.

[19] O. L. Mangasarian, "The linear complementarity problem as a separable bilinear program," *Journal of Global Optimization*, vol. 6, no. 2, pp. 153–161, 1995.

[20] Y. F. Ke and C. F. Ma, "SOR-like iteration method for solving absolute value equations," *Applied Mathematics and Computation*, vol. 311, no. C, pp. 195–202, 2017.

[21] H. Esmaeili, M. Mirzapour, and E. Mahmoodabadi, "A fast convergent two-step iterative method to solve the absolute value equation," *U.P.B. Science Bulletin, Series: Applied Mathematics and Physics*, vol. 78, no. 1, pp. 25–32, 2016.

[22] S. Batool, M. A. Noor, and K. I. Noor, "Merit functions for absolute value variational inequalities," *AIMS Mathematics*, vol. 6, no. 11, pp. 12133–12147, 2021.

[23] J. L. Lions and G. Stampacchia, "Variational inequalities," *Communications on Pure and Applied Mathematics*, vol. 20, no. 3, pp. 493–519, 1967.

[24] A. Bensoussan and J. L. Lions, *Applications des Inequations Variationnelles en Controle Et Stochastique*, Paris: Dunod, Paris, 1978.

[25] M. A. Noor, J. Iqbal, K. I. Noor, and E. Al-Said, "Generalized AOR method for solving absolute value complementarity problems," *Journal of Applied Mathematics*, vol. 2012, Article ID 743861, 2012.

[26] R. W. Cottle and G. B. Dantzig, "Complementary pivot theory of mathematical programming," *Linear Algebra and Its Applications*, vol. 1, no. 1, pp. 103–125, 1968.

[27] M. A. Noor, K. I. Noor, and S. Batool, "On generalized absolute value equations," *U.P.B. Science Bulletin, Series A*, vol. 80, pp. 63–70, 2018.

[28] X. Weng, "Fixed point iteration for local strictly pseudocontractive mappings," *American Mathematical Society*, vol. 113, no. 3, pp. 727–731, 1991.

[29] Q. H. Ansari and J. C. Yao, "A fixed point theorem and its applications to a system of variational inequalities," *Bulletin of the Australian Mathematical Society*, vol. 59, no. 3, pp. 433–442, 1999.

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Research Article

The Strong Convex Functions and Related Inequalities

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The study of convex functions is one of the most researched of the classical fields. Analysis of the geometric characteristics of these functions is a core area of research in this field; however, a paradigm shift in this research is the application of convexity in optimization theory. The Jensen-Mercer type inequalities are studied extensively in recent years. In the present paper, we extend Jensen-Mercer type inequalities for strong convex function. Some improved inequalities in Hölder sense are also derived. The previously established results are generalized and strengthened by our results.

1. Introduction and Preliminary Results

Convex functions and their consequences are useful in the establishment of different kinds of inequalities; therefore, they are considered the base of theory of inequalities in mathematical analysis. A real valued function $\psi : I \longrightarrow \mathbb{R}$ is said to be convex on the interval $I \subset \mathbb{R}$, if

$$\psi(zx + (1-z)y) \leq z\psi(x) + (1-z)\psi(y) \tag{1}$$

holds for all $x, y \in I$ and $z \in [0, 1]$. The function ψ is said to be concave if reverse of inequality (1) holds.

Convex functions are also very important in the fields of mathematical analysis, mathematical statistics, and optimization theory. These functions motivate towards a nice theory named convex analysis (see [1–3]). Convex functions have been defined in various ways by using different techniques, for example, by support function, by chords joining two points, and Jensen's inequality. Inequality (1) represents the convex function analytically and provides encouragement to define further general notions.

The study of convex functions [4–14] began with Jensen's thought-provoking concepts and interesting work over the period from 1905 to 1906. It is used in the analysis as an efficient tool for solving optimization issues. Additionally, inequalities involving convex functions are very stimulating in the development of different sections of mathematics, such as mathematical finance, economics, management sciences, and optimization theory.

If the function $\psi : I \subset \mathbb{R}$ is convex, then the inequality

$$\psi\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \psi(\alpha_1)d\alpha_1 \leq \frac{\psi(c)+\psi(d)}{2} \tag{2}$$

is called the Hermite-Hadamard inequality [15, 16].

Definition 1 (Convex set) (see [17]). A set I is considered to be convex if the line segment between any two points in I lies in I; i.e., $\forall \alpha_1, \alpha_2 \in I, \forall \in [0, 1]$

$$l\alpha_1 + (1-l)\alpha_2 \in I. \tag{3}$$

Authors [18–20] expanded on the idea of a strongly convex function by replacing the nonnegative term with a real-valued nonnegative function and defined it as follows:

Definition 2 (Strongly convex function (see [18])). A function $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is strongly convex, if

$$\psi(l\alpha_1 + (1-l)\alpha_2) \le l\psi(\alpha_1) + 1(1-l)\psi(\alpha_2) - l(1-l)M(\alpha_1 - \alpha_2) \quad (4)$$

holds for all $\alpha_1, \alpha_2 \in I$ and $l \in [0, 1]$.

Definition 3 (Riemann-Liouville fractional integral). For a function $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, the Riemann-Liouville fractional integral operator of order $\xi \le 0$ with $c \le 0$ is defined as

$$\begin{aligned} J_c^\xi \psi(\alpha_1) &= \frac{1}{\Gamma_\xi} \int_c^{\alpha_1} (\alpha_1 - l)^{\zeta-1} \psi(l) dl, \\ J_c^0 \psi(\alpha_1) &= \psi(\alpha_1). \end{aligned} \quad (5)$$

Many scholars have recently analyzed a variety of inequalities by using the Riemann-Liouville fractional integrals (see [21–24]).

Definition 4 (Hadamard fractional integral). For a function $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, the Hadamard fractional integral of order $\zeta \le \mathbb{R}^+$ for all $\alpha_1 > 1$ is defined as

$$_H J_{1,\alpha_1}^{-\zeta} \psi(\alpha_1) = \frac{1}{\Gamma_\xi} \int_1^{\alpha_1} \ln \left(\frac{\alpha_1}{l}\right)^{\zeta-1} \psi(l) dl, \quad (6)$$

where $\Gamma_\zeta = \int_0^\infty e^{-l} l^{\zeta-1} dl$.

Definition 5 (Conformable fractional integral). Let $\zeta \in (0, 1)$ and $0 \le c < d$. A function $\psi : [c, d] \longrightarrow \mathbb{R}$ is ζ-fractional integrable on $[c, d]$ if the integral

$$\int_c^d \psi(\alpha_1) d_\zeta \alpha_1 = \int_c^d \psi(\alpha_1) \alpha_1^{\zeta-1} \quad (7)$$

exists and is finite.

This paper is aimed at establishing Hermite-Jensen-Mercer type inequalities and some other inequalities including improved Hölder inequality for strong convex function.

2. New Hermite-Jensen-Mercer Type Inequalities

Theorem 6. *Let $\zeta, \xi > 0$ and $\psi : [\phi, \varphi] \longrightarrow \mathbb{R}$ be a strong convex function. Then, the inequality*

$$\begin{aligned} \psi\left(\phi + \varphi - \frac{\alpha_1 + \alpha_2}{2}\right) &\le \frac{2^{\zeta\xi/k-1} \zeta^{\xi/k} \Gamma_k(\xi + k)}{(\alpha_2 - \alpha_1)^{\zeta\xi/k}} \left[{}_k^\xi J_{(\phi+\varphi-\alpha_1+\alpha_2/2)^+}^\zeta \psi(\phi \right. \\ &\left. + \varphi - \alpha_2) + {}_k^\xi J_{(\phi+\varphi-\alpha_1+\alpha_2/2)^+}^\zeta \psi(\phi + \varphi - \alpha_2) - \frac{a}{2} M(\alpha_2 - \alpha_1) \right] \\ &\le \psi(\phi) + \psi(\varphi) - \left(\frac{\psi(\alpha_2) + \psi(\alpha_2)}{2}\right) - \frac{a}{2} \psi\left(\phi + \varphi - \frac{\alpha_1 - \alpha_2}{2}\right) \end{aligned} \quad (8)$$

holds for all $\alpha_1, \alpha_2 \in [\phi, \varphi]$.

Proof. To prove that the first inequality holds, take

$$\psi\left(\phi + \varphi - \frac{\alpha_{11} + \alpha_{21}}{2}\right) = \psi\left(\frac{2\phi + 2\varphi - \alpha_{11} - \alpha_{21}}{2}\right). \quad (9)$$

Since ψ is a strong convex function, so

$$\begin{aligned} &\psi\left(\phi + \varphi - \frac{\alpha_{11} + \alpha_{21}}{2}\right) \psi\left(\phi + \varphi - \frac{\alpha_{11} + \alpha_{21}}{2}\right) \\ &\le \left(\frac{1}{2}\right) \psi\left(\frac{\phi + \varphi - \alpha_{11}}{2}\right) + \left(\frac{1}{2}\right) \psi\left(\frac{\phi + \varphi - \alpha_{21}}{2}\right) \\ &\quad - a\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\alpha_{21} - \alpha_{11}}{2}\right)^2 \\ &\le \left(\frac{1}{2}\right)\left[\psi\left(\frac{\phi + \varphi - \alpha_{11}}{2}\right) + \psi\left(\frac{\phi + \varphi - \alpha_{21}}{2}\right) - \left(\frac{a}{2}\right)\left(\frac{\alpha_{21} - \alpha_{11}}{2}\right)^2\right]. \end{aligned} \quad (10)$$

Suppose $\alpha_{11} = l/2\alpha_1 + (2-l)/2\alpha_2$ and $\alpha_{21} = (2-l)/2\alpha_1 + l/2\alpha_2$; then, for $\alpha_1, \alpha_2 \in [\phi, \varphi]$ and $l \in [0, 1]$, we have

$$\begin{aligned} &\psi\left(\frac{\phi + \varphi - l/2\alpha_1 - 2 - l/2\alpha_2}{2}\right) + \psi\left(\frac{\phi + \varphi - 2 - l/2\alpha_1 - l/2\alpha_2}{2}\right) \\ &\quad - \frac{a}{2}\left(\frac{\alpha_1 - \alpha_2}{2}\right)^2 (1-l)^2 2\psi\left(\phi + \varphi - \frac{\alpha_{11} + \alpha_{21}}{2}\right) \le \psi\left(\frac{\phi + \varphi - \alpha_{11}}{2}\right) \\ &\quad + \psi\left(\frac{\phi + \varphi - \alpha_{21}}{2}\right) - \frac{a}{2}\left(\frac{\alpha_{21} - \alpha_{11}}{2}\right)^2. \end{aligned} \quad (11)$$

Multiplying both sides of Equation (11) with $(1-(1-l)^\zeta/\zeta)^{\xi/k-1}(1-l)^{\zeta-1}$, we get

$$\begin{aligned} &\left(2\psi\left(\phi + \varphi - \frac{\alpha_{11} + \alpha_{21}}{2}\right)\right)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1} (1-l)^{\zeta-1} \\ &\le \psi\left(\frac{\phi + \varphi - \alpha_{11}}{2}\right) + \psi\left(\frac{\phi + \varphi - \alpha_{21}}{2}\right) \\ &\quad - \frac{a}{2}\left(\frac{\alpha_{21} - \alpha_{11}}{2}\right)^2 \left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1} (1-l)^{\zeta-1}. \end{aligned} \quad (12)$$

Integrating the above inequality with respect to l over the range $[0, 1]$ and then combining the result with the integral operator yield

$$2\psi\left(\phi+\varphi-\frac{\alpha_{11}+\alpha_{21}}{2}\right)\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1}(1-l)^{\alpha-1}$$
$$\leq\int_0^1\left[\psi\left(\frac{\phi+\varphi-\alpha_{11}}{2}\right)+\psi\left(\frac{\phi+\varphi-\alpha_{21}}{2}\right)-\frac{c}{2}\left(\frac{\alpha_{21}-\alpha_{11}}{2}\right)^2\times\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}\right]dl$$
$$=\int_0^1\left[\psi\left(\frac{\phi+\varphi-\alpha_{11}}{2}\right)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}\right]dl+\int_0^1\left[\psi\left(\frac{\phi+\varphi-\alpha_{21}}{2}\right)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}\right]dl-\int_0^1\left[\frac{a}{2}\left(\frac{\alpha_{21}-\alpha_{11}}{2}\right)^2\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}\right]dl. \tag{13}$$

Now, by altering the variables, we can obtain

$$\int_{\phi+\varphi-\alpha_2}^{\phi+\varphi-(\alpha_1+\alpha_2/2)}\left[\frac{1-(2\phi+2\varphi-(\alpha_1+\alpha_2)-2l_1/(\alpha_2-\alpha_2))^\zeta}{\zeta}\right]^{\xi/k-1}\times\left(\frac{2\phi+2\varphi-(\alpha_1+\alpha_2)-2l_1}{(\alpha_2-\alpha_1)}\right)^{\zeta-1}\psi(l_1)\frac{\alpha_2-\alpha_1}{2}dl$$
$$+\int_{\phi+\varphi-\alpha_1}^{\phi+\varphi-(\alpha_1+\alpha_2/2)}\left[\frac{1-(2l_2-2\phi+2\varphi-(\alpha_1+\alpha_2)/(\alpha_2-\alpha_2))^\zeta}{\zeta}\right]^{\xi/k-1}\times\left(\frac{2l^2-2\phi+2\varphi-(\alpha_1+\alpha_2)}{(\alpha_2-\alpha_1)}\right)^{\zeta-1}\psi(l_2)\frac{\alpha_2-\alpha_1}{2}dl$$
$$+\int_{\alpha_1-\alpha_2}^{\alpha_1+\alpha_2}\left[\frac{1-(2l_3/(\alpha_2-\alpha_1))^\zeta}{\zeta}\right]^{\xi/k-1}\cdot\left(\frac{l_3}{\alpha_1-\alpha_2}\right)a\psi(l_3)(\alpha_2-\alpha_1)dl. \tag{14}$$

So,

$$\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}dl=\frac{1}{\xi/k\zeta^{\xi/k}},$$
$${}_\kappa^\xi J_{\phi^+}^\zeta\psi(\alpha_2)=\frac{1}{\kappa\Gamma_\kappa(\xi)}\int_\phi^{\alpha_2}\left(\frac{(\alpha_2-\phi)^\zeta-(l-\phi)^\zeta}{\zeta}\right)^{\xi/k-1}\left(\frac{\psi(l)}{(l-\phi)^{1-\zeta}}\right)dl,$$
$${}_\kappa^\xi J_{\varphi^-}^\zeta\psi(\alpha_2)=\frac{1}{\kappa\Gamma_\kappa(\xi)}\int_\phi^{\varphi}\left(\frac{(\varphi-\alpha_2)^\zeta-(\varphi-l)^\zeta}{\zeta}\right)^{\xi/k-1}\left(\frac{\psi(l)}{(\varphi-l)^{1-\zeta}}\right)dl. \tag{15}$$

Therefore,

$$2\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\frac{1}{\xi/k\zeta^{\xi/k}}\leq\left(\frac{2}{\alpha_2-\alpha_1}\right)^{\zeta\xi/k}\cdot\left\{\Gamma_k(\xi)_k^\xi J_{(\phi+\varphi-\alpha_1+\alpha_2/2)^+}^\zeta S(\phi+\varphi-\alpha_2)+\Gamma_k(\xi)_k^\xi J_{(\phi+\varphi-\alpha_1+\alpha_2/2)^-}^\zeta\psi(\phi+\varphi-\alpha_1)-\frac{a}{2}M(\alpha_{21}-\alpha_{11})\right\}. \tag{16}$$

As a result, the first inequality of (8) is proved.

We can prove the second inequality of (8) by using strong convexity of ψ for l over $[0,1]$.

$$\psi(\phi+\varphi-\alpha_1)=\psi(\phi+\varphi)+\psi(\alpha_1)-\frac{1}{2}aM(\phi+\varphi-\alpha_1), \tag{17}$$

$$\psi(\phi+\varphi-\alpha_2)=\psi(\phi+\varphi)+\psi(\alpha_1)-\frac{1}{2}aM(\phi+\varphi-\alpha_2), \tag{18}$$

$$\psi\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)=\psi(\phi+\varphi)+\left(\frac{l}{2}\psi\alpha_1+\frac{2-l}{2}\psi\alpha_2\right)-\frac{a}{2}M\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right), \tag{19}$$

$$\psi\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{5l}{2}\alpha_2\right)\right)=\psi(\phi+\varphi)+\left(\frac{2-l}{2}\psi\alpha_1+\frac{l}{2}\psi\alpha_2\right)-\frac{a}{2}M\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right). \tag{20}$$

Adding (17) and (20) leads to

$$\psi\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)+\psi\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)\leq2\psi(\phi+\varphi)+\left[\psi\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)+\psi\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right]-\frac{a}{2}(2\phi+2\varphi-(\alpha_2-\alpha_1)). \tag{21}$$

Multiply 9 with $(1-(1-l)^\zeta/\zeta)^{\xi/k-1}(1-l)^{\zeta-1}$, and integrating the obtained inequality w.r.t to l over [0,1] gives

$$\int_0^1 \left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}\left[\psi\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)\right.$$
$$\left.+\psi\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)\right]dl \le 2\psi(\phi+\varphi)$$
$$+\left[\psi\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)+\psi\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right]$$
$$-\frac{a}{2}[2\phi+2\varphi-(\alpha_2-\alpha_1)]\int_0^1$$
$$\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k1}(1-l)^{\zeta-1}-\left(\frac{2}{\alpha_2-\alpha_1}\right)^{\zeta\xi/k}$$
$$\times\left\{\Gamma_k(\xi){}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_1+\alpha_2/2)^+}\psi(\phi+\varphi-\alpha_2)\right.$$
$$\left.+\Gamma_k(\xi){}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_1+\alpha_2/2)^-}\psi(\phi+\varphi-\alpha_1)-\frac{a}{2}M(\alpha_{21}-\alpha_{11})\right\}dl$$
$$\le\frac{1}{\xi/k\zeta^{\xi/k}}2\psi(\phi+\varphi)-[\psi\alpha_1+\psi\alpha_2]-\frac{a}{2}[2\phi+2\varphi-(\alpha_2-\alpha_1)]. \tag{22}$$

This completes the proof. □

Remark 7. It is obvious from Theorem 6 that

(1) Theorem 2.1 of [25] is obtained if we take $a=0$, $\alpha_1 = x$, and $\alpha_2 = y$ in Theorem 6

(2) Theorem 2.1 of [26] is obtained if we take $a=0$, $k=1$, $\alpha_1=\theta$, and $\alpha_2=\vartheta$ in Theorem 6

(3) Theorem 2 of [27] is obtained by taking $a=0$, $\alpha=k=1$, $\alpha_1=\theta$, and $\alpha_2=\vartheta$ in Theorem 6

Theorem 8. *Let $\zeta,\xi>0$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a strong convex function. Then, the inequalities*

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\le\psi(\phi)+\psi(\varphi)-\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}$$
$$\times\left\{{}_k^{\xi}J^{\zeta}_{\left(\phi+\varphi-\alpha_1^+\right)}\psi(\alpha_2)+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2)^-}\psi(\alpha_1)-\frac{a}{2}M(\alpha_2-\alpha_1)\right\}$$
$$\le\psi(\phi)+\psi(\varphi)-\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)-\frac{a}{2}(\alpha_2-\alpha_1)^2 \tag{23}$$

and

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\le\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}$$
$$\times\left\{{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_1^+)}\psi(\alpha_2)+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2)}-\psi(\alpha_1)-\frac{a}{2}M(\alpha_2-\alpha_1)\right\}$$
$$\le\psi(\phi)+\psi(\varphi)-\left(\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}\right)-\frac{a}{2}(\alpha_2-\alpha_1)^2 \tag{24}$$

hold $\forall\alpha_1,\alpha_2\in[\phi,\varphi]$.

Proof. The Jensen-Mercer inequality dictates that

$$\psi\left(\phi+\varphi-\frac{\alpha_{11}+\alpha_{21}}{2}\right)\le\psi(\phi)+\psi(\varphi)-\frac{\psi(\alpha_{11})+\psi(\alpha_{21})}{2}-\frac{a}{2}M(\alpha_2-\alpha_1), \tag{25}$$

$\forall\alpha_{11},\alpha_{21}\in[\phi,\varphi]$. □

Taking $\alpha_{11}=l\alpha_1+(1-l)\alpha_2$ and $\alpha_{21}=(1-l)\alpha_1+(l)\alpha_2$ for $\alpha_1,\alpha_2\in[\phi,\varphi]$ and $l\in[0,1]$ in (25), we get

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\le\psi(\phi)+\psi(\varphi)-\frac{\psi(l\alpha_1+(1-l)\alpha_2)+\psi((1-l)\alpha_1+(l)\alpha_2)}{2}-\frac{a}{2}M(\alpha_2-\alpha_1). \tag{26}$$

Multiplying the above inequality by $(1-(1-l)^{\zeta}/\zeta)^{\xi/k}(1-l)^{\zeta-1}$ and integrating the obtained inequality with respect to l over [0,1] give

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\int_0^1\left(\frac{1-(1-la)^{\zeta}}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}dl$$
$$\le\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}$$
$$\times\left\{\psi(\phi)+\psi(\varphi)-\frac{\psi(l\alpha_1+(1-l)\alpha_2)+\psi((1-l)\alpha_1+(l)\alpha_2)}{2}\right.$$
$$\left.-\frac{a}{2}M(\alpha_2-\alpha_1)\right\}dl, \tag{27}$$

that is,

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\le\psi(\phi)+\psi(\varphi)-\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}$$
$$\times\left\{{}_k^{\xi}J^{\zeta}_{\alpha_1^+}\psi(\alpha_2)+{}_k^{\xi}J^{\zeta}_{\alpha_2^-}\psi(\alpha_1)-\frac{a}{2}M(\alpha_1+\alpha_2)\right\}. \tag{28}$$

That proves the first inequality of (24).

To prove the second inequality of (24), from the definition of strong convexity of ψ, for $l\in[0,1]$, we get

$$\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)=\psi\left(\frac{l\alpha_1+(1-l)\alpha_2+(1-l)\alpha_1+l\alpha_2}{2}\right)$$
$$\le\frac{\psi(l\alpha_1+(1-l)\alpha_2)+\psi((1-l)\alpha_1+l\alpha_2)}{2}-\frac{a}{2}M(\alpha_1-\alpha_2). \tag{29}$$

Multiplying the above inequality by $(1-(1-l)^{\zeta}/\zeta)^{\xi/k}$

$(1-l)^{\zeta-1}$ and then integrating with respect to l over [0,1], we have

$$\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1} \leq \int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1} \times\left\{\frac{\psi(l\alpha_1+(1-l)\alpha_2)+\psi((1-l)\alpha_1+l\alpha_2)}{2}-\frac{a}{2}M(\alpha_1-\alpha_2)\right\}dl, \quad (30)$$

where

$$\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)\leq\left\{{}_k^\xi J_{\alpha_1^+}^\zeta\psi(\alpha_2)+k^\xi J_{\alpha_2^-}^\zeta\psi(\alpha_1) - \frac{a}{2}M(\alpha_1+\alpha_2))\right\}\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}} \quad (31)$$

implies

$$-\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)\geq-\left\{{}_k^\xi J_{\alpha_1^+}^\zeta\psi(\alpha_2)+k^\xi J_{\alpha_2^-}^\zeta\psi(\alpha_1)\frac{a}{2}M(\alpha_1+\alpha_2)\right\}\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}. \quad (32)$$

Adding $\psi(\phi)+\psi(\varphi)$ on both side of above inequality,

$$\psi(\phi)+\psi(\varphi)-\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)\geq\psi(\phi)+\psi(\varphi) -\left\{{}_k^\xi J_{\alpha_1^+}^\zeta\psi(\alpha_2)+k^\xi J_{\alpha_2^-}^\zeta\psi(\alpha_1)\right\}\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}, \quad (33)$$

which gives (23).

To prove (24), using strong convexity of ψ, we get

$$\psi\left(\phi+\varphi-\frac{\alpha_{11}+\alpha_{21}}{2}\right)=\psi\left(\frac{\phi+\varphi-\alpha_{11}+\phi+\varphi-\alpha_{21}}{2}\right) \leq\psi\left(\frac{\phi+\varphi-\alpha_{11}}{2}\right)+\psi\left(\frac{\phi+\varphi-\alpha_{21}}{2}\right) -\frac{a}{2}M(\alpha_2-\alpha_1), \quad (34)$$

$$\forall\alpha_{11},\alpha_{21}\in[\phi,\varphi]. \quad (35)$$

Let $\alpha_{11}=l\alpha_1+(1-l)\alpha_2$ and $\alpha_{21}=(1-l)\alpha_1+l\alpha_2$; then, (34) leads

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\leq\psi\left(\frac{\phi+\varphi-(l\alpha_1+(1-l)\alpha_2)}{2}\right) +\psi\left(\frac{\phi+\varphi-((1-l)\alpha_1+l\alpha_2)}{2}\right) -\frac{a}{2}M(\alpha_1+\alpha_2). \quad (36)$$

Multiplying the above inequality by $(1-(1-l)^\zeta/\zeta)^{\xi/k}(1-l)^{\zeta-1}$ and then integrating with respect to l over [0,1], we have

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}dl \leq\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}\left\{\psi\left(\frac{\phi+\varphi-(l\alpha_1+(1-l)\alpha_2)}{2}\right) +\psi\left(\frac{\phi+\varphi-((1-l)\alpha_1+l\alpha_2)}{2}\right)-\frac{a}{2}M(\alpha_1+\alpha_2)\right\}dl, \quad (37)$$

which can be written as

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\leq\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^\xi J_{\alpha_1^+}^\zeta\psi(\alpha_2)+k^\xi J_{\alpha_2^-}^\zeta\psi(\alpha_1) -\frac{a}{2}M(\alpha_1+\alpha_2)\right\}. \quad (38)$$

If follows from the definition of strong convexity of ψ that

$$\psi(l(\phi+\varphi-\alpha_1)+(1-l)(\phi+\varphi-\alpha_2))\leq l\psi(\phi+\varphi-\alpha_1) +(1-l)\psi(\phi+\varphi-\alpha_2)-l\frac{a}{2}(1-l)M(\alpha_1-\alpha_2), \quad (39)$$

$$\psi((1-l)(\phi+\varphi-\alpha_1)+l(\phi+\varphi-\alpha_2))\leq(1-l)\psi(\phi+\varphi-\alpha_1) +l\psi(\phi+\varphi-\alpha_2)-al(1-l)M(\alpha_1-\alpha_2). \quad (40)$$

Adding the above two inequalities and with the help of Jensen-Mercer inequality, we have

$$\psi(l(\phi+\varphi-\alpha_1)+(1-l)(\phi+\varphi-\alpha_2))+\psi((1-l)(\phi+\varphi-\alpha_1) +l(\phi+\varphi-\alpha_2))\leq l\psi(\phi+\varphi-\alpha_1)+(1-l)\psi(\phi+\varphi-\alpha_2) -l(1-l)M(\alpha_1-\alpha_2)+(1-l)\psi(\phi+\varphi-\alpha_1) +l\psi(\phi+\varphi-\alpha_2)-l(1-l)M(\alpha_1-\alpha_2),$$

$$\psi(l(\phi+\varphi-\alpha_1)+(1-l)(\phi+\varphi-\alpha_2))+\psi((1-l)(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2))\leq\psi(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2)-2al(1-l)M(\alpha_2-\alpha_1)\leq 2(\psi(\phi)+\psi(\varphi))-(\psi(\alpha_1)+\psi(\alpha_2))-2aM(\alpha_2-\alpha_1). \quad (41)$$

Multiplying the above inequality by $(1-(1-l)^\zeta/\zeta)^{\xi/k}(1-l)^{\zeta-1}$ and then integrating with respect to l over [0,1], we have

$$\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}[S(l(\phi+\varphi-\alpha_1)+(1-l)(\phi+\varphi-\alpha_2))+\psi((1-l)(\phi+\varphi-\alpha_1)+l(\phi+\varphi-\alpha_2))]dl \leq 2(\psi(\phi)+\psi(\varphi))-(\psi(\alpha_1)+\psi(\alpha_2))-\frac{a}{2}M(\alpha_2-\alpha_1)\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}dl, \quad (42)$$

that is,

$$\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^\xi J_{\alpha_1^+}^\zeta\psi(\alpha_2)+{}_k^\xi J_{\alpha_2^-}^\zeta\psi(\alpha_1)-\frac{a}{2}M(\alpha_1+\alpha_2)\right\}\leq\psi(\phi)+\psi(\varphi)-\left(\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}\right)-\frac{a}{2}M(\alpha_2-\alpha_1)^2. \quad (43)$$

Combining (38) and (43) leads to (24).

Remark 9. Let a = 0, $\zeta=\xi=k=1$; then, Theorem 8 leads to

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\leq\psi(\phi)+\psi(\varphi)-\int_0^1\psi(l\alpha_1+(1-l)\alpha_2)dl\leq\psi(\phi)+\psi(\varphi)-\psi\left(\frac{\alpha_1+\alpha_2}{2}\right), \quad (44)$$

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\leq\frac{1}{\alpha_2-\alpha_2}\int_{\alpha_1}^{\alpha_2}\psi(\phi+\varphi_l)dl\leq\psi(\phi)+\psi(\varphi)-\psi\left(\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}\right), \quad (45)$$

which was proved in Theorem 2.1 of [28].

Lemma 10. *Let $\zeta,\xi>0,\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow R$ be a differentiable mapping such that $\psi'\in L[\phi,\varphi]$. Then, the inequality*

$$\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)\zeta\xi/k}\left\{{}_k^\xi J_{(\phi+\varphi-\alpha_1+\alpha_2/2)}\psi(\phi+\varphi-\alpha_1)+{}_k^\xi J_{(\phi+\varphi-\alpha_1+\alpha_2/2)}\psi(\phi+\varphi-\alpha_2)x\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\right\}=\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}^{\xi/k}\right)\times\left\{\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)-\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)\right\} \quad (46)$$

holds $\forall\alpha_1,\alpha_2\in[\phi,\varphi]$.

Proof. Suppose

$$P=(P_1-P_2)\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}, \quad (47)$$

where

$$P_1=\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)dl,$$
$$P_2=\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)dl. \quad (48)$$

Using integration by parts, we get

$$=-\frac{2}{\zeta^{\xi/k}(\alpha_2-\alpha_1)}\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)+\frac{2\zeta\xi/k}{\zeta^{\xi/k}(\alpha_2-\alpha_1)}\int_0^1\times\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)dl$$
$$=-\frac{2}{\zeta^{\xi/k}(\alpha_2-\alpha_1)}\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)+\frac{2\zeta\xi/k}{\zeta^{\xi/k}(\alpha_2-\alpha_1)^{\zeta\xi/k}+1}\int_{\phi+\varphi-\frac{\alpha_1-\alpha_2}{2}}^{\phi+\varphi-\alpha_1}\times\left(\left(\frac{\alpha_2-\alpha_2}{2}\right)^\zeta-\left(l_1-\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)^\zeta\right)^{\xi/k-1}\times\frac{\psi(l_1)}{(\alpha_1-\phi+\varphi-\alpha_1+\alpha_2/2)^{1-\zeta}}\right)dl_1$$
$$=-\frac{2}{\zeta^{\xi/k}(\alpha_2-\alpha_1)}\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)+\left(\frac{2}{y-x}\right)^{\zeta\xi/k+1}\frac{\Gamma_k(\xi+k)}{\zeta^{\xi/k-1}}{}_k^\xi J_{\phi+\varphi-\alpha_1+\alpha_2/2^+}^\zeta\psi(\phi+\varphi-\alpha_1). \quad (49)$$

Similarly, using integration by parts for P_2, we get

$$P_2 = \int_0^1 \left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k} \psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)dl$$
$$= \frac{2}{\zeta^{\xi/k}(\alpha_2-\alpha_1)}\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)$$
$$-\left(\frac{2}{y-x}\right)^{\zeta\xi/k+1}\frac{\Gamma_k(\xi+k)^\xi}{\zeta^{\xi/k-1}}\, {}_kJ^\zeta_{(\phi+\varphi-\alpha_1+\alpha_2/2)}-\psi(\phi+\varphi-\alpha_2). \tag{50}$$

Therefore, inequality (43) follows from (47)–(50). □

Remark 11.

(1) If we take $K=1$, $\alpha_1=\theta$, and $\alpha_2=\upsilon$ in Lemma 10, then we can get Lemma 3.1 of [26]

(2) If we take $\zeta=K=1$, $\alpha_1=\theta$, and $\alpha_2=\upsilon$, then Lemma 10 reduces to Lemma 1.1 of [29]

Lemma 12. *Let $\zeta,\xi>0$, $\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in\psi[\phi,\varphi]$. Then, the identity*

$$\frac{\psi(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2)}{2}-\frac{\zeta^{\xi/k\Gamma_k(\xi+k)}}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}$$
$$\times\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2)}+\psi(\phi+\varphi-\alpha_1)+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2)}+\psi(\phi+\varphi-\alpha_1)\right\}$$
$$=\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\int_0^1\left[\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right]$$
$$\times\psi'(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))dl \tag{51}$$

holds $\forall\alpha_1,\alpha_2\in[\phi,\varphi]$.

Proof. Suppose

$$P=\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\int_0^1\left[\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right]$$
$$\times\psi'(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))dl$$
$$=\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\{P_1-P_2\}. \tag{52}$$

Then, we clearly see that

$$P_1=\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\times\psi'(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))dla$$
$$=\frac{1}{\alpha^{\xi/k}}\frac{\psi(\phi+\varphi-\alpha_1)}{\alpha_2-\alpha_1}-\frac{\xi/k}{\alpha_2-\alpha_1}\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1}$$
$$\times(1-l)^{\zeta-1}\psi(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))dl$$
$$=\frac{1}{\alpha^{\xi/k}}\frac{\psi(\phi+\varphi-\alpha_1)}{\alpha_2-\alpha_1}-\frac{\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k+1}}$$
$$\cdot\left\{{}^{\zeta}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2)}+\psi(\phi+\varphi-\alpha_2)\right\}, \tag{53}$$

and

$$P_2=\int_0^1\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\times\psi'(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))dl$$
$$=-\frac{1}{\zeta^{\xi/k}}\frac{\psi(\phi+\varphi-\alpha_2)}{\alpha_2-\alpha_1}+\frac{\xi/k}{\alpha_2-\alpha_1}\int_0^1\left(\frac{1-l^\zeta}{\zeta}\right)^{\xi/k-1}$$
$$\times l^{\zeta-1}\psi(\phi+\varphi(l\alpha_1+(1-l)y))dl$$
$$=-\frac{1}{\zeta^{\xi/k}}\frac{\psi(\phi+\varphi-\alpha_2)}{\alpha_2-\alpha_1}+\frac{\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k+1}}$$
$$\cdot\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2)}-\psi(\phi+\varphi-\alpha_1)\right\}. \tag{54}$$

Therefore, identity (51) follows from (52)–(54). □

Corollary 13. *If we take $\zeta=\xi=k=1$, then Lemma 12 leads to the equality*

$$\frac{\psi(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\phi+\varphi-\alpha_2}^{\phi+\varphi-\alpha_1}\psi(la)dl$$
$$=\frac{\alpha_2-\alpha_1}{2}\int_0^1(2l-1)S'(\phi+\varphi-(l\alpha_1+(1-l)y))dl. \tag{55}$$

Remark 14. If we take $\alpha_1=\phi=\theta$ and $\alpha_2=\varphi=\upsilon$ in Corollary 13, then (55) becomes

$$\frac{\psi(\phi)+\psi(\varphi)}{2}-\frac{1}{\varphi-\phi}\int_0^1\psi(l)dl=\frac{\phi-\varphi}{2}\int_0^1(2l-1)\psi'((1-l)\phi+l\varphi)dl, \tag{56}$$

which was proved in Lemma 2.1 of [30].

Theorem 15. *Let $\zeta,\xi>0$, $\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in L[\phi,\varphi]$ and $|\psi'|^q$ is a* *convex mapping on* $[\phi, \varphi]$. *Then, the inequality*

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\cdot\left[\left(\left|\psi'(\phi)+\psi'(\varphi)\right|\right)\left(\frac{1}{\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right)-\left\{\left|\psi'(\alpha_1)\right|\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)+B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right\}+\left\{\left|\psi'(\alpha_2)\right|\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)-B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right\}-\frac{a}{4}\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}l(2-l)dl+\left(\left|\psi'(\phi)+\psi'(\varphi)\right|\right)\cdot\left(\frac{1}{\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right)-\left\{\left|\psi'(\alpha_1)\right|\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)-B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right\}+\left\{\left|\psi'(\alpha_2)\right|\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)+B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right\}-\frac{a}{4}\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}l(2-l)dl\right] \quad (57)$$

holds for all $\alpha_2, \alpha_1 \in [\phi, \varphi]$.

Proof. It follows from Lemma 10, Jensen-Mercer inequality, power mean inequality, and the convexity of function $|\psi'|^q$ that

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)\right|dl+\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)\right|dl\right\}. \quad (58)$$

Using the definition of strong convexity

$$\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left\{\left|\psi'(\phi)\right|+\left|\psi'(\varphi)\right|-\left(\frac{2-l}{2}\left|\psi'(\alpha_1|)+\frac{l}{2}\right|\psi'(\alpha_2|)\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right\}dl+\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left\{\left|\psi'(\phi)\right|+\left|\psi'(\varphi)\right|-\left(\frac{l}{2}\left|\psi'(\alpha_1|)\right|+\frac{2-l}{2}\left|\psi'(\alpha_2|)\right)\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right\}dl\right\}\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\cdot\left\{\left(\left|\psi'(\phi)\right|+\left|\psi'(\varphi)\right|\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi\xi/k/k}dl-\left(\left|\psi'(\alpha_1)\right|\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl+\left|\psi'(\alpha_2)\right|\int_0^1\cdot\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{l}{2}dl\right)-(\alpha_2-\alpha_1)^2\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{al}{2}\cdot\left(\frac{2-l}{2}\right)dl\right\}+\left\{\left(\left|\psi'(\phi)\right|+\left|\psi'(\varphi)\right|\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}dl-\left(\left|\psi'(\alpha_1)\right|\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{l}{2}dl+\left|\psi'(\alpha_2)\right|\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl\right)-(\alpha_2-\alpha_1)^2\int_0^1\cdot\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{al}{2}\left(\frac{2-l}{2}\right)dl\right\}. \quad (59)$$

Therefore, inequality (57) can be derived after some simple calculation. □

Remark 16. From Theorem 15, we clearly see that

(1) If we take $a = 0$, $\alpha_1 = x$, and $\alpha_2 = y$ in Theorem 15, then we get Theorem 3.1 of [25]

(2) If we take $a = 0$, $\zeta = k = 1$, $\alpha_1 = \theta$, and $\alpha_2 = \vartheta$ in Theorem 15, then we get Theorem 3.1 of [26]

(3) If we take $a = 0$, $\zeta = k = 1$, $\alpha_1 = \theta$, and $\alpha_2 = \vartheta$ in Theorem 15, then we get Theorem 5 of [29] in the case of $q = 1$

Theorem 17. *Let* $q > 1, \zeta, \xi > 0, \phi < 0$ *and* $\psi : [\phi, \varphi] \longrightarrow \mathbb{R}$ *be a differentiable mapping such that* $\psi' \in L[\phi, \varphi]$ *and* $|\psi'|^q$ *is a convex mapping on* $[\phi, \varphi]$. *Then, the inequality*

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}+\left\{{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)\right.\right.$$
$$\left.\left.+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|$$
$$\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\Bigg)\left(\frac{1}{\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right)^{1-1/q}$$
$$\times\left\{\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q\right)\left(\frac{1}{\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right)\right.$$
$$-\left\{\left|\psi'(\alpha_1)\right|^q\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)+B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right\}\right.$$
$$+\left\{\left|\psi'(\alpha_2)\right|^q\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)-B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right\}-\frac{a}{4}\int_0^1\right.$$
$$\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}l(2-l)dl+\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q\right)$$
$$\times\left(\frac{1}{\zeta^{\xi/k+1}}\times B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right)-\left\{\left|\psi'(\alpha_1)\right|^q\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right.\right.$$
$$\left.-B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right\}+\left\{\left|\psi'(\alpha_2)\right|^q\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)+B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right\}$$
$$\left.-\frac{a}{4}\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}l(2-l)dl\right]^{1/q} \tag{60}$$

holds for all $\alpha_2,\alpha_1\in[\phi,\varphi]$.

Proof. It follows from Lemma 10, Jensen-Mercer inequality, power-mean inequality, and the convexity of function $|\psi'|^q$ that

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)\right.\right.$$
$$\left.\left.+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)-\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\right\}\right|$$
$$\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\left(\left(\int_0^1\left(\frac{\left(1-(1-l)^{\zeta}\right.}{\zeta}\right)^{\xi/k}dl\right)^{1-1/q}\right.\right.$$
$$\times\left(\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\zeta+\frac{2-l}{2}y\right)\right)\right|^q dl\right)^{1/q}$$
$$+\left(\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}dl\right)^{1-1/q}\times\left(\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\right.$$
$$\left.\times\left|\psi'\left(\phi+\varphi-\left(\frac{2-la}{2}\zeta+\frac{l}{2}y\right)\right)\right|^q dl\right)^{1/q}\Bigg\}. \tag{61}$$

Using definition of strong convexity, we have

$$\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left\{\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q\right.\right.$$
$$-\left(\frac{2-l}{2}\left|\psi'(\alpha_1)\right|^q+\frac{l}{2}\left|\psi'(\alpha_2)\right|^q\right)-\frac{al}{2}$$
$$\times\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\}dl+\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\right.$$
$$\times\left\{\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q-\left(\frac{l}{2}\left|\psi'(\alpha_1)\right|^q+\frac{2-l}{2}\left|\psi'(\alpha_2)\right|^q\right)\right.$$
$$-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\}dl\}\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}$$
$$\times\left\{\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q\right)\int_0^1\left(\frac{1-(1-la)^{\zeta}}{\zeta}\right)^{\xi/k}dl\right.$$
$$-\left(\left|\psi'(\alpha_1)\right|^q\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl+\left|\psi'(\alpha_2)\right|^q\int_0^1\right.$$
$$\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{l}{2}dl)-(\alpha_2-\alpha_1)^2\int_0^1$$
$$\left.\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{al}{2}\left(\frac{2-l}{2}\right)dl\right\}$$
$$+\left\{\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}dl\right.$$
$$-\left(\left|\psi'(\alpha_1)\right|^q\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{l}{2}dl+\left|\psi'(\alpha_2)\right|^q\int_0^1\right.$$
$$\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl)-(\alpha_2-\alpha_1)^2\int_0^1$$
$$\left.\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{al}{2}\left(\frac{2-l}{2}\right)dl\right\}. \tag{62}$$

Making simple simplification, we get (60) from (61). □

Remark 18. Theorem 17 leads to

(1) If we take $a=0$, $\alpha_1=x$, and $\alpha_2=y$ in Theorem 17, then we get Theorem 2.12 of [25]

(2) If we take $a=0$, $k=1$, $\alpha_1=x=\phi$, and $\alpha_2=y=\varphi$ in Theorem 17, then we get Theorem 3.1 of [26]

(3) If we take $a=0$, $\zeta=k=1$, $\alpha_1=x=\phi$, and $\alpha_2=y=\varphi$ in Theorem 17, then we get Theorem 5 of [29] in the case of $q=1$

Theorem 19. *Let $q > 1, \zeta, \xi > 0, \phi < 0$ and $\psi : [\phi, \varphi] \longrightarrow \mathbb{R}$ be a differentiable mapping such that $\psi' \in L[\phi, \varphi]$ and $|\psi'|^q$ is a convex mapping on $[\phi, \varphi]$. Then, the inequality*

$$\begin{aligned}
&\left| \frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}} \left\{ {}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)} + \psi(\phi+\varphi-\alpha_1) \right.\right. \\
&\left.\left. + {}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)} + \psi(\phi+\varphi-\alpha_2) \right\} - \psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right) \right| \\
&\le \frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left(\frac{1}{\zeta^{\xi/kp+1}}B\left(\frac{\xi}{k}p+1, \frac{1}{\zeta}\right)\right)^{1/p} \\
&\times \left\{ \left(\left|\psi'(\phi)\right|^q + \left|\psi'(\varphi)\right|^q - \left(\frac{3\left|\psi'\alpha_1\right|^q + \left|\psi'\alpha_2\right|^q}{4}\right) - \frac{al}{2} \right.\right. \\
&\left. \times \left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2 \right)\Bigg)^{1/q} + \left(\left|\psi'(\phi)\right|^q + \left|\psi'(\varphi)\right|^q - \right. \\
&\left.\left. \times \left(\frac{\left|\psi^1(\alpha^1)\right| + 3\left|\psi'(\alpha_2)\right|}{}\right) - \frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2 \right)^{1/q} \right\}.
\end{aligned} \tag{63}$$

Proof. It follows from Lemma 10, Jensen-Mercer inequality, power mean inequality, and the convexity of function $|\psi'|^q$ that

$$\begin{aligned}
&\left| \frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}} \left\{ {}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)} + \psi(\phi+\varphi-\alpha_1) \right.\right. \\
&\left.\left. + {}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)} + \psi(\phi+\varphi-\alpha_2) - \psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right) \right\} \right| \\
&\le \frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{ \left(\left(\int_0^1 \left(\frac{\left(1-(1-l)^\zeta\right)}{\zeta} \right)^{\xi/kp} dl \right)^{1/p} \right.\right. \\
&\times \left(\int_0^1 \left| \psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\zeta+\frac{l}{2}y\right)\right) \right|^q dl \right)^{1/q} + \\
&\left.\left. \times \left(\int_0^1 \left| \psi'\left(\phi+\varphi-\left(\frac{l}{2}\zeta+\frac{2-l}{2}y\right)\right) \right|^q dl \right)^{1/q} \right\}.
\end{aligned} \tag{64}$$

It follows from the strong convexity of $|\psi'|^q$

$$\begin{aligned}
&\le \frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left(\int_0^1 \left(\frac{1-(1-l)^\zeta}{\zeta} \right)^{\xi/kp} dl \right)^{1/p} \times \left\{ \left(\left(\int_0^1 \left(\left|\psi'(\phi)\right|^q + \left|\psi'(\varphi)\right|^q \right.\right.\right.\right. \\
&- \left(\frac{2-l}{2}\left|\psi'(\alpha_1)\right|^q + \frac{l}{2}\left|\psi'(\alpha_2)\right|^q \right) - \frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2 \Big)\Big)^{1/q} dl \\
&+ \left(\int_0^1 \left(\left|\psi'(\phi)\right|^q + \left|\psi'(\varphi)\right|^q - \left(\frac{l}{2}\left|\psi'(\alpha_1)\right|^q + \frac{2-l}{2}\left|\psi'(\alpha_2)\right|^q \right) \right.\right. \\
&\left.\left. - \frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2 \Big)\right)^{1/q} dl \right\} \le \frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left(\frac{1}{\zeta^{\xi/kp+1}} B\left(\frac{\xi}{k}p+1, \frac{1}{\zeta}\right) \right)^{1/p} \\
&\cdot \left\{ \left(\left|\psi'(\phi)\right|^q + \left|\psi'(\varphi)\right|^q - \left(\frac{3\left|\psi'\alpha_1\right|^q + \left|\psi'\alpha_2\right|^q}{4} \right) - \frac{al}{2} \right.\right. \\
&\cdot \left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2 \Big)\Big)^{1/q} + \left(\left|\psi'(\phi)\right|^q + \left|\psi'(\varphi)\right|^q - \right. \\
&\left.\left. \cdot \left(\frac{\left|\psi'(\alpha_1)\right| + 3\left|\psi'(\alpha_2)\right|}{} \right) - \frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2 \right)^{1/q} \right\},
\end{aligned} \tag{65}$$

which completes the proof. □

Corollary 20. *Let $a = 0$ and $\xi = k = 1$. Then, Theorem 19 leads to*

$$\begin{aligned}
&\left| \frac{1}{\alpha_2-\alpha_1}\int_{\phi+\varphi-\alpha_2}^{\phi+\varphi-\alpha_1} \psi(l)dl - \psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right) \right| \\
&\le \frac{1}{2^{1/p}}\left\{ \left(\left|\psi'(\phi)\right|^q + \left|\psi'(\varphi)\right|^q - \left(\frac{3\left|\psi'\alpha_1\right|^q + \left|\psi'\alpha_2\right|^q}{4} \right) \right.\right. \\
&\left. \times -\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2 \right)\Bigg)^{1/q} \\
&+ \left(\left|\psi'(\phi)\right|^q + \left|\psi'(\varphi)\right|^q - \left(\frac{\left|\psi'(\alpha_1)\right| + 3\left|\psi'(\alpha_2)\right|}{} \right) \right. \\
&\left.\left. - \frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2 \right)^{1/q} \right\}.
\end{aligned} \tag{66}$$

Theorem 21. *Let $\zeta, \xi > 0, p, q > 1$ with $1/p + 1/q = 1, \phi < \varphi$ and $\psi : [\phi, \varphi] \longrightarrow \mathbb{R}$ be a differentiable mapping such that $\psi' \in L[\phi, \varphi]$ and $|\psi'|^q$ is a convex mapping on $[\phi, \varphi]$. Then, the inequality*

$$\begin{aligned}
&\left| \frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}} \left\{ {}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)} + \psi(\phi+\varphi-\alpha_1) + {}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)} \right.\right. \\
&\left.\left. + \psi(\phi+\varphi-\alpha_2) \right\} - \psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right) \right| \le \frac{\left(\alpha_2-\alpha_1\zeta^{\xi/k}\right)}{4} \\
&\cdot \left\{ \left(\left|\psi'(\phi)\right|^q + \left|\psi'(\varphi)\right|^q \right) \left(\frac{B(\xi/k+1, \xi/k)}{\xi^{\xi/k+1}} \right) \right. \\
&- \left(\left(\frac{B(\xi/k+1, 2/\xi) + B(\xi/k+1, 1/\xi)}{2\zeta^{\xi/k+1}} \right) \left|\psi'(\alpha_1)\right|^q \right. \\
&+ \left(\frac{B(\xi/k+1, 1/\xi) + B(\xi/k+1, 2/\xi)}{2\zeta^{\xi/k+1}} \right) \left|\psi'(\alpha_1)\right|^q \\
&- \frac{al}{2}\left(\frac{2-l}{2}\right)\left|\alpha_2-\alpha_1\right|^2 \Big)^{1/q} + \left(\left|\psi'(\phi)\right|^q + \left|\psi'(\varphi)\right|^q \right) \\
&\cdot \left(\frac{B(\xi/k+1, 1/\xi)}{\xi^{\xi/k+1}} \right) - \left(\left(\frac{B(\xi/k+1, 1/\xi) - B(\xi/k+1, 2/\xi)}{2\zeta^{\xi/k+1}} \right) \left|S'(\alpha_1)\right|^q \right. \\
&+ \left(\frac{B(\xi/k+1, 2/\xi) + B(\xi/k+1, 1/\xi)}{2\zeta^{\xi/k+1}} \right) \left|\psi'(\alpha_2)\right|^q \\
&\left. - \frac{al}{2}\left(\frac{2-l}{2}\right)\left|\alpha_2-\alpha_1\right|^2 \right)^{1/q} \Bigg\}
\end{aligned} \tag{67}$$

holds $\forall \alpha_1, \alpha_1 \in [\phi, \varphi]$.

Proof. It follows from Lemma 10, Jensen-Mercer inequality, strong convexity of $|\psi'|^q$, and Holder integral inequality that

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\right.\right.$$
$$\left.\left.+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\leq\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}$$
$$\times\left\{\left(\int_0^1 1dl\right)^{1/p}\left(\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left|\psi'(\phi+\varphi-\right.\right.\right.$$
$$\left.\left.\times\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)\right|^q dl\right)^{1/q}+\left(\int_0^1 1dl\right)^{1/p}$$
$$\left.\times\left(\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)\right|^q dl\right)^{1/q}\right\}$$
$$\leq\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}\left\{\left(\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left[|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right.\right.\right.$$
$$\left.\left.-\left(\frac{2-l}{2}|\psi'(\alpha_1)|^q+\frac{l}{2}|\psi'(\alpha_2)|^q\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right]^q\right)^{1/q}$$
$$+\left(\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left[|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right.\right.$$
$$\left.\left.-\left(\frac{l}{2}|\psi'(\alpha_1)|^q+\frac{2-l}{2}|\psi'(\alpha_2)|^q\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right]dl\right)^{1/q}\right\}. \tag{68}$$

By making necessary changes, we get (67). □

Theorem 22. *Let $\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in L[\phi,\varphi]$ and $|\psi'|$ is a convex mapping on$[\phi,\varphi]$. Then, one has*

$$\left|\frac{(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2)}{2}-\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}\right.$$
$$\left.\times\left\{{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_2)}+\psi(\phi+\varphi-\alpha_1)+{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_1)}+\psi(\phi+\varphi-\alpha_2)\right\}\right|$$
$$\leq\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\left[\left\{\left(\left||\psi'(\phi)+\psi'(\varphi)\right|\right)\frac{B((1/\xi),\xi/k+1)}{\zeta^{\xi/k+1}}\right.\right.$$
$$-\frac{|\psi'(\alpha_1)|}{\zeta^{\xi/k+1}}\left\{B_{1/2^\zeta}\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)+B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)\right\}$$
$$\left.-\frac{|\psi'(\alpha_2)|}{\zeta^{\xi/k+1}}\left\{B_{1/2^\zeta}\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-al(1-l)|\alpha_2-\alpha_1|^2\right\}\right\}$$
$$+\left\{\left(\left||\psi'(\phi)+\psi'(\varphi)\right|\right)\frac{B((1/\xi),\xi/k+1)}{\zeta^{\xi/k+1}}-\frac{|\psi'(\alpha_1)|}{\zeta^{\xi/k+1}}\right.$$
$$\times\left\{B_{1/2^\zeta}\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)\right\}-\frac{|\psi'(\alpha_2)|}{\zeta^{\xi/k+1}}\left\{B_{1/2^\zeta}\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)\right.$$
$$\left.\left.\left.+B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-al(1-l)|\alpha_2-\alpha_1|^2\right\}\right\}\right],\forall\alpha_2,\alpha_1\in[\phi,\varphi]. \tag{69}$$

Proof. By using Lemma 12 and similar arguments as in the proofs of previous theorem, we have

$$\left|\frac{\psi(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2)}{2}-\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}\right.$$
$$\left.\times\left\{{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_2)}+\psi(\phi+\varphi-\alpha_1)+{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_1)}\psi(\phi+\varphi-\alpha_2)\right\}\right|$$
$$\leq\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\int_0^1\left|\left(\frac{1-(1-la)^\zeta}{\zeta}\right)^{\xi/k}-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right|$$
$$\times\left|\psi'(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))\right|dl$$
$$\leq\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\int_0^1\left|\left(\frac{1-(1-la)^\zeta}{\zeta}\right)^{\xi/k}-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right|$$
$$\times\left\{|\psi'(\phi)|+|\psi'(\varphi)|-\left(l|\psi'(\alpha_1)|+(1-l)|(\alpha_2)|\right)\right.$$
$$\left.-al(1-l)|\alpha_2-\alpha_1|^2\right\}dl\leq\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\left[\int_0^{1/2}\left[\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right.\right.$$
$$\left.-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right]\times\left\{|\psi'(\phi)|+|\psi'(\varphi)|-\left(l|\psi'(\alpha_1)|\right.\right.$$
$$\left.\left.+(1-l)|(\alpha_2)|\right)-al(1-l)|\alpha_2-\alpha_1|^2\right\}dl+\int_{1/2}^1$$
$$\times\left[\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right]$$
$$\times\left\{|\psi'(\phi)|+|\psi'(\varphi)|-\left(l|\psi'(\alpha_1)|+(1-l)|(\alpha_2)|\right)\right.$$
$$\left.-al(1-l)|\alpha_2-\alpha_1|^2\right\}dl, \tag{70}$$

which completes the proof. □

3. New Inequalities by Improved Hölder Inequality

Theorem 23. *Let $\xi,\zeta>0,p,q>1$ with $1/p+1/q=1,\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in l[\phi,\varphi]$ and $|\psi'|^q$ is a strong convex mapping on $[\phi,\varphi]$. Then, one has*

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)\right.\right.$$
$$\left.\left.+{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|$$
$$\leq\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\left[\left\{\left(\frac{B(2/\zeta,\xi/kp+1)}{\zeta^{\xi/kp+1}}\right)^{1/p}\left(\frac{|\psi'(\phi)|^q+|\psi'(\varphi)|^q}{2}\right.\right.\right.$$
$$-\left(\frac{5}{12}|\psi'(\alpha_1)|^q+\frac{1}{12}|\psi'(\alpha_2)|^q\right)-a\frac{5}{144}(\alpha_2-\alpha_1)^2)+$$
$$\times\left(\frac{B(1/\zeta,\xi/kp+1)-B(2/\zeta,\xi/kp+1)}{\zeta^{\xi/kp+1}}\right)^{1/p}$$
$$\times\left(\frac{|\psi'(\phi)|^q+|\psi'(\varphi)|^q}{2}-\left(\frac{1}{3}|\psi'(\alpha_1)|^q+\frac{1}{6}|\psi'(\alpha_2)|^q\right)\right.$$

$$-a\frac{1}{18}(\alpha_2-\alpha_1)^2)\}+\left(\frac{B(2/\zeta,\xi/kp+1)}{\zeta^{\xi/kp+1}}\right)^{1/p}\left(\frac{|\psi'(\phi)|^q+|\psi'(\varphi)|^q}{2}\right.$$
$$-\left(\frac{1}{12}|\psi'(\alpha_1)|^q+\frac{5}{12}|\psi'(\alpha_2)|^q\right)-a\frac{5}{144}(\alpha_2-\alpha_1)^2)$$
$$+\left(\frac{B(1/\zeta,\xi/kp+1)-B(2/\zeta,\xi/kp+1)}{\zeta^{\xi/kp+1}}\right)^{1/p}\times\left(\frac{|\psi'(\phi)|^q+|\psi'(\varphi)|^q}{2}\right.$$
$$\left.\left.-\left(\frac{1}{6}|\psi'(\alpha_1)|^q+\frac{1}{3}|\psi'(\alpha_2)|^q\right)-a\frac{1}{18}(\alpha_2-\alpha_1)^2)\right\}\right]. \quad (71)$$

Proof. It follows from Lemma 10, Jensen-Mercer inequality, the convexity of $|\psi'|^q$, and Holder-Iscan integral inequality given in Theorem 1.4 of [31] that

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)\right\}\right.$$
$$\left.-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\leq\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}\left[\left\{\left(\left(\int_0^1(1-l)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\right.\right.\right.$$
$$\times\left(\int_0^1(1-l)\left|\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\right)\right)\right|^q dl\right)^{1/q}\left(\int_0^1 l\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}$$
$$\left.\times\left(\int_0^1(l)\left|\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\right)\right)\right|^q dl\right)^{1/q}\right\}$$
$$+\left\{\left(\int_0^1(1-l)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\left(\int_0^1(1-l)|\psi'(\phi\right.\right.$$
$$\left.+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\right))|^q dl\right)^{1/q}\left(\int_0^1 l\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}$$
$$\left.\left.\times\left(\int_0^1(l)\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\right)\right)\right|^q dl\right)^{1/q}\right\}\right]. \quad (72)$$

Applying definition of strong convexity,

$$\leq\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}\left[\left\{\left(\left(\int_0^1(1-l)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\left(\int_0^1(1-l)\right.\right.\right.$$
$$\times\left[|\psi'\phi|^q+|\psi'\varphi|^q-\left(\frac{2-l}{2}|\psi'\alpha_1|^q+\left|\psi'\frac{l}{2}\right|^q\right)\right.$$
$$\left.-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2]dl\right)^{1/q}$$
$$+\left(\int_0^1 l\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\left(\int_0^1(l)\left[|\psi'\phi|^q+|\psi'\varphi|^q\right.\right.$$
$$\left.\left.-\left(\frac{2-l}{2}|\psi'\alpha_1|^q+\left|\psi'\frac{l}{2}\right|^q-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right)\right]^q dl\right]^{1/q}\Bigg\}$$
$$+\left\{\left(\left(\int_0^1(1-l)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\left(\int_0^1(1-l)\left[|\psi'\phi|^q\right.\right.\right.$$
$$+|\psi'\varphi|^q-\left(\frac{l}{2}|\psi'\alpha_1|^q+\frac{2-l}{2}|\psi'\alpha_2|^q\right.$$
$$\left.\left.-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2)\right]^q dl\right)^{1/q}$$
$$\times\left(\int_0^1 l\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\left(\int_0^1(l)\left[|\psi'\phi|^q\right.\right.$$
$$+|\psi'\varphi|^q-\left(\frac{l}{2}|\psi'\alpha_1|^q+\frac{2-l}{2}|\psi'\alpha_2|^q\right)$$
$$\left.\left.\left.-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\}\right]^q dl\right)^{1/q}\right\}\right]. \quad (73)$$

By some computations, one can get the required result. □

Theorem 24. *Let $\xi,\zeta>0$, $p,q>1$ with $1/p+1q=1$, $\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in l[\phi,\varphi]$ and $|\psi'|^q$ is a strong convex mapping on $[\phi,\varphi]$. Then, one has*

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\psi(\phi+\varphi-\alpha_1)+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\psi(\phi+\varphi-\alpha_2)\right\}\right.$$
$$\left.-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\leq\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}\left[\left\{\left(\frac{B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)^{1-1/q}\right.\right.$$
$$\times\left(\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\left(\frac{B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)\right.$$
$$-\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_1)|^q\left(B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)+B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right.$$
$$+\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_2)|^q\left(B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right.\right.$$
$$-\frac{al}{4\zeta^{\xi/k+1}}\left(\frac{2-l}{2}\right)|\alpha_2-\alpha_1|^2)))^{1/q}+\left(\frac{B(1/\zeta,\xi/k+1)-B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)^{1-1/q})$$
$$\times\left(\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\left(\frac{B(1/\zeta,\xi/k+1)-B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)\right.$$
$$-\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_1)|^q\left(B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right)$$
$$+\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_2)|^q\left(B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-2B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)+B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)$$
$$-\frac{al}{4\zeta^{\xi/k+1}}\left(\frac{2-l}{2}\right)|\alpha_2-\alpha_1|^2)))^{1/q}\Bigg\}+\left\{\left(\frac{B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)^{1-1/q}\right.$$
$$\times\left(\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\left(\frac{B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)\right.$$
$$-\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_1)|^q\left(B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right)$$
$$+\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_2)|^q\left(B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right)$$
$$-\frac{al}{4\zeta^{\xi/k+1}}\left(\frac{2-l}{2}\right)|\alpha_2-\alpha_1|^2)))^{1/q}+\left(\frac{B(1/\zeta,\xi/k+1)-B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)^{1-1/q}$$
$$\times\left(\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\left(\frac{B(1/\zeta,\xi/k+1)-B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)\right.$$
$$-\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_1)|^q\left(B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-2B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)+B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)$$
$$+\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_1)|^q\left(B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right)$$
$$\left.\left.-\frac{al}{4\zeta^{\xi/k+1}}\left(\frac{2-l}{2}\right)|\alpha_2-\alpha_1|^2)))^{1/q}\right\}\right]. \quad (74)$$

Proof. It follows from Lemma 10, Jensen-Mercer inequality, the convexity of $|\psi'|^q$, and the improved power-mean integral inequality given in Theorem 1.5 of [31] that

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\right.\right.$$
$$\left.\left.+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}$$
$$\times\left\{\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)\right|dl\right.$$
$$\left.+\left\{\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)\right|dl\right\}.\right. \quad (75)$$

Using definition of strong convexity,

$$
\begin{aligned}
&\leq \frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left\{|\psi'(\phi)|+|\psi'(\varphi)|\right.\right.\\
&\quad-\left(\frac{2-l}{2}\left|\psi'(\alpha_1|\right)+\frac{l}{2}\left|\psi'(\alpha_2|\right)\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\}dl\\
&\quad+\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left\{|\psi'(\phi)|+|\psi'(\varphi)|-\left(\frac{l}{2}|\psi'(\alpha_1|\right)\right.\right.\\
&\quad+\frac{2-l}{2}|\psi'(\alpha_2|))-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\}dl\}\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\\
&\quad\times\left\{\left(|\psi'(\phi)|+|\psi'(\varphi)|\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}dl\right.\\
&\quad-\left(|\psi'(\alpha_1)|\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl+|\psi'(\alpha_2)|\int_0^1\right.\\
&\quad\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{l}{2}dl)-(\alpha_2-\alpha_1)^2\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{al}{2}\\
&\quad\times\left(\frac{2-l}{2}\right)dl\}+\left\{\left(|\psi'(\phi)|+|\psi'(\varphi)|\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}dl\right.\\
&\quad-\left(|\psi'(\alpha_1)\times|\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{l}{2}dl\right.\\
&\quad+|\psi'(\alpha_2)|\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl)-(\alpha_2-\alpha_1)^2\int_0^1\\
&\quad\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{al}{2}\left(\frac{2-l}{2}\right)dl\}.
\end{aligned}
\tag{76}
$$

□

4. Applications to Special Means

Means are important in applied and pure mathematics, especially they are used frequently in numerical approximation. In literature, they are order in the following way:

$$H\leq G\leq L\leq I\leq A. \tag{77}$$

The arithmetic mean of two numbers a, b such that a $a \neq b$ is defined as

$$A=A(a,b)=\frac{a+b}{2},\ a,b\in\mathbb{R}. \tag{78}$$

The generalized logarithmic mean is defined as follows:

$$
\begin{aligned}
L=L_r^r(a,b)&=\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)},\ r\in\mathbb{R}\,[-1,0],\ a,b\in\mathbb{R},\ a\neq b,\\
L_p(a,b)&=\left[\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^p\right],\ p\in\mathbb{R}\,[-1,0]
\end{aligned}
\tag{79}
$$

Proposition 25. *Assume $a, b > 0$ and $a < b$; then,*

$$M_p(a,b)\leq L_{1-p}^{p-1}(a,b).L_p^p(a,b)\leq A \tag{80}$$

holds for $p\in(-\infty,1)/0$ where

$$M_p(a,b)=\left[\left(\frac{a^p+b^p}{2}\right)^{1/p}\right]. \tag{81}$$

Proof. From Theorem 23, we have

$$
\begin{aligned}
\left[\left(\frac{a^p+b^p}{2}\right)^{1/p}\right]&\leq\frac{pB(\alpha)}{\alpha(b^p-a^p)}\left[\left({}_a^{CF}I^{\alpha}\psi\right)(k)+\left({}_a^{CF}I^{\alpha}\psi\right)(k)\right.\\
&\quad\left.-\frac{2(1-\alpha)}{B(\alpha)}\psi(k)\right]\leq\frac{\phi(a)+\phi(b)}{2},
\end{aligned}
\tag{82}
$$

with $\psi(x)=\phi(x)/x^{1-p}$ holds. Setting $f(x)=x$, $\alpha=1$ and $B(\alpha)=B(1)=1$ in the above theorem, we obtain

$$\left(\frac{a^p+b^p}{2}\right)^{1/p}\leq\frac{p(b-a)}{b^p-a^p}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]\leq\frac{a+b}{2}. \tag{83}$$

Now use the following:

$$L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p}. \tag{84}$$

For $p=p-1$, we have

$$L_{p-1}(a,b)=\left(\frac{b^p-a^p}{p(b-a)}\right)^{1/p-1}. \tag{85}$$

This implies that

$$L_{p-1}^{p-1}(a,b)=\frac{b^p-a^p}{p(b-a)}. \tag{86}$$

By using these means, we get

$$M_p(a,b)\leq L_{p-1}^{1-p}(a,b).L_p^p(a,b)\leq A. \tag{87}$$

By using the results of Section 3, we get some application to special means. □

Proposition 26. *Let $a,b\in\mathbb{R}^+$, $a<b$; then,*

$$\left|A(a^2,b^2)-pL_{p+1}^{1+p}(a^p,b^p)\right|\leq\frac{b^p-a^p}{p}\left[|a|C_1(a,b)+|b|C_2(a,b)\right]. \tag{88}$$

Proof. We obtain the result immediately from Theorem 23. □

Proposition 27. *Let $a, b \in \mathbb{R}^+$, $a < b$, then*

$$\left|A(a^n, b^n) - pL_{n-p+1}^{n-1+p}(a^p, b^p)\right| \le \frac{b^p - a^p}{2p}\left[\left|a^{n-1}\right|C_1(a, b) + \left|b^{n-1}\right|C_2(a, b)\right]. \tag{89}$$

Proof. We obtain the result immediately from Theorem 24. □

Data Availability

All data required for this research is included within the paper.

Conflicts of Interest

Authors of this paper declare that they have no competing interests.

Authors' Contributions

X.W. analyzed and approved the results, wrote the final version of the paper, and arranged the funding. A.H. proved the main results. M.S.S. proposed the problem and supervised this work. S.U.Z. wrote the first version of the paper.

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References

[1] C. Niculescu and L. E. Persson, *Convex Functions and Their Applications (p. 219)*, Springer, New York, 2006.

[2] J. E. Peajcariaac and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, 1992.

[3] A. W. Roberts, "Convex Functions," in *Handbook of Convex Geometry*, pp. 1081–1104, Elsevier, North-Holland, 1993.

[4] Y. Khurshid, M. A. Khan, and Y. M. Chu, "Conformable fractional integral inequalities for GG-and GA-convex *function*," *AIMS Math*, vol. 5, no. 5, pp. 5012–5030, 2020.

[5] Y. M. Chu, M. U. Awan, M. Z. Javad, and A. G. Khan, "Bounds for the remainder in Simpson's inequality via-polynomial convex functions of higher order using Katugampola fractional integrals," *Journal of Mathematics*, vol. 2020, 10 pages, 2020.

[6] P. Y. Yan, Q. Li, Y. M. Chu, S. Mukhtar, and S. Waheed, "On some fractional integral inequalities for generalized strongly modified h-convex function," *AIMS Math*, vol. 5, no. 6, pp. 6620–6638, 2020.

[7] P. Agarwal, M. Kadakal, I. Iscan, and Y. M. Chu, "Better approaches for n-times differentiable convex functions," *Mathematics*, vol. 8, no. 6, p. 950, 2020.

[8] M. K. Wang, H. H. Chu, Y. M. Li, and Y. M. Chu, "Answers to three conjectures on convexity of three functions involving complete elliptic integrals of the first kind," *Applicable Analysis and Discrete Mathematics*, vol. 14, no. 1, pp. 255–271, 2020.

[9] J. M. Shen, Z. H. Yang, W. M. Qian, W. Zhang, and Y. M. Chu, "Sharp rational bounds for the gamma function," *Mathematical Inequalities & Applications*, vol. 23, no. 3, pp. 843–853, 2020.

[10] M. U. Awan, N. Akhtar, A. Kashuri, M. A. Noor, and Y. M. Chu, "2D approximately reciprocal ?-convex functions and associated integral inequalities," *AIMS Math*, vol. 5, no. 5, pp. 4662–4680, 2020.

[11] M. B. Sun and Y. M. Chu, "Inequalities for the generalized weighted mean values of g-convex functions with applications," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 114, no. 4, pp. 1–12, 2020.

[12] M. U. Awan, N. Akhtar, S. Iftikhar, M. A. Noor, and Y. M. Chu, "New Hermite-Hadamard type inequalities for n-polynomial harmonically convex functions," *Journal of Inequalities and Applications*, vol. 2020, no. 1, Article ID 125, 2020.

[13] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, "Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities," *IEEE Access*, vol. 6, pp. 64946–64953, 2018.

[14] X. Z. Yang, G. Farid, W. Nazeer, Y. M. Chu, and C. F. Dong, "Fractional generalized Hadamard and Fejér-Hadamard inequalities for m-convex function," *AIMS Math*, vol. 5, no. 6, pp. 6325–6340, 2020.

[15] Y. C. Kwun, G. Farid, S. Ullah, W. Nazeer, K. Mahreen, and S. M. Kang, "Inequalities for a unified integral operator and associated results in fractional calculus," *IEEE Access*, vol. 7, pp. 126283–126292, 2019.

[16] S. M. Kang, G. Farid, W. Nazeer, and B. Tariq, "Hadamard and Fejér-Hadamard inequalities for extended generalized fractional integrals involving special functions," *Journal of Inequalities and Applications*, vol. 2018, no. 1, Article ID 119, 2018.

[17] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex Optimization*, Cambridge university press, 2004.

[18] A. Azócar, K. Nikodem, and G. Roa, "Fejér-type inequalities for strongly convex functions," *Annales Mathematicae Silesianae*, vol. 26, pp. 43–54, 2012.

[19] N. Merentes and K. Nikodem, "Remarks on strongly convex functions," *Aequationes mathematicae*, vol. 80, no. 1-2, pp. 193–199, 2010.

[20] K. Nikodem and Z. Pales, "Characterizations of inner product spaces by strongly convex functions," *Banach Journal of Mathematical Analysis*, vol. 5, no. 1, pp. 83–87, 2011.

[21] G. Anastassiou, M. R. Hooshmandasl, A. Ghasemi, and F. Moftakharzadeh, "Montgomery identities for fractional integrals and related fractional inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 4, pp. 1–6, 2009.

[22] X. Li, S. Qaisar, J. Nasir et al., "Some results on integral inequalities via Riemann-Liouville fractional integrals," *Journal of Inequalities and Applications*, vol. 2019, no. 1, Article ID 214, 2019.

[23] M. Z. Sarikaya and H. Ogunmez, "On new inequalities via Riemann-Liouville fractional integration," *Abstract and Applied Analysis*, vol. 2012, Article ID 428983, 10 pages, 2012.

[24] M. Bezziou and Z. Dahmani, "Random inequalities via Riemann-Liouville fractional integration," *Journal of Interdisciplinary Mathematics*, pp. 1–10, 2021.

[25] S. I. Butt, M. Umar, S. Rashid, A. O. Akdemir, and Y. M. Chu, "New Hermite-Jensen-Mercer-type inequalities via k-fractional

integrals," *Advances in Difference Equations*, vol. 2020, no. 1, Article ID 635, 2020.

[26] A. Gözpnar, "Some Hermite-Hadamard type inequalities for convex functions via new fractional conformable integrals and related inequalities," *AIP Conference Proceedings*, vol. 1991, article 020006, 2018.

[27] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

[28] M. Kian and M. Moslehian, "Refinements of the operator Jensen-Mercer inequality," *The Electronic Journal of Linear Algebra*, vol. 26, pp. 742–753, 2013.

[29] M. Z. Sarikaya and H. Yildirim, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals," *Miskolc Mathematical Notes*, vol. 17, no. 2, pp. 1049–1059, 2016.

[30] S. S. Dragomir and R. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.

[31] S. Özcan and I. Iscan, "Some new Hermite-Hadamard type inequalities for s-convex functions and their applications," *Journal of Inequalities and Applications*, vol. 2019, no. 1, Article ID 201, 2019.

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Research Article

New Subclass of Analytic Function Involving q-Mittag-Leffler Function in Conic Domains

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In this paper, we formulate the q-analogus of differential operator associated with q-Mittag-Leffler function. By using this newly defined operator, we define a new subclass $k-\mathscr{U}\mathscr{S}_{q,\gamma}^{m}(\alpha,\beta)$, of analytic functions in conic domains. We investigate the number of useful properties such that structural formula, coefficient estimates, Fekete–Szego problem and subordination result. We also highlighted some known corollaries of our main results.

1. Introduction Definition

Let $\mathscr{A}$ denote the class of functions $l(z)$ which are analytic in the open unit disk $E=\{z\in\mathbb{C}:|z|<1\}$, satisfying the condition $l(0)=0$ and $l'(0)=1$, and for every $l\in\mathscr{A}$ has the series expansion of the form

$$l(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1}$$

Let $\mathscr{S}\subset\mathscr{A}$ be the class of all functions which are univalent in E (see [1]). Also, $\mathscr{P}$ denotes the well-known Carathéodory class of functions p which are analytic in open unit disk E and has the series expansion of the form

$$p(z)=1+\sum_{n=1}^{\infty}c_nz^n, \tag{2}$$

and satisfying the condition

$$p(0)=1 \text{ and } Rep(z)>0. \tag{3}$$

For the function l given by (1) and the function g defined by $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$, the Hadamard product (convolution) $l*g$ of the functions l and g stated by

$$(l*g)z=z+\sum_{n=2}^{\infty}a_nb_nz^n. \tag{4}$$

For the analytic functions l,g, l is said to be subordinate to g (indicated as $l\prec g$), if there exists a Schwarz function

$$w(z)=\sum_{n=1}^{\infty}c_nz^n, \tag{5}$$

with

$$w(0)=0 \text{ and } |w(z)|<1, \tag{6}$$

such that

$$l(z)=g(w(z)). \tag{7}$$

Furthermore, if g is univalent in E, (see [2]); then, we have

$$l(z)\prec g(z) \text{ if and only if } l(0)=g(0) \text{ and } l(E)\subset g(E), z\in E. \tag{8}$$

The class of starlike functions of order $\alpha(\mathscr{S}^*(\alpha))$ in E and the class of convex functions of order $\alpha(\mathscr{K}(\alpha))$, $0\leq\alpha<1$, were defined as follows:

$$\mathcal{S}^*(\alpha) = \left\{ l : l \in \mathcal{S} \text{ and } \operatorname{Re}\left(\frac{zl'(z)}{l(z)}\right) > \alpha, (0 \le \alpha < 1, z \in E) \right\},$$
$$\mathcal{K}(\alpha) = \left\{ l : l \in \mathcal{S} \text{ and } \operatorname{Re}\left(\frac{z\left(zl'(z)\right)'}{l'(z)}\right) > \alpha, (0 \le \alpha < 1, z \in E) \right\}. \quad (9)$$

It should be noted that

$$\mathcal{S}^*(0) = \mathcal{S}^* \text{ and } \mathcal{K}(0) = \mathcal{K}, \quad (10)$$

where $\mathcal{S}^*$ and $\mathcal{K}$ are the well-known function classes of starlike and convex functions, respectively.

In the year of 1991, Goodman [3] introduced the class $\mathcal{UCV}$ of uniformly convex functions which was extensively studied by Ronning [4], and its characterization was given by Ma and Minda [5]. After that, Kanas and Wisniowska [6] defined the class k-uniformly convex functions (k-$\mathcal{UCV}$) and a related class $k-\mathcal{ST}$ was defined by

$$l \in k-\mathcal{UCV} \Longleftrightarrow zl' \in k-\mathcal{ST}$$
$$\Longleftrightarrow l \in \mathcal{A} \text{ and } \operatorname{Re}\left\{\frac{\left(zl'(z)\right)'}{l'(z)}\right\} > \left|\frac{zl''(z)}{l'(z)}\right|, (k \ge 0). \quad (11)$$

From different viewpoints, the various subclasses of the normalized analytic function of class $\mathcal{A}$ have been studied in the field of Geometric Function Theory. To investigate various subclasses of $\mathcal{A}$, many authors have been used the q-calculus as well as the fractional q-calculus. In 1910, Jackson [7] was among the one of few researchers who studied q-calculus operator theory on q-definite integrals and also Trjitzinsky in [8] studied about analytic theory of linear q-difference equations. Curmicheal [9] studied general theory of linear q-difference equations and the first use of q-calculus operator theory in Geometric Function Theory in a book chapter by Srivastava (see, for details, [10]). Recently, Hussain et al. discussed the some applications of q-calculus operator theory in [11], while in [12, 13], Ibrahim et al. used the notion of quantum calculus and the Hadamard product to improve an extended Sàlàgean q-differential operator and defined some new subclasses of analytic functions in open unit disk E. Govindaraj and Sivasubramanian [14] as well as Ibrahim et al. [15, 16] employed the quantum calculus and the Hadamard product to defined some new subclasses of analytic functions involving the Sàlàgean q-differential operator and the generalized symmetric Sàlàgean q-differential operator, respectively. Furthermore, Srivastava et al. [17] defined q-Noor integral operator by using q-calculus operator theory and investigated some subclasses of biunivalent functions in open unit disk.

Here, we give some basic definitions and details of the q-calculus and suppose that $0 < q < 1$.

For any nonnegative integer n, the q-integer number $[n]_q$ is defined by

$$[n]_q = \frac{1-q^n}{1-q}, [0]_q = 0, \quad (12)$$

and for any nonnegative integer n, the q-number shift factorial is defined by

$$[n]_q! = [1]_q[2]_q[3]_q \cdots [n]_q, \left([0]_q! = 1\right). \quad (13)$$

We note that when $q \longrightarrow 1-$, then $[n]! = n$.

The q-difference operator was introduced by Jackson (see in [7]). For $l \in \mathcal{A}$, the q-derivative operator or q-difference operator is defined as

$$\partial_q l(z) = \frac{l(qz) - l(z)}{z(q-1)}, z \in E, z \ne 0, q \ne 1. \quad (14)$$

It is readily deduced from (1) and (14) that

$$\partial_q z^n = [n]_q z^{n-1}, \partial_q l(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \quad (15)$$

We can observe that

$$\lim_{q \longrightarrow 1-} \partial_q l(z) = l'(z). \quad (16)$$

The familiar Mittag-Leffler function $\mathcal{H}_\alpha(z)$ introduced by Mittag-Leffler [18] and its generalization $\mathcal{H}_{\alpha,\beta}(z)$ introduced by Wiman [19] which are defined by

$$\mathcal{H}_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + 1)} z^n, (\alpha \in \mathbb{C}, \operatorname{Re}(\alpha)) > 0,$$
$$\mathcal{H}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} z^n, (\alpha, \beta \in \mathbb{C}, \operatorname{Re}(\alpha), \operatorname{Re}(\beta)) > 0. \quad (17)$$

Recently, Attiya [20] investigated some applications of Mittag-Leffler functions and generalized k-Mittag-Leffler studied by Rehman et al. in [21]. Moreover, Srivastava et al. [22, 23] introduced the generalization of Mittag-Leffler functions.

The q-Mittag-Leffler function was defined by (see [24]):

$$\mathcal{H}_{\alpha,\beta}(z,q) = \sum_{n=0}^{\infty} \frac{1}{\Gamma_q(\alpha n + \beta)} z^n, (\alpha, \beta \in \mathbb{C}, \operatorname{Re}(\alpha), \operatorname{Re}(\beta)) > 0. \quad (18)$$

The q-Mittag-Leffler function has also been investigated in [25, 26]. Since the q-Mittag-Leffler function $\mathcal{H}_{\alpha,\beta}(z,q)$ defined by (18) does not belong to the normalized analytic function class $\mathcal{A}$. Hence, we define the normalization of q-Mittag-Leffler function as

$$\mathscr{M}_{\alpha,\beta}(z,q) = z\Gamma_q(\beta)\mathscr{H}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(n-1)+\beta)} z^n, \tag{19}$$

where $z \in E, Re\alpha > 0$, $\beta \in \mathbb{C} \setminus \{0,-1,-2,-3 \cdots\}$). Corresponding to $\mathscr{M}_{\alpha,\beta}(z,q)$ and for $l \in \mathscr{A}$, we define the following q-analogous of differential operator $\mathscr{D}_q^m(\alpha,\beta)$: $\mathscr{A} \longrightarrow \mathscr{A}$ by

$$\begin{aligned} \mathscr{D}_q^0(\alpha,\beta)l(z) &= l(z) * \mathscr{M}_{\alpha,\beta}(z,q), \\ \mathscr{D}_q^1(\alpha,\beta)l(z) &= z\partial_q\left(l(z) * \mathscr{M}_{\alpha,\beta}(z,q)\right), \\ \mathscr{D}_q^2(\alpha,\beta)l(z) &= \mathscr{D}\left(\mathscr{D}^1(\alpha,\beta)l(z)\right), \\ \mathscr{D}_q^m(\alpha,\beta)l(z) &= \mathscr{D}\left(\mathscr{D}^{m-1}(\alpha,\beta)l(z)\right). \end{aligned} \tag{20}$$

We note that

$$\mathscr{D}_q^m(\alpha,\beta)l(z) = z + \sum_{n=2}^{\infty} [n]_q^m \mathscr{T}_n(\alpha,q) a_n z^n, \tag{21}$$

where

$$\mathscr{T}_n(\alpha,q) = \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(n-1)+\beta)}. \tag{22}$$

Note that

(i) For $\alpha = 0$ and $\beta = 1$, we get Salagean q-differential operator [14]

(ii) For $q \longrightarrow 1-$, $\alpha = 0$, and $\beta = 1$, we get Salagean differential operator [27]

(iii) For $m = 0$, we get $E_{\alpha,\beta}(z,q)$ (see [24])

(iv) For $m = 0$, we get $E_{\alpha,\beta}(z)$ (see [22])

Definition 1. Let $l(z) \in \mathscr{A}$, then $l(z)$ is in the class $k - \mathscr{US}_{q,\gamma}^m(\alpha,\beta)$, $\gamma \in \mathbb{C} \setminus \{0\}$, if it satisfies the condition

$$\begin{aligned} &\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z\partial_q \mathscr{D}_q^m(\alpha,\beta)l(z)}{\mathscr{D}_q^m(\alpha,\beta)l(z)} - 1\right)\right\} \\ &> k\left|\frac{1}{\gamma}\left(\frac{z\partial_q \mathscr{D}_q^m(\alpha,\beta)l(z)}{\mathscr{D}_q^m(\alpha,\beta)l(z)} - 1\right)\right|, z \in E. \end{aligned} \tag{23}$$

Remark 2.

(i) For $\alpha = 0$ and $\beta = 1$, the class $k - \mathscr{US}_{q,\gamma}^m(\alpha,\beta) = k - \mathscr{US}(q,\gamma,m)$ studied in [11]

(ii) For $m = 0, \alpha = 0, \beta = 1, q \longrightarrow 1-$, and $\gamma = 1/(1-\eta)$, $\eta \in \mathbb{C} \setminus \{1\}$, the class $k - \mathscr{US}_{q,\gamma}^m(\alpha,\beta) = \mathscr{SD}(k,\eta)$ studied in [28]

(iii) For $m = 0, \alpha = 0, \beta = 1, q \longrightarrow 1-$, and $\gamma = 2/(1-\eta)$, $\eta \in \mathbb{C} \setminus \{1\}$, the class $k - \mathscr{US}_{q,\gamma}^m(\alpha,\beta) = \mathscr{KD}(k,\eta)$, studied in [29]

(iv) For $k = 1, m = 0, \alpha = 0, \beta = 1, q \longrightarrow 1-$, and $\gamma = (1-\eta)$, $\eta \in \mathbb{C} \setminus \{1\}$, the class $k - \mathscr{US}_{q,\gamma}^m(\alpha,\beta) = \mathscr{S}_p(\eta)$ studied in [30]

(v) For $k = 1, m = 0, \alpha = 0, \beta = 1, q \longrightarrow 1-$, and $\gamma = 2/(1-\eta)$, $\eta \in \mathbb{C} \setminus \{1\}$, $k - \mathscr{US}_{q,\gamma}^m(\alpha,\beta) = \mathscr{K}_p(\eta)$, studied in [30]

2. Geometric Interpretation

A function $l(z) \in \mathscr{A}$, belongs to $k - \mathscr{US}_{q,\gamma}^m(\alpha,\beta)$ if and only if $z\partial_q \mathscr{D}_q^m(\alpha,\beta)l(z)/\mathscr{D}_q^m(\alpha,\beta)l(z)$ takes all the values in the conic domain $\Omega_{k,\gamma} = p_{k,\gamma}(E)$, such that

$$\Omega_{k,\gamma} = \gamma\Omega_k + (1-\gamma), 0 \le \gamma < 1, k \ge 0, \tag{24}$$

where

$$\Omega_k = u + iv : u^2 > k^2\left((u-1)^2 + v^2\right). \tag{25}$$

The domain $\Omega_{k,\gamma}$ is not always well defined because in general $(1,0) \notin \Omega_{k,\gamma}$ (for example, in particular $(1,0) \notin \Omega_{2,0.5}$). We see that in [31], the conic domain $\Omega_k(0,b)$ concides with $\Omega_{k,b}$ only when b is chosen according to

(i) For $k = 0$, we take $b = 0$

(ii) For $k \in (0, 1/\sqrt{2})$, we take $b \in [1/2k^2 - 1, 1)$

(iii) For $k \in [1/\sqrt{2}1/\sqrt{2}, 1]$, we take $b \in (-\infty,1)$

(iv) For $k \in (1,\infty)$, we take $b \in (-\infty,1/2k^2 - 1]$

This means that for $\Omega_{k,\gamma}$ to contain the point $(1,0)$, γ must be chosen according as follows:

$$\gamma \in \begin{cases} (0,1) & \text{if } 0 \le k \le 1, \\ \left[0, 1 - \dfrac{\sqrt{k^2-1}}{k}\right] & \text{if } k \ge 0. \end{cases} \tag{26}$$

Since $p_{k,\gamma}(z)$ is convex univalent, the above definition can be written as

$$\frac{z\partial_q \mathscr{D}_q^m(\alpha,\beta)l(z)}{\mathscr{D}_q^m(\alpha,\beta)l(z)} \prec p_{k,\gamma}(z), \tag{27}$$

where

$$p_{k,\gamma}(z)=\begin{cases}\dfrac{1+z}{1-z}, & \text{for } k=0,\\ U_1(\gamma,k), & \text{for } k=1,\\ U_2(\gamma,k), & \text{for } 0<k<1,\\ U_3(\gamma,k), & \text{for } k>1,\end{cases} \tag{28}$$

$$U_1(\gamma,k)=1+\frac{2\gamma}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{2'}, \tag{29}$$

$$U_2(\gamma,k)=1+\frac{2\gamma}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right\}, \tag{30}$$

$$U_3(\gamma,k)=1+\frac{\gamma}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right)+\frac{\gamma}{1-k^2}. \tag{31}$$

For more detail (see [32, 33]).

3. Set of Lemmas

Lemma 3. (see [34]). *Let* $p(z)=\sum_{n=1}^{\infty}p_n z^n \prec F(z)=\sum_{n=1}^{\infty}d_n z^n$ *in* E. *If* $F(z)$ *is convex univalent in* E, *then*

$$|p_n|\le |d_1|,\ n\ge 1. \tag{32}$$

Lemma 4. (see [35]). *Let* $k\in[0,\infty)$ *be fixed and let* $p_{k,\gamma}(z)$ *of the form (28). If*

$$p_{k,\gamma}(z)=1+Q_1 z+Q_2 z^2+\cdots, \tag{33}$$

where

$$Q_1=\begin{cases}\dfrac{2\gamma A^2}{1-k^2}, & 0\le k<1,\\ \dfrac{8\gamma}{\pi^2}, & k=1,\\ \dfrac{\pi^2\gamma}{4(1+t)\sqrt{t}K^2(t)\left(k^2-1\right)}, & k>1,\end{cases} \tag{34}$$

$$Q_2=\begin{cases}\dfrac{A^2+2}{3}Q_1, & 0\le k<1,\\ \dfrac{2}{3}Q_1, & k=1,\\ \dfrac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24K^2(t)(1+t)\sqrt{t}}Q_1, & k>1.\end{cases} \tag{35}$$

Lemma 5. (see [36]). *Let* $p(z)=1+\sum_{n=1}^{\infty}c_n z^n\in\mathcal{P}$ *and let* $p(z)$ *be analytic in* E *and satisfy* $\operatorname{Re}(p(z))>0$ *for* z *in* E, *then*

$$\left|c_2-\mu c_1^2\right|\le 2\max\{1,|2\mu-1|\},\forall\mu\in\mathbb{C}. \tag{36}$$

4. Main Results

Theorem 6. *Let* $l(z)\in k-\mathcal{US}_{q,\gamma}^{m}(\alpha,\beta)$. *Then,*

$$\mathcal{D}_q^m(\alpha,\beta)l(z))\prec z\exp\int_0^z\frac{p_{k,\gamma}(w(\xi))-1}{\zeta}d\xi, \tag{37}$$

where $w(z)$ *is a Schwarz function given in (5). Moreover, for* $|z|=\rho$, *we have*

$$\exp\left(\int_0^1\frac{p_{k,\gamma}(-\rho)-1}{\rho}d\rho\right)\le\left|\frac{\mathcal{D}_q^m(\alpha,\beta)l(z)}{z}\right| \le\exp\left(\int_0^1\frac{p_{k,\gamma}(\rho)-1}{\rho}d\rho\right), \tag{38}$$

where $p_{k,\gamma}(z)$ *is given by (28).*

Proof. If $l(z)\in k-\mathcal{US}_{q,\gamma}^{m}(\alpha,\beta)$, then by using (27), we obtain

$$\frac{\partial_q\mathcal{D}_q^m(\alpha,\beta)l(z)}{\mathcal{D}_q^m(\alpha,\beta)l(z)}-\frac{1}{z}=\frac{p_{k,\gamma}(w(z))-1}{z}. \tag{39}$$

Integrating (39) and after some simplification, we have

$$\mathcal{D}_q^m(\alpha,\beta)l(z)\prec z\exp\int_0^z\frac{p_{k,\gamma}(w(\xi))-1}{\zeta}d\xi. \tag{40}$$

This proves (37). We know that

$$p_{k,\gamma}(-\rho|z|)\le\operatorname{Re}\left\{p_{k,\gamma}(w(\rho z)\right\}\le p_{k,\gamma}(\rho|z|)\ (0<\rho\le 1, z\in E). \tag{41}$$

Using (40) and (41), we have

$$\int_0^1\frac{p_{k,\gamma}(-\rho|z|)-1}{\rho}d\rho\le\operatorname{Re}\int_0^1\frac{p_{k,\gamma}(w(\rho(z))-1}{\rho}d\rho \le\int_0^1\frac{p_{k,\gamma}(\rho|z|)-1}{\rho}d\rho, \tag{42}$$

for $z\in E$. From (40), we have

$$\begin{gathered}\frac{\mathcal{D}_q^m(\alpha,\beta)l(z)}{z}\prec\exp\int_0^z\frac{p_{k,\gamma}(w(\xi))-1}{\zeta}d\xi,\\ \int_0^1\frac{p_{k,\gamma}(-\rho|z|)-1}{\rho}d\rho\le\log\left|\frac{\mathcal{D}_q^m(\alpha,\beta)l(z)}{z}\right|\le\int_0^1\frac{p_{k,\gamma}(\rho|z|)-1}{\rho}d\rho,\\ \exp\int_0^1\frac{p_{k,\gamma}(-\rho|z|)-1}{\rho}d\rho\le\exp\left(\log\left|\frac{\mathcal{D}_q^m(\alpha,\beta)l(z)}{z}\right|\right)\le\exp\int_0^1\frac{p_{k,\gamma}(\rho|z|)-1}{\rho}d\rho,\end{gathered} \tag{43}$$

which implies that

$$\exp\int_0^1 \frac{p_{k,\gamma}(-\rho)-1}{\rho}d\rho \le \left|\frac{\mathcal{D}_q^m(\alpha,\beta)l(z)}{z}\right| \le \exp\int_0^1 \frac{p_{k,\gamma}(\rho)-1}{\rho}d\rho. \quad (44)$$

□

Corollary 7. (see [11]). *Let $l(z)\in k-\mathcal{US}_{q,\gamma}^m(0,1)$. Then,*

$$\left.\mathcal{D}_q^m l(z)\right) \prec z\exp\int_0^z \frac{p_{k,\gamma}(w(\xi))-1}{\zeta}d\xi, \quad (45)$$

where $w(z)$ is a Schwarz function given in (5). Moreover, for $|z|=\rho$, we have

$$\exp\left(\int_0^1 \frac{p_{k,\gamma}(-\rho)-1}{\rho}d\rho\right) \le \left|\frac{\mathcal{D}_q^m l(z)}{z}\right| \le \exp\left(\int_0^1 \frac{p_{k,\gamma}(\rho)-1}{\rho}d\rho\right). \quad (46)$$

Theorem 8. *If $l(z)\in k-\mathcal{US}_{q,\gamma}^m(\alpha,\beta)$. Then,*

$$|a_2| \le \frac{\delta}{[2]_q^m\left\{[2]_q-1\right\}\mathcal{T}_2(\alpha,q)}, \quad (47)$$

$$|a_n| \le \frac{\delta}{[n]_q^m\left\{[n]_q-1\right\}\mathcal{T}_n(\alpha,q)}\prod_{j=1}^{n-2}\left(1+\frac{\delta}{[j+1]_q-1}\right), \quad for n\ge 3, \quad (48)$$

where $\delta=|Q_1|$ with Q_1 and $\mathcal{T}_n(\alpha,q)$ are given by (34) and (22).

Proof. Let

$$\frac{z\partial_q\mathcal{D}_q^m(\alpha,\beta)l(z)}{\mathcal{D}_q^m(\alpha,\beta)l(z)} = p(z), \quad (49)$$

where $p(z)$ is the analytic in E and $p(0)=1$. Let $p(z)=1+\sum_{n=1}^{\infty}c_n z^n$ and $\mathcal{D}_q^m(\alpha,\beta)l(z)$ is given by (21). Then, (49) implies that

$$\begin{aligned} z+\sum_{n=2}^{\infty}[n]_q^{m+1}\mathcal{T}_n(\alpha,q)a_n z^n &= \left(\sum_{n=0}^{\infty}c_n z^n\right)\left(z+\sum_{n=2}^{\infty}[n]_q^m\mathcal{T}_n(\alpha,q)a_n z^n\right),\\ &= \sum_{n=0}^{\infty}c_n z^{n+1}+\left(\sum_{n=0}^{\infty}c_n z^n\right)\left(\sum_{n=2}^{\infty}[n]_q^m\mathcal{T}_n(\alpha,q)a_n z^n\right). \end{aligned} \quad (50)$$

Now comparing the coefficients of z^n, we obtain

$$\begin{aligned} [n]_q^{m+1}\mathcal{T}_n(\alpha,q)a_n &= [n]_q^m\mathcal{T}_n(\alpha,q)a_n+\sum_{j=1}^{n-1}[j]_q^m\frac{\Gamma(\beta)}{\Gamma_q(\alpha(j-1)+\beta)}a_j c_{n-j},\\ [n]_q^{m+1}\mathcal{T}_n(\alpha,q)a_n-[n]_q^m\mathcal{T}_n(\alpha,q)a_n &= \sum_{j=1}^{n-1}[j]_q^m\frac{\Gamma(\beta)}{\Gamma_q(\alpha(j-1)+\beta)}a_j c_{n-j},\\ [n]_q^m\left\{[n]_q-1\right\}\mathcal{T}_n(\alpha,q)a_n &= \sum_{j=1}^{n-1}[j]_q^m\frac{\Gamma(\beta)}{\Gamma_q(\alpha(j-1)+\beta)}a_j c_{n-j}, \end{aligned} \quad (51)$$

which implies

$$a_n = \frac{1}{[n]_q^m\left\{[n]_q-1\right\}\mathcal{T}_n(\alpha,q)}\sum_{j=1}^{n-1}[j]_q^m\frac{\Gamma(\beta)}{\Gamma_q(\alpha(j-1)+\beta)}a_j c_{n-j}. \quad (52)$$

Using the results that $|c_n|\le|Q_1|$ given in ([33]), we have

$$|a_n| \le \frac{|Q_1|}{[n]_q^m\left\{[n]_q-1\right\}\mathcal{T}_n(\alpha,q)}\sum_{j=1}^{n-1}[j]_q^m\frac{\Gamma(\beta)}{\Gamma_q(\alpha(j-1)+\beta)}|a_j|. \quad (53)$$

Let us take $\delta=|Q_1|$. Then, we have

$$|a_n| \le \frac{\delta}{[n]_q^m\left\{[n]_q-1\right\}\mathcal{T}_n(\alpha,q)}\sum_{j=1}^{n-1}[j]_q^m\frac{\Gamma(\beta)}{\Gamma_q(\alpha(j-1)+\beta)}|a_j|. \quad (54)$$

For $n=2$ in (54), we have

$$\begin{aligned} |a_2| &\le \frac{\delta}{[2]_q^m\left\{[2]_q-1\right\}\mathcal{T}_2(\alpha,q)}\sum_{j=1}^{1}[j]_q^m\frac{\Gamma(\beta)}{\Gamma_q(\alpha(j-1)+\beta)}|a_j|\\ &= \frac{\delta}{[2]_q^m\left\{[2]_q-1\right\}\mathcal{T}_2(\alpha,q)}, \end{aligned} \quad (55)$$

Hence, for $n=2$ the inequality (48) holds. To prove (48), we use mathematical induction, for $n=3$

$$|a_3| \le \frac{\delta}{[3]_q^m\left\{[3]_q-1\right\}\mathcal{T}_3(\alpha,q)}\left\{1+[2]_q^m\mathcal{T}_2(\alpha,q)|a_2|\right\}. \quad (56)$$

By using (55), we have

$$|a_3| \le \frac{\delta}{[3]_q^m\left\{[3]_q-1\right\}\mathcal{T}_3(\alpha,q)}\left\{1+[2]_q^m\mathcal{T}_2(\alpha,q)\left(\frac{\delta}{[2]_q^m\left\{[2]_q-1\right\}\mathcal{T}_2(\alpha,q)}\right)\right\}. \quad (57)$$

Therefore,

$$|a_3| \le \frac{\delta}{[3]_q^m \left\{[3]_q - 1\right\} \mathcal{T}_3(\alpha, q)} \left\{1 + \frac{\delta}{[2]_q - 1}\right\}. \tag{58}$$

Hence, (48) holds for $n = 3$. Now, we suppose that (48) is true for $n = t + 1$, that is

$$|a_t| \le \frac{\delta}{[t]_q^m \left\{[t]_q - 1\right\} \mathcal{T}_t(\alpha, q)} \prod_{j=1}^{t-2} \left(1 + \frac{\delta}{[j+1]_q - 1}\right), \quad n \ge 3. \tag{59}$$

Consider

$$\begin{aligned} |a_{t+1}| \le & \frac{\delta}{[t+1]_q^m \left\{[t+1]_q - 1\right\} \mathcal{T}_{t+1}(\alpha, q)} \\ & \times \left\{1 + [2]_q^m \mathcal{T}_2(\alpha, q)|a_2| + [3]_q^m \mathcal{T}_3(\alpha, q)|a_3| \right. \\ & \left. + [4]_q^m \mathcal{T}_4(\alpha, q)|a_4| + \cdots + [t]_q^m \mathcal{T}_t(\alpha, q)|a_t| \right\} \\ \le & \frac{\delta}{[t+1]_q^m \left\{[t+1]_q - 1\right\} \mathcal{T}_{t+1}(\alpha, q)} \\ & \cdot \left\{1 + \frac{\delta}{[2]_q - 1} + \frac{\delta}{[3]_q - 1}\left(1 + \frac{\delta}{[2]_q - 1}\right) \right. \\ & \left. + \cdots + \frac{\delta}{[t]_q - 1} \prod_{j=1}^{t-2}\left(1 + \frac{\delta}{[j+1]_q - 1}\right)\right\} \\ = & \frac{\delta}{[t+1]_q^m \left\{[t+1]_q - 1\right\} \mathcal{T}_{t+1}(\alpha, q)} \prod_{j=1}^{t-1}\left(1 + \frac{\delta}{[j+1]_q - 1}\right). \end{aligned} \tag{60}$$

Hence, (48) holds for $n = t + 1$. Hence, proof is complete. □

Corollary 9. (see [11]). *f* $l(z) \in k - \mathcal{US}_{q,\gamma}^m(0, 1)$. *Then,*

$$|a_2| \le \frac{\delta}{[2]_q^m \left\{[2]_q - 1\right\}}$$

$$|a_n| \le \frac{\delta}{[n]_q^m \left\{[n]_q - 1\right\}} \prod_{j=1}^{n-2}\left(1 + \frac{\delta}{[j+1]_q - 1}\right), \quad \text{for } n \ge 3. \tag{61}$$

Theorem 10. *Let* $0 \le k < \infty$ *be fixed and let* $l(z) \in k - \mathcal{US}_{q,\gamma}^m(\alpha, \beta)$ *with the form (1), then, for* $\mu \in \mathbb{C}$

$$\left|a_3 - \mu a_2^2\right| \le \frac{Q_1}{2[3]_q^m \mathcal{T}_3(\alpha, q)\left\{[3]_q - 1\right\}} \max\left[1, |2\nu - 1|\right], \tag{62}$$

where Q_1 *and* Q_2 *are given by (34) and (35).*

Proof. Let $l(z) \in k - \mathcal{US}_{q,\gamma}^m(\alpha, \beta)$, then there exists a Schwarz function $w(z)$ given by (5), such that

$$\frac{z\partial_q \mathcal{D}_q^m(\alpha, \beta) l(z)}{\mathcal{D}_q^m(\alpha, \beta) l(z)} \prec p_{k,\gamma}(z), z \in E \tag{63}$$

$$\frac{z\partial_q \mathcal{D}_q^m(\alpha, \beta) l(z)}{\mathcal{D}_q^m(\alpha, \beta) l(z)} = p_{k,\gamma}(w(z)), z \in E. \tag{64}$$

Let $p(z) \in \mathcal{P}$ be defined as

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots. \tag{65}$$

This gives

$$w(z) = \frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots \tag{66}$$

$$p_{k,\gamma}(w(z)) = 1 + \frac{Q_1 c_1}{2} z + \left\{\frac{Q_2 c_1^2}{4} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) Q_1\right\} z^2 + \cdots. \tag{67}$$

$$\begin{aligned} \frac{z\partial_q \mathcal{D}_q^m(\alpha, \beta) l(z)}{\mathcal{D}_q^m(\alpha, \beta) l(z)} = & 1 + [2]_q^m \mathcal{T}_2(\alpha, q)\left\{[2]_q - 1\right\} a_2 z \\ & + \left\{[3]_q^m \mathcal{T}_3(\alpha, q)\left\{[3]_q - 1\right\} a_3 \right. \\ & \left. - \left([2]_q^m \mathcal{T}_2(\alpha, q)\right)^2 \left\{[2]_q - 1\right\} a_2^2\right\} z^2. \end{aligned} \tag{68}$$

Using (67) in (64) and comparing with (68), we obtain

$$\begin{aligned} a_2 = & \frac{Q_1 c_1}{2[2]_q^m \mathcal{T}_2(\alpha, q)\left\{[2]_q - 1\right\}}, \\ a_3 = & \frac{1}{[3]_q^m \mathcal{T}_3(\alpha, q)\left\{[3]_q - 1\right\}} \left\{\frac{Q_1 c_2}{2} + \frac{c_1^2}{4}\left(Q_2 - Q_1 + \frac{Q_1^2}{\left\{[2]_q - 1\right\}}\right)\right\}. \\ a_3 - \mu a_2^2 = & \frac{1}{[3]_q^m \mathcal{T}_3(\alpha, q)\left\{[3]_q - 1\right\}} \left\{\frac{Q_1 c_2}{2} + \frac{c_1^2}{4}\left(Q_2 - Q_1 + \frac{Q_1^2}{\left\{[2]_q - 1\right\}}\right)\right\} \\ & - \mu\left(\frac{Q_1 c_1}{2[2]_q^m \mathcal{T}_2(\alpha, q)\left\{[2]_q - 1\right\}}\right)^2. \end{aligned} \tag{69}$$

For any complex number μ and after some calculation we have

$$a_3 - \mu a_2^2 = \frac{Q_1}{2[3]_q^m \mathcal{T}_3(\alpha, q)\left\{[3]_q - 1\right\}}\left\{c_2 - v c_1^2\right\}, \tag{70}$$

where

$$v = \frac{1}{2}\left\{1 - \frac{Q_2}{Q_1} - Q_1\left(\frac{1}{\left\{[2]_q - 1\right\}} - \mu\frac{[3]_q^m\left\{[3]_q - 1\right\}}{2\mathcal{T}_2(\alpha, q)\left([2]_q^m\left\{[2]_q - 1\right\}\right)^2}\right)\right\}. \tag{71}$$

Using a lemma (36) on (70), we have the required result. □

Corollary 11. (see [11]). *Let $0 \le k < \infty$ be fixed and let $l(z) \in k - \mathcal{US}_{q,\gamma}^m(0, 1)$ with the form (1.1), then, for $\mu \in \mathbb{C}$*

$$\left|a_3 - \mu a_2^2\right| \le \frac{Q_1}{2[3]_q^m\left\{[3]_q - 1\right\}} \max\left[1, |2v - 1|\right], \tag{72}$$

where

$$v = \frac{1}{2}\left\{1 - \frac{Q_2}{Q_1} - Q_1\left(\frac{1}{\left\{[2]_q - 1\right\}} - \mu\frac{[3]_q^m\left\{[3]_q - 1\right\}}{2\left([2]_q^m\left\{[2]_q - 1\right\}\right)^2}\right)\right\}. \tag{73}$$

Theorem 12. *Let $l(z) \in \mathcal{A}$ of the form (1) and satisfy the condition*

$$\sum_{n=2}^{\infty}\left\{\left\{[n]_q - 1\right\}(k + 1) + |\gamma|\right\}|\mathcal{T}_n(\alpha, q)|\left|[n]_q^m\right||a_n| \le |\gamma|, \tag{74}$$

then, $l(z) \in k - \mathcal{US}_{q,\gamma}^m(\alpha, \beta)$.

Proof. Let we note that

$$\begin{aligned}\left|\frac{z\partial_q\mathcal{D}_q^m(\alpha, \beta)l(z)}{\mathcal{D}_q^m(\alpha, \beta)l(z)} - 1\right| &= \left|\frac{z\partial_q\mathcal{D}_q^m(\alpha, \beta)l(z) - \mathcal{D}_q^m(\alpha, \beta)l(z)}{\mathcal{D}_q^m(\alpha, \beta)l(z)}\right| \\ &= \left|\frac{\sum_{n=2}^{\infty}[n]_q^m\mathcal{T}_n(\alpha, q)\left\{[n]_q - 1\right\}a_n z^n}{z + \sum_{n=2}^{\infty}[n]_q^m\mathcal{T}_n(\alpha, q)a_n z^n}\right| \\ &\le \frac{\sum_{n=2}^{\infty}\left|[n]_q^m\mathcal{T}_n(\alpha, q)\left\{[n]_q - 1\right\}\right||a_n|}{1 - \sum_{n=2}^{\infty}\left|[n]_q^m\right||\mathcal{T}_n(\alpha, q)||a_n|}.\end{aligned} \tag{75}$$

From (74), we get

$$1 - \sum_{n=2}^{\infty}\left|[n]_q^m\right||\mathcal{T}_n(\alpha, q)||a_n| > 0. \tag{76}$$

To show that $l(z) \in k - \mathcal{US}_{q,\gamma}^m(\alpha, \beta)$, it suffices that

$$\left|\frac{k}{\gamma}\left(\frac{z\partial_q\mathcal{D}_q^m(\alpha, \beta)l(z)}{\mathcal{D}_q^m(\alpha, \beta)l(z)} - 1\right)\right| - \mathrm{Re}\left\{\frac{1}{\gamma}\left(\frac{z\partial_q\mathcal{D}_q^m(\alpha, \beta)l(z)}{\mathcal{D}_q^m(\alpha, \beta)l(z)} - 1\right)\right\} \le 1. \tag{77}$$

From (Proof), we have

$$\begin{aligned}&\left|\frac{k}{\gamma}\left(\frac{z\partial_q\mathcal{D}_q^m(\alpha, \beta)l(z)}{\mathcal{D}_q^m(\alpha, \beta)l(z)} - 1\right)\right| - \mathrm{Re}\left\{\frac{1}{\gamma}\left(\frac{z\partial_q\mathcal{D}_q^m(\alpha, \beta)l(z)}{\mathcal{D}_q^m(\alpha, \beta)l(z)} - 1\right)\right\} \\ &\le \frac{k}{|\gamma|}\left|\frac{z\partial_q\mathcal{D}_q^m(\alpha, \beta)l(z)}{\mathcal{D}_q^m(\alpha, \beta)l(z)} - 1\right| + \frac{1}{|\gamma|}\left|\frac{z\partial_q\mathcal{D}_q^m(\alpha, \beta)l(z)}{\mathcal{D}_q^m(\alpha, \beta)l(z)} - 1\right| \\ &\le \frac{(k + 1)}{|\gamma|}\left|\frac{z\partial_q\mathcal{D}_q^m(\alpha, \beta)l(z)}{\mathcal{D}_q^m(\alpha, \beta)l(z)} - 1\right| \\ &= \frac{(k + 1)}{|\gamma|}\left|\frac{z\partial_q\mathcal{D}_q^m(\alpha, \beta)l(z) - \mathcal{D}_q^m(\alpha, \beta)l(z)}{\mathcal{D}_q^m(\alpha, \beta)l(z)}\right| \\ &\le \frac{(k + 1)}{|\gamma|}\left(\frac{\left|\sum_{n=2}^{\infty}[n]_q^m\mathcal{T}_n(\alpha, q)\left\{[n]_q - 1\right\}\right||a_n|}{1 - \sum_{n=2}^{\infty}\left|[n]_q^m\right||\mathcal{T}_n(\alpha, q)||a_n|}\right) \le 1.\end{aligned} \tag{78}$$

Because of (74). □

Corollary 13. (see [11]). *If a function $l(z) \in \mathcal{A}$ of the form (1) and satisfy the condition*

$$\sum_{n=2}^{\infty}[n]_q^m\left\{\left\{[n]_q - 1\right\}(k + 1) + |\gamma|\right\}|a_n| \le \gamma, \tag{79}$$

then, $l(z) \in k - \mathcal{US}_{q,\gamma}^m(0, 1)$.

Corollary 14. (see [28]). *A function $l \in \mathcal{A}$ of the form (1) belongs to $k - \mathcal{US}(1 - 2\eta)$, if*

$$\sum_{n=2}^{\infty}\left\{n(k + 1) - (k + \eta)\right\}|a_n| \le 1 - \eta, \tag{80}$$

where $0 \le \eta < 1$ and $k \ge 0$. Then, $l(z) \in k - \mathcal{US}_{q\longrightarrow 1-,1-\eta}^0(0, 1)$.

When $q \longrightarrow 1-$, then, $m = 0, \alpha = 0, \beta = 1, \gamma = 1 - \eta$, with $0 \le \eta < 1$ and $k = 0$.

Corollary 15. (see [37]). *A function $l \in \mathcal{A}$ of the form (1) is in the class $0 - \mathcal{US}(1 - \eta)$, if*

$$\sum_{n=2}^{\infty}(n - \eta)|a_n| \le 1 - \delta, 0 \le \eta < 1. \tag{81}$$

Theorem 16. *Let $l(z) \in k-\mathscr{U}\mathscr{S}_{q,\gamma}^{m}(\alpha, \beta)$. Then, $l(E)$ includes an open disk of radius*

$$\frac{[2]^{m}\mathscr{T}_{2}(\alpha, q)\left\{[2]_{q}-1\right\}}{2[2]_{q}^{m}\left\{[2]_{q}-1\right\}\mathscr{T}_{2}(\alpha, q)+\delta}, \tag{82}$$

where Q_1 is given by (34).

Proof. Let a nonzero complex number w_0, such that $l(z) \neq w_0$ for $z \in E$. Then,

$$l_{1}(z)=\frac{w_{0} l(z)}{w_{0}-l(z)}=z+\left(a_{2}+\frac{1}{w_{0}}\right) z^{2}+\cdots. \tag{83}$$

Since $l_1(z)$ is univalent, therefore

$$\left|a_{2}+\frac{1}{w_{0}}\right| \leq 2. \tag{84}$$

Now using (47), we have

$$\left|\frac{1}{w_{0}}\right| \leq 2+\frac{\delta}{[2]_{q}^{m}\left\{[2]_{q}-1\right\}\mathscr{T}_{2}(\alpha, q)}=\frac{\left(2[2]_{q}^{m}\left\{[2]_{q}-1\right\}\mathscr{T}_{2}(\alpha, q)\right)+\delta}{[2]_{q}^{m}\left\{[2]_{q}-1\right\}\mathscr{T}_{2}(\alpha, q)}. \tag{85}$$

Hence we have

$$\left|w_{0}\right| \geq \frac{[2]_{q}^{m}\left\{[2]_{q}-1\right\}\mathscr{T}_{2}(\alpha, q)}{\left(2[2]_{q}^{m}\left\{[2]_{q}-1\right\}\mathscr{T}_{2}(\alpha, q)\right)+\delta}. \tag{86}$$

When $\alpha=0$ and $\beta=1$, then we have known result [11]. □

Corollary 17. *Let $l(z) \in k-\mathscr{U}\mathscr{S}_{q,\gamma}^{m}(0, 1)$. Then, $l(E)$ includes an open disk of radius*

$$\frac{[2]^{m}\left\{[2]_{q}-1\right\}}{2[2]_{q}^{m}\left\{[2]_{q}-1\right\}+\delta}. \tag{87}$$

5. Conclusion

In this paper, we formulate the q-analogous of differential operator associated with q-Mittag-Leffler function. By applying newly defined operator, we defined and investigated a new subclass $k-\mathscr{U}\mathscr{S}_{q,\gamma}^{m}(\alpha, \beta)$, of analytic functions in conic domains. We investigated the number of useful properties such that structural formula, coefficient estimates, Fekete–Szego problem, and subordination results. We also highlighted some known consequences of our main result. For future work, one can employ the q-analogous of differential operator (21) in different classes of analytic functions such as the meromorphic and multivalent functions (see [38–42].

Data Availability

All data are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] A. W. Goodman, *Univalent Functions*, II. Polygonal Publishing House, New Jersey, 1983.

[2] S. S. Miller and P. T. Mocanu, *Differential Subordinations, Theory and Applications, Series of Monographs and Textbooks in Pure and Application Mathematics*, Marcel Dekker Inc, New York, Basel, 2000.

[3] A. W. Goodman, "On uniformly convex functions," *Annales Polonici Mathematici*, vol. 56, no. 1, pp. 87–92, 1991.

[4] F. Rønning, "Uniformly convex functions and a corresponding class of starlike functions," *Proceedings of American Mathematical Society*, vol. 118, no. 1, pp. 189–196, 1993.

[5] W. Ma and D. Minda, "Uniformly convex functions," *Annales Polonici Mathematici*, vol. 57, no. 2, pp. 165–175, 1992.

[6] S. Kanas and A. Wisniowska, "Conic regions and k-uniform convexity," *Journal of Computational and Applied Mathematics*, vol. 105, no. 1-2, pp. 327–336, 1999.

[7] F. H. Jackson, "On q-definite integrals Quart. J," *Pure and Applied Mathematics*, vol. 41, no. 15, pp. 193–203, 1910.

[8] W. J. Trjitzinsky, "Analytic theory of linear q-difference equations," *Acta Math.*, vol. 61, pp. 1–38, 1933.

[9] R. D. Carmichael, "The general theory of linear q-difference equations," *American Journal of Mathematics*, vol. 34, no. 2, pp. 147–168, 1912.

[10] H. M. Srivastava, "Univalent functions, fractional calculus, and associated generalized hypergeometric functions," in *In Univalent Functions; Fractional Calculus; and Their Applications*, H. M. Srivastava and S. Owa, Eds., Halsted Press Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[11] S. Hussain, S. Khan, M. A. Zaighum, and M. Darus, "Certain subclass of analytic functions related with conic domains and associated with Salagean q-differential operator," *AIMS Math*, vol. 2, no. 4, pp. 622–634, 2017.

[12] R. W. Ibrahim, R. M. Elobaid, and S. J. Obaiys, "On subclasses of analytic functions based on a quantum symmetric conformable differential operator with application," *Adv. Difference Equ.*, vol. 2020, no. 1, pp. 1–14, 2020.

[13] R. W. Ibrahim, R. M. Elobaid, and S. J. Obaiys, "A class of quantum Briot–Bouquet differential equations with complex coefficients," *Mathematics*, vol. 8, no. 5, p. 794, 2020.

[14] M. Govindaraj and S. Sivasubramanian, "On a class of analytic functions related to conic domains involving q-calculus," *Analysis Mathematica*, vol. 43, no. 3, pp. 475–487, 2017.

[15] R. W. Ibrahim, S. B. Hadid, and S. Momani, "Generalized Briot–Bouquet differential equation by a quantum difference

operator in a complex domain," *International Journal of Dynamics and Control*, vol. 8, no. 3, pp. 762–771, 2020.

[16] R. W. Ibrahim and D. Baleanu, "On quantum hybrid fractional conformable differential and integral operators in a complex domain," *Fsicas y Naturales. Serie A. Matemáticas*, vol. 115, no. 1, pp. 1–13, 2021.

[17] H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan, and S. Hussain, "The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator," *Stud. Univ. Babe s-Bolyai Math*, vol. 63, no. 4, pp. 419–436, 2018.

[18] G. M. Mittag-Leffler, "Sur la nouvelle fonction $E_\alpha(x)$, C R Acad," *Sci. Paris*, vol. 137, no. 2, pp. 554–558, 1903.

[19] A. Wiman, "Über den Fundamentalsatz in der Teorie der Funktionen Ea(x)," *Acta Mathematica*, vol. 29, pp. 191–201, 1905.

[20] A. A. Attiya, "Some applications of Mittag-Leffler function in the unit disk," *Univerzitet u Nišu*, vol. 30, no. 7, pp. 2075–2081, 2016.

[21] H. Rehman, M. Darus, and J. Salah, "Coefficient properties involving the generalized k-Mittag-Leffler functions," *Transyl. J. Math. Mech.(TJMM)*, vol. 9, no. 2, pp. 155–164, 2017.

[22] H. M. Srivastava, B. A. Frasin, and V. Pescar, "Univalence of integral operators involving Mittag-Leffler functions," *Appl. Math. Inf. Sci.*, vol. 11, no. 3, pp. 635–641, 2017.

[23] H. M. Srivastava and Z. Tomovski, "Fractional calculus with an integral operator containing a generalized Mittag- Leffler function in the kernel," *Applied Mathematics and Computation*, vol. 211, no. 1, pp. 198–210, 2009.

[24] S. K. Sharma and R. Jain, "On some properties of generalized q -Mittag Leffler function," *Math. Aeterna*, vol. 4, pp. 613–619, 2014.

[25] H. M. Srivastava, "Certainq-polynomial expansions for functions of several variables," *IMA Journal of Applied Mathematics*, vol. 30, no. 3, pp. 315–323, 1983.

[26] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series, Halsted Press*, Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[27] G. S. Salagean, "Subclasses of univalent functions, in: Complex Analysis, fifth Romanian–Finnish Seminar, Part 1 (Bucharest, 1981)," in *Lecture Notes in Mathematics*, Springer, Berlin, Heidelberg, 1983.

[28] S. Shams, S. R. Kulkarni, and J. M. Jahangiri, "Classes of uniformly starlike and convex functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 55, p. 2961, 2004.

[29] S. Owa, Y. Polatoglu, and E. Yavuz, "Coefficient inequalities for classes of uniformly starlike and convex functions," *J. Ineq. Pure Appl. Math*, vol. 7, no. 5, pp. 1–5, 2006.

[30] R. M. Ali, "Starlikeness associated with parabolic regions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 4, p. 570, 2005.

[31] K. I. Noor and S. N. Malik, "On a new class of analytic functions associated with conic domain," *Mathematics with Applications*, vol. 62, no. 1, pp. 367–375, 2011.

[32] S. Kanas and S. Altinkaya, "Functions of bounded variation related to domains bounded by conic sections," *Mathematica Slovaca*, vol. 69, no. 4, pp. 833–842, 2019.

[33] K. I. Noor, M. Arif, and W. Ul-Haq, "On k-uniformly close-to-convex functions of complex order," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 629–635, 2009.

[34] W. Rogosinski, "On the coefficients of subordinate functions," *Proceedings of the London Mathematical Society*, vol. s2-48, no. 1, pp. 48–82, 1945.

[35] S. J. Sim, O. S. Kwon, N. E. Cho, and H. M. Srivastava, "Some classes of analytic functions associated with conic regions," *Taiwanese Journal of Mathematics*, vol. 16, no. 1, pp. 387–408, 2012.

[36] W. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," *Proceedings of the Conference on Complex Analysis*, Z. Li, F. Ren, L. Yang, and S. Zhang, Eds., , pp. 157–169, International Press Inc., 1992.

[37] H. Selverman, "Univalent functions with negative coefficients," *Proceedings of the American Mathematical Society*, vol. 51, no. 1, pp. 109–116, 1975.

[38] I. Aldawish and R. W. Ibrahim, "Solvability of a new q-differential equation related to q-differential inequality of a special type of analytic functions," *Fractal and Fractional*, vol. 5, no. 4, p. 228, 2021.

[39] M. Arif, O. Barkub, H. M. Srivastava, S. Abdullah, and S. A. Khan, "Some Janowski type harmonic q-starlike functions associated with symmetrical points," *Mathematics*, vol. 8, no. 4, p. 629, 2020.

[40] S. Hussain, S. Khan, M. A. Zaighum, and M. Darus, "Applications of a q-Salagean type operator on multivalent functions," *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.

[41] K. I. Noor and R. S. Badar, "On a class of quantum alpha-convex functions," *Journal of Applied Mathematics & Informatics*, vol. 36, pp. 541–548, 2018.

[42] H. Tang, S. Khan, S. Hussain et al., "Hankel and Toeplitz determinant for a subclass of multivalent $ q $-starlike functions of order $ \alpha $," *AIMS Mathematics*, vol. 6, no. 6, pp. 5421–5439, 2021.

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Research Article

On the Janowski Starlikeness of the Coulomb Wave Functions

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In this article, we are interested in finding sufficient conditions on A, B, L, and η which ensure the normalized Coulomb wave function to be Janowski starlike. Sufficient conditions are also obtained for $g_{L,\eta}/z \in \mathcal{P}[A, B]$, which readily yield conditions for $g_{L,\eta}$ to be close-to-convex.

1. Introduction

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. An analytic function f is subordinate to an analytic function g (written as $f \prec g$) if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathcal{U}$, then $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. Let $\mathcal{P}[A, B]$ denote the class of analytic functions p such that $p(0) = 1$ and

$$p(z) \prec \frac{1 + Az}{1 + Bz}, -1 \le B < A \le 1, z \in \mathcal{U}. \tag{1}$$

Note that for $0 \le \beta < 1, \mathcal{P}[1 - 2\beta, -1]$ is the class of analytic functions p with $p(0) = 1$ satisfying $\operatorname{Re} p(z) > \beta$ in $\mathcal{U}$. For $-1 \le B < A \le 1$, the class $\mathcal{S}^*[A, B]$ defined by

$$\mathcal{S}^*[A, B] \coloneqq \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, z \in \mathcal{U} \right\}, \tag{2}$$

is the class of Janowski starlike functions [13]. For $0 \le \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] \coloneqq \mathcal{S}^*(\beta)$ is the usual class of starlike functions of order β:

$$\mathcal{S}^*[1-\beta, 0] \coloneqq \mathcal{S}^*_\beta = \left\{ f \in \mathcal{A} : \left| zf'(z)/f(z) - 1 \right| < 1 - \beta \right\},$$
$$\mathcal{S}^*[\beta, -\beta] \coloneqq \mathcal{S}^*[\beta] = \left\{ f \in \mathcal{A} : \left| zf'(z)/f(z) - 1 \right| < \beta \left| zf'(z)/f(z) + 1 \right| \right\}. \tag{3}$$

These classes have been studied in [6, 8]. A function $f \in \mathcal{A}$ is said to be close-to-convex of order β with respect to a function $g \in \mathcal{S}^*$ if $\operatorname{Re}\,(zf'(z)/g(z)) > \beta$. In particular case, if $f \in \mathcal{A}$ and satisfies the condition $\operatorname{Re} f'(z) > \beta$ for all z in $\mathcal{U}$, then $f(z)$ is a close-to-convex of order β.

Let ${}_1F_1$ denote the Kummer confluent hypergeometric function. The regular Coulomb wave function is defined as

$$\begin{aligned} F_{L,\eta}(z) &= z^{L+1} e^{-iz} C_L(\eta)\, {}_1F_1(L + 1 - i\eta, 2L + 2 ; 2iz) \\ &= C_L(\eta) \sum_{n\ge0} a_{L,n} z^{n+L+1}, L, \eta \in \mathbb{C}, z \in \mathbb{C}, \end{aligned} \tag{4}$$

where

$$C_L(\eta) = \frac{2^L e^{-\pi\eta/2} |\Gamma(L + 1 + i\eta)|}{\Gamma(2L + 2)} = \begin{cases} \dfrac{2^L}{(2L+1)!} \sqrt{\dfrac{2\pi \prod_{k=0}^{L} (k^2 + \eta^2)}{\eta(e^{2\pi\eta} - 1)}}, & \text{if } \eta \ne 0, \\ \dfrac{2^L L!}{(2L+1)!}, & \text{if } \eta = 0, \end{cases}$$
$$a_{L,0} = 1, a_{L,1} = \frac{\eta}{L+1}, a_{L,n} = \frac{2\eta a_{L,n-1} - a_{L,n-2}}{n(n + 2L + 1)}, n \in \{2, 3, \cdots\}, \tag{5}$$

which is the solution of following differential equation:

$$z^2\omega''(z) + \left[z^2 - 2\eta z - L(L+1)\right]\omega(z) = 0. \tag{6}$$

In this paper, we focus on the following normalized form:

$$g_{L,\eta}(z) = C_L(\eta)^{-1} z^{-L} F_{L,\eta}(z). \tag{7}$$

The function $g_{L,\eta}(z)$ satisfies the following homogenous second-order differential equation:

$$z^2 g_{L,\eta}''(z) + 2Lz g_{L,\eta}'(z) + \left(z^2 - 2\eta z - 2L\right) g_{L,\eta}(z) = 0. \tag{8}$$

Baricz [9, 10] studied the Turan-type inequalites of regular Coulomb wave functions and zeros of a cross-product of the Coulomb wave and Tricomi hypergeometric functions, respectively. Baricz et al. [11] also investigated the radii of starlikeness and convexity of regular Coulomb wave functions. Recently, Aktas [1] has studied lemniscate and exponential starlikeness of Coulomb wave functions. In some recent papers [2–5, 12], the authors have discussed certain geometric properties of some special functions. The relationships of generalized Bessel function, Bessel-Struve kernal function, and Struve function with the Janowski class have also been studied by various researchers, see [7, 14, 17, 18]. Motivated by the above papers in this subject, in this paper, our aim is to present some geometric results for the normalized regular Coulomb wave function.

The following lemmas are needed in the paper.

Lemma 1 (see [15, 16]). *Let $\Omega \subset \mathbb{C}$, and $\Psi : \mathbb{C}^2 \times \mathcal{U} \longrightarrow \mathbb{C}$ satisfy*

$$\Psi(i\rho, \sigma; z) \notin \Omega, \tag{9}$$

whenever $z \in \mathcal{U}$, ρ real, $\sigma \le -(1+\rho^2)/2$. If p is analytic in $\mathcal{U}$ with $p(0) = 1$, and $\Psi(= (p(z), zp'(z); z) \in \Omega$ for $z \in \mathcal{U}$, then $\operatorname{Re} p(z) > 0$ in $\mathcal{U}$. In the case $\Psi : \mathbb{C}^3 \times \mathcal{U} \longrightarrow \mathbb{C}$, then the condition in Lemma 1 generalized to

$$\Psi(i\rho, \sigma, u + iv; z) \notin \Omega, \tag{10}$$

ρ real, $\sigma + \mu \le 0$, and $\sigma \le -(1+\rho^2)/2$.

Lemma 2 (see [19]). *Let $f \in \mathcal{A}$. If*

$$\operatorname{Re} \frac{f(z)}{z} > 0, \tag{11}$$

then

$$\operatorname{Re} f'(z) > 0, \tag{12}$$

for $|z| < \sqrt{2} - 1$.

2. Inclusion of Generalized Coulomb Wave Function in the Janowski Class

Our first result is related with Janowski starlikeness of normalized Coulomb wave function.

Theorem 3. *Let $-1 \le B < A \le 1$ and $L, \eta \in \mathbb{C}$. Suppose that*

$$\begin{aligned} \operatorname{Re}(2L-1) \le \frac{(1+A)}{4} - (\operatorname{Im}(L))^2\left(\frac{1+A}{2+A}\right) \\ - 2(1+|\eta|+|L|) \text{for} - 1 \\ = B < A \le 1, \end{aligned} \tag{13}$$

or, for $-1 < B < A \le 1$,

$$\begin{cases} (1+2|\eta|+2|L|)\dfrac{(1+B)}{(1+A)} - \dfrac{(1+A)}{(1+B)} - \dfrac{(A-B)}{(1+A)(1+B)} < \operatorname{Re}(2L-1), \\ \dfrac{(A-B)}{2(1-B)} - \dfrac{2}{(1-B)} - (1+2|\eta|+2|L|)\dfrac{(1-B)}{2} > \operatorname{Re}(2L-1), \end{cases} \tag{14}$$

$$\begin{aligned} [2\operatorname{Im}(L)(1-AB)]^2 \le \Bigg[1 + \frac{(1+A)(1+B)}{(A-B)}\operatorname{Re}(2L-1) \\ + \frac{(1+A)^2}{(A-B)} - \frac{(1+B)^2}{(A-B)}(1+2|\eta|+2|L|)\Bigg] \\ \times \Bigg[\frac{2(1-B)}{(A-B)}\operatorname{Re}(2L-1) - 1 + \frac{4}{(A-B)} \\ + \frac{(1-B)^2}{(A-B)}(1+2|\eta|+2|L|)\Bigg]. \end{aligned} \tag{15}$$

If $(1+B)zg_{L,\eta}'(z) \ne (1+A)g_{L,\eta}(z)$, $0 \ne g_{L,\eta}(z)$ and $0 \ne g_{L,\eta}'(z)$, then $g_{L,\eta} \in \mathcal{S}^*[A,B]$.

Proof. Define an analytic function $p : \mathcal{U} \longrightarrow \mathbb{C}$ by

$$p(z) := \frac{(1-A)g_{L,\eta}(z) - (1-B)zg_{L,\eta}'(z)}{(1+B)zg_{L,\eta}'(z) - (1+A)g_{L,\eta}(z)}, p(0) = 1. \tag{16}$$

Then,

$$\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)} = \frac{(1-A) + (1+A)p(z)}{(1-B) + (1+B)p(z)}, \tag{17}$$

$$\frac{zg_{L,\eta}''(z)}{g_{L,\eta}'(z)} = \frac{2(A-B)zp'(z)}{[(1-A)+(1+A)p(z)][(1-B)+(1+B)p(z)]} - 1 + \frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}. \tag{18}$$

A rearrangement of (18) gives.

$$\left(\frac{zg_{L,\eta}'{}'(z)}{g_{L,\eta}'(z)}\right)\left(\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}\right)=\frac{2(A-B)zp'(z)}{[(1-B)+(1+B)p(z)]^2}-\frac{[(1-A)+(1+A)p(z)]}{[(1-B)+(1+B)p(z)]}+\frac{[(1-A)+(1+A)p(z)]^2}{[(1-B)+(1+B)p(z)]^2}. \tag{19}$$

Now, define a function $q_{L,\eta}:\mathscr{U}\longrightarrow\mathbb{C}$ by

$$q_{L,\eta}(z)=\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}. \tag{20}$$

This function $q_{L,\eta}$ is analytic in $\mathscr{U}$ and $q_{L,\eta}(0)=1$. Suppose that $z\neq 0$. We know that $g_{L,\eta}(z)\neq 0$. This function satisfies the following equation:

$$\frac{z^2g_{L,\eta}'{}'(z)}{g_{L,\eta}(z)}+2L\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}+\left(z^2-2\eta z-2L\right)=0. \tag{21}$$

which yields

$$\left(\frac{zg_{L,\eta}'{}'(z)}{g_{L,\eta}'(z)}\right)\left(\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}\right)+2L\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}+\left(z^2-2\eta z-2L\right)=0. \tag{22}$$

Substituting (17) and (19) in (22), we get

$$\frac{2(A-B)zp'(z)}{[(1-B)+(1+B)p(z)]^2}+\frac{[(1-A)+(1+A)p(z)]^2}{[(1-B)+(1+B)p(z)]^2}+(2L-1)\frac{[(1-A)+(1+A)p(z)]}{[(1-B)+(1+B)p(z)]}+\left(z^2-2\eta z-2L\right)=0, \tag{23}$$

or equivalently

$$zp'(z)+\left[\frac{(2L-1)}{2(A-B)}\{(1-A)+(1+A)p(z)\}\{(1-B)+(1+B)p(z)\}\right]+\frac{[(1-A)+(1+A)p(z)]^2}{2(A-B)}+\frac{\left(z^2-2\eta z-2L\right)}{2(A-B)}[(1-B)+(1+B)p(z)]^2=0. \tag{24}$$

Now setting

$$\Psi\left(p(z),zp'(z);z\right):=zp'(z)+\left[\frac{(2L-1)}{2(A-B)}\{(1-A)+(1+A)p(z)\}\cdot\{(1-B)+(1+B)p(z)\}\right]+\frac{[(1-A)+(1+A)p(z)]^2}{2(A-B)}+\frac{\left(z^2-2\eta z-2L\right)}{2(A-B)}[(1-B)+(1+B)p(z)]^2. \tag{25}$$

Then, for $\rho\in\mathbb{R}$and $\sigma=-(1+\rho^2)/2$, we get

$$\begin{aligned}\operatorname{Re}\Psi(i\rho,\sigma;z)&=\sigma+\operatorname{Re}\left[\frac{(2L-1)}{2(A-B)}\{(1-A)+(1+A)i\rho\}\cdot\{(1-B)+(1+B)i\rho\}\right]\\&\quad+\operatorname{Re}\frac{[(1-A)+(1+A)i\rho]^2}{2(A-B)}+\operatorname{Re}\left[\frac{\left(z^2-2\eta z-2L\right)}{2(A-B)}\{(1-B)+(1+B)i\rho\}^2\right]\\&\le-\frac{\left(1+\rho^2\right)}{2}+\frac{\operatorname{Re}(2L-1)}{2(A-B)}[(1-A)(1-B)-(1+A)(1+B)\rho^2]-\frac{2\operatorname{Im}(L)}{(A-B)}[1-AB]\rho\\&\quad+\frac{(1-A)^2-(1+A)^2\rho^2}{2(A-B)}+\frac{(1+|2\eta|+|2L|)}{2(A-B)}\left|\{(1-B)+(1+B)i\rho\}^2\right|\\&=-\rho^2\left[\frac{1}{2}+\frac{(1+A)(1+B)}{2(A-B)}\operatorname{Re}(2L-1)+\frac{(1+A)^2}{2(A-B)}-\frac{(1+B)^2}{2(A-B)}(1+|2\eta|+|2L|)\right]\\&\quad-\frac{2\operatorname{Im}(L)}{(A-B)}[1-AB]\rho+\left[\frac{(1-A)(1-B)}{2(A-B)}\operatorname{Re}(2L-1)-\frac{1}{2}+\frac{(1-A)^2}{2(A-B)}+\frac{(1-B)^2}{2(A-B)}(1+|2\eta|+|2L|)\right]\\&\le-\rho^2\left[\frac{1}{2}+\frac{(1+A)(1+B)}{2(A-B)}\operatorname{Re}(2L-1)+\frac{(1+A)^2}{2(A-B)}-\frac{(1+B)^2}{2(A-B)}(1+|2\eta|+|2L|)\right]\\&\quad-\frac{2\operatorname{Im}(L)}{(A-B)}[1-AB]\rho+\frac{(1-B)}{(A-B)}\operatorname{Re}(2L-1)-\frac{1}{2}+\frac{2}{(A-B)}+\frac{(1-B)^2}{2(A-B)}(1+|2\eta|+|2L|)\coloneqq Q(\rho).\end{aligned} \tag{26}$$

To get the contradiction, we have to show $Q(\rho)\le 0$ for

$\rho \in \mathbb{R}$. We split the proof into two cases. First, consider the case $B = -1 < A \le 1$. Then, the function Q becomes

$$Q(\rho) = -\rho^2 \left[\frac{2+A}{2}\right] - 2\,\mathrm{Im}\,(L)\rho + \left[\frac{2}{(1+A)}\,\mathrm{Re}\,(2L-1) - \frac{1}{2} + \frac{2}{(1+A)} + \frac{2}{(1+A)}(1+|2\eta|+|2L|)\right], \quad (27)$$

that achieve its maximum at $\rho_0 = -2\,\mathrm{Im}\,(L)/(2+A)$, and

$$Q(\rho_0) = \frac{2(\mathrm{Im}\,(L))^2}{[2+A]} + \left[\frac{2}{(1+A)}\,\mathrm{Re}\,(2L-1) - \frac{1}{2} + \frac{2}{(1+A)} + \frac{2}{(1+A)}(1+|2\eta|+|2L|)\right], \quad (28)$$

which is nonpositive if and only if

$$\mathrm{Re}\,(2L-1) \le \left[\frac{(1+A)}{4} - (\mathrm{Im}\,(L))^2\left(\frac{1+A}{2+A}\right) - 2(1+|\eta|+|L|)\right]. \quad (29)$$

Now, consider the case $-1 < B < A \le 1$. Rewriting Q in the form

$$Q(\rho) = -P\rho^2 + R\rho - S = -P\left\{\left(\rho - \frac{R}{2P}\right)^2 + \frac{4PS - R^2}{4P^2}\right\}, \quad (30)$$

where

$$\begin{aligned} P &= \left[\frac{1}{2} + \frac{(1+A)(1+B)}{2(A-B)}\,\mathrm{Re}\,(2L-1) + \frac{(1+A)^2}{2(A-B)} - \frac{(1+B)^2}{2(A-B)}(1+|2\eta|+|2L|)\right], \\ R &= -\frac{2\,\mathrm{Im}\,(L)}{(A-B)}[1-AB], \\ S &= -\left[\frac{(1-B)}{(A-B)}\,\mathrm{Re}\,(2L-1) - \frac{1}{2} + \frac{2}{(A-B)} + \frac{(1-B)^2}{2(A-B)}(1+|2\eta|+|2L|)\right]. \end{aligned} \quad (31)$$

The inequality $Q(\rho) \le 0$ holds for any real ρ, if $P > 0$, $S > 0$ and $R^2 \le 4PS$ or

$$\begin{cases} (1+|2\eta|+|2L|)\dfrac{(1+B)}{(1+A)} - \dfrac{(1+A)}{(1+B)} - \dfrac{(A-B)}{(1+A)(1+B)} < \mathrm{Re}\,(2L-1), \\ \dfrac{(A-B)}{2(1-B)} - \dfrac{2}{(1-B)} - (1+|2\eta|+|2L|)\dfrac{(1-B)}{2} > \mathrm{Re}\,(2L-1), \end{cases} \quad (32)$$

and

$$\begin{aligned} [2\,\mathrm{Im}\,(L)(1-AB)]^2 \le{}& \left[1 + \frac{(1+A)(1+B)}{(A-B)}\,\mathrm{Re}\,(2L-1) + \frac{(1+A)^2}{(A-B)} - \frac{(1+B)^2}{(A-B)}(1+|2\eta|+|2L|)\right] \\ &\times \left[\frac{2(1-B)}{(A-B)}\,\mathrm{Re}\,(2L-1) - 1 + \frac{4}{(A-B)} + \frac{(1-B)^2}{(A-B)}(1+|2\eta|+|2L|)\right], \\ \le{}& \left[1 + \frac{(1+A)(1+B)}{(A-B)}\,\mathrm{Re}\,(2L-1) + \frac{(1+A)^2}{(A-B)} - \frac{(1+B)^2}{(A-B)}(1+|2\eta|+|2L|)\right] \\ &\times \left[\frac{2(1-B)}{(A-B)}\,\mathrm{Re}\,(2L-1) - 1 + \frac{4}{(A-B)} + \frac{(1-B)^2}{(A-B)}(1+|2\eta|+|2L|)\right], \end{aligned} \quad (33)$$

that holds by hypothesis (14) and (15). Thus, in both cases, the function Ψ satisfies the hypothesis of lemma (8) and hence $\mathrm{Re}\, p(z) > 0$, or

$$\frac{(1-A)g_{L,\eta}(z) - (1-B)zg'_{L,\eta}(z)}{(1+B)zg'_{L,\eta}(z) - (1+A)g_{L,\eta}(z)} \prec \frac{1+z}{1-z}. \quad (34)$$

By definition of subordination, there exist a map ω in $\mathscr{U}$with $\omega(0) = 0$, and

$$\frac{(1-A)g_{L,\eta}(z) - (1-B)zg'_{L,\eta}(z)}{(1+B)zg'_{L,\eta}(z) - (1+A)g_{L,\eta}(z)} = \frac{1+\omega(z)}{1-\omega(z)}, \quad (35)$$

which yields

$$\frac{zg'_{L,\eta}(z)}{g_{L,\eta}(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}. \quad (36)$$

Hence,

$$\frac{zg'_{L,\eta}(z)}{g_{L,\eta}(z)} \prec \frac{1+Az}{1+Bz}. \quad (37)$$

□

If take $A = 1-2\beta$ and $B = -1$ for $0 \le \beta < 1$ in Theorem 3, we obtain following result.

Corollary 4. *Let $0 \le \beta < 1$ and$L, \eta \in \mathbb{C}$. If*

$$\mathrm{Re}\,(L) \le -\frac{1+\beta}{4} - (\mathrm{Im}\,(L))^2\left(\frac{1-\beta}{3-2\beta}\right) - (|\eta| + |L|), \quad (38)$$

then the normalized Coulomb wave function $g_{L,\eta}(z)$ is starlike of order β.

Theorem 5. *Let $-1 \le B < A \le 1$ and $L, \eta \in \mathbb{C}$ satisfy*

$$\operatorname{Re}(2L+1) \ge \begin{cases} \dfrac{(1+|2\eta|)}{(1+A)}\left(\sqrt{2(1+A^2)}+(1-A)\right), & -1 = B < A \le 3-2\sqrt{2}, \\ \dfrac{(1+|2\eta|)(1+A)}{2\sqrt{A}} \text{ and } \operatorname{Re}(2L+1) \le \dfrac{(1+|2\eta|)(1+A)}{(1-A)} (A \ne 1), & B = -1, A > 3-2\sqrt{2}, \\ \dfrac{(1+|2\eta|)(1+A)(1-B)^2}{(A-B)(1+B)} - \dfrac{(1-B)}{(1+B)}, & -1 < B \le 0, \\ \dfrac{(1+|2\eta|)(1+A)(1+B)}{(A-B)} - \dfrac{(1-B)}{(1+B)}, & B \ge 0. \end{cases} \quad (39)$$

If $(1+B)s_{L,\eta} \ne (1+A)$, then $g_{L,\eta}(z)/z \in \mathcal{P}[A,B]$.

Proof. Define a function $s_{L,\eta} : \mathcal{U} \longrightarrow \mathbb{C}$ by

$$s_{L,\eta}(z) = \frac{g_{L,\eta}(z)}{z}. \quad (40)$$

The function $s_{L,\eta}$ is analytic in $\mathcal{U}$ and $s_{L,\eta}(0) = 1$. Suppose that $z \ne 0$.. This function satisfies the following equation:

$$z^2 s_{L,\eta}''(z) + 2(L+1) z s_{L,\eta}'(z) + \left(z^2 - 2\eta z\right) s_{L,\eta}(z) = 0. \quad (41)$$

Define the analytic function p::$\mathcal{U} \longrightarrow \mathbb{C}$ by

$$p(z) = -\frac{(1-A)-(1-B)s_{L,\eta}(z)}{(1+A)-(1+B)s_{L,\eta}(z)}. \quad (42)$$

Then, simple computation yields

$$s_{L,\eta}(z) = \frac{(1-A)+(1+A)p(z)}{(1-B)+(1+B)p(z)}, \quad (43)$$

$$s_{L,\eta}'(z) = \frac{2(A-B)p'(z)}{((1-B)+(1+B)p(z))^2}, \quad (44)$$

$$s_{L,\eta}''(z) = \frac{2(A-B)((1-B)+(1+B)p(z))p''(z) - 4(1+B)(A-B)p'^2(z)}{((1-B)+(1+B)p(z))^3}. \quad (45)$$

Thus, using (43)–(45), the differential equation (41) can be rewritten as

$$z^2 p''(z) - \frac{2z^2(1+B)p'^2(z)}{(1-B)+(1+B)p(z)} + 2(L+1) z p'(z) + \left[\frac{\{(1-B)+(1+B)p(z)\}\{(1-A)+(1+A)p(z)\}}{2(A-B)}\right]\left(z^2 - 2\eta z\right) = 0. \quad (46)$$

Assume $\Omega = \{0\}$ and define $\Psi(r,s,t;z)$ by

$$\Psi(r,s,t;z) := t - \frac{2(1+B)}{(1-B)+(1+B)r} s^2 + 2(L+1)s + \left[\frac{\{(1-B)+(1+B)r\}\{(1-A)+(1+A)r\}}{2(A-B)}\right]\left(z^2 - 2\eta z\right). \quad (47)$$

It follows from (47) that $\Psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega$. To ensure $\operatorname{Re} p(z) > 0$ for $z \in \mathcal{U}$, from Lemma 1, it is enough to establish $\operatorname{Re}\Psi(i\rho, \sigma, \mu + i\nu; z) \le 0$ in $\mathcal{U}$ for any real ρ, $\sigma \le -(1+\rho^2)/2$,, and $\sigma + \mu \le 0$. Let $z = x + iy \in \mathcal{U}$.. A computation yields

$$\begin{aligned} \operatorname{Re}\Psi(i\rho, \sigma, \mu + i\nu; z) &= \mu - \frac{2(1+B)(1-B)\sigma^2}{(1-B)^2 + (1+B)^2\rho^2} + \operatorname{Re} 2(L+1)\sigma \\ &\quad + \left|\frac{\{(1-B)+(1+B)i\rho\}\{(1-A)+(1+A)i\rho\}}{2(A-B)}\right|\left(\left|z^2 - 2\eta z\right|\right). \end{aligned} \quad (48)$$

Since $\sigma \le -(1+\rho^2)/2$. Thus,

$$\begin{aligned} \operatorname{Re}\Psi(i\rho, \sigma, \mu + i\nu; z) &\le -\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2} - \frac{\left(1-B^2\right)\left(1+\rho^2\right)^2}{2\left[(1-B)^2 + (1+B)^2\rho^2\right]} \\ &\quad + \left|\frac{\{(1-B)+(1+B)i\rho\}\{(1-A)+(1+A)i\rho\}}{2(A-B)}\right|\left(\left|z^2 - 2\eta z\right|\right) \\ &\le -\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2} - \frac{\left(1-B^2\right)\left(1+\rho^2\right)^2}{2\left[(1-B)^2 + (1+B)^2\rho^2\right]} \\ &\quad + \left[\frac{|(1-B)+(1+B)i\rho||(1-A)+(1+A)i\rho|}{2(A-B)}\right](1+|2\eta|). \end{aligned} \quad (49)$$

The proof will be divided into four cases. Consider first $B = -1$, $B < A \le 3-2\sqrt{2}$. The inequality (49) reduces to

$$\begin{aligned}
&\operatorname{Re}\Psi(i\rho,\sigma,\mu+iv;z)\\
&\le -\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2}+\left[\frac{|(1-A)+(1+A)i\rho|}{(1+A)}\right](1+|2\eta|)\\
&\le -\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2}+\frac{(1+|2\eta|)}{(1+A)}[(1-A)+(1+A)|\rho|]\\
&=-\frac{\operatorname{Re}(2L+1)}{2}\rho^2+(1+|2\eta|)|\rho|+\frac{(1-A)}{(1+A)}(1+|2\eta|)-\frac{\operatorname{Re}(2L+1)}{2}\\
&=-\frac{\operatorname{Re}(2L+1)}{2}\left(|\rho|-\frac{(1+|2\eta|)}{\operatorname{Re}(2L+1)}\right)^2+\frac{(1+|2\eta|)^2}{2\operatorname{Re}(2L+1)}\\
&\quad+\frac{(1-A)}{(1+A)}(1+|2\eta|)-\frac{\operatorname{Re}(2L+1)}{2}=:G(\rho).
\end{aligned} \tag{50}$$

A quadratic function G takes nonpositive values for any ρ, if

$$\frac{(1+|2\eta|)^2}{2\operatorname{Re}(2L+1)}+\frac{(1-A)}{(1+A)}(1+|2\eta|)-\frac{\operatorname{Re}(2L+1)}{2}\le 0. \tag{51}$$

Last inequality can be rewritten as

$$-[\operatorname{Re}(2L+1)]^2+\frac{2(1-A)}{(1+A)}(1+|2\eta|)\operatorname{Re}(2L+1)+(1+|2\eta|)^2\le 0, \tag{52}$$

that holds, if

$$\operatorname{Re}(2L+1)\ge\frac{(1+|2\eta|)}{(1+A)}\left(\sqrt{(1-A)^2+(1+A)^2}-(1-A)\right), \tag{53}$$

which reduces to the assumption. Therefore, the assertion follows.

In second case, we consider $B=-1$, $A>3-2\sqrt{2}$. According to (49), we have

$$\begin{aligned}
\operatorname{Re}\Psi(i\rho,\sigma,\mu+iv;z)&\le -\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2}\\
&\quad+\left[\frac{|(1-A)+(1+A)i\rho|}{(1+A)}\right](1+|2\eta|)\\
&=-\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2}\\
&\quad+\frac{(1+|2\eta|)}{(1+A)}\left(\sqrt{(1-A)^2+(1+A)^2\rho^2}\right)=:H(\rho).
\end{aligned} \tag{54}$$

We note that the function H is even with respect to ρ, and

$$H(0)=-\frac{\operatorname{Re}(2L+1)}{2}+\frac{(1-A)}{(1+A)}(1+|2\eta|), \tag{55}$$

that satisfies $H(0)\le 0$, if

$$\operatorname{Re}(2L+1)\ge\frac{2(1-A)}{(1+A)}(1+|2\eta|). \tag{56}$$

Moreover, $\lim_{\rho\longrightarrow\infty}H(\rho)=-\infty$, and

$$H'(\rho)=-\operatorname{Re}(2L+1)\rho+\frac{(1+A)\rho(1+|2\eta|)}{\sqrt{(1-A)^2+(1+A)^2\rho^2}}, \tag{57}$$

with $H'(\rho)=0$ if and only if $\rho=0$ or

$$\rho_0^2=\frac{(1+|2\eta|)^2}{\operatorname{Re}^2(2L+1)}-\frac{(1-A)^2}{(1+A)^2}. \tag{58}$$

We observe that $\rho_0^2\ge 0$ by the inequality

$$\frac{(1+|2\eta|)^2}{\operatorname{Re}^2(2L+1)}\ge\frac{(1-A)^2}{(1+A)^2}, \tag{59}$$

$$\operatorname{Re}(2L+1)\le\frac{(1+A)}{(1-A)}(1+|2\eta|). \tag{60}$$

Additionally,

$$H''(\rho)=-\operatorname{Re}(2L+1)+\frac{(1-A)^2}{(1+A)^2}\frac{\operatorname{Re}^3(2L+1)}{(1+|2\eta|)^2}\le 0, \tag{61}$$

in view of (60). Hence, $H(\rho_0)=\max H(\rho)$, and

$$H(\rho_0)=-\frac{\operatorname{Re}(2L+1)}{2}\left[1-\left(\frac{1-A}{1+A}\right)^2\right]+\frac{(1+|2\eta|)^2}{2\operatorname{Re}(2L+1)}\le 0, \tag{62}$$

that holds if

$$\operatorname{Re}(2L+1)\ge\frac{(1+A)}{2\sqrt{A}}(1+|2\eta|). \tag{63}$$

Since,

$$\frac{(1+A)}{(1-A)}(1+|2\eta|)\ge\frac{(1+A)}{2\sqrt{A}}(1+|2\eta|)\ge\frac{(1-A)}{(1+A)}(1+|2\eta|), \tag{64}$$

holds for $3-2\sqrt{2}\le A\le 1$, then the condition (56), (60), and (63) reduce to the assumption (39). Therefore, the assertion follows. Let now $-1<B\le 0, A>B$. By the fact $(1-A)/(1+A)<(1-B)/(1+B)$, we get

$$\begin{aligned}
&|(1-A)+(1+A)i\rho||(1-B)+(1+B)i\rho|\\
&\quad=(1+A)(1+B)\sqrt{\left(\frac{1-B}{1+B}\right)^2+\rho^2}\sqrt{\left(\frac{1-A}{1+A}\right)^2+\rho^2}\\
&\quad\le(1+A)(1+B)\left[\left(\frac{1-B}{1+B}\right)^2+\rho^2\right].
\end{aligned} \tag{65}$$

Also, for $B \le 0$, we have $(1+B)/(1-B) \le 1$; therefore,

$$\frac{1+\rho^2}{(1-B)^2+(1+B)^2\rho^2} = \frac{1}{(1-B)^2}\frac{1+\rho^2}{1+((1+B)/(1-B))^2\rho^2} \ge \frac{1}{(1-B)^2}, \quad (66)$$

for any real ρ. Thus,

$$\begin{aligned}\operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z) \le & -\frac{\operatorname{Re}(2L+1)(1+\rho^2)}{2} - \frac{(1+B)(1+\rho^2)}{2(1-B)} \\ & + \frac{(1+|2\eta|)}{2(A-B)}(1+A)(1+B)\left[\left(\frac{1-B}{1+B}\right)^2+\rho^2\right] \\ = & \rho^2\left(-\frac{\operatorname{Re}(2L+1)}{2} - \frac{(1+B)}{2(1-B)} + \frac{(1+|2\eta|)}{2(A-B)}(1+A)(1+B)\right) \\ & - \frac{\operatorname{Re}(2L+1)}{2} - \frac{(1+B)}{2(1-B)} \\ & + \frac{(1+|2\eta|)}{2(A-B)(1+B)}(1+A)(1-B)^2.. \end{aligned} \quad (67)$$

Since for $B \le 0$,,

$$\begin{aligned} & -\frac{\operatorname{Re}(2L+1)}{2} - \frac{(1+B)}{2(1-B)} + \frac{(1+|2\eta|)}{2(A-B)}(1+A)(1+B) \\ & \le -\frac{\operatorname{Re}(2L+1)}{2} - \frac{(1+B)}{2(1-B)} + \frac{(1+|2\eta|)}{2(A-B)(1+B)}(1+A)(1-B)^2, \end{aligned} \quad (68)$$

and the last expression is nonpositive in view of (39); then, the assertion follows. Finally, consider $0 \le B < A \le 1$. In this case $\beta = (1+B)/(1-B) \le 1$. Hence, setting $t = \beta^2 + \rho^2$ with $t \ge \beta^2$ and using (65), we get from (49)

$$\begin{aligned}\operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z) \le & -\frac{\operatorname{Re}(2L+1)}{2}(1-\beta^2+t) - \frac{\beta(1-\beta^2+t)^2}{2t} \\ & + \frac{(1+|2\eta|)}{2(A-B)}(1+A)(1+B)t \\ = & t\left\{-\frac{\operatorname{Re}(2L+1)}{2} - \frac{\beta}{2} + \frac{(1+|2\eta|)}{2(A-B)}(1+A)(1+B)\right\} \\ & - \frac{\operatorname{Re}(2L+1)}{2}(1-\beta^2) - \frac{\beta(1-\beta^2)^2}{2t} - \beta(1-\beta^2)^2, \end{aligned} \quad (69)$$

that is nonpositive due to inequality

$$\operatorname{Re}(2L+1) \ge \frac{(1+|2\eta|)}{(A-B)}(1+A)(1+B) - \frac{(1-B)}{(1+B)}, \quad (70)$$

that is equivalent to the assumption (39). Evidently, Ψ satisfies the hypothesis of Lemma 1, and thus, $\operatorname{Re} p(z) > 0$, that is

$$-\frac{(1-A)-(1-B)s_{L,\eta}(z)}{(1+A)-(1+B)s_{L,\eta}(z)} \prec \frac{1+z}{1-z}. \quad (71)$$

Hence, there exists an analytic self-map ω of $\mathcal{U}$ with $\omega(0)=0$ such that

$$-\frac{(1-A)-(1-B)s_{L,\eta}(z)}{(1+A)-(1+B)s_{L,\eta}(z)} = \frac{1+\omega(z)}{1-\omega(z)}, \quad (72)$$

which implies that

$$\frac{g_{L,\eta}(z)}{z} \prec \frac{1+Az}{1+Bz}. \quad (73)$$

□

If we take $A = 1-2\beta$ and $B = -1$ for $0 \le \beta < 1$ in Theorem 5, we obtain following result.

Corollary 6. *Let $0 \le \beta < 1$ and $L, \eta \in \mathbb{C}$. If*

$$\begin{aligned} & \frac{(1+2|\eta|)(1-\beta)}{2\sqrt{1-2\beta}} - \frac{1}{2} \le \operatorname{Re}(L) \\ & \le \frac{(1+2|\eta|)(1-\beta)}{2\beta} - \frac{1}{2}, \left(0 \le \beta < \sqrt{2}-1\right), \end{aligned} \quad (74)$$

$$\operatorname{Re}(L) \ge \frac{1}{2} + \frac{1+2|\eta|}{2(1-\beta)}\left(\sqrt{1-2\beta+2\beta^2}+\beta\right), \left(\sqrt{2}-1 \le \beta < 1\right), \quad (75)$$

then $\operatorname{Re}(g_{L,\eta}(z)/z) > \beta$, *that is,* $z + \int_0^z (g_{L,\eta}(t)/t)dt$ *is close-to-convex of order* β.

Applying Corollary 6 for $\beta = 0$ and Lemma 2, the following result for close-to-convexity of $g_{L,\eta}(z)$ immediately follows.

Corollary 7. *Let $L, \eta \in \mathbb{C}$. If* $\operatorname{Re}(L) \ge |\eta|$, *then $g_{L,\eta}(z)$ is close-to-convex (univalent) for $|z| < \sqrt{2}-1$.*

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] İ. Aktas, "Lemniscate and exponential starlikeness of regular Coulomb wave functions," *Studia Scientiarum Mathematicarum Hungarica*, vol. 57, no. 3, pp. 372–384, 2020.

[2] İ. Aktas and Á. Baricz, "Bounds for radii of starlikeness of some $${\varvec{q}}$$ q -Bessel functions," *Results in Mathematics*, vol. 72, no. 1-2, pp. 947–963, 2017.

[3] İ. Aktas, Á. Baricz, and H. Orhan, "Bounds for radii of starlikeness and convexity of some special functions," *Turkish Journal of Mathematics*, vol. 42, no. 1, pp. 211–226, 2018.

[4] İ. Aktas, Á. Baricz, and S. Singh, "Geometric and monotonic properties of hyper-Bessel functions," *The Ramanujan Journal*, vol. 51, no. 2, pp. 275–295, 2020.

[5] İ. Aktas, Á. Baricz, and N. Yagmur, "Bounds for the radii of univalence of some special functions," *Mathematical Inequalities & Applications*, vol. 20, no. 3, pp. 825–843, 2017.

[6] R. M. Ali, R. Chandrashekar, and V. Ravichandran, "Janowski starlikeness for a class of analytic functions," *Applied Mathematics Letters*, vol. 24, no. 4, pp. 501–505, 2011.

[7] R. M. Ali, S. R. Mondal, and V. Ravichandran, "On the Janowski convexity and starlikeness of the confluent hypergeometric functions," *Bulletin of the Belgian Mathematical Society*, vol. 22, pp. 227–250, 2015.

[8] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Sufficient conditions for Janowski starlikeness," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 062925, 7 pages, 2007.

[9] Á. Baricz, "Turan type inequalities for regular Coulomb wave functions," *Journal of Mathematical Analysis and Applications*, vol. 430, no. 1, pp. 166–180, 2015.

[10] Á. Baricz, "Zeros of a cross-product of the Coulomb wave and Tricomi hypergeometric functions," *Proceedings of the American Mathematical Society*, vol. 145, no. 4, pp. 1643–1648, 2016.

[11] Á. Baricz, M. Çağlar, E. Deniz, and E. Toklu, "Radii of starlikeness and convexity of regular Coulomb wave functions," *Complex Variables*, 2016, arXiv:1605.06763v1.

[12] Á. Baricz and R. Szasz, "Close-to-convexity of some special functions and their derivatives," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 39, no. 1, pp. 427–437, 2016.

[13] W. Janowski, "Some extremal problems for certain families of analytic functions I," *Annales Polonici Mathematici*, vol. 28, no. 3, pp. 297–326, 1973.

[14] S. Kanas, S. R. Mondal, and A. D. Mohammed, "Relations between the generalized Bessel functions and the Janowski class," *Mathematical Inequalities & Applications*, vol. 21, no. 1, pp. 165–178, 2018.

[15] S. S. Miller and P. T. Mocanu, *Differential Subordinations, Monographs and Textbooks in Pure and Applied Mathematics, 225*, Dekker, New York, 2000.

[16] S. S. Miller and P. T. Mocanu, "Differential subordinations and inequalities in the complex plane," *Journal of Differential Equations*, vol. 67, no. 2, pp. 199–211, 1987.

[17] S. R. Mondal and A. D. Mohammed, "One some differential subordination involving the Bessel-Struve kernel function," *Korean Mathematical Society*, vol. 33, no. 2, pp. 445–458, 2018.

[18] S. Noreen, M. Raza, E. Deniz, and S. K. Moğlu, "On the Janowski class of generalized Struve functions," *Afrika Matematika*, vol. 30, no. 1-2, pp. 23–35, 2019.

[19] K. Yamaguchi, "On functions satisfying Re $(f(z)/z) > 0$," *Proceedings of the American Mathematical Society*, vol. 17, pp. 588–591, 1966.

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Research Article

Fractional Minkowski-Type Integral Inequalities via the Unified Generalized Fractional Integral Operator

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This paper is aimed at presenting the unified integral operator in its generalized form utilizing the unified Mittag-Leffler function in its kernel. We prove the boundedness of this newly defined operator. A fractional integral operator comprising a unified Mittag-Leffler function is used to establish further Minkowski-type integral inequalities. Several related fractional integral inequalities that have recently been published in various articles can be inferred.

1. Introduction

Integral operators are useful in the study of differential equations and in the formation of real-world problems in integral equations. They also behave like integral transformations in particular cases. In the past few decades, fractional integral operators have been defined extensively (see [1–4]). Recently, in [5] a unified integral operator is studied which has interesting consequences in the theory of fractional integral operators.

This paper is aimed at presenting a unified integral operator in the more generalized form via the unified Mittag-Leffler function introduced in [6]. The boundedness of the newly defined integral operator is studied. By taking the power function $\xi^{\beta}; \beta > 1$, a unified generalized extended fractional integral operator is deduced and analyzed to construct Minkowski-type integral inequalities. This is the extension of our previous work on Minkowski-type integral inequalities [7]. The connection of the results of this paper is established with many published results of references [7–9]. We begin by reviewing several key Minkowski-type inequalities as well as some definitions that will be useful in our subsequent work.

The well-known Minkowski inequality is given as follows:

Theorem 1. Let $\phi, \psi \in L_m[u, v]$**.** *Then for* $m \geq 1$*, we have*

$$\left(\int_u^v (\phi(\xi) + \psi(\xi))^m d\xi\right)^{1/m} \leq \left(\int_u^v \phi^m(\xi) d\xi\right)^{1/m} + \left(\int_u^v \psi^m(\xi) d\xi\right)^{1/m}. \quad (1)$$

Some more Minkowski-type inequalities are stated in the next results.

Theorem 2. ([10]). *Let* $\phi, \psi \in L_m[u, v]$*. Also* $\phi, \psi \in \mathfrak{R}^+$ *such that* $0 < k_1 \leq (\phi(\xi))/(\psi(\xi)) \leq k_2 \forall \xi \in [u, v]$*. Then for* $m \geq 1$*, the following inequality holds true*

$$\left(\int_u^v \phi^m(\xi) d\xi\right)^{1/m} + \left(\int_u^v \psi^m(\xi) d\xi\right)^{1/m} \leq \left(1 + \frac{k_2 - k_1}{(k_1 + 1)(k_2 + 1)}\right)\left(\int_u^v (\phi(\xi) + \psi(\xi))^m d\xi\right)^{1/m}. \quad (2)$$

Theorem 3. ([11]). *Under the assumptions of Theorem 2, we* *have the following inequality:*

$$\left(\int_u^v \phi^m(\xi)d\xi\right)^{2/m} + \left(\int_u^v \psi^m(\xi)d\xi\right)^{2/m} \geq \left(\frac{2+(k_1-1)(k_2+1)}{k_2}\right)\left(\int_u^v \phi^m(\xi)d\xi\right)^{2/m}\left(\int_u^v \psi^m(\xi)d\xi\right)^{2/m}. \quad (3)$$

Theorem 4. ([9]). *Let $\omega \in \mathbb{R}, \alpha, \beta, \gamma > 0, \theta > \lambda > 0$ with $s \geq 0$, $r > 0$ and $0 < k \leq r + \alpha$. Let $m \geq 1$ and $\phi, \psi \in L_m[u, v]$ be positive functions satisfying*

$$0 < k_1 \leq \frac{\phi(\xi)}{\psi(\xi)} \leq k_2, \xi \in [u, v]. \quad (4)$$

Then the following inequality holds:

$$[(\varepsilon\phi^m)(\xi;s)]^{1/m} + [(\varepsilon\psi^m)(\xi;s)]^{1/m} \leq \left(1 + \frac{k_2-k_1}{(k_1+1)(k_2+1)}\right)[(\varepsilon(\phi+\psi)^m)(\xi;s)]^{1/m}. \quad (5)$$

Theorem 5. [9]. *Let $m, n > 1$ such that $1/m + 1/n = 1$. Then under the assumptions of Theorem 4, we have*

$$[(\varepsilon\phi)(\xi;s)]^{1/m}[(\varepsilon\psi)(\xi;s)]^{1/n} \leq \left(\frac{k_2}{k_1}\right)^{1/mn}\left[\left(\varepsilon\phi^{1/m}\psi^{1/n}\right)(\xi;s)\right]. \quad (6)$$

A special function known as the Mittag-Leffler function was introduced by a Swedish mathematician Gosta Mittag-Leffler [12] by the following series:

$$E_\alpha(t) = \sum_{l=0}^{\infty} \frac{t^l}{\Gamma(\alpha l + 1)}, \quad (7)$$

where $t, \alpha \in \mathbb{C}$ and $\mathfrak{R}(\alpha) > 0$.

This function is a direct extension of the exponential function that can be used to construct solutions of fractional differential equations. Due to its wide range of applications, this function has received considerable attention in recent decades. Many researchers provided its numerous generalized forms due to its intriguing results. We refer the readers to [2, 4, 13–16] for the study of generalized versions of the Mittag-Leffler function. In [17], Bhatnagar et al. introduced the generalization of Mittag-Leffler function in the form of generalized Q function as follows:

Definition 6. The generalized Q function denoted by ($Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu})(.;.,.)$ is defined by the following series:

$$Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{a},\underline{b}) = \sum_{l=0}^{\infty} \frac{\Pi_{i=1}^n B(b_i,l)(\lambda)_{\rho l}(\theta)_{kl} t^l}{\Pi_{i=1}^n B(a_i,l)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l + \beta)}, \quad (8)$$

where $\underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\alpha, \beta, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta, a_i, b_i \in \mathbb{C}$, $k \in (0,1) \cup \mathbb{N}$ and $\min\{\mathfrak{R}(\alpha), \mathfrak{R}(\beta), \mathfrak{R}(\gamma), \mathfrak{R}(\theta), \mathfrak{R}(\lambda), \mathfrak{R}(\delta), \mathfrak{R}(\rho)\} > 0$.

Recently, in [7], we introduced the fractional integral operator associated with generalized Q function as follows:

$${}_Q I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} f(\xi;\underline{a},\underline{b}) = \int_u^\xi (\xi-t)^{\beta-1} Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(\xi-t)^\alpha;\underline{a},\underline{b}\right)f(t)dt, \quad (9)$$

$${}_Q I^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\delta,\mu,\nu} f(\xi;\underline{a},\underline{b}) = \int_\xi^v (t-\xi)^{\beta-1} Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(t-\xi)^\alpha;\underline{a},\underline{b}\right)f(t)dt. \quad (10)$$

Andrić et al. in [2] introduced an extended and generalized Mittag-Leffler function along with the corresponding fractional integral operator as follows:

Definition 7. The extended and generalized Mittag-Leffler function ($E^{\delta,\mu,k,\nu}_{\alpha,\beta,\gamma}$) is defined by the following series:

$$E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}(t;s) = \sum_{l=0}^{\infty} \frac{B_s(\lambda+lk,\theta-\lambda)(\theta)_{lk} t^l}{B(\lambda,\theta-\lambda)(\gamma)_{lr}\Gamma(\alpha l+\beta)}, \quad (11)$$

where $t, \alpha, \beta, \gamma, \theta, \lambda \in \mathbb{C}$, $\mathfrak{R}(\alpha), \mathfrak{R}(\beta), \mathfrak{R}(\gamma), \mathfrak{R}(\theta), \mathfrak{R}(\lambda) > 0$, $\mathfrak{R}(\theta) > \mathfrak{R}(\lambda)$ with $s \geq 0$, $r > 0$, $0 < k \leq r + \mathfrak{R}(\alpha)$, and $(\theta)_{lk} = (\Gamma(\theta+lk))/(\Gamma(\theta))$.

Definition 8. Let $f \in L_1[u, v]$. Then for $\xi \in [u, v]$, the fractional integral operator corresponding to (11) is defined by the following integrals:

$$\varepsilon^{\omega,\lambda,\theta,k,r}_{u^+,\alpha,\beta,\gamma} f(\xi;s) = \int_u^\xi (\xi-t)^{\beta-1} E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma,}\left(\omega(\xi-t)^\alpha;s\right)f(t)dt,$$
$$\varepsilon^{\omega,\lambda,\theta,k,r}_{v^-,\alpha,\beta,\gamma} f(\xi;s) = \int_\xi^v (t-\xi)^{\beta-1} E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma,}\left(\omega(\xi-t)^\alpha;s\right)f(t)dt. \quad (12)$$

In [5], Farid defined the unified integral operator based on the extended and generalized Mittag-Leffler function (11) as follows:

Definition 9. Let $\omega, \beta, \gamma, \lambda, \theta \in \mathbb{C}$, $\mathfrak{R}(\beta), \mathfrak{R}(\gamma) > 0$, $\mathfrak{R}(\theta) > \mathfrak{R}(\lambda) > 0$ with $s \geq 0$, $\alpha, r > 0$ and $0 < k \leq r + \alpha$. Let $\phi \in L_1[u, v]$, $0 < u < v < \infty$, be a positive function. Let $g : [u, v] \longrightarrow \mathbb{R}$ be a differentiable function, strictly increasing. Also let $\zeta(\xi)/\xi$ be an increasing function on $[u, \infty)$ and $\xi \in [u, v]$. Then the left-sided integral is defined by

$$\left({}^\zeta_g F^{\omega,\lambda,\theta,k,r}_{u^+,\alpha,\beta,\gamma}\phi\right)(\xi;s) = \int_u^\xi \frac{\zeta(g(\xi)-g(t))}{g(\xi)-g(t)} E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}\left(\omega(g(\xi)-g(t))^\alpha;s\right)g'(t)\phi(t)dt. \quad (13)$$

In [6], we have presented a further generalized unified Mittag-Leffler function and the associated fractional integral operator as follows:

Definition 10. For $\underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\underline{c} = (c_1, c_2, \cdots, c_n)$, where $a_i, b_i, c_i \in \mathbb{C}$; $i = 1, \cdots, n$ such that $\Re(a_i), \Re(b_i), \Re(c_i) > 0$. Also let , $\beta, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta, t \in \mathbb{C}$, $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\lambda), \Re(\theta)\} > 0$ and $k \in (0, 1) \cup \mathbb{N}$ with $s \geq 0$. *Let* $k + \Re(\rho) < \Re(\delta + \nu + \alpha)$ with $\mathrm{Im}\,(\rho) = \mathrm{Im}\,(\delta + \nu + \alpha)$, then the unified Mittag-Leffler function is defined as follows

$$M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t\,;\underline{a},\underline{b},\underline{c},s) = \sum_{l=0}^{\infty} \frac{\Pi^n_{i=1} B_s(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\Pi^n_{i=1} B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}} \frac{t^l}{\Gamma(\alpha l + \beta)}. \quad (14)$$

Definition 11. Let $\phi \in L_1[u, v]$. Then for $\xi \in [u, v]$, the fractional integral operators corresponding to (14) are defined by

$$I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi(\xi\,;\underline{a},\underline{b},\underline{c},s) = \int_u^{\xi} (\xi - t)^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(\xi - t)^{\alpha}\,;\underline{a},\underline{b},\underline{c},s\right)\phi(t)dt, \quad (15)$$

$$I^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi(\xi\,;\underline{a},\underline{b},\underline{c},s) = \int_{\xi}^{v} (t - \xi)^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(t - \xi)^{\alpha}\,;\underline{a},\underline{b},\underline{c},s\right)\phi(t)dt. \quad (16)$$

Fractional integral operators are used to extend different types of integral inequalities such as Opial-type inequalities [2, 18–22], Hadamard- and Fejér-Hadamard-type inequalities [23–32], Pólya-Szegö-, Chebyshev-, and Grüss-type inequalities [33–36] (see references therein), and Minkowski-type fractional inequalities [7–9]. In this paper we study Minkowski-type fractional inequalities via the unified Mittag-Leffler function.

In Section 2, we give the definition of further generalized integral operator containing the unified Mittag-Leffler function. The boundedness of this integral operator is proved under the conditions stated in the definition. In Section 3, by applying a particular fractional integral operator for the power function, Minkowski-type fractional integral inequalities are established. In Section 4, reverse Minkowski-type fractional integral inequalities are presented. The connection of these inequalities with previous work is stated in the form of remarks and corollaries.

2. Generalized Version of a Unified Integral Operator

In this section, we introduce a generalized version of a unified integral operator containing a unified Mittag-Leffler function in its kernel and also discuss its boundedness.

Definition 12. Let ω, $\underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\underline{c} = (c_1, c_2, \cdots, c_n)$, where $a_i, b_i, c_i \in \mathbb{C}$; $i = 1, \cdots, n$ such that $\Re(a_i), \Re(b_i), \Re(c_i) > 0 \forall i$. Also let , $\gamma, \mu, \lambda, \theta, t \in \mathbb{C}$, $\min\{\Re(\beta), \Re(\gamma), \Re(\mu), \Re(\lambda), \Re(\theta)\} > 0$, $\rho, \delta, \nu, \alpha > 0$ and $k \in (0, 1) \cup \mathbb{N}$. Let $k + \rho < \delta + \nu + \alpha$ with $s \geq 0$. Let $\phi \in L_1[u, v]$, $0 < u < v < \infty$ be a positive function, and let $g : [u, v] \longrightarrow \mathbb{R}$ be a differentiable and strictly increasing function. Also let $\zeta(\xi)/\xi$ be an increasing function on $[u, \infty]$ for $\xi \in [u, v]$. Then the unified integral operator in its generalized form is defined by the following integral:

$$\left({}^{\zeta}_{g}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi\right)(\xi\,;s) = \int_u^{\xi} \frac{\zeta(g(\xi) - g(t))}{g(\xi) - g(t)} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(g(\xi) - g(t))^{\alpha}\,;s\right)g'(t)\phi(t)dt. \quad (17)$$

On a particular case, by taking $\zeta(\xi) = \xi^{\beta}$; $\beta > 1$ and replacing $\mathbb{C}$ by $\mathbb{R}$, the above operator takes the following form:

$$\left({}_{g}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi\right)(\xi\,;s) = \int_u^{\xi} (g(\xi) - g(t))^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(g(\xi) - g(t))^{\alpha}\,;s\right)g'(t)\phi(t)dt, \quad (18)$$

where ω, $\underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\underline{c} = (c_1, c_2, \cdots, c_n)$, $a_i, b_i, c_i \in \mathbb{R}$; $i = 1, \cdots, n$ such that $a_i, b_i, c_i > 0 \forall i$. Also $\alpha, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta > 0$, $\beta > 1$ and $k \in (0, 1) \cup \mathbb{N}$ with $k + \rho < \delta + \nu + \alpha$ and $s \geq 0$.

Definition 13. By setting $a_i = l$, $s = 0$ and $\rho > 0$ in (18), we will get the following integral operator:

$$\left({}^{g}_{Q}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi\right)(\xi\,;s) = \int_u^{\xi} (g(\xi) - g(t))^{\beta-1} Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(g(\xi) - g(t))^{\alpha}\,;s\right)g'(t)\phi(t)dt. \quad (19)$$

Remark 14.

(i) By considering $n = 1$, $b_1 = \lambda + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, $\delta > 0$ in (17), the unified integral operator given in (13) is deduced

(ii) By considering the function g to be an identity function in (19), the fractional integral operator given in (9) is deduced

(iii) By considering $n = 1$, $b_1 = \lambda + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, and $\delta > 0$ in (18), the generalized fractional integral operator ([8], Definition 1.4) is deduced

(iv) By considering $s = 0 = \omega$ in (18), then the left-sided Riemann-Liouville fractional integral operator of a function ϕ with respect to another function g of order β given in [1, 3] is deduced

(v) By considering g to be an identity function, (18) is deduced to (15)

(vi) By considering g as identity function and setting $n = 1$, $b_1 = \lambda + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, and $\delta > 0$, in (18), the generalized fractional integral operator (21) is deduced

For simplicity, we will use the following notations throughout this paper: $M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu} := \mathbf{M}$, $I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} = \mathbf{I}$, ${}_{Q}I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} = {}_{\mathbf{Q}}\mathbf{I}$, ${}^{\zeta}_{g}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} := {}^{\zeta}\mathbf{\Omega}$, ${}_{g}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} := \mathbf{\Omega}$, ${}^{g}_{Q}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} := {}_{\mathbf{Q}}\mathbf{\Omega}$.

Next, we discuss the boundedness of the newly defined generalized form of unified fractional integral operator.

Theorem 15. *Let* $\omega \in \mathbb{R}$, $\underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\underline{c} = (c_1, c_2, \cdots, c_n)$, $a_i, b_i, c_i \in \mathbb{R}$; $i = 1, \cdots, n$ *such that* $a_i, b_i, c_i > 0$. *Also* $\alpha, \beta, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta, t \in \mathbb{R}$, $k \in (0, 1) \cup \mathbb{N}$ *and* $\min\{\alpha, \beta, \gamma, \delta, \lambda, \theta\} > 0$ *with* $k + \rho < \delta + \nu + \alpha$ *with* $s \geq 0$. *Let* $\phi \in L_1[u, v]$, $0 < u < v < \infty$ *be a positive function, and let* $g : [u, v] \longrightarrow \mathbb{R}$ *be a differentiable and strictly increasing function. Also let* ζ/ξ *be an increasing function on* $[u, \infty]$. *Then for* $\xi \in [u, v]$, *we get*

$$\left({}^{\zeta}\mathbf{\Omega}\phi\right)(\xi ; s) \leq \zeta(g(\xi) - g(u))\mathbf{M}\left(\omega(g(\xi) - g(u))^{\alpha} ; s\right)\phi_{[u,\xi]}, \tag{20}$$

$$\left({}^{\zeta}\mathbf{\Omega}\phi\right)(\xi ; s) \leq \zeta(g(v) - g(\xi))\mathbf{M}\left(\omega(g(\xi) - g(u))^{\alpha} ; s\right)\phi_{[\xi,v]}, \tag{21}$$

where $\|\phi\|_{[u,\xi]} = sup_{t\in[u,\xi]}|\phi(t)|$ *and* $\|\phi\|_{[\xi,v]} = sup_{t\in[\xi,v]}|\phi(t)|$.

Proof. According to the statement, ζ/ξ is an increasing function; therefore, the following inequality prevails:

$$\frac{\zeta(g(\xi) - g(t))}{g(\xi) - g(t)} \leq \frac{\zeta(g(\xi) - g(u))}{g(\xi) - g(u)}. \tag{22}$$

Since g is differentiable and increasing and ϕ is a positive function, so the above inequality remains preserved by multiplying it with $g'(t)\phi(t)$. Therefore, we obtain the following inequality:

$$\frac{\zeta(g(\xi) - g(t))}{g(\xi) - g(t)} g'(t)\phi(t) \leq \frac{\zeta(g(\xi) - g(u))}{g(\xi) - g(u)} g'(t)\phi(t). \tag{23}$$

Multiplying (23) by $\mathbf{M}(\omega(g(\xi) - g(t))^{\alpha} ; s)$ and integrating over $[u, \xi]$ one can get

$$\left({}^{\zeta}\mathbf{\Omega}\phi\right)(\xi ; s) \leq \frac{\zeta(g(\xi) - g(u))}{g(\xi) - g(u)} \|\phi\|_{[u,\xi]} \int_u^{\xi} \mathbf{M}\left(\omega(g(\xi) - g(t))^{\alpha} ; s\right) g'(t)dt. \tag{24}$$

Solving the above definite integral, we get

$$\left({}^{\zeta}\mathbf{\Omega}\phi\right)(\xi ; s) \leq \zeta(g(\xi) - g(u))\mathbf{M}\left(\omega(g(\xi) - g(u))^{\alpha} ; s\right)\|\phi\|_{[u,\xi]}. \tag{25}$$

Similarly, one can easily prove (21). □

3. Unified Versions of Minkowski-Type Fractional Integral Inequalities

In this section, we give proof of unified versions of generalized Minkowski-type integral inequalities.

Theorem 16. *Let* $\omega \in \mathbb{R}$, $\underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\underline{c} = (c_1, c_2, \cdots, c_n)$, $a_i, b_i, c_i \in \mathbb{R}$; $i = 1, \cdots, n$ *such that* $a_i, b_i, c_i > 0$. *Also* $\alpha, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta > 0$, $\beta > 1$ *and* $k \in (0, 1) \cup \mathbb{N}$ *with* $k + \rho < \delta + \nu + \alpha$ *with* $s \geq 0$. *Let* $g : [u, v] \longrightarrow \mathbb{R}$ *be a differentiable and strictly increasing function, and let* $\phi, \psi, \zeta_1, \zeta_2$ *be* $m-$ *power integrable and positive functions on* $[u, v]$ *such that the ratio* $\phi(\xi)/\psi(\xi)$ *is bounded above by* ζ_2 *and bounded below by* $\zeta_1 \forall \xi \in [u, v]$. *Let* $m, n > 1$ *such that* $1/m + 1/n = 1$; *then*

$$[(\mathbf{\Omega}\phi)(\xi ; s)]^{1/m}[(\mathbf{\Omega}\psi)(\xi ; s)]^{1/n} \leq \left[\left(\mathbf{\Omega}\zeta_2^{1/mn}\phi^{1/m}\psi^{1/n}\right)(\xi ; s)\right]^{1/m}\left[\left(\mathbf{\Omega}\zeta_1^{-(1/mn)}\phi^{1/m}\psi^{1/n}\right)(\xi ; s)\right]^{1/n}. \tag{26}$$

Proof. According to the statement of the theorem, we have

$$0 < \zeta_1(t) \leq \frac{\phi(t)}{\psi(t)} \leq \zeta_2(t), t \in [u, v]. \tag{27}$$

By considering the lower bound, the above inequality produces

$$\psi(t) \leq \frac{1}{\zeta_1^{1/m}} \phi^{1/m}(t)\psi^{1/n}(t). \tag{28}$$

By multiplying both sides of the above inequality with $(g(\xi) - g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi) - g(t))^{\alpha} ; s)g'(t)$ and integrating on $[u, \xi]$, we get

$$[(\mathbf{\Omega}\psi)(\xi ; s)]^{1/n} \leq \left[\left(\mathbf{\Omega}\zeta_1^{-(1/mn)}\phi^{1/m}\psi^{1/n}\right)(\xi ; s)\right]^{1/n}. \tag{29}$$

Also, by considering the upper bound of inequality (27), the following inequality holds:

$$\phi(t) \leq \zeta_2^{1/n}\phi^{1/m}(t)\psi^{1/n}(t). \tag{30}$$

Multiplying both sides of the above inequality with $(g(\xi) - g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi) - g(t))^{\alpha} ; s)g'(t)$ and integrating on $[u, \xi]$, we get the following inequality:

$$[(\mathbf{\Omega}\phi)(\xi ; s)]^{1/m} \leq \left[\left(\mathbf{\Omega}\zeta_2^{1/mn}\phi^{1/m}\psi^{1/n}\right)(\xi ; s)\right]^{1/m}. \tag{31}$$

The product of (29) and (31) results in inequality (26). □

Corollary 17. *Under the assumptions of Theorem 16 along with the condition that* $\zeta_1(\xi) = k_1$ *and* $\zeta_2(\xi) = k_2$ *with* $0 < k_1 < k_2 \forall \xi \in [u, v]$, *(26) takes the following form:*

$$[(\mathbf{\Omega}\phi)(\xi ; s)]^{1/m}[(\mathbf{\Omega}\psi)(\xi ; s)]^{1/n} \leq \left(\frac{k_2}{k_1}\right)^{1/mn}\left[\left(\mathbf{\Omega}\phi^{1/m}\psi^{1/n}\right)(\xi ; s)\right]. \tag{32}$$

Corollary 18. *Under the assumptions of above theorem and substituting* $a_i = l, s = 0$ *and* $\rho > 0$ *in (26), we get the following* *inequality:*

$$\left[\left({}_{\mathrm{Q}}\boldsymbol{\Omega}\phi\right)(\xi)\right]^{1/m}\left[\left({}_{\mathrm{Q}}\boldsymbol{\Omega}\psi\right)(\xi)\right]^{1/n} \le \left[\left({}_{\mathrm{Q}}\boldsymbol{\Omega}\zeta_2^{1/mn}\phi^{1/m}\psi^{1/n}\right)(\xi)\right]^{1/m}\left[\left({}_{\mathrm{Q}}\boldsymbol{\Omega}\zeta_1^{-(1/mn)}\phi^{1/m}\psi^{1/n}\right)(\xi)\right]^{1/n}. \tag{33}$$

Remark 19.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (32), the inequality given in [8], Theorem 5, is deduced

(ii) By setting g, the identity function in (32) the inequality [7], Theorem 4, is deduced

$$[(\mathbf{I}\phi)(\xi;s)]^{1/m}[(\mathbf{I}\psi)(\xi;s)]^{1/n} \le \left(\frac{k_2}{k_1}\right)^{1/mn}\left[\left(\mathbf{I}\phi^{1/m}\psi^{1/n}\right)(\xi;s)\right] \tag{34}$$

(iii) Under the assumptions of the Corollary 21 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2 \forall \xi\in[u,v]$ and setting the function g to be an identity function, the inequality given in [7], Corollary 1, is deduced

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (34), the Minkowski-type inequality (6) is deduced

In the proof of our next result, we will use Young's inequality for $x,y\ge 0$ with $m,n>1$ satisfying $m^{-1}+n^{-1}=1$:

$$xy \le m^{-1}x^m + n^{-1}y^n. \tag{35}$$

Also, the following inequality will be required:

$$(x+y)^m \le 2^{m-1}(x^m+y^m)\,;\, x,y\ge 0 \text{ and } m>1. \tag{36}$$

Theorem 20. *Under the assumptions of Theorem 16 the following inequality holds:*

$$(\boldsymbol{\Omega}(\phi\psi))(\xi;s) \le m^{-1}2^{m-1}\left(\boldsymbol{\Omega}\left(\frac{\zeta_2}{\zeta_2+1}\right)^m(\phi^m+\psi^m)\right)(\xi;s) + n^{-1}2^{n-1}\left(\boldsymbol{\Omega}\left(\frac{1}{\zeta_1+1}\right)^n(\phi^n+\psi^n)\right)(\xi;s). \tag{37}$$

Proof. Taking the left side of inequality (27), we obtain the following form:

$$\psi^n(t) \le (\zeta_1+1)^{-n}(\phi(t)+\psi(t))^n. \tag{38}$$

Multiplying both sides of inequality by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ and integrating over $[u,\xi]$, the above inequality gives

$$n^{-1}(\boldsymbol{\Omega}\psi^n)(\xi;s) \le n^{-1}\left(\boldsymbol{\Omega}(\zeta_1+1)^{-n}(\phi+\psi)^n\right)(\xi;s). \tag{39}$$

Also, by considering right side of inequality (27), we have the following inequality:

$$\phi^m(t) \le \left(\frac{\zeta_2}{\zeta_2+1}\right)^m(\psi(t)+\phi(t))^m. \tag{40}$$

Multiplying both sides of the above inequality by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$, integrating over $[u,\xi]$ and multiplying resulting inequality by m^{-1}, we get

$$m^{-1}(\boldsymbol{\Omega}\phi^m)(\xi;s) \le m^{-1}\left(\boldsymbol{\Omega}\left(\frac{\zeta_2}{\zeta_2+1}\right)^m(\psi+\phi)^m\right)(\xi;s). \tag{41}$$

By Young's inequality, we have

$$\phi(t)\psi(t) \le m^{-1}\phi^m(t) + n^{-1}\psi^n(t). \tag{42}$$

Multiplying both sides of the above inequality by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ and integrating over $[u,\xi]$, the above inequality takes the form

$$(\boldsymbol{\Omega}(\phi\psi))(\xi;s) \le m^{-1}(\boldsymbol{\Omega}\phi^m)(\xi;s) + n^{-1}(\boldsymbol{\Omega}\psi^n)(\xi;s). \tag{43}$$

Applying (43) to the sum of (39) and (41), we get the following inequality:

$$(\boldsymbol{\Omega}(\phi\psi))(\xi;s) \le m^{-1}\left(\boldsymbol{\Omega}\left(\frac{\zeta_2}{\zeta_2+1}\right)^m(\phi+\psi)^m\right)(\xi;s) + n^{-1}\left(\boldsymbol{\Omega}\left(\frac{1}{\zeta_1+1}\right)^n(\phi+\psi)^n\right)(\xi;s). \tag{44}$$

Inequality (37) follows by using (36) in (44). □

Corollary 21. *Under the assumptions of Theorem 20 together with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$, (37) becomes*

$$(\boldsymbol{\Omega}(\phi\psi))(\xi;s) \le m^{-1}2^{m-1}\left(\frac{k_2}{k_2+1}\right)^m(\boldsymbol{\Omega}(\phi^m+\psi^m))(\xi;s) + n^{-1}2^{n-1}\left(\frac{1}{k_1+1}\right)^n(\boldsymbol{\Omega}(\phi^n+\psi^n))(\xi;s). \tag{45}$$

Corollary 22. *Under the assumptions of the above theorem and setting $a_i=l, s=0$ and $\rho>0$ in (37), the following* *inequality holds true:*

$$\left({}_{Q}\boldsymbol{\Omega}(\phi\psi)\right)(\xi) \leq m^{-1}2^{m-1}\left({}_{Q}\boldsymbol{\Omega}\left(\frac{\zeta_2}{\zeta_2+1}\right)^m(\phi^m+\psi^m)\right)(\xi) + n^{-1}2^{n-1}\left({}_{Q}\boldsymbol{\Omega}\left(\frac{1}{\zeta_1+1}\right)^n(\phi^n+\psi^n)\right)(\xi). \tag{46}$$

Remark 23.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (45), the inequality given in [8], Theorem 6, is deduced

(ii) By setting g the identity function, (45) is deduced to the following inequality given in [7]:

$$(\mathbf{I}(\phi\psi))(\xi\,;s) \leq m^{-1}2^{m-1}\left(\frac{k_2}{k_2+1}\right)^m(\mathbf{I}(\phi^m+\psi^m))(\xi\,;s) + n^{-1}2^{n-1}\left(\frac{1}{k_1+1}\right)^n(\mathbf{I}(\phi^n+\psi^n))(\xi\,;s) \tag{47}$$

(iii) Under the assumptions of Corollary 25 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$ and setting the function g the identity function, the inequality given in [7], Corollary 2, is deduced

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (47), the Minkowski-type inequality given in [9], Theorem 3.2, is deduced

Theorem 24. *Suppose the assumptions of Theorem 16 hold; then for $m\geq 1$, the following inequalities hold:*

$$\left(\boldsymbol{\Omega}\left(\frac{1}{\zeta_2}(\phi\psi)\right)\right)(\xi\,;s) \leq \left(\boldsymbol{\Omega}\left(\frac{1}{(\zeta_1+1)(\zeta_2+1)}\right)(\phi+\psi)^2\right)(\xi\,;s) \leq \left(\boldsymbol{\Omega}\left(\frac{1}{\zeta_1}(\phi\psi)\right)\right)(\xi\,;s). \tag{48}$$

Proof. Considering right side of inequality (27), we get the following inequalities:

$$\phi(t)+\psi(t) \leq (\zeta_2(t)+1)\psi(t), \tag{49}$$

$$\zeta_2^{-1}(t)(\zeta_2(t)+1)\phi(t) \leq \phi(t)+\psi(t). \tag{50}$$

Also, from the left side of inequality (27), we have the following inequalities:

$$\phi(t)+\psi(t) \geq (\zeta_1(t)+1)\psi(t), \tag{51}$$

$$\zeta_1^{-1}(t)(\zeta_1(t)+1)\phi(t) \geq \phi(t)+\psi(t). \tag{52}$$

Combining the inequalities (49) and (51), the following inequality holds

$$(\zeta_1(t)+1)\psi(t) \leq \phi(t)+\psi(t) \leq (\zeta_2(t)+1)\psi(t). \tag{53}$$

By the combining the inequalities (50) and (52), we get

$$\zeta_2^{-1}(t)(\zeta_2(t)+1)\phi(t) \leq \phi(t)+\psi(t) \leq \zeta_1^{-1}(t)(\zeta_1(t)+1)\phi(t). \tag{54}$$

The product of the above two inequalities yields

$$\zeta_2^{-1}(t)(\phi(t)\psi(t)) \leq \left(\frac{1}{(\zeta_1(t)+1)(\zeta_2(t)+1)}\right)(\phi(t)+\psi(t))^2 \leq \zeta_1^{-1}(t)(\phi(t)\psi(t)). \tag{55}$$

Now, multiplying $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha\,;s)g'(t)$ with the above inequality and integrating over $[u,\xi]$, we get the required inequality (48). □

Corollary 25. *Under the assumptions of Theorem 24 and taking $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$, (48) takes the following form:*

$$\frac{1}{k_2}(\boldsymbol{\Omega}(\phi\psi))(\xi\,;s) \leq \frac{1}{(k_1+1)(k_2+1)}\left(\boldsymbol{\Omega}(\phi+\psi)^2\right)(\xi\,;s) \leq \frac{1}{k_1}(\boldsymbol{\Omega}(\phi\psi))(\xi\,;s). \tag{56}$$

Corollary 26. *Under the assumptions of the above theorem and considering $a_i=l, s=0$ and $\rho>0$ in (48), the following inequality holds:*

$$\left({}_{Q}\boldsymbol{\Omega}\left(\frac{1}{\zeta_2}(\phi\psi)\right)\right)(\xi) \leq \left({}_{Q}\boldsymbol{\Omega}\left(\frac{1}{(\zeta_1+1)(\zeta_2+1)}\right)(\phi+\psi)^2\right)(\xi) \leq \left({}_{Q}\boldsymbol{\Omega}\left(\frac{1}{\zeta_1}(\phi\psi)\right)\right)(\xi). \tag{57}$$

Remark 27.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (56), the inequality given in [8], Theorem 16 is deduced

(ii) By setting g the identity function, (56) gives the following inequality [7]:

$$\frac{1}{k_2}(\mathbf{I}(\phi\psi))(\xi;s) \leq \frac{1}{(k_1+1)(k_2+1)}\left(\mathbf{I}(\phi+\psi)^2\right)(\xi;s) \leq \frac{1}{k_1}(\mathbf{I}(\phi\psi))(\xi;s) \tag{58}$$

(iii) Under the assumptions of Corollary 29 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2 \forall \xi \in [u,v]$ and setting the function g to be the identity function, the inequality given in [7], Corollary 3, is deduced

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (58), the Minkowski-type inequality given in [9], Theorem 3.3, is deduced

Theorem 28. *Let the assumptions of Theorem 16 hold true. Also, let $\phi, \psi, \zeta_1, \zeta_2, f$ be $m-$power integrable and positive functions on $[u,v]$ such that $0<f(t)<\zeta_1(t)\leq \phi(t)/\psi(t)\leq \zeta_2(t) \forall t \in [u,v]$; then the following inequalities hold for $m\geq 1$:*

$$\begin{aligned}&\left[\left(\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_2-f}\right)^m\right)(\xi;s)\right]^{1/m}+\left[\left(\mathbf{\Omega}\left(\frac{\zeta_2(\phi-f\psi)}{\zeta_2-f}\right)^m\right)(\xi;s)\right]^{1/m}\\&\leq [(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}+[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}\\&\leq \left[\left(\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_1-f}\right)^m\right)(\xi;s)\right]^{1/m}+\left[\left(\mathbf{\Omega}\left(\frac{\zeta_1(\phi-f\psi)}{\zeta_1-f}\right)^m\right)(\xi;s)\right]^{1/m}.\end{aligned} \tag{59}$$

Proof. By the assumption of the theorem, we have

$$0<f(t)<\zeta_1(t)\leq \frac{\phi(t)}{\psi(t)}\leq \zeta_2(t), t\in[u,v]. \tag{60}$$

The above inequality can be arranged as follows:

$$\zeta_1(t)-f(t)\leq \frac{\phi(t)-f(t)\psi(t)}{\psi(t)}\leq \zeta_2(t)-f(t). \tag{61}$$

From which we can write

$$\frac{(\phi(t)-f(t)\psi(t))^m}{(\zeta_2(t)-f(t))^m}\leq \psi^m(t)\leq \frac{(\phi(t)-f(t)\psi(t))^m}{(\zeta_1(t)-f(t))^m}. \tag{62}$$

The following inequality follows by multiplying $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^{\alpha};s)g'(t)$ throughout the above inequality and integrating over $[u,\xi]$:

$$\left[\left(\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_2-f}\right)^m\right)(\xi;s)\right]^{1/m}\leq [(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}\leq \left[\left(\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_1-f}\right)^m\right)(\xi;s)\right]^{1/m}. \tag{63}$$

Also, from (60), one can have

$$\frac{\zeta_1(t)-f(t)}{\zeta_1(t)}\leq \frac{\phi(t)-f(t)\psi(t)}{\phi(t)}\leq \frac{\zeta_2(t)-f(t)}{\zeta_2(t)}, \tag{64}$$

which can also be written as

$$\frac{\zeta_2(t)(\phi(t)-f(t)\psi(t))}{\zeta_2(t)-f(t)}\leq \phi(t)\leq \frac{\zeta_1(t)(\phi(t)-f(t)\psi(t))}{\zeta_1(t)-f(t)}. \tag{65}$$

Taking the power m, after multiplying by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^{\alpha};s)g'(t)$ throughout the above inequality and integrating over $[u,\xi]$, one can get the following inequality:

$$\begin{aligned}\left[\left(\mathbf{\Omega}\left(\frac{\zeta_2(\phi-f\psi)}{\zeta_2-f}\right)^m\right)(\xi;s)\right]^{1/m}&\leq [(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}\\&\leq \left[\left(\mathbf{\Omega}\left(\frac{\zeta_1(\phi-f\psi)}{\zeta_1-f}\right)^m\right)(\xi;s)\right]^{1/m}.\end{aligned} \tag{66}$$

The sum of (63) and (66) produces the required inequality (59). □

Corollary 29. *Under the assumptions of Theorem 28 along with the condition that $f(\xi)=m, \zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$ in (59), the following inequality holds:*

$$\begin{aligned}&\frac{k_2+1}{k_2-m}[(\mathbf{\Omega}(\phi-m\psi)^m)(\xi;s)]^{1/m}\\&\leq [(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}+[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}\\&\leq \frac{k_1+1}{k_1-m}[(\mathbf{\Omega}(\phi-m\psi)^m)(\xi;s)]^{1/m}.\end{aligned} \tag{67}$$

Corollary 30. *Under the assumptions of the above theorem with the condition that $a_i=l, s=0$ and $\rho>0$ in (59), the following inequality holds:*

$$\begin{aligned}&\left[\left({}_{Q}\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_2-f}\right)^m\right)(\xi)\right]^{1/m}+\left[\left({}_{Q}\mathbf{\Omega}\left(\frac{\zeta_2(\phi-f\psi)}{\zeta_2-f}\right)^m\right)(\xi)\right]^{1/m}\\&\leq \left[({}_{Q}\mathbf{\Omega}\phi^m)(\xi;s)\right]^{1/m}+\left[({}_{Q}\mathbf{\Omega}\psi^m)(\xi)\right]^{1/m}\\&\leq \left[\left(\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_1-f}\right)^m\right)(\xi)\right]^{1/m}+\left[\left(\mathbf{\Omega}\left(\frac{\zeta_1(\phi-f\psi)}{\zeta_1-f}\right)^m\right)(\xi)\right]^{1/m}.\end{aligned} \tag{68}$$

Remark 31.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (67), the inequality given in [8], Theorem 8, is deduced

(ii) By setting g the identity function, (67) is deduced to the following inequality [7]:

$$\frac{k_2+1}{k_2-m}[(\mathbf{I}(\phi-m\psi)^m)(\xi;s)]^{1/m}\le[(\mathbf{I}\phi^m)(\xi;s)]^{1/m}+[(\mathbf{I}\psi^m)(\xi;s)]^{1/m}\le\frac{k_1+1}{k_1-m}[(\mathbf{I}(\phi-m\psi)^m)(\xi;s)]^{1/m} \tag{69}$$

(iii) Under the assumptions of Corollary 33 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$ and setting the function g to be the identity function, the inequality given in [7], Corollary 4, is deduced

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (69), the Minkowski-type inequality given [9], Theorem 3.4, is deduced

4. Reverse Minkowski-Type Fractional Integral Inequalities

In this section, we state and prove some reverse versions of Minkowski-type inequalities that are the generalizations of (2), (3), and (5).

Theorem 32. *Under the assumptions of Theorem 16, the following inequality holds for $m\ge 1$:*

$$[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}+[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}\le\left[\left(\mathbf{\Omega}\left(\left(\frac{\zeta_2}{1+\zeta_2}\right)^m(\phi+\psi)^m\right)\right)(\xi;s)\right]^{1/m}+\left[\left(\mathbf{\Omega}\left(\left(\frac{1}{1+\zeta_1}\right)^m(\phi+\psi)^m\right)\right)(\xi;s)\right]^{1/m}. \tag{70}$$

Proof. From (27), one can obtain the following inequality:

$$\psi^m(t)\le\frac{1}{(1+\zeta_1(t))^m}(\phi(t)+\psi(t))^m. \tag{71}$$

Multiplying both sides of inequality by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ and integrating on $[u,\xi]$, the above inequality can take the form as follows:

$$[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}\le\left[\left(\mathbf{\Omega}\left(\frac{1}{(1+\zeta_1(t))^m}(\psi+\phi)^m\right)\right)(\xi;s)\right]^{1/m}. \tag{72}$$

Also, by considering inequality (27), one can have the following inequality:

$$\phi^m(t)\le\left(\frac{\zeta_2}{1+\zeta_2(t)}\right)^m(\psi(t)+\phi(t))^m. \tag{73}$$

Multiplying both sides of the above inequality by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ and integrating over $[u,\xi]$, we can get

$$[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}\le\left[\left(\mathbf{\Omega}\left(\left(\frac{\zeta_2}{1+\zeta_2(t)}\right)^m(\psi+\phi)^m\right)\right)(\xi;s)\right]^{1/m}. \tag{74}$$

Adding (72) and (74), inequality (70) can be obtained. □

Corollary 33. *Under the assumptions of Theorem 32 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$, (70) takes the following form:*

$$[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}+[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}\le\left(1+\frac{k_2-k_1}{(k_1+1)(k_2+1)}\right)[(\mathbf{\Omega}(\phi+\psi)^m)(\xi;s)]^{1/m}. \tag{75}$$

Corollary 34. *Under the assumptions of the above theorem and taking $a_i=l, s=0$, and $\rho>0$ in (70), the following inequality is obtained:*

$$\left[\left({}_Q\mathbf{\Omega}\phi^m\right)(\xi)\right]^{1/m}+\left[\left({}_Q\mathbf{\Omega}\psi^m\right)(\xi)\right]^{1/m}\le\left[\left({}_Q\mathbf{\Omega}\left(\left(\frac{\zeta_2}{1+\zeta_2}\right)^m(\phi+\psi)^m\right)\right)(\xi)\right]^{1/m}+\left[\left({}_Q\mathbf{\Omega}\left(\left(\frac{1}{1+\zeta_1}\right)^m(\phi+\psi)^m\right)\right)(\xi)\right]^{1/m}. \tag{76}$$

Remark 35.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (70), the inequality introduced by Andric et. al [8] (Theorem 3) is generated

(ii) Taking $g:[u,v]\longrightarrow\mathbb{R}$ to be an identity function, (75) gives the following inequality [7]:

$$[(\mathbf{I}\phi^m)(\xi;s)]^{1/m}+[(\mathbf{I}\psi^m)(\xi;s)]^{1/m}\le\left(1+\frac{k_2-k_1}{(k_1+1)(k_2+1)}\right)[(\mathbf{I}(\phi+\psi)^m)(\xi;s)]^{1/m} \tag{77}$$

(iii) Under the assumptions of Corollary 37 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$ and setting the function g to be an identity function, we obtain the inequality as in [7], Corollary 5

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (77), Minkowski-type inequality (5) is deduced

Theorem 36. *Under the assumptions of Theorem 16, for $m \geq 1$, we have*

$$[(\mathbf{\Omega}\phi^m)(\xi;s)]^{2/m} + [(\mathbf{\Omega}\psi^m)(\xi;s)]^{2/m} \geq \left[\left(\mathbf{\Omega}\left(\frac{1+\zeta_2}{\zeta_2}\right)^m \phi^m\right)(\xi;s)\right]^{1/m} \left[(\mathbf{\Omega}(1+\zeta_1)^m\psi^m)(\xi;s)\right]^{1/m} - 2[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}. \quad (78)$$

Proof. Inequalities (71) and (73) from the previous theorem can be arranged in the following forms:

$$(1+\zeta_1(t))^m\psi^m(t) \leq (\phi(t)+\psi(t))^m, \quad \left(\frac{1+\zeta_2(t)}{\zeta_2(t)}\right)^m \phi^m(t) \leq (\psi(t)+\phi(t))^m. \quad (79)$$

By multiplying with $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha; s)g'(t)$ and integrating over $[u, \xi]$ and taking the power $1/m$ of the resulting inequalities, the above inequalities further take the following forms:

$$\left[(\mathbf{\Omega}(1+\zeta_1)^m\psi^m)(\xi;s)\right]^{1/m} \leq [(\mathbf{\Omega}(\phi+\psi)^m)(\xi;s)]^{1/m}, \quad (80)$$

$$\left[\left(\mathbf{\Omega}\left(\frac{1+\zeta_2}{\zeta_2}\right)^m \phi^m\right)(\xi;s)\right]^{1/m} \leq [(\mathbf{\Omega}(\psi+\phi)^m)(\xi;s)]^{1/m}. \quad (81)$$

By multiplying (80) and (81), we get the following inequality:

$$\left[(\mathbf{\Omega}(1+\zeta_1)^m\psi^m)(\xi;s)\right]^{1/m}\left[\left(\mathbf{\Omega}\left(\frac{1+\zeta_2}{\zeta_2}\right)^m \phi^m\right)(\xi;s)\right]^{1/m} \leq \left[((\mathbf{\Omega}(\psi+\phi)^m)(\xi;s))^{1/m}\right]^2. \quad (82)$$

Applying Minkowski's inequality on the term within the square brackets at the right side of the above inequality and then using $(a+b)^2 = a^2 + 2ab + b^2$, the above inequality gives the required inequality (78). □

Corollary 37. *Under the assumptions of Theorem 36 along with the condition that $\zeta_1(\xi) = k_1$ and $\zeta_2(\xi) = k_2$ with $0 < k_1 < k_2 \forall \xi \in [u, v]$, (78) takes the following form:*

$$[(\mathbf{\Omega}\phi^m)(\xi;s)]^{2/m} + [(\mathbf{\Omega}\psi^m)(\xi;s)]^{2/m} \geq \left(\frac{2+(k_1-1)(k_2+1)}{k_2}\right)[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}. \quad (83)$$

Corollary 38. *Under the assumptions of above theorem together with the condition $a_i = l$, $s = 0$, and $\rho > 0$, (78) results in the following inequality:*

$$\left[({}_Q\mathbf{\Omega}\phi^m)(\xi)\right]^{2/m} + \left[({}_Q\mathbf{\Omega}\psi^m)(\xi)\right]^{2/m} \geq \left[\left({}_Q\mathbf{\Omega}\left(\frac{1+\zeta ss_2}{\zeta_2}\right)^m \phi^m\right)(\xi)\right]^{1/m}\left[({}_Q\mathbf{\Omega}(1+\zeta_1)^m\psi^m)(\xi)\right]^{1/m} - 2\left[({}_Q\mathbf{\Omega}\phi^m)(\xi)\right]^{1/m}\left[({}_Q\mathbf{\Omega}\psi^m)(\xi)\right]^{1/m}. \quad (84)$$

Remark 39.

(i) By setting $n = 1$, $b_1 = \lambda + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, and $\delta > 0$ in (78), the inequality given in [8], Theorem 4, is deduced

(ii) By setting g the identity function in (83), the following inequality is deduced [7]:

$$[(\mathbf{I}\phi^m)(\xi;s)]^{2/m} + [(\mathbf{I}\psi^m)(\xi;s)]^{2/m} \geq \left(\frac{2+(k_1-1)(k_2+1)}{k_2}\right)[(\mathbf{I}\phi^m)(\xi;s)]^{1/m}[(\mathbf{I}\psi^m)(\xi;s)]^{1/m} \quad (85)$$

(iii) Under the assumptions of Corollary 13 along with the condition that $\zeta_1(\xi) = k_1$ and $\zeta_2(\xi) = k_2$ with $0 < k_1 < k_2 \forall \xi \in [u, v]$ and setting g the identity function, the inequality given in [7], Corollary 6, is deduced

(iv) By setting $n = 1$, $b_1 = \lambda + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, and $\delta > 0$ in (85), the Minkowski-type inequality given in [9], Theorem 2.2, is deduced

Remark 40. All results of this paper hold for the right-sided integral operator:

$$\left({}_g\Omega^{\omega,\lambda,\rho,\theta,k,n}_{\nu^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi\right)(\xi;s) = \int_\xi^\nu (g(t)-g(\xi))^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,u,\nu}\left(\omega(g(t)-g(\xi))^\alpha; s\right)g'(t)\phi(t)dt. \quad (86)$$

5. Conclusion

A generalized integral operator with the help of a unified Mittag-Leffler function is defined, and its boundedness is proved. By giving specific values to parameters and considering suitable functions involved in the kernel of this operator, various kinds of well-known integral and fractional integral operators can be reproduced. For a fractional integral operator, we have constructed several Minkowski- and reverse Minkowski-type inequalities. The particular cases of the results of this paper are connected with many already published results.

Data Availability

All data required for this research is included within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204, Elsevier, Amsterdam, Netherlands, 2006.

[2] M. Andrić, G. Farid, and J. Pečarić, "A further extension of Mittag-Leffler function," *Fractional Calculus and Applied Analysis*, vol. 21, no. 5, pp. 1377–1395, 2018.

[3] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, 1993.

[4] T. O. Salim and A. W. Faraj, "A generalization of Mittag-Leffler function and integral operator associated with fractional calculus," *Journal of Fractional Calculus and Applications*, vol. 3, no. 5, pp. 1–13, 2012.

[5] G. Farid, "A unified integral operator and further its consequences," *Open Journal of Mathematical Analysis*, vol. 4, no. 1, pp. 1–7, 2020.

[6] Y. Zhang, G. Farid, Z. Salleh, and A. Ahmad, "On a unified Mittag-Leffler function and associated fractional integral operator," *Mathematical Problems in Engineering*, vol. 2021, Article ID 6043769, 9 pages, 2021.

[7] S.-S. Zho, G. Farid, and A. Ahmad, "Fractional versions of Minkowski-type fractional integral inequalities via unified Mittag-Leffler function," *Advances in Continuous and Discrete Models*, vol. 2022, no. 1, pp. 1–10, 2022.

[8] M. Andric, G. Farid, J. Pecaric, and M. U. Siddique, "Further generalizations of Minkowski-type inequalities with extended Mittag-Leffler function," *Matematichki Bilten*, vol. 44, no. (-LXX) 2, pp. 107–117, 2020.

[9] M. Andric, G. Farid, J. Pecaric, and M. U. Siddique, "Generalized Minkowski-type fractional inequalities involving extended Mittag-Leffler function," *The Journal of the Indian Mathematical Society*, vol. 87, no. 3-4, pp. 137–147, 2020.

[10] L. Bougoffa, "On Minkowski and hardy integral inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 2, p. 60, 2006.

[11] E. Set, M. Özdemir, and S. S. Dragomir, "On the Hermite-Hadamard inequality and other integral inequalities involving two functions," *Journal of Inequalities and Applications*, vol. 2010, no. 1, Article ID 148102, 9 pages, 2010.

[12] G. M. Mittag-Leffler, "Sur la Nouvelle Fonction $E_a(x)$," *C. R. Acad. Sci. Paris*, vol. 137, no. 2, pp. 554–558, 1903.

[13] A. H. Ansari, X. Liu, and V. N. Mishra, "On Mittag-Leffler function and beyond," *Nonlinear Science Letters A*, vol. 8, no. 2, pp. 187–199, 2017.

[14] A. K. Shukla and J. C. Prajapati, "On a generalization of Mittag-Leffler function and its properties," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 797–811, 2007.

[15] G. Rahman, D. Baleanu, M. A. Qurashi, S. D. Purohit, S. Mubeen, and M. Arshad, "The extended Mittag-Leffler function via fractional calculus," *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 8, pp. 4244–4253, 2017.

[16] T. R. Prabhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernel," *Yokohama Mathematical Journal*, vol. 19, pp. 7–15, 1971.

[17] D. Bhatnagar and R. M. Pandey, "A study of some integral transforms on Q function," *South East Asian Journal of Mathematics and Mathematical Sciences*, vol. 16, no. 1, pp. 99–110, 2020.

[18] A. U. Rehman, G. Farid, and Y. Mehboob, "Mean value theorems associated to the differences of Opial-type and their fractional versions," *Fractional Differential Calculus*, vol. 10, no. 2, pp. 224–231, 2012.

[19] G. Farid, A. U. Rehman, S. Ullah, A. Nosheen, M. Waseem, and Y. Mehboob, "Opial-type inequalities for convex functions and associated results in fractional calculus," *Advances in Difference Equations*, vol. 2019, Article ID 152, 2019.

[20] M. Andric, J. Pecaric, and I. Peric, "A multiple Opial type inequality for the Riemann-Liouville fractional derivatives," *Journal of Mathematical Inequalities*, vol. 7, no. 1, pp. 139–150, 2013.

[21] M. Andric, J. Pecaric, and I. Peric, "Composition identities for the Caputo fractional derivatives and applications to Opial-type inequalities," *Mathematical Inequalities & Applications*, vol. 16, no. 3, pp. 657–670, 2013.

[22] Z. Tomovski, J. Pecaric, and G. Farid, "Weighted Opial inequalities for fractional differential and integral operators involving generalized Mittag-Leffler function," *European Journal of Pure and Applied Mathematics*, vol. 10, no. 3, pp. 419–439, 2017.

[23] B. Ahmed, A. Alsaedi, M. Kirane, and B. T. Torebek, "Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals," *Journal of Computational and Applied Mathematics*, vol. 353, pp. 120–129, 2019.

[24] E. Set and B. Celik, "Hermite-Hadamard-Fejér type inequalities for quasi-geometrically convex functions via fractional integrals," *Journal of Mathematics*, vol. 2016, Article ID 6523041, 7 pages, 2016.

[25] G. Farid, V. N. Mishra, and S. Mehmood, "Hadamard and Féjer-Hadamard type inequalities for convex and relative convex functios via an extended generalized Mittag-Leffler function," *International Journal of Analysis and Applications*, vol. 17, no. 5, pp. 892–903, 2019.

[26] G. Farid, A. U. Rehman, and Q. U. Ain, "k-fractional integral inequalities of Hadamard type for $(h-m)$-convex functions," *Computational Methods for Differential Equations*, vol. 2020, no. 1, pp. 119–140, 2020.

[27] L. N. Mishra, Q. U. Ain, G. Farid, and A. U. Rehman, "k-fractional integral inequalities for $(h-m)$-convex functions via Caputo k-fractional derivatives," *The Korean Journal of Mathematics*, vol. 27, no. 2, pp. 357–374, 2019.

[28] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

[29] M. Z. Sarikaya and H. Yildirim, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals," *Miskolc Mathematical Notes*, vol. 17, no. 2, pp. 1049–1059, 2017.

[30] S. Mehmood and G. Farid, "Fractional integrals inequalities for exponentially m-convex functions," *Open Journal of Mathematical Sciences*, vol. 4, no. 1, pp. 78–85, 2020.

[31] W. Sudsutad, S. K. Ntouyas, and J. Tariboon, "Fractional integral inequalities via Hadamards fractional integral," *Abstract and Applied Analysis*, vol. 2014, Article ID 563096, 11 pages, 2014.

[32] S. I. Butt, M. Nadeem, and G. Farid, "On Caputo fractional derivatives via exponential s-convex functions," *Turkish Journal of Science*, vol. 5, no. 2, pp. 140–146, 2020.

[33] G. Farid, A. U. Rehman, V. N. Mishra, and S. Mehmood, "Fractional integral inequalities of Grüss type via generalized Mittag-Leffler function," *International Journal of Analysis and Applications*, vol. 17, no. 4, pp. 548–558, 2019.

[34] M. Andri, G. Farid, and S. Mehmood, "Pólya-Szegö and Chebshev types inequalities via an extended generalized Mittag-Leffler function," *Mathematical Inequalities & Applications*, vol. 22, no. 4, pp. 1365–1377, 2019.

[35] A. O. Akdemir, S. I. Butt, M. Nadeem, and M. A. Ragusa, "New general variants of Chebyshev type inequalities via generalized fractional integral operators," *Mathematics*, vol. 9, no. 2, p. 122, 2021.

[36] S. K. Ntouyas, P. Agarwal, and J. Tariboon, "On Pólya-Szegö and Chebyshev types inequalities involving the Riemann-Liouville fractional integral operators," *Journal of Mathematical Inequalities*, vol. 10, no. 2, pp. 491–504, 2016.

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Research Article

On Fixed-Point Results of Generalized Contractions

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The purpose of this research article is to introduce new contractive conditions and to examine the existence and uniqueness of fixed points of self-mappings in the context of b-metric spaces by applying these different contractive conditions. Furthermore, some examples are given to illustrate its validity and superiority. Our results generalize and extend several well-known results in metric and b-metric spaces.

1. Introduction and Preliminaries

Many issues in engineering and science explained by nonlinear equations may be tackled by confining them to analogous fixed-point case. An operator sum $Gx = 0$ can be proved as fixed-point sum $Fx = x$, wherein F is a self-defining along with some relevant discipline. Fixed-point theory endows with some key modes to resolving problems ensuing from multiple offshoots of mathematical inspection such as split feasibility issues, supportive problems, equilibrium problems, and matching, as well as selective issues and such others. The theory of fixed points is the great vibrant and energetic zone of the study. This theory has previously been exposed as an excessive and major deployment for cramming nonlinear analysis. Specially, fixed-point procedures are being applied in a diversity of fields, for example, biology, chemistry, engineering, economics, physics, and game theory. Functional analysis is a very useful and important field of mathematics. Its results are supportive tools for other fields to solve many problems. Many researchers have put their efforts in obtaining these results; for further study, see [1–12] and literature. Because of Banach [4], the Banach-contraction theorem (1922) is indeed the most significant consequence in the theory of fixed points in metric spaces. This theorem promises the presence and distinctiveness of fixed points of self-mapping that satisfy the contraction condition on complete metric spaces as well as dispense a valuable approach for finding them. Since contractive condition deduces the uniform continuity of an operator f, so it was a natural question to raise the concern about existence of fixed point in the absence of continuity of f. In 1968, Kannan [12] answered this question by the introduction of Kannan contractive condition. One of the most famous generalizations of metric spaces was given by Bakhtin [3] in 1989. He presented the notion of b-metric space in which triangular inequality has been relaxed. In 1993, Czerwik [7] drew out the results in b-metric space. By accepting this idea, many researchers gave extensions of Banach's principle in b-metric space. Boriceanu [5], Czerwik [7], Bota [13], and Pacurar [13] drew out the fixed-point theorems in b-metric space. Moreover, many authors examined and derived the existence of fixed point of a contraction function in the context of b-metric spaces; for detail, see [12, 14–17] and references therein. In this paper, we have established fixed-point results in the context of b-metric spaces for two different contractive conditions. Some direct consequences from our main results are also presented. In the support of these results, examples are created.

Definition 1 (see [4]). Consider a metric space Ω with metric d. A function $G: \Omega \longrightarrow \Omega$ is known as Banach contraction on G if there exists a number $\alpha \in [0, 1)$ such that $\forall$, $a, b \in \Omega$:

$$d(Ga, Gb) \leq \alpha d(a, b). \quad (1)$$

Definition 2 (see [9]). Consider (Ω, d) a metric space and G: $\Omega \longrightarrow \Omega$ is a function if $\exists\ \alpha \in (0, 1/2)$ such that, for all $a_1, a_2 \in \Omega$, we have

$$d(Ga_1, Ga_2) \leq \alpha\{d(a_1, Ga_1) + d(a_2, Ga_2)\}. \quad (2)$$

Then, G is known as Kannan contraction.

Definition 3 (see [3]). Let Ω be any nonempty set and $w \geq 1$ be a real number. A function d: $\Omega \times \Omega \longrightarrow R^+$ is called b-metric if axioms given below are fulfilled for all $\mu,\ \nu,\ \xi \in \Omega$:

$d(\mu, \nu) \geq 0$ and $d(\nu, \mu) = 0$ iff $\mu = \nu$

$d(\mu, \nu) \geq 0$ and $d(\nu, \mu) = 0$ iff $\mu = \nu$

$d(\mu, \xi) \leq w[d(\mu, \nu) + d(\nu, \xi)]$

Then, (Ω, d) is called b-metric space.

If we take $w = 1$, then b-metric space becomes ordinary metric space. Hence, set of all metric spaces is a subset of set of all b-metric spaces.

Example 1. Let $\Omega = R$; then, $d(\omega_1, \omega_2) = (\omega_1 - \omega_2)^2$ is a b-metric space with $s = 2$.

(1) Obviously, $d(\omega_1,\ \omega_2) =\ (\omega_1 - \omega_2)^2$ is real, finite, and nonnegative.

(2) Consider $d(\omega_1, \omega_2) = 0 \Leftrightarrow (\omega_1 - \omega_2)^2 = 0 \Leftrightarrow (\omega_1 - \omega_2) = 0 \Leftrightarrow \omega_1 = \omega_2$ and $d(\omega_1, \omega_2) = (\omega_1 - \omega_2)^2 = (\omega_2 - \omega_1)^2 = d(\omega_2, \omega_1)$.

So, $d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$.

(3) To solve the triangular property in b-metric space, we will use the convexity of function, i.e.,

"If $1 < p < \infty$, then convexity of function $f(a) = a^p\ (a > 0)$ implies that $(a + b/2)^p \leq a^p/2 + b^p/b^p$, i.e.,

$$\left(\frac{a + b}{2}\right)^p \leq \frac{a^p}{2} + \frac{b^p}{b^p}$$
$$(a + b)^p \leq 2^{p-1}(a^p + b^p), \quad (3)$$

holds," and we have

$$d(\omega_1, \omega_3) = (\omega_1 - \omega_3)^2 = (\omega_1 - \omega_2 + \omega_2 - \omega_3)^2$$
$$= [(\omega_1 - \omega_2) + (\omega_2 - \omega_3)]^2. \quad (4)$$

Using (3),

$$d(\omega_1, \omega_3) \leq 2[d(\omega_1, \omega_2) + d(\omega_2, \omega_3)]. \quad (5)$$

Hence, $d(\omega_1, \omega_2) = (\omega_1 - \omega_2)^2$ is a b-metric with $s = 2$.

Example 2 (see [5]). The set l_p with $0 < p < 1$, where $l_p = \{\{x_n\} \subset \mathbb{R}: \sum_{\iota=1}^{\infty} |\mu_\iota|^p < \infty\}$, together with the function d: $l_p \times l_p \longrightarrow [0, \infty)$, is

$$d(\mu, \nu) = \left(\sum_{\iota=1}^{\infty} |\mu_\iota - \nu_\iota|^p\right)^{1/p}, \quad (6)$$

where $\mu = \{\mu_\iota\}$ and $\nu = \{\nu_\iota\} \in l_p$ is b-metric space with $w = 2^{1/p} > 1$. Notice that the abovementioned result holds with $0 < p < 1$.

Definition 4 (see [5]). Let (Ω, d) be b-metric space and $\{z_n\}$ be a sequence in Ω. Then,

(1) $\{z_n\}$ is called a convergent sequence if and only if there exists $z \in \Omega$, such that for any $r > 0\ \exists\ n(r) \in N$ such that, for all $n \geq n(r)$, we get $d(z_n, z) < r$. In this case, we write $\lim_{n \longrightarrow \infty} z_n = z$.

(2) $\{z_n\}$ is said to be a Cauchy sequence if and only if for any $r > 0\ \exists\ n(r) \in N$ such that, for each $i, j \geq n(r)$, we get $d(z_i, z_j) < r$.

(3) Ω is called complete if every Cauchy sequence in Ω is convergent in Ω.

Let Ω be a nonempty set and R: $\Omega \longrightarrow \Omega$ be a self-map. We say that $x \in \Omega$ is a fixed point of R if $R(x) = x$.

Let Ω be any set and R: $\Omega \longrightarrow \Omega$ be a self-map. For any given $x \in \Omega$, we define $R^n(x)$ inductively by $R^0(x) = x$ and $R^{n+1}(x) = R(R^n(x))$, and we recall $R^n(x)$, the n^{th} iterative of x under R. For any $x_0 \in \Omega$, the sequence $\{x_n\}_{n \geq 0} \subset \Omega$ is given by

$$x_n = Rx_{n-1} = R^n x_0, \quad n = 1, 2, \ldots \quad (7)$$

which is known as the sequence of successive approximations, where x_0 is the initial value. This is also known as the Picard iteration starting at x_0.

2. Fixed Points for Contractive Mappings

This section consists of our main results. We investigate the existence and uniqueness of fixed points of some new contractive conditions in the context of b-metric spaces. Moreover, results are supported with example.

Theorem 1. *Let (Ω, d) be a complete b-metric space with coefficient ≥ 1. Let R be a function R: $\Omega \longrightarrow \Omega$ such that*

$$d(Rx, Ry) \leq ad(x, y) + b\max\left\{\begin{array}{c} d(x, Rx), d(y, Ry), d(x, Ry), \\ d(y, Rx) \end{array}\right\}, \quad (8)$$

where $a, b > 0$ such that $a + 2sb < 1$, $\forall\, x, y \in \Omega$ and $s \geq 1$. Then, there is a unique fixed point of R.

Proof. Let $\text{ɤ}_0 \in \Omega$ and $\{\text{ɤ}_n\}_{n=1}^{\infty}$ be a sequence in Ω defined by the recursion:

$$\text{ɤ}_n = R\text{ɤ}_{n-1} = R^n\text{ɤ}_0,\ n = 1, 2, 3, 4, \ldots. \quad (9)$$

By (8) and (9),

$$d(\gamma_n, \gamma_{n+1}) = d(R\gamma_{n-1}, R\gamma_n), \tag{10}$$

$$\leq ad(\gamma_{n-1}, \gamma_n) + b \max\left\{\begin{array}{c} d(\gamma_{n-1}, R\gamma_{n-1}), d(\gamma_n, R\gamma_n), d(\gamma_{n-1}, R\gamma_n), \\ d(\gamma_n, R\gamma_{n-1}) \end{array}\right\}, \tag{11}$$

$$= ad(\gamma_{n-1}, \gamma_n) + b \max\left\{\begin{array}{c} d(\gamma_{n-1}, \gamma_n), d(\gamma_n, \gamma_{n+1}), d(\gamma_{n-1}, \gamma_{n+1}), \\ d(\gamma_n, \gamma_n) \end{array}\right\}. \tag{12}$$

So,

$$\begin{aligned} &d(\gamma_n, \gamma_{n+1}) \leq ad(\gamma_{n-1}, \gamma_n) + b \max\{d(\gamma_{n-1}, \gamma_n), d(\gamma_n, \gamma_{n+1}), d(\gamma_{n-1}, \gamma_{n+1})\}, \\ &d(\gamma_n, \gamma_{n+1}) \leq ad(\gamma_{n-1}, \gamma_n) + bM, \end{aligned} \tag{13}$$

where $M = \max\{d(\gamma_{n-1}, \gamma_n), d(\gamma_n, \gamma_{n+1}), d(\gamma_{n-1}, \gamma_{n+1}),\}$. □

Case 1. If $M = d(\gamma_{n-1}, \gamma_n)$, then

$d(\gamma_n, \gamma_{n+1}) \leq ad(\gamma_{n-1}, \gamma_n) + bd(\gamma_{n-1}, \gamma_n)$

$d(\gamma_n, \gamma_{n+1}) \leq (a+b)d(\gamma_{n-1}, \gamma_n)$

Let $k = (a+b) < 1$; then, $d(\gamma_n, \gamma_{n+1}) \leq kd(\gamma_{n-1}, \gamma_n)$ and

. . . . .

. . . . .

. . . . .

$d(\gamma_n, \gamma_{n+1}) \leq k^n d(\gamma_0, \gamma_1)$.

Case 2. If $M = d(\gamma_n, \gamma_{n+1})$, then

$$d(\gamma_n, \gamma_{n+1}) \leq ad(\gamma_{n-1}, \gamma_n) + bd(\gamma_n, \gamma_{n+1}), \tag{14}$$

$$(1-b)d(\gamma_n, \gamma_{n+1}) \leq ad(\gamma_{n-1}, \gamma_n), \tag{15}$$

$$d(\gamma_n, \gamma_{n+1}) \leq \frac{a}{(1-b)} d(\gamma_{n-1}, \gamma_n). \tag{16}$$

Let $k = a/(1-b) < 1$; then,

$$d(\gamma_n, \gamma_{n+1}) \leq kd(\gamma_{n-1}, \gamma_n), \tag{17}$$

. . . . . .

. . . . . .

. . . . . .

$$d(\gamma_n, \gamma_{n+1}) \leq k^n d(\gamma_0, \gamma_1). \tag{18}$$

Case 3. If $M = d(\gamma_{n-1}, \gamma_{n+1})$, then

$$\begin{aligned} &d(\gamma_n, \gamma_{n+1}) \leq ad(\gamma_{n-1}, \gamma_n) + bd(\gamma_{n-1}, \gamma_{n+1}), \\ &d(\gamma_n, \gamma_{n+1}) \leq ad(\gamma_{n-1}, \gamma_n) + b[s(d(n_{n-1}, \gamma_n) + d(\gamma_n, \gamma_{n+1}))], \end{aligned} \tag{19}$$

$$\begin{aligned} d(\gamma_n, \gamma_{n+1}) &\leq ad(\gamma_{n-1}, \gamma_n) + sbd(\gamma_{n-1}, \gamma_n) + sbd(\gamma_n, \gamma_{n+1}), \\ (1-sb)d(\gamma_n, \gamma_{n+1}) &\leq (a+sb)d(\gamma_{n-1}, \gamma_n), \end{aligned} \tag{20}$$

$$d(\gamma_n, \gamma_{n+1}) \leq \frac{(a+sb)}{(1-sb)} d(\gamma_{n-1}, \gamma_n). \tag{21}$$

Let $k = (a+sb)/(1-sb) < 1$; then,

$$d(ɤ_n, ɤ_{n+1}) \leq kd(ɤ_{n-1}, ɤ_n), \tag{22}$$

$$\begin{matrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{matrix}$$

$$d(ɤ_n, ɤ_{n+1}) \leq k^n d(ɤ_0, ɤ_1). \tag{23}$$

Cases 1–3 show that R is a contractive-type mapping. Let $m, n \in N$ and $m > n$; then,

$$\begin{aligned} d(ɤ_n, ɤ_m) &\leq s[d(ɤ_n, ɤ_{n+1}) + d(ɤ_{n+1}, ɤ_m)], \\ d(ɤ_n, ɤ_m) &\leq sd(ɤ_n, ɤ_{n+1}) + sd(ɤ_{n+1}, ɤ_m), \\ d(ɤ_n, ɤ_m) &\leq sd(ɤ_n, ɤ_{n+1}) + sd(ɤ_{n+1}, ɤ_m), \\ d(ɤ_n, ɤ_m) &\leq sd(ɤ_n, ɤ_{n+1}) + sd(ɤ_{n+1}, ɤ_m), \\ d(ɤ_n, ɤ_m) &\leq sd(ɤ_n, ɤ_{n+1}) + s(s[d(ɤ_{n+1}, ɤ_{n+2}) + d(ɤ_{n+2}, ɤ_m)]), \\ d(ɤ_n, ɤ_m) &\leq sd(ɤ_n, ɤ_{n+1}) + s^2 d(ɤ_{n+1}, ɤ_{n+2}) + s^2 d(ɤ_{n+2}, ɤ_m). \end{aligned} \tag{24}$$

Continue in the following way:

$$d(ɤ_n, ɤ_m) \leq sd(ɤ_n, ɤ_{n+1}) + s^2 d(ɤ_{n+1}, ɤ_{n+2}) + s^3 d(ɤ_{n+2}, ɤ_{n+3}) + \cdots + s^{m-n} d(ɤ_{m-1}, ɤ_m). \tag{25}$$

By Cases 1–3,

$$\begin{aligned} d(ɤ_n, ɤ_m) \leq sk^n d(ɤ_0, ɤ_1) &+ s^2 k^{n+1} d(ɤ_0, ɤ_1) + s^3 k^{n+2} d(ɤ_0, ɤ_1) \\ &+ \cdots + s^{m-n} k^{m-1} d(ɤ_0, ɤ_1), \\ &= d(ɤ_0, ɤ_1)\left[sk^n + s^2 k^{n+1} + s^3 k^{n+2} + \cdots + s^{m-n} k^{m-1}\right]. \end{aligned} \tag{26}$$

Since $sk^n + s^2k^{n+1} + s^3k^{n+2} + \cdots + s^{m-n}k^{m-1}$ is geometric series with common ratio less than 1, so

$$d(ɤ_n, ɤ_m) \leq \left(\frac{sk^n(1-(sk)^n)}{1-sk}\right) d(ɤ_0, ɤ_1) \leq \left(\frac{sk^n}{1-sk}\right) d(ɤ_0, ɤ_1). \tag{27}$$

As $n \longrightarrow \infty$ and $k < 1$, $(sk^n/1-sk) \longrightarrow 0$. Hence,

$$d(ɤ_n, ɤ_m) \longrightarrow 0. \tag{28}$$

Hence, $\{ɤ_n\}$ is a Cauchy sequence in Ω. Due to completeness of Ω, there exists $ɤ^* \in \Omega$ such that

$$ɤ_n \longrightarrow ɤ^*, \text{as } n \longrightarrow \infty. \tag{29}$$

Now,

$$
\begin{aligned}
&d(\gamma^*, R\gamma^*) \le s[d(\gamma^*, \gamma_{n+1}) + d(\gamma_{n+1}, R\gamma^*)],\\
&d(\gamma^*, R\gamma^*) \le sd(\gamma^*, \gamma_{n+1}) + sd(\gamma_{n+1}, R\gamma^*),\\
&d(\gamma^*, R\gamma^*) \le sd(\gamma^*, \gamma_{n+1}) + sd(R\gamma_n, R\gamma^*),\\
&d(\gamma^*, R\gamma^*) \le sd(\gamma^*, \gamma_{n+1}) + s[ad(\gamma_n, \gamma^*) + b\max]\\
&\qquad\left\{\begin{matrix} d(\gamma_n, R\gamma_n), d(\gamma^*, R\gamma^*), d(\gamma_n, R\gamma^*) \\ d(\gamma^*, R\gamma_n) \end{matrix}\right\},\\
&d(\gamma^*, R\gamma^*) \le sd(\gamma^*, \gamma_{n+1}) + s[ad(\gamma_n, \gamma^*) + bM_o],
\end{aligned}
\tag{30}
$$

where $M_o = \max\{d(\gamma_n, R\gamma_n), d(\gamma^*, R\gamma^*), d(\gamma_n, R\gamma^*), d(\gamma^*, R\gamma_n)\}$.

Case 4. If $M_o = d(\gamma_n, R\gamma_n)$, then

$$
\begin{aligned}
&d(\gamma^*, R\gamma^*) \le sd(\gamma^*, \gamma_{n+1}) + asd(\gamma_n, \gamma^*) + bsd(\gamma_n, R\gamma_n),\\
&d(\gamma^*, R\gamma^*) \le sd(\gamma^*, \gamma_{n+1}) + asd(\gamma_n, \gamma^*) + bsd(\gamma_n, \gamma_{n+1}).
\end{aligned}
\tag{31}
$$

As $n \longrightarrow \infty$ and $\gamma_n \longrightarrow \gamma^*$, so

$$d(\gamma^*, R\gamma^*) \longrightarrow 0. \tag{32}$$

Hence,

$$\gamma^* = R\gamma^*. \tag{33}$$

γ^* is a fixed point of R.

Case 5. If $M_o = d(\gamma^*, R\gamma^*)$, then

$$
\begin{aligned}
d(\gamma^*, R\gamma^*) &\le sd(\gamma^*, \gamma_{n+1}) + asd(\gamma_n, \gamma^*) + bsd(\gamma^*, R\gamma^*),\\
(1-bs)d(\gamma^*, R\gamma^*) &\le sd(\gamma^*, \gamma_{n+1}) + asds(\gamma_n, \gamma^*),\\
d(\gamma^*, R\gamma^*) &\le \frac{s}{(1-bs)} d(\gamma^*, \gamma_{n+1}) + \frac{as}{(1-bs)} d(\gamma_n, \gamma^*).
\end{aligned}
\tag{34}
$$

As $n \longrightarrow \infty$ and $\gamma_n \longrightarrow \gamma^*$, so

$$d(\gamma^*, R\gamma^*) \longrightarrow 0. \tag{35}$$

Hence, $\gamma^* = R\gamma^*$. γ^* is a fixed point of R.

Case 6. If $M_o = d(\omega_n, R\omega^*)$, then

$$
\begin{aligned}
d(\gamma^*, R\gamma^*) &\le sd(\gamma^*, \gamma_{n+1}) + asd(\gamma_n, \gamma^*) + bsd(\gamma_n, R\gamma^*),\\
d(\gamma^*, R\gamma^*) &\le sd(\gamma^*, \gamma_{n+1}) + asd(\gamma_n, \gamma^*) + bs(s[d(\gamma_n, \gamma^*) + d(\gamma^*, R\gamma^*)]),\\
(1-s^2b)d(\gamma^*, R\gamma^*) &\le sd(\gamma^*, \gamma_{n+1}) + (as + s^2b)d(\gamma_n, \gamma^*),\\
d(\gamma^*, R\gamma^*) &\le \frac{s}{(1-s^2b)} d(\gamma^*, \gamma_{n+1}) + (as + s^2b)/(1-s^2b)d(\gamma_n, \gamma^*).
\end{aligned}
\tag{36}
$$

As $n \longrightarrow \infty$ and $\gamma_n \longrightarrow \gamma^*$, so

$$d(\gamma^*, R\gamma^*) \longrightarrow 0. \tag{37}$$

Hence, $\gamma^* = R\gamma^*$.
Hence, γ^* is a fixed point of R.

Case 7. If $M_o = d(ɤ^*, Rɤ_n)$, then

$$
\begin{aligned}
&d(n^*, Rɤ^*) \leq sd(ɤ^*, ɤ_{n+1}) + asd(ɤ_n, ɤ^*) + bsd(ɤ^*, Rɤ_n),\\
&d(ɤ^*, Rɤ^*) \leq sd(ɤ^*, ɤ_{n+1}) + asd(ɤ_n, ɤ^*) + bsd(ɤ^*, ɤ_{n+1}),\\
&d(ɤ^*, Rɤ^*) \leq (s + bs)d(ɤ^*, ɤ_{n+1}) + asd(ɤ_n, ɤ^*).
\end{aligned} \tag{38}
$$

As $n \longrightarrow \infty$ and $ɤ_n \longrightarrow ɤ^*$, so

$$d(\Upsilon^*, Rɤ^*) \longrightarrow 0. \tag{39}$$

Hence, $ɤ^* = Rɤ^*$. $ɤ^*$ is a fixed point of R.

Now, assume that $ɤ'$ is another fixed point of R, i.e., $Rɤ' = ɤ'$. Consider

$$
\begin{aligned}
&d(ɤ^*, ɤ') = d(Rɤ^*, Rɤ'),\\
&d(ɤ^*, ɤ') \leq ad(ɤ^*, ɤ') + b\max\{d(ɤ^*, Rɤ^*), d(ɤ', Rɤ'), d(ɤ^*, Rɤ'), d(ɤ', Rɤ^*)\},\\
&d(ɤ^*, ɤ') \leq ad(ɤ^*, ɤ')ɤ' + b\max\{d(ɤ^*, ɤ^*), d(ɤ', ɤ'), d(ɤ^*, ɤ'), d(ɤ', ɤ^*)\},\\
&d(ɤ^*, ɤ') \leq ad(ɤ^*, ɤ') + b\max\{d(ɤ^*ɤ'), d(ɤ', ɤ^*)\},\\
&d(ɤ^*, ɤ') \leq ad(ɤ^*, ɤ') + b\,d(ɤ^*, ɤ'),
\end{aligned} \tag{40}
$$

$d(ɤ^*, ɤ') \leq (a + b)d(ɤ^*, ɤ')$, where $a, b > 0$. That is not possible; therefore, $ɤ^* = ɤ'$. Hence, $ɤ^*$ is unique fixed point of R.

Example 3. Suppose $X = \{1, 2, 3\}$; we define σ: $X \times X \longrightarrow \mathbb{R}$ as below:

$\sigma\ (1,\ 1) = \sigma\ (2,\ 2) = \sigma\ (3,\ 3) = 0$

$\sigma\ (1,\ 2) = \sigma\ (2,\ 1) = \sigma\ (2,\ 3) = \sigma\ (3,\ 2) = 1$

$\sigma\ (1,\ 3) = \sigma\ (3,\ 1) = z \geq 2$

Then, (X, σ) is a b-metric space with coefficient $z/2$, where $z \geq 2$.

Define R: $X \longrightarrow X$ by $R(x) = x^3 - 6x^2 + 11x - 4$, for all $x \in X$.

Then, all conditions of the above theorems are satisfied for $a = 1/4$, $b = 1/6$, and $2 \leq z < 4.5$. Hence, $2 \in X$ is the unique fixed point of R.

In the following, some direct consequences of Theorem 1 are as follows.

Corollary 1 (Banach-contraction theorem in b-metric space). *Let (Ω, d) be a complete b-metric space with coefficient $s \geq 1$. Let R be a self-function on Ω such that*

$$d(Rx, Ry) \leq \cdots ad(x, y), \tag{41}$$

where $0 \leq a < 1$, $\forall x, y \in \Omega$. Then, there exists a unique fixed point of R.

Proof. In Theorem 1, by putting $b = 0$, the required result will be obtained. □

Corollary 2. *Let (Ω, d) be a complete metric space and a function R: $\Omega \longrightarrow \Omega$ such that*

$$d(Rx, Ry) \leq ad(x, y) + b\max\left\{\begin{matrix} d(x, Rx), d(y, Ry), d(x, Ry), \\ d(y, Rx) \end{matrix}\right\}, \tag{42}$$

where $a, b > 0$ such that $a + 2b < 1$, $\forall x, y \in \Omega$. Then, there exists a unique fixed point of R.

Proof. Since every b-metric space is a metric space by taking $s = 1$, so the proof is obvious. □

Corollary 3 (Banach-contraction theorem in metric space). *Let (Ω, d) be a complete metric space. Let R be a function R: $\Omega \longrightarrow \Omega$ such that*

$$d(Rx, Ry) \leq ad(x, y), \tag{43}$$

where $0 \le a < 1$, $\forall\, x, y \in \Omega$. *Then, there exists a unique fixed point of R.*

Proof. In Corollary 2, by putting $b = 0$, the required will be obtained. □

Theorem 2. *Let* (Ω, d) *be a complete b-metric space. Let R be a mapping* $R\colon \Omega \longrightarrow \Omega$ *such that*

$$d(Rx, Ry) \le a \max\{d(x, Ry), d(y, Rx), d(x, y)\} + b[d(x, Rx) + d(y, Ry)], \tag{44}$$

where $a, b > 0$ *such that* $2as + 2b < 1$, $\forall x, y \in \Omega$ *and* $s \ge 1$. *Then, there exists a unique fixed point of R.*

Proof. Let $\omega_0 \in \Omega$ and $\{\omega_n\}$ be a sequence in Ω defined as

$$\omega_n = R\omega_{n-1} = R^n \omega_0, \quad n = 1, 2, 3, \ldots \tag{45}$$

By (44) and (45), we obtain that

$$\begin{aligned}
&d(\omega_n, \omega_{n+1}) = d(R\omega_{n-1}, R\omega_n),\\
&d(\omega_n, \omega_{n+1}) \le a \max\{d(\omega_{n-1}, R\omega_n), d(\omega_n, R\omega_{n-1}), d(\omega_{n-1}, \omega_n)\} + b[d(\omega_{n-1}, R\omega_{n-1}) + d(\omega_n, R\omega_n)],\\
&d(\omega_n, \omega_{n+1}) \le a \max\{d(\omega_{n-1}, \omega_{n+1}), d(\omega_n, x_n), d(\omega_{n-1}, \omega_n)\} + b[d(\omega_{n-1}, \omega_n) + d(\omega_n, \omega_{n+1})],
\end{aligned} \tag{46}$$

$$d(\omega_n, \omega_{n+1}) \le aM + b[d(\omega_{n-1}, \omega_n) + d(\omega_n, \omega_{n+1})], \tag{47}$$

where

$$M = \max\{d(\omega_{n-1}, \omega_{n+1}), d(\omega_n, \omega_n), d(\omega_{n-1}, \omega_n)\}, \tag{48}$$

$$M = \max\{d(\omega_{n-1}, \omega_{n+1}), d(\omega_{n-1}, \omega_n)\}.$$ □

Case 8. If $M = d(\omega_{n-1}, \omega_{n+1})$, then (44) becomes

$$\begin{aligned}
d(\omega_n, \omega_{n+1}) &\le ad(\omega_{n-1}, \omega_{n+1}) + b[d(\omega_{n-1}, \omega_n) + d(\omega_n, \omega_{n+1})],\\
d(\omega_n, \omega_{n+1}) &\le as[d(\omega_{n-1}, \omega_n) + d(\omega_n, \omega_{n+1})] + bd(\omega_{n-1}, \omega_n) + bd(\omega_n, \omega_{n+1}),\\
d(\omega_n, \omega_{n+1}) &\le asd(\omega_{n-1}, \omega_n) + asd(\omega_n, \omega_{n+1}) + bd(\omega_{n-1}, \omega_n) + bd(\omega_n, \omega_{n+1}),\\
(1 - b - as)d(\omega_n, \omega_{n+1}) &\le (as + b)d(\omega_{n-1}, \omega_n),\\
d(\omega_n, \omega_{n+1}) &\le (as + b/1 - b - as)d(\omega_{n-1}, \omega_n),\\
d(\omega_n, \omega_{n+1}) &\le kd(\omega_{n-1}, \omega_n),
\end{aligned} \tag{49}$$

where $k = (as + b/1 - b - as) < 1$.

Continuing in this way,

$$d(\omega_n, \omega_{n+1}) \le k^2 d(\omega_{n-2}, \omega_{n-1}), \tag{50}$$

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

$$d(\omega_n, \omega_{n+1}) \le k^n d(\omega_0, \omega_1). \tag{51}$$

Case 9. If $M = d(\omega_{n-1}, \omega_n)$, then (47) becomes

$$\begin{aligned}
d(\omega_n, \omega_{n+1}) &\le ad(\omega_{n-1}, \omega_n) + b[d(\omega_{n-1}, \omega_n) + d(\omega_n, \omega_{n+1})],\\
d(\omega_n, \omega_{n+1}) &\le ad(\omega_{n-1}, \omega_n) + bd(\omega_{n-1}, \omega_n) + bd(\omega_n, \omega_{n+1}),\\
(1 - b)d(\omega_n, \omega_{n+1}) &\le (a + b)d(\omega_{n-1}, \omega_n),\\
d(\omega_n, \omega_{n+1}) &\le \left(\frac{a + b}{1 - b}\right)d(\omega_{n-1}, \omega_n),\\
d(\omega_n, \omega_{n+1}) &\le kd(\omega_{n-1}, \omega_n),
\end{aligned} \tag{52}$$

where $k = (a + b/1 - b) < 1$.

Continuing in this way,

$$d(\omega_n, \omega_{n+1}) \le k^2 d(\omega_{n-2}, \omega_{n-1}), \tag{53}$$

. . . . . .
. . . . . .
. . . . . .

$$d(\omega_n,\omega_{n+1}) \leq k^n d(\omega_0,\omega_1). \tag{54}$$

Cases 8 and 9 show that R is a contractive-type mapping.

Now, let m and n be any two natural numbers and $m>n$; then,

$$\begin{aligned} d(\omega_n,\omega_m) &\leq s[d(\omega_n,\omega_{n+1})+d(\omega_{n+1},\omega_m)],\\ d(\omega_n,\omega_m) &\leq sd(\omega_n,\omega_{n+1})+sd(\omega_{n+1},\omega_m),\\ d(\omega_n,\omega_m) &\leq sd(\omega_n,\omega_{n+1})+s(s[d(\omega_{n+1},\omega_{n+2})+d(\omega_{n+2},\omega_m)]),\\ d(\omega_n,\omega_m) &\leq sd(\omega_n,\omega_{n+1})+s^2 d(\omega_{n+1},\omega_{n+2})+s^2 d(\omega_{n+2},\omega_m). \end{aligned} \tag{55}$$

Continuing in this way,

$$d(\omega_n,\omega_m) \leq sd(\omega_n,\omega_{n+1})+s^2 d(\omega_{n+1},\omega_{n+2})+s^3 d(\omega_{n+2},\omega_{n+3})+\cdots+s^{m-n} d(\omega_{m-1},\omega_m), \tag{56}$$

By Cases 1 and 2,

$$\begin{aligned} d(\omega_n,\omega_m) &\leq sk^n d(\omega_n,\omega_{n+1})+s^2 k^{n+1} d(\omega_{n+1},\omega_{n+2})+s^3 k^{n+2} d(\omega_{n+2},\omega_{n+3})+\cdots+s^{m-n}k^{m-1} d(\omega_{m-1},\omega_m),\\ d(\omega_n,\omega_m) &\leq d(\omega_0,\omega_1)\left[sk^n+s^2 k^{n+1}+s^2 k^{n+2}+\cdots+s^{m-n}k^{m-1}\right]. \end{aligned} \tag{57}$$

Since $sk^n+s^2 k^{n+1}+s^2 k^{n+2}+\cdots+s^{m-n}k^{m-1}$ is a geometric series with common ratio less than 1, so $d(\omega_n,\omega_m) \leq d(\omega_0,\omega_1)(sk^n(1-(sk)^n)/1-sk) \leq d(\omega_0,\omega_1)(sk^n/1-sk)$.

As $n \longrightarrow \infty$, $(sk^n/1-sk) \longrightarrow 0$, so

$$d(\omega_n,\omega_m) \longrightarrow 0. \tag{58}$$

Hence, $\{\omega_n\}$ is a Cauchy sequence in Ω. Due to completeness of Ω, there exists ω^* in Ω such that $\omega_n \longrightarrow \omega^*$. Now,

$$\begin{aligned} d(\omega^*,R\omega^*) &\leq s[d(\omega^*,\omega_{n+1})+d(\omega_{n+1},R\omega^*)],\\ d(\omega^*,R\omega^*) &\leq sd(\omega^*,\omega_{n+1})+sd(\omega_{n+1},R\omega^*),\\ d(\omega^*,R\omega^*) &\leq sd(\omega^*,\omega_{n+1})+sd(R\omega_n,R\omega^*). \end{aligned} \tag{59}$$

By (44),

$$d(\omega^*,R\omega^*) \leq sd(\omega^*,\omega_{n+1})+sa\max\{d(\omega_n,R\omega^*),d(\omega^*,R\omega_n),d(\omega_n,\omega^*)\}+sb[d(\omega_n,R\omega_n)+d(\omega^*,R\omega^*)], \tag{60}$$

$$d(\omega^*,R\omega^*) \leq sd(\omega^*,\omega_{n+1})+saM_0+sb[d(\omega_n,R\omega_n)+d(\omega^*,R\omega^*)], \tag{61}$$

where $M_0=\max\{d(\omega_n,R\omega^*),d(\omega^*,R\omega_n),d(\omega_n,\omega^*)\}$.

Case 10. If $M_0=d(\omega_n,R\omega^*)$, then (61) becomes

$$\begin{aligned} d(\omega^*,R\omega^*) &\leq sd(\omega^*,\omega_{n+1})+sa\, d(\omega_n,R\omega^*)+sb[d(\omega_n,R\omega_n)+d(\omega^*,R\omega^*)],\\ d(\omega^*,R\omega^*) &\leq sd(\omega^*,\omega_{n+1})+sab[d(\omega_n,\omega^*)+d(\omega^*,R\omega^*)]+sb[d(\omega_n,R\omega_n)+d(\omega^*,R\omega^*)],\\ d(\omega^*,R\omega^*) &\leq sd(\omega^*,\omega_{n+1})+sabd(\omega_n,\omega^*)+sabd(\omega^*,R\omega^*)+sbd(\omega_n,R\omega_n)+sbd(\omega^*,R\omega^*),\\ (1-sb-sab)d(\omega^*,R\omega^*) &\leq sd(\omega^*,\omega_{n+1})+sabd(\omega_n,\omega^*)+sbd(\omega_n,\omega_{n+1}),\\ d(\omega^*,R\omega^*) &\leq \left(\frac{s}{1-sb-sab}\right)d(\omega^*,\omega_{n+1})+\left(\frac{sab}{1-sb-sab}\right)d(\omega_n,\omega^*)+\left(\frac{sb}{1-sb-sab}\right)d(\omega_n,\omega_{n+1}). \end{aligned} \tag{62}$$

As $n \longrightarrow \infty, d(\omega^*, R\omega^*) \longrightarrow 0$. Hence,

$$\omega^* = R\omega^*. \tag{63}$$

ω^* is a fixed point of R.

Case 11. If $M_0 = d(\omega^*, R\omega_n)$, then (61) becomes

$$\begin{aligned}
&d(\omega^*, R\omega^*) \le sd(\omega^*, \omega_{n+1}) + sad(\omega^*, R\omega_n) + sb[d(\omega_n, R\omega_n) + d(\omega^*, R\omega^*)],\\
&d(\omega^*, R\omega^*) \le sd(\omega^*, \omega_{n+1}) + sab[d(\omega^*, \omega_{n+1}) + d(\omega_{n+1}, R\omega_n)] + sb[d(\omega_n, R\omega_n) + d(\omega^*, R\omega^*)],\\
&d(\omega^*, R\omega^*) \le sd(\omega^*, \omega_{n+1}) + sab\ d(\omega^*, \omega_{n+1}) + sabd(\omega_{n+1}, R\omega_n) + sbd(\omega_n, R\omega_n) + sbd(\omega^*, R\omega^*),\\
&d(\omega^*, R\omega^*) \le sd(\omega^*, \omega_{n+1}) + sab\ d(\omega^*, \omega_{n+1}) + sab\ d(\omega_{n+1}, \omega_{n+1}) + sb\ d(\omega_n, \omega_{n+1}) + sb\ d(\omega^*, R\omega^*),\\
&d(\omega^*, R\omega^*) \le sd(\omega^*, \omega_{n+1}) + sabd(\omega^*, \omega_{n+1}) + sbd(\omega_n, \omega_{n+1}) + sbd(\omega^*, R\omega^*),\\
&(1 - sb)d(\omega^*, R\omega^*) \le (s + sab)d(\omega^*, \omega_{n+1}) + sbd(\omega_n, \omega_{n+1}),\\
&d(\omega^*, R\omega^*) \le \left(\frac{s + sab}{1 - sb}\right)d(\omega^*, \omega_{n+1}) + \left(\frac{sb}{1 - sb}\right)d(\omega_n, \omega_{n+1}).
\end{aligned} \tag{64}$$

As $n \longrightarrow \infty, d(\omega^*, R\omega^*) \longrightarrow 0$. Hence,

$$\omega^* = R\omega^*. \tag{65}$$

ω^* is a fixed point of R.

Case 12. If $M_0 = d(\omega_n, \omega^*)$, then (61) becomes

$$\begin{aligned}
&d(\omega^*, R\omega^*) \le sd(\omega^*, \omega_{n+1}) + sad(\omega_n, \omega^*) + sb[d(\omega_n, R\omega_n) + d(\omega^*, R\omega^*)],\\
&d(\omega^*, R\omega^*) \le sd(\omega^*, \omega_{n+1}) + sad(\omega_n, \omega^*) + sbd(\omega_n, R\omega_n) + sbd(\omega^*, R\omega^*),\\
&d(\omega^*, R\omega^*) \le sd(\omega^*, \omega_{n+1}) + sad(\omega_n, \omega^*) + sbd(\omega_n, \omega_{n+1}) + sbd(\omega^*, R\omega^*),\\
&(1 - sb)d(\omega^*, R\omega^*) \le sd(\omega^*, \omega_{n+1}) + sad(\omega_n, \omega^*) + sbd(\omega_n, \omega_{n+1}),\\
&d(\omega^*, R\omega^*) \le \left(\frac{s}{1 - sb}\right)d(\omega^*, \omega_{n+1}) + \left(\frac{sa}{1 - sb}\right)d(\omega_n, \omega^*) + \left(\frac{sb}{1 - sb}\right)d(\omega_n, \omega_{n+1}).
\end{aligned} \tag{66}$$

As $n \longrightarrow \infty, d(\omega^*, R\omega^*) \longrightarrow 0$. Hence,

$$\omega^* = R\omega^*, \tag{67}$$

where ω^* is a fixed point of R.

For uniqueness, assume that $\omega\prime$ is another fixed point of R, i.e., $R\omega\prime = \omega\prime$. Consider

$$d(\omega^*, \omega') = d(R\omega^*, R\omega'). \tag{68}$$

By (44),

$$\begin{aligned}
&d(\omega^*, \omega') \le a \max\{d(\omega^*, R\omega'), d(\omega', R\omega^*), d(\omega^*, \omega')\} + b[d(\omega^*, R\omega^*) + d(\omega', R\omega')],\\
&d(\omega^*, \omega') \le a \max\{d(\omega^*, \omega'), d(\omega', \omega^*), d(\omega^*, \omega')\} + b[d(\omega^*, \omega^*) + d(\omega', \omega')],\\
&d(\omega^*, \omega') \le a \max\{d(\omega^*, \omega'), d(\omega', \omega^*), d(\omega^*, \omega')\} + 0,\\
&d(\omega^*, \omega') \le ad(\omega^*, \omega'),
\end{aligned} \tag{69}$$

which is a contradiction. Hence,

$$\omega^* = \omega', \tag{70}$$

where ω^* is unique fixed point of R.

Example 4. Suppose $X = \{1, 2, 3\}$, and we define $\sigma: X \times X \longrightarrow \mathbb{R}$ as below:

$$\begin{aligned} &\sigma(1,1) = \sigma(2,2) = \sigma(3,3) = 0, \\ &\sigma(1,2) = \sigma(2,1) = \sigma(2,3) = \sigma(3,2) = 1. \end{aligned} \tag{71}$$

σ (1, 3) = σ (3, 1) = $z \geq 2$.

Then, (X, σ) is a b-metric space with coefficient $z/2$, where $z \geq 2$.

Define $T: X \longrightarrow X$ by $T(x) = x^3 - 6x^2 + 11x - 3$, for all $x \in X$.

Then, all conditions of the above theorems are satisfied for $a = 1/4$, $b = 1/8$, and $2 \leq z < 3$. Hence, $3 \in X$ is the unique fixed point of T.

Corollary 4. *(Kannan-contraction theorem in b-metric space). Let (Ω, d) be a complete b-metric space. Let R be a mapping $R: \Omega \longrightarrow \Omega$ such that*

$$d(Rx, Ry) \leq b[d(x, Rx) + d(y, Ry)], \tag{72}$$

where $0 < b < 1/2$, $\forall x, y \in \Omega$, and $s \geq 1$. Then, R has a unique fixed point.

Proof. By putting $a = 0$ in Theorem 2, we get the required result. ☐

Corollary 5. *Let (Ω, d) be a complete metric space and R be a mapping $R: \Omega \longrightarrow \Omega$ such that*

$$d(Rx, Ry) \leq a \max\{d(x, Ry), d(y, Rx), d(x, y)\} + b[d(x, Rx) + d(y, Ry)], \tag{73}$$

where $a, b > 0$ such that $2a + 2b < 1$, $\forall x, y \in \Omega$. Then, there exists a unique fixed point of R.

Corollary 6. *(Kannan-contraction theorem in metric space). Let (Ω, d) be a complete metric space and a self-mapping $R: \Omega \longrightarrow \Omega$ such that*

$$d(Rx, Ry) \leq b[d(x, Rx) + d(y, Ry)]. \tag{74}$$

where $0 < b < 1/2$ and $\forall x, y \in \Omega$. Then, there exists a unique fixed point of R.

Proof. By putting $a = 0$ in Corollary 5, we get the required result. ☐

3. Conclusion

We have established and proved fixed-point results for different contractive conditions in b-metric space. To furnish, these results supportive examples are created. This generalization will be helpful for further investigation and applications. We conclude this paper by indicating, in the form of open questions, some directions for further investigation and work:

(1) Can the conditions $a + 2sb$ in Theorem 1 and $2as + 2b$ in Theorem 2 be relaxed?

(2) If the answer to 1 is yes, then what hypotheses on a and b are needed to guarantee the existence of fixed points?

(3) Can the concept of fixed point for these contractions be extended to more than one mapping?

Data Availability

The data used to support the finding of the study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publications.

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References

[1] J. Ahmad, A. S. Al-Rawashdeh, and A. Azam, "Fixed point results for {α, ξ}-expansive locally contractive mappings," *Journal of Inequalities and Applications*, vol. 2014, no. 1, pp. 1–10, 2014.

[2] H. H. Al-Sulami, N. Hussain, and J. Ahmad, "Some generalized fixed point results with applications to dynamic programming," *Journal of Function Spaces*, vol. 2020, Article ID 8130764, 8 pages, 2020.

[3] I. A. Bakhtin, "The contraction mapping principle in quasimetric spaces," *Func. An., Gos. Ped. Inst. Unianowsk*, vol. 30, pp. 26–37, 1989.

[4] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, no. 1, pp. 133–181, 1922.

[5] M. Boriceanu, *Fixed Point Theory for Multivalued Generalized Contraction on a Set with Two B-Metrics*, Studia Universitatis Babes-Bolyai, Romania, Italy, 2009.

[6] S. Czerwik, "Contraction mappings in $ b $-metric spaces," *Acta mathematica et informatica universitatis ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.

[7] A. Elmoasry, M. Shams, N. Yaqoob, N. Kausar, Y. U. Gaba, and N. Rafiq, "Numerical scheme for finding roots of interval-valued fuzzy nonlinear equation with application in optimization," *Journal of Function Spaces*, vol. 2021, Article ID 6369129, 16 pages, 2021.

[8] M. Iqbal, A. Batool, O. Ege, and M. de la Sen, "Fixed point of generalized weak contraction in metric spaces," *Journal of Function Spaces*, vol. 2021, Article ID 2042162, 8 pages, 2021.

[9] R. Kannan, "Some results on fixed points," *Bulletin of the Calcutta Mathematical Society*, vol. 60, pp. 71–76, 1968.

[10] Q. Mahmood, A. Shoaib, A. Shoaib, T. Rasham, and M. Arshad, "Fixed point results for the family of multivalued F-contractive mappings on closed ball in complete dislocated b-metric spaces," *Mathematics*, vol. 7, no. 1, p. 56, 2019.

[11] N. Sarfraz, M. Aslam, and F. Jarad, "Boundedness for commutators of rough -adic hardy operator on -adic central morrey spaces," *Journal of Function Spaces*, vol. 2021, Article ID 4886197, 5 pages, 2021.

[12] M. Zoran, V. Parvaneh, N. Mlaiki, N. Hussain, and S. Radenović, "On some new generalizations of Nadler contraction in b-metric spaces," *Cogent Mathematics & Statistics*, vol. 7, no. 1, Article ID 1760189, 2020.

[13] M. Bota, A. Molnar, and C. S. A. B. A. Varga, "On Ekeland's variational principle in b-metric spaces," *Fixed Point Theory*, vol. 12, no. 2, pp. 21–28, 2011.

[14] A. K. Dubey, R. Shukla, and R. P. Dubey, "Some fixed point results in b-metric," *Asian journal of mathematics and applications*, vol. 2014, 2014.

[15] K. Jain and J. Kaur, "Some fixed point results in b-metric spaces and b-metric-like spaces with new contractive mappings," *Axioms*, vol. 10, no. 2, p. 55, 2021.

[16] M. Kir and H. Kiziltunc, "On some well known fixed point theorems in b-metric spaces," *Turkish journal of analysis and number theory*, vol. 1, no. 1, pp. 13–16, 2013.

[17] Z. Mustafa, V. Parvaneh, M. M. M. Jaradat, and Z. Kadelburg, "Extended rectangular b-metric spaces and some fixed point theorems for contractive mappings," *Symmetry Plus*, vol. 11, no. 4, p. 594, 2019.

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Research Article

Kannan Nonexpansive Mappings on Variable Exponent Function Space of Complex Variables with Some Applications

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This paper establishes the reality of a fixed point of Kannan's prequasinorm contraction mapping on the variable exponent function space of complex variables, demonstrating that it satisfies the property (R) and possesses a prequasinormal structure. We have established the presence of a fixed point of Kannan prequasinorm nonexpansive mapping on it and Kannan prequasinorm contraction mapping in the prequasi-Banach operator ideal, created by this function space and s-numbers. Finally, we provide some applications for solutions to summable equations and illustrate instances to corroborate our findings.

1. Introduction

Many mathematicians have worked on feasible extensions to the Banach fixed point theorem since the publication of book [1] on the Banach fixed point theorem. Kannan [2] approved an instance of a class of operators that perform the same fixed point operations as contractions but with a continuous flop. Ghoncheh [3] made the first attempt to characterize Kannan operators in modular vector spaces. The variable exponent Lebesgue spaces $L_{(r)}$ contain Nakano sequence spaces. Throughout the second half of the twentieth century, it was assumed that these variable exponent spaces provided an acceptable framework for the mathematical components of several problems for which the conventional Lebesgue spaces were insufficient. Due to the relevance of these spaces and their effects, they have become a well-known and efficient tool for solving a variety of problems; nowadays, the area of $L_{(r)}(\Omega)$ spaces is a burgeoning area of research, with ramifications extending into a wide variety of mathematical specialties [4]. The study of variable exponent Lebesgue spaces $L_{(r)}$ received additional impetus from the mathematical description of non-Newtonian fluid hydrodynamics [5, 6]. Non-Newtonian fluids, also known as electrorheological fluids, have various applications ranging from military science to civil engineering and orthopedics. Operator ideal theory has a variety of applications in Banach space geometry, fixed point theory, spectral theory, and other branches of mathematics, among other branches of knowledge; for more details, see [7–13]. Bakery and Mohamed [14] investigated the concept of a prequasinorm on Nakano sequence space with a variable exponent in the range $(0, 1]$. They discussed the adequate circumstances for it to generate prequasi-Banach and closed space when endowed with a definite prequasinorm and the Fatou property of various prequasinorms on it. Additionally, they established the existence of a fixed point for Kannan prequasinorm contraction mappings on it and the prequasi-Banach operator ideal generated from s-numbers belonging to this sequence space. Also, in [15], they found some fixed points results of Kannan nonexpansive mappings on generalized Cesàro backward difference sequence space of nonabsolute type. For more recent developments in contractive mappings and the existence of fixed points of nonlinear operators in various Banach spaces, Nguyen and Tram [16] examined various fixed point results with applications to involution mappings. Dehici and Redjel [17] introduced some fixed point results for nonexpansive mappings in Banach spaces. Benavides and Ramírez [18] presented some fixed points for multivalued nonexpansive mappings.

We denote the set of complex numbers by $\mathbb{C}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ Assuming that $r = (r_a)_{n\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$, Bakery and El Dewaik [19] defined the following function space:

$$(\mathscr{H}_w((r_a)))_\psi = \left\{ f\colon f(z) = \sum_{y=0}^{\infty} \widehat{f}_y z^y \in \mathbb{C}, \text{ for every } z \in \mathbb{C} \text{ and } \psi(\varpi f) < \infty, \text{ for some } \varpi > 0 \right\}, \quad (1)$$

where

$$\psi(f) = \sum_{y=0}^{\infty} \left| \frac{\widehat{f}_y}{y+1} \right|^{r_y}. \quad (2)$$

They developed a multitude of topological and geometric characteristics for this variable exponent weighted formal power series space, as well as the prequasi-ideal construction utilizing s-number and $\mathscr{H}_w((p_n))$. Upper bounds for s-numbers of infinite series of the weighted n-th power forward shift operator on $(\mathscr{H}_w((r_a)))_\psi$ were also introduced for some entire functions. Further, they evaluated Caristi's fixed point theorem in $(\mathscr{H}_w((r_a)))_\psi$. For extra information on formal power series spaces and their behaviors, see [20–23]. The purpose of this paper is to develop an insight into how to think about the existence of a fixed point of Kannan prequasinorm contraction mapping in the prequasi-Banach special space of formal power series, where $(\mathscr{H}_w((r_a)))_\psi$ satisfies the property (R) and $(\mathscr{H}_w((r_a)))_\psi$ possesses the ψ-normal structure property. It has been established that a fixed point of the Kannan prequasinorm nonexpansive mapping exists in the prequasi-Banach special space of formal power series. Additionally, we discuss the Kannan prequasinorm contraction mapping in terms of the prequasioperator ideal. The existence of a fixed point of the Kannan prequasi norm contraction mapping in the prequasi Banach operator ideal $S_{(\mathscr{H}_w((r_a)))_\psi}$ is offered, where $S_{(\mathscr{H}_w((r_a)))_\psi}$ is the class of all bounded linear mappings between any two Banach spaces with the sequence s-numbers. Finally, we discuss several applications of solutions to summable equations and illustrate our findings with some instances.

2. Definitions and Preliminaries

Definition 2.1 (see [19]). The linear space $\mathscr{H} = \{h\colon h(z) = \sum_{y=} 0^{\infty}\widehat{h}_y z^y \in \mathbb{C}, \text{for every } z \in \mathbb{C}\}$ is called a special space of formal power series (or in short (ssfps)), if it shows the following settings:

(1) $e^{(m)} \in \mathscr{H}$, for all $m \in \mathbb{N}_0$, where $e^{(m)}(z) = \sum_{y=0}^{\infty} \widehat{e_y^{(m)}} z^y = z^m$.

(2) If $g \in \mathscr{H}$ and $|\widehat{h}_y| \le |\widehat{g}_y|$, for every $y \in \mathbb{N}_0$, then $h \in \mathscr{H}$.

(3) Suppose $h \in \mathscr{H}$; then $h_{[.]} \in \mathscr{H}$, with $h_{[.]}(z) = \sum_{b=0}^{\infty} \widehat{h_{[b/2]}} z^b$ and $[b/2]$ marks the integral part of $b/2$.

Definition 2.2 (see [19]). A subspace $\mathscr{H}_\psi$ of $\mathscr{H}$ is said to be a premodular (ssfps), if there is a function $\psi\colon \mathscr{H} \longrightarrow [0, t\infty)$ that verifies the next conditions:

(i) For $h \in \mathscr{H}$, we have $\psi(h) \ge 0$ and $h = \theta \Leftrightarrow \psi(h) = 0$, where θ is the zero function of $\mathscr{H}$.

(ii) Suppose $h \in \mathscr{H}$ and $\varpi \in \mathbb{R}$; then there is $q \ge 1$ with $\psi(\varpi h) \le |\varpi| q \psi(h)$.

(iii) Let $f, g \in \mathscr{H}$; then there is $A \ge 1$ such that $\psi(f+g) \le A(\psi(f) + \psi(g))$.

(iv) Suppose $|\widehat{f}_b| \le |\widehat{g}_b|$, for every $b \in \mathbb{N}_0$; then $\psi(f) \le \psi(g)$.

(v) There is $K_0 \ge 1$ so that $\psi(f) \le \psi(f_{[.]}) \le K_0 \psi(f)$.

(vi) $\overline{\mathfrak{F}} = \mathscr{H}_\psi$, where $\mathfrak{F}$ indicates the space of finite formal power series; that is, for $h \in \mathfrak{F}$, we have $l \in \mathbb{N}_0$ with $h(z) = \sum_{y=0}^{l} \widehat{h}_y z^y$.

(vii) One has $\xi > 0$ with $\psi(\lambda e^{(0)}) \ge \xi |\lambda| \psi(e^{(0)})$, where $\lambda \in \mathbb{R}$.

It is worth noting that the continuity of $\psi(f)$ at θ is due to condition (ii). Condition (1) in Definition 2.1 and condition (vi) in Definition 2.2 analyze the notion that $(e^{(m)})_{m\in\mathbb{N}_0}$ is a Schauder basis for $\mathscr{H}_\psi$.

The (ssfps) $\mathscr{H}_\psi$ is called a prequasinormed (ssfps) if ψ shows conditions (i)–(iii) of Definition 2.2, and if the space $\mathscr{H}$ is complete under ψ, then $\mathscr{H}_\psi$ is called a prequasi-Banach (ssfps). By L, we denote the ideal of all bounded linear operators between any arbitrary Banach spaces. Also, $L(\mathfrak{X}, \mathfrak{Y})$ marks the space of all bounded linear operators from a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$.

Definition 2.3 (see [24]). A function $s\colon L(\mathfrak{X}, \mathfrak{Y}) \longrightarrow [0,\infty)^{\mathbb{N}_0}$ is said to be an s-number, if the sequence $(s_b(B))_{a=0}^{\infty}$, for all $B \in L(\mathfrak{X}, \mathfrak{Y})$, shows the following settings:

(a) If $B \in L(\mathfrak{X}, \mathfrak{Y})$, then $\|B\| = s_0(B) \ge s_1(B) \ge s_2(B) \ge \ldots \ge 0$.

(b) $s_{b+a-1}(B_1 + B_2) \le s_b(B_1) + s_a(B_2)$, for every $B_1, B_2 \in L(\mathfrak{X}, \mathfrak{Y})$, b, $a \in \mathbb{N}_0$.

(c) The inequality $s_a(\text{ABD}) \le \|A\| s_a(B) \|D\|$ holds, if $D \in L(\mathfrak{X}_0, \mathfrak{X})$, $B \in L(\mathfrak{X}, \mathfrak{Y})$, and $A \in L(\mathfrak{Y}, \mathfrak{Y}_0)$, where $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ are arbitrary Banach spaces.

(d) Suppose $A \in L(\mathfrak{Y}, \mathfrak{Y}_0)$ and $\lambda \in \mathbb{R}$; then $s_a(\lambda A) = |\lambda| s_a(A)$.

(e) Let $\operatorname{rank}(A) \le b$; then $s_b(A) = 0$, whenever $A \in L(\mathfrak{Y}, \mathfrak{Y}_0)$.

(f) Assume that I_λ denotes the identity mapping on the λ-dimensional Hilbert space ℓ_2^λ; then $s_{r\ge\lambda}(I_\lambda) = 0$ or $s_{r<\lambda}(I_\lambda) = 1$.

Definition 2.4 (see [7]). A class $U \subseteq L$ is said to be an operator ideal if every vector $U(\mathfrak{X}, \mathfrak{Y}) = U \cap L(\mathfrak{X}, \mathfrak{Y})$ shows the following settings:

(i) $F \subseteq U$, where F is the ideal of all finite rank operators between any arbitrary Banach spaces.

(ii) $U(\mathfrak{X}, \mathfrak{Y})$ is linear space on $\mathbb{R}$.

(iii) If $D \in L(\mathfrak{X}_0, \mathfrak{X})$, $B \in U(\mathfrak{X}, \mathfrak{Y})$, and $A \in L(\mathfrak{Y}, \mathfrak{Y}_0)$, then $\mathrm{ABD} \in U(\mathfrak{X}_0, \mathfrak{Y}_0)$.

Definition 2.5 (see [10]). A function $g: U \longrightarrow [0, t\infty)$ is called a prequasinorm on the ideal U if it shows the following settings:

(1) For each $A \in L(\mathfrak{X}, \mathfrak{Y})$, $g(A) \ge 0$ and $g(A) = 0 \Leftrightarrow A = 0$.

(2) One has $M \ge 1$ with $g(\beta A) \le M|\beta| g(A)$, for all $\beta \in \mathbb{R}$ and $A \in U(\mathfrak{X}, \mathfrak{Y})$.

(3) One has $K \ge 1$ with $g(A_1 + A_2) \le K[g(A_1) + g(A_2)]$, for every $A_1, A_2 \in U(\mathfrak{X}, \mathfrak{Y})$.

(4) There is $C \ge 1$ so that if $A \in L(\mathfrak{X}_0, \mathfrak{X})$, $B \in U(\mathfrak{X}, \mathfrak{Y})$, and $D \in L(\mathfrak{Y}, \mathfrak{Y}_0)$, then $g(\mathrm{DBA}) \le C\|D\| g(B)\|A\|$, where $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ are normed spaces.

Definition 2.6 (see [19])

(a) The prequasinormed (ssfps) η on $(\mathscr{H}_w((r_n)))_\eta$ is said to be η-convex, if

$\eta(\varepsilon g + (1-\varepsilon)h) \le \varepsilon\eta(g) + (1-\varepsilon)\eta(h)$, for all $\varepsilon \in [0,1]$ and $g, h \in (\mathscr{H}_w((r_n)))_\eta$.

(b) $\{h^{(y)}\}_{y\in\mathbb{N}_0} \subseteq (\mathscr{H}_w((r_y)))_\eta$ is η-convergent to $h \in (\mathscr{H}_w((r_n)))_\eta$, if and only if $\lim_{y\longrightarrow\infty}\eta(h^{(y)} - h) = 0$. If the η-limit exists, then it is unique.

(c) $\{h^{(y)}\}_{y\in\mathbb{N}_0} \subseteq (\mathscr{H}_w((r_y)))_\eta$ is η-Cauchy, if $\lim_{x,y\longrightarrow\infty}\eta(h^{(x)} - h^{(y)}) = 0$.

(d) $Y \subset (\mathscr{H}_w((r_y)))_\eta$ is η-closed, if $\lim_{y\longrightarrow\infty}\eta(h^{(y)} - h) = 0$, where $\{h^{(y)}\}_{y\in\mathbb{N}_0} \subset Y$; then $h \in Y$.

(e) $Y \subset (\mathscr{H}_w((r_y)))_\eta$ is η-bounded, if $\delta_\eta(Y) = \sup\{\eta(g-h): g, h \in Y\} < \infty$.

(f) The η-ball of radius $l \ge 0$ and center g, for all $g \in (\mathscr{H}_w((r_y)))_\eta$, is defined as

$$\mathscr{B}_\eta(g, l) = \left\{h \in \left(\mathscr{H}_w\left(\left(r_y\right)\right)\right)_\eta : \eta(g-h) \le l\right\}. \quad (3)$$

(g) A prequasinormed (ssfps) η on $\mathscr{H}_w((r_y))$ verifies the Fatou property, if for every sequence $\{h^{(y)}\} \subseteq (\mathscr{H}_w((r_y)))_\eta$ with $\lim_{y\longrightarrow\infty}\eta(h^{(y)} - h) = 0$ and every $f \in (\mathscr{H}_w((r_y)))_\eta$,

$$\eta(f-g) \le \sup_j \inf_{y\ge j} \eta\left(f - g^{(y)}\right). \quad (4)$$

Take note that the Fatou property determined the η-balls' closedness. By ℓ_∞ and $mi_\nearrow$, we denote the space of real bounded sequences and the space of all monotonic increasing sequences of positive reals.

Lemma 2.7 (see [19]). The function $\psi(f) = [\sum_{r=0}^\infty |\hat{f}_r / r + 1|^{p_r}]^{1/K}$, for all $f \in (\mathscr{H}_w((r_y)))_\eta$, verifies the Fatou property, when $(r_n) \in mi_\nearrow \cap \ell_\infty$.

Lemma 2.8 (see [19]). If $(r_n) \in mi_\nearrow \cap \ell_\infty$, then the following settings hold:

(1) The function space $(\mathscr{H}_w((r_y)))_\eta$ is a prequasiclosed and Banach (ssfps), with

$$\psi(h) = \sum_{y=0}^\infty \left|\frac{\hat{h}_y}{y+1}\right|^{r_y}, \text{ for all } h \in \mathscr{H}_w\left(\left(r_y\right)\right). \quad (5)$$

(2) The class $(S_{(\mathscr{H}_w((r_y)))_\psi}, \Psi)$ is a prequasi-Banach and closed operator ideal, where $\Psi(P) = \psi(f_s) = \sum_{y=0}^\infty |s_y(P)/y+1|^{r_y}$, where $f_s(z) = \sum_{y=0}^\infty s_y(P) z^y \in \mathbb{C}$, for every $z \in \mathbb{C}$.

Lemma 2.9. *The following inequalities will be utilized in the continuation:*

(i) [25] If $b \ge 2$ and for each $f, g \in \mathbb{C}$, then

$$\left|\frac{f+g}{2}\right|^b + \left|\frac{f-g}{2}\right|^b \le \frac{1}{2}\left(|f|^b + |g|^b\right). \quad (6)$$

(ii) [26] Let $1 < b \le 2$, and for every $f, g \in \mathbb{C}$ with $|f| + |g| \ne 0$; then

$$\left|\frac{f+g}{2}\right|^b + \frac{b(b-1)}{2}\left|\frac{f-g}{|f|+|g|}\right|^{2-b}\left|\frac{f-g}{2}\right|^b \le \frac{1}{2}\left(|f|^b + |g|^b\right). \quad (7)$$

(iii) [27] Assume $b_y > 0$ and $f_y, g_y \in \mathbb{C}$, for each $y \in \mathbb{N}_0$; then $|f_y + g_y|^{b_y} \le 2^{K-1}(|f_y|^{b_y} + |g_y|^{b_y})$, where $K = \max\{1, sup_y b_y\}$.

3. Some Topological and Geometric Properties

In this section, first, we will talk about the uniform convexity (UUC 2) defined in [28] of the prequasinormed (ssfps) $(\mathscr{H}_w((r_a)))_\psi$.

Definition 3.1 (see [4, 29]). We define the prequasinorm ψ's uniform convexity type behavior as follows:

(1) [30] Suppose $\lambda > 0$ and $\beta > 0$. Let

$$\mathfrak{B}_1(\lambda,\beta)=\left\{(f,g)\colon\ f,g\in(\mathscr{H}_w((r_a)))_\psi,\psi(f)\le\lambda,\psi(g)\le\beta,\psi(f-g)\ge\lambda\beta\right\}. \tag{8}$$

When $\mathfrak{B}_1(\lambda,\beta)\neq\varnothing$, we put

$$V_1(\lambda,\beta)=\inf\left\{1-\frac{1}{\lambda}\psi\left(\frac{f+g}{2}\right)\colon\ (f,g)\in\mathfrak{B}_1(\lambda,\beta)\right\}. \tag{9}$$

When $\mathfrak{B}_1(\lambda,\beta)\neq\varnothing$, we put $V_1(\lambda,\beta)=1$. The function ψ holds the uniform convexity (UC) if for each $\lambda>0$ and $\beta>0$, we have $V_1(\lambda,\beta)>0$. Observe that, for all $\lambda>0$, then $\mathfrak{B}_1(\lambda,\beta)\neq\varnothing$, for very small $\beta>0$.

(2) [28] The function ψ verifies (UUC) if for every $\gamma\ge 0$ and $\beta>0$, there is $\zeta_1(\gamma,\beta)$ with

$$V_1(\lambda,\beta)>\zeta_1(\gamma,\beta)>0,\text{for }\lambda>\gamma. \tag{10}$$

(3) [28] Suppose $\lambda>0$ and $\beta>0$. Let

$$\mathfrak{B}_2(\lambda,\beta)=\left\{(f,g)\colon\ f,t\in(\mathscr{H}_w((r_a)))_\psi,\psi(f)\le\lambda,\psi(g)\le\lambda,\psi\left(\frac{f-g}{2}\right)\ge\lambda\beta\right\}. \tag{11}$$

When $\mathfrak{B}_2(\lambda,\beta)\neq\varnothing$, we put

$$V_2(\lambda,\beta)=\inf\left\{1-\frac{1}{\lambda}\psi\left(\frac{f+g}{2}\right)\colon\ (f,g)\in\mathfrak{B}_2(\lambda,\beta)\right\}. \tag{12}$$

When $\mathfrak{B}_2(\lambda,\beta)=\varnothing$, we place $V_2(\lambda,\beta)=1$. The function ψ satisfies (UC 2) if for every $\lambda>0$ and $\beta>0$, one has $V_2(\lambda,\beta)>0$. Observe that, for each $\lambda>0$, $\mathfrak{B}_2(\lambda,\beta)\neq\varnothing$, for very small $\beta>0$.

(4) [28] The function ψ verifies (UUC 2) if for all $\gamma\ge 0$ and $\beta>0$, there is $\zeta_2(\gamma,\beta)$ with

$$V_2(\lambda,\beta)>\zeta_2(\gamma,\beta)>0,\text{for }\lambda>\gamma. \tag{13}$$

(5) [30] The function ψ is strictly convex (SC), if for all $f,t\in(\mathscr{H}_w((r_a)))_\psi$ so that $\psi(f)=\psi(g)$ and $\psi((f+t)/2)=(\psi(f)+\psi(g))/2$, we get $f=g$.

We will require the following comment here and in the next: $\psi_B(f)=[\sum_{m\in B}|\widehat{f_m}/m+1|^{r_m}]^{1/K}$, for every $B\subset\mathbb{N}_0$ and $f\in(\mathscr{H}_w((r_a)))_\psi$. When $B=\varnothing$, we put $\psi_B(f)=0$.

Theorem 3.2. *The function $\psi(f)=[\sum_{a=0}^{\infty}(|\widehat{f_a}|/a+1)^{r_a}]^{1/K}$, for all $f\in(\mathscr{H}_w((r_a)))_\psi$, is (UUC2), if $(r_a)\in mi_{\nearrow}\cap\ell_\infty$ with $r_0>1$.*

Proof. Let the condition be satisfied, $b>0$, and $a>p\ge 0$. Suppose $f,g\in(\mathscr{H}_w((r_a)))_\psi$ so that

$$\begin{aligned}\psi(f)&\le a,\\ \psi(g)&\le a,\\ \psi\left(\frac{f-g}{2}\right)&\ge ab.\end{aligned} \tag{14}$$

From the definition of ψ, we have

$$\begin{aligned}ab\le\psi\left(\frac{f-g}{2}\right)&=\left[\sum_{d=0}^{\infty}(d+1)^{-r_d}\left|\frac{\widehat{f_d}-\widehat{g_d}}{2}\right|^{r_d}\right]^{\frac{1}{K}}\le\left[2^{-r_0}\sum_{d=0}^{\infty}(d+1)^{-r_d}\left|\widehat{f_d}-\widehat{g_d}\right|^{r_d}\right]^{\frac{1}{K}}\\ &\le 2^{-\frac{r_0}{K}}\left(\left[\sum_{d=0}^{\infty}(d+1)^{-r_d}|\widehat{f_d}|^{r_d}\right]^{\frac{1}{K}}+\left[\sum_{d=0}^{\infty}(d+1)^{-r_d}|\widehat{g_d}|^{r_d}\right]^{\frac{1}{K}}\right)=2^{-\frac{r_0}{K}}(\psi(f)+\psi(g))\le 2a,\end{aligned} \tag{15}$$

and this implies $b\le 2$. Consequent, let $Q=\{x\in\mathbb{N}_0\colon 1<r_x<2\}$ and $P=\{x\in\mathbb{N}_0\colon r_x\ge 2\}=\mathbb{N}_0\setminus Q$. For every $w\in(\mathscr{H}_w((r_a)))_\psi$, we get $\psi^K(w)=\psi_P^K(w)+\psi_Q^K(w)$. From the setup, one has $\psi_P(f-g/2)\ge ab/2$ or $\psi_Q(f-g/2)\ge ab/2$. Assume first $\psi_P(f-g/2)\ge ab/2$. By using Lemma 2.9, condition (i), we obtain

$$\psi_P^K\left(\frac{f+g}{2}\right)+\psi_P^K\left(\frac{f-g}{2}\right)\le\frac{\psi_P^K(f)+\psi_P^K(g)}{2}. \tag{16}$$

This explains

$$\psi_P^K\left(\frac{f+g}{2}\right)\le\frac{\psi_P^K(f)+\psi_P^K(g)}{2}-\left(\frac{ab}{2}\right)^K. \tag{17}$$

As

$$\psi_Q^K\left(\frac{f+g}{2}\right) \le \frac{\psi_Q^K(f)+\psi_Q^K(g)}{2}, \tag{18}$$

by adding inequalities 2 and 3, and from inequality 1, we have

$$\psi^K\left(\frac{f+g}{2}\right) \le \frac{\psi^K(f)+\psi^K(g)}{2} - \left(\frac{ab}{2}\right)^K \le a^K\left(1-\left(\frac{b}{2}\right)^K\right). \tag{19}$$

This gives

$$\psi\left(\frac{f+g}{2}\right) \le a\left(1-\left(\frac{b}{2}\right)^K\right)^{\frac{1}{K}}. \tag{20}$$

Next, suppose $\psi_Q((f-g)/2) \ge ab/2$. Set $B=(b/4)^K$,

$$Q_1 = \left\{d \in Q: \left|\widehat{f}_d - \widehat{g}_d\right| \le B\left(\left|\widehat{f}_d\right| + \left|\widehat{g}_d\right|\right)\right\} \text{ and } Q_2 = Q: Q_1. \tag{21}$$

As $B \le 1$ and the power function is convex,

$$\psi_{Q_1}^K\left(\frac{f-g}{2}\right) \le \sum_{d\in Q_1} B^{r_d}(d+1)^{-r_d}\left|\frac{\left|\widehat{f}_d\right| + \left|\widehat{g}_d\right|}{2}\right|^{r_d} \le \left(\frac{B}{2}\right)^{r_0}\left(\psi_{Q_1}^K(f)+\psi_{Q_1}^K(g)\right) \le \frac{B}{2}\left(\psi_Q^K(f)+\psi_Q^K(g)\right) \le \frac{B}{2}\left(\psi^K(f)+\psi^K(g)\right) \le Ba^K. \tag{22}$$

Since $\psi_Q((f-g)/2) \ge ab/2$, we get

$$\psi_{Q_2}^K\left(\frac{f-g}{2}\right) = \psi_Q^K\left(\frac{f-g}{2}\right) - \psi_{Q_1}^K\left(\frac{f-g}{2}\right) \ge a^K\left(\left(\frac{b}{2}\right)^K - \left(\frac{b}{4}\right)^K\right). \tag{23}$$

For any $d \in Q_2$, we have

$$r_0 - 1 < r_0(r_0-1) \le \cdots \le r_{d-1}(r_{d-1}-1) \le r_d(r_d-1) \text{ and } B < B^{2-r_d} < \left|\frac{\widehat{f}_d - \widehat{g}_d}{\left|\widehat{f}_d\right| + \left|\widehat{g}_d\right|}\right|^{2-r_d}. \tag{24}$$

By Lemma 2.9, condition (ii), we have

$$(d+1)^{-r_d}\left|\frac{\widehat{f}_d + \widehat{g}_d}{2}\right|^{r_d} + \frac{(r_0-1)B}{2}(d+1)^{-r_d}\left|\frac{\widehat{f}_d - \widehat{g}_d}{2}\right|^{r_d} \le \frac{1}{2}\left((d+1)^{-r_d}\left|\widehat{f}_d\right|^{r_d} + (d+1)^{-r_d}\left|\widehat{g}_d\right|^{r_d}\right). \tag{25}$$

Hence,

$$\psi_{Q_2}^K\left(\frac{f+g}{2}\right) + \frac{(r_0-1)B}{2}\psi_{Q_2}^K\left(\frac{f-g}{2}\right) \le \frac{\psi_{Q_2}^K(f)+\psi_{Q_2}^K(g)}{2}. \tag{26}$$

This investigates

$$\psi_{Q_2}^K\left(\frac{f+g}{2}\right) \le \frac{\psi_{Q_2}^K(f)+\psi_{Q_2}^K(g)}{2} - \frac{(r_0-1)B}{2}a^K\left(\left(\frac{b}{2}\right)^K - \left(\frac{b}{4}\right)^K\right). \tag{27}$$

Since

$$\psi_{Q_1}^K\left(\frac{f+g}{2}\right) \le \frac{\psi_{Q_1}^K(f)+\psi_{Q_1}^K(g)}{2}, \tag{28}$$

by adding inequalities (27) and (28), one has

$$\begin{aligned}\psi_Q^K\left(\frac{f+g}{2}\right) &\le \frac{\psi_Q^K(f)+\psi_Q^K(g)}{2} - \frac{(r_0-1)B}{2}a^K\left(\left(\frac{b}{2}\right)^K - \left(\frac{b}{4}\right)^K\right) \\ &\le \frac{\psi_Q^K(f)+\psi_Q^K(g)}{2} - \frac{(r_0-1)}{2}\left(\frac{b}{4}\right)^{2K}a^K\left(2^K-1\right) \\ &\le \frac{\psi_Q^K(f)+\psi_Q^K(g)}{2} - \frac{(r_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}a^K.\end{aligned} \tag{29}$$

Since

$$\psi_P^K\left(\frac{f+g}{2}\right) \le \frac{\psi_P^K(f)+\psi_P^K(g)}{2}, \tag{30}$$

by adding inequalities (29) and (30) and from inequality 1, we obtain

$$\psi^K\left(\frac{f+g}{2}\right)\leq\frac{\psi^K(f)+\psi^K(g)}{2}-\frac{(r_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}a^K\leq a^K\left[1-\frac{(r_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}\right]. \quad (31)$$

This implies

$$\psi\left(\frac{f+g}{2}\right)\leq a\left[1-\frac{(r_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}\right]^{\frac{1}{K}}. \quad (32)$$

It is clear that

$$1<r_0\leq K<2^K\Rightarrow 0<\frac{r_0-1}{2^K-1}<1. \quad (33)$$

By using inequalities 4 and 9 and Definition 3.1, we put

$$\zeta_2(p,b)=\min\left(1-\left(1-\left(\frac{b}{2}\right)^K\right)^{\frac{1}{K}},1-\left[1-\frac{(r_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}\right]^{\frac{1}{K}}\right). \quad (34)$$

Therefore, we have $V_2(a,b)>\zeta_2(p,b)>0$, and we conclude that ψ is (UUC2).We will examine the property (R) of the prequasinormed (ssfps) $(\mathscr{H}_w((r_a)))_\psi$ in this second part. □

Theorem 3.3. *Let $(r_a)\in mi_\nearrow\cap\ell_\infty$ with $r_0>1$; then the next setups are satisfied:*

(1) Assume that $\Lambda\subseteq(\mathscr{H}_w((r_a)))_\psi$, $\Lambda\neq\emptyset$, ψ-closed and ψ-convex, where $\psi(f)=[\sum_{a=0}^\infty(|\widehat{f_a}|/(a+1))^{r_a}]^{1/K}$, for all $f\in(\mathscr{H}_w((r_a)))_\psi$. Suppose $f\in(\mathscr{H}_w((r_a)))_\psi$ so that

$$d_\psi(f,\Lambda)=\inf\{\psi(f-g):\ g\in\Lambda\}<\infty. \quad (35)$$

Hence, one has a unique $\lambda\in\Lambda$ with $d_\psi(f,\Lambda)=\psi(f-\lambda)$.

(2) $(\mathscr{H}_w((r_a)))_\psi$ satisfies the property (R). This means that, for every decreasing sequence $\{\Lambda_x\}_{x\in\mathbb{N}_0}$ of ψ-closed and ψ-convex nonempty subsets of $(\mathscr{H}_w((r_a)))_\psi$ such that $\sup_{x\in\mathbb{N}_0}d_\psi(f,\Lambda_x)<\infty$, for some $f\in(\mathscr{H}_w((r_a)))_\psi$, then $\cap_{x\in\mathbb{N}_0}\Lambda_x\neq\emptyset$.

Proof. Suppose the setups are satisfied. To show (1), let $f\notin\Lambda$ as Λ is ψ-closed. Then, one has $A:=d_\psi(f,\Lambda)>0$. So, for every $p\in\mathbb{N}_0$, we have $g_p\in\Lambda$ with $\psi(f-g_p)<A(1+1/p)$. Assume $\{g_p/2\}$ is not ψ-Cauchy. Therefore, one obtains a subsequence $\{g_{h(p)}/2\}$ and $b_0>0$ so that $\psi((g_{h(p)}-g_{h(q)})/2)\geq b_0$, for all $p>q\geq 0$. Furthermore, one has $V_2(A(1+1/p),b_0/2A)>\xi:=\beta_2(A(1+1/p),b_0/2A)>0$, for each $p\in\mathbb{N}_0$. As

$$\max\left(h\left(f-g_{h(p)}\right),h\left(f-g_{h(q)}\right)\right)\leq A\left(1+\frac{1}{h(q)}\right), \quad (36)$$

and

$$\psi\left(\frac{g_{h(p)}-g_{h(q)}}{2}\right)\geq b_0\geq A\left(1+\frac{1}{h(q)}\right)\frac{b_0}{2A}. \quad (37)$$

Under $p>q\geq 0$, we get

$$\psi\left(f-\frac{g_{h(p)}+g_{h(q)}}{2}\right)\leq A\left(1+\frac{1}{h(q)}\right)(1-\xi). \quad (38)$$

Therefore,

$$A=d_\psi(f,\Lambda)\leq A\left(1+\frac{1}{h(q)}\right)(1-\xi), \quad (39)$$

with $q\in\mathbb{N}_0$. If we let $q\longrightarrow\infty$, we get

$$0<A\leq A\left(1+\frac{1}{h(q)}\right)(1-\xi)<A. \quad (40)$$

We have a contradiction. Then $\{g_p/2\}$ is ψ-Cauchy. As $(\mathscr{H}_w((r_a)))_\psi$ is ψ-complete, $\{g_p/2\}\psi$ converges to some g. For all $q\in\mathbb{N}_0$, we have the sequence $\{g_p+g_q/2\}\psi$ that converges to $g+g_q/2$. As Λ is ψ-closed and ψ-convex, one gets $g+g_q/2\in\Lambda$. Surely $g+g_q/2\psi$ converges to $2g$, so $2g\in\Lambda$. For $\lambda=2g$ and using Theorem 2.7, since ψ satisfies the Fatou property, we get

$$\begin{aligned}d_\psi(f,\Lambda)&\leq\psi(f-\lambda)\leq\sup_i\inf_{q\geq i}\psi\left(f-\left(g+\frac{g_q}{2}\right)\right)\leq\sup_i\inf_{q\geq i}\sup_i\inf_{p\geq i}\psi\left(f-\frac{g_p+g_q}{2}\right)\\&\leq\frac{1}{2}\sup_i\inf_{q\geq i}\sup_i\inf_{p\geq i}\left[\psi\left(f-g_p\right)+\psi\left(f-g_q\right)\right]=d_\psi(f,\Lambda).\end{aligned} \quad (41)$$

Therefore, $\psi(f-\lambda)=d_\psi(f,\Lambda)$. As ψ is (UUC2), so it is SC, which implies that λ is unique. To show (2), let $f\notin\Lambda_{p_0}$, for some $p_0\in\mathbb{N}_0$. $(d_\psi(f,\Lambda_p))_{p\in\mathbb{N}_0}\in\ell_\infty$ is increasing. Let $\lim_{p\longrightarrow\infty}d_\psi(f,\Lambda_p)=A$. Suppose $A>0$. Else $f\in\Lambda_p$, for every $p\in\mathbb{N}_0$. By using Part (1), we have one point $g_p\in\Lambda_p$ with $d_\psi(f,\Lambda_p)=\psi(f-g_p)$, for every $p\in\mathbb{N}_0$. A consistent proof will show that $\{g_p/2\}\psi$ converges to some $g\in(\mathscr{H}_w((r_a)))_\psi$. When $\{\Lambda_p\}$ are ψ-convex, decreasing, and ψ-closed, we get $2g\in\cap_{p\in\mathbb{N}_0}\Lambda_p$.

This third part discusses the prequasinormed structure's ψ-normal structure feature (ssfps) $(\mathscr{H}_w((r_a)))_\psi$. □

Definition 3.4. $(\mathscr{H}_w((r_a)))_\psi$ satisfies the ψ-normal structure property if for all nonempty ψ-bounded, ψ-convex, and

ψ-closed subset Λ of $(\mathscr{H}_w((r_a)))_\psi$ did not decrease to one point, we have $f \in \Lambda$ with

$$\sup_{g\in\Lambda}\psi(f-g) < \delta_\psi(\Lambda) := \sup\{\psi(f-g): f, g \in \Lambda\} < \infty. \tag{42}$$

Theorem 3.5. *If $(r_a) \in mi_\nearrow \cap \ell_\infty$ with $r_0 > 1$, then $(\mathscr{H}_w((r_a)))_\psi$ holds the ψ-normal structure property, where $\psi(f) = [\sum_{a=0}^{\infty}(|\widehat{f_a}|/(a+1))^{r_a}]^{1/K}$, for every $f \in (\mathscr{H}_w((r_a)))_\psi$.*

Proof. Assume the setups are satisfied. Theorem 3.2 explains that ψ is (UUC2). Let Λ be a ψ-bounded, ψ-convex, and ψ-closed subset of $(\mathscr{H}_w((r_a)))_\psi$ not decreased to unique point. Hence, $\delta_\psi(\Lambda) > 0$. Let $A = \delta_\psi(\Lambda)$. Suppose $f, g \in \Lambda$ with $f \neq g$. So $\psi((f-g)/2) = b > 0$. For all $\lambda \in \Lambda$, one obtains $\psi(f-\lambda) \le A$ and $\psi(g-\lambda) \le A$. Since Λ is ψ-convex, one has $(f+g)/2 \in \Lambda$. Hence,

$$\psi\left(\frac{f+g}{2}-\lambda\right) = \psi\left(\frac{(f-\lambda)+(g-\lambda)}{2}\right) \le A\left(1 - V_2\left(A, \frac{b}{A}\right)\right). \tag{43}$$

For all $\lambda \in \Lambda$,

$$\sup_{\lambda\in\Lambda}\psi\left(\frac{f+g}{2}-\lambda\right) \le A\left(1 - V_2\left(A, \frac{b}{A}\right)\right) < A = \delta_\psi(\Lambda). \tag{44}$$

□

4. Kannan Contraction Mapping

In the prequasinormed space, we now develop Kannan ψ-Lipschitzian mapping (ssfps). We study enough conditions on $(\mathscr{H}_w((r_a)))_\psi$ with a defined prequasinorm such that Kannan prequasinorm contraction mapping has a unique fixed point.

Definition 4.1. An operator $\mathfrak{W}: (\mathscr{H}_w((r_a)))_\psi \longrightarrow (\mathscr{H}_w((r_a)))_\psi$ is called a Kannan ψ-Lipschitzian, if there is $\kappa \ge 0$, so that

$$\psi(\mathfrak{W}f - \mathfrak{W}g) \le \kappa(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \tag{45}$$

For all $f, g \in (\mathscr{H}_w((r_a)))_\psi$, one has the following:

(1) If $\kappa \in [0, 1/2)$, then the operator $\mathfrak{W}$ is said to be Kannan ψ-contraction.

(2) If $\kappa = 1/2$, then the operator $\mathfrak{W}$ is said to be Kannan ψ-nonexpansive.

A vector $g \in (\mathscr{H}_w((r_a)))_\psi$ is called a fixed point of $\mathfrak{W}$, when $\mathfrak{W}(g) = g$.

Theorem 4.2. *If $(r_a) \in mi_\nearrow \cap \ell_\infty$ and $\mathfrak{W}: (\mathscr{H}_w((r_a)))_\psi \longrightarrow (\mathscr{H}_w((r_a)))_\psi$ is Kannan ψ-contraction mapping, where $\psi(f) = [\sum_{a=0}^{\infty}(|\widehat{f_a}|/(a+1))^{r_a}]^{1/K}$, for all $f \in (\mathscr{H}_w((r_a)))_\psi$, then $\mathfrak{W}$ has a unique fixed point.*

Proof. Let the setups be satisfied. For every $f \in (\mathscr{H}_w((r_a)))_\psi$, then $\mathfrak{W}^p f \in (\mathscr{H}_w((r_a)))_\psi$. Since $\mathfrak{W}$ is a Kannan ψ-contraction mapping, we have

$$\begin{aligned}
&\psi\left(\mathfrak{W}^{p+1}f - \mathfrak{W}^p f\right) \le \kappa\left(\psi\left(\mathfrak{W}^{p+1}f - \mathfrak{W}^p f\right) + \psi\left(\mathfrak{W}^p f - \mathfrak{W}^{p-1}f\right)\right) \Longrightarrow \\
&\psi\left(\mathfrak{W}^{p+1}f - \mathfrak{W}^p f\right) \le \frac{\kappa}{1-\kappa}\psi\left(\mathfrak{W}^p f - \mathfrak{W}^{p-1}f\right) \le \left(\frac{\kappa}{1-\kappa}\right)^2\psi\left(\mathfrak{W}^{p-1}f - \mathfrak{W}^{p-2}f\right) \le \cdots \le \left(\frac{\kappa}{1-\kappa}\right)^p\psi(\mathfrak{W}f - f).
\end{aligned} \tag{46}$$

Therefore, for every $p, q \in \mathbb{N}_0$ with $q > p$, then we get

$$\psi(\mathfrak{W}^p f - \mathfrak{W}^q f) \le \kappa\left(\psi\left(\mathfrak{W}^p f - \mathfrak{W}^{p-1}f\right) + \psi\left(\mathfrak{W}^q f - \mathfrak{W}^{q-1}f\right)\right) \le \kappa\left(\left(\frac{\kappa}{1-\kappa}\right)^{p-1} + \left(\frac{\kappa}{1-\kappa}\right)^{q-1}\right)\psi(\mathfrak{W}f - f). \tag{47}$$

So $\{\mathfrak{W}^p f\}$ is a Cauchy sequence in $(\mathscr{H}_w((r_a)))_\psi$. As the space $(\mathscr{H}_w((r_a)))_\psi$ is prequasi-Banach (ssfps). Therefore, there is $g \in (\mathscr{H}_w((r_a)))_\psi$ such that $\lim_{p\longrightarrow\infty}\mathfrak{W}^p f = g$. To prove that $\mathfrak{W}g = g$, by Theorem 2.7, ψ holds the Fatou property, and we have

$$\psi(\mathfrak{W}g - g) \le \sup_i \inf_{p\ge i}\psi\left(\mathfrak{W}^{p+1}f - \mathfrak{W}^p f\right) \le \sup_i \inf_{p\ge i}\left(\frac{\kappa}{1-\kappa}\right)^p\psi(\mathfrak{W}f - f) = 0. \tag{48}$$

Hence, $\mathfrak{W}g = g$. Then g is a fixed point of $\mathfrak{W}$. To show that the fixed point is unique, assume we have two different fixed points $f, g \in (\mathscr{H}_w((r_a)))_\psi$ of $\mathfrak{W}$. Then, one has

$$\psi(f - g) \leq \psi(\mathfrak{W}b - \mathfrak{W}t) \leq \kappa(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)) = 0. \tag{49}$$

Therefore, $f = g$. □

Corollary 4.3. *Let $(r_a) \in mi_\nearrow \cap \ell_\infty$ and $\mathfrak{W}: (\mathscr{H}_w((r_a)))_\psi \longrightarrow (\mathscr{H}_w((r_a)))_\psi$ be Kannan ψ-contraction mapping, where $\psi(f) = [\sum_{a=0}^{\infty} (|\widehat{f_a}|/(a+1))^{r_a}]^{1/K}$, for all $f \in (\mathscr{H}_w((r_a)))_\psi$; then $\mathfrak{W}$ has one and only one fixed point g with $\psi(\mathfrak{W}^p f - g) \leq \kappa(\kappa/(1-\kappa))^{p-1}\psi(\mathfrak{W}f - f)$.*

Proof. It is obvious, so it is omitted. □

Definition 4.4. Assume $(\mathscr{H}_w((r_a)))_\psi$ is a prequasinormed (ssfps) and $\mathfrak{W}: (\mathscr{H}_w((r_a)))_\psi \longrightarrow (\mathscr{H}_w((r_a)))_\psi$. The operator $\mathfrak{W}$ is called ψ sequentially continuous at $g \in (\mathscr{H}_w((r_a)))_\psi$, if and only if when $\lim_{a\longrightarrow\infty} \psi(f^{(a)} - g) = 0$, $\lim_{a\longrightarrow\infty} \psi(\mathfrak{W}f^{(a)} - \mathfrak{W}g) = 0$.

Theorem 4.5. *Let $(r_a) \in mi_\nearrow \cap \ell_\infty$ with $r_0 > 1$ and $\mathfrak{W}: (\mathscr{H}_w((r_a)))_\psi \longrightarrow (\mathscr{H}_w((r_a)))_\psi$, where $\psi(f) = \sum_{a=0}^{\infty} (|\widehat{f_a}|/(a+1))^{r_a}$, for all $f \in (\mathscr{H}_w((r_a)))_\psi$. The point $f \in (\mathscr{H}_w((r_a)))_\psi g \in (\mathscr{H}_w((r_a)))_\psi$ is the unique fixed point of $\mathfrak{W}$, if the following conditions are satisfied:*

(a) $\mathfrak{W}$ is Kannan ψ-contraction mapping.

(b) $\mathfrak{W}$ is ψ sequentially continuous at $g \in (\mathscr{H}_w((r_a)))_\psi$.

(c) One has $f \in (\mathscr{H}_w((r_a)))_\psi$ with the sequence of iterates $\{\mathfrak{W}^p f\}$ having a subsequence $\{\mathfrak{W}^{p_i} f\}$ converging to g.

Proof. Suppose the settings are verified. If g is not a fixed point of $\mathfrak{W}$, then $\mathfrak{W}g \neq g$. By conditions (b) and (c), we have

$$\begin{aligned} \lim_{p_i\longrightarrow\infty} \psi(\mathfrak{W}^{p_i} f - g) &= 0, \\ \lim_{p_i\longrightarrow\infty} \psi\left(\mathfrak{W}^{p_i+1} f - \mathfrak{W}g\right) &= 0. \end{aligned} \tag{50}$$

Since the mapping $\mathfrak{W}$ is Kannan ψ-contraction, one can see

$$\begin{aligned} 0 < \psi(\mathfrak{W}g - g) &= \psi\left(\left(\mathfrak{W}g - \mathfrak{W}^{p_i+1} f\right) + \left(\mathfrak{W}^{p_i} f - g\right) + \left(\mathfrak{W}^{p_i+1} f - \mathfrak{W}^{p_i} f\right)\right) \\ &\leq 2^{2\sup_i r_i - 2}\psi\left(\mathfrak{W}^{p_i+1} f - \mathfrak{W}g\right) + 2^{2\sup_i r_i - 2}\psi\left(\mathfrak{W}^{p_i} f - g\right) + 2^{\sup_i r_i - 1}\kappa\left(\frac{\kappa}{1-\kappa}\right)^{p_i - 1}\psi(\mathfrak{W}f - f). \end{aligned} \tag{51}$$

Since $p_i \longrightarrow \infty$, one has a contradiction. Hence, g is a fixed point of $\mathfrak{W}$. To explain that the fixed point g is unique, suppose we have two different fixed points $g, b \in (\mathscr{H}_w((r_a)))_\psi$ of $\mathfrak{W}$. Therefore, one gets

$$\psi(g - b) \leq \psi(\mathfrak{W}g - \mathfrak{W}b) \leq \kappa(\psi(\mathfrak{W}g - g) + \psi(\mathfrak{W}b - b)) = 0. \tag{52}$$

So, $g = b$. □

Theorem 4.6. *Assume $(r_a) \in (0, 1)^{\mathbb{N}_0}$ is an increasing, and $\mathfrak{W}: (\mathscr{H}_w((r_a)))_\psi \longrightarrow (\mathscr{H}_w((r_a)))_\psi$, where $\psi(f) = [\sum_{a=0}^{\infty} (|\widehat{f_a}|/(a+1))^{r_a}]^{1/r_0}$, for all $f \in (\mathscr{H}_w((r_a)))_\psi$. The point $g \in (\mathscr{H}_w((r_a)))_\psi$ is the only fixed point of $\mathfrak{W}$, if the following conditions are satisfied:*

(a) $\mathfrak{W}$ is Kannan ψ-contraction mapping.

(b) $\mathfrak{W}$ is ψ sequentially continuous at $g \in (\mathscr{H}_w((r_a)))_\psi$.

(c) One has $f \in (\mathscr{H}_w((r_a)))_\psi$ so that the sequence of iterates $\{\mathfrak{W}^p f\}$ has a subsequence $\{\mathfrak{W}^{p_i} f\}$ converging to g.

Proof. Let the conditions be verified. If g is not a fixed point of $\mathfrak{W}$, then $\mathfrak{W}g \neq g$. By conditions (b) and (c), we have

$$\begin{aligned} \lim_{p_i\longrightarrow\infty} \psi(\mathfrak{W}^{p_i} f - g) &= 0, \\ \lim_{p_i\longrightarrow\infty} \psi\left(\mathfrak{W}^{p_i+1} f - \mathfrak{W}g\right) &= 0. \end{aligned} \tag{53}$$

As the operator $\mathfrak{W}$ is Kannan ψ-contraction, one can see

$$\begin{aligned} 0 < \psi(\mathfrak{W}g - g) &= \psi\left(\left(\mathfrak{W}g - \mathfrak{W}^{p_i+1} f\right) + \left(\mathfrak{W}^{p_i} f - g\right) + \left(\mathfrak{W}^{p_i+1} f - \mathfrak{W}^{p_i} f\right)\right) \\ &\leq 2^{2r_0^{-1} - 2}\psi\left(\mathfrak{W}^{p_i+1} f - \mathfrak{W}g\right) + 2^{2r_0^{-1} - 2}\psi\left(\mathfrak{W}^{p_i} f - g\right) + 2^{r_0^{-1} - 1}\kappa\left(\frac{\kappa}{1-\kappa}\right)^{p_i - 1}\psi(\mathfrak{W}f - f). \end{aligned} \tag{54}$$

Since $p_i \longrightarrow \infty$, one obtains a contradiction. Hence, g is a fixed point of $\mathfrak{W}$. To explain that the fixed point g is unique, assume we have two different fixed points $g, b \in (\mathscr{H}_w((r_a)))_\psi$ of $\mathfrak{W}$. Then, one gets

$$\psi(g - b) \leq \psi(\mathfrak{W}g - \mathfrak{W}b) \leq \kappa(\psi(\mathfrak{W}g - g) + \psi(\mathfrak{W}b - b)) = 0. \tag{55}$$

So, $g = b$. □

Example 4.7. Pick up $\mathfrak{W}: (\mathscr{H}_w(((a+1)/(2a+4))_{a=0}^{\infty}))_{\psi} \longrightarrow (\mathscr{H}_w(((a+1)/(2a+4))_{a=0}^{\infty}))_{\psi}$, where $\psi(f) = \sum_{a\in\mathbb{N}_0}|\widehat{f_a}/(a+1)|^{(a+1)/(2a+4)}$, for all $f \in \mathscr{H}_w(((a+1)/(2a+4))_{a=0}^{\infty})$ and

$$\mathfrak{W}(f) = \begin{cases} \frac{f}{18}, & \psi(f) \in [0,1), \\ \frac{f}{20}, & \psi(f) \in [1,\infty). \end{cases} \tag{56}$$

For all $f_1, f_2 \in ((((a+1)/(2a+4))_{a=0}^{\infty}))_{\psi}$ with $\psi(f_1), \psi(f_2) \in [0,1)$, we have

$$\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) = \psi\left(\frac{f_1}{18} - \frac{f_2}{18}\right) \le \frac{1}{\sqrt[4]{17}}\left(\psi\left(\frac{17f_1}{18}\right) + \psi\left(\frac{17f_2}{18}\right)\right) = \frac{1}{\sqrt[4]{17}}(\psi(\mathfrak{W}f_1 - f_1) + \psi(\mathfrak{W}f_2 - f_2)). \tag{57}$$

For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ with $\psi(f_1), \psi(f_2) \in [1,\infty)$, we have

$$\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) = \psi\left(\frac{f_1}{20} - \frac{f_2}{20}\right) \le \frac{1}{\sqrt[4]{19}}\left(\psi\left(\frac{19f_1}{20}\right) + \psi\left(\frac{19f_2}{20}\right)\right) = \frac{1}{\sqrt[4]{19}}(\psi(\mathfrak{W}f_1 - f_1) + \psi(\mathfrak{W}f_2 - f_2)). \tag{58}$$

For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ with $\psi(f_1) \in [0,1)$ and $\psi(f_2) \in [1,\infty)$, we have

$$\begin{aligned}\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) &= \psi\left(\frac{f_1}{18} - \frac{f_2}{20}\right) \le \frac{1}{\sqrt[4]{17}}\psi\left(\frac{17f_1}{18}\right) + \frac{1}{\sqrt[4]{19}}\psi\left(\frac{19f_2}{20}\right) \le \frac{1}{\sqrt[4]{17}}\left(\psi\left(\frac{17f_1}{18}\right) + \psi\left(\frac{19f_2}{20}\right)\right) \\ &= \frac{1}{\sqrt[4]{17}}(\psi(\mathfrak{W}f_1 - f_1) + \psi(\mathfrak{W}f_2 - f_2)).\end{aligned} \tag{59}$$

Hence, $\mathfrak{W}$ is Kannan ψ-contraction mapping. By Theorem 2.7, the function ψ satisfies the Fatou property. By Theorem 4.2, the map $\mathfrak{W}$ has a unique fixed point $\theta \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$.

Let $\{f^{(n)}\} \subseteq (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ with $\lim_{n\longrightarrow\infty}\psi(f^{(n)} - f^{(0)}) = 0$, where $f^{(0)} \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ and $\psi(f^{(0)}) = 1$. Since the prequasinorm ψ is continuous, one gets

$$\lim_{n\longrightarrow\infty}\psi(\mathfrak{W}f^{(n)} - \mathfrak{W}f^{(0)}) = \lim_{n\longrightarrow\infty}\psi\left(\frac{f^{(n)}}{18} - \frac{f^{(0)}}{20}\right) = \psi\left(\frac{f^{(0)}}{180}\right) > 0. \tag{60}$$

Therefore, $\mathfrak{W}$ is not ψ sequentially continuous at $f^{(0)}$. Then, the map $\mathfrak{W}$ is not continuous at $f^{(0)}$.

If $\psi(f) = [\sum_{a\in\mathbb{N}_0}|\widehat{f_a}/(a+1)|^{(a+1)/(2a+4)}]^4$, for all $f \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})$. For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ with $\psi(f_1), \psi(f_2) \in [0,1)$, one obtains

$$\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) = \psi\left(\frac{f_1}{18} - \frac{f_2}{18}\right) \le \frac{8}{17}\left(\psi\left(\frac{17f_1}{18}\right) + \psi\left(\frac{17f_2}{18}\right)\right) = \frac{8}{17}(\psi(\mathfrak{W}f_1 - f_1) + \psi(\mathfrak{W}f_2 - f_2)). \tag{61}$$

For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\psi(f_1), \psi(f_2) \in [1,\infty)$, we have

$$\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) = \psi\left(\frac{f_1}{20} - \frac{f_2}{20}\right) \leq \frac{8}{19}\left(\psi\left(\frac{19f_1}{20}\right) + \psi\left(\frac{19f_2}{20}\right)\right) = \frac{8}{19}\left(\psi(\mathfrak{W}f_1 - f_1) + \psi(\mathfrak{W}f_2 - f_2)\right). \tag{62}$$

For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\psi(f_1) \in [0,1)$ and $\psi(f_2) \in [1,\infty)$, we have

$$\begin{aligned}\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) &= \psi\left(\frac{f_1}{18} - \frac{f_2}{20}\right) \leq \frac{8}{17}\psi\left(\frac{17f_1}{18}\right) + \frac{8}{19}\psi\left(\frac{19f_2}{20}\right) \leq \frac{8}{17}\left(\psi\left(\frac{17f_1}{18}\right) + \psi\left(\frac{19f_2}{20}\right)\right) \\ &= \frac{8}{17}\left(\psi(\mathfrak{W}f_1 - f_1) + \psi((\mathfrak{W}f_2 - f_2))\right).\end{aligned} \tag{63}$$

Therefore, the map $\mathfrak{W}$ is Kannan ψ-contraction mapping and $\mathfrak{W}^p(f) = \begin{cases} f/18^p, & \psi(f) \in [0,1), \\ f/20^p, & \psi(f) \in [1,\infty). \end{cases}$

Obviously, $\mathfrak{W}$ is ψ sequentially continuous at $\theta \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ and $\{\mathfrak{W}^p f\}$ has a subsequence $\{\mathfrak{W}^{p_i} f\}$ converging to θ. By Theorem 4.5, the point $\theta \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ is the unique fixed point of $\mathfrak{W}$.

Example 4.8. Assume $\mathfrak{W}: (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi \longrightarrow (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$, where $\psi(f) = \sqrt{\sum_{a\in\mathbb{N}_0} |\widehat{f_a}/(a+1)|^{(2a+3)/(a+2)}}$, for all $v \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})$ and

$$\mathfrak{W}(f) = \begin{cases} \dfrac{f}{4}, & \psi(f) \in [0,1), \\ \dfrac{f}{5}, & \psi(f) \in [1,\infty). \end{cases} \tag{64}$$

For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\psi(f_1), \psi(f_2) \in [0,1)$, we have

$$\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) = \psi\left(\frac{f_1}{4} - \frac{f_2}{4}\right) \leq \frac{1}{\sqrt[4]{27}}\left(\psi\left(\frac{3f_1}{4}\right) + \psi\left(\frac{3f_2}{4}\right)\right) = \frac{1}{\sqrt[4]{27}}\left(\psi(\mathfrak{W}f_1 - f_1) + \psi(\mathfrak{W}f_2 - f_2)\right). \tag{65}$$

For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\psi(f_1), \psi(f_2) \in [1,\infty)$, we have

$$\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) = \psi\left(\frac{f_1}{5} - \frac{f_2}{5}\right) \leq \frac{1}{\sqrt[4]{64}}\left(\psi\left(\frac{4f_1}{5}\right) + \psi\left(\frac{4f_2}{5}\right)\right) = \frac{1}{\sqrt[4]{64}}\left(\psi(\mathfrak{W}f_1 - f_1) + \psi(\mathfrak{W}f_2 - f_2)\right). \tag{66}$$

For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\psi(f_1) \in [0,1)$ and $\psi(f_2) \in [1,\infty)$, we have

$$\begin{aligned}\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) &= \psi\left(\frac{f_1}{4} - \frac{f_2}{5}\right) \leq \frac{1}{\sqrt[4]{27}}\psi\left(\frac{3f_1}{4}\right) + \frac{1}{\sqrt[4]{64}}\psi\left(\frac{4f_2}{5}\right) \leq \frac{1}{\sqrt[4]{27}}\left(\psi\left(\frac{3f_1}{4}\right) + \psi\left(\frac{4f_2}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}\left(\psi(\mathfrak{W}f_1 - f_1) + \psi(\mathfrak{W}f_2 - f_2)\right).\end{aligned} \tag{67}$$

Hence, $\mathfrak{W}$ is Kannan ψ-contraction mapping. From Theorem 2.7, the function ψ satisfies the Fatou property. By Theorem 4.2, the map $\mathfrak{W}$ has a unique fixed point $\theta \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$.

Let $\{f^{(n)}\} \subseteq (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ with $\lim_{n\longrightarrow\infty} \psi(f^{(n)} - f^{(0)}) = 0$, where $f^{(0)} \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ and $\psi(f^{(0)}) = 1$. Since the prequasinorm ψ is continuous, we have

$$\lim_{n\longrightarrow\infty} \psi\left(\mathfrak{W}f^{(n)} - \mathfrak{W}f^{(0)}\right) = \lim_{n\longrightarrow\infty} \psi\left(\frac{f^{(n)}}{4} - \frac{f^{(n)}}{5}\right) = \psi\left(\frac{f^{(0)}}{20}\right) > 0. \tag{68}$$

Therefore, $\mathfrak{W}$ is not ψ sequentially continuous at $f^{(0)}$. So, the map $\mathfrak{W}$ is not continuous at $f^{(0)}$.

Suppose $\psi(f) = [\sum_{a\in\mathbb{N}_0} |\widehat{f_a}/(a+1)|^{(a+1)/(2a+4)}]^4$, for all $f \in \mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty}$. For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ with $\psi(f_1), \psi(f_2) \in [0,1)$, we have

$$\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) = \psi\left(\frac{f_1}{4} - \frac{f_2}{4}\right) \leq \frac{2}{\sqrt{27}}\left(\psi\left(\frac{3f_1}{4}\right) + \psi\left(\frac{3f_2}{4}\right)\right) = \frac{2}{\sqrt{27}}\left(\psi(\mathfrak{W}f_1 - f_1) + \psi(\mathfrak{W}f_2 - f_2)\right). \tag{69}$$

For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ with $\psi(f_1), \psi(f_2) \in [1,\infty)$, we have

$$\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) = \psi\left(\frac{f_1}{5} - \frac{f_2}{5}\right) \leq \frac{1}{4}\left(\psi\left(\frac{4f_1}{5}\right) + \psi\left(\frac{4f_2}{5}\right)\right) = \frac{1}{4}\left(\psi(\mathfrak{W}f_1 - f_1) + \psi(\mathfrak{W}f_2 - f_2)\right). \tag{70}$$

For all $f_1, f_2 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ with $\psi(f_1) \in [0,1)$ and $\psi(f_2) \in [1,\infty)$, we have

$$\begin{aligned}\psi(\mathfrak{W}f_1 - \mathfrak{W}f_2) &= \psi\left(\frac{f_1}{4} - \frac{f_2}{5}\right) \leq \frac{2}{\sqrt{27}}\psi\left(\frac{3f_1}{4}\right) + \frac{1}{4}\psi\left(\frac{4f_2}{5}\right) \leq \frac{2}{\sqrt{27}}\left(\psi\left(\frac{3f_1}{4}\right) + \psi\left(\frac{4f_2}{5}\right)\right) \\ &= \frac{2}{\sqrt{27}}\left(\psi(\mathfrak{W}f_1 - \mathfrak{W}f_1) + \psi(\mathfrak{W}f_2 - \mathfrak{W}f_2)\right).\end{aligned} \tag{71}$$

So, the map $\mathfrak{W}$ is Kannan ψ-contraction mapping and $\mathfrak{W}^p(f) = \begin{cases} f/4^p, & \psi(f) \in [0,1), \\ f/5^p, & \psi(f) \in [1,\infty). \end{cases}$

Obviously, $\mathfrak{W}$ is ψ sequentially continuous at $\theta \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ and $\{\mathfrak{W}^p f\}$ has a subsequence $\{\mathfrak{W}^{p_i} f\}$ converging to θ. By Theorem 4.5, the point $\theta \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ is the unique fixed point of $\mathfrak{W}$.

Example 4.9. Let $\mathfrak{W}: (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi} \longrightarrow (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$, where $\psi(f) = [\sum_{a\in\mathbb{N}_0} |\widehat{f_a}/(a+1)|^{(a+1)/(2a+4)}]^4$, for all $f \in \mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty}$ and

$$\mathfrak{W}(f) = \begin{cases} \dfrac{1}{18}(1+f), & \widehat{f}_0 \in \left(-\infty, \dfrac{1}{17}\right), \\ \dfrac{1}{17}, & \widehat{f}_0 = \dfrac{1}{17}, \\ \dfrac{1}{18}, & \widehat{f}_0 \in \left(\dfrac{1}{17}, \infty\right). \end{cases} \tag{72}$$

For all $f, g \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_{\psi}$ with $\widehat{f}_0, \widehat{g}_0 \in (-\infty, 1/17)$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = \psi\left(\frac{1}{18}(f-g)\right) \leq \frac{8}{17}\left(\psi\left(\frac{17f}{18}\right) + \psi\left(\frac{17g}{18}\right)\right) \leq \frac{8}{17}\left(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)\right). \tag{73}$$

For all $f, g \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\widehat{f}_0, \widehat{g}_0 \in (1/17, \infty)$ and then for any $\varepsilon > 0$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = 0 \leq \varepsilon(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (74)$$

For all $f, g \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\widehat{f}_0 \in (-\infty, 1/17)$ and $\widehat{g}_0 \in (1/17, \infty)$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = \psi\left(\frac{f}{18}\right) \leq \frac{1}{17}\psi\left(\frac{17f}{18}\right) = \frac{1}{17}\psi(\mathfrak{W}f - f) \leq \frac{1}{17}(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (75)$$

Hence, $\mathfrak{W}$ is Kannan ψ-contraction mapping. Clearly, $\mathfrak{W}$ is ψ sequentially continuous at $1/17 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$, and there is $f \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\widehat{f}_0 \in (-\infty, 1/17)$ such that the sequence of iterates $\{\mathfrak{W}^p f\} = \left\{\sum_{n=1}^{p} 1/18^n + 1/18^p f\right\}$ has a subsequence $\{\mathfrak{W}^{p_i} f\} = \left\{\sum_{n=1}^{p} 1/18^n + 1/18^p f\right\}$ converging to 1/17. By Theorem 4.5, the map $\mathfrak{W}$ has one fixed point $1/17 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$. Note that $\mathfrak{W}$ is not continuous at $1/17 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$.

If $\psi(f) = \sum_{a\in\mathbb{N}_0} |\widehat{f_a}/(a+1)|^{(a+1)/(2a+4)}$, $f \in \mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty}$. For all $f, g \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\widehat{f}_0, \widehat{g}_0 \in (-\infty, 1/17)$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = \psi\left(\frac{1}{18}(f - g)\right) \leq \frac{1}{\sqrt[4]{17}}\left(\psi\left(\frac{17f}{18}\right) + \psi\left(\frac{17g}{18}\right)\right)$$
$$\leq \frac{1}{\sqrt[4]{17}}(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (76)$$

For all $f, g \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\widehat{f}_0, \widehat{g}_0 \in (1/17, \infty)$ and then for any $\varepsilon > 0$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = 0 \leq \varepsilon(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (77)$$

For all $f, g \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\widehat{f}_0 \in (-\infty, 1/17)$ and $\widehat{g}_0 \in (1/17, \infty)$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = \psi\left(\frac{f}{18}\right) \leq \frac{1}{\sqrt[4]{17}}\psi\left(\frac{17f}{18}\right) = \frac{1}{\sqrt[4]{4}17}\psi(\mathfrak{W}f - f) \leq \frac{1}{\sqrt[4]{17}}(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (78)$$

So, $\mathfrak{W}$ is Kannan ψ-contraction mapping. By Theorem 2.7, the function ψ satisfies the Fatou property. By Theorem 4.2, the map $\mathfrak{W}$ has a unique fixed point $1/17 \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$.

Example 4.10. Let $\mathfrak{W}: (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi \longrightarrow (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$, where $\psi(f) = \sum_{a\in\mathbb{N}_0} |\widehat{f_a}/(a+1)|^{(a+1)/(2a+4)}$, for all $v \in \mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty}$ and

$$\mathfrak{W}(f) = \begin{cases} \frac{1}{4}(z + tf), & \widehat{f}_0 \in \left(-\infty, \frac{1}{3}\right), \\ \frac{1}{3}z, & \widehat{f}_0 = \frac{1}{3}, \\ \frac{1}{4}z, & \widehat{f}_0 \in \left(\frac{1}{3}, \infty\right). \end{cases} \quad (79)$$

For all $f, g \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\widehat{f}_0, \widehat{g}_0 \in (-\infty, 1/3)$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = \psi\left(\frac{1}{4}(f - g)\right) \leq \frac{2}{\sqrt{27}}\left(\psi\left(\frac{3f}{4}\right) + \psi\left(\frac{3g}{4}\right)\right)$$
$$\leq \frac{2}{\sqrt{27}}(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (80)$$

For all $f, g \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\widehat{f}_0, \widehat{g}_0 \in (1/3, \infty)$ and then for any $\varepsilon > 0$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = 0 \leq \varepsilon(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (81)$$

For all $f, g \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\widehat{f}_0 \in (-\infty, 1/3)$ and $\widehat{g}_0 \in (1/3, \infty)$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = \psi\left(\frac{f}{4}\right) \le \frac{1}{\sqrt{27}}\psi\left(\frac{3f}{4}\right) = \frac{1}{\sqrt{27}}\psi(\mathfrak{W}f - f) \le \frac{1}{\sqrt{27}}(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (82)$$

Hence, $\mathfrak{W}$ is Kannan ψ-contraction mapping. Evidently, $\mathfrak{W}$ is ψ sequentially continuous at $1/3z \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ and there is $f \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$ with $\hat{f}_0 \in (-\infty, 1/3)$ such that the sequence of iterates $\{\mathfrak{W}^p f\} = \{\sum_{n=1}^{p} 1/4^n z + 1/4^p f\}$ has a subsequence $\{\mathfrak{W}^{p_i} f\} = \{\sum_{n=1}^{p_i} 1/4^n z + 1/4^{p_i} f\}$ converging to $1/3z$. By Theorem 4.5, the map $\mathfrak{W}$ has one fixed point $1/3z \in (\mathscr{H}_w((a+1)/(2a+4))_{a=0}^{\infty})_\psi$. Note that $\mathfrak{W}$ is not continuous at $1/3z \in (\mathscr{H}_w((2a+3/a+2)_{a=0}^{\infty}))_\psi$.

If $\psi(f) = \sum_{a\in\mathbb{N}_0} |\widehat{f_a}/(a+1)|^{(2a+3)/(a+2)}$, $f \in \mathscr{H}_w((2a+3/a+2)_{a=0}^{\infty})$. For all $f, g \in (\mathscr{H}_w((2a+3/a+2)_{a=0}^{\infty}))_\psi$ with $\hat{f}_0, \hat{g}_0 \in (-\infty, 1/3)$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = \psi\left(\frac{1}{4}(f-g)\right) \le \frac{1}{\sqrt[4]{27}}\left(\psi\left(\frac{3f}{4}\right) + \psi\left(\frac{3g}{4}\right)\right)$$
$$\le \frac{1}{\sqrt[4]{27}}(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (83)$$

For all $f, g \in (\mathscr{H}_w((2a+3/a+2)_{a=0}^{\infty}))_\psi$ with $\hat{f}_0, \hat{g}_0 \in (1/3, \infty)$ and then for any $\varepsilon > 0$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = 0 \le \varepsilon(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (84)$$

For all $f, g \in (\mathscr{H}_w((2a+3/a+2)_{a=0}^{\infty}))_\psi$ with $\hat{f}_0 \in (-\infty, 1/3)$ and $\hat{g}_0 \in (1/3, \infty)$, we have

$$\psi(\mathfrak{W}f - \mathfrak{W}g) = \psi\left(\frac{f}{4}\right) \le \frac{1}{\sqrt[4]{27}}\psi\left(\frac{3f}{4}\right) = \frac{1}{\sqrt[4]{27}}\psi(\mathfrak{W}f - f) \le \frac{1}{\sqrt[4]{27}}(\psi(\mathfrak{W}f - f) + \psi(\mathfrak{W}g - g)). \quad (85)$$

Hence, $\mathfrak{W}$ is Kannan ψ-contraction mapping. By Theorem 2.7, the function ψ satisfies the Fatou property. By Theorem 4.2, the map $\mathfrak{W}$ has a unique fixed point $1/3z \in (\mathscr{H}_w((2a+3/a+2)_{a=0}^{\infty}))_\psi$.

5. Kannan Nonexpansive Mapping

We examine enough conditions on the prequasinormed (ssfps) $(\mathscr{H}_w((r_a)))_\psi$ for the Kannan prequasinorm nonexpansive mapping on it to have a fixed point in this section.

Lemma 5.1. *Allow the prequasinormed (ssfps) $(\mathscr{H}_w((r_a)))_\psi$ to validate the (R) and ψ-quasi-normal properties. If Ξ is a nonempty ψ-bounded, ψ-convex, and ψ-closed subset of $(\mathscr{H}_w((r_a)))_\psi$, suppose $\mathfrak{W}: \Xi \longrightarrow \Xi$ is a Kannan ψ-nonexpansive mapping. For $y > 0$, let $G_y = \{f \in \Xi: \psi(f - \mathfrak{W}(f)) \le y\} \ne \varnothing$. Let*

$$\Xi_y = \cap\left\{\mathscr{B}_\psi(\lambda, y): \mathfrak{W}(G_y) \subset \mathscr{B}_\psi(\lambda, y)\right\} \cap \Xi. \quad (86)$$

Then, Ξ_y is a nonempty, ψ-convex, ψ-closed subset of Ξ and

$$\mathfrak{W}(\Xi_y) \subset \Xi_y \subset G_y,$$
$$\delta_\psi(\Xi_y) \le y. \quad (87)$$

Proof. As $\mathfrak{W}(G_y) \subset \Xi_y$, this gives $\Xi_y \ne \varnothing$. As the ψ-balls are ψ-convex and ψ-closed, one has Ξ_y is a ψ-closed and ψ-convex subset of Ξ. To prove that $\Xi_y \subset G_y$, suppose $f \in \Xi_y$. If $\psi(f - \mathfrak{W}(f)) = 0$, we get $f \in G_y$. Otherwise, suppose $\psi(f - \mathfrak{W}(f)) > 0$. Let

$$\lambda = \sup\left\{\psi(\mathfrak{W}(w) - \mathfrak{W}(f)): w \in G_y\right\}. \quad (88)$$

From the definition of λ, $\mathfrak{W}(G_y) \subset \mathscr{B}_\psi(\mathfrak{W}(f), \lambda)$. So, $\Xi_y \subset \mathscr{B}_\psi(\mathfrak{W}(f), \lambda)$, and we have $\psi(f - \mathfrak{W}(f)) \le \lambda$. Assume $\eta > 0$. Hence, there is $w \in G_y$ so that $\lambda - \eta \le \psi(\mathfrak{W}(w) - \mathfrak{W}(f))$. Then,

$$\psi(f - \mathfrak{W}(f)) - \eta \le \lambda - \eta \le \psi(\mathfrak{W}(w) - \mathfrak{W}(f)) \le \frac{1}{2}(\psi(f - \mathfrak{W}(f))) + \psi(w - \mathfrak{W}(w)) \le \frac{1}{2}(\psi(f - \mathfrak{W}(f)) + y). \quad (89)$$

Since η is randomly positive, we obtain $\psi(f-\mathfrak{W}(f))\le y$, so we have $f\in G_y$. As $\mathfrak{W}(G_y)\subset \Xi_y$, one has $\mathfrak{W}(\Xi_y)\subset \mathfrak{W}(G_y)\subset \Xi_y$; this indicates that Ξ_y is $\mathfrak{W}$-invariant, consequent to prove that $\delta_\psi(\Xi_y)\le y$, as

$$\psi(\mathfrak{W}(f)-\mathfrak{W}(g))\le \frac{1}{2}(\psi(f-\mathfrak{W}(f))+\psi(g-\mathfrak{W}(g))). \tag{90}$$

For every $f,g\in G_y$, let $f\in G_y$. Hence, $\mathfrak{W}(G_y)\subset \mathscr{B}_\psi(\mathfrak{W}(f),y)$. The definition of Ξ_y provides $\Xi_y\subset \mathscr{B}_\psi(\mathfrak{W}(f),y)$. Hence, $\mathfrak{W}(f)\in \cap_{g\in\Xi_y}\mathscr{B}_\psi(g,y)$. So, one has $\psi(g-w)\le y$, for every $g,w\in\Xi_y$, this means $\delta_\psi(\Xi_y)\le y$. This completes the proof. □

Theorem 5.2. *Let the prequasinormed (ssfps) $(\mathscr{H}_w((r_a)))_\psi$ verify the ψ-quasi-normal property and the (R) property. Assume Ξ is a nonempty, ψ-convex, ψ-closed, and ψ-bounded subset of $(\mathscr{H}_w((r_a)))_\psi$. Suppose $\mathfrak{W}:\Xi\longrightarrow\Xi$ is a Kannan ψ-nonexpansive mapping. Hence, $\mathfrak{W}$ has a fixed point.*

Proof. Suppose $d_0=\inf\{\psi(f-\mathfrak{W}(f)): f\in\Xi\}$ and $d_t=d_0+1/t$, for every $t\ge 1$. By using the definition of d_0, we have $G_{d_t}=\{f\in\Xi: \psi(f-\mathfrak{W}(f))\le d_t\}\ne\varnothing$, with $t\ge1$. Assume Ξ_{d_t} is described as in Lemma 5.1. Obviously, $\{\Xi_{d_t}\}$ is a decreasing sequence of nonempty ψ-bounded, ψ-closed, and ψ-convex subsets of Ξ. The property (R) proves that $\Xi_\infty=\cap_{t\ge1}\Xi_{d_t}\ne\varnothing$. Assume $f\in\Xi_\infty$; one can see $\psi(f-\mathfrak{W}(f))\le d_t$, for every $t\ge1$. Suppose $p\longrightarrow\infty$; we have $\psi(f-\mathfrak{W}(f))\le d_0$; this gives $\psi(f-\mathfrak{W}(f))=d_0$. Therefore, $G_{d_0}\ne\varnothing$. We have $d_0=0$. Else, $d_0>0$; this gives that $\mathfrak{W}$ fails to have a fixed point. Let Ξ_{d_0} be defined as in Lemma 5.1. As $\mathfrak{W}$ misses to have a fixed point and Ξ_{d_0} is $\mathfrak{W}$-invariant, so Ξ_{d_0} has more than one point, which implies, $\delta_\psi(\Xi_{d_0})>0$. By the ψ-quasinormal property, there is $f\in\Xi_{d_0}$ so that

$$\psi(f-g)<\delta_\psi\left(\Xi_{d_0}\right)\le d_0. \tag{91}$$

□

For all $g\in\Xi_{d_0}$, by Lemma 5.1, we have $\Xi_{d_0}\subset G_{d_0}$. By definition of Ξ_{d_0}, then $\mathfrak{W}(f)\in G_{d_0}\subset\Xi_{d_0}$. Obviously, we have

$$\psi(f-\mathfrak{W}(f))<\delta_\psi\left(\Xi_{d_0}\right)\le d_0, \tag{92}$$

which contradicts the definition of d_0. So $d_0=0$, which implies that any point in G_{d_0} is a fixed point of $\mathfrak{W}$; that is, $\mathfrak{W}$ has a fixed point in Ξ.

In view of Theorems 3.3, 3.5, and 5.2, it is easy to conclude the following theorem.

Theorem 5.3. *If $(r_a)\in mi_\nearrow\cap\ell_\infty$ with $r_0>1$, Ξ is a nonempty, ψ-convex, ψ-closed, and ψ-bounded subset of $(\mathscr{H}_w((r_a)))_\psi$, where $\psi(f)=[\sum_{a=0}^\infty(|\widehat{f_a}|/(a+1))^{r_a}]^{1/K}$, for every $(\mathscr{H}_w((r_a)))_\psi$, and $\mathfrak{W}:\Xi\longrightarrow\Xi$ is a Kannan ψ-nonexpansive mapping. Then, $\mathfrak{W}$ has a fixed point.*

Example 5.4. Let $\mathfrak{W}:\Xi\longrightarrow\Xi$ with $\mathfrak{W}(f)=\begin{cases} f/4, & \psi(f)\in[0,1),\\ f/5, & \psi(f)\in[1\quad,t\infty),\end{cases}$ where $\Xi=\left\{f\in(\mathscr{H}_w((2a+3/a+2)_{a=0}^\infty))_\psi: \widehat{f}_0=\widehat{f}_1=0\right\}$ and $\psi(f)=\sqrt{\sum_{a\in\mathbb{N}_0}|\widehat{f_a}/(a+1)|^{(2a+3)/(a+2)}}$, for all $f\in(\mathscr{H}_w((2a+3/a+2)_{a=0}^\infty))_\psi$. In view of example 4.8, the map $\mathfrak{W}$ is Kannan ψ-contraction mapping. This implies Kannan ψ-nonexpansive mapping. Evidently, Ξ is a nonempty, ψ-convex, ψ-closed, and ψ-bounded subset of $(\mathscr{H}_w((2a+3/a+2)_{a=0}^\infty))_\psi$. By Theorem 5.3, the map $\mathfrak{W}$ has one fixed point $(f=\theta)$ in Ξ.

6. Kannan Contraction Mappings on the Operator Ideal

We study in this section the presence of a fixed point for the Kannan prequasinorm contraction mapping in the prequasi-Banach operator ideal defined by the $(\mathscr{H}_w((r_a)))_\psi$ and s-numbers.

Notations 6.1. [19]

$$S_{\mathscr{H}}:=\{S_{\mathscr{H}}(\mathfrak{X},\mathfrak{Y});\ \mathfrak{X}\text{ and }\mathfrak{Y}\text{ are Banach Spaces}\},\text{ where}$$
$$S_{\mathscr{H}}(\mathfrak{X},\mathfrak{Y}):=\left\{P\in L(\mathfrak{X},\mathfrak{Y}): f_s\in\mathscr{H},\text{ where } f_s(z)=\sum_{y=0}^\infty s_y(P)z^y\in\mathbb{C},\text{ for any } z\in\mathbb{C}\right\}. \tag{93}$$

Definition 6.2. If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, a prequasinorm Ψ on the ideal $S_{(\mathscr{H}_w((r_a)))_\psi}$, where $\Psi(W)=\psi(f_s)$ and $f_s(z)=\sum_{a=0}^\infty s_a(W)z^a$ converge for any $z\in\mathbb{C}$, satisfies the Fatou property if for every sequence $\{W_a\}_{a\in\mathbb{N}_0}\subseteq S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$ with $\lim_{a\longrightarrow\infty}\Psi(W_a-W)=0$ and any $V\in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$,

$$\Psi(V-W)\le\sup_a\inf_{i\ge a}\Psi(V-W_i). \tag{94}$$

Theorem 6.3. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces. The prequasinorm $\Psi(W)=[\sum_{a=0}^\infty(s_a(W)/(a+1))^{r_a}]^{1/K}$, for all $W\in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$ does not satisfy the Fatou property, if $(r_a)\in mi_\nearrow\cap\ell_\infty$.*

Proof. Let the condition be satisfied and $\{W_p\}_{p\in\mathbb{N}_0}\subseteq S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$ with $\lim_{p\longrightarrow\infty}\Psi(W_p-W)=0$. By Theorem 2.8, the space $S_{(\mathscr{H}_w((r_a)))_\psi}$ is a prequasiclosed ideal, and then $W \in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$. Hence, for all $V \in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$, we have

$$\begin{aligned}\Psi(V-W) &= \left[\sum_{a=0}^{\infty}\left(\frac{s_a(V-W)}{a+1}\right)^{r_a}\right]^{\frac{1}{K}} \le \left[\sum_{a=0}^{\infty}\left(\frac{s_{[a/2]}(V-W_i)}{a+1}\right)^{r_a}\right]^{\frac{1}{K}} + \left[\sum_{a=0}^{\infty}\left(\frac{s_{[a/2]}(W-W_i)}{a+1}\right)^{r_a}\right]^{\frac{1}{K}} \\ &\le 2^{\frac{1}{K}}\sup_p \inf_{i\ge p}\left[\sum_{a=0}^{\infty}\left(\frac{s_a(V-W_i)}{a+1}\right)^{r_a}\right]^{\frac{1}{K}} = 2^{\frac{1}{K}}\sup_p \inf_{i\ge p}\Psi(V-W_i).\end{aligned} \tag{95}$$

Hence, Ψ does not satisfy the Fatou property. □

Definition 6.4. Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces. For the prequasinorm Ψ on the ideal $S_{(\mathscr{H}_w((r_a)))_\psi}$, where $\Psi(W)=\psi(f_s)$, where $f_s(z)=\sum_{a=0}^{\infty} s_a(W)z^a$ converges for any $z\in\mathbb{C}$, an operator $G: S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y}) \longrightarrow S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$ is called a Kannan Ψ-Lipschitzian, if there is $\kappa\ge 0$, so that

$$\Psi(GW-GA)\le\kappa(\Psi(GW-W)+\Psi(GA-A)), \tag{96}$$

for all $W, A\in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$. An operator G is said to be

(1) Kannan Ψ-contraction, when $\kappa\in[0,1/2)$.

(2) Kannan Ψ-nonexpansive, when $\kappa=1/2$.

Definition 6.5. Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces. For the prequasinorm Ψ on the ideal $S_{(\mathscr{H}_w((r_a)))_\psi}$, where $\Psi(W)=\psi(f_s)$, where $f_s(z)=\sum_{a=0}^{\infty} s_a(W)z^a$ converges for any $z\in\mathbb{C}$, $G: S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y}) \longrightarrow S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$ and $B\in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$. The operator G is said to be Ψ sequentially continuous at B, if and only if when $\lim_{p\longrightarrow\infty}\Psi(W_p-B)=0$, $\lim_{p\longrightarrow\infty}\Psi(GW_p-GB)=0$.

Theorem 6.6. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces. Let $(r_a)\in mi_{\nearrow}\cap\ell_\infty$ and $G: S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y}) \longrightarrow S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$, where $\Psi(W)=[\sum_{a=0}^{\infty}(s_a(W)/(a+1))^{r_a}]^{1/K}$, for all $W\in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$. The point $A\in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$ is the unique fixed point of G, if the following conditions are satisfied:*

(a) G is Kannan Ψ-contraction mapping.

(b) G is Ψ sequentially continuous at a point $A\in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$.

(c) There is $B\in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$ so that the sequence of iterates $\{G^pB\}$ has a subsequence $\{G^{p_i}B\}$ converging to A.

Proof. Let the conditions be verified. If A is not a fixed point of G, then $GA\ne A$. From conditions (b) and (c), we have

$$\begin{aligned}\lim_{p_i\longrightarrow\infty}\Psi(G^{p_i}B-A)&=0,\\ \lim_{p_i\longrightarrow\infty}\Psi\left(G^{p_i+1}B-GA\right)&=0.\end{aligned} \tag{97}$$

Since G is Kannan Ψ-contraction mapping, one can see

$$\begin{aligned}0<\Psi(GA-A)&=\Psi\left(\left(GA-G^{p_i+1}B\right)+\left(G^{p_i}B-A\right)+\left(G^{p_i+1}B-G^{p_i}B\right)\right)\\ &\le 2^{\frac{1}{K}}\Psi\left(G^{p_i+1}B-GA\right)+2^{\frac{2}{K}}\Psi(G^{p_i}B-A)+2^{\frac{2}{K}}\kappa\left(\frac{\kappa}{1-\kappa}\right)^{p_i-1}\Psi(GB-B).\end{aligned} \tag{98}$$

As $p_i\longrightarrow\infty$, we have a contradiction. Therefore, A is a fixed point of G. To show that the fixed point A is unique. Let us have two different fixed points $A, D\in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$ of G. Therefore, one has

$$\Psi(A-D)\le\Psi(GA-GD)\le\kappa(\Psi(GA-A)+\Psi(GD-D))=0. \tag{99}$$

So, $A=D$. □

Example 6.7. Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces; $G: S_{(\mathscr{H}_w(((a+1)/(a+2))_{a=0}^{\infty}))_\psi}(\mathfrak{X},\mathfrak{Y}) \longrightarrow S_{(\mathscr{H}_w(((a+1)/(a+2))_{a=0}^{\infty}))_\psi}(\mathfrak{X},\mathfrak{Y})$, where $\Psi(W)=\sum_{a=0}^{\infty}(s_a(W)/(a+1))^{a+1/a+2}$, for every $W\in S_{(\mathscr{H}_w(((a+1)/(a+2))_{a=0}^{\infty}))_\psi}(\mathfrak{X},\mathfrak{Y})$ and

$$G(W)=\begin{cases}\dfrac{W}{26}, & \Psi(W)\in[0,1),\\[2ex] \dfrac{W}{37}, & \Psi(W)\in[1,\infty).\end{cases} \tag{100}$$

For all $W_1, W_2\in S_{(\mathscr{H}_w(((a+1)/(a+2))_{a=0}^{\infty}))_\psi}$ with $\Psi(W_1),\Psi(W_2)\in[0,1)$, we have

$$\Psi(GW_1-GW_2)=\Psi\left(\frac{W_1}{26}-\frac{W_2}{26}\right)\leq\frac{2}{5}\left(\Psi\left(\frac{25W_1}{26}\right)+\Psi\left(\frac{25W_2}{26}\right)\right)=\frac{2}{5}\left(\Psi(GW_1-W_1)+\Psi(GW_2-W_2)\right). \tag{101}$$

For all $W_1, W_2 \in S_{(\mathscr{H}_w(((a+1)/(a+2))_{a=0}^{\infty}))_\psi}$ with $\Psi(W_1), \Psi(W_2) \in [1,\infty)$, we have

$$\Psi(GW_1-GW_2)=\Psi\left(\frac{W_1}{37}-\frac{W_2}{37}\right)\leq\frac{1}{3}\left(\Psi\left(\frac{36W_1}{37}\right)+\Psi\left(\frac{36W_2}{37}\right)\right)=\frac{1}{3}\left(\Psi(GW_1-W_1)+\Psi(GW_2-W_2)\right). \tag{102}$$

For all $W_1, W_2 \in S_{(\mathscr{H}_w(((a+1)/(a+2))_{a=0}^{\infty}))_\psi}$ with $\Psi(W_1) \in [0,1)$ and $\Psi(W_2) \in [1,\infty)$, we have

$$\begin{aligned}\Psi(GW_1-GW_2)&=\Psi\left(\frac{W_1}{26}-\frac{W_2}{37}\right)\leq\frac{2}{5}\Psi\left(\frac{25W_1}{26}\right)+\frac{1}{3}\Psi\left(\frac{36W_2}{37}\right)\leq\frac{2}{5}\left(\Psi\left(\frac{25W_1}{26}\right)+\Psi\left(\frac{36W_2}{37}\right)\right)\\&=\frac{2}{5}\left(\Psi(GW_1-W_1)+\Psi(GW_2-W_2)\right).\end{aligned} \tag{103}$$

Hence, G is Kannan Ψ-contraction mapping and $G^p(W)=\begin{cases}W/26^p, & \Psi(W)\in[0,1),\\ W/37^p, & \Psi(W)\in[0,1).\end{cases}$

Evidently, G is Ψ sequentially continuous at the zero operator $\Theta \in S_{(\mathscr{H}_w(((a+1)/(a+2))_{a=0}^{\infty}))_\psi}$ and $\{G^pW\}$ has a subsequence $\{G^{p_i}W\}$ converging to Θ. By Theorem 6.6, the zero operator $\Theta \in S_{(\mathscr{H}_w(((a+1)/(a+2))_{a=0}^{\infty}))_\psi}$ is the only fixed point of G. Assume $\{W^{(n)}\}\subseteq S_{(\mathscr{H}_w(((a+1)/(a+2))_{a=0}^{\infty}))_\psi}$ with $\lim_{n\longrightarrow\infty}\Psi(W^{(n)}-W^{(0)})=0$, where $W^{(0)} \in S_{(\mathscr{H}_w(((a+1)/(a+2))_{a=0}^{\infty}))_\psi}$ and $\Psi(W^{(0)})=1$. Since the prequasinorm Ψ is continuous, one obtains

$$\lim_{n\longrightarrow\infty}\Psi\left(GW^{(n)}-GW^{(0)}\right)=\lim_{n\longrightarrow\infty}\Psi\left(\frac{W^{(n)}}{26}-\frac{W^{(0)}}{37}\right)=\Psi\left(\frac{11W^{(0)}}{962}\right)>0. \tag{104}$$

Therefore, G is not Ψ sequentially continuous at $W^{(0)}$. Then, the map G is not continuous at $W^{(0)}$.

Example 6.8. If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, $G: S_{(\mathscr{H}_w(((2a+1)/(a+3))_{a=0}^{\infty}))_\psi}(\mathfrak{X},\mathfrak{Y}) \longrightarrow S_{(\mathscr{H}_w(((2a+1)/(a+3))_{a=0}^{\infty}))_\psi}(\mathfrak{X},\mathfrak{Y})$, where $\Psi(W)=\sqrt{\sum_{a=0}^{\infty}(s_a(W)/(a+1))^{2a+1/a+3}}$, for every $W \in S_{(\mathscr{H}_w(((2a+1)/(a+3))_{a=0}^{\infty}))_\psi}(\mathfrak{X},\mathfrak{Y})$ and

$$G(W)=\begin{cases}\dfrac{W}{263170}, & \Psi(W)\in[0,1),\\ \dfrac{W}{263171}, & \Psi(W)\in[1,\infty).\end{cases} \tag{105}$$

For all $W_1, W_2 \in S_{(\mathscr{H}_w(((2a+1)/(a+3))_{a=0}^{\infty}))_\psi}$ with $\Psi(W_1), \Psi(W_2) \in [0,1)$, we have

$$\begin{aligned}\Psi(GW_1-GW_2)&=\Psi\left(\frac{W_1}{263170}-\frac{W_2}{263170}\right)\leq\frac{\sqrt{2}}{\sqrt[6]{263169}}\left(\Psi\left(\frac{263169W_1}{263170}\right)+\Psi\left(\frac{263169W_2}{263170}\right)\right)\\&=\frac{\sqrt{2}}{\sqrt[6]{263169}}\left(\Psi(GW_1-W_1)+\Psi(GW_2-W_2)\right).\end{aligned} \tag{106}$$

For all $W_1, W_2 \in S_{(\mathscr{H}_w(((2a+1)/(a+3))_{a=0}^{\infty}))_\psi}$ with $\Psi(W_1), \Psi(W_2) \in [1,\infty)$, we have

$$\begin{aligned}\Psi(GW_1-GW_2)&=\Psi\left(\frac{W_1}{263171}-\frac{W_2}{263171}\right)\leq\frac{\sqrt{2}}{\sqrt[6]{263170}}\left(\Psi\left(\frac{263170W_1}{263171}\right)+\Psi\left(\frac{263170W_2}{263171}\right)\right)\\&=\frac{\sqrt{2}}{\sqrt[6]{263170}}\left(\Psi(GW_1-W_1)+\Psi(GW_2-W_2)\right).\end{aligned} \tag{107}$$

For all $W_1, W_2 \in S_{(\mathscr{H}_w(((2a+1)/(a+3))_{a=0}^{\infty}))_\psi}$ with $\Psi(W_1) \in [0,1)$ and $\Psi(W_1) \in [1,\infty)$, we have

$$\begin{aligned}\Psi(GW_1 - GW_2) &= \Psi\left(\frac{W_1}{263170} - \frac{W_2}{263171}\right) \leq \frac{\sqrt{2}}{\sqrt{[6]}263169}\Psi\left(\frac{263169W_1}{263170}\right) + \frac{\sqrt{2}}{\sqrt{[6]}263170}\Psi\left(\frac{263170W_2}{263171}\right)\\ &\leq \frac{\sqrt{2}}{\sqrt{[6]}263169}\left(\Psi\left(\frac{263169W_1}{263170}\right) + \Psi\left(\frac{263170W_2}{263171}\right)\right) = \frac{\sqrt{2}}{\sqrt{[6]}263169}\left(\Psi(GW_1 - W_1) + \Psi(GW_2 - W_2)\right).\end{aligned} \tag{108}$$

Hence, G is Kannan Ψ-contraction mapping and $G^p(W) = \begin{cases} W/263170^p, & \Psi(W) \in [0,1), \\ W/263171^p, & \Psi(W) \in [1 \quad ,t\infty). \end{cases}$

Obviously, G is Ψ sequentially continuous at the zero operator $\Theta \in S_{(\mathscr{H}_w(((2a+1)/(a+3))_{a=0}^{\infty}))_\psi}$ and $\{G^p W\}$ has a subsequence $\{G^{p_i} W\}$ converging to Θ. By Theorem 6.6, the zero operator $\Theta \in S_{(\mathscr{H}_w(((2a+1)/(a+3))_{a=0}^{\infty}))_\psi}$ is the only fixed point of G. Suppose $\{W^{(n)}\} \subseteq S_{(\mathscr{H}_w(((2a+1)/(a+3))_{a=0}^{\infty}))_\psi}$ with $\lim_{n\to\infty} \Psi(W^{(n)} - W^{(0)}) = 0$, where $W^{(0)} \in S_{(\mathscr{H}_w(((2a+1)/(a+3))_{a=0}^{\infty}))_\psi}$ and $\Psi(W^{(0)}) = 1$. Since the prequasinorm Ψ is continuous, one gets

$$\lim_{n\longrightarrow\infty} \Psi\left(GW^{(n)} - GW^{(0)}\right) = \lim_{n\longrightarrow\infty} \Psi\left(\frac{W^{(n)}}{263170} - \frac{W^{(0)}}{263171}\right) = \Psi\left(\frac{W^{(0)}}{69258712070}\right) > 0. \tag{109}$$

Therefore, G is not Ψ sequentially continuous at $W^{(0)}$. Then, the map G is not continuous at $W^{(0)}$.

7. Application to Nonlinear Summable Equations

Numerous authors have examined nonlinear summable equations such as (10); see [31–33]. This section is dedicated to locating a solution to (10) in $(\mathscr{H}_w((r_a)))_\psi$, where $(r_a) \in mi_\nearrow \cap \ell_\infty$ and $\psi(g) = [\sum_{a=0}^{\infty} (|\widehat{g}_a|/(a+1))^{r_a}]^{1/K}$, for every $g \in (\mathscr{H}_w((r_a)))_\psi$. Take a look at the equations that are summable:

$$\widehat{g}_a = \widehat{p}_a + \sum_{m=0}^{\infty} A(a,m) f(m, \widehat{g_m}), \tag{110}$$

and assume $W\colon (\mathscr{H}_w((r_a)))_\psi \longrightarrow (\mathscr{H}_w((r_a)))_\psi$ defined by

$$W(g) = \sum_{a=0}^{\infty}\left(\widehat{p}_a + \sum_{m=0}^{\infty} A(a,m) f(m, \widehat{g_m})\right) z^a. \tag{111}$$

Theorem 7.1. *The summable equation (10) has one solution in $(\mathscr{H}_w((r_a)))_\psi$, if $A\colon \mathbb{N}_0^2 \longrightarrow \mathbb{C}$, $f\colon \mathbb{N}_0 \times \mathbb{C} \longrightarrow \mathbb{C}$, $\widehat{p}\colon \mathbb{N}_0 \longrightarrow \mathbb{C}$, $\widehat{t}\colon \mathbb{N}_0 \longrightarrow \mathbb{C}$, and for every $a \in \mathbb{N}_0$, we have $\kappa \in [0,1/2)$, with*

$$\left|\sum_{m\in\mathbb{N}_0} A(a,m)\left(f(m,\widehat{g_m}) - f(m,\widehat{t_m})\right)\right|^{r_a} \leq \kappa^K\left[\left|\widehat{p}_a - \widehat{g}_a + \sum_{m=0}^{\infty} A(a,m)\left(f(m,\widehat{g_m})\right)\right|^{r_a} + \left|\widehat{p}_a - \widehat{t}_a + \sum_{m=0}^{\infty} A(a,m)\left(f(m,\widehat{t_m})\right)\right|^{r_a}\right]. \tag{112}$$

Proof. Let the setups be verified. Consider the mapping $W\colon (\mathscr{H}_w((r_a)))_\psi \longrightarrow (\mathscr{H}_w((r_a)))_\psi$ defined by (11). We have

$$\begin{aligned}\psi(Wg - Wt) &= \left[\sum_{a\in\mathbb{N}_0}\left(\frac{\left|\sum_{m\in\mathbb{N}_0} A(a,m)\left[f(m,\widehat{g_m}) - f(m,\widehat{t_m})\right]\right|}{a+1}\right)^{r_a}\right]^{\frac{1}{K}}\\ &\leq \kappa\left(\left[\sum_{a\in\mathbb{N}_0}\left(\frac{\left|\widehat{p}_a - \widehat{g}_a + \sum_{m=0}^{\infty} A(a,m) f(m,\widehat{g_m})\right|}{a+1}\right)^{r_a}\right]^{\frac{1}{K}} + \left[\sum_{a\in\mathbb{N}_0}\left(\frac{\left|\widehat{p}_a - \widehat{t}_a + \sum_{m=0}^{\infty} A(a,m) f(m,\widehat{t_m})\right|}{a+1}\right)^{r_a}\right]^{\frac{1}{K}}\right)\\ &= \kappa\left(\psi(Wg - g) + \psi(Wt - t)\right).\end{aligned} \tag{113}$$

According to Theorem 4.2, one obtains a unique solution of equation (10) in $(\mathscr{H}_w((r_a)))_\psi$. □

Example 7.2. Assume the function space $(\mathscr{H}_w((a+1)/(a+2))_{a=0}^{\infty})_\psi$, where $\psi(f)=\sum_{a\in\mathbb{N}_0}(|\widehat{f_a}|/a+1)^{a+1/a+2}$, for all $f\in\mathscr{H}_w((a+1)/(a+2))_{a=0}^{\infty}$. Consider the summable equation

$$\widehat{g}_a = 5^{-(2a+3i)} + \sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos|\widehat{g}_a|}{\sinh|\widehat{g}_a|+\sin ma+1}\right)^q, \tag{114}$$

where $q>0$ and $i^2=-1$ and let $W:(\mathscr{H}_w((a+1/a+2))_{a=0}^{\infty})_\psi \longrightarrow (\mathscr{H}_w((a+1/a+2))_{a=0}^{\infty})_\psi$ defined by

$$W(g)=\sum_{a=0}^{\infty}\left(5^{-(2a+3i)}+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos|\widehat{g}_a|}{\sinh|\widehat{g}_a|+\sin ma+1}\right)^q\right)z^a. \tag{115}$$

It is easy to see that

$$\begin{aligned}&\left|\sum_{m=0}^{\infty}(-1)^{ai}\left(\frac{\cos|\widehat{g}_a|}{\sinh|\widehat{g}_a|+\sin ma+1}\right)^q\left((-1)^{3m}-(-1)^{3m}\right)\right|^{\frac{a+1}{a+2}}\\&\leq\frac{1}{3}\left|5^{-(2a+3i)}-\widehat{g}_a+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos|\widehat{g}_a|}{\sinh|\widehat{g}_a|+\sin ma+1}\right)^q\right|^{\frac{a+1}{a+2}}\\&+\frac{1}{3}\left|5^{-(2a+3i)}-\widehat{t}_a+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos|\widehat{t}_a|}{\sinh|\widehat{t}_a|+\sin ma+1}\right)^q\right|^{\frac{a+1}{a+2}}.\end{aligned} \tag{116}$$

By Theorem 7.1, the summable equation (114) has one solution in $(\mathscr{H}_w((a+1/a+2))_{a=0}^{\infty})_\psi$.

Example 7.3. Given the function space $(\mathscr{H}_w((2a+1/a+3))_{a=0}^{\infty})_\psi$, where $\psi(g)=\sqrt{\sum_{a\in\mathbb{N}_0}(|\widehat{g}_a|/a+1)^{2a+1/a+3}}$, for all $g\in\mathscr{H}_w((2a+1/a+3))_{a=0}^{\infty}$, consider the summable equation (12). It is easy to see that

$$\begin{aligned}&\left|\sum_{m=0}^{\infty}(-1)^{ai}\left(\frac{\cos|\widehat{g}_a|}{\sinh|\widehat{g}_a|+\sin ma+1}\right)^q\left((-1)^{3m}-(-1)^{3m}\right)\right|^{2a+1/a+3}\\&\leq\frac{1}{9}\left|5^{-(2a+3i)}-\widehat{g}_a+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos|\widehat{g}_a|}{\sinh|\widehat{g}_a|+\sin ma+1}\right)^q\right|^{2a+1/a+3}\\&+\frac{1}{9}\left|5^{-(2a+3i)}-\widehat{t}_a+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos|\widehat{t}_a|}{\sinh|\widehat{t}_a|+\sin ma+1}\right)^q\right|^{2a+1/a+3}.\end{aligned} \tag{117}$$

By Theorem 7.1, the summable equation (114) has one solution in $(\mathscr{H}_w((a+1/a+2))_{a=0}^{\infty})_{\psi}$.

Example 7.4. Given the function space $(\mathscr{H}_w((2a+3/a+2))_{a=0}^{\infty})_{\psi}$, where $\psi(f)=\sqrt{\sum_{a\in\mathbb{N}_0}(|\widehat{f_a}|/a+1)^{2a+3/a+2}}$, for all $f\in\mathscr{H}_w((2a+3/a+2))_{a=0}^{\infty}$, consider the summable equation (114) with $a\geq 2$ and let $W:\Xi\longrightarrow\Xi$, where $\Xi=\{f\in\mathscr{H}_w((2a+3/a+2))_{a=0}^{\infty}: \widehat{f}_0=\widehat{f}_1=0\}$, defined by

$$W(f)=\sum_{a=2}^{\infty}\left(5^{-(2a+3i)}+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos\left|\widehat{f_a}\right|}{\sinh\left|\widehat{f_a}\right|+\sin ma+1}\right)^{q}\right)z^{a}. \tag{118}$$

Clearly, Ξ is a nonempty, ψ-convex, ψ-closed, and ψ-bounded subset of $(\mathscr{H}_w((2a+3/a+2))_{a=0}^{\infty})_{\psi}$. It is easy to see that

$$\begin{aligned}&\left|\sum_{m=0}^{\infty}(-1)^{ai}\left(\frac{\cos|\widehat{g}_a|}{\sinh|\widehat{g}_a|+\sin ma+1}\right)^{q}\left((-1)^{3m}-(-1)^{3m}\right)\right|^{2a+3/a+2}\\&\leq\frac{1}{9}\left|5^{-(2a+3i)}-\widehat{g}_a+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos|\widehat{g}_a|}{\sinh|\widehat{g}_a|+\sin ma+1}\right)^{q}\right|^{2a+3/a+2}\\&+\frac{1}{9}\left|5^{-(2a+3i)}-\widehat{t}_a+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos|\widehat{t}_a|}{\sinh|\widehat{t}_a|+\sin ma+1}\right)^{q}\right|^{2a+3/a+2}.\end{aligned} \tag{119}$$

By Theorem 7.1 and Theorem 5.3, the summable equation (114) with $a\geq 2$ has a solution in Ξ.

In this part, we search for a solution to nonlinear matrix (120) at $D\in S_{(\mathscr{H}_w((r_a)))_{\psi}}(\mathfrak{X},\mathfrak{Y})$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, $(r_a)\in mi\nearrow\cap\ell_{\infty}$, and $\Psi(G)=[\sum_{a=0}^{\infty}(s_a(G)/a+1)^{r_a}]^{1/K}$, for all $G\in S_{(\mathscr{H}_w((r_a)))_{\psi}}(\mathfrak{X},\mathfrak{Y})$. Consider the summable equation

$$s_a(G)=s_a(P)+\sum_{m=0}^{\infty}A(a,m)f(m,s_m(G)), \tag{120}$$

and suppose $W: S_{(\mathscr{H}_w((r_a)))_{\psi}}(\mathfrak{X},\mathfrak{Y})\longrightarrow S_{(\mathscr{H}_w((r_a)))_{\psi}}(\mathfrak{X},\mathfrak{Y})$ defined by

$$W(G)=\sum_{a=0}^{\infty}s_a(P)+\sum_{m=0}^{\infty}A(a,m)f(m,s_m(G))z^{a}. \tag{121}$$

Theorem 7.5. *The summable equation (120) has one solution in $S_{(\mathscr{H}_w((r_a)))_{\psi}}(\mathfrak{X},\mathfrak{Y})$, if the following conditions are satisfied:*

(a) $A:\mathbb{N}_0^2\longrightarrow\mathbb{C}$, $f:\mathbb{N}_0\times[0,t\infty)\longrightarrow\mathbb{C}$, $P\in L(\mathfrak{X},\mathfrak{Y})$, $T\in L(\mathfrak{X},\mathfrak{Y})$, and for every $a\in\mathbb{N}_0$, one has $\kappa\in[0,t1n/2)$, with

$$\begin{aligned}&\left|\sum_{m\in\mathbb{N}_0}A(a,m)\left(f(m,s_m(G))-f(m,s_m(T))\right)\right|^{r_a}\\&\leq\kappa^{K}\left[\left|s_a(P)-s_a(G)+\sum_{m=0}^{\infty}A(a,m)f(m,s_m(G))\right|^{r_a}+\left|s_a(P)-s_a(T)+\sum_{m=0}^{\infty}A(a,m)f(m,s_m(T))\right|^{r_a}\right].\end{aligned} \tag{122}$$

(b) W is Ψ sequentially continuous at a point $D\in S_{(\mathscr{H}_w((r_a)))_{\psi}}(\mathfrak{X},\mathfrak{Y})$.

(c) There is $B\in S_{(\mathscr{H}_w((r_a)))_{\psi}}(\mathfrak{X},\mathfrak{Y})$ so that the sequence of iterates $\{W^pB\}$ has a subsequence $\{W^{p_i}B\}$ converging to D.

Proof. Suppose the settings are verified. Consider the mapping $W: S_{(\mathscr{H}_w((r_a)))_{\psi}}(\mathfrak{X},\mathfrak{Y})\longrightarrow S_{(\mathscr{H}_w((r_a)))_{\psi}}(\mathfrak{X},\mathfrak{Y})$ defined by (16). We have

$$\psi(WG-WT)=\left[\sum_{a\in\mathbb{N}_0}\left(\frac{\left|\sum_{m\in\mathbb{N}_0}A(a,m)\left(f\left(m,s_m(G)\right)-f\left(m,s_m(T)\right)\right)\right|}{a+1}\right)^{r_a}\right]^{1/K}$$
$$\leq\kappa\left[\sum_{a\in\mathbb{N}_0}\left(\frac{\left|s_a(P)-s_a(G)+\sum_{m=0}^{\infty}A(a,m)f\left(m,s_m(G)\right)\right|}{a+1}\right)^{r_a}\right]^{1/K}$$
$$+\kappa\left[\sum_{a\in\mathbb{N}_0}\left(\frac{\left|s_a(P)-s_a(T)+\sum_{m=0}^{\infty}A(a,m)f\left(m,s_m(T)\right)\right|}{a+1}\right)^{r_a}\right]^{1/K}$$
$$=\kappa(\psi(WG-G)+\psi(WT-T)). \tag{123}$$

In view of Theorem 6.6, one obtains a unique solution of (120) at $D\in S_{(\mathscr{H}_w((r_a)))_\psi}(\mathfrak{X},\mathfrak{Y})$. ☐

Example 7.6. Assume the function space $(\mathscr{H}_w((a+1/a+2))_{a=0}^{\infty})_\psi$, where $\psi(g)=\sum_{a\in\mathbb{N}_0}(|\widehat{g}_a|/a+1)^{a+1/a+2}$, for all $g\in\mathscr{H}_w((a+1/a+2))_{a=0}^{\infty}$. Consider the nonlinear difference equation

$$\widehat{g}_a=e^{-(2a+3i)}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3mi-a)\cos^p|\widehat{g_{a-2}}|}{\sinh^q|\widehat{g_{a-1}}|+\sin ma+1}, \tag{124}$$

where $\widehat{g_{-2}},\widehat{g_{-1}},p,q>0$, $i^2=-1$, and let $W\colon(\mathscr{H}_w((a+1/a+2))_{a=0}^{\infty})_\psi\longrightarrow(\mathscr{H}_w((a+1/a+2))_{a=0}^{\infty})_\psi$ defined by

$$W(g)=\sum_{a=0}^{\infty}\left(e^{-(2at+n3qi)}+\sum_{m=0}^{\infty}\frac{\tan(2mt+n1)\cosh(3mtin-qa)\cos^p|\widehat{g_{a-2}}|}{\sinh^q|\widehat{g_{a-1}}|+\sin ma+1}\right)z^a. \tag{125}$$

It is easy to see that

$$\left|\sum_{m=0}^{\infty}\frac{\cosh(3mi-a)\cos^p|\widehat{g_{a-2}}|}{\sinh^q|\widehat{g_{a-1}}|+\sin ma+1}(\tan(2m+1)-\tan(2m+1))\right|^{a+1/a+2}$$
$$\leq\frac{1}{5}\left|e^{-(2a+3i)}-\widehat{g}_a+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3mi-a)\cos^p|\widehat{g_{a-2}}|}{\sinh^q|\widehat{g_{a-1}}|+\sin ma+1}\right|^{a+1/a+2}$$
$$+\frac{1}{5}\left|e^{-(2a+3i)}-\widehat{t}_a+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3mi-a)\cos^p|\widehat{t_{a-2}}|}{\sinh^q|\widehat{t_{a-1}}|+\sin ma+1}\right|^{a+1/a+2}. \tag{126}$$

By Theorem 7.1, the nonlinear difference equation (124) has one solution in $(\mathscr{H}_w((a+1/a+2))_{a=0}^{\infty})_\psi$.

Example 7.7. Given the function space $(\mathscr{H}_w((2a+1/a+3))_{a=0}^{\infty})_\psi$, where $\psi(g)=\sum_{a\in\mathbb{N}_0}(|\widehat{g}_a|/a+1)^{a+1/a+2}$, for all $g\in\mathscr{H}_w((2a+1/a+3))_{a=0}^{\infty}$, consider the nonlinear difference equation (17). It is easy to see that

$$\left|\sum_{m=0}^{\infty}\frac{\cosh(3mi-a)\cos^{p}|\widehat{g_{a-2}}|}{\sinh^{q}|\widehat{g_{a-1}}|+\sin ma+1}(\tan(2m+1)-\tan(2m+1))\right|^{2a+1/a+3} \leq \frac{1}{25}\left|e^{-(2a+3i)}-\widehat{g}_a+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3mi-a)\cos^{p}|\widehat{g_{a-2}}|}{\sinh^{q}|\widehat{g_{a-1}}|+\sin ma+1}\right|^{2a+1/a+3} + \frac{1}{25}\left|e^{-(2a+3i)}-\widehat{t}_a+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3mi-a)\cos^{p}|\widehat{t_{a-2}}|}{\sinh^{q}|\widehat{t_{a-1}}|+\sin ma+1}\right|^{2a+1/a+3}. \quad (127)$$

By Theorem 7.1, the nonlinear difference equation (124) has one solution in $(\mathscr{H}_w((2a+1/a+3))_{a=0}^{\infty})_{\psi}$.

Example 7.8. Given the function space $(\mathscr{H}_w((2a+3/a+2))_{a=0}^{\infty})_{\psi}$, where $\psi(g)=\sum_{a\in\mathbb{N}}(|\widehat{g}_a|/a+1)^{a+1/a+2}$, for all $g\in\mathscr{H}_w((2a+3/a+2))_{a=0}^{\infty}$, consider the nonlinear difference equation (124) with $a\geq 1$ and let $W:\Xi\longrightarrow\Xi$, where $\Xi=\{g\in\mathscr{H}_w((2a+3/a+2))_{a=0}^{\infty}:\ \widehat{g}_0=0\}$, defined by

$$W(g)=\sum_{a=1}^{\infty}\left(e^{-(2at+n3qi)}+\sum_{m=0}^{\infty}\frac{\tan(2mt+n1)\cosh(3mtin-qa)\cos^{p}|\widehat{g_{a-2}}|}{\sinh^{q}|\widehat{g_{a-1}}|+\sin ma+1}\right)z^{a}. \quad (128)$$

Clearly, Ξ is a nonempty, ψ-convex, ψ-closed, and ψ-bounded subset of $(\mathscr{H}_w((2a+3/a+2))_{a=0}^{\infty})_{\psi}$. It is easy to see that

$$\left|\sum_{m=0}^{\infty}\frac{\cosh(3mi-a)\cos^{p}|\widehat{g_{a-2}}|}{\sinh^{q}|\widehat{g_{a-1}}|+\sin ma+1}(\tan(2m+1)-\tan(2m+1))\right|^{2a+1/a+3} \leq \frac{1}{25}\left|e^{-(2a+3i)}-\widehat{g}_a+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3mi-a)\cos^{p}|\widehat{g_{a-2}}|}{\sinh^{q}|\widehat{g_{a-1}}|+\sin ma+1}\right|^{2a+1/a+3} + \frac{1}{25}\left|e^{-(2a+3i)}-\widehat{t}_a+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3mi-a)\cos^{p}|\widehat{t_{a-2}}|}{\sinh^{q}|\widehat{t_{a-1}}|+\sin ma+1}\right|^{2a+1/a+3}. \quad (129)$$

By Theorem 7.1 and Theorem 5.3, the nonlinear difference equation (124) with $a\geq 1$ has a solution in Ξ.

8. Conclusion

This paper studies the existence of a fixed point for Kannan's prequasinorm contractive mappings in function spaces of complex variables. We have studied the existence of fixed points of Kannan prequasinorm nonexpansive mapping and the existence of Kannan's prequasinorm contractive mapping in the prequasi-Banach operator ideal created by this function space and s-numbers. We have also presented some applications of summable equations. Several numerical experiments were introduced to illustrate our results. Moreover, some successful applications to the existence of solutions of nonlinear difference equations are discussed. This paper has several advantages for researchers, such as studying the fixed points of any contraction mappings on this prequasinormed function space, which is a generalization of the quasinormed function space, examining the eigenvalue problem in these new settings and noting that the closed operator ideals are certain to play an important function in the principle of Banach lattices.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final version of the paper.

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References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] R. Kannan, "Some results on fixed points-II," *The American Mathematical Monthly*, vol. 76, no. 4, pp. 405–408, 1969.

[3] S. J. H. Ghoncheh, "Some fixed point theorems for Kannan mapping in the modular spaces," *Ciencia e Natura*, vol. 37, pp. 462–466, 2015.

[4] L. Diening, P. Harjulehto, P. Hästö, and M. Ružiĉka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Berlin, Germany, 2011.

[5] K. R. Rajagopal and M. Růzǐčka, "On the modeling of electrorheological materials," *Mechanics Research Communications*, vol. 23, no. 4, pp. 401–407, 1996.

[6] M. Ružiĉka, "Electrorheological fluids. Modeling and mathematical theory," in *Lecture Notes in Mathematics*p. 1748, Springer, Berlin, Germany, 2000.

[7] A. Pietsch, *Operator Ideals*, " North-Holland Publishing Company, Amsterdam-New York-Oxford, 1980.

[8] A. Pietsch, "Small ideals of operators," *Studia Mathematica*, vol. 51, no. 3, pp. 265–267, 1974.

[9] B. M. Makarov and N. Faried, "Some properties of operator ideals constructed by s numbers (In Russian)," pp. 206–211, Academy of Science, Siberian section, Novosibirsk, Russia, 1977.

[10] N. Faried and A. A. Bakery, "Small operator ideals formed by s numbers on generalized Cesáro and Orlicz sequence spaces," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 357, 2018.

[11] T. Yaying, B. Hazarika, and M. Mursaleen, "On sequence space derived by the domain of q-Cesàro matrix in ℓ_p space and the associated operator ideal," *Journal of Mathematical Analysis and Applications*, vol. 493, Article ID 124453, 2021.

[12] M. Mursaleen and A. K. Noman, "Compactness by the Hausdorff measure of noncompactness," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 8, pp. 2541–2557, 2010.

[13] M. Mursaleen and A. K. Noman, "Compactness of matrix operators on some new difference sequence spaces," *Linear Algebra and Its Applications*, vol. 436, no. 1, pp. 41–52, 2012.

[14] A. A. Bakery and O. S. K. Mohamed, "Kannan prequasi contraction maps on Nakano sequence spaces," *Journal of Functional Spaces*, vol. 2020, Article ID 8871563, 10 pages, 2020.

[15] A. A. Bakery and O. S. K. Mohamed, "Kannan nonexpansive maps on generalized Cesàro backward difference sequence space of non-absolute type with applications to summable equations," *Journal of Inequalities and Applications*, vol. 103, 2021.

[16] L. V. Nguyen and N. T. N. Tram, "Fixed point results with applications to involution mappings," *Journal of Nonlinear Variable Analysis*, vol. 4, pp. 415–426, 2020.

[17] A. Dehici and N. Redjel, "Some fixed point results for nonexpansive mappings in Banach spaces," *Journal of Nonlinear Variable Analysis*, 2020.

[18] T. D. Benavides and P. L. Ramírez, "Fixed points for multivalued non-expansive mappings," *Applied Set-Valued Analyse Optimation*, vol. 2, pp. 143–157, 2020.

[19] A. A. Bakery and M. H. El Dewaik, "A generalization of Caristi's fixed point theorem in the variable exponent weighted formal power series space," *Journal of Function Spaces*, vol. 2021, Article ID 9919420, 18 pages, 2021.

[20] A. L. Shields, "Weighted shift operators and analytic function theory," in *Topics of Operator Theory*vol. 13, Providence, RI, USA, Amer. Math., 1974.

[21] K. Hedayatian, "On cyclicity in the space $H^p(\beta)$," *Taiwanese Journal of Mathematics*, vol. 8, no. 3, pp. 429–442, 2004.

[22] H. Emamirad and G. S. Heshmati, "Chaotic weighted shifts in Bargmann space," *Journal of Mathematical Analysis and Applications*, vol. 308, no. 1, pp. 36–46, 2005.

[23] N. Faried, A. Morsy, and Z. A. Hassanain, "s-numbers of shift operators of formal entire functions," *Journal of Approximation Theory*, vol. 176, pp. 15–22, 2013.

[24] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge University Press, New York, NY, USA, 1986.

[25] J. A. Clarkson, "Uniformly convex spaces," *Transactions of the American Mathematical Society*, vol. 40, no. 3, pp. 396–414, 1936.

[26] K. Sundaresan, "Uniform convexity of Banach spaces l({p_{i}})," *Studia Mathematica*, vol. 39, no. 3, pp. 227–231, 1971.

[27] B. Altay and F. Başar, "Generalization of the sequence space $\ell(p)$ derived by weighted means," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 147–185, 2007.

[28] M. A. Khamsi and W. M. Kozlowski, *Fixed Point Theory in Modular Function Spaces*, Birkhauser, New York, NY, USA, 2015.

[29] J. Musielak, *Orlicz Spaces and Modular Spaces*, Springer-Verlag, Berlin, Germany, 1983.

[30] H. Nakano, *Topology of Linear Topological Spaces*, Maruzen Co. Ltd., Tokyo, Japan, 1951.

[31] P. Salimi, Abdul Latif, and N. Hussain, "Modified α-ψ-contractive mappings with applications," *Fixed Point Theory and Applications*, vol. 151, 2013.

[32] R. P. Agarwal, N. Hussain, and M.-A. Taoudi, "Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 245872, 15 pages, 2012.

[33] N. Hussain, A. R. Khan, and R. P. Agarwal, "Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces," *Journal of Nonlinear Convex Analysis*, vol. 11, no. 3, pp. 475–489, 2010.

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Research Article

A New Measure of Quantum Starlike Functions Connected with Julia Functions

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In a complex domain, the investigation of the quantum differential subordinations for starlike functions is newly considered by few research studies. In this note, we arrange a set of necessary conditions utilizing the concept of the quantum differential subordinations for starlike functions related to the set of parametric Julia functions. Our method is based on the usage of quantum Jack lemma, where this lemma is generalized recently by the quantum derivative (Jackson calculus). We illustrate a starlike formula dominated by different types of Julia functions. The sufficient conditions are computed in the quantum and the Julia fractional parameters. We indicate a relationship between these two parameters.

1. Introduction

The notion of differential subordination and superordination (DSS) shows a dynamic model in the investigation of geometric possessions of holomorphic functions in the open unit disk. Lindelof first presented it, while Littlewood [1] did the extraordinary exertion in this area of study. Numerous investigators added information in the application of DSS. Antiquity and the improvement of mechanisms in the area connected with DSS are concisely designated and incorporated in the hardcover by Miller and Mocanu [2]. The main growth in the area of derivative of DSS began by Miller et al. [3]. Generally, the concept is defined for univalent function ω by

$$\varphi \prec \psi \Leftrightarrow \varphi(0) = \psi(0), \tag{1}$$

and $\varphi(\omega) \subset \psi(\omega)$. In general, if there is a function with the properties $\omega(0) = 0, |\omega(\xi)| < |\xi|$, satisfying $\varphi(\xi) = \psi(\omega(\xi))$, then

$$\varphi(\xi) \prec \psi(\xi), \tag{2}$$

where $\xi \in \sqcup := \{\xi \in \mathbb{C}: |\xi| < 1\}$.

Ismail et al. [4] presented a class of complex functions for each fractional number q, $0 < q < 1$ as the class of analytic functions φ on the open unit disk ($\sqcup$), $\varphi(0) = 0, \varphi'(0) = 1$, and $|\varphi(q\xi)| \le |\varphi(\xi)|$ on $\sqcup$. This class is investigated, as well as the links between it and other analytic function classes. Agrawal and Sahoo [5] extended this notion by suggesting the q-starlike functions family in a logical order. Srivastava et al. [6] explored the link between the Janowski functions and several known types of q-starlike functions. The Janowski functions are a novel subclass of q-starlike functions that they introduced and presented. Recent investigations can be located in works by Mahmood et al. [7] and Ul-Haq et al. [8].

Parametric Julia functions are usually utilized to determine the upper bound solutions of different types of differential equations of a complex variable [4–11]. In the recent study, we shall extend this concept applying the quantum calculus (Jackson calculus) and employ it to define special classes of analytic function types normalized analytically in the open unit disk $(\varphi(0) = 0, \varphi'(0) = 1)$ and dominated by different kinds of the parametric Julia functions. Our method is based on the quantum Jack lemma.

2. Quantum Starlike Formula

The effort of Ma and Minda [12] in this area of studies is not minor as they considered the normalized analytic function $p(0) = 1$ and the condition of a positive real part $\Re(p\prime(\xi)) > 0$. They have formulated the famous subclasses for starlike and convex functions, as follows, respectively:

$$\mathcal{S}^*(p) = \left\{\varphi \in \Delta: \frac{\xi\varphi\prime(\xi)}{\varphi(\xi)} \prec p(\xi), \xi \in \sqcup\right\},$$
$$\mathcal{C}(p) = \left\{\varphi \in \Delta: \frac{\xi\varphi\prime\prime(\xi)}{\varphi\prime(\xi)} + 1 \prec p(\xi), \xi \in \sqcup\right\}, \quad (3)$$

where Δ indicates the class of normalized function $\varphi(0) = 0 = \varphi\prime(0) - 1$.

Quantum calculus (QC) is the novel part of mathematical analysis and its applications and is correspondingly significant for its appearances, both in physics and in mathematics as well. Jackson [13,14] formulated the functions of q-differentiation and q-integration and decorated their meanings for the first stage. Later, Ismail et al. [4] contributed the indication of q-calculus in geometric function theory.

Nowadays, different classes of Ma and Minda are suggested and developed, using QC by researchers. For instant, Seoudy and Aouf [15] introduced subclass of quantum starlike functions involving q-derivative. Recently, Zainab et al. [16] presented a sufficient condition for q-starlikeness using a special curve. In addition, different differential and integral operators are generalized utilizing QC [17–20].

Definition 1. Jackson derivative is indicated in the following difference operator:

$$\left(\partial_q h\right)(\xi) = \frac{h(\xi) - h(q\xi)}{\xi(1-q)}, \quad q \in (0,1), \quad (4)$$

such that

$$\partial_q(\xi^\nu) = \left(\frac{1-q^\nu}{1-q}\right)\xi^{\nu-1}. \quad (5)$$

Moreover, Maclaurin's series representation takes the sum

$$\left(\partial_q h\right)(\xi) = \sum_{\ell=0}^{\infty} h_\ell [\ell]_q \xi^{\ell-1}, \quad (6)$$

where

$$[\ell]_q := \frac{1-q^\ell}{1-q}. \quad (7)$$

Note that

$$\lim_{q\longrightarrow 1^-}\left(\partial_q h\right)(\xi) = h\prime(\xi). \quad (8)$$

The multiplication rule takes the following formula:

$$\partial_q(f(\xi)g(\xi)) = g(\xi)\partial_q f(\xi) + f(q\xi)\partial_q g(\xi)$$
$$= g(q\xi)\partial_q f(\xi) + f(\xi)\partial_q g(\xi). \quad (9)$$

We proceed to define our q-starlike class using the q-parametric Julia functions and connecting with the subclass of normalized functions in $\sqcup$ (Figure 1):

$$J_1^{(\beta)}(\xi) = 1 + \xi - \beta\xi^3 \quad (10)$$

$$J_2^{(\beta)}(\xi) = \left(1 + \xi - \beta\xi^2\right)^2, \quad (11)$$

$$J_3^{(\beta)}(\xi) = 1 + \xi - \beta\xi^2, \quad (\beta \in \mathbb{C}, \xi \in \sqcup) \quad (12)$$

Definition 2. For a normalized function $\varphi(\xi) \in \Delta$ of the formula

$$\varphi(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n, \quad \xi \in \sqcup, \quad (13)$$

the q-starlike is defined by the subordination formula:

$$\frac{\xi\left(\partial_q\varphi\right)(\xi)}{\varphi(\xi)} \prec J_i^{(\beta)}(\xi)$$
$$(i = 1, 2, 3, q \in (0,1), \beta \in \mathbb{C}). \quad (14)$$

We denote the subclass of these functions by $\Delta_q^{(\beta)}$, where

$$\left(\partial_q\varphi\right) = 1 + \sum_{n=2}^{\infty} \varphi_n\left(\frac{1-q^n}{1-q}\right)\xi^n. \quad (15)$$

Moreover, a function $\varphi \in \Delta$ is called q-bounded turning if it satisfies the inequality

$$\partial_q\varphi(\xi) \prec J_i^{(\beta)}(\xi). \quad (16)$$

We denote this class by $\mathbb{B}_q^{(\beta)}$.

We aim to find the range of β in terms of q satisfying the inequality (14). For this purpose, we need the following result.

Lemma 1 (see [21]). *Let ϖ be analytic in $\sqcup$, such that $\varpi(0) = 0$. Then, the upper value of ϖ on the circle $|\xi| = 1$ at the point $\xi_0 = re^{i\theta}, \theta \in [-\pi, \pi], q \in (0,1)$, is*

$$\xi_0\left(\partial_q\varpi(\xi_0)\right) = \mu\varpi(\xi_0), \quad \mu \geq 1. \quad (17)$$

3. Results

In this section, we shall illustrate the sufficient conditions on functions $\varphi \in \Delta$ to be in $\Delta_q^{(\beta)}$.

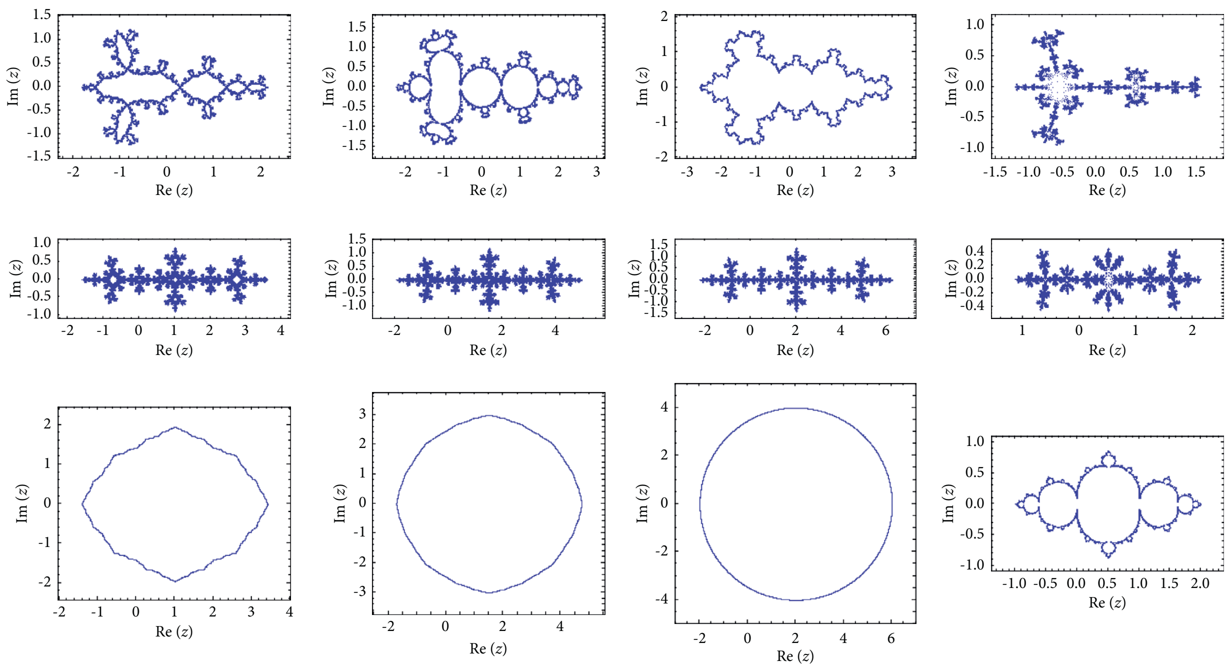

FIGURE 1: Plot of $J_i^{(\beta)}, i = 1, 2, 3$ for $\beta = 1/2, 1/3, 1/4$ and $\beta = 1$, respectively. The plot is connected when $\beta \in (0, 1]$; otherwise, it is disconnected.

Theorem 1. *Let the function $\rho \in \sqcup$, such that $\rho(0) = 1$ and*

$$1 + \xi\left(\partial_q \rho(\xi)\right) \prec \sqrt{1+\xi}, \quad \xi \sqcup. \tag{18}$$

If one of the cases

$$\beta \neq \frac{q+1}{q^2-q+1}, \beta < \frac{q+1+\sqrt{2}}{q^2-q+1}, \beta > \frac{q+1-\sqrt{2}}{q^2-q+1}, \quad q \in (0,1), \tag{19}$$

holds, then

$$\rho(\xi) \prec J_1^{(\beta)}(\xi) = 1 + \xi - \beta\xi^3. \tag{20}$$

Proof. Define a function ϱ as follows:

$$\varrho(\xi) \coloneqq 1 + \xi\left(\partial_q \rho(\xi)\right). \tag{21}$$

By the assumption (18) and the definition of the subordination, we have

$$1 + \xi\left(\partial_q \rho(\xi)\right) = \sqrt{1 + \varpi(\xi)}, \quad \varpi(0) = 0, |\varpi(\xi)| \le |\xi| < 1, \tag{22}$$

which leads to

$$\varpi(\xi) = \varrho^2(\xi) - 1. \tag{23}$$

We aim to show that $|\varrho^2(\xi) - 1| < 1$ for some values $\xi_0 \in \sqcup$, such that

$$\rho(\xi) = 1 + \varpi(\xi) - \beta\varpi^3(\xi). \tag{24}$$

Assume not; then, the above conclusion implies that

$$\varrho(\xi) = 1 + \xi\left(\partial_q\left[1 + \varpi(\xi) - \beta\varpi^3(\xi)\right]\right). \tag{25}$$

By using the rules of Jackson derivative, we obtain

$$\begin{aligned} \partial_q \varpi^2(\xi) &= \partial_q \varpi(\xi)\left[\varpi(\xi) + \varpi(q\xi)\right], \\ \partial_q \varpi^3(\xi) &= \partial_q \varpi(\xi)\left[\varpi^2(\xi) + \varpi(\xi)\varpi(q\xi) + \varpi^2(q\xi)\right]. \end{aligned} \tag{26}$$

Consequently, we get

$$\varrho(\xi) = 1 + W_q(\xi)\left(\xi\partial_q \varpi(\xi)\right), \tag{27}$$

where

$$W_q(\xi) \coloneqq \varpi(\xi) + \varpi(q\xi) - \beta\left(\varpi^2(\xi) + \varpi(\xi)\varpi(q\xi) + \varpi^2(q\xi)\right). \tag{28}$$

But

$$\varpi(q\xi) = \varpi(\xi) - (1-q)\xi\left(\partial_q \varpi(\xi)\right). \tag{29}$$

Hence, this yields

$$W_q(\xi) = \varpi(\xi)\left[2 - 3\beta\varpi(\xi)\right] + \xi\partial_q\varpi(\xi)\left[-(1-q) + (3-q)\varpi(\xi) - \beta(1-q)^2\xi\partial_q\varpi(\xi)\right]. \tag{30}$$

Suppose that there exists a point $\xi_0 \in ⊔$, such that

$$\max_{|\xi|\le|\xi_0|} |\varpi(\xi)| = |\varpi(\xi_0)| = 1, \quad \xi_0(\partial_q\varpi(\xi_0)) = \mu\varpi(\xi_0), \quad \mu \ge 1. \tag{31}$$

By Jack Lemma 1 and by letting $\varpi(\xi_0) = e^{i\theta}$, we have

$$\begin{aligned}
|W_q(\xi_0)| &= \left|\varpi(\xi)[2-3\beta\varpi(\xi)] + \xi\partial_q\varpi(\xi)\left[-(1-q)+\beta(3-q)\varpi(\xi)-\beta(1-q)^2\xi\partial_q\varpi(\xi)\right]\right|_{\xi=\xi_0} \\
&= \left|e^{i\theta}\left[2-3\beta e^{i\theta}\right] + \mu e^{i\theta}\left[-(1-q)+\beta(3-q)e^{i\theta}-\beta(1-q)^2\mu e^{i\theta}\right]\right| \\
&\ge \Re\left(e^{i\theta}\left[2-3\beta e^{i\theta}\right] + \mu e^{i\theta}\left[-(1-q)+\beta(3-q)e^{i\theta}-\beta(1-q)^2\mu e^{i\theta}\right]\right) \\
&= \cos(\theta)[2-3\beta\cos(\theta)] + \mu\cos(\theta)\left[-(1-q)+\beta(3-q)\cos(\theta)-\beta(1-q)^2\mu\cos(\theta)\right] \\
&= \beta\left[-3+\mu(3-q)-\mu(1-q)^2\right]\cos^2(\theta) + [2-\mu(1-q)]\cos(\theta) \\
&= \beta\left[\mu\left(3-q-(1-q)^2\right)-3\right]\cos^2(\theta) + [2-\mu(1-q)]\cos(\theta).
\end{aligned} \tag{32}$$

Accordingly, we conclude that

$$\begin{aligned}
|\varrho(\xi)^2-1|_{\xi=\xi_0} &= \left|\left(1+W_q(\xi)\left(\xi\partial_q\varpi(\xi)\right)\right)^2-1\right|_{\xi=\xi_0} \\
&\ge \left|\left(W_q(\xi_0)\left(\xi_0\partial_q\varpi(\xi_0)\right)\right)^2-1\right| \\
&= \left|\left(\mu\beta\left[\mu\left(3-q-(1-q)^2\right)-3\right]\cos^3(\theta)+\mu[2-\mu(1-q)]\cos^2(\theta)\right)^2-1\right| \\
&\ge \left|\left(\beta\left[\left(3-q-(1-q)^2\right)-3\right]+[1+q]\right)^2-1\right| \ge 1,
\end{aligned} \tag{33}$$

provided one of the following cases holds

$$\Upsilon \le -\sqrt{2}, \Upsilon = 0, \Upsilon \ge \sqrt{2}, \tag{34}$$

where

$$\Upsilon := \beta\left[\left(3-q-(1-q)^2\right)-3\right]+[1+q]. \tag{35}$$

Hence, we obtain one of the following arguments:

$$\beta = \frac{q+1}{q^2-q+1}, \beta \ge \frac{q+1+\sqrt{2}}{q^2-q+1}, \beta \le \frac{q+1-\sqrt{2}}{q^2-q+1}, \tag{36}$$

which are all contradict (19), that is

$$\rho(\xi) \prec J_1^{(\beta)}(\xi) = 1+\xi-\beta\xi^3. \tag{37}$$

□

As a special case, we have the following result.

Corollary 1. *Let $\varphi \in \Delta$ be satisfied the subordination:*

$$1+\xi\left(\partial_q\left(\frac{\xi\partial_q\varphi(\xi)}{\varphi(\xi)}\right)\right) \prec \sqrt{1+\xi}, \quad \xi \in ⊔. \tag{38}$$

If one of the cases in (21) is occurred, then $\varphi \in \Delta_q^{(\beta)}$.

Proof. Assume

$$\rho(\xi) = \left(\frac{\xi\partial_q\varphi(\xi)}{\varphi(\xi)}\right). \tag{39}$$

Obviously, $\rho(0) = 1$. Thus, in virtue of Theorem 1, we have $\varphi \in \Delta_q^{(\beta)}$. □

Similarly, by assuming $\rho(\xi) = \partial_q\varphi(\xi), \varphi \in \Delta$, we have the following result.

Corollary 2. *Let $\varphi \in \Delta$ be the satisfied subordination:*

$$1+\xi\left(\partial_q\varphi(\xi)\right) \prec \sqrt{1+\xi}, \quad \xi \in ⊔. \tag{40}$$

If one of the cases in (21) is occurred, then $\varphi \in \mathbb{B}_q^{(\beta)}$.

Theorem 2. *Let the function $h \in ⊔$, such that $h(0) = 1$ and*

$$1+\xi\left(\partial_q h(\xi)\right) \prec \sqrt{1+\xi}, \quad \xi \in ⊔, \tag{41}$$

if one of the cases

$$\beta \ne \frac{1}{q+1}, \beta \ne 2, \tag{42}$$

where for $0.418341 < q < 1$,

$$\frac{\left(0.5\left(2q+\sqrt{(2q+3)^2-10.3784(q+1)}\right)+3)\right)}{q+1} \geq \beta \geq \frac{\left(0.5\left(2q-\sqrt{(2q+3)^2-5.62159(q+1)}\right)+3\right)}{q+1}, \tag{43}$$

and for $0<q<0.418341$,

$$\frac{\left(0.5\left(2q+\sqrt{(2q+3)^2-10.3784(q+1)}\right)+3\right)}{q+1} \geq \beta \geq \frac{\left(0.5\left(2q-\sqrt{(2q+3)^2-10.3784(q+1)}\right)+3\right)}{q+1}, \tag{44}$$

hold; then,

$$h(\xi) \prec J_2^{(\beta)}(\xi) = \left(1+\xi-\beta\xi^2\right)^2. \tag{45}$$

Proof. Define a function p as follows:

$$p(\xi) := 1+\xi\left(\partial_q h(\xi)\right). \tag{46}$$

By the assumption (41) and the meninges of the subordination, we have

$$1+\xi\left(\partial_q h(\xi)\right) = \sqrt{1+w(\xi)}, \quad w(0)=0, |w(\xi)| \leq |\xi| < 1, \tag{47}$$

which yields

$$w(\xi) = p^2(\xi)-1. \tag{48}$$

We have to prove that

$$|w(\xi)| = \left|p^2(\xi)-1\right| < 1, \tag{49}$$

for some values $\xi_0 \in ⊔$, such that

$$h(\xi) = \left[1+w(\xi)-\beta w^2(\xi)\right]^2. \tag{50}$$

Assume not; then, the above conclusion imposes

$$p(\xi) = 1+\xi\left(\partial_q\left[1+w(\xi)-\beta w^2(\xi)\right]^2\right). \tag{51}$$

By using the rules of Jackson derivative and the facts

$$\begin{aligned} w(q\xi) &= w(\xi)-(1-q)\xi\partial_q w(\xi), \\ \partial_q w^2(\xi) &= \partial_q w(\xi)\left[2w(\xi)-(1-q)\xi\partial_q w(\xi)\right], \end{aligned} \tag{52}$$

we obtain

$$\begin{aligned} &\partial_q\left[1+w(\xi)-\beta w^2(\xi)\right]^2 = \left[1+w(\xi)-\beta w^2(\xi)\right]\partial_q\left[1+w(\xi)-\beta w^2(\xi)\right] \\ &\quad +\left[1+w(q\xi)-\beta w^2(q\xi)\right]\partial_q\left[1+w(\xi)-\beta w^2(\xi)\right] = 2\left[1+w(\xi)-\beta w^2(\xi)\right] \\ &\left(\partial_q w(\xi)-\beta\partial_q w(\xi)\left[2w(\xi)-(1-q)\xi\partial_q w(\xi)\right]\right) = 2\left[1+w(\xi)-\beta w^2(\xi)\right] \\ &\partial_q w(\xi)\left(1-\beta\left[2w(\xi)-(1-q)\xi\partial_q w(\xi)\right]\right). \end{aligned} \tag{53}$$

Consequently, we get

$$p(\xi) = 1+2\xi\left[1+w(\xi)-\beta w^2(\xi)\right]\partial_q w(\xi)\left(1-\beta\left[2w(\xi)-(1-q)\xi\partial_q w(\xi)\right]\right). \tag{54}$$

Suppose that there exists a point $\xi_0 \in ⊔$, such that

$$\max_{|\xi|\le|\xi_0|} |w(\xi)| = |w(\xi_0)| = 1,$$
$$\xi_0\left(\partial_q w(\xi_0)\right) = \mu w(\xi_0), \quad \mu \ge 1. \tag{55}$$

We aim to show that

$$|w(\xi)| = \left|p^2(\xi) - 1\right| < 1. \tag{56}$$

Our method is based on Jack Lemma 1. Assume not. Then, by consuming $w(\xi_0) = e^{i\theta}$, we get

$$\begin{aligned}
\left|p^2(\xi_0) - 1\right| &= \left|\left(1 + 2\xi\left[1 + w(\xi) - \beta w^2(\xi)\right]\partial_q w(\xi)\left(1 - \beta\left[2w(\xi) - (1-q)\xi\partial_q w(\xi)\right]\right)\right)^2 - 1\right|_{\xi=\xi_0} \\
&\ge \left|4\left[1 + e^{i\theta} - \beta e^{2i\theta}\right]^2\left(\mu e^{i\theta}\left(1 - \beta\left[2e^{i\theta} - (1-q)\mu e^{i\theta}\right]\right)\right)^2 - 1\right| \\
&\ge \Re\left(4\left[1 + e^{i\theta} - \beta e^{2i\theta}\right]^2\left(\mu e^{i\theta}\left(1 - \beta\left[2e^{i\theta} - (1-q)\mu e^{i\theta}\right]\right)\right)^2 - 1\right) \\
&= 4\left[1 + \cos(\theta) - \beta\cos^2(\theta)\right]^2(\mu\cos(\theta)(1 - \beta[2\cos(\theta) - (1-q)\mu\cos(\theta)]))^2 - 1 \ge 1.
\end{aligned} \tag{57}$$

Then, the solution when $\cos(\theta) = 1$ of

$$\left|4\left[1 + \cos(\theta) - \beta\cos^2(\theta)\right]^2(\mu\cos(\theta)(1 - \beta[2\cos(\theta) - (1-q)\mu\cos(\theta)]))^2 - 1\right| \ge 1 \tag{58}$$

brings one of the following cases:

$$⅄ = 0, ⅄ \ge \sqrt{2}, ⅄ \le -\sqrt{2}, \tag{59}$$

where

$$⅄: = 4(2-\beta)^2(1 - \beta(1+q))^2. \tag{60}$$

Hence, we obtain one of the following arguments:

$$\beta = \frac{1}{q+1}, \beta = 2, \tag{61}$$

and for $0.418341 < q < 1$,

$$\begin{aligned}
&\frac{\left(\left(0.52q + \sqrt{(2q+3)^2 - 10.3784(q+1)}\right) + 3\right)}{q+1} \le \beta \\
&\le \frac{\left(0.5\left(2q - \sqrt{(2q+3)^2 - 5.62159(q+1)}\right) + 3\right)}{q+1}.
\end{aligned} \tag{62}$$

Moreover, for $0 < q < 0.418341$, we have

$$\begin{aligned}
&\frac{\left(0.5\left(2q + \sqrt{(2q+3)^2 - 10.3784(q+1)}\right) + 3\right)}{q+1} \le \beta \\
&\le \frac{\left(0.5\left(2q - \sqrt{(2q+3)^2 - 10.3784(q+1)}\right) + 3\right)}{q+1}.
\end{aligned} \tag{63}$$

All the above inequalities contradict the assumptions of the theorem, which lead to

$$h(\xi) \prec J_2^{(\beta)}(\xi) = \left(1 + \xi - \beta\xi^2\right)^2. \tag{64}$$

□

Corollary 3. *Let $\varphi \in \Delta$ be the satisfied subordination:*

$$1 + \xi\left(\frac{\xi\left(\partial_q\varphi(\xi)\right)}{\varphi(\xi)}\right) \prec \sqrt{1+\xi}. \tag{65}$$

If one of the assumptions of Theorem 2 is occurred, then $\varphi \in \Delta_q^{(\beta)}$.

Proof. Assume

$$p(\xi) = \left(\frac{\xi\partial_q\varphi(\xi)}{\varphi(\xi)}\right). \tag{66}$$

Obviously, $p(0) = 1$. Thus, according to Theorem 2, we get $\varphi \in \Delta_q^{(\beta)}$. □

In the same manner of the above result, we obtain the next one when $p(\xi) = \partial_q\varphi(\xi), \varphi \in \Delta$.

Corollary 4. *Let $\varphi \in \Delta$ be the satisfied subordination:*

$$1 + \xi\left(\partial_q\varphi(\xi)\right) \prec \sqrt{1+\xi}. \tag{67}$$

If one of the assumptions of Theorem 2 is occurred, then $\varphi \in \mathbb{B}_q^{(\beta)}$.

Theorem 3. *Let the function $g \in ⊔$, such that $g(0) = 1$ and*

$$1 + \xi\left(\partial_q g(\xi)\right) \prec \sqrt{1+\xi}, \quad \xi \in ⊔. \tag{68}$$

If one of the cases

$$\beta \neq \frac{1}{q+1};$$
$$\beta \geq -\frac{1.18921\sqrt{\left(1/(q+1)^2\right)}q + 1.18921\sqrt{\left(1/(q+1)^2\right)} - 1}{(q+1)}, \quad 0 < q < 1, \tag{69}$$
$$\beta \leq \frac{1.18921\sqrt{\left(1/(q+1)^2\right)}q + 1.18921\sqrt{\left(1/(q+1)^2\right)} + 1}{(q+1)}, \quad 0 < q < 1,$$

holds, then

$$g(\xi) \prec J_3^{(\beta)}(\xi) = \left(1 + \xi - \beta\xi^2\right). \tag{70}$$

Proof. Define a function σ as follows:

$$\sigma(\xi) \coloneqq 1 + \xi\left(\partial_q g(\xi)\right). \tag{71}$$

By the assumption (68) and the meninges of the subordination, we have

$$1 + \xi\left(\partial_q g(\xi)\right) = \sqrt{1 + u(\xi)}, \quad u(0) = 0, |u(\xi)| \leq |\xi| < 1, \tag{72}$$

which yields

$$u(\xi) = \sigma^2(\xi) - 1. \tag{73}$$

We have to prove that

$$|u(\xi)| = \left|\sigma^2(\xi) - 1\right| < 1, \tag{74}$$

for some values $\xi_0 \in \sqcup$, such that

$$g(\xi) = \left[1 + u(\xi) - \beta u^2(\xi)\right]. \tag{75}$$

Assume not; then, the above conclusion imposes

$$\sigma(\xi) = 1 + \xi\left(\partial_q\left[1 + u(\xi) - \beta u^2(\xi)\right]\right). \tag{76}$$

By employing the rules of Jackson derivative and the facts

$$u(q\xi) = u(\xi) - (1-q)\xi\partial_q u(\xi),$$
$$\partial_q u^2(\xi) = \partial_q u(\xi)\left[2u(\xi) - (1-q)\xi\partial_q u(\xi)\right], \tag{77}$$

we obtain

$$\begin{aligned}\partial_q\left[1 + u(\xi) - \beta u^2(\xi)\right] &= \left(\partial_q u(\xi) - \beta\partial_q u(\xi)\left[2u(\xi) - (1-q)\xi\partial_q u(\xi)\right]\right)\\ &= \partial_q u(\xi)\left(1 - \beta\left[2u(\xi) - (1-q)\xi\partial_q u(\xi)\right]\right).\end{aligned} \tag{78}$$

Following the above structure, we get

$$\sigma(\xi) = 1 + \xi\partial_q u(\xi)\left(1 - \beta\left[2u(\xi) - (1-q)\xi\partial_q u(\xi)\right]\right). \tag{79}$$

Suppose that there exists a point $\xi_0 \in \sqcup$, such that

$$\max_{|\xi| \leq |\xi_0|} |u(\xi)| = \left|u(\xi_0)\right| = 1, \tag{80}$$

$$\xi_0\left(\partial_q u(\xi_0)\right) = \mu u(\xi_0), \quad \mu \geq 1. \tag{81}$$

We aim to show that

$$|u(\xi)| = \left|\sigma^2(\xi) - 1\right| < 1. \quad (82)$$

Our method is based on Jack Lemma 1. Assume not. Then, by consuming $u(\xi_0) = e^{i\theta}$, we get

$$\begin{aligned}
\left|\sigma^2(\xi_0) - 1\right| &= \left|\left(1 + \xi\partial_q u(\xi)\left(1 - \beta\left[2u(\xi) - (1-q)\xi\partial_q u(\xi)\right]\right)\right)^2 - 1\right|_{\xi=\xi_0} \\
&\geq \left|\left(\xi\partial_q u(\xi)\left(1 - \beta\left[2u(\xi) - (1-q)\xi\partial_q u(\xi)\right]\right)\right)^2 - 1\right|_{\xi=\xi_0} \\
&= \left|\left(\xi_0\partial_q u(\xi_0)\left(1 - \beta\left[2u(\xi_0) - (1-q)\xi_0\partial_q u(\xi_0)\right]\right)\right)^2 - 1\right| \\
&= \left|\left(\mu e^{i\theta}\left(1 - \beta\left[2e^{i\theta} - (1-q)\mu e^{i\theta}\right]\right)\right)^2 - 1\right| \geq \Re\left(\left(\mu e^{i\theta}\left(1 - \beta\left[2e^{i\theta} - (1-q)\mu e^{i\theta}\right]\right)\right)^2 - 1\right) \\
&= (\mu\cos(\theta)(1 - \beta[2\cos(\theta) - (1-q)\mu\cos(\theta)]))^2 - 1 \geq 1.
\end{aligned} \quad (83)$$

Thus, for $\cos(\theta) = 1, \mu = 1$, the solution of

$$\left|(\mu\cos(\theta)(1 - \beta[2\cos(\theta) - (1-q)\mu\cos(\theta)]))^2 - 1\right| \geq 1, \quad (84)$$

provided one of the following cases:

$$\curlyvee = 0, \curlyvee \geq \sqrt{2}, \curlyvee \leq -\sqrt{2}, \quad (85)$$

where

$$\curlyvee := (1 - \beta(1+q))^2. \quad (86)$$

Hence, we obtain one of the following arguments:

$$\begin{aligned}
\beta &= \frac{1}{q+1}; \\
\beta &\leq -\frac{1.18921\sqrt{\left(1/(q+1)^2\right)}q + 1.18921\sqrt{\left(1/(q+1)^2\right)} - 1}{(q+1)}, \quad 0 < q < 1, \\
\beta &\geq \frac{1.18921\sqrt{\left(1/(q+1)^2\right)}q + 1.18921\sqrt{\left(1/(q+1)^2\right)} + 1}{(q+1)}, \quad 0 < q < 1.
\end{aligned} \quad (87)$$

All the above inequalities contradict the assumptions of the theorem, which mean that

$$g(\xi) \prec J_3^{(\beta)}(\xi) = \left(1 + \xi - \beta\xi^2\right). \quad (88)$$

□

Corollary 5. *Let $\varphi \in \Delta$ be the satisfied subordination:*

$$1 + \xi\left(\frac{\xi\left(\partial_q\varphi(\xi)\right)}{\varphi(\xi)}\right) \prec \sqrt{1+\xi}. \quad (89)$$

If one of the assumptions of Theorem 3 occurred, then $\varphi \in \Delta_q^{(\beta)}$.

Proof. Assume

$$\sigma(\xi) = \left(\frac{\xi\partial_q\varphi(\xi)}{\varphi(\xi)}\right). \quad (90)$$

Obviously, $\sigma(0) = 1$. Thus, according to Theorem 3, we obtain $\varphi \in \Delta_q^{(\beta)}$. □

Similarly, for $\sigma(\xi) = \partial_q\varphi(\xi)$, we have the following consequence.

Corollary 6. *Let $\varphi \in \Delta$ be the satisfied subordination:*

$$1 + \xi\left(\partial_q\varphi(\xi)\right) \prec \sqrt{1+\xi}. \quad (91)$$

If one of the assumptions of Theorem 3 is occurred, then $\varphi \in \mathbb{B}_q^{(\beta)}$.

4. Conclusion

From above, we investigate the sufficient conditions to obtain the q-subordination of the q-starlike class

$$\left(\frac{\xi\partial_q\varphi(\xi)}{\varphi(\xi)}\right) \prec \mathbb{J}(\xi), \quad \xi \in \sqcup, \quad (92)$$

where $\mathbb{J}(\xi) = J_i^{(\beta)}$, $i = 1, 2, 3$. Differential inequalities are illustrated, involving the q-differential subordination. Nice geometric presentation is included describing the connected

Julia functions of different orders. Our class is called 2D parametric subclass of analytic function, and β is given in terms of q. Note that the case of 1D parametric subclass is given by

$$\mathbb{J}(\xi) = \frac{1+\xi}{1-q\xi}. \tag{93}$$

It is studied in [21], while null parametric subclass is formulated by

$$\mathbb{J}(\xi) = 1 + \frac{4}{3}\xi + \frac{2}{3}\xi^2, \tag{94}$$

and it is investigated in [16].

For future works, one can suggest any types of parametric analytic functions (geometric functions) in the open unit disk. The above q-differential subordination formula can be suggested to study the solution of many classes of generalized differential equations such as the class of Briot-Bouquet differential equation (2).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly to writing this article and read and agreed to the published version of the manuscript.

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References

[1] J. E. Littlewood, *Lectures on the Theory of Functions*, Springer, London, UK, 1944.

[2] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, CRC Press, Boca Raton, FL, USA, 2000.

[3] S. S. Miller, P. T. Mocanu, and M. Reade, "Bazilevic functions and generalized convexity," *Revue Roumaine de Mathématique Pures et Appliquées*, vol. 19, pp. 213–224, 1974.

[4] M. E. H. Ismail, E. Merkes, and S. David, ""A generalization of starlike functions." Complex Variables," *Theory and Application: International Journal*, vol. 14, no. 1-4, pp. 77–84, 1990.

[5] S. Agrawal and S. K. Sahoo, "A generalization of starlike functions of order alpha," *Hokkaido Mathematical Journal*, vol. 46, pp. 15–27, 2017.

[6] H. Srivastava, M. Tahir, B. Khan, Q. Ahmad, and N. Khan, "Some general classes of q-starlike functions associated with the Janowski functions," *Symmetry*, vol. 11, no. 2, p. 292, 2019.

[7] S. Mahmood, M. Jabeen, H. M. Srivastava, R. Manzoor, and S. M. Riaz, "Some coefficient inequalities of q-starlike functions associated with conic domain defined by q-derivative," *Journal of Function Spaces*, vol. 2018, Article ID 8492072, 13 pages, 2018.

[8] M. Ul-Haq, M. Raza, M. Arif, Q. Khan, and H. Tang, "Q-analogue of differential subordinations," *Mathematics*, vol. 7, no. 8, p. 724, 2019.

[9] R. W. Ibrahim and B.. Dumitru, "Analytic studies of a class of Langevin differential equations dominated by a class of Julia fractal functions," *Kragujevac journal of mathematics*, vol. 48, no. 4, pp. 577–590, 2024.

[10] R. W. Ibrahim and D. Baleanu, "Analytic solution of the Langevin differential equations dominated by a multibrot fractal set," *Fractal and Fractional*, vol. 5, no. 2, p. 50, 2021.

[11] Z. E. Abdulnaby, R. W. Ibrahim, and R. W. Ibrahim, "On a subclass of analytic functions of fractal power with negative coefficients," *Series Iii-Matematics, Informatics, Physics*, vol. 13, no. 2, pp. 387–398, 2020.

[12] W. C. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, Z. Li, F. Ren, L. Yang, and S. Zhang, Eds., International Press, Cambridge, MA, USA, January 1992.

[13] F. H. Jackson, "XI–On q-functions and a certain difference operator," *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2, pp. 253–281, 1909.

[14] D. O. Jackson, T. Fukuda, O. Dunn, and E. Majors, "On q-definite integrals," *The Quarterly Journal of Pure and Applied Mathematics*, 1910.

[15] T. M. Seoudy and M. K. Aouf, "Coefficient estimates of new classes of q-starlike and q-convex functions of complex order," *Journal of Mathematical Inequalities*, vol. 10, no. 1, pp. 135–145, 2016.

[16] S. Zainab, A. Shakeel, M. Imran, N. Muhammad, H. Naz, and M. Arif, "Sufficiency criteria for-starlike functions associated with cardioid," *Journal of Function Spaces*, vol. 2021, Article ID 9999213, 9 pages, 2021.

[17] R. W. Ibrahim, R. M. Elobaid, and S. J. Obaiys, "Geometric inequalities via a symmetric differential operator defined by quantum calculus in the open unit disk," *Journal of Function Spaces*, vol. 2020, Article ID 6932739, 8 pages, 2020.

[18] R. W. Ibrahim, R. M. Elobaid, and S. J. Obaiys, "A class of quantum Briot-Bouquet differential equations with complex coefficients," *Mathematics*, vol. 8, no. 5, p. 794, 2020.

[19] R. W. Ibrahim and B. Dumitru, "On quantum hybrid fractional conformable differential and integral operators in a complex domain," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, vol. 115, pp. 1–13, 2021.

[20] R. W. Ibrahim, "Geometric process solving a class of analytic functions using q-convolution differential operator," *Journal of Taibah University for Science*, vol. 14, no. 1, pp. 670–677, 2020.

[21] A. Cetinkaya and P. G. Yasar, "q-Harmonic mappings for which analytic part is q-convex functions of complex order," *Hacettepe Journal of Mathematics and Statistics*, vol. 47, no. 4, pp. 813–820, 2018.

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Research Article

Ostrowski Type Inequalities for s-Convex Functions via q-Integrals

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The new outcomes of the present paper are q-analogues (q stands for quantum calculus) of Hermite-Hadamard type inequality, Montgomery identity, and Ostrowski type inequalities for s-convex mappings. Some new bounds of Ostrowski type functionals are obtained by using Hölder, Minkowski, and power mean inequalities via quantum calculus. Special cases of new results include existing results from the literature.

1. Introduction

Integral inequalities provide a notable role in both pure and applied mathematics in the light of their wide applications in numerous regular and human sociologies, while convexity hypothesis has stayed a significant apparatus in the foundation of the theory of integral inequalities. The classical inequalities are helpful in numerous down-to-earth issues. In recent years, many authors (see [1–12]) proved numerous inequalities associated with the functions of bounded variation, Lipschitzian, monotone, absolutely continuous, convex functions, s-convex, h-convex, and n-times differentiable mappings with error estimates. Integral inequalities have been studied extensively by several researchers either in classical analysis or in the quantum one. In many practical problems, it is important to bound one quantity by another quantity. The classical inequalities including Hermite-Hadamard and Ostrowski type inequalities are very useful for this purpose (see [13–24]). Ostrowski type inequalities are well known to study the upper bounds for approximation of the integral average by the value of the function. In [25], Dragomir and Fitzpatrick have constructed Hermite-Hadamard's inequality which is specified to s-convex functions in the second sense as follows:

Theorem 1. *Suppose $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is an s-convex function in second sense, $s \in (0, 1)$, and suppose $\wp, \upsilon \in \mathbb{R}^+, \wp<\upsilon$. If $\Phi' \in L^1([\wp,\upsilon])$, then the integral inequality is valid:*

$$2^{s-1}\Phi\left(\frac{\wp+\upsilon}{2}\right) \le \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(w)dw \le \frac{\Phi(\wp)+\Phi(\upsilon)}{s+1}, \tag{1}$$

where $\mathbb{R}^+ = \{w \in \mathbb{R} \mid w \ge 0\}$.

The following Montgomery equality is established by Alomari (see [26]):

Lemma 2. *Assume that $\Phi : J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is differentiable function on $(\wp, \upsilon)$ in which $\wp, \upsilon \in J$ for $\wp<\upsilon$. If $\Phi' \in L[\wp,\upsilon]$, then we have the equality:*

$$\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)d\zeta = \frac{(w-\wp)^2}{\upsilon-\wp}\int_0^1 \zeta\Phi'(\zeta w + (1-\zeta)\wp) \cdot d\zeta - \frac{(\upsilon-w)^2}{\upsilon-\wp}\int_0^1 \zeta\Phi'(\zeta w + (1-\zeta)\upsilon)d\zeta, \tag{2}$$

for each $w \in [\wp,\upsilon]$.

By using Lemma 2, Alomari et al. in [26] had proved the Ostrowski type inequality, which holds for s-convex mappings in second sense as follows:

Theorem 3. *Assume $\Phi : J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a differentiable on $(\wp, \upsilon)$ and $\Phi' \in L[\wp,\upsilon]$ such that $\wp, \upsilon \in J$ for $\wp<\upsilon$. If $|\Phi'|$ is s-convex mapping in the second sense on $[\wp, \upsilon]$ unique $s \in (0, 1]$ and $|\Phi'(w)| \leq M, w \in [\wp,\upsilon]$, then the following result holds:*

$$\left|\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)d\zeta\right| \leq \frac{M}{\upsilon-\wp}\left[\frac{(w-\wp)^2 + (\upsilon - w)^2}{s+1}\right], \quad (3)$$

for each $w \in [\wp,\upsilon]$.

Theorem 4. *Suppose that $\Phi : J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is the differentiable on $(\wp, \upsilon)$ and $\Phi' \in L[\wp,\upsilon]$, where $\wp, \upsilon \in J$ with $\wp<\upsilon$. If absolute value of $(\Phi')^m$ is s-convex function in the second sense in $[\wp, \upsilon]$ for unique $s \in (0, 1]$, $m > 1$, $n = m/m - 1$ and $|\Phi'(w)| \leq M, w \in [\wp,\upsilon]$, then following integral inequality holds:*

$$\left|\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)d\zeta\right| \leq \frac{M}{(1+n)^{1/n}}\left(\frac{2}{s+1}\right)^{1/m}\left[\frac{(w-\wp)^2 + (\upsilon - w)^2}{\upsilon-\wp}\right], \quad (4)$$

for each $w \in [\wp,\upsilon]$.

Theorem 5. *Suppose that $\Phi : J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is differentiable on $(\wp, \upsilon)$ and $\Phi' \in L[\wp,\upsilon]$, in which $\wp, \upsilon \in J$ for $\wp<\upsilon$. If the absolute value of $(\Phi')^m$ is s-convex function in $[\wp, \upsilon]$ for static $s \in (0, 1]$, $m \geq 1$ and $|\Phi'(w)| \leq M, w \in [\wp,\upsilon]$, then the following integral inequality holds:*

$$\left|\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)d\zeta\right| \leq M\left(\frac{2}{s+1}\right)^{1/m}\left[\frac{(w-\wp)^2 + (\upsilon - w)^2}{2(\upsilon-\wp)}\right], \quad (5)$$

for each $w \in [\wp,\upsilon]$.

Theorem 6. *Suppose $\Phi : J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ be the differentiable on $(\wp, \upsilon)$ and $\Phi' \in L[\wp,\upsilon]$, in which $\wp, \upsilon \in J$ for $\wp<\upsilon$. If absolute value of $(\Phi')^m$ is a s-convex mapping in second sense on $[\wp, \upsilon]$ for static $s \in (0, 1]$, $m > 1$ and $n = m/m - 1$, we have*

$$\begin{aligned}\left|\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)d\zeta\right| &\leq \frac{2^{(s-1)/m}}{(1+n)^{1/n}(\upsilon-\wp)} \\ &\cdot \left[(w-\wp)^2\left|\Phi'\left(\frac{w+\wp}{2}\right)\right| + (\upsilon - w)^2\left|\Phi'\left(\frac{\upsilon+w}{2}\right)\right|\right],\end{aligned} \quad (6)$$

for each $w \in [\wp,\upsilon]$.

The renowned mathematician Euler started the investigation of q-calculus in the eighteenth century by presenting Newton's work of limitless series. This subject has gotten extraordinary consideration by numerous specialists, and consequently, it is considered an in-corporative subject among math and material science. In the mid-20th century, Jackson (1910) has begun a symmetric investigation of calculus and presented q-distinct integrals. The subject of quantum analytic has various applications in different spaces of arithmetic and physical science like number hypothesis, combinatorics, symmetrical polynomials, essential hyper-mathematical functions, quantum theory, and mechanics and in the hypothesis of relativity. Quantum calculus can be seen as a scaffold among arithmetic and material science. It has been shown that quantum calculus is a subfield of the more general mathematical field of time scales calculus. Time scales provide a unified framework for studying dynamic equations on both discrete and continuous domains. In [27, 28], q-Bernoulli and dynamic inequalities associated with Leibniz integral rule on time scales were studied. In studying quantum calculus, we are concerned with a specific time scale, called the q-time scale. The study of q-integral inequalities is also of great importance. Integral inequalities have been studied extensively by several researchers either in classical analysis or in the quantum one.

The following q-Hermite-Hadamard and q-Ostrowski type integral inequalities were proved by Tariboon and Ntouyas (see Theorems 3.2 and 3.5 [29]):

Theorem 7. *Let $\Phi : J \longrightarrow \mathbb{R}$ be a q-differentiable function with $D_q\Phi$ continuous on $[\wp, \upsilon]$ and $0 < q < 1$. Then, we have*

$$\Phi\left(\frac{\wp+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)\,{}_{\wp}d_q\zeta \leq \frac{\Phi(\wp) + q\Phi(\upsilon)}{q+1}. \quad (7)$$

Theorem 8. *Suppose $\Phi : J \longrightarrow \mathbb{R}$, where $[\wp,\upsilon] \subseteq \mathbb{R}$ is an interval, be a q-differentiable in open interval $\wp, \upsilon$ belonging to interior I for $\wp<\upsilon$. If $|D_q\Phi(w)| \leq M$ for all $w \in [\wp,\upsilon]$ and $0 < q < 1$, then the integral inequality is valid:*

$$\begin{aligned}&\left|\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)\,{}_{\wp}d_q\zeta\right| \\ &\leq M\left[\frac{2q}{1+q}\left(\frac{w - (((3q-1)\wp + (1+q)\upsilon)/4q)}{\upsilon-\wp}\right)^2 + \left(\frac{(-q^2+6q-1)}{8q(1+q)}\right)\right],\end{aligned} \quad (8)$$

for all $w \in [\wp,\upsilon]$. The least value of constant on RHS of inequality (8) is $(-q^2 + 6q - 1)/8q(1 + q)$.

The following q-Ostrowski type integral inequalities for convex functions were proved by Noor et al. (see [30]):

Theorem 9. *Let $\Phi : J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ be q-differentiable mapping for $(\wp, \upsilon)$ and $D_q\Phi \in L[\wp,\upsilon]$, in which $\wp, \upsilon \in J$ for $\wp<\upsilon$. If $|D_q\Phi|$ is convex mapping $[\wp, \upsilon]$ for some static $q \in (0, 1)$ and $|D_q\Phi(w)| \leq M, w \in [\wp,\upsilon]$, then we have the following q*

-integral inequality:

$$\left|\frac{1}{q}\left(\Phi(w)-\frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)_{\wp}d_q\zeta\right)\right| \leq \frac{M}{\upsilon-\wp}\left[\frac{(w-\wp)^2+(\upsilon-w)^2}{q+1}\right], \tag{9}$$

for each $w\in[\wp,\upsilon]$.

Theorem 10. *Assume that* $\Phi: J\subset\mathbb{R}^{+}\longrightarrow\mathbb{R}$ *is* q*-differentiable mapping on* $(\wp,\upsilon)$ *and* $D_q\Phi\in L[\wp,\upsilon]$, *in which* $\wp,\upsilon\in J$ *for* $\wp<\upsilon$. *If* $|D_q\Phi|^m$ *is a convex function in second sense on* $[\wp,\upsilon]$ *unique* $q\in(0,1)$, $m>1$, $n=m/m-1$, *and* $|D_q\Phi(w)|\leq M$, $w\in[\wp,\upsilon]$, *then we have the* q*-integral inequality:*

$$\begin{aligned}&\left|\frac{1}{q}\left(\Phi(w)-\frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)_{\wp}d_q\zeta\right)\right|\\&\quad\leq\frac{M}{([n+1])^{1/n}}\left[\frac{(w-\wp)^2+(\upsilon-w)^2}{\upsilon-\wp}\right],\end{aligned} \tag{10}$$

for each $w\in[\wp,\upsilon]$.

The aim of this work is to find q-analogues of Hermite-Hadmard and Ostrowski type integral inequalities for functions whose q-derivatives are s-convex in the second sense. An interesting feature of our results is that they provide new estimates and good approximation on such types of inequalities involving q-integrals.

2. Basic Essentials

2.1. Convex Function. Let Φ be the function; it is said to be convex function on interval$J$ if

$$\Phi(\Omega w+(1-\Omega)\rho)\leq\Omega\Phi(w)+(1-\Omega)\Phi(\rho) \tag{11}$$

holds for all $w,\rho\in J$ and $\Omega\in[0,1]$.

In [31], s-convex functions in the second sense have been introduced by Hudzik and Maligranda as follows:

2.2. s-Convex Function. A mapping $\Phi:\mathbb{R}^{+}\longrightarrow\mathbb{R}$ is said to be s-convex if

$$\Phi(\Omega w+(1-\Omega)\rho)\leq\Omega^s\Phi(w)+(1-\Omega)^s\Phi(\rho), \tag{12}$$

for each $w,\rho\in\mathbb{R}^{+}$, $\Omega\in 0,1]$ and for unique $s\in(0,1]$.

2.3. q-Derivative [32]. For a continuous mapping $\Phi:[\wp,\upsilon]\longrightarrow\mathbb{R}q$-derivative at $w\in[\wp,\upsilon]$ is

$$_{\wp}D_q\Phi(w)=\frac{\Phi(w)-\Phi(qw+(1-q)\wp)}{(1-q)(w-\wp)}\quad w\neq\wp. \tag{13}$$

Also, for $n\geq 1$, one may find the following evaluations:

$$\begin{aligned}&(w-\wp)_q^n=(w-\wp)(w-q\wp)\left(w-q^2\wp\right)\cdots\left(w-q^{n-1}\wp\right),\\&(\wp-w)_q^n=(\wp-qw)\left(\wp-q^2w\right)\cdots\left(\wp-q^{n-1}w\right),\\&D_q(w-\wp)_q^n=[n](w-\wp)_q^{n-1},\\&D_q(\wp-w)_q^n=-[n](\wp-qw)_q^{n-1},\\&(\wp-qw)_q^n=-\frac{1}{[n+1]}D_q(\wp-w)_q^{n+1},\\&D_q(\wp-w)_q^n=-[n](\wp-qw)_q^{n-1},\\&\int(\wp-w)_q^n d_qw=-\frac{q\left(\wp-q^{-1}w\right)_q^{n+1}}{[n+1]}\quad(\wp\neq-1).\end{aligned} \tag{14}$$

Here,

$$[n]=\frac{q^n-1}{q-1}, \tag{15}$$

and also, we have

$$(1-\wp)_q^n=\prod_{j=0}^{n}\left(1-q^j\wp\right). \tag{16}$$

2.4. q-Antiderivative [32]. Suppose that $\Phi:[\wp,\upsilon]\longrightarrow\mathbb{R}$ be the continuous mapping. Then, q-definite integral on $[\wp,\upsilon]$ is stated as

$$\int_{\wp}^{w}\Phi(\zeta)_{\wp}d_q\zeta=(1-q)(w-\wp)\sum_{n=0}^{\infty}q^n\Phi(q^nw+(1-q^n)\wp), \tag{17}$$

for $w\in[\wp,\upsilon]$.

2.5. The Formula of q-Integration by Parts [29]. Let $\Phi,g:[\wp,\upsilon]\longrightarrow\mathbb{R}$ be the continuous functions $\wp\in\mathbb{R}$ and $w,c\in[\wp,\upsilon]$, Then, the formula of q-integration by parts is stated as

$$\begin{aligned}\int_c^w\Phi(\zeta)_{\wp}D_qg(\zeta)d_q\zeta&=\Phi(w)g(w)-\Phi(c)g(c)\\&\quad-\int_c^w g(q\zeta+(1-q)\wp)_{\wp}D_q\Phi(\zeta)d_q\zeta.\end{aligned} \tag{18}$$

Theorem 11. **q-Hölder Inequality** ([4], Theorem 2). *Let* Φ *and* g *be* q*-integrable on* $[\wp,\upsilon]$ *and* $0<q<1$ *and* $(1/n)+(1/m)=1$ *with* $m>1$; *then, one may obtain the following:*

$$\int_{\wp}^{\upsilon}|\Phi(\zeta)g(\zeta)|_{\wp}d_q\zeta\leq\left\{\int_{\wp}^{\upsilon}|\Phi(\zeta)|^n{}_{\wp}d_q\zeta\right\}^{1/n}\left\{\int_{\wp}^{\upsilon}|g(\zeta)|^m{}_{\wp}d_q\zeta\right\}^{1/m}. \tag{19}$$

Using (19), the following is valid.

2.6. q-Minkowski's Inequality. Let $\wp, \upsilon \in \mathbb{R}$ and $n > 1$ be a real number then for continuous functions $\Phi, g : [\wp,\upsilon] \longrightarrow \mathbb{R}$,

$$\left\{\int_{\wp}^{\upsilon} |(\Phi(\zeta) + g(\zeta))|^n {}_{\wp}d_q\zeta\right\}^{1/n} \leq \left\{\int_{\wp}^{\upsilon} |\Phi(\zeta)|^n {}_{\wp}d_q\zeta\right\}^{1/n} + \left\{\int_{\wp}^{\upsilon} |g(\zeta)|^n {}_{\wp}d_q\zeta\right\}^{1/n}. \quad (20)$$

Proof.

$$\begin{aligned}\int_{\upsilon}^{\wp} |(\Phi + g)(\zeta)|^n d_q\zeta &= \int_{\upsilon}^{\wp} |(\Phi + g)(\zeta)|^{n-1} |(\Phi + g)(\zeta)| d_q\zeta \\ &\leq \int_{\upsilon}^{\wp} |(\Phi + g)(\zeta)|^{n-1} |\Phi(\zeta)| d_q\zeta + \int_{\upsilon}^{\wp} |(\Phi + g)(\zeta)|^{n-1} \\ &\quad \cdot |g(\zeta)| d_q\zeta \leq \left\{\int_{\upsilon}^{\wp} |\Phi(\zeta)|^n d_q\zeta\right\}^{1/n} \left\{\int_{\upsilon}^{\wp} |(\Phi + g)(\zeta)|^{m(n-1)} d_q\zeta\right\}^{1/n} \\ &\quad + \left\{\int_{\upsilon}^{\wp} |g(\zeta)|^n d_q\zeta\right\}^{1/n} \left\{\int_{\upsilon}^{\wp} |(\Phi + g)(\zeta)|^{m(n-1)} d_q\zeta\right\}^{1/m} \\ &= \left[\left\{\int_{\upsilon}^{\wp} |\Phi(\zeta)|^n d_q\zeta\right\}^{1/n} + \left\{\int_{\upsilon}^{\wp} |g(\zeta)|^n d_q\zeta\right\}^{1/n}\right] \\ &\quad \cdot \left[\left\{\int_{\upsilon}^{\wp} |(\Phi + g)(\zeta)|^{m(n-1)} d_q\zeta\right\}^{1/m}\right],\end{aligned} \quad (21)$$

which gives the required result for positive real numbers m, n such that $(1/m) + (1/n) = 1$.

The classical power mean inequality for integrals has the following form for q-integral. □

2.7. q-Power Mean Inequality. Let $(1/n) + (1/m) = 1$ for real numbers $n, m > 1$. Let $\wp, \upsilon \in \mathbb{R}$ and $\Phi, g : [\wp,\upsilon] \longrightarrow \mathbb{R}$ be continuous functions; then,

$$\int_{\wp}^{\upsilon} |\Phi(\zeta) g(\zeta)|_{\wp} d_q\zeta \leq \left\{\int_{\wp}^{\upsilon} |\Phi(\zeta)|_{\wp} d_q\zeta\right\}^{1-(1/m)} \cdot \left\{\int_{\wp}^{\upsilon} |\Phi(\zeta)||g(\zeta)|^m {}_{\wp}d_q\zeta\right\}^{1/m}. \quad (22)$$

Proposition 12. [33]. *For each $k, r \in \mathbb{N}(or\, \mathbb{Z}\, q \in \mathbb{R}^{\times})$, we have*

$$[k + r]_q = [k]_q + q^k [r]_q. \quad (23)$$

3. Main Results

3.1. q-Hermite-Hadamard Inequality

Theorem 13. *Suppose $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a s-convex mapping in the second sense, in which $s, q \in (0, 1)$, and let $\wp, \upsilon \in \mathbb{R}^+$, $\wp<\upsilon$. If $D_q\Phi \in L([a, b])$, then the integral inequality is valid:*

$$2^{s-1}\Phi\left(\frac{\wp+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon} \Phi(\zeta)_{\wp} d_q\zeta \leq \frac{\Phi(\upsilon) q\left(1 - \left(1 - q^{-1}\right)^{s+1}\right) + \Phi(\wp)}{[s + 1]}. \quad (24)$$

Proof. By definition of s-convex functions,

$$\begin{aligned}&\Phi(\zeta\wp+(1-\zeta)\upsilon) \leq \zeta^s\Phi(\wp) + (1-\zeta)^s\Phi(\upsilon) \\ &\int_0^1 \Phi(\zeta\wp+(1-\zeta)\upsilon)_0 d_q\zeta \leq \Phi(\wp)\int_0^1 \zeta^s{}_0 d_q\zeta + \Phi(\upsilon)\int_0^1 (1-\zeta)^s{}_0 d_q\zeta, \\ &\int_0^1 \Phi(\zeta\wp+(1-\zeta)\upsilon)_0 d_q\zeta = \frac{(1-q)(\upsilon-\wp)}{\upsilon-\wp}\sum_{n=0}^{\infty} q^n\Phi(q^n\wp+(1-q^n)\upsilon) \\ &= \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon} \Phi(\zeta)_0 d_q\zeta \leq \Phi(\wp)\int_0^1 \zeta^s{}_0 d_q\zeta + \Phi(\upsilon)\int_0^1 (1-\zeta)^s{}_0 d_q\zeta \\ &= \frac{\Phi(\wp) + q\left(1 - \left(1 - q^{-1}\right)^{s+1}\right)\Phi(\upsilon)}{[s + 1]}.\end{aligned} \quad (25)$$

Hence,

$$\frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon} \Phi(\zeta)_0 d_q\zeta \leq \frac{\Phi(\wp) + q\left(1 - \left(1 - q^{-1}\right)^{s+1}\right)\Phi(\upsilon)}{[s + 1]}. \quad (26)$$

Let $w = \zeta\wp+(1-\zeta)\upsilon$ and $\zeta = \zeta\upsilon + (1-\zeta)\wp$ in $\Phi((w + \zeta)/2) \leq ((\Phi(w) + \Phi(\zeta))/2^s)$ to get

$$\begin{aligned}&\Phi\left(\frac{\zeta\wp+(1-\zeta)\upsilon + \zeta\upsilon + (1-\zeta)\wp}{2}\right) \\ &\leq \frac{\Phi(\zeta\wp+(1-\zeta)\upsilon) + \Phi(\zeta\upsilon + (1-\zeta)\wp)}{2^s}, \\ &\Phi\left(\frac{\wp+\upsilon}{2}\right) \leq \frac{1}{2^s}\left(\int_0^1 \Phi(\zeta\wp+(1-\zeta)\upsilon)_0 d_q\zeta + \int_0^1 \Phi(\zeta\upsilon + (1-\zeta)\wp)_0 d_q\zeta\right) \\ &= \frac{1}{2^s}\left(\frac{1}{\wp-\upsilon}\int_{\upsilon}^{\wp} \Phi(\zeta)_0 d_q\zeta + \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon} \Phi(\zeta)_0 d_q\zeta\right),\end{aligned} \quad (27)$$

$$2^{s-1}\Phi\left(\frac{\wp+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon} \Phi(\zeta)_0 d_q\zeta, \quad (28)$$

From (26) and (28), the desired result is

$$2^{s-1}\Phi\left(\frac{\wp+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon} \Phi(\zeta)_{\wp} d_q\zeta \leq \frac{\Phi(\wp) + q\left(1 - \left(1 - q^{-1}\right)^{s+1}\right)\Phi(\upsilon)}{[s + 1]}. \quad (29)$$

□

3.2. q-Ostrowski Type Inequalities. To prove some q-Ostrowski type inequalities, it needs to establish the following Montgomery identity for q-integrals:

Lemma 14. *Let $\Phi : J \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a q-differentiable on J° in which $\wp, \upsilon \in J$ for $\wp<\upsilon$. If $D_q\Phi \in L[\wp,\upsilon]$, we have the following q-integral equality which is valid:*

$$\frac{1}{q}\left[\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)\,{}_{\wp}d_q\zeta\right] = \frac{(w-\wp)^2}{\upsilon-\wp}\int_0^1 \zeta D_q\Phi(\zeta w + (1-\zeta)\wp)\cdot{}_0\, d_q\zeta - \frac{(\upsilon-w)^2}{\upsilon-\wp}\int_0^1 \zeta D_q\Phi(\zeta w + (1-\zeta)\upsilon)\,{}_0d_q\zeta, \tag{30}$$

for each $w \in [\wp,\upsilon]$.

By using Lemma 14, we have constructed the following Ostrowski type inequalities, which hold for s-convex functions in the second sense:

Theorem 15. *Let $\Phi : J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a q-differentiable mapping on J° and $D_q\Phi \in L[\wp,\upsilon]$, in which $\wp, \upsilon \in J$ for $\wp<\upsilon$. If the absolute value of $D_q\Phi(w)$ is s-convex in second sense on $[\wp, \upsilon]$ for unique $s \in (0, 1]$ and $D_q\Phi(w)$ is bounded by M, $w \in [\wp,\upsilon]$, we have been seeing that the following q-integral inequality is valid:*

$$\left|\frac{1}{q}\left[\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)\,{}_{\wp}d_q\zeta\right]\right| \le M\frac{(w-\wp)^2 + (\upsilon-w)^2}{\upsilon-\wp}\cdot\left[-\frac{q}{[s+1]}\left(\left(1-q^{-1}\right)_q^{s+1} + \frac{q\left(1-q^{-1}\right)_q^{s+2}}{[s+2]} - \frac{q}{[s+2]}\right) + \frac{1}{[s+2]}\right], \tag{31}$$

for each $w \in [\wp,\upsilon]$.

Proof. Since $|D_q\Phi|$ is s-convex function in the second sense on $[\wp, \upsilon]$, therefore, Lemma 14 gives the following:

$$\begin{aligned}
&\left|\frac{1}{q}\left[\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)\,{}_{\wp}d_q\zeta\right]\right| \\
&\le \frac{(w-\wp)^2}{\upsilon-\wp}\int_0^1 \zeta\left|D_q\Phi(\zeta w + (1-\zeta)\wp)\right|{}_0d_q\zeta + \frac{(\upsilon-w)^2}{\upsilon-\wp}\int_0^1 \zeta\left|D_q\Phi(\zeta w + (1-\zeta)\upsilon)\right|{}_0d_q\zeta \\
&\le \frac{(w-\wp)^2}{\upsilon-\wp}\int_0^1 \zeta_q^{s+1}\left|D_q\Phi(w)\right|{}_0d_q\zeta + \int_0^1 \zeta(1-\zeta)_q^s\left|D_q\Phi(\wp)\right|{}_0d_q\zeta \\
&\quad + \frac{(\upsilon-w)^2}{\upsilon-\wp}\int_0^1 \zeta_q^{s+1}\left|D_q\Phi(w)\right|{}_0d_q\zeta + \int_0^1 \zeta(1-\zeta)_q^s\left|D_q\Phi(\upsilon)\right|{}_0d_q\zeta \\
&= \frac{M(w-\wp)^2}{\upsilon-\wp}\left[\int_0^1 \zeta_q^{s+1}\,{}_0d_q\zeta + \int_0^1 \zeta(1-\zeta)_{q0}^s d_q\zeta\right] + \frac{(\upsilon-w)^2}{\upsilon-\wp}\left[\int_0^1 \zeta_q^{s+1}\,{}_0d_q\zeta + \int_0^1 \zeta(1-\zeta)_{q0}^s d_q\zeta\right] \\
&= M\frac{(w-\wp)^2 + (\upsilon-w)^2}{\upsilon-\wp}\left[\int_0^1 \zeta^{s+1}\,{}_0d_q\zeta + \int_0^1 \zeta(1-\zeta)_{q0}^s d_q\zeta\right],
\end{aligned}$$

$$\int_0^1 \zeta_q^{s+1}\,{}_0d_q\zeta = \frac{1}{[s+2]}, \tag{32}$$

$$\begin{aligned}
&= -\frac{q}{[s+1]}\int_0^1 \zeta D_q\left(1-q^{-1}\zeta\right)_q^{s+1}\,{}_0d_q\zeta \\
&= -\frac{1}{q[s+1]}\left[\left|\zeta\left(1-q^{-1}\zeta\right)_q^{s+1}\right|_0^1 - \int_0^1 (1-\zeta)_q^{s+1}.1_0d_q\zeta\right] \\
&= -\frac{q}{[s+1]}\left[\left(1-q^{-1}\right)_q^{s+1} + \frac{q\left(1-q^{-1}\right)_q^{s+2}}{[s+2]} - \frac{q}{[s+2]}\right] \\
&= -\frac{q}{[s+1]}\left[\left(1-q^{-1}\right)_q^{s+1} + \frac{q\left(1-q^{-1}\right)_q^{s+2}}{[s+2]} - \frac{q}{[s+2]}\right] + \frac{1}{[s+2]} \\
&= M\frac{(w-\wp)^2 + (\upsilon-w)^2}{\upsilon-\wp}\cdot\left[-\frac{q}{[s+1]}\left(\left(1-q^{-1}\right)_q^{s+1} + \frac{q\left(1-q^{-1}\right)_q^{s+2}}{[s+2]} - \frac{q}{[s+2]}\right) + \frac{1}{[s+2]}\right].
\end{aligned} \tag{33}$$

□

Theorem 16. *Suppose that $\Phi : J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a q-differentiable on J° and $D_q\Phi \in L[\wp,\upsilon]$, in which $\wp, \upsilon \in J$ for $\wp<\upsilon$. If $|D_q\Phi|^m$ is a s-convex function in second sense on $[\wp, \upsilon]$ for some static $s \in (0, 1], m > 1, n = m/m-1$ and $D_q\Phi(w)$ is bounded by M, $w \in [\wp,\upsilon]$, then the q-integral inequality is valid:*

$$\left|\frac{1}{q}\left[\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)\,{}_{\wp}d_q\zeta\right]\right| \le \frac{M}{[n+1]^{1/n}}\left[\frac{1 + q\left(1 - \left(1-q^{-1}\right)^{s+1}\right)}{[s+1]}\right]^{1/m} \times \left[\frac{(w-\wp)^2 + (\upsilon-w)^2}{\upsilon-\wp}\right], \tag{34}$$

for each $w \in [\wp,\upsilon]$.

Proof. From Lemma 14 and keeping in view the well-known q-analogue of Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{q}\left[\Phi(w) - \frac{1}{\upsilon-\wp}\int_{\wp}^{\upsilon}\Phi(\zeta)\,{}_{\wp}d_q\zeta\right]\right| \le \frac{(w-\wp)^2}{\upsilon-\wp}\int_0^1 \zeta\left|D_q\Phi(\zeta w + (1-\zeta)\wp)\right|\cdot{}_0\, d_q\zeta + \frac{(\upsilon-w)^2}{\upsilon-\wp}\int_0^1 \zeta\left|D_q\Phi(\zeta w + (1-\zeta)\upsilon)\right|{}_0d_q\zeta \\
&\le \frac{(w-\wp)^2}{\upsilon-\wp}\left(\int_0^1 \zeta_{q0}^n d_q\zeta\right)^{1/n}\left(\int_0^1 \left|D_q\Phi(\zeta w + (1-\zeta)\wp)\right|^m{}_0d_q\zeta\right)^{1/m} \\
&\quad + \frac{(\upsilon-w)^2}{\upsilon-\wp}\int_0^1\left(\int_0^1 \zeta_{q0}^n d_q\zeta\right)^{1/n}\left(\int_0^1 \left|D_q\Phi(\zeta w + (1-\zeta)\upsilon)\right|^m{}_0d_q\zeta\right)^{1/m},
\end{aligned}$$

$$\int_0^1 \left|D_q\Phi(\zeta w+(1-\zeta)\wp)\right|^m {}_0d_q\zeta \le \int_0^1 \zeta_q^s \left|D_q\Phi(w)\right|^m {}_0d_q\zeta + \int_0^1 (1-\zeta)_q^s \left|D_q\Phi(\wp)\right|^m {}_0d_q\zeta \le M^m\left(\left|\frac{\zeta_q^{s+1}}{[s+1]}\right|_0^1 - \left|\frac{q(1-q^{-1}\zeta)_q^{s+1}}{[s+1]}\right|_0^1\right) = M^m\left(\frac{1}{[s+1]} - \frac{q(1-q^{-1})_q^{s+1}}{[s+1]} + \frac{q}{[s+1]}\right) = M^m\left(\frac{1+q\left(1-(1-q^{-1})_q^{s+1}\right)}{[s+1]}\right),$$

$$\int_0^1 \left|D_q\Phi(\zeta w+(1-\zeta)\upsilon)\right|^m {}_0d_q\zeta \le \int_0^1 \zeta_q^s \left|D_q\Phi(w)\right|^m {}_0d_q\zeta + \int_0^1 (1-\zeta)_q^s \left|D_q\Phi(\upsilon)\right|^m {}_0d_q\zeta \le M^m\left(\left|\frac{\zeta_q^{s+1}}{[s+1]}\right|_0^1 - \left|\frac{q(1-q^{-1}\zeta)_q^{s+1}}{[s+1]}\right|_0^1\right) = M^m\left(\frac{1}{[s+1]} - \frac{q(1-q^{-1})_q^{s+1}}{[s+1]} + \frac{q}{[s+1]}\right) = M^m\left(\frac{1+q\left(1-(1-q^{-1})_q^{s+1}\right)}{[s+1]}\right) \le M\left(\frac{1}{[1+n]}\right)^{1/n}\left(\frac{1+q\left(1-(1-q^{-1})_q^{s+1}\right)}{[s+1]}\right)^{1/m}\left[\frac{(w-\wp)^2+(\upsilon-w)^2}{\upsilon-\wp}\right]. \quad (35)$$

It completes the proof. □

Theorem 17. *Let $\Phi : J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a q-differentiable mapping on J° such as $D_q\Phi \in L[\wp,\upsilon]$, in which $\wp, \upsilon \in J$ for $\wp<\upsilon$. If the absolute value of $(D_q\Phi(w))^m$ is a s -convex mapping in the second sense on $[\wp,\upsilon]$ for unique $s \in (0,1]$, $m \ge 1$, and $|D_q\Phi(w)| \le M$, $w \in [\wp,\upsilon]$, we have seen that the q-integral inequality is valid:*

$$\left|\frac{1}{q}\left[\Phi(w) - \frac{1}{\upsilon-\wp}\int_\wp^\upsilon \Phi(\zeta)\,{}_\wp d_q\zeta\right]\right| \le M\left(\frac{(w-\wp)^2+(\upsilon-w)^2}{\upsilon-\wp}\right) \cdot \left(\frac{1}{[2]}\right)^{1-(1/m)}\left[-\frac{q}{[s+1]}\left(\left(1-q^{-1}\right)_q^{s+1} + \frac{q(1-q^{-1})_q^{s+2}}{[s+2]} - \frac{q}{[s+2]}\right)\right]^{1/m}, \quad (36)$$

for each $w \in [\wp,\upsilon]$.

Proof. Lemma 14 and keeping in view the well-known q-analogue of power-mean inequality, we have

$$\left|\frac{1}{q}\left[\Phi(w) - \frac{1}{\upsilon-\wp}\int_\wp^\upsilon \Phi(\zeta)\,{}_\wp d_q\zeta\right]\right| \le \frac{(w-\wp)^2}{\upsilon-\wp}\int_0^1 \zeta\left|D_q\Phi(\zeta w+(1-\zeta)\wp)\right|_0 d_q\zeta + \frac{(\upsilon-w)^2}{\upsilon-\wp}\int_0^1 \zeta\left|D_q\Phi(\zeta w+(1-\zeta)\upsilon)\right|_0 d_q\zeta \le \frac{(w-\wp)^2}{\upsilon-\wp}\left(\int_0^1 \zeta_0 d_q\zeta\right)^{1-(1/m)} \cdot \left(\int_0^1 \zeta\left|D_q\Phi(\zeta w+(1-\zeta)\wp)\right|^m {}_0d_q\zeta\right)^{1/m} + \frac{(\upsilon-w)^2}{\upsilon-\wp}\left(\int_0^1 \zeta_0 d_q\zeta\right)^{1-(1/m)} \cdot \left(\int_0^1 \zeta\left|D_q\Phi(\zeta w+(1-\zeta)\upsilon)\right|^m {}_0d_q\zeta\right)^{1/m},$$

$$\int_0^1 \zeta\left|D_q\Phi(\zeta w+(1-\zeta)\wp)\right|^m {}_0d_q\zeta \le \int_0^1 \zeta_q^{s+1}\left|D_q\Phi(w)\right|^m {}_0d_q\zeta + \int_0^1 \zeta(1-\zeta)_q^s\left|D_q\Phi(\wp)\right|^m {}_0d_q\zeta \le M^m\left(\int_0^1 \zeta_0 d_q\zeta\right)^{1-(1/m)}\left[\int_0^1 \zeta_q^{s+1}{}_0d_q\zeta + \int_0^1 \zeta(1-\zeta)_{q0}^s d_q\zeta\right] = M\frac{(w-\wp)^2+(\upsilon-w)^2}{\upsilon-\wp} \cdot \left(\int_0^1 \zeta_0 d_q\zeta\right)^{1-(1/m)}\left[\int_0^1 \zeta^{s+1}{}_0d_q\zeta + \int_0^1 \zeta(1-\zeta)_{q0}^s d_q\zeta\right],$$

$$\int_0^1 \zeta_q^{s+1}{}_0d_q\zeta = \frac{1}{[s+2]},$$

$$-\frac{q}{[s+1]}\int_0^1 \zeta D_q\left(1-q^{-1}\zeta\right)_q^{s+1}{}_0d_q\zeta = -\frac{q}{[s+1]} \cdot \left[\left|\zeta\left(1-q^{-1}\zeta\right)_q^{s+1}\right|_0^1 - \int_0^1 (1-\zeta)_q^{s+1}.1_0 d_q\zeta\right] = -\frac{q}{[s+1]}\left[\left(1-q^{-1}\right)_q^{s+1} + \frac{q(1-q^{-1})_q^{s+2}}{[s+2]} - \frac{q}{[s+2]}\right] = -\frac{q}{[s+1]}\left[\left(1-q^{-1}\right)_q^{s+1} + \frac{q(1-q^{-1})_q^{s+2}}{[s+2]} - \frac{q}{[s+2]}\right] = M\left(\frac{(w-\wp)^2+(\upsilon-w)^2}{\upsilon-\wp}\right)\left(\frac{1}{[2]}\right)^{1-(1/m)} \cdot \left[-\frac{q}{[s+1]}\left(\left(1-q^{-1}\right)_q^{s+1} + \frac{q(1-q^{-1})_q^{s+2}}{[s+2]} - \frac{q}{[s+2]}\right)\right]^{1/m}. \quad (37)$$

It completes the proof. □

Theorem 18. *Suppose that $\Phi : J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a q-differentiable mapping on J° such that $D_q\Phi \in L[\wp,\upsilon]$, in which $\wp, \upsilon \in J$ for $\wp<\upsilon$. If $|D_q\Phi|^m$ is s-convex function in second sense on $[\wp,\upsilon]$ for some $s \in (0,1]$, $q>1$ and $m>1$ and $n = m/m-1$, then the q-integral inequality is valid:*

$$\left|\frac{1}{q}\left[\Phi(w) - \frac{1}{\upsilon-\wp}\int_\wp^\upsilon \Phi(\zeta)\,{}_\wp d_q\zeta\right]\right| \le \frac{2^{(s-1/m)}}{[1+n]^{1/n}(\upsilon-\wp)}\left[(w-\wp)^2\left|D_q\Phi\left(\frac{w+\wp}{2}\right)\right| + (\upsilon-w)^2\left|D_q\Phi\left(\frac{\upsilon+w}{2}\right)\right|\right], \quad (38)$$

for each $w \in [\wp,\upsilon]$.

Proof. Lemma 3.1 and keeping in view the familiar q-analogue of Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{q}\left[\Phi(w)-\frac{1}{\wp+v}\int_{\wp}^{v}\Phi(\zeta)_{\wp}d_q\zeta\right]\right| \\
&\quad\le \frac{(w-\wp)^2}{v-\wp}\int_0^1\zeta\left|D_q\Phi(\zeta w+(1-\zeta)\wp)\right|_0 d_q\zeta \\
&\qquad+\frac{(v-w)^2}{v-\wp}\int_0^1\zeta\left|D_q\Phi(\zeta w+(1-\zeta)v)\right|_0 d_q\zeta \\
&\quad\le \frac{(w-\wp)^2}{v-\wp}\left(\int_0^1\zeta_{q0}^n d_q\zeta\right)^{1/n}\left(\int_0^1\left|D_q\Phi(\zeta w+(1-\zeta)\wp)\right|^m{}_0d_q\zeta\right)^{1/m} \\
&\qquad+\frac{(v-w)^2}{v-\wp}\int_0^1\left(\int_0^1\zeta_{q0}^n d_q\zeta\right)^{1/n}\left(\int_0^1\left|D_q\Phi(\zeta w+(1-\zeta)v)\right|^m{}_0d_q\zeta\right)^{1/m}, \\
&\qquad\qquad\int_0^1\left|D_q\Phi(\zeta w+(1-\zeta)\wp)\right|^m d\zeta\le 2^{s-1}\left|D_q\Phi\left(\frac{w+\wp}{2}\right)\right|^m \\
&\int_0^1\left|D_q\Phi(\zeta w+(1-\zeta)v)\right|^m d\zeta\le 2^{s-1}\left|D_q\Phi\left(\frac{v+w}{2}\right)\right|^m \\
&\quad\le \frac{2^{(s-1/m)}}{[1+n]^{1/n}(v-\wp)}\left[(w-\wp)^2\left|D_q\Phi\left(\frac{w+\wp}{2}\right)\right|+(v-w)^2\left|D_q\Phi\left(\frac{v+w}{2}\right)\right|\right].
\end{aligned} \tag{39}$$

□

Remark 19. In Theorem 13, if we choose $q=1$, then (24) diminishes the inequality (1) of Theorem 1.

Remark 20. In Theorem 13, if we choose $s=1$, then (24) diminishes the inequality (7) of Theorem 7.

Remark 21. In Theorem 15, if we fixed $q=1$, then (31) reduces the inequality (3) of Theorem 3.

Remark 22. In Theorem 15, if we take $s=1$, then (31) diminishes the inequality (9) of Theorem 9.

Remark 23. In Theorem 16, if we take $q=1$, then (34) reduces the inequality (4) of Theorem 4.

Remark 24. In Theorem 16, if we choose $s=1$, then (34) diminishes the inequality (10) of Theorem 10.

Remark 25. In Theorem 17, if we take $q=1$, then (36) diminishes the inequality (5) of Theorem 5.

Remark 26. In Theorem 18, if we take $q=1$, then (38) diminishes the inequality (6) of Theorem 6.

4. Conclusion

By the virtue of q-calculus, some integral inequalities are proved, which provides a method to study more properties of q-integrals via other classes of integral inequalities. q-Hermite-Hadmard and q-Ostrowski type integral inequalities have provided new estimates and good approximations in comparison with existing Hermite-Hadamard and Ostrowski inequalities. In similar fashion, the same methods can be applied to other inequalities, including Simpson's and trapezoidal inequalities for different classes of s-convex functions.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that there is no conflict of interest.

References

[1] S. S. Dragomir, "On the Ostrowski's integral inequality for mappings with bounded variation and applications," *Mathematical Inequalities and Applications*, vol. 4, no. 2, pp. 59–66, 1998.

[2] S. S. Dragomir and S. Wang, "A New Inequality Of Ostrowski's Type In L_1 Norm And Applications To Some Special Means And To Some Numerical Quadrature Rules," *Tamkang Journal of Mathematics*, vol. 28, no. 3, pp. 239–244, 1997.

[3] Z. Liu, "Some companions of an Ostrowski type inequality and application," *JIPAM - Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 2, p. 52, 2009.

[4] N. Alp and M. Z. Sarikaya, "Q-hardy type inequalities for quantum integrals," *Adv. Difference Equ.*, vol. 2021, article 355, 2021.

[5] B. G. Pachpatte, "On an inequality of Ostrowski type in three independent variables," *Journal of Mathematical Analysis and Applications*, vol. 249, no. 2, pp. 583–591, 2000.

[6] A. O. Akdemir, S. I. Butt, M. Nadeem, and M. A. Ragusa, "New general variants of Chebyshev type inequalities via generalized fractional integral operators," *Mathematics*, vol. 9, no. 2, p. 122, 2021.

[7] S. I. Butt, M. Nadeem, and G. Farid, "On Caputo fractional derivatives via exponential sconvex functions," *Turkish Journal of Science*, vol. 5, no. 2, pp. 140–146, 2020.

[8] B. G. Pachpatte, "On a new Ostrowski type inequality in two independent variables," *Tamkang Journal of Mathematics*, vol. 32, no. 1, pp. 45–49, 2001.

[9] M. Z. Sarikaya, "On the Ostrowski type integral inequality," *Acta Mathematica Universitatis Comenianae*, vol. 79, no. 1, pp. 129–134, 2010.

[10] N. Ujević, "Sharp inequalities of Simpson type and Ostrowski type," *Computers & Mathematcs with Applications*, vol. 48, no. 1-2, pp. 145–151, 2004.

[11] Z. Lü, "On sharp inequalities of Simpson type and Ostrowski type in two independent variables," *Computers & Mathematcs with Applications*, vol. 56, no. 8, pp. 2043–2047, 2008.

[12] M. Alomari and M. Darus, "Some Ostrowski type inequalities for convex functions with applications," *Research Group in Mathematical Inequalities and Applications*, vol. 13, no. 1, 2010.

[13] M. Adil Khan, S. Begum, Y. Khurshid, and Y. M. Chu, "Ostrowski type inequalities involving conformable fractional integrals," *Journal of Inequalities and Applications*, vol. 2018, Article ID 70, 2018.

[14] A. Ekinci and N. EroÄŸlu, "New generalizations for s-convex functions via conformable fractional integrals," *Turkish Journal of Inequalities*, vol. 4, no. 2, pp. 1–9, 2020.

[15] F. Usta, H. Budak, M. Sarikaya Zeki, and E. Set, "On generalization of trapezoid type inequalities for s-convex functions with generalized fractional integral operators," *Filomat*, vol. 32, no. 6, pp. 2153–2171, 2018.

[16] M. A. Ardic, A. O. Akdemir, and E. Set, "New Ostrowski like inequalities for GG-convex and GA-convex functions," *Mathematical Inequalities and Applications*, vol. 19, pp. 1159–1168, 1998.

[17] A. Ekinci, M. E. Özdemir, and S. E. T. Erhan, "New integral inequalities of Ostrowski type for quasi-convex functions with applications," *Turkish Journal of Science*, vol. 5, no. 3, pp. 290–304, 2020.

[18] M. A. Khan, Y. Chu, T. U. Khan, and J. Khan, "Some new inequalities of Hermite-Hadamard type for s-convex functions with applications," *Open Mathematics*, vol. 15, no. 1, pp. 1414–1430, 2017.

[19] M. Adil Khan, M. Hanif, Z. Abdul Hameed Khan, K. Ahmad, and Y. M. Chu, "Association of Jensen's inequality for s-convex function with Csiszár divergence," *Journal of Inequalities and Applications*, vol. 2019, Article ID 162, 2019.

[20] S. I. Butt, M. Nadeem, and G. Farid, "On Caputo fractional derivatives via exponential s-convex functions," *Turkish Journal of Science*, vol. 5, no. 2, pp. 140–146, 2020.

[21] Y. Khurshid, M. Adil Khan, and Y. M. Chu, "Ostrowski type inequalities involving conformable integrals via preinvex functions," *AIP Advances*, vol. 10, no. 5, article 055204, 2020.

[22] A. Ostrowski, "Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert," *Commentarii Mathematici Helvetici*, vol. 10, pp. 226-227, 1937.

[23] A. Ekinci, "Inequalities for convex functions on time scales," *TWMS Journal of Applied and Engineering Mathematics*, vol. 9, no. 1, pp. 64–72, 2019.

[24] S. Shaimardan, *Hardy-Type Inequalities Quantum Calculus, [Ph.D. Thesis]*, Luleå University of Technology, 2018.

[25] S. S. Dragomir and S. Fitzpatrick, "The hadamard inequalities for s-convex functions in the second sense," *Demonstratio Mathematica*, vol. 32, no. 4, pp. 687–696, 1999.

[26] M. Alomari, M. Darus, S. S. Dragomir, and P. Cerone, "Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense," *Applied Mathematics Letters*, vol. 23, no. 9, pp. 1071–1076, 2010.

[27] M. Alomari, "Q-Bernoulli inequality," *Turkish Journal of Science*, vol. 3, no. 1, pp. 32–39, 2018.

[28] A. A. El-Deeb and S. Rashid, "On some new double dynamic inequalities associated with Leibniz integral rule on time scales," *Adv. Difference Equ.*, vol. 1, pp. 1–22, 2021.

[29] J. Tariboon and S. K. Ntouyas, "Quantum integral inequalities on finite intervals," *Journal of Inequalities and Applications*, vol. 2014, Article ID 121, 2014.

[30] M. A. Noor, M. U. Awan, and K. I. Noor, "Quantum Ostrowski inequalities for q-differentiable convex functions," *Journal of Mathematical Inequalities*, vol. 4, no. 10, pp. 1013–1018, 2007.

[31] H. Hudzik and L. Maligranda, "Some remarks on s-convex functions," *Aequationes Mathematicae*, vol. 48, no. 1, pp. 100–111, 1994.

[32] V. Kac and P. Cheung, *Quantum Calculus*, Springer-Verlag, New York, 2002.

[33] H. Gauchman, "Integral inequalities in q-calculus," *Computers & Mathematics with Applications*, vol. 47, no. 2-3, pp. 281–300, 2004.

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Research Article

Some Improvements of Jensen's Inequality via 4-Convexity and Applications

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The intention of this note is to investigate some new important estimates for the Jensen gap while utilizing a 4-convex function. We use the Jensen inequality and definition of convex function in order to achieve the required estimates for the Jensen gap. We acquire new improvements of the Hölder and Hermite–Hadamard inequalities with the help of the main results. We discuss some interesting relations for quasi-arithmetic and power means as consequences of main results. At last, we give the applications of our main inequalities in the information theory. The approach and techniques used in the present note may simulate more research in this field.

1. Introduction

The theory of convex functions performs an extremely significant and consequential role in several areas of pure and applied sciences. Due to its numerous and extensive applications, the concept of convex functions has been extended and generalized in many directions. The most important and elegant aspect of the class of convex functions, which attracted many researchers, is its deep relation with theory of inequalities [1–3]. In the literature, there are several well-known inequalities which are the direct consequences and applications of convexity [4, 5]. In this respect, some of the noted inequalities associated with the class of convex functions are majorization, Hermite–Hadamard and Jensen–Mercer inequalities [6]. Among these inequalities, one of the considerable and vital inequalities which are studied very widely in the literature is the Jensen inequality. This celebrated inequality reads as follows:

Theorem 1. *Assume that I is an interval of real numbers and Ψ is a convex function on I. If $y_j \in I$ and $w_j > 0$ for $j = 1, 2, \cdots, n$ with $W = \sum_{j=1}^{n} w_j$, then*

$$\Psi\left(\frac{1}{W}\sum_{j=1}^{n} w_j y_j\right) \le \frac{1}{W}\sum_{j=1}^{n} w_j \Psi\left(y_j\right). \tag{1}$$

Inequality (1) will be true in the reverse direction, if the function Ψ is concave on I.

The Jensen inequality has multitudinous applications in Mathematics [7–11], Statistics [12], Economics [13] and Information Theory [14], etc. The most interesting and attractive applications of this inequality is that it generalized the classical convexity. Moreover, there are several inequalities which are the direct consequences of this inequality such as Ky Fan, Cauchy, Hermite–Hadamard and Hölder inequalities. Due to the vast applications of the Jensen inequality, many researchers dedicated their work to this inequality. This inequality has been extended, improved, and refined in multidirections by using different techniques and principals. For some more extensive literature concerning to the Jensen inequality, see [15, 16].

2. Main Results

In the present part, we discuss the main results. Let us begin this section with the following theorem, in which we acquire an upper bound for the Jensen gap.

Theorem 2. *Assume that I is an interval in* $\mathbb{R}$, $x_i \in I$ *and* $p_i > 0$ *for* $i = 1, 2, \cdots, n$ *with* $P_n := \sum_{i=1}^n p_i$ *and* $\bar{x} = 1/P_n \sum p_i x_i$. *If* Ψ *is a twice differentiable function such that* Ψ *is 4-convex on I, then*

$$\frac{1}{P_n}\sum_{i=1}^n p_i \Psi(x_i) - \Psi(\bar{x}) \le \frac{1}{6P_n}\sum_{i=1}^n p_i (\bar{x} - x_i)^2 \left(2\Psi''(\bar{x}) + \Psi''(x_i)\right). \tag{2}$$

Inequality (2) will be true in the opposite direction, if the function Ψ *is 4-concave.*

Proof. Without misfortune of sweeping statement, assume that $\bar{x} \neq x_i$ for $i = 1, 2, \cdots, n$. Utilizing integration by parts, we have

$$\begin{aligned}
&\frac{1}{P_n}\sum_{i=1}^n p_i (\bar{x} - x_i)^2 \int_0^1 t\Psi''(t\bar{x} + (1-t)x_i)dt \\
&\quad = \frac{1}{P_n}\sum_{i=1}^n p_i (\bar{x} - x_i)^2 \left(\frac{t}{\bar{x} - x_i}\Psi'(t\bar{x} + (1-t)x_i)\Big|_0^1 \right. \\
&\qquad \left. - \frac{1}{\bar{x} - x_i}\int_0^1 \Psi'(t\bar{x} + (1-t)x_i)dt\right) \\
&\quad = \frac{1}{P_n}\sum_{i=1}^n p_i (\bar{x} - x_i)^2 \left(\frac{\Psi'(\bar{x})}{\bar{x} - x_i} - \frac{t}{(\bar{x} - x_i)^2}\Psi(t\bar{x} + (1-t)x_i)\Big|_0^1\right) \\
&\quad = \frac{1}{P_n}\sum_{i=1}^n p_i (\bar{x} - x_i)^2 \left(\frac{\Psi'(\bar{x})}{\bar{x} - x_i} - \frac{t}{(\bar{x} - x_i)^2}(\Psi(\bar{x}) - \Psi(x_i))\right) \\
&\quad = \frac{1}{P_n}\sum_{i=1}^n p_i (\bar{x} - x_i)\Psi'(\bar{x}) - \frac{1}{P_n}\sum_{i=1}^n p_i (\Psi(\bar{x}) - \Psi(x_i)) \\
&\quad = \frac{1}{P_n}\sum_{i=1}^n p_i \Psi(x_i) - \Psi(\bar{x}),
\end{aligned} \tag{3}$$

which implies that

$$\frac{1}{P_n}\sum_{i=1}^n p_i \Psi(x_i) - \Psi(\bar{x}) = \frac{1}{P_n}\sum_{i=1}^n p_i (x_i - \bar{x})^2 \int_0^1 t\Psi''(t\bar{x} + (1-t)x_i)dt. \tag{4}$$

Since, the function Ψ is 4-convex on I. Therefore, using the definition of convex function on the right hand side of (4), we receive

$$\begin{aligned}
&\frac{1}{P_n}\sum_{i=1}^n p_i \Psi(x_i) - \Psi(\bar{x}) \\
&\quad \le \frac{1}{P_n}\sum_{i=1}^n p_i (x_i - \bar{x})^2 \int_0^1 \left(t^2\Psi''(\bar{x}) + t(1-t)\Psi''(x_i)\right)dt \\
&\quad = \frac{1}{P_n}\sum_{i=1}^n p_i (\bar{x} - x_i)^2 \left(\Psi''(\bar{x})\int_0^1 t^2 dt + \Psi''(x_i)\int_0^1 \left(t - t^2\right)dt\right) \\
&\quad = \frac{1}{P_n}\sum_{i=1}^n p_i (\bar{x} - x_i)^2 \left(\frac{1}{3}\Psi''(\bar{x}) + \frac{1}{6}\Psi''(x_i)\right),
\end{aligned} \tag{5}$$

which is equivalent to (2). □

The integral version of (2) is stated in the following theorem.

Theorem 3. *Assume that I is an interval in* $\mathbb{R}, \Psi : I \longrightarrow \mathbb{R}$ *that is a twice differentiable function such that* Ψ *is 4-convex and* $\phi, \varphi : [a, b] \longrightarrow I$ *are integrable functions with* $\varphi \ge 0$ *on* $[a, b]$. *Also, assume that* $\Psi \circ \phi : [a, b] \longrightarrow \mathbb{R}$ *is an integrable function,* $\bar{\varphi} := \int_a^b \varphi(x)dx > 0$ *and* $\overline{\phi} := \frac{1}{\bar{\varphi}}\int_a^b \varphi(x)\phi(x)dx$. *Then*

$$\begin{aligned}
&\frac{1}{\bar{\varphi}}\int_a^b \varphi(x)\Psi \circ \phi(x)dx - \Psi\left(\overline{\phi}\right) \\
&\quad \le \frac{1}{6\bar{\varphi}}\int_a^b \varphi(x)\left(\overline{\phi} - \phi(x)\right)^2 \left(2\Psi''\left(\overline{\phi}\right) + \Psi''(\phi(x))\right)dx.
\end{aligned} \tag{6}$$

Inequality (6) will be true in the reverse sense, if Ψ *is a 4-concave function.*

In the next theorem, we acquire a lower bound for the Jensen gap while utilizing the Jensen inequality.

Theorem 4. *Assume that all the suppositions of Theorem 2 are valid, then*

$$\frac{1}{P_n}\sum_{i=1}^n p_i \Psi(x_i) - \Psi(\bar{x}) \ge \frac{1}{2P_n}\sum_{i=1}^n p_i (\bar{x} - x_i)^2 \Psi''\left(\frac{2\bar{x} + x_i}{3}\right). \tag{7}$$

Inequality (7) will become true in the reverse direction, if the function Ψ *is 4-concave.*

Proof. Since, the function Ψ is 4-convex on I. Therefore, applying integral Jensen's inequality on the right of (4), we get

$$\begin{aligned}\frac{1}{P_n}\sum_{i=1}^{n}p_i\Psi(x_i)\Psi(\bar{x})&\\ &=\frac{1}{P_n}\sum_{i=1}^{n}p_i(\bar{x}-x_i)^2\left(\frac{\int_0^1 t\Psi''(t\bar{x}+(1-t)x_i)dt}{\int_0^1 tdt}\right)\\ &\geq\frac{1}{2P_n}\sum_{i=1}^{n}p_i(\bar{x}-x_i)^2\Psi''\left(\frac{\int_0^1 t(t\bar{x}+(1-t)x_i)dt}{\int_0^1 tdt}\right)\\ &=\frac{1}{2P_n}\sum_{i=1}^{n}p_i(\bar{x}-x_i)^2\Psi''\left(\frac{\bar{x}\int_0^1 t^2dt+x_i\int_0^1(t-t^2)dt}{\int_0^1 tdt}\right)\\ &=\frac{1}{2P_n}\sum_{i=1}^{n}p_i(\bar{x}-x_i)^2\Psi''\left(\frac{2\bar{x}+x_i}{3}\right),\end{aligned}\tag{8}$$

which is the required inequality. □

The analogous inequality of (7) is given in the following theorem.

Theorem 5. *Suppose that all the hypotheses of Theorem 3 are true. Then*

$$\begin{aligned}&\frac{1}{\bar{\varphi}}\int_a^b\varphi(x)\Psi\circ\phi(x)dx-\Psi(\bar{\phi})\\ &\quad\geq\frac{1}{2\bar{\varphi}}\int_a^b\varphi(x)\left(\bar{\phi}-\phi(x)\right)^2\Psi''\left(\frac{2\bar{\phi}+\phi(x)}{3}\right)dx.\end{aligned}\tag{9}$$

If the function Ψ is 4-concave, then the inequality (9) holds in the opposite direction.

3. Numerical Experiments

In this section, we are going to provide some simple examples to show how sharp our estimates for the Jensen gap.

Example 6. Consider the functions $\Psi(x)=x^4$, $\phi(x)=x$ and $\varphi(x)=1$ for *all* $x\in[0,1]$. Then, $\Psi''(x)=12x^2\geq 0$ and $\Psi''''(x)=24>0$ *on*$[0,1]$. This verifies that the function Ψ is convex as well as 4-convex. Now, utilizing (6) for $\Psi(x)=x^4$, $\phi(x)=x$, $\varphi(x)=1$, $a=0$, and $b=1$, we get

$$0.1375<0.15.\tag{10}$$

Using above functions with the given interval in inequality (4) in [17], we acquire

$$0.1375<0.25.\tag{11}$$

From inequalities (10) and (11) it is clear that the bounds given in (6) provide a good and better estimate for the Jensen gap. Moreover, the inequality (10) shows that the value of the obtained estimate for the Jensen gap given in (6) is very close to the value of the Jensen gap.

Example 7. Consider the functions $\Psi(x)=(1-x)^5$, $\varphi(x)=1$, and $\phi(x)=1$ for all $x\in[0,1]$. Then, $\Psi''(x)=20(1-x)^3$and $\Psi''''(x)=120(1-x)$. Clearly, both Ψ'' and Ψ'''' are nonnegative on [0,1]. This shows that the function $\Psi(x)=(1-x)^5$ is convex as well as 4-convex. Utilizing $\Psi(x)=(1-x)^5$, $\Psi''(x)=20(1-x)^3$, $\varphi(x)=1$ and $\phi(x)=1$ in (6), we obtain

$$0.1345<0.1666.\tag{12}$$

Now, using the chosen functions in the inequality (4) in [17], we acquire

$$0.1345<0.4166.\tag{13}$$

From (12) and (13) it is clear that the inequality (6) provides an efficient and superior estimate as compared to the inequality (4) in [17].

Example 8. Assume that the functions $\Psi(x)=\exp x$, $\varphi(x)=1$, and $\phi(x)=x$ are defined on [0,1]. Then, $\Psi''(x)=\exp x$ and $\Psi''''(x)=\exp x$. Obviously, both the functions Ψ'' and Ψ'''' are nonnegative on $\in[0,1]$. This confirms the convexity and 4-convexity of the function $\Psi(x)=\exp x$. Choosing $\Psi(x)=\exp x$, $\Psi''(x)=\exp x$, $\varphi(x)=1$ and $\phi(x)=x$ in (6), we obtain

$$0.0695<0.0704.\tag{14}$$

Now, using the given functions in the inequality (4) in [17], we acquire

$$0.0695<0.0996.\tag{15}$$

Again, from (14) and (15), it is obvious that the estimate provided by inequality (6) for the Jensen gap is better than the estimate provided by inequality (4) in [17]. Moreover, the value of the estimate for the Jensen gap in (6) is very close to the value of the Jensen gap.

Remark 9. The authors in [17] compared the value of estimate for the Jensen gap in the inequality (4) with the value of the estimates for the Jensen gap in inequalities (5) and (8) in [18]. From the comparison, the authors declared that the estimate for the Jensen gap in inequality (4) in [17] is better than the estimates for the Jensen gap in inequalities (5) and (8) in [18]. Hence from this, we can also conclude that our estimate for the Jensen gap may be better than the estimates for the Jensen gap in (5) and (8) in [18].

4. Applications for Classical Inequalities

This section is devoted to the consequences of main results. In this section, we obtain some improvements for the Hölder and Hermite–Hadamard inequalities with the help of our main results. Furthermore, we acquire different relations for the power and quasi-arithmetic means with the utilization of our obtained results.

In the following proposition, we give an improvement for the Hölder inequality with the help of Theorem 2.

Proposition 10. *Let $m_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ and $m_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ be two positive n-tuples and $p, q > 1$, such that $p \notin (2, 3)$. If $1/p + 1/q = 1$, then*

$$\left(\sum_{i=1}^{n}\zeta_i^p\right)^{1/p}\left(\sum_{i=1}^{n}\gamma_i^q\right)^{1/q} - \sum_{i=1}^{n}\gamma_i\zeta_i \le \left[\frac{p(p-1)}{6}\sum_{i=1}^{n}\gamma_i^q\left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^q} - \zeta_i\gamma_i^{-q/p}\right)^2 \times \left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^q} - \zeta_i\gamma_i^{-q/p}\right)\right]^{1/p}\left(\sum_{i=1}^{n}\gamma_i^q\right)^{\frac{1}{q}}. \tag{16}$$

Proof. Since the function $\Psi = x^p$ is convex as well as 4-convex on $(0,\infty)$ for all $p > 1, p \notin (2, 3)$. Therefore, utilizing (2) by choosing $\Psi(x) = x^p$ and $p_i = \gamma_i^q, x_i = \zeta_i\gamma_i^{-q/p}$ for all $i \in \{1, 2, \cdots, n\}$ and then taking power $1/p$, we get

$$\left(\left(\sum_{i=1}^{n}\zeta_i^p\right)\left(\sum_{i=1}^{n}\gamma_i^q\right)^{p-1} - \left(\sum_{i=1}^{n}\gamma_i\zeta_i\right)^p\right)^{1/p} \le \left[\frac{p(p-1)}{6}\sum_{i=1}^{n}\gamma_i^q\left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^q} - \zeta_i\gamma_i^{-(q/p)}\right)^2 \times \left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^q} - \zeta_i\gamma_i^{-(q/p)}\right)\right]^{1/p}\left(\sum_{i=1}^{n}\gamma_i^q\right)^{1/q}. \tag{17}$$

As the inequality

$$a^l - b^l \le (a-b)^l \tag{18}$$

holds, for all $a, b \ge 0$ and $l \in [0, 1]$, thus using (18) for $a = (\sum_{i=1}^{n}\zeta_i^p)(\sum_{i=1}^{n}\gamma_i^q)^{p-1}$, $b = (\sum_{i=1}^{n}\gamma_i\zeta_i)^p$ and $l = 1/p$, we obtain

$$\left(\sum_{i=1}^{n}\zeta_i^p\right)^{1/l}\left(\sum_{i=1}^{n}\gamma_i^q\right)^{1/q} - \sum_{i=1}^{n}\gamma_i\zeta_i \le \left(\left(\sum_{i=1}^{n}\zeta_i^p\right)\left(\sum_{i=1}^{n}\gamma_i^q\right)^{p-1} - \left(\sum_{i=1}^{n}\gamma_i\zeta_i\right)\right)^{1/p}. \tag{19}$$

Now, comparing (17) and (19), we acquire (16). □

Another consequence of Theorem 2 is given in the following corollary, in which we provide a relation for the Hölder inequality.

Corollary 11. *Let $m_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ and $m_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ be two positive n-tuples, $0 < p < 1$ and $q = p/(p-1)$ such that $1/p \notin (2, 3)$. If $1/p + 1/q = 1$, then*

$$\sum_{i=1}^{n}\gamma_i\zeta_i - \left(\sum_{i=1}^{n}\zeta_i^p\right)^{1/p}\left(\sum_{i=1}^{n}\gamma_i^q\right)^{1/q} \le \frac{1-p}{6p^2}\sum_{i=1}^{n}\gamma_i^q\left(\frac{\sum_{i=1}^{n}\zeta_i^p}{\sum_{i=1}^{n}\gamma_i^q} - \zeta_i^p\gamma_i^{-q}\right)^2 \times \left(\frac{\sum_{i=1}^{n}\zeta_i^p}{\sum_{i=1}^{n}\gamma_i^q} - \zeta_i^p\gamma_i^{-q}\right). \tag{20}$$

Proof. For $p \in (0, 1)$ such that $1/p \notin (2, 3)$, the function $\Psi(x) = x^{1/p}$ is convex as well as 4-convex on $[0,\infty]$. Therefore, utilizing (2) by choosing $\Psi(x) = x^{1/p}, p_i = \gamma_i{}^q$, and $x_i = \gamma_i{}^{-q}\zeta_i{}^p$, we get (20). □

As a consequence of Theorem 4, we acquire another improvement for the Hölder inequality which is stated in the following corollary.

Corollary 12. *Assume that $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ are two n-tuples such that $\gamma_i, \zeta_i > 0$ for all $i \in \{1, 2, \cdots, n\}$. If $p \in (0, 1)$ and $q = p/(p-1)$ such that $1/p \notin (2, 3)$, then*

$$\sum_{i=1}^{n}\gamma_i\zeta_i - \left(\sum_{i=1}^{n}\zeta_i^p\right)^{1/p}\left(\sum_{i=1}^{n}\gamma_i^q\right)^{1/q} \ge \frac{1-p}{2p^2}\sum_{i=1}^{n}\gamma_i^q\left(\frac{\sum_{i=1}^{n}\zeta_i^p}{\sum_{i=1}^{n}\gamma_i^q} - \zeta_i^p\gamma_i^{-q}\right)^2 \times \left(\frac{2\sum_{i=1}^{n}\zeta_i^p + \zeta_i^p\gamma_i^{-q}\sum_{i=1}^{n}\gamma_i^q}{3\sum_{i=1}^{n}\gamma_i^q}\right). \tag{21}$$

Proof. The function $\Psi(x) = x^{1/p}$ is both convex and 4-convex for $x \ge 0$ with $p \in (0, 1)$ such that $1/p \notin (2, 3)$. Therefore, inequality (21) can easily be acquired by putting $\Psi(x) = x^{1/p}, p_i = \gamma_i{}^q$, and $x_i = \gamma_i{}^{-q}\zeta_i^p$ in (7). □

Now, we recall the definition of power mean.

Definition 13. Let $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be two n-tuples such that $\gamma_i, \zeta_i \in (0,\infty)$ for all $i \in \{1, 2, \cdots, n\}$ with $\bar{\gamma} = \sum_{i=1}^{n}\gamma_i$. Then, the power mean of order $p \in \mathbb{R}$ is defined by

$$M_p(m_1, m_2) = \begin{cases} \left(\frac{1}{\bar{\gamma}}\sum_{i=1}^{n}\gamma_i\zeta_i^p\right)^{1/p}, & p \ne 0 \\ \left(\prod_{i=1}^{n}\zeta_i^{\gamma_i}\right)^{\frac{1}{\bar{\gamma}}}, & p = 0. \end{cases} \tag{22}$$

As a consequence of Theorem 2, in the following corollary, we give bound for the power mean.

Corollary 14. *Let $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be two positive n-tuples such that $\bar{\gamma} = \sum_{i=1}^{n}\gamma_i$ and r and t be* *nonzero real numbers. Then the following statements are true:*

(i) If $r > 0$ with $t \geq 3r$ or $2r \geq t \geq r$ or $0 > t$, then

$$M_r^r(m_1, m_2) - M_r^t(m_1, m_2) \leq \frac{t(t-r)}{6r^2\bar{\gamma}} \sum_{i=1}^{n} \gamma_i \left(M_r^r(m_1, m_2) - \zeta_i^r\right)^2 \times \left(M_r^{t-2r}(m_1, m_2) + \zeta_i^{t-2}\right). \tag{23}$$

(ii) If $r < 0$ with $t \leq 3r$ or $2r \leq t \leq r$ or $0 < t$, then (23) holds.

(iii) If $r > 0$ with $3r > t > 2r$ or $r < 0$ with $3r < t < 2r$, then (23) holds in the reverse direction.

Proof.

(i) For $x > 0$, the function $\Psi(x) = x^{t/r}$ is 4-convex with the given conditions.

Therefore, using (2) by taking $\Psi(x) = x^{t/r}$, $p_i = \gamma_i$, and $x_i = \zeta_i^r$, we obtain (23)

(ii) If the given conditions are hold, then the function $\Psi(x) = x^{t/r}$ will be 4-convex on $(0,\infty)$. Thus, (23) can easily be obtained by adopting the procedure of (i)

(iii) The function $\Psi(x) = x^{t/r}$ is 4-concave on $(0,\infty)$ for the given values of r and t. Therefore, we can get the reverse inequality in (23) by adopting the procedure of (i).

□

In the following result, we present an application of Theorem 4.

Corollary 15. *Let m_1, m_2, and $\bar{\gamma}$ be the same as that of Corollary 14 and $r, t \in \mathbb{R} - \{0\}$. Then*

(A) If the conditions given in (i) and (ii) are satisfied, then

$$M_t^t(m_1, m_2) - M_r^t(m_1, m_2) \geq \frac{t(t-r)}{2\bar{\gamma}r^2} \sum_{i=1}^{n} \gamma_i \left(M_r^r(m_1, m_2) - \zeta_i^r\right)^2 \cdot \left(\frac{2M_r^r(m_1, m_2) + \zeta_i^{t-2}}{3}\right)^{t/r-2}. \tag{24}$$

(B) If the conditions in (iii) are fulfilled, then (24) holds in the reverse direction.

Proof.

(A) Since, the function $\Psi(x) = x^{t/r}$ is 4-convex on $(0,\infty)$ for the conditions given in (i) and (ii) of Corollary 14. Therefore, using (7) for $\Psi(x) = x^{t/r}$, $p_i = \gamma_i$, and $x_i = \zeta_i^r$, we obtain (24)

(B) If the condition on r and t mentioned in (iii) of Corollary 14 is true, then the function $\Psi(x) = x^{t/r}$ will be 4-concave for $x > 0$. Thus, utilizing (7) while choosing $\Psi(x) = x^{t/r}$, $p_i = \gamma_i$, and $x_i = \zeta_i^r$, we obtain the reverse inequality in (24).

□

In the following corollary, we obtain an interesting relation for different means as a consequence of Theorem 2.

Corollary 16. *Let $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be positive n-tuples with $\bar{\gamma} = \sum_{i=1}^{n} \gamma_i$. Then*

$$\frac{M_1(m_1, m_2)}{M_0(m_1, m_2)} \leq \exp\left[\frac{1}{6\bar{\gamma}} \sum_{i=1}^{n} \gamma_i (M_1(m_1, m_2) - \zeta_i)^2 \left(M_1^{-2}(m_1, m_2) + \zeta_i^{-2}\right)\right]. \tag{25}$$

Proof. Let $\Psi(x) = -\ln x$, $x > 0$. Then, $\Psi''(x) = 1/x^2$ and $\Psi''(x) = 6/x^4$. Clearly, both $\Psi''(x)$ and $\Psi''''(x)$ are positive for all $x > 0$. This confirms that the function $\Psi(x)$ is convex as well as 4-convex on $(0,\infty)$. Therefore, putting $\Psi(x) = -\ln x$, $p_i = \gamma_i$, and $x_i = \zeta_i$ in (2), we acquire (25). □

In the following corollary, a relation for distinct means is obtain with the help of Theorem 2.

Corollary 17. *Let hypotheses of Corollary 16 hold. Then*

$$M_t^t(m_1, m_2) - M_0(m_1, m_2) \leq \frac{1}{6\bar{\gamma}} \sum_{i=1}^{n} \gamma_i (M_1(m_1, m_2) - In\zeta_i)^2 (2M_0(m_1, m_2) + \zeta_i). \tag{26}$$

Proof. Consider function $\Psi(x) = \exp x$, $x \in \mathbb{R}$. Then clearly, $\Psi''(x) = \exp x > 0$ and $\Psi''''(x) = \exp x > 0$. This shows that the given function is convex as well as 4-convex. Thus, applying (2) by choosing $p_i = \gamma_i$, $x_i = \ln \zeta_i$, and $\Psi(x) = \exp x$, we get (26). □

An application of Theorem 4 is acquired in the below corollary.

Corollary 18. *Suppose that all the assumptions of Corollary 16 are true, then*

$$\frac{M_1(m_1, m_2)}{M_0(m_1, m_2)} \geq \exp\left[\frac{1}{2\bar{\gamma}} \sum_{i=1}^{n} \gamma_i (M_1(m_1, m_2) - \zeta_i)^2 \left(\frac{3}{2M_1(m_1, m_2) + \zeta_i}\right)^2\right]. \tag{27}$$

Proof. Put $\Psi(x) = -\ln x$, $p_i = \gamma_i$, and $x_i = \zeta_i$ in (7), we get (27). □

The following is another relation for distinct means which is the consequence of Theorem 4.

Corollary 19. *Let the hypotheses of Corollary 16 be fulfilled. Then*

$$\begin{aligned} M_1(m_1, m_2) - M_0(m_1, m_2) \\ \geq \frac{1}{2\bar{\gamma}} \sum_{i=1}^{n} \gamma_i (M_0(m_1, m_2) - In\zeta_i)^2 \exp \\ \cdot \left(\frac{2M_0(m_1, m_2) + In\zeta_i}{3} \right). \end{aligned} \tag{28}$$

Proof. Utilizing (7) for $p_i = \gamma_i$, $x_i = \ln \zeta_i$, and $\Psi(x) = \exp x$, we get (28).

Now, we give the definition of quasiarithmetic mean. □

Definition 20. Let $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be positive n-tuples with $\bar{\gamma} = \sum_{i=1}^{n} \gamma_i$ and φ be strictly monotonic continuous function. Then the quasi-arithmetic mean is defined as

$$M_\varphi(m_1, m_2) = \varphi^{-1} \left(\frac{1}{\bar{\gamma}} \sum_{i=1}^{n} \gamma_i \varphi(\zeta_i) \right). \tag{29}$$

In the following corollary, we obtain a relation for the quasi-arithmetic mean with the help of Theorem 2.

Corollary 21. *Let $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be positive n-tuples with $\bar{\gamma} = \sum_{i=1}^{n} \gamma_i$. Also, let φ be strictly monotonic continuous function and $\Psi^\circ \varphi^{-1}$ be 4-convex on $(0,\infty)$. Then*

$$\begin{aligned} \frac{1}{\bar{\gamma}} \sum_{i=1}^{n} \gamma_i \Psi(\zeta_i) - \Psi(M_\varphi(m_1, m_2)) \\ \leq \frac{1}{6\bar{\gamma}} \sum_{i=1}^{n} \gamma_i (\varphi(M_\varphi(m_1, m_2)) - \varphi(\zeta_i))^2 \\ \times \left(2(\Psi \circ \varphi^{-1})'' (\varphi(M_\varphi(m_1, m_2))) + (\Psi \circ \varphi^{-1})'' \varphi(\zeta_i) \right). \end{aligned} \tag{30}$$

Proof. Since, the function $\Psi \circ \varphi^{-1}$ is 4-convex on $(0,\infty)$. Therefore, choosing $\Psi = \Psi \circ \varphi^{-1}$, $p_i = \gamma_i$, and $x_i = \phi(\zeta_i)$ in (2), we obtain (30). □

As an application of Theorem 4, in the following corollary, we present a relation for the quasi-arithmetic mean.

Corollary 22. *Let the hypotheses of Corollary 21 hold. Then*

$$\begin{aligned} \frac{1}{\bar{\gamma}} \sum_{i=1}^{n} \gamma_i \Psi(\zeta_i) - \Psi(M_\varphi(m_1, m_2)) \\ \geq \frac{1}{2\bar{\gamma}} \sum_{i=1}^{n} \gamma_i (\varphi(M_\varphi(m_1, m_2)) - \varphi(\zeta_i))^2 \\ \times \left((\Psi \circ \varphi^{-1})'' \left(\frac{2\varphi(M_\varphi(m_1, m_2)) + \varphi(\zeta_i)}{3} \right) \right). \end{aligned} \tag{31}$$

Proof. Using (7) for $\Psi = \Psi \circ \varphi^{-1}$, $p_i = \gamma_i$, and $x_i = \varphi(\zeta_i)$, we obtain (31). □

In the following corollaries, we present some improvements for the Hermite–Hadamard inequalities with the support of our main results.

Corollary 23. *Let $\Psi : [a, b] \longrightarrow \mathbb{R}$ be a 4-convex function. Then*

$$\begin{aligned} \frac{1}{b-a} \int_a^b \Psi(x) dx - \Psi\left(\frac{a+b}{2}\right) \\ \leq \frac{1}{6(b-a)} \int_a^b \left(\frac{a+b}{2} - x\right)^2 \left(2\Psi''\left(\frac{a+b}{2}\right) + \Psi''(x)\right) dx. \end{aligned} \tag{32}$$

If the function Ψ is 4-concave, then the inequality (32) holds in the reverse direction.

Proof. Since, the function Ψ is 4-convex on $[a, b]$. Therefore, using (6) for $\varphi(x) = 1$ and $\phi(x) = x$, we get (32). □

Corollary 24. *Assume that the function $\Psi : [a, b] \longrightarrow R$ is 4-convex, then*

$$\begin{aligned} \frac{1}{b-a} \int_a^b \Psi(x) dx - \Psi\left(\frac{a+b}{2}\right) \\ \geq \frac{1}{2(b-a)} \int_a^b \left(\frac{a+b}{2} - x\right)^2 \Psi''\left(\frac{a+b+x}{3}\right) dx. \end{aligned} \tag{33}$$

Inequality (33) will be true in the opposite sense, if the function Ψ is 4-concave.

Proof. Inequality (33) can easily be deduced by choosing $\phi(x) = x$ and $\varphi(x) = 1$ in (9). □

Remark 25. The integral version of the above discrete improvements of Hölder and Hermite–Hadamard inequalities and relations for different means can easily be achieved by using Theorems 3 and 5.

5. Applications in Information Theory

In this part of the note, we are going to discuss some applications of the main inequalities in information theory. These applications involve some bounds for different divergences, the Bhattacharyya coefficient and the Shannon entropy.

Definition 26. Let Φ be a real valued function defined on $[a, b] \subset R$ and $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be two n-tuples such that $\zeta_i/\gamma_i \in [a, b]$ and $\gamma_i > 0$ for all $i \in \{1, 2, \cdots, n\}$. Then the Csisźar divergence is defined by

$$D_c(m_1, m_2) = \sum_{i=1}^{n} \gamma_i \Phi\left(\frac{\zeta_i}{\gamma_i}\right). \quad (34)$$

Theorem 27. *Assume that the function $\Phi : [a, b] \longrightarrow R$ is 4-convex and $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ are two n-tuples such that $\sum_{i=1}^n \zeta_i / \sum_{i=1}^n \gamma_i, \zeta_i/\gamma_i \in [a, b]$ and $\gamma_i > 0$ for all $i \in \{1, 2, \cdots, n\}$, then*

$$D_c(m_1, m_2) - \Phi\left(\frac{\sum_{i=1}^n \zeta_i}{\sum_{i=1}^n \gamma_i}\right) \sum_{i=1}^{n} \gamma_i \leq \frac{1}{6} \sum_{i=1}^{n} \gamma_i \left(\frac{\sum_{i=1}^n \zeta_i}{\sum_{i=1}^n \gamma_i} - \frac{\zeta_i}{\gamma_i}\right)^2 \left(2\Phi''\left(\frac{\sum_{i=1}^n \zeta_i}{\sum_{i=1}^n \gamma_i}\right) + \Phi''\left(\frac{\zeta_i}{\gamma_i}\right)\right). \quad (35)$$

Proof. All the hypotheses of this theorem are same as that of Theorem 2.

Thus, using (2) by taking $\Psi = \Phi$, $p_i = \gamma_i / \sum_{i=1}^n \gamma_i$, and $x_i = \zeta_i/\gamma_i$, we obtain (35). □

Theorem 28. *Suppose that all the assumptions of Theorem 27 are satisfied, then*

$$D_c(m_1 - m_2) - \Phi\left(\frac{\sum_{i=1}^n \zeta_i}{\sum_{i=1}^n \gamma_i}\right) \sum_{i=1}^{n} \gamma_i \geq \frac{1}{2} \sum_{i=1}^{n} \gamma_i \left(\frac{\sum_{i=1}^n \zeta_i}{\sum_{i=1}^n \gamma_i} - \frac{\zeta_i}{\gamma_i}\right)^2 \Phi''\left(2\frac{\sum_{i=1}^n \zeta_i}{3\sum_{i=1}^n \gamma_i} + \frac{\zeta_i}{3\gamma_i}\right). \quad (36)$$

Proof. Since, the function Φ is 4-convex. Therefore, utilizing (7) for $\Psi = \Phi$, $p_i = \gamma_i / \sum_{i=1}^n \gamma_i$, and $x_i = \zeta_i/\gamma_i$, we get (36). □

Definition 29. For any $\delta \in [0, \infty)$ such that $\delta \neq 1$ and arbitrary positive probability distributions $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, the Rényi divergence is defined by

$$D_{re}(m_1, m_2) = \frac{1}{\delta - 1} \log\left(\sum_{i=1}^{n} \gamma_i^{\delta} \zeta_i^{1-\delta}\right). \quad (37)$$

Corollary 30. *Assume that $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ are two positive probability distributions and $\delta > 1$, then*

$$D_{re}(m_1, m_2) - \frac{1}{\delta - 1} \sum_{i=1}^{n} \gamma_i^{\delta} \log\left(\frac{\gamma_i}{\zeta_i}\right)^{\delta-1} \leq \frac{1}{6} \sum_{i=1}^{n} \gamma_i \left(\sum_{i=1}^{n} \gamma_i^{\delta} \zeta_i^{1-\delta} - \left(\frac{\gamma_i}{\zeta_i}\right)^{\delta-1}\right)^2 \times \left(\frac{2}{\delta - 1}\left(\sum_{i=1}^{n} \gamma_i^{\delta} \zeta_i^{1-\delta}\right)^{-2} + \frac{1}{\delta - 1}\left(\frac{\gamma_i}{\zeta_i}\right)^{2-2\delta}\right). \quad (38)$$

Proof. Let $\Phi(x) = 1/(\delta - 1) \log x$, $x > 0$. Then, $\Phi''(x) = 1/((\delta - 1)x^2)$ and $\Phi''''(x) = 6/((\delta - 1)x^4)$. Clearly, $\Phi''(x) > 0$ and $\Phi''''(x) > 0$ for all $x \in (0, \infty)$. Thus, this verifies that the function $\Phi(x) = 1/(\delta - 1) \log x$ is convex as well as 4-convex on $(0, \infty)$. Therefore, using (2) for $\Phi(x) = 1/(\delta - 1) \log x$, $p_i = \gamma_i$, and $x_i = (\gamma_i/\zeta_i)^{\delta-1}$, we get (38). □

Corollary 31. *Let the hypotheses of Corollary 30 hold. Then*

$$D_{re}(m_1, m_2) - \frac{1}{\delta - 1} \sum_{i=1}^{n} \gamma_i^{\delta} \log\left(\frac{\gamma_i}{\zeta_i}\right)^{\delta-1} \geq \frac{1}{2(\delta - 1)} \sum_{i=1}^{n} \gamma_i \left(\sum_{i=1}^{n} \gamma_i^{\delta} \zeta_i^{1-\delta} - \left(\frac{\gamma_i}{\zeta_i}\right)^{\delta-1}\right)^2 \times \left(\frac{3}{2\sum_{i=1}^n \gamma_i^{\delta} \zeta_i^{1-\delta} + (\gamma_i/\zeta_i)^{\delta-1}}\right)^2. \quad (39)$$

Proof. Using $\Phi(x) = 1/(\delta - 1) \log x$, $p_i = \gamma_i$ and $x_i = (\gamma_i/\zeta_i)^{\delta-1}$ in (7), we get (39). □

Definition 32. Let $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be a probability distribution with positive entries. Then the Shannon entropy is defined by

$$E_s(m_2) = -\sum_{i=1}^{n} \zeta_i \log \zeta_i. \quad (40)$$

Corollary 33. *Suppose that $m_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ is a probability distribution such that $\gamma_i > 0$ for all $i \in \{1, 2, \cdots, n\}$, then*

$$\log n - E_s(m_2) \leq \frac{1}{6} \sum_{i=1}^{n} \gamma_i \left(n - \frac{1}{\gamma_i}\right)^2 \left(\frac{2}{\gamma_i^2} + \gamma_i^2\right). \quad (41)$$

Proof. Since the function $\Phi(x) = -\log x$ is both convex and 4-convex on $(0, \infty)$ because $\Phi(x) = 1/x^2$ and $\Phi''''(x) = 6/x^4$ are positive for $x > 0$, therefore (41) can easily be obtained by putting $\Phi(x) = -\log x$ and $\zeta_i = 1$ $(i = 1, 2, \cdots, n)$ in (35). □

Corollary 34. *Assume that the conditions of Corollary 33 are fulfilled, then*

$$\log n - E_s(m_2) \geq \frac{1}{2} \sum_{i=1}^{n} \gamma_i \left(n - \frac{1}{\gamma_i}\right)^2 \left(\frac{3\gamma_i}{2n\gamma_i + 1}\right)^2. \quad (42)$$

Proof. Utilizing the function $\Phi(x) = -\log x$ and $\zeta_i = 1$ $(1, 2, \cdots, n)$ in (36), we obtain (42). □

Definition 35. Let $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be arbitrary probability distributions such that $\gamma_i, \zeta_i > 0$ for all $i \in \{1, 2, \cdots, n\}$. Then, the Kullback–Liebler divergence is defined by

$$D_{kl}(m_1, m_2) = \sum_{i=1}^{n} \zeta_i \log \left(\frac{\zeta_i}{\gamma_i} \right). \tag{43}$$

Corollary 36. *Assume that* $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ *and* $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ *are positive probability distributions, then*

$$D_{kl}(m_1, m_2) \le \frac{1}{6} \sum_{i=1}^{n} \gamma_i \left(1 - \frac{\zeta_i}{\gamma_i} \right)^2 \left(2 + \frac{\gamma_i}{\zeta_i} \right). \tag{44}$$

Proof. The function $\Phi(x) = x \log x$ is convex and 4-convex on $(0,\infty)$ because $\Phi''(x) = 1/x > 0$ and $\Phi''''(x) = 2/x^3 > 0$ for all $x \in (0,\infty)$. Thus, using (35) by taking $\Phi(x) = x \log x$, we get (44). □

Corollary 37. *Let the postulates of Corollary 36 be true. Then*

$$D_{kl}(m_1, m_2) \ge \frac{3}{2} \sum_{i=1}^{n} \frac{(\gamma_i - \zeta_i)^2}{2\gamma_i - \zeta_i}. \tag{45}$$

Proof. Substituting $\Phi(x) = x \log x$, $x > 0$ in (36), we acquire (45). □

Definition 38. For any positive probability distributions $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, the Bhattacharyya coefficient is defined as

$$C_b(m_1, m_2) = \sum_{i=1}^{n} \sqrt{\gamma_i \zeta_i}. \tag{46}$$

Corollary 39. *Assume that* $m_1 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ *and* $m_2 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ *are positive probability distributions; then*

$$1 - C_b(m_1 - m_2) \le \frac{1}{24} \sum_{i=1}^{n} \gamma_i \left(1 - \frac{\zeta_i}{\gamma_i} \right)^2 \left(2 + \left(\frac{\gamma_i}{\zeta_i} \right)^{3/2} \right). \tag{47}$$

Proof. If $\Phi(x) = -\sqrt{x}, x > 0$, then $\Phi''(x) = (1/4)x^{-3/2}$ and $\Phi''''(x) = (15/16)x^{-7/2}$. Thus, this shows that both Φ'' and Φ'''' are positive on $(0,\infty)$. Hence, this confirms the convexity as well as 4-convexity of the function $\Phi(x) = -\sqrt{x}$. Therefore, using (35) by choosing $\Phi(x) = -x$, we obtain (47). □

Corollary 40. *Suppose that the assumptions of Corollary 39 hold, then*

$$1 - C_b(m_1, m_2) \ge \frac{1}{2} \sum_{i=1}^{n} \frac{(\gamma_i - \zeta_i)^2}{\gamma_i} \left(\frac{3\gamma_i}{2\gamma_i + \zeta_i} \right)^{3/2}. \tag{48}$$

Proof. Using $\Phi(x) = -\sqrt{x}, x > 0$, in (36), we obtain (48). □

Remark 41. The analogous form of above discrete forms for different divergences, Shannon entropy and Bhattacharyya coefficient, can easily be obtained by utilizing Theorems 3 and 5.

6. Conclusion

There are extensive literature devoted to the Jensen inequality concerning different refinements, extensions, and improvements. Also, there are many bounds obtained for the Jensen gap which provides many interesting and valuable estimates for the Jensen gap. In this note, we proposed a novel technique of obtaining of some significant estimates for Jensen's gap while utilizing a 4-convex function. We obtained the required estimates for the Jensen gap by utilizing the definition of convex function and the famous Jensen inequality. For the support of our main results, we provided some examples for taking some particular convex functions. We presented some consequences of the main results in which some new important improvements for the Hölder and Hermite–Hadamard inequalities are acquired. Furthermore, for some more consequences of the main results, we obtain several relations for power and quasiarithmetic means. Applications of the main results are discussed in the information theory. These applications give many interesting estimates for several divergences, Bhattacharyya coefficient and Shannon entropy.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

References

[1] M. Adeel, K. A. Khan, Đ. Pečarić, and J. Pečarić, "Levinson type inequalities for higher order convex functions via Abel-Gontscharoff interpolation," *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 430, 2019.

[2] K. Ahmad, M. Adil Khan, S. Khan, A. Ali, and Y. M. Chu, "New estimation of Zipf-Mandelbrot and Shannon entropies via refinements of Jensen's inequality," *AIP Advances*, vol. 11, no. 1, article 015147, pp. 1–10, 2021.

[3] C. P. Niculescu and L. E. Persson, "Convex functions and their applications. A contemporary approach," in *CMS Books in Mathematics*, Springer–Verlage, New York, 2006.

[4] S. I. Butt, M. Klaričić Bakula, and J. Pečarić, "Steffensen-Grüss inequality," *Journal of Inequalities and Applications*, vol. 15, no. 2, p. 810, 2021.

[5] N. Siddique, M. Imran, K. A. Khan, and J. Pečarić, "Majorization inequalities via Green functions and Fink's identity with applications to Shannon entropy," *Journal of Inequalities and Applications*, vol. 2020, no. 1, Article ID 192, 2020.

[6] J. Pečarić, L. E. Persson, and Y. L. Tong, *Convex Functions, Partial Ordering and Statistical Applications*, Academic Press, 1992.

[7] M. Adeel, K. Ali Khan, Đ. Pečarić, and J. Pečarić, "Estimation of f–divergence and Shannon entropy by Levinson type inequalities for higher–order convex functions via Taylor polynomial," *Journal of Mathematical and Computational Science*, vol. 21, no. 4, pp. 322–334, 2020.

[8] M. Adil Khan, Đ. Pečaric, and J. Pečarić, "Bounds for Shannon and Zipf-Mandelbrot entropies," *Mathematicsl Methods in the Applied Sciences*, vol. 40, no. 18, pp. 7316–7322, 2017.

[9] M. J. Cloud, B. C. Drachman, and L. P. Lebedev, *Inequalities with Applications to Engineering*, Springer, Cham Heidelberg New York Dordrecht London, 2014.

[10] A. Waheed, A. U. Rehman, M. I. Qureshi, F. A. Shah, K. A. Khan, and G. Farid, "On Caputo *k*-fractional derivatives and associated inequalities," *IEEE Access*, vol. 7, pp. 32137–32145, 2019.

[11] L. Horváth, Đ. Pečarić, and J. Pečarić, "Estimations of f- and Rényi divergences by using a cyclic refinement of the Jensen's inequality," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 3, pp. 933–946, 2019.

[12] S. Leorato, "A refined Jensen's inequality in Hilbert spaces and empirical approximations," *Journal of Multivariate Analysis*, vol. 100, no. 5, pp. 1044–1060, 2009.

[13] Q. Lin, "Jensen inequality for superlinear expectations," *Statistics & Probability Letters*, vol. 151, pp. 79–83, 2019.

[14] S. I. Butt, N. Mehmood, Đ. Pečarić, and J. Pečarić, "New bounds for Shannon, relative and Mandelbrot entropies via Abel-Gontscharoff interpolating polynomial," *Mathematical Inequalities & Applications*, vol. 22, no. 4, pp. 1283–1301, 2019.

[15] M. Sababheh, "Improved Jensen's inequality," *Mathematical Inequalities & Applications*, vol. 20, no. 2, pp. 389–403, 2017.

[16] Z.-G. Xiao, H. M. Srivastava, and Z. H. Zhang, "Further refinements of the Jensen inequalities based upon samples with repetitions," *Mathematical and Computer Modelling*, vol. 51, no. 5-6, pp. 592–600, 2010.

[17] M. Adil Khan, S. Khan, and Y. Chu, "A new bound for the Jensen gap with applications in information theory," *IEEE Access*, vol. 8, pp. 98001–98008, 2020.

[18] D. Costarelli and R. Spigler, "How sharp is the Jensen inequality?," *Journal of Inequalities and Applications*, vol. 2015, no. 1, Article ID 69, 2015.

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Research Article

On a Subclass of Analytic Functions That Are Starlike with Respect to a Boundary Point Involving Exponential Function

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In the present exploration, the authors define and inspect a new class of functions that are regular in the unit disc $\mathfrak{D} ≔ \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, by using an adapted version of the interesting analytic formula offered by Robertson (unexploited) for starlike functions with respect to a boundary point by subordinating to an exponential function. Examples of some new subclasses are presented. Initial coefficient estimates are specified, and the familiar Fekete-Szegö inequality is obtained. Differential subordinations concerning these newly demarcated subclasses are also established.

1. Introduction and Preliminary Results

Let $\mathscr{H}$ be the class comprising of all holomorphic functions in the unit disc $\mathfrak{D} ≔ \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$. Also, let $\mathscr{A}$ signify the subclass of $\mathscr{H}$ entailing of functions $h \in \mathscr{A}$ be of the form

$$h(\varsigma) = \varsigma + \sum_{n=2}^{\infty} a_n \varsigma^n, \quad \varsigma \in \mathfrak{D}, \tag{1}$$

with the normalization $h(0) = h'(0) - 1 = 0$. Denote by $\mathscr{S}$, the subclass of $\mathscr{A}$ comprising univalent functions. Two conversant subclasses of $\mathscr{A}$ are familiarized by Robertson [1], are defined with their analytical description as

$$\begin{aligned} \mathscr{S}^*(\alpha) &≔ \left\{ h \in \mathscr{A} : \Re\left(\frac{\varsigma h'(\varsigma)}{h(\varsigma)}\right) > \alpha, \quad \varsigma \in \mathfrak{D} \right\}, \\ \mathscr{C}(\alpha) &≔ \left\{ h \in \mathscr{A} : \Re\left(1 + \frac{\varsigma h''(\varsigma)}{h'(\varsigma)}\right) > \alpha, \quad \varsigma \in \mathfrak{D} \right\}, \end{aligned} \tag{2}$$

and are correspondingly known as starlike and convex functions of order $\alpha (0 \le \alpha < 1)$. It is well known that $\mathscr{S}^*(\alpha) \subset \mathscr{S}$ and $\mathscr{C}(\alpha) \subset \mathscr{S}$. In interpretation of Alexander's relation, $h \in \mathscr{C}(\alpha) \Leftrightarrow \varsigma h'(\varsigma) \in \mathscr{S}^*(\alpha)$ for $\varsigma \in \mathfrak{D}$. For $\alpha = 0$, the class $\mathscr{S}^* ≔ \mathscr{S}^*(0)$ condenses to the well-known class of normalized starlike univalent functions, and $\mathscr{C} ≔ \mathscr{C}(0)$ reduces to the normalized convex univalent functions.

A function $f \in \mathscr{H}$ is subordinate to $g \in \mathscr{H}$ written as $f \prec g$ if there exists $\omega \in \mathscr{H}$ with $\omega(0) = 0$ and $\omega(\mathfrak{D}) \subset \mathfrak{D}$ such that $f(\varsigma) = g(\omega(\varsigma))$ for every $\varsigma \in \mathfrak{D}$. In precise, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathfrak{D}) \subset g(\mathfrak{D})$.

Let $\mathscr{P}$ symbolize the class of functions $p \in \mathscr{H}$ with the normalization $p(0) = 1$, i.e., of the form

$$p(\varsigma) = 1 + \sum_{n=1}^{\infty} p_n \varsigma^n, \quad \varsigma \in \mathfrak{D}, \tag{3}$$

and such that $\Re p(\varsigma) > 0$ for $\varsigma \in \mathfrak{D}$. Functions in $\mathscr{P}$ are called familiarly as the Carathéodory class of functions. Ma and Minda [2] proposed a appropriate subclass of $\mathscr{P}$ denoted by $\mathscr{P}^*(1)$ comprising of all Φ that is univalent in $\mathfrak{D}$ with

$$\Phi(0)=1\,;\Phi'(0)>0, \quad (4)$$

$\Phi(\mathfrak{D})$ is symmetric with respect to the real axis

(2) Starlike with respect to 1

He also represented the class $\Phi\in\mathscr{P}^*(1)$ by

$$\Phi(\varsigma)=1+\sum_{n=1}^{\infty}B_n\varsigma^n, B_1>0\,;\varsigma\in\mathfrak{D}. \quad (5)$$

The class $\mathscr{P}^*(1)$ plays a vital part in defining generalized form of holomorphic functions. Ma and Minda [2] considered the function $\Phi\in\mathscr{P}^*(1)$ and defined $\mathscr{S}^*(\Phi)$ as the class of all $h\in\mathscr{A}$ such that $\varsigma h'(\varsigma)/h(\varsigma)\prec\Phi(\varsigma)$ for $\varsigma\in\mathfrak{D}$. The above functions defined are called as functions of Ma and Minda kind. Observe that $\mathscr{S}^*(\alpha)=\mathscr{S}^*(\Phi)$ with $\Phi(\varsigma)=(1+(1-2\alpha)\varsigma)/(1-\varsigma), \varsigma\in\mathfrak{D}$.

There are recent articles ([3–6]) where subclasses of $\mathscr{A}$ were defined by using subordination satisfying the relation $\varsigma h'(\varsigma)/h(\varsigma)\prec\Phi(\varsigma)$ for $\varsigma\in\mathfrak{D}$ (see also [7, 8]). In particular, the exponential function $\Phi_e(\varsigma)=e^{\varsigma}:=\exp(\varsigma)$, an entire function in $\mathbb{C}$ has positive real part in $\mathfrak{D}$, $\Phi_e(0)=1$, $\Phi_e'(0)=1$, and $\Phi_e(\mathfrak{D})=\{w\in\mathbb{C}:|\log w|<1\}$, is symmetric with respect to the real axis and starlike with respect to 1. Further, $\Phi_e\in\mathscr{P}^*(1)$ and therefore, it is now to make a remark that the class

$$\mathscr{S}_e=\left\{f\in\mathscr{A}:\frac{\varsigma f'(\varsigma)}{f(\varsigma)}\prec\Phi_e(\varsigma)=e^{\varsigma},\quad \varsigma\in\mathfrak{D}\right\} \quad (6)$$

is well defined. For an attractive study on starlike functions connected with the exponential function, an individual can refer to Mendiratta et al. [9, 10] (see also the works of [11–13]).

We recall the class of close-to-convex functions denoted by $\mathscr{K}$ introduced and studied by Kaplan [14]. A function $h\in\mathscr{H}$ is called to be close-to-convex if and only if there exist a function $\psi\in\mathscr{C}$ and $\beta\in(-\pi/2,\pi/2)$ such that

$$\Re\left(\frac{e^{i\beta}h'(\varsigma)}{\psi'(\varsigma)}\right)>0,\quad \varsigma\in\mathfrak{D}. \quad (7)$$

Remarking at this time that even though starlikeness of a fixed order has been discussed and well thought-out in detail in countless articles in excess of a elongated stage of period, class of univalent functions $g\in\mathscr{H}$ that maps $\mathfrak{D}$ onto Ω, starlike domain with reverence to a boundary point is still a conception that is not exclusively explored. Robertson [15] recognized this examination and introduced a new subclass

$$\mathscr{G}^*=\left\{g\in\mathscr{H}:\Re\left(e^{i\delta}g(\varsigma)\right)>0\,;\delta\in\mathbb{R};\forall\varsigma\in\mathfrak{D}\right\}, \quad (8)$$

with

$$g(0)=1,\quad g(1):=\lim_{r\longrightarrow 1^-}g(r)=0, \quad (9)$$

and maps (univalently) $\mathfrak{D}$ onto a domain starlike with respect to the origin. Presume in addition that the constant function $g\equiv 1\in\mathscr{G}^*$, in addition, Robertson through a conjecture that $\mathscr{G}^*$ coincides with the class $\mathscr{G}$ of all $g\in\mathscr{H}$ of the structure

$$g(\varsigma)=1+\sum_{n=1}^{\infty}\vartheta_n\varsigma^n,\quad \varsigma\in\mathfrak{D}, \quad (10)$$

such that

$$\Re\left(\frac{2\varsigma g'(\varsigma)}{g(\varsigma)}+\frac{1+\varsigma}{1-\varsigma}\right)>0,\quad \varsigma\in\mathfrak{D}, \quad (11)$$

proving that $\mathscr{G}\subset\mathscr{G}^*$. Definitely, in the same article Robertson shown that if $g\in\mathscr{G}$ and $g\not\equiv 1$, then $g\in\mathscr{K}$ and so univalent in $\mathfrak{D}$. It is importance of citing that (11) was identified by much erstwhile by Styer [16]. This surmise of Robertson that $\mathscr{G}^*$ coincide with the class $\mathscr{G}$ was soon after proved by Lyzzaik [17], where he established that $\mathscr{G}^*\subset\mathscr{G}$.

A different analytical categorization of starlike functions with respect to a boundary point was proposed by Lecko [18] proving the necessity. The sufficiency part of the categorization was afterwards proved by Lecko and Lyzzaik [19] (see [[20], Chapter VII] as well). Encouraged by the article of Robertson [15], Aharanov et al. [21] (see also [22]) investigated about the class of functions that are spirallike with respect to a boundary point. Let

$$\mathrm{P}(\varsigma\,;M):=\frac{4\varsigma}{\left(\sqrt{(1-\varsigma)^2+4\varsigma/M}+1-\varsigma\right)^2},\ \sqrt{1}:=1,\quad \varsigma\in\mathfrak{D}, \quad (12)$$

be the Pick function. By using the Pick function $\mathrm{P}(\varsigma\,;M)$, the author in [23] considered another closely related class to $\mathscr{G}$, the family $\mathscr{G}(M)$, $M>1$, comprising of all $g\in\mathscr{H}$ of the form (10) such that

$$\Re\left(\frac{2\varsigma g'(\varsigma)}{g(\varsigma)}+\frac{\varsigma\mathrm{P}'(\varsigma\,;M)}{\mathrm{P}(\varsigma\,;M)}\right)>0,\quad \varsigma\in\mathfrak{D}. \quad (13)$$

In [24], Todorov established a structural formula and coefficient estimates by associating $\mathscr{G}$ with a functional $f(\varsigma)/1-\varsigma$ for $\varsigma\in\mathfrak{D}$. For $g\in\mathscr{H}$ in (10), Obradovič and Owa [25] and Silverman and Silvia [26] separately introduced the classes

$$\mathscr{G}_\alpha=\left\{\Re\left(\frac{\varsigma g'(\varsigma)}{g(\varsigma)}+(1-\alpha)\frac{1+\varsigma}{1-\varsigma}\right)>0,\quad \varsigma\in\mathfrak{D}\right\}, \quad (14)$$

where $\alpha \in [0,1)$. The authors in [26] confirmed a remarkable fact that for each $\alpha \in [0,1)$, the class $\mathcal{G}_\alpha$ is a subclass of $\mathcal{G}^*$. Clearly, $\mathcal{G}_{1/2} = \mathcal{G}$ and appealing coefficient inequalities of $\mathcal{G}$ were established in [27].

For $g \in \mathcal{H}$ assumed as in (10) and $-1 < E \le 1\,; -E < F \le 1$, Jakubowski and Włodarczyk [28] defined the class $\mathcal{G}(E,F)$ as

$$\mathfrak{R}(J(\varsigma)) > 0, \quad \varsigma \in \mathfrak{D}, \tag{15}$$

where

$$J(\varsigma) = \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+E\varsigma}{1-F\varsigma}. \tag{16}$$

By desirable quality of the initiative proposed in [2], Mohd and Darus in [29] presented a new class $\mathcal{S}_b^*(\Phi)$, where $\Phi \in \mathcal{P}^*(1)$, of all $g \in \mathcal{H}$ of the form (10) such that

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma} \prec \Phi(\varsigma), \quad \varsigma \in \mathfrak{D}. \tag{17}$$

An additional appealing class on the above direction was in recent times analyzed by Lecko et al. [30].

The most important intend of the present article is to illustrate and do a organized inquiry of the function class defined as below.

Definition 1. For $g \in \mathcal{H}$ and as assumed in (10), we let a new class $\mathcal{G}_e$ as

$$\mathcal{G}_e = \left\{ g \in \mathcal{H} : \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma} \prec e^{\varsigma}, \quad \varsigma \in \mathfrak{D} \right\}. \tag{18}$$

Remark 2. Note that the condition (18) is well defined, for

$$p(\varsigma) := \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma}, \quad \varsigma \in \mathfrak{D} \tag{19}$$

is holomorphic in $\mathfrak{D}$.

Based on the description of the class $\mathcal{G}_e$ and on the analytical characterization of the class $\mathcal{G}^*$ of starlike functions with respect to a boundary point, we can prepare the next result.

2. Representation Theorem and Coefficient Results

Let us start the section with the following representation theorem which in fact offers a handy procedure to build functions in our new class $\mathcal{G}_e$.

Theorem 3. *A function $g \in \mathcal{G}_e$ if and only if there exists $p \in \mathcal{H}$ such that $p \prec \Phi_e$ and*

$$g(\varsigma) = (1-\varsigma)\exp\left(\frac{1}{2}\int_0^{\varsigma} \frac{p(\zeta)-1}{\zeta}\, d\zeta\right), \quad \varsigma \in \mathfrak{D}. \tag{20}$$

Proof. Let us suppose that $g \in \mathcal{G}_e$, then, a function p defined by (19) is holomorphic and satisfies $p \prec \Phi_e$. Also, (19) can be rewritten in the type

$$\frac{2g'(\varsigma)}{g(\varsigma)} + \frac{2}{1-\varsigma} = \frac{p(\varsigma)-1}{\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{21}$$

This upon integration give

$$\log\frac{(g(\varsigma))^2}{(1-\varsigma)^2} = \int_0^{\varsigma} \frac{p(\zeta)-1}{\zeta}\, d\zeta, \quad \varsigma \in \mathfrak{D}, \quad \log 1 := 0. \tag{22}$$

This in essence gives

$$(g(\varsigma))^2 = (1-\varsigma)^2 \exp\left(\int_0^{\varsigma} \frac{p(\zeta)-1}{\zeta}\, d\zeta\right), \quad \varsigma \in \mathfrak{D}, \tag{23}$$

which imply (20).□ □

Let us presume $p \prec \Phi_e$. By defining a function g as in (20), and by observing that $p(0)=1$, it is noticeable that g is holomorphic in $\mathfrak{D}$. A working out shows that g satisfies (21); so, (19). Thus, $g \in \mathcal{G}_e$, which ends the confirmation of the theorem.

Let Ψ_e be a holomorphic function which is the solution of the differential equation (see also [[10], p. 367])

$$\frac{\varsigma \Psi_e'(\varsigma)}{\Psi_e(\varsigma)} = e^{\varsigma}, \quad \varsigma \in \mathfrak{D}, \quad \Psi_e(0) = 0, \quad \Psi_e'(0) = 1, \tag{24}$$

i.e.,

$$\Psi_e(\varsigma) = \varsigma \exp\left(\int_0^{\varsigma} \frac{e^{\zeta}-1}{\zeta}\, d\zeta\right) = \varsigma + \varsigma^2 + \frac{3}{4}\varsigma^3 + \frac{17}{36}\varsigma^4 + \cdots, \quad \varsigma \in \mathfrak{D}. \tag{25}$$

Next, we present few examples for the class $\mathcal{G}_e$.

Example 4.

(1) For a specified $A \in \mathbb{R}$ and $\varsigma \in \mathfrak{D}$, let us name

$$p_A(\varsigma) := 1 + A\varsigma, \qquad g_A(\varsigma) := (1-\varsigma)\exp\left(\frac{A\varsigma}{2}\right), \quad \varsigma \in \mathfrak{D}. \tag{26}$$

Note down that $g_A \in \mathscr{H}$ with $g_A(0)=1$. Observe that

$$\frac{2\varsigma g_{A'}(\varsigma)}{g_A(\varsigma)} + \frac{1+\varsigma}{1-\varsigma} = p_A(\varsigma), \quad \varsigma \in \mathfrak{D}. \tag{27}$$

We finish that $g_A \in \mathscr{G}_e$ for $|A| \le 1-1/\mathrm{e}$.

(2) Given $-1 < A \le 1$ and $-A < B < 1$, define

$$w = p_{A,B}(\varsigma) := \frac{1+A\varsigma}{1-B\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{28}$$

Then, we identify that $p_{A,B}(\mathfrak{D})$ is an open disk symmetrical with respect to the real axis centered at $(1+AB)/(1-B^2)$ of radius $(A+B)/(1-B^2)$. In particular, for $B=A$, this disk is given by

$$\left| w - \frac{1+A^2}{1-A^2} \right| < \frac{2A}{1-A^2}, \tag{29}$$

with diametric end points $x_L := (1-|A|)/(1+|A|)$ and $x_R := (1+|A|)/(1-|A|)$. Since $x_L \ge 1/\mathrm{e}$ and $x_R \le \mathrm{e}$ iff $|A| \le (\mathrm{e}-1)/(\mathrm{e}+1)$, we perceive that then $p_{A,A} \prec \Phi_e$. As a result, a function $g \in \mathscr{H}$ with $g(0)=1$ defined by

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma} = p_{A,A}(\varsigma), \quad \varsigma \in \mathfrak{D}, \tag{30}$$

i.e., the function

$$g(\varsigma) = \frac{1-\varsigma}{1-A\varsigma}, \quad \varsigma \in \mathfrak{D}, \tag{31}$$

belongs to the class $\mathscr{G}_e$ for $|A| \le (\mathrm{e}-1)/(\mathrm{e}+1)$.

Theorem 5. *Let $0<r<1$. If $g \in \mathscr{G}_e$, then*

(i)

$$\sqrt{\frac{-\Psi_e(-r)}{r}}(1-r) \le |g(\varsigma)| \le \sqrt{\frac{\Psi_e(-r)}{r}}(1+r), \quad |\varsigma| = r. \tag{32}$$

(ii)

$$\left| \arg \frac{g(\varsigma_0)}{(1-\varsigma_0)^2} \right| \le \frac{1}{2} \max_{|\varsigma|=r} \arg \frac{\Psi_e(\varsigma)}{\varsigma}, |\varsigma_0| = r, \quad \arg 1 := 0. \tag{33}$$

Proof. Let $g \in \mathscr{G}_e$.

(i) Describe the function

$$h(\varsigma) := \frac{\varsigma (g(\varsigma))^2}{(1-\varsigma)^2}, \quad \varsigma \in \mathfrak{D}. \tag{34}$$

Obviously, h is a holomorphic function in $\mathfrak{D}$, and an uncomplicated working out yields

$$\frac{\varsigma h'(\varsigma)}{h(\varsigma)} = \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{35}$$

It is straightforward to witness from the above that $g \in \mathscr{G}_e$ if and only if

$$\frac{\varsigma h'(\varsigma)}{h(\varsigma)} \prec \mathrm{e}^{\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{36}$$

By the result of Corollary 1′ of [2], we obtain

$$-\Psi_e(-r) \le |h(\varsigma)| \le \Psi_e(r), \quad |\varsigma| = r, \tag{37}$$

i.e., by using (34),

$$-\Psi_e(-r) \le \left| \frac{\varsigma (g(\varsigma))^2}{(1-\varsigma)^2} \right| \le \Psi_e(r), \quad |\varsigma| = r, \tag{38}$$

which gives (32).

(ii) By (36), a function h defined by (34) belongs to $\mathscr{S}^*(\Phi_e)$. Due to Corollary 3′ of [2], the inequality

$$\left| \arg \frac{h(\varsigma_0)}{\varsigma_0} \right| \le \max_{|\varsigma|=r} \arg \frac{\Psi_e(\varsigma)}{\varsigma}, \quad |\varsigma_0| = r \tag{39}$$

is valid. Using now (34) in turn yields (33).□ □

Next, we ascertain some coefficient results for the class $g \in \mathscr{G}_e$. Let $\mathscr{B} := \{\omega \in \mathscr{H} : |\omega(\varsigma)| \le 1, \varsigma \in \mathfrak{D}\}$ and $\mathscr{B}_0$ be the subclass of $\mathscr{B}$ consisting of functions ω such that $\omega(0)=0$. We comment at this time that the elements of $\mathscr{B}_0$ are termed as Schwarz functions.

We will pertain two lemmas below to prove our main results.

Lemma 6. (see [2]). *If $p \in \mathscr{P}$ is of the form (3), then for $\mu \in \mathbb{C}$,*

$$\left| p_2 - \mu p_1^2 \right| \le 2 \max \{1, |2\mu - 1|\}. \tag{40}$$

In particular, if μ is a real number, then

$$\left| p_2 - \mu p_1^2 \right| \le \begin{cases} -4\mu + 2, & \mu \le 0, \\ 2, & 0 \le \mu \le 1, \\ 4\mu - 2, & \mu \ge 1. \end{cases} \tag{41}$$

When $\mu < 0$ or $\mu > 1$, the equality holds true if and only if $p(\varsigma) = (1+\varsigma)/(1-\varsigma) =: \mathscr{L}(\varsigma), \varsigma \in \mathfrak{D}$, or one of its rotations. If $0 < \mu < 1$, then the equality holds true if and only if $p(\varsigma = \mathscr{L}(\varsigma^2), \varsigma \in \mathfrak{D}$, or one of its rotations. If $\mu = 0$, the equality holds true if and only if

$$p(\varsigma) = \frac{1}{2}(1+\lambda)\mathscr{L}(\varsigma) + \frac{1}{2}(1-\lambda)\mathscr{L}(-\varsigma), \quad \varsigma \in \mathfrak{D}, \tag{42}$$

where $0 \le \lambda \le 1$, or one of its rotations. If $\mu = 1$, then the equality holds true if p is a reciprocal of one of the functions such that the equality holds true in the case when $\mu = 0$.

Lemma 7. (see [31]). *If $p \in \mathscr{P}$ is of the form (3) and $\beta(2\beta - 1) \le \delta \le \beta$, then*

$$|p_3 - 2\beta p_1 p_2 + \delta p_1^3| \le 2. \tag{43}$$

At the moment, we are in a position to state the theorem which give a few better bounds for early coefficients and the Fekete-Szegö inequalities for $f \in \mathscr{G}_e$.

Theorem 8. *If $g \in \mathscr{G}_e$ is of the form (10), then*

$$|\vartheta_1 + 1| \le \frac{1}{2}, \tag{44}$$

$$|\vartheta_1| \le \frac{3}{2}, \tag{45}$$

$$|2\vartheta_2 - \vartheta_1^2 + 1| \le \frac{1}{2}, \tag{46}$$

$$|\vartheta_2| \le \frac{3}{4}, \tag{47}$$

$$|3\vartheta_3 - 3\vartheta_1\vartheta_2 + \vartheta_1^3 + 1| \le \frac{1}{2}, \tag{48}$$

and for $\delta \in \mathbb{R}$,

$$|\vartheta_2 - \delta\vartheta_1^2| \le \frac{1}{4}(\max\{1, |\delta - 1|\} + 2|2\delta - 1| + 4|\delta|). \tag{49}$$

Inequalities (44), (45), (46), (47), and (48) are sharp.

Proof. In view of (18), there exists $\omega \in \mathscr{B}_0$ such that

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma} = \Phi_e(\omega(\varsigma)) = \exp(\omega(\varsigma)), \quad \varsigma \in \mathfrak{D}. \tag{50}$$

By an application of (10), one can easily obtain with simple computation that

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma} = 1 + 2(\vartheta_1 + 1)\varsigma + 2(2\vartheta_2 - \vartheta_1^2 + 1)\varsigma^2 + 2(3\vartheta_3 - 3\vartheta_1\vartheta_2 + \vartheta_1^3 + 1)\varsigma^3 + \cdots, \quad \varsigma \in \mathfrak{D}. \tag{51}$$

Define the function p by

$$p(\varsigma) = \frac{1+\omega(\varsigma)}{1-\omega(\varsigma)} = 1 + p_1\varsigma + p_2\varsigma^2 + \cdots, \quad \varsigma \in \mathfrak{D}. \tag{52}$$

Clearly, $p \in \mathscr{P}$. Moreover,

$$\omega(\varsigma) = \frac{p(\varsigma) - 1}{p(\varsigma) + 1} = \frac{p_1}{2}\varsigma + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\varsigma^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{2} + \frac{p_1^3}{8}\right)\varsigma^3 + \cdots, \quad \varsigma \in \mathfrak{D}. \tag{53}$$

Hence,

$$\exp(\omega(\varsigma)) = 1 + \omega(\varsigma) + \frac{(\omega(\varsigma))^2}{2} + \frac{(\omega(\varsigma))^3}{6} + \cdots = 1 + \frac{p_1\varsigma}{2} + \left(\frac{p_2}{2} - \frac{p_1^2}{8}\right)\varsigma^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{4} + \frac{p_1^3}{48}\right)\varsigma^3 + \cdots, \quad \varsigma \in \mathfrak{D}. \tag{54}$$

□ □

Substituting (51) and (54) into (50), by comparing the corresponding coefficients, we obtain

$$2(\vartheta_1 + 1) = \frac{p_1}{2}, \tag{55}$$

$$2(2\vartheta_2 - \vartheta_1^2 + 1) = \frac{p_2}{2} - \frac{p_1^2}{8}, \tag{56}$$

$$2(3\vartheta_3 - 3\vartheta_1\vartheta_2 + \vartheta_1^3 + 1) = \frac{p_3}{2} - \frac{p_1 p_2}{4} + \frac{p_1^3}{48}. \tag{57}$$

Since (e.g., ([[32]], Vol. I, p. 80)),

$$|p_n| \le 2, \quad n \in \mathbb{N}. \tag{58}$$

From (55), we obtain (44). Rewriting (55) as $\vartheta_1 = p_1/4 - 1$, (45) easily follows. Further, (56) together with (40) yields

$$|2(2\vartheta_2 - \vartheta_1^2 + 1)| = \left|\frac{p_2}{2} - \frac{p_1^2}{8}\right| \le 1, \tag{59}$$

which proves (46).

Upon applying (55) for ϑ_1 in (56), we get

$$4\vartheta_2 = \frac{p_2}{2} - p_1. \tag{60}$$

Hence, by applying (41), we obtain (47).

An application of (43) in (57) gives

$$|6\vartheta_3 - 6\vartheta_1\vartheta_2 + 2\vartheta_1^3 + 2| = \left|\frac{p_3}{2} - \frac{p_1 p_2}{4} + \frac{p_1^3}{48}\right| \le 1, \tag{61}$$

i.e., the inequality (48).

Using (60) and making use of the expression for ϑ_1 and in turn by applying (41) and (58), we get

$$\left|\vartheta_2 - \delta\vartheta_1^2\right| \le \frac{1}{8}\left(\left|p_2 - \frac{\delta}{2}p_1^2\right| + 2|2\delta - 1||p_1| + 8|\delta|\right), \quad \delta \in \mathbb{R}, \tag{62}$$

which leads to the inequality (49).

Equalities in (44) and (45) hold for the function $p = \mathscr{L}$; in (46) for the function $p(\varsigma) = \mathscr{L}(\varsigma^2), \varsigma \in \mathfrak{D}$, in (47) for the function $p(\varsigma) = \mathscr{L}(-\varsigma), \varsigma \in \mathfrak{D}$ and in (48) for the function $p(\varsigma) = \mathscr{L}(\varsigma^3), \varsigma \in \mathfrak{D}$.

3. Differential Subordination Results Involving $\mathscr{G}_e$

In this segment, we derive certain differential subordination result concerning the class $\mathscr{G}_e$.

To demonstrate differential subordination results, we recollect the next lemma (see ([[33]], Theorem 8.4 h, p. 132)).

Q is starlike univalent in $\mathfrak{D}$, or

h is convex univalent in $\mathfrak{D}$

Lemma 9. *Suppose q is univalent in $\mathfrak{D}$,θ and φ be holomorphic in a domain D containing $q(\mathfrak{D})$ with $\varphi(w) \neq 0$ when $w \in q(\mathfrak{D})$. Let $Q(\varsigma) := \varsigma q'(\varsigma)\varphi(q(\varsigma))$ and $h(\varsigma) := \theta(q(\varsigma)) + Q(\varsigma)$ for $\varsigma \in \mathfrak{D}$. Suppose that either*

Assume also that
(iii)

$$\mathfrak{R}\frac{\varsigma h'(\varsigma)}{Q(\varsigma)} > 0, \quad \varsigma \in \mathfrak{D}. \tag{63}$$

If $p \in \mathscr{H}$ with $p(0) = q(0)$,$p(\mathfrak{D}) \subset D$, and

$$\theta(p(\varsigma)) + \varsigma p'(\varsigma)\varphi(p(\varsigma)) \prec \theta(q(\varsigma)) + \varsigma q'(\varsigma)\varphi(q(\varsigma)), \quad \varsigma \in \mathfrak{D}, \tag{64}$$

then $p \prec q$ and q are the best dominant.

Theorem 10. *Let $g \in \mathscr{H}$ and $g(0) = 1$. If g satisfies the subordination condition,*

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma} \prec 1 + \varsigma, \quad \varsigma \in \mathfrak{D}. \tag{65}$$

Then,

$$p(\varsigma) := \frac{(g(\varsigma))^2}{(1-\varsigma)^2} \prec e^{\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{66}$$

Proof. Let $D := \mathbb{C} \setminus \{0\}$. Let $\theta(w) := 1, w \in \mathbb{C}$ and $\varphi(w) := 1/w, w \in D$. Note that $\Phi_e(\mathfrak{D}) \subset D$ and θ and φ are holomorphic in D. Thus,

$$Q(\varsigma) := \varsigma\Phi_e'(\varsigma)\varphi(\Phi_e(\varsigma)) = \frac{\varsigma{\Phi_e}'(\varsigma)}{\Phi_e(\varsigma)} = \varsigma, \quad \varsigma \in \mathfrak{D} \tag{67}$$

is well defined and holomorphic. Clearly, Q is a univalent starlike function and so for a function $h(\varsigma) := \theta(\Phi_e(\varsigma)) + Q(\varsigma) = 1 + Q(\varsigma),\varsigma \in \mathfrak{D}$, we achieve

$$\mathfrak{R}\frac{\varsigma h'(\varsigma)}{Q(\varsigma)} = \mathfrak{R}\frac{\varsigma Q'(\varsigma)}{Q(\varsigma)} = 1 > 0, \quad \varsigma \in \mathfrak{D}. \tag{68}$$

Hence, for any function p belonging to $\mathscr{H}$ with $p(0) = \Phi_e(0) = 1$ such that $p(\mathfrak{D}) \subset D$, i.e., for p nonvanishing in $\mathfrak{D}$, by applying Lemma 9, we infer that from the subordination

$$1 + \frac{\varsigma p'(\varsigma)}{p(\varsigma)} \prec 1 + \frac{\varsigma\Phi_{e'}(\varsigma)}{\Phi_e(\varsigma)} = 1 + \varsigma, \quad \varsigma \in \mathfrak{D}, \tag{69}$$

it follows the subordination $p \prec \Phi_e$.□ □

Next, we at this time take $g \in \mathscr{H}$ with $g(0) = 1$ and $g(\varsigma)$ be nonzero for $\varsigma \in \mathfrak{D}$ satisfying (65). Let a function p be taken as in (66). Then, one can notice that $p(0) = \Phi_e(0) = 1$, $p(\varsigma) \neq 0$, for $\varsigma \in \mathfrak{D}$, and p is holomorphic. Since

$$1 + \frac{\varsigma p'(\varsigma)}{p(\varsigma)} = \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma}, \quad \varsigma \in \mathfrak{D}, \tag{70}$$

from (69), the conclusion (66) follows, which complete the proof.

Theorem 11. *Let $g \in \mathscr{H}$ with $g(0) = 1$. If g satisfies*

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma} \prec e^{\varsigma} + \varsigma, \quad \varsigma \in \mathfrak{D}, \tag{71}$$

then

$$p(\varsigma) := \varsigma\left(\frac{g(\varsigma)}{1-\varsigma}\right)^2\left(\int_0^{\varsigma}\left(\frac{g(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} \prec e^{\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{72}$$

Proof. Let $D := \mathbb{C} \setminus \{0\}$. Let $\phi(w) := w, w \in \mathbb{C}$, and $\psi(w) := 1/w, w \in D$. Note that $\Phi_e(\mathfrak{D}) \subset D$ and ϕ and ψ are holomorphic in D. Thus, the function Q defined by (67), i.e., the identity function, is univalent starlike. Hence, for a function $h(\varsigma) := \theta(\Phi_e(\varsigma)) + Q(\varsigma) = \Phi_e(\varsigma) + Q(\varsigma),\varsigma \in \mathfrak{D}$, we obtain

$$\begin{aligned}\mathfrak{R}\frac{\varsigma h'(\varsigma)}{Q(\varsigma)} &= \mathfrak{R}\frac{\varsigma\Phi_{e'}(\varsigma)}{Q(\varsigma)} + \mathfrak{R}\frac{\varsigma Q'(\varsigma)}{Q(\varsigma)} \\ &= \mathfrak{R}\Phi_e(\varsigma) + \mathfrak{R}\frac{\varsigma Q'(\varsigma)}{Q(\varsigma)} > 0, \quad \varsigma \in \mathfrak{D}.\end{aligned} \tag{73}$$

Thus, for any function $p \in \mathscr{H}$ with $p(0) = \Phi_e(0) = 1$ such that $p(\mathfrak{D}) \subset D$, i.e., $p(\varsigma) \neq 0$ for $\varsigma \in \mathfrak{D}$, by applying Lemma 9, we deduce that from the subordination

$$p(\varsigma) + \frac{\varsigma p'(\varsigma)}{p(\varsigma)} \prec \Phi_e(\varsigma) + \frac{\varsigma \Phi_{e'}(\varsigma)}{\Phi_e(\varsigma)} = e^{\varsigma} + \varsigma, \quad \varsigma \in \mathfrak{D}, \tag{74}$$

it follows the subordination $p \prec \Phi_e$.□ □

Let now take $g \in \mathscr{H}$ with $g(0) = 1$ and $g(\varsigma) \neq 0$ for $\varsigma \in \mathfrak{D}$ satisfying (65). Define a function p as in (72). We see that

$$\begin{aligned} p(0) &= \lim_{\varsigma \longrightarrow 0} \varsigma \left(\frac{g(\varsigma)}{1-\varsigma}\right)^2 \left(\int_0^{\varsigma} \left(\frac{g(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} \\ &= (g(0))^2 \lim_{\varsigma \longrightarrow 0} \varsigma \left(\int_0^{\varsigma} \left(\frac{g(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} = 1 = \Phi_e(0), \end{aligned} \tag{75}$$

$p(\varsigma)$=0 for $\varsigma \in \mathfrak{D}$ and p is holomorphic. Since

$$p(\varsigma) + \frac{\varsigma p'(\varsigma)}{p(\varsigma)} = \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma}, \quad \varsigma \in \mathfrak{D}, \tag{76}$$

from (74), (71) follows which completes the proof.

Data Availability

No data sets were used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] M. S. Robertson, "On the theory of univalent functions," *The Annals of Mathematics*, vol. 37, no. 2, pp. 374–408, 1936.

[2] W. C. Ma and D. Minda, "A unified treatment of some special classes of unvalent functions," *Proceedings of the Conference on Complex Analysis (Tianjin; June 19–23, 1992)*, Z. Li, F. Ren, L. Yang, and S. Zhang, Eds., , pp. 157–169, International Press, Cambridge, Massachusetts, 1994.

[3] J. Dziok, R. K. Raina, and J. Sokół, "On α -convex functions related to shell-like functions connected with Fibonacci numbers," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 996–1002, 2011.

[4] J. Dziok, R. K. Raina, and J. Sokół, "Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers," *Computers & Mathematcs with Applications*, vol. 61, no. 9, pp. 2605–2613, 2011.

[5] J. Dziok, R. K. Raina, and J. Sokół, "On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers," *Mathematical and Computer Modelling*, vol. 57, no. 5-6, pp. 1203–1211, 2013.

[6] J. Dziok, R. K. Raina, and J. Sokół, "Differential subordinations and α-convex functions," *Scientia*, vol. 33, no. 3, pp. 609–620, 2013.

[7] R. K. Raina and J. Sokól, "Some properties related to a certain class of starlike functions," *Comptes Rendus Mathematique*, vol. 353, no. 11, pp. 973–978, 2015.

[8] R. K. Raina and J. Sokól, "Quelques proprietes liees a une classe de fonctions etoilees," *Comptes Rendus Mathematique*, vol. 353, no. 11, pp. 973–978, 2015.

[9] K. Khatter, V. Ravichandran, and S. S. Kumar, "Starlike functions associated with exponential function and the lemniscate of Bernoulli," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, no. 1, pp. 233–253, 2019.

[10] R. Mendiratta, S. Nagpal, and V. Ravichandran, "On a subclass of strongly starlike functions associated with exponential function," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 38, no. 1, pp. 365–386, 2015.

[11] S. Çakmak, E. Yaşar, and S. Yalçın, "Convolutions of harmonic mappings convex in the horizontal direction," *Journal of Function Spaces*, vol. 2021, Article ID 2949573, 9 pages, 2021.

[12] G. Murugusundaramoorthy, A. Shakeel, and M. A. Kutbi, "Coefficient bounds of Kamali-type starlike functions related with a Limacon- shaped domain," *Journal of Function Spaces*, vol. 2021, Article ID 4395574, 10 pages, 2021.

[13] A. E. Shammaky and T. M. Seoudy, "Certain subclass of -valent functions associated with a new extended Ruscheweyh operator related to conic domains," *Journal of Function Spaces*, vol. 2021, Article ID 7298104, 8 pages, 2021.

[14] W. Kaplan, "Close-to-convex schlicht functions," *The Michigan Mathematical Journal*, vol. 1, no. 2, pp. 169–185, 1952.

[15] M. S. Robertson, "Univalent functions starlike with respect to a boundary point," *Journal of Mathematical Analysis and Applications*, vol. 81, no. 2, pp. 327–345, 1981.

[16] D. Styer, "On weakly starlike multivalent functions," *Journal d'Analyse Mathématique*, vol. 26, no. 1, pp. 217–233, 1973.

[17] A. Lyzzaik, "On a conjecture of M. S. Robertson,," *Proceedings of the American Mathematical Society*, vol. 91, no. 1, pp. 108–110, 1984.

[18] A. Lecko, "On the class of functions starlike with respect to a boundary point," *Journal of Mathematical Analysis and Applications*, vol. 261, no. 2, pp. 649–664, 2001.

[19] A. Lecko and A. Lyzzaik, "A note on univalent functions starlike with respect to a boundary point," *Journal of Mathematical Analysis and Applications*, vol. 282, no. 2, pp. 846–851, 2003.

[20] A. Lecko, *Some Methods in the Theory of Univalent Functions*, Oficyna Wdawnicza Politechniki Rzeszowskiej, Rzeszów, 2005.

[21] D. Aharonov, M. Elin, and D. Shoikhet, "Spiral-like functions with respect to a boundary point," *Journal of Mathematical Analysis and Applications*, vol. 280, no. 1, pp. 17–29, 2003.

[22] A. Lecko, "δ-Spirallike functions with respect to a boundary point," *The Rocky Mountain Journal of Mathematics*, vol. 38, no. 3, pp. 979–992, 2008.

[23] Z. J. Jakubowski, "On properties of the pick function and some applications of them," *Acta Universitatis Purkynianae*, vol. 42, pp. 51–62, 1999.

[24] P. G. Todorov, "On the univalent functions starlike with respect to a boundary point," *Proceedings of the American Mathematical Society*, vol. 97, no. 4, pp. 602–604, 1986.

[25] M. Obradović and S. Owa, "On some classes of close-to-convex functions and its applications," *Bulletin of the Institute of Mathematics, Academia Sinica*, vol. 16, no. 2, pp. 123–133, 2012.

[26] H. Silverman and E. M. Silvia, "Subclasses of univalent functions starlike with respect to a boundary point," *Houston Journal of Mathematics*, vol. 16, no. 2, pp. 289–299, 1990.

[27] A. S. Abdullah, R. M. Ali, and V. Singh, "On functions starlike with respect to a boundary point," *Annales Universitatis Mariae Curie - Skl0d0wska Lublin - Polonia*, vol. 50, pp. 7–15, 1996.

[28] Z. J. Jakubowski and A. Włodarczyk, "On some classes of functions of Robertson type," *Annales Universitatis Mariae Curie - Skl0d0wska Lublin - Polonia*, vol. LIX, pp. 27–42, 2005.

[29] M. H. Mohd and M. Darus, "Starlike function with respect to a boundary point defined by subordination," *Advances in Mathematics: Scientific Journal*, vol. 1, no. 1, pp. 15–21, 2012.

[30] A. Lecko, G. Murugusundaramoorthy, and S. Sivasubramanian, "On a class of analytic functions related to Robertson's formula and subordination," *Boletín de la Sociedad Matemática Mexicana*, vol. 27, no. 1, 2021.

[31] R. M. Ali, "Coefficients of the inverse of strongly starlike functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 26, 2003.

[32] A. W. Goodman, *Univalent Functions*, Mariner, Tampa, Florida, 1983.

[33] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, Theory and Applications, Dekker, Basel, 2000.

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Research Article

Properties of Functions with Symmetric Points Involving Subordination

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We study a new subclass of functions with symmetric points and derive an equivalent formulation of these functions in term of subordination. Moreover, we find coefficient estimates and discuss characterizations for functions belonging to this new class. We also obtain distortion and growth results. We relate our results with the existing literature of the subject.

1. Introduction and Definitions

Let $\mathscr{H}(\Delta)$ represent analytic functions f in the disc $\Delta := \{z : |z|<1\}$ and $\mathscr{A} \subset \mathscr{H}(\Delta)$ be defined as:

$$\mathscr{A} := \left\{ f \in \mathscr{H}(\Delta) : f(z) = z + a_2 z^2 + \cdots (z \in \Delta) \right\}. \tag{1}$$

Let $\mathcal{Q}$ denote "Carathéodory functions" h such that $h(0) = 1$, $\operatorname{Re}(h(z)) > 0$ and $h(z) = 1 + h_1 z + h_2 z^2 + \cdots, z \in \Delta$. The Möbius function $k_0(z) = (1+z)/(1-z) \in \mathcal{Q}$ or its rotation acts as an extremal function for the class $\mathcal{Q}$ and maps Δ onto $\operatorname{Re}(k_0(z)) > 0$. Recall that $\mathcal{Q}(\varepsilon) \subset \mathcal{Q}, 0 \le \varepsilon < 1$ consists of functions $h \in \mathcal{Q}$ such that $\operatorname{Re}(h(z)) > \varepsilon$ in Δ. For $f, g \in \mathscr{H}$, we say that the function f is subordinate to g and write $f \prec g$, if for

$$\Phi \in \mathscr{H}(\Delta), \text{with } \Phi(0) = 0 \text{ and } |\Phi(z)| < 1, f(z) = g(\Phi(z)). \tag{2}$$

For a univalent function g, $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. For reference, see [1]. Applying subordination, Janowski [2] defined the class $\mathcal{Q}[A, A_1]$ for $-1 \le A_1 < A \le 1$. A function $h \in \mathcal{Q}[A, A_1]$, if

$$h(z) = \frac{1 + A\Phi(z)}{1 + A_1 \Phi(z)} \prec \frac{1 + Az}{1 + A_1 z} (z \in \Delta). \tag{3}$$

Geometrically, the image $h(\Delta)$ lies inside the disk centered on $Im(z) = 0$, and diameter ends at $h(-1)$ and $h(1)$. Clearly, $\mathcal{Q}[A, A_1] \subset \mathcal{Q}((1-A)/(1-A_1))$. The class $\mathcal{Q}[A, A_1]$ is related with the class $\mathcal{Q}$ as: $h \in \mathcal{Q}$ iff, we write

$$\frac{(A+1)h(z) - (A-1)}{(A_1+1)h(z) - (A_1-1)} \in \mathcal{Q}[A, A_1]. \tag{4}$$

Also Λ_k, $k \ge 0$ is given by $\Lambda_k = \{\Phi = u + iv : u > k\sqrt{(u-1)^2 + v^2}\}$, represents various plane curves for the specific values of k. Let $\mathcal{S}$ be the class of complex-valued injective functions and $\mathcal{S}^*$ represents the class of starlike functions whereas $\mathscr{C}$ denotes the class of convex functions. A function $f \in \mathcal{S}$ is close-to-convex, if and only if there exists a function $g \in \mathscr{C}$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} \in \mathcal{Q}, z \in \Delta. \tag{5}$$

We denote the class of close-to-convex functions by $\mathscr{K}$. This class was introduced by Kaplan in [3]. Sakaguchi (see [4]) defined the class $\mathcal{S}_{SP}$ as:

Definition 1. Let $f \in \mathcal{S}$. Then, $f \in \mathcal{S}_{SP}$, if

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) > 0, (z \in \Delta). \quad (6)$$

For $f \in \mathcal{A}$, $f \in \mathcal{C}_{\mathrm{SP}}$ [5] iff $zf' \in \mathcal{S}_{\mathrm{SP}}$, where $\mathcal{C}_{\mathrm{SP}}$ is the class of convex functions with respect to symmetric points. Various authors studied the class $\mathcal{C}_{\mathrm{SP}}$ and its subclasses, for detail, see [3, 6–8]. Obviously, it represents the univalent functions. Moreover, it includes the class of convex and odd starlike functions, see [4]. This and other classes are investigated in the literature of the subject; for example, see [9–14].

Definition 2. Let f be analytic in Δ defined by (1). We say that $f \in \mathcal{K}_{SP}(\varepsilon)$, $0 \le \varepsilon < 1$, if for $g \in \mathcal{S}^*(1/2)$ we have

$$\operatorname{Re}\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) < -\varepsilon, (z \in \Delta). \quad (7)$$

For more details, see [15]. We see that $\mathcal{K}_{\mathrm{SP}}(0) = \mathcal{K}_{\mathrm{SP}}$, where $\mathcal{K}_{\mathrm{SP}}$ is the class of functions defined in [16]. We study a new class $\mathcal{K}_{\mathrm{SP}}(\varepsilon, \eta)$ involving $g \in S^*(1/2)$.

Definition 3. Let $f \in \mathcal{A}$. Then, $f \in \mathcal{K}_{\mathrm{SP}}(\varepsilon, \eta)$, $0 \le \varepsilon < 1$, $0 \le \eta \le 1$ if for $g \in \mathcal{S}^*(1/2)$ we have

$$\operatorname{Re}\left(\frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)}\right) < -\varepsilon (z \in \Delta), \quad (8)$$

where $G(z) = g(z)g(-z)/z$. By a simple calculations, we see that (8) is equivalent to

$$\left|\frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} + 1\right| < \left|\frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} - 1 + 2\varepsilon\right|, (z \in \Delta). \quad (9)$$

From [16], we have the following lemma.

Lemma 4. *For $g \in \mathcal{S}^*(1/2)$ such that*

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, (z \in \Delta), \quad (10)$$

if we put

$$zG(z) = g(z)g(-z) = z^2 + \sum_{k=1}^{\infty} c_{k+1} z^{2k+1}, \quad (11)$$

where

$$c_k = 2b_{2k-1} - 2b_2 b_{2k-2} + .. \cdots + (-1)^k 2b_{k-1}b_{k+1} + (-1)^{k+1} b_k^2, \quad (12)$$

then $G \in \mathcal{S}^*$.

Remark 5. Since $g \in \mathcal{S}^*(1/2)$, then Lemma 4 proves that $G \in \mathcal{S}^*$. Also from (8), we see that $\mathcal{K}_{\mathrm{SP}}(\varepsilon, \eta)$ contains close-to-convex functions.

2. Main Results

In the following theorem, we have an equivalent formulation of condition (9) in terms of subordination.

Theorem 6. *A function $f \in \mathcal{K}_{SP}(\varepsilon, \eta)(0 \le \varepsilon < 1, 0 \le \eta \le 1)$ iff for $g \in \mathcal{S}^*(1/2)$, we write*

$$\frac{-zf'(z) - \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} \prec \frac{1 + (1-2\varepsilon)z}{1-z}, (z \in \Delta), \quad (13)$$

where $G(z) = g(z)g(-z)/z$.

Proof. Let $f \in \mathcal{K}_{\mathrm{SP}}(\varepsilon, \eta)$. Then, for $g \in \mathcal{S}^*(1/2)$, we write

$$\left|\frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} + 1\right| < \left|\frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} - 1 + 2\varepsilon\right|, (z \in \Delta), \quad (14)$$

or

$$\operatorname{Re}\left(\frac{-zf'(z) - \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)}\right) > \varepsilon, (z \in \Delta), \quad (15)$$

where $G(z) = g(z)g(-z)/z$. Using subordination, we write

$$H(z) = \frac{-zf'(z) - \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} \prec \frac{1 + (1-2\varepsilon)z}{1-z}, z \in \Delta, \quad (16)$$

because $k_0(\varepsilon, z) = (1 + (1-2\varepsilon)z)/(1-z) \in \mathcal{S}$ and $H(0) = k_0(\varepsilon, 0) = 1$, where $G(z) = g(z)g(-z)/z$. Conversely, we assume that (13) holds. Then, there exists Φ with $\Phi(0) = 0$ and $|\Phi(z)| < 1$ such that

$$\frac{-zf'(z) - \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} = \frac{1 + (1-2\varepsilon)\Phi(z)}{1-\Phi(z)}. \quad (17)$$

Hence, using $|\Phi(z)| < 1$, we obtain (9) equivalent to (8), so $f \in \mathcal{K}_{\mathrm{SP}}(\varepsilon, \eta)$. □

Now, we prove sufficient conditions for $f \in \mathcal{K}_{\mathrm{SP}}(\varepsilon, \eta)$.

Theorem 7. *Let $g \in \mathcal{S}^*(1/2)$ be a function given by (10) and $0 \le \varepsilon < 1$, $0 \le \eta \le 1$. If f defined by (1) satisfies.*

$$2\sum_{k=2}^{\infty}[k + \eta k(k-1)]|a_k| + (|1-2\varepsilon| + 1)\sum_{k=2}^{\infty}[2\eta(k-1) + 1]|c_k| \le 2(1-\varepsilon), \quad (18)$$

where the coefficients c_k are given by (12), then $f \in \mathcal{K}_{SP}(\varepsilon, \eta)$.

In particular, if

$$\sum_{k=2}^{\infty}[k+\eta k(k-1)]|a_k| \le 1-\varepsilon, \tag{19}$$

then $f \in \mathscr{K}_{SP}(\varepsilon,\eta)$.

Proof. We set $F(\eta,z)=zf'(z)+\eta z^2 f''(z)$ for f given by (1), $zG(z)=g(z)g(-z)$ where g is defined by (10) and have

$$\begin{aligned}\Lambda &= \left|F(\eta,z)-(1-\eta)G(z)+\eta zG'(z)\right| \\ &\quad - \left|F(\eta,z)-(1-2\eta)\left[(1-\eta)G(z)+\eta zG'(z)\right]\right| \\ &= \left|\sum_{k=2}^{\infty}[k+\eta k(k-1)]a_k z^k - \sum_{k=2}^{\infty}[2\eta(k-1)+1]c_k z^{2k-1}\right| \\ &\quad - \left|(2-2\varepsilon)z+\sum_{k=2}^{\infty}[k+\eta k(k-1)]a_k z^k \right.\\ &\quad \left.+ (1-2\varepsilon)\sum_{k=2}^{\infty}[2\eta(k-1)+1]c_k z^{2k-1}\right|.\end{aligned} \tag{20}$$

Hence, for $z \in \Delta$, we have the inequality

$$\begin{aligned}\Lambda &\le \sum_{k=2}^{\infty}[k+\eta k(k-1)]|a_k||z|^k + \sum_{k=2}^{\infty}[2\eta(k-1)+1]|c_k||z|^{2k-1} \\ &\quad -(2-2\varepsilon)|z| + \sum_{k=2}^{\infty}[k+\eta k(k-1)]|a_k||z|^k \\ &\quad + |1-2\varepsilon|\sum_{k=2}^{\infty}[2\eta(k-1)+1]|c_k||z|^{2k-1} \\ &= -(2-2\varepsilon)|z| + \sum_{k=2}^{\infty}2[k+\eta k(k-1)]|a_k||z|^k \\ &\quad + (|1-2\varepsilon|+1)\sum_{k=2}^{\infty}[2\eta(k-1)+1]|c_k||z|^{2k-1} \\ &\le -(2-2\varepsilon) + \sum_{k=2}^{\infty}2[k+\eta k(k-1)]|a_k| + (|1-2\varepsilon|+1) \\ &\quad \cdot \sum_{k=2}^{\infty}[2\eta(k-1)+1]|c_k| \le 0.\end{aligned} \tag{21}$$

From these calculations, we see that $\Lambda < 0$. Also by (20), we can write

$$\begin{aligned}&\left|F(\eta,z)-\left[(1-\eta)G(z)+\eta zG'(z)\right]\right| \\ &< \left|F(\eta,z)+(1-2\eta)\left[(1-\eta)G(z)+\eta zG'(z)\right]\right|,\end{aligned} \tag{22}$$

which is equivalent to (9) and (8). Thus, $f \in \mathscr{K}_{SP}(\varepsilon,\eta)$, and it completes the proof. □

The next theorem deals with the coefficient estimates $f \in \mathscr{K}_{SP}(\varepsilon,\eta)$.

Theorem 8. *Let* $0 \le \varepsilon < 1$ *and* $0 \le \eta \le 1$. *Suppose that* f *given by (1) and* $g \in S^*(1/2)$ *given by (10) are such that (8) holds. Then, for* $k=2,3,\cdots$ *we have*

$$\begin{aligned}&2k^2\{1+\eta(2k-1)\}^2 - 2(1-\varepsilon)^2 \le (1-\varepsilon) \\ &\quad \cdot \sum_{k=2}^{n} 2(2k-1)\eta_k^2|a_{2k-1}||c_{2k-1}| + (|2\varepsilon-1|+1)\eta_k^2|c_{2k-1}|^2,\end{aligned} \tag{23}$$

where $\eta_k = 1+2\eta(k-1)$ *and* c_k *is defined by (12). In particular, if* $g(z)=z$, *then*

$$k(1+\eta(2k-1)|a_{2k}| \le 1-\varepsilon. \tag{24}$$

Proof. If $f \in \mathscr{K}_{SP}(\varepsilon,\eta)$ for some $g \in S^*(1/2)$, then (9) holds. Using Lemma 4, with $\alpha=\beta=1$, we have

$$\frac{zf'(z)+\eta z^2 f''(z)}{(1-\eta)G(z)+\eta zG'(z)} = \frac{1+(2\varepsilon-1)z\Phi(z)}{1+z\Phi(z)}, z \in \Delta, \tag{25}$$

where Φ is an analytic function in Δ, $|\Phi(z)| \le 1$ for $z \in \Delta$, and G is given by (11). Then

$$\begin{aligned}&\left[\eta z^2 f''(z)+zf'(z)-(2\varepsilon-1)\left\{(1-\eta)G(z)+\eta zG'(z)\right\}\right] \\ &\quad \cdot z\Phi(z) = (1-\eta)G(z)+\eta zG'(z)-zf'(z)-\eta z^2 f''(z).\end{aligned} \tag{26}$$

Now, $z\Phi(z)=\sum_{k=1}^{\infty}s^k z^k$. We see that $|\Phi(z)| \le |z|$ for $z \in \Delta$. Thus

$$\begin{aligned}&\left[(2-2\varepsilon)z+\sum_{k=2}^{\infty}[k+\eta k(k-1)]a_k z^k-(2\varepsilon-1)\sum_{k=2}^{\infty}[2\eta(k-1)+1]c_k z^{2k-1}\right] \\ &\quad \cdot \sum_{k=1}^{\infty}s^k z^k = \sum_{k=2}^{\infty}[2\eta(k-1)+1]c_k z^{2k-1} - \sum_{k=2}^{\infty}[k+\eta k(k-1)]a_k z^k.\end{aligned} \tag{27}$$

Equating coefficients in (27), for $k \ge 2$, we can also write

$$\begin{aligned}&\left[(2-2\varepsilon)z+\sum_{k=1}^{n-1}2k\eta_{2k}a_{2k}z^{2k}+\sum_{k=2}^{n}\eta_k[(2k-1)a_{2k-1}-(2\varepsilon-1)c_{2k-1}]z^{2k-1}\right] \\ &\quad \cdot z\Phi(z) = \sum_{k=2}^{n}\eta_k[c_{2k-1}-(2k-1)a_{2k-1}]z^{2k-1} - \sum_{k=1}^{k}2k\eta_{2k}a_{2k}z^{2k} \\ &\quad + \sum_{k=2k+1}^{\infty}c_k z^k,\end{aligned} \tag{28}$$

where $\eta_k = 1+2\eta(k-1)$ and $\eta_{2k} = 1+\eta(2k-1)$.

Then, we square and integrate along $|z| = r < 1$. After using the fact $|\Phi(z)| \le |z| < 1$, we obtain

$$\begin{aligned}\sum_{k=2}^{n} |\eta_k[c_{2k-1} - (2k-1)a_{2k-1}]|^2 r^{4k-2} &+ \sum_{k=1}^{k} |2k\eta_{2k}a_{2k}|^2 r^{4k} \\ &+ \sum_{k=2k+1}^{\infty} |c_k|^2 r^{2k} < |2-2\varepsilon|^2 r^2 + \sum_{k=1}^{n-1} |2k\eta_{2k}a_{2k}|^2 r^{4k} \\ &+ \sum_{k=2}^{n} |\eta_k[(2k-1)a_{2k-1} - (2\varepsilon-1)c_{2k-1}]|^2 r^{4k-2}.\end{aligned} \tag{29}$$

Letting $r \longrightarrow 1$, we have

$$\begin{aligned}\sum_{k=2}^{n} |\eta_k[c_{2k-1} - (2k-1)a_{2k-1}]|^2 + \sum_{k=1}^{k} |2k\eta_{2k}a_{2k}|^2 < |2-2\varepsilon|^2 \\ + \sum_{k=1}^{k-1} |2k\eta_{2k}a_{2k}|^2 + \sum_{k=2}^{n} |\eta_k[(2k-1)a_{2k-1} - (2\varepsilon-1)c_{2k-1}]|^2.\end{aligned} \tag{30}$$

Hence,

$$\begin{aligned}&4k^2\{1+\eta(2k-1)\}^2 - 4(1-\varepsilon)^2 \\ &\le \sum_{k=2}^{n}\left[|\eta_k[(2k-1)a_{2k-1} - (2\varepsilon-1)c_{2k-1}]|^2 - |\eta_k[c_{2k-1} - (2k-1)a_{2k-1}]|^2\right] \\ &\le 2(1-\varepsilon)\sum_{k=2}^{n} 2\eta_k^2 |a_{2k-1}c_{2k-1}| + (|2\varepsilon-1|+1)\eta_k^2 |c_{2k-1}|^2.\end{aligned} \tag{31}$$

Thus, we have the inequality (23) which finishes the proof. □

In the following theorem, we prove the growth and distortion theorems for f in the class $\mathcal{K}_{\mathrm{SP}}(\varepsilon, \eta)$.

Theorem 9. *If $f \in \mathcal{K}_{SP}(\varepsilon, \eta)$, where $0 \le \varepsilon < 1$ and $0 \le \eta \le 1$, then*

$$\chi_1(r, \varepsilon) \le |f'(z)| \le \chi_2(r, \varepsilon), \tag{32}$$

where

$$\begin{aligned}\chi_1(r, \varepsilon) &= \frac{1-(1-2\varepsilon)r}{(1+r)(1+r^2)} - \frac{(-1)^{1/\eta}(1-\varepsilon)\{1+(1+r)\operatorname{arctanh} r}{r^{1/\eta}(1+r)}, \\ \chi_2(r, \varepsilon) &= \frac{1+(1-2\varepsilon)r}{(1-r)(1-r^2)} - \frac{(-1)^{1/\eta}(1-\varepsilon)}{4r^{1/\eta}} \\ &\quad \cdot \ln\left(\frac{1+r}{1-r}\right) - \frac{(1-\varepsilon)\{(-1)^{1/\eta}(1-\varepsilon)+1\}}{2r^{1/\eta}(1-r)^2}.\end{aligned} \tag{33}$$

Also, we have

$$\chi_3(r, \varepsilon, \eta) \le |f(z)| \le \chi_4(r, \varepsilon, \eta) + \chi_5(r, \varepsilon, \eta), \tag{34}$$

where

$$\begin{aligned}\chi_3(r, \varepsilon, \eta) &= 1-\varepsilon) \ln \frac{1+r}{\sqrt{1+r^2}} \\ &\quad + \varepsilon \arctan r - \frac{\eta}{\eta-1} r^{-(1/\eta)+1} - (-1)^{1/\eta} \ln|1+r|, \\ \chi_4(r, \varepsilon, \eta) &= \frac{\varepsilon}{2} \ln \frac{1+r}{1-r} + \frac{(2-\varepsilon)r}{1-r} - \frac{\eta(-1)^{1/\eta}(1-\varepsilon)}{4(\eta-1)} \\ &\quad \cdot \left[\ln\left(\frac{1+r}{1-r}\right) r^{-(1/\eta)+1} - \ln \frac{|1+r|^{(-1)^{1/\eta}}}{|1-r|}\right], \\ \chi_5(r, \varepsilon, \eta) &= -\frac{1-\varepsilon}{2}\left[\frac{\eta}{\eta-1}\left[(-1)^{1/\eta}+1\right] r^{-(1/\eta)+1} \right. \\ &\quad \left. - \left[(-1)^{1/\eta}+1\right]\right] \ln|1-r|,\end{aligned} \tag{35}$$

and $|z| = r$, $0 \le r < 1$.

Proof. If $f \in \mathcal{K}_{\mathrm{SP}}(\varepsilon, \eta)$, then for $g \in S^*(1/2)$, (8) holds. It follows from Lemma 4 that G in (11) is an odd starlike function. Then

$$\frac{r}{1+r^2} \le |G(z)| \le \frac{r}{1-r^2}, \ |z| = r, 0 \le r < 1, \tag{36}$$

$$\frac{1-r^2}{(1+r^2)^2} \le |G'(z)| \le \frac{1+r^2}{(1-r^2)^2}, \ |z| = r, 0 \le r < 1. \tag{37}$$

For detail, see [17]. From (8), we obtain a function h with real part greater then ε such that

$$\frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} = h(z), z \in \Delta. \tag{38}$$

It is known, see [18], that

$$\frac{1-(1-2\varepsilon)r}{1+r} \le |h(z)| \le \frac{1+(1-2\varepsilon)r}{1-r}, \ |z| = r, 0 \le r < 1. \tag{39}$$

Thus, from (36), (38), and (39), we obtain (32). From (32) for $z = te^{i\theta}$, we have

$$|f(z)| = \left|\int_0^z f'(s)ds\right| \le \int_0^r \left|f'\left(te^{i\theta}\right)\right| \cdot dt \le \int_0^r \left[\frac{1+(1-2\varepsilon)t}{(1-t)^2(1+t)} - \frac{(-1)^{1/\eta}(1-\varepsilon)}{4t^{1/\eta}} \cdot \ln\left(\frac{1+r}{1-r}\right) - \frac{(1-\varepsilon)\left\{(-1)^{1/\eta}(1-\varepsilon)+1\right\}}{2t^{1/\eta}(1-t)^2}\right] \cdot dt = \frac{\varepsilon}{2}\ln\frac{1+r}{1-r} + (2-\varepsilon)\frac{r}{1-r} - \frac{\eta(-1)^{1/\eta}(1-\varepsilon)}{4(\eta-1)} \cdot \left[\ln\left(\frac{1+r}{1-r}\right)r^{-(1/\eta)+1} - \ln\frac{|1+r|^{(-1)^{1/\eta}}}{|1-r|}\right] - \frac{1-\varepsilon}{2}\left[\frac{\eta}{\eta-1}\left[(-1)^{1/\eta}+1\right]r^{-(1/\eta)+1} - \left[(-1)^{1/\eta}+1\right]\right]\ln|1-r|. \tag{40}$$

This gives us the right-hand side of the inequality (34). To prove the left-hand side of the inequality (34), we must show that it holds for the nearest point $f(z_o)$ from zero, where $|z_o| = r$ and $0 < r < 1$. Moreover, we have $|f(z)| \ge |f(z_o)|$ for $|z| = r$. Since $f \in \mathscr{K}$, we know that the function f is univalent in the unit disc Δ. We conclude that the original image of the line segment

$$|f(z)| = \int_{f(C)} |d\Phi| = \int_C \left|f'(z)\right||d(z)| \ge \int_0^r \left[\frac{1-(1-2\varepsilon)t}{(1+t)(1+t^2)} - \frac{(-1)^{1/\eta}(1-\varepsilon)\{1+(1+r)\operatorname{arctanh} t}{t^{1/\eta}(1+t)}\right] \cdot dt = (1-\varepsilon)\ln\frac{1+r}{\sqrt{1+r^2}} + \varepsilon \arctan r - \frac{\eta}{\eta-1}r^{-(1/\eta)+1} - (-1)^{1/\eta}\ln|1+r|. \tag{41}$$

This finishes the proof of the inequality (34). □

3. Conclusions

In this research, we studied a new subclass of functions with symmetric points and derived an equivalent formulation of these functions in terms of subordination. Moreover, we determined coefficient estimates and discussed characterizations for functions belonging to this new class. We also obtained distortion and growth results. We observed that our findings are related with the existing literature of the subject.

Data Availability

There is no data available.

Disclosure

The research is performed as part of the employment of the "Mirpur University of Science and Technology, Mirpur-10250 (AJK), Pakistan."

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Marcel Dekker, Inc., New York, 2000.

[2] W. Janowski, "Some extremal problems for certain families of analytic functions I," *Annales Polonici Mathematici*, vol. 28, no. 3, pp. 297–326, 1973.

[3] T. N. Shanmugam, C. Ramachandram, and V. Ravichandran, "Fekete-Szegö problem for subclasses of starlike functions with respect to symmetric points," *Bulletin of the Korean Mathematical Society*, vol. 43, no. 3, pp. 589–598, 2006.

[4] K. Sakaguchi, "On a certain univalent mapping," *Journal of the Mathematical Society of Japan*, vol. 11, pp. 72–75, 1959.

[5] H. Silverman, "Integral means for univalent functions with negative coefficients," *Houston Journal of Mathematics*, vol. 23, no. 1, pp. 169–174, 1997.

[6] R. Parvatham and S. Radha, "On starlike and close-to-convex functions with respect to *k*-symmetric points," *Indian Journal of Pure and Applied Mathematics*, vol. 17, no. 9, pp. 1114–1122, 1986.

[7] V. Ravichandran, "Starlike and convex functions with respect to conjugate points," *Acta Mathematica Academiae Paedagogicae Nyıregyháziensis*, vol. 20, no. 1, pp. 31–37, 2004.

[8] Z.-G. Wang, "A new subclass of quasi-convex functions with respect to *k*-symmetric points," *Lobachevskii Journal of Mathematics*, vol. 19, pp. 41–50, 2005.

[9] S. Z. H. Bukhari, T. Bulboaca, and M. S. Shabbir, "Subordination and superordination results for analytic functions with respect to symmetrical points," *Quaestiones Mathematicae*, vol. 41, no. 1, pp. 65–79, 2018.

[10] S. Z. H. Bukhari, M. Nazir, and M. Raza, "*k*-symmetric points - some generalizations of the class of analytic functions with respect to k-symmetric points," *Maejo International Journal of Science and Technology*, vol. 1, no. 45, pp. 1–12, 2016.

[11] S. Z. H. Bukhari, M. Raza, and M. Nazir, "*k*-symmetric points - some generalizations of the class of analytic functions with respect to k-symmetric points," *Hacettepe Journal of Mathematics and Statistics*, vol. 1, no. 45, pp. 1–14, 2016.

[12] B. Seker and N. E. Cho, "A subclass of close-to-convex functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 42, no. 4, pp. 373–379, 2013.

[13] H. M. Srivastava, S. Z. H. Bukhari, and M. Nazir, "A subclass of alpha-convex function with respect to $(2j, k)$-symmetric conjugate points," *Bulletin of the Iranian Mathematical Society*, vol. 44, pp. 1227–1242, 2018.

[14] Z.-G. Wang and D.-Z. Chen, "On a subclass of close-to-convex functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 38, no. 2, pp. 95–101, 2009.

[15] J. Kowalczyk and E. Les-Bomba, "On a subclass of close-to-convex functions," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1147–1151, 2010.

[16] C. Gao and S. Zhaou, “On a class of analytic functions related to the starlike functions,” *Kyungpook Mathematical Journal*, vol. 45, pp. 123–130, 2005.

[17] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, 1983.

[18] A. W. Goodman, *Univalent Functions*, vol. 1, Mariner Publishing House, Tampa, Florida, 1983.

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Research Article

On Certain Products of Complex Intuitionistic Fuzzy Graphs

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A complex intuitionistic fuzzy set (CIFS) can be used to model problems that have both intuitionistic uncertainty and periodicity. A diagram composed of nodes connected by lines and labeled with specific information may be used to depict a wide range of real-life and physical events. Complex intuitionistic fuzzy graphs (CIFGs) are a broader type of diagram that may be used to manipulate data. In this paper, we define the key operations direct, semistrong, strong, and modular products for complex intuitionistic fuzzy graphs and look at some interesting findings. Further, the strong complex intuitionistic fuzzy graph is defined, and several significant findings are developed. Furthermore, we study the behavior of the degree of a vertex in the modular product of two complex intuitionistic fuzzy graphs.

1. Introduction

Obscurity is a common occurrence in everyday life. This is not a world of precise calculations and ideas. For human intellect, this judgment mistake is extremely tough. To tackle this problem, a variety of mathematical techniques and ideas, such as fuzzy sets and complex fuzzy sets, have been developed. A group system with uncertain information was used to create the complex fuzzy logic. Due to the elastic potential of advanced intuitionistic fuzzy sets (IFSs) to control unreliability, this event is considered wonderfully great for humanistic logic underlying wrong reality and infinite knowledge. Because it allows for more erroneous information to be given, this theory is a cornerstone of classical complex fuzzy sets because it provides for more suitable answers to a range of situations. In cases when we must deal with relatively limited alternatives, such as yes or no, these specialized sets generated beneficial models. Another essential feature of this knowledge is that it allows man to evaluate the negative and positive elements of erroneous ideas. To deal with uncertainty, Zadeh [1] introduced the fuzzy set theory. Following that, a number of academics looked into the theory of fuzzy sets and fuzzy logic in order to deal with a variety of real-world problems involving an uncertain and ambiguous environment. Atanassov [2] came up with the concept of intuitionistic fuzzy sets (IFSs), which are fuzzy sets with a new component. With the addition of the degree of truth and falsity membership, the concept of IFSs has become more relevant and vivid. The applications of these sets have gotten a lot of attention in fields like multicriteria decision-making and image processing. Furthermore, when data is phase-shifted, the ambiguity and uncertainty in the data come from everyday life. As a result, taking this information into account is theoretically insufficient, and information is lost as a result of the procedure. To handle this uncertainty, Ramot et al. [3] introduced the elongated form the fuzzy set by including a phase term part, called complex fuzzy set. The competency of complex fuzzy logic in the sense of membership has a very significant role to address concrete problems. It is not only a vital source for measuring unevenness but also very effective mode to deal with ambiguous ideas. Besides its usefulness, we still have massive problems regarding the physical properties of complex membership-based functions. It is highly demanding to design an additional theory of complex fuzzy set in the sense of set knotty members. This logic is straight development of conventional fuzzy logic that naturally develops problem basing fuzzy logic which is not suitable for the artificial function of membership. This certain set has core role in various applications especially in modern commanding systems which foreshadows periodic phenomenon wherein a series of fuzzy variables are interlinked in a complicated way and it cannot be properly run by fuzzy operations. Owing to the feature of handling the information regarding both periodicity and uncertainty, the complex fuzzy sets gained the special attentions in the latest trends of fuzzy sets, bipolar fuzzy sets, and IFSs. By using these models, both periodicity and uncertainty may be presented in a single set. Atanassov [2] added new components to the concept of a fuzzy set that specifies the degree of nonmembership. Fuzzy sets provide membership degrees, while IFSs provide both membership and nonmembership degrees, which are more or less independent of one another. The sole requirement is that the total of the two degrees is less than one. IFSs have been used in economics, chemistry, medicine, engineering, and computer science. Therefore, the studies regarding complex fuzzy sets got a broad spectrum both in theoretical aspects as well as application aspects. Many researchers investigated the extensive applications of complex fuzzy sets in signal processing applications, time series, solar activity, and forecasting problems (for instance, see [4–8]). Complex numbers and complex fuzzy sets were utilized by Buckley [9]. Alkouri and Salleh [10] developed the concept of complex Atanassov's intuitionistic fuzzy relation and complex Atanassov's intuitionistic fuzzy sets. Rosenfeld coined the term "fuzzy subgroups" and established a connection between group theory and fuzzy set theory. As a result, several academics developed fuzzy algebraic structures based on fuzzy sets, intuitionistic fuzzy sets (IFSs), and CIFs (for detail, see [11–18]). Several real and tangible circumstances can be illustrated using a diagram composed of a collection of nodes with lines joining specific pairs of these nodes. The nodes could select individuals, with lines connecting pairs of friends, or primary health care facilities, with lines representing beneficiaries' streets or roads in the region. Fuzzy graph modeling is a useful mathematical tool for solving combinatorial issues in a variety of fields, such as image capturing, computer network, electric network, operations research, social science, road network, topology, optimization, algebra, computer science, environmental science, and scheduling problem. Fuzzy graph theory has an intuitive and aesthetic appeal because of the diagrammatic representation. Due to the natural presence of vagueness and ambiguity, fuzzy graphical models are far superior to graphical models. We needed fuzzy set theory at first to deal with numerous complicated phenomena that had inadequate information. Based on Zadeh's fuzzy connection, Kauffman [19] was the first to coin the term "fuzzy graph." Rosenfeld [20] went on to invent fuzzy vertex, fuzzy edge, and theoretical fuzzy graph ideas like routes, connectedness, and cycle, among other things. Following Mordeson and Chang-Shyh's [21] discussion of fuzzy graph operations, Bhutani and Battou's [22] research of M-strong fuzzy graphs was published. Following that, Eslahchi and Onagh [23], Gani and Malarvizhi [24], Mordeson and Nair [25], and Mathew and Sunitha [26] propose a slew of concepts and definitions, primarily under the headings of vertex strength of fuzzy graphs, fuzzy trees, isomorphism on fuzzy graphs, fuzzy subgraphs, fuzzy paths, and complement of a fuzzy graph. Because the membership function was insufficient to express the complexity of object features, a nonmembership function was created. By combining the nonmembership and hesitation qualities, Atanassov [2] constructed the intuitionistic fuzzy set theory, which was an elaboration of the basic set theory. This idea has been used to a variety of domains, including computer programming, medical fields, decision-making problems, marketing evaluation, and banking issues. In 2006, Parvathi and Karunambigai [27] proposed an intuitionistic fuzzy graph as a variant of Atanassov's IFG. Thirunavukarasu et al. [28] built on this concept by incorporating complex fuzzy graphs. Shannon and Atanassov [29] defined and discussed intuitionistic fuzzy graphs. Later on, a number of authors worked on intuitionistic fuzzy graphs and made several important contributions to the subject (for instance, see [30–33]). Sahoo and Pal discussed different types of products on intuitionistic fuzzy graphs in [34]. Using the concept of a complex intuitionistic fuzzy set, Yaqoob et al. [35] constructed complex intuitionistic fuzzy graphs (CIFGs).

This paper's structure is as follows: the second section dives into some basic definitions. In Section 3, we define the direct product of two CIFGs. We define strong CIFG. We show that the direct product of two CIFGs is a CIFG as well. At the end of this section, we show that if the direct product of two CIFGs is strong, then at least one of them is strong. In Section 4, we define the semidirect product of two CIFGs. This section demonstrates that the semidirect product of two CIFGs is also a CIFG. At the end of this section, we demonstrate that if the semidirect product of two CIFGs is strong, then at least one of them is strong. The strong product of two CIFGs is defined in the fifth section. We demonstrate that the strong direct product of two CIFGs is CIFG. Furthermore, we demonstrated that if the strong product of two CIFGs is strong, then at least one of them is strong. In the last section of this paper, we define the modular product of two CIFGs and examine some intriguing results. We also investigate how the degree of vertex behaves in the modular product of two CIFGs.

2. Preliminaries

We go over some basic definitions that will assist us in our future discussions.

Definition 1 [1]. A fuzzy set (FS) X of a nonempty set A is a function, $X : A \longrightarrow [0, 1]$.

Definition 2 [2]. An intuitionistic fuzzy set (IFS) X of a universe of discourse A is a triplet of the form $X = \{\langle a, m_X(a), n_X(a)\rangle \mid a \in A\}$, where the functions $m_X(a)$: $A \longrightarrow [0, 1]$ and $n_X(a)$: $A \longrightarrow [0, 1]$ are the membership function (degree of truthfulness) and nonmembership functions (degree of falsity), respectively. These functions must fulfill the condition $0 \le m_X(a) + n_X(a) \le 1$.

Definition 3 [36]. The object of the form

$$X = \left\{ \left(a, m_X(a)e^{i\alpha_X(a)}, n_X(a)e^{i\beta_X(a)} \right): a \in A \right\}, \quad (1)$$

is a complex intuitionistic fuzzy set (CIFS) defined on universe of discourse A.

Here,

$$i = \sqrt{-1}, m_X(a), n_X(a) \in [0,1], \alpha_X(a), \beta_X(a) \in [0,2\pi] \text{ and } 0 \le m_X(a) + n_X(a) \le 1. \tag{2}$$

Definition 4 [27]. An intuitionistic fuzzy graph is of the form $G = (B, C, X, Y)$ on the crisp graph G^* with vertex set B and edge set C, where

(1) $B = \{b_1, b_2, \cdots, b_n\}$ and $X = (m_X, n_X)$ such that $m_X : B \longrightarrow [0,1]$ and $n_X : B \longrightarrow [0,1]$ denote the membership value (MV) and nonmembership value (NMV) of the element $b_i \in B$, respectively, such that $m_X(b_i) + n_X(b_i) \le 1$ for all $b_i \in B$

(2) $C \subseteq B \times B$ and $Y = (m_Y, n_Y)$ where $m_Y : C \longrightarrow [0,1]$ and $n_Y : C \longrightarrow [0,1]$ are defined by $m_Y(b_i, b_j) \le m_X(b_i) \wedge m_X(b_j)$ and $n_Y(b_i, b_j) \le n_X(b_i) \vee n_X(b_j)$ such that $m_Y(b_i, b_j) + n_Y(b_i, b_j) \le 1, \forall (b_i, b_j) \in C$

Definition 5 [35]. A complex intuitionistic fuzzy graph (CIFG) with an underlaying vertex set B and edge set C is defined to be a pair $\mathbb{G} = (B, C, X, Y)$, where X is a CIFS on B and Y is a CIFS on $C \subseteq B \times B$ such that

$$m_Y(b_i, b_j)e^{i\alpha_Y(b_i,b_j)} \le \min\{m_X(b_i), m_X(b_j)\}e^{i\min\{\alpha_X(b_i),\alpha_X(b_j)\}},$$
$$n_Y(b_i, b_j)e^{i\beta_Y(b_i,b_j)} \le \max\{m_X(b_i), n_X(b_j)\}e^{i\max\{\beta_X(b_i),\beta_X(b_j)\}}, \tag{3}$$

for all $b_i, b_j \in B$.

Definition 6 [35]. Let $G = (B, C, X, Y)$ be the given CIFG. The degree of a vertex b_i in G is defined by

$$\deg_G(b_i) = \left(\sum_{(b_i,b_j)\in C} m_Y(b_i, wb_j)e^{i\alpha_Y(b_i,b_j)}, \sum_{(b_i,b_j)\in C} n_Y(b_i b_j)e^{i\beta_Y(b_i,b_j)}\right). \tag{4}$$

3. Direct Product of Two CIFGs

Definition 7. The direct product of two CIFGs, $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ such that $B_1 \cap B_2 = \phi$ is defined to be CIFG $G_1 o G_2 = (B, C, X_1 o X_2, Y_1 o Y_2)$ where

$$\begin{aligned} B &= B_1 \times B_2 \text{and} C = C_1 \times C_2 \\ &= \left\{\left(b_{i_1}, b_{j_1}\right), \left(b_{i_2}, b_{j_2}\right)\right) \Big| \left(b_{i_1}, b_{i_2}\right) \in C_1, \left(b_{j_1}, b_{j_2}\right) \in C_2\right\}. \end{aligned} \tag{5}$$

The MV and NMV for the vertex (b_{i_1}, b_{i_2}) in $G_1 o G_2$ are given by

$$\begin{aligned} \left(m_{X_1}e^{i\alpha_{X_1}} o m_{X_2}e^{i\alpha_{X_2}}\right)(b_{i_1}, b_{i_2}) &= m_{X_1}(b_{i_1})e^{i\alpha_{X_1}(b_{i_1})} \wedge m_{X_2}(b_{i_2})e^{i\alpha_{X_2}(b_{i_2})} \\ &= \min\{m_{X_1}(b_{i_1}), m_{X_2}(b_{i_2})\}e^{i\min\{\alpha_{X_1}(b_{i_1}),\alpha_{X_2}(b_{i_2})\}}, \\ \left(n_{X_1}e^{i\beta_{X_1}} o n_{X_2}e^{i\beta_{X_2}}\right)(b_{i_1}, b_{i_2}) &= n_{X_1}(b_{i_1})e^{i\beta_{X_1}(b_{i_1})} \wedge n_{X_2}(b_{i_2})e^{i\beta_{X_2}(b_{i_2})} \\ &= \max\{n_{X_1}(b_{i_1}), n_{X_2}(b_{i_2})\}e^{i\max\{\beta_{X_1}(b_{i_1}),\beta_{X_2}(b_{i_2})\}}. \end{aligned} \tag{6}$$

The NM and NMV for the edge $(u = (b_{i_1}, b_{j_1}), v = (b_{i_2}, b_{j_2}))$ in $G_1 \Pi G_2$ are given by

$$\begin{aligned} \left(m_{Y_1}e^{i\alpha_{Y_1}} \Pi m_{Y_2}e^{i\alpha_{Y_2}}\right)(u, v) &= m_{Y_1}(u)e^{i\alpha m_{Y_1}(u)} \wedge m_{Y_2}(v)e^{i\alpha m_{Y_2}(v)} \\ &= \min\{m_{Y_1}(u), m_{Y_2}(v)\}e^{i\min\{\alpha_{Y_1}(u),\alpha_{Y_2}(v)\}}, \end{aligned} \tag{7}$$

$$\begin{aligned} \left(n_{Y_1}e^{i\beta_{Y_1}} \Pi n_{Y_2}e^{i\beta_{Y_2}}\right)(u, v) &= n_{Y_1}(u)e^{i\beta_{Y_1}(u)} \vee n_{Y_2}(v)e^{i\beta_{Y_2}(v)} \\ &= \max\{n_{Y_1}(u), n_{Y_2}(v)\}e^{i\max\{\beta_{Y_1}(u),\beta_{Y_2}(v)\}}. \end{aligned} \tag{8}$$

Now we define the strong CIFG.

Definition 8. A CIFG $G = (B, C, X, Y)$ is called strong CIFG if

$$\begin{aligned} m_Y(u, v)e^{i\alpha_Y(u,v)} &= \left\{m_X(u)e^{i\alpha_X(u)} \wedge m_X(v)e^{i\alpha_X(v)}\right\} \\ &= \min\{m_X(u), m_X(v)\}e^{i\min\{m_X(u),m_X(v)\}}, \end{aligned} \tag{9}$$

$$\begin{aligned} n_Y(u, v)e^{i\beta_B(u,v)} &= \left\{n_X(u)e^{i\beta_X(u)} \vee n_X(v)e^{i\beta_X(v)}\right\} \\ &= \max\{n_X(u), n_X(v)\}e^{i\max\{\beta_X(u),\beta_X(v)\}}. \end{aligned} \tag{10}$$

Theorem 9. *Let $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ be two strong CIFGs; then, $G_1 \Pi G_2$ is also a strong CIFG.*

Proof. As $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ are strong CIFGs, so by (9) and (10), we have

$$\begin{cases} \left(m_{Y_1}(u_1, v_1)e^{i\alpha_{Y_1}(u_1,v_1)} = \min\{m_{X_1}(u_1), m_{X_1}(v_1)\}e^{i\min\{\alpha_{X_1}(u_1),\alpha_{X_1}(v_1)\}},\right. \\ \left(n_{Y_1}(u_1, v)e^{i\beta Y_1(u_1,v_1)} = \max\{n_{X_1}(u_1), n_{X_1}(v_1)\}e^{i\max\{\beta_{X_1}(u_1),\beta_{X_1}(v_1)\}},\right. \end{cases}$$
$$\begin{cases} \left(m_{Y_2}(u_2, v_2)e^{i\alpha_{Y_2}(u_2,v_2)} = \min\{m_{X_2}(u_2), m_{X_2}(v_2)\}e^{i\min\{\alpha_{X_2}(u_2),\alpha_{X_2}(v_2)\}},\right. \\ \left(n_{Y_2}(u_2, v_2)e^{i\beta Y_2(u_2,v_2)} = \max\{n_{X_2}(u_2), n_{X_2}(v_2)\}e^{i\max\{\beta_{X_2}(u_2),\beta_{X_2}(v_2)\}},\right. \end{cases} \tag{11}$$

for all $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$.

Now from (7) and (8), we have

$$\begin{aligned} &\left(m_{Y_1}e^{i\alpha Y_1} \Pi m_{Y_2}e^{i\alpha Y_2}\right)\left(\left(b_{i_1}, b_{j_1}\right), \left(b_{i_2}, b_{j_2}\right)\right) \\ &= m_{Y_1}(b_{i_1}, b_{i_1})e^{i\alpha_{Y_1}(b_{i_1},b_{j_1})} \wedge m_{Y_2}\left(b_{j_2}, v_{j_2}\right)e^{i\alpha_{Y_2}(b_{i_2},b_{j_2})} \end{aligned}$$

$$= \left[m_{X_1}\left(b_{i_1}\right)e^{i\alpha X_1\left(b_{i_1}\right)} \wedge m_{X_1}\left(b_{j_1}\right)e^{i\alpha X_1\left(b_{j_1}\right)}\right]$$
$$\wedge \left[m_{X_2}\left(b_{i_2}e^{i\alpha X_2} \wedge m_{X_2}\left(b_{j_2}\right)e^{i\alpha X_2\left(b_{j_2}\right)}\right]$$
$$= \left[m_{X_1}\left(b_{i_1}\right)e^{i\alpha X_1\left(b_{i_1}\right)} \wedge m_{X_2}\left(b_{j_2}\right)e^{i\alpha X_2\left(b_{j_2}\right)}\right]$$
$$\wedge \left[m_{X_1}\left(b_{j_2}\right)e^{i\alpha X_1\left(b_{j_2}\right)} \wedge m_{X_2}\left(b_{j_2}\right)e^{i\alpha X_2\left(b_{j_2}\right)}\right]$$
$$= \left(m_{X_1}e^{i\alpha X_1} \Pi m_{x_2}e^{i\alpha X_2}\right)\left(b_{i_1}, b_{j_1}\right)$$
$$\wedge \left(m_{X_1}e^{i\alpha X_1} \Pi m_{X_2}e^{i\alpha X_2}\right)\left(b_{i_2}, b_{j_2}\right). \quad (12)$$

In addition for nonmembership,

$$\left(n_{Y_1}e^{i\beta Y_1} \mathrm{o} n_{Y_2}e^{i\beta Y_2}\right)\left(\left(b_{i_2}, b_{j_1}\right), \left(b_{i_2}, b_{j_2}\right)\right)$$
$$= n_{Y_1}\left(b_{i_1}, b_{j_1}\right)e^{i\beta_{Y_1}\left(b_{i_1}, b_{j_1}\right)} \vee n_{Y_2}\left(b_{i_2}, b_{j_2}\right)e^{i\beta_{Y_2}\left(b_{i_2}, b_{j_2}\right)}$$
$$= \left[n_{X_1}\left(b_{i_1}\right)e^{i\beta X_1\left(b_{i_1}\right)} \vee n_{X_1}\left(b_{i_2}\right)e^{i\alpha X_1\left(b_{j_1}\right)}\right]$$
$$\vee \left[n_{X_2}\left(b_{i_2}e^{i\beta X_2} \wedge n_{X_2}\left(b_{j_2}\right)e^{i\beta X_2\left(b_{j_2}\right)}\right]$$
$$= \left[n_{X_1}\left(b_{i_1}\right)e^{i\beta X_1\left(b_{i_1}\right)} \vee n_{X_2}\left(b_{j_2}\right)e^{i\beta X_2\left(b_{j_2}\right)}\right]$$
$$\vee \left[n_{X_1}\left(b_{j_2}\right)e^{i\beta X_1\left(b_{j_2}\right)} \wedge n_{X_2}\left(b_{j_2}\right)e^{i\beta X_2\left(b_{j_2}\right)}\right]$$
$$= \left(n_{X_1}e^{i\beta X_1} \Pi n_{X_2}e^{i\beta X_2}\right)\left(b_{i_1}, b_{j_1}\right)$$
$$\vee \left(n_{X_1}e^{i\alpha X_1} \Pi n_{X_2}e^{i\beta X_2}\right)\left(b_{i_2}, b_{j_2}\right). \quad (13)$$

This completes the proof. □

Theorem 10. *Let $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ be two CIFGs, such that $G_1 \Pi G_2$ is strong; then, at least one of G_1 or G_2 must be strong CIFG.*

Proof. Suppose G_1 and G_2 are not strong CIFGs. Thus, there exists at least one $(b_{i_1}, b_{j_1}) \in B_1, (b_{i_2}, b_{j_2}) \in B_2$ such that

$$m_{Y_1}\left(b_{i_1}, b_{j_1}\right)e^{i\alpha_{Y_1}\left(b_{i_1}, b_{j_1}\right)} < m_{X_1}\left(b_{i_1}\right)e^{i\alpha_{X_1}\left(b_{i_1}\right)} \wedge m_{X_1}\left(b_{j_1}\right)e^{i\alpha_{X_1}\left(b_{j_2}\right)},$$
$$n_{Y_1}\left(b_{i_1}, b_{j_1}\right)e^{i\beta_{Y_1}\left(b_{i_1}, b_{j_1}\right)} < n_{X_1}\left(b_{i_1}\right)e^{i\beta_{X_1}\left(b_{j_1}\right)} \vee n_{X_1}\left(b_{j_1}\right)e^{i\beta_{X_1}\left(b_{j_1}\right)},$$
$$m_{Y_2}\left(b_{i_2}, b_{j_2}\right)e^{i\alpha_{Y_2}\left(b_{i_2}, b_{j_2}\right)} < m_{X_2}\left(b_{i_2}\right)e^{i\alpha_{X_2}\left(b_{i_2}\right)} \wedge m_{X_2}\left(b_{j_2}\right)e^{i\alpha_{X_2}\left(b_{j_2}\right)},$$
$$n_{Y_2}\left(b_{i_2}, b_{j_2}\right)e^{i\beta_{Y_2}\left(b_{i_2}, b_{j_2}\right)} < n_{X_2}\left(b_{i_2}\right)e^{i\beta_{X_2}\left(b_{j_2}\right)} \vee n_{X_2}\left(b_{j_2}\right)e^{i\beta_{X_2}\left(b_{j_2}\right)}. \quad (14)$$

Let $(u = (b_{i_1}, b_{j_1}), v = (b_{i_2}, b_{j_2})) \in B_1 \times B_2$; then,

$$\left(m_{Y_1}e^{i\alpha_{Y_1}} \Pi m_{Y_2}e^{i\alpha_{Y_2}}\right)((u, v)) = m_{Y_1}(u)e^{i\alpha_{Y_1}(u)} \wedge \left(m_{Y_2}(v)e^{i\alpha_{Y_2}(v)}\right.$$
$$< \left[m_{X_1}\left(b_{i_1}\right)e^{i\alpha_{X_1}\left(b_{i_1}\right)} \wedge m_{X_1}\left(b_{j_1}\right)e^{i\alpha_{X_1}\left(b_{j_1}\right)}\right]$$
$$\wedge \left[m_{X_2}\left(b_{i_2}\right)e^{i\alpha_{X_2}\left(b_{i_2}\right)} \wedge m_{X_2}\left(b_{j_2}\right)e^{i\alpha_{X_2}\left(b_{i_2}\right)}\right]$$
$$= \left[m_{X_1}\left(b_{i_1}\right)e^{i\alpha_{X_1}\left(b_{i_1}\right)} \wedge m_{X_2}\left(b_{i_1}\right)e^{i\alpha_{A_2}\left(b_{j_1}\right)}\right]$$
$$\wedge \left[m_{A_1}\left(b_{i_2}\right)e^{i\alpha_{A_1}\left(b_{i_1}\right)} \wedge m_{A_2}\left(b_{i_2}\right)e^{i\alpha_{A_2}\left(b_{i_2}\right)}\right]$$
$$= \left(m_{X_1}e^{i\alpha_{X_1}} \Pi m_{X_2}e^{i\alpha_{X_2}}\right)\left(b_{i_1}, b_{j_1}\right)$$
$$\wedge \left(m_{X_1}e^{i\alpha_{X_1}} \Pi m_{X_2}e^{i\alpha_{X_2}}\right)\left(b_{i_2}, b_{j_2}\right). \quad (15)$$

Therefore,

$$\left(m_{Y_1}e^{i\alpha_{Y_1}} \Pi m_{Y_2}e^{i\alpha_{Y_2}}\right)\left(u = \left(b_{i_1}, b_{j_1}\right), v = \left(b_{i_2}, b_{j_2}\right)\right)$$
$$< \left(m_{X_1}e^{i\alpha_{X_1}} \Pi m_{X_2}e^{i\alpha_{X_2}}\right)\left(b_{i_1}, b_{j_1}\right) \quad (16)$$
$$\wedge \left(m_{X_1}e^{i\alpha_{X_1}} \Pi m_{X_2}e^{i\alpha_{X_2}}\right)\left(b_{i_2}, b_{j_2}\right).$$

Again, let $(u = (b_{i_1}, b_{j_1}), v = (b_{i_2}, b_{j_2})) \in B_1 \times B_2$; then,

$$\left(n_{Y_1}e^{i\beta_{Y_1}} \Pi n_{Y_2}e^{i\beta_{Y_2}}\right)((u, v)) = \left(n_{Y_1}(u)e^{i\beta_{Y_1}(u)} \vee \left(n_{Y_2}(v)e^{i\beta_{Y_2}(v)}\right.\right.$$
$$< \left[n_{X_1}\left(b_{i_1}\right)e^{i\beta_{X_1}\left(b_{i_1}\right)} \vee n_{X_1}\left(b_{j_1}\right)e^{i\beta_{X_1}\left(b_{j_1}\right)}\right]$$
$$\vee \left[n_{X_2}\left(b_{i_2}\right)e^{i\beta_{X_2}\left(b_{i_2}\right)} \vee n_{X_2}\left(b_{j_2}\right)e^{i\beta_{X_2}\left(b_{i_2}\right)}\right]$$
$$= \left[n_{X_1}\left(b_{i_1}\right)e^{i\beta_{X_1}\left(b_{i_1}\right)} \vee n_{X_2}\left(b_{i_1}\right)e^{i\beta_{X_2}\left(b_{j_1}\right)}\right]$$
$$\vee \left[n_{X_1}\left(b_{i_2}\right)e^{i\beta_{X_1}\left(b_{i_1}\right)} \vee n_{X_2}\left(b_{i_2}\right)e^{i\beta_{X_2}\left(b_{i_2}\right)}\right]$$
$$= \left(n_{X_1}e^{i\beta_{X_1}} \Pi n_{X_2}e^{i\beta_{X_2}}\right)\left(b_{i_1}, b_{j_1}\right)$$
$$\vee \left(n_{X_1}e^{i\beta_{X_1}} \Pi n_{X_2}e^{i\beta_{X_2}}\right)\left(b_{i_2}, b_{j_2}\right). \quad (17)$$

Therefore,

$$\left(n_{Y_1}e^{i\beta_{Y_1}} \Pi n_{Y_2}e^{i\beta_{Y_2}}\right)\left(u = \left(b_{i_1}, b_{j_1}\right), v = \left(b_{i_2}, b_{j_2}\right)\right)$$
$$< \left(n_{X_1}e^{i\beta_{X_1}} \Pi n_{X_2}e^{i\beta_{X_2}}\right)\left(b_{i_1}, b_{j_1}\right) \quad (18)$$
$$\vee \left(n_{X_1}e^{i\beta_{X_1}} \Pi n_{X_2}e^{i\beta_{X_2}}\right)\left(b_{i_2}, b_{j_2}\right).$$

This shows that $G_1 \Pi G_2$ is not strong, which is contradiction.

This completes the proof. □

4. Semistrong Product of Two CIFGs

Definition 11. The semistrong product of two CIFGs $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ such that $B_1 \cap B_2 = \phi$ is defined to be the CIFG as $G_1 ✦ G_2 = (B, C, X_1 ✦$

$X_2, Y_1 ✦ Y_2)$, where $B = B_1 \times B_2$ and $C = \{((b_{i_1}, b_{i_2}), (b_{j_1}, b_{j_2})) \mid (b_{i_1}, b_{j_1}) \in B_1$ and $b_{i_2} = b_{j_2}$ or $(b_{i_1}, b_{j_1}) \in B_1$ and $(b_{i_2}, b_{j_2}) \in B_2\}$.

The MV and NMV of the vertex (u, v) in $G_1 ✦ G_2$ are given as

$$\left(m_{X_1} e^{i\alpha_{X_1}} ✦ m_{X_2} e^{i\alpha_{X_2}}\right)(u, v) = m_{X_1}(u) e^{i\alpha_{X_1}(u)} \wedge m_{X_2}(v) e^{i\alpha_{X_2}(v)} = \min\left\{m_{X_1}(u), m_{X_2}(v)\right\} e^{i \min\left\{\alpha_{X_1}(u), \alpha_{X_2}(v)\right\}}, \tag{19}$$

$$\left(n_{X_1} e^{i\beta_{X_1}} ✦ n_{X_2} e^{i\beta_{X_2}}\right)(u, v) = n_{X_1}(u) e^{i\beta_{X_1}(u)} \vee n_{X_2}(v) e^{i\beta_{X_2}(v)} = \max\left\{n_{X_1}(u), n_{X_2}(v)\right\} e^{i \max\left(\beta_{X_1}(u), \beta_{X_2}(v)\right\}}. \tag{20}$$

The MV and NMV for the edge $(u, v_1), (u, v_2)$ and $(u_1, v_1), (u_2, v_2) \in C$ in $G_1 ✦ G_2$ are given by as follows:

$$\left(m_{Y_1} e^{i\alpha_{Y_1}} ✦ m_{Y_2} e^{i\alpha_{B_2}}\right)((u, v_1), (u, v_2)) = m_{X_1}(u) e^{i\alpha_{A_1}(u)} \wedge m_{Y_2}(v_1, v_2) e^{i\alpha_{Y_2}(v_1, v_2)}, \tag{21}$$

$$\left(m_{Y_1} e^{i\alpha_{B_1}} ✦ m_{Y_2} e^{i\alpha_{B_2}}\right)((u_1, v_1), (u_2, v_2)) = m_{Y_1}(u_1, u_2) e^{i\alpha_{B_1}(u)} \wedge m_{Y_2}(v_1, v_2) e^{i\alpha_{Y_2}(v_1, v_2)}, \tag{22}$$

$$\left(m_{Y_1} e^{i\alpha_{B_1}} ✦ m_{Y_2} e^{i\alpha_{Y_2}}\right)((u, v_1), (u, v_2)) = m_{X_1}(u) e^{i\alpha_{X_1}(u)} \wedge m_{Y_2}(v_1, v_2) e^{i\alpha_{Y_2}(v_1, v_2)}, \tag{23}$$

$$\left(m_{Y_1} e^{i\alpha_{Y_1}} ✦ m_{Y_2} e^{i\alpha_{Y_2}}\right)((u_1, v_1), (u_2, v_2)) = m_{Y_1}(u_1, u_2) e^{i\alpha_{Y_1}(u)} \wedge m_{Y_2}(v_1, v_2) e^{i\alpha_{Y_2}(v_1, v_2)}. \tag{24}$$

Theorem 12. *If $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ are strong CIFGs, then $G_1 ✦ G_2$ is also strong.*

Proof. If $((u, v_1), (u, v_2)) \in C$, then using (19) and (20), we have

$$\begin{aligned}
&\left(m_{Y_1} e^{i\alpha_{Y_1}} ✦ m_{Y_2} e^{i\alpha_{Y_2}}\right)((u, v_1), (u, v_2)) \\
&= m_{X_1}(u) e^{i\alpha_{X_1}(u)} \wedge m_{X_2}(v_1, v_2) e^{i\alpha_{Y_2}(v_1, v_2)} \\
&= m_{X_1}(u) e^{i\alpha_{X_1}(u)} \wedge \left(m_{X_2}(v_1)^{e_{i\alpha} X_2(v_1)} \wedge m_{X_2}(v_2)^{e_{i\alpha} X_2(v_2)}\right) \\
&= \left(m_{X_1}(u) e^{i\alpha_{X_1}(u)} \wedge m_{X_2}(v_1) e^{i\alpha_{X_2}(v_1)}\right) \\
&\wedge \left(m_{X_2}(u) e^{i\alpha_{A_2}(u)} \wedge m_{X_2}(v_2) e^{i\alpha_{A_2}(v_2)}\right) \\
&= \left(m_{X_1} e^{i\alpha_{X_1}} ✦ m_{X_2} e^{i\alpha_{X_2}}\right)(u, v_1) \\
&\wedge \left(m_{X_1} e^{i\alpha_{X_1}} ✦ m_{X_2} e^{i\alpha_{A_2}}\right)(u, v_2).
\end{aligned} \tag{25}$$

Similar, we can show that $(n_{B_1} e^{i\beta B_1} ✦ n_{B_2} e^{i\beta B_2})((u, v_1), (u, v_2)) = (n_{A_1} e^{i\alpha A_1} ✦ n_{A_2} e^{i\alpha A_2})(u, v_1) \vee (n_{A_1} e^{i\alpha A_1} ✦ n_{A_2} e^{i\alpha A_2})(u, v_2)$.

Again, if $((u_1, v_1), (u_2, v_2)) \in E$, then using (22) and (24), we have

$$\begin{aligned}
&\left(m_{Y_1} e^{i\alpha_{Y_1}} ✦ m_{Y_2} e^{i\alpha_{Y_2}}\right)((u_1, v_1), (u_2, v_2)) \\
&= m_{Y_1}(u_1, v_1) e^{i\beta_{Y_1}(u_1, v_2)} \wedge m_{Y_2}(u_2, v_2) e^{i\beta_{Y_2}(u_2, v_2)} \\
&= \left(m_{X_1}(u_1) e^{i\alpha_{X_1}(u_1)} \wedge m_{X_2}(v_1) e^{i\alpha_{X_2}(v_1)}\right) \\
&\wedge \left(m_{X_2}(u_2) e^{i\alpha_{X_2}(u_2)} \wedge m_{X_2}(v_2) e^{i\alpha_{X_2}(v_2)}\right) \\
&= \left(m_{X_1} e^{i\alpha_X} ✦ m_{X_2} e^{i\alpha_{X_2}}\right)(u_1, v_1) \\
&\wedge \left(m_{X_1} e^{i\alpha_{A_1}} ✦ m_{X_2} e^{i\alpha X_2}\right)(u_2, v_2).
\end{aligned} \tag{26}$$

Similarly, we can show that

$$\left(n_{Y_1} e^{i\beta_{Y_1}} ✦ n_{Y_2} e^{i\beta_{Y_2}}\right)((u_1, v_1), (u_2, v_2)) = \left(n_{X_1} e^{i\beta_{X_1}} ✦ n_{X_2} e^{i\beta_{X_2}}\right)(u_1, v_1) \vee \left(n_{X_1} e^{i\beta_{X_1}} ✦ n_{X_2} e^{i\beta X_2}\right)(u_2, v_2). \tag{27}$$

This completes the proof. □

Theorem 13. *If $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ are two CIFG, such that $G_1 ✦ G_2$ is strong, then at least one of G_1 or G_2 must be strong.*

5. Strong Product of Two CIFGs

Definition 14. The strong product of two CIFGs is $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ such that $B_1 \cap B_2 = \phi$ is defined to be CIFG$G_1 \otimes G_2 = (B, C, X_1 \otimes X_2, Y_1 \otimes Y_2)$ where $B = B_1 \times B_2$ and $C = \{(u, v_1), u, v_2) \mid u \in C_1, (v_1, v_2) \in C_2\} \cup \{(u_1, v), u_2, v) \mid u_1), u, v_2) \in C_1\} \cup \{((u_1, u_2), (v_1, v_2)) \mid (u_1, u_2) \in C_1, (v_1, v_2) \in C_2\}$. The MV and NMV for the vertex (u, v) in $G_1 \otimes G_2$ are given by

$$\left(m_{X_1} e^{i\alpha_{X_1}} \otimes m_{X_2} e^{i\alpha_{X_2}}\right)(u, v) = m_{X_1}(u) e^{i\alpha_{X_1}(u)} \wedge m_{X_2}(v) e^{i\alpha_{A_2}(v)}, \tag{28}$$

$$\left(n_{X_1} e^{i\beta_{X_1}} \otimes n_{X_2} e^{i\beta_{X_2}}\right)(u, v) = n_{X_1}(u) e^{i\beta_{X_1}(u)} \vee n_{X_2}(v) e^{i\beta_{X_2}(v)}. \tag{29}$$

The MV and NMV for edges in $G_1 \otimes G_2$ are given by

$$\begin{cases} \left(m_{Y_1}e^{i\alpha_{Y_1}} \otimes m_{Y_2}e^{i\alpha_{Y_2}}\right)((u, v_1), (u, v_2)) = & m_{X_1}(u)e^{i\alpha_{X_1}(u)} \wedge m_{Y_2}(v_1, v_2)e^{i\alpha_{Y_2}(v_1,v_2)}, \\ \left(m_{Y_1}e^{i\alpha_{Y_1}} \otimes m_{Y_2}e^{i\alpha_{Y_2}}\right)((u_1, v), (u_2, v)) = & m_{Y_1}(u_1, u_2)e^{i\alpha_{Y_1}((u_1,u_2))} \wedge m_{X_2}(v)e^{i\alpha_{X_2}(v)}, \\ \left(m_{Y_1}e^{i\alpha Y_1} \otimes m_{Y_2}e^{i\alpha_{Y_2}}\right)((u_1, v_1), (u_2, v_2)) = & m_{Y_1}(u_1, v_2)e^{i\alpha_{Y_1}((u_1,v_2))} \wedge m_{Y_2}(u_1, v_2)e^{i\alpha_{Y_2}(u_1,v_2)}, \end{cases} \tag{30}$$

$$\begin{cases} \left(n_{Y_1}e^{i\beta_{Y_1}} \otimes n_{Y_2}e^{i\beta_{Y_2}}\right)((u, v_1), (u, v_2)) = & n_{X_1}(u)e^{i\beta_{X_1}(u)} \vee n_{Y_2}(v_1, v_2)e^{i\beta_{Y_2}(v_1,v_2)}, \\ \left(n_{Y_1}e^{i\beta_{Y_1}} \otimes n_{Y_2}e^{i\beta_{Y_2}}\right)((u_1, v), (u_2, v)) = & n_{Y_1}(u_1, u_2)e^{i\beta_{Y_1}((u_1,u_2))} \vee n_{X_2}(v)e^{i\beta_{x_2}(v)}, \\ \left(n_{Y_1}e^{i\beta_{Y_1}} \otimes n_{Y_2}e^{i\beta_{Y_2}}\right)((u_1, v_1), (u_2, v_2)) = & n_{Y_1}(u_1, v_2)e^{i\beta_{Y_1}((u_1,v_2))} \vee n_{Y_2}(u_1, v_2)e^{i\beta_{Y_2}(u_1,v_2)}. \end{cases} \tag{31}$$

Theorem 15. *If $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ are strong CIFGs, then $G_1 \otimes G_2$ is also strong.*

Theorem 16. *If $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ are two CIFGs, such that $G_1 \otimes G_2$ is strong, then at least one of G_1 or G_2 must be strong.*

Definition 17. A CIFG $G = (B, C, X, Y)$ is said to be complete if

$$\begin{aligned} m_Y(u, v)e^{i\alpha_Y(u,v)} &= m_X(u)e^{i\alpha_X(u)} \wedge m_X(v)e^{i\alpha_X(v)} \\ n_Y(u, v)e^{i\beta_Y(u,v)} &= n_X(u)e^{i\alpha_X(u)} \wedge n_X(v)e^{i\alpha_X(v)}. \end{aligned} \tag{32}$$

for all $u, v \in B$.

Theorem 18. *If $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ are two CIFGs, then $G_1 \otimes G_2$ is complete.*

Proof. As a strong product of CIFGs is CIFG, and every pair of vertices is adjacent. If $((u, v_1), (u, v_2)) \in C$, then using (29) and (30), we have

$$\begin{aligned} &\left(m_{Y_1}e^{i\alpha_{Y_1}} \otimes m_{Y_2}e^{i\alpha_{Y_2}}\right)((u, v_1), (u, v_2)) \\ &\quad = m_{X_1}(u)e^{\alpha_{X_1}(u)} \wedge m_{Y_2}(v_1, v_2)e^{\alpha_{Y_2}(v_1,v_2)} \\ &\quad = m_{X_1}(u)e^{\alpha_{X_1}(u)} \wedge \left(m_{X_2}(v_1)e^{i\alpha_{X_2}(v_1)} \wedge m_{X_2(v_2)}e^{i\alpha_{X_2}(v_2)}\right) \\ &\quad = \left(m_{X_1}(u)e^{i\alpha_{X_1}(u)} \wedge m_{X_2}(v_1)e^{i\alpha_{X_2}(v_1)}\right) \\ &\quad \wedge \left(m_{X_1}(u)e^{i\alpha_{X_1}(u)} \wedge m_{X_2}(v_2)e^{i\alpha_{X_2}(v_2)}\right) \\ &\quad = \left(m_{X_1}e^{i\alpha_{X_1}} \otimes m_{X_2}e^{i\alpha_{X_2}}\right)(u, v_1) \\ &\quad \wedge \left(m_{X_1}e^{i\alpha_{X_1}} \otimes m_{X_2}e^{i\alpha_{X_2}}\right)(u, v_1). \end{aligned} \tag{33}$$

And by (31), it follows that

$$\begin{aligned} &\left(n_{Y_1}e^{i\beta_{Y_1}} \otimes n_{Y_2}e^{i\beta_{Y_2}}\right)((u.v_1), (u, v_2)) \\ &\quad = n_{X_1}(u)e^{\beta_{X_1}(u)} \vee n_{Y_1}(v_1, v_2)e^{\beta_{Y_2}(v_1,v_2)} \\ &\quad = n_{X_1}(u)e^{\beta_{X_1}(u)} \vee n_{Y_1}(v_1, v_2)e^{\beta_{Y_2}(v_1,v_2)} \\ &\quad = n_{X_1}(u)e^{\beta_{X_1}(u)} \vee \left[n_{X_2}(v_1)e^{i\beta_{X_2}(v_1)} \vee n_{X_2(v_2)}e^{i\beta_{X_2}(v_2)}\right] \\ &\quad = \left(n_{X_1}(u)e^{i\beta_{X_1}(u)} \vee n_{X_2}(v_1)e^{i\beta_{X_2}(v_1)}\right) \\ &\quad \vee \left(n_{X_1}(u)e^{i\beta_{X_1}(u)} \vee n_{X_2}(v_2)e^{i\beta_{X_2}(v_2)}\right) \\ &\quad = \left(n_{X_1}e^{i\beta_{X_1}} \otimes n_{X_2}e^{i\beta_{X_2}}\right)(u, v_1) \\ &\quad \vee \left(n_{X_1}e^{i\beta_{X_1}} \otimes n_{X_2}e^{i\beta_{X_2}}\right)(u, v_1). \end{aligned} \tag{34}$$

If $((u_1, v), (u_2, v)) \in C$, then

$$\begin{aligned} &\left(m_{Y_1}e^{i\alpha_{Y_1}} \otimes m_{Y_2}e^{i\alpha_{Y_2}}\right)((u_1, v)(u_2, v)) \\ &\quad = m_{Y_1}(u_1, v_2)e^{i\alpha_{Y_1}(u_1,v_2)} \wedge m_{X_2}(v)e^{i\alpha_{X_2}(v)} \\ &\quad = \left(m_{X_1}(u_1)e^{i\alpha_{X_1}(u_1)} \wedge m_{X_1}(v_1)e^{i\alpha_{X_2}(u_2)}\right) \\ &\quad \wedge \left(m_{X_2}(v)e^{i\alpha_{X_2}(v)} \wedge m_{X_2}(v)e^{i\alpha_{X_2}(v)}\right) \\ &\quad = \left(m_{X_1}e^{i\alpha_{X_1}} \otimes m_{X_2}e^{i\alpha_{X_2}}\right)(u_1, v) \\ &\quad \wedge \left(m_{X_1}e^{i\alpha_{X_1}} \otimes m_{X_2}e^{i\alpha X_2}\right)(u_2, v). \end{aligned} \tag{35}$$

Similarly,

$$\begin{aligned} \left(n_{Y_1}e^{i\beta_{Y_1}} \otimes n_{Y_2}e^{i\beta_{Y_2}}\right)((u_1, v)(u_2, v)) &= \left(n_{X_1}e^{i\beta_{X_1}} \otimes n_{X_2}e^{i\beta_{X_2}}\right)(u_1, v), \\ &\vee \left(n_{X_1}e^{i\beta_{X_1}} \otimes n_{X_2}e^{i\beta_{X_2}}\right)(u_2, v). \end{aligned} \tag{36}$$

Again if $((u_1, v_1), (u_2, v_2)) \in C$, then

$$\begin{aligned}
&\left(m_{Y_1}e^{i\alpha_{Y_1}} \otimes m_{Y_2}e^{i\alpha_{Y_2}}\right)((u_1, v_1)(u_2, v_2)) \\
&= m_{Y_1}(u_1, v_1)e^{i\alpha_{Y_1}(u_1,v_1)} \wedge m_{Y_2}(v_1, v_2)e^{i\alpha_{Y_2}(u_2,v_2)} \\
&= \left(m_{X_1}(u_1)e^{i\alpha_{X_1}(u_1)} \wedge m_{X_1}(v_2)e^{i\alpha_{X_2}(v_2)}\right) \\
&\wedge \left(m_{X_2}(v_1)e^{i\alpha_{X_2}(v_1)} \wedge m_{X_2}(v_2)e^{i\alpha_{X_2}(v_2)}\right) \\
&= \left(m_{X_1}(u_1)e^{i\alpha_{X_1}(u_1)} \wedge m_{X_2}(u_1)e^{i\alpha_{X_2}(v_1)}\right) \\
&\wedge \left(m_{X_1}(u_2)e^{i\alpha_{X_2}(u_2)} \wedge m_{X_2}(v_2)e^{i\alpha_{X_2}(v_2)}\right) \\
&= \left(m_{X_1}e^{i\alpha_{X_1}} \otimes m_{X_2}e^{i\alpha_{X_2}}\right)(u_1, v_1) \\
&\wedge \left(m_{X_1}e^{i\alpha_{X_1}} \otimes m_{X_2}e^{i\alpha_{X_2}}\right)(u_2, v_2).
\end{aligned} \tag{37}$$

Similarly,

$$\begin{aligned}
\left(n_{Y_1}e^{i\beta_{Y_1}} \otimes n_{B_2}e^{i\beta_{Y_2}}\right)((u_1, v_1)(u_2, v_2)) &= \left(n_{X_1}e^{i\beta_{X_1}} \otimes n_{X_2}e^{i\beta_{X_2}}\right)(u_1, v_1), \\
&\vee\left(n_{X_1}e^{i\beta_{X_1}} \otimes n_{X_2}e^{i\beta_{X_2}}\right)(u_2, v_2).
\end{aligned} \tag{38}$$

This completes the proof. □

6. Modular Product of CIFGs

Definition 19. Let $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ be two CIFGs with underlying vertex sets B_1 and B_2 and edge sets C_1 and C_2, respectively. Then, modular product of G_1 and G_2 is $G_1eG_2 = (B_1eB_2, C_1eC_2, X_1eX_2, X_1eX_2)$ with underlying vertex set $B_1eB_2 = \{(x_1, y_1) | x_1 \in B_1, y_1 \in B_2\}$ and underlying edge set $C_1eC_2 = \{(x_1, y_1)x_2, y_2 | (x_1, x_2) \in C_1, (y_1, y_2) \in C_2$ or $(x_1, x_2) \notin C_1, (y_1, y_2) \notin C_2\}$ with

$$\begin{aligned}
m_{X_1}e^{i\alpha_{X_1}} e m_{X_2}e^{i\alpha_{X_2}}(x_1, y_1) &= m_{X_1}(x_1)e^{i\alpha_{X_1}}(x_1) \wedge m_{X_2}(y_1)e^{i\alpha_{X_2}(y_1)} \\
&= \min\{m_{X_1}(x_1), m_{X_2}(y_1)\}e^{i\min\{\alpha_{X_1}(x_1), \alpha_{X_2}(y_1)\}}, \\
\left(n_{X_1}e^{i\beta_{X_1}} e n_{X_2}e^{i\beta_{X_2}}\right)(x_1, y_1) &= n_{X_1}(x_1)e^{i\beta_{X_1}(x_1)} \vee n_{X_2}(y_1)e^{i\beta_{X_2}(y_1)} \\
&= \max\{n_{X_1}(x_1), n_{X_2}(y_1)\}e^{i\max\{\beta_{X_1}(x_1), \beta_{X_2}(y_1)\}}.
\end{aligned} \tag{39}$$

Here, $x_1 \in B_1$ and $y_1 \in B_2$.

The MV and NMV for edges in G_1eG_2 are given by

$$m_{Y_1}e^{i\alpha_{Y_1}} e m_{Y_2}e^{i\alpha_{Y_2}}((x_1, y_1), (x_2, y_2)) = \begin{cases} m_{Y_1}(x_1, x_2)e^{i\alpha_{Y_1}(x_1,x_2)} \wedge m_{Y_2}(x_1, x_2)e^{i\alpha_{Y_2}(x_1,x_2)}, \\ \text{if } (x_1, x_2) \in C_1, (y_1 y_2) \in C_2, \\ m_{X_1}(x_1)e^{i\alpha_{X_1}(x_1)}(x_1) \wedge m_{X_1}(x_2)e^{i\alpha_{X_1}(x_2)} \wedge m_{X_2}(y_1)e^{i\alpha_{X_2}(y_1)} \wedge m_{Y_2}(y_2)e^{i\alpha_{Y_2}(x_1)}, \\ \text{if } (x_1, x_2) \notin C_1, (y_1, y_2) \notin C_2, \end{cases} \tag{40}$$

$$n_{Y_1}e^{i\beta_{Y_1}} e n_{Y_2}e^{i\beta_{Y_2}}((x_1, y_1), (x_2, y_2)) = \begin{cases} n_{Y_1}(x_1, x_2)e^{i\beta_{Y_1}(x_1,x_2)} \vee n_{Y_2}(x_1, x_2)e^{i\beta_{Y_2}(x_1,x_2)}, \\ \text{if } (x_1, x_2) \in C_1, (y_1, y_2) \in C_2, \\ n_{X_1}(x_1)e^{i\beta_{X_1}(x_1)}(x_1) \vee n_{X_1}(x_2)e^{i\beta_{X_1}(x_2)} \vee n_{X_2}(y_1)e^{i\beta_{X_2}(y_1)} \vee n_{Y_2}(y_2)e^{i\beta_{Y_2}(x_1)}, \\ \text{if } (x_1, x_2) \notin C_1, (y_1, y_2) \notin C_2. \end{cases} \tag{41}$$

Theorem 20. *Let $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, Y_2, Y_2)$ be the CIFGs; then, modular product $G_1 \odot G_2$ is also a CIFGs.*

Proof. Let $G_1 = (B_1, C_1, X_1, Y_1)$ and $G_2 = (B_2, C_2, X_2, Y_2)$ be the CIFGs. We have to prove that $G_1 \odot G_2$ is CIFG. By the definition, $X_1 \odot X_2$ is CIFS on $B_1 \odot B_2$ and $Y_1 \odot Y_2$ is CIFS on $C_1 \odot C_2$. From (40) and (41), we have

$$\begin{aligned}
&m_{Y_1}e^{i\alpha_{Y_1}} e m_{Y_2}e^{i\alpha_{Y_2}}((x_1, y_1), (x_2, y_2)) \\
&= m_{Y_1}(x_1, x_2)e^{i\alpha_{Y_1}(x_1,x_2)} \\
&\wedge m_{Y_2}(x_1, x_2)e^{i\alpha_{Y_2}(x_1,x_2)} \text{if } (x_1, x_2) \in C_1, y_1 y_2 \in C_2 \\
&\le m_{X_1}(x_1)e^{i\alpha_{X_1}(x_1)}(x_1) \wedge m_{X_1}(x_2)e^{i\alpha_{X_1}(x_2)} \\
&\wedge m_{X_2}(y_1)e^{i\alpha_{X_2}(y_1)} \wedge m_{Y_2}(y_2)e^{i\alpha_{Y_2}(x_1)}.
\end{aligned} \tag{42}$$

Since G_1 and G_2 are CIFGs,

$$\begin{aligned}
&n_{Y_1}e^{i\beta_{Y_1}} \odot n_{Y_2}e^{i\beta_{Y_2}}((x_1, y_1), (x_2, y_2)) \\
&= n_{Y_1}(x_1, x_2)e^{i\beta_{Y_1}(x_1,x_2)} \vee n_{Y_2}(x_1, x_2)e^{i\beta_{Y_2}(x_1,x_2)} \\
&\qquad \text{if } (x_1, x_2) \in C_1, (y_1 y_2) \in C_2 \\
&\le n_{X_1}(x_1)e^{i\beta_{X_1}(x_1)}(x_1) \vee n_{X_1}(x_2)e^{i\beta_{X_1}(x_2)} \\
&\vee n_{X_2}(y_1)e^{i\beta_{X_2}(y_1)} \vee n_{Y_2}(y_2)e^{i\beta_{Y_2}(x_1)}.
\end{aligned} \tag{43}$$

Since G_1 and G_2 are CIFGs,

$$\begin{aligned}
&m_{Y_1}e^{i\alpha_{Y_1}} \odot m_{Y_2}e^{i\alpha_{Y_2}}((x_1, y_1), (x_2, y_2)) \\
&= m_{X_1}(x_1)e^{i\alpha_{X_1}(x_1)}(x_1) \wedge m_{X_1}(x_2)e^{i\alpha_{X_1}(x_2)} \\
&\wedge m_{X_2}(y_1)e^{i\alpha_{X_2}(y_1)} \\
&\wedge m_{Y_2}(y_2)e^{i\alpha_{Y_2}(x_1)} \text{if } (x_1, x_2) \notin C_1, (y_1, y_2) \notin C_2 \\
&= m_{Y_1}e^{i\alpha_{Y_1}} \odot m_{Y_2}e^{i\alpha_{Y_2}}((x_1, y_1), (x_2, y_2)).
\end{aligned} \tag{44}$$

Since G_1 and G_2 are CIFGs,

$$\begin{aligned}
&n_{Y_1}e^{i\beta_{Y_1}} \odot n_{Y_2}e^{i\beta_{Y_2}}((x_1, y_1), (x_2, y_2)) \\
&= n_{X_1}(x_1)e^{i\beta_{X_1}(x_1)}(x_1) \vee n_{X_1}(x_2)e^{i\beta_{X_1}(x_2)} \\
&\vee n_{X_2}(y_1)e^{i\beta_{X_2}(y_1)} \\
&\vee n_{Y_2}(y_2)e^{i\beta_{Y_2}(x_1)} \text{if } (x_1, x_2) \notin C_1, (y_1, y_2) \notin C_2 \\
&= n_{Y_1}e^{i\beta_{Y_1}} \odot n_{Y_2}e^{i\beta_{Y_2}}((x_1, y_1), (x_2, y_2)).
\end{aligned} \tag{45}$$

Hence,

$$\begin{aligned}
&m_{Y_1}e^{i\alpha_{Y_1}}\odot m_{Y_2}e^{i\alpha_{Y_2}}((x_1,y_1),(x_2,y_2))\\
&\quad\le m_{X_1}e^{i\alpha_{X_1}}\odot m_{X_2}e^{i\alpha_{X_2}}(x_1,y_1)\wedge m_{X_1}e^{i\alpha_{X_1}}\odot m_{X_2}e^{i\alpha_{X_2}}(x_2,y_2),\\
&n_{Y_1}e^{i\beta_{Y_1}}\odot n_{Y_2}e^{i\beta_{Y_2}}((x_1,y_1),(x_2,y_2))\\
&\quad\le n_{X_1}e^{i\beta_{X_1}}\odot m_{X_2}e^{i\beta_{X_2}}(x_1,y_1)\vee n_{X_1}e^{i\beta_{X_1}}\odot n_{X_2}e^{i\beta_{X_2}}(x_2,y_2).
\end{aligned}\tag{46}$$

Hence, $G_1 e G_2$ is CIFG. □

Theorem 21. *Let $G_1=(B_1,C_1,X_1,Y_1)$ and $G_2=(B_2,C_2,X_2,Y_2)$ be strong CIFGs. Then, modular product $G_1\odot G_2$ is also a strong CIFG.*

Theorem 22. *Let $G_1=(B_1,C_1,X_1,Y_1)$ and $G_2=(B_2,C_2,X_2,Y_2)$ be two complete CIFGs.*

(1) If $m_{Y_1}(x)\le m_{Y_2}(x),\alpha_{Y_1}(x)\le\alpha_{Y_2}(x)$ and $n_{Y_1}(x)\ge n_{Y_2}(x),\beta_{Y_1}(x)\ge\beta_{Y_2}(x)$, then $deg_{G_1eG_2}(x,y)=deg_{G_1}(x)$

(2) If $m_{Y_1}(x)\ge m_{Y_2}(x),\alpha_{Y_1}(x)\ge\alpha_{Y_2}(x)$ and $n_{Y_1}(x)\le n_{Y_2}(x),\beta_{Y_1}(x)\le\beta_{Y_2}(x)$, then $deg_{G_1eG_2}(x,y)=deg_{G_2}(x)$

Proof.

(1) Let $G_1=(B_1,C_1,X_1,Y_1)$ and $G_2=(B_2,C_2,X_2,Y_2)$ be two complete CIFGs. The degree of vertex $(x,y)\in B_1\odot B_2$ is $\deg_{G_1\odot G_2}(x,y)=(\deg_{1_{G_1\odot G_1}}(x,y),\deg_{2_{G_1\odot G_1}}(x,y))$. From (4), we have

$$\begin{aligned}
d_{1G_1\odot G_2}(x_1,y_1)&=\sum_{(x_1x_2)\in C_1,(y_1y_2)\in C_2}\left(m_{Y_1}(x_1,x_2)\wedge m_{Y_2}(y_1,y_2)\right)e^{\alpha_{Y_1}(x_1,x_2)\wedge m_{Y_2}(y_1,y_2)}\\
&\quad+\sum_{(x_1x_2)\notin C_1,(y_1y_2)\notin C_2}\left(m_{X_1}(x_1)\wedge m_{X_1}(x_2)\wedge m_{X_2}(y_1)\wedge m_{X_2}(y_1)\right)e^{\alpha_{X_1}(x_1)\wedge\alpha_{X_1}(x_2)\wedge\alpha_{X_2}(y_1)\wedge\alpha_{X_2}(y_2)}\\
&=\sum_{(x_1x_2)\in C_1,(y_1y_2)\in C_2}\left(m_{Y_1}(x_1,x_2)\wedge m_{Y_2}(y_1,y_2)\right)e^{\alpha_{Y_1}(x_1,x_2)\wedge\alpha_{Y_2}(y_1,y_2)}.
\end{aligned}\tag{47}$$

Since both G_1 and G_2 are complete CIFGs, it follows that $\deg_{1_{G_1\odot G_2}}(x_1,y_1)=\deg_{1_{G_1\odot G_2}}(x_1)$.

Similarly,

$$\begin{aligned}
\deg_{1G_1\odot G_2}(x_1,y_1)&=\sum_{(x_1,x_2)\in C_1,(y_1,y_2)\in C_2}\left(m_{Y_1}(x_1,x_2)\wedge m_{Y_2}(y_1,y_2)\right)e^{\alpha_{Y_1}(x_1,x_2)\wedge m_{Y_2}(y_1,y_2)}\\
&\quad+\sum_{(x_1,x_2)\notin C_1,(y_1,y_2)\notin C_2}\left(m_{X_1}(x_1)\wedge m_{X_1}(x_2)\wedge m_{X_2}(y_1)\wedge m_{X_2}(y_1)\right)e^{\alpha_{X_1}(x_1)\wedge\alpha_{X_1}(x_2)\wedge\alpha_{X_2}(y_1)\wedge\alpha_{X_2}(y_2)}\\
&=\sum_{(x_1,x_2)\in C_1,(y_1,y_2)\in C_2}\left(m_{Y_1}(x_1,x_2)\wedge m_{Y_2}(y_1,y_2)\right)e^{\alpha_{Y_1}(x_1,x_2)\wedge\alpha_{Y_2}(y_1,y_2)}.
\end{aligned}\tag{48}$$

Since both G_1 and G_2 are complete CIFGs, it follows that $\deg_{1_{G_1\odot G_2}}(x_1,y_1)=\deg_{1_{G_1\odot G_2}}(x_1)$.

(2) Omitted

□

7. Conclusions

Graphs are a strong and adaptable data structure for representing real-world connections between different types of data. Graph operations take existing graphs and build new ones. In this investigation, we looked at some interesting results from the key operations direct, semistrong, strong, and modular products for complex intuitionistic fuzzy graphs. A strong complex intuitionistic fuzzy graph is also defined, as well as a number of noteworthy findings. We also look at how a vertex's degree behaves in the modular product of two complex intuitionistic fuzzy graphs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. Atanassov, "Intuitionistic fuzzy sets," in *VII ITKR's Session*, V. Sgurev, Ed., Sofia, 1983.

[3] D. Ramot, R. Milo, M. Friedman, and A. Kandel, "Complex fuzzy sets," *IEEE Transactions on Fuzzy Systems*, vol. 10, pp. 171–186, 2002.

[4] C. Li, T. Wu, and F. T. Chan, "Self-learning complex neuro-fuzzy system with complex fuzzy sets and its application to adaptive image noise canceling," *Neurocomputing*, vol. 94, pp. 121–139, 2012.

[5] C. Li and W. Chiang, "Complex fuzzy computing to time series prediction A multi-swarm PSO learning approach," in *Asian Conference on Intelligent Information and Database Systems*, pp. 242–251, Springer, Berlin, Heidelberg, 2011.

[6] P. Thirunavukarasu, R. Suresh, and P. Thamilmani, "Applications of complex fuzzy sets," *JP Journal of Applied Mathematics*, vol. 6, no. 1, pp. 5–22, 2013.

[7] M. Xueling, J. Zhan, M. Khan, M. Zeeshan, S. Anis, and A. S. Awan, "Complex fuzzy sets with applications in signals," *Computational and Applied Mathematics*, vol. 38, no. 4, pp. 1–34, 2019.

[8] L. Chunshien and T. W. Chiang, "Complex neurofuzzy ARIMA forecasting a new approach using complex fuzzy sets," *IEEE Transactions on Fuzzy Systems*, vol. 21, no. 3, pp. 567–584, 2012.

[9] J. Buckley, "Fuzzy complex numbers," *Fuzzy Sets and Systems*, vol. 33, no. 3, pp. 333–345, 1989.

[10] A. M. Alkouri and A. R. Salleh, "Complex Atanassov's intuitionistic fuzzy relation," *In Abstract and Applied Analysis*, vol. 2013, pp. 287–382, 2013.

[11] W. J. Liu, "Fuzzy invariant subgroups and fuzzy ideals," *Fuzzy Sets and Systems*, vol. 8, pp. 133–139, 1982.

[12] M. Gulzar, F. D. D. Alghazzawi, and M. H. Mateen, "A note on complex fuzzy subfield," *Indonesian Journal of Electrical Engineering and Computer Science*, vol. 21, pp. 1048–1056, 2021.

[13] M. Gulzar, D. Alghazzawi, M. H. Mateen, and N. Kausar, "A certain class of t-intuitionistic fuzzy subgroups," *IEEE access*, vol. 8, pp. 163260–163268, 2020.

[14] U. M. Swamy and K. L. N. Swamy, "Fuzzy prime ideals of rings," *Journal of Mathematical Analysis and Applications*, vol. 134, no. 1, pp. 94–103, 1988.

[15] M. Gulzar, M. H. Mateen, D. Alghazzawi, and N. Kausar, "A novel applications of complex intuitionistic fuzzy sets in group theory," *IEEE Access*, vol. 8, pp. 196075–196085, 2020.

[16] M. Gulzar, M. H. Mateen, Y. M. Chu, D. Alghazzawi, and G. Abbas, "Generalized direct product of complex intuitionistic fuzzy subrings," *International Journal of Computational Intelligence Systems*, vol. 14, no. 1, pp. 582–593, 2021.

[17] K. Hur, S. Y. Jang, and H. W. Kang, "Intuitionistic fuzzy subgroups and cosets," *Honam Mathematical Journal*, vol. 26, pp. 17–41, 2004.

[18] A. Rosenfeld, "Fuzzy groups," *Journal of Mathematical Analysis and Applications*, vol. 35, pp. 512–517, 1971.

[19] A. Kauffman, *Introductionà la théorie des sous-ensembles flous à l'usage des ingenieurs*, 1973.

[20] A. Rosenfeld, "Fuzzy graphs," in *In Fuzzy sets and their applications to cognitive and decision processes*, pp. 77–95, Academic press, 1975.

[21] N. J. Mordeson and P. Chang-Shyh, "Operations on fuzzy graphs," *Information Sciences*, vol. 79, pp. 159–170, 1994.

[22] K. R. Bhutani and A. Battou, "On M-strong fuzzy graphs," *Information Sciences*, vol. 155, no. 1-2, pp. 103–109, 2003.

[23] C. Eslahchi and B. N. Onagh, "Vertex-strength of fuzzy graphs," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, 9 pages, 2006.

[24] A. N. Gani and J. Malarvizhi, "Isomorphism on fuzzy graphs," *International Journal of Computational and Mathematical Sciences*, vol. 2, no. 4, pp. 190–196, 2008.

[25] J. N. Mordeson and P. S. Nair, "Fuzzy graphs and fuzzy hypergraphs," *Physica*, vol. 46, 2012.

[26] S. Mathew and M. S. Sunitha, "Types of arcs in a fuzzy graph," *Information Sciences*, vol. 179, no. 11, pp. 1760–1768, 2009.

[27] R. Parvathi and M. G. Karunambigai, "Intuitionistic fuzzy graphs," in *Computational Intelligence, Theory and applications*, pp. 18–20, International Conference in Germany, 2006.

[28] P. Thirunavukarasu, R. Suresh, and K. K. Viswanathan, "Energy of a complex fuzzy graph," *Int. J. Math. Sci. Eng. Appl.*, vol. 10, pp. 243–248, 2016.

[29] A. Shannon and K. T. Atanassov, "A first step to a theory of the intuitionistic fuzzy graphs," in *Proceeding of Fuzzy Based Expert System*, D. Lakov, Ed., pp. 59–61, Springer, Berlin, 1994.

[30] S. Zeng, M. Shoaib, S. Ali, F. Smarandache, H. Rashmanlou, and F. Mofidnakhaei, "Certain properties of single-valued neutrosophic graph with application in food and agriculture organization," *International Journal of Computational Intelligence Systems*, vol. 14, no. 1, pp. 1516–1540, 2021.

[31] A. N. Gani and S. S. Begum, "Degree, order and size in intuitionistic fuzzy graphs," *International JournalAlgorithm of Computer Mathematics*, vol. 3, no. 3, pp. 11–16, 2010.

[32] S. Sahoo and M. Pal, "Intuitionistic fuzzy competition graphs," *Journal of Applied Mathematics and Computing*, vol. 52, no. 1, pp. 37–57, 2016.

[33] S. Sahoo and M. Pal, "Intuitionistic fuzzy tolerance graphs with application," *Journal of Applied Mathematics and Computing*, vol. 55, no. 1, pp. 495–511, 2017.

[34] S. Sahoo and M. Pal, "Different types of products on intuitionistic fuzzy graphs," *Pacific Science Review A: Natural Science and Engineering*, vol. 17, no. 3, pp. 87–96, 2015.

[35] N. Yaqoob, M. Gulistan, S. Kadry, and H. A. Wahab, "Complex intuitionistic fuzzy graphs with application in cellular network provider companies," *Mathematics*, vol. 7, no. 1, p. 35, 2019.

[36] A. M. Alkouri and A. R. Salleh, "Complex intuitionistic fuzzy sets," *AIP Conf. Proc.*, vol. 1482, pp. 464–470, 2012.

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Research Article

Fourth Hankel Determinant for a Subclass of Starlike Functions Based on Modified Sigmoid

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In our present investigation, we obtain the improved third-order Hankel determinant for a class of starlike functions connected with modified sigmoid functions. Further, we investigate the fourth-order Hankel determinant, Zalcman conjecture, and also evaluate the fourth-order Hankel determinants for 2-fold, 3-fold, and 4-fold symmetric starlike functions.

1. Introduction and Motivation

Denoted by $\mathscr{A}$, the class of functions f which are analytic in

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}, \tag{1}$$

and are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n (\forall z \in \mathbb{U}). \tag{2}$$

Also, let $\mathscr{S}$ be a subclass of class $\mathscr{A}$, containing all univalent functions in $\mathbb{U}$, and be normalized by the conditions

$$f(0) = 0, \quad f'(0) = 1. \tag{3}$$

In 1916, working on the coefficients a_n of class $\mathscr{S}$, Bieberbach conjectured that

$$|a_n| \leq n (n = 2, 3, \cdots,), \tag{4}$$

which was proved by De Branges in 1984 (see [1]). From 1916 to 1984, for some subclasses of $\mathscr{S}$, many researchers used different techniques and established a number of results. The classes $\mathscr{S}^*$, $\mathscr{K}$, and $\mathscr{R}$, namely, the classes of convex, starlike, and bounded turning functions, respectively, are some major subclasses of the class $\mathscr{S}$. These classes are defined as follows:

$$\mathcal{S}^* = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0\ (\forall z \in \mathbb{U}) \right\},$$
$$\mathcal{K} = \left\{ f \in \mathcal{S} : \Re\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right) > 0\ (\forall z \in \mathbb{U}) \right\}, \tag{5}$$
$$\mathcal{R} = \left\{ f \in \mathcal{S} : \Re\left[f'(z)\right] > 0\ (\forall z \in \mathbb{U}) \right\}.$$

Furthermore, we say that an analytic function $f_1(z)$ is subordinated to $f_2(z)$ in $\mathbb{U}$ and is symbolically written as

$$f_1(z) \prec f_2(z)\ (\forall z \in \mathbb{U}), \tag{6}$$

if there exists a Schwartz function $u(z)$ with properties that

$$|u(z)| \le 1,$$
$$u(0) = 1, \tag{7}$$

such that

$$f_1(z) = f_2(u(z)). \tag{8}$$

Moreover, if $f_2(z)$ is in the class $\mathcal{S}$. Due to [2, 3], we get the following equivalence relation

$$f_1(0) = f_2(0),$$
$$f_1(\mathbb{U}) \subseteq f_2(\mathbb{U}). \tag{9}$$

Now, by using the principle of subordination, a generalized set of the classes $\mathcal{S}^*$,$\mathcal{K}$, and $\mathcal{R}$ are given, respectively, as follows:

$$\mathcal{S}^*(\psi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \psi(z) = \frac{1+z}{1-z}\ (\forall z \in \mathbb{U}) \right\}, \tag{10}$$

$$\mathcal{K}(\psi) = \left\{ f \in \mathcal{S} : \frac{\left(zf'(z)\right)'}{f'(z)} \prec \psi(z) = \frac{1+z}{1-z}\ (\forall z \in \mathbb{U}) \right\}, \tag{11}$$

$$\mathcal{R}(\psi) = \left\{ f \in \mathcal{S} : f'(z) \prec \psi(z) = \frac{1+z}{1-z}\ (\forall z \in \mathbb{U}) \right\}. \tag{12}$$

By changing the right-hand side in (10), several familiar classes can be obtained such as if we keep $\psi(z) = (1 + Az)/(1 + Bz)$, we get the class of starlike functions associated with the Janowski functions (see [4]). Moreover, if we take

$$A = 1,$$
$$B = 1 - 2\alpha\ (0 < \alpha < 1), \tag{13}$$

then, we have a class of starlike functions of order α (see [5]). Also, for the choice of $\psi(z) = 1 + (2/\pi^2)(\log((1+\sqrt{z})/(1-\sqrt{z})))^2$, a corresponding class of starlike functions is obtained, which was introduced by Ronning (see [6]). Furthermore, if we take $\psi(z) = \sqrt{1+z}$, the class of starlike functions related with the lemniscate of Bernoulli domain is resulted which was introduced and investigated by Jangteng et al. [7, 8]. Next, if we take $\psi(z) = 1 + \sin(z)$, the family of starlike functions connected with the sine function is obtained (see [9]). Mendiratta et al. [10] obtained a subclass of strongly starlike functions associated with exponential functions for the choice of $\psi(z) = e^z$. Sharma et al. [11] derived a class of starlike functions associated with a cardioid domain by taking $\psi(z) = 1 + (4/3)z + (2/3)z^2$.

Moreover, several more subclasses of starlike functions have recently been presented in [12–15] through selecting specific functions for ψ, like functions associated with conic domains, shell-like curves associated with Fibonacci numbers, and functions related with Bell numbers.

Lately, based on the techniques of Ma and Minda [16], Goel and Kumar in [17] defined the class $\mathcal{S}^*_{SG}$, based on the subordination principle, as follows:

$$\frac{zf'(z)}{f(z)} \prec \frac{2}{1+e^{-z}}\ (\forall z \in \mathbb{U}), \tag{14}$$

and studied its various important geometric properties.

For a function f of the form (2), Pommerenke [18, 19] defined Hankel determinant $H_{q,n}(f)$, parameter q, with $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{15}$$

The growth of $H_{q,n}(f)$ for fixed integer q and n was evaluated for different subfamilies of univalent functions. Jangteng et al. [7, 20] investigated the sharp bound of the determinant $H_{2,2}(f) = |a_2a_4 - a_3^2|$ for each of the classes $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{R}$, while sharp estimation for the family of close-to-convex functions is still unknown (see [21]). On the other hand, Krishna et al. [22] proved the best estimate of $|H_{2,2}(f)|$ for the class of Bazilevič functions. More detailed work on $H_{2,2}(f)$ can be seen in [23–27] and also the references cited therein.

The determinant,

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \tag{16}$$

is known as third-order Hankel determinant and the estimation of this determinant $|H_{3,1}(f)|$ is more difficult and hence a potential attraction for a lot of researchers who focus on this field. In 1966–1967, Pommerenke defined this Hankel determinant but was an open problem till 2010. In 2010, Babalola [28] was the first researcher who worked on $H_{3,1}(f)$ and successfully obtained the upper bound of $|H_{3,1}(f)|$ for the functions belonging to the classes $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{R}$. A few mathematicians further expanded on this work to include other subclasses of holomorphic and univalent functions (see for example [29–35]). Zaprawa [36] enhanced their work in 2017 by demonstrating

$$|H_{3,1}(f)| \leq \begin{cases} 1, & (f \in \mathcal{S}^*), \\ \frac{49}{540}, & (f \in \mathcal{K}), \\ \frac{41}{60}, & (f \in \mathcal{R}), \end{cases} \tag{17}$$

and asserted that these inequalities are still nonsharp. Additionally, for the sharpness, he thought about the subfamilies of $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{R}$ comprising of functions with m-fold symmetry and acquired the sharp bounds. Recently, in 2018, Kowalczyk et al. [37] and Lecko et al. [38] got the sharp inequalities which are

$$\begin{aligned} |H_{3,1}(f)| &\leq \frac{4}{135}, \\ |H_{3,1}(f)| &\leq \frac{1}{9}, \end{aligned} \tag{18}$$

for the classes $\mathcal{K}$ and $\mathcal{S}^*(1/2)$, respectively, where the symbol $\mathcal{S}^*(1/2)$ indicates the family of starlike functions of order 1/2.

The main goal of this paper is to investigate the necessary and sufficient conditions for functions to get into the class $\mathcal{S}^*_{SG}$ in the form of coefficient inequality, convolution results, and the essential third-order Hankel determinant for this class in (6) and also for its 2-, 3-, and 4-fold symmetric functions.

2. A Set of Lemmas

Let $\mathcal{P}$ be the family of functions p that are holomorphic in $\mathbb{U}$ with $\mathfrak{Re}(p(z)) > 0$ and its series form is as follow:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n (\forall z \in \mathbb{U}). \tag{19}$$

Lemma 1. *If $p \in \mathcal{P}$ and it is of the form ((19)), then,*

$$|c_n| \leq 2(n \geq 1), \tag{20}$$

$$|c_{n+k} - \delta c_n c_k| \leq 2(0 \leq \delta \leq 1), \tag{21}$$

$$\left|c_2 - \xi c_1^2\right| \leq 2 \max\{1 ; |2\xi - 1|\}(\xi \in \mathbb{C}). \tag{22}$$

Further results related to Lemma 1 can be found in [39, 40].

Lemma 2 (see [41]). *Let $p \in P$ have the series expansion of the form ((19)). Then, for $x, \sigma, \rho \in \bar{\mathbb{U}} = \mathbb{U} \cup \{1\}$,*

$$\begin{aligned} 2c_2 &= c_1^2 + x\left(4 - c_1^2\right), \\ 4c_3 &= c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\sigma, \\ 8c_4 &= c_1^4 + \left(4 - c_1^2\right)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4\left(4 - c_1^2\right) \\ &\quad \cdot \left(1 - |x|^2\right)\left[c(x-1)\sigma + \bar{x}\sigma^2 - \left(1 - |\sigma|^2\right)\rho\right]. \end{aligned} \tag{23}$$

Lemma 3 (see [42]). *Let m, n, l satisfy the inequalities $0 < m < 1$, $0 < r < 1$, and*

$$\begin{aligned} &8r(1-r)\left[(mn - 2l)^2 + (m(r+m) - n)^2\right] \\ &\quad + m(1-m)(n - 2rm)^2 \leq 4m^2(1-m)^2 r(1-r). \end{aligned} \tag{24}$$

If $p \in \mathcal{P}$ and has power series (19), then,

$$\left|l p_1^4 + r p_2^2 + 2m p_1 p_3 - \frac{3}{2} n p_1^2 p_2 - p_4\right| \leq 2. \tag{25}$$

3. Improve Upper Bound $H_{3,1}(f)$ for the Class $\mathcal{S}^*_{SG}$

To prove Theorem 6, we need the following two lemmas (Lemma 4 and Lemma 5).

Lemma 4 (see [43]). *If $f \in \mathcal{S}^*_{SG}$ and is of the form (2), then,*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{4}. \tag{26}$$

Lemma 5 (see [43]). *Let $f \in \mathcal{S}^*_{SG}$ be of the form (2), and then,*

$$|a_2 a_3 - a_4| \leq \frac{1}{6}. \tag{27}$$

We now state and prove Theorem 6.

Theorem 6. *Let $f \in \mathcal{S}^*_{SG}$ be of the form (2), and then,*

$$\begin{aligned} |a_6| &\leq \frac{21\,797}{57\,600} \simeq 0.378\,42, \\ |a_7| &\leq \frac{1424\,429}{2073\,600} \simeq 0.686\,94, \\ \left|a_2 a_4 - a_3^2\right| &\leq \frac{1}{16}. \end{aligned} \tag{28}$$

The result is sharp for function

$$f(z) = z \exp\left(\int_0^z \frac{e^{t^2} - 1}{t\left(e^{t^2} + 1\right)} dt\right) = z + \frac{1}{4} z^3 + \cdots. \tag{29}$$

Proof. Since $f \in \mathcal{S}^*_{SG}$, then, there exists a Schwarz function $w(z)$, given in (7) such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1+e^{-w(z)}}. \tag{30}$$

Let

$$\Psi(w(z)) = \frac{2}{1+e^{-w(z)}}, \tag{31}$$

$$k(z) = 1 + c_1 z + c_2 z^2 + \cdots = \frac{1+w(z)}{1-w(z)}. \tag{32}$$

Obviously, the function $k(z) \in \mathscr{P}$ and

$$w(z) = \frac{k(z)-1}{k(z)+1}. \tag{33}$$

This gives

$$w(z) = \frac{k(z)-1}{k(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}. \tag{34}$$

From (31) and (34), we have

$$\begin{aligned} \frac{2}{1+e^{-w(z)}} = {} & 1 + \frac{1}{4}c_1 z + \left(\frac{1}{4}c_2 - \frac{1}{8}c_1^2\right)z^2 + \left(\frac{11}{192}c_1^3 - \frac{1}{4}c_2 c_1 + \frac{1}{4}c_3\right)z^3 \\ & + \left(-\frac{3}{128}c_1^4 + \frac{11}{64}c_1^2 c_2 - \frac{1}{4}c_3 c_1 - \frac{1}{8}c_2^2 + \frac{1}{4}c_4\right)z^4 + \cdots, \end{aligned} \tag{35}$$

while

$$\begin{aligned} \frac{zf'(z)}{f(z)} = {} & 1 + a_2 z + \left(2a_3 - a_2^2\right)z^2 + \left(a_2^3 - 3a_2 a_3 + 3a_4\right)z^3 \\ & + \left(-a_2^4 + 4a_2^2 a_3 - 4a_2 a_4 - 2a_3^2 + 4a_5\right)z^4 \\ & + \left(5a_6 - 5a_2 a_5 + a_2^5 - 5a_3 a_4 - 5a_2^3 a_3 + 5a_2^2 a_4 + 5a_2 a_3^2\right)z^5 \\ & + \begin{pmatrix} 6a_7 - 6a_2 a_6 + 6a_2^2 a_5 - 6a_3 a_5 + 12 a_2 a_3 a_4 - a_2^6 - 6a_2^3 a_4 \\ -3a_4^2 + 2a_3^3 - 9a_2^2 a_3^2 + 6a_2^4 a_3 \end{pmatrix} z^6 \\ & + \cdots. \end{aligned} \tag{36}$$

□

On equating coefficients of (35) and (36), we get

$$a_2 = \frac{1}{4}c_1, \tag{37}$$

$$a_3 = \frac{1}{8}c_2 - \frac{1}{32}c_1^2, \tag{38}$$

$$a_4 = \frac{7}{1152}c_1^3 - \frac{5}{96}c_2 c_1 + \frac{1}{12}c_3, \tag{39}$$

$$a_5 = -\frac{17}{18\,432}c_1^4 + \frac{7}{384}c_1^2 c_2 - \frac{1}{24}c_3 c_1 - \frac{3}{128}c_2^2 + \frac{1}{16}c_4, \tag{40}$$

$$\begin{aligned} a_6 = {} & -\frac{257}{1843\,200}c_1^5 - \frac{107}{23\,040}c_1^3 c_2 + \frac{31}{1920}c_3 c_1^2 + \frac{139}{7680}c_1 c_2^2 \\ & - \frac{11}{320}c_4 c_1 - \frac{19}{480}c_3 c_2 + \frac{1}{20}c_5, \end{aligned} \tag{41}$$

$$\begin{aligned} a_7 = {} & \frac{33\,599}{132\,710\,400}c_1^6 - \frac{73}{2211\,840}c_1^4 c_2 - \frac{79}{17\,280}c_1^3 c_3 \\ & - \frac{1451}{184\,320}c_1^2 c_2^2 + \frac{109}{7680}c_4 c_1^2 + \frac{47}{1440}c_1 c_2 c_3 - \frac{7}{240}c_5 c_1 \\ & + \frac{55}{9216}c_2^3 - \frac{13}{384}c_4 c_2 - \frac{5}{288}c_3^2 + \frac{1}{24}c_6. \end{aligned} \tag{42}$$

Now from (41), we have

$$\begin{aligned} |a_6| = {} & \left| \begin{matrix} \frac{1}{96}\left(c_5 - \frac{9}{10}c_1 c_4\right) + \frac{19}{480}\left(c_5 - c_1 c_4\right) + \frac{31}{1920}c_1^2\left(c_3 - \frac{107}{372}c_1 c_2\right) \\ -\frac{257}{1843200}c_1^5 + \frac{139}{7680}c_1 c_2 \end{matrix} \right| \\ \le {} & \frac{1}{96}\left|c_5 - \frac{9}{10}c_1 c_4\right| + \frac{19}{480}\left|c_5 - c_1 c_4\right| + \frac{31}{1920}|c_1|^2\left|c_3 - \frac{107}{372}c_1 c_2\right| \\ & + \frac{257}{1843200}|c_1|^5 + \frac{139}{7680}|c_1||c_2|. \end{aligned} \tag{43}$$

By applying (20) and (21) to above we get

$$|a_6| \le \frac{21\,797}{57\,600}. \tag{44}$$

Now from (42), we have

$$\begin{aligned} |a_7| = {} & \left| \begin{matrix} \frac{1}{24}\left(c_6 - \frac{5}{12}c_3^2\right) - \frac{13}{384}c_2\left(c_4 - \frac{55}{312}c_2^2\right) - \frac{7}{240}c_1\left(c_5 - \frac{109}{224}c_1 c_4\right) \\ + \frac{47}{1440}c_1 c_2\left(c_3 - \frac{1451}{6016}c_1 c_2\right) + \frac{33\,599}{132\,710\,400}c_1^6 - \frac{73}{2211\,840}c_1^4 c_2 - \frac{79}{17\,280}c_1^3 c_3 \end{matrix} \right| \\ \le {} & \frac{1}{24}\left|c_6 - \frac{5}{12}c_3^2\right| + \frac{13}{384}|c_2|\left|c_4 - \frac{55}{312}c_2^2\right| + \frac{7}{240}|c_1|\left|c_5 - \frac{109}{224}c_1 c_4\right| + \frac{47}{1440}|c_1||c_2| \\ & \cdot \left|c_3 - \frac{1451}{6016}c_1 c_2\right| + \frac{33\,599}{132\,710\,400}|c_1|^6 + \frac{73}{2211\,840}|c_1|^4|c_2| + \frac{79}{17\,280}|c_1|^3|c_3|. \end{aligned} \tag{45}$$

By applying (20) and (21) to the above, we get

$$|a_7| \le \frac{1424\,429}{2073\,600}. \tag{46}$$

Now, from (37)–(39), we have

$$\left|a_2 a_4 - a_3^2\right| = \left|\frac{5}{9216}c_1^4 - \frac{1}{192}c_1^2 c_2 + \frac{1}{48}c_3 c_1 - \frac{1}{64}c_2^2\right|. \tag{47}$$

Using Lemma 2, we get

$$\left|a_2a_4 - a_3^2\right| = \left| -\frac{7}{9216}c_1^4 - \frac{1}{192}c_1^2x^2\left(4-c_1^2\right) - \frac{1}{256}x^2\left(4-c_1^2\right)^2 + \frac{1}{96}c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z \right|. \tag{48}$$

Let $|x| = y, y \in [0,1], c_1 = c, c \in [0,2]$, and $|z| = 1$, along with triangle inequality, and we have

$$\left|a_2a_4 - a_3^2\right| \le \frac{7}{9216}c^4 + \frac{1}{192}c^2y^2\left(4-c^2\right) + \frac{1}{256}y^2\left(4-c^2\right)^2 + \frac{1}{96}c\left(4-c^2\right)\left(1-y^2\right) = G(c,y). \tag{49}$$

Differentiating (49) partially with respect to y, we have

$$\frac{\partial G(c,y)}{\partial y} = \frac{\left(c^2-8c+12\right)\left(4-c^2\right)y}{384} > 0, \tag{50}$$

showing that $G(c,y)$ is an increasing function in interval $y \in [0,1], c \in [0,2]$, so the maximum is attained at $y = 1$, that is,

$$\max G(c,y) = G(c,1) = \frac{7}{9216}c^4 + \frac{1}{192}c^2\left(4-c^2\right) + \frac{1}{256}\left(4-c^2\right)^2 = F(c). \tag{51}$$

Now

$$\begin{aligned} F'(c) &= \frac{7}{2304}c^3 + \frac{1}{96}c\left(4-c^2\right) - \frac{1}{96}c^3 - \frac{1}{64}c\left(4-c^2\right) \\ &= \frac{7}{2304}c^3 - \frac{1}{192}c\left(4-c^2\right) - \frac{1}{96}c^3, \end{aligned} \tag{52}$$

since $F'(c) = 0$ has root at $c = 0$ and also

$$F''(c) = -\frac{1}{48} < 0, \tag{53}$$

so the maximum is attained at $c = 0$; therefore, we have

$$F(c) = \frac{1}{16}. \tag{54}$$

Hence,

$$\left|a_2a_4 - a_3^2\right| \le F(c) = \frac{1}{16}. \tag{55}$$

For the third Hankel determinant, we need the following result.

Lemma 7 (see [17]). *Let $f \in \mathcal{S}_{SG}^*$ be of the form (2). Then,*

$$\begin{aligned} |a_2| &\le \frac{1}{2}, \\ |a_3| &\le \frac{1}{4}, \\ |a_4| &\le \frac{1}{6}, \\ |a_5| &\le \frac{1}{8}. \end{aligned} \tag{56}$$

Theorem 8. *Let $f \in \mathcal{S}_{SG}^*$ be of the form (2). Then,*

$$\left|H_{3,1}(f)\right| \le \frac{43}{576} \simeq 0.07465. \tag{57}$$

Proof. Since from (16), we have

$$H_{3,1}(f) = a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right), \tag{58}$$

by applying triangle inequality, we obtain

$$\left|H_{3,1}(f)\right| \le |a_3|\left|a_2a_4 - a_3^2\right| + |a_4|\left|a_4 - a_2a_3\right| + |a_5|\left|a_3 - a_2^2\right|. \tag{59}$$

Now, using Lemmas 4–7 and Theorem 6 in conjunction with (59), we can get the required result. □

4. Bounds of $H_{4,1}(f)$ for the Class $\mathcal{S}_{SG}^*$

In recent years, researchers has started to evaluate the fourth-order Hankel determinant for different subclasses of analytic functions. The trend of finding the fourth-order Hankel determinant in geometric function theory started in 2018, when Arif et al. [44] studied and obtained the upper bound for the class of bounded turning functions. Recently Zhang and Tang [31] studied the fourth-order Hankel determinant for a subclass of starlike functions associated with the sine function. Inspired from the recent research going on and from the above works, we discuss here the fourth-order Hankel determinant for the class $\mathcal{S}_{SG}^*$.

From (15), we can write $H_{4,1}(f)$ as

$$H_{4,1}(f) = a_7H_{3,1}(f) - a_6\delta_1 + a_5\delta_2 - a_4\delta_3, \tag{60}$$

where

$$\delta_1 = a_3\left(a_2a_5 - a_3a_4\right) - a_4\left(a_5 - a_2a_4\right) + a_6\left(a_3 - a_2^2\right), \tag{61}$$

$$\delta_2 = a_3\left(a_3a_5 - a_4^2\right) - a_5\left(a_5 - a_2a_4\right) + a_6\left(a_4 - a_2a_3\right), \tag{62}$$

$$\delta_3 = a_4\left(a_3a_5 - a_4^2\right) - a_5\left(a_2a_5 - a_3a_4\right) + a_6\left(a_4 - a_2a_3\right). \tag{63}$$

Theorem 9. *Let $f \in \mathcal{S}^*_{SG}$ be of the form (2), and then,*

$$|a_2a_5 - a_3a_4| \le \frac{5}{36}. \tag{64}$$

Proof. From equations (37), (38), (39), and (40), we get

$$\begin{aligned}|a_2a_5 - a_3a_4| &= \left|-\frac{1}{24\,576}c_1^5 + \frac{5}{2304}c_1^3c_2 - \frac{1}{128}c_3c_1^2 + \frac{1}{1536}c_1c_2^2 + \frac{1}{64}c_4c_1 - \frac{1}{96}c_3c_2\right| = \left|\frac{5}{2304}c_1^3\left(c_2 - \frac{3}{360}c_1^2\right) + \frac{1}{64}c_1\left(c_4 - \frac{1}{2}c_1c_3\right) - \frac{1}{96}c_2\left(c_3 - \frac{1}{16}c_1c_2\right)\right| \\ &\le \frac{5}{2304}|c_1|^3\left|c_2 - \frac{3}{360}c_1^2\right| + \frac{1}{64}|c_1|\left|c_4 - \frac{1}{2}c_1c_3\right| + \frac{1}{96}|c_2|\left|c_3 - \frac{1}{16}c_1c_2\right|.\end{aligned} \tag{65}$$

Now, making use of (20), (21), and (22) in conjunction with (65), we can get the required result. □

Theorem 10. *Let $f \in \mathcal{S}^*_{SG}$ be of the form (2), and then,*

$$|a_5 - a_2a_4| \le \frac{1}{8}. \tag{66}$$

Proof. From equations (37), (39), and (40), we get

$$|a_5 - a_2a_4| = \frac{1}{16}\left|\frac{5}{128}c_1^4 - \frac{1}{2}c_1^2c_2 + c_3c_1 + \frac{3}{8}c_2^2 - c_4\right|. \tag{67}$$

Applying Lemma 3 to the last term, we get the required result. □

Theorem 11. *Let $f \in \mathcal{S}^*_{SG}$ be of the form (2), and then,*

$$|a_3a_5 - a_4^2| \le \frac{7405}{82\,944}. \tag{68}$$

Proof. From equations (37), (39), and (40), we get

$$\begin{aligned}|a_3a_5 - a_4^2| &= \left|-\frac{43}{5308\,416}c_1^6 - \frac{23}{442\,368}c_1^4c_2 + \frac{1}{3456}c_1^3c_3 + \frac{11}{36\,864}c_1^2c_2^2 - \frac{1}{512}c_4c_1^2 + \frac{1}{288}c_1c_2c_3 - \frac{3}{1024}c_2^3 + \frac{1}{128}c_4c_2 - \frac{1}{144}c_3^2\right| = \left|-\frac{43}{5308\,416}c_1^6\right. \\ &\quad - \frac{23}{442\,368}c_1^4c_2 - \frac{1}{144}c_3\left(c_3 - \frac{1}{2}c_1c_2\right) - \frac{3}{1024}c_2^2\left(c_2 - \frac{11}{108}c_1^2\right) + \frac{1}{128}c_4\left(c_2 - \frac{1}{4}c_1^2\right) \\ &\quad \left. + \frac{1}{3456}c_1^3\left(c_3 - \frac{23}{128}c_1c_2\right)\right| \le \frac{43}{5308\,416}|c_1|^6 + \frac{23}{442\,368}|c_1|^4|c_2| + \frac{1}{144}|c_3|\left|c_3 - \frac{1}{2}c_1c_2\right| \\ &\quad + \frac{3}{1024}|c_2|^2\left|c_2 - \frac{11}{108}c_1^2\right| + \frac{1}{128}|c_4|\left|c_2 - \frac{1}{4}c_1^2\right| + \frac{1}{3456}|c_1|^3\left|c_3 - \frac{23}{128}c_1c_2\right|.\end{aligned} \tag{69}$$

Now, making use of (20), (21), and (22) in conjunction with (69), we can get the required result. □

Theorem 12. *Let $f \in \mathcal{S}^*_{SG}$ be of the form (2), and then,*

$$|H_{4,1}(f)| \le \frac{16\,431\,024\,581}{119\,439\,360\,000} \simeq 0.137\,57. \tag{70}$$

Proof. From (15), we have

$$H_{4,1}(f) = a_7H_{3,1}(f) - a_6\delta_1 + a_5\delta_2 - a_4\delta_3, \tag{71}$$

where δ_1,δ_2, and δ_3 are defined in (61), (62), and (63), respectively. Now, using triangle inequalities, we have

$$|H_{4,1}(f)| \le |a_7||H_{3,1}(f)| + |a_6||\delta_1| + |a_5||\delta_2| + |a_4||\delta_3|, \tag{72}$$

since

$$\begin{aligned}|\delta_1| &= \left|a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6\left(a_3 - a_2^2\right)\right| \\ &\le |a_3||a_2a_5 - a_3a_4| + |a_4||a_5 - a_2a_4| + |a_6|\left|a_3 - a_2^2\right|.\end{aligned} \tag{73}$$

By applying Lemmas 4 and 7 and Theorems 6, 9, and 10, we get

$$|\delta_1| \le \frac{34\,597}{230\,400}. \tag{74}$$

And also,

$$\begin{aligned}|\delta_2| &= \left|a_3\left(a_3a_5 - a_4^2\right) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3)\right| \\ &\le |a_3|\left|a_3a_5 - a_4^2\right| + |a_5||a_5 - a_2a_4| + |a_6||a_4 - a_2a_3|.\end{aligned} \tag{75}$$

Using Lemmas 5 and 7 and Theorems 6, 10, and 11 we get

$$|\delta_2| \le \frac{837\,853}{8294\,400}. \tag{76}$$

Also, again,

$$\begin{aligned}|\delta_3| &= \left|a_4\left(a_3a_5 - a_4^2\right) - a_5\left(a_2a_5 - a_3a_4\right) + a_6\left(a_4 - a_2a_3\right)\right| \\ &\le |a_4|\left|a_3a_5 - a_4^2\right| + |a_5||a_2a_5 - a_3a_4| + |a_6||a_4 - a_2a_3|,\end{aligned} \tag{77}$$

Using Lemmas 5 and 7 and Theorems 6, 9, and 11, we get

$$|\delta_3| \le \frac{1185\,817}{12\,441\,600}. \tag{78}$$

Now, using the values of (74), (76), and (78) along with Theorem 6 and Lemma 7 to (72), we get the desired the estimate. □

5. Zalcman Conjecture for Class $\mathcal{S}^*_{SG}$

One of the main conjectures in the geometric function theory, suggested by Lawrence Zalcman in 1960, is that the coefficients of class $\mathcal{S}$ satisfy the inequality

$$|a_n^2 - a_{2n-1}| \le (n-1)^2. \tag{79}$$

Only the well-known Koebe function $k(z) = z/(1-z)^2$ and its rotations have equality in the above form. For the popular Fekete-Szego inequality, when $n = 2$, the equality holds. Recently, Khan et al. [43] evaluated the Zalcman conjecture for the class of starlike functions with respect to symmetric points associated with the sine function. Many researchers have studied the Zalcman function in the literature [45–47].

Theorem 13. *Let $f \in \mathcal{S}^*_{SG}$ be of the form (2), and then,*

$$\left|a_5 - a_3^2\right| \le \frac{1}{8}. \tag{80}$$

The result is sharp for function

$$f(z) = z\exp\left(\int_0^z \frac{e^{t^4} - 1}{t\left(e^{t^4} + 1\right)}dt\right) = z + \frac{1}{8}z^5 + \cdots. \tag{81}$$

Proof. From equations (37) and (40), we get

$$\left|a_5 - a_3^2\right| = \frac{1}{16}\left|-\frac{35}{1152}c_1^4 + \frac{5}{12}c_1^2c_2 - \frac{2}{3}c_3c_1 - \frac{5}{8}c_2^2 + c_4\right|. \tag{82}$$

Using Lemma 3 and equation (82), we can get the required result. □

6. Bounds of $H_{4,1}(f)$ for 2-Fold, 3-Fold, and 4-Fold Symmetric Functions

Let $m \in \mathbb{N} = \{1, 2, 3, \cdots,\}$. It is called m-fold symmetric if a rotation of domain $\mathbb{U}$ about the origin through an angle $2\pi/m$ carries itself on the domain $\mathbb{U}$. It is obvious that in $\mathbb{U}$, an analytic function f is m-fold symmetric if

$$f\left(e^{\frac{2\pi}{m}}z\right) = e^{\frac{2\pi}{m}}f(z)(\forall z \in \mathbb{U}). \tag{83}$$

The set of m-fold symmetric univalent functions with the following series:

$$f(z) = z + \sum_{k=2}^{\infty} a_{mk+1}z^{mk+1}(\forall z \in \mathbb{U}), \tag{84}$$

is referred to as $\mathcal{S}^{(m)}$.

The subclass $\mathcal{S}^{*(m)}_{SG}$ is a collection of m-fold symmetric starlike functions associated with the modified sigmoid function. More precisely, an analytic function f of the form (84) belongs to class $\mathcal{S}^{*(m)}_{SG}$ if and only if

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-((p(z)-1)/p(z)+1)}}\left(p \in \mathcal{P}^{(m)}\right), \tag{85}$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk}z^{mk}(\forall z \in \mathbb{U})\right\}. \tag{86}$$

Theorem 14. *If $f \in \mathcal{S}^{*(2)}_{SG}$ and be of the form (84), then,*

$$\left|a_3a_7 - a_5^2\right| \le \frac{751}{21\,504}. \tag{87}$$

Proof. Since $f \in \mathcal{S}^{*(2)}_{SG}$, therefore, there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-((p(z)-1)/p(z)+1)}}. \tag{88}$$

Using the series forms (84) and (86), when $m = 2$ in the above relation, we have

$$a_3 = \frac{1}{8}c_2, \tag{89}$$

$$a_5 = \frac{1}{16}c_4 - \frac{3}{128}c_2^2, \tag{90}$$

$$a_7 = \frac{55}{10\,752}c_2^3 - \frac{13}{448}c_4c_2 + \frac{1}{28}c_6. \tag{91}$$

Now, using (89), (90), and (91), we get

$$\begin{aligned}\left|a_3a_7-a_5^2\right| &= \left|\frac{31}{344\,064}c_2^4-\frac{5}{7168}c_2^2c_4+\frac{1}{224}c_6c_2-\frac{1}{256}c_4^2\right| \\ &= \left|\frac{31}{344\,064}c_2^4+\frac{1}{224}c_2\left(c_6-\frac{5}{32}c_2c_4\right)-\frac{1}{256}c_4^2\right| \\ &\le \frac{31}{344\,064}|c_2|^4+\frac{1}{224}|c_2|\left|c_6-\frac{5}{32}c_2c_4\right|+\frac{1}{256}|c_4|^2.\end{aligned} \tag{92}$$

Now, using (20) and (21) to the above, we get the required result. □

Theorem 15. *If $f \in \mathcal{S}_{SG}^{*(2)}$ and be of the form (84), then,*

$$\left|a_5-a_3^2\right| \le \frac{1}{8}. \tag{93}$$

Proof. Using (89) and (90), we have

$$\left|a_5-a_3^2\right| = \frac{1}{16}\left|c_4-\frac{5}{8}c_2^2\right|. \tag{94}$$

Using (21) to the above, we get the required result. □

Theorem 16. *If $f \in \mathcal{S}_{SG}^{*(2)}$ and be of the form (84), then,*

$$\left|H_{4,1}(f)\right| \le \frac{751}{172\,032}. \tag{95}$$

Proof. Since $f \in \mathcal{S}_{SG}^{*(2)}$, therefore, $a_2 = a_4 = a_4 = 0$ and we have

$$H_{4,1}(f) = \left(a_5-a_3^2\right)\left(a_3a_7-a_5^2\right). \tag{96}$$

Then,

$$\left|H_{3,1}(f)\right| = \left|a_5-a_3^2\right|\left|a_3a_7-a_5^2\right|, \tag{97}$$

Using (87) and (93), we get the required result. □

Theorem 17. *If $f \in \mathcal{S}_{SG}^{*(3)}$ and be of the form (84), then,*

$$\left|H_{4,1}(f)\right| \le \frac{1}{432}. \tag{98}$$

Proof. Since $f \in \mathcal{S}_{SG}^{*(3)}$, therefore, there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1+e^{-((p(z)-1)/p(z)+1)}}. \tag{99}$$

Using the series forms (84) and (86), when $m = 3$ in the above relation, we have

$$\begin{aligned}a_4 &= \frac{1}{12}c_3, \\ a_7 &= \frac{1}{24}c_6-\frac{5}{288}c_3^2.\end{aligned} \tag{100}$$

Now,

$$H_{4,1}(f) = a_4^2\left(a_4^2-a_7\right). \tag{101}$$

Therefore,

$$\left|H_{4,1}(f)\right| = \left|-\frac{1}{3456}c_3^2\left(c_6-\frac{7}{12}c_3^2\right)\right| = \frac{1}{3456}|c_3|^2\left|c_6-\frac{7}{12}c_3^2\right|. \tag{102}$$

Using (20) and (21), we get the desired result. □

Theorem 18. *If $f \in \mathcal{S}_{SG}^{*(4)}$ and be of the form (84), then,*

$$\left|H_{4,1}(f)\right| \le \frac{1}{64}. \tag{103}$$

Proof. Since $f \in \mathcal{S}_{SG}^{*(4)}$, therefore, there exists a function $p \in \mathcal{P}^{(4)}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1+e^{-((p(z)-1)/p(z)+1)}}. \tag{104}$$

Using the series forms (84) and (86), when $m = 4$ in the above relation, we have

$$a_5 = \frac{1}{16}c_4. \tag{105}$$

Since $f \in \mathcal{S}_{SG}^{*(4)}$, therefore, $a_2 = a_3 = a_4 = a_6 = a_7 = 0$ and we have

$$\begin{aligned}H_{4,1}(f) &= a_5^2, \\ H_{4,1}(f) &= \frac{1}{256}c_4^2.\end{aligned} \tag{106}$$

Now,

$$\left|H_{4,1}(f)\right| = \frac{1}{256}|c_4|^2. \tag{107}$$

Using (20) to the above, we get the required result. □

7. Conclusion

In our present investigation, we have obtained the improved third-order Hankel determinant for a class of starlike functions connected with modified sigmoid functions. Furthermore, we have investigated the fourth-order Hankel determinant and Zalcman conjecture and also evaluated fourth-order Hankel determinants for 2-fold, 3-fold, and 4-fold symmetric starlike functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors equally contributed to this manuscript and approved the final version.

References

[1] L. De Branges, "A proof of the Bieberbach conjecture," *Acta Mathematica*, vol. 154, no. 1-2, pp. 137–152, 1985.

[2] S. S. Miller and P. T. Mocanu, "Second order differential inequalities in the complex plane," *Journal of Mathematical Analysis and Applications*, vol. 65, no. 2, pp. 289–305, 1978.

[3] S. S. Miller, P. T. Mocanu, and Differential Subordinations, *Theory and Applications, Series on Monographs and Textbooks in Pure and Appl. Math. No. 225*, Marcel Dekker Inc., New York, NY, USA, 2000.

[4] W. Janowski, "Extremal problems for a family of functions with positive real part and for some related families," *Annales Polonici Mathematici*, vol. 23, no. 2, pp. 159–177, 1970.

[5] M. S. Robertson, "Certain classes of starlike functions," *Michigan Mathematical Journal*, vol. 32, no. 2, pp. 135–140, 1985.

[6] F. Ronning, "Uniformly convex functions and a corresponding class of starlike functions," *Proceedings of American Mathematical Society*, vol. 118, no. 1, pp. 189–196, 1993.

[7] A. Jangteng, S. A. Halim, and M. Darus, "Coefficient inequality for starlike and convex functions," *International Journal of Mathematical Analysis*, vol. 1, pp. 619–625, 2007.

[8] J. Sokół, "Radius problems in the class SL," *Applied Mathematics and Computation*, vol. 214, no. 2, pp. 569–573, 2009.

[9] N. E. Cho, V. Kumar, S. Kumar, and V. Ravichandran, "Radius problems for starlike functions associated with the sine function," *Iranian Mathematical Society*, vol. 45, no. 1, pp. 213–232, 2019.

[10] R. Mendiratta, S. Nagpal, and V. Ravichandran, "On a subclass of strongly starlike functions associated with exponential function," *Bulletin of the Australian Mathematical Society*, vol. 38, no. 1, pp. 365–386, 2015.

[11] K. Sharma, N. K. Jain, and V. Ravichandran, "Starlike functions associated with a cardioid," *Afrika Matematika*, vol. 27, no. 5-6, pp. 923–939, 2016.

[12] N. E. Cho, S. Kumar, V. Kumar, V. Ravichandran, and H. M. Srivastava, "Starlike functions related to the Bell numbers," *Symmetry*, vol. 11, no. 2, 2019.

[13] J. Dzoik, R. K. Raina, and J. Sokół, "On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers," *Mathematical and Computer Modelling*, vol. 57, no. 5-6, pp. 1203–1211, 2013.

[14] S. Kanas and D. Răducanu, "Some class of analytic functions related to conic domains," *Mathematica Slovaca*, vol. 64, no. 5, pp. 1183–1196, 2014.

[15] S. Kumar and V. Ravichandran, "A subclass starlike functions associated with rational function," *Southeast Asian Bulletin of Mathematics*, vol. 40, pp. 199–212, 2016.

[16] W. Ma and M. Minda, "A unified treatment of some special classes of univalent functions," *Proceedings of the Conference on Complex Analysis*, Z. Li, F. Ren, L. Yang, and S. Zhang, Eds., , pp. 157–169, International Press, Cambridge, MA, USA, 1992.

[17] P. Goel and S. Kumar, "Certain class of starlike functions associated with modified sigmoid function," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, pp. 957–991, 2020.

[18] C. Pommerenke, "On the coefficients and Hankel determinants of univalent functions," *Journal of the London Mathematical Society*, vol. s1-41, no. 1, pp. 111–122, 1966.

[19] C. Pommerenke, "On the Hankel determinants of univalent functions," *Mathematika*, vol. 14, no. 1, pp. 108–112, 1967.

[20] A. Jangteng, S. A. Halim, and M. Darus, "Coefficient inequality for a function whose derivative has a positive real part," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, pp. 1–5, 2006.

[21] D. Răducanu and P. Zaprawa, "Deuxieme determinant de Hankel pour les fonctions presque convexes," *Comptes Rendus Mathematique*, vol. 355, no. 10, pp. 1063–1071, 2017.

[22] D. V. Krishna and T. RamReddy, "Second Hankel determinant for the class of Bazilevic functions," *Universitatis Babeș-Bolyai*, vol. 60, pp. 413–420, 2015.

[23] D. Bansal, "Upper bound of second Hankel determinant for a new class of analytic functions," *Applied Mathematics Letters*, vol. 26, no. 1, pp. 103–107, 2013.

[24] S. K. Lee, V. Ravichandran, and S. Supramaniam, "Bounds for the second Hankel determinant of certain univalent functions," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.

[25] M. S. Liu, J. F. Xu, and M. Yang, "Upper bound of second Hankel determinant for certain subclasses of analytic functions," *Abstract and Applied Analysis*, vol. 2014, Article ID 603180, 10 pages, 2014.

[26] J. W. Noonan and D. K. Thomas, "On the second Hankel determinant of areally mean p-valent functions," *Transactions of the American Mathematical Society*, vol. 223, pp. 337–346, 1976.

[27] H. Orhan, N. Magesh, and J. Yamini, "Bounds for the second Hankel determinant of certain bi-univalent functions," *Turkish Journal of Mathematics*, vol. 40, pp. 679–687, 2016.

[28] K. O. Babalola, "On $H_3(1)$ Hankel determinant for some classes of univalent functions," http://arxiv.org/abs/0910.3779.

[29] M. Arif, K. I. Noor, and M. Raza, "Hankel determinant problem of a subclass of analytic functions," *Journal of Inequalities and Applications*, vol. 2012, no. 1, 2012.

[30] L. Shi, M. G. Khan, B. Ahmad, W. K. Mashwani, P. Agarwal, and S. Momani, "Certain coefficient estimate problems for three-leaf-type starlike functions," *Fractal and Fractional*, vol. 5, no. 4, p. 137, 2021.

[31] H. Y. Zhang and H. A. Tang, "A study of fourth-order Hankel determinants for starlike functions connected with the sine function," *Journal of Function Spaces*, vol. 2021, Article ID 9991460, 8 pages, 2021.

[32] Q. Hu, H. M. Srivastava, B. Ahmad et al., "A subclass of multivalent Janowski type q-starlike functions and its consequences," *Symmetry*, vol. 13, no. 7, p. 1275, 2021.

[33] M. G. Khan, B. Ahmad, N. Khan et al., "Applications of Mittag-Leffer type Poisson distribution to a subclass of analytic functions involving conic-type regions," *Journal of Function Spaces*, vol. 2021, Article ID 4343163, 9 pages, 2021.

[34] M. Raza and S. N. Malik, "Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.

[35] G. Shanmugam, B. A. Stephen, and K. O. Babalola, "Third Hankel determinant for α-starlike functions," *Gulf Journal of Mathematics*, vol. 2, pp. 107–113, 2014.

[36] P. Zaprawa, "Third Hankel determinants for subclasses of univalent functions," *Mediterranean Journal of Mathematics*, vol. 14, no. 1, 2017.

[37] B. Kowalczyk, A. Lecko, and Y. J. Sim, "The sharp bound for the Hankel determinant of the third kind for convex functions," *Bulletin of the Australian Mathematical Society*, vol. 97, no. 3, pp. 435–445, 2018.

[38] A. Lecko, Y. J. Sim, and B. Śmiarowska, "The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2," *Complex Analysis and Operator Theory*, vol. 13, no. 5, pp. 2231–2238, 2019.

[39] F. Keogh and E. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proceedings of the American Mathematical Society*, vol. 20, no. 1, pp. 8–12, 1969.

[40] C. Pommerenke, *Univalent Functions, Math, Lehrbucher, vandenhoeck and Ruprecht, Gottingen*, 1975.

[41] R. J. Libera and E. J. Złotkiewicz, "Early coefficients of the inverse of a regular convex function," *Proceedings of the American Mathematical Society*, vol. 85, no. 2, pp. 225–230, 1982.

[42] V. Ravichandran and S. Verma, "Borne pour le cinquieme coefficient des fonctions etoilees," *Comptes Rendus Mathematique*, vol. 353, no. 6, pp. 505–510, 2015.

[43] G. K. Khan, B. Ahmad, G. Murugusundaramoorthy, R. Chinram, and W. K. Mashwani, "Applications of modified Sigmoid functions to a class of starlike functions," *Journal of Function Spaces*, vol. 2020, Article ID 8844814, 8 pages, 2020.

[44] M. Arif, L. Rani, M. Raza, and P. Zaprawa, "Fourth Hankel determinant for the family of functions with bounded turning," *Bulletin of the Korean Mathematical Society*, vol. 55, pp. 1703–1711, 2018.

[45] D. Bansal and S. Janusz, "Zalcman conjecture for some subclass of analytic functions," *Journal of Fractional Calculus and Application*, vol. 8, no. 1, pp. 1–5, 2017.

[46] J. E. Brown and A. Tsao, "On the Zalcman conjecture for starlike and typically real functions," *Mathematische Zeitschrift*, vol. 191, no. 3, pp. 467–474, 1986.

[47] W. C. Ma, "The Zalcman conjecture for close-to-convex functions," *Proceedings of the American Mathematical Society*, vol. 104, no. 3, pp. 741–744, 1988.

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Research Article

Starlikeness of Normalized Bessel Functions with Symmetric Points

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Bessel functions are related with the known Bessel differential equation. In this paper, we determine the radius of starlikeness for starlike functions with symmetric points involving Bessel functions of the first kind for some kinds of normalized conditions. Our prime tool in these investigations is the Mittag-Leffler representation of Bessel functions of the first kind.

1. Introduction and Definitions

Let $\mathbb{E}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and $\mathbb{E}(0,1) \subset \mathbb{E}(z_0, r)$ denote the interior of the unit circle with center at origin. Suppose that $\mathscr{A}$ represent functions f in $\mathbb{E}(0,1)$:

$$f(z) = z + a_2 z^2 + \cdots. \tag{1}$$

Obviously, $f(0) = 0$ along with $f'(0) = 1$. The subclass $\mathcal{S} \subset A$ only contains univalent (one-to-one) functions and $\mathcal{S}^* \subset \mathcal{S}$ represent the set of starlike functions. A function f for which $f(\mathbb{E}(0,1))$ is star-shaped is starlike if $\operatorname{Re}\{zf'(z)/f(z)\} > 0$. Also, $f \in \mathcal{S}_s$ if

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in \mathbb{E}(0,1), \tag{2}$$

and $f \in \mathcal{S}_s(\eta)$ if

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > \eta, \quad 0 \le \eta < 1, z \in \mathbb{E}(0,1). \tag{3}$$

Let

$$r^*(f) = \sup\left\{r > 0 : \operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0, z \in \mathbb{E}(0,r)\right\},$$
$$r^*_\eta(f) = \sup\left\{r > 0 : \operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > \eta, z \in \mathbb{E}(0,r)\right\} \tag{4}$$

be the radii of the classes defined above. We note that r^* is the maximum value of the radius such that $f(\mathbb{E}(0, r^*(f))) \in \mathcal{S}^*$ and r^*_η is the maximum value of the radius such that $f(\mathbb{E}(0, r^*_\eta(f))) \in \mathcal{S}^*$ with symmetric points. Consider the following representation of the function b_μ as in [1], which satisfies the well-known Bessel differential equation:

$$b_\mu(z) = \sum_{j\ge 0} \frac{(-1)^j}{j!\Gamma(j+\mu+1)}\left(\frac{z}{2}\right)^{2j+\mu} = \sum_{j\ge 0 j!} \frac{(-1)^j}{(j+\mu)!}\left(\frac{z}{2}\right)^{2j+\mu}, \tag{5}$$

where $z, \mu \in \mathbb{C}$ such that $\mu \ne -1, -2, -3, \cdots$. Observe that $b_\mu(z) \notin A$. Thus, we consider the following normalizations:

$$f_\mu(z) = \left[2^\mu \Gamma(\mu+1) b_\mu(z)\right]^{1/\mu}, \quad \mu \ne 0, \tag{6}$$

$$k_\mu(z) = 2^\mu \Gamma(\mu+1) z^{1-\mu} b_\mu(z), \tag{7}$$

$$\ell_\mu(z) = 2^\mu \Gamma(\mu+1) z^{1-(\mu/2)} b_\mu(\sqrt{z}). \tag{8}$$

Clearly, the function $f_\mu, k_\mu, \ell_\mu \in \mathscr{A}$. We see that

$$f_\mu(z) = \exp\left[\frac{1}{\mu}\log\left(2^\mu \Gamma(\mu+1) b_\mu(z)\right)\right]. \tag{9}$$

The geometric behavior and properties of the functions f_μ, k_μ, and ℓ_μ were studied by Brown, Kreyszig, Robertson, and many others (for detail, see [2–4] and also the references therein). The related problems were also studied in [3, 5–8] with references therein. We study the radius problems for the functions f_μ, k_μ, and ℓ_μ starlike with symmetric points. Mittag-Leffler expansion for Bessel functions is used as a prime tool along with the conclusion that the specific positive roots of the Dini functions are always smaller than the related zeros $b_\mu(z)$, for reference, see [9].

2. Preliminaries

Lemma 1. *Let $f : \mathbb{E}(0,1) \longrightarrow \mathbb{C}$ be a transcendental function having the following expansion:*

$$f(z) = z\prod_{j\geq 1}\left(1 - \frac{z}{z_j}\right), \tag{10}$$

where $z_j : |z_j| > 1$ have the same argument. For a univalent function f in $\mathbb{E}(0,1)$, we have

$$\sum_{j\geq 1}\frac{1}{|z_j| - 1} \leq 1. \tag{11}$$

This result holds if and only if $f \in \mathcal{S}^*$, and each of its derivatives is close to convex in the open unit disk $\mathbb{E}(0,1)$. Furthermore, for z_j', the zeroes of the derivative of f, f, and f' are univalent in $\mathbb{E}(0,1)$ and for $\mathbb{E}(0,1), f(\mathbb{E}(0,1))$ is a convex-shaped if and only if

$$\sum_{j\geq 1}\frac{1}{|z_j'| - 1} \leq 1. \tag{12}$$

Lemma 2. *The function*

$$\ell_\mu(z) = 2^\mu \Gamma(\mu+1) z^{1-\mu/2} b_\mu(\sqrt{z}) \in \mathcal{S}^*(\eta) \tag{13}$$

and each of its derivative is close to convex in $\mathbb{E}(0,1)$ if and only if $\mu > \mu_0(\eta)$, where $\mu_0(\eta) \simeq 0.5623\cdots$ is a unique zero of $\ell_\mu(1) = 0$ on $(-1,\infty)$.

The proof of Lemma 1 and Lemma 2 is found in [10].

Lemma 3. *The function*

$$f_\mu(z) = \left(2^\mu \Gamma(\mu+1) b_\mu(z)\right)^{1/\mu} \in \mathcal{S}^*(\eta) \tag{14}$$

in $\mathbb{E}(0,1)$ if and only if $\mu > \mu_1(\eta)$, where $0 < \mu_1(\eta) < \infty$ is the unique solution of

$$(1-\eta)\mu b_\mu(1) = b_{\mu+1}(1). \tag{15}$$

In particular, $f_\mu \in \mathcal{S}^*$ in $\mathbb{E}(0,1)$ if and only if $\mu > \mu_1(0)$, where $\mu_1(0) \simeq 0.3908\cdots$ is a unique zero of

$$\mu b_\mu(1) = b_{\mu+1}(1). \tag{16}$$

Lemma 4. *The function*

$$k_\mu(z) = 2^\mu \Gamma(\mu+1) b_\mu(z) z^{1-\mu} \in \mathcal{S}^*(\eta), \quad 0 \leq \eta < 1, \tag{17}$$

in $\mathbb{E}(0,1)$ if and only if $\mu > \mu_2(\eta)$, where $\mu_2(\eta)$ is the unique zero of

$$(1-\eta) b_\mu(1) = b_{\mu+1}(1), \tag{18}$$

lies in $(\tilde{\mu},\infty)$, where $\tilde{\mu} \simeq -0.7745\cdots$ is the unique root of $b_{\mu,1} = 1$ and $b_{\mu,1}$ is the first positive zero of b_μ. In particular, $k_\mu \in \mathcal{S}^*$ in $\mathbb{E}(0,1)$ if and only if $\mu > \mu_2(0)$, where $\mu_2(0) \simeq -0.3397\cdots$ is a unique zero of

$$b_\mu(1) = b_{\mu+1}(1). \tag{19}$$

The proof of Lemma 3 and Lemma 4 can be seen in [11].

Lemma 5. *If $z \in \mathbb{C}$ and $\eta \in \mathbb{R} : \eta > |z|$, then*

$$\frac{|z|}{\eta - |z|} \geq \operatorname{Re}\left(\frac{z}{\eta - z}\right). \tag{20}$$

For the detail of the above Lemma 5, we refer to [3].

3. Main Results

Theorem 6. *Let $1 > \eta \geq 0$, and $\mu \in (-1,0)$. Then, $r_\eta^*(I_\mu)$, is a unique positive zero of*

$$z I_\mu'(z) - \eta\mu I_\mu(z) = 0, \tag{21}$$

where $I_\mu(z) = i^{-\mu} b_\mu(iz)$. Moreover, if $\mu > 0$, then $r_\eta^*(b_\mu)$ is the least positive zero of

$$z b_\mu'(z) - \eta\mu b_\mu(z) = 0. \tag{22}$$

Proof. Using Lemma 3, we see that the function $f_\mu(z) = (2^\mu \Gamma(\mu+1) b_\mu(z))^{1/\mu} \in \mathcal{S}^*(\eta)$ in $\mathbb{E}(0,1)$ with respect to z iff $\mu > \mu_1(\eta)$, where $\mu_1(\eta)$ is a unique zero of

$$(1-\eta)\mu b_{\mu}(1)=b_{\mu+1}(1), \tag{23}$$

lies in $(0,\infty)$. Suppose $b_{\mu,j}$ is the jth positive root of $b_{\mu}(z)$. By using infinite product representation,

$$b_{\mu}(z)=\frac{1}{\Gamma(1+\mu)}\left(\frac{z}{2}\right)^{\mu}\prod_{j\geq 1}\left(1-\frac{z^2}{b_{\mu,j}^2}\right). \tag{24}$$

Also, as given in [12], we see that f_{μ} has the following form:

$$\begin{aligned} f_{\mu}(z)&=z\prod_{j\geq 1}\left(1-\frac{z}{b_{\mu,j}^2}\right)\\ &=\left[\sum_{j\geq 0}\frac{(-1)^j\Gamma(1+\mu)}{j!\Gamma(\mu+j+1)}z^{\mu+2j}\right]^{1/\mu}\\ &=z-\frac{1}{4\mu(\mu+1)}z^3+\cdots,\quad \mu\neq 0. \end{aligned} \tag{25}$$

From Lemma 3, we see that for $\mu>\mu^*\simeq -0.7745\cdots$ the unique value of the root of $f_{\mu}(1)=0$ or $b_{\mu,1}=1$, we have $b_{\mu,1}>1$, and $b_{\mu,j}>1, j=1,2,\cdots$. The above result is immediate, if $b_{\mu,1}$ is increasing on $(-1,\infty)$. Using (24) and (25), we can write

$$\begin{aligned} -b_{\mu}(z)&=b_{\mu}(-z),\\ -b_{\mu}'(z)&=b_{\mu}'(-z). \end{aligned} \tag{26}$$

Also, from (6), we have

$$\begin{aligned} -f_{\mu}(z)&=f_{\mu}(-z),\\ f_{\mu}'(-z)&=(-1)f_{\mu}'(z), \end{aligned} \tag{27}$$

which in the context of $zb_{\mu}'(z)-\mu b_{\mu}(z)=-zb_{\mu+1}(z)$ is equivalent to the Mittag-Leffler representation:

$$\frac{1}{b_{\mu}(z)}b_{\mu+1}(z)=\sum_{j\geq 1}\frac{2z}{b_{\mu,j}^2-z^2}. \tag{28}$$

Consequently,

$$\frac{b_{\mu+1}(-z)}{b_{\mu}(-z)}=-\frac{b_{\mu+1}(z)}{b_{\mu}(z)}. \tag{29}$$

In view of (6), (25), and (27), we can write

$$\frac{zf_{\mu}'(-z)}{f_{\mu}(-z)}=-1+\frac{1}{\mu}\sum_{j\geq 1}\frac{2z^2}{b_{\mu,j}^2-z^2}=-=-\frac{1}{\mu}\frac{zb_{\mu}'(z)}{b_{\mu}(z)}. \tag{30}$$

Using Lemma 1, we find that for $j\in\mathbb{N}, \mu=-1$, and $z\in\mathbb{E}(0,b_{\mu,1})$, the following inequality

$$\frac{|z|^2}{b_{\mu,j}^2-|z|^2}\geq \operatorname{Re}\left(\frac{z^2}{b_{\mu,j}^2-z^2}\right) \tag{31}$$

implies that

$$\operatorname{Re}\frac{zf_{\mu}'(-z)}{f_{\mu}(-z)}\leq -1+\frac{1}{\mu}\sum_{j\geq 1}\frac{2|z|^2}{b_{\mu,j}^2-|z|^2}=-\frac{|z|f_{\mu}'(|z|)}{f_{\mu}(|z|)}. \tag{32}$$

When $|z|<b_{\mu,1}$, we observe that

$$\begin{aligned} \operatorname{Re}\left\{\frac{zf_{\mu}'(-z)}{f_{\mu}(-z)}\right\}&\leq -1+\frac{1}{\mu}\sum_{j\geq 1}\frac{2r^2}{b_{\mu,j}^2-r^2}\\ &\leq -1+\frac{1}{\mu}\sum_{j\geq 1}\frac{2}{b_{\mu,j}^2-1}\\ &=-\frac{f_{\mu}'(1)}{f_{\mu}(1)}. \end{aligned} \tag{33}$$

As in [12], we see that $b_{\mu,j}'>0$ on $(0,\infty)$ for a fixed $j\in N$. Thus, $f_{\mu}'(1)/f_{\mu}(1)$ is increasing on $(0,\infty)$, and $-(f_{\mu}'(1)/f_{\mu}(1))$ is decreasing on $(0,\infty)$. Also,

$$-\frac{f_{\mu}'(1)}{f_{\mu}(1)}<-\eta\Leftrightarrow \mu<\mu_1(\eta), \tag{34}$$

where $\mu_1(\eta)$ is the unique zero of

$$f_{\mu}'(1)=\eta f_{\mu}(1)\text{ or }\eta\mu b_{\mu}(1)=b_{\mu}'(1)\text{ or }(1-\eta)\mu b_{\mu}(1)=b_{\mu+1}(1). \tag{35}$$

We also note that

$$\frac{zf_{\mu}'(-z)}{f_{\mu}(-z)}=1-\frac{b_{\mu+1}(z)}{\mu b_{\mu}(z)}, \tag{36}$$

when $\mu\in(-1,0)$. For $-1<\mu<-\infty$, the Dini function $zb_{\mu}'(z)+\eta b_{\mu}(z)$ has real roots except a pair of complex conjugate roots (for detail, see [1]). Thus,

$$f_{\mu}(-z)\in\mathcal{S}^*(\eta),\quad \eta\in(-1,0), \tag{37}$$

in $\mathbb{E}(0,1)$ if and only if $\mu<\mu_1(\eta)$. Considering (5), (30), (33), and (36), we have

$$\begin{aligned} \frac{2zf_{\mu}'(z)}{f_{\mu}(z)-f_{\mu}(-z)}&=\frac{2zf_{\mu}(z)b_{\mu}'(z)}{\mu\left(f_{\mu}(z)+f_{\mu}(z)\right)b_{\mu}(z)}\\ &=\frac{1}{\mu}\frac{zb_{\mu}'(z)}{b_{\mu}(z)}\\ &=1-\frac{1}{\mu}\sum_{j\geq 1}\frac{2z^2}{b_{\mu,j}^2-z^2}. \end{aligned} \tag{38}$$

Also, from (38), it is obvious that

$$\operatorname{Re}\frac{2zf'_\mu(z)}{f_\mu(z)-f_\mu(-z)} \geq 1-\frac{1}{\mu}\sum_{j\geq1}\frac{2|z|^2}{b^2_{\mu,j}-|z|^2} = \frac{|z|f'_\mu(|z|)}{f_\mu(|z|)} = \operatorname{Re}\frac{zf'_\mu(z)}{f_\mu(z)}. \tag{39}$$

As in [1], for $1>\eta\geq0$ and $\mu\in(-1,0)$, $r^*_\eta(I_\mu)$ is the unique value of positive zero of $zI'_\mu(z)-\eta\mu I_\mu(z)=0$. Moreover, if $\mu>0$, then we have $r^*_\eta(b_\mu)$ which is the least positive zero of $zb'_\mu(z)-\eta\mu b_\mu(z)=0$. □

Theorem 7. *If $\mu>-1$, then $r^*_\eta(k_\mu)>0$ is the smallest zero of*

$$zb'_\mu(z)+(1-\eta-\mu)b_\mu(z)=0, \tag{40}$$

where $k_\mu(z)=2^\mu\Gamma(\mu+1)b_\mu(z)(z)^{1-\mu}$.

Proof. By Lemma 4, the function

$$k_\mu(z)=2^\mu\Gamma(\mu+1)\left[b_\mu(z)\right]^{1-\mu}\in\mathcal{S}^*(\eta), \tag{41}$$

for $\eta\in(-1,0)$in the open unit disk $\mathbb{E}(0,1)$ if and only if $\mu<\mu_2(\eta)$, where $\mu_2(\eta)$ is the unique value of the zero of the following equation:

$$(1-\eta)b_\mu(1)=b_{\mu+1}(1). \tag{42}$$

Suppose that $b_{\mu,j}$ is the jth positive zero of $b_\mu(z)$ given by (24) and (25). Consider the normalization (7) such that

$$k_\mu(-z)=(-1)k_\mu(z). \tag{43}$$

We write

$$\frac{zk'_\mu(-z)}{k_\mu(-z)}=-\frac{zk'_\mu(z)}{k_\mu(z)}=-1+\mu-\partial\left(b_\mu\right)(z)=-1+\frac{zb_{\mu+1}(z)}{b_\mu(z)}. \tag{44}$$

For $\mu>-1$ and $r=|z|<b_{\mu,1}$, we see that

$$\operatorname{Re}\frac{zk'_\mu(-z)}{k_\mu(-z)}\leq-\frac{zk'_\mu(\mathrm{r})}{k_\mu(r)}=-1+\sum_{j\geq1}\frac{2|z|^2}{b^2_{\mu,j}-|z|^2}=-\frac{k'_\mu(1)}{k_\mu(1)}. \tag{45}$$

For detail, we refer to [12]. Since the function $b'_{\mu,j}>0$ on $(-1,\infty)$ for fixed $j\in\mathbb{N}$, thus $k'_\mu(1)/k_\mu(1)$ is increasing on $(\tilde{\mu},\infty)$, and $-(k'_\mu(1)/k_\mu(1))$ is decreasing on $(\tilde{\mu},\infty)$ and $-(k'_\mu(1)/k_\mu(1))<-\eta$ if and only if $\mu<\mu_2(\eta)$, where $\mu_2(\eta)$ is the unique value of the root of

$$\begin{aligned} k'_\mu(1)=\eta k_\mu(1)\text{or}(1-\mu-\eta)b_\mu(1)+b'_\mu(1) &= 0\text{ or}(1-\eta)b_\mu(1)\\ &= b_{\mu+1}(1). \end{aligned} \tag{46}$$

Thus,

$$\begin{aligned}\frac{zb'_\mu(-z)}{b_\mu(-z)} &= -1+\operatorname{Re}\sum_{j\geq1}\frac{2z^2}{b^2_{\mu,j}-z^2}\\ &\leq -1+\sum_{j\geq1}\frac{2|z|^2}{b^2_{\mu,j}-|z|^2}\\ &= -\frac{|z|k'_\mu(|z|)}{k_\mu(|z|)}\\ &= -\operatorname{Re}\frac{zk'_\mu(z)}{k_\mu(z)},\end{aligned} \tag{47}$$

and equality holds for $|z|=r$. The above inequality implies that the function $k_\mu(-z)\in\mathcal{S}^*(\eta),\eta\in(-1,0)$, in $\mathbb{E}(0,1)$ if and only if $\mu<\mu_2(\eta)$. Considering normalization in (7), we can write

$$\begin{aligned}k_\mu(-z)=(-1)k_\mu(z)\frac{2zk'_\mu(z)}{k_\mu(z)-k_\mu(-z)} &= \frac{zb'_\mu(z)}{b_\mu(z)}+(1-\mu)\\ &= 1-\sum_{j\geq1}\frac{2z^2}{b^2_{\mu,j}-z^2}.\end{aligned} \tag{48}$$

From (48), it is known that

$$\begin{aligned}\operatorname{Re}\left\{\frac{2zk'_\mu(z)}{k_\mu(z)-k_\mu(-z)}\right\} &= 1-\operatorname{Re}\sum_{j\geq1}\frac{2z^2}{b^2_{\mu,j}-z^2}\\ &\geq 1-\sum_{j\geq1}\frac{2|z|^2}{b^2_{\mu,j}-|z|^2}\\ &= \frac{|z|k'_\mu(|z|)}{k_\mu(|z|)}\\ &= \operatorname{Re}\frac{zk'_\mu(z)}{k_\mu(z)}.\end{aligned} \tag{49}$$

For1 $>\eta\geq0$ and $\mu>-1$, we see that $r^*_\eta(k_\mu)$ is the least positive zero of the following equation:

$$zb'_\mu(z)+(1-\eta-\mu)b_\mu(z)=0, \tag{50}$$

where $k_\mu(z)=2^\mu\Gamma(\mu+1)b_\mu(z)(z)^{1-\mu}$. □

Theorem 8. *If $\mu > -1$ and $1 > \eta \geq 0$, then $r^*_\eta(\ell_\mu)$ is the least positive zero of the differential equation $zb'_\mu(z) + (2 - 2\eta - \mu) b_\mu(z) = 0$, where $\ell_\mu(z) = 2^\mu \Gamma(\mu+1) b_\mu(\sqrt{z})(z)^{1-\mu/2}$.*

Proof. Assume that $b_{\mu,j}$ is the jth positive zero of $b_\mu(z)$ given by (24) and (25). Considering the normalization given in (7), we write

$$\ell_\mu(-z) = 2^\mu \Gamma(\mu+1) b_\mu\left(\sqrt{-z}\right)(-z)^{1-(\mu/2)}. \tag{51}$$

Since

$$\begin{aligned}\ell_\mu(z) &= 2^\mu \Gamma(\mu+1) b_\mu\left(\sqrt{z}\right)(z)^{1-(\mu/2)} \\ &= \sum_{j\geq 0} \frac{(-1)^j \Gamma(1+\mu)}{4^j j! \Gamma(\mu+j+1)} z^{1+j} \\ &= z \prod_{j\geq 1}\left(1 - \frac{z}{b^2_{\mu,j}}\right),\end{aligned} \tag{52}$$

so we can write

$$\operatorname{Re} \frac{z\ell'_\mu(z)}{\ell_\mu(z)} = 1 - \frac{\mu}{2} + \operatorname{Re} \frac{1}{2\sqrt{z}} \frac{zb'_\mu\left(\sqrt{z}\right)}{b_\mu\left(\sqrt{z}\right)} = 1 - \operatorname{Re} \sum_{j\geq 1} \frac{z}{b^2_{\mu,j} - z}. \tag{53}$$

This result shows that

$$\begin{aligned}\operatorname{Re} \frac{z\ell'_\mu(-z)}{\ell_\mu(-z)} &= -1 + \frac{\mu}{2} - \frac{1}{2\sqrt{-z}} \frac{-zb'_\mu\left(\sqrt{-z}\right)}{b_\mu\left(\sqrt{-z}\right)} \\ &= -1 + \sum_{j\geq 1} \frac{-z}{b^2_{\mu,j} + z},\end{aligned} \tag{54}$$

or

$$\begin{aligned}\operatorname{Re} \frac{z\ell'_\mu(-z)}{\ell_\mu(-z)} &= -1 - \operatorname{Re} \sum_{j\geq 1} \frac{z}{b^2_{\mu,j} + z} \\ &\leq -1 + \sum_{j\geq 1} \frac{|z|}{b^2_{\mu,j} - |z|} \\ &= -\frac{|z|\ell'_\mu(|z|)}{\ell_\mu(|z|)} \\ &= -\operatorname{Re} \frac{z\ell'_\mu(z)}{\ell_\mu(z)}.\end{aligned} \tag{55}$$

The equality holds for $r = |z| = z$. The principle of minimum value for harmonic functions along with (7) shows that

$$\operatorname{Re} \frac{z\ell'_\mu(z)}{\ell_\mu(z)} \geq 1 - \sum_{j\geq 1} \frac{|z|}{b^2_{\mu,j} - |z|} \tag{56}$$

is valid if and only if $|z| < b_{\mu,1}$, and $b_{\mu,1}$ is the minimum positive root of the equation

$$rb'_\mu(r) + (2-\mu) b_\mu(r) = 0. \tag{57}$$

Thus, we have

$$\begin{aligned}\frac{\ell'_\mu(-1)}{\ell_\mu(-1)} &= -1 + \sum_{j\geq 1} \frac{1}{1 - \left((-1)/b^2_{\mu,j}\right)} \frac{1}{b^2_{\mu,j}} \\ &= -1 - \sum_{j\geq 1} \frac{1}{b^2_{\mu,j} + 1} \leq 0,\end{aligned} \tag{58}$$

$$\begin{aligned}\frac{\ell'_\mu(-z)}{\ell_\mu(-z)} &= -\frac{1}{z} - \sum_{j\geq 1} \frac{1}{b^2_{\mu,j} + z} \text{ or } \frac{\partial}{\partial \mu}\left(\frac{\ell'_\mu(-z)}{\ell_\mu(-z)}\right) \\ &= -\sum_{j\geq 1} \frac{2b_{\mu,j}(\partial/\partial\mu) b_{\mu,j}}{\left(z + b^2_{\mu,j}\right)^2},\end{aligned} \tag{59}$$

or

$$\frac{\partial}{\partial \mu}\left(\frac{\ell'_\mu(-1)}{\ell_\mu(-1)}\right) = -\sum_{j\geq 1} \frac{2b_{\mu,j}(\partial/\partial\mu) b_{\mu,j}}{\left(1 + b^2_{\mu,j}\right)^2} \leq 0, \tag{60}$$

since $b'_{\mu,j} > 0$ on $(-1, \infty)$ for a fixed $j \in \mathbb{N}$. Thus, by using (58) and (60), we see that ℓ_μ satisfies (7) and by applying Lemma 1 and Lemma 2, we obtain that $\ell_\mu(-z) \in \mathcal{S}^*$ and decreasing on $(-1, \infty)$ and by considering (8), we can write

$$\begin{aligned}z\ell'_\mu(z) &= 2^\mu \Gamma(\mu+1)\left[\frac{zb'_\mu\left(\sqrt{z}\right)}{2\sqrt{z} z^{-1+(\mu/2)}} + \left(1 - \frac{\mu}{2}\right) \frac{b_\mu\left(\sqrt{z}\right)}{z^{-1+(\mu/2)}}\right] \\ &= \sum_{j\geq 0} \frac{(-1)^j (j+1) \Gamma(1+\mu)}{4^j j! \Gamma(\mu+j+1) z^{-j-1}}.\end{aligned} \tag{61}$$

We also write

$$\ell_\mu(z) - \ell_\mu(-z) = \sum_{j\geq 0}\left(1 - (-1)^{1+j}\right) \frac{(-1)^j \Gamma(1+\mu)}{4^j j! \Gamma(j+\mu+1)} z^{1+j}. \tag{62}$$

We can write

$$\begin{aligned}\frac{2z\ell'_\mu(z)}{\ell_\mu(z) - \ell_\mu(-z)} &= \frac{\sum_{j\geq 0}\left((j+1)(-1)^j \Gamma(1+\mu) z^{j+1}\right)/\left(4^j j! \Gamma(\mu+j+1)\right)}{\sum_{j\geq 0}\left(1 - (-1)^{1+j}\right)\left((-1)^j \Gamma(1+\mu) z^{j+1}\right)/\left(4^j j! \Gamma(1+j+\mu)\right)} \\ &= \frac{2}{1 + (-i)^\mu b_\mu\left(i\sqrt{z}\right)/b_\mu\left(\sqrt{z}\right)} \frac{z\ell'_\mu(z)}{\ell_\mu(z)}.\end{aligned} \tag{63}$$

Since

$$b_\mu\left(\sqrt{-z}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\sqrt{-z}/2\right)^{\mu+2j}}{\Gamma(1+j)\Gamma(1+\mu+j)} = \sum_{j=0}^{\infty} \frac{(i)^\mu (-1)^{2j}}{\Gamma(j+1)\Gamma(1+\mu+j)} \left(\frac{\sqrt{z}}{2}\right)^{\mu+2j}, \quad (64)$$

so we have

$$1 + \frac{b_\mu\left(i\sqrt{z}\right)}{b_\mu\left(\sqrt{z}\right)}(-i)^\mu = \frac{\sum_{j=0}^{\infty}\left((-1)^j\left(1+(-1)^j\right)/\Gamma(j+1)\Gamma(1+\mu+j)\right)\left(\sqrt{z}/2\right)^{\mu+2j}}{\sum_{j=0}^{\infty}\left((-1)^j/\Gamma(1+j)\Gamma(\mu+j+1)\right)\left(\sqrt{z}/2\right)^{\mu+2j}}. \quad (65)$$

From (63) along with (65), we see that

$$\begin{aligned}
\operatorname{Re} \frac{2z\ell_\mu'(z)}{\ell_\mu(z)-\ell_\mu(-z)} &= \operatorname{Re} \frac{z\ell_\mu'(z)}{\ell_\mu(z)}\left[1+\frac{b_\mu\left(i\sqrt{z}\right)}{b_\mu\left(\sqrt{z}\right)}(-i)^\mu\right]^{-1} \\
&= \operatorname{Re} \frac{z\ell_\mu'(z)}{\ell_\mu(z)}\left[1-\frac{b_\mu\left(i\sqrt{z}\right)(-i)^\mu}{b_\mu\left(\sqrt{z}\right)}+\cdots\right] \\
&= \operatorname{Re} \frac{z\ell_\mu'(z)}{\ell_\mu(z)} \frac{\sum_{j=0}^{\infty}\left((-1)^j/\Gamma(j+1)\Gamma(1+\mu+j)\right)\left(\sqrt{z}/2\right)^{\mu+2j}\left(1+(-1)^j\right)}{\sum_{j=0}^{\infty}\left((-1)^j/\Gamma(j+1)\Gamma(1+\mu+j)\right)\left(\sqrt{z}/2\right)^{\mu+2j}} \\
&\geq 1-\sum_{j\geq 1}\frac{2|z|}{b_{\mu,j}^2-|z|} = \frac{|z|\ell_\mu'(|z|)}{\ell_\mu(|z|)}.
\end{aligned} \quad (66)$$

As in [11], we observe that

$$\operatorname{Re} \frac{2z\ell_\mu'(z)}{\ell_\mu(z)-\ell_\mu(-z)} \geq \operatorname{Re} \frac{z\ell_\mu'(z)}{\ell_\mu(z)} \geq 1-\sum_{j\geq 1}\frac{2|z|}{b_{\mu,j}^2-|z|} = \frac{|z|\ell_\mu'(|z|)}{\ell_\mu(|z|)}. \quad (67)$$

For $1>\eta\geq 0$ and $\mu>-1$, $r_\eta^*(\ell_\mu)$ is the least positive zero of

$$zb_\mu'(z) + (2-2\eta-\mu)b_\mu(z) = 0, \quad (68)$$

where $\ell_\mu(z) = 2^\mu \Gamma(\mu+1) b_\mu\left(\sqrt{z}\right)(z)^{1-\mu/2}$. □

4. Conclusion

The class of Bessel functions is originated as a solution of the well-known Bessel differential equation. We studied the radius problems of starlike functions with symmetric points involving Bessel functions under some kind of normalized conditions. We used the Mittag-Leffler representation of Bessel functions and derived our main results.

Data Availability

There is no data available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Uni. Press, Cambridge, 2nd edition, 1944.

[2] R. K. Brown, "Univalent solutions of $W'' + pW = 0$," *Journal Canadien De Mathematiques*, vol. 14, pp. 69–78, 1962.

[3] R. Szász, "On starlikeness of Bessel functions of the first kind," in *Proc. 8th Joint Conf. Math & Comp. Sci.*, Comp. Sci., Komarno, Slovakia, 2010.

[4] R. Szász and P. A. Kupán, "About the univalence of the Bessel functions," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 54, no. 1, pp. 127–132, 2009.

[5] A. Saliu, K. I. Noor, S. Hussain, and M. Darus, "On Bessel functions related with certain classes of analytic functions with respect to symmetrical points," *Journal of Mathematics*, vol. 2021, Article ID 6648710, 9 pages, 2021.

[6] A. Baricz, "Geometric properties of generalized Bessel functions," *Universitatis Debreceniensis*, vol. 73, pp. 155–178, 2008.

[7] A. Baricz and S. Ponnusamy, "Starlikeness and convexity of generalized Bessel functions," *Integral Transforms and Special Functions*, vol. 21, no. 9, pp. 641–653, 2010.

[8] B. A. Frasin, F. Yousef, T. al-Hawary, and I. Aldawish, "Application of generalized Bessel functions to classes of analytic functions," *Afrika Matematika*, vol. 32, no. 3-4, pp. 431–439, 2021.

[9] M. E. H. Ismail and M. E. Muldoon, "Bounds for the small real and purely imaginary zeros of Bessel and related functions," *Methods and Applications of Analysis*, vol. 2, no. 1, pp. 1–21, 1995.

[10] S. M. Shah and S. Y. Trimble, "Entire functions with univalent derivatives," *Journal of Mathematical Analysis and Applications*, vol. 33, no. 1, pp. 220–229, 1971.

[11] A. Baricz, M. Çağlar, and E. Deniz, "Starlikeness of Bessel functions and their derivatives," *Mathematical Inequalities & Applications*, vol. 19, no. 2, pp. 439–449, 2016.

[12] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Math. Functions*, Cambridge Univ. Press, Cambridge, 2010.

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Research Article

On the Chebyshev Polynomial for a Certain Class of Analytic Univalent Functions

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In this work, by considering the Chebyshev polynomial of the first and second kind, a new subclass of univalent functions is defined. We obtain the coefficient estimate, extreme points, and convolution preserving property. Also, we discuss the radii of starlikeness, convexity, and close-to-convexity.

1. Introduction

Let Δ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathscr{A}$ be the class of analytic functions in Δ, satisfying the normalized conditions:

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= 1. \end{aligned} \tag{1}$$

Thus, each $f \in \mathscr{A}$ has the following Taylor expansion:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

Furthermore, by $\mathscr{S}$, we shall denote the family of all functions in $\mathscr{A}$ that are univalent in Δ. Denote by $\mathscr{N}$ the subclass of $\mathscr{A}$ consisting of functions with negative coefficients of the type:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, (a_k \geq 0, z \in \Delta), \tag{3}$$

see [1].

Many researchers deal with orthogonal polynomials of Chebyshev, see [2, 3] and [4]. The Chebyshev polynomials of first kind and the second kind are defined by

$$\begin{aligned} T_k(t) &= \cos k\theta, \\ U_k(t) &= \frac{\sin (k+1)\theta}{\sin \theta}, \end{aligned} \tag{4}$$

respectively, where $-1 < t < 1$, $t = \cos \theta$, and k is the degree of polynomial.

The polynomial in (1) is connected by the following relations:

$$\frac{dT_k(t)}{dt} = kU_{k-1}(t), T_k(t) = U_k(t) - kU_{k-1}(t), \tag{5}$$

$$2T_k(t) = U_k(t) - U_{k-2}(t). \tag{6}$$

We note that if $t = \cos \theta$, $(-\pi/3 < \theta < \pi/3)$, then

$$H(z, t) = \frac{1}{1 - 2z \cos \theta + z2} = 1 + \sum_{k=1}^{\infty} \frac{\sin (k+1)\theta}{\sin \theta} z^k, (z \in \Delta). \tag{7}$$

Also, we have

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots, (z \in \Delta, -1 < t < 1), \quad (8)$$

where

$$Uk - 1(t) = \frac{\sin(k \arccos t)}{\sqrt{1-t^2}}, (k \in \mathbb{N}), \quad (9)$$

are the Chebyshev polynomials of the second kind, see [5, 6] and [7].

The generating function of the first kind of Chebyshev polynomial $T_k(t)$, $t \in [-1,1]$ is given by

$$\sum_{k=0}^{\infty} T_k(t) z^k = \frac{1 - tz}{1 - 2tz + z^2}. \quad (10)$$

For more details, see [8, 9] and [10].

For two functions f and g, analytic in Δ, we say that f is subordinate to g in Δ, written

$$f(z) \prec g(z), (z \in \Delta), \quad (11)$$

if there exists a Schwarz function w which is analytic in Δ with

$$w(0) = 0, \quad |w(z)| < 1, (z \in \Delta), \quad (12)$$

such that $f(z) = g(w(z))$, $(z \in \Delta)$, see [11].

Also, if g is univalent in Δ, then

$$f(z) \prec g(z), (z \in \Delta) \Longleftrightarrow f(0) = g(0), f(\Delta) \subset g(\Delta). \quad (13)$$

Furthermore, if $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$, then the Hadamard product (or covolution) of f and g is defined by

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \quad (14)$$

Now, we consider the following functions which are connected with the Chebyshev polynomial of the first and second kind:

$$\mathscr{C}_1(z) = 1 + (1 + \cos\theta)z - \frac{1 - tz}{1 - 2tz + z2}, \quad (15)$$

$$\mathscr{C}_2(z) = 1 + (2\cos\theta + 1)z - H(z,t), \quad (16)$$

$$Q(z) = [(\mathscr{C}_1 * \mathscr{C}_1) * (\mathscr{C}_2 * \mathscr{C}_2) * f](z), \quad (17)$$

where $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathscr{N}$ and "$*$" denotes the Hadamard product.

With a simple calculation we conclude that $Q(z)$ belongs to $\mathscr{N}$ and it is of the form:

$$Q(z) = z - \sum_{k=2}^{\infty} \left[\frac{\sin(k+1)\theta}{\sin\theta} T_k(t)\right]^2 a_k z^k, \quad (18)$$

where $-\pi/3 < \theta < \pi/3$ and $t = \cos\theta$.

Definition. For $M = \alpha + (\beta - \alpha)(1 - \gamma)$, $-1 \le \beta < \alpha \le 1$, $0 < \gamma < 1$, and $0 \le \lambda \le 1$, we say that $Q(z)$ of the form (18) is a member of $\mathscr{E}_{\gamma}^{\lambda}(\alpha,\beta)$ if the following subordination relation holds

$$\frac{zQ'(z)}{f_\lambda(z)} \prec \frac{1 + Mz}{1 + \alpha z}, \quad (19)$$

where $f_\lambda(z) = (1-\lambda)z + \lambda f(z)$, $f(z) \in \mathscr{N}$.

Equation (19) is equivalent to the following inequality:

$$\left|\frac{\left(zQ'(z)/f_\lambda(z)\right) - 1}{M - \alpha z\left(Q'(z)/f_\lambda(z)\right)}\right| < 1. \quad (20)$$

2. Main Results

In this section, we introduce a sharp coefficient bound for the class $\mathscr{E}_{\gamma}^{\lambda}(\alpha,\beta)$. Also, the convolution preserving property is investigated.

Theorem 1. *The function $Q(z)$ of form (18) belongs to $\mathscr{E}_{\gamma}^{\lambda}(\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty} \left[\left(\left(\sqrt{k}\frac{\sin(k+1)\theta}{\sin\theta} T_k(t)\right)^2 - \lambda\right)(1-\alpha) + \lambda(\beta-\alpha)(1-\gamma)\right] a_k \le (\beta - \alpha)(1-\gamma). \quad (21)$$

Proof. Let the inequality (21) holds and $z \in \partial\Delta = \{z \in \mathbb{C} : |z| = 1\}$. We have to prove that (19) or equivalently (20) holds true. But we have

$$\begin{aligned}
Y &= \left|zQ'(z) - f_\lambda(z)\right| - \left|Mf_\lambda(z) - \alpha zQ'(z)\right| \\
&= \left|z - \sum_{k=2}^{\infty}\left[\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right]^2 ka_kz^k - (1-\lambda)z - \lambda\right. \\
&\quad \cdot\left.\left(z - \sum_{k=2}^{\infty}a_kz^k\right)\right| - \left|M\left((1-\lambda)z + \lambda\left(z - \sum_{k=2}^{\infty}a_kz^k\right)\right)\right. \\
&\quad \left. - \alpha z + \sum_{k=2}^{\infty}\left[\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right]^2 \alpha ka_kz^k\right| \\
&= \left|-\sum_{k=2}^{\infty}\left[\left(\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right)2k - \lambda\right]a_kz^k\right| \\
&\quad - \left|(M-\alpha)z - \sum_{k=2}^{\infty}\left[\lambda M - \alpha k\left(\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right)^2\right]a_kz^k\right|.
\end{aligned} \tag{22}$$

By putting $z \in \partial\Delta$ and

$$\lambda M - \alpha k\left(\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right)^2 = \lambda(M-\alpha) - \left[k\left(\frac{\sin (k+1)\theta}{\sin\theta}T_k(\theta)\right)^2 - \lambda\right]\alpha, \tag{23}$$

the above expression reduces to

$$Y \le \left|\sum_{k=2}^{\infty}\left[\left(k\left(\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right)^2 - \lambda\right)(1-\alpha) + \lambda(M-\alpha)\right]a_k - (M-\alpha)\right|. \tag{24}$$

Since $H - \alpha = (\beta-\alpha)(1-\gamma)$, by using inequality (21), we get $Y \le 0$, so $Q \in \mathscr{E}_\gamma^\lambda(\alpha,\beta)$.

To prove the converse, let $Q \in \mathscr{E}_\gamma^\lambda(\alpha,\beta)$, thus

$$\left|\frac{\left(zQ'(z)/f_\lambda(z)\right) - 1}{M - \alpha z\left(Q'(z)/f_\lambda(z)\right)}\right| = \frac{\left|z - \sum_{k=2}^{\infty}\left(\sqrt{k}(\sin (k+1)\theta/\sin\theta)T_k(t)\right)^2 a_kz^k - (1-\lambda)z + \lambda f(z)\right|}{\left|M((1-\lambda)z + \lambda f(z)) - \alpha z\left(1 - \sum_{k=2}^{\infty}\left(\sqrt{k}(\sin (k+1)\theta/\sin\theta)T_k(t)\right)^2 a_kz^k - 1\right)\right|} < 1, \tag{25}$$

for all $z \in \Delta$. By $\operatorname{Re}(z) \le |z|$ for all $z \in \Delta$, we have

$$\operatorname{Re}\left\{\frac{\sum_{k=2}^{\infty}\left[\left(\sqrt{k}(\sin (k+1)\theta/\sin\theta)T_k(t)\right)^2 - \lambda\right]a_kz^k}{(M-\alpha)z - \sum_{k=2}^{\infty}\left[\lambda M - \alpha\left(\sqrt{k}(\sin (k+1)\theta/\sin\theta)T_k(t)\right)^2\right]a_kz^k}\right\} < 1. \tag{26}$$

By letting $z \longrightarrow 1$, through positive values and choose the values of z such that $zQ'(z)/f_\lambda(z)$ is real, we have

$$\sum_{k=2}^{\infty}\left[\left(\left(\sqrt{k}\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right)^2 - \lambda\right)(1-\alpha) + t(M-\alpha)\right]a_k \le M - \alpha, \tag{27}$$

and this completes the proof. □

Remark. We note that the function:

$$V(z) = z - \frac{(\beta-\alpha)(1-\gamma)}{\left[\left(\sqrt{2}(\sin 3\theta/\sin\theta)\cos 2\theta\right)^2 - \lambda\right](1-\alpha) + \lambda(\beta-\alpha)(1-\gamma)}z^2, \tag{28}$$

shows that the inequality (21) is sharp.

Theorem 2. *Let*

$$Q_1(z) = z - \sum_{k=2}^{\infty}\left[\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right]^2 a_kz^k, \tag{29}$$

$$Q_2(z) = z - \sum_{k=2}^{\infty}\left[\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right]^2 b_kz^k, \tag{30}$$

be in the class $\mathscr{E}_\gamma^\lambda(\alpha,\beta)$, *then* $(Q_1 * Q_2)(z)$ *belongs to* $\mathscr{E}_\gamma^\lambda(\alpha,\tilde{\beta})$, *where*

$$\tilde{\beta} \le \alpha + \frac{(\beta-\alpha)2(1-\gamma)X(1-\alpha)}{(X(1-\alpha) + \lambda(\beta-\alpha)(1-\gamma))^2 - \lambda(1-\gamma)^2(\beta-\alpha)^2}, \tag{31}$$

$$X = \left(\sqrt{k}\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right)^2 - \lambda. \tag{32}$$

Proof. It is sufficient to show that

$$\sum_{k=2}^{\infty}\left[\left(\left(\sqrt{k}\frac{\sin (k+1)\theta}{\sin\theta}T_k(t)\right)^2 - \lambda\right)\left(\frac{1-\alpha}{(\beta-\alpha)(1-\gamma)}\right) + \lambda\right]a_kb_k \le 1. \tag{33}$$

By using the Cauchy-Schwarz inequality, from (21), we obtain

$$\sum_{k=2}^{\infty}\left[\left(\left(\sqrt{k}\frac{\sin{(k+1)\theta}}{\sin\theta}T_k(t)\right)^2-\lambda\right)\left(\frac{1-\alpha}{(\beta-\alpha)(1-\gamma)}\right)+\lambda\right]\sqrt{a_kb_k}\le 1. \tag{34}$$

Here, we find the largest k such that

$$\sum_{k=2}^{\infty}\left[\left(\left(\sqrt{k}\frac{\sin{(k+1)\theta}}{\sin\theta}T_k(t)\right)^2-\lambda\right)\left(\frac{1-\alpha}{(\beta-\alpha)(1-\gamma)}\right)-\lambda\right]\sqrt{a_kb_k}\le 1, \tag{35}$$

or equivalently for $k\ge 2$,

$$\sqrt{a_kb_k}\le\frac{[X(1-\alpha)+\lambda(\beta-\alpha)(1-\gamma)]\left(\tilde{\beta}-\alpha\right)}{[X(1-\alpha)+\lambda(\beta-\alpha)(1-\gamma)](\beta-\alpha)}, \tag{36}$$

where X is given by (32).
This inequality holds if

$$\frac{(\beta-\alpha)(1-\gamma)}{X(1-\alpha)+\lambda(\beta-\alpha)(1-\gamma)}\le\frac{[X(1-\alpha)+\lambda(\beta-\alpha)(1-\gamma)]\left(\tilde{\beta}-\alpha\right)}{\left[X(1-\alpha)+\lambda\left(\tilde{\beta}-\alpha\right)(1-\gamma)\right](\beta-\alpha)}, \tag{37}$$

or equivalently

$$\tilde{\beta}\le\alpha+\frac{(\beta-\alpha)^2(1-\gamma)X(1-\alpha)}{(X(1-\alpha)+\lambda(\beta-\alpha)(1-\gamma))^2-\lambda(1-\gamma)^2(\beta-\alpha)^2}, \tag{38}$$

where X is given by (32), so the proof is complete. □

3. Geometric Properties of $\mathscr{E}_{\gamma}^{\lambda}(\alpha,\beta)$

In this section, we show that the class $\mathscr{E}_{\gamma}^{\lambda}(\alpha,\beta)$ is a convex set. Also, the radii of starlikeness, convexity, and close-to-convexity are obtained.

Theorem 3. *The class $\mathscr{E}_{\gamma}^{\lambda}(\alpha,\beta)$ is a convex set.*

Proof. It is enough to prove that if for $j=1,2,\cdots,m$,

$$Q_j(z)=z-\sum_{k=2}^{\infty}\left[\frac{\sin{(k+1)\theta}}{\sin\theta}\right]^k a_{k,j}z^k, \tag{39}$$

be in $\mathscr{E}_{\gamma}^{\lambda}(\alpha,\beta)$, then the function

$$F(z)=\sum_{j=1}^{m}d_jQ_j(z), \tag{40}$$

is also in $\mathscr{E}_{\gamma}^{\lambda}(\alpha,\beta)$, where $\sum_{j=1}^{m}d_j=1$. But, we have

$$\begin{aligned}F(z)&=z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}\left(\frac{\sin{(k+1)\theta}}{\sin\theta}T_k(t)\right)^2 d_ja_{k,j}\right)z^k,\\&=z-\sum_{k=2}^{\infty}\left(\frac{\sin{(k+1)\theta}}{\sin\theta}T_k(t)\right)^2\left(\sum_{j=1}^{m}d_ja_{k,j}\right)z^k.\end{aligned} \tag{41}$$

Since by Theorem 1,

$$\begin{aligned}&\sum_{k=2}^{\infty}\left[\left(\left(\sqrt{k}\frac{\sin{(k+1)\theta}}{\sin\theta}T_k(t)\right)^2-\lambda\right)(1-\alpha)+\lambda(\beta-\alpha)(1-\gamma)\right]\left(\sum_{j=1}^{m}d_ja_{k,j}\right)\\&=\sum_{j=1}^{m}\left(\sum_{k=2}^{\infty}\left[\left(\left(\sqrt{k}\frac{\sin{(k+1)\theta}}{\sin\theta}T_k(t)\right)^2-\lambda\right)\right.\right.\\&\quad\left.\cdot(1-\alpha)+\lambda(\beta-\alpha)(1-\gamma)]a_{k,j}\right)d_j\\&<\sum_{j=1}^{m}(\beta-\alpha)(1-\gamma)d_j=(\beta-\alpha)(1-\gamma)\left(\sum_{j=1}^{m}d_j\right)\\&=(\beta-\alpha)(1-\gamma),\end{aligned} \tag{42}$$

so, $F(z)\in\mathscr{E}_{\gamma}^{\lambda}(\alpha,\beta)$. Hence, the proof is complete. □

Theorem 4. *Let $f\in\mathscr{E}_{\gamma}^{\lambda}(\alpha,\beta)$, then*

(i) f is a starlike of order θ_1 ($\cos\theta_1<1$) in $|z|<R_1$ where

$$\begin{aligned}R_1=\inf_k&\left\{\left[\left(\left(\sqrt{k}\frac{\sin{(k+1)\theta}}{\sin\theta}T_k(t)\right)^2-\lambda\right)\right.\right.\\&\left.\left.\cdot\left(\frac{1-\alpha}{(\beta-\alpha)(1-\gamma)}\right)+\lambda\right]\left(\frac{1-\theta_1}{k-\theta_1}\right)\right\}^{1/k-1}\end{aligned} \tag{43}$$

(ii) f is convex of order θ_2 ($0\le\theta_2<1$) in $|z|<R_2$ where

$$\begin{aligned}R_2=\inf_k&\left\{\left[\left(\left(\sqrt{k}\frac{\sin{(k+1)\theta}}{\sin\theta}T_k(t)\right)^2-\lambda\right)\right.\right.\\&\left.\left.\cdot\left(\frac{1-\alpha}{(\beta-\alpha)(1-\gamma)}\right)+\lambda\right]\left(\frac{1-\theta_2}{k(k-\theta_2)}\right)\right\}^{1/k-1}\end{aligned} \tag{44}$$

(iii) f is close-to-convex of order θ_3 ($0\theta_3<1$) in $|z|<R_3$, where

$$R_3 = \inf_k \left\{ \left[\left(\left(\sqrt{k} \frac{\sin (k+1)\theta}{\sin \theta} T_k(t) \right)^2 - \lambda \right) \cdot \left(\frac{1-\alpha}{(\beta-\alpha)(1-\gamma)} \right) + \lambda \right] \left(\frac{1-\theta_3}{k} \right) \right\}^{1/k-1} \quad (45)$$

Proof.

(i) For $0 \le \theta_1 < 1$, we need to show that

$$\left| \frac{zf'}{f} - 1 \right| < 1 - \theta_1 \quad (46)$$

In other words, it is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{k=2}^{\infty}(k-1)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} a_k z^{k-1}} \right| \le \frac{\sum_{k=2}^{\infty}(k-1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}} < 1 - \theta_1, \quad (47)$$

$$\sum_{k=2}^{\infty} \left(\frac{k-\theta_1}{1-\theta_1} \right) a_k |z|^{k-1} < 1. \quad (48)$$

By (21), it is easy to see that above inequality holds if

$$|z|^{k-1} \le \left[\left(\left(\sqrt{k} \frac{\sin (k+1)\theta}{\sin \theta} T_k(t) \right)^2 - \lambda \right) \left(\frac{1-\alpha}{(\beta-\alpha)(1-\gamma)} \right) + \lambda \right] \left(\frac{1-\theta_1}{k-\theta_1} \right). \quad (49)$$

This completes the proof of (i).

(ii) Since f is convex if and only if zf' is starlike, we get the required result (ii)

(iii) We must show that $|f'(z) - 1| \le 1 - \theta_3$. But

$$|f'(z) - 1| = \left| \sum_{k=2}^{\infty} k a_k z^{k-1} \right| \le \sum_{k=2}^{\infty} k a_k |z|^{k-1} \quad (50)$$

Thus, $|f'(z) - 1| < 1 - \theta_3$ if $\sum_{k=2}^{\infty}(k/1-\theta_3) a_k |z|^{k-1} \le 1$. But by Theorem 1, the above inequality holds true, if

$$|z|^{k-1} \le \left[\left(\left(\sqrt{k} \frac{\sin (k+1)\theta}{\sin \theta} T_k(t) \right)^2 - \lambda \right) \left(\frac{1-\alpha}{(\beta-\alpha)(1-\gamma)} \right) + \lambda \right] \left(\frac{1-\theta_3}{k} \right). \quad (51)$$

Hence, the proof is complete. □

4. Conclusions

Univalent functions have always been the main interests of many researchers in geometric function theory. Many studies recently related to Chebyshev polynomials revolved around classes of analytic normalized univalent functions. In this particular work, the geometric properties are obtained for functions in more general class using the Chebyshev polynomials associated with a convolution structure. In this paper, when the parameters being complex numbers could be subject to further investigation. Also, by changing the operator and extending, it may be for future studies.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to and approved the final manuscript.

References

[1] P. L. Duren, *Univalent Functions, Grundlehren der mathematischen Wissenschaften 259*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[2] Ş. Altinkaya and S. Yalçin, "On the Chebyshev polynomial bounds for classes of univalent functions," *Khayyam Journal of Mathematics*, vol. 2, no. 1, pp. 1–5, 2016.

[3] Ş. Altinkaya and S. Yalçin, "On the Chebyshev coefficients for a general subclass of univalent functions," *Turkish Journal of Mathematics*, vol. 42, no. 6, pp. 2885–2890, 2018.

[4] J. Mason, "Chebyshev polynomial approximations for the L-membrane eigenvalue problem," *SIAM Journal on Applied Mathematics*, vol. 15, no. 1, pp. 172–186, 1967.

[5] S. Bulut and N. Magesh, "On the sharp bounds for a comprehensive class of analytic and univalent functions by means of Chebyshev polynomials," *Khayyam Journal of Mathematics*, vol. 2, no. 2, pp. 194–200, 2016.

[6] S. Bulut, N. Magesh, and C. Abiram, "A comprehensive class of analytic bi-univalent functions by means of Chebyshev polynomials," *Journal of Fractional Calculus and Applications*, vol. 8, no. 2, pp. 32–39, 2017.

[7] S. Bulut, N. Magesh, and V. K. Balaji, "Initial bounds for analytic and biunivalent functions by means of Chebyshev polynomials," *Journal of Classical Analysis*, vol. 11, no. 1, pp. 83–89, 2012.

[8] E. Doha, "The first and second kind Chebyshev coefficients of the moments for the general order derivative on an infinitely differentiable function," *International Journal of Computer Mathematics*, vol. 51, no. 1-2, pp. 21–35, 1994.

[9] J. Dziok, R. K. Raina, and J. Sokół, "Application des polynomes de Chebyshev a des classes de fonctions analytiques," *Comptes Rendus Mathematique*, vol. 353, no. 5, pp. 433–438, 2015.

[10] F. M. Saker and E. Dogan, "On initial chebyshev polynomial coefficient problem for certain subclass of bi-univalent functions," *Communications in Mathematics and Applications*, vol. 11, no. 1, pp. 57–64, 2020.

[11] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, CRC Press, 2000.

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Research Article

A New Parametric Differential Operator of p-Valently Analytic Functions

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Newly, numerous investigations are considered utilizing the idea of parametric operators (integral and differential). The objective of this effort is to formulate a new 2D-parameter differential operator (PDO) of a class of multivalent functions in the open unit disk. Consequently, we formulate the suggested operator in some interesting classes of analytic functions to study its geometric properties. The recognized class contains some recent works.

1. Introduction

In analysis, a parametric differential operator (PDO) is a differential operator of a dependent variable with respect to another dependent variable that is engaged when both variables formulate on an independent third variable, typically supposed as "time." We shall use this idea to consider the PDO of a complex variable to discuss its properties in the opinion of the geometric function theory (GFT). The field of differential operators is investigated in GFT early by the well-known Salagean differential operator and the Ruscheweyh derivative. Later, these operators are generalized by different types of parameters using a 1D-parameter fractional differential operator [1] and 2D-parameter fractional differential operator [2]. Recently, using the class of normalized functions $\psi \in \Sigma$

$$\psi(\zeta) = \zeta + \sum_{n=2}^{\infty} \psi_n \zeta^n, \zeta \in \Delta \coloneqq \{\zeta \in \mathbb{C} : |\zeta| < 1\}. \quad (1)$$

Ibrahim and Jay [3] presented PDO of the following form: for $\alpha \in [0, 1]$

$$\mathscr{P}^0 \psi(\zeta) = \psi(\zeta),$$
$$\mathscr{P}^\alpha \psi(\zeta) = \frac{\rho_1(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\psi(\zeta) + \frac{\rho_0(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\left(\zeta\psi'(\zeta)\right). \quad (2)$$

The functions $\rho_1, \rho_0 : [0,1] \times \Delta \longrightarrow \Delta$ are analytic in Δ satisfying $\rho_1(\alpha,\zeta) \neq -\rho_0(\alpha,\zeta)$.

$$\lim_{\alpha\longrightarrow 0}\rho_1(\alpha,\zeta) = 1, \lim_{\alpha\longrightarrow 1}\rho_1(\alpha,\zeta) = 0, \rho_1(\alpha,\zeta) \neq 0, \forall\zeta \in \Delta, \alpha \in (0,1), \quad (3)$$

$$\lim_{\alpha\longrightarrow 0}\rho_0(\alpha,\zeta) = 0, \lim_{\alpha\longrightarrow 1}\rho_0(\alpha,\zeta) = 1, \rho_0(\alpha,\zeta) \neq 0, \forall\zeta \in \Delta, \alpha \in (0,1). \quad (4)$$

More studies are given by Ibrahim and Baleanu [4, 5] using (2) to present a hybrid diff-integral operator and a quantum hybrid operator, respectively.

In this effort, we generalize (2) by considering another class of analytic functions denoting by $\Sigma_\wp$ and constructing by

$$\psi(\zeta) = \zeta^\wp + \sum_{n=\wp+1}^{\infty} \psi_n \zeta^n, \wp \in \mathbb{N}, \tag{5}$$

which are analytic in Δ. Recently, different investigations are presented studying the geometric behavior of this class (see [6–9]).

The Hadamard product [10, 11] for two functions in $\Sigma_\wp$ is given by the series

$$\begin{aligned}(\psi * \varphi)(\zeta) &= \left(\zeta^\wp + \sum_{n=\wp+1}^{\infty} \psi_n \zeta^n\right) * \left(\zeta^\wp + \sum_{n=\wp+1}^{\infty} \varphi_n \zeta^n\right) \\ &= \zeta^\wp + \sum_{n=\wp+1}^{\infty} (\psi_n \varphi_n) \zeta^n \in \Sigma_\wp.\end{aligned} \tag{6}$$

Definition 1. For a function $\psi \in \Sigma_\wp$, PDO is defined as follows:

$$\begin{aligned}&\mathcal{Q}^0\psi(\zeta) = \psi(\zeta)\\ &\mathcal{Q}^\alpha\psi(\zeta) = \left(\frac{\rho_1(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)\psi(\zeta) + \left(\frac{\rho_0(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)\left(\frac{\zeta}{\wp}\right)\psi'(\zeta)\\ &\quad = \zeta^\wp + \sum_{n=\wp+1}^{\infty} \psi_n \left(\frac{\rho_1(\alpha,\zeta)+(n/\wp)\rho_0(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)\zeta^n,\\ &\mathcal{Q}^{2\alpha}\psi(\zeta) = \mathcal{Q}(\mathcal{Q}^\alpha\psi(\zeta)) = \zeta^\wp + \sum_{n=\wp+1}^{\infty} \psi_n \left(\frac{\rho_1(\alpha,\zeta)+(n/\wp)\rho_0(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)^2\zeta^n, \vdots\\ &\mathcal{Q}^{m\alpha}\psi(\zeta) = \mathcal{Q}^\alpha\left[\mathcal{Q}^{(m-1)\alpha}\psi(\zeta)\right] = \zeta^\wp + \sum_{n=\wp+1}^{\infty} \psi_n \left(\frac{\rho_1(\alpha,\zeta)+(n/\wp)\rho_0(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)^m\zeta^n,\\ &\mathcal{Q}^{m\alpha}\psi(\zeta) = \zeta^\wp + \sum_{n=\wp+1}^{\infty} \psi_n \Lambda_n^m \zeta^n,\\ &(\zeta \in \Delta, \wp \in \mathbb{N}, \alpha \in [0,1], m \in \mathbb{N}),\end{aligned} \tag{7}$$

where

$$\Lambda_n = \frac{\rho_1(\alpha,\zeta)+(n/\wp)\rho_0(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}. \tag{8}$$

ρ_1 and ρ_0 are defined in (3) and (4), respectively.

Remark 2.

(i) It is clear that $\mathcal{Q}^{m\alpha}\psi(\zeta) \in \Sigma_\wp$, and it is a generalization of (2) ($\wp = 1$)

(ii) The integral operator that corresponds to $\mathcal{Q}^{m\alpha}\psi(\zeta)$ is

$$\mathcal{L}^{m\alpha}\psi(\zeta) = \zeta^\wp + \sum_{n=\wp+1}^{\infty} \frac{\psi_n}{\Lambda_n^m}\zeta^n. (\zeta \in \Delta, \wp \in \mathbb{N}, \alpha \in [0,1]), \tag{9}$$

where

$$(\mathcal{Q}^{m\alpha} * \mathcal{L}^{m\alpha})\psi(\zeta) = (\mathcal{L}^{m\alpha} * \mathcal{Q}^{m\alpha})\psi(\zeta) = \psi(\zeta). \tag{10}$$

Moreover, we have the following property:

Proposition 3 (semigroup property). *Consider the PDO; then for ψ and $\varphi \in \Sigma_\wp$*

$$\mathcal{Q}^{m\alpha}[a\,\psi(\zeta) + b\,\varphi(\zeta)] = a\,\mathcal{Q}^{m\alpha}\psi(\zeta) + b\mathcal{Q}^{m\alpha}\varphi(\zeta), a, b \in \mathbb{R}. \tag{11}$$

Proof. Let $m = 1$; the definition of $\mathcal{Q}^{m\nu}$ implies

$$\begin{aligned}\mathcal{Q}^\alpha[a\,\psi(\zeta)+b\,\varphi(\zeta)] &= \left(\frac{\rho_1(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)[a\,\psi(\zeta)+b\,\varphi(\zeta)]\\ &\quad + \left(\frac{\rho_0(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)\left(\frac{\zeta}{\wp}\right)[a\,\psi(\zeta)+b\,\varphi(\zeta)]'\\ &= a\left(\left(\frac{\rho_1(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)\psi(\zeta)\right.\\ &\quad \left.+ \left(\frac{\rho_0(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)\left(\frac{\zeta}{\wp}\right)\psi'(\zeta)\right)\\ &\quad + b\left(\left(\frac{\rho_1(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)\varphi(\zeta)\right.\\ &\quad \left.+ \left(\frac{\rho_0(\alpha,\zeta)}{\rho_1(\alpha,\zeta)+\rho_0(\alpha,\zeta)}\right)\left(\frac{\zeta}{\wp}\right)\varphi'(\zeta)\right)\\ &= a\mathcal{Q}^\alpha\psi(\zeta) + b\mathcal{Q}^\alpha\varphi(\zeta).\end{aligned} \tag{12}$$

Hence, for all m, we have the desired assertion. □

Our study is about the following class:

Definition 4. A function $\psi \in \Sigma_\wp$ is called in the class $\Sigma_\wp^\alpha(\sigma, p)$ if it satisfies the inequality

$$\frac{(1-\sigma)}{\zeta^\wp}[\mathcal{Q}^{m\alpha}\psi(\zeta)] + \left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)[\mathcal{Q}^{m\alpha}\psi(\zeta)]' \prec p(\zeta) = \frac{\mu\zeta+1}{\nu\zeta+1}, \tag{13}$$

$$(\zeta \in \Delta, \alpha, \sigma \in [0,1], -1 \le \nu < \mu \le 1, \wp \in \mathbb{N}), \tag{14}$$

where the symbol $\prec$ presents the subordination symbol [12] and p is convex univalent in Δ.

For example

$$p(\zeta) = \frac{\mu\zeta+1}{\nu\zeta+1} = Y_{\mu,\nu}(\zeta), \tag{15}$$

which is univalent convex in Δ, and it is the extreme function in the set

$$\mathcal{P} := \left\{p \in \Delta : p(\zeta) = 1 + \sum_{n=1}^{\infty} p_i \zeta^n\right\}. \tag{16}$$

Define a functional $\Psi : \Delta \longrightarrow \Delta$, as follows:

$$\Psi(\zeta) := \frac{(1-\sigma)}{\zeta^{\wp}} \left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right] + \left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right) \left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]', \quad (17)$$

$$= 1 + \sum_{n=\wp+1}^{\infty} \Psi_{n+\wp}\psi_n \zeta^{n-\wp}, \zeta \in \Delta, \quad (18)$$

where

$$\Psi_{n+\wp} = \left(1 + \frac{n-1}{\wp}\sigma\right) \Lambda_n^m. \quad (19)$$

Shortly, by Definition 4 we say

$$\Psi(\zeta) \prec Y_{\mu,\nu}(\zeta) := \frac{\mu\zeta + 1}{\nu\zeta + 1}, \zeta \in \Delta. \quad (20)$$

Our aim is to study the operator formula Ψ. We recall the following results:

Lemma 5 (see [12]). *Let two analytic functions $f(\zeta)$ and $g(\zeta)$ be convex univalent defined in Δ such that $f(0) = g(0)$. Moreover, for a constant $c \neq 0, \mathfrak{R}(c) \geq 0$, the subordination*

$$f(\zeta) + (1/c) f'(\zeta) \prec g(\zeta) \quad (21)$$

implies

$$f(\zeta) \prec g(\zeta). \quad (22)$$

Lemma 6 (see [12]). *Define the general class of holomorphic functions*

$$\mathbb{H}[a, n] = \left\{h : h(\zeta) = a + a_n\zeta^n + a_{n+1}\zeta^{n+1} + \cdots\right\}, \quad (23)$$

where $a \in \mathbb{C}$ and n is a positive integer. If $c \in \mathbb{R}$, then

$$\mathfrak{R}\left\{h(\zeta) + c\zeta h'(\zeta)\right\} > 0 \Rightarrow \mathfrak{R}(h(\zeta)) > 0. \quad (24)$$

Moreover, if $c > 0$ and $h \in \mathbb{H}[1, n]$, then there are fixed numbers $\ell_1 > 0$ and $\ell_2 > 0$ with the inequality

$$h(\zeta) + c\zeta h'(\zeta) \prec \left(\frac{1+\zeta}{1-\zeta}\right)^{\ell_1},$$
$$h(\zeta) \prec \left(\frac{1+\zeta}{1-\zeta}\right)^{\ell_2}. \quad (25)$$

Lemma 7 (see [13]). *Let $\hbar, p \in \mathbb{H}[a, n]$, where p is convex univalent in Δ and for $\Bbbk_1, \Bbbk_2 \in \mathbb{C}, \Bbbk_2 \neq 0$; then*

$$\Bbbk_1\hbar(\zeta) + \Bbbk_2\zeta\hbar'(\zeta) \prec \Bbbk_1 p(\zeta) + \Bbbk_2\zeta p'(\zeta) \longrightarrow \hbar(\zeta) \prec p(\zeta). \quad (26)$$

Lemma 8 (see [14]). *Let $h, p \in \mathbb{H}[a, n]$, where p is convex univalent in Δ such that $h(\zeta) + \Bbbk\zeta h'(\zeta)$ is univalent; then*

$$p(\zeta) + \Bbbk\zeta p'(\zeta) \prec h(\zeta) + \Bbbk\zeta h'(\zeta) \longrightarrow p(\zeta) \prec h(\zeta). \quad (27)$$

Lemma 9 (see [15]). *Let $\hbar, y, g \in \mathbb{H}[a, n]$, and g is convex univalent in Δ such that $\hbar \prec g$ and $y \prec g$; then*

$$\Bbbk\hbar + (1-\Bbbk)y \prec g, \Bbbk \in [0, 1]. \quad (28)$$

2. The Results

In this section, we illustrate our main results concerning the class $\Sigma_{\wp}^{\alpha}(\sigma, p)$ for some special $p(\zeta), \zeta \in \Delta$.

2.1. General Properties

Theorem 10. *Suppose that $\psi \in \Sigma_{\wp}^{\alpha}(\sigma, p)$. If $\mathfrak{R}\{\Psi(\zeta)\} > 0$, then the coefficient bounds of Ψ satisfy the inequality*

$$\frac{|\Psi_n|}{2} \leq \int_0^{2\pi} |e^{-in\theta}|\, d\mathfrak{M}(\theta), \quad (29)$$

where $d\mathfrak{M}$ is a probability measure. Also, if

$$\mathfrak{R}\left(e^{i\chi}\Psi(\zeta)\right) > 0, \zeta \in \Delta, \chi \in \mathbb{R}, \quad (30)$$

then $\psi \in \Sigma_{\wp}^{\alpha}(\sigma, (\nu\zeta + 1)/(\nu\zeta + 1))$, that is

$$\Psi(\zeta) \approx \frac{\mu\zeta + 1}{\nu\zeta + 1}, \zeta \in \Delta. \quad (31)$$

Proof. By the assumption, we have

$$\mathfrak{R}(\Psi(\zeta)) = \mathfrak{R}\left(1 + \sum_{n=\wp+1}^{\infty} \Psi_n\zeta^n\right) > 0. \quad (32)$$

Thus, the Carathéodory positivist technique yields

$$|\psi_n| \leq 2\int_0^{2\pi} |e^{-in\theta}| d\mathfrak{M}(\theta), \quad (33)$$

where $d\mathfrak{M}$ is a probability measure. In addition, if

$$\mathfrak{R}\left(e^{i\chi}\Psi(\zeta)\right) > 0, \zeta \in \Delta, \chi \in \mathbb{R}, \quad (34)$$

then according to Theorem 1.6 in [10] and for fixed $\chi \in \mathbb{R}$, we have

$$\Psi(\zeta) \approx p(\zeta) = \frac{\mu\zeta + 1}{\nu\zeta + 1}, \zeta \in \Delta. \quad (35)$$

Hence, $\psi \in \Sigma_{\wp}^{\alpha}(\sigma, (\nu\zeta + 1)/(\nu\zeta + 1))$.

The next results show the sufficient and necessary conditions for the sandwich behavior of the functional $\Psi(\zeta) = (1 - \sigma/\zeta^{\wp}) \left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right] + (\sigma/\wp\zeta^{\wp-1}) \left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]'$. □

Theorem 11. *Let the following assumptions hold*

$$\frac{\sigma\zeta[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]'' + (-2\sigma\wp+2\sigma+\wp)[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]' + (\sigma-1)(\wp-1)\wp[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]}{\wp\zeta^{\wp-1}} \prec p_2(\zeta) + \zeta p_2{}'(\zeta), \tag{36}$$

where $p_2(0) = 1$ and convex in Δ. Moreover, let $\Psi(\zeta)$ be univalent in Δ such that $\Psi \in \mathbb{H}[p_1(0), 1] \cap \mathbb{Q}$, where $\mathbb{Q}$ represents the set of all injection analytic functions f with $\lim_{\zeta\in\partial\Delta} f \neq \infty$ and

$$p_1(\zeta) + \zeta\, p_1{}'(\zeta) \prec \frac{\sigma\zeta[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]'' + (-2\sigma\wp+2\sigma+\wp)[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]'\big) + (\sigma-1)(\wp-1)\wp[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]}{\wp\zeta^{\wp-1}}. \tag{37}$$

Then

$$p_1(\zeta) \prec \Psi(\zeta) \prec p_2(\zeta), \tag{38}$$

and $p_1(\zeta)$ is the best subdominant, and $p_2(\zeta)$ is the best dominant.

Proof. Since

$$\Psi(\zeta) + \zeta\Psi'(\zeta) = \frac{\sigma\zeta[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]'' + (-2\sigma\wp+2\sigma+\wp)[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]'\big) + (\sigma-1)(\wp-1)\wp[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]}{\wp\zeta^{\wp-1}}, \tag{39}$$

then we obtain the next double inequality

$$p_1(\zeta) + \zeta p_1{}'(\zeta) \prec \Psi(\zeta) + \zeta\Psi'(\zeta) \prec p_2(\zeta) + \zeta p_2{}'(\zeta). \tag{40}$$

Thus, Lemmas 7 and 8 imply the desired assertion. □

Theorem 12. *Let p be a univalent convex function in Δ such that $p(0) = 0$ and*

$$[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] \prec p(\zeta), [\mathcal{L}^{m\,\alpha}\psi(\zeta)] \prec p(\zeta). \tag{41}$$

Then

$$[\mathcal{A}^{m\,\alpha}\psi(\zeta)] \coloneqq \Bbbk[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] + (1-\Bbbk)[\mathcal{L}^{m\,\alpha}\psi(\zeta)] \prec p(\zeta), \Bbbk \in [0, 1]. \tag{42}$$

Proof. By the definition of $[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]$ and $[\mathcal{L}^{m\,\alpha}\psi(\zeta)]$, clearly we have $[\mathcal{A}^{m\,\alpha}\psi(\zeta)] \in \Sigma_\wp$. Hence, a direct application of Lemma 9, we obtain the result. □

2.2. Inclusion Properties. In this part, we deal with the inclusion properties.

Theorem 13. *For $\sigma_2 \le \sigma_1 < 0$ and $\psi \in \Sigma_\wp$, then*

$$\Sigma^\alpha_\wp(\sigma_2, p) \subset \Sigma^\alpha_\wp(\sigma_1, p). \tag{43}$$

Proof. Let $\psi \in \Sigma^\alpha_\wp(\sigma_2, p)$. Define the analytic function in Δ, as follows:

$$\phi(\zeta) = \zeta^{-\wp}[\mathcal{Q}^{m\,\alpha}\psi(\zeta)], \tag{44}$$

satisfying $\phi(0) = 1$. A computation gives

$$\frac{(1-\sigma_2)}{\zeta^\wp}[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] + \left(\frac{\sigma_2}{\wp\zeta^{\wp-1}}\right)[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]' = \phi(\zeta) + \frac{\sigma_2}{\wp}\left(\zeta\phi'(\zeta)\right). \tag{45}$$

Consequently, we get the inequality

$$\phi(\zeta) + \frac{\sigma_2}{\wp}\left(\zeta\phi'(\zeta)\right) \prec \frac{\mu\zeta+1}{\nu\zeta+1}. \tag{46}$$

Applying Lemma 5 with $\sigma_2/\wp > 0$ gives

$$\phi(\zeta) \prec \frac{\mu\zeta+1}{\nu\zeta+1}. \tag{47}$$

Since $0 < \sigma_1/\sigma_2 < 1$ and $Y_{\mu,\nu}(\zeta)$ is convex univalent in Δ, we arrive at the inequality

$$\begin{aligned}
&\frac{(1-\sigma_1)}{\zeta^{\wp}}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]+\left(\frac{\sigma_1}{\wp\zeta^{\wp-1}}\right)\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]' \\
&= (1-\sigma_1)\phi(\zeta)+\left(\frac{\sigma_1}{\wp\zeta^{\wp-1}}\right)\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]' \\
&= (1-\sigma_1)\phi(\zeta)+\frac{\sigma_1}{\wp}\left(\zeta\phi'(\zeta)+\wp\phi(\zeta)\right), \\
&= (1-\sigma_1)\phi(\zeta)+\frac{\sigma_1}{\wp}\left(\zeta\phi'(\zeta)+\wp\phi(\zeta)\right)+\left(\frac{\sigma_1}{\sigma_2}\phi(\zeta)-\frac{\sigma_1}{\sigma_2}\phi(\zeta)\right) \\
&= \frac{\sigma_1}{\sigma_2}(1-\sigma_2)\phi(\zeta)+\frac{\sigma_2}{\wp}\left(\zeta\phi'(\zeta)+\wp\phi(\zeta)\right)+\left(1-\frac{\sigma_1}{\sigma_2}\right)\phi(\zeta) \\
&= \frac{\sigma_1}{\sigma_2}\left[\frac{(1-\sigma_2)}{\zeta^{\wp}}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]+\frac{\sigma_2}{\wp\zeta^{\wp-1}}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]'\right] \\
&\quad+\left(1-\frac{\sigma_1}{\sigma_2}\right)\phi(\zeta)\prec\frac{\mu\zeta+1}{\nu\zeta+1}=Y_{\mu,\nu}(\zeta).
\end{aligned} \tag{48}$$

Hence, by Definition 4, we conclude that $\psi\in\Sigma_{\wp}^{\alpha}(\sigma_1,p)$. □

Theorem 14. *Let*

$$\Psi(\zeta)=\frac{(1-\sigma)}{\zeta^{\wp}}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]'. \tag{49}$$

Then

$$\begin{aligned}
&\frac{\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]'}{\zeta^{\wp}}\hbar_1+\frac{\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]}{\zeta^{\wp-1}}\left[\hbar_1+(1+\wp)\hbar_2+\hbar_2\right] \\
&\quad+\hbar_2\zeta^{2-\wp}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]''\prec\left(\frac{1+\zeta}{1-\zeta}\right)^{\ell_1}\Rightarrow\Psi(\zeta)\prec\left(\frac{1+\zeta}{1-\zeta}\right)^{\ell_2},
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
&\ell_1>0,\ell_2>0, \\
&\hbar_1=1-\sigma, \\
&\hbar_2=\frac{\sigma}{\wp}, \\
&\wp>0.
\end{aligned} \tag{51}$$

Proof. A calculation implies that

$$\begin{aligned}
\Psi(\zeta)+\zeta\Psi'(\zeta)&=\frac{(1-\sigma)}{\zeta^{\wp}}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]' \\
&\quad+\zeta\left(\frac{(1-\sigma)}{\zeta^{\wp}}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]'\right)' \\
&=\frac{\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]'}{\zeta^{\wp}}\hbar_1+\frac{\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]}{\zeta^{\wp-1}}\left[\hbar_1+(1+\wp)\hbar_2+\hbar_2\right] \\
&\quad+\hbar_2\zeta^{2-\wp}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]''\prec\left(\frac{1+\zeta}{1-\zeta}\right)^{\ell_1}.
\end{aligned} \tag{52}$$

According to Lemma 6 joining the value $c=1$, we get

$$\Psi(\zeta)\prec\left(\frac{1+\zeta}{1-\zeta}\right)^{\ell_2}. \tag{53}$$

□

Corollary 15. *Let $\Psi(\zeta)$ be assumed as in Theorem14. If the subordination*

$$\begin{aligned}
&\frac{\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]'}{\zeta^{\wp}}\hbar_1+\frac{\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]}{\zeta^{\wp-1}}\left[\hbar_1+(1+\wp)\hbar_2+\hbar_2\right] \\
&\quad+\hbar_2\zeta^{2-\wp}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]''\prec\left(\frac{1+\zeta}{1-\zeta}\right),
\end{aligned} \tag{54}$$

where $\ell_1>0,\ell_2>0,\hbar_1=1-\sigma,\hbar_2=\sigma/\wp,\wp>0$ holds, then $\psi\in\Sigma_{\wp}^{\alpha}(\sigma,(1+\zeta)/(1-\zeta))$.

Proof. Taking, $\ell_1=\ell_2=1$ in Theorem 14 implies that $\Psi(\zeta)\prec(1+\zeta)/(1-\zeta)$. Consequently, we have $\psi\in\Sigma_{\wp}^{\alpha}(\sigma,(1+\zeta)/(1-\zeta))$. □

Theorem 16. *Let $\psi\in\Sigma_{\wp}^{\alpha}(\sigma,p)$ and $f\in\Sigma_{\wp}$. If*

$$\Re\left(\frac{\mathcal{Q}^{m\,\alpha}\psi(\zeta)}{\zeta^{\wp}}\right)>\frac{1}{2}, \tag{55}$$

then $\psi\times f\in\Sigma_{\wp}^{\alpha}(\sigma,p)$.

Proof. A convolution product indicates that

$$\begin{aligned}
&\frac{(1-\sigma)}{\zeta^{\wp}}\left[\mathcal{Q}^{m\,\alpha}(\psi(\zeta)\times f(\zeta))\right]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)\left[\mathcal{Q}^{m\,\alpha}(\psi(\zeta)\times f(\zeta))\right]' \\
&=(1-\sigma)\left[\frac{\mathcal{Q}^{m\,\alpha}(\psi(\zeta)}{\zeta^{\wp}}\times\frac{\mathcal{Q}^{m\,\alpha}f(\zeta)}{\zeta^{\wp}}\right)\Bigg] \\
&\quad+\frac{\sigma}{\wp}\left(\frac{\left[\mathcal{Q}^{m\,\alpha}f(\zeta)\right)\right]'}{\zeta^{\wp-1}}\times\frac{\mathcal{Q}^{m\,\alpha}f(\zeta)}{\zeta^{\wp}}\right) \\
&=\left[(1-\sigma)\left[\frac{\mathcal{Q}^{m\,\alpha}(\psi(\zeta)}{\zeta^{\wp}}\right]+\frac{\sigma}{\wp\zeta^{\wp-1}}\left[\mathcal{Q}^{m\,\alpha}f(\zeta)\right)\right]'\right]\times\frac{\mathcal{Q}^{m\,\alpha}f(\zeta)}{\zeta^{\wp}} \\
&=\Psi(\zeta)\times\frac{\mathcal{Q}^{m\,\alpha}f(\zeta)}{\zeta^{\wp}},
\end{aligned}\tag{56}$$

where $\Psi(\zeta)\prec Y_{\mu,\nu}(\zeta)$. In view of real inequality (55), we get that $(\mathcal{Q}^{m\,\alpha}f(\zeta)/\zeta^{\wp})$ has the Herglotz integral formula [11].

$$\frac{\mathcal{Q}^{m\,\alpha}f(\zeta)}{\zeta^{\wp}}=\int_{|\tau|=1}\frac{d\varsigma(\tau)}{1-\tau\zeta},\tag{57}$$

where $d\varsigma$ conforms the probability measure on the unit circle $|\tau|=1$ and

$$\int_{|\tau|=1}d\varsigma(\tau)=1.\tag{58}$$

But, $Y_{\mu,\nu}(\zeta)$ is convex in Δ; then we have

$$\begin{aligned}
&\frac{(1-\sigma)}{\zeta^{\wp}}\left[\mathcal{Q}^{m\,\alpha}(\psi(\zeta)*f(\zeta))\right]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)\left[\mathcal{Q}^{m\,\alpha}(\psi(\zeta)*f(\zeta))\right]' \\
&=\Theta(\zeta)*\frac{\mathcal{Q}^{m\,\alpha}f(\zeta)}{\zeta^{\wp}} \\
&=\int_{|\tau|=1}\Theta(\tau\zeta)d\varsigma(\tau)\prec Y_{\mu,\nu}(\zeta).
\end{aligned}\tag{59}$$

Thus, $\psi\times f\in\Sigma_{\wp}^{\alpha}(\sigma,p)$. □

2.3. Fekete-Szegö Inequality. In this section, we obtain the Fekete-Szegö relation coefficient estimates for the class $\Sigma_{\wp}^{\alpha}(\sigma,p)$. Let Ω be the class of functions of the form

$$\varpi(\zeta)=1+\varpi_1\zeta+\varpi_2\zeta^2+\cdots,\tag{60}$$

in the open unit disk Δ satisfying $|\varpi(z)|<1$. To prove our results, we need the following lemma.

Lemma 17 (see *[16]*). *If $\varpi\in\Omega$, then for any complex number ρ*

$$|\varpi_2-\rho\varpi_1^2|\le\max\{1,|\rho|\}.\tag{61}$$

The result is sharp for the functions given by $\varpi(\zeta)=\zeta$ or $\varpi(\zeta)=\zeta^2$.

Theorem 18. *Let the function ψ be formulated by ((5)). Then, $\psi\in\Sigma_{\wp}^{\alpha}(\sigma,p)$ and*

$$\left|\psi_{\wp+2}-\rho\psi_{\wp+1}^2\right|\le\left(\frac{(\mu-\nu)\wp}{[\wp+(\wp+1)\sigma]\Lambda_{\wp+2}^m}\right)\max\{1,|\aleph|\},\tag{62}$$

where

$$\aleph=\left(\nu+\frac{\rho(\mu-\nu)[\wp+(\wp+1)\sigma]\Lambda_{\wp+2}^m}{\wp(1+\sigma)^2\Lambda^2 m_{\wp+1}}\right).\tag{63}$$

Proof. Since $\psi\in\Sigma_{\wp}^{\alpha}(\sigma,p)$, we have

$$\frac{(1-\sigma)}{\zeta^{\wp}}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]'\prec p(\zeta)=\frac{\mu\zeta+1}{\nu\zeta+1}.\tag{64}$$

In addition, there is a Schwarz function $\varpi(\zeta)=1+\varpi_1\zeta+\varpi_2\zeta^2+\cdots$ in Ω such that

$$\begin{aligned}
&\frac{(1-\sigma)}{\zeta^{\wp}}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]'\prec p(w(\zeta))p(w(\zeta)) \\
&=\frac{1+\mu w(\zeta)}{1+\nu w(\zeta)} \\
&=1+(\mu-\nu)\varpi_1\zeta+\left[(\mu-\nu)\varpi_2-\nu(\mu-\nu)\varpi_1^2\right]\zeta^2+\cdots.
\end{aligned}\tag{65}$$

Now by (18), we have

$$\begin{aligned}
&\frac{(1-\sigma)}{\zeta^{\wp}}\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)\left[\mathcal{Q}^{m\,\alpha}\psi(\zeta)\right]' \\
&=1+\sum_{n=\wp+1}^{\infty}\left(1+\frac{n-1}{\wp}\sigma\right)\Lambda_n^m\psi_n\zeta^{n-\wp},\zeta\in\Delta,
\end{aligned}\tag{66}$$

where Λ_n^m is given by (19). Equating the coefficients of ζ and ζ^2, we get

$$(1+\sigma)\Lambda_{\wp+1}^m\psi_{\wp+1}=(\mu-\nu)\varpi_1,\tag{67}$$

$$\left(1+\frac{\wp+1}{\wp}\sigma\right)\Lambda_{\wp+2}^m\psi_{\wp+2}=(\mu-\nu)\varpi_2-\nu(\mu-\nu)\varpi_1^2,\tag{68}$$

$$\left(\frac{\wp+(\wp+1)\sigma}{\wp}\right)\Lambda_{\wp+2}^m\psi_{\wp+2}=(\mu-\nu)\varpi_2-\nu(\mu-\nu)\varpi_1^2.\tag{69}$$

From (67) and (69), we get

$$\psi_{\wp+1} = \frac{(\mu-\nu)\varpi_1}{(1+\sigma)\Lambda^m_{\wp+1}},$$
$$\psi_{\wp+2} = \frac{(\mu-\nu)\wp}{[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}}\varpi_2 - \frac{\nu(\mu-\nu)\wp}{[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}}\varpi_1^2. \tag{70}$$

For any $\rho \in \mathbb{C}$, we get

$$\begin{aligned}\psi_{\wp+2}-\rho\psi^2_{\wp+1} &= \frac{(\mu-\nu)\wp}{[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}}\varpi_2 - \frac{\nu(\mu-\nu)\wp}{[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}}\varpi_1^2 \\ &- \left(\frac{(\mu-\nu)\varpi_1}{(1+\sigma)\Lambda^m_{\wp+1}}\right)^2 = \frac{(\mu-\nu)\wp}{[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}} \\ &\cdot \left[\varpi_2 - \left(\nu + \frac{\rho(\mu-\nu)[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}}{\wp(1+\sigma)^2\Lambda^{2m}_{\wp+1}}\right)\varpi_1^2\right] \\ &= \frac{(\mu-\nu)\wp}{[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}}\left(\varpi_2 - \aleph\varpi_1^2\right),\end{aligned} \tag{71}$$

where

$$\aleph = \left(\nu + \frac{\rho(\mu-\nu)[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}}{\wp(1+\sigma)^2\Lambda 2m_{\wp+1}}\right). \tag{72}$$

By applying Lemma 17, we get

$$\left|\psi_{\wp+2}-\rho\psi^2_{\wp+1}\right| \le \left(\frac{(\mu-\nu)\wp}{[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}}\right)\max\{1,|\aleph|\}. \tag{73}$$

The result is sharp for the function

$$\frac{(1-\sigma)}{\zeta^\wp}[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]' = p\left(\zeta^2\right) = \frac{\mu\zeta^2+1}{\nu\zeta^2+1}, \tag{74}$$

or

$$\frac{(1-\sigma)}{\zeta^\wp}[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]' = p(\zeta) = \frac{\mu\zeta+1}{\nu\zeta+1}. \tag{75}$$

□

Remark 19. By fixing $\rho = 1$ in Theorem18, we get

$$\left|\psi_{\wp+2}-\psi^2_{\wp+1}\right| \le \left(\frac{(\mu-\nu)\wp}{[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}}\right)\max\{1,|\aleph|\}, \tag{76}$$

where

$$\aleph = \left(\nu + \frac{(\mu-\nu)[\wp+(\wp+1)\sigma]\Lambda^m_{\wp+2}}{\wp(1+\sigma)^2\Lambda 2m_{\wp+1}}\right). \tag{77}$$

From Definition 4, a function $\psi \in \Sigma_\wp$ is said to be in the class $\Sigma^\alpha_\wp(\sigma,p)$ if it satisfies the inequality (13); then we have

$$\left|\frac{\Psi(\zeta)-1}{\mu-\nu\Psi(\zeta)}\right| < 1, \tag{78}$$

where

$$\Psi(\zeta) = \frac{(1-\sigma)}{\zeta^\wp}[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]+\left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]' \tag{79}$$

is as given in (17).

Now, we obtain coefficient estimates for $f \in \Sigma^\alpha_\wp(\sigma,p)$.

Theorem 20. *Let the function ψ be defined by ((5)). Then, $\psi \in \Sigma^\alpha_\wp(\sigma,p)$ if*

$$\sum_{n=p+1}^{\infty}\Psi_{n,\wp}[1+\nu]\psi_n \le |\mu-\nu|, \tag{80}$$

$$\Psi_{n,\wp} = \left(1+\frac{n-1}{\wp}\sigma\right)\Lambda^m_n, \tag{81}$$

where Λ^m_n is given by (8).

Proof. Suppose ψ satisfies (80). Then, for $|\zeta| = r < 1$

$$\begin{aligned}|\Psi(\zeta)-1| - |\mu-\nu\Psi(\zeta)| &= \left|\sum_{n=p+1}^{\infty}\Psi_{n,\wp}\psi_n\zeta^{n-p}\right| - \beta|(\mu-\nu) \\ &+ \nu\sum_{n=p+1}^{\infty}\Psi_{n,\wp}\,|\,\psi_n\zeta^{n-p}| \\ &\le \sum_{n=p+1}^{\infty}\Psi_{n,\wp}|\psi_n| - |\mu-\nu| \\ &+ \sum_{n=p+1}^{\infty}\Psi_{n,\wp}\nu|\psi_n| \\ &= \sum_{n=p+1}^{\infty}\Psi_{n,\wp}[1+\nu]|\psi_n| - |\mu-\nu| \le 0.\end{aligned} \tag{82}$$

□

3. An Application

In this section, we consider the suggested class $\Sigma^\alpha_\wp(\sigma,(1+\zeta)/(1-\zeta))$ for all $\alpha \in [0,1]$.

Theorem 21. *Consider the class of analytic functions $\Sigma_{\wp}^{\alpha}(\sigma, (1+\zeta)/(1-\zeta))$. Then, the solution of the differential equation corresponds to this class is*

$$[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] = c_1\zeta^{\frac{(\sigma-1)\wp}{\sigma}} + \zeta^{\wp}\left(\frac{2\wp\zeta_2F_1(1,(\sigma+\wp)/\sigma,\wp/\sigma+2,\zeta))+1}{(\sigma+\wp)}\right), \quad (83)$$

where ${}_2F_1(a,b,c;\zeta)$ represents the hypergeometric function.

Proof. Suppose that $\psi \in \Sigma_{\wp}^{\alpha}(\sigma,(1+\zeta)/(1-\zeta))$. Then, it satisfies the differential equation

$$\frac{(1-\sigma)}{\zeta^{\wp}}[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] + \left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]' = \frac{\omega(\zeta)+1}{1-\omega(\zeta)}, \quad (84)$$

where $\omega(0)=0$ and $|\omega|<1$. This leads to the solution

$$[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] = \zeta^{((\sigma-1)\wp)/\sigma}\int_0^{\zeta} -\wp z^{\wp/\sigma-1}\left(\frac{\omega(z)+1}{\sigma(\omega(z)-1)}\right)dz. \quad (85)$$

To find the upper solution, we let $\omega(\zeta)=\zeta$. Thus, we have the differential equation

$$\frac{(1-\sigma)}{\zeta^{\wp}}[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] + \left(\frac{\sigma}{\wp\zeta^{\wp-1}}\right)[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]' = \frac{\zeta+1}{1-\zeta}. \quad (86)$$

Rewrite the above equation as follows:

$$[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]' + \frac{\wp(1-\sigma)}{\sigma\zeta}[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] = \left(\frac{\wp\zeta^{\wp-1}}{\sigma}\right)\left(\frac{1+\zeta}{1-\zeta}\right). \quad (87)$$

Multiplying the above equation by the functional

$$T(\zeta) = \exp\left(\int \frac{\wp(\sigma+\zeta-\sigma\zeta-1)}{\sigma\zeta(\zeta-1)}d\zeta\right), \quad (88)$$

we obtain

$$\zeta^{\wp(1/\sigma-1)}[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]' - \frac{[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]\left(\wp\zeta^{\wp(1/\sigma-1)-1(\sigma+\zeta-\sigma\zeta-1)}\right)}{\sigma(1-\zeta)} = \left(\frac{\wp\zeta^{\wp/\sigma-1}}{\sigma}\right)\left(\frac{1+\zeta}{1-\zeta}\right). \quad (89)$$

Hence, it follows the solution (26). □

Example 1. For

(i) $\wp=1, \sigma=0.5, c_1=0$, the solution is

$$[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] = -\zeta - \frac{4(\zeta+\log(1-\zeta))}{\zeta} \quad (90)$$

(ii) $\wp=1, \sigma=0.25, c_1=0$; the solution becomes

$$[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] = -\zeta - \frac{4\left(2\zeta^3+3\zeta^2+6\zeta+6\log(1-\zeta)\right)}{3\zeta^3} \quad (91)$$

(iii) $\wp=2, \sigma=0.5, c_1=0$; then the solution is given by the formula

$$[\mathcal{Q}^{m\,\alpha}\psi(\zeta)] = -\zeta^2 - \frac{4\left(2\zeta^3+3\zeta^2+6\zeta+6\log(1-\zeta)\right)}{3\zeta^2}. \quad (92)$$

4. Conclusion

Commencing overhead, we formulated a new parametric differential operator for a certain class of multivalently analytic functions. We investigated some geometric conducts of the operator connecting with the Janowski function, which is convex univalent in the open unit disk. As an application, we presented the formula of the suggested class involving the operator. For future works, one can generalize the suggested fractional operator using various classes of analytic functions such as meromorphic and harmonic functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally and significantly to writing this article. All authors read and agreed to the published version of the manuscript.

References

[1] S. Eker and S. Owa, "Certain classes of analytic functions involving Salagean operator," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 1, pp. 12–22, 2009.

[2] R. W. Ibrahim, "On generalized Srivastava-Owa fractional operators in the unit disk," *Advances in Difference Equations*, vol. 2011, no. 1, 10 pages, 2011.

[3] R. W. Ibrahim and J. M. Jahangiri, "Conformable differential operator generalizes the Briot-Bouquet differential equation in a complex domain," *AIMS Mathematics*, vol. 4, no. 6, pp. 1582–1595, 2019.

[4] R. W. Ibrahim and D. Baleanu, "On a combination of fractional differential and integral operators associated with a class of normalized functions," *AIMS Mathematics*, vol. 6, no. 4, pp. 4211–4226, 2021.

[5] R. W. Ibrahim and D. Baleanu, "On quantum hybrid fractional conformable differential and integral operators in a complex domain," *Revista de la Real Academia de Ciencias Exactas, Fìsicas y Naturales. Serie A. Matemáticas*, vol. 115, no. 1, pp. 1–13, 2021.

[6] B. Ahmad, M. G. Khan, M. K. Aouf, W. K. Mashwani, Z. Salleh, and H. Tang, "Applications of a new-difference operator in Janowski-type meromorphic convex functions," *Journal of Function Spaces*, vol. 2021, 10 pages, 2021.

[7] H.-Y. Zhang and H. Tang, "Fourth Toeplitz determinants for starlike functions defined by using the sine function," *Journal of Function Spaces*, vol. 2021, 7 pages, 2021.

[8] N. M. Alarifi and R. W. Ibrahim, "A new class of analytic normalized functions structured by a fractional differential operator," *Journal of Function Spaces*, vol. 2021, 9 pages, 2021.

[9] M. K. Aouf, A. O. Mostafa, and S. M. Madian, "Certain classes of meromorphic p-valent functions associated with Mittag-Leffler function," *Proceedings of the Pakistan Academy of Sciences: A. Physical and Computational Sciences*, vol. 58, no. 1, pp. 97–106, 2021.

[10] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, Les Presses De L'Universite De Montreal, Montreal, 1982.

[11] P. L. Duren, "Univalent functions," in *Grundlehren der Mathematischen Wissenschaften*, vol. 259, Springer-Verlag, 1983, ISBN 0-387-90795-5.

[12] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, CRC Press, 2000.

[13] T. N. Shanmugam, S. Sivasubramanian, and H. M. Srivastava, "Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations," *Integral Transforms and Special Functions*, vol. 17, no. 12, pp. 889–899, 2006.

[14] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations," *Complex Variables*, vol. 48, no. 10, pp. 815–826, 2003.

[15] G. Dong and M.-S. Liu, "On certain subclass of Bazilevic functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 8, no. 1, pp. 1–11, 2007.

[16] F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proceedings of American Mathematical Society*, vol. 20, no. 1, pp. 8–12, 1969.

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Research Article

A Comprehensive Family of Biunivalent Functions Defined by k-Fibonacci Numbers

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By using k-Fibonacci numbers, we present a comprehensive family of regular and biunivalent functions of the type $g(z)=z+\sum_{j=2}^{\infty} d_j z^j$ in the open unit disc $\mathfrak{D}$. We estimate the upper bounds on initial coefficients and also the functional of Fekete-Szegö for functions in this family. We also discuss few interesting observations and provide relevant connections of the result investigated.

1. Introduction and Notations

Let $\mathbb{C}$ be the set of all complex numbers and the disc $\{z \in \mathbb{C} : |z| < 1\}$ be symbolized by $\mathfrak{D}$. Let $\mathbb{N} = \mathbb{N}_0 \setminus \{0\} := \{1, 2, 3, \cdots\}$ and $\mathbb{R}$ be the collection of all real numbers. We denote the set of all normalized regular functions in $\mathfrak{D}$ that have the series of the form

$$g(z) = z + \sum_{j=2}^{\infty} d_j z^j, \quad (1)$$

by $\mathscr{A}$ and the symbol $\mathscr{S}$ stands for set of all functions of $\mathscr{A}$ that are univalent (or Schlicht) in $\mathfrak{D}$. As per the popular Koebe theorem (see [1]), every function $g \in \mathscr{S}$ has an inverse function given by

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + \left(2d_2^2 - d_3\right)\omega^3 - \left(5d_2^3 - 5d_2 d_3 + d_4\right)\omega^4 + \cdots, \quad (2)$$

such that $z = g^{-1}(g(z))$ and $\omega = g(g^{-1}(\omega)), |\omega| < r_0(g)$, $r_0(g) \geq 1/4, z, \omega \in \mathfrak{D}$.

A function g of $\mathscr{A}$ is called biunivalent (or bi-Schlicht) in $\mathfrak{D}$ if both g and g^{-1} are univalent (or Schlicht) in $\mathfrak{D}$. Let Σ stands for the set of biunivalent (or bi-Schlicht) functions having the form (1). Historically, investigations of the family Σ begun five decades ago by Lewin [2] and Brannan and Clunie [3]. Later, Tan [4] found some coefficient estimates for biunivalent functions. In 1986 [5], Brannan and Taha introduced certain well-known subfamilies of Σ in $\mathfrak{D}$. Many interesting results related to initial bounds for some special families of Σ have appeared in [6–8].

In 2007, the concept of k-Fibonacci number sequence $\{F_{k,j}\}_{j=0}^{\infty}$, $k \in \mathbb{R}^+$ was examined by Falcón and Plaza [9] and is given by

$$\begin{aligned} F_{k,0} &= 0, \\ F_{k,1} &= 1, \\ F_{k,j+1} &= kF_{k,j} + F_{k,j-1}, \end{aligned} \quad (3)$$

where $j \in \mathbb{N}$ and

$$F_{k,j} = \frac{(k-t_k)^j - t_k^j}{\sqrt{k^2+4}} \text{ with } t_k = \frac{k-\sqrt{k^2+4}}{2}. \quad (4)$$

$F_{1,j} = F_j$ is the well-known Fibonacci number sequence. Özgür and Sokól in 2015 [10] proved that if

$$\tilde{p}_k(z) = \frac{1+t_k^2 z^2}{1-kt_k z - t_k^2 z^2}, \quad (5)$$

then,

$$\begin{aligned}\tilde{p}_k(z) &= 1 + (F_{k,0}+F_{k,2})t_k z + (F_{k,1}+F_{k,3})t_k^2 z^2 + \cdots \\ &= 1 + k\,t_k z + (k^2+2)t_k^2 z^2 + \cdots,\end{aligned} \quad (6)$$

where t_k is as in (4) and $z \in \mathfrak{D}$. Further, if $\tilde{p}_k(z) = 1 + \Sigma \tilde{p}_{k,j} z^j$, then, we have

$$\tilde{p}_{k,j} = t_k^j \left(F_{k,j-1} + F_{k,j+1}\right), \quad j \in \mathbb{N}. \quad (7)$$

Fibonacci polynomials, Pell-Lucas polynomials, Gegenbauer polynomials, Chebyshev polynomials, Horadam polynomials, Fermat-Lucas polynomials, and generalizations of them are potentially important in many branches such as architecture, physics, combinatorics, number theory, statistics, and engineering. Additional information is associated with these polynomials one can go through [11–13]. More details about the very popular functional of Fekete-Szegö for biunivalent functions based on k-Fibonacci numbers can be found in [14–20].

The recent research trends are the outcomes of the study of functions in Σ based on any one of the above-mentioned polynomials, which can be seen in the recent papers [21–28]. Generally, interest was shown to estimate the first two coefficient bounds and the functional of Fekete-Szegö for some subfamilies of Σ.

For functions g and f regular in $\mathfrak{D}$, g is said to subordinate f, if there is a Schwarz function ψ in $\mathfrak{D}$, such that $\psi(0) = 0$, $|\psi(z)| < 1$, and $g(z) = f(\psi(z))$, $z \in \mathfrak{D}$. This subordination is indicated as $g \prec f$. In particular, if $f \in \mathcal{S}$, then $g(z) \prec f(z) \Leftarrow g(0) = f(0)$ and $g(\mathfrak{D}) \subset f(\mathfrak{D})$.

Inspired by the recent articles and the new trends on functions in Σ, we present a comprehensive family of Σ defined by using k-Fibonacci numbers as given by (3) with $F_{k,j}$ as in (4).

Throughout this paper, $g^{-1}(\omega) = f(\omega)$ is as in (2), $T_k = k - (k^2+2)t_k$, t_k is as in (4), and $\tilde{p}_k$ is as in (5).

Definition 1. A function $g \in \Sigma$ having the power series (1) is said to be in the family $SR\mathfrak{S}_\Sigma^\tau(\gamma, \mu, \tilde{p}_k)$, if

$$\begin{aligned}&\frac{z\left(g'(z)\right)^\tau + \mu z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} \prec \tilde{p}_k(z), \quad z \in \mathfrak{D} \\ &\frac{\omega\left(g'(\omega)\right)^\tau + \mu z^2 g''(\omega)}{\gamma g(\omega) + (1-\gamma)\omega} \prec \tilde{p}_k(\omega), \quad \omega \in \mathfrak{D},\end{aligned} \quad (8)$$

where $\tau \geq 1$, $0 \leq \gamma \leq 1$, and $\mu \geq 0$.

Remark 2. The function families $SR\mathfrak{S}_\Sigma^\tau(\gamma, 0, \tilde{p}_k)$ and $SR\mathfrak{S}_\Sigma^1(\gamma, \mu, \tilde{p}_k)$ were investigated by Frasin et al. [29].

It is interesting to note that (i) $\gamma = 1$, (ii) $\gamma = 0$, and (iii) $\mu = 1$ lead the family $SR\mathfrak{S}_\Sigma^\tau(\gamma, \mu, \tilde{p}_k)$ to various subfamilies, as illustrated in the following:

(1) $SR\mathfrak{S}_\Sigma^\tau(1, \mu, \tilde{p}_k) \equiv L_\Sigma^\tau(\mu, \tilde{p}_k)$ is the family of functions $g \in \Sigma$ satisfying

$$\begin{aligned}&\frac{z\left(g'(z)\right)^\tau}{g(z)} + \mu\left(\frac{z^2 g''(z)}{g(z)}\right) \prec \tilde{p}_k(z) \text{ and } \frac{\omega\left(f'(\omega)\right)^\tau}{f(\omega)} \\ &\quad + \mu\left(\frac{\omega^2 f''(\omega)}{f(\omega)}\right) \prec \tilde{p}_k(\omega), \quad z, \omega \in \mathfrak{D}\end{aligned} \quad (9)$$

(2) $SR\mathfrak{S}_\Sigma^\tau(0, \mu, \tilde{p}_k) \equiv K_\Sigma^\tau(\mu, \tilde{p}_k)$ is the set of functions $g \in \Sigma$ satisfying

$$\begin{aligned}&\left(g'(z)^\tau + \mu z g''(z) \prec \tilde{p}_k(z) \text{ and } \left(f'(\omega)\right)^\tau \\ &\quad + \mu\omega f''(\omega) \prec \tilde{p}_k(\omega), \quad z, \omega \in \mathfrak{D}\end{aligned} \quad (10)$$

(3) $SR\mathfrak{S}_\Sigma^\tau(\gamma, 1, \tilde{p}_k) \equiv M_\Sigma^\tau(\gamma, \tilde{p}_k)$ is the collection of functions $g \in \Sigma$ satisfying

$$\begin{aligned}&\frac{z\left(g'(z)\right)^\tau + z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} \prec \tilde{p}_k(z), \quad z \in \mathfrak{D}, \\ &\frac{\omega\left(f'(\omega)\right)^\tau + \omega^2 f''(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} \prec \tilde{p}_k(\omega), \quad \omega \in \mathfrak{D}\end{aligned} \quad (11)$$

Remark 3. We note that (i) $L_\Sigma^\tau(1, \tilde{p}_k) \equiv M_\Sigma^\tau(1, \tilde{p}_k)$ and (ii) $K_\Sigma^\tau(1, \tilde{p}_k) \equiv M_\Sigma^\tau(0, \tilde{p}_k)$.

Remark 4.

(i) When $\tau = 1$, the family $K^1_\Sigma(\mu, \tilde{p}_k)$ was introduced by Frasin et al. [30]

(ii) The family $L^1_\Sigma(0, \tilde{p}_k) \equiv \mathcal{S}^*_\Sigma(\tilde{p}_k))$ was mentioned by Güney et al. [18], when $\mu = 0$ and $\tau = 1$

(iii) For $\mu = 0$ and $k = 1$, the class $L^\tau_\Sigma(0, \tilde{p}_1) \equiv \mathcal{S}_\Sigma(\tilde{p}_1)$ was investigated by Magesh et al. [31]

We now state the following lemma, which we will be using in the proof of our theorem.

Lemma 5 (see [32]). *If $p \in P$, where P is the collection of regular functions p in $\mathfrak{D}$, satisfying $\Re(p(z)) > 0, z \in \mathfrak{D}$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots, z \in \mathfrak{D}$, then $|p_i| \le 2$, for each i.*

In the next section, we derive the estimates for $|d_2|$,$|d_3|$ and obtain the Fekete-Szegö [33] inequalities for functions in the class $SR\mathfrak{S}^\tau_\Sigma(\gamma, \mu, \tilde{p}_k)$.

2. Coefficient Bounds and Fekete-Szegö Functional

In this section, we offer to get the upper bounds on initial coefficients and find the functional of Fekete-Szegö for functions $\in SR\mathfrak{S}^\tau_\Sigma(\gamma, \mu, \tilde{p}_k)$.

Theorem 6. *Let $\tau \ge 1$, $0 \le \gamma \le 1$, and $\mu \ge 0$. If $g(z)$ of the form (1) in the family $SR\mathfrak{S}^\tau_\Sigma(\gamma, \mu, \tilde{p}_k)$, then*

$$|d_2| \le \frac{k\sqrt{k}\,|t_k|}{\sqrt{|(\gamma^2 + (\tau-\gamma)(2\tau+1) + 2\mu(3-\gamma))k^2 t_k + (2(\mu+\tau)-\gamma)^2 T_k|}}, \tag{12}$$

$$|d_3| \le \frac{k|t_k|}{3(2\mu+\tau)-\gamma} + \frac{k^3 t_k^2}{|(\gamma^2 + (\tau-\gamma)(2\tau+1) + 2\mu(3-\gamma))k^2 t_k + (2(\mu+\tau)-\gamma)^2 T_k|}, \tag{13}$$

and for $\delta \in \mathbb{R}$,

$$\left|d_3 - \delta d_2^2\right| \le \begin{cases} \dfrac{k\,|\,t_k\,|}{3(2\mu+\tau)-\gamma} & ;|1-\delta| \le J, \\ \dfrac{k^3 t_k^2\,|1-\delta|}{\left|(\gamma^2 + (\tau-\gamma)(2\tau+1) + 2\mu(3-\gamma))k^2 t_k + (2(\mu+\tau)-\gamma)^2 T_k\right|} & ;|1-\delta| \ge J, \end{cases} \tag{14}$$

where

$$J = \frac{1}{3(2\mu+\tau)-\gamma} \cdot \left[\gamma^2 + (\tau-\gamma)(2\tau+1) + 2\mu(3-\gamma) + (2(\mu+\tau)-\gamma)^2 \left|\frac{T_k}{k^2 t_k}\right|\right]. \tag{15}$$

Proof. Let the function $g \in SR\mathfrak{S}^\tau_\Sigma(\gamma, \mu, \tilde{p}_k)$. Then, from Definition 1, we have

$$\frac{z\left(g'(z)\right)^\tau + \mu z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} \prec \tilde{p}_k(u(z)), \quad z \in \mathfrak{D}, \tag{16}$$

$$\frac{\omega\left(f'(\omega)\right)^\tau + \mu\omega^2 f''(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} \prec \tilde{p}_k(v(\omega)), \quad \omega \in \mathfrak{D}. \tag{17}$$

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and $p \prec \tilde{p}_k$. Then, there exists a regular function u with $|u(z)| < 1$ in $\mathfrak{D}$ and $p(z) = \tilde{p}_k(u(z))$. Therefore, the function $m(z)$ is in the class P, where

$$m(z) = \frac{1+u(z)}{1-u(z)} = 1 + u_1 z + u_2 z^2 + \cdots. \tag{18}$$

So it follows that

$$u(z) = \frac{m(z)-1}{m(z)+1} = \frac{u_1}{2} z + \left(u_2 - \frac{u_1^2}{2}\right)\frac{z^2}{2} + \left(u_3 - u_1 u_2 + \frac{u_1^3}{4}\right)\frac{z^3}{2} + \cdots, \tag{19}$$

$$\begin{aligned} \tilde{p}_k(u(z)) &= 1 + \tilde{p}_{k,1}\left(\frac{u_1 z}{2} + \left(u_2 - \frac{u_1^2}{2}\right)\frac{z^2}{2} + \cdots\right) \\ &\quad + \tilde{p}_{k,2}\left(\frac{u_1 z}{2} + \left(u_2 - \frac{u_1^2}{2}\right)\frac{z^2}{2} + \cdots\right)^2 + \cdots \\ &= 1 + \frac{\tilde{p}_{k,1} u_1 z}{2} + \left(\frac{1}{2}\left(u_2 - \frac{u_1^2}{2}\right)\tilde{p}_{k,1} + \frac{u_1^2}{4}\tilde{p}_{k,2}\right) z^2 + \cdots. \end{aligned} \tag{20}$$

Similarly, it follows that

$$\tilde{p}_k(\nu(\omega)) = 1 + \frac{\tilde{p}_{k,1}\nu_1\omega}{2} + \left(\frac{1}{2}\left(\nu_2 - \frac{\nu_1^2}{2}\right)\tilde{p}_{k,1} + \frac{\nu_1^2}{4}\tilde{p}_{k,2}\right)\omega^2 + \cdots, \tag{21}$$

where ν is a regular function such that $|\nu(\omega)| < 1$ in $\mathfrak{D}$ such that $p(\omega) = \tilde{p}_k(\nu(\omega))$ and the function $l(\omega)$ is in the class P, where

$$l(\omega) = \frac{1 + \nu(\omega)}{1 - \nu(\omega)} = 1 + \nu_1\omega + \nu_2\omega^2 + \cdots. \tag{22}$$

By virtue of (14), (15), (18), and (19), we obtain

$$(2(\mu + \tau) - \gamma)\, d_2 = \frac{u_1 k\, t_k}{2}, \tag{23}$$

$$(3(2\mu + \tau) - \gamma)d_3 + \left(\gamma^2 - 2\gamma(\mu + \tau) + 2\tau(\tau - 1)\right)d_2^2 = \frac{1}{2}\left(u_2 - \frac{u_1^2}{2}\right)kt_k + \frac{u_1^2}{4}\left(k^2 + 2\right)t_k^2, \tag{24}$$

$$-(2(\mu + \tau) - \gamma)d_2 = \frac{\nu_1 k\, t_k}{2}, \tag{25}$$

$$(3(2\mu + \tau) - \gamma)\left(2d_2^2 - d_3\right) + \left(\gamma^2 - 2\gamma(\mu + \tau) + 2\tau(\tau - 1)\right)d_2^2 = \frac{1}{2}\left(\nu_2 - \frac{\nu_1^2}{2}\right)k\, t_k + \frac{\nu_1^2}{4}\left(k^2 + 2\right)t_k^2. \tag{26}$$

From (21) and (23), we get

$$u_1 = -\nu_1, \tag{27}$$

and also,

$$2(2(\mu + \tau) - \gamma)^2 d_2^2 = \frac{\left(u_1^2 + \nu_1^2\right)k^2\, t_k^2}{4}. \tag{28}$$

If we add (26) and (24), then we obtain

$$2\left(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma)\right)d_2^2 = \frac{1}{2}(u_2 + \nu_2)k\, t_k - \frac{1}{4}\left(k\, t_k - \left(k^2 + 2\right)t_k^2\right)\left(u_1^2 + \nu_1^2\right). \tag{29}$$

Substituting the value of $(u_1^2 + \nu_1^2)$ from (26) in (27), we get

$$d_2^2 = \frac{k^3\, t_k^2(u_2 + \nu_2)}{4\left[\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma))k^2\, t_k + (2(\mu + \tau) - \gamma)^2 T_k\right]}, \tag{30}$$

which gets (10), on using Lemma 5.

On using (25) in the subtraction of (24) from (26), we arrive at

$$d_3 = d_2^2 + \frac{k\, t_k(u_2 - \nu_2)}{4(3(2\mu + \tau) - \gamma)}. \tag{31}$$

Then, in view of Lemma 5 and equation (28), (29) reduces to (11).

From (28) and (29), for $\delta \in \mathbb{R}$, we can easily compute that

$$\left|d_3 - \delta d_2^2\right| = k\,|t_k| \left|\left(T(\delta) + \frac{1}{4(3(2\mu + \tau) - \gamma)}\right)u_2 + \left(T(\delta) - \frac{1}{4(3(2\mu + \tau) - \gamma)}\right)\nu_2\right|, \tag{32}$$

where

$$T(\delta) = \frac{(1 - \delta)k^2 t_k}{4\left[(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma))k^2\, t_k + (2(\mu + \tau) - \gamma)^2 T_k\right]}. \tag{33}$$

In view of (4), we find that

$$\left|d_3 - \delta d_2^2\right| \le \begin{cases} \dfrac{k\,|\,t_k\,|}{(3(2\mu + \tau) - \gamma)} & ;0 \le |T(\delta)| \le \dfrac{1}{4(3(2\mu + \tau) - \gamma)}, \\ 4k\,|t_k|\,|T(\delta)| & ;|T(\delta)| \ge \dfrac{1}{4(3(2\mu + \tau) - \gamma)}, \end{cases} \tag{34}$$

which enable us to conclude (12) with J as in (13). Theorem 6 is proved. □

Remark 7. By taking $\tau = 1$ in the above theorem, we obtain a result of Frasin et al. ([29], Corollary 3.4) and if we let $\mu = 0$ in the above theorem, we get another result of Frasin et al. ([29], Corollary 3.7).

Remark 8. Allowing $k = \gamma = 1$ and $\mu = 0$ in the above theorem, we have Theorem 2.3 of Magesh et al. [31].

Remark 9. Letting $\tau = \gamma = 1$ and $\mu = 0$ in the Theorem 6, we obtain two results of Güney et al. ([18], Corollary 10 and Corollary 23). Further, if we take $k = 1$, we get results of Güney et al. ([17], Corollary 1 and Corollary 4).

In Section 3, few interesting consequences and relevant observations of the main result are mentioned.

3. Outcome of the Main Result

By setting (i) $\gamma = 1$, (ii) $\gamma = 0$, and (iii) $\mu = 1$ in our main theorem, we obtain the following results, respectively.

Corollary 10. *If the function* $g \in L^{\tau}_{\Sigma}(\mu, \tilde{p}_k)$, *then*

$$|d_2| \le \frac{k\sqrt{k}\,|t_k|}{\sqrt{|(\tau(2\tau-1)+4\mu)k^2 t_k + (2(\mu+\tau)-1)^2 T_k|}},$$
$$|d_3| \le \frac{k|t_k|}{3(2\mu+\tau)-1} + \frac{k^3 t_k^2}{|(\tau(2\tau-1)+4\mu)k^2 t_k + (2(\mu+\tau)-1)^2 T_{k.}|}, \tag{35}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \le \begin{cases} \dfrac{k|t_k|}{3(2\mu+\tau)-1} & ;|1-\delta| \le J_1, \\ \dfrac{k^3 t_k^2 \,|\, 1-\delta \,|}{|(\tau(2\tau-1)+4\mu)k^2 t_k + (2(\mu+\tau)-1)^2 T_k\,|} & ;|1-\delta| \ge J_1, \end{cases} \tag{36}$$

where

$$J_1 = \frac{1}{3(2\mu+\tau)-1}\left(\tau(2\tau-1)+4\mu+(2(\mu+\tau)-1)^2\left|\frac{T_k}{k^2 t_k}\right|\right). \tag{37}$$

Remark 11.

(i) By taking $\mu = 0$ and $k = 1$ in the above corollary, we obtain Theorem 2.3 of Magesh et al. [31]

(ii) By allowing $\mu = 0$ and $\tau = 1$ in the above corollary, we get two results Güney et al. ([18], Corollary 10 and Corollary 23)

Corollary 12. *If the function* $g \in K^{\tau}_{\Sigma}(\mu, \tilde{p}_k)$, *then*

$$|d_2| \le \frac{k\sqrt{k}\,|t_k|}{\sqrt{|(\tau(2\tau+1)+6\mu)k^2 t_k + 4(\mu+\tau)^2 T_k|}},$$
$$|d_3| \le \frac{k^3 t_k^2}{|(\tau(2\tau+1)+6\mu)k^2 t_k + 4(\mu+\tau)^2 T_k|} + \frac{k|t_k|}{3(2\mu+\tau)}, \tag{38}$$

and for some $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \le \begin{cases} \dfrac{k|t_k|}{3(2\mu+\tau)} & ;|1-\delta| \le J_2, \\ \dfrac{k^3 t_k^2 |1-\delta|}{|(\tau(2\tau+1)+6\mu)k^2 t_k + 4(\mu+\tau)^2 T_k|} & ;|1-\delta| \ge J_2, \end{cases} \tag{39}$$

where

$$J_2 = \frac{1}{3(2\mu+\tau)}\left[\tau(2\tau+1)+6\mu+4\left(\mu+\tau^2\right)\left|\frac{T_k}{k^2 t_k^2}\right|\right]. \tag{40}$$

Remark 13. For $\tau = 1$, Corollary 12 reduces to a result of Frasin et al. ([30], Corollary 3.6). Further, allowing $k = 1$, we get Corollary 10 of Altnkaya [22].

Corollary 14. *If the function* $g \in M^{\tau}_{\Sigma}(\gamma, \tilde{p}_k)$, *then*

$$|d_2| \le \frac{k\sqrt{k}\,|\,t_k\,|}{\sqrt{|((1-\gamma)^2+(\tau-\gamma)(2\tau+1)+5-\gamma)k^2 t_k + (2(1+\tau)-\gamma)^2 T_k\,|}},$$
$$|d_3| \le \frac{k|t_k|}{3(2+\tau)-\gamma} + \frac{k^3 t_k^2}{|(4\mu+1)k^2 t_k + (2\mu+1)^2 T_k|}, \tag{41}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \le \begin{cases} \dfrac{k\,|\,t_k\,|}{3(2+\tau)-\gamma} & ;|1-\delta| \le J_3, \\ \dfrac{k^3 t_k^2 \,|\, 1-\delta \,|}{|((1-\gamma)^2+(\tau-\gamma)(2\tau+1)+5-\gamma)k^2 t_k + (2(1+\tau)-\gamma)^2 T_k|} & ;|1-\delta| \ge J_3, \end{cases} \tag{42}$$

where

$$J_3 = \frac{1}{3(2+\tau)-\gamma}\left(\left((1-\gamma)^2+(\tau-\gamma)(2\tau+1)+5-\gamma\right)+(2(1+\tau)-\gamma)^2\left|\frac{T_k}{k^2\, t_k}\right|\right). \tag{43}$$

4. Conclusion

A comprehensive family of biunivalent (or bi-Schlicht) functions is introduced by using k-Fibonacci numbers. Bounds of the first two coefficients $|d_2|$ and $|d_3|$ and the celebrated Fekete-Szegö functional have been fixed for this family. Through corollaries of our main result, we have highlighted many interesting new consequences.

A comprehensive family examined in this research paper could inspire further research related to other aspects such as a comprehensive family using q-derivative operator, a meromorphic biunivalent function family associated with Al-Oboudi differential operator, and a comprehensive family using integrodifferential operator.

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors confirm that there are no competing interests regarding the publication of this manuscript.

Authors' Contributions

The authors contributed equally in the preparation of this manuscript and have approved the final version of the manuscript.

References

[1] P. L. Duren, *Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259*, Springer-Verlag, New York, 1983.

[2] M. Lewin, "On a coefficient problem for bi-univalent functions," *Proceedings of the American Mathematical Society*, vol. 18, no. 1, pp. 63–68, 1967.

[3] D. A. Brannan and J. G. Clunie, *Aspects of Contemporary Complex Analysis, Proceedings of the NATO Advanced Study Institute Held at University of Durhary*, Academic press, Newyork, 1979.

[4] D. L. Tan, "Coefficient estimates for bi-univalent functions," *Chinese Annals of Mathematics, Series A*, vol. 5, pp. 559–568, 1984.

[5] D. A. Brannan and T. S. Taha, "On some classes of bi-univalent functions," *Mathematical Analysis and Its Applications. Kuwait, pp 53-60, KFAS Proceedings Series, Vol. 3 (1985)*, S. M. Mazhar, A. Hamoui, and N. S. Faour, Eds., Pergamon Press (Elsevier Science Limited), Oxford, 1988, see also Studia Univ. Babes-Bolyai Math., 31(2) (1986), 70-77.

[6] B. A. Frasin, "Coefficient bounds for certain classes of bi-univalent functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 3, pp. 383–389, 2014.

[7] B. A. Frasin and M. K. Aouf, "New subclasses of bi-univalent functions," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1569–1573, 2011.

[8] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1188–1192, 2010.

[9] S. Falcón and A. Plaza, "On the Fibonacci _k_ -numbers," *Chaos, Solitons & Fractals*, vol. 32, no. 5, pp. 1615–1624, 2007.

[10] N. Y. Özgür and J. Sokół, "On starlike functions connected with k -Fibonacci numbers," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 38, no. 1, pp. 249–258, 2015.

[11] P. Filipponi and A. F. Horadam, "Derivative sequences of Fibonacci and Lucas polynomials," *Applications of Fibonacci Numbers, vol 4 (1990), pp 99-108, Proceedings of the Fourth International Conference on Fibonacci Numbers and their Applications*, , no. 1991, pp. 99–108, Wake Forest University, Winston-Salem, North Carolina; Springer (Kluwer Academic Publishers), Dordrecht, Boston and London.

[12] P. Filipponi and A. F. Horadam, "Second derivative sequence of Fibonacci and Lucas polynomials," *The Fibonacci Quarterly*, vol. 31, pp. 194–204, 1993.

[13] T.-T. Wang and W.-P. Zhang, "Some identities involving Fibonacci, Lucas polynomials and their applications," *Bulletin mathématique de la société des sciences mathématiques de Roumanie*, vol. 55, no. 103, pp. 95–103, 2012.

[14] J. Dziok, R. K. Raina, and J. Sokół, "Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers," *Computers & Mathematics with Applications*, vol. 61, no. 9, pp. 2605–2613, 2011.

[15] J. Dziok, R. K. Raina, and J. Sokól, "On α-convex functions related to shell-like functions connected with Fibonacci numbers," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 996–1002, 2011.

[16] H. Ö. Güney, "Coefficient bounds for analytic bi-Bazilevič functions related to shell-like curves connected with Fibonacci numbers," *Sahand Communications in Mathematical Analysis*, vol. 16, no. 1, pp. 149–160, 2019.

[17] H. Ö. Güney, G. Murugusundaramoorthy, and J. Sokól, "Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers," *Acta Universitatis Sapientiae, Mathematica*, vol. 10, no. 1, pp. 70–84, 2018.

[18] H. Ö. Güney, G. Murugusundaramoorthy, and J. Sokól, "Certain subclasses of bi-univalent functions related to k-Fibonacci numbers," *Communications*, vol. 68, no. 2, pp. 1909–1921, 2019.

[19] R. K. Raina and J. Sokól, "Fekete-Szegö problem for some starlike functions related to shell-like curves," *Mathematica Slovaca*, vol. 66, no. 1, pp. 135–140, 2016.

[20] J. Sokól, R. K. Raina, and N. Yilmaz Özgür, "Applications of Fibonacci numbers for the starlike analytic functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 1, no. 1036, pp. 121–127, 2015.

[21] I. Aldawish, T. Al-Hawary, and B. A. Frasin, "Subclasses of bi-univalent functions defined by Frasin differential operator," *Mathematics*, vol. 8, no. 5, p. 783, 2020.

[22] S. Altnkaya, "Bounds for a new subclass of bi-univalent functions subordinate to the Fibonacci numbers," *Turkish Journal of Mathematics*, vol. 44, no. 2, pp. 553–560, 2020.

[23] A. Amourah, B. A. Frasin, and T. Abdeljawad, "Fekete-Szegö inequality for analytic and biunivalent functions subordinate to Gegenbauer polynomials," *Journal of Function Spaces*, vol. 2021, Article ID 5574673, 7 pages, 2021.

[24] S. M. El-Deeb, T. Bulboača, and B. M. El-Matary, "Maclaurin coefficient estimates of bi-univalent functions connected with the q-derivative," *Mathematics*, vol. 8, no. 3, p. 418, 2020.

[25] H. M. Srivastava, Ş. Altınkaya, and S. Yalçın, "Certain subclasses of bi-univalent functions associated with the Horadam

polynomials," *Iranian journal of science and technology, transactions A: Science*, vol. 43, no. 4, pp. 1873–1879, 2019.

[26] S. R. Swamy, "Coefficient bounds for Al-Oboudi type bi-univalent functions based on a modified sigmoid activation function and Horadam polynomials," *Earthline Journal of Mathematical Sciences*, vol. 7, no. 2, pp. 251–270, 2021.

[27] S. R. Swamy, S. Bulut, and Y. Sailaja, "Some special families of holomorphic and Salagean type bi-univalent functions associated with Horadam polynomials involving modified sigmoid activation function," *Hacettepe Journal of Mathematics and Statistics*, vol. 50, no. 3, pp. 710–720, 2021.

[28] S. R. Swamy, A. K. Wanas, and Y. Sailaja, "Some special families of holomorphic and Sălăgean type bi-univalent functions associated with (m, n)-Lucas polynomials," *Communications in Mathematics and Applications*, vol. 11, no. 4, pp. 1–12, 2020.

[29] B. A. Frasin, S. R. Swamy, and Y. Sailaja, "Coefficient bounds for Al-Oboudi type bi-univalent functions connected with a modified sigmoid activated function and k-Fibonacci numbers (preprint)".

[30] B. A. Frasin, S. R. Swamy, and J. Nirmala, "Some special families of holomorphic and Al-Oboudi type bi-univalent functions related to k-Fibonacci numbers involving modified sigmoid activation function," *Afrika Matematika*, vol. 32, no. 3-4, pp. 631–643, 2021.

[31] N. Magesh, V. K. Balaji, and C. Abirami, "Certain classes of bi-univalent functions related to Shell-like curves connected with Fibonacci numbers," https://arxiv.org/abs/1810.06216.

[32] C. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[33] M. Fekete and G. Szegö, "Eine Bemerkung Über Ungerade Schlichte Funktionen," *Journal of the London Mathematical Society*, vol. s1-8, no. 2, pp. 85–89, 1933.

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Research Article

The Spectrum of Mapping Ideals of Type Variable Exponent Function Space of Complex Variables with Some Applications

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The topological and geometric behaviors of the variable exponent formal power series space, as well as the prequasi-ideal construction by s-numbers and this function space of complex variables, are investigated in this article. Upper bounds for s-numbers of infinite series of the weighted nth power forward and backward shift operator on this function space are being investigated, with applications to some entire functions.

1. Introduction

Operator ideal theory has various applications in the geometry of Banach spaces, xed point theory, spectral theory, and other areas of mathematics, among other areas of knowledge. Throughout the article, we will adhere to the etymological conventions listed below. If any other sources are used, we will make a note of them.

1.1. Conventions 1.1. $\mathbb{N}=\{0,1,2,\cdots\}$.$\mathbb{C}$: complex number space

$\mathbb{R}^{\mathbb{N}}$: the space of all real sequences

ℓ_∞: the space of bounded real sequences

ℓ^r: the space of r-absolutely summable real sequences

c_0: the space of null real sequences

$e_l=(0,0,\cdots,1,0,0,\cdots)$, as 1 lies at the l^{th} coordinate, for all $l\in\mathbb{N}$

$\mathscr{F}$: the space of each sequence with finite nonzero coordinates

card ($\mathfrak{G}$): the number of elements of the set $\mathfrak{G}$

$mi\nearrow$: the space of all monotonic increasing sequences of positive reals

L: the ideal of all bounded linear operators between any arbitrary Banach spaces

F: the ideal of finite rank operators between any arbitrary Banach spaces

Λ: the ideal of approximable operators between any arbitrary Banach spaces

L_c: the ideal of compact operators between any arbitrary Banach spaces

$L(\mathfrak{X},\mathfrak{Y})$: the space of all bounded linear operators from a Banach space X into a Banach space Y

$L(\mathfrak{X})$: the space of all bounded linear operators from a Banach space X into itself

$F(\mathfrak{X},\mathfrak{Y})$: the space of finite rank operators from a Banach space X into a Banach space Y

$F(\mathfrak{X})$: the space of finite rank operators from a Banach space X into itself

$\Lambda(\mathfrak{X},\mathfrak{Y})$: the space of approximable operators from a Banach space X into a Banach space Y

$\Lambda(\mathfrak{X})$: the space of approximable operators from a Banach space X into itself

$L_c(\mathfrak{X},\mathfrak{Y})$: the space of compact operators from a Banach space X into a Banach space Y

$L_c(\mathfrak{X})$: the space of compact operators from a Banach space X into itself

$(s_a(G))_{a\in\mathbb{N}}$: the sequence of s-numbers of the bounded linear operator G

$(\alpha_a(G))_{a\in\mathbb{N}}$: the sequence of approximation numbers of the bounded linear operator G

$(s_a(G))_{a\in\mathbb{N}}$: the sequence of Kolmogorov numbers of the bounded linear operator G

S_v: the operator ideals formed by the sequence of s-numbers in any sequence space V

S_V^{app}: the operator ideals formed by the sequence of approximation numbers in any sequence space V

S_V^{Kol}: the operator ideals formed by the sequence of Kolmogorov numbers in any sequence space V

1.2. Notations 1.2 (see [1]). $S_{\mathscr{H}} \coloneqq \{S_{\mathscr{H}}(\mathfrak{X},\mathfrak{Y})\,;\,\mathfrak{X}$ and $\mathfrak{Y}$ are Banach Spaces$\}$, where

$$S_{\mathscr{H}}(\mathfrak{X},\mathfrak{Y}) \coloneqq \left\{P\in L(\mathfrak{X},\mathfrak{Y}): f_s\in\mathscr{H}, \text{where } f_s(z)=\sum_{n=0}^{\infty} s_n(P)z^n \text{ converges for any } z\in\mathbb{C}\right\} \tag{1}$$

$S_{\mathscr{H}}^{\mathrm{app}} \coloneqq \{S_{\mathscr{H}}^{\mathrm{app}}(\mathfrak{X},\mathfrak{Y})\,;\,\mathfrak{X}$ and $\mathfrak{Y}$ are Banach Spaces$\}$, where

$$S_{\mathscr{H}}^{\mathrm{app}}(\mathfrak{X},\mathfrak{Y}) \coloneqq \left\{P\in L(\mathfrak{X},\mathfrak{Y}): f_{\mathrm{app}}\in\mathscr{H}, \text{where } f_{\mathrm{app}}(z)=\sum_{n=0}^{\infty} \alpha_n(P)z^n \text{ converges for any } z\in\mathbb{C}\right\} \tag{2}$$

$S_{\mathscr{H}}^{\mathrm{Kol}}\,;\,\{S_{\mathscr{H}}^{\mathrm{Kol}}(\mathfrak{X},\mathfrak{Y})\,;\,\mathfrak{X}$ and $\mathfrak{Y}$ are Banach Spaces$\}$, where

$$S_{\mathscr{H}}^{\mathrm{Kol}}(\mathfrak{X},\mathfrak{Y}) \coloneqq \left\{P\in L(\mathfrak{X},\mathfrak{Y}): f_{\mathrm{Kol}}\in\mathscr{H}, \text{where } f_{\mathrm{Kol}}(z)=\sum_{n=0}^{\infty} d_n(P)z^n \text{ converges for any } z\in\mathbb{C}\right\} \tag{3}$$

$(S_{\mathscr{H}_\rho})^\lambda \coloneqq \{(S_{\mathscr{H}_\rho})^\lambda(\mathfrak{X},\mathfrak{Y})\,;\,\mathfrak{X}$ and $\mathfrak{Y}$ are Banach Spaces$\}$,
where

$$\left(S_{\mathscr{H}_\rho}\right)^\lambda(\mathfrak{X},\mathfrak{Y}) \coloneqq \left\{T\in L(\mathfrak{X},\mathfrak{Y}): f_\lambda\in\mathscr{H}_\rho, \text{where } f_\lambda(z)=\sum_{n=0}^{\infty} \lambda_n(T)z^n \text{ converges for any } z\in\mathbb{C} \text{ and } \|T-\lambda_l(T)I\|=0, \text{for every } l\in\mathbb{N}\right\} \tag{4}$$

Several operator ideals in the class of Banach or Hilbert spaces are defined by sequences of real numbers. L_c, for example, is produced by $(d_a(G))_{a\in\mathbb{N}}$ and c_0. Pietsch [2] looked into the quasi-ideals $S_{\ell^t}^{\mathrm{app}}$, for $0<t<\infty$. He demonstrated how ℓ^2 and ℓ^1 yield the ideals of Hilbert Schmidt operators and nuclear operators between Hilbert spaces, respectively. In addition, he proved that $\bar{F}=S_{\ell^t}$, for $1\le t<\infty$, and S_{ℓ^t} is a simple Banach space. Pietsch [3] explained that S_{ℓ^t}, where $0<t<\infty$, is small. Makarov and Faried [4] showed that for any Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$ with dim $(\mathfrak{X})$ = dim $(\mathfrak{Y})=\infty$, then for every $r>t>0$, one has $S_{\ell^t}^{\mathrm{app}}(\mathfrak{X},\mathfrak{Y})\subsetneq S_{\ell^r}^{\mathrm{app}}(\mathfrak{X},\mathfrak{Y})\subsetneq L(\mathfrak{X},\mathfrak{Y})$. The concept of prequasi-ideal was developed by Faried and Bakery [5], who elaborated on the concept of quasi-ideal. They investigated some geometric and topological properties of the spaces $S_{\mathrm{ces}(t)}$ and S_{ℓ_M}. According to the spectral decomposition theorem [2], for $A\in L_c(H)$, where H is a Hilbert space, one has $A(y)=\sum_{a=0}^{\infty}\alpha_a\langle y, r_a\rangle w_a$, where $\{r_a\}$ and $\{w_a\}$ are orthonormal families in H. Suppose $(t_a)_{a\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ be decreasing and $D:(\eta_a)\longrightarrow(t_a\eta_a)$ be the diagonal operator on ℓ^p with $p\ge 1$. Therefore, $s_a(D)=t_a$. Shields [6] investigated an indication to the weighted shift operators as formal power series in unilateral shifts and formal Laurent series in bilateral shifts. Hedayatian [7] offered the space of formal power series with power r, $\mathscr{H}^r((ba))$, where $((ba))$ is a sequence of positive numbers with $b_0=1$ and $r>0$. By the space $\mathscr{H}^p((ba))$, he meant that the set of all formal power series $\sum_{a=0}^{\infty}\widehat{f_a}z^a$ with $\sum_{a=0}^{\infty}$

$|b_a f_a \wedge|^p < \infty$. He studied cyclic vectors for the H forward shift operator and supercyclic vectors for the backward shift operator on the space $\mathscr{H}^p((b_a))$.

However, Emamirad and Heshmati [8] explored the idea of functions evident on the Bargmann space by $f(z)=\sum_{a=0}^{\infty} c_a(z^a/\sqrt{a!})$ with $\|f\|=\sum_{a=0}^{\infty}|c_a|^2<\infty$, where $\{z^a/\sqrt{a!} : a \in \mathbb{N}\}$ is an orthonormal basis. Faried et al. [9] introduced the upper bounds for s-numbers of infinite series of the weighted nth power forward shift operator on $\mathscr{H}^r((ba))$, for $1 \le r < \infty$, with some applications to some entire functions.

The paper is arranged as follows. In Section 3, we offer the definition of the space $\mathscr{H}_{p(.)}$ with definite function ρ. We introduce the sufficient conditions on $\mathscr{H}_{p(.)}$ to generate premodular special space of formal power series. This gives that $\mathscr{H}_{p(.)}$ is a prequasinormed space. In Section 4, firstly, we give the sufficient conditions on $\mathscr{H}_{p(.)}$ such that the class $S_{\mathscr{H}_{p(.)}}$ generates an operator ideal. Secondly, we explain enough settings (not necessary) on $(\mathscr{H}_{p(.)})_\rho$, so that $\bar{F}=S_{(\mathscr{H}_{p(.)})_\rho}$. This shows the nonlinearity of s-type $(\mathscr{H}_{p(.)})_\rho$ spaces which gives an answer of Rhoades [10] open problem. Thirdly, we investigate the conditions on $(\mathscr{H}_{p(.)})_\rho$ such that the prequasi-ideal $S_{(\mathscr{H}_{p(.)})_\rho}$ are Banach and closed. Fourthly, we examine the sufficient conditions on $(\mathscr{H}_{p(.)})_\rho$ such that $S_{(\mathscr{H}_{p(.)})_\rho}$ is strictly contained for different powers. We show the smallness of $S_{(\mathscr{H}_{p(.)})_\rho}$. Fifthly, we investigate the simpleness of $S_{(\mathscr{H}_{p(.)})_\rho}$. Sixthly, we present the enough setup on $(\mathscr{H}_{p(.)})_\rho$ such that the class L with its sequence of eigenvalues in $(\mathscr{H}_{p(.)})_\rho$ equals $S_{(\mathscr{H}_{p(.)})_\rho}$. In Section 5, we estimate the upper bounds for s-numbers of infinite series of the weighted nth power forward and backward shift operator on $\mathscr{H}_{p(.)}$ with approaches to some entire functions.

2. Definitions and Preliminaries

Definition 1 (see [11]). A function $s : L(\mathfrak{X},\mathfrak{Y}) \longrightarrow [0,\infty)^{\mathbb{N}}$ is called an s-number, if the sequence $(s_b(B))_{a=0}^{\infty}$, for all $B \in L(\mathfrak{X},\mathfrak{Y})$, shows the following settings:

(a) If $B \in L(\mathfrak{X},\mathfrak{Y})$, then $\|B\| = s_0(B) \ge s_1(B) \ge s_2(B) \ge \cdots \ge 0$

(b) $s_{b+a-1}(B_1+B_2) \le s_b(B_1)+s_a(B_2)$, for every $B_1, B_2 \in L(\mathfrak{X},\mathfrak{Y})$, $b, a \in \mathbb{N}$

(c) The inequality $s_a(ABD) \le \|A\| s_a(B) \|D\|$ holds, if $D \in L(\mathfrak{X}_0,\mathfrak{X})$, $B \in L(\mathfrak{X},\mathfrak{Y})$ and $A \in L(\mathfrak{Y},\mathfrak{Y}_0)$, where $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ are arbitrary Banach spaces

(d) Suppose $A \in L(\mathfrak{X}_0,\mathfrak{X})$ and $\lambda \in \mathbb{R}$, then $s_a(\lambda A)=|\lambda| s_a(A)$

(e) Let rank $(A) \le b$ then $s_b(A)=0$, whenever $A \in L(\mathfrak{X}_0,\mathfrak{X})$

(f) Assume I_λ indicates the identity operator on the λ -dimensional Hilbert space ℓ_2^λ, then $s_{r\ge\lambda}(I_\lambda)=0$ or $s_{r<\lambda}(I_\lambda)=1$

Consider the following examples of s-numbers:

(i) The bth approximation number, $\alpha_b(A)$, where

$$\alpha_b(A)=\inf\{\|A-B\| : B \in L(X,Y) \text{ and rank } (B) \le b\} \tag{5}$$

(ii) The bth Kolmogorov number, $d_b(A)$, where

$$d_b(A)=\inf_{\dim Y \le b} \sup_{\|u\|\le 1} \inf_{v \in Y} \|Au-v\|. \tag{6}$$

Remark 2 (see [11]). If $B \in Lc(H)$, where H be a Hilbert space, then all the s-numbers equal the eigenvalues of $|B|$, where $|B|=\sqrt{B*B}$.

Lemma 3 (see [2]). *If $B \in L(\mathfrak{X}_0,\mathfrak{X})$ and $B \notin \Lambda(\mathfrak{X}_0,\mathfrak{X})$, then $D \in L(\mathfrak{X})$ and $M \in L(\mathfrak{Y})$ with $MBD_{eb}=e_b$, for each $b \in \mathbb{N}$.*

Definition 4 (see [2]). A Banach space $\mathfrak{Y}$ is said to be simple if $L(\mathfrak{Y})$ has one and only one nontrivial closed ideal.

Theorem 5 (see [2]). *If D is a Banach space with* dim $(D)=\infty$, *then*

$$F(D) \subsetneq \Lambda(D) \subsetneq Lc(D) \subsetneq L(D). \tag{7}$$

Definition 6. (see [2]). A class $U \subseteq L$ is said to be an operator ideal if every vector $U(X,Y)=U \cap L(X,Y)$ shows the following settings:

(i) $F \subseteq U$

(ii) $U(\mathfrak{X},\mathfrak{Y})$ is linear space on $\mathbb{R}$

(iii) If $D \in L(\mathfrak{X}_0,\mathfrak{X})$, $B \in U(\mathfrak{X},\mathfrak{Y})$ and $A \in L(\mathfrak{Y},\mathfrak{Y}_0)$ then, $ABD \in U(\mathfrak{X}_0,\mathfrak{Y}_0)$

Definition 7 (see [5]). A function $g : U \longrightarrow [0,\infty)$ is called a prequasinorm on the ideal U if it shows the next settings:

(1) For each $A \in L(\mathfrak{X},\mathfrak{Y})$, $g(A) \ge 0$ and $g(A)=0 \Leftrightarrow A=0$

(2) One has $M \ge 1$ with $g(\beta A) \le M|\beta| g(A)$, for all $\beta \in \mathbb{R}$ and $A \in U(\mathfrak{X},\mathfrak{Y})$

(3) One has $K \ge 1$ with $g(A_1+A_2) \le K[g(A1)+g(A2)]$, for every $A_1, A_2 \in U(\mathfrak{X},\mathfrak{Y})$

(4) There exists $C \ge 1$ so that if $A \in L(\mathfrak{X}_0,\mathfrak{X})$, $B \in U(\mathfrak{X},\mathfrak{Y})$ and $D \in L(\mathfrak{Y},\mathfrak{Y}_0)$ then $g(DBA) \le C\|D\| g(B) \|A\|$, where $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ are normed spaces

Theorem 8 (see [5]). *Suppose g is a quasinorm on the ideal U, then g is a prequasinorm on the ideal U.*

Theorem 9 (see [12]). *Assume s-type $\mathscr{V}v := \{f = (sr(T)) \in \mathbb{R}^{\mathbb{N}} : T \in L(\mathfrak{X}, \mathfrak{Y}) and\, v(f) < \infty\}$. If Svv is an operator ideal, then we have*

(1) $\mathscr{F} \subset$ *s-type* $\mathscr{V}v$

(2) *Assume* $(s_r(T_1))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$ *and* $(s_r(T_2))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$, *then* $(s_r(T_1 + T_2))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$

(3) *Suppose* $\lambda \in \mathbb{R}$ *and* $(s_r(T))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$, *then* $|\lambda| (s_r(T))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$

(4) *The sequence space* $\mathscr{V}_v$ *is solid. i.e., when* $(s_r(G))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$ *and* $s_r(T) \le s_r(G)$, *for every* $r \in \mathbb{N}$ *and* $T, G \in L(X, Y)$, *then* $(s_r(T))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$

Lemma 10 (see [13]). *If $\{\xi i\}_{i\in\Psi}$ is a bounded family of $\mathbb{R}$. We have*

$$\sup_{\mathrm{card}\,(G)=a+1} \inf_{i\in G} \xi_i = \inf_{\mathrm{card}\,(G)=a} \sup_{i\notin G} \xi_i. \tag{8}$$

Lemma 11 (see [14]). *If $(r_a), (t_a) \in \mathbb{R}^{\mathbb{N}}$ and $(q_a) \in (0,\infty)^{\mathbb{N}}$, with $K = \max\{1, 2^{\varpi_{q-1}}\}$ and $\varpi_q = \max\{1, \sup_a q_a\}$, then*

$$|r_a + t_a|^{q_a} \le K(|r_a|^{q_a} + |t_a|^{q_a}). \tag{9}$$

Definition 12 (see [1]). The linear space of formal power series

$$\mathscr{H} = \left\{ f : f(z) = \sum_{n=0}^{\infty} \widehat{f}_n z^n \text{ converges for any } z \in \mathbb{C}, \right\}, \tag{10}$$

is called a special space of formal power series (or in short (ssfps)), if it shows the following settings:

(1) $e^{(m)} \in \mathscr{H}$, for all $m \in \mathbb{N}$, where $e^{(m)}(z) = \sum_{n=0}^{\infty} e_n^{\widehat{(m)}} z^n = z^m$

(2) If $g \in \mathscr{H}$ and $|\widehat{f}_n| \le |\widehat{g}_n|$, for all $n \in \mathbb{N}$, then $f \in \mathscr{H}$

(3) Suppose $f \in \mathscr{H}$, then $f_{[.]} \in \mathscr{H}$, where $f_{[.]}(z) = \sum_{b=0}^{\infty} \widehat{f}_{[b/2]} z^b$ and $[b/2]$ marks the integral part of $b/2$

Theorem 13 (see [1]). *If $\mathscr{H}$ is a (ssfps), then $S_{\mathscr{H}}$ is an operator ideal.*

By $\mathfrak{F}$, we explain the space of finite formal power series, i.e, for $f \in \mathfrak{F}$, one has $l \in \mathbb{N}$ with $f(z) = \sum_{n=0}^{l} \widehat{f}_n z^n$.

Definition 14 (see [1]). A subspace $\mathscr{H}\rho$ of the (ssfps) is called a premodular (ssfps), if there is a function $\rho : \mathscr{H} \longrightarrow [0,\infty)$ verifies the next conditions:

(i) For $f \in \mathscr{H}$, we have $\rho(f) \ge 0$ and $f = \theta \Leftrightarrow \rho(f) = 0$, where θ is the zero function of $\mathscr{H}$

(ii) Suppose $f \in \mathscr{H}$ and $\lambda \in \mathbb{R}$, then there is $l \ge 1$ with $\rho(\lambda f) \le |\lambda| l \rho(f)$

(iii) Let $f, g \in \mathscr{H}$, then there is $K \ge 1$ such that $\rho(f + g) \le K(\rho(f) + \rho(g))$

(iv) Suppose $|\widehat{f}_b| \le |\widehat{g}_b|$, for every $b \in \mathbb{N}$, then $\rho(f) \le \rho(g)$

(v) There is $K_0 \ge 1$ so that $\rho(f) \le \rho(f[.]) \le K_0 \rho(f)$

(vi) $\bar{\mathfrak{F}} = \mathscr{H}_\rho$

(vii) one has $\xi > 0$ with $\rho(\lambda e^{(0)}) \ge \xi |\lambda| \rho(e^{(0)})$, where $\lambda \in \mathbb{R}$

Note that the continuity of $\rho(f)$ at θ comes from condition (ii). Condition (1) in Definition 12 and condition (vi) in Definition 14 investigate that $(e^{(m)})_{m \in \mathbb{N}}$ is a Schauder basis of $\mathscr{H}\rho$.

The (ssfps) $\mathscr{H}\rho$ is called a prequasinormed (ssfps) if ρ shows the conditions (i)–(iii) of Definition 14, and if the space H is complete under ρ, then $\mathscr{H}\rho$ is called a prequasi-Banach (ssfps).

Theorem 15 (see [1]). *Every premodular (ssfps) $\mathscr{H}\rho$ is a prequasinormed (ssfps).*

Definition 16 (see [1]). Assume $\mathscr{H}\rho$ is a prequasinormed (ssfps). An operator $V_z : \mathscr{H}\rho \longrightarrow \mathscr{H}\rho$ is called forward shift, if $V_z f = zf$, for all $f \in \mathscr{H}\rho$, where $V_z f(z) = \sum_{n=0}^{\infty} \widehat{f}_n z^{n+1}$ converges for every $z \in \mathbb{C}$ and $\rho(V_z f) < \infty$.

Definition 17 (see [1]). Suppose $\mathscr{H}\rho$ is a prequasinormed (ssfps). An operator $B_z\, \mathscr{H}\rho \longrightarrow \mathscr{H}\rho$ is called backward shift, if $B_z f(z) = (f(z) - f(0))/z$, for all $f \in \mathscr{H}\rho$, where $B_z f(z) = \sum_{n=0}^{\infty} \widehat{f}_{n+1} z^n$ converges for every $z \in \mathbb{C}$ and $\rho(B_z f) < \infty$.

Definition 18 (see [9]). By using the power series of an entire function $g(z) = \sum_{m=0}^{\infty} a_m z^m$, the shift operator $V_{g(z)}$ is defined as

$$V_{g(z)}(f(z)) = \left(\sum_{m=0}^{\infty} a_m V_z^m \right)(f(z)). \tag{11}$$

Definition 19 (see [9]). By using the power series of an entire function $g(z) = \sum_{m=0}^{\infty} a_m z^m$, the shift operator $B_{g(z)}$ is defined as

$$B_{g(z)}(f(z)) = \left(\sum_{m=0}^{\infty} a_m B_z^m \right)(f(z)). \tag{12}$$

3. Main Results

3.1. The Space of Functions $(\mathscr{H}_{p(.)})_\rho$. We define in this section the space $(\mathscr{H}_{p(.)})_\rho$ under the function ρ and give enough conditions on it to create pre-modular (ssfps) which implies that is a prequasi-Banach (ssfps).

If $p=(p_v)_{v\in\mathbb{N}}(0,\infty)^{\mathbb{N}}$, we define the new space of functions:

$$\left(\mathscr{H}_{p(.)}\right)_\rho=\left\{f: f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v \text{ converges for any } z\in\mathbb{C} \text{ and } \rho(\zeta f)<\infty, \text{for some } \zeta>0\right\}, \tag{13}$$

where

$$\rho(f)=\sum_{v=0}^{\infty}\frac{1}{p_v}\left|\widehat{f}_v\right|^{p_v}. \tag{14}$$

If $(p_v)\in\ell_\infty$, one has

$$\begin{aligned}\left(\mathscr{H}_{p(.)}\right)_\rho&=\left\{f: f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v \text{ converges for any } z\in\mathbb{C} \text{ and } \rho(\zeta f)<\infty, \text{for some } \zeta>0\right\}\\&=\left\{f: f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v \text{ converges for any } z\in\mathbb{C} \text{ and } \sum_{v=0}^{\infty}\frac{1}{p_v}\left|\zeta\widehat{f}_v\right|^{p_v}<\infty, \text{for some } \zeta>0\right\}\\&=\left\{f: f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v \text{ converges for any } z\in\mathbb{C} \text{ and } \inf_v|\zeta|^{p_v}\sum_{v=0}^{\infty}\frac{1}{p_v}\left|\zeta\widehat{f}_v\right|^{p_v}<\infty, \text{for some } \zeta>0\right\}\\&=\left\{f: f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v \text{ converges for any } z\in\mathbb{C} \text{ and } \sum_{v=0}^{\infty}\frac{1}{p_v}\left|\widehat{f}_v\right|^{p_v}<\infty\right\}\\&=\left\{f: f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v \text{ converges for any } z\in\mathbb{C} \text{ and } \rho(\zeta f)<\infty, \text{for any } \zeta>0\right\}.\end{aligned} \tag{15}$$

Theorem 20. *Consider* $(p_v)\in mi_{\nearrow}\cap\ell_\infty$ *with* $p_0>0$, *one has* $(\mathscr{H}_{p(.)})_\rho$ *is a premodular Banach (ssfps).*

Proof (1-i). Let $f,g\in\mathscr{H}_{p(.)}$. Therefore, $f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v$ and $g(z)=\sum_{v=0}^{\infty}\widehat{g}_v z^v$ converge for any $z\in\mathbb{C}$. Then, $(f+g)(z)=\sum_{v=0}^{\infty}(\widehat{f}_v+\widehat{g}_v)z^v$ converges for any $z\in\mathbb{C}$. From $(p_v)\in\ell_\infty$, we have-$\sum_{v=0}^{\infty}(1/p_v)|\widehat{f}_v+\widehat{g}_v|^{p_v}\le K(\sum_{v=0}^{\infty}(1/p_v)|\widehat{f}_v|^{p_v}+\sum_{v=0}^{\infty}(1/p_v)|\widehat{g}_v|^{p_v})<\infty$,so $f+g\in\mathscr{H}_{p(.)}$.

(1-ii) Let $\lambda\in\mathbb{R}$ and $f\in\mathscr{H}p_{(.)}$. Therefore, $f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v$ converges for any $z\in\mathbb{C}$. Then, $(\lambda f)(z)=\sum_{v=0}^{\infty}\lambda\widehat{f}_v z^v$ converges for any $z\in\mathbb{C}$. From $(p_v)\in\ell_\infty$, we have$\sum_{v=0}^{\infty}(1/p_v)|\lambda\widehat{f}_v|^{p_v}\le\sup_v|\lambda|^{p_v}\sum_{v=0}^{\infty}(1/p_v)|\widehat{f}_v|^{p_v}<\infty$.

So $\lambda f\in\mathscr{H}_{p(.)}$. Therefore, from conditions (1-i) and (1-ii), the space $\mathscr{H}_{p(.)}$ is linear. To prove $e^{(m)}\in\mathscr{H}p_{(.)}$,

for all $m\in\mathbb{N}$, where $e^{(m)}(z)=\sum_{v=0}^{\infty}e_v^{\widehat{(m)}}z^v=z^m$ and $\sum_{v=0}^{\infty}(1/p_v)|e_v^{\widehat{(m)}}|^{p_v}=1/p_m$

(2) Assume $|\widehat{f_v}|\le|\widehat{g_v}|$, for all $v\in\mathbb{N}$ and $g\in\mathscr{H}_{p(.)}$. Then,converges for any $z\in\mathbb{C}$

One has

$$\sum_{v=0}^{\infty}\frac{1}{p_v}|f\wedge_v|^{p_v}\le\sum_{v=0}^{\infty}\frac{1}{p_v}|g\wedge_v|^{p_v}<\infty. \tag{16}$$

So, $f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v$ and $z\in\mathbb{C}$ and $\rho(f)<\infty$. Hence, $f\in\mathscr{H}_{p(\cdot)}$

(3) Let $f\in\mathscr{H}_{p(\cdot)}$ and $(p_v)\in mi\nearrow\cap\ell_\infty$ with $p_0>0$. Then, $f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v$ converges for any $z\in\mathbb{C}$

and $\rho(f)<\infty$. One has

$$\rho\left(f_{[\cdot]}\right)=\sum_{v=0}^{\infty}\frac{1}{p_v}\left|f_{[v/2]}\wedge\right|^{p_v}=\sum_{v=0}^{\infty}\frac{1}{p_{2v}}|f_v\wedge|^{p_{2v}}+\sum_{v=0}^{\infty}\frac{1}{p_{2v+1}}|f_v\wedge|^{p_{2v+1}}\le 2\sum_{v=0}^{\infty}\frac{1}{p_v}|f_v\wedge|^{pv}=2\rho(f) \tag{17}$$

Hence, $f_{[\cdot]}(z)=\sum_{v=0}^{\infty}\widehat{f_{[v/2]}}z^v$ converges for any $z\in\mathbb{C}$ and $\rho(f_{[\cdot]})<\infty$. Then $f_{[\cdot]}\in\mathscr{H}_{p(\cdot)}$.

(i) Obviously, if $f\in\mathscr{H}_{p(.)}$, one gets $\rho(f)\ge 0$ and $\rho(f)=0\Leftrightarrow f=\theta$

(ii) There is $l=\max\{1,\sup_v|\eta|^{pv-1}\}\ge 1$, for all $\eta\in\mathbb{R}\setminus\{0\}$ and $l\ge 1$, for $\eta=0$ so that

$$\rho(\eta f)=\sum_{v=0}^{\infty}\frac{1}{p_v}|\eta f_v\wedge|^{p_v}\le\sup_v|\eta|^{p_v}\sum_{v=0}^{\infty}\frac{1}{p_v}|f_v\wedge|^{p_v}\le l|\eta|\rho(f), \tag{18}$$

for all $f\in\mathscr{H}_{p(.)}$

(iii) There is $K=\max\{1,2^{\sup_v p_v-1}\}\ge 1$ so that

$$\rho(\eta f)=\sum_{v=0}^{\infty}\frac{1}{p_v}\left|\eta\widehat{f_v}\right|\le\sup_v|\eta|^{p_v}\sum_{v=0}^{\infty}\frac{1}{p_v}|f_v\wedge|^{p_v}\le l|\eta|\rho(f), \tag{19}$$

for every $f,g\in\mathscr{H}_{p(.)}$

(iv) Obviously from the proof part (2).

(v) From the proof part (3), one has $K_0=2\ge 1$

(vi) Clearly, $\bar{\mathfrak{F}}=\mathscr{H}_{p(.)}$

(vii) One has ζ with $0<\zeta\le\eta^{p_0-1}$ with $\rho(\eta e^{(0)})\ge\zeta|\eta|\rho(e^{(0)})$, for each $\eta\neq 0$ and $\zeta>0$, when $\eta=0$. Therefore, the space $(\mathscr{H}_{p(.)})_\rho$ is a premodular (ssfps). To show that $(\mathscr{H}_{p(.)})_\rho$ is a premodular Banach (ssfps), we suppose $f^{(i)}$ to be a Cauchy sequence in $(\mathscr{H}_{p(.)})_\rho$, then for every $\varepsilon\in(0,1)$, there is $i_0\in N$ such that for all $i,j\ge i_0$, one gets

$$\rho\left(f^{(i)}-f^{(j)}\right)=\sum_{v=0}^{\infty}\frac{1}{p_v}\left|f_v^{(i)}\wedge-f_v^{(j)}\wedge\right|^{p_v}<\varepsilon^{\varpi_p} \tag{20}$$

For $i,j\ge i_0$ and $v\in\mathbb{N}$, we have

$$\left|\widehat{f_v^{(i)}}-\widehat{f_v^{(j)}}\right|<\varepsilon. \tag{21}$$

So, $(\widehat{f_v^{(j)}})$ is a Cauchy sequence in $\mathbb{R}$, for fixed $v\in\mathbb{N}$, hence $\lim_{j\longrightarrow\infty}\widehat{f_v^{(j)}}=\widehat{f_v^{(0)}}$, for fixed $v\in\mathbb{N}$.

Therefore, $\rho(f^{(i)}-f^{(0)})<\varepsilon^{\varpi}p$, for every $i\ge i_0$. Finally, to show that $f^{(0)}\in\mathscr{H}_{p(.)}$, we have

$$\rho\left(f^{(0)}\right)=\rho\left(f^{(0)}-f^{(i)}+f^{(i)}\right)\le K\left(\rho\left(f^{(i)}-f^{(0)}\right)+\rho\left(f^{(i)}\right)\right)<\infty. \tag{22}$$

Hence, $f^{(0)}\in\mathscr{H}_{p(.)}$. Then, the space $(\mathscr{H}_{p(.)})_\rho$ is a premodular Banach (ssfps).

In view of Theorems 15 and 20, we conclude the following theorem. □

Theorem 21. *If $(p_v)\in mi\nearrow\cap\ell_\infty$ with $p_0>0$, then the space $(H_{p(.)})_\rho$ is a prequasi-Banach (ssfps), where $\rho(f)=\sum_{v=0}^{\infty}(1/p_v)|f_v\wedge|^{p_v}$, for all $f\in\mathscr{H}_{p(\cdot)}$.*

Theorem 22. *Suppose $(p_v)\in mi\nearrow\cap\ell_\infty$ with $p_0>0$, one has $(H_{p(.)})_\rho$ is a prequasiclosed (ssfps), where*

$$\rho(f)=\sum_{v=0}^{\infty}\frac{1}{p_v}|f_v\wedge|^{p_v},\ \text{for all}\ f\in\mathscr{H}_{p(\cdot)}. \tag{23}$$

Proof. According to Theorem 21, the space $(H_{p(.)})_\rho$ is a prequasinormed (ssfps). To explain that $(H_{p(.)})_\rho$ is a prequasiclosed (ssfps), let $\{f^{(i)}\}_{i=0}^{\infty}\in(\mathscr{H}_{p(\cdot)})_\rho$ and $\lim_{i\longrightarrow\infty}\rho(f^{(i)}-f^{(0)})=0$, we have for all $\varepsilon\in(0,1)$, there is $i_0\in\mathbb{N}$ such that for all $i\ge i_0$, one gets

$$\varepsilon>\rho\left(f^{(i)}-f^{(0)}\right)=\left[\sum_{a=0}^{\infty}\frac{1}{p_a}\left|f_a^{(i)}\wedge-f_a^{(0)}\wedge\right|^{p_a}\right]^{1/\varpi_p}. \tag{24}$$

So, for $i\ge i_0$ and $a\in\mathbb{N}$, we have $|\widehat{f_a^{(i)}}-\widehat{f_a^{(0)}}|<\varepsilon$.Therefore, $(\widehat{f_a^{(i)}})$ is a convergent sequence in $\mathbb{R}$, for fixed $a\in\mathbb{N}$. Then, $\lim_{i\longrightarrow\infty}\widehat{f_a^{(i)}}=\widehat{f_a^{(0)}}$ for fixed $a\in\mathbb{N}$. Finally to prove that $f^{(0)} \in (\mathscr{H}_{p(.)})_\rho$, we have

$$\rho\left(f^{(0)}\right) = \rho\left(f^{(0)} - f^{(i)} + f^{(i)}\right) \le \rho\left(f^{(i)} - f^{(0)}\right) + \rho\left(f^{(i)}\right) < \infty, \tag{25}$$

this gives $f^{(0)} \in (\mathscr{H}_{p(.)})_\rho$ which shows that $(\mathscr{H}_{p(.)})_\rho$ is a pre-quasiclosed (ssfps). □

4. Properties of Operator Ideal

Throughout this section, some geometric and topological properties of the prequasi-ideals formed by s-numbers and $(\mathscr{H}_{p(.)})_\rho$ are presented.

4.1. Ideal of Finite Rank Operators. In this part, enough settings (not necessary) on $(\mathscr{H}_{p(.)})_\rho$ so that $\bar{F} = S_{(\mathscr{H}_{p(.)})_\rho}$ are given. This explains the nonlinearity of the s-type $(\mathscr{H}_{p(.)})_\rho$ spaces (Rhoades open problem [10]).

In view of Theorems 13 and 20, we conclude the next theorem.

Theorem 23. *Consider $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then $S_{\mathscr{H}_{p(.)}}$ is an operator ideal.*

Theorem 24. *If $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then $\bar{F} = S_{(\mathscr{H}_{p(.)})_\rho}$, where*

$$\rho(f) = \sum_{v=0}^{\infty} \frac{1}{p_v} \left|\widehat{f_v}\right|, \text{for every } f \in \mathscr{H}_{p(\cdot)}. \tag{26}$$

Proof. Clearly, $\overline{F(\mathfrak{X}, \mathfrak{Y})} \subset S_{(\mathscr{H}_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y})$, since the space $S_{(\mathscr{H}_{p(.)})_\rho}$ is an operator ideal. Therefore, we have to show that $S_{(\mathscr{H}_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y}) \subseteq \overline{F(\mathfrak{X}, \mathfrak{Y})}$. By letting $T \in S_{(\mathscr{H}_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y})$, then, $f_s \in (H_{p(.)})_\rho$, with $f_s(z) = \sum_{v=0}^{\infty} s_v(T) z^v$ converges for any $z \in \mathbb{C}$. So, $\rho(f_s) < \infty$, fix $\varepsilon \in (0, 1)$, we have $m \in \mathbb{N} - \{0\}$ with $\rho(f_s - \sum_{v=0}^{m-1} e^{(v)}) < \varepsilon/4$. As $(s_v(T))_{v\in\mathbb{N}}$ is decreasing, we have

$$\sum_{v=m+1}^{2m} \frac{1}{p_v}(s_{2m}(T))^{p_v} \le \sum_{v=m+1}^{2m} \frac{1}{p_v}(s_v(T))^{p_v} \le \sum_{v=m}^{\infty} \frac{1}{p_v}(s_v(T))^{p_v} < \frac{\varepsilon}{4}. \tag{27}$$

Therefore, we have $A \in F_{2m}(\mathfrak{X}, \mathfrak{Y})$, rank $A \le 2m$ and

$$\sum_{v=2m+1}^{3m} \frac{1}{pv} \|T - A\|^{p_v} \le \sum_{v=m+1}^{2m} \frac{1}{p_v} \|T - A\|^{p_v} < \frac{\varepsilon}{4}. \tag{28}$$

As $(p_v) \in \ell_\infty$, then

$$\sum_{v=0}^{m} \frac{1}{p_v} \|T - A\|^{p_v} < \frac{\varepsilon}{4}. \tag{29}$$

Since $T - A \in S_{(\mathscr{H}_{p(.)})_p}(\mathfrak{X}, \mathfrak{Y})$, then $h_s \in (\mathscr{H}_{p(.)})_p$, where $h_s(z) := \sum_{v=0}^{\infty} s_v(T - A) z^v$ converges for any $z \in \mathbb{C}$. Because (p_v) is increasing and from the inequalities (27)–(29), we get

$$\begin{aligned} d(T, A) = \rho(h_s) &= \sum_{v=0}^{3m-1} \frac{1}{p_v}(s_v(T - A))^{p_v} + \sum_{v=3m}^{\infty} \frac{1}{p_v}(s_v(T - A))^{p_v} \\ &\le \sum_{v=0}^{3m} \frac{1}{p_v} \|T - A\|^{p_v} + \sum_{v=m}^{\infty} \frac{1}{p_{v+2m}}(s_{v+2m}(T - A))^{p_{v+2m}} \\ &\le 3\sum_{v=0}^{m} \frac{1}{p_v} \|T - A\|^{p_v} + \sum_{v=m}^{\infty} \frac{1}{p_v}(s_v(T))^{p_v} < \varepsilon. \end{aligned} \tag{30}$$

Since $I_2 \in S_{(\mathscr{H}_{(1/(n+1))})_p}(\mathfrak{X}, \mathfrak{Y})$ but the condition $(p_v) \in mi\nearrow \cap \ell_\infty$ is not verified which explain a negative example of the converse statement. This finishes the proof.

We can reformulate Theorem 24 as follows: if $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then every compact operators can be approximated by finite rank operators and the converse is not always true. □

4.2. Banach and Closed Prequasi-Ideal. In this part, enough settings on $(\mathscr{H}_{p(.)})_\rho$ so that the prequasioperator ideal $S_{\mathscr{H}_\rho}$ is Banach and closed are investigated.

Theorem 25. *Assume $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then the function $g(P) = \rho(f_s)$ is a prequasinorm on $S_{(\mathscr{H}_{p(.)})_\rho}$, where $f_s(z) = \sum_{v=0}^{\infty} s_v(P) z^v$ converges for any $z \in \mathbb{C}$ and*

$$\rho(f_s) = \sum_{v=0}^{\infty} \frac{1}{p_v} s_v(P)^{p_v}, \text{for every } f_s \in \mathscr{H}_{p(\cdot)}. \tag{31}$$

Proof. One has g verifies the next setups:

(1) Let $P \in S_{(\mathscr{H}_{p(\cdot)})_\rho}(\mathfrak{X}, \mathfrak{Y})$, $g(P) = \rho(f_s) \ge 0$ and $g(P) = \rho(f_s) = 0 \Leftrightarrow s_v(P) = 0$, for all $v \in N \Leftrightarrow P = 0$

(2) There is $l \ge 1$ with $g(\lambda P) = \rho(\lambda f_s) \le l|\lambda|\rho(f_s) = l|\lambda| g(P)$, for every $P \in S_{(\mathscr{H}_{p(\cdot)})_\rho}(\mathfrak{X}, \mathfrak{Y})$ and $\lambda \in \mathbb{R}$

(3) One has $KK_0 \ge 1$, for $P_1, P_2 \in S_{(\mathscr{H}_{p(\cdot)})_\rho}(\mathfrak{X}, \mathfrak{Y})$. Then, $f1_s(z) = \sum_{v=0}^{\infty} s_v(P_1) z^v$ and $f2_s(z) = \sum_{v=0}^{\infty} s_v(P_2) z^v$ converge for any $z \in \mathbb{C}$. Therefore, for $h_s(z) =: \sum_{v=0}^{\infty} s_v(P_1 + P_2) z^v$, one has

$$\begin{aligned} g(P_1 + P_2) = \rho(h_s) &\le \rho\left((f1_s)_{[\cdot]} + (f2_s)_{[\cdot]}\right) \le K\left(\rho\left((f1_s)_{[\cdot]}\right)\right. \\ &\left.+ \rho\left((f2_s)_{[\cdot]}\right)\right) \le KK_0(g(P_1) + g(P_2)) \end{aligned} \tag{32}$$

(4) We have $C \geq 1$, let $A \in L(\mathfrak{X}_0, \mathfrak{X}), B \in S_{(\mathscr{H}_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y})$ and $D \in L(\mathfrak{Y}, \mathfrak{Y}_0)$. Then, $f_s(z) = \sum_{v=0}^{\infty} s_v(B) z^v$ converges for all $z \in \mathbb{C}$. Then, for $h_s(z) =: \sum_{v=0}^{\infty} s_v(DBA) z^v$, one has

$$g(DBA) = \rho(h_s) \leq \rho(\|A\|\|D\| f_s) \leq C\|A\| g(B)\|D\| \quad (33)$$

□

Theorem 26. *Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, and $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then $(S_{(\mathscr{H}_{p(.)})_\rho}, g)$ is a prequasi-Banach operator ideal, where $g(P) = \rho(f_s), f_s(z) = \sum_{v=0}^{\infty} s_v(P) z^v$ converges for any $z \in \mathbb{C}$ and*

$$\rho(f_s) = \sum_{v=0}^{\infty} \frac{1}{p_v} s_v(P)^{p_v}, \text{for every } f_s \in \mathscr{H}_{p(\cdot)}. \quad (34)$$

Proof. As $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, one has the function $g(P) = \rho(f_s)$ is a prequasinorm on $S_{(\mathscr{H}_{p(\cdot)})_\rho}$. Let (P_m) be a Cauchy sequence in $S_{(\mathscr{H}_{p(\cdot)})_\rho}(\mathfrak{X}, \mathfrak{Y})$. Therefore, $f_s^{(m)} \in (\mathscr{H}_{p(\cdot)})_\rho$ and $f_s^{(m)}(z) = \sum_{v=0}^{\infty} s_v(P_m) z^v$ converges for any $z \in \mathbb{C}$. Suppose $h_s(z) =: \sum_{v=0}^{\infty} s_v(P_i - P_j) z^v$, then from parts (iv) and (vii) of Definition 14 and since $L(\mathfrak{X}, \mathfrak{Y}) \supseteq S_{(\mathscr{H}_{p(\cdot)})_\rho}(\mathfrak{X}, \mathfrak{Y})$, we have

$$\begin{aligned} g(P_i - P_j) &= \rho(h_s) \geq \rho\left(s_0(P_i - P_j) e^{(0)}\right) = \rho\left(\|P_i - P_j\| e^{(0)}\right) \\ &\geq \xi \|P_i - P_j\| \rho\left(e^{(0)}\right), \end{aligned} \quad (35)$$

then $(P_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L(\mathfrak{X}, \mathfrak{Y})$. Since the space $L(\mathfrak{X}, \mathfrak{Y})$ is a Banach space, there is $P \in L(\mathfrak{X}, \mathfrak{Y})$ with $\lim_{m\longrightarrow\infty} \|P_m - P\| = 0$ and as $f_s^{(m)} \in (\mathscr{H}_{p(.)})_\rho$, for every $m \in \mathbb{N}$. Hence, by using Theorem 25 and the continuity of ρ at θ, we have

$$\begin{aligned} g(P) &= g(P - P_m + P_m) \leq KK_0(g(P_m - P) + g(P_m)) \\ &= KK_0 \rho\left(\|P_m - P\| \sum_{m=-}^{\infty} e^{(m)}\right) + KK_0 \rho\left(f_s^{(m)}\right) < \varepsilon, \end{aligned} \quad (36)$$

so $f_s \in (\mathscr{H}_{p(.)})_\rho$, which implies $P \in S_{(\mathscr{H}_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y})$. □

Theorem 27. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, and $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then $(S_{(\mathscr{H}_{p(.)})_\rho}, g)$ is a prequasiclosed operator ideal, where $g(P) = \rho(f_s), f_s(z) = \sum_{v=0}^{\infty} s_v(P) z^v$ converges for any $z \in \mathbb{C}$ and*

$$\rho(f_s) = \sum_{v=0}^{\infty} \frac{1}{p_v} s_v(P)^{p_v}, \text{for every } f_s \in \mathscr{H}_{p(\cdot)}. \quad (37)$$

Proof. As $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, so the function $g(P) = \rho(f_s)$ is a prequasinorm on $S_{(\mathscr{H}_{p(.)})_\rho}$. Let $P_m \in S_{(\mathscr{H}_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y})$, with $m \in \mathbb{N}$ and $\lim_{m\longrightarrow\infty} g(P_m - P) = 0$. Then, $f_s^{(m)} \in (\mathscr{H}_{p(.)})_\rho$ and $f_s^{(m)}(z) = \sum_{v=0}^{\infty} s_v(P_m) z^v$ converges for any $z \in \mathbb{C}$. Suppose $h_s(z) =: \sum_{v=0}^{\infty} s_v(P_i - P_j) z^v$, then from parts (iv) and (vii) of Definition 14 and since $L(\mathfrak{X}, \mathfrak{Y}) \supseteq S_{(\mathscr{H}_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y})$, one obtains

$$\begin{aligned} g(P - P_j) &= \rho(h_s) \geq \rho\left(s_0(P - P_j) e^{(0)}\right) = \rho\left(\|P - P_j\| e^{(0)}\right) \\ &\geq \xi \|P - P_j\| \rho\left(e^{(0)}\right), \end{aligned} \quad (38)$$

then $(P_m)_{m\in\mathbb{N}}$ is a convergent sequence in $L(\mathfrak{X}, \mathfrak{Y})$. Since the space $L(\mathfrak{X}, \mathfrak{Y})$ is a Banach space, then there is $P \in L(\mathfrak{X}, \mathfrak{Y})$ with $\lim_{m\longrightarrow\infty} \|P_m - P\| = 0$ and as $f_s^{(m)} \in (\mathscr{H}_{p(.)})_\rho$, for every $m \in \mathbb{N}$, by using Theorem 25 and the continuity of ρ at θ, one has

$$\begin{aligned} g(P) &= g(P - P_m + P_m) \leq KK_0(g(P_m - P) + g(P_m)) \\ &= KK_0 \rho\left(\|P_m - P\| \sum_{m=0}^{\infty} e^{(m)}\right) + KK_0 \rho\left(f_s^{(m)}\right) < \varepsilon, \end{aligned} \quad (39)$$

hence, $f_s \in (\mathscr{H}_{p(.)})_\rho$, which gives $P \in S_{(\mathscr{H}_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y})$. □

According to Theorem 9, we introduce the following properties of the s-type $(\mathscr{H}_{p(.)})_\rho$.

Theorem 28. *For s-type $(H_{p(.)})_\rho := \{(s_v(T)) \in \mathbb{R}^{\mathbb{N}} : T \in S_{(\mathscr{H}_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y})\}$. The next settings are verified.*

(1) We have s-type $(\mathscr{H}_{p(.)})_\rho \supset \mathscr{F}$

(2) Suppose $(s_r(T_1))_{r=0}^{\infty} \in$ s-type $(\mathscr{H}_{p(\cdot)})_\rho$ and $(s_r(T_2))_{r=0}^{\infty} \in$ s-type $(\mathscr{H}_{p(\cdot)})_\rho$, then $(s_r(T_1 + T_2))_{r=0}^{\infty} \in$ s-type $(\mathscr{H}_{p(.)})_\rho$

(3) One has $\lambda \in \mathbb{R}$ and $(s_r(T))_{r=0}^{\infty} \in$ s-type $(\mathscr{H}_{p(\cdot)})_\rho$, then $|\lambda|(s_r(T))_{r=0}^{\infty} \in s -$ type $(\mathscr{H}_{p(.)})_\rho$

(4) The s-type $(\mathscr{H}_{p(.)})_\rho$ is solid

4.3. Small Prequasi-Banach Ideal. We introduce here some inclusion relations concerning the space $S_{(\mathscr{H}_{p(\cdot)})_\rho}$ for different (p_v).

Theorem 29. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and $(p_v), (q_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$ and $p_v < q_v$, for all $v \in \mathbb{N}$, we have*

$$S_{(\mathscr{H}_{p(\cdot)})_\rho}(\mathfrak{X}, \mathfrak{Y}) \subsetneq S_{(\mathscr{H}_{p(\cdot)})_\rho}(\mathfrak{X}, \mathfrak{Y}) \subsetneq L(\mathfrak{X}, \mathfrak{Y}). \quad (40)$$

Proof. Assume $T \in S(\mathscr{H}_{p(.)})_\rho(\mathfrak{X}, \mathfrak{Y})$. Therefore, $f_s \in (\mathscr{H}_{p(\cdot)})_\rho$ and $f_s(z) = \sum_{v=0}^{\infty} s_v(T) z^v$ converges for any $z \in \mathbb{C}$. Then,

$$\sum_{v=0}^{\infty} \frac{1}{q_v}(s_v(T))^{q_v} < \sum_{v=0}^{\infty} \frac{1}{p_v}(s_v(T))^{q_v} < \infty, \quad (41)$$

hence, $T \in S(\mathscr{H}_{q(.)})_\rho(\mathfrak{X}, \mathfrak{Y})$. Next, by taking T with $s_v(T) = (p_v/(v+1))^{1/p_v}$, one has $T \notin S(\mathscr{H}_{p(.)})_\rho(\mathfrak{X}, \mathfrak{Y})$ and $T \in S(\mathscr{H}_{q(.)})_\rho(\mathfrak{X}, \mathfrak{Y})$. Clearly, $S_{(H_{q(.)})_\rho}(\mathfrak{X}, \mathfrak{Y}) \subset L(\mathfrak{X}, \mathfrak{Y})$. Again, by choosing $s_v(T) = (q_v/(v+1))^{1/q_v}$, one has $T \notin S(\mathscr{H}_{q(.)})_\rho(\mathfrak{X}, \mathfrak{Y})$ and $T \in L(\mathfrak{X}, \mathfrak{Y})$. This finishes the proof.

In this part, we examine the sufficient setting for which $S^{app}_{(\mathscr{H}_{p(\cdot)})_\rho}$ is small. □

Theorem 30. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$. *Assume $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then $S^{app}_{(\mathscr{H}_{p(\cdot)})_\rho}$ is small.*

Proof. Obviously, the space $(S^{\text{app}}_{(\mathscr{H}_{p(\cdot)})_\rho}, g)$ generates a prequasi-Banach operator ideal, with $g(T) = \sum_{v=0}^{\infty}(1/p_v)(\alpha_v(T))^{p_v}$. Let $S^{\text{app}}_{(\mathscr{H}_{p(\cdot)})_\rho}(\mathfrak{X}, \mathfrak{Y}) = L(\mathfrak{X}, \mathfrak{Y})$. Hence, there is $C > 0$ with $g(T) \le C\|T\|$, for all $T \in L(\mathfrak{X}, \mathfrak{Y})$. According to Dvoretzky's theorem [15] with $r \in \mathbb{N}$, there are quotient spaces $\mathfrak{X}/\lambda_r$ and subspaces η_r of $\mathfrak{Y}$ that operated onto ℓ_2^r by isomorphisms D_r and B_r with $\|D_r\|\|D_r^{-1}\| \le 2$ and $\|B_r\|\|B_r^{-1}\| \le 2$. Suppose I_r be the identity operator on ℓ_2^r, ζ_r be the quotient operator from $\mathfrak{X}$ onto $\mathfrak{X}/\lambda_r$ and J_r be the natural embedding operator from η_r into $\mathfrak{Y}$. Let h_a be the Bernstein numbers [16], we have

$$\begin{aligned} 1 = h_a(I_r) &= h_a\left(B_r B_r^{-1} I_r D_r D_r^{-1}\right) \le \|B_r\| h_a\left(B_r^{-1} I_r D_r\right)\left\|D_r^{-1}\right\| \\ &= \|B_r\| h_a\left(J_r B_r^{-1} I_r D_r\right)\left\|D_r^{-1}\right\| \le \|B_r\| d_a\left(J_r B_r^{-1} I_r D_r\right)\left\|D_r^{-1}\right\| \\ &= \|B_r\| d_a\left(J_r B_r^{-1} I_r D_r \zeta_r\right)\left\|D_r^{-1}\right\| \le \|B_r\| \alpha_a\left(J_r B_r^{-1} I_r D_r \zeta_r\right)\left\|D_r^{-1}\right\|, \end{aligned} \quad (42)$$

for $0 \le j \le r$. Then for $l \ge 1$, one has

$$\begin{aligned} \frac{1}{p_j} &\le \left(\|B_r\|\left\|D_r^{-1}\right\|\right)^{p_j} \frac{1}{p_j}\left(\alpha_j\left(J_r B_r^{-1} I_r D_r \zeta_r\right)\right)^{p_j} \Rightarrow \frac{1}{p_j} \\ &\le l\|B_r\| \frac{1}{p_j}\left(\alpha_j\left(J_r B_r^{-1} I_r D_r \zeta_r\right)\right)^{p_j}\left\|D_r^{-1}\right\| \Rightarrow \sum_{j=0}^{r} \frac{1}{p_j} \\ &\le l\|B_r\|\left\|D_r^{-1}\right\| \sum_{j=0}^{r} \frac{1}{p_j}\left(\alpha_j\left(J_r B_r^{-1} I_r D_r \zeta_r\right)\right)^{p_j} \Rightarrow \sum_{j=0}^{r} \frac{1}{p_j} \\ &\le lC\|B_r\|\left\|D_r^{-1}\right\| g\left(J_r B_r^{-1} I_r D_r \zeta_r\right) \Rightarrow \sum_{j=0}^{r} \frac{1}{p_j} \\ &\le lC\|B_r\|\left\|D_r^{-1}\right\|\left\|J_r B_r^{-1} I_r D_r \zeta_r\right\| \Rightarrow \sum_{j=0}^{r} \frac{1}{p_j} \\ &\le lC\|B_r\|\left\|D_r^{-1}\right\|\left\|J_r B_r^{-1}\right\|\|I_r\|\|D_r \zeta_r\| \\ &= lC\|B_r\|\left\|D_r^{-1}\right\|\left\|B_r^{-1}\right\|\|I_r\|\|D_r\| \Rightarrow \sum_{j=0}^{r} \frac{1}{p_j} \le 4lC. \end{aligned} \quad (43)$$

As $r \longrightarrow \infty$, we get $\sum_{j=0}^{\infty} 1/p_j < \infty$. Since $\sum_{j=0}^{\infty} a/p_j \ge 1/\sup p_j \sum_{j=0}^{\infty} 1 = \infty$. Hence, the space $S^{\text{app}}_{(\mathscr{H}_{p(\cdot)})_\rho}$ is small.

By the same manner, we can easily conclude the next theorem. □

Theorem 31. *Assume $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$. *Suppose $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then $S^{Kol}_{(\mathscr{H}_{p(\cdot)})_\rho}$ is small.*

4.4. Simple Prequasi-Ideal. In this part, we offer enough settings on $(\mathscr{H}_{p(\cdot)})_\rho$ so that the space $S_{(\mathscr{H}_{p(\cdot)})_\rho}$ is simple.

Theorem 32. *Let $(p_v), (q_v) \in mi\nearrow \cap \ell_\infty$ with $1 \le p_v < q_v$, for every $v \in \mathbb{N}$, then*

$$L\left(S_{(\mathscr{H}_{q(\cdot)})_\rho}, S_{(\mathscr{H}_{p(\cdot)})_\rho}\right) = \Lambda\left(S_{(\mathscr{H}_{q(\cdot)})_\rho}, S_{(\mathscr{H}_{p(\cdot)})_\rho}\right). \quad (44)$$

Proof. Consider $T \in L(S_{(\mathscr{H}_{q(\cdot)})_\rho}, S_{(\mathscr{H}_{p(\cdot)})_\rho})$ and $T \notin \Lambda(S_{(\mathscr{H}_{q(\cdot)})_\rho}, S_{(\mathscr{H}_{p(\cdot)})_\rho})$. According to Lemma 3, one has $G \in L(S_{(\mathscr{H}_{q(\cdot)})_\rho})$ and $B \in L(S_{(\mathscr{H}_{p(\cdot)})_\rho})$ with $BTGI_m = I_m$. For every $m \in \mathbb{N}$, one obtains

$$\begin{aligned} \|I_m\|_{S_{(\mathscr{H}_{p(\cdot)})_\rho}} &= \left(\sum_{v=0}^{\infty} \frac{1}{p_v}(s_v(I_m))^{p_v}\right)^{1/\sup p_v} = \left(\sum_{v=0}^{m-1} \frac{1}{p_v}\right)^{1/\sup p_v} \\ &\le \|BTG\|\|I_m\|_{S_{(\mathscr{H}_{q(\cdot)})_\rho}} \le \left(\sum_{v=0}^{\infty} \frac{1}{q_v}(s_v(I_m))^{q_v}\right)^{1/\sup q_v} \\ &= \left(\sum_{v=0}^{m-1} \frac{1}{q_v}\right)^{1/\sup q_v}. \end{aligned} \quad (45)$$

This defies Theorem 29. □

Corollary 33. *Let $(p_v), (q_v) \in mi\nearrow \cap \ell_\infty$ with $1 \le p_v < q_v$, for each $v \in \mathbb{N}$, then*

$$L\left(S_{(\mathscr{H}_{q(\cdot)})_\rho}, S_{(\mathscr{H}_{p(\cdot)})_\rho}\right) = L_C\left(S_{(\mathscr{H}_{q(\cdot)})_\rho}, S_{(\mathscr{H}_{p(\cdot)})_\rho}\right). \quad (46)$$

Proof. Clearly, as $\Lambda \subseteq L_c$. □

Theorem 34. *Assume $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 \ge 1$, then $S_{(\mathscr{H}_{p(.)})_\rho}$ is simple.*

Proof. Suppose $T \in L_C(S_{(\mathscr{H}_{p(\cdot)})_\rho})$ and $T \notin \Lambda(S_{(\mathscr{H}_{p(\cdot)})_\rho})$. In view of Lemma 3, we have $G, B \in L(S_{(\mathscr{H}_{p(\cdot)})_\rho})$ so as to $BTGI_k = I_k$. One gets $I_{S_{(\mathscr{H}_{p(\cdot)})_\rho}} \in L_C(S_{(\mathscr{H}_{p(\cdot)})_\rho})$. Therefore, $L(S_{(\mathscr{H}_{p(\cdot)})_\rho}) = L_C(S_{(\mathscr{H}_{p(\cdot)})_\rho})$. This implies one and only one nontrivial closed ideal $\Lambda(S_{(\mathscr{H}_{p(\cdot)})_\rho})$ in $L(S_{(\mathscr{H}_{p(\cdot)})_\rho})$. □

4.5. Spectrum of Prequasi-Ideal. In this part, we introduce enough settings on $(\mathscr{H}_{p(\cdot)})_\rho$ so that the class L with sequence of eigenvalues in $(\mathscr{H}_{p(\cdot)})_\rho$ equals $S_{(\mathscr{H}_{p(\cdot)})_\rho}$.

Theorem 35. *If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* dim $(\mathfrak{X})$ = dim $(\mathfrak{Y}) = \infty$. *Suppose $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, we have*

$$\left(S_{(\mathscr{H}_{p(\cdot)})_\rho}\right)^\lambda(\mathfrak{X}, \mathfrak{Y}) = S_{(\mathscr{H}_{p(\cdot)})_\rho}(\mathfrak{X}, \mathfrak{Y}). \quad (47)$$

Proof. Let $T \in (S_{(\mathscr{H}_{p(\cdot)})_\rho})^\lambda(\mathfrak{X}, \mathfrak{Y})$, then $f_\lambda \in ((\mathscr{H}_{p(\cdot)})_\rho)$, where $f_\lambda(z) = \sum_{v=0}^\infty \lambda_v(T) z^v$ converges for all $z \in \mathbb{C}$ with $\rho(f_\lambda) = \sum_{v=0}^\infty (1/p_v)|\lambda_v(T)|^{p_v} < \infty$, and $\|T - \lambda_v(T)I\| = 0$ for all $v \in \mathbb{N}$. We have $T = \lambda_v(T)I$, with $v \in \mathbb{N}$, hence $s_v(T) = s_v(\lambda_v(T)I) = |\lambda_v(T)|$, with $v \in \mathbb{N}$. As a result, $f_s \in (H_{p(.)})_\rho$, then $T \in S_{(H_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y})$. Secondly, assume $T \in S_{(H_{p(.)})_\rho}(\mathfrak{X}, \mathfrak{Y})$. Hence, $f_s \in (H_{p(.)})_\rho$, where $f_s(z) = \sum_{v=0}^\infty s_v(T) z^v$ converges for all $z \in \mathbb{C}$ with $\rho(f_s) = \sum_{v=0}^\infty (1/p_v)|s_v(T)|^{p_v} < \infty$. One has

$$\sum_{v=0}^\infty \frac{1}{p_v}|s_v(T)|^{p_v} \ge \frac{1}{\sup_v p_v}\sum_{v=0}^\infty [s_v(T)]^{p_v}. \quad (48)$$

Therefore, $\lim_{v\longrightarrow\infty} s_v(T) = 0$. Let $\|T - s_v(T)I\|^{-1}$ exists, for all $v \in \mathbb{N}$. Hence, $\|T - s_v(T)I\|^{-1}$ exists and bounded, for all $v \in \mathbb{N}$. Therefore, $\lim_{v\longrightarrow\infty} \|T - s_v(T)I\|^{-1} = \|T\|^{-1}$ exists and bounded. By using the prequasioperator ideal of $(S_{(\mathscr{H}_{p(\cdot)})_\rho}, g)$, one has

$$I = TT^{-1} \in S_{(\mathscr{H}_{p(\cdot)})_\rho}(\mathfrak{X}, \mathfrak{Y}) \Rightarrow (s_v(I))_{v=0}^\infty \in \left(\mathscr{H}_{p(\cdot)}\right)_\rho \Rightarrow \lim_{v\longrightarrow\infty} s_v(I) = 0. \quad (49)$$

Since $\lim_{v\longrightarrow\infty} s_v(I) = 1$. Hence, $\|T - s_v(T)I\| = 0$, for all $v \in \mathbb{N}$. This gives $T \in (S_{(\mathscr{H}_{p(\cdot)})_\rho})^\lambda(\mathfrak{X}, \mathfrak{Y})$.

This shows the proof. □

5. Weighted Shift Operators on $(\mathscr{H}_{p(\cdot)})_\rho$

In this section, we present the upper bounds of s-numbers for infinite series of the weighted nth power forward and backward shift operator on $\mathscr{H}_{p(.)}$ with applications to some entire functions.

Theorem 36. *Assume $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then $V_z \in L((\mathscr{H}_{p(.)})_\rho)$ with*

$$\|V_z\| = \sup_r \left(\frac{p_r}{p_{r+1}}\right)^{1/\varpi_p}, \quad (50)$$

where $\rho(f) = [\sum_{r=0}^\infty (1/p_r)|f_r \wedge|^{p_r}]^{1/\varpi_p}$, for all $f \in (\mathscr{H}_{p(.)})_\rho$.

Proof. Suppose the setups are verified. For $f \in (H_{p(.)})_\rho$. Since $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then $\rho(V_z f) = \rho(zf) = [\sum_{r=0}^\infty (1/p_{r+1})|f_r \wedge|^{p_{r+1}}]^{1/\varpi_p} \le [\sum_{r=0}^\infty (1/p_{r+1})|f_r \wedge|^{p_{r+1}}]^{1/\varpi_p} \le \sup_r (p_r/p_{r+1})^{1/\varpi_p} [\sum_{r=0}^\infty (1/p_r)|f_r \wedge|^{p_{r+1}}]^{1/\varpi_p} = \sup_r (p_r/p_{r+1})^{1/\varpi_p} \rho(f)$.

Therefore, $V_z \in L((\mathscr{H}_{p(\cdot)})_p)$ with $\|V_z\| \le \sup_r (p_r/p_{r+1})^{1/\varpi_p}$. Since $V_z \in L((\mathscr{H}_{p(.)})_\rho)$. Then, there is $A > 0$ with $\rho(V_z f) \le A\rho(f)$, for all $f \in (\mathscr{H}_{p(.)})_\rho$. Hence, $\rho(V_z e^{(r)}) \le A\rho(e^{(r)})$, one gets $\sup_r (p_r/p_{r+1})^{1/\varpi_p} \le \|V_z\|$.

This completes the proof. □

Theorem 37. *Consider $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then $B_z \in L((\mathscr{H}_{p(.)})_\rho)$ with*

$$\|B_z\| = \sup_r \left(\frac{p_{r+1}}{p_r}\right)^{1/\varpi_p}, \quad (51)$$

where $\rho(f) = [\sum_{r=0}^\infty (1/p_r)|f_r \wedge|^{p_r}]^{1/\varpi_p}$, for every $f \in (\mathscr{H}_{p(.)})_\rho$.

Proof. Let the given settings hold for every $f \in (\mathscr{H}_{p(.)})_\rho$. Since $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 > 0$, then

$$\rho(B_z f) = \left[\sum_{r=0}^\infty \frac{1}{p_r}|f_{r+1}\wedge|^{p_r}\right]^{1/\varpi_p} \le \sup_r \left(\frac{p_{r+1}}{p_r}\right)^{1/\varpi_p} \left[\sum_{r=0}^\infty \frac{1}{p_{r+1}}|f_{r+1}\wedge|^{p_r}\right]^{1/\varpi_p} \le \sup_r \left(\frac{p_{r+1}}{p_r}\right)^{1/\varpi_p} \left[\sum_{r=0}^\infty \frac{1}{p_r}|f_r\wedge|^{p_r}\right]^{1/\varpi_p} = \sup_r \left(\frac{p_{r+1}}{p_r}\right)^{1/\varpi_p} \rho(f). \quad (52)$$

Therefore, $B_z \in L((\mathscr{H}_{p(\cdot)})_\rho)$ with $\|B_z\| \le \sup_r (p_{r+1}/p_r)^{1/\varpi_p}$. Since $B_z \in L((\mathscr{H}_{p(\cdot)})_\rho)$. Then, there is $A > 0$ with $\rho(B_z f) \le A\rho(f)$, for all $f \in (\mathscr{H}_{p(\cdot)})_\rho$. Hence, $\rho(B_z e^{(r+1)}) \le A\rho(e^{(r+1)})$, then $\sup_r (p_{r+1}/p_r)^{1/\varpi_p} \le \|B_z\|$. This completes the proof. □

Theorem 38. *Let $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 \ge 1$. Suppose $\limsup_{v\longrightarrow\infty}(1/\sqrt[v]{p_v}) = 1$, then every function in $(\mathscr{H}_{p(\cdot)})_\rho$ is analytic on the open unit disc $\mathbb{D}$. Moreover, the convergence in $(\mathscr{H}_{p(\cdot)})_\rho$ implies the uniform convergence on compact subsets of $\mathbb{D}$, where $\rho(f) = [\sum_{r=0}^{\infty}(1/p_r)|f_r\wedge|^{p_r}]^{1/\varpi_p}$, for any $f \in (\mathscr{H}_{p(\cdot)})_\rho$.*

Proof. Suppose $\limsup_{v\longrightarrow\infty}(1/\sqrt[v]{p_v}) = 1$, and $f \in (\mathscr{H}_{p(\cdot)})_\rho$.Therefore, $f(z) = \sum_{v=0}^{\infty}\widehat{f}_v z^v$ converges for every $z \in \mathbb{C}$ and $\rho(f) = [\sum_{v=0}^{\infty}(1/p_v)|f_v\wedge|^{p_v}]^{1/\varpi_p} < \infty$. Hence, $\limsup_{v\longrightarrow\infty}\sqrt[v]{(1/p_v)|f_v\wedge|^{p_v}} < 1$. We have

$$\limsup_{v\longrightarrow\infty}\sqrt[v]{|f_v\wedge|^{p_v}} < \frac{1}{\limsup_{v\longrightarrow\infty}(1/\sqrt[v]{p_v})} = 1. \tag{53}$$

Since $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 \ge 1$, we obtain $\limsup_{v\longrightarrow\infty}\sqrt[v]{|\widehat{f_v}|}|z| < |z| < |1|$, for all $z \in \mathbb{D}$. Hence, $f(z) = \sum_{v=0}^{\infty}\widehat{f}_v z^v$ converges for every complex value of $z \in D$. Assume A is a compact subset of $\mathbb{D}$ and $f^k(z) \in A$, for all $k \in \mathbb{N}$. Let f^k converges to $f \in (\mathscr{H}_{p(\bullet)})_\rho$, we have

$$\begin{aligned}
\left|f^k(z) - f(z)\right| &= \left|\sum_{v=0}^{\infty}\left(\widehat{f_v^k} - \widehat{f}_v\right)z^v\right| \le \sum_{v=0}^{\infty}\left|\widehat{f_v^k} - \widehat{f}_v\right||z^v| \\
&\le \left[\sum_{v=0}^{\infty}\frac{1}{p_v}\left|f_v^k\wedge - f_v\wedge\right|^{p_v}\right]^{1/\varpi_p}\left[\sum_{v=0}^{\infty}p_v^{q_v}|z|^{vq_v}\right]^{1/\varpi_p} \\
&= \left[\sum_{v=0}^{\infty}p_v^{q_v}|z|^{vq_v}\right]^{1/\varpi_p}\rho\left(f^k - f\right),
\end{aligned} \tag{54}$$

where $(q_v) \in mi\nearrow \bigcap \ell_\infty$ with $q_0 \ge 1$ and $(1/p_v) + (1/q_v) = 1$, for all $v \in \mathbb{N}$. Clearly, $\limsup_{v\longrightarrow\infty} p_v^{q_v/v}|z|^{q_v} < 1$, then $[\sum_{v=0}^{\infty}p_v^{q_v}|z|^{vq_v}]^{1/\varpi_p} < \infty$. So $\lim_{k\longrightarrow\infty}f^k(z) = f(z) \in A$. □

Theorem 39. *Assume V_z is the forward shift operator on $(\mathscr{H}_{p(\cdot)})_\rho$, with $\rho(f) = [\sum_{r=0}^{\infty}(1/p_r)|f_r\wedge|^{p_r}]^{1/\varpi_p}$, for all $f \in (\mathscr{H}_{p(\cdot)})_\rho$. Then,*

$$\sup_{card\xi=r+1}\inf_{k\in\xi}\left(\frac{p_k}{p_{k+n}}\right)^{1/\varpi_p}\frac{1}{A_n} \le s_r(V_z^n) \le \sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\left(\frac{p_k}{p_{k+n}}\right)^{1/\varpi_p}, \tag{55}$$

where $A_n = [\sum_{k=0}^{\infty}(1/p_k)|f_k\wedge|^{p_k}]^{1/\varpi_p}/[\sum_{k=\xi}^{\infty}(1/p_k)|f_k\wedge|^{p_{k+n}}]^{1/\varpi_p}$..

Proof. Let card $\xi = r+1$ and as $V_z^n f \in (\mathscr{H}_{p(\cdot)})_\rho$, for all $f \in (\mathscr{H}_{p(\cdot)})_\rho$, where $f(z) = \sum_{k=0}^{\infty}\widehat{f_k}z^k$ converges for every $z \in \mathbb{C}$ and $\rho(f) = [\sum_{k=0}^{\infty}(1/p_k)|\overline{f_k}|^{p_k}]^{1/\varpi_p} < \infty$. Hence, $V_z^n f(z) = \sum_{k=0}^{\infty}\widehat{f_k}z^{k+n}$ and $\rho(V_z^n f) = [\sum_{k=0}^{\infty}(a/p_{k+n})|f_k\wedge|^{p_{k+n}}]^{1/\varpi_p} < \infty$ .Assume P_ξ is and operator on $(\mathscr{H}_{p(\cdot)})_\rho$ with rank $P_\xi = r+1$ defined by

$$\left(p_\xi g\right)(z) = P_\xi\left(\sum_{k=0}^{\infty}\widehat{f_k}z^{k+n}\right) = \sum_{k\in\xi}^{i}\widehat{f_k}z^{k+n}. \tag{56}$$

Since $\rho(P_\xi g) = [\sum_{k\in\xi}(1/p_{k+n})|f_k\wedge|^{p_{k+n}}]^{1/\varpi_p} \le [\sum_{k=0}^{\infty}(1/p_{k+n})|f_k\wedge|^{p_{k+n}}]^{1/\varpi_p} = \rho(g)$. This implies $\|P_\xi\| \le 1$.

Define an operator S_z^n by $(S_z^n h)(z) = S_z^n(\sum_{k\in\xi}\widehat{f_k}z^{k+n}) = \sum_{k=0}^{\infty}\widehat{f_k}z^k$, then

$$\rho(S_z^n h) = \left[\sum_{k=0}^{\infty}\frac{1}{p_k}|f_k\wedge|^{p_k}\right]^{1/\varpi_p} \le U_n\left[\sum_{k\in\xi}\frac{1}{p_{k+n}}|f_k\wedge|^{p_{k+n}}\right]^{1/\varpi_p} = U_n\rho(h). \tag{57}$$

Hence, $\|S_z^n\| \le U_n$, where

$$1 \le U_n = \frac{\left[\sum_{k=0}^{\infty}(1/p_k)|f_k\wedge|^{p_k}\right]^{1/\varpi_p}}{\left[\sum_{k\in\xi}|1/p_{k+n}|f_k\wedge|^{p_{k+n}}\right]^{1/\varpi_p}} < \infty. \tag{58}$$

Therefore, the identity map will be $I_{r+1} = P_\xi V_z^n S_z^n$, according to the definition of s-numbers, we have

$$\begin{gathered}
s_r(I_{r+1}) = 1 \le \|P_\xi\|s_r(V_z^n)\|s_z^n\| \le s_r(V_z^n)\|s_z^n\| \Rightarrow \\
s_r(V_z^n) \ge \frac{1}{\|S_z^n\|} \le \frac{1}{U_n} = \frac{\left[\sum_{k\in\xi}|1/p_{k+n}|f_k\wedge|^{p_{k+n}}\right]^{1/\varpi_p}}{\left[\sum_{k=0}^{\infty}(1/p_k)|f_k\wedge|^{p_k}\right]^{1/\varpi_p}} \ge \inf_{k\in\xi}\left(\frac{p_k}{p_{k+n}}\right)^{1/\varpi_p}\frac{1}{A_n}.
\end{gathered} \tag{59}$$

This inequality is satisfied for all card, $\xi = r+1$ and one has

$$s_r(V_z^n) \ge \sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\left(\frac{p_k}{p_{k+n}}\right)^{1/\varpi_p}\frac{1}{A_n}. \tag{60}$$

On the other hand, let ξ be a subset of $\mathbb{N}$ with card $\xi = r$. Define the finite rank map R_z^n by $(R_z{}^n v)(z) = R_z^n(\sum_{k=0}^{\infty}\widehat{f}_k z^k) = \sum_{k\in\xi}\widehat{f}_k z^{k+n}$. In view of the definition of approximation numbers, we have

$$s_r(V_z^n) \le \alpha_r(V_z^n) \le \|V_z^n - R_z^n\| \le \sup_{|f(z)|\neq 0} \frac{|(V_z^n - R_z^n)f(z)|}{|f(z)|}$$
$$= \sup_{|f(z)|\neq 0} \frac{\left|\sum_{k\notin\xi}\widehat{f}_k z^{k+n}\right|}{|f(z)|} \le \sup_{|f(z)|\neq 0} \frac{\left[\sum_{k\notin\xi}(1/p_{k+n})\left|\widehat{f}_k\right|^{p_{k+n}}\right]^{1/\varpi_p}}{|f(z)|}$$
$$\le \sup_{k\notin\xi}\left(\frac{p_k}{p_{k+n}}\right)^{1/\varpi_p}. \quad (61)$$

This inequality is verified for every card $\xi = r$ and by using Lemma 10, one has

$$\sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\left(\frac{p_k}{p_{k+n}}\right)^{1/\varpi_p}\frac{1}{A_n} \le s_r(V_z^n) \le \inf_{\text{card }\xi=r}\sup_{k\notin\xi}\left(\frac{p_k}{p_{k+n}}\right)^{1/\varpi_p}$$
$$= \sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\left(\frac{p_k}{p_{k+n}}\right)^{1/\varpi_p}. \quad (62)$$

This completes the proof. □

Theorem 40. *If B_z is the backward shift operator on $(\mathscr{H}_{p(.)})_\rho$, with $\rho(f) = [\sum_{r=0}^{\infty}(1/p_r)|\widehat{f}_r|^{p_r}]^{1/\varpi_p}$, for all $f \in (\mathscr{H}_{p(.)})_\rho$ Then,*

$$\sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\left(\frac{p_{k+n}}{p_k}\right)^{1/\varpi_p}\frac{1}{G_n} \le s_r(B_z^n) \le \sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\left(\frac{p_{k+n}}{p_k}\right)^{1/\varpi_p}, \quad (63)$$

where $G_n = [\sum_{k=0}^{\infty}(1/p_k)|f_k\wedge|^{p_k}]^{1/\varpi_p}/[\sum_{k\in\xi}|1/p_{k+n}|f_k\wedge|^{p_{k+n}}]^{1/\varpi_p}$.

Proof. Assume card $\xi = r+1$ and since $B_z^n f \in (\mathscr{H}_{p(.)})_\rho$, for every $f \in (\mathscr{H}p_{(.)})_\rho$, where $f(z) = \sum_{k=0}^{\infty}\widehat{f}_k z^k$ converges for any $z \in \mathbb{C}$ and $\rho(f) = [\sum_{k=0}^{\infty}(1/p_k)|f_k\wedge|^{p_k}]^{1/\varpi_p} < \infty$. Therefore, $B_z^n f(z)\sum_{k=0}^{\infty}\widehat{f}_{k+n}z^k$ and $\rho(B_z^n f) = [\sum_{k=0}^{\infty}(1/p_k)|f_k\wedge|^{p_k}]^{1/\varpi_p} < \infty$ .Suppose P_ξ is an operator on $(\mathscr{H}_{p(.)})_\rho$ with rank $P_\xi = r+1$ evident by

$$(p_\xi g)(z) = P_\xi\left(\sum_{k=0}^{\infty}\widehat{f}_{k+n}z^k\right) = \sum_{k\in\xi}^{\infty}\widehat{f}_{k+n}z^k. \quad (64)$$

As $\rho(P\xi g) = [\sum_{k\in\xi}(1/p_k)|\widehat{f}_{k+n}|^{p_k}]^{1/\varpi_p} \le [\sum_{k=0}^{\infty}(1/p_k)|f_{k+n}\wedge|^{p_k}]^{1/\varpi_p} = p(g)$. This implies $\|p_\xi\| \le 1$. Define an operator S_z^n *by* $(S_z^n h)(z) = S_z^n(\sum_{k\in\xi}\widehat{f}_{k+n}z^k) = \sum_{k=0}^{\infty}\widehat{f}_k z^k$, one gets

$$\rho(S_z^n h) = \left[\sum_{k=0}^{\infty}\frac{1}{p_k}\left|\widehat{f}_k\right|^{p_k}\right]^{1/\varpi_p} \le U_n\left[\sum_{k\in\xi}\left|\widehat{f}_{k+n}\right|^{p_k}\right]^{1/\varpi_p} = U_n\rho(h). \quad (65)$$

Therefore, $\|S_z^n\| \le U_n$, where $1 \le U_n = [\sum_{k=0}^{\infty}(1/p_k)|\widehat{f}_k|^{p_k}]^{1/\varpi_p}/[\sum_{k\in\xi}|(1/p_k)|\widehat{f}_{k+n}|^{p_k}]^{1/\varpi_p} < \infty$. Hence, the identity operator will be $I_{r+1} = P_\xi B_z^n S_z^n$, in view of the definition of s-numbers, one has

$$s_r(I_{r+1}) = 1 \le \|P_\xi\| s_r(B_z^n)\|S_z^n\| \le s_r(B_z^n)\|S_z^n\| \Rightarrow$$
$$s_r(B_z^n) \ge \frac{1}{\|S_z^n\|} \ge \frac{1}{U_n} = \frac{\left[\sum_{k\in\xi}|(1/p_k)\left|\widehat{f}_{k+n}\right|^{p_k}\right]^{1/\varpi_p}}{\left[\sum_{k=0}^{\infty}(1/p_k)\left|\widehat{f}_k\right|^{p_k}\right]^{1/\varpi_p}} \ge \inf_{k\in\xi}\left(\frac{p_{k+n}}{p_k}\right)^{1/\varpi_p}\frac{1}{G_n}. \quad (66)$$

This inequality is confirmed for all card $\xi = r+1$, and we have

$$s_r(B_z^n) \ge \sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\left(\frac{p_{k+n}}{p_k}\right)^{1/\varpi_p}\frac{1}{G_n}. \quad (67)$$

On the other hand, suppose ξ is a subset of $\mathbb{N}$ with card $\xi = r$. Define the finite rank operator R_z^n by $(R_z^n v)(z) = R_z^n(\sum_{k=0}^{\infty}\widehat{f}_k z^k) = \sum_{k\in\xi}\widehat{f}_{k+n}z^k$. From the definition of approximation numbers, one gets

$$s_r(B_z^n) \le \alpha_r(B_z^n) \le \|B_z^n - R_z^n\| \le \sup_{|f(z)|\neq 0}\frac{|(B_z^n - R_z^n)f(z)|}{|f(z)|}$$
$$= \sup_{|f(z)|\neq 0}\frac{\left|\sum_{k\notin\xi}\widehat{f}_{k+n}z^k\right|}{|f(z)|} \le \sup_{|f(z)|\neq 0}\frac{\left[\sum_{k\notin\xi}1/p_k\left|\widehat{f}_{k+n}\right|^{p_k}\right]^{1/\varpi_p}}{|f(z)|}$$
$$\le \sup_{k\notin\xi}\left(\frac{p_{k+n}}{p_k}\right)^{1/\varpi_p}. \quad (68)$$

This inequality is satisfied for any card $\xi = r$ and from Lemma 10, we have

$$\sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\left(\frac{p_{k+n}}{p_k}\right)^{1/\varpi_p}\frac{1}{G_n} \le s_r(B_z^n) \le \inf_{\text{card }\xi=r}\sup_{k\in\xi}\left(\frac{p_{k+n}}{p_k}\right)^{1/\varpi_p}$$
$$= \sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\left(\frac{p_{k+n}}{p_k}\right)^{1/\varpi_p}. \quad (69)$$

This finishes the proof. □

Next, the upper and lower bounds of norm $\sum_{m=0}^{\infty} c_m V_z^m$ on the space $(\mathscr{H}p_{(.)})_\rho$ have been explained.

Theorem 41. *The effect of* $\sum_{m=0}^{\infty} c_m V_z^m$ *on the space* $(\mathscr{H}_{p(.)})_p$, *where* $\rho(f) = [\sum_{r=0}^{\infty} 1/p_r |\widehat{f}_r|^{p_r}]^{1/\varpi_p}$, *for all* $f \in (\mathscr{H}_{p(.)})_p$, $(c_m)_{m=0}^{\infty} \in \ell^{(pm)/\varpi_p}$, *and* $(p_v) \in mi\nearrow \cap \ell_\infty$ *with* $p_0 \geq 1$, *we have*

$$\sup_k \left[\sum_{m=0}^{\infty} |c_m|^{p_{m+k}} \frac{p_k}{p_{k+m}}\right]^{1/\varpi_p} \leq \left\|\sum_{m0}^{\infty} c_m V_z^m\right\| \leq \sup_{m,k} \left(\frac{p_k}{p_{k+m}}\right)^{1/\varpi_p} \sum_{m=0}^{\infty} |c_m|^{p_m/\varpi_p}. \tag{70}$$

Proof. Assume $f \in (\mathscr{H}_{p(.)})_\rho$, we have $\sum_{m=0}^{\infty} c_m V_z^m f(z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} c_m \widehat{f}_k z^{k+m}$. Then,

$$\left\|\sum_{m=0}^{\infty} c_m V_z^m\right\| \geq \frac{\rho\left(\sum_{m=0}^{\infty} c_m V_z^m e^{(k)}\right)}{\rho\left(e^{(k)}\right)} = \left[\frac{\sum_{m=0}^{\infty} (1/p_{m+k}) |c_m|^{p_{m+k}}}{1/p_k}\right]^{1/\varpi_p} \geq \sup_k \left[\sum_{m=0}^{\infty} |c_m|^{p_{m+k}} \frac{p_k}{p_{k+m}}\right]^{1/\varpi_p}. \tag{71}$$

Since ρ satisfies the triangle inequality, we get

$$\begin{aligned}\left\|\sum_{m=0}^{\infty} c_m V_z^m\right\| &= \sup_{\rho(f)\neq 0} \frac{\rho(\sum_{m=0}^{\infty} c_m V_z^m f)}{\rho(f)} \\ &\leq \sup_{\rho(f)\neq 0} \frac{\sum_{m=0}^{\infty} \left[\sum_{k=0}^{\infty} (1/p_{m+k}) \left(|c_m| \left|\widehat{f}_k\right|\right)^{p_{m+k}}\right]^{1/\varpi_p}}{\left[\sum_{k=0}^{\infty} (1/p_k) \left|\widehat{f}_k\right|^{p_k}\right]^{1/\varpi_p}} \\ &\leq \sup_{m,k} \left(\frac{p_k}{p_{k+m}}\right)^{1/\varpi_p} \frac{\sum_{m=0}^{\infty} \left[\sum_{k=0}^{\infty} (1/p_k) \left(|c_m| \left|\widehat{f}_k\right|\right)^{p_{m+k}}\right]^{1/\varpi_p}}{\left[\sum_{k=0}^{\infty} (1/p_k)(1/p_k) \left|\widehat{f}_k\right|^{p_k}\right]^{1/\varpi_p}} \\ &\leq \sup_{m,k} \left(\frac{p_k}{p_{k+m}}\right)^{1/\varpi_p} \sum_{m=0}^{\infty} |c_m|^{p_m/\varpi_p}.\end{aligned} \tag{72}$$

Next the upper and lower bounds of norm $\sum_{m=0}^{\infty} c_m B_z^m$ on the space $(\mathscr{H}_{p(.)})_\rho$ have been investigated. □

Theorem 42. *The effect of* $\sum_{m=0}^{\infty} c_m B_z^m$ *on the space* $(\mathscr{H}_{p(.)})_\rho$, *where* $\rho(f) = [\sum_{r=0}^{\infty} 1/p_r |\widehat{f}_r|^{p_r}]^{1/\varpi_p}$, *for all* $f \in (\mathscr{H}_{p(.)})_\rho$, $(c_m)_{m=0}^{\infty} \in \ell^{(p_m)/\varpi_p}$, *and* $(p_v) \in mi\nearrow \cap \ell_\infty$ *with* $p_0 \geq 1$, *we have*

$$\sup_k \left[\sum_{m=0}^{\infty} |c_m|^{p_k} \frac{p_{k+m}}{p_k}\right]^{1/\varpi_p} \leq \left\|\sum_{m=0}^{\infty} c_m B_z^m\right\| \leq \sup_{m,k} \left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p} \sum_{m=0}^{\infty} |c_m|^{p_m/\varpi_p}. \tag{73}$$

Proof. Suppose $f \in (\mathscr{H}_{p(.)})_\rho$, one has $\sum_{m=0}^{\infty} c_m B_z^m f(z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} c_m \widehat{f}_{k+m} z^k$. We have

$$\left\|\sum_{m=0}^{\infty} c_m B_z^m\right\| \geq \frac{\rho\left(\sum_{m=0}^{\infty} c_m B_z^m e^{(k)}\right)}{\rho\left(e^{(k)}\right)} = \left[\frac{\sum_{m=0}^{\infty} (1/p_{k-m}) |c_m|^{p_{k-m}}}{1/p_k}\right]^{1/\varpi_p} \geq \sup_k \left[\sum_{m=0}^{\infty} |c_m|^{p_k} \frac{p_{k+m}}{p_k}\right]^{1/\varpi_p}. \tag{74}$$

As ρ verifies the triangle inequality, one can see

$$\begin{aligned}\left\|\sum_{m=0}^{\infty} c_m B_z^m\right\| &= \sup_{\rho(f)\neq 0} \frac{\rho(\sum_{m=0}^{\infty} c_m B_z^m f)}{\rho(f)} \\ &\leq \sup_{\rho(f)\neq 0} \frac{\sum_{m=0}^{\infty} \left[\sum_{k=0}^{\infty} (1/p_k) \left(|c_m| \left|\widehat{f}_{k+m}\right|\right)^{p_k}\right]^{1/\varpi_p}}{\left[\sum_{k=0}^{\infty} (1/p_k) \left|\widehat{f}_k\right|^{p_k}\right]^{1/\varpi_p}} \\ &\leq \sup_{m,k} \left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p} \frac{\sum_{m=0}^{\infty} \left[\sum_{k=0}^{\infty} (1/p_{k+m}) \left(|c_m| \left|\widehat{f}_{k+m}\right|\right)^{p_k}\right]^{1/\varpi_p}}{\left[\sum_{k=0}^{\infty} (1/p_k) \left|\widehat{f}_k\right|^{p_k}\right]^{1/\varpi_p}} \\ &\leq \sup_{m,k} \left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p} \sum_{m=0}^{\infty} |c_m|^{p_m/\varpi_p}.\end{aligned} \tag{75}$$

The following theorem indicates an upper estimation to the s-numbers of $\sum_{m=0}^{\infty} c_m V_z^m$ on the space $(\mathscr{H}p_{(.)})_\rho$. □

Theorem 43. *The effect of* $\sum_{m=0}^{\infty} c_m V_z^m$ *on the space* $(\mathscr{H}p_{(.)})_\rho$, *where* $p(f) = [\sum_{r=0}^{\infty} 1/p_r |\widehat{f}_r|^{p_r}]^{1/\varpi_p}$, *for all* $f \in (\mathscr{H}p_{(.)})_\rho$, *the s -numbers of this operator are presented by*

$$s_r\left(\sum_{m=0}^{\infty} c_m V_z^m\right) \leq \sup_{\text{card } \xi = r+1} \inf_{k\in\xi} \sup_m \left(\frac{p_k}{p_{k+m}}\right)^{1/\varpi_p} \sum_{m=0}^{\infty} |c_m|^{p_m/\varpi_p}, \tag{76}$$

for all $(c_m)_{m=0}^{\infty} \in \ell^{(p_m)/\varpi_p}$ *and* $(p_v) \in mi_\nearrow \cap \ell_\infty$ *with* $p_0 \geq 1$.

Proof. Let ξ be a subset of $\mathbb{N}$ and card $\xi = r$. By using the definition of s-numbers. Define the finite rank operator R by $R f(z) = R(\sum_{k=0}^{\infty} \widehat{f}_k z^k) = \sum_{k\in\xi} \sum_{m=0}^{k} c_m \widehat{f}_{k-m} z^k$. In view of the definition of approximation numbers and since ρ satisfies the triangle inequality, we have

$$s_r\left(\sum_{m=0}^{\infty} c_m V_z^m\right) \le \alpha_r\left(\sum_{m=0}^{\infty} c_m V_z^m\right) \le \left\|\sum_{m=0}^{\infty} c_m V_z^m - R\right\| \le \sup_{\rho(f)\neq 0} \frac{\rho(\sum_{m=0}^{\infty} c_m V_z^m f - Rf)}{\rho(f)} \le \sup_{\rho(f)\neq 0} \frac{\sum_{m=0}^{\infty}\left[\sum_{k\notin\xi}(1/p_{m+k})\left(|c_m|\left|\widehat{f}_k\right|\right)^{p_{k+m}}\right]^{1/\varpi_p}}{\rho(f)} \le \sup_{k\notin\xi,m}\left(\frac{p_k}{p_{k+m}}\right)^{1/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p}. \quad (77)$$

This inequality is verified for every card $\xi = r$, and one has

$$s_r\left(\sum_{m=0}^{\infty} c_m V_z^m\right) \le \inf_{\text{card }\xi=r}\sup_{k\notin\xi,m}\left(\frac{p_k}{p_{k+m}}\right)^{1/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p} = \sup_{\text{card }\xi=r+1}\inf_{k\notin\xi}\sup_{m}\left(\frac{p_k}{p_{k+m}}\right)^{1/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p}. \quad (78)$$

This implies the proof. □

The next theorem investigates an upper estimation to the s-numbers of $\sum_{m=0}^{\infty} c_m B_z^m$ on the space $(\mathscr{H}_{p(.)})_\rho$.

Theorem 44. *Acting $\sum_{m=0}^{\infty} c_m B_z^m$ on the space $(\mathscr{H}_{p(.)})_\rho$, where $\rho(f) = [\sum_{r=0}^{\infty}(1/p_r)|\widehat{f}_r|^{p_r}]^{1/\varpi_p}$, for every $f \in (\mathscr{H}_{p(.)})_\rho$, the s-numbers of this operator satisfy*

$$s_r\left(\sum_{m=0}^{\infty} c_m B_z^m\right) \le \sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\sup_{m}\left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p}, \quad (79)$$

for all $(c_m)_{m=0}^{\infty} \in \ell^{(p_m)/\varpi_p}$ and $(p_v) \in mi\nearrow \cap \ell_\infty$ with $p_0 \ge 1$.

Proof. Assume ξ is a subset of $\mathbb{N}$ and card $\xi = r$. From the definition of s-numbers. Define the finite rank operator R by $Rf(z) = R(\sum_{k=0}^{\infty}\widehat{f}_k z^k) = \sum_{k\in\xi}\sum_{m=0}^{k} c_m \widehat{f}_{k-m} z^k$. From the definition of approximation numbers and as ρ verifies the triangle inequality, one has

$$s_r\left(\sum_{m=0}^{\infty} c_m B_z^m\right) \le \alpha_r\left(\sum_{m=0}^{\infty} c_m B_z^m\right) \le \left\|\sum_{m=0}^{\infty} c_m B_z^m - R\right\| \le \sup_{\rho(f)\neq 0} \frac{\rho(\sum_{m=0}^{\infty} c_m B_z^m f - Rf)}{\rho(f)} \le \sup_{\rho(f)\neq 0} \frac{\sum_{m=0}^{\infty}\left[\sum_{k\notin\xi}(1/p_k)\left(|c_m|\left|\widehat{f}_{k+m}\right|\right)^{p_k}\right]^{1/\varpi_p}}{\rho(f)} \le \sup_{k\notin\xi,m}\left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p}. \quad (80)$$

This inequality is satisfied for all card $\xi = r$, and we have

$$s_r\left(\sum_{m=0}^{\infty} c_m B_z^m\right) \le \inf_{\text{card }\xi=r}\sup_{k\notin\xi,m}\left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p} = \sup_{\text{card }\xi=r+1}\inf_{k\notin\xi}\sup_{m}\left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p}. \quad (81)$$

This completes the proof. □

The following theorems are direct consequences of Theorem 43 and Definition 18, for some entire functions, for example, the exponential and the sine functions.

Theorem 45. *Let $(p_v) \in mi_\nearrow \cap \ell_\infty$ with $p_0 \ge 1$. Assume B_{e^z} is a shift operator on $(\mathscr{H}_{p(.)})_\rho$, where for $\rho(f) = [\sum_{r=0}^{\infty}(1/p_r)|\widehat{f}_r|^{p_r}]^{1/\varpi_p}$, for all $f \in (\mathscr{H}_{p(.)})_\rho$ and $e^z = \sum_{m=0}^{\infty} z^m/m!$. The upper estimation of the s-number of V_{e^z} is given by*

$$s_r(V_{e^z}) \le \sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\sup_{m}\left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p}\sum_{m=0}^{\infty}\left(\frac{1}{m!}\right)^{p_m/\varpi_p}. \quad (82)$$

Theorem 46. *Let $(p_v) \in mi_\nearrow \cap \ell_\infty$ with $p_0 \ge 1$. Suppose $B_{\sin(z)}$ is a shift operator on $(\mathscr{H}_{p(.)})_\rho$, where $\rho(f) = [\sum_{r=0}^{\infty}(1/p_r)|\widehat{f}_r|^{p_r}]^{1/\varpi_p}$, for all $f \in (\mathscr{H}_{p(.)})_\rho$ and $\sin(z) = \sum_{m=0}^{\infty}(-1)^m(z^{2m+1}/(2m+1)!)$. The upper estimation of the s-numbers of $V_{\sin(z)}$ is presented by*

$$s_r\left(V_{\sin(z)}\right) \le \sup_{\text{card}\xi=r+1}\inf_{k\in\xi}\sup_{m}\left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p}\sum_{m=0}^{\infty}\left(\frac{1}{(2m+1)!}\right)^{p_m/\varpi_p}. \quad (83)$$

The following theorems are direct consequences of Theorem 44 and Definition 19, for some entire functions, for example, the exponential and the sine functions.

Theorem 47. *Assume* $(p_v) \in mi_{\nearrow} \cap \ell_\infty$ *with* $p_0 \geq 1$. *Suppose* B_{e^z} *is a shift operator on* $(\mathscr{H}_{p(.)})_\rho$, *where* $\rho(f) = [\sum_{r=0}^{\infty}(1/p_r)|\widehat{f}_r|^{p_r}]^{1/\varpi_p}$, *for every* $f \in (\mathscr{H}_{p(.)})_\rho$ *and* $e^z = \sum_{m=0}^{\infty} z^m/m!$. *The upper estimation of the s-numbers of* B_{e^z} *is pretended by*

$$s_r(B_{e^z}) \leq \sup_{\text{card } \xi=r+1} \inf_{k\in\xi} \sup_{m} \left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p} \sum_{m=0}^{\infty} \left(\frac{1}{m!}\right)^{p_m/\varpi_p}. \quad (84)$$

Theorem 48. *Suppose* $(p_v) \in mi_{\nearrow} \cap \ell_\infty$ *with* $p_0 \geq 1$. *Assume* $B_{\sin(z)}$ *is a shift operator on* $(\mathscr{H}_{p(.)})_\rho$, *where* $\rho(f) = [\sum_{r=0}^{\infty}(1/p_r)|\widehat{f}_r|^{p_r}]^{1/\varpi_p}$, *for every* $f \in (\mathscr{H}_{p(.)})_\rho$ *and* $\sin(z) = \sum_{m=0}^{\infty}(-1)^m (z^{2m+1}/(2m+1)!)$. *The upper estimation of the* $B_{\sin(z)}$ *is presented by*

$$s_r\left(B_{\sin(z)}\right) \leq \sup_{\text{card } \xi=r+1} \inf_{k\in\xi} \sup_{m} \left(\frac{p_{k+m}}{p_k}\right)^{1/\varpi_p} \sum_{m=0}^{\infty} \left(\frac{1}{(2m+1)!}\right)^{p_m/\varpi_p}. \quad (85)$$

Data Availability

No data were used.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

[1] A. A. Bakery and M. H. El Dewaik, "A generalization of Caristi's fixed point theorem in the variable exponent weighted formal power series space," *Journal of Function Spaces*, vol. 2021, Article ID 9919420, 18 pages, 2021.

[2] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1980.

[3] A. Pietsch, "Small ideals of operators," *Studia Mathematica*, vol. 51, no. 3, pp. 265–267, 1974.

[4] B. M. Makarov and N. Faried, "Some properties of operator ideals constructed by *s* numbers (In Russian)," in *Theory of Operators in Functional Spaces*, pp. 206–211, Academy of Science, Siberian section, Novosibirsk, Russia, 1977.

[5] N. Faried and A. A. Bakery, "Small operator ideals formed by *s* numbers on generalized Cesáro and Orlicz sequence spaces," *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.

[6] A. L. Shields, "Weighted shift operators and analytic function theory," in *Topics of Operator Theory, in: Mathematical Surveys Monographs*, vol. 13, American Mathematical Society, Providence, RI, USA, 1974.

[7] K. Hedayatian, "On cyclicity in the space $H^p(\beta)$," *Taiwanese Journal of Mathematics*, vol. 8, no. 3, pp. 429–442, 2004.

[8] H. Emamirad and G. S. Heshmati, "Chaotic weighted shifts in Bargmann spaces," *Journal of mathematical analysis and applications*, vol. 308, no. 1, pp. 36–46, 2005.

[9] N. Faried, A. Morsy, and Z. A. Hassanain, "*s*-numbers of shift operators of formal entire functions," *Journal of Approximation Theory*, vol. 176, pp. 15–22, 2013.

[10] B. E. Rhoades, "Operators of $A - p$ type," *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, vol. 59, no. 3-4, pp. 238–241, 1975.

[11] A. Pietsch, *Eigenvalues and S-Numbers*, Cambridge University Press, New York, NY, USA, 1986.

[12] A. A. Bakery and A. R. Abou Elmatty, "A note on Nakano generalized difference sequence space," *Advances in Difference Equations*, vol. 2020, no. 1, 17 pages, 2020.

[13] N. Faried, Z. Abd El Kader, and A. A. Mehanna, "*s*-numbers of polynomials of shift operators on ℓ^p $1 \leq p \leq \infty$," *Journal of the Egyptian Mathematical Society*, vol. 1, pp. 31–37, 1993.

[14] B. Altay and F. Başar, "Generalization of the sequence space $\ell(p)$ derived by weighted means," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 147–185, 2007.

[15] A. Pietsch, *Operator Ideals*, VEB Deutscher Verlag derWissenschaften, Berlin, 1978.

[16] A. Pietsch, "*s*-numbers of operators in Banach spaces," *Studia Mathematica*, vol. 51, no. 3, pp. 201–223, 1974.

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Research Article

Solution of Linear and Quadratic Equations Based on Triangular Linear Diophantine Fuzzy Numbers

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This paper is introducing a new concept of triangular linear Diophantine fuzzy numbers (TLDFNs) in a generic way. We first introduce the concept of TLDFNs and then study the arithmetic operations on these numbers. We find a method for the ranking of these TLDFNs. At the end, we formulate the linear and quadratic equations of the types $A + X = B$,$A \cdot X + B = C$, and $A \cdot X^2 + B \cdot X + C = D$ where the elements A, B, C, and D are TLDFNs. We provide a procedure for the solution of these equations using $(\langle s, t \rangle, \langle u, v \rangle)$-cut and also provide the examples.

1. Introduction

In 1965, Zadeh [1] introduced a new notion of fuzzy set theory. Fuzzy set (FS) theory has been widely acclaimed as offering greater richness in applications than ordinary set theory. Zadeh popularized the concept of fuzzy sets for the first time. There is an area of FS theory, in which the arithmetic operations on FNs play an essential part known as fuzzy equations (FEQs). Fuzzy equations were studied by Sanchez [2], by using extended operations. Accordingly, a profuse number of researchers like Biacino and Lettieri [3], Buckley [4], and Wasowski [5] have studied several approaches to solve FEQs. In [6–9], Buckley and Qu introduced several techniques to evaluate the fuzzy equations of the type $A \cdot X + B = C$ and $A \cdot X^2 + B \cdot X + C = D$, where A, B, C, D, and X are fuzzy numbers (FNs). Jiang [10] studied an approach to solve simultaneous linear equations that coefficients are fuzzy numbers.

Intuitionistic fuzzy sets [11, 12], neutrosophic sets [13, 14], and bipolar fuzzy sets [15] are the generalizations of the fuzzy sets. There are several mathematicians who solved linear and quadratic equations based on intuitionistic fuzzy sets, neutrosophic sets, and bipolar fuzzy sets. Banerjee and Roy [16] studied the intuitionistic fuzzy linear and quadratic equations, Chakraborty et al. [17] studied arithmetic operations on generalized intuitionistic fuzzy number and its applications to transportation problem, Rahaman et al. [18] introduced the solution techniques for linear and quadratic equations with coefficients as Cauchy neutrosophic numbers, and Akram et al. [19–23] introduced some methods for solving the bipolar fuzzy system of linear equations, also see [24–26].

Linear Diophantine fuzzy set [27] is a new generalization of fuzzy set, intuitionistic fuzzy set, neutrosophic set, and bipolar fuzzy set which was introduced by Riaz and Hashmi in 2019. After the introduction of this concept, several mathematicians were attracted towards this concept and worked in this area. Riaz and others studied the decision-making problems related to linear Diophantine fuzzy Einstein aggregation operators [28], spherical linear Diophantine fuzzy sets [29], and linear Diophantine fuzzy relations [30]. Almagrabi et al. [31] introduced a new approach to q-linear Diophantine fuzzy emergency decision support system for COVID-19. Kamac [32] studied linear Diophantine fuzzy algebraic structures.

Motivated by the work of Buckley and Qu [7], we solve the linear and quadratic equations with more generalized fuzzy numbers. As the linear Diophantine fuzzy set, [27] is the more generalized form of fuzzy sets so we studied the linear and quadratic equations based on linear Diophantine fuzzy numbers. In linear Diophantine fuzzy sets, we use the reference parameters, which allow us to choose the grades without any limitation; this helps us in obtaining better results.

In Section 2, we provided the fundamental definitions related to fuzzy sets and linear Diophantine fuzzy sets. In Section 3, we define linear Diophantine fuzzy numbers, in particular, triangular linear Diophantine fuzzy number. Also defined some basic operations on LDF numbers. In Section 4, we provide the ranking of LDF numbers, and in Section 5, we solved linear and quadratic equations based on LDF numbers.

2. Preliminaries and Basic Definitions

This section is devoted to review some indispensable concepts, which are very beneficial to develop the understanding of the prevalent model.

Definition 1 (see [1]). *Let X be a classical set, $\mu_{\mathfrak{M}}$ be a function from X to $[0, 1]$. The MF (membership function) $\mu_{\mathfrak{M}}(\vartheta)$ of a FS (fuzzy set) $\mathfrak{M}$ is defined by*

$$\mathfrak{M} = \{(\vartheta, \mu_{\mathfrak{M}}(\vartheta)) \mid \vartheta \in X \text{ and } \mu_{\mathfrak{M}}(\vartheta) \in 0, 1]\}. \tag{1}$$

Definition 2 (see [33]). *Let $\mathfrak{M}$ be a fuzzy subset of universal set X . Then, $\mathfrak{M}$ is called convex FS if $\forall r, s \in X$ and $\lambda \in [0, 1]$ we have*

$$\mu_{\mathfrak{M}}(\lambda r + (1-\lambda)s) \geq \min\{\mu_{\mathfrak{M}}(r), \mu_{\mathfrak{M}}(s)\}. \tag{2}$$

Definition 3 (see [1]). *A fuzzy set $\mathfrak{M}$ is said to be normalized if $h(\mathfrak{M}) = 1$.*

Definition 4. An α -level set of a FS $\mathfrak{M}$ is defined as

$$\mathfrak{M}^{\alpha} = \{\vartheta \in X : \mu_{\mathfrak{M}}(\vartheta) \geq \alpha\} \text{ for each } \alpha \in (0, 1]. \tag{3}$$

Definition 5 (see [33]). *A fuzzy subset $\mathfrak{M}$ defined on a set $\mathbb{R}$ (of real numbers) is said to be a FN (fuzzy number) if $\mathfrak{M}$ satisfies the following axioms:*

(a) $\mathfrak{M}$ is continuous: $\mu_{\mathfrak{M}}(t)$ is a continuous function from $\mathbb{R} \longrightarrow [0, 1]$

(b) $\mathfrak{M}$ is normalized: there exists $t \in \mathbb{R}$ such that $\mu_{\mathfrak{M}}(t) = 1$

(c) Convexity of $\mathfrak{M}$: i.e., $\forall t, u, w \in \mathbb{R}$, if $t \leq u \leq w$, then $\mu_{\mathfrak{M}}(u) \geq \min\{\mu_{\mathfrak{M}}(t), \mu_{\mathfrak{M}}(w)\}$

(d) Boundness of support: i.e., $\exists S \in \mathbb{R}$ and $\forall t \in \mathbb{R}$, if $|t| \geq S$, then $\mu_{\mathfrak{M}}(t) = 0$

We denote the set of all FNs by $F_{ns}(\mathbb{R})$.

Now, we study the idea of LDFSs (linear Diophantine fuzzy sets) and their fundamental operations.

Definition 6 (see [27]). *Let X be the universe. A LDFS $\pounds_{\mathfrak{R}}$ on X is defined as follows:*

$$\pounds_{\mathfrak{R}} = \{(\vartheta, \langle \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)\rangle, \langle \alpha(\vartheta), \beta(\vartheta)\rangle) : \vartheta \in X\} \tag{4}$$

where $\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta), \alpha(\vartheta), \beta(\vartheta) \in [0, 1]$ such that

$$\begin{aligned} &0 \leq \alpha(\vartheta)\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) + \beta(\vartheta)\mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \leq 1, \quad \forall \vartheta \in X, \\ &0 \leq \alpha(\vartheta) + \beta(\vartheta) \leq 1. \end{aligned} \tag{5}$$

The hesitation part can be written as

$$\xi \pi_{\mathfrak{R}} = 1 - (\alpha(\vartheta)\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) + \beta(\vartheta)\mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)), \tag{6}$$

where ξ is the reference parameter.

We write in short $\pounds_{\mathfrak{R}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}}\rangle, \langle \alpha, \beta\rangle)$ or $\pounds_{\mathfrak{R}} = \langle\langle \mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}}\rangle, \langle \alpha, \beta\rangle\rangle$ for

$$\pounds_{\mathfrak{R}} = \{(\vartheta, \langle \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)\rangle, \langle \alpha(\vartheta), \beta(\vartheta)\rangle) : \vartheta \in X\}. \tag{7}$$

Definition 7 (see [27]). *An absolute LDFS on X can be written as*

$$^{1}\pounds_{\mathfrak{R}} = \{(\vartheta, \langle 1, 0\rangle, \langle 1, 0\rangle) : \vartheta \in X\}, \tag{8}$$

and empty or null LDFS can be expressed as

$$^{0}\pounds_{\mathfrak{R}} = \{(\vartheta, \langle 0, 1\rangle, \langle 0, 1\rangle) : \vartheta \in X\}. \tag{9}$$

Definition 8 (see [27]). *Let $\pounds_{\mathfrak{R}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}}\rangle, \langle \alpha, \beta\rangle)$ and $\pounds_{\mathfrak{P}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{P}}\rangle, \langle \gamma, \delta\rangle)$ be two LDFSs on the reference set X and $\vartheta \in X$. Then,*

(i) $\pounds^{c}_{\mathfrak{R}} = (\langle \mathfrak{N}^{\nu}_{\mathfrak{R}}, \mathfrak{M}^{\tau}_{\mathfrak{R}}\rangle, \langle \beta, \alpha\rangle)$

(ii) $\pounds_{\mathfrak{R}} = \pounds_{\mathfrak{P}} \Leftrightarrow \mathfrak{M}^{\tau}_{\mathfrak{R}} = \mathfrak{M}^{\tau}_{\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{R}} = \mathfrak{N}^{\nu}_{\mathfrak{P}}, \alpha = \gamma, \beta = \delta$

(iii) $\pounds_{\mathfrak{R}} \subseteq \pounds_{\mathfrak{P}} \Leftrightarrow \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \leq \mathfrak{M}^{\tau}_{\mathfrak{P}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \geq \mathfrak{N}^{\nu}_{\mathfrak{P}}(\vartheta)$, $\alpha(\vartheta) \leq \gamma(\vartheta), \beta(\vartheta) \geq \delta(\vartheta)$

(iv) $\pounds_{\mathfrak{R}} \cup \pounds_{\mathfrak{P}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{R}\cup\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{R}\cap\mathfrak{P}}\rangle, \langle \alpha \vee \gamma, \beta \wedge \delta\rangle)$

(v) $\pounds_{\mathfrak{R}} \cap \pounds_{\mathfrak{P}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{R}\cap\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{R}\cup\mathfrak{P}}\rangle, \langle \alpha \wedge \gamma, \beta \vee \delta\rangle)$

where

$$\begin{aligned} &\mathfrak{M}^{\tau}_{\mathfrak{R}\cup\mathfrak{P}}(\vartheta) = \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \vee \mathfrak{M}^{\tau}_{\mathfrak{P}}(\vartheta), \mathfrak{M}^{\tau}_{\mathfrak{R}\cap\mathfrak{P}}(\vartheta) = \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \wedge \mathfrak{M}^{\tau}_{\mathfrak{P}}(\vartheta), \\ &\mathfrak{N}^{\nu}_{\mathfrak{R}\cap\mathfrak{P}}(\vartheta) = \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \wedge \mathfrak{N}^{\nu}_{\mathfrak{P}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}\cup\mathfrak{P}}(\vartheta) = \mathfrak{N}_{\mathfrak{R}}(\vartheta) \vee \mathfrak{N}^{\nu}_{\mathfrak{P}}(\vartheta). \end{aligned} \tag{10}$$

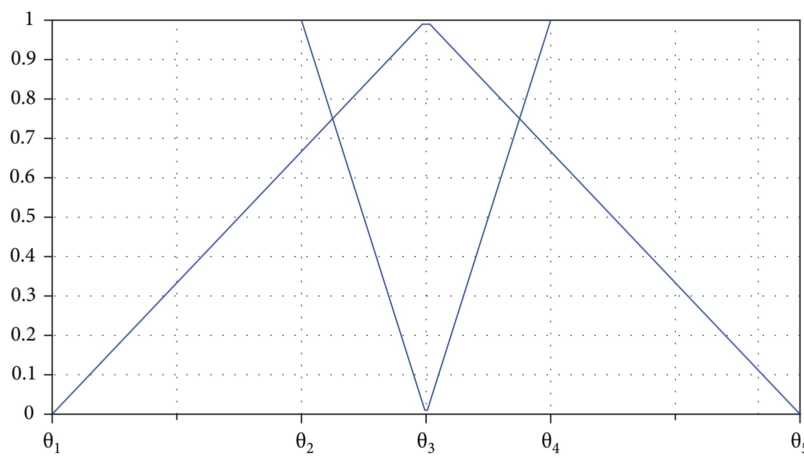

FIGURE 1: The figure of $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5)$.

Definition 9 (see [27]). *Let* $\pounds_{\mathfrak{R}} = \{(\vartheta, \langle \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)\rangle, \langle \alpha(\vartheta), \beta(\vartheta)\rangle) : \vartheta \in X\}$ *be an LDFS. For any constants* $s,t,u,v \in [0, 1]$ *such that* $0 \le su + tv \le 1$ *with* $0 \le u + v \le 1$, *define the* $(\langle s, t\rangle, \langle u, v\rangle)$ *-cut of* $\pounds_{\mathfrak{R}}$ *as follows:*

$$(\pounds_{\mathfrak{R}})^{\langle s,t\rangle}_{\langle u,v\rangle} = \{\vartheta \in X : \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \ge \mathrm{s}, \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \le \mathrm{t}, \alpha(\vartheta) \ge u, \beta(\vartheta))\rangle \le \mathrm{v}\}. \tag{11}$$

3. Triangular LDF Numbers

Here, in this section, we provide definitions and arithmetic operations on LDF numbers (LDFNs).

Definition 10. A LDF number $\pounds_{\mathfrak{R}}$ *is*

(i) a LDF fuzzy subset of the real line $\mathbb{R}$

(ii) normal, i.e., there is any $\vartheta_0 \in \mathbb{R}$ such that $\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta_0) = 1, \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta_0) = 0, \alpha(\vartheta_0) = 1, \beta(\vartheta_0) = 0$

(iii) convex for the membership functions $\mathfrak{M}^{\tau}_{\mathfrak{R}}$ and α, i.e.,

$$\begin{gathered}\mathfrak{M}^{\tau}_{\mathfrak{R}}(\lambda\vartheta_1 + (1-\lambda)\vartheta_2) \ge \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta_1), \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta_2)\} \quad \forall \vartheta_1, \vartheta_2 \in \mathbb{R}, \lambda \in [0,1],\\ \alpha(\lambda\vartheta_1 + (1-\lambda)\vartheta_2) \ge \min\{\alpha(\vartheta_1), \alpha(\vartheta_2)\} \quad \forall \vartheta_1, \vartheta_2 \in \mathbb{R}, \lambda \in [0,1]\end{gathered} \tag{12}$$

(iv) concave for the nonmembership functions $\mathfrak{N}^{\nu}_{\mathfrak{R}}$ and β, i.e.,

$$\begin{gathered}\mathfrak{N}^{\nu}_{\mathfrak{R}}(\lambda\vartheta_1 + (1-\lambda)\vartheta_2) \le \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta_1), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta_2)\} \quad \forall \vartheta_1, \vartheta_2 \in \mathbb{R}, \lambda \in [0,1],\\ \beta(\lambda\vartheta_1 + (1-\lambda)\vartheta_2) \le \max\{\beta(\vartheta_1), \beta(\vartheta_2)\} \quad \forall \vartheta_1, \vartheta_2 \in \mathbb{R}, \lambda \in 0,1].\end{gathered} \tag{13}$$

We now provide the 4 types of triangular LDF numbers.

Definition 11. Let $\pounds_{\mathfrak{R}}$ *be a LDFS on* $\mathbb{R}$ *with the following membership functions* ($\mathfrak{M}^{\tau}_{\mathfrak{R}}$ *and* α) *and nonmembership functions* ($\mathfrak{N}^{\nu}_{\mathfrak{R}}$ *and* β)

$$\begin{gathered}\mathfrak{M}^{\tau}_{\mathfrak{R}}(x) = \begin{cases} \dfrac{x-\vartheta_1}{\vartheta_3-\vartheta_1}, & \vartheta_1 \le x \le \vartheta_3,\\ \dfrac{\vartheta_5-x}{\vartheta_5-\vartheta_3}, & \vartheta_3 \le x \le \vartheta_5,\\ 0, & \text{otherwise,}\end{cases}\\ \mathfrak{N}^{\nu}_{\mathfrak{R}}(x) = \begin{cases} \dfrac{\vartheta_3-x}{\vartheta_3-\vartheta_2}, & \vartheta_2 \le x \le \vartheta_3,\\ \dfrac{x-\vartheta_3}{\vartheta_4-\vartheta_3}, & \vartheta_3 \le x \le \vartheta_4,\\ 0, & \text{otherwise,}\end{cases}\\ \alpha(x) = \begin{cases} \dfrac{x-\vartheta'_2}{\vartheta'_3-\vartheta'_2}, & \vartheta'_2 \le x \le \vartheta'_3,\\ \dfrac{\vartheta'_4-x}{\vartheta'_4-\vartheta'_3}, & \vartheta'_3 \le x \le \vartheta'_4,\\ 0, & \text{otherwise,}\end{cases}\\ \beta(x) = \begin{cases} \dfrac{\vartheta'_3-x}{\vartheta'_3-\vartheta'_1}, & \vartheta'_1 \le x \le \vartheta'_3,\\ \dfrac{x-\vartheta'_3}{\vartheta'_5-\vartheta'_3}, & \vartheta'_3 \le x \le \vartheta'_5,\\ 0, & \text{otherwise,}\end{cases}\end{gathered} \tag{14}$$

where $\vartheta'_1 \le \vartheta'_2 \le \vartheta'_3 \le \vartheta'_4 \le \vartheta'_5$ for all $x \in \mathbb{R}$. Then, $\pounds_{\mathfrak{R}}$ is called

(i) a triangular LDFN of type-1 if $\vartheta_3 = \vartheta'_3$ and $\vartheta_1 \le \vartheta_2 \le \vartheta_3 \le \vartheta_4 \le \vartheta_5$

(ii) a triangular LDFN of type-2 if $\vartheta_3 \ne \vartheta'_{3'}$ and $\vartheta_1 \le \vartheta_2 \le \vartheta_3 \le \vartheta_4 \le \vartheta_5$

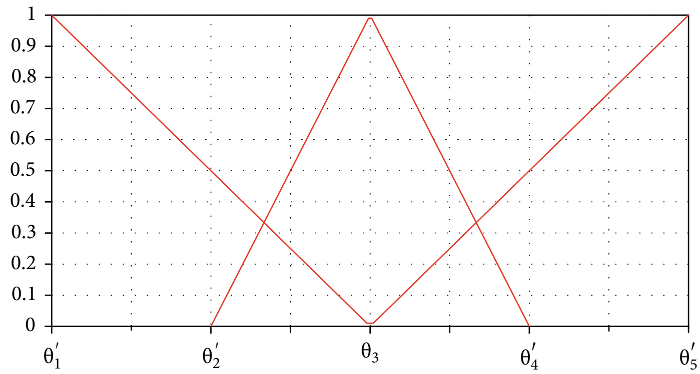

Figure 2: The figure of $(\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5')$.

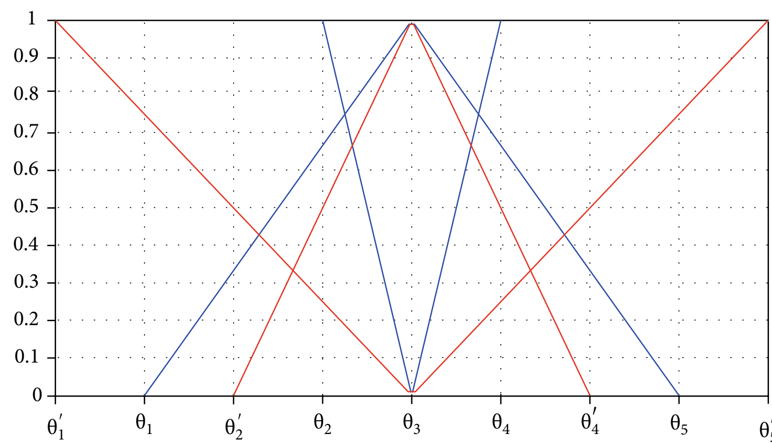

Figure 3: The figure of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$.

(iii) a triangular LDFN of type-3 if $\vartheta_3 = \vartheta_3'$ and $\vartheta_2 \le \vartheta_1 \le \vartheta_3 \le \vartheta_5 \le \vartheta_4$

(iv) a triangular LDFN of type-4 if $\vartheta_3 \ne \vartheta_3'$ and $\vartheta_2 \le \vartheta_1 \le \vartheta_3 \le \vartheta_5 \le \vartheta_4$

Throughout the paper, we consider only triangular LDFN of type-1 and we write this type as triangular LDFN (TLDFN). This TLDFN is denoted by

$$\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5), \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5'). \end{cases} \tag{15}$$

The figure of $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5)$ is shown in Figure 1. The figure of $(\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5')$ is shown in Figure 2. The figure of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is shown in Figure 3.

Remark 12. If we take $\vartheta_1' = \vartheta_2' = \vartheta_1 = \vartheta_2$ and $\vartheta_4' = \vartheta_5' = \vartheta_1 = \vartheta_2$, then both type-1 and type-3 become the same.

Definition 13. Consider a TLDFN $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{smallmatrix} \right.$. Then,

(i) s-cut set of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is a crisp subset of $\mathbb{R}$, which is defined as follows:

$$\begin{aligned} \pounds^s_{\mathfrak{R}_{\text{TLDFN}}} &= \{x \in X : \mathfrak{M}^{\tau}_{\mathfrak{R}}(x) \ge s\} = \left[\underline{\mathfrak{M}^{\tau}_{\mathfrak{R}}(s)}, \mathfrak{M}^{\bar{\tau}}_{\mathfrak{R}}(s)\right] \\ &= [\vartheta_1 + s(\vartheta_3 - \vartheta_1), \vartheta_5 - s(\vartheta_5 - \vartheta_3)], \end{aligned} \tag{16}$$

(ii) t-cut set of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is a crisp subset of $\mathbb{R}$, which is defined as follows:

$$\begin{aligned} \pounds^t_{\mathfrak{R}_{\text{TLDFN}}} &= \{x \in X : \mathfrak{N}^{\nu}_{\mathfrak{R}}(x) \le t\} = \left[\underline{\mathfrak{N}^{\nu}_{\mathfrak{R}}(t)}, \mathfrak{N}^{\bar{\nu}}_{\mathfrak{R}}(t)\right] \\ &= [\vartheta_3 - t(\vartheta_3 - \vartheta_2), \vartheta_3 + t(\vartheta_4 - \vartheta_3)], \end{aligned} \tag{17}$$

(iii) u-cut set of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is a crisp subset of $\mathbb{R}$, which is defined as follows:

$$\begin{aligned}\pounds^{u}_{\mathfrak{R}_{\text{TLDFN}}} &= \{x \in X : \alpha(x) \geq u\} = \left[\underline{\alpha(u)}, \bar{\alpha}(u)\right] \\ &= \left[\vartheta_2' + u\left(\vartheta_3 - \vartheta_2'\right), \vartheta_4' - u\left(\vartheta_4' - \vartheta_3\right)\right],\end{aligned} \tag{18}$$

(iv) v-cut set of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is a crisp subset of $\mathbb{R}$, which is defined as follows:

$$\begin{aligned}\pounds^{v}_{\mathfrak{R}_{\text{TLDFN}}} &= \{x \in X : \beta(x) \leq v\} = \left[\underline{\beta(v)}, \bar{\beta}(v)\right] \\ &= \left[\vartheta_3 - v\left(\vartheta_3 - \vartheta_1'\right), \vartheta_3 + v\left(\vartheta_5' - \vartheta_3\right)\right].\end{aligned} \tag{19}$$

We can denote the $(\langle s, t\rangle, \langle u, v\rangle)$-cut of $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ by

$$\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right)^{\langle s,t\rangle}_{\langle u,v\rangle} = \begin{cases} \left(\left[\underline{\mathfrak{M}^{\tau}_{\mathfrak{R}}(s)}, \mathfrak{M}^{\bar{\tau}}_{\mathfrak{R}}(s)\right], \left[\underline{\mathfrak{N}^{v}_{\mathfrak{R}}(t)}, \mathfrak{N}^{\bar{v}}_{\mathfrak{R}}(t)\right]\right), \\ \left(\left[\underline{\alpha(u)}, \bar{\alpha}(u)\right], \left[\underline{\beta(v)}, \bar{\beta}(v)\right]\right). \end{cases} \tag{20}$$

We denote the set of all TLDFN on $\mathbb{R}$ by $\pounds_{\mathfrak{R}_{\text{TLDFN}}}(\mathbb{R})$. The arithmetic operations based on extension principle are defined as follows.

Definition 14. Let $\pounds_{\mathfrak{R}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{v}_{\mathfrak{R}}\rangle, \langle \alpha, \beta\rangle)$ *and* $\pounds_{\mathfrak{P}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{P}}, \mathfrak{N}^{v}_{\mathfrak{P}}\rangle, \langle \gamma, \delta\rangle)$ *be two TLDFN on* $\mathbb{R}$ *. Then,*

(i) $\pounds_{\mathfrak{R}} + \pounds_{\mathfrak{P}} = \{ \sup_{t=x+y} \{\min \{\mathfrak{M}^{\tau}_{\mathfrak{R}}(x), \mathfrak{M}^{\tau}_{\mathfrak{P}}(y)\}\} \inf_{t=x+y} \{\max \{\mathfrak{N}^{v}_{\mathfrak{R}}(x), \mathfrak{N}^{v}_{\mathfrak{P}}(y)\}\} \sup_{t=x+y} \{\min \{\alpha(x), \gamma(y)\}\} \inf_{t=x+y} \{\max \{\beta(x), \delta(y)\}\}$

(ii) $\pounds_{\mathfrak{R}} - \pounds_{\mathfrak{P}} = \{ \sup_{t=x-y} \{\min \{\mathfrak{M}^{\tau}_{\mathfrak{R}}(x), \mathfrak{M}^{\tau}_{\mathfrak{P}}(y)\}\} \inf_{t=x-y} \{\max \{\mathfrak{N}^{v}_{\mathfrak{R}}(x), \mathfrak{N}^{v}_{\mathfrak{P}}(y)\}\} \sup_{t=x-y} \{\min \{\alpha(x), \gamma(y)\}\} \inf_{t=x-y} \{\max \{\beta(x), \delta(y)\}\}$

(iii) $\pounds_{\mathfrak{R}} \times \pounds_{\mathfrak{P}} = \{ \sup_{t=x\times y} \{\min \{\mathfrak{M}^{\tau}_{\mathfrak{R}}(x), \mathfrak{M}^{\tau}_{\mathfrak{P}}(y)\}\} \inf_{t=x\times y} \{\max \{\mathfrak{N}^{v}_{\mathfrak{R}}(x), \mathfrak{N}^{v}_{\mathfrak{P}}(y)\}\} \sup_{t=x\times y} \{\min \{\alpha(x), \gamma(y)\}\} \inf_{t=x\times y} \{\max \{\beta(x), \delta(y)\}\}$

(iv) $\pounds_{\mathfrak{R}} \div \pounds_{\mathfrak{P}} = \{ \sup_{t=x\div y} \{\min \{\mathfrak{M}^{\tau}_{\mathfrak{R}}(x), \mathfrak{M}^{\tau}_{\mathfrak{P}}(y)\}\} \inf_{t=x\div y} \{\max \{\mathfrak{N}^{v}_{\mathfrak{R}}(x), \mathfrak{N}^{v}_{\mathfrak{P}}(y)\}\} \sup_{t=x\div y} \{\min \{\alpha(x), \gamma(y)\}\} \inf_{t=x\div y} \{\max \{\beta(x), \delta(y)\}\}$

Definition 15. A TLDFN $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ *is said to be positive if and only if* $\vartheta_1 \geq 0$ *and* $\vartheta_1' \geq 0$.

Definition 16. Two TLDFNs $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ *and* $\xi_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \\ (\delta_1', \delta_2', \delta_3, \delta_4', \delta_5') \end{cases}$ *are said to be equal if and only if* $\vartheta_1 = \delta_1, \vartheta_2 = \delta_2, \vartheta_3 = \delta_3, \vartheta_4 = \delta_4, \vartheta_5 = \delta_5, \vartheta_1' = \delta_1'$, $\vartheta_2' = \delta_2', \vartheta_4' = \delta_4'$, *and* $\vartheta_5' = \delta_5'$.

We now define the arithmetic operations on TLDFNs using the concept of interval arithmetic.

Definition 17. Consider two positive TLDFNs $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5) \\ (\vartheta_1',\vartheta_2',\vartheta_3,\vartheta_4',\vartheta_5') \end{smallmatrix} \right.$ *and* $\xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\delta_1,\delta_2,\delta_3,\delta_4,\delta_5) \\ (\delta_1',\delta_2',\delta_3,\delta_4',\delta_5') \end{smallmatrix} \right.$, *then,*

(i) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} + \xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\vartheta_1+\delta_1,\vartheta_2+\delta_2,\vartheta_3+\delta_3,\vartheta_4+\delta_4,\vartheta_5+\delta_5) \\ (\vartheta_1'+\delta_1',\vartheta_2'+\delta_2',\vartheta_3+\delta_3,\vartheta_4'+\delta_4',\vartheta_5'+\delta_5') \end{smallmatrix} \right.$

(ii) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} - \xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\vartheta_1-\delta_5,\vartheta_2-\delta_4,\vartheta_3-\delta_3,\vartheta_4-\delta_2,\vartheta_5-\delta_1) \\ (\vartheta_1'-\delta_5',\vartheta_2'-\delta_4',\vartheta_3-\delta_3,\vartheta_4'-\delta_2',\vartheta_5'-\delta_1') \end{smallmatrix} \right.$

(iii) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} \times \xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\vartheta_1\delta_1,\vartheta_2\delta_2,\vartheta_3\delta_3,\vartheta_4\delta_4,\vartheta_5\delta_5) \\ (\vartheta_1'\delta_1',\vartheta_2'\delta_2',\vartheta_3\delta_3,\vartheta_4'\delta_4',\vartheta_5'\delta_5') \end{smallmatrix} \right.$

(iv) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} \div \xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} ((\vartheta_1/\delta_5),(\vartheta_2/\delta_4),(\vartheta_3/\delta_3),(\vartheta_4/\delta_2),(\vartheta_5/\delta_1)) \\ ((\vartheta_1'/\delta_5'),(\vartheta_2'/\delta_4'),(\vartheta_3/\delta_3),(\vartheta_4'/\delta_2'),(\vartheta_5'/\delta_1')) \end{smallmatrix} \right.$

(v) $k \times \pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} \left\{ \begin{smallmatrix} (k\vartheta_1,k\vartheta_2,k\vartheta_3,k\vartheta_4,k\vartheta_5) \\ (k\vartheta_1',k\vartheta_2',k\vartheta_3,k\vartheta_4',k\vartheta_5') \end{smallmatrix} \right. \text{if } k > 0 \\ \left\{ \begin{smallmatrix} (k\vartheta_5,k\vartheta_4,k\vartheta_3,k\vartheta_2,k\vartheta_1) \\ (k\vartheta_5',k\vartheta_4',k\vartheta_3,k\vartheta_2',k\vartheta_1') \end{smallmatrix} \right. \text{if } k < 0 \end{cases}$

4. Ranking Function of TLDFNs

There are many methods for defuzzification such as the centroid method, mean of interval method, and removal area method. In this paper, we have used the concept of the mean of interval method to find the value of the membership and nonmembership function of TLDFN.

Consider a TLDFN

$$\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5), \\ (\vartheta_1',\vartheta_2',\vartheta_3,\vartheta_4',\vartheta_5'). \end{cases} \tag{21}$$

The $(\langle s, t\rangle, \langle u, v\rangle)$-cut of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is

$$(\pounds_{\mathfrak{R}})^{\langle s,t\rangle}_{\langle u,v\rangle} = \{\vartheta \in X : \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \geq s, \mathfrak{N}^{v}_{\mathfrak{R}}(\vartheta) \leq t, \alpha(\vartheta) \geq u, \beta(\vartheta)\rangle \leq v\}, \tag{22}$$

where

$$
\begin{aligned}
\underline{\mathfrak{M}^{\tau}_{\mathfrak{R}}(s)} &= \vartheta_1 + s(\vartheta_3 - \vartheta_1),\\
\mathfrak{M}^{\bar{\tau}}_{\mathfrak{R}}(s) &= \vartheta_5 - s(\vartheta_5 - \vartheta_3),\\
\underline{\mathfrak{N}^{\nu}_{\mathfrak{R}}(t)} &= \vartheta_3 - t(\vartheta_3 - \vartheta_2),\\
\mathfrak{N}^{\bar{\nu}}_{\mathfrak{R}}(t) &= \vartheta_3 + t(\vartheta_4 - \vartheta_3),\\
\underline{\alpha(u)} &= \vartheta'_2 + u\left(\vartheta_3 - \vartheta'_2\right),\\
\alpha(\bar{u}) &= \vartheta'_4 - u\left(\vartheta'_4 - \vartheta_3\right),\\
\underline{\beta(\nu)} &= \vartheta_3 - \nu\left(\vartheta_3 - \vartheta'_1\right),\\
\beta(\bar{\nu}) &= \vartheta_3 + \nu\left(\vartheta'_5 - \vartheta_3\right).
\end{aligned}
\tag{23}
$$

Now, by the mean of $(\langle s,t\rangle,\langle u,\nu\rangle)$ -cut method, the representation of membership functions is

$$
\begin{aligned}
R_{\mathfrak{M}^{\tau}_{\mathfrak{R}}}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) &= \frac{1}{2}\int_0^1 \left(\underline{\mathfrak{M}^{\tau}_{\mathfrak{R}}(s)} + \mathfrak{M}^{\bar{\tau}}_{\mathfrak{R}}(s)\right)ds\\
&= \frac{1}{2}\int_0^1 (\vartheta_1 + s(\vartheta_3 - \vartheta_1) + \vartheta_5 - s(\vartheta_5 - \vartheta_3))ds\\
&= \frac{1}{2}\left[\vartheta_1 + \frac{1}{2}(\vartheta_3 - \vartheta_1) + \vartheta_5 - \frac{1}{2}(\vartheta_5 - \vartheta_3)\right]\\
&= \frac{\vartheta_1 + 2\vartheta_3 + \vartheta_5}{4},\\
R_{\alpha}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) &= \frac{1}{2}\int_0^1 \left(\underline{\alpha(u)} + \alpha(\bar{u})\right)ds\\
&= \frac{1}{2}\int_0^1 \left(\vartheta'_2 + u\left(\vartheta_3 - \vartheta'_2\right) + \vartheta'_4 - u\left(\vartheta'_4 - \vartheta_3\right)\right)du\\
&= \frac{1}{2}\left[\vartheta'_2 + \frac{1}{2}\left(\vartheta_3 - \vartheta'_2\right) + \vartheta'_4 - \frac{1}{2}\left(\vartheta'_4 - \vartheta_3\right)\right]\\
&= \frac{\vartheta'_2 + 2\vartheta_3 + \vartheta'_4}{4}.
\end{aligned}
\tag{24}
$$

Now, by the mean of $(\langle s,t\rangle,\langle u,\nu\rangle)$ -cut method, the representation of nonmembership functions is

$$
\begin{aligned}
R_{\mathfrak{N}^{\nu}_{\mathfrak{R}}}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) &= \frac{1}{2}\int_0^1 \left(\underline{\mathfrak{N}^{\nu}_{\mathfrak{R}}(t)} + \mathfrak{N}^{\bar{\nu}}_{\mathfrak{R}}(t)\right)dt\\
&= \frac{1}{2}\int_0^1 (\vartheta_3 - t(\vartheta_3 - \vartheta_2) + \vartheta_3 + t(\vartheta_4 - \vartheta_3))dt\\
&= \frac{1}{2}\left[2\vartheta_3 - \frac{1}{2}(\vartheta_3 - \vartheta_2) + \frac{1}{2}(\vartheta_4 - \vartheta_3)\right]\\
&= \frac{\vartheta_2 + 2\vartheta_3 + \vartheta_4}{4},\\
R_{\beta}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) &= \frac{1}{2}\int_0^1 \left(\underline{\beta(\nu)} + \beta(\bar{\nu})\right)d\nu\\
&= \frac{1}{2}\int_0^1 \left(\vartheta_3 - \nu\left(\vartheta_3 - \vartheta'_1\right) + \vartheta_3 + \nu\left(\vartheta'_5 - \vartheta_3\right)\right)'d\nu\\
&= \frac{1}{2}\left[2\vartheta_3 - \frac{1}{2}\left(\vartheta_3 - \vartheta'_1\right) + \frac{1}{2}\left(\vartheta'_5 - \vartheta_3\right)\right]\\
&= \frac{\vartheta'_1 + 2\vartheta_3 + \vartheta'_5}{4}.
\end{aligned}
\tag{25}
$$

Now,

$$
\begin{aligned}
R\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) &= \frac{R_{\mathfrak{M}^{\tau}_{\mathfrak{R}}}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) + R_{\alpha}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) + R_{\mathfrak{N}^{\nu}_{\mathfrak{R}}}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) + R_{\beta}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right)}{4}\\
&= \frac{((\vartheta_1 + 2\vartheta_3 + \vartheta_5)/4) + \left(\left(\vartheta'_2 + 2\vartheta_3 + \vartheta'_4\right)/4\right) + ((\vartheta_2 + 2\vartheta_3 + \vartheta_4)/4) + \left(\left(\vartheta'_1 + 2\vartheta_3 + \vartheta'_5\right)/4\right)}{4}\\
&= \frac{\vartheta_3}{2} + \frac{\vartheta_1 + \vartheta_2 + \vartheta_4 + \vartheta_5 + \vartheta'_1 + \vartheta'_2 + \vartheta'_4 + \vartheta'_5}{16}.
\end{aligned}
\tag{26}
$$

Consider two positive TLDFNs $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \left\{\begin{matrix}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5)\\(\vartheta'_1,\vartheta'_2,\vartheta_3,\vartheta'_4,\vartheta'_5)\end{matrix}\right.$ and $\xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{\begin{matrix}(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5)\\(\delta'_1,\delta'_2,\delta_3,\delta'_4,\delta'_5)\end{matrix}\right.$, then

(i) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} \prec \xi_{\mathfrak{R}_{\text{TLDFN}}}$ iff $R(\pounds_{\mathfrak{R}_{\text{TLDFN}}}) < R(\xi_{\mathfrak{R}_{\text{TLDFN}}})$

(ii) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} \succ \xi_{\mathfrak{R}_{\text{TLDFN}}}$ iff $R(\pounds_{\mathfrak{R}_{\text{TLDFN}}}) > R(\xi_{\mathfrak{R}_{\text{TLDFN}}})$

(iii) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \xi_{\mathfrak{R}_{\text{TLDFN}}}$ iff $R(\pounds_{\mathfrak{R}_{\text{TLDFN}}}) = R(\xi_{\mathfrak{R}_{\text{TLDFN}}})$

5. Solution of LDF Equations

5.1. Solution of $A + X = B$ by Using the Method of $(\langle s,t\rangle,\langle u,\nu\rangle)$-Cut. Let A, B, and X be the LDFNs and let $A = \left\{\begin{matrix}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5)\\(\vartheta'_1,\vartheta'_2,\vartheta_3,\vartheta'_4,\vartheta'_5)\end{matrix}\right.$ and $B = \left\{\begin{matrix}(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5)\\(\delta'_1,\delta'_2,\delta_3,\delta'_4,\delta'_5)\end{matrix}\right.$. Then,

$$
A + X = B \tag{27}
$$

is a LDF equation (LDFE). Let $X \approx \left\{\begin{matrix}(x_1,x_2,x_3,x_4,x_5)\\(x'_1,x'_2,x_3,x'_4,x'_5)\end{matrix}\right.$. Then, $X = B - A$ in general is not the solution of Equation (27).

Let

$$
\begin{aligned}
A^{\langle s,t\rangle}_{\langle u,\nu\rangle} &= \left\{\begin{matrix}\left(\left[\underline{\mathfrak{M}^{\tau}_{A}(s)}, \mathfrak{M}^{\bar{\tau}}_{A}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{A}(t)}, \mathfrak{N}^{\bar{\nu}}_{A}(t)\right]\right),\\ \left(\left[\underline{\alpha_A(u)}, \alpha_A(\bar{u})\right], \left[\underline{\beta_A(\nu)}, \beta_A(\bar{\nu})\right]\right),\end{matrix}\right.\\
B^{\langle s,t\rangle}_{\langle u,\nu\rangle} &= \left\{\begin{matrix}\left(\left[\underline{\mathfrak{M}^{\tau}_{B}(s)}, \mathfrak{M}^{\bar{\tau}}_{B}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{B}(t)}, \mathfrak{N}^{\bar{\nu}}_{B}(t)\right]\right),\\ \left(\left[\underline{\alpha_B(u)}, \alpha_B(\bar{u})\right], \left[\underline{\beta_B(\nu)}, \beta_B(\bar{\nu})\right]\right),\end{matrix}\right.\\
X^{\langle s,t\rangle}_{\langle u,\nu\rangle} &= \left\{\begin{matrix}\left(\left[\underline{\mathfrak{M}^{\tau}_{X}(s)}, \mathfrak{M}^{\bar{\tau}}_{X}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{X}(t)}, \mathfrak{N}^{\bar{\nu}}_{X}(t)\right]\right),\\ \left(\left[\underline{\alpha_X(u)}, \alpha_X(\bar{u})\right], \left[\underline{\beta_X(\nu)}, \beta_X(\bar{\nu})\right]\right)\end{matrix}\right.
\end{aligned}
\tag{28}
$$

represent the $(\langle s,t\rangle,\langle u,\nu\rangle)$-cuts of A, B, and X, respectively, in the given (27). Substituting these into Equation (27), we get

$$
A^{\langle s,t\rangle}_{\langle u,\nu\rangle} + X^{\langle s,t\rangle}_{\langle u,\nu\rangle} = B^{\langle s,t\rangle}_{\langle u,\nu\rangle}. \tag{29}
$$

By comparing the $(\langle s,t\rangle,\langle u,v\rangle)$ -cuts of A, B, and X, we get

$$\begin{gathered}\left[\underline{\mathfrak{M}_A^{\tau}(s)},\mathfrak{M}_A^{\bar{\tau}}(s)\right]+\left[\underline{\mathfrak{M}_X^{\tau}(s)},\mathfrak{M}_X^{\bar{\tau}}(s)\right]=\left[\underline{\mathfrak{M}_B^{\tau}(s)},\mathfrak{M}_B^{\bar{\tau}}(s)\right],\\ \left[\underline{\mathfrak{N}_A^{\nu}(t)},\mathfrak{N}_A^{\bar{\nu}}(t)\right]+\left[\underline{\mathfrak{N}_X^{\nu}(t)},\mathfrak{N}_X^{\bar{\nu}}(t)\right]=\left[\underline{\mathfrak{N}_B^{\nu}(t)},\mathfrak{N}_B^{\bar{\nu}}(t)\right],\\ \left[\underline{\alpha_A(u)},\overline{\alpha}_A(u)\right]+\left[\underline{\alpha_X(u)},\overline{\alpha}_X(u)\right]=\left[\underline{\alpha_B(u)},\overline{\alpha}_B(u)\right],\\ \left[\underline{\beta_A(v)},\overline{\beta}_A(v)\right]+\left[\underline{\beta_X(v)},\overline{\beta}_X(v)\right]=\left[\underline{\beta_B(v)},\overline{\beta}_B(v)\right].\end{gathered}\tag{30}$$

Now,

$$\begin{gathered}\underline{\mathfrak{M}_X^{\tau}(s)}=\underline{\mathfrak{M}_B^{\tau}(s)}-\underline{\mathfrak{M}_A^{\tau}(s)},\ \mathfrak{M}_X^{\bar{\tau}}(s)=\mathfrak{M}_B^{\bar{\tau}}(s)-\mathfrak{M}_A^{\bar{\tau}}(s),\\ \underline{\mathfrak{N}_X^{\nu}(t)}=\underline{\mathfrak{N}_B^{\nu}(t)}-\underline{\mathfrak{N}_A^{\nu}(t)},\ \mathfrak{N}_X^{\bar{\nu}}(t)=\mathfrak{N}_B^{\bar{\nu}}(t)-\mathfrak{N}_A^{\bar{\nu}}(t),\\ \underline{\alpha_X(u)}=\underline{\alpha_B(u)}-\underline{\alpha_A(u)},\ \overline{\alpha}_X(u)=\overline{\alpha}_B(u)-\overline{\alpha}_A(u),\\ \underline{\beta_X(v)}=\underline{\beta_B(v)}-\underline{\beta_A(v)},\ \overline{\beta}_X(v)=\overline{\beta}_B(v)-\overline{\beta}_A(v).\end{gathered}\tag{31}$$

Then, the solution of the equation $A+X=B$ exists iff

(1) $\underline{\mathfrak{M}_X^{\tau}(s)}$ is monotonically increasing in $0\le s\le 1$

(2) $\mathfrak{M}_X^{\bar{\tau}}(s)$ is monotonically decreasing in $0\le s\le 1$

(3) $\underline{\mathfrak{N}_X^{\nu}(t)}$ is monotonically decreasing in $0\le t\le 1$

(4) $\mathfrak{N}_X^{\bar{\nu}}(t)$ is monotonically increasing in $0\le t\le 1$

(5) $\underline{\alpha_X(u)}$ is monotonically increasing in $0\le u\le 1$

(6) $\overline{\alpha}_X(u)$ is monotonically decreasing in $0\le u\le 1$

(7) $\underline{\beta_X(v)}$ is monotonically decreasing in $0\le v\le 1$

(8) $\overline{\beta}_X(v)$ is monotonically increasing in $0\le v\le 1$

(9) $\underline{\mathfrak{M}_X^{\tau}(1)}=\mathfrak{M}_X^{\bar{\tau}}(1)=\underline{\mathfrak{N}_X^{\nu}(0)}=\mathfrak{N}_X^{\bar{\nu}}(0)=\underline{\alpha_X(1)}=\overline{\alpha}_X(1)=\underline{\beta_X(0)}=\overline{\beta}_X(0)$.

Example 1. Consider the equation $A+X=B$, where

$$\begin{gathered}A=\begin{cases}(3,5,7,10,15),\\ (2,6,7,8,11),\end{cases}\\ B=\begin{cases}(1,6,11,15,24),\\ (3,9,11,13,22).\end{cases}\end{gathered}\tag{32}$$

The $(\langle s,t\rangle,\langle u,v\rangle)$-cuts of A, B, and X are

$$\begin{gathered}A_{\langle u,v\rangle}^{\langle s,t\rangle}=\begin{cases}([3+4s,15-8s],[7-2t,7+3t]),\\ ([6+u,8-u],[7-5v,7+4v]),\end{cases}\\ B_{\langle u,v\rangle}^{\langle s,t\rangle}=\begin{cases}([1+10s,24-13s],[11-5t,11+4t]),\\ ([9+2u,13-2u],[11-8v,11+11v]),\end{cases}\\ X_{\langle u,v\rangle}^{\langle s,t\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}_X^{\tau}(s)},\mathfrak{M}_X^{\bar{\tau}}(s)\right],\left[\underline{\mathfrak{N}_X^{\nu}(t)},\mathfrak{N}_X^{\bar{\nu}}(t)\right]\right),\\ \left(\left[\underline{\alpha_X(u)},\overline{\alpha}_X(u)\right],\left[\underline{\beta_X(v)},\overline{\beta}_X(v)\right]\right),\end{cases}\end{gathered}\tag{33}$$

respectively. The $(\langle s,t\rangle,\langle u,v\rangle)$-cut equation is

$$A_{\langle u,v\rangle}^{\langle s,t\rangle}+X_{\langle u,v\rangle}^{\langle s,t\rangle}=B_{\langle u,v\rangle}^{\langle s,t\rangle}.\tag{34}$$

By comparing the $(\langle s,t\rangle,\langle u,v\rangle)$-cuts of A, B, and X, we get

$$\begin{gathered}\underline{\mathfrak{M}_X^{\tau}(s)}=\underline{\mathfrak{M}_B^{\tau}(s)}-\underline{\mathfrak{M}_A^{\tau}(s)}=-2+6s,\\ \mathfrak{M}_X^{\bar{\tau}}(s)=\mathfrak{M}_B^{\bar{\tau}}(s)-\mathfrak{M}_A^{\bar{\tau}}(s)=9-5s,\\ \underline{\mathfrak{N}_X^{\nu}(t)}=\underline{\mathfrak{N}_B^{\nu}(t)}-\underline{\mathfrak{N}_A^{\nu}(t)}=4-3t,\\ \mathfrak{N}_X^{\bar{\nu}}(t)=\mathfrak{N}_B^{\bar{\nu}}(t)-\mathfrak{N}_A^{\bar{\nu}}(t)=4+t,\\ \underline{\alpha_X(u)}=\underline{\alpha_B(u)}-\underline{\alpha_A(u)}=3+u,\\ \overline{\alpha}_X(u)=\overline{\alpha}_B(u)-\overline{\alpha}_A(u)=5-u,\\ \underline{\beta_X(v)}=\underline{\beta_B(v)}-\underline{\beta_A(v)}=4-3v,\\ \overline{\beta}_X(v)=\overline{\beta}_B(v)-\overline{\beta}_A(v)=4+7v.\end{gathered}\tag{35}$$

It is easy to see that $\underline{\mathfrak{M}_X^{\tau}(s)}$, $\mathfrak{N}_X^{\bar{\nu}}(t)$, $\underline{\alpha_X(u)}$, and $\overline{\beta}_X(v)$ are increasing and $\mathfrak{M}_X^{\bar{\tau}}(s)$, $\underline{\mathfrak{N}_X^{\nu}(t)}$, $\overline{\alpha}_X(u)$, and $\underline{\beta_X(v)}$ are decreasing in $0\le s,t,u,v\le 1$. Also,

$$\begin{gathered}\underline{\mathfrak{M}_X^{\tau}(1)}=\mathfrak{M}_X^{\bar{\tau}}(1)=\underline{\mathfrak{N}_X^{\nu}(0)}=\mathfrak{N}_X^{\bar{\nu}}(0)=\underline{\alpha_X(1)}\\ =\overline{\alpha}_X(1)=\underline{\beta_X(0)}=\overline{\beta}_X(0)=4.\end{gathered}\tag{36}$$

This shows that the solution of $A+X=B$ exists with $(\langle s,t\rangle,\langle u,v\rangle)$ -cut. The solution is

$$X=\begin{cases}(-2,1,4,5,9),\\ (1,3,4,5,11).\end{cases}\tag{37}$$

The solution in continuous form is

$$\mathfrak{M}_{\mathfrak{R}}^{\tau}(x)=\begin{cases}\dfrac{2+x}{6}, & -2\le x\le 4,\\ \dfrac{9-x}{5}, & 4\le x\le 9,\\ 0, & \text{otherwise},\end{cases}$$

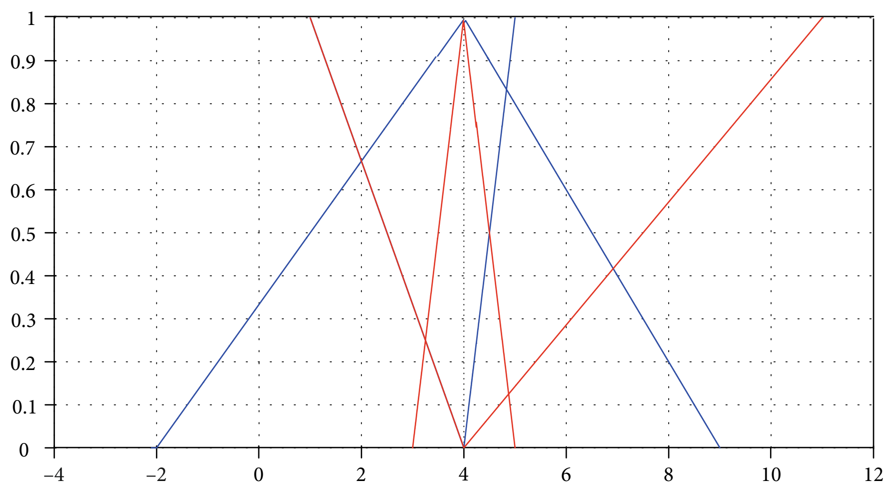

FIGURE 4: The graph of the solution obtained in Example 1.

$$\mathfrak{N}^{\nu}_{\mathfrak{R}}(x) = \begin{cases} \frac{4-x}{3}, & 1 \le x \le 4, \\ -4+x, & 4 \le x \le 5, \\ 0, & \text{otherwise}, \end{cases}$$

$$\alpha(x) = \begin{cases} -3+x, & 3 \le x \le 4, \\ 5-x, & 4 \le x \le 5, \\ 0, & \text{otherwise}, \end{cases}$$

$$\beta(x) = \begin{cases} \frac{4-x}{3}, & 1 \le x \le 4, \\ \frac{-4+x}{7}, & 4 \le x \le 11, \\ 0, & \text{otherwise}. \end{cases} \quad (38)$$

The graph of the solution is given in Figure 4.

5.2. Solution of $A \cdot X + B = C$ by Using the Method of $(\langle s,t\rangle, \langle u,v\rangle)$-Cut. Let A, B, C, and X be the LDFNs and let $A = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta'_1, \vartheta'_2, \vartheta_3, \vartheta'_4, \vartheta'_5) \end{cases}$, $B = \begin{cases} (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \\ (\delta'_1, \delta'_2, \delta_3, \delta'_4, \delta'_5) \end{cases}$, and $C = \begin{cases} (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) \\ (\eta'_1, \eta'_2, \eta_3, \eta'_4, \eta'_5) \end{cases}$. Then,

$$A \cdot X + B = C \quad (39)$$

is a LDF equation (LDFE). Let $X \approx \begin{cases} (x_1, x_2, x_3, x_4, x_5) \\ (x'_1, x'_2, x_3, x'_3, x'_5) \end{cases}$. Then, $X = (C-B)/A$ in general is not the solution of Equation (39).

Let

$$\begin{aligned}
A^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} \left(\left[\underline{\mathfrak{M}^{\tau}_{A}(s)}, \mathfrak{M}^{\bar{\tau}}_{A}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{A}(t)}, \mathfrak{N}^{\bar{\nu}}_{A}(t)\right]\right), \\ \left(\left[\underline{\alpha_A(u)}, \alpha_{\bar{A}}(u)\right], \left[\underline{\beta_A(v)}, \beta_{\bar{A}}(v)\right]\right), \end{cases} \\
B^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} \left(\left[\underline{\mathfrak{M}^{\tau}_{B}(s)}, \mathfrak{M}^{\bar{\tau}}_{B}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{B}(t)}, \mathfrak{N}^{\bar{\nu}}_{B}(t)\right]\right), \\ \left(\left[\underline{\alpha_B(u)}, \alpha_{\bar{B}}(u)\right], \left[\underline{\beta_B(v)}, \beta_{\bar{B}}(v)\right]\right), \end{cases} \\
C^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} \left(\left[\underline{\mathfrak{M}^{\tau}_{C}(s)}, \mathfrak{M}^{\bar{\tau}}_{C}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{C}(t)}, \mathfrak{N}^{\bar{\nu}}_{C}(t)\right]\right), \\ \left(\left[\underline{\alpha_C(u)}, \alpha_{\bar{C}}(u)\right], \left[\underline{\beta_C(v)}, \beta_{\bar{C}}(v)\right]\right), \end{cases} \\
X^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} \left(\left[\underline{\mathfrak{M}^{\tau}_{X}(s)}, \mathfrak{M}^{\bar{\tau}}_{X}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{X}(t)}, \mathfrak{N}^{\bar{\nu}}_{X}(t)\right]\right), \\ \left(\left[\underline{\mathfrak{M}^{\tau}_{X}(s)}, \mathfrak{M}^{\bar{\tau}}_{X}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{X}(t)}, \mathfrak{N}^{\bar{\nu}}_{X}(t)\right]\right) \end{cases}
\end{aligned} \quad (40)$$

represent the $(\langle s,t\rangle, \langle u,v\rangle)$-cuts of A, B, C, and X, respectively, in the given (39). Substituting these into Equation (39), we get

$$A^{\langle s,t\rangle}_{\langle u,v\rangle} \cdot X^{\langle s,t\rangle}_{\langle u,v\rangle} + B^{\langle s,t\rangle}_{\langle u,v\rangle} = C^{\langle s,t\rangle}_{\langle u,v\rangle}. \quad (41)$$

By comparing the $(\langle s,t\rangle, \langle u,v\rangle)$-cuts of A, B, C, and X, we get

$$\begin{aligned}
&\left[\underline{\mathfrak{M}^{\tau}_{A}(s)}, \mathfrak{M}^{\bar{\tau}}_{A}(s)\right] \cdot \left[\underline{\mathfrak{M}^{\tau}_{X}(s)}, \mathfrak{M}^{\bar{\tau}}_{X}(s)\right] + \left[\underline{\mathfrak{M}^{\tau}_{B}(s)}, \mathfrak{M}^{\bar{\tau}}_{B}(s)\right] = \left[\underline{\mathfrak{M}^{\tau}_{C}(s)}, \mathfrak{M}^{\bar{\tau}}_{C}(s)\right], \\
&\left[\underline{\mathfrak{N}^{\nu}_{A}(t)}, \mathfrak{N}^{\bar{\nu}}_{A}(t)\right] \cdot \left[\underline{\mathfrak{N}^{\nu}_{X}(t)}, \mathfrak{N}^{\bar{\nu}}_{X}(t)\right] + \left[\underline{\mathfrak{N}^{\nu}_{B}(t)}, \mathfrak{N}^{\bar{\nu}}_{B}(t)\right] = \left[\underline{\mathfrak{N}^{\nu}_{C}(t)}, \mathfrak{N}^{\bar{\nu}}_{C}(t)\right], \\
&\left[\underline{\alpha_A(u)}, \alpha_{\bar{A}}(u)\right] \cdot \left[\underline{\alpha_X(u)}, \alpha_{\bar{X}}(u)\right] + \left[\underline{\alpha_B(u)}, \alpha_{\bar{B}}(u)\right] = \left[\underline{\alpha_C(u)}, \alpha_{\bar{C}}(u)\right], \\
&\left[\underline{\beta_A(v)}, \beta_{\bar{A}}(v)\right] \cdot \left[\underline{\beta_X(v)}, \beta_{\bar{X}}(v)\right] + \left[\underline{\beta_B(v)}, \beta_{\bar{B}}(v)\right] = \left[\underline{\beta_C(v)}, \beta_{\bar{C}}(v)\right].
\end{aligned} \quad (42)$$

Now,

$$\begin{aligned}
\underline{\mathfrak{M}^{\tau}_X(s)} &= \frac{\underline{\mathfrak{M}^{\tau}_C(s)} - \underline{\mathfrak{M}^{\tau}_B(s)}}{\underline{\mathfrak{M}^{\tau}_A(s)}},\\
\mathfrak{M}^{\bar{\tau}}_X(s) &= \frac{\mathfrak{M}^{\bar{\tau}}_C(s) - \mathfrak{M}^{\bar{\tau}}_B(s)}{\mathfrak{M}^{\bar{\tau}}_A(s)},\\
\underline{\mathfrak{N}^{\nu}_X(t)} &= \frac{\underline{\mathfrak{N}^{\nu}_C(t)} - \underline{\mathfrak{N}^{\nu}_B(t)}}{\underline{\mathfrak{N}^{\nu}_A(t)}},\\
\mathfrak{N}^{\bar{\nu}}_X(t) &= \frac{\mathfrak{N}^{\bar{\nu}}_C(t) - \mathfrak{N}^{\bar{\nu}}_B(t)}{\mathfrak{N}^{\bar{\nu}}_A(t)},\\
\underline{\alpha_X(u)} &= \frac{\underline{\alpha_C(u)} - \underline{\alpha_B(u)}}{\underline{\alpha_A(u)}},\\
\overline{\alpha_X}(u) &= \frac{\overline{\alpha_C}(u) - \overline{\alpha_B}(u)}{\overline{\alpha_A}(u)},\\
\underline{\beta_X(v)} &= \frac{\underline{\beta_C(v)} - \underline{\beta_B(v)}}{\underline{\beta_A(v)}},\\
\overline{\beta_X}(v) &= \frac{\overline{\beta_C}(v) - \overline{\beta_B}(v)}{\overline{\beta_A}(v)}.
\end{aligned} \tag{43}$$

Then, the solution of the equation $A \cdot X + B = C$ exists iff

(1) $\underline{\mathfrak{M}^{\tau}_X(s)}$ is monotonically increasing in $0 \le s \le 1$

(2) $\mathfrak{M}^{\bar{\tau}}_X(s)$ is monotonically decreasing in $0 \le s \le 1$

(3) $\underline{\mathfrak{N}^{\nu}_X(t)}$ is monotonically decreasing in $0 \le t \le 1$

(4) $\mathfrak{N}^{\bar{\nu}}_X(t)$ is monotonically increasing in $0 \le t \le 1$

(5) $\underline{\alpha_X(u)}$ is monotonically increasing in $0 \le u \le 1$

(6) $\overline{\alpha_X}(u)$ is monotonically decreasing in $0 \le u \le 1$

(7) $\underline{\beta_X(v)}$ is monotonically decreasing in $0 \le v \le 1$

(8) $\overline{\beta_X}(v)$ is monotonically increasing in $0 \le v \le 1$

$$\underline{\mathfrak{M}^{\tau}_X(1)} = \mathfrak{M}^{\bar{\tau}}_X(1) = \underline{\mathfrak{N}^{\nu}_X(0)} = \mathfrak{N}^{\bar{\nu}}_X(0) = \underline{\alpha_X(1)} = \overline{\alpha_X}(1) = \underline{\beta_X(0)} = \overline{\beta_X}(0). \tag{44}$$

Example 2. Consider the equation $A \cdot X + B = C$, where

$$\begin{aligned}
A &= \begin{cases} (1,2,5,7,10),\\ (1,3,5,6,11), \end{cases}\\
B &= \begin{cases} (4,6,8,10,15),\\ (4,5,8,11,19), \end{cases}\\
C &= \begin{cases} (1,4,18,38,65),\\ (1,5,18,29,85). \end{cases}
\end{aligned} \tag{45}$$

The $(\langle s,t\rangle, \langle u,v\rangle)$-cuts of A, B, C, and X are

$$\begin{aligned}
A^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} ([1+4s,10-5s],[5-3t,5+2t]),\\ ([3+2u,6-u],[5-4v,5+6v]), \end{cases}\\
B^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} ([4+4s,15-7s],[8-2t,8+2t]),\\ ([5+3u,11-3u],[8-4v,8+11v]), \end{cases}\\
C^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} ([1+17s,65-47s],[18-14t,18+20t]),\\ ([5+13u,29-11u],[18-17v,18+67v]), \end{cases}\\
X^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} \left(\left[\underline{\mathfrak{M}^{\tau}_X(s)}, \mathfrak{M}^{\bar{\tau}}_X(s)\right], \left[\underline{\mathfrak{N}^{\nu}_X(t)}, \mathfrak{N}^{\bar{\nu}}_X(t)\right]\right),\\ \left(\left[\underline{\alpha_X(u)}, \overline{\alpha_X}(u)\right], \left[\underline{\beta_X(v)}, \overline{\beta_X}(v)\right]\right), \end{cases}
\end{aligned} \tag{46}$$

respectively. The $(\langle s,t\rangle, \langle u,v\rangle)$-cut equation is

$$A^{\langle s,t\rangle}_{\langle u,v\rangle} \cdot X^{\langle s,t\rangle}_{\langle u,v\rangle} + B^{\langle s,t\rangle}_{\langle u,v\rangle} = C^{\langle s,t\rangle}_{\langle u,v\rangle}. \tag{47}$$

By comparing the $(\langle s,t\rangle, \langle u,v\rangle)$-cuts of A, B, C, and X, we get

$$\begin{aligned}
\underline{\mathfrak{M}^{\tau}_X(s)} &= \frac{(1+17s)-(4+4s)}{1+4s} = \frac{-3+13s}{1+4s},\\
\mathfrak{M}^{\bar{\tau}}_X(s) &= \frac{(65-47s)-(15-7s)}{10-5s} = \frac{10-8s}{2-s},\\
\underline{\mathfrak{N}^{\nu}_X(t)} &= \frac{(18-14t)-(8-2t)}{5-3t} = \frac{10-12t}{5-3t},\\
\mathfrak{N}^{\bar{\nu}}_X(t) &= \frac{(18+20t)-(8+2t)}{5+2t} = \frac{10+18t}{5+2t},\\
\underline{\alpha_X(u)} &= \frac{(5+13u)-(5+3u)}{3+2u} = \frac{10u}{3+2u},\\
\overline{\alpha_X}(u) &= \frac{(29-11u)-(11-3u)}{6-u} = \frac{18-8u}{6-u},\\
\underline{\beta_X(v)} &= \frac{(18-17v)-(8-4v)}{5-4v} = \frac{10-13v}{5-4v},\\
\overline{\beta_X}(v) &= \frac{(18+67v)-(8+11v)}{5+6v} = \frac{10+56v}{5+6v}.
\end{aligned} \tag{48}$$

It is easy to see that $\underline{\mathfrak{M}^{\tau}_X(s)}, \mathfrak{N}^{\bar{\nu}}_X(t), \underline{\alpha_X(u)}$, and $\overline{\beta_X}(v)$ are increasing and $\mathfrak{M}^{\bar{\tau}}_X(s), \underline{\mathfrak{N}^{\nu}_X(t)}, \overline{\alpha_X}(u)$, and $\underline{\beta_X(v)}$ are decreasing in $0 \le s, t, u, v \le 1$. Also,

$$\begin{aligned}
\underline{\mathfrak{M}^{\tau}_X(1)} &= \mathfrak{M}^{\bar{\tau}}_X(1) = \underline{\mathfrak{N}^{\nu}_X(0)} = \mathfrak{N}^{\bar{\nu}}_X(0) = \underline{\alpha_X(1)} = \overline{\alpha_X}(1)\\
&= \underline{\beta_X(0)} = \overline{\beta_X}(0) = 2.
\end{aligned} \tag{49}$$

This shows that the solution of $A \cdot X + B = C$ exists with $(\langle s,t\rangle, \langle u,v\rangle)$ -cut. The solution is

$$X = \begin{cases} (-3,-1,2,4,5),\\ (-3,0,2,3,6). \end{cases} \tag{50}$$

The solution in continuous form is

$$\mathfrak{M}^{\tau}_{\mathfrak{R}}(x)=\begin{cases}\dfrac{3+x}{13-4x}, & -3\le x\le 2,\\ \dfrac{-10+2x}{-8+x}, & 2\le x\le 5,\\ 0, & \text{otherwise},\end{cases}$$
$$\mathfrak{N}^{\nu}_{\mathfrak{R}}(x)=\begin{cases}\dfrac{-10+5x}{-12+3x}, & -1\le x\le 2,\\ \dfrac{-10+5x}{18-2x}, & 2\le x\le 4,\\ 0, & \text{otherwise},\end{cases}$$
$$\alpha(x)=\begin{cases}\dfrac{3x}{10-2x}, & 0\le x\le 2,\\ \dfrac{-18+6x}{-8+x}, & 2\le x\le 3,\\ 0, & \text{otherwise},\end{cases}$$
$$\beta(x)=\begin{cases}\dfrac{-10+5x}{-13+4x}, & -3\le x\le 2,\\ \dfrac{-10+5x}{56-6x}, & 2\le x\le 6,\\ 0, & \text{otherwise}.\end{cases} \tag{51}$$

The graph of the solution is given in Figure 5.

5.3. Solution of $A\cdot X^2+B\cdot X+C=D$ by Using the Method of α-Cut. Let A, B, C, D, and X be the LDFNs and let

$$A=\begin{cases}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5),\\ \left(\vartheta_1',\vartheta_2',\vartheta_3,\vartheta_4',\vartheta_5'\right),\end{cases}$$
$$B=\begin{cases}(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5),\\ \left(\delta_1',\delta_2^1,\delta_3,\delta_4',\delta_5'\right),\end{cases}$$
$$C=\begin{cases}(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5),\\ \left(\eta_1',\eta_2',\eta_3,\eta_4',\eta_5'\right),\end{cases}$$
$$D=\begin{cases}(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5),\\ \left(\zeta_1',\zeta_2',\zeta_3,\zeta_4',\zeta_5'\right).\end{cases} \tag{52}$$

Then,

$$A\cdot X^2+B\cdot X+C=D \tag{53}$$

is a LDF equation (LDFE). Let $X\approx\begin{cases}(x_1,x_2,x_3,x_4,x_5)\\ (x_1',x_2',x_3,x_4',x_5')\end{cases}$. Let

$$A^{\langle s,t\rangle}_{\langle u,v\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_A(s)},\overline{\mathfrak{M}^{\tau}_A(s)}\right],\left[\underline{\mathfrak{N}^{\nu}_A(t)},\overline{\mathfrak{N}^{\nu}_A(t)}\right]\right),\\ \left(\left[\underline{\alpha_A(u)},\overline{\alpha_A(u)}\right],\left[\underline{\beta_A(v)},\overline{\beta_A(v)}\right]\right),\end{cases}$$
$$B^{\langle s,t\rangle}_{\langle u,v\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_B(s)},\overline{\mathfrak{M}^{\tau}_B(s)}\right],\left[\underline{\mathfrak{N}^{\nu}_B(t)},\overline{\mathfrak{N}^{\nu}_B(t)}\right]\right),\\ \left(\left[\underline{\alpha_B(u)},\overline{\alpha_B(u)}\right],\left[\underline{\beta_B(v)},\overline{\beta_B(v)}\right]\right),\end{cases}$$
$$C^{\langle s,t\rangle}_{\langle u,v\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_C(s)},\overline{\mathfrak{M}^{\tau}_C(s)}\right],\left[\underline{\mathfrak{N}^{\nu}_C(t)},\overline{\mathfrak{N}^{\nu}_C(t)}\right]\right),\\ \left(\left[\underline{\alpha_C(u)},\overline{\alpha_C(u)}\right],\left[\underline{\beta_C(v)},\overline{\beta_C(v)}\right]\right),\end{cases}$$
$$D^{\langle s,t\rangle}_{\langle u,v\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_D(s)},\overline{\mathfrak{M}^{\tau}_D(s)}\right],\left[\underline{\mathfrak{N}^{\nu}_D(t)},\overline{\mathfrak{N}^{\nu}_D(t)}\right]\right),\\ \left(\left[\underline{\alpha_D(u)},\overline{\alpha_D(u)}\right],\left[\underline{\beta_D(v)},\overline{\beta_D(v)}\right]\right),\end{cases}$$
$$X^{\langle s,t\rangle}_{\langle u,v\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_X(s)},\overline{\mathfrak{M}^{\tau}_X(s)}\right],\left[\underline{\mathfrak{N}^{\nu}_X(t)},\overline{\mathfrak{N}^{\nu}_X(t)}\right]\right),\\ \left(\left[\underline{\alpha_X(u)},\overline{\alpha_X(u)}\right],\left[\underline{\beta_X(v)},\overline{\beta_X(v)}\right]\right)\end{cases} \tag{54}$$

represent the $(\langle s,t\rangle,\langle u,v\rangle)$-cuts of A, B, C, D, and X, respectively, in the given (53). Substituting these into Equation (53), we get

$$A^{\langle s,t\rangle}_{\langle u,v\rangle}\cdot\left(X^{\langle s,t\rangle}_{\langle u,v\rangle}\right)^2+B^{\langle s,t\rangle}_{\langle u,v\rangle}\cdot X^{\langle s,t\rangle}_{\langle u,v\rangle}+C^{\langle s,t\rangle}_{\langle u,v\rangle}=D^{\langle s,t\rangle}_{\langle u,v\rangle}. \tag{55}$$

By comparing the $(\langle s,t\rangle,\langle u,v\rangle)$ -cuts of A, B, C, D, and X, we get

$$\left\{\left[\underline{\mathfrak{M}^{\tau}_A(s)},\overline{\mathfrak{M}^{\tau}_A(s)}\right]\cdot\left[\underline{\mathfrak{M}^{\tau}_X(s)},\overline{\mathfrak{M}^{\tau}_X(s)}\right]^2+\left[\underline{\mathfrak{M}^{\tau}_B(s)},\overline{\mathfrak{M}^{\tau}_B(s)}\right]\cdot\left[\underline{\mathfrak{M}^{\tau}_X(s)},\overline{\mathfrak{M}^{\tau}_X(s)}\right]=\left[\underline{\mathfrak{M}^{\tau}_D(s)},\overline{\mathfrak{M}^{\tau}_D(s)}\right]+\left[\underline{\mathfrak{M}^{\tau}_C(s)},\overline{\mathfrak{M}^{\tau}_C(s)}\right],\right.$$
$$\left\{\left[\underline{\mathfrak{N}^{\nu}_A(t)},\overline{\mathfrak{N}^{\nu}_A(t)}\right]\cdot\left[\underline{\mathfrak{N}^{\nu}_X(t)},\overline{\mathfrak{N}^{\nu}_X(t)}\right]^2+\left[\underline{\mathfrak{N}^{\nu}_B(t)},\overline{\mathfrak{N}^{\nu}_B(t)}\right]\cdot\left[\underline{\mathfrak{N}^{\nu}_X(t)},\overline{\mathfrak{N}^{\nu}_X(t)}\right]=\left[\underline{\mathfrak{N}^{\nu}_D(t)},\overline{\mathfrak{N}^{\nu}_D(t)}\right]+\left[\underline{\mathfrak{N}^{\nu}_C(t)},\overline{\mathfrak{N}^{\nu}_C(t)}\right],\right.$$
$$\left\{\left[\underline{\alpha_A(u)},\overline{\alpha_A(u)}\right]\cdot\left[\underline{\alpha_X(u)},\overline{\alpha_X(u)}\right]^2+\left[\underline{\alpha_B(u)},\overline{\alpha_B(u)}\right]\cdot\left[\underline{\alpha_X(u)},\overline{\alpha_X(u)}\right]=\left[\underline{\alpha_D(u)},\overline{\alpha_D(u)}\right]+\left[\underline{\alpha_C(u)},\overline{\alpha_C(u)}\right],\right.$$
$$\left\{\left[\underline{\beta_A(v)},\overline{\beta_A(v)}\right]\cdot\left[\underline{\beta_X(v)},\overline{\beta_X(v)}\right]^2+\left[\underline{\beta_B(v)},\overline{\beta_B(v)}\right]\cdot\left[\underline{\beta_X(v)},\overline{\beta_X(v)}\right]=\left[\underline{\beta_D(v)},\overline{\beta_D(v)}\right]+\left[\underline{\beta_C(v)},\overline{\beta_C(v)}\right].\right. \tag{56}$$

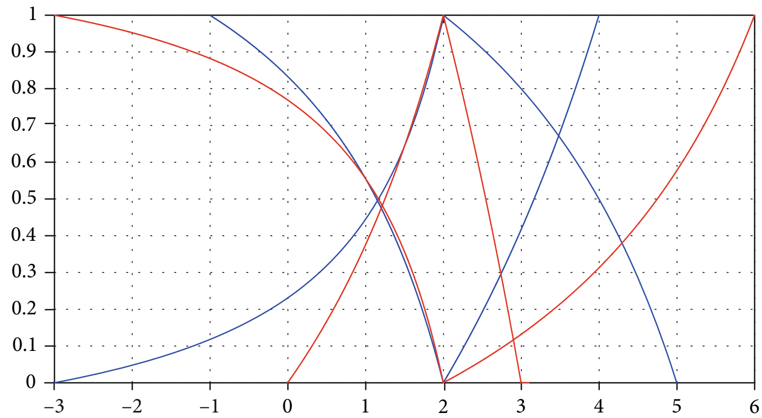

FIGURE 5: The graph of the solution obtained in Example 2.

Now,

$$\underline{\mathfrak{M}_X^{\tau}(s)} = \frac{-\underline{\mathfrak{M}_B^{\tau}(s)} \pm \sqrt{\underline{\mathfrak{M}_B^{\tau}(s)}^2 - 4\left(\underline{\mathfrak{M}_A^{\tau}(s)}\right)\left(\underline{\mathfrak{M}_C^{\tau}(s)} - \underline{\mathfrak{M}_D^{\tau}(s)}\right)}}{2\underline{\mathfrak{M}_A^{\tau}(s)}},$$

$$\overline{\mathfrak{M}_X^{\tau}}(s) = \frac{-\overline{\mathfrak{M}_B^{\tau}}(s) \pm \sqrt{\overline{\mathfrak{M}_B^{\tau}}(s)^2 - 4\left(\overline{\mathfrak{M}_A^{\tau}}(s)\right)\left(\overline{\mathfrak{M}_C^{\tau}}(s) - \overline{\mathfrak{M}_D^{\tau}}(s)\right)}}{2\overline{\mathfrak{M}_A^{\tau}}(s)},$$

$$\underline{\mathfrak{N}_X^{\nu}(t)} = \frac{-\underline{\mathfrak{N}_B^{\nu}(t)} \pm \sqrt{\underline{\mathfrak{N}_B^{\nu}(t)}^2 - 4\left(\underline{\mathfrak{N}_A^{\nu}(t)}\right)\left(\underline{\mathfrak{N}_C^{\nu}(t)} - \underline{\mathfrak{N}_D^{\nu}(t)}\right)}}{2\underline{\mathfrak{N}_A^{\nu}(t)}},$$

$$\overline{\mathfrak{N}_X^{\nu}}(t) = \frac{-\overline{\mathfrak{N}_B^{\nu}}(t) \pm \sqrt{\overline{\mathfrak{N}_B^{\nu}}(t)^2 - 4\left(\overline{\mathfrak{N}_A^{\nu}}(t)\right)\left(\overline{\mathfrak{N}_C^{\nu}}(t) - \overline{\mathfrak{N}_D^{\nu}}(t)\right)}}{2\overline{\mathfrak{N}_A^{\nu}}(t)},$$

$$\underline{\alpha_X(u)} = \frac{-\underline{\alpha_B(u)} \pm \sqrt{\underline{\alpha_B(u)}^2 - 4\left(\underline{\alpha_A(u)}\right)\left(\underline{\alpha_C(u)} - \underline{\alpha_D(u)}\right)}}{2\underline{\alpha_A(u)}},$$

$$\overline{\alpha_X}(u) = \frac{-\overline{\alpha_B}(u) \pm \sqrt{\overline{\alpha_B}(u)^2 - 4(\overline{\alpha_A}(u))(\overline{\alpha_C}(u) - \overline{\alpha_D}(u))}}{2\overline{\alpha_A}(u)},$$

$$\underline{\beta_X(v)} = \frac{-\underline{\beta_B(v)} \pm \sqrt{\underline{\beta_B(v)}^2 - 4\left(\underline{\beta_A(v)}\right)\left(\underline{\beta_C(v)} - \underline{\beta_D(v)}\right)}}{2\underline{\beta_A(v)}},$$

$$\overline{\beta_X}(v) = \frac{-\overline{\beta_B}(v) \pm \sqrt{\overline{\beta_B}(v)^2 - 4(\overline{\beta_A}(v))(\overline{\beta_C}(v) - \overline{\beta_D}(v))}}{2\overline{\beta_A}(v)}. \tag{57}$$

Then, the solution of the equation $A \cdot X^2 + B \cdot X + C = D$ exists iff

(1) $\underline{\mathfrak{M}_X^{\tau}(s)}$ is monotonically increasing in $0 \le s \le 1$

(2) $\overline{\mathfrak{M}_X^{\tau}}(s)$ is monotonically decreasing in $0 \le s \le 1$

(3) $\underline{\mathfrak{N}_X^{\nu}(t)}$ is monotonically decreasing in $0 \le t \le 1$

(4) $\overline{\mathfrak{N}_X^{\nu}}(t)$ is monotonically increasing in $0 \le t \le 1$

(5) $\underline{\alpha_X(u)}$ is monotonically increasing in $0 \le u \le 1$

(6) $\overline{\alpha_X}(u)$ is monotonically decreasing in $0 \le u \le 1$

(7) $\underline{\beta_X(v)}$ is monotonically decreasing in $0 \le v \le 1$

(8) $\overline{\beta_X}(v)$ is monotonically increasing in $0 \le v \le 1$

(9) $\underline{\mathfrak{M}_X^{\tau}(1)} = \overline{\mathfrak{M}_X^{\tau}}(1) = \underline{\mathfrak{N}_X^{\nu}(0)} = \overline{\mathfrak{N}_X^{\nu}}(0) = \underline{\alpha_X(1)} = \overline{\alpha_X}(1) = \underline{\beta_X(0)} = \overline{\beta_X}(0)$

TABLE 1: $\langle s, t\rangle$-cuts of A, B, C, D, and X.

$\langle s, t\rangle$-cuts	X	A	B	C	D	$C-D$
$\vartheta_1 + s(\vartheta_3 - \vartheta_1)$	$\underline{\mathfrak{M}_X^{\tau}(s)}$	$4+3s$	$2+3s$	$1+3s$	$1+5s$	$-2s$
$\vartheta_5 - s(\vartheta_5 - \vartheta_3)$	$\overline{\mathfrak{M}_X^{\tau}}(s)$	$10-3s$	$8-3s$	$7-3s$	$12-6s$	$-5+3s$
$\vartheta_3 - t(\vartheta_3 - \vartheta_2)$	$\underline{\mathfrak{N}_X^{\nu}(t)}$	$7-2t$	$5-t$	$4-2t$	$6-3t$	$-2+t$
$\vartheta_3 + t(\vartheta_4 - \vartheta_3)$	$\overline{\mathfrak{N}_X^{\nu}}(t)$	$7+2t$	$5+t$	$4+t$	$6+2t$	$-2-t$

Example 3. Consider the equation $A \cdot X^2 + B \cdot X + C = D$, where

$$A = \begin{cases} (4, 5, 7, 9, 10), \\ (2, 6, 7, 8, 13), \end{cases}$$
$$B = \begin{cases} (2, 4, 5, 6, 8), \\ (4, 4, 5, 6, 7), \end{cases}$$
$$C = \begin{cases} (1, 2, 4, 5, 7), \\ (1, 3, 4, 5, 7), \end{cases}$$
$$D = \begin{cases} (1, 3, 6, 8, 12), \\ (1, 4, 6, 8, 11). \end{cases} \tag{58}$$

The $\langle s, t\rangle$-cuts of A, B, C, D, and X are given in Table 1.

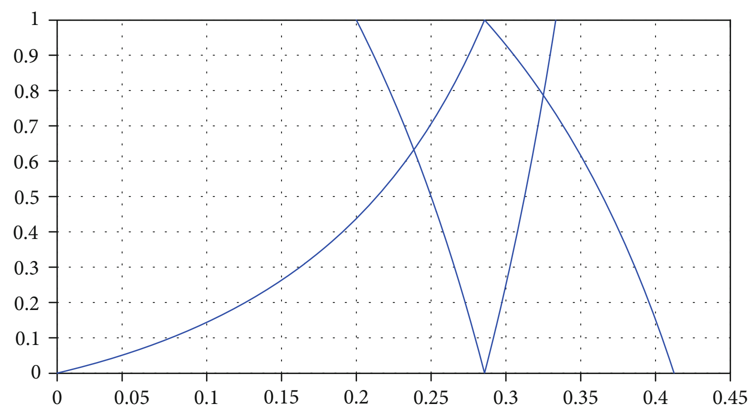

Figure 6: Solution by $\langle s, t\rangle$-cut.

By comparing the $\langle s, t\rangle$-cuts of A, B, C, D, and X, we get

$$
\begin{aligned}
\underline{\mathfrak{M}_X^{\tau}(s)} &= \frac{-(2+3s)+\sqrt{(2+3s)^2-4(4+3s)(-2s)}}{2(4+3s)},\\
\mathfrak{M}_X^{\bar{\tau}}(s) &= \frac{-(8-3s)+\sqrt{(8-3s)^2-4(10-3s)(-5+3s)}}{2(10-3s)},\\
\underline{\mathfrak{N}_X^{\nu}(t)} &= \frac{-(5-t)+\sqrt{(5-t)^2-4(7-2t)(-2+t)}}{2(7-2t)},\\
\mathfrak{N}_X^{\bar{\nu}}(t) &= \frac{-(5+t)+\sqrt{(5+t)^2-4(7+2t)(-2-t)}}{2(7+2t)}.
\end{aligned}
\tag{59}
$$

The graph obtained by $\langle s, t\rangle$-cut is shown in Figure 6. The $\langle u, v\rangle$-cuts of A, B, C, D, and X are given in Table 2. By comparing the $\langle u, v\rangle$-cuts of A, B, C, D, and X, we get

$$
\begin{aligned}
\underline{\alpha(u)} &= \frac{-(4+u)+\sqrt{(4+u)^2-4(6+u)(-1-u)}}{2(6+u)},\\
\alpha(\bar{u}) &= \frac{-(6-u)+\sqrt{(6-u)^2-4(8-u)(-3+u)}}{2(8-u)},\\
\underline{\beta(v)} &= \frac{-(5-v)+\sqrt{(5-v)^2-4(7-5v)(-2+2v)}}{2(7-5v)},\\
\beta(\bar{v}) &= \frac{-(5+2v)+\sqrt{(5+2v)^2-4(7+6v)(-2-2v)}}{2(7+6v)}.
\end{aligned}
\tag{60}
$$

The graph obtained by $\langle u, v\rangle$-cut is shown in Figure 7. It is easy to see that $\underline{\mathfrak{M}_X^{\tau}(s)}$, $\mathfrak{N}_X^{\bar{\nu}}(t)$, $\underline{\alpha_X(u)}$, and $\beta_X(\bar{v})$ are increasing and $\mathfrak{M}_X^{\bar{\tau}}(s)$, $\underline{\mathfrak{N}_X^{\nu}(t)}$, $\alpha_X(\bar{u})$, and $\underline{\beta_X(v)}$ are decreasing in $0 \le s, t, u, v \le 1$. Also,

Table 2: $\langle u, v\rangle$-cuts of A, B, C, D, and X.

$\langle u, v\rangle$-cuts	X	A	B	C	D	$C-D$
$\vartheta_2' + u(\vartheta_3 - \vartheta_2')$	$\underline{\alpha(u)}$	$6+u$	$4+u$	$3+u$	$4+2u$	$-1-u$
$\vartheta_4' - u(\vartheta_4' - \vartheta_3)$	$\alpha(\bar{u})$	$8-u$	$6-u$	$5-u$	$8-2u$	$-3+u$
$\vartheta_3 - v(\vartheta_3 - \vartheta_1')$	$\underline{\beta(v)}$	$7-5v$	$5-v$	$4-3v$	$6-5v$	$-2+2v$
$\vartheta_3 + v(\vartheta_5' - \vartheta_3)$	$\beta(\bar{v})$	$7+6v$	$5+2v$	$4+3v$	$6+5v$	$-2-2v$

$$
\begin{aligned}
\underline{\mathfrak{M}_X^{\tau}(1)} &= \mathfrak{M}_X^{\bar{\tau}}(1) = \underline{\mathfrak{N}_X^{\nu}(0)} = \mathfrak{N}_X^{\bar{\nu}}(0) = \underline{\alpha_X(1)} = \alpha_X(\bar{1})\\
&= \underline{\beta_X(0)} = \beta_X(\bar{0}) = 0.2857.
\end{aligned}
\tag{61}
$$

This shows that the solution of $A \cdot X^2 + B \cdot X + C = D$ exists with $(\langle s, t\rangle, \langle u, v\rangle)$ -cut. The solution is

$$
X = \begin{cases} \left(0, \frac{1}{5}, \frac{2}{7}, \frac{1}{3}, \frac{-4+\sqrt{66}}{10}\right) \\ \left(0, \frac{-2+\sqrt{10}}{6}, \frac{2}{7}, \frac{-3+\sqrt{33}}{8}, \frac{-7+\sqrt{257}}{26}\right) \end{cases} = \begin{cases} (0, 0.2, 0.2857, 0.3333, 0.4124), \\ (0, 0.1937, 0.2857, 0.3431, 0.3474). \end{cases}
\tag{62}
$$

The solution in continuous form is

$$
\mathfrak{M}_{\mathfrak{R}}^{\tau}(x) = \begin{cases} -\dfrac{2x(2x+1)}{3x^2+3x-2}, & 0 \le x \le 0.2857, \\ \dfrac{10x^2+8x-5}{3(x^2+x-1)}, & 0.2857 \le x \le 0.4124, \\ 0, & \text{otherwise}, \end{cases}
$$

$$
\mathfrak{N}_{\mathfrak{R}}^{\nu}(x) = \begin{cases} \dfrac{7x-2}{2x-1}, & 0.2 \le x \le 0.2857, \\ -\dfrac{7x-2}{2x-1}, & 0.2857 \le x \le 0.3333, \\ 0, & \text{otherwise}, \end{cases}
$$

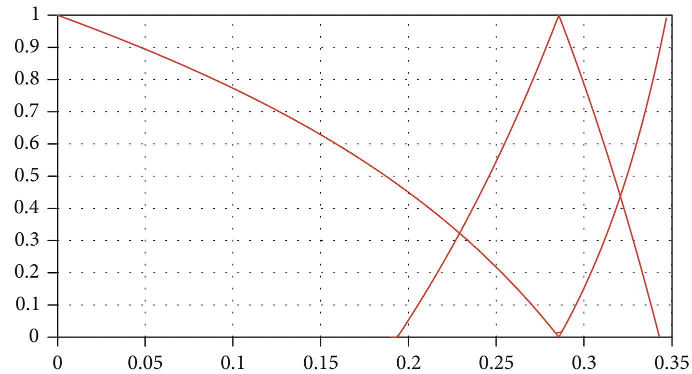

FIGURE 7: Solution by $\langle u, v\rangle$-cut.

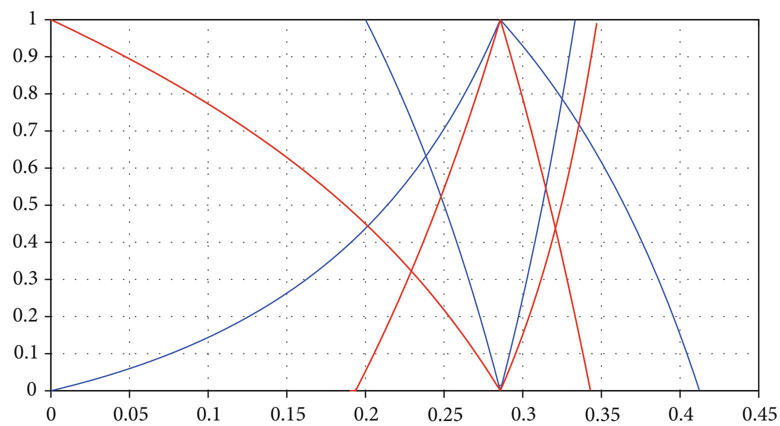

FIGURE 8: The graph of the solution obtained in Example 3.

$$\alpha(x) = \begin{cases} \dfrac{-6x^2 - 4x + 1}{x^2 + x - 1}, & 0.1937 \le x \le 0.2857, \\ \dfrac{8x^2 + 6x - 3}{x^2 + x - 1}, & 0.2857 \le x \le 0.3431, \\ 0, & \text{otherwise}, \end{cases}$$

$$\beta(x) = \begin{cases} \dfrac{7x^2 + 5x - 2}{5x^2 + x - 2}, & 0 \le x \le 0.2857, \\ \dfrac{-7x^2 - 5x + 2}{2(3x^2 + x - 1)}, & 0.2857 \le x \le 0.3474, \\ 0, & \text{otherwise}. \end{cases} \quad (63)$$

The graph of the solution is given in Figure 8.

6. Conclusion

In this paper, we have defined the linear Diophantine fuzzy numbers, in particular triangular linear Diophantine fuzzy number, and present some properties related to them. After finding the ranking function of triangular linear Diophantine fuzzy number, our study has focussed on the linear Diophantine fuzzy equations. We used the more general approach to solve LDF equations that is the method of $(\langle s, t\rangle, \langle u, v\rangle)$-cut. In LDF sets, there is no limitation to take the grades like in intuitionistic fuzzy sets, Pythagorean fuzzy sets, and q-rung orthopair fuzzy sets. The linear Diophantine fuzzy numbers may have several applications, like in linear programming, transportation problems, assignment problems, and shortest route problems. Our future work may be on the following topics:

(i) LDF linear programming problems

(ii) LDF assignment problems and transportation problems

(iii) LDF shortest path problems

(iv) Numerical solutions of linear and nonlinear LDF equations

Data Availability

No data were used to support this study.

Disclosure

The statements made and views expressed are solely the responsibility of the author.

Conflicts of Interest

The authors of this paper declare that they have no conflict of interest.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] E. Sanchez, "Solution of fuzzy equations with extend operations," *Fuzzy Sets and Systems*, vol. 12, pp. 248–273, 1984.

[3] L. Biacino and A. Lettieri, "Equations with fuzzy numbers," *Information Sciences*, vol. 47, no. 1, pp. 63–76, 1989.

[4] J. J. Buckley, "Solving fuzzy equations," *Fuzzy Sets and Systems*, vol. 50, no. 1, pp. 1–14, 1992.

[5] J. Wasowski, "On solutions to fuzzy equations," *Control and Cybern*, vol. 26, pp. 653–658, 1997.

[6] J. J. Buckley and Y. Qu, "Solving linear and quadratic fuzzy equations," *Fuzzy Sets and Systems*, vol. 38, no. 1, pp. 43–59, 1990.

[7] J. J. Buckley and Y. Qu, "On using α-cuts to evaluate fuzzy equations," *Fuzzy Sets and Systems*, vol. 38, no. 3, pp. 309–312, 1999.

[8] J. J. Buckley and Y. Qu, "Solving fuzzy equations: a new solution concept," *Fuzzy Sets and Systems*, vol. 39, no. 3, pp. 291–301, 1991.

[9] J. J. Buckley and Y. Qu, "Solving systems of linear fuzzy equations," *Fuzzy Sets and Systems*, vol. 43, no. 1, pp. 33–43, 1991.

[10] H. Jiang, "The approach to solving simultaneous linear equations that coefficients are fuzzy numbers," *Journal of National University of Defence Technology(Chinese)*, vol. 3, pp. 96–102, 1986.

[11] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[12] K. T. Atanassov, "More on intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 33, no. 1, pp. 37–45, 1989.

[13] F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability*, American Research Press, Rehoboth, NM, 1999.

[14] F. Smarandache, "Neutrosophy and neutrosophic logic," in *First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico*, Gallup, NM 87301, USA, 2002.

[15] W.-R. Zhang, "Bipolar fuzzy sets," in *1998 IEEE International Conference on Fuzzy Systems Proceedings. IEEE World Congress on Computational Intelligence (Cat. No. 98CH36228)*, vol. 1, pp. 835–840, Anchorage, AK, USA, 1998.

[16] S. Banerjee and T. K. Roy, "Intuitionistic fuzzy linear and quadratic equations," *Journal of Information and Computing Science*, vol. 10, no. 4, pp. 291–310, 2015.

[17] D. Chakraborty, D. K. Jana, and T. K. Roy, "Arithmetic operations on generalized intuitionistic fuzzy number and its applications to transportation problem," *Opsearch*, vol. 52, no. 3, pp. 431–471, 2015.

[18] M. Rahaman, S. P. Mondal, B. Chatterjee, and S. Alam, "The solution techniques for linear and quadratic equations with coefficients as Cauchy neutrosphic numbers," *Granular Computing*, pp. 1–19, 2021.

[19] M. Akram, M. Ali, and T. Allahviranloo, "Solution of complex bipolar fuzzy linear system," in *International Online Conference on Intelligent Decision Science*, vol. 1301, pp. 899–927, Istanbul, Turkey, 2021.

[20] M. Akram, T. Allahviranloo, W. Pedrycz, and M. Ali, "Methods for solving *LR*-bipolar fuzzy linear systems," *Soft Computing*, vol. 25, no. 1, pp. 85–108, 2021.

[21] M. Akram, M. Ali, and T. Allahviranloo, "Certain methods to solve bipolar fuzzy linear system of equations," *Computational and Applied Mathematics*, vol. 39, no. 3, pp. 1–28, 2020.

[22] M. Akram, G. Muhammad, and T. Allahviranloo, "Bipolar fuzzy linear system of equations," *Computational and Applied Mathematics*, vol. 38, no. 2, pp. 1–29, 2019.

[23] M. Akram, G. Muhammad, and N. Hussain, "Bipolar fuzzy system of linear equations with polynomial parametric form," *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 6, pp. 8275–8287, 2019.

[24] A. Chakraborty, S. Maity, S. Jain, S. P. Mondal, and S. Alam, "Hexagonal fuzzy number and its distinctive representation, ranking, defuzzification technique and application in production inventory management problem," *Granul Computing*, vol. 6, no. 3, pp. 507–521, 2021.

[25] R. Jafari, W. Yu, S. Razvarz, and A. Gegov, "Numerical methods for solving fuzzy equations: a survey," *Fuzzy Sets and Systems*, vol. 404, pp. 1–22, 2021.

[26] A. K. Shaw and T. K. Roy, "Some arithmetic operations on triangular intuitionistic fuzzy number and its application on reliability evaluation," *International Journal of Fuzzy Mathematics and Systems*, vol. 2, no. 4, pp. 363–382, 2012.

[27] M. Riaz and M. R. Hashmi, "Linear Diophantine fuzzy set and its applications towards multi-attribute decision-making problems," *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 4, pp. 5417–5439, 2019.

[28] A. Iampan, G. S. García, M. Riaz, H. M. Athar Farid, and R. Chinram, "Linear Diophantine fuzzy Einstein aggregation operators for multi-criteria decision-making problems," *Journal of Mathematics*, vol. 2021, Article ID 5548033, 31 pages, 2021.

[29] M. Riaz, M. R. Hashmi, D. Pamucar, and Y. M. Chu, "Spherical linear Diophantine fuzzy sets with modeling uncertainties in MCDM," *Computer Modeling in Engineering & Sciences*, vol. 126, no. 3, pp. 1125–1164, 2021.

[30] S. Ayub, M. Shabir, M. Riaz, M. Aslam, and R. Chinram, "Linear Diophantine fuzzy relations and their algebraic properties with decision making," *Symmetry*, vol. 13, no. 6, p. 945, 2021.

[31] A. O. Almagrabi, S. Abdullah, M. Shams, Y. D. Al-Otaibi, and S. Ashraf, "A new approach to q-linear Diophantine fuzzy emergency decision support system for COVID19," *Journal of Ambient Intelligence and Humanized Computing*, no. article 3130, pp. 1–27, 2021.

[32] H. Kamac, "Linear Diophantine fuzzy algebraic structures," *Journal of Ambient Intelligence and Humanized Computing*, vol. 12, no. 11, pp. 10353–10373, 2021.

[33] A. Ebrahimnejad and J. L. Verdegay, *Fuzzy Sets-Based Methods and Techniques for Modern Analytics*, vol. 364 of Studies in Fuzziness and Soft Computing, , Springer International Publishing, 2018.

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Research Article

Faber Polynomial Coefficient Bounds for m-Fold Symmetric Analytic and Bi-univalent Functions Involving q-Calculus

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In our present investigation, by applying q-calculus operator theory, we define some new subclasses of m-fold symmetric analytic and bi-univalent functions in the open unit disk $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$ and use the Faber polynomial expansion to find upper bounds of $|a_{mk+1}|$ and initial coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ as well as Fekete-Szego inequalities for the functions belonging to newly defined subclasses. Also, we highlight some new and known corollaries of our main results.

1. Introduction, Definitions, and Motivation

Let $\mathscr{A}$ denote the class of all analytic functions $f(z)$ in the open unit disk $\mathscr{U} = \{z : |z| < 1\}$ and have the series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

By $\mathscr{S}$, we mean the subclass of $\mathscr{A}$ consisting of univalent functions. The inverse f^{-1} of univalent function f can be defined as

$$\begin{aligned} &f^{-1}(f(z)) = z,\ z \in \mathscr{U}, \\ &f(f^{-1}(w)) = w, \quad |w| < r_0(f), r_0(f) \geq \frac{1}{4}, \end{aligned} \tag{2}$$

where

$$\begin{aligned} g_1(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 \\ - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + . \cdots \end{aligned} \tag{3}$$

According to the Koebe one-quarter theorem [1], an analytic function f is called bi-univalent in $\mathscr{U}$ if both f and f^{-1} are univalent in $\mathscr{U}$. Let Σ denote the class all bi-univalent functions in $\mathscr{U}$. For $f \in \Sigma$, Lewin [2] showed that $|a_2| < 1.51$ and Brannan and Cluni [3] proved that $|a_2| \leq \sqrt{2}$. Netanyahu [4] showed that $\max |a_2| = 4/3$. Brannan and Taha [5] introduced a certain subclass of bi-univalent functions for class Σ. In recent years, Srivastava et al. [6], Frasin and Aouf [7], Altinkaya and Yalcin [8, 9], and Hayami and Owa [10] studied the various subclasses of analytic and bi-univalent function. For a brief history, see [11].

In [12], Faber introduced Faber polynomials, and after that, Gong [13] studied Faber polynomials in geometric function theory. In their published works, some contributions have been made to finding the general coefficient bounds $|a_n|$ by applying Faber polynomial expansions. By using Faber polynomial expansions, very little work has been done for the coefficient bounds $|a_n|$ for $n \geq 4$ of Maclaurin's series. For more studies, see [14–17].

A domain $\mathscr{U}$ is said to be m-fold symmetric if

$$f\left(e^{i(2\pi/m)} z\right) = e^{i(2\pi/m)} f(z), \quad z \in \mathscr{U}, f \in \mathscr{A}, m \in \mathbb{N}. \tag{4}$$

The univalent function $h(z)$ maps the unit disk $\mathcal{U}$ into a region with m-fold symmetry and can be defined as

$$h(z) = \sqrt[m]{f(z^m)}, \quad f \in \mathcal{S}. \tag{5}$$

A function f is said to be m-fold symmetric [18] if it has the series expansion of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}. \tag{6}$$

The class of all m-fold symmetric univalent functions is denoted by $\mathcal{S}^m$, and for $m = 1$, then $\mathcal{S}^m = \mathcal{S}$.

In [19], Srivastava et al. proved the inverse f_m^{-1} series expansion for $f \in \Sigma_m$, which is given as follows:

$$\begin{aligned} g(w) = f_m^{-1}(w) = w - a_{m+1}w^{m+1} + \left((m+1)a_{m+1}^2 - a_{2m+1}\right)w^{2m+1} \\ - \left\{\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right)\Big\} w^{3m+1} \\ +. \cdots \end{aligned} \tag{7}$$

Here, we will denote m-fold symmetric bi-univalent functions by Σ_m. For $m = 1$, equation (7) coincides with equation (3) of the class Σ. The coefficient problem for $f \in \Sigma_m$ is one of the favorite subjects of geometric function theory in these days (see [20–23]).

The quantum (or q-) calculus has great importance because of its applications in several fields of mathematics, physics, and some related areas. The importance of q -derivative operator (D_q) is pretty recognizable by its applications in the study of numerous subclasses of analytic functions. Initially, in 1908, Jackson [24] introduced a q -derivative operator and studied its applications. Further, in [25], Ismail et al. defined a class of q-starlike functions; after that, Srivastava [26] studied q-calculus in the context of univalent function theory; also, numerous mathematicians studied q-calculus in the context of univalent function theory. Further, the q-analogue of the Ruscheweyh differential operator was defined by Kanas and Raducanu [27] and Arif et al. [28] discussed some of its applications for multivalent functions while Zhang et al. in [29] studied q-starlike functions related with the generalized conic domain. Srivastava et al. published the articles (see [30, 31]) in which they studied the class of q-starlike functions. For some more recent investigations about q-calculus, we may refer to [32–34].

For a better understanding of the article, we recall some concept details and definitions of the q-difference calculus. Throughout the article, we presume that

$$0 < q < 1. \tag{8}$$

Definition 1. The q-factorial $[n]_q!$ is defined as

$$[n]_q! = \prod_{k=1}^{n} [k]_q \quad (n \in \mathbb{N}), \tag{9}$$

and the q-generalized Pochhammer symbol $[t]_{n,q}$, $t \in \mathbb{C}$, is defined as

$$[t]_{n,q} = [t]_q[t+1]_q[t+2]_q \cdots [t+n-1]_q \quad (n \in \mathbb{N}). \tag{10}$$

Remark 2. For $n = 0$, then $[n]_q! = 1$, and $[t]_{n,q} = 1$.

Definition 3. The q-number $[t]_q$ for $q \in (0,1)$ is defined as

$$[t]_q = \begin{cases} \dfrac{1-q^t}{1-q} & (t \in \mathbb{C}), \\ \displaystyle\sum_{k=0}^{n-1} q^k & (t = n \in \mathbb{N}). \end{cases} \tag{11}$$

Definition 4 (see [24]). The q-derivative (or q-difference) operator D_q of a function f is defined, in a given subset of $\mathbb{C}$, by

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{12}$$

provided that $f'(0)$ exists.

From Definition 4, we can observe that

$$\lim_{q \longrightarrow 1^-} (D_q f)(z) = \lim_{q \longrightarrow 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z), \tag{13}$$

for a differentiable function f in a given subset of $\mathbb{C}$. It is also known from (1) and (12) that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{14}$$

Here, in this paper, we use the q-difference operator to define new subclasses of m-fold symmetric analytic and bi-univalent functions and then apply the Faber polynomial expansion technique to determine the general coefficient bounds $|a_{mk+1}|$ and initial coefficient bounds $|a_{m+1}|$ and $|a_{2m+1}|$ as well as Fekete-Szego inequalities.

Definition 5. A function $f \in \Sigma_m$ is said to be in the class $\mathcal{R}_b(\varphi, m, q)$ if and only if

$$\begin{aligned} 1 + \frac{1}{b}\left(D_q f(z) - 1\right) \prec \varphi(z), \\ 1 + \frac{1}{b}\left(D_q g(w) - 1\right) \prec \varphi(w), \end{aligned} \tag{15}$$

where $\varphi \in \mathcal{P}$, $b \in \mathbb{C} \setminus \{0\}$, and$z, w \in \mathcal{U}$, and$g(w) = f_m^{-1}(w)$ is defined by (7).

Remark 6. For $q \longrightarrow 1^-$ and $m = 1$, then the class $\mathscr{R}_b(\varphi, m, q)$ reduces into the class $\mathscr{R}_b(\varphi)$ introduced by Hamidi and Jahangiri in [35].

Definition 7. A function $f \in \Sigma_m$ is said to be in the class $\mathscr{S}^*_{\Sigma_m}(\varphi, q)$ if and only if

$$\frac{zD_qf(z)}{f(z)} \prec \varphi(z), \quad \frac{wD_qg(w)}{g(w)} \prec \varphi(w), \tag{16}$$

where $\varphi \in \mathscr{P}$, $b \in \mathbb{C} \setminus \{0\}$, and$z, w \in \mathscr{U}$, and$g(w) = f_m^{-1}(w)$ is defined by (7).

Remark 8. For $q \longrightarrow 1^-$, $m = 1$, and $\varphi(z) = (1 + Az)/(1 + Bz)$, then the class $S^*_{\Sigma_m}(\varphi, q)$ reduces into the class $\mathscr{S}(A, B)$, introduced by Hamidi and Jahangiri in [36].

2. Main Results

Using the Faber polynomial expansion of functions $f \in \mathscr{A}$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as [15] given by

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots) w^n, \tag{17}$$

for an expansion of K_{n-1}^{-n} (see [37]). In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2}K_1^{-2} = -a_2, \quad \frac{1}{3}K_2^{-3} = 2a_2^2 - a_3, \quad \frac{1}{4}K_3^{-4} = -\left(5a_2^3 - 5a_2a_3 + a_4\right). \tag{18}$$

In general, for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of K_{n-1}^p is as (see [15])

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}E_{n-1}^2 + \frac{p!}{(p-3)!3!}E_{n-1}^3 + \cdots + \frac{p!}{(p-n+1)!(n-1)!}E_{n-1}^{n-1}, \tag{19}$$

where $E_{n-1}^p = E_{n-1}^p(a_2, a_3, \cdots)$, and by [37],

$$E_{n-1}^m(a_2, \cdots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \cdots (a_n)^{\mu_{n-1}}}{\mu_1!, \cdots, \mu_{n-1}!}, \quad \text{for } m \leq n, \tag{20}$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, \cdots, \mu_n$ satisfying

$$\mu_1 + \mu_2 + \cdots + \mu_n = m, \quad \mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} = n - 1. \tag{21}$$

Evidently, $E_{n-1}^{n-1}(a_2, \cdots, a_n) = a_2^{n-1}$ (see [14]), or equivalently,

$$E_n^m(a_1, a_2, \cdots, a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1} \cdots (a_n)^{\mu_n}}{\mu_1!, \cdots, \mu_n!}, \quad \text{for } m \leq n, \tag{22}$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, \cdots, \mu_n$ satisfying

$$\mu_1 + \mu_2 + \cdots + \mu_n = m, \quad \mu_1 + 2\mu_2 + \cdots + (n)\mu_n = n. \tag{23}$$

It is clear that $E_n^n(a_1, \cdots, a_n) = E_1^n$, and the first and last polynomials are $E_n^n = a_1^n$ and $E_n^1 = a_n$.

Similarly, using the Faber polynomial expansion of functions $f \in \mathscr{A}$ of the form (6), that is,

$$f(z) = z + \sum_{k=1}^{\infty} K_k^{1/m}(a_2, a_3, \cdots, a_{k+1}) z^{mk+1}. \tag{24}$$

The coefficients of its inverse map $g = f_m^{-1}$ may be expressed as

$$g(z) = f_m^{-1}(z) = w + \sum_{k=1}^{\infty} \frac{1}{(mk+1)} K_k^{-(mk+1)} \cdot (a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}) w^{mk+1}. \tag{25}$$

Theorem 9. *For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathscr{R}_b(\varphi, m, q)$be given by (6), and if$a_{mj+1} = 0$, $1 \leq j \leq k - 1$, then*

$$|a_{mk+1}| \leq \frac{2|b|}{1 + mk}, \quad \textit{for } k \geq 2. \tag{26}$$

Proof. By definition, for the function $f \in \mathscr{R}_b(\varphi, m, q)$ of the form (6), we have

$$1 + \frac{1}{b}\left(D_qf(z) - 1\right) = 1 + \sum_{k=1}^{\infty} \frac{[1 + mk]_q}{b} a_{mk+1} z^{mk}, \tag{27}$$

and for its inverse map $g = f_m^{-1}$, we have

$$1 + \frac{1}{b}\left(D_qg(w) - 1\right) = 1 + \sum_{k=1}^{\infty} \frac{[1 + mk]_q}{b} A_{mk+1} w^{mk}, \tag{28}$$

where

$$A_{mk+1} = \frac{1}{mk+1} K_k^{-(mk+1)}(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}), \quad k \geq 1. \tag{29}$$

On the other hand, since $f \in \mathscr{R}_b(\varphi, m, q)$ and $g = f_m^{-1} \in \mathscr{R}_b(\varphi, m, q)$ by definition, we have

$$p(z) = c_1 z^m + c_2 z^{2m} + \cdots = \sum_{k=1}^{\infty} c_k z^{mk},$$
$$q(w) = d_1 w^m + d_2 w^{2m} + \cdots = \sum_{k=1}^{\infty} d_k w^{mk}, \tag{30}$$

where

$$\varphi(p(z)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_l K_k^l (c_1, c_2, \cdots, c_k) z^{mk}, \tag{31}$$

$$\varphi(q(w)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_l K_k^l (d_1, d_2, \cdots, d_k) w^{mk}. \tag{32}$$

Comparing the coefficients of (27) and (31), we have

$$\frac{1}{b}[1 + mk]_q a_{mk+1} = \sum_{l=1}^{k-1} \varphi_l K_k^l (c_1, c_2, \cdots, c_k). \tag{33}$$

Similarly, comparing coefficients of (28) and (32), we have

$$\frac{1}{b}[1 + mk]_q A_{mk+1} = \sum_{l=1}^{k-1} \varphi_l K_k^l (d_1, d_2, \cdots, d_k). \tag{34}$$

Note that for $a_{mj+1} = 0$, $1 \le j \le k-1$, we have

$$A_{mk+1} = -a_{mk+1}, \tag{35}$$

and so

$$\frac{1}{b}[1 + mk]_q a_{mk+1} = \varphi_1 c_k, \tag{36}$$

$$-\frac{1}{b}[1 + mk]_q a_{mk+1} = \varphi_1 d_k. \tag{37}$$

Now taking the absolute of (36) and (37) and using the fact that $|\varphi_1| \le 2$, $|c_k| \le 1$, and $|d_k| \le 1$, we have

$$|a_{mk+1}| \le \frac{|b|}{[1 + mk]_q} |\varphi_1 c_k| = \frac{|b|}{[1 + mk]_q} |\varphi_1 d_k|,$$
$$|a_{mk+1}| \le \frac{2|b|}{[1 + mk]_q}, \tag{38}$$

which completes the proof of Theorem 9. □

For $m = 1$ and $k = n - 1$, in Theorem 9, we obtain the following corollary.

Corollary 10. *For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathscr{R}_b(\varphi, q)$, and if $a_{j+1} = 0$, $1 \le j \le n$, then*

$$|a_n| \le \frac{2|b|}{[n]_q}, \quad \textit{for } n \ge 3. \tag{39}$$

For $q \longrightarrow 1^-$, $m = 1$, and $k = n - 1$, in Theorem 9, we obtain the following known corollary.

Corollary 11 (see [35]). *For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathscr{R}_b(\varphi)$, and if $a_{j+1} = 0$, $1 \le j \le n$, then*

$$|a_n| \le \frac{2|b|}{n}, \quad \textit{for } n \ge 3. \tag{40}$$

Theorem 12. *For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathscr{B}_b(\varphi, m, q)$ be given by (6), and then*

$$|a_{m+1}| \le \begin{cases} \dfrac{2|b|}{[m+1]_q}, & \textit{if } |b| < \dfrac{8}{(m+1)[2m+1]_q}, \\ \sqrt{\dfrac{8|b|}{(m+1)[2m+1]_q}}, & \textit{if } |b| \ge \dfrac{8}{(m+1)[2m+1]_q}, \end{cases}$$
$$|a_{2m+1}| \le \begin{cases} \dfrac{2|b|}{[2m+1]_q} + \dfrac{2(m+1)|b|^2}{\left([m+1]_q\right)^2}, & \textit{if } |b| < \dfrac{2}{[2m+1]_q}, \\ \dfrac{4|b|}{[2m+1]_q}, & \textit{if } |b| \ge \dfrac{2}{[2m+1]_q}, \end{cases}$$
$$\left|a_{2m+1} - (m+1)a_{m+1}^2\right| \le \frac{4|b|}{[2m+1]_q},$$
$$\left|a_{2m+1} - \frac{(m+1)}{2}a_{m+1}^2\right| \le \frac{2|b|}{[2m+1]_q}. \tag{41}$$

Proof. Replacing k by 1 and 2 in (33) and (34), respectively, we have

$$\frac{1}{b}[m + 1]_q a_{m+1} = \varphi_1 c_1, \tag{42}$$

$$\frac{1}{b}[2m + 1]_q a_{2m+1} = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{43}$$

$$-\frac{1}{b}[m + 1]_q a_{m+1} = \varphi_1 d_1, \tag{44}$$

$$\frac{1}{b}[2m + 1]_q \left\{(m + 1)a_{m+1}^2 - a_{2m+1}\right\} = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{45}$$

From (42) and (44), we have

$$|a_{m+1}| \le \frac{|b|}{[m + 1]_q} |\varphi_1 c_1| = \frac{|b|}{[m + 1]_q} |\varphi_1 d_1| \le \frac{2|b|}{[m + 1]_q}. \tag{46}$$

Adding (43) and (45), we have

$$a_{m+1}^2 = \frac{b\{\varphi_1(c_2+d_2)+\varphi_2(c_1^2+d_1^2)\}}{(m+1)[2m+1]_q}. \quad (47)$$

Taking the absolute value (47), we have

$$|a_{m+1}| \le \sqrt{\frac{8|b|}{(m+1)[2m+1]_q}}. \quad (48)$$

Now, the bounds given for $|a_{m+1}|$ can be justified since

$$|b| < \sqrt{\frac{8|b|}{(m+1)[2m+1]_q}}, \quad \text{for } |b| < \frac{8}{(m+1)[2m+1]_q}. \quad (49)$$

From (43), we have

$$|a_{2m+1}| = \frac{|b||\varphi_1 c_2+\varphi_2 c_1^2|}{[2m+1]_q} \le \frac{4|b|}{[2m+1]_q}. \quad (50)$$

Next, we subtract (45) from (43), and we have

$$\frac{2[2m+1]_q}{b}\left\{a_{2m+1}-\frac{(m+1)}{2}a_{m+1}^2\right\} = \varphi_1(c_2-d_2)+\varphi_2(c_1^2-d_1^2) = \varphi_1(c_2-d_2), \quad (51)$$

or

$$a_{2m+1} = \frac{(m+1)}{2}a_{m+1}^2 + \frac{\varphi_1 b(c_2-d_2)}{2[2m+1]_q}. \quad (52)$$

After some simple calculation and by taking the absolute, we have

$$|a_{2m+1}| \le \frac{|\varphi_1||b||c_2-d_2|}{2(2m+1)} + \frac{(m+1)}{2}|a_{m+1}^2|. \quad (53)$$

Using the assertion (46), we have

$$|a_{2m+1}| \le \frac{2|b|}{[2m+1]_q} + \frac{2(m+1)|b|^2}{\left([m+1]_q\right)^2}. \quad (54)$$

From (50) and (54), we note that

$$\frac{2|b|}{[2m+1]_q} + \frac{2(m+1)|b|^2}{\left([m+1]_q\right)^2} \le \frac{4|b|}{[2m+1]_q}, \quad \text{if } |b| < \frac{2}{[2m+1]_q}. \quad (55)$$

Now, we rewrite (45) as

$$\frac{1}{b}[2m+1]_q\{(m+1)a_{m+1}^2 - a_{2m+1}\} = \varphi_1 d_2 + \varphi_2 d_1^2. \quad (56)$$

Taking the absolute value, we have

$$\left|a_{2m+1} - (m+1)a_{m+1}^2\right| \le \frac{4|b|}{[2m+1]_q}. \quad (57)$$

Finally, from (51), we have

$$\frac{2[2m+1]_q}{b}\left\{a_{2m+1}-\frac{(m+1)}{2}a_{m+1}^2\right\} = \varphi_1(c_2-d_2). \quad (58)$$

Taking the absolute value, we have

$$\left|a_{2m+1}-\frac{(m+1)}{2}a_{m+1}^2\right| \le \frac{2|b|}{[2m+1]_q}. \quad (59)$$

□

For $m=1$ and $k=n-1$, in Theorem 12, we obtain the following corollary.

Corollary 13. *For $b \in \mathbb{C}\setminus\{0\}$, let $f \in \mathscr{B}_b(\varphi, q)$ be given by (1), and then*

$$|a_2| \le \begin{cases} \dfrac{2|b|}{[2]_q}, & \text{if } |b| < \dfrac{4}{[3]_q}, \\ \sqrt{\dfrac{4|b|}{[3]_q}}, & \text{if } |b| \ge \dfrac{4}{[3]_q}, \end{cases}$$

$$|a_3| \le \begin{cases} \dfrac{2|b|}{[3]_q} + \dfrac{4|b|^2}{\left([2]_q\right)^2}, & \text{if } |b| < \dfrac{2}{[3]_q}, \\ \dfrac{4|b|}{[3]_q}, & \text{if } |b| \ge \dfrac{2}{[3]_q}, \end{cases} \quad (60)$$

$$\left|a_3 - 2a_{m+1}^2\right| \le \frac{4|b|}{[3]_q},$$

$$\left|a_{2m+1} - a_{m+1}^2\right| \le \frac{2|b|}{[3]_q}.$$

For $q \longrightarrow 1^-$, $m=1$, and $k=n-1$, in Theorem 12, we obtain the following corollary.

Corollary 14 (see [35]). *For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathscr{B}_b(\varphi)$ be given by (1), and then*

$$\begin{aligned} |a_2| &\le \begin{cases} |b|, & \text{if } |b| < \dfrac{4}{3}, \\ \sqrt{\dfrac{4|b|}{3}}, & \text{if } |b| \ge \dfrac{4}{3}, \end{cases} \\ |a_3| &\le \begin{cases} \dfrac{2|b|}{3} + |b|^2, & \text{if } |b| < \dfrac{2}{3}, \\ \dfrac{4|b|}{3}, & \text{if } |b| \ge \dfrac{2}{3}, \end{cases} \\ \left|a_3 - 2a_2^2\right| &\le \frac{4|b|}{3}, \\ \left|a_3 - a_2^2\right| &\le \frac{2|b|}{3}. \end{aligned} \tag{61}$$

Theorem 15. *Let $f \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$be given by (6), and if$a_{mj+1} = 0$, $1 \le j \le k-1$, then*

$$|a_{mk+1}| \le \frac{2}{[mk]_q}, \quad \text{for } k \ge 2. \tag{62}$$

Proof. By definition, for the function $f \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$ of the form (6), we have

$$\frac{zD_q f(z)}{f(z)} = 1 - \sum_{k=1}^{\infty} F_k(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}) z^{mk}, \tag{63}$$

where the first few coefficients of $F_k(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1})$ are

$$\begin{aligned} F_1 &= -a_{m+1}, \\ F_2 &= a_{m+1}^2 - (m+1)a_{2m+1}, \\ F_3 &= \left\{-a_{m+1}^3 + (2m+1)a_{m+1}a_{2m+1} - (2m+1)a_{3m+1}\right\}. \end{aligned} \tag{64}$$

In general,

$$\begin{aligned} &F_k(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}) \\ &= \sum_{i_1 + 2i_2 + \cdots k i_{nk} = k} \left\{A(i_1, i_2, i_2, \cdots, i_k)(a_{m+1})^{i_1}(a_{2m+1})^{i_2} \cdots (a_{mk+1})^{i_k}\right\}, \end{aligned} \tag{65}$$

where

$$A(i_1, i_2, i_2, \cdots, i_k) = (-1)^{(k)+2i_1+\cdots(k+1)i_k} \frac{(i_1 + i_2 + i_2 \cdots + i_k - 1)!k}{(i_1!)(i_2!) \cdots (i_k!)}. \tag{66}$$

For the inverse map $g = f_m^{-1} \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$, we obtain

$$\frac{zD_q g(w)}{g(w)} = 1 - \sum_{k=1}^{\infty} F_k(b_{m+1}, b_{2m+1}, \cdots, b_{mk+1}) w^{mk}, \tag{67}$$

where

$$A_{mk+1} = \frac{1}{mk+1} K_k^{-(mk+1)}(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}), \quad k \ge 1. \tag{68}$$

On the other hand, since $f \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$ and $g = f_m^{-1} \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$ by definition, we have

$$\begin{aligned} p(z) &= c_1 z^m + c_2 z^{2m} + \cdots = \sum_{k=1}^{\infty} c_k z^{mk}, \\ q(w) &= d_1 w^m + d_2 w^{2m} + \cdots = \sum_{k=1}^{\infty} d_k w^{mk}, \end{aligned} \tag{69}$$

where

$$\varphi(p(z)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_l K_k^l(c_1, c_2, \cdots, c_k) z^{mk}, \tag{70}$$

$$\varphi(q(w)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_l K_k^l(d_1, d_2, \cdots, d_k) w^{mk}. \tag{71}$$

Comparing the coefficients of (63) and (70), we have

$$-[mk]_q a_{mk+1} = \sum_{l=1}^{k-1} \varphi_l K_k^l(c_1, c_2, \cdots, c_k). \tag{72}$$

Similarly, comparing the coefficients of (67) and (71), we have

$$-[mk]_q b_{mk+1} = \sum_{l=1}^{k-1} \varphi_l K_k^l(d_1, d_2, \cdots, d_k). \tag{73}$$

Note that for $a_{mj+1} = 0$, $1 \le j \le k-1$, we have

$$A_{mk+1} = -a_{mk+1}, \tag{74}$$

and so

$$-[mk]_q a_{mk+1} = \varphi_1 c_k, \tag{75}$$

$$[mk]_q a_{mk+1} = \varphi_1 d_k. \tag{76}$$

Taking the absolute values of (75) and (76) and using the fact that $|\varphi_1| \le 2$, $|c_k| \le 1$, and $|d_k| \le 1$, we have

$$|a_{mk+1}| \le \frac{1}{[mk]_q}|\varphi_1 c_k| = \frac{1}{[mk]_q}|\varphi_1 d_k|,$$
$$|a_{mk+1}| \le \frac{2}{[mk]_q}. \tag{77}$$

Hence, Theorem 15 is complete. □

For $q \longrightarrow 1^-$, $m = 1$, and $k = n-1$, in Theorem 15, we obtain the following corollary.

Corollary 16. *$f \in \mathcal{S}^*(\varphi)$, and if $a_{j+1} = 0$, $1 \le j \le n$, then*

$$|a_n| \le \frac{2}{n-1}, \quad \text{for } n \ge 3. \tag{78}$$

Theorem 17. *Let $f \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$ be given by (6), and then*

$$|a_{m+1}| \le \frac{2}{[m]_q},$$
$$|a_{2m+1}| \le \frac{4(m+1)}{m[2m]_q} + \frac{2}{[2m]_q},$$
$$\left|a_{2m+1} - \frac{[m]_q(2m+1)}{[2m]_q}a_{m+1}^2\right| \le \frac{4}{[2m]_q},$$
$$\left|a_{2m+1} - \frac{[m]_q(m+1)}{[2m]_q}a_{m+1}^2\right| \le \frac{2}{[2m]_q}. \tag{79}$$

Proof. Replacing k by 1 and 2 in (72) and (73), respectively, we have

$$[m]_q a_{m+1} = \varphi_1 c_1, \tag{80}$$

$$[2m]_q a_{2m+1} - [m]_q a_{m+1}^2 = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{81}$$

$$-[m]_q a_{m+1} = \varphi_1 d_1, \tag{82}$$

$$[m]_q(2m+1)a_{m+1}^2 - [2m]_q a_{2m+1} = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{83}$$

From (80) and (82), we have

$$|a_{m+1}| \le \frac{1}{[m]_q}|\varphi_1 c_1| = \frac{1}{[m]_q}|\varphi_1 d_1| \le \frac{2}{[m]_q}. \tag{84}$$

Adding (81) and (83), we have

$$a_{m+1}^2 = \frac{\varphi_1(c_2 + d_2) + \varphi_2\left(c_1^2 + d_1^2\right)}{2m[m]_q}. \tag{85}$$

Taking the absolute value (85), we have

$$|a_{m+1}| \le \frac{2}{\sqrt{m[m]_q}}. \tag{86}$$

Next, we subtract (83) from (81), and we have

$$\left\{2[2m]_q a_{2m+1} - 2[m]_q(m+1)a_{m+1}^2\right\} = \varphi_1(c_2 - d_2) + \varphi_2\left(c_1^2 - d_1^2\right), \tag{87}$$

or

$$a_{2m+1} = \frac{[m]_q(m+1)}{[2m]_q}a_{m+1}^2 + \frac{\varphi_1(c_2 - d_2)}{2[2m]_q}. \tag{88}$$

After some simple calculation of (88) and by taking the absolute, we have

$$|a_{2m+1}| \le \frac{|\varphi_1||c_2 - d_2|}{2[2m]_q} + \frac{[m]_q(m+1)}{[2m]_q}\left|a_{m+1}^2\right|. \tag{89}$$

Using the assertion (86), we have

$$|a_{2m+1}| \le \frac{4(m+1)}{m[2m]_q} + \frac{2}{[2m]_q}. \tag{90}$$

For the third part, we rewrite (83) as

$$\left|[m]_q(2m+1)a_{m+1}^2 - [2m]_q a_{2m+1}\right| = \left|\varphi_1 d_2 + \varphi_2 d_1^2\right|. \tag{91}$$

Taking the absolute value, we have

$$\left|a_{2m+1} - \frac{[m]_q(2m+1)}{[2m]_q}a_{m+1}^2\right| \le \frac{4}{[2m]_q}. \tag{92}$$

Finally, from (87), we have

$$2[2m]_q\left|a_{2m+1} - \frac{[m]_q(m+1)}{[2m]_q}a_{m+1}^2\right| = |\varphi_1(c_2 - d_2)|. \tag{93}$$

Taking the absolute value, we have

$$\left|a_{2m+1} - \frac{[m]_q(m+1)}{[2m]_q}a_{m+1}^2\right| \le \frac{2}{[2m]_q}. \tag{94}$$

□

For $q \longrightarrow 1^-$, $m = 1$, and $k = n-1$, in Theorem 17, we get the following corollary.

Corollary 18. *Let $f \in \mathcal{S}^*(\varphi)$ be given by (1), and then*

$$\begin{aligned} &|a_2| \le 2, \\ &|a_3| \le 5, \\ &\left|a_3 - \frac{3}{2}a_2^2\right| \le 2, \\ &\left|a_3 - a_2^2\right| \le 1. \end{aligned} \tag{95}$$

3. Conclusion

In this paper, we have applied q-calculus operator theory to define some new subclasses of m-fold symmetric analytic and bi-univalent functions in open unit disk $\mathcal{U}$ and used the Faber polynomial expansion to find upper bounds $|a_{mk+1}|$ and initial coefficient bounds $|a_{m+1}|$ and $|a_{2m+1}|$ as well as Fekete-Szego inequalities for the functions belonging to newly defined subclasses of m-fold symmetric analytic and bi-univalent function. Also, we highlighted some new and known consequences of our main results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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References

[1] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, NY, USA, 1983.

[2] M. Lewin, "On a coefficient problem for bi-univalent functions," *Proceedings of the American Mathematical Society*, vol. 18, no. 1, pp. 63–68, 1967.

[3] D. A. Brannan and J. Cluni, "Aspects of contemporary complex analysis," *Proceedings of the NATO Advanced Study Institute Held at University of Durham*, Academic Press, New York, NY, USA, 1979.

[4] E. Netanyahu, "The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$," *Archive for Rational Mechanics and Analysis*, vol. 32, pp. 100–112, 1967.

[5] D. A. Brannan and T. S. Taha, "On some classes of bi-univalent function," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 31, no. 2, pp. 70–77, 1986.

[6] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1188–1192, 2010.

[7] B. A. Frasin and M. K. Aouf, "New subclasses of bi-univalent functions," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1569–1573, 2011.

[8] S. Altinkaya and S. Yalcin, "Coefficient bounds for a subclass of bi-univalent functions," *Journal of Pure and Applied Mathematics*, vol. 6, no. 2, pp. 180–185, 2015.

[9] S. Altinkaya and S. Yalcin, "Bornes des coefficients des developpements en polynomes de Faber d'une sous- classe de fonctions bi-univalentes," *Comptes Rendus Mathematique*, vol. 353, no. 12, pp. 1075–1080, 2015.

[10] T. Hayami and S. Owa, "Coefficient bounds for bi-univalent functions," *Panamerican Mathematical Journal*, vol. 22, no. 4, pp. 15–26, 2012.

[11] S. Hussain, S. Khan, M. A. Zaighum, M. Darus, and Z. Shareef, "Coefficients bounds for certain subclass of bi-univalent functions associated with Ruscheweyh q-differential operator," *Journal of Complex Analysis*, vol. 2017, Article ID 2826514, 9 pages, 2017.

[12] G. Faber, "Über polynomische Entwickelungen," *Mathematische Annalen*, vol. 57, no. 3, pp. 389–408, 1903.

[13] S. Gong, "The Bieberbach conjecture, translated from the 1989 Chinese original and revised by the author," in *AMS/IP Studies in Advanced Mathematics*, Journal of the American Mathematical Society, Providence, RI, USA, 1999.

[14] H. Airault, "Remarks on Faber polynomials," *International Mathematical Forum*, vol. 3, no. 9, pp. 449–456, 2008.

[15] H. Airault and H. Bouali, "Differential calculus on the Faber polynomials," *Bulletin des Sciences Mathematiques*, vol. 130, no. 3, pp. 179–222, 2006.

[16] S. Bulut, "Estimations des coefficients polynomes de Faber pour une sous-classe complete de fonctions analytiques bi-univalentes," *Comptes Rendus Mathematique*, vol. 352, no. 6, pp. 479–484, 2014.

[17] S. Yalcin, S. Khan, and S. Hussain, "Faber polynomial coefficients estimates of bi-univalent functions associated with generalized Salagean q-differential operator," *Konuralp Journal of Mathematics*, vol. 7, no. 1, pp. 25–32, 2019.

[18] W. Koepf, "Coefficients of symmetric functions of bounded boundary rotation," *Proceedings of the American Mathematical Society*, vol. 105, no. 2, pp. 324–329, 1989.

[19] H. M. Srivastava, S. Sivasubramanian, and R. Sivakumar, "Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions," *Tbilisi Mathematical Journal*, vol. 7, no. 2, pp. 1–10, 2014.

[20] A. Akgul, "A new general subclass of m-fold symmetric bi-univalent functions given by subordination," *Turkish Journal of Mathematics*, vol. 43, no. 3, pp. 1688–1698, 2019.

[21] A. Akgul, "On the coefficient estimates of analytic and bi-univalent m-fold symmetric functions," *Mathematica Aeterna*, vol. 7, no. 3, pp. 253–260, 2017.

[22] S. G. Hamidi and J. M. Jahangiri, "Unpredictability of the coefficients of m-fold symmetric bi-starlike functions," *International Journal of Mathematics*, vol. 25, no. 7, pp. 1450064–1450068, 2014.

[23] S. Sümer Eker, "Coefficient bounds for subclasses of m-fold symmetric bi-univalent functions," *Turkish Journal of Mathematics*, vol. 40, no. 3, pp. 641–646, 2016.

[24] F. H. Jackson, "On q-functions and a certain difference operator," *Transactions of the Royal Society of Edinburgh*, vol. 46, pp. 253–281, 1909.

[25] M. E. H. Ismail, E. Merkes, and D. Styer, "A generalization of starlike functions," *Complex Variables, Theory and Application: An International Journal*, vol. 14, no. 1-4, pp. 77–84, 1990.

[26] H. M. Srivastava, "Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis," *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 44, pp. 327–344, 2020.

[27] S. Kanas and D. Raducanu, "Some class of analytic functions related to conic domains," *Mathematica Slovaca*, vol. 64, no. 5, pp. 1183–1196, 2014.

[28] M. Arif, M. Raza, K. I. Noor, and S. N. Malik, "On strongly Bazilevic functions associated with generalized Robertson functions," *Mathematical and Computer Modelling*, vol. 54, no. 5-6, pp. 1608–1612, 2011.

[29] X. Zhang, S. Khan, S. Hussain, H. Tang, and Z. Shareef, "New subclass of q-starlike functions associated with generalized conic domain," *AIMS Mathematics*, vol. 5, no. 5, pp. 4830–4848, 2020.

[30] H. M. Srivastava, B. Khan, N. Khan, and Q. A. Ahmad, "Coefficient inequalities for q-starlike functions associated with the Janowski functions," *Hokkaido Mathematical Journal*, vol. 48, no. 2, pp. 407–425, 2019.

[31] H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, and N. Khan, "Some general classes of q-starlike functions associated with the Janowski functions," *Symmetry*, vol. 11, no. 2, p. 292, 2019.

[32] A. Akgul and F. M. Sakar, "A certain subclass of bi-univalent analytic functions introduced by means of the q-analogue of Noor integral operator and Horadam polynomials," *Turkish Journal of Mathematics*, vol. 43, no. 5, pp. 2275–2286, 2019.

[33] H. Tang, S. Khan, S. Hussain, and N. Khan, "Hankel and Toeplitz determinant for a subclass of multivalent q-starlike functions of order α," *AIMS Mathematics*, vol. 6, no. 6, pp. 5421–5439, 2021.

[34] Z.-G. Wang, S. Hussain, M. Naeem, T. Mahmood, and S. Khan, "A subclass of univalent functions associated with q-analogue of Choi-Saigo-Srivastava operator," *Hacettepe Journal of Mathematics and Statistics*, vol. 49, no. 4, pp. 1471–1479, 2020.

[35] S. G. Hamidi and J. M. Jahangiri, "Faber polynomial coefficient estimates for bi-univalent functions defined by subordinations," *Bulletin of the Iranian Mathematical Society*, vol. 41, no. 5, pp. 1103–1119, 2014.

[36] S. G. Hamidi and J. M. Jahangiri, "Polynomes de Faber et coefficients des fonctions bi-subordonnees," *Comptes Rendus Mathematique*, vol. 354, no. 4, pp. 365–370, 2016.

[37] H. Airault, "Symmetric sums associated to the factorizations of Grunsky coefficients," in *Conference, Groups and Symmetries*, pp. 3–16, Montreal, Canada, April 2007.

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Research Article

On the Algebraic Characteristics of Fuzzy Sub e-Groups

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Fuzzy set is a modern tool for depicting uncertainty. This paper introduces the concept of fuzzy sub e-group as an extension of fuzzy subgroup. The concepts of identity and inverse are generalized in fuzzy sub e-groups. Every fuzzy subgroup is proven to be a fuzzy sub e-group, but the converse is not true. Various properties of fuzzy sub e-groups are established. Moreover, the concepts of proper fuzzy sub e-group and super fuzzy sub e-group are discussed. Further, the concepts of fuzzy e-coset and normal fuzzy sub e-group are presented. Finally, we describe the effect of e-group homomorphism on normal fuzzy sub e-groups.

1. Introduction

Many decades ago, researchers developed an algebraic structure and named it as a group. Various properties of groups are proposed later on. In group theory, every group contains a unique identity element and every element has a unique inverse. In 2018, Saeid et al. [1] generalized the notion of groups to a new algebraic structure as e-groups. They generalized the notion of the identity of a group. Instead of choosing a single element as an identity element, Saeid et al. [1] considered a subset of the main set as an identity set. So, in an e-group, the identity element needs not to be unique. They proved that every group is an e-group, but the converse is not true. They defined homomorphism on e-groups in a different manner. E-group is an important tool for classifying isotopes. It is also a physical background in the unified Gauge theory.

Uncertainty is a massive component in the life of a person. In 1965, in his pioneer paper, Zadeh [2] first defined fuzzy set to handle uncertainty in real-life problems. In 1971, utilizing the concept of fuzzy set, Rosenfeld [3] first defined fuzzy subgroup. In 1979, using the t-norm concept of the fuzzy subgroup was restructured by Anthony and Sherwood [4, 5]. In 1981, the idea of fuzzy level subgroup was introduced by Das [6]. In 1988, Choudhury et al. [7] proved various properties of fuzzy homomorphism. In 1990, Dixit et al. [8] discussed the union of fuzzy subgroups and fuzzy level subgroups. The concept of antifuzzy subgroups was proposed by Biswas [9]. In 1992, Ajmal and Prajapati [10] developed fuzzy cosets and fuzzy normal subgroups. Chakraborty and Khare [11] studied various properties of fuzzy homomorphism. Ajmal [12] also studied the homomorphism of fuzzy subgroups. Later, many researchers studied various properties of fuzzy subgroups [13–16]. In 2015, Tarnauceanu [17] classified fuzzy normal subgroups of finite groups. In 2016, Onasanya [18] reviewed some antifuzzy properties of fuzzy subgroups. In 2018, Shuaib and Shaheryar [19] introduced omicron fuzzy subgroups. In 2018, Addis [20] developed fuzzy homomorphism theorems on groups. In 2019, Bhunia and Ghorai [21] studied (α, β)-Pythagorean fuzzy subgroups. In 2021, Bhunia et al. [22, 23] developed Pythagorean fuzzy subgroups. Abuhijleh et al. [24] worked on complex fuzzy subgroups in 2021. Alolaiyan et al. [25] studied algebraic structure of (α, β)

-complex fuzzy subgroups. Alolaiyan et al. [26] developed bipolar fuzzy subrings in 2021. In 2021, Talafha et al. [27] studied fuzzy fundamental groups and fuzzy folding of fuzzy Minkowski space.

In a fuzzy subgroup, the identity element of the group has the highest membership value. Also, in a fuzzy subgroup, the membership value of an element and its inverse are equal. But in an e-group, there is no unique identity element and no direct concept of the inverse of elements. Till now, no fuzzification has been done for e-groups. So, it is a challenge for us to fuzzify e-groups. In this study, we construct the concept of fuzzy sub e-groups. Here, we show that every fuzzy subgroup is a fuzzy sub e-group, but the converse is not true. So, the idea of fuzzy sub e-group is a much more generalized concept. In the study of isotopes, we notice that isotopes decay through neutron emission. So, the membership degree of these unstable neutrons must lie in $[0, 1]$. Therefore, fuzzy sub e-group will be more efficient in studying these unstable isotopes rather than a crisp e-group.

This paper is arranged in the following order. In Section 2, we recall some important concepts. In Section 3, utilizing the concept of e-groups, we generalize fuzzy subgroups. We develop the concept of fuzzy sub e-group and show that any fuzzy subgroup is also a fuzzy sub e-group. We also prove many algebraic properties of fuzzy sub e-groups. Further, we define fuzzy e-coset and normal fuzzy sub e-group in Section 4. Moreover, in Section 5, we show that after e-group homomorphism, a fuzzy sub e-group remains a fuzzy sub e-group. Finally, the conclusion is given in Section 6.

2. Preliminaries

Here, we will go over some basic definitions and concepts, which will be useful in the following sections.

Definition 1 (see [2]). A fuzzy set (FS) (D, κ) on a crisp set D is an object having the form $(D, \kappa) = \{(d, \kappa(d)) | d \in D\}$, where $\kappa \longrightarrow [0, 1]$ is the membership function.

Definition 2 (see [3]). Let (D, κ) be a FS on a group D. Then, (D, κ) is referred to be a fuzzy subgroup (FSG) of D if the following conditions hold:

(i) $\kappa(d_1 d_2) \geq \kappa(d_1) \wedge \kappa(d_2) \forall d_1, d_2 \in D$

(ii) $\kappa(d^{-1}) \geq \kappa(d) \forall d \in D$

Definition 3 (see [2]). Let (D, κ) be a FS on D. Then, for any $a \in [0, 1]$, the set $\kappa_a = \{d \mid d \in D, \kappa(d) \geq a\}$ is called a-cut of (D, κ).

Clearly, κ_a is a subset of D.

Proposition 4 (see [28]). *Let h be a mapping from D_1 into D_2. Let (D_1, κ_1) and (D_2, κ_2) be the two FSs on D_1 and D_2, respectively. Then, $(D_2, h(\kappa_1))$ and $(D_1, h^{-1}(\kappa_2))$ are FSs on D_2 and D_1, respectively, where for all $d_2 \in D_2$*

$$h(\kappa_1)(d_2) = \begin{cases} \vee\{\kappa_1(d_1) | d_1 \in D_1, h(d_1) = d_2\}, \\ \text{when } h^{-1}(d_2) \neq \varnothing, \\ 0, \text{elsewhere}, \end{cases} \quad (1)$$

and for all $d_1 \in D_1$, $(h^{-1}(\kappa_2))(d_1) = \kappa_2(h(d_1))$.

Definition 5 (see [1]). Let D be a nonempty crisp set and $L \subseteq D$. Then, $(D, \circ, L)$ is an e-group, where $\circ$ is the binary operation on D, which meets the following criteria:

(i) $d_1 \circ (d_2 \circ d_3) = (d_1 \circ d_2) \circ d_3 \forall d_1, d_2, d_3 \in D$

(ii) For every $d \in D$, $\exists$ an element $l \in L$ such that $d \circ l = l \circ d = d$

(iii) For every $d_1 \in D$, $\exists$ an element $d_2 \in D$ such that $d_1 \circ d_2$ and $d_2 \circ d_1 \in L$

Definition 6 (see [1]). Let $(D_1, \circ_1, L_1)$ and $(D_2, \circ_2, L_2)$ be the two e-groups. If a mapping $h : D_1 \longrightarrow D_2$ meets the following criteria, it is referred to as a homomorphism:

(i) $h(L_1) \subseteq L_2$

(ii) $h(d_1 \circ_1 d_2) = h(d_1) \circ_2 h(d_2) \forall d_1, d_2 \in D_1$

3. Fuzzy Sub e-Group and Its Properties

In this section, fuzzy sub e-group is briefly described as a generalization of fuzzy subgroup. The notions of identity and inverse are generalized in fuzzy sub e-group. We investigate its properties. We define super fuzzy sub e-group. We check whether union and intersections of fuzzy sub e-group are fuzzy sub e-groups.

Definition 7. A FS (D, κ) is referred to be a fuzzy sub e-group of an e-group $(D, \circ, L)$ if the following conditions hold:

(i) $\kappa(d_1 \circ d_2) \geq \kappa(d_1) \wedge \kappa(d_2) \forall d_1, d_2 \in D$

(ii) $\kappa(l) \geq \kappa(d) \forall l \in L$ and $d \in D/L$

where $\kappa : D \longrightarrow [0, 1]$ is the membership function.

Example 8. Let $D = \{d_1, d_2, d_3\}$ and $L = \{d_1, d_2\}$. Define a binary operation $\circ$ on D as below.

o	d_1	d_2	d_3
d_1	d_3	d_1	d_2
d_2	d_1	d_2	d_3
d_3	d_2	d_3	d_1

(2)

Then, $(D, \circ, L)$ is an e-group.

Here, we assign membership degrees to the elements of D by $\kappa(d_1) = \kappa(d_3) = 0.8$ and $\kappa(d_2) = 0.9$.

Now, $\kappa(d_1 \circ d_3) = \kappa(d_2) = 0.9 > 0.8 = \kappa(d_1) \wedge \kappa(d_3)$.

Similarly, we can check for other elements of D.

Therefore, $\kappa(d_i \circ d_j) \geq \kappa(d_i) \wedge \kappa(d_j)$ for all $d_i, d_j \in D$.

Also, $\kappa(d_1) = 0.8 = \kappa(d_3)$ and $\kappa(d_2) = 0.9 > 0.8 = \kappa(d_3)$.

Thus, (D, κ) forms a fuzzy sub e-group of the e-group $(D,\circ,L)$.

Theorem 9. *Let $(D,\circ,L)$ stand for an e-group and (D, κ) be a FS on D. Then, (D, κ) is referred to be a fuzzy sub e-group of $(D,\circ,L)$ if for all d_1 and $d_2 \in D$, $\kappa(d_1 \circ d_2') \geq \kappa(d_1) \wedge \kappa(d_2)$ for some $d_2' \in D$ such that $d_2 \circ d_2'$, $d_2' \circ d_2 \in L$.*

Proof. Let (D, κ) stand for a fuzzy sub e-group of $(D,\circ,L)$.

Then, for all d_1 and d_2 in D, $\kappa(d_1 \circ d_2) \geq \kappa(d_1) \wedge \kappa(d_2)$.

Let $d_1, d_2 \in D$. Then, $\exists$ some $d_2' \in D$ such that $d_2 \circ d_2'$ and $d_2' \circ d_2 \in L$.

Therefore, $\kappa(d_1 \circ d_2') \geq \kappa(d_1) \wedge \kappa(d_2') \geq \kappa(d_1) \wedge \kappa(d_2)$.

Conversely, assume that $\forall d_1, d_2 \in D$, $\kappa(d_1 \circ d_2') \geq \kappa(d_1) \wedge \kappa(d_2)$ for some $d_2' \in D$ such that $d_2 \circ d_2'$ and $d_2' \circ d_2 \in L$. Let $d_1 \in D$, $d_2 \in D/L$.

Here, $d_2 \circ d_2' \in E$, then $\kappa(d_2 \circ d_2') \geq \kappa(d_2) \wedge \kappa(d_2) = \kappa(d_2)$.

Therefore, $\kappa(l) \geq \kappa(d)$, where $l \in L$ and $d \in D/L$.

Let $d_2 = (d_2')'$ for all $d_2, d_2' \in D$ such that $d_2 \circ d_2'$ and $d_2' \circ d_2 \in L$.

Therefore, $(d_1 \circ d_2) = \kappa(d_1 \circ (d_2')') \geq \kappa(d_1) \wedge \kappa(d_2') \geq \kappa(d_1) \wedge \kappa(d_2) \forall d_1, d_2 \in D$.

Hence, the e-group $(D,\circ,L)$ has a fuzzy sub e-group (D, κ). □

Remark 10. The above theorem gives the necessary and sufficient condition for a FS of an e-group to be a fuzzy sub e-group.

Now, we will demonstrate that any FSG within a group D is also a fuzzy sub e-group of the e-group $(D,\circ,\{l\})$, where l is the group's identity element. But the converse needs not to be true.

Theorem 11. *Any fuzzy subgroup of a group is a fuzzy sub e-group.*

Proof. Let (D, κ) stand for a FSG of a group D.

Then, $\kappa(d_1 \circ d_2) \geq \kappa(d_1) \wedge \kappa(d_2)$ and $\kappa(d_1^{-1}) = \kappa(d_1)$ for all d_1 and $d_2 \in D$.

So, the first condition of fuzzy sub e-group is satisfied.

As D is a group, then there is a unique identity element l in D.

Since (D, κ) is a FSG of D, $\kappa(l) \geq \kappa(d) \forall d \in D$.

Now, we take $L = \{l\}$; then, $L \subseteq D$.

Therefore, $\kappa(l) \geq \kappa(d)$, where $l \in L$ and $d \in D/L$.

Hence, the FSG (D, κ) of D is a fuzzy sub e-group of the e-group $(D, \cdot, L)$. □

Example 12. Let $D = \{d_1, d_2, d_3, d_4\}$ and $L = \{d_1, d_2\}$.

Define $\circ$ on D as binary operation by the following:

o	d_1	d_2	d_3	d_3
d_1	d_1	d_1	d_1	d_1
d_2	d_1	d_2	d_3	d_4
d_3	d_1	d_3	d_1	d_1
d_4	d_1	d_4	d_3	d_4

(3)

Then, $(D,\circ,L)$ is an e-group.

Now, we assign a membership value to each of the elements of D by the following:

$$\begin{aligned} \kappa(d_1) &= 0.8, \\ \kappa(d_2) &= 0.9, \\ \kappa(d_3) &= 0.6, \\ \kappa(d_4) &= 0.7. \end{aligned} \quad (4)$$

Now, we can verify that (D, κ) is a fuzzy sub e-group of the e-group $(D,\circ,L)$.

But $|L| > 1$. So, $(D, \circ)$ is not a group. Hence, the FS (D, κ) is not a FSG.

Remark 13. A fuzzy sub e-group of an e-group is not necessarily a FSG.

Definition 14. A fuzzy sub e-group of an e-group which is not a FSG is said to be a proper fuzzy sub e-group.

The fuzzy sub e-group (D, κ) in Example 12. is a proper fuzzy sub e-group.

Now, we will check about union and intersection of fuzzy sub e-groups.

Theorem 15. *Intersection of fuzzy sub e-groups of an e-group is also a fuzzy sub e-group of that e-group.*

Proof. Let (D, κ_1) and (D, κ_2) be the two fuzzy sub e-groups of an e-group $(D,\circ,L)$.

Then, $\forall d_1, d_2 \in D$, $\kappa_1(d_1 \circ d_2) \geq \kappa_1(d_1) \wedge \kappa_1(d_2)$ and $\kappa_1(l) \geq \kappa_1(d)$, where $l \in L$ and $d \in D/L$.

Also, $\forall d_1, d_2 \in D$, $\kappa_2(d_1 \circ d_2) \geq \kappa_2(d_1) \wedge \kappa_2(d_2)$ and $\kappa_2(l) \geq \kappa_2(d)$, where $l \in L$ and $d \in D/L$.

Let (D, κ) be the intersection of (D, κ_1) and (D, κ_2), where $\kappa = \kappa_1 \cap \kappa_2$ is given by $\kappa(d) = \kappa_1(d) \wedge \kappa_2(d) \forall d \in D$.

Now for all $d_1, d_2 \in D$,

$$\begin{aligned}\kappa(d_1 \circ d_2) &= \kappa_1(d_1 \circ d_2) \wedge \kappa_2(d_1 \circ d_2)\\ &\geq (\kappa_1(d_1) \wedge \kappa_1(d_2)) \wedge (\kappa_2(d_1) \wedge \kappa_2(d_2))\\ &= (\kappa_1(d_1) \wedge \kappa_2(d_1)) \wedge (\kappa_1(d_2) \wedge \kappa_2(d_2))\\ &= \kappa(d_1) \wedge \kappa(d_2).\end{aligned} \quad (5)$$

Therefore, $\kappa(d_1 \circ d_2) \geq \kappa(d_1) \wedge \kappa(d_2)$ for all $d_1, d_2 \in D$.

Again for $l \in L$ and $d \in D/L$, we have the following:

$$\kappa(l) = \kappa_1(l) \wedge \kappa_2(l) \geq \kappa_1(d) \wedge \kappa_2(d) = \kappa(d). \quad (6)$$

Therefore, (D, κ) is a fuzzy sub e-group of the e-group $(D,\circ,L)$.

Hence, the intersection of two fuzzy sub e-groups of an e-group is also a fuzzy sub e-group of that e-group. □

Corollary 16. *Intersection of any fuzzy sub e-groups of an e-group $(D,\circ,L)$ is also a fuzzy sub e-group of that e-group $(D,\circ,L)$.*

Remark 17. Union of two fuzzy sub e-groups of an e-group may not be a fuzzy sub e-group of that e-group.

Example 18. Let us take the e-group $(D,\circ,L)$, where $D = \mathbb{Z}$, $L = 2\mathbb{Z}$, and $\circ$ is the addition of integers.

Let (D, κ_1) and (D, κ_2) be the two fuzzy sub e-groups of the e-group $(D,\circ,L)$, where κ_1 and κ_2 are presented by the following:

$$\kappa_1(d) = \begin{cases} 0.6, & \text{when } d \in 2\mathbb{Z},\\ 0.3, & \text{when } d \in 5\mathbb{Z}/2\mathbb{Z},\\ 0, & \text{elsewhere,}\end{cases}$$
$$\kappa_2(d) = \begin{cases} 0.8, & \text{when } d \in 2\mathbb{Z},\\ 0.2, & \text{when } d \in 3\mathbb{Z}/2\mathbb{Z},\\ 0, & \text{elsewhere.}\end{cases} \quad (7)$$

Let (D, κ) be the union of (D, κ_1) and (D, κ_2), where $\kappa = \kappa_1 \cup \kappa_2$ is given by $\kappa(d) = \kappa_1(d) \vee \kappa_2(d) \forall d \in D$.

Therefore,

$$\kappa(d) = \begin{cases} 0.8, & \text{when } d \in 2\mathbb{Z},\\ 0.3, & \text{when } d \in 5\mathbb{Z}/2\mathbb{Z},\\ 0.2, & \text{when } d \in 3\mathbb{Z}/(2\mathbb{Z} \cap 5\mathbb{Z}),\\ 0, & \text{elsewhere.}\end{cases} \quad (8)$$

Now, $\kappa(5 + (-4)) = \kappa(1) = 0$, but $\kappa(5) \wedge \kappa(-4) = \min\{0.3, 0.8\} = 0.3$.

So, $\kappa(5 + (-4)) \ngeq \kappa(5) \wedge \kappa(-4)$.

Hence, (D, κ) is not a fuzzy sub e-group of the e-group $(D,\circ,L)$.

Definition 19. Let (D, κ_1) and (D, κ_2) be the two fuzzy sub e-groups of an e-group $(D,\circ,L)$ such that $\kappa_2(l) \geq \kappa_1(l)$ for all $l \in L$ and $\kappa_2(d) \leq \kappa_1(d)$ for all $d \in D/L$, then (D, κ_2) is referred to be a super fuzzy sub e-group of (D, κ_1).

Example 20. In Example 12., we take another FS (D, κ_1) on $(D,\circ,L)$, where

$$\begin{aligned}\kappa_1(d_1) &= 0.85,\\ \kappa_1(d_2) &= 0.93,\\ \kappa_1(d_3) &= 0.57,\\ \kappa_1(d_4) &= 0.68.\end{aligned} \quad (9)$$

Then, we can simply verify that $(D,\circ,L)$ has a fuzzy sub e-group (D, κ_1).

Now, we can see that for all $l \in L$, $\kappa_1(l) \geq \kappa(l)$ and for all $d \in D/L$, $\kappa_1(d) \leq \kappa(d)$.

Hence, (D, κ_1) is a super fuzzy sub e-group of (D, κ).

Theorem 21. *Let (D, κ) stand for a fuzzy sub e-group of an e-group $(D,\circ,L)$. Then, the set $K = \{d \mid d \in D, \kappa(d) = p\}$ forms a sub e-group $(K,\circ,L)$ of the e-group $(D,\circ,L)$, where $p = \wedge\{\kappa(l) \mid l \in L\}$.*

Proof. Given $K = \{d \mid d \in D, \kappa(d) = p\}$, where $p = \wedge\{\kappa(l) | l \in L\}$.

To show that the e-group $(D,\circ,L)$ has a sub e-group $(K,\circ,L)$, we have to show that $(K,\circ,L)$ itself forms an e-group.

Since $(D,\circ,L)$ is an e-group, the associative law holds.

Clearly, K is a subset of D. Then, associative law also holds in K. Instead of showing the other two conditions of e-group, we will show that for all k_1 and $k_2 \in K$, $\exists$ a $k_2' \in K$ such that $k_1 \circ k_2'$ and $k_2' \circ k_1 \in L$.

Let k_1, k_2, and $k_2' \in K$. Then, $\kappa(k_1) = \kappa(k_2) = \kappa(k_2') = p$.

Since (D, κ) is a fuzzy sub e-group of $(D,\circ,L)$, by Theorem 9, we have the following:

$$\kappa\left(k_1 \circ k_2'\right) \geq \kappa(k_1) \wedge \kappa(k_2) = p. \quad (10)$$

Similarly, we can show that $\kappa(k_2' \circ k_1) \geq p$.

Since $p = \wedge\{\kappa(l) | l \in L\}$, $k_1 \circ k_2'$ and $k_2' \circ k_1 \in L$.

Hence, $(K,\circ,L)$ forms a sub e-group of $(D,\circ,L)$. □

4. Normal Fuzzy Sub e-Group and Level Fuzzy Sub e-Group

This section will describe fuzzy e-cosets and normal fuzzy sub e-groups. We will also introduce the concept of level fuzzy sub e-groups.

Definition 22. Let (D, κ) stand for a fuzzy sub e-group of an e-group $(D,\circ,L)$. Then, $\forall s, d \in D$, the left fuzzy e-coset $s\kappa = \kappa(l)_{\{s\}} \circ \kappa$ is defined by $s\kappa(d) = \kappa(s' \circ d)$ and the right fuzzy e-coset $\kappa s = \kappa \circ \kappa(l)_{\{s\}}$ is defined by $\kappa s(d) = \kappa(d \circ s')$, where l is any element of L and $s' \in D$ such that $s \circ s'$ and $s' \circ s \in L$.

If a left fuzzy e-coset is also a right fuzzy e-coset, then we will simply call it is a fuzzy e-coset.

Definition 23. Let (d, κ) stand for a fuzzy sub e-group of an e-group $(D,\circ,L)$. Then, (D, κ) forms a normal fuzzy sub e-group of the e-group $(D,\circ,L)$ if every left fuzzy e-coset of (D, κ) is a right fuzzy e-coset of (D, κ) in $(D,\circ,L)$.

Equivalently, $s\kappa = \kappa s$ for all $s \in D$.

Example 24. Let us take the e-group $(\mathbb{Z},+,2\mathbb{Z})$. Now, $(\mathbb{Z}, \kappa)$ forms a fuzzy sub e-group on $\mathbb{Z}$, where κ is presented by the following:

$$\kappa(z) = \begin{cases} 0.9, & \text{when } z \in 2\mathbb{Z}, \\ 0.6, & \text{elsewhere.} \end{cases} \quad (11)$$

Let us take $s = 3 \in \mathbb{Z}$.

Then, $\forall d \in \mathbb{Z}$, the left fuzzy e-coset (3κ) is presented by $(3\kappa)(d) = \kappa(3' + d) = \kappa(-3 + d)$ and the right fuzzy e-coset $(\kappa 3)$ is presented by $(\kappa 3)(d) = \kappa(d + 3') = \kappa(d - 3)$.

Since addition is commutative on $\mathbb{Z}$, then $(3\kappa) = (\kappa 3)$. Similarly, we can check for other elements of $\mathbb{Z}$. Hence, $(\mathbb{Z}, \kappa)$ forms a normal fuzzy sub e-group of $(\mathbb{Z},+,2\mathbb{Z})$.

In the next theorem, we represent the necessary and sufficient condition for a fuzzy sub e-group to be a normal fuzzy sub e-group.

Theorem 25. *Let (D, κ) stand for a fuzzy sub e-group of an e-group $(D,\circ,L)$. Then, (D, κ) forms a normal fuzzy sub e-group of the e-group $(D,\circ,L)$ iff $\kappa(d_1 \circ d_2) = \kappa(d_2 \circ d_1) \forall d_1, d_2 \in D$.*

Proof. Let (D, κ) stand for a normal fuzzy sub e-group of the e-group $(D,\circ,L)$.

Then, every left fuzzy e-coset of (D, κ) is also a right fuzzy e-coset of (D, κ) in $(D,\circ,L)$.

Therefore, $(t\kappa) = (\kappa t)$ for all $t \in D$. That is $(t\kappa)(d_2) = (\kappa t)(d_2) \forall d_2, t \in D$.

This suggests that $\kappa(t' \circ d_2) = \kappa(d_2 \circ t') \forall t, t', d_2 \in D$ such that $t \circ t', t' \circ t \in L$.

Now, we put $t' = d_1 \in D$. Therefore, $\kappa(d_1 \circ d_2) = \kappa(d_2 \circ d_1) \forall d_1, d_2 \in D$.

Conversely, let $\kappa(d_1 \circ d_2) = \kappa(d_2 \circ d_1) \forall d_1, d_2 \in D$.

Let $s \in D$. Since $(D,\circ,L)$ is an e-group, then $\exists$ a $s' \in D$ such that $s \circ s', s' \circ s \in L$.

Suppose $s' = d_1$. Then, $\kappa(s' \circ d_2) = \kappa(d_2 \circ s') \forall d_2, s' \in D$.

Therefore, $(s\kappa)(d_2) = (\kappa s)(d_2) \forall s, d_2 \in D$. Thus, $(s\kappa) = (\kappa s) \forall s \in D$.

Hence, (D, κ) forms a normal fuzzy sub e-group of $(D,\circ,L)$. □

Theorem 26. *Let (D, κ) stand for a fuzzy sub e-group of an e-group $(D,\circ,L)$. Then, the a-cut κ_a of (D, κ) forms a sub e-group $(\kappa_a,\circ,L)$ of the e-group $(D,\circ,L)$, where $a \leq \wedge\{\kappa(l) \mid l \in L\}$.*

Proof. We have $\kappa_a = \{d \mid d \in D, \kappa(d) \geq a\}$, where $a \in [0, 1]$ and $a \leq \wedge\{\kappa(l) \mid l \in L\}$.

Clearly, κ_a is nonempty as $L \subseteq \kappa_a$.

To show that $(\kappa_a,\circ,L)$ is an e-subgroup of $(D,\circ,L)$, we need to prove that for $d_1, d_2, d_2' \in \kappa_a$, $d_1 \circ d_2' \in \kappa_a$, where $d_2 \circ d_2', d_2' \circ d_2 \in L$.

Let $d_1, d_2, d_2' \in \kappa_a$ and $d_2 \circ d_2', d_2' \circ d_2 \in L$. Then, $\kappa(d_1) \geq a$ and $\kappa(d_2) \geq a$.

Since (D, κ) forms a fuzzy sub e-group of the e-group $(D,\circ,L)$,

$$\kappa\left(d_1 \circ d_2'\right) \geq \kappa(d_1) \wedge \kappa\left(d_2'\right) \geq \kappa(d_1) \wedge \kappa(d_2) \geq a \wedge a = a. \quad (12)$$

Therefore, $d_1 \circ d_2' \in \kappa_a$, where $d_2 \circ d_2', d_2' \circ d_2 \in L$. Hence, $(\kappa_a,\circ,L)$ forms a sub e-group of the e-group $(D,\circ,L)$. □

Definition 27. The sub e-group $(\kappa_a,\circ,L)$ of the e-group $(D,\circ,L)$ is referred to as a level fuzzy sub e-group of (D, κ).

Example 28. We consider the fuzzy sub e-group (D, κ) of the e-group $(D,\circ,L)$ in Example 12..

Choose $a = 0.65$. Then, $\kappa_a = \{d_1, d_2, d_4\}$ and $L = \{d_1, d_2\}$.

Clearly, $\kappa_a \subseteq D$.

We can easily check that $(\kappa_a,\circ,L)$ is a sub e-group of the e-group $(D,\circ,L)$.

Therefore, $(\kappa_a,\circ,L)$ is a level fuzzy sub e-group of (D, κ).

5. Homomorphism of Fuzzy Sub e-Groups

We shall demonstrate some important theorems on fuzzy sub e-group homomorphism in this section.

Theorem 29. *Let $(D_1, \circ_1, L_1)$ and $(D_2, \circ_2, L_2)$ be the two e-groups. Let h be a bijective homomorphism from $(D_1, \circ_1, L_1)$ to $(D_2, \circ_2, L_2)$ and (D_2, κ) be a fuzzy sub e-group of $(D_2, \circ_2, L_2)$. Then, $(D_1, h^{-1}(\kappa))$ forms a fuzzy sub e-group of $(D_1, \circ_1, L_1)$.*

Proof. Let d_1 and l_1 be the two elements of D_1. Now,

$$\begin{aligned}\left(h^{-1}(\kappa)\right)(d_1 \circ_1 l_1) &= \kappa(h(d_1 \circ_1 l_1)) = \kappa(h(d_1) \circ_2 h(l_1)) \\ &\geq \kappa(h(d_1)) \wedge \kappa(h(l_1)) \\ &= \left(h^{-1}(\kappa)\right)(d_1) \wedge \left(h^{-1}(\kappa)\right)(l_1).\end{aligned} \quad (13)$$

Therefore, $(h^{-1}(\kappa))(d_1 \circ_1 l_1) \geq (h^{-1}(\kappa))(d_1) \wedge (h^{-1}(\kappa))(l_1)$ for all d_1 and $l_1 \in D_1$.

Let $l \in L_1$ and $d \in D_1/L_1$.

Since h is a homomorphism, $h(l) \in L_2$ as $h(L_1) \subseteq L_2$. Now,

$$\left(h^{-1}(\kappa)\right)(l) = \kappa(h(l)) \geq \kappa(h(d)) = \left(h^{-1}(\kappa)\right)(d).. \quad (14)$$

□

Therefore, $(h^{-1}(\kappa))(l) \geq (h^{-1}(\kappa))(d)$ for all $l \in L_1$ and $d \in D_1/L_1$.

Hence, $(D_1, h^{-1}(\kappa))$ forms a fuzzy sub e-group of $(D_1, \circ_1, L_1)$.

Theorem 30. *Let $(D_1, \circ_1, L_1)$ and $(D_2, \circ_2, L_2)$ be the two e-groups. Let h be a homomorphism from $(D_1, \circ_1, L_1)$ to $(D_2, \circ_2, L_2)$ and (D_1, κ) be a fuzzy sub e-group of $(D_1, \circ_1, L_1)$. Then, $(D_2, h(\kappa))$ forms a fuzzy sub e-group of $(D_2, \circ_2, L_2)$.*

Proof. Let d_2 and l_2 be the two elements of D_2.

If either $d_2 \notin h(D_1)$ or $l_2 \notin h(D_1)$ then,

$$(h(\kappa))(d_2) \wedge (h(\kappa))(l_2) = 0 \leq (h(\kappa))(d_2 \circ_2 l_2). \tag{15}$$

Suppose $d_2 = h(d_1)$ and $l_2 = h(l_1)$ for some $d_1, l_1 \in D_1$. Now,

$$\begin{aligned}(h(\kappa))(d_2 \circ_2 l_2) &= \vee\{\kappa(p) | h(p) = d_2 \circ_2 l_2\} \\ &\geq \vee\{\kappa(d_1 \circ_1 l_1) | d_1, l_1 \in D_1, h(d_1) = d_2, h(l_1) = l_2\} \\ &\geq \vee\{\kappa(d_1) \wedge \kappa(l_1) | d_1, l_1 \in D_1, h(d_1) = d_2, h(l_1) = l_2\} \\ &= (\vee\{\kappa(d_1) | d_1 \in D_1, h(d_1) = d_2\}) \wedge \\ &\quad \cdot (\vee\{\kappa(l_1) | l_1 \in D_1, h(l_1) = l_2\}) \\ &= (h(\kappa))(d_2) \wedge (h(\kappa))(l_2).\end{aligned} \tag{16}$$

Therefore $(h(\kappa))(d_2 \circ_2 l_2) \geq (h(\kappa))(d_2) \wedge (h(\kappa))(l_2)$ for all d_2 and $l_2 \in D_2$.

Let $l_2 \in L_2$ and $d_2 \in D_2/L_2$.

Since h is a homomorphism, $h(L_1) \subseteq L_2$. Now,

$$\begin{aligned}(h(\kappa))(l_2) &= \vee\{\kappa(l_1) | l_1 \in D_1, h(l_1) = l_2\} \\ &\geq \vee\{\kappa(d_1) | d_1 \in D_1, h(d_1) = d_2\} = (h(\kappa))(d_2).\end{aligned} \tag{17}$$

Therefore, $(h(\kappa))(l_2) \geq (h(\kappa))(d_2)$ for all $l_2 \in L_2$ and $d_2 \in D_2/L_2$.

Hence, $(D_2, h(\kappa))$ forms a fuzzy sub e-group of $(D_2, \circ_2, L_2)$. □

Theorem 31. *Let $(D_1, \circ_1, L_1)$ and $(D_2, \circ_2, L_2)$ be the two e-groups. Let h be a bijective homomorphism from $(D_1, \circ_1, L_1)$ to $(D_2, \circ_2, L_2)$ and (D_2, κ) be a normal fuzzy sub e-group of $(D_2, \circ_2, L_2)$. Then, $(D_1, h^{-1}(\kappa))$ forms a normal fuzzy sub e-group of $(D_1, \circ_1, L_1)$.*

Proof. From Theorem 29, we can say that $(D_1, h^{-1}(\kappa))$ forms a fuzzy sub e-group of $(D_1, \circ_1, L_1)$.

Since (D_2, κ) forms a normal fuzzy sub e-group of $(D_2, \circ_2, L_2)$, $\kappa(d_2 \circ_2 l_2) = \kappa(l_2 \circ_2 d_2)$ for all $d_2, l_2 \in D_2$.

Let d_1 and l_1 be the two elements of D_1. Then,

$$\begin{aligned}\left(h^{-1}(\kappa)\right)(d_1 \circ_1 l_1) &= \kappa(h(d_1 \circ_1 l_1)) = \kappa(h(d_1) \circ_2 h(l_1)) \\ &= \kappa(h(l_1) \circ_2 h(d_1)) = \kappa(h(l_1 \circ_1 d_1)) \\ &= \left(h^{-1}(\kappa)\right)(l_1 \circ_1 d_1).\end{aligned} \tag{18}$$

Therefore, $(h^{-1}(\kappa))(d_1 \circ_1 l_1) = (h^{-1}(\kappa))(l_1 \circ_1 d_1)$ for all d_1 and $l_1 \in D_1$.

Hence, $(D_1, h^{-1}(\kappa))$ forms a normal fuzzy sub e-group of $(D_1, \circ_1, L_1)$. □

Theorem 32. *Let $(D_1, \circ_1, L_1)$ and $(D_2, \circ_2, L_2)$ be the two e-groups. Let h be a bijective homomorphism from $(D_1, \circ_1, L_1)$ to $(D_2, \circ_2, L_2)$ and (D_1, κ) be a normal fuzzy sub e-group of $(D_1, \circ_1, L_1)$. Then, $(D_2, h(\kappa))$ forms a normal fuzzy sub e-group of $(D_2, \circ_2, L_2)$.*

Proof. From Theorem 30, we can say that $(D_2, h(\kappa))$ is a fuzzy sub e-group of $(D_2, \circ_2, L_2)$.

Since (D_1, κ) forms a normal fuzzy sub e-group of $(D_1, \circ_1, L_1)$, $\kappa(d_1 \circ_1 l_1) = \kappa(l_1 \circ_1 d_1)$ for all $d_1, l_1 \in D_1$.

Let d_2 and l_2 be the two elements of D_2.

Suppose that there are unique d_1 and $l_1 \in D_1$, such that $d_2 = h(d_1)$ and $l_2 = h(l_1)$. Now,

$$\begin{aligned}(h(\kappa))(d_2 \circ_2 l_2) &= \vee\{\kappa(p) | h(p) = d_2 \circ_2 l_2\} \\ &= \vee\{\kappa(d_1 \circ_1 l_1) | d_1, l_1 \in D_1, h(d_1) = d_2, h(l_1) = l_2\} \vee \\ &\quad \cdot \{\kappa(l_1 \circ_1 d_1) | d_1, l_1 \in D_1, h(d_1) = d_2, h(l_1) = l_2\} \\ &= \vee\{\kappa(p) | h(p) = l_2 \circ_2 d_2\} = (h(\kappa))(l_2 \circ_2 d_2).\end{aligned} \tag{19}$$

Therefore, $(h(\kappa))(d_2 \circ_2 l_2) = (h(\kappa))(l_2 \circ_2 d_2) \forall d_2, l_2 \in D_2$.

Hence, $(D_2, h(\kappa))$ forms a normal fuzzy sub e-group of $(D_2, \circ_2, L_2)$. □

6. Conclusion

In this paper, we presented a brief demonstration of fuzzy sub e-groups and its properties. A condition is given for a FS of an e-group to be a fuzzy sub e-group. We have demonstrated that any fuzzy sub e-group forms a fuzzy subgroup. However, the reverse is not always true. Therefore, fuzzy sub e-group is the generalization of fuzzy subgroup. We have presented the difference between FSG and fuzzy sub e-group. We have discussed about the idea of normal fuzzy sub e-groups and level fuzzy e-subgroups. Finally, we have explained the effect of e-group homomorphism on fuzzy sub e-groups. In future, we will work on important theorems like Lagrange's theorem and Sylow theorem in fuzzy sub e-groups.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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References

[1] A. B. Saeid, A. Rezaei, and A. Radfar, "A generalization of groups," *Atti della Accademia Peloritana dei Pericolanti*, vol. 96, no. 1, pp. 1–11, 2018.

[2] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[3] A. Rosenfeld, "Fuzzy groups," *Journal of Mathematical Analysis and Applications*, vol. 35, no. 3, pp. 512–517, 1971.

[4] J. M. Anthony and H. Sherwood, "Fuzzy groups redefined," *Journal of Mathematical Analysis and Applications*, vol. 69, no. 1, pp. 124–130, 1979.

[5] J. M. Anthony and H. Sherwood, "A characterization of fuzzy subgroups," *Fuzzy Sets and Systems*, vol. 7, no. 3, pp. 297–305, 1982.

[6] P. S. Das, "Fuzzy groups and level subgroups," *Journal of Mathematical Analysis and Applications*, vol. 84, no. 1, pp. 264–269, 1981.

[7] F. P. Choudhury, A. B. Chakraborty, and S. S. Khare, "A note on fuzzy subgroups and fuzzy homomorphism," *Journal of Mathematical Analysis and Applications*, vol. 131, no. 2, pp. 537–553, 1988.

[8] V. N. Dixit, R. Kumar, and N. Ajmal, "Level subgroups and union of fuzzy subgroups," *Fuzzy Sets and Systems*, vol. 37, no. 3, pp. 359–371, 1990.

[9] R. Biswas, "Fuzzy subgroups and anti fuzzy subgroups," *Fuzzy Sets and Systems*, vol. 35, no. 1, pp. 121–124, 1990.

[10] N. Ajmal and A. S. Prajapati, "Fuzzy cosets and fuzzy normal subgroups," *Information Sciences*, vol. 64, no. 1-2, pp. 17–25, 1992.

[11] A. B. Chakraborty and S. S. Khare, "Fuzzy homomorphism and algebraic structures," *Fuzzy Sets and Systems*, vol. 59, no. 2, pp. 211–221, 1993.

[12] N. Ajmal, "Homomorphism of fuzzy groups, correspondence theorem and fuzzy quotient groups," *Fuzzy Sets and Systems*, vol. 61, no. 3, pp. 329–339, 1994.

[13] J. G. Kim, "Orders of fuzzy subgroups and fuzzy p-subgroups," *Fuzzy Sets and Systems*, vol. 61, no. 2, pp. 225–230, 1994.

[14] A. K. Ray, "On product of fuzzy subgroups," *Fuzzy Sets and Systems*, vol. 105, no. 1, pp. 181–183, 1999.

[15] M. Gulfam, M. K. Mahmood, F. Smarandache, and S. Ali, "New Dombi aggregation operators on bipolar neutrosophic set with application in multi-attribute decision-making problems," *Journal of Intelligent Fuzzy Systems*, vol. 40, no. 3, pp. 5043–5060, 2021.

[16] M. K. Mahmood, S. Zeng, M. Gulfam, S. Ali, and Y. Jin, "Bipolar neutrosophic dombi aggregation operators with application in multi-attribute decision making problems," *IEEE Access*, vol. 8, pp. 156600–156614, 2020.

[17] M. Tarnauceanu, "Classifying fuzzy normal subgroups of finite groups," *Iranian Journal of Fuzzy Systems*, vol. 12, no. 2, pp. 107–115, 2015.

[18] B. O. Onasanya, "Review of some anti fuzzy properties of some fuzzy subgroups," *Annals of Fuzzy Mathematics and Informatics*, vol. 11, no. 6, pp. 899–904, 2016.

[19] U. Shuaib and M. Shaheryar, "On some characterizations of o-fuzzy subgroups," *International Journal of Mathematics and Computer Science*, vol. 13, no. 2, pp. 119–131, 2018.

[20] G. M. Addis, "Fuzzy homomorphism theorems on groups," *The Korean Journal of Mathematics*, vol. 26, no. 3, pp. 373–385, 2018.

[21] S. Bhunia and G. Ghorai, "A new approach to fuzzy group theory using (α, β)-Pythagorean fuzzy sets," *Songklanakarin Journal of Science and Technology*, vol. 43, no. 1, pp. 295–306, 2021.

[22] S. Bhunia, G. Ghorai, and Q. Xin, "On the characterization of Pythagorean fuzzy subgroups," *AIMS Mathematics*, vol. 6, no. 1, pp. 962–978, 2021.

[23] S. Bhunia, G. Ghorai, and Q. Xin, "On the fuzzification of Lagrange's theorem in α, β-Pythagorean fuzzy environment," *AIMS Mathematics*, vol. 6, no. 9, pp. 9290–9308, 2021.

[24] E. A. Abuhijleh, M. Massa'deh, A. Sheimat, and A. Alkouri, "Complex fuzzy groups based on Rosenfeld's approach," *WBSEAS Transactions on Mathematics*, vol. 20, no. 1, pp. 368–377, 2021.

[25] H. Alolaiyan, H. A. Alshehri, M. H. Mateen, D. Pamucar, and M. Gulzar, "A novel algebraic structure of (α, β)-complex fuzzy subgroups," *Entropy*, vol. 23, no. 8, p. 992, 2021.

[26] H. Alolaiyan, M. H. Mateen, D. Pamucar, M. K. Mahmmod, and F. Arslan, "A certain structure of bipolar fuzzy subrings," *Symmetry*, vol. 13, no. 8, p. 1397, 2021.

[27] A. G. Talafha, S. M. Alqaraleh, A. E. el-Ahmady, and J. al Jaraden, "On fuzzy fundamental groups and fuzzy folding of fuzzy Minkowski space," *Advances in Fuzzy Systems*, vol. 2021, Article ID 7747394, 2021.

[28] J. N Mordeson, K. R Bhutani, and A. Rosenfeld, *Fuzzy Group Theory*, vol. 182, Springer-Verlag, Netherlands, 2005.

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Research Article

Commutators of the Fractional Hardy Operator on Weighted Variable Herz-Morrey Spaces

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In the present paper, our aim is to establish the boundedness of commutators of the fractional Hardy operator and its adjoint operator on weighted Herz-Morrey spaces with variable exponents $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(w)$.

1. Introduction

Hardy operators and related commutators play an indispensable role in the theory of partial differential equations [1, 2] and the characterization of function spaces [3–5]. Without going into much details, let us first define the fractional Hardy operators [3]

$$Hg(z) = \frac{1}{|z|^{n-\beta}} \int_{|t|\le|z|} g(t)dt, \quad H^* g(z) = \int_{|t|>|z|} \frac{g(t)}{|t|^{n-\beta}} dt, z \in \mathbb{R}^n/\{0\} \tag{1}$$

and related commutators:

$$\left[b, H_\beta\right] g = bHg - H(bg), \quad \left[b, H_\beta^*\right] g = bH^* g - H^*(bg). \tag{2}$$

It is important to note that taking $\beta = 0$ in (1), we get multidimensional Hardy operator defined and studied in [6, 7]. Also, (1) reduces to the one dimensional Hardy operator [8] if we choose $\beta = 0$ and $n = 1$. Here, we cite some important literature with regards to the study of Hardy-type operators on different function spaces which include [9–15].

The new development of variable exponent commenced with the work of Kov'aˇcik and R'akosn'ık in [16], where a class of function spaces having variable exponent was defined, and basic properties of variable exponent Lebesgue space were explored. Recently, the theory of variable exponent analysis is modeled in terms of the boundedness of the Hardy Littlewood maximal operator M [17–21]:

$$Mg(z) = \sup_{B:\text{ball}, z\in B} \frac{1}{|B|} \int_{|B|} \int_B |g(t)| dt. \tag{3}$$

Besides, Muckenhoupt A_p theory [22] is generalized in the recent span of time with regard to variable exponent spaces ([23–28]). By taking into account the generalization of function spaces with variable exponents and the same with weights, many results like duality, boundedness of sublinear operators, the wavelet characterization, and commutators of fractional and singular integrals have been studied [29–38].

Recently, authors have studied generalized Herz space in terms of both Muckenhoupt weights and variable exponent [39–41]. Moreover, an idea of combining two function spaces to develop a new one is also an interesting problem in Harmonic analysis. One such problem is considered in [42] in which Herz-Morrey space was defined. Although, the weighted versions of Herz-Morrey spaces were introduced recently in [43, 44].

In this piece of work, our main focus is on establishing the boundedness of commutators of fractional Hardy operators on a class of function spaces called the weighted Herz-Morrey space with variable exponents. We seek to find the boundedness of these commutators with symbol functions in BMO (bounded mean oscillation) spaces. In establishing such a boundedness, we make use of the boundedness of the fractional integral operator I_β

$$I_\beta(g)(z)=\int_{\mathbb{R}^n}\frac{g(t)}{|z-t|^{n-\beta}}dt \tag{4}$$

on weighted Lebesgue space which was done in [39].

In the rest of this paper, the symbol C expresses a constant whose value may differ at all of its occurrences. The Greek letter χ_S denotes the characteristics function of a sphere S where S is a measurable subset of R^n and $|S|$ represents its Lebesgue measure. Before turning to our key results, let us first define the relevant variable exponent function spaces.

2. Preliminaries

Let us consider a measurable function $p(\cdot)$ on $\mathbb{R}^n$ having range $[1,\infty)$. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable function f such that

$$L^{p(\cdot)}(\mathbb{R}^n)=\left\{f:\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\sigma}\right)^{p(x)}dx<\infty,\text{for some }\sigma>0\right\}. \tag{5}$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ turns out to be Banach function space under the norm:

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}=\inf\left\{\sigma>0:\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\sigma}\right)^{p(x)}dx\le 1\right\}. \tag{6}$$

We denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot)$: $\mathbb{R}^n\longrightarrow(1,\infty)$ such that

$$1<p_-\le p(x)\le p_+<\infty, \tag{7}$$

where

$$p_-:=\operatorname{essinf}_{x\in\mathbb{R}^n}p(x),\, p_+:=\operatorname{esssup}_{x\in\mathbb{R}^n}p(x). \tag{8}$$

Definition 1. Suppose $p(\cdot)$ is a real valued function on $\mathbb{R}^n$. We say that

(i) $C^{\log}_{loc}(\mathbb{R}^n)$ is the set of all local log-Holder continuous functions $p(\cdot)$ satisfying

$$|p(x)-p(y)|\lesssim\frac{-C}{\log(|x-y|)},\quad |x-y|<\frac{1}{2},\quad x,y\in\mathbb{R}^n. \tag{9}$$

(ii) $\mathscr{C}^{\log}_0(\mathbb{R}^n)$ is the set of all local log-Holder continuous function $p(\cdot)$ satisfying at the origin

$$|p(x)-p(0)|\lesssim\frac{C}{\log(|e+(1/|x|)|)},\quad |x-y|<\frac{1}{2},\quad x\in\mathbb{R}^n. \tag{10}$$

(iii) $\mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ is the set of all log-Holder continuous functions satisfying at infinity

$$|p(x)-p_\infty|\le\frac{C_\infty}{\log(e+|x|)},\quad x\in\mathbb{R}^n. \tag{11}$$

(iv) $\mathscr{C}^{\log}(\mathbb{R}^n)=\mathscr{C}^{\log}_\infty\cap\mathscr{C}^{\log}_{loc}$ denotes the set of all global log-Holder continuous functions $p(\cdot)$.

It was proved in [21] that if $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)\cap C^{\log}(\mathbb{R}^n)$, then Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Suppose $w(x)$ is a weight function on $\mathbb{R}^n$, which is non-negative and locally integrable on $\mathbb{R}^n$. Let $L^{p(\cdot)}(w)$ be the space of all complex-valued functions f on $\mathbb{R}^n$ such that$fw^{1/p(\cdot)}\in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(w)$ is a Banach function space equipped with the norm:

$$\|f\|_{L^{p(\cdot)}(w)}=\left\|fw^{\frac{1}{p(\cdot)}}\right\|_{L^{p(\cdot)}}. \tag{12}$$

Benjamin Muckenhoupt introduced the theory of $A_p(1<p<\infty)$ weights on $\mathbb{R}^n$ in [22]. Recently, in [39, 40], Izuki and Noi generalized the Muckenhoupt A_p class by taking p as a variable.

Definition 2. Let $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$. A weight w is an $A_{p(\cdot)}$ weight if

$$\sup_B\frac{1}{|B|}\left\|w^{1/p(\cdot)}\chi_B\right\|_{L^{p(\cdot)}}\left\|w^{-1/p(\cdot)}\chi_B\right\|_{L^{p'(\cdot)}}<\infty. \tag{13}$$

In [25], the authors proved that $w\in A_{p(\cdot)}$ if and only if M is bounded on the space $L^{p(\cdot)}$.

Remark 3 (see [39]). Suppose $p(\cdot),q(\cdot)\in\mathscr{P}(\mathbb{R}^n)\cap\mathscr{C}^{\log}(\mathbb{R}^n)$ and $p(\cdot)\le q(\cdot)$, then we have

$$A_1\subset A_{p(\cdot)}\subset A_{q(\cdot)}. \tag{14}$$

Definition 4. Suppose $p_1(\cdot),p_2(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ and $\beta\in(0,n)$ such that $1/p_2(x)=1/p_1(x)-\beta/n$. A weight w is said to be $A(p_1(\cdot),p_2(\cdot))$ weight if

$$\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_B\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})'}\le C|B|^{1-\frac{\beta}{n}}. \tag{15}$$

Definition 5 (see [39]). Suppose $p_1(\cdot), p_2(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $\beta \in (0, n)$ such that $1/p_2(x) = 1/p_1(x) - \beta/n$. Then, $w \in A_{(p_1(\cdot),p_2(\cdot))}$ if and only if $w^{p_2(\cdot)} \in A_{1+p_2(\cdot)/p_1'(\cdot)}$.

Now, we define the variable exponent weighted Morrey-Herz space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)$. Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $A_k = B_k/B_{k-1}$, and $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$.

Definition 6. Let w be a weight on $\mathbb{R}^n$, $\lambda \in [0,\infty)$, $q \in (0,\infty)$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $\alpha(\cdot)$: $\mathbb{R}^n \longrightarrow \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)$ is the set of all measurable functions which is given by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w) = \left\{ f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n/\{0\}, w) \colon \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)} < \infty \right\}, \tag{16}$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)} = \sup_{k_0 \in Z} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q} \|f\chi_k\|^q_{L^{p(\cdot)}(w)} \right)^{1/q}. \tag{17}$$

Obviously, $M\dot{K}^{\alpha(\cdot),0}_{q,p(\cdot)}(w) = \dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(w)$ is the weighted Herz space with variable exponent (see [30]). Here, it is important to refer to some of the pioneering studies of the Herz space with constant exponents made in [45, 46].

3. Some Useful Lemmas

We start this section with some useful lemmas that will be helpful in proving our main results.

Lemma 7 (see [47]). *If X is Banach function space, then*

(i) The associated space X' is also Banach function space

(ii) $\|\cdot\|_{(X')'}$ and $\|\cdot\|_X$ are equivalent

(iii) If $g \in X$ and $f \in X'$, then

$$\int_{\mathbb{R}^n} |f(x)g(x)| \leq \|g\|_X \|f\|_{X'} \tag{18}$$

is the generalized Hölder inequality.

Lemma 8 (see [39]). *Suppose X is a Banach function space. Then, we have that for all balls B,*

$$1 \leq \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'}. \tag{19}$$

Lemma 9 (see [28, 39]). *Let X be a Banach function space. Suppose that the Hardy Littlewood maximal operator M is weakly bounded on X; that is,*

$$\left\| \chi_{\{M_f > \sigma\}} \right\| X \lesssim \sigma^{-1} \|f\|_X \tag{20}$$

is true for $\sigma > 0$ and for all $f \in X$. Then, we have

$$\sup_{B:ball} \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'} < \infty. \tag{21}$$

Lemma 10 (see [39, 48]).

(1) $X(\mathbb{R}^n, W)$ is Banach function space equipped with the norm

$$\|f\|_{X(\mathbb{R}^n,W)} = \|fw\|_X, \tag{22}$$

where

$$X(\mathbb{R}^n, W) = \{f \in M : fW \in X :\}. \tag{23}$$

(2) The associate space $X'(\mathbb{R}^n, W^{-1})$ is also a Banach function space

Lemma 11 (see [39]). *Let X be a Banach function space. Assume that M is bounded on X', then there exists a constant $\delta \in (0, 1)$ for all $B \subset \mathbb{R}^n$ and $E \subset B$,*

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \lesssim \left(\frac{|E|}{|B|} \right)^\delta. \tag{24}$$

The paper [16] shows that $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space and the associated space $L^{p'(\cdot)}(R^n)$ with equivalent norm.
Remark 12. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, and by comparing the Lebesgue space $L^{p(\cdot)}(w^{p(\cdot)})$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$ with the definition of $X(\mathbb{R}^n, W)$, we have

(1) If we take $W = w$ and $X = L^{p(\cdot)}(\mathbb{R}^n)$, then we get $L^{p(\cdot)}(\mathbb{R}^n, w) = L^{p(\cdot)}(w^{p(\cdot)})$

(2) If we consider $W = w^{-1}$ and $X = L^{p'(\cdot)}(\mathbb{R}^n)$, then we have $L^{p'(\cdot)}(w^{-p'(\cdot)}) = L^{p'(\cdot)}(\mathbb{R}^n, w^{-1})$

By virtue of Lemma 10, we get $(L^{p(\cdot)}(\mathbb{R}^n, w))' = (L^{p(\cdot)}(w^{p(\cdot)}))' = L^{p'(\cdot)}(w^{-p'(\cdot)}) = L^{p'(\cdot)}(\mathbb{R}^n.w^{-1})$. Next, in view of Lemma 11 and Remark 12, we have the following Lemma.

Lemma 13 (see [41]). *Let $p(\cdot) \in \mathscr{P}(R^n) \cap \mathscr{C}\log(\mathbb{R}^n)$ be a Log Hölder continuous function both at infinity and at origin, if $w^{p2}(\cdot) \in A_{p2(\cdot)}$ implies $w^{-p_2'} \in A_{p_2'(\cdot)}$. Thus, the Hardy Littlewood operator is bounded on $L^{p_2'(\bullet)}(w^{p_2'(\bullet)})$, and there exist constants $\delta_1, \delta_2 \in (0, 1)$ such that*

$$\begin{aligned} \frac{\|\chi_E\|_{L^{p_2(\bullet)}(w^{p_2(\cdot)})}}{\|\chi_B\|_{L^{p_2(\bullet)}(w^{p_2(\cdot)})}} &= \frac{\|\chi_E\|_{(L^{p_2'(\bullet)} w^{-p_2'(\bullet)})'}}{\|\chi_B\|_{(L^{p_2'(\bullet)} w^{-p_2'(\bullet)})'}} \lesssim \left(\frac{|E|}{|B|} \right)^{\delta_1}, \\ \frac{\|\chi_E\|_{(L^{p_2(\bullet)} w^{p_2(\cdot)})'}}{\|\chi_B\|_{(L^{p_2(\bullet)} w^{p_2(\cdot)})'}} &\lesssim \left(\frac{|E|}{|B|} \right)^{\delta_1}, \end{aligned} \tag{25}$$

for all balls B and all measurable sets $E \subset B$.

Lemma 14 (see [39]). *Let $p_1(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap \mathscr{C}^{\log}(R^n)$ and $0 < \beta < n/p_{1+}$ and $1/p_2(\cdot) = 1/p_1(\bullet) - \beta/n$. If $\in A(p_1(\cdot), p_2(\cdot))$, then I^β is bounded from $L^{p_1(\cdot)}(w^{P_1(\cdot)})$ to $L^{p_2(\cdot)}(w^{P_2(\cdot)})$.*

4. Main Results and their Proofs

Definition 15. Let $f \in L^1_{loc}(Rn)$ and set

$$\|b\|_{BMO} = \sup_B \int_B |b(x) - b_B| dx, \quad (26)$$

where the supremum is taken all over the balls $B \in \mathbb{R}^n$ and $b_B = |B|^{-1} \int_B b(y)dy$. The function b is a bounded mean oscillation if $\|b\|_{BMO} < \infty$ and $BMO(\mathbb{R}^n)$ consist of all $f \in L^1_{loc}(\mathbb{R}^n)$ with $BMO(\mathbb{R}^n) < \infty$. For a comprehensive review of the BMO space, we suggest the reader to follow the books [49, 50].

Lemma 16. *Let $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and w be an $A_{q(\cdot)}$ weight. Then, for all $b \in BMO$ and all $l, i \in \mathbb{Z}$ with $l > i$, we have*

$$\|b\|_{BMO} \sim \sup_{B:Ball} \frac{1}{\|\chi_B\|_{L^{q(\bullet)}(w^{q(\cdot)})}} \|(b - b_B)\chi_B\|_{L^{q(\bullet)}(w^{q(\cdot)})}, \quad (27)$$

$$\left\|(b - b_{B_i})\chi_{B_l}\right\|_{L^{q(\bullet)}(w^{q(\cdot)})} \le C(l - i)\|b\|_{BMO} \left\|\chi_{B_l}\right\|_{L^{q(\bullet)}(w^{q(\cdot)})}. \quad (28)$$

Proof. First part of this lemma is a consequence of [[41], Theorem 18]. Next, we will prove (28), for all $l, i \in \mathbb{Z}$ with $l > i$

$$\begin{aligned}\left\|(b - b_{B_i})\chi_{B_l}\right\|_{L^{q(\bullet)}(w^{q(\cdot)})} &\le C\left\|\left(|b - b_{B_l}| + |b - b_{B_l}|\right)\chi_{B_l}\right\|_{L^{q(\bullet)}(w^{q(\cdot)})} \\ &\le C\left\{\left\|(b - b_{B_l})\chi_{B_l}\right\|_{L^{q(\bullet)}(w^{q(\cdot)})} + \left\|(b_{Bl} - b_{B_i})\chi_{B_l}\right\|_{L^{q(\bullet)}(w^{q(\cdot)})}\right\}. \end{aligned} \quad (29)$$

In the view of (27), we have

$$\left\|(b - b_{B_i})\chi_{B_l}\right\|_{L^{q(\bullet)}(w^{q(\cdot)})} \le C\|b\|_{BMO}\left\|\chi_{B_l}\right\|_{L^{q(\bullet)}(w^{q(\cdot)})}. \quad (30)$$

Also, it is easy to see that

$$\begin{aligned}|b_{Bl} - b_{B_i}| &\le \sum_{n=1}^{l-1} |b_{n+1} - b_n| \\ &\le \sum_{n=1}^{l-1} \frac{1}{|B_n|}\int_{B_n} |b_{n+1} - b(x)| dx \\ &\le C\sum_{n=1}^{l-1} \frac{1}{|B_{n+1}|}\int_{B_n} |b_{n+1} - b(x)| dx \\ &= C(l - i)\|b\|_{BMO(\mathbb{R}^n)}. \end{aligned} \quad (31)$$

Combining (29), (30), and (31), we get (28). □

Proposition 17. *Let $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 < p < \infty$, and $0 \le \lambda < \infty$. If $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$, then*

$$\begin{aligned}\|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(w^{q(\cdot)})} &= \sup_{k_0 \in Z} 2^{-k_0\lambda p} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\cdot)p}\left\|f\chi_j\right\|^p_{L^{q(\bullet)}(w^{q(\cdot)})} \\ &\le \max\left\{\sup_{\substack{k_0 \in Z\\ k_0<0}} 2^{-k_0\lambda p}\left(\sum_{j=-\infty}^{k_0} 2^{j\alpha(\cdot)p}\left\|f\chi_j\right\|^p_{L^{q(\bullet)}(w^{q(\cdot)})}\right), \sup_{\substack{k_0 \in Z\\ k_0<0}} \cdot\left(2^{-k_0\lambda p}\left(\sum_{j=-\infty}^{-1} 2^{j\alpha(\cdot)p}\left\|f\chi_j\right\|^p_{L^{q(\bullet)}(w^{q(\cdot)})}\right) + 2^{-k_0\lambda p}\left(\sum_{j=0}^{k_0} 2^{j\alpha(\infty)p}\left\|f\chi_j\right\|^p_{L^{q(\bullet)}(w^{q(\cdot)})}\right)\right)\right\}. \end{aligned} \quad (32)$$

Proof. The proof is similar to the proof of Proposition 17 in [44]. So, we omit the details. □

Theorem 18. *Let $0 < p_1 \le p_2 < \infty$, $q_2(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$, and $q_1(\cdot)$ be such that $1/q_1(\cdot) = 1/q_2(\bullet) - \beta/n$..*

Also, let $w^{q2}(\cdot) \in A_1$, $b \in BMO(\mathbb{R}^n)$, $\lambda > 0$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{C}^{\log}(\mathbb{R}^n)$ be log Hölder continuous at the origin, with $\alpha(0) \le \alpha(\infty) < \lambda + n\delta_2 - \beta$, where $0 < \delta_2 < 1$, then

$$\left\|[b, H_\beta]f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(w^{q_2(\cdot)})} \le C\|b\|_{BMO}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_2(\cdot)})}. \quad (33)$$

Proof. For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})$, if we denote $f_l = f \cdot \chi_l = f \cdot \chi_{A_l}$, and for each $l \in \mathbb{Z}$,

$$f(x) = \sum_{1=-\infty}^{\infty} f(x) \cdot \chi_l(x) = \sum_{1=-\infty}^{\infty} f_l(x), \quad (34)$$

then it is not difficult to see that

$$\begin{aligned}\left|[b, H_\beta]f(x)\chi_j(x)\right| &\le \frac{1}{|x|^{n-\beta}}\int_{B_j} |(b(x) - b(y))f(y)| dy \cdot \chi_j(x) \\ &\le 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\int_{B_j} |(b(x) - b(y))f(y)| dy \cdot \chi_j(x) \\ &\le 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\int_{B_j} \left|(b(x) - b_{B_l})f(y)\right| dy \cdot \chi_j(x) \\ &\quad + 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\int_{B_j} \left|(b(x) - b_{B_l})f(y)\right| dy \cdot \chi_j(x) \\ &= E_1 + E_2. \end{aligned} \quad (35)$$

The generalized Hölder inequality (Lemma 7) yields the following inequality for E_1:

$$E_1 = 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \int_{B_l} \left| \left(b(x) - b_{B_l}\right) f(y) \right| dy \cdot \chi_j(x) \leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \left| \left(b(x) - b_{B_l}\right) \right| \cdot \chi_j(x) \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \times \left\| \chi_{B_l} \right\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}. \tag{36}$$

Applying the norm on both sides and using Lemma 16, we get

$$\begin{aligned} \|E_1\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))} &\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \left\| \left(b(x) - b_{B_l} \cdot \chi_{B_j}\right) \right\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))} \cdot \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \left\| \chi_{B_l} \right\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \\ &\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} (j-l) \|b\|_{BMO} \left\| \chi_{B_l} \right\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'} \cdot \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \left\| \chi_{B_l} \right\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}. \end{aligned} \tag{37}$$

Now, we turn to estimate E_2. For this, we have

$$\begin{aligned} E_2 &\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \left\| \left(b(y) - b_{B_l} \cdot \chi_l\right) \right\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \cdot \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \chi_j(x) \\ &\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \left\| \left(b(y) - b_{B_l} \cdot \chi_{B_l}\right) \right\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \cdot \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \chi_j(x). \end{aligned} \tag{38}$$

Similar to the estimation for E_1, we take the norm on both sides of above inequality and use Lemma 16 to obtain

$$\begin{aligned} \|E_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} &\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \left\| \left(b(y) - b_{B_l}\right) \cdot \chi_l \right\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \cdot \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \left\| \chi_j \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\ &\leq 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \|b\|_{BMO} \left\| \chi_{B_l} \right\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \cdot \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \left\| \chi_{B_j} \right\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))}. \end{aligned} \tag{39}$$

Hence, from inequalities (35), (37), and (39), one has $\|[b, H_\beta] f \chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \leq 2^{-j(n-\beta)} \|b\|_{BMO} \sum_{l=-\infty}^{j} (j-1)$ $\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_{B_j}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))} \|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}$, which by virtue of Lemma 9 reduces to

$$\left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \leq 2^{j\beta} \|b\|_{BMO} \sum_{l=-\infty}^{j} (j-l) \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \left\| \chi_{B_l} \right\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \left\| \chi_{B_j} \right\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}. \tag{40}$$

Now using Lemma 13, we learn

$$\begin{aligned} \left\| [b, H_\beta] f \chi_j \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} &\leq 2^{j\beta} \|b\|_{BMO} \sum_{l=-\infty}^{j} (j-l) \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \times \frac{\left\| \chi_{B_l} \right\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}{\left\| \chi_{B_l} \right\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}} \frac{\left\| \chi_{B_l} \right\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}{\left\| \chi_{B_j} \right\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}} \\ &\leq 2^{j\beta} \|b\|_{BMO} \sum_{l=-\infty}^{j} (j-l) 2^{(l-j)n\delta_2} \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \times \frac{\left\| \chi_{B_l} \right\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}{\left\| \chi_{B_l} \right\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}. \end{aligned} \tag{41}$$

In the definition of the fraction integral I_β, we replace f by χ_{Bl} to obtain

$$I_\beta\left(\chi_{B_l}\right)(x) \geq C 2^{l\beta} \chi_{B_l}(x), \tag{42}$$

from which we infer that

$$\chi_{B_l}(x) \leq C 2^{-l\beta} I\beta\left(\chi_{B_l}\right)(x). \tag{43}$$

Taking the norm on both sides and using Lemmas 14 and 9, respectively, we get

$$\begin{aligned} \left\| \chi_{B_l} \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} &\leq C 2^{l\beta} \left\| I_\beta\left(\chi_{B_l}\right) \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\ &\leq C 2^{l\beta} \left\| \chi_{B_l} \right\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \\ &\leq C 2^{l(n-\beta)} \left\| \chi_{B_l} \right\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}. \end{aligned} \tag{44}$$

In view of Lemmas 8 and 9, the use of (44) into (41) results in the following inequality:

$$\begin{aligned}
&\left\|\left[b, H_\beta\right] f \chi_j\right\|_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)} \\
&\leq C\|b\|_{BMO} \sum_{l=-\infty}^{j} 2^{l(n-\beta)} 2^{j\beta}(j-l) 2^{(l-j) n\delta_2}\left\|f_l\right\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)} \\
&\quad\times\left(\left\|\chi_l\right\|_{\left(L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)\right)}\left\|\chi_l\right\|_{\left(L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)\right)'}\right)^{-1} \\
&\leq C\|b\|_{BMO} \sum_{l=-\infty}^{j} 2^{(j-l)(\beta-n\delta_2)}(j-l)\left\|f_l\right\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)} \\
&\quad\times\left(2^{-\ln}\left\|\chi_l\right\|_{\left(L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)\right)}\left\|\chi_l\right\|_{\left(L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)\right)'}\right)^{-1} \\
&\leq C\|b\|_{BMO} \sum_{l=-\infty}^{j} 2^{(j-l)(\beta-n\delta_2)}(j-l)\left\|f_l\right\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}.
\end{aligned} \tag{45}$$

Now, by virtue of the condition $p_1 \leq p_2$ and Proposition 17, we have

$$\begin{aligned}
&\left\|\left[b, H_\beta\right] f \chi_j\right\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}\left(w^{q_2(\cdot)}\right)} \\
&\leq \max\left\{\sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1}\left(\sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1}\left\|\left[b, H_\beta\right] f \chi_j\right\|^{p_1}_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}\right), \sup_{\substack{k_0\in Z\\ k_0\geq 0}}\right. \\
&\quad\times\left(2^{-k_0\lambda p_1}\left(\sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1}\left\|\left[b, H_\beta\right] f \chi_j\right\|^{p_1}_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}\right)\right. \\
&\quad\left.\left.+2^{-k_0\lambda p_1}\left(\sum_{j=0}^{k_0} 2^{j\alpha(\infty)p_1}\left\|\left[b, H_\beta\right] f \chi_j\right\|^{p_1}_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}\right)\right)\right\} \\
&= \max\left\{X_1, X_2, X_3\right\},
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
X_1 &= \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1}\left(\sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1}\left\|\left[b, H_\beta\right] f \chi_j\right\|^{p_1}_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}\right), \\
X_2 &= \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda p_1}\left(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)p_1}\left\|\left[b, H_\beta\right] f \chi_j\right\|^{p_1}_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}\right), \\
X_3 &= \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda p_1}\left(\sum_{j=0}^{k_0} 2^{j\alpha(0)p_1}\left\|\left[b, H_\beta\right] f \chi_j\right\|^{p_1}_{L^{q_2(\cdot)}\left(w^{q_2(\cdot)}\right)}\right).
\end{aligned} \tag{47}$$

To estimate X_1, X_2, and X_3, we make use of the conditions on $\alpha(\cdot)$, such that for $l < 0$, we have

$$\begin{aligned}
\left\|f_l\right\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)} &= 2^{-l\alpha(0)}\left(2^{j\alpha(0)p_1}\left\|f_l\right\|^{p_1}_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}\right)^{\frac{1}{p_1}} \\
&\leq 2^{-l\alpha(0)}\left(\sum_{1=-\infty}^{l} 2^{i\alpha(0)p_1}\left\|f_i\right\|^{p_1}_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}\right)^{\frac{1}{p_1}} \\
&\leq 2^{l(\lambda-\alpha(0))} 2^{-l\lambda}\left(\sum_{1=-\infty}^{l} 2^{i\alpha(\cdot)p_1}\left\|f_i\right\|^{p_1}_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}\right)^{\frac{1}{p_1}} \\
&\leq C 2^{l(\lambda-\alpha(0))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}\left(w^{q_1(\cdot)}\right)},
\end{aligned} \tag{48}$$

and for $l \geq 0$, we obtain

$$\begin{aligned}
\left\|f_l\right\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)} &= 2^{-l\alpha(\infty)}\left(2^{l\alpha(\infty)p_1}\left\|f_l\right\|^{p_1}_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}\right)^{\frac{1}{p_1}} \\
&\leq 2^{-l\alpha(\infty)}\left(\sum_{1=-\infty}^{l} 2^{i\alpha(\infty)p_1}\left\|f_i\right\|^{p_1}_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}\right)^{\frac{1}{p_1}} \\
&\leq 2^{l(\lambda-\alpha(\infty))} 2^{-l\lambda}\left(\sum_{1=-\infty}^{l} 2^{i\alpha(\cdot)p_1}\left\|f_i\right\|^{p_1}_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}\right)^{\frac{1}{p_1}} \\
&\leq C 2^{l(\lambda-\alpha(\infty))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}.
\end{aligned} \tag{49}$$

In order to estimate X_1, we need to use $\alpha(0) \leq \alpha(\infty) < n\delta_2 + \lambda - \beta$.

$$\begin{aligned}
X_1 &\leq \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \\
&\quad\cdot\left(\sum_{l=-\infty}^{j}(j-l) 2^{(j-l)(\beta-n\delta_2)}\|b\|_{BMO}\left\|f_l\right\|_{L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}\right)^{p_1} \\
&\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \\
&\quad\cdot\left(\sum_{l=-\infty}^{j}(j-l) 2^{(j-l)(\beta-n\delta_2)} 2^{l(\lambda-\alpha(0))}\|b\|_{BMO}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}\right)^{p_1} \\
&\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1}\left(\sum_{l=-\infty}^{j}(j-l) 2^{(j-l)(\beta-n\delta_2)} 2^{l(\lambda-\alpha(0))}\right)^{p_1} \\
&\quad\cdot\|b\|^{p_1}_{BMO}\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}\left(w^{q_1(\cdot)}\right)}
\end{aligned}$$

$$\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\lambda p_1} \left(\sum_{l=-\infty}^{j} (j-l) 2^{(l-j)(-\beta+n\delta_2-\alpha(0)+\lambda)} \right)^{p_1}$$
$$\cdot \|b\|_{BMO}^{p_1} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})}$$
$$\leq C \|b\|_{BMO}^{p_1} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})}. \tag{50}$$

The result of X_2 is similar to that of X_1. Next, we will estimate X_3 below

$$X_3 \leq \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\infty)p_1} \cdot \left(\sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta-n\delta_2)} \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{p_1}$$
$$\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\infty)p_1} \cdot \left(\sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta-n\delta_2)} 2^{l(\lambda-\alpha(\infty))} \|b\|_{BMO} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_2(\cdot)})} \right)^{p_1}$$
$$\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\infty)p_1} \left(\sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta-n\delta_2)} 2^{l(\lambda-\alpha(\infty))} \right)^{p_1} \cdot \|b\|_{BMO}^{p_1} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\lambda p_1} \left(\sum_{l=-\infty}^{j} (j-l) 2^{(l-j)(-\beta+n\delta_2-\alpha(\infty)+\lambda)} \right)^{p_1} \cdot \|b\|_{BMO}^{p_1} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq C \|b\|_{BMO}^{p_1} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_2(\cdot)})}. \tag{51}$$

Finally, we combine the estimates for $X_i (i = 1, 2, 3)$, to have the desired result. □

Theorem 19. *Let $p_1, p_2, q_1(\cdot), q_2(\cdot), \beta, \alpha(\cdot)$ and w be as in Theorem 18. In addition, if $\lambda - n\delta_1 < \alpha(0) \leq \alpha(\infty)$, where $1 < \delta_1 < 0$, then*

$$\left\| \left[b, H^*_\beta \right] f \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(w^{q_2(\cdot)})} \leq C \|b\|_{BMO} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_2(\cdot)})}. \tag{52}$$

Proof. We write

$$\begin{aligned}\left[b, H^*_\beta \right] f(x)\chi_j(x) &\leq \int_{\mathbb{R}^n \backslash B_j} |y|^{\beta-n} |(b(x)-b(y)) f(y)| dy \cdot \chi_j(x) \\ &\leq \sum_{l=j+1}^{\infty} \int_{B_l} |y|^{\beta-n} |(b(x)-b(y)) f(y)| dy \cdot \chi_j(x) \\ &\leq \sum_{l=j+1}^{\infty} \int_{B_l} |y|^{\beta-n} \left| b(x) - b_{B_l} \right| f(y) dy \cdot \chi_j(x) + \\ &\leq \sum_{l=j+1}^{\infty} \int_{B_l} |y|^{\beta-n} \left| \left(b(y) - b_{B_l} \right) f(y) \right| dy \cdot \chi_j(x) \\ &= F_1 + F_2.\end{aligned} \tag{53}$$

We estimate F_1 and F_2 separately. A use of generalized inequality results in the following:

$$\begin{aligned}F_1 &\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \int_{B_l} \left| \left(b(x) - b_{B_l} \right) f(y) \right| dy \cdot \chi_j(x) \\ &\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \|f_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \left\| \chi_{B_l} \right\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})'} \\ &\quad \cdot \left| b(x) - b_{B_l} \right| \cdot \chi_j.\end{aligned} \tag{54}$$

Applying the weighted Lebesgue space norm on both sides and using Lemma 16, we obtain

$$\begin{aligned}\|F_1\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} &\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \left\| \left(b(x) - b_{B_j} \right) \chi_j \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\ &\quad \cdot \|f_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \left\| \chi_{B_l} \right\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})'} \\ &\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \|b\|_{BMO} \left\| \left(b(x) - b_{B_j} \right) \chi_j \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\ &\quad \cdot \|f_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \left\| \chi_{B_l} \right\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})'}.\end{aligned} \tag{55}$$

Similarly,

$$\begin{aligned}F_2 &\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \int_{B_l} \left| \left(b(y) - b_{B_l} \right) f(y) \right| dy \cdot \chi_j(x) \\ &\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \left\| b(y) - b_{B_l} \cdot \chi_j \right\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \\ &\quad \cdot \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})'} \cdot \chi_j(x)(x).\end{aligned} \tag{56}$$

In view of the weighted Lebesgue norm and Lemma 16, we get

$$\|F_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \le C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \left\| \left(b(y) - b_{B_l}\right) \cdot \chi_j \right\|_{\left(L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)\right)'} \cdot \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \left\|\chi_j\right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \le C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} (l-j) \|b\|_{BMO} \|\chi_l\|_{\left(L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)\right)'} \cdot \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \left\|\chi_j\right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}. \quad (57)$$

Hence, from (53), (55), and (57), we obtain

$$\left\| \left[b, H_\beta^*\right] f \chi_j \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \le \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} (l-j) \|b\|_{BMO} \left\|\chi_j\right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \times \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \left\|\chi_{B_l}\right\|_{\left(L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)\right)'} \le \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} (l-j) \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \times \left\|\chi_{B_l}\right\|_{\left(L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)\right)'} \|\chi_l\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \frac{\left\|\chi_j\right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_l\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}} \le \sum_{l=j+1}^{\infty} 2^{n\delta(j-1)} 2^{-l(n-\beta)} (l-j) \|b\|_{BMO} \|f_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \times \left\|\chi_{B_l}\right\|_{\left(L^{q_1(\cdot)}\left(w^{q_1(\cdot)}\right)\right)'} \|\chi_l\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}. \quad (58)$$

Using the condition of $A(q_1(\cdot), q_2(\cdot))$ weights given in the Definition 4, the above inequality reduces to

$$\left\| \left[b, H_\beta^*\right] f \chi_j \right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \le \sum_{l=j+1}^{\infty} 2^{n\delta_1(j-l)} (l-j) \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}. \quad (59)$$

Next, the condition $p_1 < p_2$ and Proposition 17 help us to write

$$\left\| \left[b, H_\beta^*\right] f \chi_j \right\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(w^{q_2(\cdot)})} = \max\{Y_1, Y_2, Y_3\}, \quad (60)$$

where

$$Y_1 = \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1} \left(\sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \left\| \left[b, H_\beta^*\right] f \cdot \chi_j \right\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right),$$
$$Y_2 = \sup_{\substack{k_0 \in Z \\ k_0 \ge 0}} 2^{-k_0\lambda p_1} \left(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)p_1} \left\| \left[b, H_\beta^*\right] f \cdot \chi_j \right\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right),$$
$$Y_3 = \sup_{\substack{k_0 \in Z \\ k_0 \ge 0}} 2^{-k_0\lambda p_1} \left(\sum_{j=0}^{k_0} 2^{j\alpha(\infty)p_1} \left\| \left[b, H_\beta^*\right] f \cdot \chi_j \right\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right). \quad (61)$$

Lastly, in view of the condition $-n\delta_1 + \lambda < \alpha(0) \le \alpha(\infty)$, we estimate $Y_i, i = 1, 2, 3$, as we estimated $X_i, i = 1, 2, 3$, in Theorem 18. Hence, we finish the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

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References

[1] R. C. Brown and D. B. Hinton, "A weighted Hardy's inequality and nonoscillatory differential equations," *Quaestiones Mathematicae*, vol. 15, no. 2, pp. 197–212, 1992.

[2] R. C. Brown and D. B. Hinton, "Some one variable weighted norm inequalities and their applications to sturm-liouville and other differential operators," *International Series of Numerical Mathematics*, vol. 157, pp. 61–76, 2008.

[3] Z. W. Fu, Z. G. Liu, S. Z. Lu, and H. Wong, "Characterization for commutators of n−dimensional fractional Hardy operators," *Science in China (Scientia Sinica)*, vol. 50, no. 10, pp. 1418–1426, 2007.

[4] Z. Fu and S. Lu, "A characterization of λ−central BMO spaces," *Frontiers of Mathematics in China*, vol. 8, article 229238, 2013.

[5] S. Shi and S. Lu, “Characterization of the central Campanato space via the commutator operator of Hardy type,” *Journal of Mathematical Analysis and Applications*, vol. 429, no. 2, pp. 713–732, 2015.

[6] W. G. Faris, “Weak Lebesgue spaces and quantum mechanical binding,” *Duke Mathematical Journal*, vol. 43, no. 2, article 365373, 1976.

[7] M. Chris and L. Grafakos, “Best constant for two nonconvolution inequalities,” *Mathematische Zeitschrift*, vol. 123, pp. 1687–1693, 1995.

[8] G. H. Hardy, “Note on a theorem of Hilbert,” *Mathematische Zeitschrift*, vol. 6, no. 3-4, pp. 314–317, 1920.

[9] A. G. Bliss, “An integral inequality,” *Journal of the London Mathematical Society*, vol. s1-5, no. 1, pp. 40–46, 1930.

[10] Z. W. Fu, L. Grafakos, S. Z. Lu, and F. Y. Zhao, “Sharp bounds for m-linear Hardy and Hilbert operators,” *Houston Journal of Mathematics*, vol. 38, no. 1, pp. 225–244, 2012.

[11] A. Hussain and G. Gao, “Some new estimates for the commutators of n-dimensional Hausdorff operator,” *Applied Mathematics-A Journal of Chinese Universities*, vol. 29, no. 2, pp. 139–150, 2014.

[12] L.-E. Persson and S. G. Samko, “A note on the best constants in some hardy inequalities,” *Journal of Mathematical Inequalities*, vol. 9, no. 2, pp. 437–447, 2007.

[13] A. Hussain and N. Sarfraz, “Optimal weak type estimates for p-Adic Hardy operators,” *Analysis and Applications*, vol. 12, no. 1, pp. 29–38, 2020.

[14] Y. Mizuta, A. Nekvinda, and T. Shimomura, “Optimal estimates for the fractional Hardy operator,” *Studia Mathematica*, vol. 227, no. 1, pp. 1–19, 2015.

[15] T. L. Yee and K. P. Ho, “Hardy’s inequalities and integral operators on Herz-Morrey spaces,” *Open Mathematics*, vol. 18, no. 1, pp. 106–121, 2020.

[16] O. Kováčik and J. Rákosník, “On spaces $L^{p(x)}$ and $W^{k,p(x)}$,” *Czechoslovak mathematical journal*, vol. 41, no. 4, pp. 592–618, 1991.

[17] D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Perez, “The boundedness of classical operators on variable L^p spaces,” *Annales-Academiae Scientiarum Fennicae Mathematica*, vol. 31, no. 1, pp. 239–264, 2006.

[18] D. Cruz-Uribe, L. Diening, and A. Fiorenza, “A new proof of the boundedness of maximal operators on variable Lebesgue spaces,” *Bollettino della Unione Matematica Italiana*, vol. 2, no. 1, pp. 151–173, 2009.

[19] L. Diening, L. Harjulehto, P. Hasto, and P. Ruicka, “Lebesgue and Sobolev spaces with variable exponents,” vol. 2011, Springer, Heidelberg, 2017.

[20] D. Cruz-Uribe and D. V. Fiorenza, “Variable Lebesgue spaces. Foundations and harmonic analysis,” Springer Science & Business Media, Heidelberg, 2013.

[21] D. Cru-Uribe, A. Fiorenza, and C. Neugebauer, “The maximal function on variable L^p spaces,” *Annales-Academiae Scientiarum Fennicae Mathematica*, vol. 28, pp. 223–238, 2003.

[22] B. Muckenhoupt, “Weighted norm inequalities for the Hardy maximal function,” *Transactions of the American Mathematical Society*, vol. 165, pp. 207–226, 1972.

[23] L. Diening and P. Hästö, *Muckenhoupt weights in variable exponent spaces*, 2008, http://www.problemsolving.fi/pp/p75submit.pdf.

[24] D. Cruz-Uribe, L. Diening, and P. Hästö, “The maximal operator on weighted variable Lebesgue spaces,” *Fractional Calculus and Applied Analysis*, vol. 14, no. 3, pp. 361–374, 2011.

[25] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer, “Weighted norm inequalities for the maximal operator on variable Lebesgue spaces,” *Journal of Mathematical Analysis and Applications*, vol. 394, no. 2, pp. 744–760, 2012.

[26] D. Cruz-Uribe and L. A. Wang, “Extrapolation and weighted norm inequalities in the variable Lebesgue spaces,” *Transactions of the American Mathematical Society*, vol. 369, no. 2, pp. 1205–1235, 2017.

[27] M. Izuki, E. Nakai, and Y. Sawano, “Wavelet characterization and modular inequalities for weighted Lebesgue spaces with variable exponent,” *Annales-Academiae Scientiarum Fennicae Mathematica*, vol. 40, pp. 551–571, 2015.

[28] M. Izuki, “Remarks on Muckenhoupt weights with variable exponent,” *Journal of Analysis and Applications*, vol. 2, no. 1, pp. 27–41, 2013.

[29] M. Izuki, “Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization,” *Analysis Mathematica*, vol. 36, no. 1, pp. 33–50, 2010.

[30] A. Almeida and D. Drihem, “Maximal, potential and singular type operators on Herz spaces with variable exponents,” *Journal of Mathematical Analysis and Applications*, vol. 394, no. 2, pp. 781–795, 2012.

[31] S. Samko, “Variable exponent Herz spaces,” *Mediterranean Journal of Mathematics*, vol. 10, no. 4, pp. 2007–2025, 2013.

[32] J. L. Wu and W. J. Zhao, “Boundedness for fractional Hardy-type operator on variable-exponent Herz Morrey spaces,” *Kyoto Journal of Mathematics*, vol. 56, no. 4, pp. 831–845, 2016.

[33] A. Hussain and G. Gao, “Status of day care laparoscopic appendectomy in developing countries,” *International Scholarly Research Notices*, vol. 2014, 5 pages, 2014.

[34] A. Meskhi, H. Rafeiro, and M. A. Zaighum, “Central Calderón–Zygmund operators on Herz-type Hardy spaces of variable smoothness and integrability,” *Annals of Functional Analysis*, vol. 9, no. 3, pp. 310–321, 2018.

[35] K. P. Ho, “Extrapolation to Herz spaces with variable exponents and applications,” *Revista Matemática Complutense*, vol. 33, no. 2, pp. 437–463, 2020.

[36] F. Gürbüz, “Some estimates for generalized commutators of rough fractional maximal and integral¨ operators on generalized weighted Morrey spaces,” *Canadian Mathematical Bulletin*, vol. 60, no. 1, pp. 131–145, 2017.

[37] F. Gürbüz, S. Ding, H. Han, and P. Long, “Characterizations of rough fractional-type integral Op-¨ erators on variable exponent vanishing Morrey-type spaces,” in *Topics in Contemporary Mathematical Analysis and Applications*, H. Dutta, Ed., pp. 95–123, CRC Press, 2021.

[38] F. Gürbüz, S. Ding, H. Han, and P. Long, “Norm inequalities on variable exponent vanishing Morrey¨ type spaces for the rough singular type integral operators,” *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 2020, pp. 1–19, 2020.

[39] M. Izuki and T. Noi, “Boundedness of fractional integrals on weighted Herz spaces with variable exponent,” *Journal of Inequalities and Applications*, vol. 2016, 15 pages, 2016.

[40] M. Izuki and T. Noi, “An intrinsic square function on weighted Herz spaces with variable exponent,” *Journal of Mathematical Inequalities*, vol. 11, pp. 799–816, 2017.

[41] M. Izuki and T. Noi, “Two weighted Herz Spaces with variable exponents,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 1, pp. 169–200, 2020.

[42] M. Izuki, “Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponent,” *Mathematical Sciences Research Journal*, vol. 13, pp. 243–253, 2009.

[43] L. Wang and L. Shu, “Boundedness of some sublinear operators on weighted variable Herz-Morrey spaces,” *Journal of Mathematical Inequalities*, vol. 12, pp. 31–42, 2007.

[44] S. R. Wang and J. S. Xu, “Weighted norm inequality for bilinear Calderon-Zygmund operators on HerzMorrey spaces with variable exponents,” *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 23, 2019.

[45] C. S. Herz, “Lipschitz spaces and Bernstein’s theorem on absolutely convergent Fourier transforms,” *Journal of Mathematics and Mechanics*, vol. 18, pp. 283–323, 1968.

[46] S. Lu and L. Xu, “Boundedness of rough singular integral operatorson the homogeneous Morrey-Herz spaces,” *Hokkaido Mathematical Journal*, vol. 34, no. 2, pp. 299–314, 2005.

[47] C. Bennett and R. C. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[48] A. Y. Karlovich and I. M. Spitkovsky, “The Cauchy singular integral operator on weighted variable Lebesgue spaces,” in *Concrete Operators, Spectral Theory, Operators in Harmonic Analysis and Approximation*, vol. 236, pp. 275–291, Springer, Basel, 2014.

[49] L. Grafakos, *Modern Fourier analysis*, Springer, 2nd edition, 2008.

[50] Y. Sawano, G. Di Fazio, and D. I. Hakim, “Introduction and applications to integral operators and PDE’s,” in *Monographs and Research Notes in Mathematics*, CRC Press, Boca Raton, FL, 2020.

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Research Article

Coefficient Bounds of Kamali-Type Starlike Functions Related with a Limacon-Shaped Domain

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In this article, we familiarize a subclass of Kamali-type starlike functions connected with limacon domain of bean shape. We examine certain initial coefficient bounds and Fekete-Szegö inequalities for the functions in this class. Analogous results have been acquired for the functions f^{-1} and $\xi/f(\xi)$ and also found the upper bound for the second Hankel determinant $a_2a_4 - a_3^2$.

1. Introduction

Denote by $\mathscr{A}$ the class of analytic functions

$$f(\xi) = \xi + a_2\xi^2 + a_3\xi^3 + \cdots, \tag{1}$$

in the open unit disk $U = \{\xi : |\xi| < 1\}$. The Hankel determinants $\mathscr{H}_j(n)$, $(n = 1, 2, 3, \cdots; j = 1, 2, 3, \cdots)$ of f are denoted by

$$\mathscr{H}_j(n) = \begin{bmatrix} a_n & a_{n+1} & \cdot & \cdot & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \cdot & \cdot & a_{n+j} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+j-1} & a_{n+j} & \cdot & \cdot & a_{n+2j-2} \end{bmatrix}, \tag{2}$$

where $a_1 = 1$. Hankel determinants are beneficial, for example, in viewing that whether the certain coefficient functionals related to functions are bounded in U or not and do they carry the sharp bounds, see [1]. The applications of Hankel inequalities in the study of meromorphic functions can be seen in [2, 3]. In 1966, Pommerenke [4] inspected $|\mathscr{H}_j(n)|$ of univalent functions and $p-$valent functions as well as starlike functions. In [5], it is evidenced that the Hankel determinants of univalent functions satisfy

$$|\mathscr{H}_j(n)| < kn^{-((1/2)+\beta)j+3/2}\,; (n = 1, 2, 3, \cdots; j = 1, 2, 3, \cdots), \tag{3}$$

where $\beta > 0.00025$ and k depends only on j. Later, Hayman [6] demonstrated that $|\mathscr{H}_2(n)| < A_n^{1/2}$, $(n = 1, 2, 3, \cdots;$ A an absolute constant) for univalent functions. Further, the Hankel determinant bounds of univalent functions with a positive Hayman index α were determined by Elhosh [7], of p-valent functions were seen in [8–10], and of close-to-convex and k-fold symmetric functions were discussed in [11]. Lately, several authors have explored the bounds $|\mathscr{H}_j(n)|$, of functions belonging to various subclasses of univalent and multivalent functions (for details, see [6, 12–27]). The Hankel determinant $\mathscr{H}_2(1) = a_3 - a_2^2$ is the renowned Fekete-Szegö Functional (see [28, 29]) and $H_2(2)$; second, Hankel determinant is presumed by $\mathscr{H}_2(2) = a_2a_4 - a_3^2$.

An analytic function f_1 is subordinate to an analytic function f_2, written as $f_1 \prec f_2$, if there is an analytic function $w : U \longrightarrow U$ with $w(0) = 0$, satisfying $f_1(\xi) = f_2(w(\xi))$.

Let $\mathscr{P}$ be the class of functions with positive real part consisting of all analytic functions $p : U \longrightarrow \mathbb{C}$ satisfying $p(0) = 1$ and $\text{Re}p(z) > 0$.

Ma and Minda [30] amalgamated various subclasses of starlike and convex functions which are subordinate to a function $\psi \in \mathscr{P}$ with $\psi(0) = 1, \psi'(0) > 0, \psi$ maps U onto a region starlike with respect to 1 and symmetric with respect to real axis and familiarized the classes as below:

$$\mathcal{S}^*(\psi) = \left\{ f \in \mathcal{A} : \frac{\xi f'(\xi)}{f(\xi)} \prec \psi \right\} \text{and}\, \mathscr{C}(\psi) = \left\{ f \in \mathcal{A} : 1 + \frac{\xi f''(\xi)}{f'(\xi)} \prec \psi \right\}. \tag{4}$$

By choosing ψ satisfying Ma-Minda conditions and that maps U on to some precise regions like parabolas, cardioid, lemniscate of Bernoulli, and booth lemniscate in the right-half of the complex plane, several fascinating subclasses of starlike and convex functions are familiarized and studied. Raina and Sokół [31] considered the class $\mathcal{S}^*(\psi)$ for $\psi(z) = \xi + \sqrt{1 + \xi^2}$ and established some remarkable inequalities (also see [32] and references cited therein). Gandhi in [33] considered a class $\mathcal{S}^*(\psi)$ with $\psi = \beta e^{\xi} + (1 - \beta)(1 + \xi)$, $0 \leq \beta \leq 1$, a convex combination of two starlike functions. Further, coefficient inequalities of functions linked with petal type domains were widely discussed by Malik et al. ([34], see also references cited therein). The region bounded by the cardioid specified by the equation

$$\left(9x^2 + 9y^2 - 18x + 5\right)^2 - 16\left(9x^2 + 9y^2 - 6x + 1\right) = 0, \tag{5}$$

was studied in [35]. Lately, Masih and Kanas [36] introduced novel subclasses $\text{ST}_{L(s)}$ and $\text{CV}_{L(s)}$ of starlike and convex functions, respectively. Geometrically, they consist of functions $f \in A$ such that $\xi f'(\xi)/f(\xi)$ and $(\xi f'(\xi))'/f'(\xi)$, respectively, are lying in the region bounded by the limacon

$$\left[(u - 1)^2 + v^2 - s^4\right]^2 = 4s^2\left[\left(u - 1 + s^2\right)^2 + v^2\right], \quad \text{where}\, 0 < s \leq \frac{1}{\sqrt{2}}. \tag{6}$$

Lately, Yuzaimi et al. [37] defined a region bounded by the bean-shaped limacon region as below:

$$\Omega(U) = \left\{ w = x + iy : \left(4x^2 + 4y^2 - 8x - 5\right)^2 + 8\left(4x^2 + 4y^2 - 12x - 3\right) = 0 \right\}, s \in [-1, 1] \setminus \{0\}. \tag{7}$$

Suppose that

$$\varphi(\xi) \colon U \longrightarrow \mathbb{C}, \tag{8}$$

is the function defined by

$$\varphi(\xi) = 1 + \sqrt{2}\xi + \frac{1}{2}\xi^2, \tag{9}$$

is preferred so that the limacon is in the bean shape [37]. Motivated by this present work and other aforesaid articles, the goal in this paper is to examine some coefficient inequalities and bounds on Hankel determinants of the Kamali-type class of starlike functions satisfying the conditions as given in Definition 1.

Definition 1. Let $\varphi : U \longrightarrow \mathbb{C}$ be analytic and for $0 \leq \vartheta \leq 1$, we let the class $\mathscr{M}(\vartheta, \varphi)$ *as*

$$\mathscr{M}(\vartheta, \varphi) = \left\{ f \in A : \frac{\vartheta \xi^3 f'''(\xi) + (1 + 2\vartheta) f''(\xi)\xi^2 + \xi f'(\xi)}{\vartheta \xi^2 f''(\xi) + \xi f'(\xi)} \prec \varphi(\xi), \xi \in U \right\}, \tag{10}$$

where $\varphi(\xi) = 1 + \sqrt{2}\xi + (1/2)\xi^2$ as in (9).

We include the following results which are needed for the proofs of our main results.

Lemma 2 see [38]. *Suppose that* $p(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots$, $\mathfrak{R}(p_1) > 0$, $\xi \in U$, *then*

$$|c_n| \leq 2(n = 1, 2, 3, \cdots), \quad \left|c_2 - v c_1^2\right| \leq 2 \max\{1, |2v - 1|\}, \tag{11}$$

and the outcome is sharp for the functions formulated by

$$p(\xi) = \frac{1 + \xi^2}{1 - \xi^2}, \quad p(\xi) = \frac{1 + \xi}{1 - \xi}. \tag{12}$$

Lemma 3 see [30]. *Suppose that* $p_1(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots$, $\mathfrak{R}(p_1) > 0$, $\xi \in U$. *Then,*

(i) For $v < 0$ *or* $v > 1$, *we have*

$$|c_2 - v c_1^2| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0, \\ 2 & \text{if } 0 \leq v \leq 1, \\ 4v - 2 & \text{if } v \geq 1. \end{cases} \tag{13}$$

Equality occurs when $p_1(\xi) = (1 + \xi)/(1 - \xi)$ *or one of its rotations.*

(ii) For $v \in (0, 1)$, *the equality exists when* $p_1(\xi) = (1 + \xi^2)/(1 - \xi^2)$ *or one of its rotations*

(iii) For $v = 0$, *the equality happens when*

$$p_1(\xi) = \left(\frac{1}{2} + \frac{1}{2}\vartheta\right)\frac{1+\xi}{1-\xi} + \left(\frac{1}{2} - \frac{1}{2}\vartheta\right)\frac{1-\xi}{1+\xi} (0 \le \vartheta \le 1), \tag{14}$$

or one of its rotations.

Lemma 4 see [39]. *If $p \in \mathscr{P}$ and is given by $p(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots$ then*

$$2c_2 = c_1^2 + x\left(4 - c^2\right), \tag{15}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\xi\right), \tag{16}$$

for some x, ξ with $|x| \le 1$ and $|\xi| \le 1$.

Theorem 5. *Let the function $f \in \mathscr{M}(\vartheta, \varphi)$ be given by (1) then*

$$\begin{aligned} |a_2| &\le \frac{1}{\sqrt{2}(\vartheta+1)}, \\ |a_3| &\le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|\frac{5}{2\sqrt{2}}\right|\right\} = \frac{5}{12(2\vartheta+1)}. \end{aligned} \tag{17}$$

Proof. Since $f \in M(\vartheta, \varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(\xi)| < 1$ in U such that

$$\frac{\vartheta\xi^3 f'''(\xi) + (1+2\vartheta)f''(\xi)\xi^2 + \xi f'(\xi)}{\vartheta\xi^2 f''(\xi) + \xi f'(\xi)} = \varphi(w(\xi)). \tag{18}$$

Define the function p_1 by

$$p_1(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} = 1 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \cdots, \tag{19}$$

or, equivalently

$$\begin{aligned} w(\xi) &= \frac{p_1(\xi) - 1}{p_1(\xi) + 1} \\ &= \frac{1}{2}\left[c_1\xi + \left(c_2 - \frac{c_1^2}{2}\right)\xi^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\xi^3 + \cdots\right], \end{aligned} \tag{20}$$

then p_1 is analytic in U with $p_1(0) = 1$ and has a positive real part in U. By using (20) together with (9), it is evident that

$$\begin{aligned} \varphi(w(\xi)) = \varphi\left(\frac{p_1(\xi) - 1}{p_1(\xi) + 1}\right) &= 1 + \frac{c_1\xi}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{c_1^2}{8}\right)\xi^2 \\ &+ \left\{\frac{1}{\sqrt{2}}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{c_1}{4}\left(c_2 - \frac{c_1^2}{2}\right)\right\}\xi^3 + \cdots. \end{aligned} \tag{21}$$

Since

$$\begin{aligned} &\frac{\vartheta\xi^3 f'''(\xi) + (1+2\vartheta)f''(\xi)\xi^2 + \xi f'(\xi)}{\vartheta\xi^2 f''(\xi) + \xi f'(\xi)} \\ &= 1 + 2(\vartheta+1)a_2\xi + \left(-4(\vartheta+1)^2 a_2^2 + 6(2\vartheta+1)a_3\right)\xi^2 \\ &\quad + \left[8(\vartheta+1)^3 a_2^3 - 18\left(2\vartheta^2 + 3\vartheta + 1\right)a_2a_3 + 12(3\vartheta+1)a_4\right]\xi^3 + \cdots, \end{aligned} \tag{22}$$

and equating coefficients of ξ, ξ^2, ξ^3 from (21) to (22), we get

$$a_2 = \frac{c_1}{2\sqrt{2}(\vartheta+1)}, \tag{23}$$

$$a_3 = \frac{1}{24(2\vartheta+1)}\left[c_1^2\left(\frac{5}{2} - \sqrt{2}\right) + 2\sqrt{2}c_2\right], \tag{24}$$

$$a_4 = \frac{1}{192(3\vartheta+1)}\left[c_1^3\left(\frac{11}{\sqrt{2}} - 8\right) + 4\left(2 - \sqrt{2}\right)c_1c_2 + 8\sqrt{2}c_3\right]. \tag{25}$$

Now by applying Lemma 2, we get

$$|a_2| = \frac{1}{\sqrt{2}(\vartheta+1)}, \tag{26}$$

and also,

$$\begin{aligned} |a_3| &= \frac{1}{24(2\vartheta+1)}\left|2\sqrt{2}c_2 + c_1^2\left(\frac{5}{2} - \sqrt{2}\right)\right| \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left|c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}}\right)\right| \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left|c_2 - \kappa c_1^2\right|, \end{aligned} \tag{27}$$

where $\kappa = 1/2(1 - (5/2\sqrt{2}))$. Now by applying Lemma 2, we get

$$|a_3| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|\frac{5}{2\sqrt{2}}\right|\right\} = \frac{5}{12(2\vartheta+1)}. \tag{28}$$

To show these bounds are sharp, we define the function $K_{\phi_n}(\xi)$, $\phi_n = q(\xi^{n-1}) (n = 2, 3, 4, \cdots)$ with $K_{\phi_n}(0) = 0 = K'_{\phi_n}(0) - 1$ by

$$\frac{\vartheta\xi^3 K'_{\phi_n}(\xi) + (1+2\vartheta)K'_{\phi_n}(\xi)\xi^2 + \xi K'_{\phi_n}(\xi)}{\vartheta\xi^2 K'_{\phi_n}(\xi) + \xi K'_{\phi_n}(\xi)} = \varphi\left(\xi^{n-1}\right). \tag{29}$$

Clearly, the function $K_{\phi_n} \in \mathscr{M}(\vartheta, \varphi)$. This completes the proof. □

Theorem 6. *Let the function $f \in \mathscr{M}(\vartheta, \varphi)$ be given by (1) and for any $\varpi \in \mathbb{C}$ then*

$$|a_3 - \varpi a_2^2| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5}{2\sqrt{2}} + \varpi \frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{30}$$

Proof. Let the function $f \in \mathscr{M}(\vartheta, \varphi)$ be given by (1), as in Theorem 5, from (23) to (24), we have

$$\begin{aligned} a_3 - \varpi a_2^2 &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}}\right)\right] - \varpi \frac{c_1^2}{8(\vartheta+1)^2} \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}} + \varpi\frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \aleph c_1^2\right], \end{aligned} \tag{31}$$

where $\aleph = 1/2(1 - (5/2\sqrt{2}) + \varpi(3(2\vartheta+1)/\sqrt{2}(\vartheta+1)^2))$. Now by Lemma 2, we get

$$\left|a_3 - \varpi a_2^2\right| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5}{2\sqrt{2}} + \varpi\frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{32}$$

□

The result is sharp.

In particular, by taking $\varpi = 1$, we get

$$|a_3 - a_2^2| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5}{2\sqrt{2}} + \frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{33}$$

Theorem 7. *Let the function $f \in \mathscr{A}$ be given by (1) belongs to the class $\mathscr{M}(\vartheta, \varphi)(0 \le \vartheta \le 1)$. Then, for any real number μ, we have*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{5(1+\vartheta)^2 - 3(1+2\vartheta)\varpi}{12(2\vartheta+1)(1+\vartheta)^2} & \mu \le \delta_1, \\ \dfrac{\sqrt{2}}{6(2\vartheta+1)} & \delta_1 \le \mu \le \delta_2, \\ \dfrac{3(1+2\vartheta)\varpi - 5(1+\vartheta)^2}{12(2\vartheta+1)(1+\vartheta)^2} & \mu \ge \delta_2, \end{cases} \tag{34}$$

where for convenience

$$\begin{aligned} \delta_1 &= \frac{\left(5 - 2\sqrt{2}\right)(1+\vartheta)^2}{3(1+2\vartheta)}, \\ \delta_2 &= \frac{\left(5 + 2\sqrt{2}\right)(1+\vartheta)^2}{3(1+2\vartheta)}, \end{aligned} \tag{35}$$

$$|a_3 - \mu a_2^2| + \frac{(1+\vartheta)^2 + (1+2\vartheta)\mu}{1+2\vartheta}|a_2|^2 \le \frac{1}{2(1+2\vartheta)}.$$

If $\delta_3 \le \mu \le \delta_2$, then

$$|a_3 - \mu a_2^2| + \frac{3(1+\vartheta)^2 - (1+2\vartheta)\mu}{1+2\vartheta}|a_2|^2 \le \frac{1}{2(1+2\vartheta)}. \tag{36}$$

These results are sharp.

Proof. Between (23) and (24) and (31), we have

$$\begin{aligned} a_3 - \varpi a_2^2 &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}}\right)\right] - \varpi\frac{c_1^2}{8(\vartheta+1)^2} \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2 - 3\varpi(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \hbar c_1^2\right], \end{aligned} \tag{37}$$

where $\hbar = 1/2(1 - ((5(\vartheta+1)^2 - 3\varpi(2\vartheta+1))/2\sqrt{2}(\vartheta+1)^2))$. Our result now follows by virtue of Lemma 3. To show that these bounds are sharp, we define the function $K_{\phi_n}(n = 2, 3, \cdots)$ by

$$\begin{aligned} &\frac{\vartheta\xi^3 K_{\phi_n}''(\xi) + (1+2\vartheta)K_{\phi_n}''(\xi)\xi^2 + \xi K_{\phi_n}'(\xi)}{\vartheta\xi^2 K_{\phi_n}''(\xi) + \xi K_{\phi_n}'(\xi)} = \phi_n\left(\xi^{n-1}\right), \\ &K_{\phi_n}(0) = 0 = K_{\phi_n}'(0) - 1, \end{aligned} \tag{38}$$

and the functions F_η and $G_\eta (0 \le \eta \le 1)$ by

$$\begin{aligned} &\frac{\vartheta\xi^3 F_\eta''(\xi) + (1+2\vartheta)F_\eta''(\xi)\xi^2 + \xi F_\eta'(\xi)}{\vartheta\xi^2 F_\eta'(\xi) + \xi F_\eta'(\xi)} = \phi\left(\frac{\xi(\xi+\eta)}{1+\eta\xi}\right)F_\eta(0) \\ &= 0 = F_\eta'(0) - 1, \\ &\frac{\vartheta\xi^3 G_\eta''(\xi) + (1+2\vartheta)G_\eta''(\xi)\xi^2 + \xi G_\eta'(\xi)}{\vartheta\xi^2 G_\eta''(\xi) + \xi G_\eta'(\xi)} = \phi\left(\frac{-\xi(\xi+\eta)}{1+\eta\xi}\right)G_\eta(0) \\ &= 0 = G_\eta'(0) - 1. \end{aligned} \tag{39}$$

Clearly, the functions $K_{\phi_n} = \varphi(\xi^{n-1}), F_\eta, G_\eta \in \mathscr{M}(\vartheta, \varphi)$. Also, we write $K_\phi = K_{\phi_2} = 1 + \sqrt{2}\xi + (1/2)\xi^2$. If $\mu < \delta_1$ or μ

$> \delta_2$, then the equality holds if and only if f is K_ϕ or one of its rotations. When $\delta_1 < \mu < \delta_2$, then the equality holds if and only if f is $K_{\phi_3} = \varphi(\xi^2) = 1 + \sqrt{2}\xi^2 + (1/2)\xi^4$ or one of its rotations. If $\mu = \delta_1$, then the equality holds if and only if f is F_η or one of its rotations. If $\mu = \delta_2$, then the equality holds if and only if f is G_η or one of its rotation. □

2. Coefficient Estimates for the Function f^{-1}

Theorem 8. *If $f \in \mathcal{M}(\vartheta, \varphi)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the inverse function of f with $|w| < r_0$ where r_0 is the greater than the radius of the Koebe domain of the class $\mathcal{M}(\vartheta, \varphi)$, then for any complex number μ, we have*

$$|d_2| \le \frac{1}{\sqrt{2}(\vartheta+1)},$$
$$|d_3| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5(\vartheta+1)^2 + 12(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{40}$$

Also, for any complex number μ, we have

$$|d_3 - \mu d_2^2| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|\frac{5(\vartheta+1)^2 + (12+6\mu)(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{41}$$

The result is sharp. In particular,

$$|d_3 - d_2^2| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|\frac{5(\vartheta+1)^2 + 18(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{42}$$

Proof. Since

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n, \tag{43}$$

is the inverse function of f, we have

$$f^{-1}(f(\xi)) = f\left(f^{-1}(\xi)\right) = \xi. \tag{44}$$

From equations (23) to (24), we get

$$\xi + (a_2 + d_2)\xi^2 + (a_3 + 2a_2 d_2 + d_3)\xi^3 + \cdots = \xi. \tag{45}$$

Equating the coefficients of ξ and ξ^2 on both sides of (45) and simplifying, we get

$$d_2 = -a_2 = -\frac{c_1}{2\sqrt{2}(\vartheta+1)},$$

$$\begin{aligned} d_3 = 2a_2^2 - a_3 &= \frac{c_1^2}{4(\vartheta+1)^2} - \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}}\right)\right] \\ &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}} - \frac{6(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2 + 12(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right]. \end{aligned} \tag{46}$$

By applying Lemma 2, we get

$$|d_2| \le \frac{1}{\sqrt{2}(\vartheta+1)},$$
$$|d_3| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5(\vartheta+1)^2 + 12(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{47}$$

For any complex number μ, consider

$$\begin{aligned} d_3 - \mu d_2^2 &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2 + 12(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &\quad - \mu\frac{c_1^2}{8(1+\vartheta)^2} = -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \rho^* c_1^2\right], \end{aligned} \tag{48}$$

where

$$\rho^* = \frac{1}{2}\left(1 - \frac{5(\vartheta+1)^2 + 3(4+\mu)(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right). \tag{49}$$

Taking modulus on both sides of (49) and applying Lemma 2, we get the estimate as stated in (41). This completes the proof of Theorem 8. □

3. The Logarithmic Coefficients

The logarithmic coefficients e_n of f defined in U are given by

$$\log\frac{f(\xi)}{\xi} = 2\sum_{n=1}^{\infty} e_n \xi^n. \tag{50}$$

Using series expansion of $\log(1+\xi)$ on the left hand side of (50) and equating various coefficients give

$$e_1 = \frac{a_2}{2}, \tag{51}$$

$$e_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right). \tag{52}$$

Theorem 9. *Let $f \in \mathcal{M}(\vartheta, \varphi)$ with logarithmic coefficients given by (51) and (52). Then,*

$$|e_1| \le \frac{1}{2\sqrt{2}(\vartheta+1)},$$

$$|e_2| \le \frac{\sqrt{2}}{12(2\vartheta+1)} \max\left\{1, \left|\frac{5(\vartheta+1)^2+3(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}, \tag{53}$$

and for any $\nu \in \mathbb{C}$, then

$$|e_2 - \nu e_1^2| \le \frac{\sqrt{2}}{12(2\vartheta+1)} \max\left\{1, \left|-\frac{5(\vartheta+1)^2+(3+\nu)(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{54}$$

These inequalities are sharp. In particular, for $\nu = 1$, we get

$$|e_2 - e_1^2| \le \frac{\sqrt{2}}{12(2\vartheta+1)} \max\left\{1, \left|-\frac{5(\vartheta+1)^2+4(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{55}$$

Proof. Using (23) and (24) in (51) and (52) and after simplification, one may have

$$e_1 = \frac{c_1}{4\sqrt{2}(\vartheta+1)}, \tag{56}$$

$$e_2 = \frac{\sqrt{2}}{24(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2+3(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right]. \tag{57}$$

To determine the bounds for e_2, we express (57) in the form

$$e_2 = \frac{\sqrt{2}}{24(2\vartheta+1)}\left[c_2 - \mu^* c_1^2\right], \tag{58}$$

where

$$\mu^* = \left(1 - \frac{5(\vartheta+1)^2+3(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right), \tag{59}$$

then by applying Lemma 2, we get

$$|e_2| \le \frac{\sqrt{2}}{12(2\vartheta+1)} \max\left\{1, \left|\frac{5(\vartheta+1)^2+3(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{60}$$

For any $\nu^* \in \mathbb{C}$, from (56) to (57), we have

$$\begin{aligned} e_2 - \nu e_1^2 &= \frac{\sqrt{2}}{24(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2+3(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &\quad - \nu\frac{c_1^2}{32(\vartheta+1)^2} = \frac{\sqrt{2}}{24(2\vartheta+1)} \\ &\quad \cdot \left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2+(3+\nu)(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &= \frac{\sqrt{2}}{24(2\vartheta+1)}\left[c_2 - \mu_1^* c_1^2\right], \end{aligned} \tag{61}$$

where

$$\mu_1^* = \frac{1}{2}\left(1 - \frac{5(\vartheta+1)^2+(3+\nu)(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right). \tag{62}$$

An application of Lemma 2 gives the desired estimate. □

4. Coefficients Associated with $\xi/f(\xi)$

In this section, we determine the coefficient bounds and Fekete-Szegö problem associated with the function $H(\xi)$ given by

$$H(\xi) = \frac{\xi}{f(\xi)} = 1 + \sum_{n=1}^{\infty} u_n \xi^n \xi \in U), \tag{63}$$

where $f \in \mathcal{M}(\vartheta, \varphi)$.

Theorem 10. *Let $f \in \mathcal{M}(\vartheta, \varphi)$ and $H(\xi)$ are given by (63). Then*

$$|u_1| \le \frac{1}{\sqrt{2}(\vartheta+1)},$$

$$|u_2| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5}{2\sqrt{2}} + \frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{64}$$

The results are sharp.

Proof. By routine calculation, one may have

$$H(\xi) = \frac{\xi}{f(\xi)} = 1 - a_2\xi + \left(a_2^2 - a_3\right)\xi^2 + \left(a_2^3 + 2a_2a_3 - a_4\right)\xi^3 + \cdots. \tag{65}$$

Comparing the coefficients of ξ and ξ^2 on both sides of (63) and (65), we get

$$u_1 = -a_2, \tag{66}$$

$$u_2 = a_2^2 - a_3. \tag{67}$$

Using (23) and (24) in (66) and (67), we obtain

$$u_1 = -\frac{c_1}{2\sqrt{2}(\vartheta+1)}, \tag{68}$$

By Lemma 2, we get

$$|u_1| \le \frac{1}{\sqrt{2}(\vartheta+1)}. \tag{69}$$

Now,

$$\begin{aligned} u_2 &= \frac{c_1^2}{8(\vartheta+1)^2} - \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1-\frac{5}{2\sqrt{2}}\right)\right] \\ &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1-\frac{5}{2\sqrt{2}}+\frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \aleph^* c_1^2\right], \end{aligned} \tag{70}$$

where $\aleph^* = 1/2(1-(5/2\sqrt{2}) + (3(2\vartheta+1)c_1^2/\sqrt{2}(\vartheta+1)^2))$. Again by using Lemma 2, we get

$$|u_2| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5}{2\sqrt{2}} + \frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{71}$$

For any $\nu \in \mathbb{C}$, between (68) and (70), we get

$$\begin{aligned} u_2 - \nu u_1^2 &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1-\frac{5}{2\sqrt{2}}+\frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &\quad - \nu\frac{c_1^2}{8(\vartheta+1)^2} = -\frac{\sqrt{2}}{12(2\vartheta+1)} \\ &\quad \cdot\left[c_2 - \frac{c_1^2}{2}\left(1-\frac{5}{2\sqrt{2}}+\frac{3(1-\nu)(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right)\right. \\ &\quad \left. + \nu\frac{3(2\vartheta+1)c_1^2}{4\sqrt{2}(\vartheta+1)^2}\right] - = -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \aleph_1^* c_1^2\right]. \end{aligned} \tag{72}$$

That is,

$$|u_2 - \nu u_1^2| = \frac{1}{2(2+\lambda)}\left|c_2 - \aleph^* c_1^2\right|, \tag{73}$$

where

$$\aleph_1^* = \frac{1}{2}\left(1-\frac{5}{2\sqrt{2}}+\frac{3(1-\nu)(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right). \tag{74}$$

The result follows by application of Lemma 2 and therefore completes the proof. □

5. Second Hankel Inequality for $f \in \mathcal{M}(\vartheta, \varphi)$

Theorem 11. *Let the function $f \in \mathcal{M}(\vartheta,\varphi)$ be given by (1), then*

$$|a_2 a_4 - a_3^2| \le \frac{1}{18(2\vartheta+1)^2}. \tag{75}$$

Proof. Since $f \in \mathcal{M}(\vartheta,\varphi)$, there exists an analytic function w with $w(0)=0$ and $|w(\xi)|<1$ in U such that,

$$\frac{\vartheta\xi^3 f'''(\xi) + (1+2\vartheta)f''(\xi)\xi^2 + \xi f'(\xi)}{\vartheta\xi^2 f''(\xi) + \xi f'(\xi)} = \varphi(w(\xi)). \tag{76}$$

Therefore, between (23), (24), and (25), we get

$$\begin{aligned} a_2 a_4 - a_3^2 = \frac{\sqrt{2}}{768}\Bigg[& c_1^4\Bigg\{\left(6\vartheta^2+4\vartheta+1\right)\left(\frac{-2}{3}+\frac{2\sqrt{2}}{3}\right) \\ & - \frac{1}{3\sqrt{2}}\left(3\vartheta^2+4\vartheta+1\right) + \left(12\vartheta^2+4\vartheta+1\right) \\ & \cdot\left(\frac{-2}{3}+\frac{1}{3\sqrt{2}}\right) - \frac{2\sqrt{2}(4\vartheta+1)}{3}\Bigg\} \\ & + \frac{2c_2c_1^2}{3}\Bigg\{2\left(12\vartheta^2+4\vartheta+1\right)+4\left(6\vartheta^2+4\vartheta+1\right) \\ & \cdot\left(-\sqrt{2}+\frac{1}{2}\right)\Bigg\} + 8\sqrt{2}c_1c_3(2\vartheta+1)^2 \\ & - \frac{16\sqrt{2}c_2^2}{3}\left(3\vartheta^2+4\vartheta+1\right)\Bigg]. \end{aligned} \tag{77}$$

By writing

$$d_1 = \frac{8\sqrt{2}}{(\vartheta+1)(3\vartheta+1)}, \tag{78}$$

$$d_2 = \frac{4\left\{\left(18\vartheta^2+8\vartheta+2\right) - 2\sqrt{2}\left(6\vartheta^2+4\vartheta+1\right)\right\}}{3(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}, \tag{79}$$

$$d_3 = -\frac{16\sqrt{2}}{3(2\vartheta+1)^2}, \tag{80}$$

$$d_4 = \frac{1}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left\{(6\vartheta^2+4\vartheta+1)\left(\frac{-2}{3}+\frac{2\sqrt{2}}{3}\right) - (3\vartheta^2+4\vartheta+1)\frac{1}{3\sqrt{2}} + (12\vartheta^2+4\vartheta+1)\left(\frac{-2}{3}+\frac{1}{3\sqrt{2}}\right) - \frac{2\sqrt{2}(4\vartheta+1)}{3}\right\} = \frac{1}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2} \cdot \left(\frac{28-16\sqrt{2}}{3\sqrt{2}}\vartheta^2 - \frac{11}{3}\vartheta - \frac{4}{3}\right), \tag{81}$$

and $T = \sqrt{2}/768$, we have

$$\left|a_2a_4 - a_3^2\right| = T\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right|. \tag{82}$$

From (15) to (16), it follows that

$$\left|a_2a_4 - a_3^2\right| = \frac{T}{4}\left|c^4(d_1+2d_2+d_3+4d_4) + 2xc^2(4-c^2) \cdot (d_1+d_2+d_3) + (4-c^2)x^2(-d_1c^2+d_3(4-c^2)) + 2d_1c(4-c^2)(1-|x|^2)\right|. \tag{83}$$

Replacing $|x|$ by μ and then substituting the values of d_1, d_2, d_3, and d_4 from (81) yield

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| \le{}& \frac{T}{4(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2} \cdot \left[c^4\left|(4\vartheta+1) + \frac{4}{3\sqrt{2}}(12\vartheta^2+4\vartheta+1) - \frac{4}{3\sqrt{2}}(\vartheta+1)(3\vartheta+1)\right| + 16\sqrt{2}(2\vartheta+1)^2 \cdot c(4-c^2)(1-\mu^2) + 2\mu c^2(4-c^2) \cdot \left(\frac{4}{3}(12\vartheta^2+4\vartheta+1) + \frac{4}{3}(6\vartheta^2+4\vartheta+1)\right) - \mu^2(4-c^2)\left(\frac{8\sqrt{2}}{3}(6\vartheta^2+4\vartheta+1)c^2 + \frac{64\sqrt{2}}{3}(3\vartheta+1)(\vartheta+1)\right)\right] \\ ={}& \frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left[\frac{c^4}{3}\left|-2\sqrt{2}(4\vartheta+1) + \frac{1}{\sqrt{2}} \cdot (12\vartheta^2+4\vartheta+1) - \frac{1}{\sqrt{2}}(\vartheta+1)(3\vartheta+1)\right| + 4\sqrt{2}(2\vartheta+1)^2c(4-c^2) - 4\sqrt{2}(2\vartheta+1)^2 \cdot c(4-c^2)\mu^2 + \frac{2}{3}\mu c^2(4-c^2)(2(12\vartheta^2+4\vartheta+1) + (12\vartheta^2+8\vartheta+2)) - \frac{2\sqrt{2}}{3}\mu^2(4-c^2)(c^2(6\vartheta^2+4\vartheta+1) + 8(3\vartheta^2+4\vartheta+2))\right] \\ ={}& \frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2} \cdot \left[\frac{c^4}{3}\left|-2\sqrt{2}(4\vartheta+1) + \frac{9}{\sqrt{2}}\vartheta^2\right| + 4\sqrt{2}(2\vartheta+1)^2c(4-c^2) + \frac{1}{3}\mu c^2(4-c^2)(36\vartheta^2+16\vartheta+4) - \frac{2\sqrt{2}}{3}\mu^2(4-c^2) \cdot (c^2(6\vartheta^2+4\vartheta+1) + 6(2\vartheta+1)^2c + 8(3\vartheta^2+4\vartheta+2))\right] \\ \equiv{}& F(c,\mu,\vartheta).\end{aligned} \tag{84}$$

Differentiating $F(c,\mu,\vartheta)$ in (84) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = \frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left[\frac{1}{3}c^2(4-c^2)(36\vartheta^2+16\vartheta+4) - \frac{4\sqrt{2}}{3}\mu(4-c^2)(c^2(6\vartheta^2+4\vartheta+1) + 6(2\vartheta+1)^2c + 8(3\vartheta^2+4\vartheta+2))\right]. \tag{85}$$

It is clear from (85) that $\partial F/\partial\mu > 0$; thus, $F(c,\mu,\vartheta)$ is an increasing function of μ for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$. So, the maximum of $F(c,\mu,\vartheta)$ occurs at $\mu = 1$ and

$$\max F(c,\mu,\vartheta) = F(c,1,\vartheta) \equiv G(c,\vartheta). \tag{86}$$

Note that

$$\begin{aligned}G(c,\vartheta) ={}& \frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left[\frac{c^4}{3}\left(\left|-2\sqrt{2}(4\vartheta+1) + (12\vartheta^2+4\vartheta+1)\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(\vartheta+1)(3\vartheta+1)\right| - 2(12\vartheta^2+4\vartheta+1) - \frac{1}{2}(24\vartheta^2+16\vartheta+4) + 2\sqrt{2}(1+4\vartheta+6\vartheta^2)\right) + \frac{8}{3}c^2((12\vartheta^2+4\vartheta+1) + (12\vartheta^2+8\vartheta+2) - \sqrt{2}(12\vartheta^2+8\vartheta+2)) + \frac{64\sqrt{2}}{3}(3\vartheta+1)(\vartheta+1)\right] \\ ={}& \frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2} \cdot \left[\frac{c^4}{3}\left(\left|-2\sqrt{2}(4\vartheta+1) + \frac{9\vartheta^2}{\sqrt{2}}\right| - 4(9\vartheta^2+4\vartheta+1) + 2\sqrt{2}(1+4\vartheta+6\vartheta^2)\right) + \frac{8}{3}c^2(3(8\vartheta^2+4\vartheta+1) - \sqrt{2}(12\vartheta^2+8\vartheta+2)) + \frac{64\sqrt{2}}{3}(3\vartheta+1)(\vartheta+1)\right].\end{aligned} \tag{87}$$

Differentiating $G(c, \vartheta)$ partially with respect to c yields

$$G'(c,\vartheta)=\frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left[\frac{4c^3}{3}\left(\left|-2\sqrt{2}(4\vartheta+1)+\frac{9\vartheta^2}{\sqrt{2}}\right|-4\left(9\vartheta^2+4\vartheta+1\right)+2\sqrt{2}\left(1+4\vartheta+6\vartheta^2\right)\right)+\frac{16c}{3}\left(3\left(8\vartheta^2+4\vartheta+1\right)-\sqrt{2}\left(12\vartheta^2+8\vartheta+2\right)\right)\right]. \tag{88}$$

If $G'(c, \vartheta) = 0$ then its root is $c = 0$. Also, we have

$$G''(c,\vartheta)=\frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left[4c^2\left(\left|-2\sqrt{2}(4\vartheta+1)+\frac{9\vartheta^2}{\sqrt{2}}\right|-4\left(9\vartheta^2+4\vartheta+1+22\sqrt{2}\left(1+4\vartheta+6\vartheta^2\right)\right)\right)+\frac{16}{3}\left(3\left(8\vartheta^2+4\vartheta+1\right)-\sqrt{2}\left(12\vartheta^2+8\vartheta+2\right)\right)\right], \tag{89}$$

is negative for $c = 0$, which means that the function $G(c, \vartheta)$ can take the maximum value at $c = 0$, also which is

$$|a_2a_4-a_3^2|\le G(0,\vartheta)=\frac{1}{18(2\vartheta+1)^2}. \tag{90}$$

□

Data Availability

No data is used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] D. G. Cantor, "Power series with integral coefficients," *Bulletin of the American Mathematical Society*, vol. 69, no. 3, pp. 362–367, 1963.

[2] R. Vein and P. Dale, "Determinants and Their Applications in Mathematical Physics," in *Applied Mathematical Sciences*, vol. 134, Springer, New York, 1999.

[3] R. Wilson, "Determinantal criteria for meromorphic functions," *Proceedings of the London Mathematical Society*, vol. -s3-4, no. 1, pp. 357–374, 1954.

[4] C. Pommerenke, "On the coefficients and Hankel determinants of univalent functions," *Journal of the London Mathematical Society*, vol. s1-41, no. 1, pp. 111–122, 1966.

[5] C. Pommerenke, "On the Hankel determinants of univalent functions," *Mathematika*, vol. 14, no. 1, pp. 108–112, 1967.

[6] D. Bansal, "Upper bound of second Hankel determinant for a new class of analytic functions," *Applied Mathematics Letters*, vol. 26, no. 1, pp. 103–107, 2013.

[7] M. M. Elhosh, "On the second Hankel determinant of univalent functions," *Bull. Malays. Math. Soc.*, vol. 9, no. 1, pp. 23–25, 1986.

[8] J. W. Noonan and D. K. Thomas, "On the Hankel determinants of areally mean p-valent functions," *Proceedings of the London Mathematical Society*, vol. s3-25, no. 3, pp. 503–524, 1972.

[9] J. W. Noonan, "Coefficient differences and Hankel determinants of areally mean p-valent functions," *Proceedings of American Mathematical Society*, vol. 46, pp. 29–37, 1974.

[10] J. W. Noonan and D. K. Thomas, "On the second Hankel determinant of areally mean p-valent functions," *Transactions of the American Mathematical Society*, vol. 223, pp. 337–346, 1976.

[11] M. M. Elhosh, "On the second Hankel determinant of close-to-convex functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 9, no. 2, pp. 67-68, 1986.

[12] M. Arif, K. I. Noor, and M. Raza, "Hankel determinant problem of a subclass of analytic functions," *Journal of Inequalities and Applications*, vol. 2012, Article ID 22, 2012.

[13] T. Hayami and S. Owa, "Generalized Hankel determinant for certain classes," *International Journal of Mathematics and Analysis*, vol. 4, no. 49-52, pp. 2573–2585, 2010.

[14] T. Hayami and S. Owa, "Applications of Hankel determinant for p-valently starlike and convex functions of order a," *Far East Journal of Applied Mathematics*, vol. 46, no. 1, pp. 1–23, 2010.

[15] T. Hayami and S. Owa, "Hankel determinant for p-valently starlike and convex functions of order α," *General Math*, vol. 17, no. 4, pp. 29–44, 2009.

[16] A. Janteng, S. A. Halim, and M. Darus, "Hankel determinant for starlike and convex functions," *International Journal of Mathematics and Analysis*, vol. 1, no. 13-16, pp. 619–625, 2007.

[17] A. K. Mishra and P. Gochhayat, "Second Hankel determinant for a class of analytic functions defined by fractional derivative," *International Journal of Mathematics and Mathematical Sciences*, vol. 2008, Article ID 153280, 10 pages, 2008.

[18] N. Mohamed, D. Mohamad, and S. Cik Soh, "Second Hankel determinant for certain generalized classes of analytic functions," *International Journal of Mathematics and Analysis*, vol. 6, no. 17-20, pp. 807–812, 2012.

[19] G. Murugusundaramoorthy and N. Magesh, "Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant," *Bulletin of Mathematical Analysis and Applications*, vol. 1, no. 3, pp. 85–89, 2009.

[20] K. I. Noor, "Higher order close-to-convex functions," *Japanese Journal of Mathematics*, vol. 37, no. 1, pp. 1–8, 1992.

[21] K. I. Noor, "On the Hankel determinant problem for strongly close-to-convex functions," *Journal of Natural Geometry*, vol. 11, no. 1, pp. 29–34, 1997.

[22] K. I. Noor, "On certain analytic functions related with strongly close-to-convex functions," *Applied Mathematics and Computation*, vol. 197, no. 1, pp. 149–157, 2008.

[23] K. I. Noor and S. A. Al-Bany, "On Bazilevic functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 10, no. 1, 88 pages, 1987.

[24] K. I. Noor, "On analytic functions related with functions of bounded boundary rotation," *Commentarii Mathematici Universitatis Sancti Pauli*, vol. 30, no. 2, pp. 113–118, 1981.

[25] K. I. Noor, "On meromorphic functions of bounded boundary rotation," *Caribb. J. Math.*, vol. 1, no. 3, pp. 95–103, 1982.

[26] K. I. Noor, "Hankel determinant problem for the class of functions with bounded boundary rotation," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 28, no. 8, pp. 731–739, 1983.

[27] K. I. Noor and I. M. A. Al-Naggar, "On the Hankel determinant problem," *Journal of Natural Geometry*, vol. 14, no. 2, pp. 133–140, 1998.

[28] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, "The Fekete-Szegö coefficient functional for transforms of analytic functions," *Bulletin of the Iranian Mathematical Society*, vol. 35, no. 2, pp. 119–142, 2009.

[29] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Coefficient bounds for _p_ -valent functions," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 35–46, 2007.

[30] W. C. Ma and D. Minda, "A united treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, Tianjin, 1992.

[31] R. K. Raina and J. Sokół, "On coefficient estimates for a certain class of starlike functions, Hacet," *Journal of Mathematics and Statistics*, vol. 44, no. 6, pp. 1427–1433, 2015.

[32] G. Murugusundaramoorthy and T. Bulboacă, "Hankel determinants for new subclasses of analytic functions related to a shell shaped region," *Mathematics*, vol. 8, no. 6, article 1041, 2020.

[33] S. Gandhi, "Radius Estimates for Three Leaf Function and Convex Combination of Starlike Functions," in *Mathematical Analysis I: Approximation Theory. ICRAPAM 2018*, vol. 306 of Springer Proceedings in Mathematics & Statistics, Springer, Singapore.

[34] S. N. Malik, S. Mahmood, M. Raza, S. Farman, and S. Zainab, "Coefficient inequalities of functions associated with petal type domains," *Mathematics*, vol. 6, no. 12, p. 298, 2018.

[35] K. Sharma, N. K. Jain, and V. Ravichandran, "Starlike functions associated with a cardioid," *Afrika Matematika*, vol. 27, pp. 923–939, 2016.

[36] V. S. Masih and S. Kanas, "Subclasses of starlike and convex functions associated with the Limaçon domain," *Symmetry*, vol. 12, no. 6, p. 942, 2020.

[37] Y. Yuzaimi, A. H. Suzeini, and A. B. Akbarally, "Subclass of starlike functions associated with a limacon," *AIP Conference Proceedings*, vol. 1974, article 030023, 2018.

[38] P. L. Duren, *Univalent Functions. Grundlehren der Mathematischen Wissenschaften*, vol. 259, Springer, New York, 1983.

[39] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications. California Monographs in Mathematical Sciences*, University of California Press, Berkeley, 1958.

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Research Article

Inequalities on Generalized Sasakian Space Forms

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In this paper, we find the second variational formula for a generalized Sasakian space form admitting a semisymmetric metric connection. Inequalities regarding the stability criteria of a compact generalized Sasakian space form admitting a semisymmetric metric connection are established.

1. Introduction

The harmonic maps have aspects from both Riemannian's geometry and analysis. Harmonic mappings are considered a vast field, and because of the minimization of energy due to its dual nature, it has many applications in the field of mathematics, physics, relativity, engineering, geometry, crystal liquid, surface matching, and animation. Some particular examples of harmonic maps are geodesics, immersion, and solution of the Laplace equation. In physics, p-harmonic maps were studied in image processing. Exponential harmonic maps were discussed in the field of gravity. Due to generalized properties, F-harmonic maps have many applications in cosmology. Harmonic maps have played a significant role in Finsler's geometry. On complex manifolds, we have interesting and useful outcomes of harmonic maps (for details, see [1, 2]).

During the past years, harmonicity on almost contact metric manifolds has been considered a parallel to complex manifolds ([3–5]). The identity map on a Riemannian manifold with a compact domain becomes a trivial case of the harmonicity. However, the stability and second variation theory are complex and remarkable here. In [6], a Laplacian upon functions with its first eigenvalue is used to explain stability on Einstein's manifolds. From [7, 8], we know about the stability-based classification of a Riemannian that simply connected irreducible spaces with a compact domain.

From [6], we know a well-known result about the stability of S^{2n+1}. Further in [5], identity map stability upon a compact domain of the Sasakian space form was explained by Gherge et al. (see also [9]). Considering the generalization of Sasakian space forms, Alegre et al. presented the generalized Sasakian space forms [10]. Therefore, we naturally study the identity map stability upon a compact domain of generalized Sasakian space forms, as discussed in some results in [11]. One of the most important terms in differential geometry is connection. Research on manifolds is incomplete without the notion of connection. In manifold theory, from the relation of metric and connection, we have a very important notion known as curvature tensor. The concept of a semisymmetric metric connection was initiated by Friedmann and Schouten in 1932 [12, 13]. Semisymmetric metric connections have many applications in the field of Riemannian manifolds and are useful to study many physical problems. In the current paper, we compute the stability criteria of a generalized Sasakian space form admitting a semisymmetric metric connection.

After recollecting the essential facts about harmonic maps between Riemannian manifolds in Section 2, we explain generalized Sasakian space forms throughout Section 3. In Section 4, we give the main results for a second variational formula and establish the inequalities for the identity map stability criteria upon a compact domain generalized Sasakian space form admitting a semisymmetric metric connection.

2. Harmonic Maps on Riemannian Manifolds

We can view harmonic maps on Riemannian manifolds as the generalization of geodesics that is the case of a one-dimensional domain and range as Euclidean space. In common, a map is known as harmonic if its Laplacian becomes zero and is known as totally geodesic if its Hessian becomes zero. In this present section, the basic facts of the harmonic maps theory [14, 15] are provided. Consider a smooth map $\psi : (S, g) \longrightarrow (Q, h)$. Let the dimension of the Riemannian manifold (S, g) be s and the dimension of (Q, h) be q. The function $e(\psi)$: $S \longrightarrow [0,\infty)$ that is smooth can be considered as the energy density of ψ and is expressed as

$$e(\psi)_p = \frac{1}{2} Tr_g(\psi^* h)(p) = \frac{1}{2}\sum_{i=1}^{s} h\left(\psi_{*p}u_i, \psi_{*p}u_i\right), \quad (1)$$

at a point $p \in S$ and for any orthonormal basis $\{u_1, \cdots, u_s\}$ of T_pS. Considering the compact domain of a Riemannian manifold S, we take the energy density integral as the energy $E(\psi)$ of ψ; that is, we have

$$E(\psi) = \int_S e(\psi) v_g, \quad (2)$$

where the volume measure is represented by v_g that is related to the metric g on manifold S. In the set $C^\infty(S, Q)$ of all smooth maps from (S, g) to (Q, h), a critical point of the energy E is named as a harmonic map. That is, for any smooth variation $\psi_t \in C^\infty(S, Q)$ of $\psi(t \in (-\varepsilon, \varepsilon))$ with $\psi_0 = \psi$, we can take

$$\frac{d}{dt} E(\psi_t)\bigg|_{t=0} = 0. \quad (3)$$

Now, we consider (S, g) as a compact Riemannian manifold and take a map $\psi : (S, g) \longrightarrow (Q, h)$ that is harmonic. We consider smooth variation $\psi_{r,t}$ through constraints $r, t \in (-\varepsilon, \varepsilon)$ satisfying $\psi_{0,0} = \psi$. Respective variational vector fields are represented through W and Z. Therefore, we can define Hessian H_ψ for a harmonic map ψ through the following relation:

$$H_\psi(W, Z) = \frac{\partial^2}{\partial r \partial t}\left(E\left(\psi_{r,t}\right)\right)\bigg|_{(r,t)=(0,0)}. \quad (4)$$

The expression regarding the second variation of E is as follows ([6, 16]):

$$H_\psi(W, Z) = \int_P h\left(J_\psi(W), Z\right) v_g, \quad (5)$$

where J_ψ is the second order operator that is self-adjoint upon the space $\Gamma(\psi^{-1}(TQ))$ of variation vector fields and is represented as

$$J_\psi(U) = -\sum_{i=1}^{s}\left(\nabla^{\sim}{}_{u_i}\nabla^{\sim}{}_{u_i} - \nabla^{\sim}{}_{\nabla_{u_i}u_i}\right)U - \sum_{i=1}^{s} R^Q(U, d\psi(u_i))d\psi(u_i), \quad (6)$$

for $U \in \Gamma(\psi^{-1}(TQ))$ and any local orthonormal frame $\{u_1, \cdots, u_s\}$ on S. Here, R^Q shows the curvature tensor of (Q, h), and $\tilde{\nabla}$ illustrates the pull-back connection of ψ along with the Levi-Civita connection of Q.

We compute the dimension of the biggest subspace of $\Gamma(\psi^{-1}(TQ))$ where the Hessian H_ψ has values that are negative definite known as the index of a harmonic map $\psi : (S, g) \longrightarrow (Q, h)$. Therefore, if the index of harmonic map ψ is zero, then it is stable; otherwise, it is unstable.

An operator $\bar{\Delta}_\psi$ is represented by

$$\bar{\Delta}_\psi U = -\sum_{i=1}^{s}\left(\nabla^{\sim}{}_{u_i}\nabla^{\sim}{}_{u_i} - \nabla^{\sim}{}_{\nabla_{u_i}u_i}\right)U, \qquad V \in \Gamma\left(\phi^{-1}(TQ)\right). \quad (7)$$

It is named the rough Laplacian. We consider the spectra of J_ψ; because of the Hodge de Rham Kodaira theory, this spectra is constructed as a discrete set of infinite number of eigenvalues with finite multiplicities with no accumulation points.

3. Generalized Sasakian Space Forms

Generalized Sasakian space forms have the generalized curvature expression that combines the curvature expessions of Sasakian, Kenmotsu, and Cosymplectic space forms. Due to a generalized curvature expression, generalized Sasakian space forms have very useful and interesting properties. The current unit presents basics of almost contact metric manifolds particularly of generalized Sasakian space forms [17].

A Riemannian manifold P^{2n+1} with odd dimensions is known as an almost contact manifold if a $(1, 1)$-tensor field φ exists on P and ξ and a vector field η and a 1-form exist so that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{8}$$

Further, φ and η satisfy $\varphi(\xi) = 0$ and $\eta o \varphi = 0$. A compatible metric g on any almost contact manifold is defined as

$$g(\varphi W_1, \varphi W_2) = g(W_1, W_2) - \eta(W_1)\eta(W_2), \tag{9}$$

for any vector fields W_1, W_2 on manifold P known as an almost contact metric manifold. An almost contact metric manifold becomes a contact metric manifold if for a fundamental 2-form Ω, we have $d\eta = \Omega$, and $\Omega(W_1, W_2) = g(W_1, \varphi W_2)$ for $W_1, W_2 \in \Gamma(TP)$. Like the parallel condition of integrability for almost complex manifolds, the almost contact metric structure on P becomes normal when

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0. \tag{10}$$

The Nijenhuis torsion of φ is represented by $[\varphi, \varphi]$ and is defined as

$$\begin{aligned} [\varphi, \varphi](Y_1, Y_2) = \varphi^2[Y_1, Y_2] + [\varphi Y_1, \varphi Y_2] \\ - \varphi[\varphi Y_1, Y_2] - \varphi[Y_1, \varphi Y_2]. \end{aligned} \tag{11}$$

A Sasakian manifold is a normal contact metric manifold, and if $d\eta = 0$, a normal almost contact metric manifold is known as the Kenmotsu manifold with

$$d\Omega(Y_1, Y_2, Y_3) = \frac{2}{3}\sigma_{(Y_1,Y_2,Y_3)}\{\eta(Y_1)\phi(Y_2, Y_3)\}, \quad Y_1, Y_2, Y_3 \in \Gamma(TP), \tag{12}$$

where the cyclic sum is represented by σ. A real space form is a Riemannian manifold with a constant sectional curvature c, and its curvature tensor is represented by the following relation:

$$R(Y_1, Y_2)Y_3 = c\{g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2\}, \tag{13}$$

where Y_1, Y_2, and Y_3 are vector fields on P. An almost contact metric manifold $P(\varphi, \xi, \eta, g)$ can be identified as a generalized Sasakian space form provided that there are three functions f_1, f_2, f_3 upon P so as the curvature tensor on P is represented with the following relation:

$$\begin{aligned} R(V_1, V_2)V_3 = f_1\{g(V_2, V_3)V_1 - g(V_1, V_3)V_2\} \\ + f_2\{g(V_1, \phi V_3)\phi V_2 - g(V_2, \phi V_3)\phi V_1 \\ + 2g(V_1, \phi V_2)\phi V_3\} + f_3\{\eta(V_1)\eta(V_3)V_2 \\ - \eta(V_2)\eta(V_3)V_1 + g(V_1, V_3)\eta(V_2)\xi \\ - g(V_2, V_3)\eta(V_1)\xi\}, \end{aligned} \tag{14}$$

provided that vector fields V_1, V_2, and V_3 are on P, see [10]. In particular, if $f_1 = (c+3)/4$ and $f_2 = f_3 = (c-1)/4$, then P can be identified as a Sasakian space form. $f_1 = (c-3)/4$ and $f_2 = f_3 = (c+1)/4$ can lead to a Kenmotsu-space form [10, 18].

The semisymmetric metric connection ∇' and the Levi Civita connection ∇ defined on contact metric manifold (P^{2m+1}, g) are related by the following expression that is obtained by Yano [19] and is represented as

$$\nabla'_{W_1} W_2 = \nabla_{W_1} W_2 + \eta(W_2)W_1 - g(W_1, W_2)\xi, \tag{15}$$

where W_1 and W_2 are vector fields on P. As mentioned in [20], we have the following relation of the curvature tensor R with respect to the Levi-Civita connection ∇ and the curvature tensor R' regarding the semisymmetric metric connection ∇' of the generalized Sasakian space form.

$$\begin{aligned} R'(V_1, V_2)V_3 = R(V_1, V_2)V_3 \\ + \{g(\phi V_2, V_3)V_1 - g(\phi V_1, V_3)V_2 \\ + g(V_2, V_3)\phi V_1 - g(V_1, V_3)\phi V_2\} \\ + \{\eta(V_2)V_1 - \eta(V_1)V_2\}\eta(V_3) \\ + \{g(V_2, V_3)\eta(V_1) - g(V_1, V_3)\eta(V_2)\}\xi, \end{aligned} \tag{16}$$

taking vector fields V_1, V_2, V_3, on P.

4. Stability on Generalized Sasakian Space Forms with Semisymmetric Metric Connection

Identity maps are always harmonic maps, but here, the second variational formula is not a trivial case. In this section, with the help of the second variational formula, we derive the inequalities for the stability criteria on the generalized Sasakian space forms with a semisymmetric metric connection. Consider the identity map on a compact generalized Sasakian space form $M(\varphi, \xi, \eta, g)$ that is ($\phi = 1_M$). Then, the second variation formula is ([2]) as follows:

$$H_{1_M}(V, V) = \int_M h(\bar{\Delta} V, V)v_g - \sum_{i=1}^{2n+1}\int_M h(R(V, u_i)u_i, V)v_g, \tag{17}$$

where $V \in \Gamma(TM)$ and $\{u_1, \cdots, u_{2n+1}\}$ represents the local orthonormal frame on TM.

The rough Laplacian defined by (7) upon a generalized Sasakian manifold M^{2n+1} admitting a semisymmetric metric connection can be computed by the following lemma.

Lemma 1. *For a generalized Sasakian space form admitting semisymmetric metric connection, the rough Laplacian in* *the adopted frame field* $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi\}$ *is given by*

$$\acute{\bar{\Delta}} Y = \bar{\Delta} Y + 2trB_Y - g(tr\nabla, Y)\xi - (2 \operatorname{div} Y)\xi + 2\eta(Y)\xi - 2Y + \phi Y + \sum g(e_i, Y)\phi e_i + \sum g(e_i, \phi Y)e_i, \tag{18}$$

where $B_Y(V, W) = \eta(\nabla_V Y)W$.

Proof. Let ∇ and ∇ represent the semisymmetric connection and the Levi Civita connection on the generalized Sasakian space form, respectively. Therefore, it can be computed as

$$\begin{aligned}\acute{\nabla}_V \acute{\nabla}_V Y &= \nabla_V \acute{\nabla}_V Y + \eta\left(\acute{\nabla}_V Y\right)V - g\left(V, \acute{\nabla}_V Y\right)\xi = \nabla_V \nabla_V Y + \nabla_V(\eta(Y)V) \\ &\quad - \nabla_V(g(V, Y)\xi) + \eta(\nabla_V Y)V + \eta(V)\eta(Y)V - g(V, Y)V \\ &\quad - g(Y, \nabla_V V)\xi - g(V, \nabla_V Y)\xi - g(V, Y)\nabla_V \xi.\end{aligned} \tag{19}$$

We have $\nabla_V(\eta(Y)V) = \nabla_V(g(\xi, Y)V)$. Then, from equation (19), we have

$$\begin{aligned}\acute{\nabla}_V \acute{\nabla}_V Y &= \nabla_V \nabla_V Y + g(\nabla_V \xi, Y)V - \nabla_V(g(V, Y)\xi) + \eta(Y)\nabla_V V \\ &\quad + 2\eta(\nabla_V Y)V + \eta(V)\eta(Y)V - g(V, Y)V - g(\nabla_V V, Y)\xi \\ &\quad - g(V, \nabla_V Y)\xi - g(V, Y)\nabla_V \xi = \nabla_V \nabla_V Y + g(\nabla_V \xi, Y)V \\ &\quad + \eta(Y)\nabla_V V + 2\eta(\nabla_V Y)V + \eta(V)\eta(Y)V - g(V, Y)V \\ &\quad - 2g(\nabla_V V, Y)\xi - 2g(V, \nabla_V Y)\xi - 2g(V, Y)\nabla_V \xi \\ &= \nabla_V \nabla_V Y + g(V, \phi Y)V + \eta(Y)\nabla_V V + 2\eta(\nabla_V Y)V \\ &\quad - 2g(\nabla_V V, Y)\xi - 2g(V, \nabla_V Y)\xi - 2g(V, Y)\nabla_V \xi \\ &= \nabla_V \nabla_V Y + g(V, \phi Y)V + \eta(Y)\nabla_V V + 2\eta(\nabla_V Y)V \\ &\quad - 2g(\nabla_V V, Y)\xi - 2g(V, \nabla_V Y)\xi - 2g(V, Y)\nabla_V \xi \\ &= \nabla_V \nabla_V Y + g(V, \phi Y)V + \eta(Y)\nabla_V V + 2\eta(\nabla_V Y)V \\ &\quad - 2g(\nabla_V V, Y)\xi - 2g(V, \nabla_V Y)\xi + 2g(V, Y)\phi V \\ &\quad - 2g(V, Y)V + 2\eta(V)g(V, Y)\xi.\end{aligned} \tag{20}$$

Also, we have

$$\begin{aligned}\acute{\nabla}_V \acute{\nabla}_V Y - \acute{\nabla}_{\nabla_V V} Y &= \nabla_V \nabla_V Y - \nabla_{\nabla_V V} Y + g(V, \phi Y)V \\ &\quad + 2\eta(\nabla_V Y)V - g(\nabla_V V, Y)\xi - 2g(V, \nabla_V Y)\xi \\ &\quad + 2g(V, Y)\phi V - 2g(V, Y)V + 2\eta(V)g(V, Y)\xi.\end{aligned} \tag{21}$$

Take into account that $B_Y(V, W) = \eta(\nabla_V Y)W$. Then, in an adopted frame field $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi\}$, we arrived at

$$\acute{\bar{\Delta}} Y = \bar{\Delta} Y + 2trB_Y - g(tr\nabla, Y)\xi - (2divY)\xi + 2\eta(Y)\xi - 2Y + \phi Y + \sum g(e_i, Y)\phi e_i + \sum g(e_i, \phi Y)e_i. \tag{22}$$

□

Theorem 2. *The second variation formula for the identity map on the generalized Sasakian space form admitting a semisymmetric connection is expressed as*

$$\begin{aligned}H_{1_M}(Y, Y) &= \int_M h(\bar{\Delta} Y, Y)v_g - (3f_2 + 2nf_1 - f_3 - 2n + 3)\int_M h(Y, Y)v_g \\ &\quad + (3f_2 + (2n-1)f_3 - 2n + 3)\int_M \eta(Y)\eta(Y)v_g.\end{aligned} \tag{23}$$

Proof.

$$H_{1_M}(Y, Y) = \int_M h\left(\bar{\Delta}' Y, Y\right)v_g - \sum_{i=1}^{2n+1}\int_M h\left(R'(Y, u_i)u_i, Y\right)v_g,$$

$$\begin{aligned}h\left(\acute{\Delta} Y, Y\right) &= h\left(\bar{\Delta} Y, Y\right) + 2h\left(tr\bar{B}_Y, Y\right) - h(tr\nabla, Y)h(\xi, Y) \\ &\quad - (2divY)h(\xi, Y) + 2\eta(Y)h(\xi, Y) - 2h(Y, Y) + h(\phi Y, Y) \\ &\quad + \sum h(e_i, Y)h(\phi e_i, Y) + \sum h(e_i, \phi Y)h(e_i, Y), \left(\bar{\Delta}' Y, Y\right) \\ &= h\left(\bar{\Delta} Y, Y\right) + 2h(trB_Y, Y) - h(tr\nabla, Y)h(\xi, Y) \\ &\quad - (2divY)h(\xi, Y) + 2\eta(Y)h(\xi, Y) - 2h(Y, Y) + h(\phi Y, Y) \\ &\quad + \sum h(e_i, Y)h(\phi e_i, Y) + \sum h(e_i, \phi Y)h(e_i, Y),\end{aligned} \tag{24}$$

since $\int_M \operatorname{div}(Y) = 0$, over a compact domain M, by Green's formula and $\eta(\nabla_{e_i} Y) = h(\nabla_{e_i} Y, \xi) = e_i h(Y, \xi) - h(Y, \nabla_{e_i}\xi) = 0$, similarly, $h(tr\nabla, Y)h(\xi, Y) = 0$. Therefore, we have

$$\int_M h\left(\bar{\Delta}' Y, Y\right)v_g = \int_M h(\bar{\Delta} Y, Y)v_g + 2\int_M \eta^2(Y)v_g - 2\int_M h(Y, Y)v_g. \tag{25}$$

Now, we consider a φ-adapted orthonormal local frame $\{e_i, \varphi e_i, \xi\}$. After that, we have

$$\begin{aligned}\sum_{i=1}^{2n+1} h(R(e_i, Y)e_i, Y) &= (f_1 - 3f_2)\sum_{i=1}^{n}\left\{h(Y, e_i)^2 + h(Y, \varphi e_i)^2\right\} \\ &\quad - [(2n+1)f_1 - f_3]h(Y, Y) \\ &\quad + [(2n-1)f_3 + f_1]h(Y, \xi)^2,\end{aligned} \tag{26}$$

and thus, we have

$$\begin{aligned}\sum_{i=1}^{2n+1} h(R(e_i, Y)e_i, Y) &= -[3f_2 + 2nf_1 - f_3]h(Y, Y) \\ &\quad + [3f_2 + (2n-1)f_3]h(Y, \xi)^2,\end{aligned} \tag{27}$$

and with semisymmetric metric connection, it can be written as

$$\sum_{i=1}^{2n+1} h\left(R'(Y,e_i)e_i,Y\right) = \sum_{i=1}^{2n+1} h(R(Y,e_i)e_i,Y) - (2n-1)h(Y,Y) + (2n-1)\eta^2(Y) = [3f_2+2nf_1-f_3]h(Y,Y) - [3f_2+(2n-1)f_3]h(Y,\xi)^2 - (2n-1)h(Y,Y) + (2n-1)\eta^2(Y). \tag{28}$$

From (24) and (28), we have acquired the result of ((24)). □

Proposition 3. *Consider a compact generalized Sasakian space form M admitting a semisymmetric metric connection. The identity map 1_M is weakly stable, if $(3f_2+2nf_1-f_3-2n+3)\le 0$ and $(3f_2+(2n-1)f_3-2n+3)\ge 0$.*

Proof. We can easily prove that

$$\int_M h\left(\bar{\Delta}V,V\right)v_g = \int_M h\left(\nabla^{\sim}V,\nabla^{\sim}V\right)v_g, \quad V\in\Gamma(TM). \tag{29}$$

□ □

Now, the second variation formula with respect to a semisymmetric connection becomes

$$H_{1_M}(Y,Y) = \int_M h\left(\nabla^{\sim}Y,\nabla^{\sim}Y\right) - (3f_2+2nf_1-f_3-2n+3)\int_M h(Y,Y)v_g + (3f_2+(2n-1)f_3-2n+3)\int_M \eta(Y)\eta(Y)v_g. \tag{30}$$

Therefore, for the inequalities $(3f_2+2nf_1-f_3-2n+3)\le 0$ and $(3f_2+(2n-1)f_3-2n+3)\ge 0$, the identity map is weakly stable.

Corollary 4. *Let M be the Kenmotsu space form admitting a semisymmetric metric connection; then, the identity map on its compact domain is stable if $(3n-7/n+1)\le c\le((7(n-1))/(n+1))$.*

On the Kenmotsu space form M, $f_1=((c-3)/4)$, $f_2=f_3=((c+1)/4)$ [10]. And $(3f_2+2nf_1-f_3-2n+3)\le 0$ implies $c\le((7(n-1))/(n+1))$, and $(3f_2+(2n-1)f_3-2n+3)\ge 0$ implies $c\ge((3n-7)/(n+1))$. Then, by the above results, the identity of the 1_M map becomes stable for the values of $c\in[((3n-7)/(n+1)),((7(n-1))/(n+1))]$.

5. Conclusion

The 2nd variational formula for a generalized Sasakian space form admitting a semisymmetric metric connection has been successfully obtained in this work. All results in this work are novel where inequalities concerning the stability criteria of a compact generalized Sasakian space form admitting a semisymmetric metric connection have been established. Further research works can be conducted depending on all our obtained results in this paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

References

[1] A. Lichnerowichz, "Aplications harmoniques et varietes kahleriennes," in *Conferenze di Matematica*, pp. 341–402, 1969.

[2] H. Urakawa, *Calculus of Variation and Harmonic Maps*, American Mathematical Society, Providence. R.I, 1993.

[3] E. Boeckx and C. Gherghe, "Harmonic maps and cosymplectic manifolds," *Journal of the Australian Mathematical Society*, vol. 76, no. 1, pp. 75–92, 2004.

[4] D. Chinea, "Harmonicity on maps between almost contact metric manifolds," *Acta Mathematica Hungarica*, vol. 126, no. 4, pp. 352–365, 2010.

[5] C. Gherghe, S. Ianus, and A. M. Pastore, "CR manifolds, harmonic maps and stability," *Journal of Geometry*, vol. 71, no. 1, pp. 42–53, 2001.

[6] R. Smith, "The second variation formula for harmonic mappings," *Proceedings of the American Mathematical Society*, vol. 47, no. 1, pp. 229–236, 1975.

[7] Y. Ohnita, "Stability of harmonic maps and standard minimal immersions," *Tohoku Mathematical Journal*, vol. 38, pp. 259–267, 1986.

[8] H. Urakawa, "The first eigenvalue of the Laplacian for a positively curved homogeneous Riemannian manifold," *Compositio Mathematica*, vol. 59, pp. 57–71, 1986.

[9] C. Gherghe, S. Ianus, and A. M. Pastore, "Harmonic maps, harmonic morphism and stability," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 43, pp. 247–254, 2000.

[10] P. Alegre, D. E. Blair, and A. Carriazo, "Generalized Sasakian-space-forms," *Israel Journal of Mathematics*, vol. 141, no. 1, pp. 157–183, 2004.

[11] N. A. Rehman, "Stability on generalized Sasakian space forms," *Math Reports*, vol. 17, no. 67, pp. 57–64, 2015.

[12] A. Friedmann and J. A. Schouten, "Über die Geometrie der halbsymmetrischen Übertragungen," *Mathematische Zeitschrift*, vol. 21, no. 1, pp. 211–223, 1924.

[13] H. A. Hayden, "Sub-Spaces of a Space with Torsion," *Proceedings of the London Mathematical Society*, vol. s2-34, no. 1, pp. 27–50, 1932.

[14] J. Eells and J. H. Simpson, "Harmonic mappings of Riemannian manifolds," *American Journal of Mathematics*, vol. 86, no. 1, pp. 109–160, 1964.

[15] J. Eells and L. Lemaire, "A report on harmonic maps," *Bulletin of the London Mathematical Society*, vol. 20, pp. 385–524, 1988.

[16] E. Mazet, "La formule de la variation seconde de l' energie au voisinage d' une application harmonique," *Journal of Differential Geometry*, vol. 9, pp. 531–535, 1974.

[17] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, vol. 203 of Progress in Mathematics, Birkhauser, Boston, MA, second edition edition, 2010.

[18] P. K. Pandey, R. S. Gupta, and A. Sharfuddin, "B.Y. Chen's inequalities for bi-slant submanifolds in Kenmotsu space form," *Demonstratio Mathematica*, vol. 43, no. 4, pp. 887–898, 2010.

[19] K. Yano, "On semi symmetric metric connection," *Revue Roumaine de Mathématique Pures et Appliquées*, vol. 15, pp. 1570–1586, 1970.

[20] A. Singh and S. Kishor, "On a semi symmetric metric connection in generalized Sasakian space forms," *Global Journal of Pure and Applied Mathematics*, vol. 9, pp. 6407–6419, 2017.

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Research Article

A New Class of Analytic Normalized Functions Structured by a Fractional Differential Operator

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Newly, the field of fractional differential operators has engaged with many other fields in science, technology, and engineering studies. The class of fractional differential and integral operators is considered for a real variable. In this work, we have investigated the most applicable fractional differential operator called the Prabhakar fractional differential operator into a complex domain. We express the operator in observation of a class of normalized analytic functions. We deal with its geometric performance in the open unit disk.

1. Introduction

The class of complex fractional operators (differential and integral) is investigated geometrically by Srivastava et al. [1] and generalized into two-dimensional fractional parameters by Ibrahim for a class of analytic functions in the open unit disk [2]. These operators are consumed to express different classes of analytic functions, fractional analytic functions [3] and differential equations of a complex variable, which are called fractional algebraic differential equations studding the Ulam stability [4, 5].

We carry on our investigation in the field of complex fractional differential operators. In this investigation, we formulate an arrangement of the fractional differential operator in the open unit disk refining the well-known Prabhakar fractional differential operator. We apply the recommended operator to describe new generalized classes of fractional analytic functions including the Briot-Bouquet types. Consequently, we study the classes in terms of the geometric function theory.

2. Methods

Our methods are divided into two subsections, as follows.

2.1. Geometric Methods. In this place, we clarify selected notions in the geometric function theory, which are situated in [6–8].

Definition 1. Let $\cup := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Two analytic functions g_1, g_2 in $\cup$ are called subordinated denoting by $g_1 \prec g_2$ or $g_1(z) \prec g_2(z), z \in \mathbb{U}$, if there exits an analytic function $\omega, |w| \le |z| < 1$ having the formula

$$g_1(z) = g_2(w(z)), \quad z \in \cup. \tag{1}$$

g_1 is majorized by g_2 denoting by $g_1 \ll g_2$ if and only if

$$g_1(z) = w(z) g_2(z), \quad z \in \cup; \tag{2}$$

equivalently, the coefficient inequality is held $|a_n| \le |b_n|$, respectively.

There is a deep construction between subordination and majorization [9] in $\cup$ for selected distinct classes comprising the convex class ($\mathscr{C}$):

$$1+\Re\left(\frac{zg''(z)}{g'(z)}\right)>0,\quad z\in\cup,\tag{3}$$

and starlike functions $(\mathcal{S}^*)$

$$\Re\left(\frac{zg'}{g(z)}\right)>0,\quad z\in\cup.\tag{4}$$

Definition 2. We present a class of analytic functions by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\quad z\in\cup.\tag{5}$$

This class is denoted by Λ and known as the class of univalent functions which is normalized by $f(0)=f'(0)-1=0$.

Associated with the terms $\mathcal{S}^*$ and $\mathcal{C}$, we present the term $\mathcal{P}$ of all analytic functions p in $\cup$ with a positive real part in $\cup$ and $p(0)=1$.

Two analytic functions f, g are called convoluted, denoting by $f*g$ if and only if

$$(f*g)(z)=\left(\sum_{n=0}^{\infty}a_nz^n\right)*\left(\sum_{n=0}^{\infty}g_nz^n\right)=\sum_{n-0}^{\infty}a_ng_nz^n.\tag{6}$$

Definition 3. The generalized Mittag-Leffler function is defined by [10–12]

$$\Xi_{\nu,\mu}^{\wp}(z)=\sum_{n=0}^{\infty}\frac{(\wp)_n}{\Gamma(\nu n+\mu)}\frac{z^n}{n!},\tag{7}$$

where $(\vartheta)_n$ represents the Pochhammer symbol and

$$\Xi_{\nu,\mu}^{1}(z):=\Xi_{\nu,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\nu n+\mu)}\cdot\left((\wp)_0=1,(\wp)_n=\wp(\wp+1)\cdots(\wp+n-1)\right).\tag{8}$$

Note that $\Xi_{\nu,u}^{\wp}(z)$ is an ultimate traditional generalization of the function e^z, where $\Xi_{1,1}^{1}(z)=e^z$.

Moreover, it can be formulated by the Fox-Write hypergeometric function, as follows:

$$\Xi_{\nu,\mu}^{\wp}(z)=\left(\frac{1}{\Gamma(\wp)}\right)_1\Psi_1\left[\begin{matrix}(\wp,1)\\ \nu,\mu\end{matrix};z\right].\tag{9}$$

2.2. Complex Prabhakar Operator (CPO). The Prabhakar integral operator is defined for analytic function

$$\psi(z)\in\mathcal{H}[0,1]=\left\{\psi(z)=\psi_1z+\psi_2z^2+\cdots,z\in\cup\right\}\tag{10}$$

by the formula [13, 14]

$$P_{\alpha,\beta}^{\gamma,\omega}\psi(z)=\int_0^z(z-\zeta)^{\beta-1}\Xi_{\alpha,\beta}^{\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\psi(\zeta)d\zeta=\left(\psi\cdot\varrho_{\alpha,\beta}^{\gamma,\omega}\right)(z)\ (\alpha,\beta,\gamma,\omega\in\mathbb{C},z\in\cup).\tag{11}$$

Moreover [13, 14],

$$\varrho_{\alpha,\beta}^{\gamma,\omega}(z)\coloneqq z^{\beta-1}\Xi_{\alpha,\beta}^{\gamma}(\omega z^{\alpha}),\quad \Xi_{\alpha,\beta}^{\gamma}(\chi)=\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)\Gamma(\alpha n+\beta)}\frac{\chi^n}{n!}.\tag{12}$$

For example, let $\psi(z)=z^{\varsigma-1}$, then (see [15], Corollary 2.3)

$$P_{\alpha,\beta}^{\gamma,\omega}z^{\varsigma-1}=\int_0^z(z-\zeta)^{\beta-1}\Xi_{\alpha,\beta}^{\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\zeta^{\varsigma-1}\right)d\zeta=\Gamma(\varsigma)z^{\beta+\varsigma-1}\Xi_{\alpha,\beta+\varsigma}^{\gamma}(\omega z^{\alpha}).\tag{13}$$

The Prabhakar derivative can be computed by the formula [13]

$$_kD_{\alpha,\beta}^{\gamma,\omega}f(\chi)=\frac{d^k}{d\chi^k}\left(P_{\alpha,k-\beta}^{-\gamma,\omega}f(\chi)\right).\tag{14}$$

Definition 4. Let $\psi\in\Lambda$. Then the complex Prabhakar differential operator (CPFDO) of (13) is formulated in terms of the Riemann-Liouville derivative, as follows:

$$^{\mathcal{R}}_{k}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\psi(z)=\frac{d^k}{dz^k}\int_0^z(z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\psi(\zeta)d\zeta=\frac{d^k}{dz^k}\left(P_{\alpha,k-\beta}^{-\gamma,\omega}\psi(z)\right),\tag{15}$$

and in terms of the Caputo derivative, as follows:

$$^{\mathcal{C}}_{k}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\psi(z)=\int_0^z(z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\frac{d^k}{d\zeta^k}\psi(\zeta)\right)d\zeta=P_{\alpha,k-\beta}^{-\gamma,\omega}\left(\frac{d^k}{dz^k}\psi(z)\right).\tag{16}$$

Note that

$$^{\mathcal{C}}_{k}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\psi(z)={}^{\mathcal{R}}_{k}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\psi(z)-\sum_{m=0}^{k-1}z^{m-\beta}\Xi_{\alpha,m-\beta}^{-\gamma}[\omega z^{\alpha}]\psi^{(m)}(0).\tag{17}$$

For example, let $\psi(z)=z^{\varepsilon}, \varepsilon \geq 1$, then in virtue of [15] (Corollary 2.3), we conclude that

$$
\begin{aligned}
{}_{1}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}(z^{\varepsilon}) &= \int_{0}^{z}(z-\zeta)^{1-\beta-1}\Xi_{\alpha,1-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\frac{d}{d\zeta}\psi(\zeta)\right)d\zeta: \\
&= \int_{0}^{Z}(z-\zeta)^{\mu-1}\Xi_{\alpha,\mu}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\frac{d}{d\zeta}(\zeta^{\varepsilon})\right)d\zeta \\
&= \varepsilon\int_{0}^{z}\zeta^{\varepsilon-1}(z-\zeta)^{\mu-1}\Xi_{\alpha,\mu}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]d\zeta \\
&= \Gamma(\varepsilon+1)z^{\mu+\varepsilon-1}\Xi_{\alpha,\mu+\varepsilon}^{-\gamma}[\omega z^{\alpha}], \quad \mu:=1-\beta.
\end{aligned} \tag{18}
$$

In general, we have

$$
\begin{aligned}
{}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}(z^{\varepsilon}) &= \int_{0}^{z}(z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\frac{d^{k}}{d\zeta^{k}}(\zeta^{\varepsilon})\right)d\zeta \\
&= \int_{0}^{z}(z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\frac{d}{d\zeta}(\zeta^{\varepsilon})\right)d\zeta \\
&= (1-k+\varepsilon)_{k}\int_{0}^{z}\zeta^{\varepsilon-k}(z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]d\zeta \\
&= (1-k+\varepsilon)_{k}\int_{0}^{z}\zeta^{(\varepsilon-k+1)-1} \\
&\quad \cdot (z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]d\zeta: \\
&= (\nu)_{k}\int_{0}^{z}\zeta^{\nu-1}(z-\zeta)^{\mu-1}\Xi_{\alpha,\mu}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]d\zeta \\
&= (\nu)_{k}\Gamma(\nu)z^{\nu+\mu-1}\Xi_{\alpha,\mu+\nu}^{-\gamma}[\omega z^{\alpha}],
\end{aligned} \tag{19}
$$

where $\mu := k-\beta$, $\nu := \varepsilon-k+1$, and $(\nu)_{k}=\Gamma(1+\varepsilon)/\Gamma(1+\varepsilon-k)$. Hence, we obtain

$$
\begin{aligned}
{}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}(z^{\varepsilon}) &= \Gamma(1+\varepsilon)z^{\nu+\mu-1}\Xi_{\alpha,\mu+\nu}^{-\gamma}[\omega z^{\alpha}] \\
&= \Gamma(k+\nu)z^{\nu+\mu-1}\Xi_{\alpha,\mu+\nu}^{-\gamma}[\omega z^{\alpha}].
\end{aligned} \tag{20}
$$

We have the following property.

Proposition 5. *Let $\psi \in \Lambda$. Define a functional ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}: \cup \longrightarrow \cup$ by*

$$
{}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega} := \left(\frac{z^{\beta}}{\Xi_{\alpha,2-\beta}^{-\gamma}[wz^{\alpha}]}\right)\left({}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\right). \tag{21}
$$

Then ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi = {}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega} * \psi \in \Lambda(\alpha,\beta,\gamma,\omega \in \mathbb{C}, z \in \cup)$.

Proof. Let $\psi \in \Lambda$. Then a computation implies

$$
\begin{aligned}
{}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z) &= \left(\frac{z^{\beta}}{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)\left({}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right) \\
&= \left(\frac{z^{\beta}}{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)\left({}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\left(z+\sum_{n=2}^{\infty}\psi_{n}z^{n}\right)\right) \\
&= \left(\frac{z^{\beta}}{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)\left({}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta^{z}}^{\gamma,\omega}+\sum_{n=2}^{\infty}\psi_{n}{}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}z^{n}\right) \\
&= \left(\frac{z^{\beta}}{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)\left(\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]z^{1-\beta}\right. \\
&\quad \left.+\sum_{n=2}^{\infty}\psi_{n}\Gamma(n+1)z^{n-\beta}\Xi_{\alpha,n+1-\beta}^{-\gamma}[\omega z^{\alpha}]\right) \\
&= z+\sum_{n=2}^{\infty}\left(\psi_{n}\Gamma(n+1)\frac{\Xi_{\alpha,n+1-\beta}^{-\gamma}[\omega z^{\alpha}]}{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)z^{n}: \\
&= z+\sum_{n=2}^{\infty}\psi_{n}\delta_{n}z^{n} = \left(z+\sum_{n=2}^{\infty}\delta_{n}z^{n}\right) \\
&\quad * \left(z+\sum_{n=2}^{\infty}\psi_{n}z^{n}\right) = \left({}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega} * \psi\right)(z),
\end{aligned} \tag{22}
$$

where $\delta_{n} := \Gamma(n+1)\Xi_{\alpha,n+1-\beta}^{-\gamma}[\omega z^{\alpha}]/\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]$. This indicates that ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi \in \Lambda$. □

We call ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}$ the normalized complex Prabhakar operator (NCPO) in the open unit disk. Since ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega} \in \Lambda$, then we can study it in view of the geometric function theory.

Our aim is to formulate it in terms of some well-known classes of analytic functions. It is clear that δ_{n} is a complex connection (coefficient) of the operator and it is a constant when $\alpha=0$.

Remark 6. The integral operator corresponding to the fractional differential operator ${}_{k}^{C}\Delta_{\alpha\beta}^{\gamma\omega}$ is expanded by the series

$$
{}_{k}^{C}\Upsilon_{\alpha,\beta}^{\gamma,\omega}\psi(z) = z+\sum_{n=2}^{\infty}\left(\psi_{n}\frac{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}{\Gamma(n+1)\Xi_{\alpha,n+1-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)z^{n}. \tag{23}
$$

It is clear that

$$
\left({}_{k}^{C}\Upsilon_{\alpha,\beta}^{\gamma,\omega} * {}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\right)\psi(z) = \left({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega} * {}_{k}^{C}\Upsilon_{\alpha,\beta}^{\gamma,\omega}\right)\psi(z) = \psi(z). \tag{24}
$$

The linear convex combination of the operators ${}_{k}^{C}\Upsilon_{\alpha,\beta}^{\gamma,\omega}$ and ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}$ can be recognized by the formula

$$
{}_{k}^{\complement}\sum_{\alpha,\beta}^{\gamma,\omega}\psi(z) = \complement\,{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z) + (1-\complement)\,{}_{k}^{C}\Upsilon_{\alpha,\beta}^{\gamma,\omega}\psi(z), \tag{25}
$$

where $\complement \in [0,1]$. Clearly, ${}_{k}^{\complement}\sum_{\alpha,\beta}^{\gamma,\omega}\psi(z) \in \Lambda$, *where* $\psi \in \Lambda$.

2.3. Subclasses of NCPO. In terms of the NCPO, we formulate the next classes.

Definition 7. A function $\psi \in \Lambda$ is considered to be in the class ${}_{k}^{c}S_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$ if and only if

$$ {}_{k}^{c}S_{\alpha,\beta}^{*\gamma,\omega}(\sigma) = \left\{ \psi \in \Lambda : \frac{z\left({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \prec \sigma(z),\ \sigma(0) = 1 \right\}. \tag{26} $$

We shall deal with the conditions of a function ψ to be in ${}_{k}^{c}S_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$ whenever $\sigma \in C$ is convex as well as nonconvex.

Definition 8. A function $\psi \in \Lambda$ is considered to be in the class ${}_{k}^{c}J_{\alpha,\beta}^{\gamma,\omega}(\mathfrak{A}, \mathfrak{B}, \flat)$ if and only if

$$ {}_{k}^{c}J_{\alpha,\beta}^{\gamma,\omega}(\mathfrak{A}, \mathfrak{B}, \flat) = \left\{ \psi \in \Lambda : 1 + \frac{1}{\flat}\left(\frac{2{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z) - {}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(-z)} \right) \prec \frac{1 + \mathfrak{A}z}{1 + \mathfrak{B}z} \right\}. \tag{27} $$

We request the next result, which can be located in [6].

Lemma 9. *Define the class of analytic functions as follows: for $\varrho \in \mathbb{C}$ and a positive integer n*

$$ \mathbb{H}[\psi, n] = \left\{ \psi : \psi(z) = \varrho + \varrho_n z^n + \varrho_{n+1} z^{n+1} + \cdots \right\}. \tag{28} $$

(i) Let $\ell \in \mathbb{R}$. Then $\mathfrak{R}(\psi(z) + \ell z\psi'(z)) > 0 \longrightarrow \mathfrak{R}(\psi(z)) > 0$. In addition, if $\ell > 0$ and $\psi \in \mathbb{H}[1, n]$, then there are constants $\wp > 0$ and $\kappa > 0$ such that $\kappa = \kappa(\ell, \wp, n)$ and

$$ \psi(z) + \ell z\psi'(z) \prec \left(\frac{1+z}{1-z}\right)^{\kappa} \longrightarrow \psi(z) \prec \left(\frac{1+z}{1-z}\right)^{\wp} \tag{29} $$

(ii) Let $c \in [0, 1)$ and $\psi \in \mathbb{H}[1, n]$. Then there exists a fixed real number $\ell > 0$ so that

$$ \mathfrak{R}\left(\psi^2(z) + 2\psi(z) \cdot z\psi'(z)\right) > c \longrightarrow \mathfrak{R}(\psi(z)) > \ell \tag{30} $$

(iii) Let $\psi \in \mathbb{H}[\psi, n]$ with $\mathfrak{R}(\psi) > 0$. Then

$$ \mathfrak{R}\left(\psi(z) + z\psi'(z) + z^2\psi''(z)\right) > 0 \tag{31} $$

or for $\aleph : \cup \longrightarrow R$ such that

$$ \mathfrak{R}\left(\psi(z) + \aleph(z)\frac{z\psi'(z)}{\psi(z)}\right) > 0. \tag{32} $$

Then $\mathfrak{R}(\psi(z)) > 0$.

3. Results

Our results are as follows.

Theorem 10. *Let $\psi \in \Lambda$. If one of the next inequalities is considered,*

(i) ${}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)$ is of bounded turning function

(ii) $({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z))' \prec (1 + z/1 - z)^{\kappa}$, $\kappa > 0$, $z \in \cup$

(iii) $\mathfrak{R}(({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)')({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)/z)) > c/2$, $c \in [0, 1)$, $z \in \cup$

(iv) $\mathfrak{R}((z{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z))'' - ({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z))' + 2({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)/z)) > 0$

(v) $\mathfrak{R}((z({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z))'/{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)) + 2({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)/z)) > 1$

then ${}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)/z \in \mathscr{P}(\lambda)$ for some $\lambda \in [0, 1)$.

Proof. Define a function ρ as follows:

$$ \rho = \frac{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}, \quad z \in \cup. \tag{33} $$

Then a computation implies that

$$ z\rho'(z) + \rho(z) = \left({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'. \tag{34} $$

In virtue of the first inequality, we have that ${}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)$ is of bounded turning function, which leads to $\mathfrak{R}(z\rho'(z) + \rho(z)) > 0$. Therefore, Lemma 9(i) indicates that $\mathfrak{R}(\rho(z)) > 0$ which gives the first part of the theorem. Consequently, the second part is confirmed. In virtue of Lemma 9(i), we have a fixed real number $\ell > 0$ such that $\kappa = \kappa(\ell)$ and

$$ \frac{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\ell}. \tag{35} $$

This implies that

$$ \mathfrak{R}\left(\frac{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right) > \lambda, \quad \lambda \in [0, 1). \tag{36} $$

Suppose that

$$\Re\left(\rho^2(z)+2\rho(z)\cdot z\rho'(z)\right) = 2\Re\left(\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\left(\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'-\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{2z}\right)\right)>\varsigma. \tag{37}$$

According to Lemma 9(ii), there exists a fixed real number $\ell>0$ satisfying $\Re(\rho(z))>\ell$ and

$$\rho(z)=\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\in\mathscr{P}(\lambda),\quad \lambda\in[0,1). \tag{38}$$

It follows from (37) that $\Re({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z))')>0$; consequently, by Noshiro-Warschawski and Kaplan theorems, ${}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)$ is univalent and of bounded turning function in $\cup$. Taking the derivative (33), then we get

$$\begin{aligned}\Re\left(\rho(z)+z\rho'(z)+z^2\rho''(z)\right) &= \Re\left(z\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''-\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'\right.\\ &\left.+2\left(\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right)\right)>0.\end{aligned} \tag{39}$$

Hence, Lemma 9(ii) implies $\Re({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)/z)>0$.

The logarithmic differentiation of (33) yields

$$\Re\left(\rho(z)+\frac{z\rho'(z)}{\rho(z)}+z^2\rho''(z)\right) = \Re\left(\frac{z\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}+2\left(\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right)-1\right)>0. \tag{40}$$

Hence, Lemma 9(iii) implies, where $\aleph(z)=1$,

$$\Re\left(\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right)>0. \tag{41}$$

□

The next results show the upper bound of the operator ${}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)$ utilizing the exponential integral in the open unit disk provided that the function $\psi\in{}_k^C S_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$.

Theorem 11. *Suppose that $\psi\in{}_k^C S_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$, where $\sigma(z)$ is convex in $\cup$. Then*

$${}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\prec z\ \exp\left(\int_0^z\frac{\sigma(\Psi(\omega))-1}{\omega}d\omega\right), \tag{42}$$

where $\Psi(z)$ is analytic in $\cup$, with $\Psi(0)=0$ and $|\Psi(z)|<1$. Also, for $|z|=\xi$, ${}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)$ satisfies the inequality

$$\exp\left(\int_0^1\frac{\sigma(\Psi(\xi))-1}{\xi}\right)d\xi \le \left|\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right| \le \exp\left(\int_0^1\frac{\sigma(\Psi(\xi))-1}{\xi}\right)d\xi. \tag{43}$$

Proof. By the hypothesis, we receive the following conclusion:

$$\left(\frac{z\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\prec\sigma(z),\quad z\in\cup. \tag{44}$$

This gives the occurrence of a Schwarz function with $\Psi(0)=0$ and $|\Psi(z)|<1$ such that

$$\left(\frac{z\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)=\sigma(\Psi(z)),\quad z\in\cup. \tag{45}$$

That is,

$$\left(\frac{\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)-\frac{1}{z}=\frac{\sigma(\Psi(z))-1}{z}. \tag{46}$$

Integrating the above equality, we get

$$\log\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)-\log(z)=\int_0^z\left(\frac{\sigma(\Psi(\omega))-1}{\omega}\right)d\omega. \tag{47}$$

Consequently, we get

$$\log\left(\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right)=\int_0^z\frac{\sigma(\Psi(\omega))-1}{\omega}d\omega. \tag{48}$$

By the definition of subordination, we arrive at the following inequality

$${}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\prec z\ \exp\left(\int_0^z\frac{\sigma(\Psi(\omega))-1}{\omega}d\omega\right). \tag{49}$$

Note that the function $\sigma(z)$ plots the disk $0<|z|<\xi<1$ onto a reign, which is convex and symmetric with respect to the real axis. That is,

$$\sigma(-\xi|z|)\le\Re(\sigma(\Psi(\xi z)))\le\sigma(\xi|z|),\quad \xi\in(0,1), \tag{50}$$

then we have the inequalities

$$\sigma(-\xi)\le\sigma(-\xi|z|),\sigma(\xi|z|)\le\sigma(\xi)\,; \tag{51}$$

consequently, we get

$$\begin{aligned}&\int_0^1 \frac{\sigma(\Psi(-\xi|z|))-1}{\xi}d\xi \\ &\le \Re\left(\int_0^1 \frac{\sigma(\Psi(\xi))-1}{\xi}d\xi\right) \le \int_0^1 \frac{\sigma(\Psi(\xi|z|))-1}{\xi}d\xi.\end{aligned} \tag{52}$$

In view of Equation (48), we obtain the general log-inequality

$$\begin{aligned}\int_0^1 \frac{\sigma(\Psi(-\xi|z|))-1}{\xi}d\xi \le \log\left|\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right| \\ \le \int_0^1 \frac{\sigma(\Psi(\xi|z|))-1}{\xi}d\xi\,;\end{aligned} \tag{53}$$

that is,

$$\begin{aligned}&\exp\left(\int_0^1 \frac{\sigma(\Psi(-\xi|z|))-1}{\xi}d\xi\right) \\ &\le \left|\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right| \le \exp\left(\int_0^1 \frac{\sigma(\Psi(\xi|z|))-1}{\chi}d\xi\right).\end{aligned} \tag{54}$$

Hence, we have

$$\begin{aligned}&\exp\left(\int_0^1 \frac{\sigma(\Psi(-\xi))-1}{\xi}d\xi\right) \\ &\le \left|\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right| \le \exp\left(\int_0^1 \frac{\sigma(\Psi(\xi))-1}{\xi}d\xi\right).\end{aligned} \tag{55}$$

□

Proceeding, we illustrate the sufficient condition of ψ to be in the class ${}_k^C S_{\alpha,\beta}^{*\gamma,\omega}\psi(\sigma)$, where σ is convex univalent satisfying $\sigma(0)=1$.

Theorem 12. *If $\psi \in \Lambda$ satisfies the inequality*

$$\begin{aligned}&\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\left(2+\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''}{\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}\right. \\ &\left.-\left(\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\right) \prec \sigma(z),\end{aligned} \tag{56}$$

then $\psi \in {}_k^C S_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$.

Proof. The proof directly comes from [6] (Theorem 3.1a). Taking

$$p(z) = \frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}, \tag{57}$$

and $P(z)=1$ in the inequality

$$p(z) + P(z)\cdot\left(zp'(z)\right) \prec \sigma(z), \tag{58}$$

then we obtain

$$\begin{aligned}&p(z) + P(z)\cdot\left(zp'(z)\right) \\ &= \frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \times \left(2+\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''}{\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}\right. \\ &\left.-\left(\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\right) \prec \sigma(z).\end{aligned} \tag{59}$$

This implies that

$$p(z) = \frac{z\left({}_k^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \prec \sigma(z), \quad \sigma \in \mathscr{C}, \tag{60}$$

that is $\psi \in {}_k^C S_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$. □

Corollary 13. *Let the assumption of Theorem 12 hold. Then*

$$\begin{aligned}&\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \times \left(1+\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''}{\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}\right. \\ &\left.-\left(\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\right) \ll \sigma'(z).\end{aligned} \tag{61}$$

Proof. Let

$$p(z) = \frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}. \tag{62}$$

In view of Theorem 12, we have

$$\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \prec \sigma(z), \tag{63}$$

where $\sigma \in C$. Then by [9] (Theorem 3), we get $p'(z) \ll \sigma'(z)$ for some $z \in \cup$, where

$$\begin{aligned}p'z = &\left(\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\left(1+\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''}{\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}\right. \\ &\left.-\left(\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\right).\end{aligned} \tag{64}$$

□

It is well known that the function $\sigma(z) = e^{\epsilon z}$, $1 < |\epsilon| \le \pi/2$ is not convex in $\cup$, where the domain $\sigma(\cup)$ is lima-bean (see [6] (P123)). One can obtain the same result of Theorem 12 as follows.

Theorem 14. *If $\psi \in \Lambda$ satisfies the inequality*

$$1 + \frac{z\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''}{{}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)'} \prec e^{\epsilon z}, \quad 1 < |\epsilon| \le \frac{\pi}{2}, \tag{65}$$

then $\psi \in {}_{k}^{C}S_{\alpha,\beta}^{*\gamma,\omega}(e^{\epsilon z})$.

Proof. Let

$$p(z) := \frac{z\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}. \tag{66}$$

Then a computation implies

$$\begin{aligned} p(z) + \frac{zp'(z)}{p(z)} &= \left(\frac{z\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right) + \frac{\left(z\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'/{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)\left(1 + z\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''/\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)' - \left(z\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'/{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)\right)}{z\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'/{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \\ &= \left(1 + \frac{z\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''}{{}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)'}\right) \prec e^{\epsilon z}. \end{aligned} \tag{67}$$

This implies that [6] (P123)

$$p(z) = \frac{z\left({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \prec e^{\epsilon z}; \tag{68}$$

that is, $\psi \in {}_{k}^{C}S_{\alpha,\beta}^{*\gamma,\omega}(e^{\epsilon z})$. □

Theorem 15. *If $\psi \in {}_{k}^{c}J_{\alpha,\beta}^{\gamma,\omega}(\mathfrak{A}, \mathfrak{B}, \flat)$ then the function*

$$\mathbb{B}(z) = \frac{1}{2}[\psi(z) - \psi(-z)], \quad z \in \cup, \tag{69}$$

satisfies

$$\begin{aligned} &1 + \frac{1}{\flat}\left(\frac{{}_{k+1}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\mathbb{B}(z)}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\mathbb{B}(z)} - 1\right) \prec \frac{1 + \mathfrak{A}z}{1 + \mathfrak{B}z}, \\ &\Re\left(\frac{z\mathbb{B}(z)'}{\mathbb{B}(z)}\right) \ge \frac{1 - \eth^2}{1 + \eth^2}, \quad |z| = \eth < 1. \end{aligned} \tag{70}$$

Proof. Let $\psi \in {}_{k}^{c}J_{\alpha,\beta}^{\gamma,\omega}(\mathfrak{A}, \mathfrak{B}, \flat)$. Then there occurs a function $J(z)$ such that

$$\begin{aligned} \flat(J(z) - 1) &= \left(\frac{2{}_{k+1}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z) - {}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(-z)}\right), \\ \flat(J(-z) - 1) &= \left(\frac{2{}_{k+1}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(-z)}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(-z) - {}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right). \end{aligned} \tag{71}$$

This confirms that

$$1 + \frac{1}{\flat}\left(\frac{{}_{k+1}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\mathbb{B}(z)}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\mathbb{B}(z)} - 1\right) = \frac{J(z) + J(-z)}{2}. \tag{72}$$

However, J satisfies

$$J(z) \prec \frac{1 + \mathfrak{A}z}{1 + \mathfrak{B}z}, \tag{73}$$

which is univalent, then we get

$$1 + \frac{1}{\flat}\left(\frac{{}_{k+1}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\mathbb{B}(z)}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\mathbb{B}(z)} - 1\right) \prec \frac{1 + \mathfrak{A}z}{1 + \mathfrak{B}z}. \tag{74}$$

Also, $\mathbb{B}(z)$ is starlike in $\cup$ which implies that

$$\hbar(z) := \frac{z\mathbb{B}(z)'}{\mathbb{B}(z)} \prec \frac{1 - z^2}{1 + z^2}. \tag{75}$$

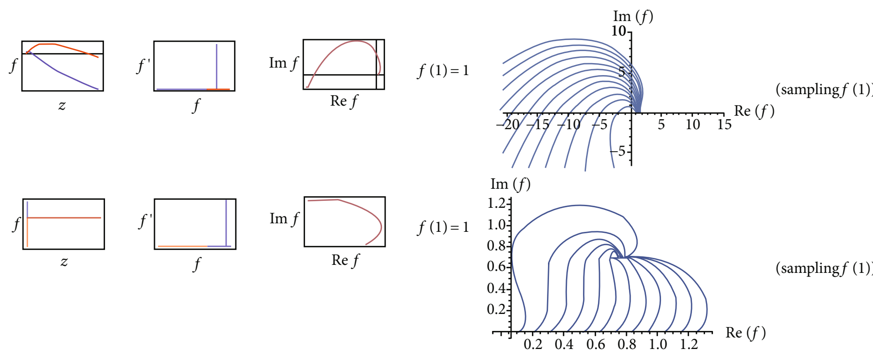

Figure 1: Plot of the solution for $zf(z/f(z))$ and $f(z)+zf(z/f(z))$, respectively.

Hence, a Schwarz function $\Bbbk \in \cup$, $|k(z)| \leq |z| < 1$, $\Bbbk(0)=0$ gets

$$\hbar(z) \prec \frac{1-\Bbbk(z)^2}{1+\Bbbk(z)^2}, \tag{76}$$

which leads to

$$\Bbbk^2(\zeta) = \frac{1-\hbar(\zeta)}{1+\hbar(\zeta)}, \quad \zeta \in \cup, \zeta, |\zeta| = r < 1. \tag{77}$$

A calculation yields

$$\left|\frac{1-\hbar(\zeta)}{1+\hbar(\zeta)}\right| = |\Bbbk(\zeta)|^2 \leq |\zeta|^2. \tag{78}$$

Therefore, we get the following inequality: or

$$\left|\hbar(\zeta) - \frac{1+|\zeta|^4}{1-|\zeta|^4}\right|^2 \leq \frac{4|\zeta|^4}{\left(1-|\zeta|^4\right)^2} \tag{79}$$

$$\left|\hbar(\zeta) - \frac{1+|\zeta|^4}{1-|\zeta|^4}\right| \leq \frac{2|\zeta|^2}{\left(1-|\zeta|^4\right)}. \tag{80}$$

Thus, we have

$$\Re(\hbar(z)) \geq \frac{1-\eth^2}{1+\eth^2}, \quad |\zeta| = \eth < 1. \tag{81}$$

This completes the assertion of the theorem. □

Example 16.

(i) Let

$$\frac{zf'(z)}{f(z)} := \frac{z\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\Psi(z)\right)'}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\Psi(z)},$$
$${}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\Psi(z) = \frac{z}{(1-z)^2}, \quad \Psi \in \Lambda. \tag{82}$$

Then the solution of $zf'(z)/f(z) = ((1+z)/(1-z))$ is formulated, as follows:

$$\left({}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\Psi(z)\right) = \frac{z}{(1-z)^2}, \quad \Psi \in \Lambda. \tag{83}$$

Moreover, the solution of the equation

$$f(z) + \frac{zf'(z)}{f(z)} = \left(\frac{1+z}{1-z}\right) \tag{84}$$

is approximated to $f(z) = z/(1-z)$.

(ii) The solution of $zf'(z)/f(z) = ((1+z)/(1-z))^{0.25}$ is given in terms of the hypergeometric function, as follows (see Figure 1):

$$f(z) = c\exp\left(1.8(z+1)\left(\frac{z+1}{1-z}\right)^{0.25} \cdot \frac{F_1(1.25;0.25,1;2.25;0.5z+0.5,z+1)}{z(2.25F_1(1.25;0.25,1;2.25;0.5z+0.5,z+1)+(z+1)F_1(2.25;0.25,2;3.25;0.5z+0.5,z+1)+(0.125z+0.125)F_1(2.25;1.25,1;3.25;0.5z+0.5z+1))}\right). \tag{85}$$

4. Conclusion

The Prabhakar fractional differential operator in the complex plane is formulated for a class of normalized function in the open unit disk. We formulated the modified operator in two classes of analytic functions to investigate its geometric behavior. Differential inequalities are formulated to include them. Examples showed the behavior of solutions and the formula. The suggested operators can be utilized to formulate some classes of analytic functions or to generalize other types of differential operators such a conformable, quantum, or fractal operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally and significantly to writing this article. All authors read and agreed to the published version of the manuscript.

References

[1] H. M. Srivastava and S. Owa, Eds., *Univalent Functions, Fractional Calculus, and Their Applications*, Chichester: Ellis Horwood, New York; Toronto: Halsted Press, 1989.

[2] R. W. Ibrahim, "On generalized Srivastava-Owa fractional operators in the unit disk," *Advances in Difference Equations*, vol. 2011, no. 1, Article ID 55, 2011.

[3] H. M. Srivastava, M. Darus, and R. W. Ibrahim, "Classes of analytic functions with fractional powers defined by means of a certain linear operator," *Integral Transforms and Special Functions*, vol. 22, no. 1, pp. 17–28, 2011.

[4] R. W. Ibrahim, "Ulam stability for fractional differential equation in complex domain," *Abstract and Applied Analysis*, vol. 2012, Article ID 649517, 8 pages, 2012.

[5] R. W. Ibrahim, "Generalized Ulam-Hyers stability for fractional differential equations," *International Journal of mathematics*, vol. 23, no. 5, article 1250056, 2012.

[6] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, CRC Press, 2000.

[7] A. W. Goodman, "An invitation on the study of univalent and multivalent functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2, no. 2, pp. 163–186, 1979.

[8] T. H. MacGregor, "Majorization by univalent functions," *Duke Mathematical Journal*, vol. 34, no. 1, 1967.

[9] D. M. Campbell, "Majorization-subordination theorems for locally univalent functions, II," *Canadian Journal of Mathematics*, vol. 25, no. 2, pp. 420–425, 1973.

[10] H. M. Srivastava, "Some families of Mittag-Leffler type functions and associated operators of fractional calculus (survey)," *Turkic World Mathematical Society (TWMS) Journal of Pure and Applied Mathematics*, vol. 7, pp. 123–145, 2016.

[11] A. K. Shukla and J. C. Prajapati, "On a generalization of Mittag-Leffler function and its properties," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 797–811, 2007.

[12] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "Mittag-Leffler functions and their applications," *Journal of Applied Mathematics*, vol. 2011, Article ID 298628, 51 pages, 2011.

[13] T. R. Prabhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernel," *Yokohama Mathematical Journal*, vol. 19, pp. 7–15, 1971.

[14] R. Garra, R. Gorenflo, F. Polito, and Z. Tomovski, "Hilfer-Prabhakar derivatives and some applications," *Applied Mathematics and Computation*, vol. 242, pp. 576–589, 2014.

[15] A. A. Kilbas, M. Saigo, and R. K. Saxena, "Generalized Mittag-Leffler function and generalized fractional calculus operators," *Integral Transforms and Special Functions*, vol. 15, no. 1, pp. 31–49, 2004.

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Research Article

Convolutions of Harmonic Mappings Convex in the Horizontal Direction

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In this paper, we establish some results concerning the convolutions of harmonic mappings convex in the horizontal direction with harmonic vertical strip mappings. Furthermore, we provide examples illustrated graphically with the help of Maple to illuminate the results.

1. Introduction

For real-valued harmonic functions u and v in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$, the complex-valued continuous function $f = u + iv$ is said to be harmonic and can be expressed as $f = h + \bar{g}$, where h and g are analytic in $\mathbb{E}$. Let H be the class of harmonic mappings $f = h + \bar{g}$ normalized by $h(0) = g(0) = h'(0) - 1 = 0$ and have the following power series representations:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \qquad g(z) = \sum_{m=1}^{\infty} b_m z^m. \tag{1}$$

We call h the analytic part and g the coanalytic part of f, respectively. The Jacobian of $f = h + \bar{g}$ is given by $J_f = |h'|^2 - |g'|^2$. Lewy's theorem [1] implies that $f \in H$ is locally univalent and sense-preserving if and only if $J_f > 0$ in $\mathbb{E}$. The condition $J_f > 0$ is equivalent to that dilatation $\omega(z) = g'(z)/h'(z)$ satisfying $|\omega(z)| < 1$ for all $z \in \mathbb{E}$ (see [2, 3]).

We denote by $\mathbb{S}_H$ the class of all harmonic, sense-preserving, and univalent mappings $f = h + \bar{g}$ in $\mathbb{E}$, which are normalized by the condition $h(0) = g(0) = 0$ and $h'(0) = 1$. Let $\mathbb{S}_H^0$ be the subset of all $f \in \mathbb{S}_H$ in which $g'(0) = 0$. Further, let $\mathbb{K}_H, \mathbb{C}_H$ (resp., $\mathbb{K}_H^0, \mathbb{C}_H^0$) be the subset of $\mathbb{S}_H$ (resp., $\mathbb{S}_H^0$) whose images are convex and close-to-convex domains. A domain Ω is said to be convex in the horizontal direction (CHD) if the intersection of Ω with each horizontal line is connected (or empty). A function $f = h + \bar{g} \in \mathbb{S}_H$ is said to be a CHD mapping if f maps $\mathbb{E}$ onto a CHD domain. Let $\mathbb{S}_{\text{CHD}}$ be the subset of $\mathbb{C}_H$ which consist of CHD mappings. The following basic theorem of Clunie and Sheil-Small [2] is known as shear construction that constructs harmonic mappings with prescribed dilatations onto a domain convex in one direction.

Theorem 1 (see [2]). *A locally univalent harmonic mapping $f = h + \bar{g}$ in $\mathbb{E}$ is a univalent mapping of $\mathbb{E}$ onto a domain convex in a direction ϕ if and only if $h - e^{2i\phi} g$ is a conformal univalent mapping of $\mathbb{E}$ onto a domain convex in the direction of ϕ.*

Let $f * F = h * H + \overline{g * G}$ be the convolution of two harmonic functions $f = h + \bar{g}$ and $F = H + \bar{G}$ where the operator $*$ is convolution (or Hadamard product) of two power series.

There are several research papers in recent years which investigate the convolution of harmonic univalent functions. In particular, Dorff [4] and Dorff et al. [5] studied the convolution of harmonic univalent mappings in the right half-plane. For some recent investigations involving convolution of harmonic mappings, we refer the reader to [6–13].

Let $F_{\mathfrak{a}} = H_{\mathfrak{a}} + \overline{G_{\mathfrak{a}}}$ sheared by $H_{\mathfrak{a}} - G_{\mathfrak{a}} = z/(1-z)$ with the dilatation $\omega_{\mathfrak{a}} = (\mathfrak{a}+z)/(1+\mathfrak{a}z)$, where $\mathfrak{a} \in (-1,1)$. Using shear construction of Clunie and Sheil-Small [2], we have

$$H_{\mathfrak{a}}(z) = \frac{1/(1-\mathfrak{a})z - 1/2z^2}{(1-z)^2} = \frac{1}{2}\left[\frac{z}{1-z} + \frac{1+\mathfrak{a}}{1-\mathfrak{a}}\frac{z}{(1-z)^2}\right], \tag{2}$$

$$G_{\mathfrak{a}}(z) = \frac{\mathfrak{a}/(1-\mathfrak{a})z + 1/2z^2}{(1-z)^2} = \frac{1}{2}\left[\frac{-z}{1-z} + \frac{1+\mathfrak{a}}{1-\mathfrak{a}}\frac{z}{(1-z)^2}\right]. \tag{3}$$

It is clear that by setting $\mathfrak{a} = 0$ in (2) and (3), we obtain $F_0 = H_0 + \overline{G_0}$ which satisfy the conditions $H_0 - G_0 = z/(1-z)$ and $\omega(z) = z$, studied by Liu and Li [8]. Wang et al. [14] also studied convolutions of this mapping. Note that $F_{\mathfrak{a}}$ is a CHD mapping.

Recently, Liu and Li [8] introduced the following generalized harmonic univalent mappings:

$$\begin{aligned} P_\delta(z) = H_\delta(z) + \overline{G_\delta(z)} &= \frac{1}{1+\delta}\left[\frac{\delta z}{(1-z)^2} + \frac{z}{1-z}\right] \\ &+ \frac{1}{1+\delta}\overline{\left[\frac{\delta z}{(1z)^2}\frac{z}{1z}\right]}, \end{aligned} \tag{4}$$

where $\delta > 0$ and $z \in \mathbb{E}$. Obviously, $P_1(z) = F_0(z)$. If $f = h + \bar{g} \in \mathbb{S}_H$, then

$$P_\delta * f = \frac{\delta z h' + h}{1+\delta} + \overline{\frac{\delta z g' g}{1+\delta}}. \tag{5}$$

Also, $P_\delta(z)$ maps $\mathbb{E}$ onto the domain $\{u+iv : v^2 > -[(2\delta)/(1+\delta)u + (1/(1+\delta)^2)],\ \delta > 0\}$ which is a CHD domain. Very recently, Yasar and Ozdemir [15] studied convolutions of these generalized harmonic mappings.

Let $f_\gamma = h_\gamma + \overline{g_\gamma} \in \mathbb{S}^0_{\mathrm{CHD}}$ with

$$h_\gamma - g_\gamma = \frac{1}{2i\sin\gamma}\log\left(\frac{1+ze^{i\gamma}}{1+ze^{-i\gamma}}\right), \tag{6}$$

where $\pi/2 \le \gamma < \pi$.

In this paper, we investigate the conditions under which the convolutions of harmonic mappings P_δ, f_γ, and F_a with prescribed dilatations are univalent and CHD provided that the convolutions are locally univalent and sense-preserving. Furthermore, we provide two examples illustrated graphically with the help of Maple to illuminate our results.

2. Preliminary Results

Lemma 2 (see [16]). *Let f be an analytic fuction in $\mathbb{E}$ with $f(0) = 0$ and $f'(0) \neq 0$ and let*

$$\varphi(z) = \frac{z}{(1+ze^{i\theta_1})(1+ze^{i\theta_2})}, \tag{7}$$

where $\theta_1, \theta_2 \in \mathbb{R}$. If

$$\operatorname{Re}\left(\frac{zf'(z)}{\varphi(z)}\right) > 0, \tag{8}$$

then f is convex in the horizontal direction.

Lemma 3 (see [17]). *Let φ and G be analytic in $\mathbb{E}$ with $\varphi'(0) = G(0) = 0$. If φ is convex and G is starlike, then for each function F analytic in $\mathbb{E}$ and satisying* Re $(F(z)) > 0$, *we have*

$$\operatorname{Re}\left(\frac{(\varphi * FG)(z)}{(\varphi * G)(z)}\right) > 0 (z \in \mathbb{E}). \tag{9}$$

Lemma 4 ([18], Cohn's rule). *Given a polynomial*

$$p(z) = p_0(z) = a_{k,0}z^k + a_{k-1,0}z^{k-1} + \cdots + a_{1,0}z + a_{0,0}\ (a_{k,0} \neq 0) \tag{10}$$

of degree k, let

$$\begin{aligned} p^*(z) = p_0^*(z) = z^k\overline{p\left(\frac{1}{\bar{z}}\right)} &= \overline{a_{k,0}} + \overline{a_{k1,0}}z + \cdots + \overline{a_{1,0}}z^{k-1} \\ &+ \overline{a_{0,0}}z^k\ (a_{k,0} \neq 0). \end{aligned} \tag{11}$$

Denote by r and s the number of zeros of $p(z)$ inside the unit circle and on it, respectively. If $|a_{0,0}| < |a_{k,0}|$, then

$$p_1(z) = \frac{\overline{a_{k,0}}p(z) - a_{0,0}p^*(z)}{z} \tag{12}$$

is of degree $k-1$ with $r_1 = r - 1$ and $s_1 = s$ the number of zeros of $p_1(z)$ inside the unit circle and on it, respectively.

Lemma 5. *Let $P_\delta = H_\delta + \overline{G_\delta}$ be defined by (4) and $f_\gamma = h_\gamma + \overline{g_\gamma}$ be defined by (6) with dilatation $\omega = g'_\gamma/h'_\gamma$. Then the dilatation of $P_\delta * f_\gamma$ is given by*

$$\tilde{\omega}(z) = \frac{\omega(1-\omega)\left[(\delta-1) - 2z\cos\gamma - (\delta+1)z^2\right] + \delta z\omega'\left(1+2z\cos\gamma+z^2\right)}{(1-\omega)\left[(\delta+1) + 2z\cos\gamma - (\delta-1)z^2\right] + \delta z\omega'\left(1+2z\cos\gamma+z^2\right)}. \tag{13}$$

Proof. Since $h_\gamma - g_\gamma = 1/(2i\sin\gamma)\log((1+ze^{i\gamma})/(1+ze^{-i\gamma}))$ $(\pi/2 \le \gamma < \pi)$ and $g'_\gamma = \omega h'_\gamma$, then $g''_\gamma = \omega' h'_\gamma + \omega h''_\gamma$. We immediately get

$$h'_\gamma = \frac{1}{(1-\omega)(1+ze^{i\gamma})(1+ze^{-i\gamma})}, \tag{14}$$

$$h''_\gamma = -\frac{2(\cos\gamma + z)(1-\omega) - \omega'\left(1+2z\cos\gamma+z^2\right)}{(1-\omega)^2(1+ze^{i\gamma})^2(1+ze^{-i\gamma})^2}. \tag{15}$$

From (4), we have

$$\begin{aligned}\tilde{\omega}(z) &= \frac{\left(G_\delta * g_\gamma\right)''}{\left(H_\delta * h_\gamma\right)''} = \frac{\left(\delta z g'_\gamma - g_\gamma\right)''}{\left(\delta z h'_\gamma + h_\gamma\right)''} = \frac{(\delta-1)g'_\gamma + \delta z g''_\gamma}{(\delta+1)h'_\gamma + \delta z h''_\gamma} = \frac{(\delta-1)\omega h'_\gamma + \delta z\left(\omega' h'_\gamma + \omega h''_\gamma\right)}{(\delta+1)h'_\gamma + \delta z h''_\gamma} \\ &= \frac{\omega(1-\omega)\left[(\delta-1) - 2z\cos\gamma - (\delta+1)z^2\right] + \delta z\omega'\left(1+2z\cos\gamma+z^2\right)}{(1-\omega)\left[(\delta+1) + 2z\cos\gamma - (\delta-1)z^2\right] + \delta z\omega'\left(1+2z\cos\gamma+z^2\right)}.\end{aligned} \tag{16}$$

□

Lemma 6. *Let $P_\delta = H_\delta + \overline{G_\delta}$ be defined by (4) and $f_\gamma = h_\gamma + \overline{g_\gamma}$ be defined by (6). If $P_\delta * f_\gamma$ is locally univalent and sense-preserving, then $P_\delta * f_\gamma$ is univalent and convex in the horizontal direction.*

Proof. Let

$$\begin{aligned} F_1 &= (H_\delta - G_\delta) * \left(h_\gamma + g_\gamma\right) \\ &= H_\delta * h_\gamma + H_\delta * g_\gamma - G_\delta * h_\gamma - G_\delta * g_\gamma, \\ F_2 &= (H_\delta + G_\delta) * \left(h_\gamma - g_\gamma\right) \\ &= H_\delta * h_\gamma - H_\delta * g_\gamma + G_\delta * h_\gamma - G_\delta * g_\gamma. \end{aligned} \tag{17}$$

Thus,

$$H_\delta * h_\gamma - G_\delta * g_\gamma = \frac{1}{2}(F_1 + F_2). \tag{18}$$

By Theorem 1, we need to prove that $1/2(F_1+F_2)$ is convex in the horizontal direction. Since

$$h_\gamma - g_\gamma = \frac{1}{2i\sin\gamma}\log\left(\frac{1+ze^{i\gamma}}{1+ze^{-i\gamma}}\right)\left(\frac{\pi}{2} \le \gamma < \pi\right), \tag{19}$$

we have

$$\begin{aligned} zF'_1 &= (H_\delta - G_\delta) * \left[z\left(h'_\gamma + g'_\gamma\right)\right] \\ &= (H_\delta - G_\delta) * \left[z\left(h'_\gamma - g'_\gamma\right)\left(\frac{h'_\gamma + g'_\gamma}{h'_\gamma - g'_\gamma}\right)\right] \\ &= \frac{2z}{(1+\delta)(1-z)} * \frac{z}{(1+ze^{i\gamma})(1+ze^{-i\gamma})}\left(\frac{1+\omega_\gamma}{1-\omega_\gamma}\right) \\ &= \frac{2zp_1(z)}{(1+\delta)(1+ze^{i\gamma})(1+ze^{-i\gamma})}, \end{aligned} \tag{20}$$

where $p_1(z) = (1+\omega_\gamma)/(1-\omega_\gamma)$ satisfies the condition Re $(p_1(z)) > 0$. Thus, we have

$$\operatorname{Re}\left\{\frac{zF'_1}{2z/[(1+\delta)(1+ze^{i\gamma})(1+ze^{-i\gamma})]}\right\} = \operatorname{Re}\{p_1(z)\} > 0. \tag{21}$$

Now, we consider

$$\begin{aligned} zF'_2 &= \left[z\left(H'_\delta + G'_\delta\right) * \left(h_\gamma - g_\gamma\right)\right] \\ &= \left[z\left(H'_\delta - G'_\delta\right)\frac{H'_\delta + G'_\delta}{H'_\delta - G'_\delta}\right] * \left(h_\gamma - g_\gamma\right) \\ &= \left[z\left(H'_\delta - G'_\delta\right)\left(\frac{1+\omega_\delta}{1-\omega_\delta}\right)\right] * \left(h_\gamma - g_\gamma\right) \\ &= \frac{2zp_2(z)}{(1+\delta)(1-z)^2} * \left(h_\gamma - g_\gamma\right), \end{aligned} \tag{22}$$

where $p_2(z) = (1+\omega_\delta)/(1-\omega_\delta)$ satisfies the condition Re $\{p_2(z)\} > 0$. Using the fact that

$$\psi(z) * \frac{z}{(1-z)^2} = z\psi'(z) \tag{23}$$

and $h_\gamma - g_\gamma$ is convex, by Lemma 3, we have

$$\begin{aligned} &\operatorname{Re}\left\{\frac{zF'_2}{z/[(1+ze^{i\gamma})(1+ze^{-i\gamma})]}\right\} \\ &= \operatorname{Re}\left\{\frac{\left(h_\gamma - g_\gamma\right) * p_2(z)\left[2z/\left((1+\delta)(1-z)^2\right)\right]}{z\left(h'_\gamma - g'_\gamma\right)}\right\} \\ &= \operatorname{Re}\left\{\frac{\left(h_\gamma - g_\gamma\right) * p_2(z)\left[2z/\left((1+\delta)(1-z)^2\right)\right]}{\left(h_\gamma - g_\gamma\right) * z/(1-z)^2}\right\} > 0. \end{aligned} \tag{24}$$

□

Finally, using Lemma 2, we obtain that $F_1 + F_2$ is convex in the horizontal direction.

Lemma 7. *Let $f_\gamma = h_\gamma + g_\gamma \in \mathbb{S}^0_{CHD}$ be given by (6) with dilatation $\omega = g'_\gamma/h'_\gamma$ and $F_\mathfrak{a} = H_\mathfrak{a} + \overline{G_\mathfrak{a}}$ be a mapping defined by (2) and (3). Then the dilatation of $F_\mathfrak{a} * f_\gamma$ is given by*

$$\tilde{W}(z) = \frac{2\omega(1-\omega)\left[\mathfrak{a} - (1-\mathfrak{a})z\cos\gamma - z^2\right] + (1+\mathfrak{a})z\omega'\left(1+2z\cos\gamma+z^2\right)}{2(1-\omega)[1+(1-\mathfrak{a})z\cos\gamma - \mathfrak{a}z^2] + (1+\mathfrak{a})z\omega'(1+2z\cos\gamma+z^2)}. \quad (25)$$

Proof. From (2) and (3), we have

$$\tilde{W}(z) = \frac{\left(G_\mathfrak{a} * g_\gamma\right)'}{\left(H_\mathfrak{a} * h_\gamma\right)'} = \frac{\left((1+\mathfrak{a})zg'_\gamma - (1-\mathfrak{a})g_\gamma\right)'}{\left((1+\mathfrak{a})zh'_\gamma + (1-\mathfrak{a})h_\gamma\right)'}$$

$$= \frac{2\mathfrak{a}g'_\gamma + (1+\mathfrak{a})zg''_\gamma}{2h'_\gamma + (1+\mathfrak{a})zh''_\gamma} = \frac{2\mathfrak{a}\omega h'_\gamma + (1+\mathfrak{a})z\left(\omega' h'_\gamma + \omega h''_\gamma\right)}{2h'_\gamma + (1+\mathfrak{a})zh''_\gamma}. \quad (26)$$

Using (14) and (15), then we obtain the dilatation of $F_\mathfrak{a} * f_\gamma$ as follows:

$$\tilde{W}(z) = \frac{2\omega(1-\omega)\left[\mathfrak{a} - (1-\mathfrak{a})z\cos\gamma - z^2\right] + (1+\mathfrak{a})z\omega'\left(1+2z\cos\gamma+z^2\right)}{2(1-\omega)[1+(1-\mathfrak{a})z\cos\gamma - \mathfrak{a}z^2] + (1+\mathfrak{a})z\omega'(1+2z\cos\gamma+z^2)}. \quad (27)$$

□

Lemma 8 ([14], Lemma 2.4). *Let $F_\mathfrak{a} = H_\mathfrak{a} + \overline{G_\mathfrak{a}}$ be a mapping defined by (2), (3) and $f_\gamma = h_\gamma + \overline{g_\gamma} \in \mathbb{S}^0_{CHD}$ be defined by (6). If $F_\mathfrak{a} * f_\gamma$ is locally univalent and sense-preserving, then $F_\mathfrak{a} * f_\gamma$ is univalent and convex in the horizontal direction.*

Lemma 9 ([19], Gauss-Lucas theorem). *Let $T(z)$ be a nonconstant polynomial with complex coefficients. Then, the zeros of the derivative $T'(z)$ are contained in the convex hull of the set of the zeros of $T(z)$.*

Lemma 10. *Let*

$$q(z) = z^{k+1} + \frac{2(k+1)\cos\gamma}{k+2}z^k + \frac{k}{k+2}z^{k-1} - \frac{2}{k+2}e^{-i\theta}z - \frac{2\cos\gamma}{k+2}e^{-i\theta} \quad (28)$$

be a complex polynomial of degree $k+1$, where $\theta \in \mathbb{R}$, $k \in \mathbb{N}^+$, and $\pi/2 \le \gamma < \pi$. Then, all zeros of $q(z)$ lie in the closed unit disk $|z| \le 1$.

Proof. Note that $q(z) = 1/(k+2)T'(z)$, where

$$T(z) = \left(z^k - e^{-i\theta}\right)\left(1+2z\cos\gamma+z^2\right). \quad (29)$$

It is obvious that the roots of $(z^k - e^{-i\theta})$ lie on the unit circle. Also, $-\cos\gamma \pm i\sin\gamma$ which are the roots of $(1+2z\cos\gamma+z^2)$ lie on the unit circle as well. Hence, the result follows from Lemma 9. □

3. Main Results

Theorem 11. *Let $P_\delta = H_\delta + \overline{G_\delta} \in \mathbb{S}_{CHD}$ be a mapping given by (4) and $f_\gamma = h_\gamma + \overline{g_\gamma} \in \mathbb{S}^0_{CHD}$ be given by (6) with the dilatation $\omega_k = g'_\gamma/h'_\gamma = e^{i\theta}z^k (\theta \in \mathbb{R}, k \in \mathbb{N}^+)$. Then $P_\delta * f_\gamma$ is univalent and convex in the horizontal direction.*

Proof. By Lemma 6, we need to prove that the dilatation $\tilde{\omega}$ of $P_\delta * f_\gamma$ satisfies $|\tilde{\omega}| < 1$ for all $z \in \mathbb{E}$. Substituting $\omega = e^{i\theta}z^k$ in (13), we yield

$$\tilde{\omega}(z) = \frac{e^{i\theta}z^k\left(1-e^{i\theta}z^k\right)\left[(\delta-1) - 2z\cos\gamma - (\delta+1)z^2\right] + \delta k e^{i\theta}z^k\left(1+2z\cos\gamma+z^2\right)}{\left(1-e^{i\theta}z^k\right)[(\delta+1)+2z\cos\gamma-(\delta-1)z^2] + \delta k e^{i\theta}z^k(1+2z\cos\gamma+z^2)} = e^{2i\theta}z^k\frac{t(z)}{t^*(z)}, \quad (30)$$

where

$$t(z)=z^{k+2}+\frac{2\cos\gamma}{1+\delta}z^{k+1}+\frac{1-\delta}{1+\delta}z^{k}+\frac{\delta(k-1)-1}{1+\delta}e^{-i\theta}z^{2}+\frac{2(k\delta-1)\cos\gamma}{1+\delta}e^{-i\theta}z+\frac{\delta(1+k)-1}{1+\delta}e^{-i\theta}, \quad (31)$$

$$t^{*}(z)=1+\frac{2\cos\gamma}{1+\delta}z+\frac{1-\delta}{1+\delta}z^{2}+\frac{\delta(k-1)-1}{1+\delta}e^{i\theta}z^{k}+\frac{2(k\delta-1)\cos\gamma}{1+\delta}e^{i\theta}z^{k+1}+\frac{\delta(1+k)-1}{1+\delta}e^{i\theta}z^{k+2}. \quad (32)$$

If we substitute $\delta=2/k$ into (30), then $t(z)/t^{*}(z)=e^{-i\theta}$, and it is clear that $|\tilde{\omega}|<1$ for all $z\in\mathbb{E}$. Now, we need to show that $|\tilde{\omega}|<1$ for $0<\delta<2/k$. Obviously, if z_0 is a zero $t(z)$, then $1/\overline{z_0}$ is zero of $t^{*}(z)$. Then, we may write

$$\tilde{\omega}(z)=e^{2i\theta}z^{k}\frac{(z+A_1)(z+A_2)\cdots(z+A_{k+2})}{(1+\overline{A_1}z)(1+\overline{A_2}z)\cdots(1+\overline{A_{k+2}}z)}. \quad (33)$$

Using Lemma 4, we only need to show that all zeros of (31) lie in the closed unit disk for $0<\delta<2/k$. Since $|a_{0,0}|=|(\delta(1+k)-1)/(1+\delta)e^{-i\theta}|=|(\delta(1+k)-1)/(1+\delta)|<|a_{k+2,0}|=1$ for $0<\delta<2/k$, thus we have

$$t_1(z)=\frac{\overline{a_{k+2,0}}t(z)-a_{0,0}t^{*}(z)}{z}=\frac{\delta(k+2)(2-k\delta)}{(1+\delta)^{2}}\cdot\left(z^{k+1}+\frac{2(k+1)\cos\gamma}{k+2}z^{k}+\frac{k}{k+2}z^{k-1}-\frac{2}{k+2}e^{-i\theta}z-\frac{2\cos\gamma}{k+2}e^{-i\theta}\right). \quad (34)$$

By Lemma 10, we know that all zeros of

$$q(z)=z^{k+1}+\frac{2(k+1)\cos\gamma}{k+2}z^{k}+\frac{k}{k+2}z^{k-1}-\frac{2}{k+2}e^{-i\theta}z-\frac{2\cos\gamma}{k+2}e^{-i\theta} \quad (35)$$

lie inside the closed disk. Then, by Cohn's rule, $t(z)$ given by (31) has all its zeros in the closed unit disk. The proof is complete. □

Theorem 12. *Let $F_{\mathfrak{a}}$ be a mapping given by (2) and $f_\gamma=h_\gamma+g_\gamma\in\mathbb{S}^{0}_{CHD}$ be a mapping given by (6) with the dilatation $\omega_k=g'_\gamma/h'_\gamma=e^{i\theta}z^{k}(\theta\in\mathbb{R},k\in\mathbb{N}^{+})$. Then, $F_{\mathfrak{a}}*f_\gamma$ is univalent and convex in the horizontal direction for $-1<\mathfrak{a}\le(2-k)/(2+k)$.*

Proof. By Lemma 8, we need to prove that $F_{\mathfrak{a}}*f_\gamma$ is locally univalent and sense-preserving, i.e., the dilatation $\tilde{W}$ of $F_{\mathfrak{a}}*f_\gamma$ satisfies $|\tilde{W}(z)|<1$ for all $z\in\mathbb{E}$. Substituting $\omega=e^{i\theta}z^{k}$ in (25),

$$\tilde{W}(z)=e^{2i\theta}z^{k}\times\frac{z^{k+2}+(1-\mathfrak{a})\cos\gamma z^{k+1}-\mathfrak{a}z^{k}+((1+\mathfrak{a})k-2)/2e^{-i\theta}z^{2}+[(k-1)+\mathfrak{a}(k+1)]\cos\gamma e^{-i\theta}z+((k+2)\mathfrak{a}+k)/2e^{-i\theta}}{1+(1-\mathfrak{a})\cos\gamma z-\mathfrak{a}z^{2}+((1+\mathfrak{a})k-2)/2e^{i\theta}z^{k}+[(k-1)+\mathfrak{a}(k+1)]\cos\gamma e^{i\theta}z^{k+1}+((k+2)\mathfrak{a}+k)/2e^{i\theta}z^{k+2}}=e^{2i\theta}z^{k}\frac{u(z)}{u^{*}(z)}, \quad (36)$$

where

$$u(z)=z^{k+2}+(1-\mathfrak{a})\cos\gamma z^{k+1}-\mathfrak{a}z^{k}+((1+\mathfrak{a})k-2)/2e^{-i\theta}z^{2}+[(k-1)+\mathfrak{a}(k+1)]\cos\gamma e^{-i\theta}z+((k+2)\mathfrak{a}+k)/2e^{-i\theta},$$

$$u^{*}(z)=1+(1-\mathfrak{a})\cos\gamma z-\mathfrak{a}z^{2}+((1+\mathfrak{a})k-2)/2e^{i\theta}z^{k}+[(k-1)+\mathfrak{a}(k+1)]\cos\gamma e^{i\theta}z^{k+1}+((k+2)\mathfrak{a}+k)/2e^{i\theta}z^{k+2}. \quad (37)$$

If we substitute $\mathfrak{a}=(2-k)/(2+k)$ into (36), we yield

$$\tilde{W}(z)=e^{2i\theta}z^{k}\frac{z^{k+2}+2k/(k+2)\cos\gamma z^{k+1}-(2-k)/(k+2)z^{k}-(2-k)/(k+2)e^{-i\theta}z^{2}+2k/(k+2)\cos\gamma e^{-i\theta}z+e^{-i\theta}}{1+2k/(k+2)\cos\gamma z-(2-k)/(k+2)z^{2}-(2-k)/(k+2)e^{i\theta}z^{k}+2k/(k+2)\cos\gamma e^{i\theta}z^{k+1}+e^{i\theta}z^{k+2}}=e^{i\theta}z^{k}. \quad (38)$$

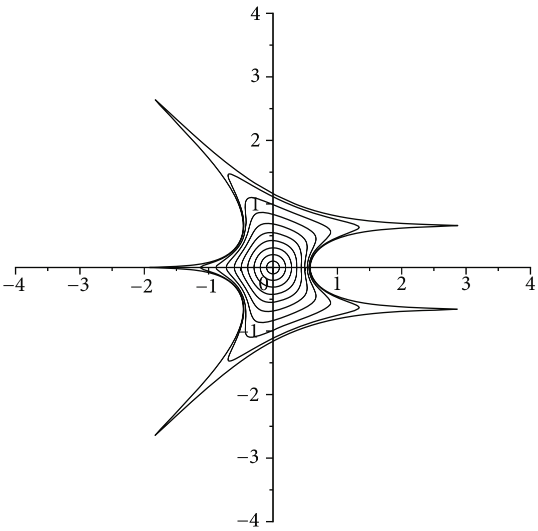

Figure 1: Image of $f_{\Pi/2}$.

Hence, $|\tilde{W}(z)| = |e^{i\theta}z^k| < 1$.

Next, we will show that $|\tilde{W}(z)| < 1$ for all $-1 < \mathfrak{a} < (2-k)/(2+k)$. If z_0 is a zero of $u(z)$, then $1/\overline{z_0}$ is zero of $u^*(z)$; hence,

$$\tilde{W}(z) = e^{2i\theta}z^k \frac{u(z)}{u^*(z)} = e^{2i\theta}z^k \cdot \frac{(z+A_1)(z+A_2)\cdots(z+A_{k+2})}{\left(1+\overline{A_1}z\right)\left(1+\overline{A_2}z\right)\cdots\left(1+\overline{A_{k+2}}z\right)}. \tag{39}$$

By Lemma 4, we need to show that all zeros of $u(z)$ lie inside or on the unit disk for $-1 < \mathfrak{a} < (2-k)/(2+k)$. Since

$$\left|a_{0,0}\right| = \left|\frac{(k+2)\mathfrak{a}+k}{2}e^{-i\theta}\right| < 1 = \left|a_{k+2,0}\right| \text{ for } -1 < \mathfrak{a} < \frac{2-k}{2+k}, \tag{40}$$

from (12), we have

$$\begin{aligned} u_1(z) &= \frac{\overline{a_{k+2,0}}u(z) - a_{0,0}u^*(z)}{z} \\ &= -\frac{(k+2)(1+\mathfrak{a})[(k+2)\mathfrak{a}+k-2]}{4} \cdot \left(z^{k+1} + \frac{2(k+1)\cos\gamma}{k+2}z^k + \frac{k}{k+2}z^{k-1} - \frac{2}{k+2}e^{-i\theta}z^k - \frac{2\cos\gamma}{k+2}e^{-i\theta}\right) \\ &= -\frac{(k+2)(1+\mathfrak{a})[(k+2)\mathfrak{a}+k-2]}{4}q(z), \end{aligned} \tag{41}$$

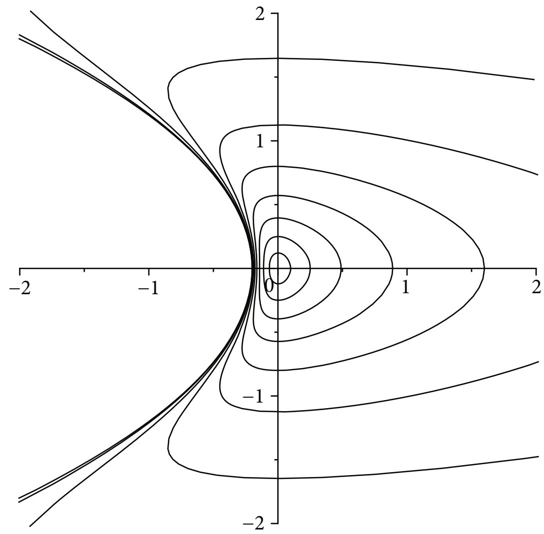

Figure 2: Image of $P_{2/3}$.

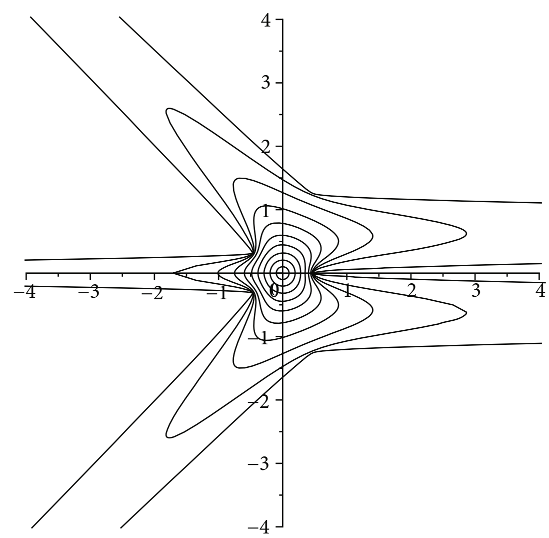

Figure 3: Image of $P_{2/3} * f_{\Pi/2}$.

where

$$q(z) = z^{k+1} + \frac{2(k+1)\cos\gamma}{k+2}z^k + \frac{k}{k+2}z^{k-1} - \frac{2}{k+2}e^{-i\theta}z - \frac{2\cos\gamma}{k+2}e^{-i\theta}. \tag{42}$$

Because $(k+2)(1+\mathfrak{a})[(k+2)\mathfrak{a}+k-2]/4 \neq 0$ for $-1 < \mathfrak{a} < (2-k)/(2+k)$, it follows that both $u_1(z)$ and $q(z)$ have the same zeros. By Lemma 10, we know that all zeros of $q(z)$ lie inside the closed unit disk. Then, by Cohn's rule, we know that all zeros $u(z)$ lie inside or on the boundary of the unit disk. The proof is completed. □

Theorem 13. *Let $F_0 = H_0 + \overline{G_0} \in \mathbb{S}^0_{CHD}$ be a harmonic mapping with $H_0 - G_0 = z/(1-z)$ and dilatation $G_0'(z)/H_0'(z) = z$. Let $f_{\pi/2} = h_{\pi/2} + \overline{g_{\pi/2}} \in \mathbb{S}_{CHD}$ be a mapping defined by (6)*

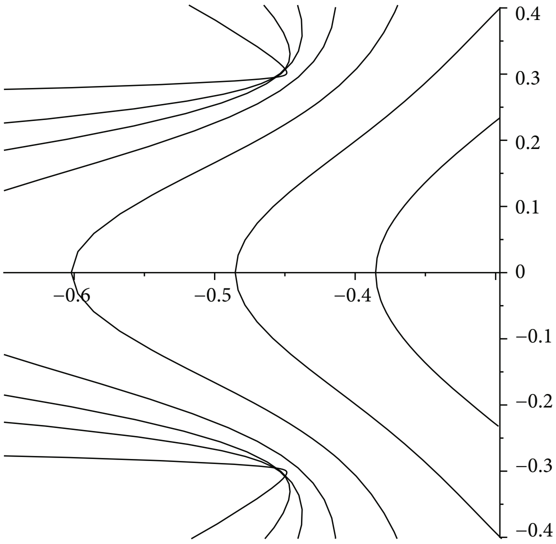

Figure 4: Image of $P_{3/4} * f_{\Pi/2}$.

with $\gamma = \pi/2$ and dilatation $\omega_\mu(z) = (\mu + z^2)/(1 + \mu z^2)$, $-1 < \mu < 1$. Then the mapping $F_0 * f_{\pi/2}$ is univalent and convex in the horizontal direction.

Proof. Since $f_{\pi/2} = h_{\pi/2} + \overline{g_{\pi/2}} \in \mathbb{S}_{\mathrm{CHD}}$ is a mapping defined by (6) with $\gamma = \pi/2$, we have

$$h_{\pi/2}(z) - g_{\pi/2}(z) = \frac{1}{2i} \log\left(\frac{1 + iz}{1 - iz}\right). \tag{43}$$

Therefore, we know that

$$\begin{aligned} \widetilde{W}(z) &= \frac{(G_0 * g_{\pi/2})'}{(H_0 * h_{\pi/2})'} = \frac{\left(zg'_{\pi/2} - g_{\pi/2}\right)'}{\left(zh'_{\pi/2} + h_{\pi/2}\right)'} \\ &= \frac{zg''_{\pi/2}}{2h'_{\pi/2} + zh''_{\pi/2}} = z\frac{\omega_\mu h''_{\pi/2} + \omega'_\mu h'_{\pi/2}}{2h'_{\pi/2} + zh''_{\pi/2}}. \end{aligned} \tag{44}$$

Substituting

$$\begin{aligned} h'_{\pi/2}(z) &= \frac{1}{\omega_\mu (1 + z^2)}, \\ h''_{\pi/2}(z) &= \frac{\omega'_\mu (1 + z^2) - 2z\omega_\mu}{\left(1 - \omega_\mu\right)^2 (1 + z^2)^2}, \end{aligned} \tag{45}$$

into (44) yields

$$\widetilde{W}(z) = z\frac{\omega_\mu^2 - \left(\omega_\mu - 1/2\omega'_\mu z\right) + 1/2\omega'_\mu 1/z}{1/z - \left(\omega_\mu - 1/2\omega'_\mu z\right) 1/z + 1/2\omega'_\mu z^2}. \tag{46}$$

Setting $\omega_\mu(z) = (\mu + z^2)/(1 + \mu z^2)$ in the above equation, we get $\widetilde{W}(z) = z^2$, and hence, $|\widetilde{W}(z)| < 1$ for all $z \in \mathbb{E}$. □

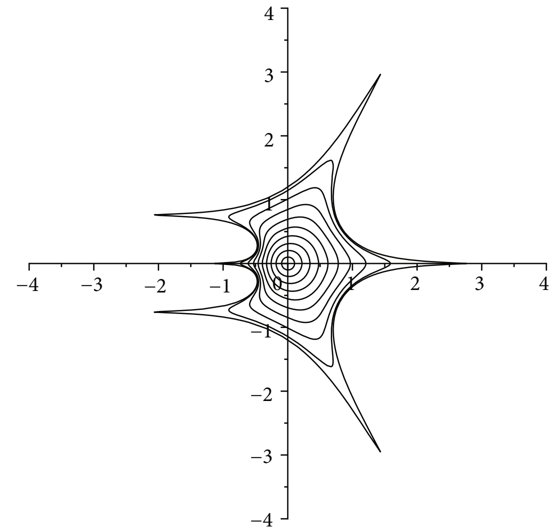

Figure 5: Image of $f_{2\Pi/3}$.

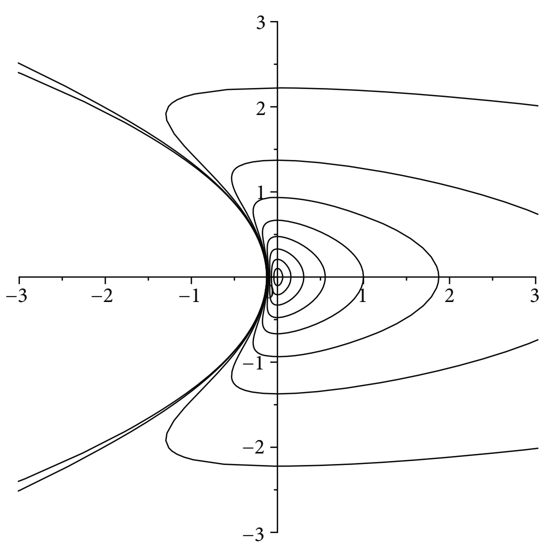

Figure 6: Image of $F_{-1/3}$.

Example 14. Suppose $f_\gamma = h_\gamma + \overline{g_\gamma} \in \mathbb{S}^0_{\mathrm{CHD}}$ be given by ((6)). If we set $\gamma = \pi/2$ and $\omega_1 = -z^3$ then by shear construction of Clunie and Sheil-Small [2], *we have*

$$\begin{aligned} h_\gamma(z) &= \frac{1}{6}\log(1 + z) - \frac{i}{4}\log\left(\frac{1 + iz}{1 - iz}\right) \\ &\quad + \frac{1}{4}\log(1 + z^2) - \frac{1}{3}\log(1 - z + z^2), \\ g_\gamma(z) &= \frac{1}{6}\log(1 + z) + \frac{i}{4}\log\left(\frac{1 + iz}{1 - iz}\right) \\ &\quad + \frac{1}{4}\log(1 + z^2) - \frac{1}{3}\log(1 - z + z^2). \end{aligned} \tag{47}$$

Recall that, if $f = h + \overline{g} \in \mathbb{S}_H$, then

$$P_\delta * f = \frac{\delta zh' + h}{1 + \delta} + \overline{\frac{\delta zg' g}{1 + \delta}}. \tag{48}$$

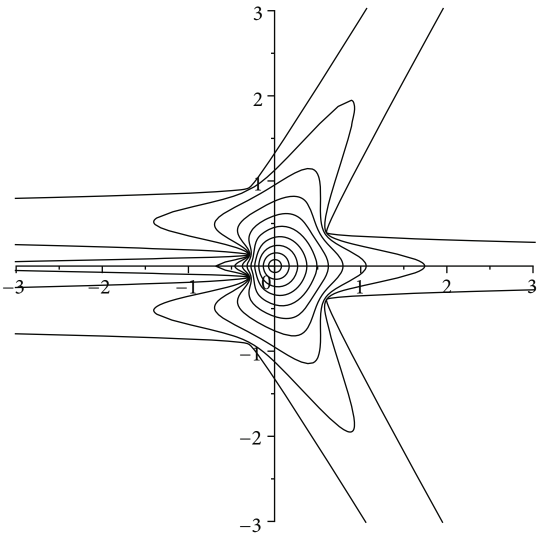

Figure 7: Image of $F_{-1/3} * f_{2\Pi/3}$.

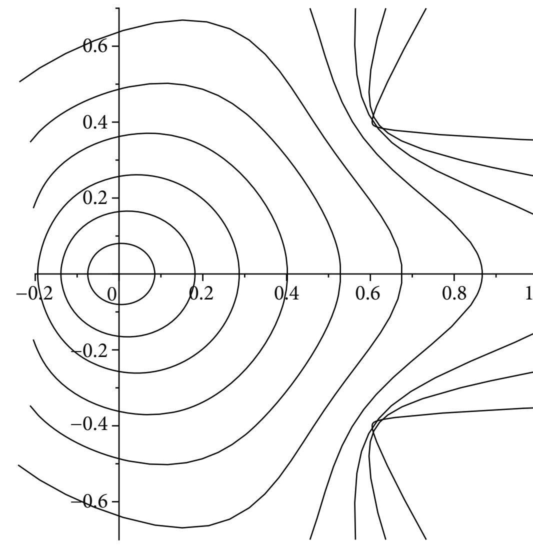

Figure 8: Image of $F_{-1/4} * f_{2\Pi/3}$.

So, we have

$$\begin{aligned} P_\delta * f_\gamma &= \frac{1}{1+\delta}\left[\delta z h_\gamma'(z) + h_\gamma(z)\right] + \frac{1}{1+\delta}\overline{\left[\delta z g_\gamma'(z) g_\gamma(z)\right]} \\ &= \frac{1}{1+\delta}\left[\frac{\delta z}{(1+z^3)(1+z^2)} + \frac{1}{6}\log(1+z) - \frac{i}{4}\log \right. \\ &\quad \left. \cdot \left(\frac{1+iz}{1-iz}\right) + \frac{1}{4}\log\left(1+z^2\right) - \frac{1}{3}\log\left(1-z+z^2\right)\right] \\ &\quad + \frac{1}{1+\delta}\overline{\left[\frac{\delta z^4}{(1+z^3)(1+z^2)}\frac{1}{6}\log(1+z)\frac{i}{4}\log\left(\frac{1+iz}{1iz}\right)\right.} \\ &\quad \overline{\left.\cdot\frac{1}{4}\log\left(1+z^2\right) + \frac{1}{3}\log\left(1z+z^2\right)\right]} \\ &= \operatorname{Re}\left\{\frac{1}{1+\delta}\left[\frac{\delta z\left(1-z^3\right)}{(1+z^3)(1+z^2)} - \frac{i}{2}\log\left(\frac{1+iz}{1-iz}\right)\right]\right\} \\ &\quad + i\operatorname{Im}\left\{\frac{1}{1+\delta}\left[\frac{\delta z}{1+z^2} + \frac{1}{3}\log(1+z)\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\log\left(1+z^2\right) - \frac{2}{3}\log\left(1-z+z^2\right)\right]\right\}. \end{aligned} \tag{49}$$

Now, in view of Theorem 11, if we set the parameter $\delta = 2/3$, then $P_\delta * f_\gamma$ is univalent and CHD. Also, if we choose $\delta = 3/4$, then $P_\delta * f_\gamma$ is not guaranteed to be univalent. The images of $|z| = r < 1$ under $f_{\Pi/2}, P_{2/3}, P_{2/3} * f_{\Pi/2}$ and $P_{3/4} * f_{\Pi/2}$ are shown in Figures 1–4.

Example 15. Suppose $f_\gamma = h_\gamma + \overline{g_\gamma} \in \mathbb{S}^0_{\mathrm{CHD}}$ be given by (6). If we set $\gamma = 2\pi/3$ and $\omega_2 = z^4$, then calculations lead to

$$\begin{aligned} h_\gamma(z) &= \frac{1}{12}\log(1+z) + \frac{1}{4}\log\left(\frac{1+z^2}{1-z}\right) \\ &\quad - \frac{1}{6}\log\left(1-z+z^2\right) - \frac{i\sqrt{3}}{6}\log\left(\frac{1+ze^{(2\pi/3)i}}{1+ze^{(-2\pi/3)i}}\right), \\ g_\gamma(z) &= \frac{1}{12}\log(1+z) + \frac{1}{4}\log\left(\frac{1+z^2}{1-z}\right) \\ &\quad - \frac{1}{6}\log\left(1-z+z^2\right) + \frac{i\sqrt{3}}{6}\log\left(\frac{1+ze^{(2\pi/3)i}}{1+ze^{(-2\pi/3)i}}\right). \end{aligned} \tag{50}$$

If $f = h + \overline{g} \in \mathbb{S}_H$, then

$$F_\alpha * f = \frac{1}{2}\left[\frac{(1+\alpha)zh'}{1-\alpha} + h\right] + \frac{1}{2}\overline{\left[\frac{(1+\alpha)zg'}{1\alpha}g\right]}. \tag{51}$$

So, we have

$$\begin{aligned} F_\alpha * f_\gamma &= \frac{1}{2}\left[\frac{(1+\alpha)zh_\gamma'}{1-\alpha} + h_\gamma\right] + \frac{1}{2}\overline{\left[\frac{(1+\alpha)zg_\gamma'}{1\alpha}g_\gamma\right]} \\ &= \frac{1}{2}\left[\frac{(1+\alpha)z}{(1-\alpha)(1-z+z^2)(1-z^4)} + \frac{1}{12}\log(1+z)\right. \\ &\quad + \frac{1}{4}\log\left(\frac{1+z^2}{1-z}\right) - \frac{1}{6}\log\left(1-z+z^2\right) \\ &\quad \left. - \frac{i\sqrt{3}}{6}\log\left(\frac{1+ze^{(2\pi/3)i}}{1+ze^{(-2\pi/3)i}}\right)\right] \\ &\quad + \frac{1}{2}\overline{\left[\frac{(1+\alpha)z^5}{(1\alpha)(1z+z^2)(1z^4)}\frac{1}{12}\log(1+z)\frac{1}{4}\log\left(\frac{1+z^2}{1z}\right)\right.} \\ &\quad \overline{\left. + \frac{1}{6}\log\left(1z+z^2\right)\frac{i\sqrt{3}}{6}\log\left(\frac{1+ze^{(2\pi/3)i}}{1+ze^{(2\pi/3)i}}\right)\right]} \\ &= \operatorname{Re}\left\{\frac{1}{2}\left[\frac{(1+\alpha)z\left(1+z^4\right)}{(1-\alpha)(1-z+z^2)(1-z^4)}\right.\right. \\ &\quad \left.\left. - \frac{i\sqrt{3}}{3}\log\left(\frac{1+ze^{(2\pi/3)i}}{1+ze^{(-2\pi/3)i}}\right)\right]\right\} \\ &\quad + i\operatorname{Im}\left\{\frac{1}{2}\left[\frac{(1+\alpha)z}{(1-\alpha)(1-z+z^2)} + \frac{1}{6}\log(1+z)\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\log\left(\frac{1+z^2}{1-z}\right) - \frac{1}{3}\log\left(1-z+z^2\right)\right]\right\}. \end{aligned} \tag{52}$$

Now, if we set the parameter $\alpha = -1/3$, in view of Theorem 11, $F_\alpha * f_\gamma$ is univalent and CHD. If we choose $\alpha = -1/4$, then $F_\alpha * f_\gamma$ is not guaranteed to be univalent (see Figures 5–8).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

References

[1] H. Lewy, "On the non-vanishing of the Jacobian in certain one-to-one mappings," *Bulletin of the American Mathematical Society*, vol. 42, no. 10, pp. 689–693, 1936.

[2] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiæ Scientiarum Fennicæ Mathematica*, vol. 9, pp. 3–25, 1984.

[3] P. Duren, *Harmonic Mapping in the Plane*, Cambridge University Press, Cambridge, 2004.

[4] M. Dorff, "Convolutions of planar harmonic convex mappings," *Complex Variables, Theory and Application: An International Journal*, vol. 45, no. 3, pp. 263–271, 2001.

[5] M. Dorff, M. Nowak, and M. Woloszkiewicz, "*Convolutions of harmonic convex mappings*," *Complex Variables and Elliptic Equations*, vol. 57, no. 5, pp. 489–503, 2012.

[6] M. Dorff, S. G. Hamidi, J. M. Jahangiri, and E. Yasar, "Convolutions of planar harmonic strip mappings," *Complex Variables and Elliptic Equations*, pp. 1–18, 2020.

[7] Y. Li and Z. Liu, "Convolutions of harmonic right half-plane mappings," *Open Mathematics*, vol. 14, no. 1, pp. 789–800, 2016.

[8] Z. Liu and Y. Li, "Corrigendum to "the properties of a new subclass of harmonic univalent mappings"," *Abstract and Applied Analysis*, vol. 2015, Article ID 794108, 2015.

[9] Z. Liu, Y. Jiang, and Y. Sun, "Convolutions of harmonic half-plane mappings with harmonic vertical strip mappings," *Filomat*, vol. 31, no. 7, pp. 1843–1856, 2017.

[10] Z. Liu, Z. Wang, A. Rasila, and Y. Jiang, "Convolutions of harmonic right half-plane mappings with harmonic strip mappings," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 3, pp. 1199–1212, 2019.

[11] C. Singla, S. Gupta, and S. Singh, "An alternative approach to convolutions of harmonic mappings," in *Mathematical Analysis and Computing. ICMAC 2019*, R. N. Mohapatra, S. Yugesh, G. Kalpana, and C. Kalaivani, Eds., vol. 344 of Springer Proceedings in Mathematics & Statistics, pp. 623–633, Springer, Singapore, 2021.

[12] E. Yasar, "On convolutions of slanted half-plane mappings," *Journal of Taibah University for Science*, vol. 15, no. 1, pp. 71–76, 2021.

[13] R. Kumar, S. Gupta, S. Singh, and M. Dorff, "An application of Cohn's rule to convolutions of univalent harmonic mapping," *Rocky Mountain Journal of Mathematics*, vol. 46, no. 2, pp. 559–570, 2016.

[14] Z. Wang, Z. Liu, and Y. Li, "On convolutions of harmonic univalent mappings convex in the direction of the real axis," *Journal of Applied Analysis & Computation*, vol. 6, no. 1, pp. 145–155, 2016.

[15] E. Yasar and O. Ozdemir, "Convolutions of a subclass of harmonic univalent mappings," *TWMS Journal of Applied and Engineering Mathematics*, vol. 10, no. 2, pp. 353–359, 2020.

[16] C. Pommerenke, "*On starlike and close-to-convex functions*," *Proceedings of the London Mathematical Society*, vol. 13, pp. 290–304, 1963.

[17] P. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.

[18] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, Oxford University Press, Oxford, 2002.

[19] M. Marden, *Geometry of Polynomials*, American Mathematical Society, 1966.

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Research Article

Numerical Scheme for Finding Roots of Interval-Valued Fuzzy Nonlinear Equation with Application in Optimization

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In this research article, we propose efficient numerical iterative methods for estimating roots of interval-valued trapezoidal fuzzy nonlinear equations. Convergence analysis proves that the order of convergence of numerical schemes is 3. Some real-life applications are considered from optimization as numerical test problems which contain interval-valued trapezoidal fuzzy quantities in parametric form. Numerical illustrations are given to show the dominance efficiency of the newly constructed iterative schemes as compared to existing methods in literature.

1. Introduction

One of the ancient problems of science and engineering in general and in mathematics is to approximate roots of a nonlinear equation. The nonlinear equations play a major role in the field of engineering, mathematics, physics, chemistry, economics, medicines finance, and in optimization. Many times the particular realization of such type of nonlinear problems involves imprecise and nonprobabilistic uncertainties in the parameter, where the approximations are known due to expert knowledge or due to some experimental data. Due to these reasons, several real-world applications contain vagueness and uncertainties. Therefore, in most of real-world problems, the parameter involved in the system or variables of the nonlinear functions are presented by a fuzzy number or interval-valued trapezoidal fuzzy number. The concept of fuzzy numbers and arithmetic operation with fuzzy numbers were first introduced and investigated in [1–8]. Hence, it is necessary to approximate the roots of fuzzy nonlinear equation.

$$F(r) = c. \tag{1}$$

The standard analytical technique like the Buckley and Qu method [9–12] is not suitable for solving the equations like

$$ar^6 + br^4 - cr^3 + dr - e = f,\ r + \cos(r) = g,\ r\ln(r) + e^r - \frac{1}{1+r^2} + \tan(r) = h, \tag{2}$$

where a, b, c, d, e, f, g,and h are fuzzy numbers and r is a fuzzy variable.

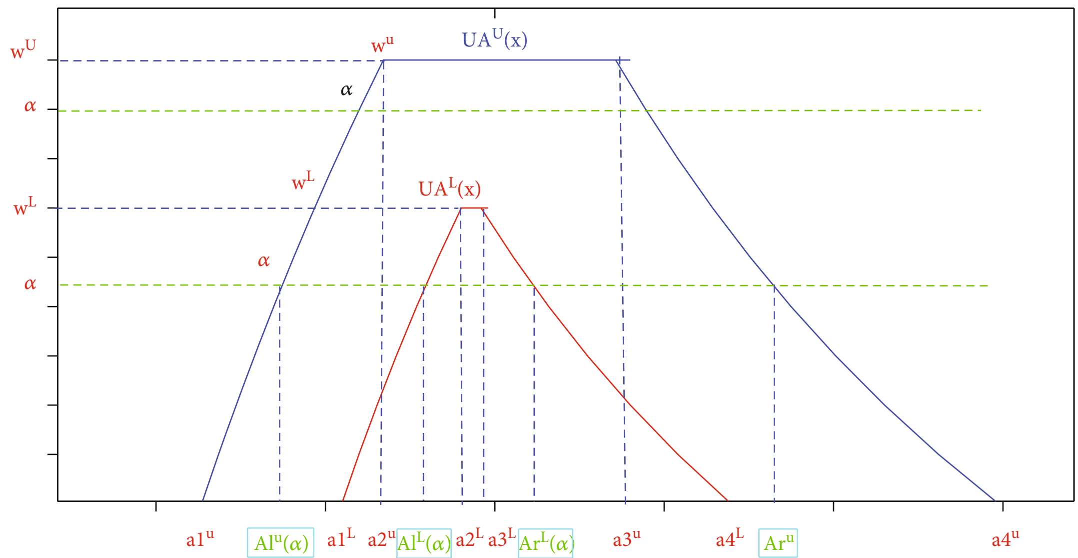


Figure 1: Alpha-cut level of interval-valued trapezoidal fuzzy number **A**.

We therefore look towards numerical iterative schemes which approximate the roots of fuzzy nonlinear equations. To approximate roots of fuzzy nonlinear equations, Abbsbandy and Asady [13] used Newton's method, Allahviranloo and Asari [14] used the Newton-Raphson method, Mosleh [15] used the Adomian decomposition method, and Ibrahim et al. give the Levenberg-Marquest method (see also [16–23]).

This research article is aimed at proposing efficient higher order iterative method as compared to well-known classical method, such as the Newton-Raphson method. Numerical test results, CPU time, and log of residual show the dominance efficiency of our newly constructed method over the classical Newton's method.

This paper is organized in five sections. In Section 2, we recall some fundamental results of interval-valued trapezoidal fuzzy numbers. In Section 3, we propose numerical iterative scheme for approximating roots of interval-valued trapezoidal fuzzy nonlinear equations and its convergence analysis. In Section 4, we illustrate some real-world applications from optimization as numerical test examples to show the performance and efficiency of the constructed method and conclusions in the last section. Section 5 is a conclusion section.

2. Preliminaries

Definition 1. A fuzzy number is a fuzzy set like $r\quad : \mathrm{R} \longrightarrow I = [0, 1]$ which satisfies [24–27].

(1) r is upper semicontinuous

(2) $r(a) = 0$ outside some interval $[a_1, a_2]$

(3) There are real numbers b_1, b_2 such that $a_1 \le b_1 \le b_2 \le a_2$ and

  (i) $r(a)$ is monotonic increasing on $[a_1, b_1]$

  (ii) $r(a)$ is monotonic decreasing on $[b_2, a_2]$

  (iii) $r(a) = 1$, for $b_1 \le a \le b_2$

We denote by E, the set of all fuzzy numbers. An equivalent parametric form is also given in [19] as follows.

Definition 2 [28]. A fuzzy number r in parametric form is a pair (r^L, r^U) of function $r^L(\tau), r^U(\tau), 0 \le \tau \le 1$, which satisfies the following requirements:

(1) $r^L(\tau)$ is a bounded monotonic increasing left continuous function

(2) $r^U(\tau)$ is a bounded monotonic decreasing left continuous function

(3) $r^L(\tau) \le r^U(\tau), 0 \le \tau \le 1$

A popular fuzzy number is the generalized interval-valued trapezoidal fuzzy number A, denoted by $A = (a_1, a_2, a_3, a_4 ; \widehat{w}), 0 < \widehat{w} < 1$, a fuzzy number with membership function as follows:

$$A(r) = \begin{cases} \widehat{w}\,\dfrac{r - a_1}{a_2 - a_1} & \text{if } a_1 < r < a_2, \\ \widehat{w} & \text{if } a_2 \le r \le a_3, \\ \widehat{w}\,\dfrac{a_4 - r}{a_4 - a_3} & a_3 < r < a_4, \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

Assume $F_{\mathrm{TN}}(\widehat{w})$ be the family of all $\widehat{w}$-trapezoidal fuzzy number, i.e.,

$$F_{\mathrm{TN}}(\widehat{w}) = \left\{ \begin{matrix} A = (a_1, a_2, a_3, a_4 ; \widehat{w}), a_1 \le a_2 \le a_3 \le a_4 ; \\ 0 < \widehat{w} < 1 \end{matrix} \right\}. \tag{4}$$

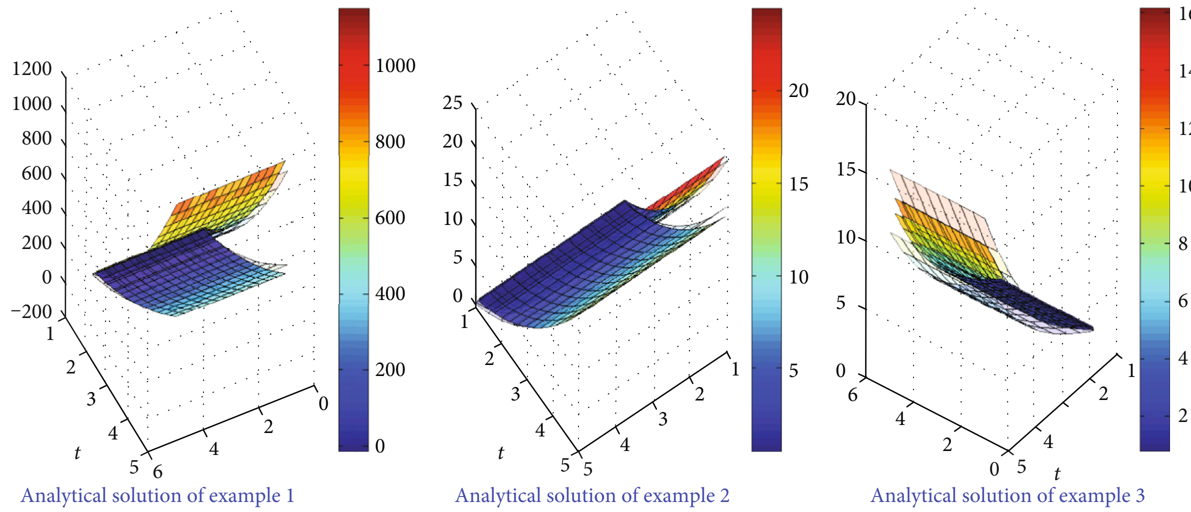


Figure 2: Analytical solution of Examples 1–3.

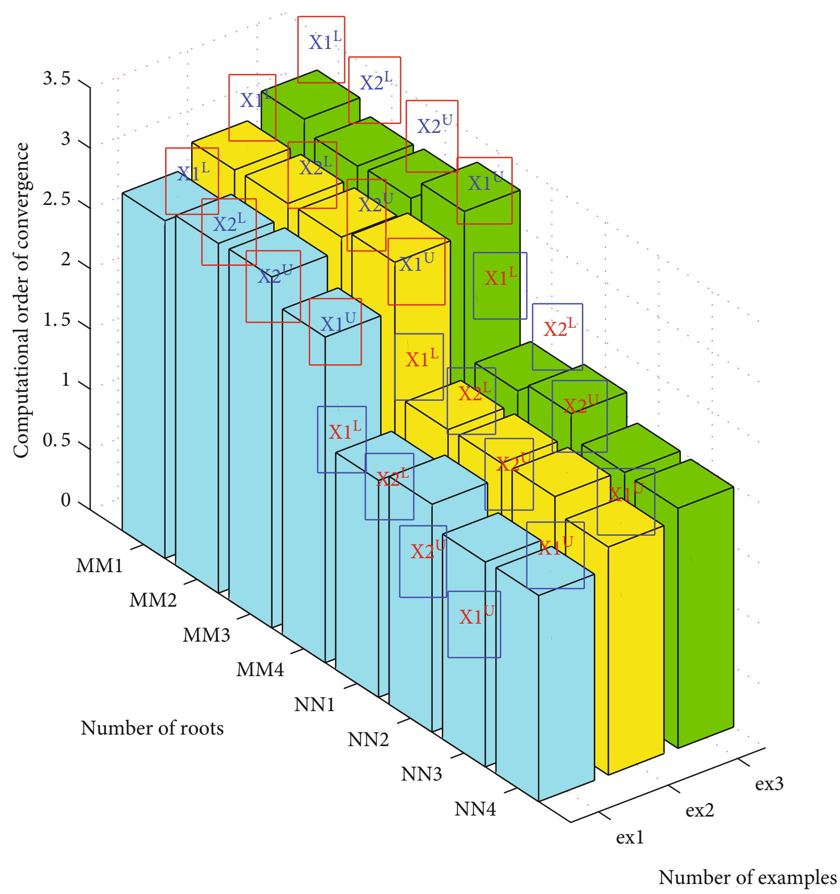


Figure 3: Computational order of convergence.

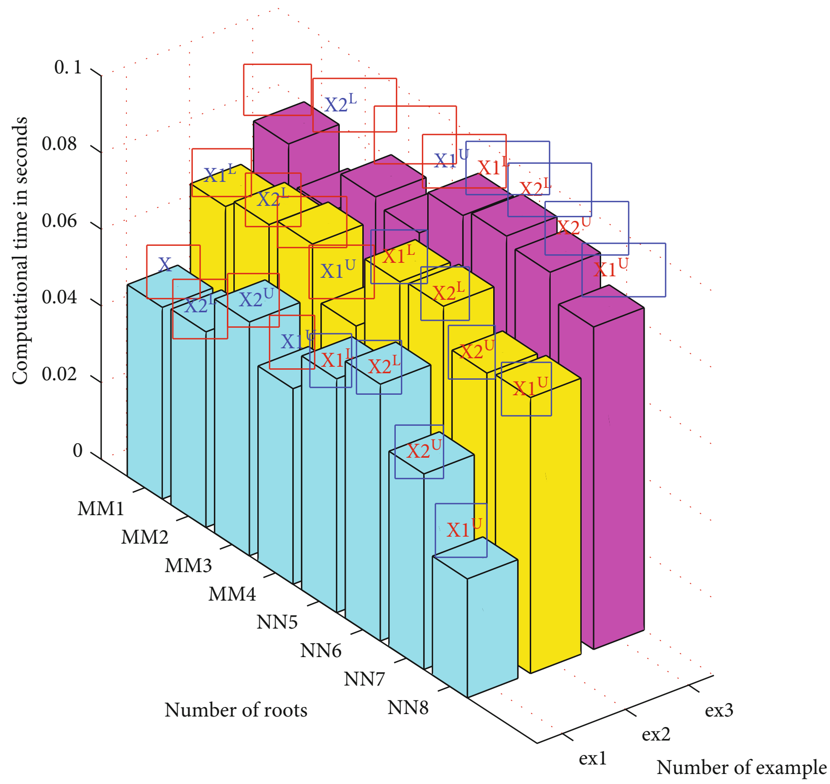


Figure 4: Computational time in seconds.

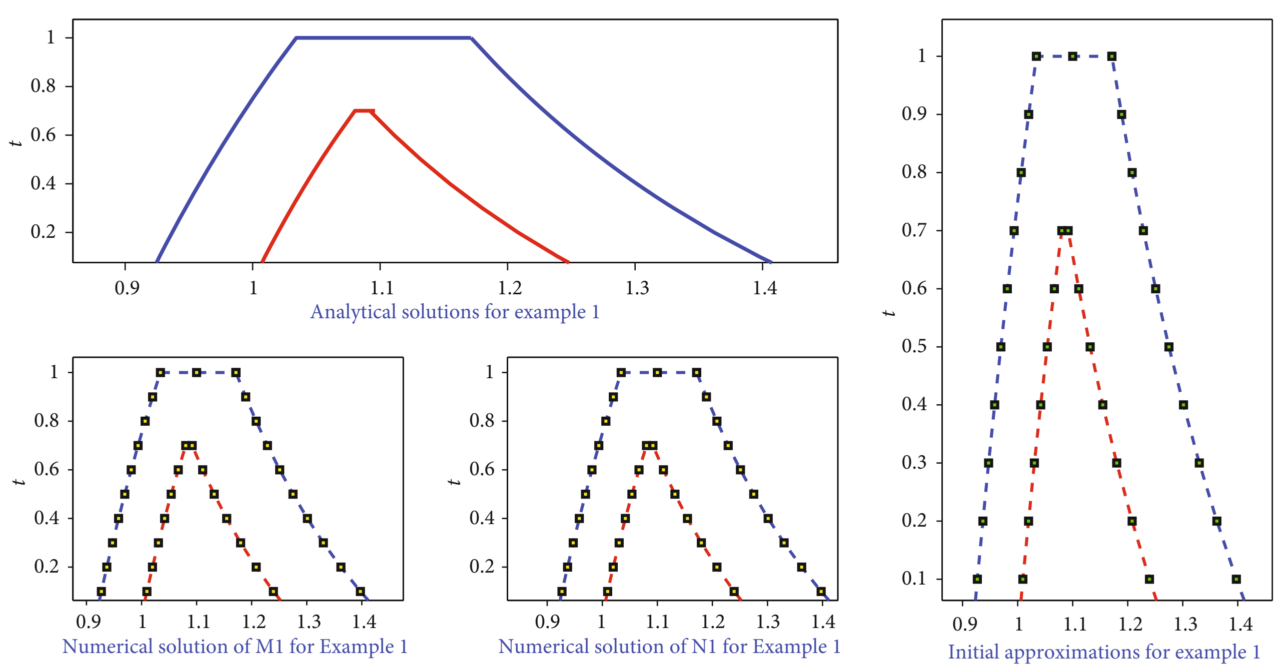


Figure 5: Initial guessed values, analytical, and numerical approximate solution.

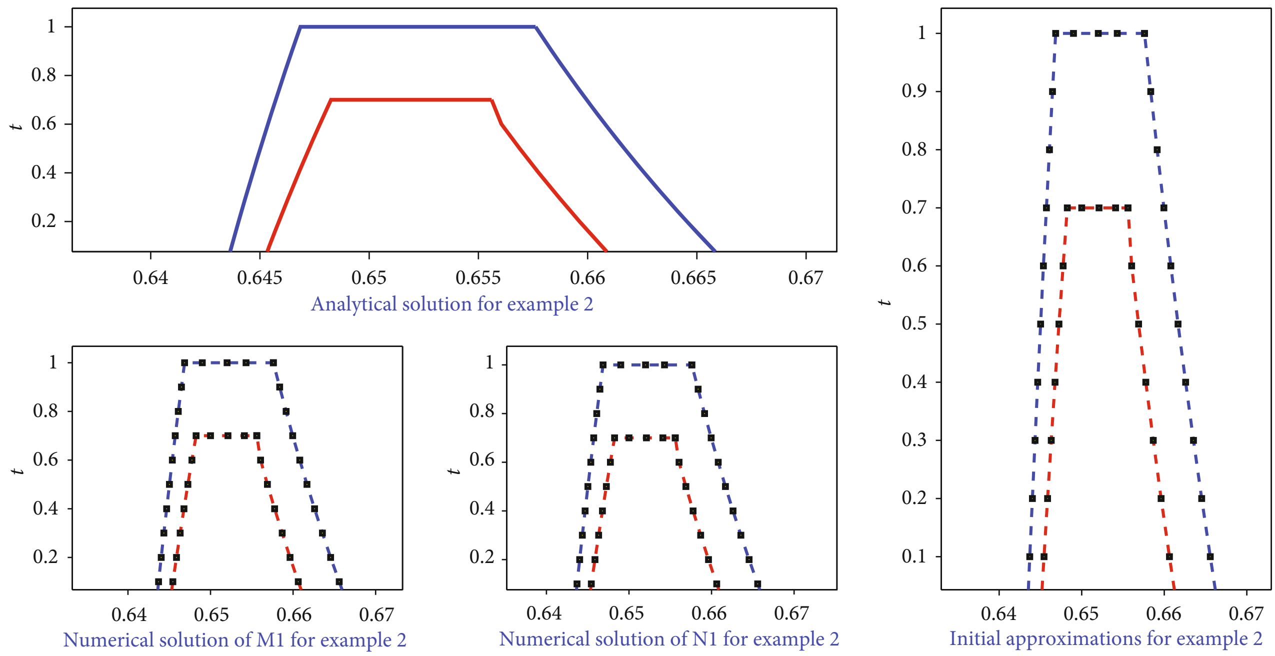


FIGURE 6: Initial guessed values, analytical, and numerical approximate solution.

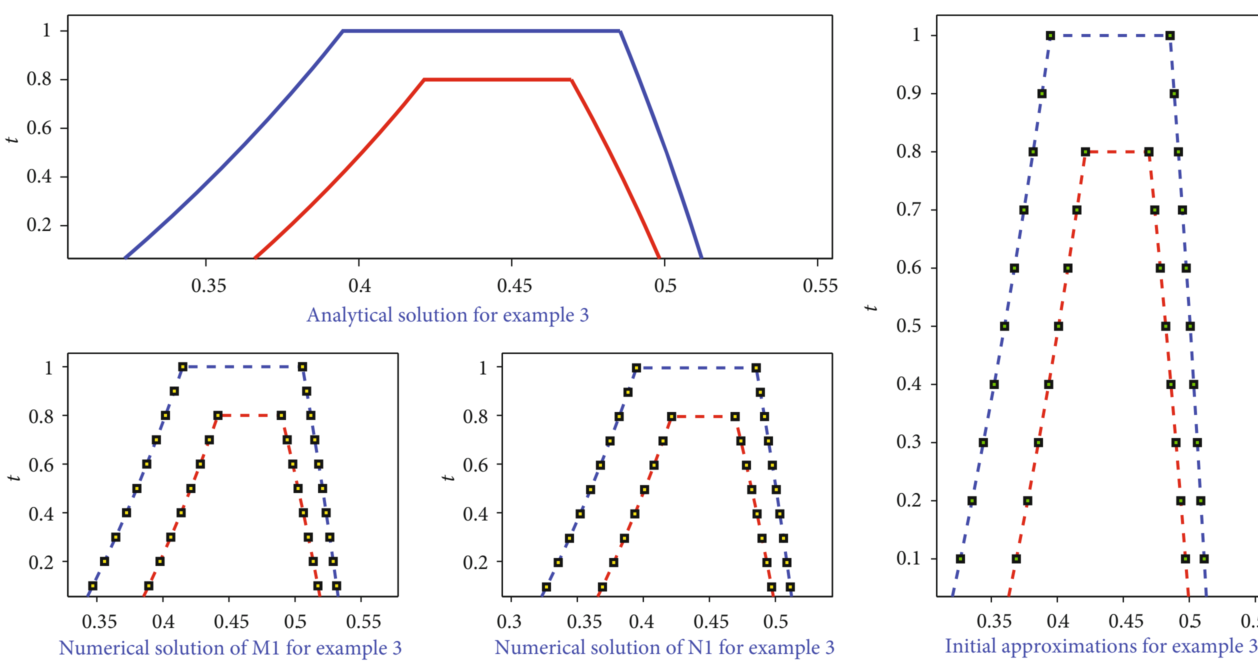
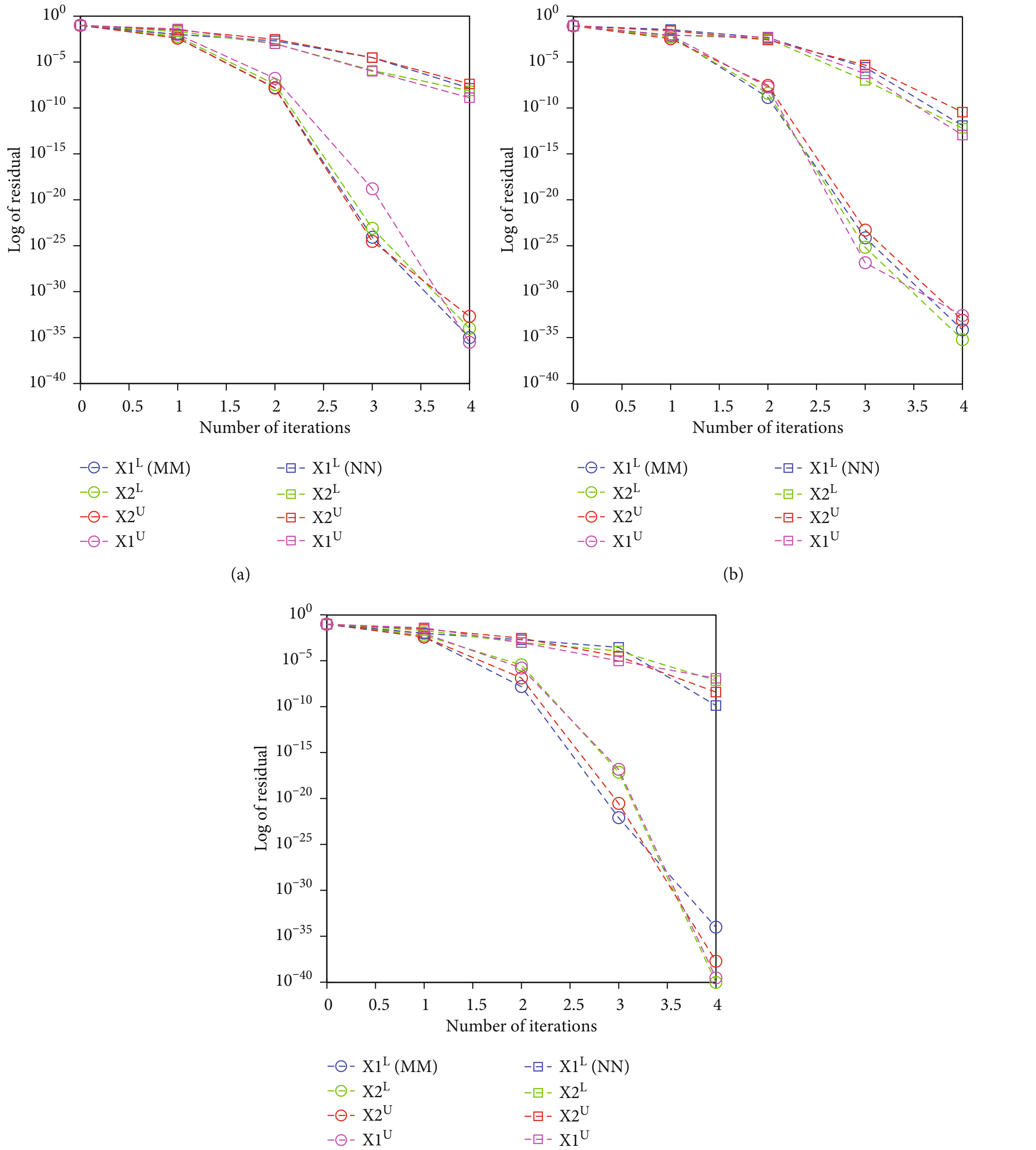


FIGURE 7: Initial guessed values, analytical, and numerical approximate solution.



(c)

Figure 8: Error graph of iterative methods MM and NN.

Step 1. Transform $F(r,\tau)=c$ into $\begin{cases} F_l^L(r_l^L, r_t^L, r_l^U, r_t^U, \tau) = c_l^L(\tau), \\ F_t^L(r_l^L, r_t^L, r_l^U, r_t^U, \tau) = c_t^L(\tau), \\ F_l^U(r_l^L, r_t^L, r_l^U, r_t^U, \tau) = c_l^U(\tau), \\ F_t^U(r_l^L, r_t^L, r_l^U, r_t^U, \tau) = c_t^U(\tau). \end{cases}$ $\forall \tau \in [0,1]$.

Step 2. Solve $\begin{cases} F_l^L(r_l^L, r_t^L, r_l^U, r_t^U, \tau) = c_l^L(\tau), \\ F_t^L(r_l^L, r_t^L, r_l^U, r_t^U, \tau) = c_t^L(\tau), \\ F_l^U(r_l^L, r_t^L, r_l^U, r_t^U, \tau) = c_l^U(\tau), \\ F_t^U(r_l^L, r_t^L, r_l^U, r_t^U, \tau) = c_t^U(\tau). \end{cases}$ for $\tau=0$ and $\tau=1$ to obtain initial guess value.

Step 3. Evaluate $F(r,\tau)=c$ at initial guess point and compute Jacobian matrix $\mathbf{J}_*, \mathbf{J}_{**}$.

Step 4. Use MM to compute next iteration

$$\begin{cases} \mathbf{y}_n(\tau) = \mathbf{r}_n(\tau) - (\mathbf{F}'(\mathbf{r}_n(\tau)))^{-1}\mathbf{F}(\mathbf{r}_n(\tau)), \\ \mathbf{r}_{n+1}(\tau) = \mathbf{y}_n(\tau) - Z * (\mathbf{F}'(\mathbf{r}_n(\tau)))^{-1}\mathbf{F}(\mathbf{r}_n(\tau)), \end{cases} \quad \forall \tau \in [0,1],$$

where

$Z = (4\mathbf{J}_{**} - 2\mathbf{J}_*)^{-1} * (\mathbf{J}_* - \mathbf{J}_{**})$.

Step 5. For given $\epsilon>0$, if (i) $\mathbf{e}_n = \|\mathbf{F}(\mathbf{r},\tau)\| < \epsilon$ and (ii) $\mathbf{e}_n = \|\mathbf{r}_{n+1}(\tau) - \mathbf{r}_n(\tau)\| < \epsilon$, then stop.

Step 6. Set $k=k+1$ and go to step 1.

Algorithm 1: (MM method).

Definition 3 [29]. Let $A^L \in F_{\mathrm{TN}}(w\wedge^L)$ and $A^U \in F_{\mathrm{TN}}(w\wedge^U)A$ level $(w\wedge^L, w\wedge^U)-$ interval-valued trapezoidal fuzzy number A, denoted by

$$\mathbf{A} = [A^L, A^U] = \langle (a_1^L, a_2^L, a_3^L, a_4^L; w\wedge^L), (a_1^U, a_2^U, a_3^U, a_4^U; w\wedge^U) \rangle, \quad (5)$$

is an interval-valued fuzzy number on set R with

$$A^L(r) = \begin{cases} w\wedge^L \dfrac{r-a_1^L}{a_2^L-a_1^L} & \text{if } a_1^L < r < a_2^L, \\ w\wedge^L & \text{if } a_2^L \le r \le a_3^L, \\ w\wedge^L \dfrac{a_4^L-r}{a_4^L-a_3^L} & a_3^L < r < a_4^L, \\ 0 & \text{otherwise}, \end{cases} \text{(lower trapezoidal fuzzy number)}$$

$$A^U(r) = \begin{cases} w\wedge^U \dfrac{r-a_1^U}{a_2^U-a_1^U} & \text{if } a_1^U < r < a_2^U, \\ w\wedge^U & \text{if } a_2^U \le r \le a_3^U, \\ w\wedge^U \dfrac{a_4^U-r}{a_4^U-a_3^U} & a_3^U < r < a_4^U, \\ 0 & \text{otherwise}. \end{cases} \text{(upper trapezoidal fuzzy number)} \quad (6)$$

where $a_1^L \le a_2^L \le a_3^L \le a_4^L$, $a_1^U \le a_2^U \le a_3^U \le a_4^U$, $0 \le w\wedge^L \le w\wedge^U \le 1$, $a_1^U \le a_1^L$, and $a_1^L \le a_4^U$. This interval-valued trapezoidal fuzzy number is shown in Figure 1. Moreover, $A^L(r) \le A^U(r)$, which means the grade of membership $r \in \mathbf{A} = [A^L(r), A^U(r)]$, and the latest and greatest grade of membership at r are $A^L(r)$ and $A^U(r)$, respectively. We therefore denote the family of all interval-valued trapezoidal fuzzy number by $F(w\wedge^L, w\wedge^U) = \mathbf{A} = [A^L(r), A^U(r)]$, i.e.,

$$F(w\wedge^L, w\wedge^U) = \mathbf{A} = [A^L(r), A^U(r)] = \{\langle (a_1^L, a_2^L, a_3^L, a_4^L; w\wedge^L), (a_1^U, a_2^U, a_3^U, a_4^U; w\wedge^U) \rangle : a_1^U \le a_1^L; a_4^L \le a_4^U$$

$$A^L(r) \in F_{\mathrm{TN}}(w\wedge^L), A^U(r) \in F_{\mathrm{TN}}(w\wedge^U), 0 \le w\wedge^L \le w\wedge^U \le 1. \quad (7)$$

Table 1

τ	MM $\|\mathbf{r}_{n+1}-\mathbf{r}_n\|$	MM $\|\mathbf{F}(\mathbf{r}_n)\|$	NN $\|\mathbf{r}_{n+1}-\mathbf{r}_n\|$	NN $\|\mathbf{F}(\mathbf{r}_n)\|$
0.0	3.1e-47	1.2e-36	4.1e-10	1.4e-8
0.1	1.1e-43	3.1e-32	3.1e-10	1.1e-8
0.2	1.4e-45	1.1e-32	0.1e-20	8.2e-7
0.3	5.6e-46	3.4e-34	1.7e-10	7.2e-9
0.4	1.3e-41	1.4e-32	1.5e-16	5.5e-9
0.5	4.4e-41	1.7e-32	7.2e-11	4.1e-9
0.6	6.1e-49	7.1e-37	3.2e-11	3.1e-9
0.7	8.1e-47	9.1e-33	4.1e-10	6.2e-9
0.8	1.3e-45	4.1e-32	6.3e-11	7.5e-9
0.9	1.6e-46	6.4e-33	7.8e-11	8.1e-9
1	7.7e-47	6.8e-33	5.1e-12	3.4e-9

Definition 4 [29]. A $(w\wedge^L, w\wedge^U)$ is said to be nonnegative $F(w\wedge^L, w\wedge^U)$ iff $a_1^U \ge 0$ and denoted by $F^+(w\wedge^L, w\wedge^U)$.

Definition 5 [30]. Two $(w\wedge^L, w\wedge^U)-$ interval-valued trapezoidal fuzzy numbers.

$A = \langle (a_1^L, a_2^L, a_3^L, a_4^L; w\wedge^L), (a_1^U, a_2^U, a_3^U, a_4^U; w\wedge^U) \rangle$ and $B = \langle (b_1^L, b_2^L, b_3^L, b_4^L; w\wedge^L), (b_1^U, b_2^U, b_3^U, b_4^U; w\wedge^U) \rangle$ are said to

Table 2: Analytical solution for Example 1.

τ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
r_l^U	0.9172	0.9268	0.9368	0.9473	0.9581	0.9694	0.9812	0.9936	1.0065	1.0200	1.0342
r_l^L	1.0000	1.0096	1.0197	1.0304	1.0418	1.0538	1.0666	1.0802	1.0947	1.1102	1.1269
r_t^L	1.2749	1.2394	1.2080	1.1800	1.1547	1.1318	1.1109	1.0918	1.0743	1.0581	1.0430
r_t^U	1.4371	1.3974	1.3620	1.3301	1.3012	1.2749	1.2507	1.2285	1.2080	1.1890	1.1712

be equal iff $A = B$, i.e., $a_i^L = b_i^L$ and $a_i^U = b_i^U$ for all $i = 1, 2, 3, 4$.

Definition 6. [30]. Extend addition, scalar multiplication, and extend multiplication in $(w\wedge^L, w\wedge^U)$ interval-valued trapezoidal fuzzy number are defined as if $A = \langle(a_1^L, a_2^L, a_3^L, a_4^L ; w\wedge^L), (a_1^U, a_2^U, a_3^U, a_4^U ; w\wedge^U)\rangle$ and $B = \langle(b_1^L, b_2^L, b_3^L, b_4^L ; w\wedge^L), (b_1^U, b_2^U, b_3^U, b_4^U ; w\wedge^U)\rangle \in F(w\wedge^L, w\wedge^U)$ and $k \in$ R; then,

$$A \oplus B = \left\langle \begin{array}{c} \left(a_1^L + b_1^L, a_2^L + b_2^L, a_3^L + b_3^L, a_4^L + b_4^L ; w\wedge^L\right) \\ \left(a_1^U + b_1^U, a_2^U + b_2^U, a_3^U + b_3^U, a_4^U + b_4^U ; w\wedge^U\right) \end{array} \right\rangle,$$

$$kA = \begin{cases} \left\langle \left(ka_1^L, ka_2^L, ka_1^L, ka_1^L ; w\wedge^L\right), \left(ka_1^U, ka_2^U, ka_1^U, ka_1^U ; w\wedge^U\right) \right\rangle, k > 0, \\ \left\langle \left(ka_4^L, ka_3^L, ka_2^L, ka_1^L ; w\wedge^L\right), \left(ka_4^U, ka_3^U, ka_2^U, ka_1^U ; w\wedge^U\right) \right\rangle, k < 0, \\ \left\langle (0, 0, 0, 0 ; w\wedge^L), (0, 0, 0, 0 ; w\wedge^U) \right\rangle, k = 0, \end{cases}$$

$$A \otimes B = \left\langle \begin{array}{c} \left(a_1^L * b_1^L, a_2^L * b_2^L, a_3^L * b_3^L, a_4^L * b_4^L ; w\wedge^L\right) \\ \left(a_1^U * b_1^U, a_2^U * b_2^U, a_3^U * b_3^U, a_4^U * b_4^U ; w\wedge^U\right) \end{array}, a_1^U, b_1^U \geq 0 \right\rangle. \tag{8}$$

Definition 7 [28]. Let $\sigma, 0 \in$ R. The signed distance between σ and 0 is $d(\sigma, 0) = \sigma$.

Definition 8 [31]. Let $A \in F(w\wedge^L, w\wedge^U)$; then, alpha-cut set of **A** denotes and is defined by

$$\begin{aligned} \mathbf{A}(\alpha) &= \left[A^L(\alpha), A^U(\alpha)\right] \\ &= \left\{ \left[A_l^U(\alpha), A_l^L(\alpha)\right] \cup \left[A_t^L(\alpha), A_t^U(\alpha)\right] ; \right. \\ &\quad 0 \leq \alpha \leq w\wedge^L \left[A_l^U(\alpha), A_t^U(\alpha)\right] ; w\wedge^L \leq \alpha \leq w\wedge^U, \end{aligned} \tag{9}$$

where

$$\begin{aligned} A_l^L(\alpha) &= a_1^L + \left(a_2^L - a_1^L\right) \frac{\alpha}{w\wedge^L}, \\ A_t^L(\alpha) &= a_4^L + \left(a_4^L - a_3^L\right) \frac{\alpha}{w\wedge^L}, \\ A_l^U(\alpha) &= a_1^U + \left(a_2^U - a_1^U\right) \frac{\alpha}{w\wedge^U}, \\ A_t^U(\alpha) &= a_4^U + \left(a_4^U - a_3^U\right) \frac{\alpha}{w\wedge^U}, \end{aligned} \tag{10}$$

Table 3

τ	MM		NN	
	$\|\mathbf{r}_{n+1} - \mathbf{r}_n\|$	$\|\mathbf{F}(\mathbf{r}_n)\|$	$\|\mathbf{r}_{n+1} - \mathbf{r}_n\|$	$\|\mathbf{F}(\mathbf{r}_n)\|$
0.0	4.0e-49	4.0e-33	1.7e-13	1.7e-11
0.1	5.6e-47	5.6e-32	3.4e-13	1.2e-11
0.2	3.1e-43	8.8e-33	6.1e-14	3.6e-11
0.3	5.4e-48	6.1e-36	5.1e-13	7.1e-11
0.4	6.1e-49	7.9e-35	4.4e-14	5.6e-12
0.5	3.7e-48	4.0e-33	4.3e-13	3.4e-11
0.6	1.6e-48	8.2e-33	1.7e-13	4.6e-11
0.7	3.6e-49	4.3e-33	3.7e-13	6.7e-11
0.8	1.2e-49	3.6e-34	7.1e-14	5.6e-11
0.9	7.5e-46	8.9e-34	8.7e-14	1.2e-11
1	8.1e-49	6.1e-35	9.1e-14	5.1e-11

3. Construction of Iterative Scheme (MM)

In order to approximate the roots of interval-valued trapezoidal fuzzy nonlinear equation $F(r) = c$, we propose the following two-step iterative scheme as follows:

$$\begin{cases} F_l^L\left(r_l^L, r_t^L, r_l^U, r_t^U, \tau\right) = c_l^L(\tau), \\ F_t^L\left(r_l^L, r_t^L, r_l^U, r_t^U, \tau\right) = c_t^L(\tau), \\ F_l^U\left(r_l^L, r_t^L, r_l^U, r_t^U, \tau\right) = c_l^U(\tau), \\ F_t^U\left(r_l^L, r_t^L, r_l^U, r_t^U, \tau\right) = c_t^U(\tau). \end{cases} \forall \tau \in [0, 1]. \tag{11}$$

Suppose that $r = (\alpha_l^L, \alpha_t^L, \alpha_l^U, \alpha_t^U)$ is the solution of above system and $\mathbf{r}_0 = (r_{l0}^L, r_{t0}^L, r_{l0}^U, r_{t0}^U)$ is approximate solutions of the system, t denote the alpha-cut parameter; then,

$$\begin{cases} \alpha_l^L(\tau) = r_{l0}^L(\tau) + h_1(\tau), \\ \alpha_l^L(\tau) = r_{t0}^L(\tau) + k_1(\tau), \\ \alpha_l^U(\tau) = r_{l0}^U(\tau) + h_2(\tau), \\ \alpha_t^U(\tau) = r_{t0}^U(\tau) + k_2(\tau). \end{cases} \tag{12}$$

By using Taylor's series of $F_l^L, F_t^L, F_l^U, F_t^U$ about $(r_{l0}^L(\tau), r_{t0}^L(\tau), r_{l0}^U(\tau), r_{t0}^U(\tau))$, then we have the following:

$$\begin{cases} F_l^L(\alpha_l^L,\alpha_t^L,\alpha_l^U,\alpha_t^U,\tau) = F_l^L(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{l0}^U,\tau) + h_1 F^L_{l\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_1 F^L_{l\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + h_2 F^L_{l\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_2 F^L_{l\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + O((h_1,k_1,h_2,k_2)^2), \\ F_t^L(\alpha_l^L,\alpha_t^L,\alpha_l^U,\alpha_t^U,\tau) = F_t^L(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{l0}^U,\tau) + h_1 F^L_{t\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_1 F^L_{t\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + h_2 F^L_{t\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_2 F^L_{t\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + O((h_1,k_1,h_2,k_2)^2), \\ F_l^U(\alpha_l^L,\alpha_t^L,\alpha_l^U,\alpha_t^U,\tau) = F_l^U(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{l0}^U,\tau) + h_1 F^U_{l\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_1 F^U_{l\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + h_2 F^U_{l\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_2 F^U_{l\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + O((h_1,k_1,h_2,k_2)^2), \\ F_t^U(\alpha_l^L,\alpha_t^L,\alpha_l^U,\alpha_t^U,\tau) = F_t^U(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{l0}^U,\tau) + h_1 F^U_{t\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_1 F^U_{t\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + h_2 F^U_{t\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_2 F^U_{t\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + O((h_1,k_1,h_2,k_2)^2). \end{cases} \tag{13}$$

If $(r_{l0}^L(\tau), r_{t0}^L(\tau), r_{l0}^U(\tau), r_{t0}^U(\tau))$ are close to $(\alpha_l^L(\tau), \alpha_l^L(\tau), \alpha_l^U(\tau), \alpha_t^U(\tau))$, then $h_1(\tau), k_1(\tau), h_2(\tau), k_2(\tau)$ are small enough. Assume all partial derivatives of $h_1(\tau), k_1(\tau), h_2(\tau), k_2(\tau)$ exist and bounded; then, we have the following:

$$\begin{cases} F_l^L(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{l0}^U,\tau) + h_1 F^L_{l\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_1 F^L_{l\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + h_2 F^L_{l\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_2 F^L_{l\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) = c_l^L(\tau), \\ F_t^L(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{l0}^U,\tau) + h_1 F^L_{t\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_1 F^L_{t\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + h_2 F^L_{t\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_2 F^L_{t\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) = c_t^L(\tau), \\ F_l^U(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{l0}^U,\tau) + h_1 F^U_{l\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_1 F^U_{l\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + h_2 F^U_{l\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_2 F^U_{l\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) = c_l^L(\tau), \\ F_t^U(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{l0}^U,\tau) + h_1 F^U_{t\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_1 F^U_{t\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + h_2 F^U_{t\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) + k_2 F^U_{t\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) = c_t^L(\tau). \end{cases} \tag{14}$$

Since $h_1(\tau), k_1(\tau), h_2(\tau), k_2(\tau)$ are unknown quantities, they are obtained by solving the following equations:

$$J_*(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau)\begin{bmatrix} h_1(\tau) \\ k_1(\tau) \\ h_2(\tau) \\ k_2(\tau) \end{bmatrix} = \begin{bmatrix} c_l^L(\tau) - F_l^L(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) \\ c_t^L(\tau) - F_t^L(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) \\ c_l^U(\tau) - F_l^U(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) \\ c_t^U(\tau) - F_t^U(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) \end{bmatrix}, \tag{15}$$

where

$$\mathbf{J}_* = \begin{bmatrix} F^L_{l\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^L_{l\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^L_{l\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^L_{l\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) \\ F^L_{t\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^L_{t\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^L_{t\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^L_{t\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) \\ F^U_{l\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^U_{l\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^U_{l\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^U_{l\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) \\ F^U_{t\,r^L_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^U_{t\,r^L_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^U_{t\,r^U_{l0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) & F^U_{t\,r^U_{t0}}(r_{l0}^L,r_{t0}^L,r_{l0}^U,r_{t0}^U,\tau) \end{bmatrix}.$$

$$\begin{bmatrix} y_{l0}^L(\tau) \\ y_{t0}^L(\tau) \\ y_{l0}^U(\tau) \\ y_{t0}^U(\tau) \end{bmatrix} = \begin{bmatrix} r_{l0}^L(\tau) \\ r_{t0}^L(\tau) \\ r_{l0}^U(\tau) \\ r_{t0}^U(\tau) \end{bmatrix} + \begin{bmatrix} h_1(\tau) \\ k_1(\tau) \\ h_2(\tau) \\ k_2(\tau) \end{bmatrix},$$

$$\begin{bmatrix} r_{l1}^{L}(\tau) \\ r_{t1}^{L}(\tau) \\ r_{l1}^{U}(\tau) \\ r_{t1}^{U}(\tau) \end{bmatrix} = \begin{bmatrix} y_{l0}^{L}(\tau) \\ y_{t0}^{L}(\tau) \\ y_{l0}^{U}(\tau) \\ y_{t0}^{U}(\tau) \end{bmatrix} + Z * \begin{bmatrix} h_1(\tau) \\ k_1(\tau) \\ h_2(\tau) \\ k_2(\tau) \end{bmatrix},$$

$$Z = \left(4J_{**}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) - 2J_{*}\left(r_{l0}^{L}, r_{t0}^{L}, r_{l0}^{U}, r_{t0}^{U}, \tau\right)\right)^{-1} * \left(J_{*}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) - J_{**}\left(r_{l0}^{L}, r_{t0}^{L}, r_{l0}^{U}, r_{t0}^{U}, \tau\right)\right),$$

$$\mathbf{J}_{**} = \begin{bmatrix} F^{L}_{l\,y_{l0}^{L}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{L}_{l\,y_{t0}^{L}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{L}_{l\,y_{l0}^{U}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{L}_{l\,y_{t0}^{U}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) \\ F^{L}_{t\,y_{l0}^{L}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{L}_{t\,y_{t0}^{L}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{L}_{t\,y_{l0}^{U}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{L}_{t\,y_{t0}^{U}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) \\ F^{U}_{l\,y_{l0}^{L}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{U}_{l\,y_{t0}^{L}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{U}_{l\,y_{l0}^{U}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{U}_{l\,y_{t0}^{U}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) \\ F^{U}_{t\,y_{l0}^{L}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{U}_{t\,y_{t0}^{L}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{U}_{t\,y_{l0}^{U}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) & F^{U}_{t\,y_{t0}^{U}}\left(y_{l0}^{L}, y_{t0}^{L}, y_{l0}^{U}, y_{t0}^{U}, \tau\right) \end{bmatrix}. \quad (16)$$

$\mathbf{J}_{*} = J_{*}(r_{\ln}^{L}, r_{tn}^{L}, r_{\ln}^{U}, r_{tn}^{U}, \tau)$, $\mathbf{J}_{**} = J_{**}(y_{\ln}^{L}, y_{tn}^{L}, y_{\ln}^{U}, y_{tn}^{U}, \tau)$, and the next approximation for $r_{l}^{L}(\tau), r_{t}^{L}(\tau), r_{l}^{U}(\tau), r_{t}^{U}(\tau)$ is found by using recursive scheme as follows:

$$\begin{bmatrix} r^{L}_{l\,(n+1)}(\tau) \\ r^{L}_{t\,(n+1)}(\tau) \\ r^{U}_{l\,(n+1)}(\tau) \\ r^{U}_{t\,(n+1)}(\tau) \end{bmatrix} = \begin{bmatrix} y^{L}_{l\,(n)}(\tau) \\ y^{L}_{t\,(n)}(\tau) \\ y^{U}_{l\,(n)}(\tau) \\ y^{U}_{t\,(n)}(\tau) \end{bmatrix} + Z * \begin{bmatrix} h_{1n}(\tau) \\ k_{1n}(\tau) \\ h_{2n}(\tau) \\ k_{2n}(\tau) \end{bmatrix},$$

$$\begin{matrix} y^{L}_{l\,(n)}(\tau) \\ y^{L}_{t\,(n)}(\tau) \\ y^{U}_{l\,(n)}(\tau) \\ y^{U}_{t\,(n)}(\tau) \end{matrix} = \begin{bmatrix} r^{L}_{l\,(n)}(\tau) \\ r^{L}_{t\,(n)}(\tau) \\ r^{U}_{l\,(n)}(\tau) \\ r^{U}_{t\,(n)}(\tau) \end{bmatrix} + \begin{bmatrix} h_{1n}(\tau) \\ k_{1n}(\tau) \\ h_{2n}(\tau) \\ k_{2n}(\tau) \end{bmatrix},$$

$$Z = \left(4J_{**}\left(y^{L}_{l\,(n)}, y^{L}_{t\,(n)}, y^{U}_{l\,(n)}, y^{U}_{t\,(n)}, \tau\right) - 2J_{*}\left(r^{L}_{l\,(n)}, r^{L}_{t\,(n)}, r^{U}_{l\,(n)}, r^{U}_{t\,(n)}, \tau\right)\right)^{-1} * \left(J_{*}\left(y^{L}_{l\,(n)}, y^{L}_{t\,(n)}, y^{U}_{l\,(n)}, y^{U}_{t\,(n)}, \tau\right) - J_{**}\left(r^{L}_{l\,(n)}, r^{L}_{t\,(n)}, r^{U}_{l\,(n)}, r^{U}_{t\,(n)}, \tau\right)\right),$$

$$J_{*}\left(r_{\ln}^{L}, r_{tn}^{L}, r_{\ln}^{U}, r_{tn}^{U}, \tau\right) \begin{bmatrix} h_{1n}(\tau) \\ k_{1n}(\tau) \\ h_{2n}(\tau) \\ k_{2n}(\tau) \end{bmatrix} = \begin{bmatrix} c_{l}^{L}(\tau) - F_{l}^{L}\left(r_{\ln}^{L}, r_{tn}^{L}, r_{\ln}^{U}, r_{tn}^{U}, \tau\right) \\ c_{t}^{L}(\tau) - F_{t}^{L}\left(r_{\ln}^{L}, r_{tn}^{L}, r_{\ln}^{U}, r_{tn}^{U}, \tau\right) \\ c_{l}^{U}(\tau) - F_{l}^{U}\left(r_{\ln}^{L}, r_{tn}^{L}, r_{\ln}^{U}, r_{tn}^{U}, \tau\right) \\ c_{t}^{U}(\tau) - F_{t}^{U}\left(r_{\ln}^{L}, r_{tn}^{L}, r_{\ln}^{U}, r_{tn}^{U}, \tau\right) \end{bmatrix},$$

$$\mathbf{J}_* = \begin{bmatrix} F^L_{l\,r^L_{\ln}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^L_{l\,r^L_{tn}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^L_{l\,r^U_{\ln}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^L_{l\,r^U_{tn}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) \\ F^L_{t\,r^L_{\ln}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^L_{t\,r^L_{tn}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^L_{t\,r^U_{\ln}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^L_{t\,r^U_{tn}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) \\ F^U_{l\,r^L_{\ln}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^U_{l\,r^L_{tn}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^U_{l\,r^U_{\ln}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^U_{l\,r^U_{tn}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) \\ F^U_{t\,r^L_{\ln}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^U_{t\,r^L_{tn}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^U_{t\,r^U_{\ln}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) & F^U_{t\,r^U_{tn}}(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn}, \tau) \end{bmatrix},$$

$$\mathbf{J}_{**} = \begin{bmatrix} F^L_{l\,y^L_{\ln}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^L_{l\,y^L_{tn}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^L_{l\,y^U_{\ln}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^L_{l\,y^U_{tn}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) \\ F^L_{t\,y^L_{\ln}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^L_{t\,y^L_{tn}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^L_{t\,y^U_{\ln}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^L_{t\,y^U_{tn}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) \\ F^U_{l\,y^L_{\ln}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^U_{l\,y^L_{tn}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^U_{l\,y^U_{\ln}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^U_{l\,y^U_{tn}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) \\ F^U_{t\,y^L_{\ln}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^U_{t\,y^L_{tn}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^U_{t\,y^U_{\ln}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) & F^U_{t\,y^U_{tn}}(y^L_{\ln}, y^L_{tn}, y^U_{\ln}, y^U_{tn}, \tau) \end{bmatrix}, \forall \tau \in [0, 1]. \quad (17)$$

For initial guess, one can use the fuzzy number

$$\mathbf{r}_0 = \left(r^U_{l0}(0), r^L_{l0}(0)\right), r^L_{l0}(1), r^L_{t0}(1), r^U_{l0}(1), r^U_{t0}(1), r^L_{t0}(0) r^U_{t0}(0)), \quad (18)$$

Remark 9. Sequence $\{(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn})\}_{n=0}^{\infty}$ converges to $(\alpha^L_l, \alpha^L_t, \alpha^U_l, \alpha^U_t)$ iff $\forall \tau \in [0, 1]$, $\lim_{n\longrightarrow\infty} r^L_{\ln}(\tau) = \alpha^L_l(\tau)$, $\lim_{n\longrightarrow\infty} r^L_{tn}(\tau) = \alpha^L_t(\tau)$, $\lim_{n\longrightarrow\infty} r^U_{\ln}(\tau) = \alpha^U_l(\tau)$, and $\lim_{n\longrightarrow\infty} r^U_{tn}(\tau) = \alpha^U_t(\tau)$.

Lemma 10. *Let $F(\alpha^L_l, \alpha^L_t, \alpha^U_l, \alpha^U_t) = (c^L_l, c^L_t, c^U_l, c^U_t)$, and if the sequence of $\{(r^L_{\ln}, r^L_{tn}, r^U_{\ln}, r^U_{tn})\}_{n=0}^{\infty}$ converges to $(\alpha^L_l, \alpha^L_t, \alpha^U_l, \alpha^U_t)$ according to Newton's method, then*

$$\lim_{n\longrightarrow\infty} \mathbf{P}_n = 0, \quad (19)$$

where

$$\mathbf{P}_n = \sup_{0\le\tau\le 1} \max \{h_{1n}(\tau), k_{1n}(\tau), h_{2n}(\tau), k_{2n}(\tau)\}. \quad (20)$$

Proof. It is obvious, because for all $\forall \tau \in [0, 1]$ in convergent case,

$$\lim_{n\longrightarrow\infty} h_{1n}(\tau) = \lim_{n\longrightarrow\infty} k_{1n}(\tau) = \lim_{n\longrightarrow\infty} h_{2n}(\tau) = \lim_{n\longrightarrow\infty} k_{2n}(\tau) = 0. \quad (21)$$

Hence, it is proved. □

Finally, it is shown that under certain condition, the MM method for fuzzy equation $\mathbf{F}(\mathbf{r}) = 0$ is cubic convergent. In compact form for $\mathbf{F}(\mathbf{r}) = 0$, the MM method can be written as follows:

$$\begin{cases} \mathbf{y}_n(\tau) = \mathbf{r}_n(\tau) - \left(\mathbf{F}'(\mathbf{r}_n(\tau))\right)^{-1} \mathbf{F}(\mathbf{r}_n(\tau)), \\ \mathbf{r}_{n+1}(\tau) = \mathbf{y}_n(\tau) - Z * \left(\mathbf{F}'(\mathbf{r}_n(\tau))\right)^{-1} \mathbf{F}(\mathbf{r}_n(\tau)), \end{cases} \forall \tau \in [0, 1], \quad (22)$$

where

$$Z = (4\mathbf{J}_{**} - 2\mathbf{J}_*)^{-1} * (\mathbf{J}_* - \mathbf{J}_{**}). \quad (23)$$

Theorem 11. *Let $F : H \subseteq R^n \longrightarrow R^n$, be u-times Frěchet differential function on a convex set H containing the root $\boldsymbol{\alpha}$ of $\mathbf{F}(\mathbf{r}) = 0$; then, the MM method has cubic convergence and satisfies the following error equation.*

$$\mathbf{e}_{n+1} = 2 * \left((\mathbf{A}_2)^2 - \frac{1}{2}\mathbf{A}_3 - 4(\mathbf{A}_2)^2\right)(\mathbf{e}_n)^3 + \left\|\mathbf{O}(\mathbf{e}_n)^4\right\|, \quad (24)$$

where $\mathbf{A}_n = 1/2! * \mathbf{F}(\mathbf{r}_n, \tau)/\mathbf{F}'(\mathbf{r}_n, \tau)$, $n = 2, 3, \cdots$.

Proof. Let $\mathbf{e}_n = \mathbf{r}_n - \alpha$ and $\mathbf{e}_{n+1} = \mathbf{r}_{n+1} - \boldsymbol{\alpha}$, then by Taylor series of $\mathbf{F}(\mathbf{r}_n, \tau)$ in the neighborhood of $\boldsymbol{\alpha}$ if $\mathbf{J}_*$ and $\mathbf{J}_{**}$ exist. Then,

$$\mathbf{F}(\mathbf{r}, \tau) = \mathbf{F}(\mathbf{r}_n, \tau) + \mathbf{F}'(\mathbf{r}_n, \tau)(\mathbf{r} - \mathbf{r}_n) + \frac{1}{2!}\mathbf{F}''(\mathbf{r}_n, \tau)(\mathbf{r} - \mathbf{r}_n)^2 + . \cdots \quad (25)$$

and $\mathbf{F}(\mathbf{r}, \boldsymbol{\alpha}) = 0$.

$$\mathbf{F}(\mathbf{r}_n, \tau) = \mathbf{F}'(\mathbf{r}, \xi)\left(\mathbf{e}_n + \mathbf{A}_2(\mathbf{e}_n)^2 + \mathbf{A}_3(\mathbf{e}_n)^3\right) + \left\|\mathbf{O}(\mathbf{e}_n)^4\right\|. \quad (26)$$

Table 4: Analytical solution for Example 2.

τ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
r_l^U	0.6433	0.6437	0.6440	0.6443	0.6446	0.6450	0.6453	0.6457	0.6460	0.6464	0.6468
r_l^L	0.6450	0.6454	0.6458	0.6463	0.6467	0.6472	0.6477	0.6482	0.6487	0.6493	0.6498
r_t^L	0.6616	0.6606	0.6596	0.6586	0.6577	0.6568	0.6560	0.6552	0.6544	0.6537	0.6530
r_t^U	0.6666	0.6655	0.6645	0.6635	0.6625	0.6616	0.6607	0.6599	0.6591	0.6583	0.6576

This gives

$$\left(\mathbf{F}'(\mathbf{r}_n,\tau)\right)^{-1}\mathbf{F}(\mathbf{r}_n,\tau)=\mathbf{e}_n+\mathbf{A}_2(\mathbf{e}_n)^2+(2\mathbf{A}_2+2\mathbf{A}_3)(\mathbf{e}_n)^3+\cdots,$$
$$\mathbf{y}_n-\boldsymbol{\alpha}=\mathbf{A}_2(\mathbf{e}_n)^2+(-2\mathbf{A}_2+2\mathbf{A}_3)(e_n)^3+.\cdots \tag{27}$$

Expanding $\mathbf{F}'(\mathbf{y}_n,\tau)$ about $\boldsymbol{\alpha}$, we have the following:

$$\mathbf{F}'(\mathbf{y}_n,\tau)=1+2(\mathbf{A}_2)^2(\mathbf{e}_n)^2+2\left(-2(\mathbf{A}_2)^2+2\mathbf{A}_3\right)(\mathbf{e}_n)^3+\cdots,$$

$$Z*\left(\mathbf{F}'(\mathbf{r}_n,\tau)\right)^{-1}\mathbf{F}(\mathbf{r}_n,\tau)=-\mathbf{A}_2(\mathbf{e}_n)^2+\left(4(\mathbf{A}_2)^2-\frac{3}{2}\mathbf{A}_3-4(\mathbf{A}_2)^2\right)(\mathbf{e}_n)^3+\cdots,$$

$$\mathbf{r}_{n+1}-\boldsymbol{\alpha}=\mathbf{y}_n-\boldsymbol{\alpha}-\mathbf{A}_2(\mathbf{e}_n)^2+\left(4(\mathbf{A}_2)^2-\frac{3}{2}\mathbf{A}_3-4(\mathbf{A}_2)^2\right)(\mathbf{e}_n)^3+\cdots,$$

$$\mathbf{e}_{n+1}=2*\left((\mathbf{A}_2)^2-\frac{1}{2}\mathbf{A}_3-4(\mathbf{A}_2)^2\right)(\mathbf{e}_n)^3+\left\|\mathbf{O}(e_n)^4\right\|. \tag{28}$$

Hence, the theorem is proved. □

A well-known existing method in literature for solving triangular fuzzy nonlinear equation is classical Newton Raphson's method. Interval-valued trapezoidal fuzzy version of well-known Newton method [13] (abbreviated as NN) for finding roots of interval-valued trapezoidal fuzzy nonlinear equation is as follows:

$$\begin{bmatrix} r^L_{l\,(n+1)}(\tau)\\ r^L_{t\,(n+1)}(\tau)\\ r^U_{l\,(n+1)}(\tau)\\ r^U_{t\,(n+1)}(\tau)\end{bmatrix}=\begin{bmatrix} r^L_{l\,(n)}(\tau)\\ r^L_{t\,(n)}(\tau)\\ r^U_{l\,(n)}(\tau)\\ r^U_{t\,(n)}(\tau)\end{bmatrix}+\begin{bmatrix} h_{1n}(\tau)\\ k_{1n}(\tau)\\ h_{2n}(\tau)\\ k_{2n}(\tau)\end{bmatrix}, \tag{29}$$

Table 5

τ	MM $\|\mathbf{r}_{n+1}-\mathbf{r}_n\|$	MM $\|\mathbf{F}(\mathbf{r}_n)\|$	NN $\|\mathbf{r}_{n+1}-\mathbf{r}_n\|$	NN $\|\mathbf{F}(\mathbf{r}_n)\|$
0.0	2.8*e*-43	5.4*e*-33	1.1*e*-9	1.6*e*-7
0.1	3.8*e*-44	3.0*e*-33	7.6*e*-10	3.4*e*-7
0.2	4.7*e*-43	5.8*e*-33	3.5*e*-10	4.2*e*-7
0.3	9.1*e*-43	2.7*e*-33	6.1*e*-11	6.1*e*-7
0.4	6.2*e*-50	5.6*e*-39	5.1*e*-12	5.8*e*-7
0.5	3.6*e*-51	6.1*e*-38	4.6*e*-11	6.5*e*-7
0.6	4.8*e*-50	7.8*e*-39	3.1*e*-11	8.9*e*-7
0.7	9.1*e*-51	3.6*e*-39	4.2*e*-11	1.3*e*-7
0.8	8.1*e*-48	8.4*e*-40	7.7*e*-11	4.1*e*-7
0.9	9.8*e*-49	3.5*e*-39	5.1*e*-11	5.6*e*-7
1	6.1*e*-52	4.0*e*-38	3.3*e*-11	1.2*e*-7

where

$$J_*\left(r^L_{\ln},r^L_{tn},r^U_{\ln},r^U_{tn},\tau\right)\begin{bmatrix} h_{1n}(\tau)\\ k_{1n}(\tau)\\ h_{2n}(\tau)\\ k_{2n}(\tau)\end{bmatrix}=\begin{bmatrix} c^L_l(\tau)-F^L_l\left(r^L_{\ln},r^L_{tn},r^U_{\ln},r^U_{tn},\tau\right)\\ c^L_t(\tau)-F^L_t\left(r^L_{\ln},r^L_{tn},r^U_{\ln},r^U_{tn},\tau\right)\\ c^U_l(\tau)-F^U_l\left(r^L_{\ln},r^L_{tn},r^U_{\ln},r^U_{tn},\tau\right)\\ c^U_t(\tau)-F^U_t\left(r^L_{\ln},r^L_{tn},r^U_{\ln},r^U_{tn},\tau\right)\end{bmatrix}. \tag{30}$$

4. Numerical Applications

Here, we present examples to illustrate the performance and efficiency of MM and NN methods for approximating roots of interval-valued trapezoidal fuzzy nonlinear equations. Examples 1–3 are considered from Buckley and Qu [9]. All the computations are performed using CAS Maple 18 with 64 digits floating point arithmetic with stopping criteria as follows. Analytical, numerical approximate solutions, computational order of convergence [32], computational time in second, and residual error graph of interval-valued trapezoidal fuzzy nonlinear equation used in Examples 1–3 are shown in Figures 2–8(a) and 8(c), respectively. Algorithm 1 shows the implementation of MM iterative method on CAS Maple18.

$$(i)\ e_n=\|F(r,\tau)\|<\epsilon\quad (ii)\ e_n=\|r_{n+1}(\tau)-r_n(\tau)\|<\epsilon, \tag{31}$$

where e_n represents the absolute error. We take $\epsilon=10^{-15}$.

In Figure 2, left shows analytical solution of interval-valued trapezoidal fuzzy nonlinear equation used in Example

Table 6: Analytical solution for Example 3.

τ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
r_l^U	0.3173	0.3266	0.3355	0.3440	0.3522	0.3600	0.3675	0.3747	0.3817	0.3883	0.3947
r_l^L	0.3600	0.3690	0.3775	0.3857	0.3935	0.4009	0.4080	0.4148	0.4213	0.4276	0.4336
r_t^L	0.5005	0.4971	0.4935	0.4898	0.4860	0.4821	0.4780	0.4738	0.4694	0.4648	0.4601
r_t^U	0.5137	0.5112	0.5087	0.5060	0.5033	0.5005	0.4977	0.4947	0.4917	0.4886	0.4854

1, center shows for Example 2, and right shows for Example 3, respectively.

Figure 3 shows computational order of convergence of iterative methods MM and NN for finding roots of interval-valued trapezoidal fuzzy nonlinear equations used in Examples 1–3, respectively.

In Figure 3, MM1-MM4 and NN1-NN4 show computational order of convergence of iterative method MM and NN for approximating roots of interval-valued trapezoidal fuzzy nonlinear equations used in Examples 1–3, respectively.

Figure 4 shows computational time in seconds of iterative methods MM and NN for finding roots of interval-valued trapezoidal fuzzy nonlinear equations used in Examples 1–3, respectively.

In Figure 4, MM1-MM4 and NN1-NN4 show computational time in seconds of iterative method MM and NN for finding roots of interval-valued trapezoidal fuzzy nonlinear equation used in Examples 1–3, respectively.

Example 1 Application in optimization (a profit maximization problem). A corporation company wishes to invest one million dollar $A_1 = \langle(10, 20, 30, 40\,; 2/3), (5, 15, 35, 43\,; 1)\rangle$ at fuzzy interest rate r to earn maximum profit, so that after a year, they may withdraw 25000\$ $S_1 = \langle(45, 55, 75, 95\,; 2/3), (80, 90, 110, 120\,; 1)\rangle$ approximately and after two years 900000\$ $S_2 = \langle(10, 15, 20, 25\,; 2/3), (5, 10, 25, 47\,; 1)\rangle$ left. Find r so that A_1 will be sufficient to cover S_1 and S_2. where r is an interval-valued trapezoidal fuzzy number whose support lies between $[0, 1]$ After a year, the amount in the account will be

$$(A_1 - S_1) + A_1 r(\tau). \tag{32}$$

At the end of second year, total amount left is

$$(A_1 - S_1) + A_1 * r(\tau) + ((A_1 - S_1) - A_1 * r(\tau))r(\tau), \tag{33}$$

or

$$A_1\left(r(\tau)^2\right) + B * r(\tau) + D, \tag{34}$$

where $B = 2A_1 - S_1$ and $D = A_1 - S_1$. Therefore, we have to solve

$$A_1 * (r(\tau))^2 + B * r(\tau) + D = S_2, \tag{35}$$

or

$$A_1 * \left(r(\tau)^2\right) + B * r(\tau) = C, \tag{36}$$

where $C = S_2 - D$. For fuzzy interest rate substituting values of A_1, B, and C in above equation, we have the following:

$$\begin{aligned} &\left\langle \left(10, 20, 30, 40\,; \frac{2}{3}\right), (5, 15, 35, 45\,; 1) \right\rangle \left(r(\tau)^2\right) \\ &\quad + \left\langle \left(50, 60, 70, 80\,; \frac{2}{3}\right), (45, 55, 75, 95\,; 1) \right\rangle r(\tau) \\ &= \left\langle \left(80, 90, 110, 120\,; \frac{2}{3}\right), (75, 85, 115, 125\,; 1) \right\rangle. \end{aligned} \tag{37}$$

Without any loss of generality, assume that r is positive; then, the parametric form of this equation is as follows:

$$\begin{aligned} &\{\langle(10 + 15\tau, 40 - 15\tau), (5 + 10\tau, 45 - 10\tau)\rangle(r(\tau))^2 \\ &\quad + \langle(50 + 15\tau, 80 - 15\tau), (45 + 10\tau, 95 - 20\tau)\rangle r(\tau) \\ &= \langle(80 + 15\tau, 120 - 15\tau), (75 + 10\tau, 125 - 10\tau)\rangle, \end{aligned}$$

$$\begin{cases} (10 + 15\tau)\left(r_l^L(\tau)\right)^2 + (50 + 15\tau)r_l^L(\tau) = (80 + 15\tau), \\ (40 - 15\tau)\left(r_t^L(\tau)\right)^2 + (80 - 15\tau)r_t^L(\tau) = (120 - 15\tau), \\ (5 + 10\tau)\left(r_l^U(\tau)\right)^2 + (45 + 10\tau)r_l^U(\tau) = (75 + 10\tau), \\ (45 - 10\tau)\left(r_t^U(\tau)\right)^2 + (95 - 20\tau)r_t^U(\tau) = (125 - 10\tau). \end{cases} \tag{38}$$

Table 1 clearly shows the dominance behavior of MM over NN in terms of absolute error on the same number of iterations $n = 4$ for Example 1.

Table 2 shows analytical solutions for Example 1.

Figure 5 shows initial guessed values, analytical, and numerical approximate solution graph of iterative methods MM and NN for interval-valued trapezoidal fuzzy nonlinear equation used in Example 1.

To obtain initial guess, we use above system for $\tau = 0$ and $\tau = 1$; therefore,

$$\begin{cases} 10\left(r_l^L\right)^2(0) + 50r_l^L(0) = 80, \\ 40\left(r_t^L\right)^2(0) + 80r_t^L(0) = 120, \\ 5\left(r_l^U\right)^2(0) + 45r_l^U(0) = 75, \\ 45\left(r_t^U\right)^2(0) + 95r_t^U(0) = 125, \end{cases}$$
$$\begin{cases} 25\left(r_l^L\right)^2(1) + 65r_l^L(1) = 95, \\ 25\left(r_t^L\right)^2(1) + 65r_t^L(1) = 105, \\ 15\left(r_l^U\right)^2(1) + 55r_l^U(1) = 85, \\ 35\left(r_t^U\right)^2(1) + 75r_t^U(1) = 115. \end{cases} \tag{39}$$

Consequently, $r_l^U(0) = 0.5$, $r_l^L(0) = 0.5$, $r_t^L(0) = 0.5$, $r_t^U(0) = 0.5$, and $r_l^U(0) = r_l^L(0) = r_t^L(0) = r_t^U(0) = 1/2$. After 4 iterations, we obtain the solution with the maximum error less than 10^{-30}. Now suppose r is negative, we have the following:

$$\begin{cases} (10 + 15\tau)\left(r_l^L(\tau)\right)^2 + (50 + 15\tau)r_t^L(\tau) = (80 + 15\tau), \\ (40 - 15\tau)\left(r_t^L(\tau)\right)^2 + (80 - 15\tau)r_l^L(\tau) = (120 - 15\tau), \\ (5 + 10\tau)\left(r_l^U(\tau)\right)^2 + (45 + 10\tau)r_t^U(\tau) = (75 + 10\tau), \\ (45 - 10\tau)\left(r_t^U(\tau)\right)^2 + (95 - 20\tau)r_l^U(\tau) = (125 - 10\tau). \end{cases} \tag{40}$$

For $\tau = 0$, we have $r_l^L(0) > r_t^L(0)$; therefore negative root does not exist.

Example 2. Consider the interval-valued trapezoidal fuzzy nonlinear equation

$$\left\langle \left(0.65, 0.73, 0.87, 0.95\,;\,\frac{2}{3}\right), (0.6, 0.7, 0.9, 1\,;\,1) \right\rangle (r(\tau))^2 + \left\langle \left(0.25, 0.33, 0.47, 0.55\,;\,\frac{2}{3}\right), (0.2, 0.3, 0.5, 0.6\,;\,1) \right\rangle r(\tau) = \left\langle \left(0.45, 0.53, 0.67, 0.75\,;\,\frac{2}{3}\right), (0.4, 0.5, 0.7, 0.8\,;\,1) \right\rangle. \tag{41}$$

Without any loss of generality, assume that r is positive; then, the parametric form of this equation is as follows:

$$\begin{aligned} &\{\langle(0.65 + 0.12\tau, 0.95 - 0.12\tau), (0.6 + 0.1\tau, 1 - 0.1\tau)\rangle \\ &\quad \cdot (r(\tau))^2 + \langle(0.25 + 0.12\tau, 0.55 - 0.12\tau), \\ &\quad (0.2 + 0.1\tau, 0.6 - 0.1\tau)\rangle r(\tau) \\ &= \langle(0.45 + 0.12\tau, 0.75 - 0.12\tau), (0.4 + 0.1\tau, 0.8 - 0.1\tau)\rangle, \end{aligned} \tag{42}$$

or

$$\begin{cases} (0.65 + 0.12\tau)\left(r_l^L(\tau)\right)^2 + (0.25 + 0.12\tau)r_l^L(\tau) = (0.45 + 0.12\tau), \\ (0.95 - 0.12\tau)\left(r_t^L(\tau)\right)^2 + (0.55 - 0.12\tau)r_t^L(\tau) = (0.75 - 0.12\tau). \\ (0.6 + 0.1\tau)\left(r_l^U(\tau)\right)^2 + (0.2 + 0.1\tau)r_l^U(\tau) = (0.4 + 0.1\tau), \\ (1 - 0.1\tau)\left(r_t^U(\tau)\right)^2 + (0.6 - 0.1\tau)r_t^U(\tau) = (0.8 - 0.1\tau). \end{cases} \tag{43}$$

Table 3 clearly shows the dominance behavior of MM over NN in terms of absolute error on the same number of iterations $n = 4$ for Example 2.

Table 4 shows analytical solutions for Example 2.

Figure 6 shows initial guessed values, analytical, and numerical approximate solution graph of iterative methods MM and NN for interval-valued trapezoidal fuzzy nonlinear equation used in Example 2.

To obtain initial guess, we use above system for $\tau = 0$ and $\tau = 1$; therefore,

$$\begin{cases} 0.65\left(r_l^L\right)^2(0) + 0.2r_l^L(0) = 0.45, \\ 0.95\left(r_t^L\right)^2(0) + 0.55r_t^L(0) = 0.75, \\ 0.6\left(r_l^U\right)^2(0) + 0.2r_l^U(0) = 0.4, \\ 1.0\left(r_t^U\right)^2(0) + 0.6r_t^U(0) = 0.8, \end{cases}$$
$$\begin{cases} 0.77\left(r_l^L\right)^2(1) + 0.37r_l^L(1) = 0.57, \\ 0.83\left(r_t^L\right)^2(1) + 0.43r_t^L(1) = 0.63, \\ 0.7\left(r_l^U\right)^2(1) + 0.3r_l^U(1) = 0.5, \\ 0.9\left(r_t^U\right)^2(1) + 0.5r_t^U(1) = 0.7. \end{cases} \tag{44}$$

Consequently, $r_l^U(0) = 0.6$, $r_l^L(0) = 0.6$, $r_t^L(0) = 0.6$, $r_t^U(0) = 0.6$, and $r_l^U(0) = r_l^L(0) = r_t^L(0) = r_t^U(0) = 1/2$. After 4 iterations, we obtain the solution with the maximum error less than 10^{-30}. Now suppose r is negative, we have

$$\begin{cases} (0.65 + 0.12\tau)\left(r_l^L(\tau)\right)^2 + (0.25 + 0.12\tau)r_l^L(\tau) = (0.45 + 0.12\tau), \\ (0.95 - 0.12\tau)\left(r_t^L(\tau)\right)^2 + (0.55 - 0.12\tau)r_t^L(\tau) = (0.75 - 0.12\tau), \\ (0.6 + 0.1\tau)\left(r_l^U(\tau)\right)^2 + (0.2 + 0.1\tau)r_l^U(\tau) = (0.4 + 0.1\tau), \\ (1 - 0.1\tau)\left(r_t^U(\tau)\right)^2 + (0.6 - 0.1\tau)r_t^U(\tau) = (0.8 - 0.1\tau). \end{cases} \tag{45}$$

For $\tau = 0$, we have $r_l^L(0) > r_t^L(0)$, therefore, negative root does not exist.

Example 3. Consider the interval-valued trapezoidal fuzzy nonlinear equation

$$\left\langle \left(0.45, 0.53, 0.67, 0.75\,;\,\frac{2}{3}\right), (0.4, 0.5, 0.7, 0.8\,;\,1) \right\rangle (r(\tau))^3 + \left\langle \left(0.65, 0.73, 0.87, 0.95\,;\,\frac{2}{3}\right), (0.6, 0.7, 0.9, 1\,;\,1) \right\rangle \sin\,(r(\tau)) = \left\langle \left(0.25, 0.33, 0.47, 0.55\,;\,\frac{2}{3}\right), (0.2, 0.3, 0.5, 0.6\,;\,1) \right\rangle. \tag{46}$$

Without any loss of generality, assume that r is positive; then, the parametric form of this equation is as follows:

$$\begin{aligned} &\{\langle(0.45 + 0.12\tau, 0.75 - 0.12\tau), (0.4 + 0.1\tau, 0.8 - 0.1\tau)\rangle \\ &\quad \cdot (r(\tau))^3 + \langle(0.65 + 0.12\tau, 0.95 - 0.12\tau), \\ &\quad (0.6 + 0.1\tau, 1.0 - 0.1\tau)\rangle r(\tau) \\ &= \langle(0.25 + 0.12\tau, 0.55 - 0.12\tau), (0.2 + 0.1\tau, 0.6 - 0.1\tau)\rangle, \end{aligned} \tag{47}$$

or

$$\begin{cases} (0.45 + 0.12\tau) r_l^L(\tau)^3 + (0.65 + 0.12\tau)\,\sin\,\left(r_l^L(\tau)\right) = (0.25 + 0.12\tau), \\ (0.75 - 0.12\tau)\left(r_t^L(\tau)\right)^3 + (0.95 - 0.12\tau)\,\sin\,\left(r_t^L(\tau)\right) = (0.55 - 0.12\tau), \\ (0.4 + 0.1\tau)\left(r_l^U(\tau)\right)^3 + (0.6 + 0.1\tau)\,\sin\,\left(r_l^U(\tau)\right) = (0.2 + 0.1\tau), \\ (0.8 - 0.1\tau)\left(r_t^U(\tau)\right)^3 + (1.0 - 0.1\tau)\,\sin\,\left(r_t^U(\tau)\right) = (0.6 - 0.1\tau). \end{cases} \tag{48}$$

Table 5 clearly shows the dominance behavior of MM over NN in terms of absolute error on the same number of iterations $n = 4$ for Example 3.

Table 6 shows analytical solutions for Example 3.

Figure 7 shows initial guessed values, analytical, and numerical approximate solution graph of iterative methods MM and NN for interval-valued trapezoidal fuzzy nonlinear equation used in Example 3.

To obtain initial guess, we use above system for $\tau = 0$ and $\tau = 1$; therefore,

$$\begin{cases} 0.45\left(r_l^L\right)^3(0) + 0.65\,\sin\,\left(r_l^L(0)\right) = 0.25, \\ 0.75\left(r_t^L\right)^3(0) + 0.95\,\sin\,\left(r_t^L(0)\right) = 0.55, \\ 0.4\left(r_l^U\right)^3(0) + 0.6\,\sin\,\left(r_l^U(0)\right) = 0.2, \\ 0.8\left(r_t^U\right)^3(0) + 1\,\sin\,\left(r_t^U(0)\right) = 0.6, \end{cases}$$
$$\begin{cases} 0.57\left(r_l^L\right)^3(1) + 0.77\,\sin\,\left(r_l^L(1)\right) = 0.37, \\ 0.63\left(r_t^L\right)^3(1) + 0.83\,\sin\,\left(r_t^L(1)\right) = 0.43, \\ 0.5\left(r_l^U\right)^3(1) + 0.7\,\sin\,\left(r_l^U(1)\right) = 0.3, \\ 0.7\left(r_t^U\right)^3(1) + 0.9\,\sin\,\left(r_t^U(1)\right) = 0.5. \end{cases} \tag{49}$$

Consequently, $r_l^U(0) = 0.5$, $r_l^L(0) = 0.3$, $r_t^L(0) = 0.5$, $r_t^U(0) = 0.3$, and $r_l^U(0) = r_l^L(0) = r_t^L(0) = r_t^U(0) = 1/2$. After 4 iterations, we obtain the solution with the maximum error less than 10^{-30}. Now suppose r is negative, we have

$$\begin{cases} (0.45 + 0.12\tau)\left(r_l^L(\tau)\right)^3 + (0.65 + 0.12\tau)\,\sin\,\left(r_t^L(\tau)\right) = (0.25 + 0.12\tau), \\ (0.75 - 0.12\tau)\left(r_t^L(\tau)\right)^3 + (0.95 - 0.12\tau)\,\sin\,\left(r_l^L(\tau)\right) = (0.55 - 0.12\tau), \\ (0.4 + 0.1\tau)\left(r_l^U(\tau)\right)^3 + (0.6 + 0.1\tau)\,\sin\,\left(r_t^U(\tau)\right) = (0.2 + 0.1\tau), \\ (0.8 - 0.1\tau)\left(r_t^U(\tau)\right)^3 + (1.0 - 0.1\tau)\,\sin\,\left(r_l^U(\tau)\right) = (0.6 - 0.1\tau). \end{cases} \tag{50}$$

For $\tau = 0$, we have hence $r_l^L(0) > r_t^L(0)$; therefore, negative root does not exist.

Figures 8(a)–8(c) show residual falls for iterative methods MM and NN for interval-valued trapezoidal fuzzy nonlinear equation used in Examples 1–3, respectively.

5. Conclusion

In this research paper, we constructed highly efficient two-step numerical iterative method to approximate roots of interval-valued trapezoidal fuzzy nonlinear equations. A set of real-life applications from optimization are considered as a numerical test examples showing the practical performance and dominance efficiency of MM over NN method on the same number of iterations. From Tables 1–6 and Figures 1–8, we observe that numerical results of MM methods are better in terms of absolute error and CPU time as compared to NN method. Considering the same ways as in this article, we can establish higher order and efficient numerical iterative methods for solving system of fuzzy nonlinear equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally in the preparation of this manuscript.

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References

[1] S. S. L. Chang and L. A. Zadeh, "On fuzzy mapping and control," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 2, pp. 30–34, 1972.

[2] D. Dubois and H. Prade, "Operations on fuzzy numbers," *International Journal of Systems Science*, vol. 9, no. 6, pp. 613–626, 1978.

[3] M. Mizumoto and K. Tanaka, "Some properties of fuzzy numbers," in *Advances in fuzzy set theory and applications*, pp. 153–164, North-Holland, Amsterdam-New York, 1979.

[4] M. Mizumoto and K. Tanaka, "The four operations of arithmetic on fuzzy numbers," *Systems Computers and Controls*, vol. 7, no. 5, pp. 73–81, 1976.

[5] M. Keyanpour and T. Akbarian, "Solving intuitionistic fuzzy nonlinear equations," *Journal of Fuzzy set Value Analysis*, vol. 2014, 2014.

[6] S. Nahmias, "Fuzzy variables," *Fuzzy Sets and Systems*, vol. 12, pp. 97–111, 1978.

[7] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[8] L. A. Zadeh, "The concept of a linguistic variable and its application to approximation reasoning," *Information Sciences*, vol. 3, pp. 199–249, 1975.

[9] J. J. Buckley and Y. Qu, "Solving linear and quadratic fuzzy equations," *Fuzzy Sets and Systems*, vol. 38, no. 1, pp. 43–59, 1990.

[10] J. J. Buckley and Y. Qu, "On using α-cuts to evaluate fuzzy equations," *Fuzzy Sets and Systems*, vol. 38, no. 3, pp. 309–312, 1990.

[11] J. J. Buckley and Y. Qu, "Solving fuzzy equations: a new solution concept," *Fuzzy Sets and Systems*, vol. 39, no. 3, pp. 291–301, 1991.

[12] J. J. Buckley and Y. Qu, "Solving systems of linear fuzzy equations," *Fuzzy Sets and Systems*, vol. 43, no. 1, pp. 33–43, 1991.

[13] S. Abbasbandy and B. Asady, "Newton's method for solving fuzzy nonlinear equations," *Applied Mathematics and Computation*, vol. 159, no. 2, pp. 349–356, 2004.

[14] T. Allahviranloo and S. Asari, "Numerical solution of fuzzy polynomial by Newton-Raphson method," *Journal of Applied Mathematics, Islamic Azad University of Lahijan*, vol. 7, no. 4, pp. 17–44, 2011.

[15] M. Mosleh, "Solution of dual fuzzy polynomial equations by modified Adomian decomposition method," *Fuzzy Information and Engineering*, vol. 5, no. 1, pp. 45–56, 2013.

[16] Y. J. Cho, N. J. Huang, and S. M. Kang, "Nonlinear equations for fuzzy mappings in probabilistic normed spaces," *Fuzzy Sets and Systems*, vol. 110, no. 1, pp. 115–122, 2000.

[17] J. E. Dennis and R. B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, Hoboken, NJ, USA, 1983.

[18] J. Fang, "On nonlinear equations for fuzzy mappings in probabilistic normed spaces," *Fuzzy Sets and Systems*, vol. 131, no. 3, pp. 357–364, 2002.

[19] R. Goetschel and W. Voxman, "Elementary fuzzy calculus," *Fuzzy Sets and Systems*, vol. 18, no. 1, pp. 31–43, 1986.

[20] I. M. Sulaiman, M. Mamat, M. Y. Waziri, M. A. Mohamed, and F. S. Mohamad, "Solving fuzzy nonlinear equation via Levenberg-Marquardt method," *Far East Journal of Mathematical Science*, vol. 103, pp. 1547–1558, 2018.

[21] J. Ma and G. Feng, "An approach to H_∞ control of fuzzy dynamic systems," *Fuzzy Sets and Systems*, vol. 137, no. 3, pp. 367–386, 2003.

[22] A. Koam, M. Akram, G. Muhammad, and N. Hussain, "LU decomposition scheme for solving -polar fuzzy system of linear equations," *Mathematical Problems in Engineering*, vol. 2020, 19 pages, 2020.

[23] M. Akram, G. Muhammad, A. N. A. Koam, and N. Hussain, "Iterative methods for solving a system of linear equation in bipolar fuzzy environment," *Mathematics*, vol. 7, no. 728, pp. 1–25, 2019.

[24] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Application*, Academic Press, New York, NY, USA, 1980.

[25] H. J. Zimmermann, *Fuzzy Sets Theory and Its Application*, Kluwer Academic Press, Dordrecht, Netherlands, 1991.

[26] M. Saqib, M. Akram, and S. Bashir, "Certain efficient iterative methods for bipolar fuzzy system of linear equations," *Journal of Intelligent & Fuzzy Systems*, vol. 39, no. 3, pp. 3971–3985, 2020.

[27] M. Akram, G. Muhammad, and T. Allahviranloo, "Bipolar fuzzy linear system of equations," *Computational and Applied Mathematics*, vol. 38, no. 69, pp. 1–29, 2019.

[28] J. Chiang, "Fuzzy linear programming based on statistical confidence interval and interval-valued fuzzy set," *European Journal of Operational Research*, vol. 129, no. 1, pp. 65–86, 2001.

[29] J. S. Yao and F. T. Lin, "Constructing a fuzzy flow-shop sequencing model based on statistical data," *International Journal of Approximate Reasoning*, vol. 29, no. 3, pp. 215–234, 2002.

[30] S. H. Wei and S. M. Chen, "Fuzzy risk analysis based on interval-valued fuzzy numbers," *Expert Systems with Applications*, vol. 36, no. 2, pp. 2285–2299, 2009.

[31] B. Farhadinia, "Sensitivity analysis in interval-valued trapezoidal fuzzy number linear programming problems," *Applied Mathematical Modelling*, vol. 38, no. 1, pp. 50–62, 2014.

[32] S. Akram, M. Shams, N. Rafiq, and N. A. Mir, "On the stability of Weierstrass type method with King's correction for finding all roots of non-linear function with engineering application," *Applied Mathematical Sciences*, vol. 14, no. 10, pp. 461–473, 2020.

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Research Article

A Study on Nash-Collative Differential Game of *N*-Players for Counterterrorism

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In this work, by using the Nash-collative approach for a differential game problem between *N*-governments and terrorist organizations, we study governments' cooperation and the role of each government for counterterrorism. Furthermore, we discuss the intertemporal strategic interaction of governments and terrorist organizations, where all world governments have to cooperate to fight terrorism. Also, we study the necessary conditions for finding the optimal strategies for each government to fight terrorism; we discuss the existence of the solution of the formulated problem and the stability set of the first kind of the optimal strategies.

1. Introduction

It is clear that the world suffers from many serious problems, and the problem of terrorism is one of the most important and serious problems suffered by the local and international communities throughout the ages, especially in recent times. Because of this, countries are controlled, their wealth is plundered, chaos and ignorance spread among peoples, the country's political and religious identities are lost, individuals are destroyed intellectually, they become truly aimless, corruption prevails, and states fall. To face this problem and uproot terrorism from its roots, cooperation must be made between the governments of different countries and the different governments of the same state. This cooperation extends to the members of all societies, coordination occurs between the governments of different countries, and the public good must be upheld over personal interests. Governments have taken security measures to combat terrorism, such as freezing the assets of terrorist organizations and invading their territories to assassinate terrorists. The measures take into account the reactions of terrorists.

The strength of a terrorist organization changes over time, as terrorists are recruited by existing terrorists, and the rate of terrorist recruitment is affected by their actions and the government's antiterrorist actions. The strength of the organization is evaluated by its resources and activities, such as arming, funding, and the expertise of technology.

The government derives its benefits from reducing terrorist resources and activities, in addition to demonstrating that these terrorist organizations are indiscriminate, but they incur costs through fighting terrorism. However, terrorist organizations try to maximize their power in terms of scale and terrorist attacks. Consequently, this study investigates how to help governments fight terrorism.

In this research, we present and study this problem, explain the cooperation of governments with each other, and formulate this problem as a differential game between different governments and terrorist organizations. To clarify the cooperation of governments, to find solutions and to derive optimal strategies for combating terrorism, we will provide a Nash-collative approach to infer the necessary conditions for finding the optimal strategies to combat terrorism. Also, we study the existence of the solution, find it, and study its stability.

The global reputation and optimal control of terrorism was discussed by Caulkins et al. [1]. They proved that success in fighting terrorism relies on community opinion, and the efficiency of water and fire strategies were studied by Caulkins et al. [2]. The first approach implementing a fuzzy differential game to guard territory was discussed by Hsia et al. [3, 4]. They considered the problem to guard territory as a differential game with fuzziness in the distance between the evader and the one guarding, and they discussed that the strategy for this problem is fuzzy. A parametric study of a Nash-collative differential game was discussed by Youness et al. [5]. A differential game as a large-scale problem was discussed by Youness et al. [6]. They presented a Nash approach to solve it. Furthermore, they studied Nash and min–max zero-sum approaches to get the optimal strategy of the differential game problem with fuzzy on the minimum of the objective function [7–9]. Nova et al. [10] studied the Stackelberg and Nash approaches of a differential game in addition to the sensitivity analysis. Cross-country strategic connectivity has been introduced to fight terrorism by Roy and Paul [11]. They analyzed the responses of equilibrium (in terms of defense, R&D, and preemption) to a possible terrorist strafe in a two-country framework using a multistep game with incomplete information. A min–max approach of a differential game was discussed to get the optimal solution of the government and the terror organization by Megahed [12–14]. He studied two problems of governments' visions and terrorist organization and proved that governmental activities are important for fighting terrorism and discussed the Stackelberg differential game with E-differentiable function and E-convex set. In [15], Wrzaczek et al. discussed models of differential terror queue games, as terrorist organizations seek to increase the rates of attacks over time, but at the same time, the government is developing its antiterror activities; in [16], Megahed introduced the Stackelberg approach for counterterrorism.

2. Nash-Collative Differential Game

Definition 1. A Nash-collative game is a Nash-equilibrium game in which some players make coalitions.

2.1. N + M Differential Games. Consider that we have $N+M$ players; the N players cost

$$\begin{aligned} J_i(u_1, u_2, \cdots u_N, v_1, v_2, \cdots, v_M) &= \phi_i\left(x\left(t_f\right)\right) + \int_{t_0}^{t_f} e^{-\rho t} I_i(t, x(t), u_1, \\ &\quad u_2, \cdots u_N, v_1, v_2, \cdots, v_M) dt, \quad i = 1, 2, \cdots N, \end{aligned} \tag{1}$$

and the M players cost

$$\begin{aligned} J'_j(u_1, u_2, \cdots u_N, v_1, v_2, \cdots, v_M) &= \phi'_i\left(x\left(t_f\right)\right) + \int_{t_0}^{t_f} e^{-\eta t} I'_j(t, x(t), u_1, \\ &\quad u_2, \cdots u_N, v_1, v_2, \cdots, v_M) dt, \quad j = 1, 2, \cdots M, \end{aligned} \tag{2}$$

subject to

$$\dot{x}(t) = f(x(t), u_1, u_2, \cdots u_N, v_1, v_2, \cdots, v_M), x(t_0) = x_0, \tag{3}$$

where $x(t) \in R^n$ is the state trajectory of the game and $x(t_0)$ is the initial state for all players.

$u_1(t) \in R^{s_1}, u_2(t) \in R^{s_2}, \cdots, u_N(t) \in R^{s_N}$ and $v_1(t) \in R^{m_1}$, $v_2(t) \in R^{m_2}, \cdots, v_M(t) \in R^{m_M}$ denote the control or the decision of N and M players, respectively, which is taken to be a piecewise continuous function of time.

$f : [t_0, t_f] \times R^n \times R^m \times R^s \longrightarrow R^n$, $I_i : [t_0, t_f] \times R^n \times R^m \times R^s \longrightarrow R^n$, $I_j : [t_0, t_f] \times R^n \times R^m \times R^s \longrightarrow R^n$ are C^1; $\phi_i(x(t_f))$ is the terminal payoff of the player $i, i = 1, 2, . \cdots, N$, and $\phi'_i(x(t_f))$ is terminal payoff of the player $j, j = 1, 2, . \cdots M$; and $I_i(t, x(t), u_1, u_2, \cdots u_N, v_1, v_2, \cdots, v_M)$ is the running payoff of the player $i, i = 1, 2, . \cdots, N$, and $I'_j(t, x(t), u_1, u_2, \cdots u_N, v_1, v_2, \cdots, v_M)$ is the running payoff of the player $j, j = 1, 2, . \cdots, M$.

Definition 2. The admissible control $v^* = (v_1^*, \cdots, v_M^*)$ is said to be a Nash-collative optimal solution if and only if for all admissible $v = (v_1, v_2, \cdots, v_M)$, we have

$$\begin{aligned} &J_i(u_1^*, u_2^*, \cdots, u_N^*, v_1^*, v_2^*, \cdots, v_M^*) \\ &\quad \le J_i(u_1, u_2, \cdots, u_N, v_1^*, v_2^*, \cdots, v_M^*), \quad i = 1, 2, \cdots, N, \\ &J'_j(u_1^*, u_2^*, \cdots, u_N^*, v_1^*, v_2^*, \cdots, v_M^*) \\ &\quad \le J'_j(u_1^*, u_2^*, \cdots, u_N^*, v_1, v_2, \cdots, v_M), \quad j = 1, 2, \cdots, M. \end{aligned} \tag{4}$$

Definition 3. Pareto optimal solution: let the controls $v^* = (v_1^*, v_2^*, \cdots, v_M^*)$ be admissible. If there exists $w' \in R^M$, with $w'_j > 0, j = 1, 2 \cdots, M$, and $\sum_{j=1}^M w'_i = 1$ and $w \in R^N, w_i > 0, i = 1, 2, \cdots, N, \sum_{i=1}^N w_i = 1$ such that for all $v = (v_1, v_2, \cdots, v_M)$, we have

$$\begin{aligned} &\sum_{i=1}^N w_i J_i(u_1^*, u_2^*, \cdots, u_N^*, v_1^*, v_2^*, \cdots, v_M^*) \\ &\quad \le \sum_{i=1}^N w_i J_i(u_1, u_2, \cdots, u_N, v_1^*, v_2^*, \cdots, v_M^*), \\ &\sum_{j=1}^M w'_i J'_j(u_1^*, u_2^*, \cdots, u_N^*, v_1^*, v_2^*, \cdots, v_M^*) \\ &\quad \le \sum_{j=1}^M w'_i J'_j(u_1^*, u_2^*, \cdots, u_N^*, v_1, v_2, \cdots, v_M). \end{aligned} \tag{5}$$

From the concept of coalitions, we suppose that N of these players agree to form a coalition and play against the other players outside the coalition. Let $u = (u_1, u_2, \cdots, u_N) \in R^s$, $s = \sum_{i=1}^{N} s_i$ be the composite control for the players in the coalition and $v = (v_1, v_2, \cdots, v_M) \in R^m$, $m = \sum_{j=1}^{M} m_j$ be the composite control outside the coalition. The problem can be formulated as follows: find (u^*, v^*) that solves the problems

$$\min_u J(u, v^*) = \Phi(x(t_f)) + \int_{t_0}^{t_f} e^{-\rho t} I(x(t), u, v^*) dt, \quad (6)$$

where

$$I = \sum_{i=1}^{N} w_i I_i, \; \Phi(x(t_f)) = \sum_{i=1}^{N} w_i \Phi_i(x(t_f)), \; \sum_{i=1}^{N} w_i = 1, \; w_i \geq 0, \quad (7)$$

$$\min_v J'(u^*, v) = \Phi'(x(t_f)) + \int_{t_0}^{t_f} e^{-\eta t} I(x(t), u^*, v) dt, \quad (8)$$

subject to

$$\dot{x} = f(x(t), u^*, v), x(t_0) = x_0, \quad (9)$$

where

$$I' = \sum_{j=1}^{M} w'_j I'_j, \; \Phi'(x(t_f)) = \sum_{j=1}^{M} w'_j \Phi_j(x(t_f)), \; \sum_{j=1}^{M} w'_j = 1, \quad w'_j \geq 0. \quad (10)$$

Theorem 4. *If (x^*, u^*, v^*) is an open-loop Nash-equilibrium solution for the problems (6), (8), and (9), then there exist continuous costate functions $\lambda(t)$: $[t_0, t_f] \longrightarrow R^n$, $q(t)$: $[t_0, t_f] \longrightarrow R^n$ and the Hamiltonian functions*

$$H(x, u, v^*, w_1, \cdots, w_N, \lambda) = I(x, u, v^*) + \lambda f(x, u, v^*),$$
$$H'(x, u^*, v, w'_1, \cdots, w'_M, q(t)) = I'(x, u^*, v) + q(t) f(x, u^*, v), \quad (11)$$

where $u^* = (u_1^*, u_2^*, \cdots, u_N^*)$ and $v^* = (v_1^*, v_2^*, \cdots, v_M^*)$ such that the following relations are satisfied:

$$\dot{\lambda}(t) = \rho\lambda - \frac{\partial H(x, u*, v, w_1, \cdots, w_N \lambda)}{\partial x},$$
$$\lambda(t_f) = \sum_{i=1}^{N} w_i \frac{\partial \phi_i(x(t_f))}{\partial x}, \quad (12)$$

$$\dot{q}(t) = \eta q(t) - \frac{\partial H'(x, u, v^*, w'_1, \cdots, w'_M, q(t))}{\partial x},$$
$$q(t_f) = \sum_{j=1}^{M} w'_i \frac{\partial \phi'_i(x(t_f))}{\partial x}, \quad (13)$$

$$H(x, u^*, v^*, w_1, \cdots, w_N, \lambda) \leq H(x, u, v^*, w_1, \cdots, w_N, \lambda),$$
$$H'(x, u^*, v^*, w'_1, \cdots, w'_M, q(t)) \leq H'(x, u^*, v, w'_1, \cdots, w'_M, q(t)). \quad (14)$$

For the proof, see [7].

3. Stability

The problem is stable if it is persistent with regard to the data; that is, the problem is stable if when we change the problem "a little," the solution changes only a little.

3.1. Stability with respect to the Parameters w and w'. We study the stability of the problem with respect to the weights w and w' for the Nash-collative differential game to show the role of each player in the coalition.

Definition 5. The stability Nash-collative differential game is denoted by

$$B(w, w') = \{\Lambda : \Lambda = (w, w') \in R^{N+M}\}, \quad (15)$$

such that the solution of the problem exists, where $w = (w_1, w_2, \cdots, w_N)$, $\sum_{i=1}^{N} w_i = 1$, and $w' = (w'_1, \cdots, w'_M)$, $\sum_{i=1}^{M} w'_j = 1$.

Definition 6. Suppose that $(w, w') \in B(w, w')$ with the corresponding the Nash-collative differential game; the stability set of the first kind for the Nash-collative differential game is defined by

$$S(u*, v*) = \{\Lambda = (w, w') \in R^{N+M} \mid (u*, v*) \text{ is a solution of the problems } (6), (7), (8)\}. \quad (16)$$

Lemma 7. *If the cost functionals $J_i(u, v, w)$ and $J'_j(u, v, w')$*

$$J(u, v, w) = \sum_{i=1}^{N} w_i J_i(u_1, \cdots, u_N, v_1, \cdots, v_M),$$
$$J'(u, v, w') = \sum_{j=1}^{M} w'_j J'_j(u_1, \cdots, u_N, v_1, \cdots, v_M), \quad (17)$$

are linear with respect to w_i and w'_j, respectively, then $S(u^*, v^*)$ is convex.

Proof. Suppose that $\Lambda_1 = (w_1, w_1{}')$, $\Lambda_2 = (w_2, w'_2) \in S$, then

$$J(u^*, v^*, w_1) \leq J(u, v^*, w_1),$$
$$J'(u^*, v^*, w'_1) \leq J'(u, v^*, w'_1), \quad (18)$$

$$J(u^*, v^*, w_2) \le J(u, v^*, w_2),$$
$$J'\left(u^*, v^*, w_2'\right) \le J'\left(u, v^*, w_2'\right). \tag{19}$$

By multiplying both sides of the above inequalities by α and $(1-\alpha)$, respectively, and adding them, then

$$J(u^*, v^*, \alpha w_1 + (1-\alpha)w_2) \le J(u, v^*, \alpha w_1 + (1-\alpha)w_2),$$
$$J'\left(u^*, v^*, \alpha w_1' + (1-\alpha)w_2'\right) \le J'\left(u, v^*, \alpha w_1' + (1-\alpha)w_2'\right), \tag{20}$$

then $S(u^*, v^*)$ is convex with respect to the weight parameters w and w'. □ □

Remark 8. If the cost functionals $J(u, v, w), J'(u, v, w')$ are continuous on the weight space R^{N+M}, where $0 \le w \le 1$ and $0 \le w' \le 1$, then the stability of the first kind $S(u^*, v^*)$ is closed.

4. Counterterrorism Problem

In this section, we discuss the governments and terrorist problems and find the optimal strategies for the government to fight terrorism.

4.1. Governments' Problem. Consider the problem of N -governments and one terrorist organization where the cost functional of the government i, $i = 1, 2, \cdots, N$, is

$$\max_{u_i} J_i = \int_0^\infty e^{-\rho_i t}[\gamma_i h_i(u_i, v) - c_i x_i(t) - kv(t) - \alpha_i u_i(t)]dt, \quad i = 1, 2, \cdots, N, \tag{21}$$

and the cost functional of the terrorist organization is

$$\max_{v} J_0 = \int_0^\infty e^{-\rho t}[\sigma_i x_i(t) + \beta v]dt, \tag{22}$$

subject to

$$\dot{x}_i = r_i x_i(t) - h_i(u_i(t), v(t)), \tag{23}$$

where $x(t)$ is the state trajectory of the game which is the stock of the terrorist organization which includes the financial capital, network supporters, and weapons. u is the strategy of the governments, v is strategy of the terrorist organization, $h(u, v)$ is the interaction between the government and the terrorist organization (harvest function), and ρ is the positive decreasing rate of the running payoff. Since the governments are cooperative, then there exist $w_i \in R^s$, $w_i \ge 0, \sum_{i=1}^N w_i = 1$

$$J = \sum_{i=1}^N w_i J_i = \int_0^\infty \sum_{i=1}^N e^{-\rho_i t} w_i[\gamma_i h_i(u_i, v) - c_i x_i(t) - kv(t) - \alpha_i u_i(t)], \tag{24}$$

$$\dot{x}_i = r_i x_i(t) - h_i(u_i, v), \tag{25}$$

suppose that the decreasing rate (ρ_i) are the same for all governments.

Then, the Hamiltonian function of the Nash-collative problem

$$H = \sum_{i=1}^N w_i \gamma_i h_i(u_i, v) - \sum_{i=1}^N w_i c_i x_i(t) - kv(t) - \sum_{i=1}^N \alpha_i w_i u_i + \lambda_i(r_i x_i - h_i(u_i(t), v(t))). \tag{26}$$

Since in counterterrorism, the governments have to maximize the Hamiltonian, we obtain the necessary conditions

$$\frac{\partial H}{\partial u_i} = (w_i \gamma_i - \lambda_i)\frac{\partial h_i(u_i, v)}{\partial u_i} - \alpha_i w_i = 0 \Rightarrow u_i^* = u_i^*(w_i, \lambda_i, \alpha_i, v). \tag{27}$$

Consider the harvest function $h_i(u_i, v) = u_i^\delta v^\varepsilon, 0 < \delta < \varepsilon < 1$

$$\frac{\partial h_i}{\partial u_i} = \delta u_i^{\delta-1} v^\varepsilon \Rightarrow u_i = \left[\frac{\alpha_i w_i}{w_i \gamma_i \delta - \delta \lambda_i}\right]^{1/(\delta-1)} v^{-\varepsilon/(\delta-1)}. \tag{28}$$

In (28), when terrorist organizations become active, governments double their activities in order to root out terrorism; this is evident in Figure 1.

The costate variable has to follow the differential equation

$$\dot{\lambda}_i = \rho\lambda_i - \frac{\partial H}{\partial x_i} = (\rho + r_i)\lambda_i - w_i c_i. \tag{29}$$

With the optimal control $u_i^*(v)$, the harvest function is

$$h(u_i, v) = \left[\frac{\alpha_i w_i}{w_i \gamma_i \delta - \delta \lambda_i}\right]^{\delta/(\delta-1)} v^{-\varepsilon/(\delta-1)}. \tag{30}$$

4.2. The Terrorist Organization Problem. Consider the payoff functional of the terrorist organization

$$\max_{v} J_0 = \int_0^\infty e^{-\rho t}[\sigma_i x_i t + \beta v]dt, \tag{31}$$

and the state trajectory

$$\dot{x}_i = r_i x_i(t) - h_i(u_i(t), v(t)). \tag{32}$$

The Hamiltonian function of the terrorism

$$H_0 = \sigma_i x_i(t) + \beta v + \eta_i(r_i x_i - h_i(u_i, v)). \tag{33}$$

As the optimal control of the terrorists have to maximize the Hamiltonian function

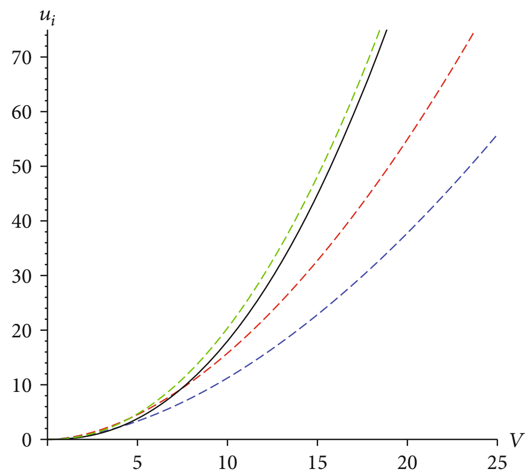

Figure 1: The relation between the strategies of governments u_i and the strategy of terrorism v.

$$\frac{\partial H_0}{\partial v} = \beta - \eta_i \frac{\partial h_i(u_i, v)}{\partial v} = 0 \Rightarrow v^* = v^*(u_i). \quad (34)$$

The costate variable for the terrorist problem

$$\dot{\eta}_i = \mu\eta_i - \frac{\partial H_0}{\partial x_i} = (\mu - r_i)\eta_i - \sigma_i. \quad (35)$$

4.3. Existence. Now, we discuss the existence of the solution for the following system of differential equation; then, we have to integrate equations (36), (37), and (38) on the interval $[0, t]$,

$$\dot{x}_i = r_i x_i(t) - h_i(u_i, v), \quad (36)$$

$$\dot{\lambda}_i = (\rho + r_i)\lambda_i - w_i c_i, \quad (37)$$

$$\dot{\eta}_i = (\mu - r_i)\eta_i - \sigma_i, \quad (38)$$

with the initial conditions $x(0) = x_0$, $\lambda_i(0) = 0$, and $\eta_i(0) = 0$, then

$$x_i = x_0 + \int_0^t (r_i x_i(t) - h_i(u_i, v))\,dt, \quad (39)$$

$$\lambda_i = \int_0^t (\rho + r_i)\lambda_i\,dt - w_i c_i t, \quad (40)$$

$$\eta_i = \int_0^t (\mu - r_i)\eta_i\,dt - \sigma_i t. \quad (41)$$

By differentiating the equations (39), (40), and (41), we get

$$\begin{aligned} \dot{x}_i &= r_i x_i(t) - h_i(u_i, v), \\ \dot{\lambda}_i &= (\rho + r_i)\lambda_i - w_i c_i, \\ \dot{\eta}_i &= (\mu - r_i)\eta_i - \sigma_i, \end{aligned} \quad (42)$$

putting $t = 0$ in the equations (39), (40), and (41); we get $x(0) = x_0$, $\lambda_i(0) = 0$, and $\eta_i(0) = 0$, then the solution exists.

Proposition 9. *The optimal control of the differential game problems (6), (9), and (8) with the initial conditions $x_i(0) = x_0$, $\lambda_i(0) = 0$, and $\eta_i(0) = 0$ are*

$$\begin{aligned} u_i &= \left(\frac{\alpha_i w_i}{w_i\gamma_i\delta - \lambda_i\delta}\right)^{(\varepsilon-1)/(1-\delta-\varepsilon)} \left(\frac{\beta}{\varepsilon\eta_i}\right)^{\varepsilon/(\varepsilon+\delta-1)}, \\ v &= \left(\frac{\beta}{\varepsilon\eta_i}\right)^{(\delta-1)/(1-\varepsilon-\delta)} \left(\frac{\alpha_i w_i}{w_i\gamma_i\delta - \delta\lambda_i}\right)^{\delta/(1-\varepsilon-\delta)}, \end{aligned} \quad (43)$$

and the harvest function

$$h_i(u_i^*, v^*) = \left(\frac{\alpha_i w_i}{w_i\gamma_i\delta - \lambda_i\delta}\right)^{\delta(2\varepsilon-1)/(1-\delta-\varepsilon)} \left(\frac{\beta}{\varepsilon\eta_i}\right)^{1/(\varepsilon+\delta-1)}. \quad (44)$$

Proof. According to equations (27) and (28), we have

$$u_i = \left[\frac{\alpha_i w_i}{w_i\gamma_i\delta - \delta\lambda_i}\right]^{1/(\delta-1)} v^{-\varepsilon/(\delta-1)}. \quad (45)$$

Also, from (34) and $\partial h(u, v)/\partial v = \varepsilon u_i^{\delta} v^{\varepsilon-1}$, we have

$$v = \left(\frac{\beta}{\varepsilon\eta_i}\right)^{1/(\varepsilon-1)} u_i^{-\delta/(\varepsilon-1)}. \quad (46)$$

By solving equations (45) and (46), we get

$$\begin{aligned} u_i &= \left(\frac{\alpha_i w_i}{w_i\gamma_i\delta - \lambda_i\delta}\right)^{(\varepsilon-1)/(1-\delta-\varepsilon)} \left(\frac{\beta}{\varepsilon\eta_i}\right)^{\varepsilon/(\varepsilon+\delta-1)}, \\ v &= \left(\frac{\beta}{\varepsilon\eta_i}\right)^{(\delta-1)/(1-\varepsilon-\delta)} \left(\frac{\alpha_i w_i}{w_i\gamma_i\delta - \delta\lambda_i}\right)^{\delta/(1-\varepsilon-\delta)}, \end{aligned} \quad (47)$$

with the harvest function

$$h_i(u_i^*, v^*) = \left(\frac{\alpha_i w_i}{w_i\gamma_i\delta - \lambda_i\delta}\right)^{\delta(2\varepsilon-1)/(1-\delta-\varepsilon)} \left(\frac{\beta}{\varepsilon\eta_i}\right)^{1/(\varepsilon+\delta-1)}. \quad (48)$$

The solution of this problem is stable when the weight parameter (w_i) is greater than λ_i/γ_i where λ_i is the costate vector of the government i and γ_i is the cost coefficient of the harvest function in the payoff of the government i. Also, the role of each government (w_i), to fight terrorism, is greater than λ_i/γ_i. Thus, the stability set of the first kind for this problem is defined as

$$S(u^*, v^*) = \left\{\Lambda = \left(w_i \in R^N, w_i > \frac{\lambda_i}{\gamma_i}\right)\right\}, \quad (49)$$

such that

$$u_i = \left(\frac{\alpha_i w_i}{w_i \gamma_i \delta - \lambda_i \delta}\right)^{(\varepsilon-1)/(1-\delta-\varepsilon)} \left(\frac{\beta}{\varepsilon \eta_i}\right)^{\varepsilon/(\varepsilon+\delta-1)},$$
$$v = \left(\frac{\beta}{\varepsilon \eta_i}\right)^{(\delta-1)/(1-\varepsilon-\delta)} \left(\frac{\alpha_i w_i}{w_i \gamma_i \delta - \delta \lambda_i}\right)^{\delta/(1-\varepsilon-\delta)}, \quad (50)$$

is the optimal strategy of the counterterrorism problems (21), (22), and (23).

In (46), when governments play their role to the fullest, the government's strategy reaches stability, and this is evident in the red and black curves, also, when the government gives up its role, the governments change their strategies and take strong measures to combat terrorism; this is evident in Figure 2.

In (47), when the government performs its role, the organizations increase their activities until they stabilize at a certain level, this is evident in the red, black, and blue curves, but when the government gives up its role, the terrorist organization is active at a high level, this is evident in the blue curve; this is evident in Figure 3.

In (48), when governments fulfill its role fully, the harvest function is greatest; this is evident in Figure 4. □

Proposition 10. *The state trajectory (the source stock of the terrorist) and the costate variables* λ_i *and* η_i *of governments and terrorists are*

$$x_i(t) = \left(x_i(0) - \frac{h_i(u_i, v)}{r_i}\right) e^{r_i t} + \frac{h_i(u_i, v)}{r_i},$$
$$\lambda_i(t) = \frac{w_i c_i}{\rho + r_i} - \frac{w_i c_i}{\rho + r_i} e^{(\rho + r_i)t}, \quad (51)$$
$$\eta_i(t) = \frac{\sigma_i}{\mu - r_i} - \frac{\sigma_i}{\mu - r_i} e^{(\mu - r_i)t}.$$

Proof. Since the dynamic system of source stock is

$$\dot{x}_i(t) = r_i x_i - h(u_i, v). \quad (52)$$

For the constant strategies and the harvest function, then

$$x_i(t) e^{-r_i t} = \frac{h_i(u_i, v)}{r_i} e^{-r_i t} + \text{constant}\,(c), \quad (53)$$

since $x_i(t) \longrightarrow x_i(0)$ as $t \longrightarrow 0$, then $c = x_i(0) - h_i(u_i, v)/r_i$ and

$$x_i(t) = \left(x_i(0) - \frac{h_i(u_i, v)}{r_i}\right) e^{r_i t} + \frac{h_i(u_i, v)}{r_i}. \quad (54)$$

Since the costate variable for the government is

$$\dot{\lambda}_i = (\rho + r_i)\lambda_i - w_i c_i. \quad (55)$$

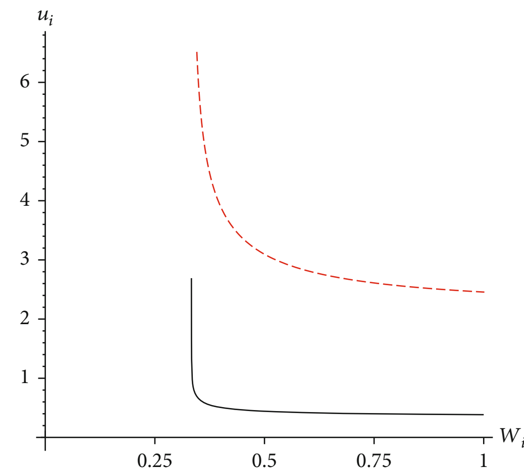

FIGURE 2: The relation between the government's strategies u_i and the weight parameter w_i.

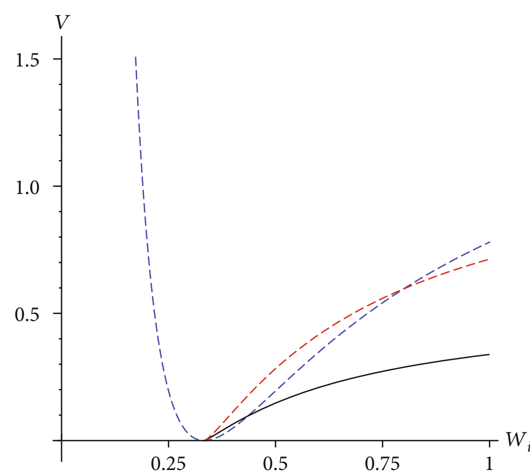

FIGURE 3: The relation between the strategy of terrorist v and the weight parameter of each government w_i.

The solution of this differential equation

$$\lambda_i e^{-(\rho + r_i)t} = \frac{w_i c_i}{\rho + r_i} e^{-(\rho + r_i)t} + C\,(\text{constant}), \quad (56)$$

since $\lambda_i(t) \longrightarrow 0$ as $t \longrightarrow 0$, then $C = -(w_i c_i/(\rho + r_i))$ and

$$\lambda_i(t) = \frac{w_i c_i}{\rho + r_i} - \frac{w_i c_i}{\rho + r_i} e^{(\rho + r_i)t}. \quad (57)$$

Also, the costate variable of the terrorist is

$$\dot{\eta}_i = (\mu - r_i)\eta_i - \sigma_i. \quad (58)$$

Then,

$$\eta_i e^{-(\mu - r_i)t} = +\frac{\sigma}{\mu - r_i} e^{-(\mu - r_i)t} + c\,(\text{constant}), \quad (59)$$

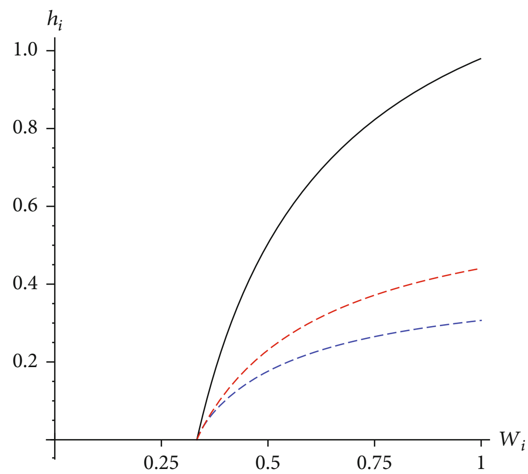

FIGURE 4: The relation between the harvest function $h(u_i, v)$ and weight parameters w_i. □

since $\eta_i(0) = 0$, then $c = -\sigma_i/(\mu - r_i)$ and

$$\eta_i(t) = \frac{\sigma_i}{\mu - r_i} - \frac{\sigma_i}{\mu - r_i} e^{(\mu - r_i)t}. \tag{60}$$

□ □

Remark 11. As shown in (54), the resource stock is increasing with time when the initial stock $x_i(0)$ is greater than

$$\frac{1}{r_i}\left(\frac{\alpha_i w_i}{w_i\gamma_i\delta - \lambda_i\delta}\right)^{\delta(2\varepsilon-1)/(1-\delta-\varepsilon)}\left(\frac{\beta}{\varepsilon\eta_i}\right)^{1/(\varepsilon+\delta-1)}. \tag{61}$$

Remark 12. The resource stock $x_i(t)$ in the duration time $[0, \infty]$ is

$$x_i(t) = \frac{1}{r_i}\left(\frac{\alpha_i w_i}{w_i\gamma_i\delta - \lambda_i\delta}\right)^{\delta(2\varepsilon-1)/(1-\delta-\varepsilon)}\left(\frac{\beta}{\varepsilon\eta_i}\right)^{1/(\varepsilon+\delta-1)}. \tag{62}$$

The resource stock $x_i(t)$ directly decays when the role of each government w_i is sufficiently larger.

Lemma 13. *For the optimal strategies u_i for the Nash-collative, v for the terrorist, and the harvest function $h_i(u_i, v)$, the objective values of the cooperative governments (J) and the terrorist J_0 are*

$$J = \sum_{i=1}^{N} w_i J_i, \tag{63}$$

$$\begin{aligned} J_i &= \frac{1}{\rho}(\gamma_i h_i(u_i, v) - kv - \alpha_i u_i) - \frac{c_i}{\rho} h_i(u_i, v) \\ &\quad - \frac{1}{\rho - r_i}\left(x_i(0) + \frac{h_i(u_i, v)}{r_i}\right), \\ J_0 &= \frac{\sigma_i}{\mu - r_i}\left(x_i(0) - \frac{h_i(u_i, v)}{r_i}\right) + \frac{\sigma_i}{\mu r_i} h(u_i, v) + \frac{\beta}{\mu} v. \end{aligned} \tag{64}$$

Proof. Since the objective function of government i is

$$J_i = \int_0^\infty e^{-\rho_i t}[\gamma_i h_i(u_i, v) - c_i x_i(t) - kv(t) - \alpha_i u_i(t)]dt, \quad i = 1, 2, \cdots, N, \tag{65}$$

and the state rejector is

$$x_i(t) = \left(x_i(0) - \frac{h_i(u_i, v)}{r_i}\right)e^{r_i t} + \frac{h_i(u_i, v)}{r_i}. \tag{66}$$

By substituting from (66) in (65) and integrating it, we have

$$\begin{aligned} J_i &= \frac{1}{\rho}(\gamma_i h_i(u_i, v) - kv - \alpha_i u_i) - \frac{c_i}{\rho} h_i(u_i, v) \\ &\quad - \frac{1}{\rho - r_i}\left(x_i(0) + \frac{h_i(u_i, v)}{r_i}\right). \end{aligned} \tag{67}$$

Also, by substituting from (66) in (68), we have

$$J_0 = \frac{\sigma_i}{\mu - r_i}\left(\mathrm{x}_i(0) - \frac{h_i(u_i, v)}{r_i}\right) + \frac{\sigma_i}{\mu r_i} h(u_i, v) + \frac{\beta}{\mu} v, \tag{68}$$

where u_i, v, and $h_i(u_i, v)$ are defined in Proposition 9. □

5. Conclusions

In this intervention, the cooperation of governments is studied for fighting terrorism by using the Nash-collative approach; the necessity for finding the optimal strategies is derived and we proved that the solution exists and found it. Also, we showed that the optimal strategies are stable when the weight parameter (w_i) is greater than (λ_i/γ_i). Finally, we derived the objective value of each government and its role (w_i) in counterterrorism, the objective of the terrorist organization, and its stock recourse.

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors confirm no competing interests.

Authors' Contributions

The authors contributed to the draft of the manuscript; they read and approved the final manuscript.

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References

[1] J. P. Caulkins, G. Feichtinger, D. Grass, and G. Tragler, "Optimal control of terrorism and global reputation: a case study with novel threshold behavior," *Operations Research Letters*, vol. 37, no. 6, pp. 387–391, 2009.

[2] J. P. Caulkins, G. Feichtinger, D. Grass, and G. Tragler, "Optimizing counterterror operations: should one fight with "fire "or "water"?," *Computers & Operations Research*, vol. 35, no. 6, pp. 1855–1874, 2008.

[3] K. H. Hsia and J. G. Hsie, "A first approach to fuzzy differential game problem: guarding a territory," *Fuzzy Sets and Systems*, vol. 55, no. 2, pp. 157–167, 1993.

[4] I. C. Hung, K. H. Hisa, and L. W. Chen, "Fuzzy differential game of guarding a movable territory," *Information Sciences*, vol. 91, no. 1-2, pp. 113–131, 1993.

[5] E. Youness, J. B. Hughes, and N. A. El-Kholy, "Parametric Nash coalitive differential games," *Mathematical and Computer Modelling*, vol. 26, no. 2, pp. 97–105, 1997.

[6] E. Youness and A. E.-M. A. Megahed, "A study on large scale continuous differential games," *Bulletin of the Australian Mathematical Society*, vol. 94, no. 3, pp. 359–368, 2002.

[7] E. Youness and A. E.-M. A. Megahed, "A study on fuzzy differential game," *Le Matematche*, vol. VI, pp. 97–107, 2001.

[8] S. Hegazy, A. E.-M. A. Megahed, E. Youness, and A. Elbanna, "Min-max zero-sum two persons fuzzy continuous differential games," *International Journal of Applied Mechanics*, vol. 21, no. 1, pp. 1–16, 2008.

[9] E.-M. A. Abd and S. H. Megahed, "Min-max zero two persons continuous differential game with fuzzy control," *African Journal of Clinical and Experimental Microbiology*, vol. 2, no. 2, pp. 86–98, 2013.

[10] A. J. Nova, G. Feichtinger, and G. Leitmann, "A differential game related to terrorism: Nash and Stackelberg strategies," *Journal of Optimization Theory and Applications*, vol. 144, pp. 533–555, 2010.

[11] A. Roy and J. A. Paul, "Terrorism deterrence in a two country framework: strategic interactions between R&D, defense and pre-emption," *Annals of Operations Research*, vol. 211, no. 1, pp. 399–432, 2013.

[12] E.-M. A. Abd and A. Megahed, "A differential game related to terrorism: min-max zero-sum two persons differential game," *Neural Computing and Applications*, vol. 30, no. 3, pp. 865–870, 2018.

[13] A. E.-M. A. Megahed, "The development of a differential game related to terrorism: min-max differential game," *Journal of Egyptian Mathematical Socity*, vol. 25, pp. 306–312, 2017.

[14] A. E.-M. A. Megahed, "A differential game related to terrorism: Stackelberg differential game of E-differentiable and E-convex function," *European Journal of Pure and Applied Mathematics*, vol. 12, no. 2, pp. 654–668, 2019.

[15] S. Wrzaczek, E. H. Kaplan, J. P. Caulkins, A. Seidl, and G. Feichtinger, "Differential terror queue games," *Dynamic Games and Applications*, vol. 7, no. 4, pp. 578–593, 2017.

[16] A. E.-M. A. Megahed, "The Stackelberg differential game for counter-terrorism," *Quality & Quantity*, vol. 53, no. 1, pp. 207–220, 2019.

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Research Article

Construction on the Degenerate Poly-Frobenius-Euler Polynomials of Complex Variable

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In this paper, we introduce degenerate poly-Frobenius-Euler polynomials and derive some identities of these polynomials. We give some relationships between degenerate poly-Frobenius-Euler polynomials and degenerate Whitney numbers and Stirling numbers of the first kind. Moreover, we define degenerate poly-Frobenius-Euler polynomials of complex variables and then we derive several properties and relations.

1. Introduction

Recently, many mathematicians, namely, Carlitz [1, 2], Kim and Kim [3–5], Kim et al. [6–9], Muhiuddin et al. [10–12], and Sharma et al. [13–15] have introduced and studied various degenerate versions of special polynomials and numbers like degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Daehee polynomials, degenerate Fubini polynomials, and degenerate Stirling numbers of the first and second kinds.

The classical Frobenius-Euler polynomials $\mathbb{H}_n^{(\alpha)}(x;u)$ ($u\in\mathbb{C}$ with $u\neq 1$) of order α are defined by means of the following generating function (see [16, 17]):

$$\left(\frac{1-u}{e^z-u}\right)^{\alpha} e^{\zeta z}=\sum_{j=0}^{\infty}\mathbb{H}_j^{(\alpha)}(\zeta;u)\frac{z^j}{j!}. \tag{1}$$

At the point $\zeta=0$, $\mathbb{H}_j^{(\alpha)}(u)=\mathbb{H}_j^{(\alpha)}(0;u)$ are called j^{th} Frobenius-Euler numbers of order α.

The poly-Frobenius-Euler polynomials due to Kurt [16] are defined as follows:

$$\frac{(1-u)Li_k(1-e^{-z})}{z(e^z-u)}e^{\zeta z}=\sum_{j=0}^{\infty}\mathbb{H}_j^{(k)}(\zeta;u)\frac{z^j}{j!}. \tag{2}$$

When $\zeta=0$, $\mathbb{H}_j^{(k)}(u)=\mathbb{H}_j^{(k)}(0;u)$ are called the poly-Frobenius-Euler numbers.

For any $\lambda\in\mathbb{R}$ (or $\mathbb{C}$), $\mathbb{R}$ and $\mathbb{C}$ being, respectively, the sets of real numbers and complex numbers, degenerate version of the exponential function $e_\lambda^\zeta(z)$ is defined as follows (see [3, 4, 6, 18]):

$$e_\lambda^\zeta(z)\coloneqq(1+\lambda z)^{\zeta/\lambda}=\sum_{j=0}^{\infty}(\zeta)_{j,\lambda}\frac{z^j}{j!}, \tag{3}$$

where $(\zeta)_{0,\lambda}=1$ and $(\zeta)_{j,\lambda}=\zeta(\zeta-\lambda)\cdots(\zeta-(j-1)\lambda)$ for $j\geq 1$ (see [1, 2, 4–10, 18]). It follows from Equation (3) that $\lim_{\lambda\longrightarrow 0}e_\lambda^\zeta(z)=e^{\zeta z}$. Note that $e_\lambda^1(z)\coloneqq e_\lambda(z)$.

Carlitz [1, 2] introduced the degenerate Euler polynomials as follows:

$$\frac{2}{e_\lambda(z)+1}e_\lambda^x(z)=\sum_{j=0}^{\infty}E_{j,\lambda}(x)\frac{z^j}{j!}. \quad (4)$$

In the case when $\zeta=0$, $E_{j,\lambda}=E_{j,\lambda}(0)$ are called the degenerate Euler numbers.

Note that

$$\lim_{\lambda\longrightarrow 0}E_j(\zeta\,;\lambda)=E_j(\zeta). \quad (5)$$

The degenerate Frobenius-Euler polynomials are defined by the following (see [6]):

$$\frac{1-u}{(1+\lambda z)^{1/\lambda}-u}(1+\lambda z)^{\zeta/\lambda}=\sum_{j=0}^{\infty}h_{j,\lambda}(\zeta\mid u)\frac{z^j}{j!}. \quad (6)$$

At the value $\zeta=0$, $h_{j,\lambda}(u)=h_{j,\lambda}(0\mid u)$ are called the degenerate Frobenius-Euler numbers.

It is readily seen that

$$\lim_{\lambda\longrightarrow 0}h_{j,\lambda}(\zeta\mid u)=\mathbb{H}_j(\zeta\mid u),\quad (j\geq 0). \quad (7)$$

Recently, Kim et al. [9] introduced the modified degenerate polyexponential function which is defined by the following:

$$Ei_{k,\lambda}(\zeta)=\sum_{j=1}^{\infty}\frac{(1)_{j,\lambda}\zeta^j}{(j-1)!j^k},\quad (|\zeta|<1,k\in\mathbb{Z}). \quad (8)$$

Here and in the following, let $\mathbb{Z}$ denote the set of integers.

We note that

$$Ei_{1,\lambda}(\zeta)=\sum_{j=1}^{\infty}\frac{(1)_{j,\lambda}\zeta^j}{j!}=e_\lambda(\zeta)-1. \quad (9)$$

The degenerate poly-Genocchi polynomials are defined as follows (see [9]):

$$\frac{2Ei_{k,\lambda}(\log_\lambda(1+z))}{e_\lambda(z)+1}e_\lambda^\zeta(z)=\sum_{j=0}^{\infty}G_{j,\lambda}^{(k)}(\zeta)\frac{z^j}{j!},\quad (k\in\mathbb{Z}). \quad (10)$$

Letting $\zeta=0$, $G_{j,\lambda}^{(k)}=G_{j,\lambda}^{(k)}(0)$ are called the poly-Genocchi numbers.

The degenerate Daehee polynomials $D_{j,\lambda}(\zeta)$ are defined as follows (see [8]):

$$\frac{\log_\lambda(1+z)}{z}(1+z)^\zeta=\sum_{j=0}^{\infty}D_{j,\lambda}(\zeta)\frac{z^j}{j!}. \quad (11)$$

$D_{j,\lambda}=D_{j,\lambda}(0)$ are called the degenerate Daehee numbers.

For $i\geq 0$, the degenerate Stirling numbers of the first kind are defined by means of the following generating function (see [4]):

$$\frac{1}{i!}(\log_\lambda(1+z))^i=\sum_{j=i}^{\infty}S_{1,\lambda}(j,i)\frac{z^j}{j!}. \quad (12)$$

Note that $\lim_{\lambda\longrightarrow 0}S_{1,\lambda}(j,k)=S_1(j,k)$ are the Stirling numbers of the first kind given by the following (see [3, 18]):

$$\frac{1}{i!}(\log\,(1+z))^i=\sum_{j=i}^{\infty}S_1(j,i)\frac{z^j}{j!},\quad (i\geq 0). \quad (13)$$

For $i\geq 0$, the degenerate Stirling numbers of the second kind are defined by means of the following generating function (see [18]):

$$\frac{1}{i!}(e_\lambda(z)-1)^i=\sum_{j=i}^{\infty}S_{2,\lambda}(j,i)\frac{z^j}{j!}. \quad (14)$$

We note that $\lim_{\lambda\longrightarrow 0}S_{2,\lambda}(j,k)=S_1(j,k)$ are the Stirling numbers of the second kind given by the following (see [3–7, 18]):

$$\frac{1}{i!}(e^z-1)^i=\sum_{j=i}^{\infty}S_2(j,i)\frac{z^j}{j!},\quad (i\geq 0). \quad (15)$$

The subsequent content of this paper is organized as follows: In Section 2, we define the degenerate poly-Frobenius-Euler polynomials and numbers by using the modified degenerate polyexponential functions and derive some properties and relations of these polynomials. In Section 3, we consider the degenerate poly-Frobenius-Euler polynomials of a complex variable and then we derive several properties and relations. Also, we examine the results derived in this study.

2. Degenerate Poly-Frobenius-Euler Numbers and Polynomials

In this section, we define degenerate poly-Frobenius-Euler numbers and polynomials and investigate some properties of these polynomials. We begin following the definition as follows.

Definition 1. We consider the degenerate poly-Frobenius-Euler polynomials are defined by means of the following generating function:

$$\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u))z)}{z(e_\lambda(z)-u)}e_\lambda^\zeta(z)=\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)\frac{z^j}{j!}, \quad (16)$$

where $\lambda, u \in \mathbb{C}$ with $u \neq 1$ and $k \in \mathbb{Z}$.

Upon setting, $\zeta=0$, $\mathbb{H}_{j,\lambda}^{(k)}(u)=\mathbb{H}_{j,\lambda}^{(k)}(0;u)$ are called the degenerate poly-Frobenius-Euler numbers, where $\log_\lambda(z)=1/z(z^\lambda-1)$ is the compositional inverse of $e_\lambda(z)$ satisfying

$$\log_\lambda(e_\lambda(z))=e_\lambda(\log_\lambda(z))=z. \quad (17)$$

Adjusting $k=1$ in Equation (16), we get the following:

$$\frac{1-u}{e_\lambda(z)-u}e_\lambda^\zeta(z)=\sum_{j=0}^{\infty}h_{j,\lambda}(\zeta;u)\frac{z^j}{j!}, \quad (18)$$

where $h_{j,\lambda}(\zeta;u)$ are called the degenerate Frobenius-Euler polynomials (see [6]).

Obviously,

$$\begin{aligned}\lim_{\lambda\longrightarrow 0}\left(\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)e_\lambda^\zeta(z)&=\sum_{j=0}^{\infty}\lim_{\lambda\longrightarrow 0}\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)\frac{z^j}{j!}\\&=\frac{Ei_k(\log(1+(1-u)z))}{z(e^z-u)}e^{\zeta z}\\&=\sum_{j=0}^{\infty}\mathbb{H}_{j}^{(k)}(\zeta;u)\frac{z^j}{j!},\end{aligned} \quad (19)$$

where $\mathbb{H}_{j}^{(k)}(\zeta;u)$ are called the type 2 poly-Frobenius-Euler polynomials.

Theorem 2. *Let $j \geq 0$. Then, we have the following:*

$$\begin{aligned}&\sum_{p=0}^{j}\binom{j}{p}\sum_{s=0}^{p}\frac{(1)_{s+1,\lambda}}{(s+1)^{k-1}}S_{1,\lambda}(p+1,s+1)(\zeta)_{j-p,\lambda}\frac{(1-u)^{p+1}}{p+1}\\&\quad=\sum_{s=0}^{j}\binom{j}{s}\mathbb{H}_{j-s,\lambda}^{(k)}(\zeta;u)(1)_{s,\lambda}-u\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u).\end{aligned} \quad (20)$$

Proof. Using Equation (16), we see that

$$\begin{aligned}&\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z}e_\lambda^\zeta(z)\\&=e_\lambda(z)\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)\frac{z^j}{j!}-u\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)\frac{z^j}{j!}\\&=\sum_{s=0}^{\infty}(1)_{s,\lambda}\frac{z^s}{s!}\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)\frac{z^j}{j!}-u\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)\frac{z^j}{j!}\\&=\sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\binom{j}{s}\mathbb{H}_{j-s,\lambda}^{(k)}(\zeta;u)(1)_{s,\lambda}-u\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)\right)\frac{z^j}{j!}.\end{aligned} \quad (21)$$

On the other hand,

$$\begin{aligned}&\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z}e_\lambda^\zeta(z)\\&=\left(\sum_{j=0}^{\infty}(\zeta)_{j,\lambda}\frac{z^j}{j!}\right)\frac{1}{z}\left(\sum_{s=1}^{\infty}\frac{(1)_{s,\lambda}(\log_\lambda(1+(1-u)z))^s}{(s-1)!s^k}\right)\\&=\left(\sum_{j=0}^{\infty}(\zeta)_{j,\lambda}\frac{z^j}{j!}\right)\frac{1}{z}\left(\sum_{s=0}^{\infty}\frac{(1)_{s+1,\lambda}}{(s+1)^{k-1}}\sum_{l=s+1}^{\infty}S_{1,\lambda}(l,s+1)\frac{(1-u)^l z^l}{l!}\right)\\&=\left(\sum_{j=0}^{\infty}(\zeta)_{j,\lambda}\frac{z^j}{j!}\right)\left(\sum_{l=0}^{\infty}\sum_{s=0}^{l}\frac{(1)_{s+1,\lambda}}{(s+1)^{k-1}}S_{1,\lambda}(l+1,s+1)\frac{(1-u)^{l+1}}{l+1}\frac{z^l}{l!}\right)\\&=\sum_{j=0}^{\infty}\left(\sum_{p=0}^{j}\binom{j}{p}\sum_{s=0}^{p}\frac{(1)_{s+1,\lambda}}{(s+1)^{k-1}}S_{1,\lambda}(p+1,s+1)(\zeta)_{j-p,\lambda}\frac{(1-u)^{p+1}}{p+1}\right)\frac{z^j}{j!}.\end{aligned} \quad (22)$$

In view of Equation (22), we complete the proof. □

Theorem 3. *Let $j \geq 0$. Then, we have the following:*

$$\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)=\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda}^{(k)}(u)(\zeta)_{r,\lambda}. \quad (23)$$

Proof. In Equation (16), we observe that

$$\begin{aligned}\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)\frac{z^j}{j!}&=\left(\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)e_\lambda^\zeta(z)\\&=\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(u)\frac{z^j}{j!}\sum_{r=0}^{\infty}(\zeta)_{r,\lambda}\frac{z^r}{r!}\\&=\sum_{j=0}^{\infty}\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda}^{(k)}(u)(\zeta)_{r,\lambda}\frac{z^j}{j!}.\end{aligned} \quad (24)$$

By Equations (16) and (24), we require at the desired result. □

Theorem 4. *Let $j \geq 0$. Then,*

$$\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)=\sum_{q=0}^{j}\binom{j}{q}\sum_{r=0}^{q}\frac{(1)_{r+1,\lambda}}{(r+1)^k}\frac{S_1(q+1,r+1)(1-u)^q}{q+1}\mathbb{H}_{j-q,\lambda}(\zeta;u). \tag{25}$$

Proof. By using Equations (14) and (16), we see that

$$\begin{aligned}\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)\frac{z^j}{j!}&=\left(\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)e_\lambda^{\zeta}(z)\\&=\frac{e_\lambda^{\zeta}(z)}{z(e_\lambda(z)-u)}\sum_{r=1}^{\infty}\frac{(1)_{r,\lambda}(\log_\lambda(1+(1-u)z))^r}{(r-1)!r^k}\\&=\frac{e_\lambda^{\zeta}(z)}{z(e_\lambda(z)-u)}\sum_{r=0}^{\infty}\frac{(1)_{r+1,\lambda}(\log_\lambda(1+(1-u)z))^{r+1}}{r!(r+1)^k}\\&=\frac{e_\lambda^{\zeta}(z)}{z(e_\lambda(z)-u)}\sum_{r=0}^{\infty}\frac{(1)_{r+1,\lambda}}{(r+1)^k}\sum_{j=r+1}^{\infty}S_{1,\lambda}(j,r+1)\frac{((1-u)t)^n}{n!}\\&=\frac{1-u}{e_\lambda(z)-u}e_\lambda^{\zeta}(z)\sum_{r=0}^{\infty}\frac{(1)_{r+1,\lambda}}{(r+1)^k}\sum_{r=j}^{\infty}\frac{S_{1,\lambda}(j+1,r+1)(1-u)^j}{j+1}\frac{z^j}{j!}\\&=\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}(\zeta;u)\frac{z^j}{j!}\sum_{l=0}^{\infty}\sum_{r=0}^{l}\frac{(1)_{r+1,\lambda}}{(r+1)^k}\frac{S_{1,\lambda}(l+1,r+1)(1-u)^l}{l+1}\frac{z^l}{l!}\\&=\sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}\binom{j}{q}\sum_{r=0}^{q}\frac{(1)_{r+1,\lambda}}{(r+1)^k}\frac{S_{1,\lambda}(q+1,r+1)(1-u)^q}{q+1}\mathbb{H}_{j-q,\lambda}(\zeta;u)\right)\frac{z^j}{j!}.\end{aligned} \tag{26}$$

In view of Equation (26), we complete the proof. □

Corollary 5. *For $k\in\mathbb{Z}$ and $j \geq 0$. Then,*

$$\mathbb{H}_{j,\lambda}^{(k)}(u)=\sum_{q=0}^{j}\binom{j}{q}\sum_{r=0}^{q}\frac{1}{(r+1)^k}\frac{S_{1,\lambda}(q+1,r+1)(1-u)^q}{q+1}\mathbb{H}_{j-q,\lambda}(u). \tag{27}$$

Corollary 6. *For $j \geq 0$. Then,*

$$\mathbb{H}_{j,\lambda}^{(k)}(\zeta;u)=\sum_{q=0}^{j}\binom{j}{q}\sum_{r=0}^{q}\frac{S_{1,\lambda}(q+1,r+1)(1-u)^q}{q+1}\mathbb{H}_{j-q,\lambda}(\zeta;u). \tag{28}$$

Corollary 7. *On setting $u=-1$ and $k=1$ and using Equation (4), Theorem 3 to get*

$$E_{j,\lambda}(x)=\sum_{q=0}^{j}\binom{j}{l}\sum_{r=0}^{q}\frac{S_{1,\lambda}(q+1,r+1)2^q}{q+1}E_{j-q,\lambda}(\zeta),\quad (j\geq 0). \tag{29}$$

It is well known from [7] that

$$\left(\frac{z}{\log(1+z)}\right)^r(1+z)^{\zeta-1}=\sum_{j=0}^{\infty}B_j^{(j-r+1)}(\zeta)\frac{z^j}{j!},\quad (r\in\mathbb{C}), \tag{30}$$

where $B_j^{(r)}(\zeta)$ are called the higher-order Bernoulli polynomials which are given by the generating function (see [3, 16]):

$$\left(\frac{z}{e^z-1}\right)^r e^{\zeta z}=\sum_{j=0}^{\infty}B_j^{(r)}(\zeta)\frac{z^j}{j!}. \tag{31}$$

Theorem 8. *For $j \geq 0$. Then, we have the following:*

$$\mathbb{H}_{j,\lambda}^{(2)}(u)=\sum_{l=0}^{j}\binom{j}{l}\frac{(1-u)^l B_l^l}{l+1}\mathbb{H}_{j-l,\lambda}(u). \tag{32}$$

Proof. In Equation (8), we note that

$$\begin{aligned}&\frac{d}{d\zeta}Ei_{k,\lambda}(\log_\lambda(1+(1-u)x))\\&\quad=\frac{d}{d\zeta}\sum_{j=1}^{\infty}\frac{(\log_\lambda(1+(1-u)\zeta))^j}{(j+1)!j^k}\\&\quad=\frac{1-u}{(1+(1-u)\zeta)\log_\lambda(1+(1-u)\zeta)}\sum_{j=1}^{\infty}\frac{(\log_\lambda(1+(1-u)\zeta))^j}{(j+1)!j^{k-1}}\\&\quad=\frac{1-u}{(1+(1-u)\zeta)\log_\lambda(1+(1-u)\zeta)}Ei_{k-1,\lambda}(\log_\lambda(1+(1-u)\zeta)).\end{aligned} \tag{33}$$

From Equation (33), for $k \geq 1$, we have the following:

$$\begin{aligned}\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(u)\frac{\zeta^j}{j!}&=\frac{(1-u)^{k-1}}{\zeta(e_\lambda(\zeta)-u)}\int_0^{\zeta}\frac{1}{(1+(1-u)t)\log_\lambda(1+(1-u)z)}\\&\quad\times\underbrace{\int_0^z\frac{1}{(1+(1-u)z)\log_\lambda(1+(1-u)z)}\cdots}_{k-2-\text{times}}\\&\quad\cdot\int_0^z\frac{z}{(1+(1-u)z)\log_\lambda(1+(1-u)z)}dzdz\cdots dz.\end{aligned} \tag{34}$$

For $k \geq 2$ in the above expression, we have the following:

$$\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(2)}(u)\frac{\zeta^j}{j!}=\frac{(1-u)}{\zeta(e_\lambda(\zeta)-u)}\int_0^{\zeta}\frac{(1-u)z}{(1+(1-u)z)\log_\lambda(1+(1-u)z)}, \tag{35}$$

$$\left(\sum_{j=0}^{\infty}H_{j,\lambda}(u)\frac{\xi^j}{j!}\right)\left(\sum_{l=0}^{\infty}\frac{(1-u)^l B_l^l}{l+1}\frac{\xi^l}{l!}\right)=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l}\frac{(1-u)^l B_l^l}{l+1}H_{j-l,\lambda}(u)\right)\frac{\xi^j}{j!} \tag{36}$$

In view of Equations (35) and (36), we obtain at the desired result. □

Theorem 9. *Let $j \geq 0$. Then, we have the following:*

$$\mathbb{H}_{j,\lambda}^{(k)}(\zeta+\eta;u)=\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda}^{(k)}(\zeta;u)(\eta)_{r,\lambda}. \tag{37}$$

Proof. Using Equation (16), we get the following:

$$\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta+\eta\,;u)\frac{z^j}{j!}=\left(\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)e_\lambda^{\zeta+\eta}(z)$$
$$=\left(\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta\,;u)\frac{z^j}{j!}\right)\left(\sum_{r=0}^{\infty}(\eta)_{r,\lambda}\frac{z^r}{r!}\right)$$
$$=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda}^{(k)}(\zeta\,;u)(\eta)_{r,\lambda}\right)\frac{z^j}{j!}. \tag{38}$$

Thus, by Equation (38), we complete the proof. □

Theorem 10. *Let $j\geq 0$. Then, we have the following:*

$$\mathbb{H}_{j,\lambda}^{(k)}(\zeta+1\,;u)=\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda}^{(k)}(\zeta\,;u)(1)_{r,\lambda}. \tag{39}$$

Proof. In Equation (16), we see that

$$\sum_{j=0}^{\infty}\left[\mathbb{H}_{j,\lambda}^{(k)}(\zeta+1\,;u)-\mathbb{H}_{j,\lambda}^{(k)}(\zeta\,;u)\right]\frac{z^j}{j!}$$
$$=\left(\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)e_\lambda^{\zeta}(z)[e_\lambda(z)-1]$$
$$=\sum_{j=0}^{\infty}\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda}^{(k)}(\zeta\,;u)(1)_{r,\lambda}\frac{z^j}{j!}-\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta\,;u)\frac{z^j}{j!}. \tag{40}$$

Comparing the coefficients of z^j on both sides, we get the result. □

Theorem 11. *Let $j\geq 0$. Then,*

$$\mathbb{H}_{j,\lambda}^{(k)}(\zeta\,;u)=\sum_{r=0}^{j}\sum_{q=0}^{r}\binom{j}{r}(\zeta)_q S_{2,\lambda}(r,q)\mathbb{H}_{j-r,\lambda}^{(k)}(u). \tag{41}$$

Proof. From Equation (16), we see that

$$\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta\,;u)\frac{z^j}{j!}=\left(\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)e_\lambda^{\zeta}(z)$$
$$=\left(\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)[e_\lambda(z)-1+1]^{\zeta}$$
$$=\left(\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)\left(\sum_{q=0}^{\infty}(\zeta)_q\sum_{l=q}^{\infty}S_{2,\lambda}(l,q)\frac{z^l}{l!}\right)$$
$$=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\sum_{q=0}^{r}\binom{j}{r}(\zeta)_q S_{2,\lambda}(r,q)\mathbb{H}_{j-r,\lambda}^{(k)}(u)\right)\frac{z^j}{j!}. \tag{42}$$

By Equation (42). We complete the proof. □

Theorem 12. *Let $j\geq 0$. Then,*

$$\mathbb{H}_{j,\lambda}^{(k)}(\zeta+\alpha\mid u)=\sum_{n=0}^{j}\sum_{l=0}^{n}\binom{j}{n}u^l(\zeta)_l W_{u,\alpha}(n,l\,;\lambda)\mathbb{H}_{j-n,\lambda}^{(k)}(u). \tag{43}$$

Proof. By changing ζ by $\zeta u+\alpha$ in Equation (16), we get the following:

$$\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta+\alpha\,;u)\frac{z^j}{j!}=\left(\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)e_\lambda^{\alpha}(t)e_\lambda^{\zeta u}(z)$$
$$=\left(\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)e_\lambda^{\alpha}(t)[e_\lambda^{u}(z)-1+1]^{\zeta}$$
$$=\left(\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)e_\lambda^{\alpha}(z)\left(\sum_{l=0}^{\infty}\binom{\zeta}{l}[e_\lambda^{u}(z)-1)^l\right)$$
$$=\left(\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\right)\left(e_\lambda^{\alpha}(z)\sum_{l=0}^{\infty}u^l(\zeta)_l\mathrm{frac}(e_\lambda^{u}(z)-1)^l l!u^l\right)$$
$$=\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(u)\frac{z^j}{j!}\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}u^l(\zeta)_l W_{u,\alpha}(n,l\,;\lambda)\frac{z^n}{n!}\right)$$
$$=\sum_{j=0}^{\infty}\left(\sum_{n=0}^{j}\sum_{l=0}^{n}\binom{j}{n}u^l(\zeta)_l W_{u,\alpha}(n,l\,;\lambda)\mathbb{H}_{j-n,\lambda}^{(k)}(u)\right)\frac{z^j}{j!}. \tag{44}$$

Therefore, by Equations (16) and (44), we obtain the result. □

3. Degenerate Unipoly-Frobenius-Euler Numbers and Polynomials

In this section, we introduce degenerate unipoly-Frobenius-Euler polynomials by using degenerate unipoly polynomials and derive some important properties of these polynomials.

In [3], Kim and Kim introduced unipoly function. In the view of [9], the degenerate unipoly function is defined by Dolgy and Khan [19] as follows:

$$u_{k,\lambda}(\zeta\mid p)=\sum_{j=1}^{\infty}p(j)\frac{(1)_{j,\lambda}\zeta^j}{j^k}. \tag{45}$$

Note that, we have the following:

$$u_{k,\lambda}\left(\zeta\mid\frac{1}{\Gamma}\right)=Ei_{k,\lambda}(\zeta) \tag{46}$$

is the modified degenerate polylogarithm function.

It is clear that

$$\lim_{\lambda\longrightarrow 0}u_{k,\lambda}(\zeta\mid p)=\sum_{j=1}^{\infty}\lim_{\lambda\longrightarrow 0}p(i)\frac{(1)_{i,\lambda}\zeta^i}{i^k}=u_k(\zeta\mid p)=\sum_{j=1}^{\infty}p(i)\frac{\zeta^i}{i^k},\quad(k\in\mathbb{Z}) \tag{47}$$

are called the unipoly function attached to polynomials $p(\zeta)$ (see [20]).

From Equation (47), we have the following:

$$u_k(\zeta \mid 1) = \sum_{j=1}^{\infty} \frac{\zeta^j}{j^k} = Li_k(\zeta)\ (\text{see}[9]) \tag{48}$$

is the ordinary polylogarithm function.

By using Equations (45) and (16), we define the degenerate unipoly-Frobenius-Euler polynomials as follows:

$$\frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z) \mid p)}{z(e_\lambda(z)-u)} e_\lambda^{\zeta}(z) = \sum_{j=0}^{\infty} \mathbb{H}_{j,\lambda,p}^{(k)}(\zeta\,;u)\frac{z^j}{j!}. \tag{49}$$

At the special value $\zeta = 0$, $\mathbb{H}_{j,\lambda,p}^{(k)}(u) = \mathbb{H}_{j,\lambda,p}^{(k)}(0\,;u)$ are called the degenerate unipoly-Frobenius-Euler numbers.

Theorem 13. *Let $j \geq 0$. Then, we have the following:*

$$\mathbb{H}_{j,\lambda,1/\Gamma}^{(k)}(\zeta\,;\eta) = \mathbb{H}_{j,\lambda}^{(k)}(\zeta\,;u), \quad (k \in \mathbb{Z}). \tag{50}$$

Proof. Let us take $p(j) = 1/\Gamma\lambda$. Then, we have the following:

$$\begin{aligned}
\sum_{j=0}^{\infty} \mathbb{H}_{j,\lambda,1/\Gamma}^{(k)}(\zeta\,;u)\frac{z^j}{j!} &= \frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z) \mid 1/\Gamma p)}{z(e_\lambda(z)-u)} e_\lambda^{\zeta}(z) \\
&= \frac{1}{z(e_\lambda(z)-u)} \sum_{r=1}^{\infty} \frac{(1)_{r,\lambda}(\log_\lambda(1+(1-u)z))^r}{r^k(r+1)!} e_\lambda^{\zeta}(z) \\
&= \frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)} e_\lambda^{\zeta}(z) \\
&= \sum_{j=0}^{\infty} \mathbb{H}_{j,\lambda}^{(k)}(\zeta\,;u)\frac{z^j}{j!}.
\end{aligned} \tag{51}$$

In view of Equation (51), we complete the proof. □

Theorem 14. *Let $j \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then,*

$$\mathbb{H}_{j,\lambda,p}^{(k)}(\zeta\,;u) = \sum_{s=0}^{j}\sum_{r=0}^{s}\binom{j}{s}\frac{p(r+1)(1)_{r+1,\lambda}(r+1)!S_{1,\lambda}(r+1,s+1)\mathbb{H}_{j-s,\lambda}(u)(1-u)^s}{(r+1)^k(s+1)}. \tag{52}$$

Proof. Using Equation (49), we have the following:

$$\begin{aligned}
\sum_{j=0}^{\infty} \mathbb{H}_{j,\lambda,p}^{(k)}(u)\frac{z^j}{j!} &= \frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z) \mid p)}{z(e_\lambda(z)-u)} \\
&= \frac{1}{z(e_\lambda(z)-u)} \sum_{r=1}^{\infty} \frac{p(r)(1)_{r,\lambda}}{r^k}(\log_\lambda(1+(1-u)z))^r \\
&= \frac{1}{z(e_\lambda(z)-u)} \sum_{r=0}^{\infty} \frac{p(r+1)(1)_{r+1,\lambda}(r+1)!}{(r+1)^k} \sum_{l=r+1}^{\infty} S_{1,\lambda}(r+1,l)(1-u)^l\frac{z^l}{l!} \\
&= \left(\sum_{j=0}^{\infty} \mathbb{H}_{j,\lambda}(u)\frac{z^j}{j!}\right)\left(\sum_{r=0}^{\infty}\sum_{l=0}^{r} \frac{p(r+1)(1)_{r+1,\lambda}(r+1)!}{(r+1)^k}\frac{S_{1,\lambda}(r+1,l+1)}{l+1}\frac{z^l}{l!}\right) \\
&= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{\infty}\sum_{r=0}^{s}\binom{j}{s}\frac{p(r+1)(1)_{r+1,\lambda}(r+1)!S_{1,\lambda}(r+1,s+1)\mathbb{H}_{j-s,\lambda}(u)(1-u)^s}{(r+1)^k(s+1)}\right)\frac{z^j}{j!}.
\end{aligned} \tag{53}$$

Therefore, by Equations (49) and (53), we get the result. □

Corollary 15. *Let $j \geq 0$. Then, we have the following:*

$$\mathbb{H}_{j,\lambda,1/\Gamma}^{(k)}(u) = \mathbb{H}_{j,\lambda}^{(k)}(u) = \sum_{s=0}^{n}\sum_{r=0}^{s}\binom{j}{l}\frac{S_{1,\lambda}(r+1,s+1)\mathbb{H}_{j-s,\lambda}(u)}{(r+1)^{k-1}(s+1)}. \tag{54}$$

Theorem 16. *Let $j \geq 0$. Then, we have the following:*

$$\mathbb{H}_{j,\lambda,p}^{(k)}(\zeta\,;u) = \sum_{r=0}^{j}\sum_{s=0}^{r}\binom{j}{r}H_{j-r,\lambda,p}^{(k)}(u)(\zeta)_{(r)}S_{2,\lambda}(r,s). \tag{55}$$

Proof. By Equation (49), we have the following:

$$\begin{aligned}
\sum_{j=0}^{\infty} H_{j,\lambda}^{(k,p)}(x\,;u)\frac{z^j}{j!} &= \frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z) \mid p)}{z(e_\lambda(z)-u)}(e_\lambda(z)-1+1)^{\zeta} \\
&= \frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z) \mid p)}{z(e_\lambda(z)-u)}\sum_{l=0}^{\infty}\binom{\zeta}{l}(e_\lambda(z)-1)^l \\
&= \left(\sum_{j=0}^{\infty} H_{j,\lambda,p}^{(k)}(u)\frac{z^j}{j!}\right)\left(\sum_{l=0}^{\infty}(\zeta)_l\sum_{m=l}^{\infty}S_{2,\lambda}(m,l)\frac{z^m}{m!}\right) \\
&= \sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\sum_{s=0}^{r}\binom{n}{r}H_{j-r,\lambda,p}^{(k)}(u)(\zeta)_{(r)}S_{2,\lambda}(r,s)\right)\frac{z^j}{j!}.
\end{aligned} \tag{56}$$

By Equation (56), we obtain the result. □

Theorem 17. *Let $j \geq 0$. Then,*

$$\mathbb{H}_{j,\lambda,p}^{(k)}(\zeta\,;u) = j\mathbb{H}_{j-1,\lambda,p}^{(k)}(\zeta\,;u). \tag{57}$$

Proof. When we consider Equation (49), we see that

$$\Delta_\lambda\left(\sum_{j=0}^{\infty} \mathbb{H}_{n,\lambda,p}^{(k)}(\zeta\,;u)\frac{z^j}{j!}\right) = \Delta_\lambda\left(\frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z) \mid p)}{z(e_\lambda(z)-u)}(1+\lambda z)^{\nu/\lambda}\right), \tag{58}$$

and then we have the following:

$$\begin{aligned}
\sum_{j=0}^{\infty} \Delta_\lambda \mathbb{H}_{j,\lambda,p}^{(k)}(\zeta\,;u)\frac{z^j}{j!} &= \frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\Delta_\lambda e_\lambda^{\zeta}(z) \\
&= \frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)} e_\lambda^{\zeta}(z)z \\
&= \sum_{j=0}^{\infty} \mathbb{H}_{j,\lambda,p}^{(k)}(\zeta\,;u)\frac{z^{j+1}}{j!}.
\end{aligned} \tag{59}$$

Therefore, by Equation (59), we complete the proof. □

Theorem 18. *Let $j \geq 0$ and $k \in \mathbb{Z}$. Then, we have the following:*

$$\frac{\partial}{\partial \zeta} \mathbb{H}_{j,\lambda,p}^{(k)}(\zeta\,;u) = \sum_{r=0}^{j} \binom{j}{r} \mathbb{H}_{j-r,\lambda,p}^{(k)}(\zeta\,;u)(1)_{r,\lambda}. \tag{60}$$

Proof. In Equation (49), we consider that

$$\begin{aligned}
&\frac{\partial}{\partial \zeta}\left(\sum_{j=0}^{\infty} \mathbb{H}_{j,\lambda,p}^{(k)}(\zeta\,;u)\frac{z^j}{j!}\right) \\
&= \frac{\partial}{\partial \zeta}\left(\frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z)\mid p)}{z(e_\lambda(z)-u)}(1+\lambda z)^{\zeta/\lambda}\right)\sum_{j=0}^{\infty}\frac{\partial}{\partial \zeta}\mathbb{H}_{j,\lambda,p}^{(k)}(\zeta\,;u)\frac{z^j}{j!} \\
&= \frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\frac{\partial}{\partial \zeta}(1+\lambda z)^{\zeta/\lambda} \\
&= \frac{u_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}(1+\lambda z)^{\zeta/\lambda}(1+\lambda z)^{1/\lambda} \\
&= \left(\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda,p}^{(k)}(\zeta\,;u)\frac{z^j}{j!}\right)\left(\sum_{r=0}^{\infty}(1)_{r,\lambda}\frac{z^r}{r!}\right) \\
&= \sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda,p}^{(k)}(\zeta\,;u)(1)_{r,\lambda}\right)\frac{z^j}{j!}.
\end{aligned} \tag{61}$$

By Equation (61), we complete the proof. □

4. Degenerate Poly-Frobenius-Euler Polynomials of Complex Variables

In this section, we define the Frobenius-Euler polynomials of the complex variables. We consider the degenerate cosine function and degenerate sine function. Using the degenerate cosine function and the degenerate sine function, we introduce the degenerate cosine poly-Frobenius-Euler polynomials and degenerate sine poly-Frobenius-Euler polynomials.

In [5], Kim et al. defined the degenerate sine $\sin_\lambda z$ and cosine $\cos_\lambda z$ functions by the following:

$$\begin{aligned}
\sin_\lambda^{(\zeta)}(z) &= \frac{e_\lambda^{ix}(z) - e_\lambda^{-i\zeta}(z)}{2i}, \\
\cos_\lambda^{(\zeta)}(z) &= \frac{e_\lambda^{i\zeta}(z) + e_\lambda^{-i\zeta}(z)}{2},
\end{aligned} \tag{62}$$

where $i=\sqrt{-1}$. Note that $\lim_{\lambda \longrightarrow 0}\sin_\lambda^{(\zeta)}(z) = \sin \zeta z$ and $\lim_{\lambda \longrightarrow 0}\cos_\lambda^{(\zeta)}(z) = \cos \zeta z$. From Equation (62), it is readily seen that

$$e_\lambda^{i\zeta}(z) = \cos_\lambda^{(\zeta)}(z) + i\sin_\lambda^{(\zeta)}(z). \tag{63}$$

By these functions in Equation (62), the degenerate sine-polynomials $S_{j,\lambda}(\zeta,\eta)$ and degenerate cosine-polynomials $C_{j,\lambda}(\zeta,\eta)$ are introduced by Kim et al. [5] as follows:

$$\sum_{j=0}^{\infty} S_{j,\lambda}(\zeta,\eta)\frac{z^j}{j!} = e_\lambda^{\zeta}(z)\sin_\lambda^{(\eta)}(z), \tag{64}$$

$$\sum_{j=0}^{\infty} C_{j,\lambda}(\zeta,\eta)\frac{z^j}{j!} = e_\lambda^{\zeta}(z)\cos_\lambda^{(\eta)}(z). \tag{65}$$

Several properties of these polynomials in Equations (64) and (65) were studied and investigated by Kim et al. [5]. Also, by means of these functions, Kim et al. [5] introduced the degenerate Euler and Bernoulli polynomials of complex variable and investigate some of their properties. Motivated and inspired by these considerations above, we define degenerate poly-Frobenius-Euler polynomials of complex variable as follows:

$$\begin{aligned}
\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta+i\eta\,;u)\frac{z^j}{j!} &= \frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}e_\lambda^{(\zeta+i\eta)}(z) \\
&= \frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}e_\lambda^{\zeta}(z)\left[\cos_\lambda^{(\eta)}(z) + i\,\sin_\lambda^{(\eta)}(z)\right],
\end{aligned} \tag{66}$$

$$\begin{aligned}
\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{(k)}(\zeta-i\eta\,;u)\frac{z^j}{j!} &= \frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}e_\lambda^{(\zeta-i\eta)}(z) \\
&= \frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}e_\lambda^{\zeta}(z)\left[\cos_\lambda^{(\eta)}(z) - i\,\sin_\lambda^{(\eta)}(z)\right].
\end{aligned} \tag{67}$$

From Equations (66) and (67), we get the following:

$$\begin{aligned}
&\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}e_\lambda^{\zeta}(z)\;\cos_\lambda^{(\eta)}(z) \\
&\quad= \sum_{j=0}^{\infty}\frac{\mathbb{H}_{j,\lambda}^{(k)}(\zeta+i\eta\,;u)+\mathbb{H}_{j,\lambda}^{(k)}(\zeta-i\eta\,;u)}{2}\frac{z^j}{j!}, \\
&\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}e_\lambda^{\zeta}(z)\;\sin_\lambda^{(\eta)}(z) \\
&\quad= \sum_{j=0}^{\infty}\frac{\mathbb{H}_{j,\lambda}^{(k)}(\zeta+i\eta\,;u)-\mathbb{H}_{j,\lambda}^{(k)}(\zeta-i\eta\,;u)}{2i}\frac{z^j}{j!}.
\end{aligned} \tag{68}$$

Now, we define the degenerate cosine poly-Frobenius-Euler polynomials and the degenerate sine poly-Frobenius-Euler polynomials, respectively, as follows:

$$\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{[k,c]}(\zeta,\eta\,;u)\frac{z^j}{j!} = \frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}e_\lambda^{\zeta}(z)\;\cos_\lambda^{(\eta)}(z), \tag{69}$$

$$\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{[k,s]}(\zeta,\eta\,;u)\frac{z^j}{j!} = \frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}e_\lambda^{\zeta}(z)\;\sin_\lambda^{(\eta)}(z). \tag{70}$$

Theorem 19. *The following results hold true:*

$$\mathbb{H}_{j,\lambda}^{[k,c]}(\zeta,\eta\,;u)=\frac{1}{2}\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda}^{[k,c]}(\zeta,\eta\,;u)\sum_{l=0}^{r}\binom{r}{l}(\zeta)_{r-l,\lambda}\left((i\eta)_{l,\lambda}+(-i\eta)_{l,\lambda}\right),\tag{71}$$

$$\mathbb{H}_{j,\lambda}^{[k,s]}(\zeta,\eta\,;u)=\frac{1}{2i}\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda}^{[k,s]}(\zeta,\eta\,;u)\sum_{l=0}^{r}\binom{r}{l}(\zeta)_{r-l,\lambda}\left((i\eta)_{l,\lambda}-(-i\eta)_{l,\lambda}\right).\tag{72}$$

Proof. From Equation (69), we have the following:

$$\begin{aligned}\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{[k,c]}(\zeta,\eta\,;u)\frac{z^j}{j!}&=\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}e_\lambda^{\zeta}(z)\ \cos_\lambda^{(\eta)}(z)\\&=\frac{Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\frac{1}{2}\sum_{r=0}^{\infty}\sum_{l=0}^{r}\binom{r}{l}(\zeta)_{r-l,\lambda}\left((i\eta)_{l,\lambda}+(-i\eta)_{l,\lambda}\right)\frac{z^r}{r!}\\&=\sum_{j=0}^{\infty}\left(\frac{1}{2}\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda}^{[k,c]}(\zeta,\eta\,;u)\sum_{l=0}^{r}\binom{r}{l}(\zeta)_{r-l,\lambda}\left((i\eta)_{l,\lambda}+(-i\eta)_{l,\lambda}\right)\right)\frac{z^j}{j!}.\end{aligned}\tag{73}$$

By Equation (73), we get Equation (71). Similarly, by using Equations (70) and (3), we can easily find the result (Equation (72)). ☐

Theorem 20. *The following results hold true:*

$$\mathbb{H}_{j,\lambda}^{[k,c]}(\zeta_1+\zeta_2,;u)=\sum_{m=0}^{j}\binom{j}{m}\sum_{r=0}^{m}S_{2,\lambda}^{(\zeta_1)}(m,r)(\zeta_2)_r\mathbb{H}_{j,\lambda}^{[k,c]}(0,\eta\,;u),\tag{74}$$

$$\mathbb{H}_{j,\lambda}^{[k,s]}(\zeta_1+\zeta_2,;u)=\sum_{m=0}^{j}\binom{j}{m}\sum_{r=0}^{m}S_{2,\lambda}^{(\zeta_1)}(m,r)(\zeta_2)_r\mathbb{H}_{j,\lambda}^{[k,s]}(0,\eta\,;u).\tag{75}$$

Proof. From Equations (3) and (69), we have the following:

$$\begin{aligned}\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{[k,c]}(\zeta_1+\zeta_2,;u)\frac{z^j}{j!}&=\frac{e_\lambda^{(\zeta_1+\zeta_2)}(z)Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\cos_\lambda^{(\eta)}(z)\\&=\sum_{m=0}^{\infty}\sum_{r=0}^{m}S_{2,\lambda}^{(\zeta_1)}(m,r)(\zeta_2)_r\frac{z^m}{m!}\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{[k,c]}(0,\eta\,;u)\frac{z^j}{j!}\\&=\sum_{j=0}^{\infty}\left(\sum_{m=0}^{j}\binom{j}{m}\sum_{r=0}^{m}S_{2,\lambda}^{(\zeta_1)}(m,r)(\zeta_2)_r\mathbb{H}_{j,\lambda}^{[k,c]}(0,\eta\,;u)\right)\frac{z^j}{j!}.\end{aligned}\tag{76}$$

In view of Equation (76), we get Equation (74). Similarly, by using Equations (3) and (70), we require at the desired result (Equation (75)). ☐

Theorem 21. *The following results hold true:*

$$\begin{aligned}\mathbb{H}_{j,\lambda}^{[k,c]}(\zeta_1+\zeta_2,\eta_1+\eta_2\,;u)&=\sum_{r=0}^{j}\binom{j}{r}\\&\cdot\left[\mathbb{H}_{j-r,\lambda}^{[k,c]}(\zeta_1,\eta_1\,;u)C_{r,\lambda}(\zeta_2,\eta_2)-\mathbb{H}_{j,\lambda}^{[k,s]}(\zeta_1,\eta_1\,;u)S_{r,\lambda}(\zeta_2,\eta_2)\right],\end{aligned}\tag{77}$$

$$\begin{aligned}\mathbb{H}_{j,\lambda}^{[k,s]}(\zeta_1+\zeta_2,\eta_1+\eta_2\,;u)&=\sum_{r=0}^{j}\binom{j}{r}\\&\cdot\left[\mathbb{H}_{j-r,\lambda}^{[k,s]}(\zeta_1,\eta_1\,;u)C_{r,\lambda}(\zeta_2,\eta_2)-\mathbb{H}_{j,\lambda}^{[k,c]}(\zeta_1,\eta_1\,;u)S_{m,\lambda}(\zeta_2,\eta_2)\right].\end{aligned}\tag{78}$$

Proof. From Equation (69), we have the following:

$$\begin{aligned}\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{[k,c]}(\zeta_1+\zeta_2,\eta_1+\eta_2\,;u)\frac{z^j}{j!}&=\frac{e_\lambda^{(\zeta_1+\zeta_2)}(z)Ei_{k,\lambda}(\log_\lambda(1+(1-u)z))}{z(e_\lambda(z)-u)}\\&\cdot\left[\cos_\lambda^{(\eta_1)}(z)\ \cos_\lambda^{(\eta_2)}(z)-\sin_\lambda^{(\eta_1)}(z)\ \sin_\lambda^{(\eta_2)}(z)\right]\\&=\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{[k,c]}(\zeta_1,\eta_1\,;u)\frac{z^j}{j!}\sum_{r=0}^{\infty}C_{r,\lambda}(\zeta_2,\eta_2)\frac{z^r}{r!}\\&-\sum_{j=0}^{\infty}\mathbb{H}_{j,\lambda}^{[k,s]}(\zeta_1,\eta_1\,;u)\frac{z^j}{j!}\sum_{r=0}^{\infty}S_{r,\lambda}(\zeta_2,\eta_2)\frac{z^r}{r!}\\&=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\binom{j}{r}\mathbb{H}_{j-r,\lambda}^{[k,c]}(\zeta_1,\eta_1\,;u)C_{r,\lambda}(\zeta_2,\eta_2)\right.\\&\left.-\mathbb{H}_{j,\lambda}^{[k,s]}(\zeta_1,\eta_1\,;u)S_{m,\lambda}(\zeta_2,\eta_2)\right)\frac{z^j}{j!}.\end{aligned}\tag{79}$$

Comparing the coefficients of z on both sides, we get Equation (77). Similarly, by using Equation (70), we can easily get Equation (78). ☐

Corollary 22. *On setting $\zeta_1=\zeta_2=\zeta$ and $\eta_1=\eta_2=\eta$ in Theorem 16, we have the following:*

$$\begin{aligned}\mathbb{H}_{j,\lambda}^{[k,c]}(2\zeta,2\eta\,;u)&=\sum_{r=0}^{j}\binom{j}{r}\left[\mathbb{H}_{j-r,\lambda}^{[k,c]}(\zeta,\eta\,;u)C_{r,\lambda}(\zeta,\eta)-\mathbb{H}_{j,\lambda}^{[k,s]}(\zeta,\eta\,;u)S_{r,\lambda}(\zeta,\eta)\right],\\\mathbb{H}_{j,\lambda}^{[k,s]}(2\zeta,2\eta\,;u)&=\sum_{r=0}^{j}\binom{j}{r}\left[\mathbb{H}_{j-r,\lambda}^{[k,s]}(\zeta,\eta\,;u)C_{r,\lambda}(\zeta,\eta)-\mathbb{H}_{j,\lambda}^{[k,c]}(\zeta,\eta\,;u)S_{m,\lambda}(\zeta,\eta)\right].\end{aligned}\tag{80}$$

5. Conclusions

In this paper, we defined the degenerate poly-Frobenius-Euler polynomials by employing the modified degenerate polyexponential functions. We have established some identities and relations between degenerate Whitney numbers and degenerate Stirling numbers of the first kind. Also, we have established addition formulas and derivative formulas of degenerate poly-Frobenius-Euler polynomials. In the last section, we have defined degenerate poly-Frobenius-Euler polynomials of complex variables and then we have derived several properties and relations.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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References

[1] L. Carlitz, "Degenerate Stirling," *Bernoulli and Eulerian numbers. Utilitas Math.*, vol. 15, pp. 51–88, 1979.

[2] L. Carlitz, "A degenerate Staudt-Clausen theorem," *Archiv der Mathematik*, vol. 7, no. 1, pp. 28–33, 1956.

[3] D. S. Kim and T. Kim, "A note on polyexponential and unipoly functions," *Russian Journal of Mathematical Physics*, vol. 26, no. 1, pp. 40–49, 2019.

[4] D. S. Kim and T. Kim, "A note on a new type of degenerate Bernoulli numbers," *Russian Journal of Mathematical Physics*, vol. 27, no. 2, pp. 227–235, 2020.

[5] D. S. Kim, T. Kim, and H. Lee, "A note on degenerate Euler and Bernoulli polynomials of complex variable," *Symmetry.*, vol. 11, no. 9, p. 1168, 2019.

[6] T. Kim, H. I. Kwon, and J. J. Seo, "On the degenerate Frobenius-Euler polynomials," July 2015, http://arxiv.org/abs/1507.04846v1.

[7] D. S. Kim, T. Kim, and J. J. Seo, "Higher-order Bernoulli, Frobenius-Euler and Euler polynomials," *Journal of Computational Analysis and Applications*, vol. 17, no. 1, pp. 147–155, 2014.

[8] T. Kim, D. S. Kim, H.-Y. Kim, and J. Kwon, "Some results on degenerate Daehee and Bernoulli numbers and polynomials," *Advances in Difference Equations*, vol. 311, 13 pages, 2020.

[9] T. Kim, D. S. Kim, J. Kwon, and H. Y. Kim, "A note on degenerate Genocchi and poly-Genocchi numbers and polynomials," *Journal of Inequalities and Applications*, vol. 2020, no. 1, 2020.

[10] G. Muhiuddin, W. A. Khan, and U. Duran, "Two variable type 2 Fubini polynomials," *Mathematics.*, vol. 9, no. 281, pp. 1–13, 2021.

[11] G. Muhiuddin, W. A. Khan, U. Duran, and D. al-Kadi, "Some identities of the degenerate multi-poly-Bernoulli polynomials of complex variable," *Journal of Function Spaces*, vol. 2021, Article ID 7172054, 8 pages, 2021.

[12] G. Muhiuddin, W. A. Khan, U. Duran, and D. Al-Kadi, "A new class of higher-order hypergeometric Bernoulli polynomials associated with Lagrange-Hermite polynomials," *Symmetry.*, vol. 13, no. 4, p. 648, 2021.

[13] S. K. Sharma, W. A. Khan, and C. S. Ryoo, "A parametric kind of the degenerate Fubini numbers and polynomials," *Mathematics.*, vol. 8, no. 3, p. 405, 2020.

[14] S. Kumar Sharma, W. A. Khan, S. Araci, and S. S. Ahmed, "New type of degenerate Daehee polynomials of the second kind," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[15] S. K. Sharma, W. A. Khan, S. Araci, and S. S. Ahmed, "New construction of type 2 degenerate central Fubini polynomials with their certain properties," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[16] B. Kurt, "Poly-Frobenius-Euler polynomials," *Proceedings of the International Conferences on Numerical Analysis and Applied Mathematics. AIP Conference Proceedings*, vol. 1863, p. 300019-1-300019-3, 2016.

[17] B. Kurt and Y. Simsek, "On the generalized Apostol-type Frobenius-Euler polynomials," *Adv. Difference Equ.*, vol. 2013, no. 1, pp. 1–9, 2013.

[18] T. Kim, "A note on degenerate Stirling polynomials of the second kind," *Proceedings of the Jangjeon Mathematical Society*, vol. 20, no. 3, pp. 319–331, 2017.

[19] D. V. Dolgy and W. A. Khan, "A note on type-two degenerate poly-Changhee polynomials of the second kind," *Symmetry.*, vol. 13, no. 4, p. 579, 2021.

[20] M. Kaneko, "Poly-Bernoulli numbers," *Journal de Théorie des Nombres de Bordeaux*, vol. 9, no. 1, pp. 221–228, 1997.

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Research Article

Boundedness for Commutators of Rough p-Adic Hardy Operator on p-Adic Central Morrey Spaces

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In the present article we obtain the boundedness for commutators of rough p-adic Hardy operator on p-adic central Morrey spaces. Furthermore, we also acquire the boundedness of rough p-adic Hardy operator on Lebesgue spaces.

1. Introduction

The classical Hardy operator for a non-negative function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$is given as

$$\mathscr{H}f(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0. \tag{1}$$

In [1], Hardy defined the above operator which satisfies

$$\|\mathscr{H}f\|_{L^r(\mathbb{R}^+)} \leq \frac{r}{r-1}\|f\|_{L^r(\mathbb{R}^+)}, \quad 1 < r < \infty. \tag{2}$$

The constant $r/(r-1)$ in (2) is sharp. In [2], Faris extended the Hardy operator in $\mathbb{R}^n$ by

$$Hf(\mathbf{x}) = \frac{1}{|\mathbf{x}|^n}\int_{(B(0,|\mathbf{x}|))} f(\mathbf{t})d\mathbf{t}. \tag{3}$$

In this day and age, the Hardy operator has received a relentless consideration, see for example [3–7]. Moreover, the publications [8–12] and the references therein will do world of good to comprehend the Hardy type operators.

The past few years has seen an immense attention towards mathematical physics [13, 14] along with harmonic analysis in the p-adic field [15–23]. Furthermore, the applications of p-adic analysis are seen mainly in string theory [24], quantum gravity [25, 26], quantum mechanics [14] and spring glass theory [27, 28].

Suppose p is a prime number, $r \in \mathbb{Q}$, we introduce the p-adic norm $|r|_p$ by a rule

$$|0|_p = 0, |r|_p = p^{-\alpha}, \tag{4}$$

where the integer $\alpha = \alpha(r)$ is defined by the following notation

$$r = p^{\alpha}m/n, \tag{5}$$

integers m, n and p are coprime to each other. $|\cdot|_p$ has many properties of a real norm together with

$$|r+s|_p \leq \max\left\{|r|_p, |s|_p\right\}. \tag{6}$$

We denote the completion of $\mathbb{Q}$ in the norm $|\cdot|_p$ by $\mathbb{Q}_p$. Any nonzero p-adic number can be written in series form as (see [14]):

$$r = p^{\alpha}\sum_{i=0}^{\infty}\gamma_i p^i, \tag{7}$$

where $\gamma_i, \alpha \in \mathbb{Z}, \gamma_i \in \mathbb{Z}/p\mathbb{Z}_p, \gamma_0 \neq 0$. The series (7) is convergent as $|p^\alpha \gamma_i p^i|_p = p^{-\alpha-i}$.

The space $\mathbb{Q}_p^n$ contains all n-tuples of $\mathbb{Q}_p$. The norm on this space is

$$|\mathbf{r}|_p = \max_{1\le k\le n} |r_k|_p. \tag{8}$$

Represent by $B_\alpha(\mathbf{a})$ the ball with radius p^α and center at $\mathbf{a}$ and $S_\alpha(\mathbf{a})$ its sphere:

$$B_\alpha(\mathbf{a}) = \left\{\mathbf{r} \in \mathbb{Q}_p^n : |\mathbf{r}-\mathbf{a}|_p \le p^\alpha\right\}, S_\alpha(\mathbf{a}) = \left\{\mathbf{r} \in \mathbb{Q}_p^n : |\mathbf{r}-\mathbf{a}|_p = p^\alpha\right\}. \tag{9}$$

Since $\mathbb{Q}_p^n$ is a locally compact Hausdorff space, then there exists the Haar measure $d\mathbf{x}$ on additive group $\mathbb{Q}_p^n$ and is normalized by

$$\int_{B_0(\mathbf{0})} d\mathbf{x} = |B_0(\mathbf{0})|_H = 1, \tag{10}$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. Moreover, it is not hard to see that $|B_\gamma(\mathbf{a})|_H = p^{n\gamma}$ and $|S_\gamma(\mathbf{a})|_H = p^{n\gamma}(1-p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

Suppose $L^s(\mathbb{Q}_p^n)(1 \le s<\infty)$ is the space of all complex-valued functions f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{L^s(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(\mathbf{x})|^s d\mathbf{x}\right)^{1/s} < \infty. \tag{11}$$

In what follows author in [29] introduced the Hardy operator in the p-adic field as for $f \in L_{loc}(\mathbb{Q}_p^n)$, we have

$$H^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} f(\mathbf{t}) d\mathbf{t}. \tag{12}$$

For better understanding of Hardy type operators in the p-adic field we refer the publications [12, 29–32] and the references therein. From here on, we discuss the rough kernel version of an operator which is also considered an important topic in analysis, see for instance [20, 33–37]. In [10], Fu et al. studied the roughness of Hardy operator in the real field. In the p-adic setting, the rough Hardy operator and its commutator are defined and studied in [20]. Suppose $f : \mathbb{Q}_p^n \longrightarrow \mathbb{R}$, $b : \mathbb{Q}_p^n \longrightarrow \mathbb{R}$ and $\Omega : S_0 \longrightarrow \mathbb{R}$ are measurable mappings, then

$$\begin{aligned} H_\Omega^p f(\mathbf{x}) &= \frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} \Omega\left(|\mathbf{t}|_p \mathbf{t}\right) f(\mathbf{t}) d\mathbf{t} \\ H_\Omega^{p,b} f(\mathbf{x}) &= \frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|)_p} (b(\mathbf{x}) - b(\mathbf{t}))\Omega\left(|\mathbf{t}|_p \mathbf{t}\right) f(\mathbf{t}) d\mathbf{t}, \end{aligned} \tag{13}$$

respectively, whenever

$$\begin{aligned} &\int_{B(\mathbf{0},|\mathbf{x}|_p)} \left|\Omega\left(|\mathbf{t}|_p \mathbf{t}\right) f(\mathbf{t})\right| d\mathbf{t} < \infty \\ &\int_{B(\mathbf{0},|\mathbf{x}|_p)} \left|b(\mathbf{t})\Omega\left(|\mathbf{t}|_p \mathbf{t}\right) f(\mathbf{t})\right| d\mathbf{t} < \infty. \end{aligned} \tag{14}$$

In [20], authors showed the weighted estimates of $H_\Omega^{p,b}$ on two weighted Herz-Morrey spaces. In the present article, we acquire the $\lambda -$ *central* bounded mean oscillations $(C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n))$ estimate of $H_\Omega^{p,b}$ on p-adic central Morrey spaces. In addition, we open up our results with a lemma which shows the boundedness of rough p-adic Hardy operator on Lebesgue spaces. Throughout this paper, we have no intention to obtain the best constants in the inequalities. The occurrence of a letter C does not mean a same constant, its value may vary at different positions.

Definition 1 [32]. Suppose $\lambda \in \mathbb{R}$ and $1 < r < \infty$. The p-adic space $\dot{B}^{r,\lambda}(\mathbb{Q}_p^n)$ is defined as follows

$$\|f\|_{\dot{B}^{r,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} |f(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} < \infty, \tag{15}$$

where $B_\gamma = B_\gamma(0)$. Interestingly $\dot{B}^{r,\lambda}(\mathbb{Q}_p^n)$ reduces to $\{0\}$ for $-1/r > \lambda$.

Definition 2 [32]. Suppose $\lambda < 1/n$ and $1 < r < \infty$. The p-adic space $C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)$ is given by

$$\|f\|_{C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} \left|f(\mathbf{x}) - f_{B_\gamma}\right|^r d\mathbf{x}\right)^{1/r} < \infty, \tag{16}$$

where $f_{B_\gamma} = 1/|B_\gamma|_H \int_{B_\gamma} f(\mathbf{x}) d\mathbf{x}$, $|B_\gamma|_H$ is the Haar measure of B_γ.

Remark 3. If $\lambda = 0$, then $C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)$ is reduced to $CMO^r(\mathbb{Q}_p^n)$ (see [29]).

2. Boundedness for Commutators of Rough p-Adic Hardy Operator on Central Morrey Spaces

In the present section $(C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n))$ estimates of $H_\Omega^{p,b}$ on central Morrey spaces in the p-adic field are obtained. However, to prove the result we need few lemmas.

Lemma 4 [32]. *Let $b \in C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)$ and $i, j \in \mathbb{Z}$, $\lambda \geq 0$. Then*

$$\left|b_{B_i} - b_{B_j}\right| \leq p^n |i-j| \|b\|_{C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)} \max\left\{|B_i|_H^{\lambda}, |B_j|_H^{\lambda}\right\}. \tag{17}$$

Lemma 5. *Suppose $1 < s < \infty$ and $1/s + 1/s' = 1$. Then the inequality*

$$\|H_\Omega^p f\|_{L^s(\mathbb{Q}_p^n)} \leq C\|f\|_{L^s(\mathbb{Q}_p^n)} \tag{18}$$

holds for all $f \in L^s_{loc}(\mathbb{Q}_p^n)$ and $\Omega \in L^s(S_0)$.

Proof. Firstly, we set

$$\tilde{f}(\mathbf{x}) = \frac{1}{1-p^n} \int_{|\xi_p|=1} f\left(|\mathbf{x}|_p^{-1}\xi\right) d\xi, \mathbf{x} \in \mathbb{Q}_p^n. \tag{19}$$

Obviously $\tilde{f}(\mathbf{x}) = \tilde{f}(|\mathbf{x}|_p^{-1})$. In what follows we take this function a radial function on p-adic Lebesgue space. It is not hard to see that

$$H_\Omega^p\left(\tilde{f}\right)(\mathbf{x}) = H_\Omega^p(f)(\mathbf{x}). \tag{20}$$

In [29], it is shown that $\|\tilde{f}\|_{L^s(\mathbb{Q}_p^n)} \leq \|f\|_{L^s(\mathbb{Q}_p^n)}$. Therefore,

$$\frac{\|H_\Omega^p f\|_{L^s(\mathbb{Q}_p^n)}}{\|f\|_{L^s(\mathbb{Q}_p^n)}} \leq \frac{\left\|H_\Omega^p \tilde{f}\right\|_{L^s(\mathbb{Q}_p^n)}}{\left\|\tilde{f}\right\|_{L^s(\mathbb{Q}_p^n)}}. \tag{21}$$

This implies that $\tilde{f} = f$ providing f is a radial function. Consequently, the norm of an operator H_Ω^p along with its restriction to the function $\hat{f}$ have the same operator norm. So, we assume f to be a radial function in the rest of the proof.

By the change of p-adic variables $\mathbf{t} = |\mathbf{x}|_p^{-1}\mathbf{y}$, we have

$$\begin{aligned}\|H_\Omega^p f\|_{L^s(\mathbb{Q}_p^n)} &= \left(\int_{\mathbb{Q}_p^n} \left|\frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} \Omega\left(|\mathbf{t}|_p \mathbf{t}\right) f(\mathbf{t}) d\mathbf{t}\right|^s d\mathbf{x}\right)^{1/s} \\ &= \left(\int_{\mathbb{Q}_p^n} \left|\int_{B(\mathbf{0},1)} \Omega(|\mathbf{y}|_p \mathbf{y}) f\left(|\mathbf{x}|_p^{-1}\mathbf{y}\right) d\mathbf{y}\right|^s d\mathbf{x}\right)^{1/s}.\end{aligned} \tag{22}$$

Now by using Minkowski's inequality and Hölder's inequality $(1/s + 1/s' = 1)$, we get

$$\begin{aligned}\|H_\Omega^p f\|_{L^s(\mathbb{Q}_p^n)} &\leq \int_{B(\mathbf{0},1)} \Omega\left(|\mathbf{y}|_p \mathbf{y}\right) \left(\int_{\mathbb{Q}_p^n} \left|f\left(|\mathbf{y}|_p^{-1}\mathbf{x}\right)\right|^s d\mathbf{x}\right)^{1/s} d\mathbf{y} \\ &\leq \left(\int_{B(\mathbf{0},1)} \Omega\left(|\mathbf{y}|_p \mathbf{y}\right) |\mathbf{y}|_p^{-n/s} d\mathbf{y}\right) \|f\|_{L^s(\mathbb{Q}_p^n)} \\ &= \left(\sum_{j=-\infty}^{0} \int_{S_j} \Omega(p^j \mathbf{y}) p^{-nj/s} d\mathbf{y}\right) \|f\|_{L^s(\mathbb{Q}_p^n)} \\ &\leq \sum_{j=-\infty}^{0} p^{-jn/s} \left(\int_{S_j} \left|\Omega(p^j \mathbf{y})\right|^s d\mathbf{y}\right)^{1/s} \left(\int_{S_j} dy\right)^{1/s'} \\ &\quad \cdot \|f\|_{L^s(\mathbb{Q}_p^n)}.\end{aligned} \tag{23}$$

We handle the first part of sum as follows

$$\int_{S_j} \left|\Omega(p^j \mathbf{y})\right|^s d\mathbf{y} = \int_{|\mathbf{z}|_p=1} |\Omega(\mathbf{z})|^s p^{jn} d\mathbf{z} = Cp^{jn}. \tag{24}$$

Hence inequality (23) takes the following form

$$\|H_\Omega^p f\|_{L^s(\mathbb{Q}_p^n)} \leq C\|f\|_{L^s(\mathbb{Q}_p^n)}, \tag{25}$$

which completes the proof of a lemma.

Now, we turn towards our key result.

Theorem 6. *Suppose $1 < r_1 < \infty$, $r_1' < r_2 < \infty$, $n(1/r_2 - 1/r_1) < n/r_1$, $1/r_1 + 1/r_2 = 1/r$, $-1/r_1 < \lambda_1 < 0$, $\lambda = \lambda_1 + \lambda_2$ and $0 \leq \lambda_2 < 1/n$. If $r_1' < s < \infty$, then the below inequality*

$$\left\|H_\Omega^{p,b} f\right\|_{\dot{B}^{r,\lambda}(\mathbb{Q}_p^n)} \leq C\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}, \tag{26}$$

holds for $b \in CMO^{\max\{r_2, sr_1'/(s-r_1'), \lambda_2\}}(\mathbb{Q}_p^n)$ and $\Omega \in L^s(S_0)$.

Proof. We suppose $f \in \dot{B}^{r_1,\lambda_1}(\mathbb{Q}_p^n)$. We also take $\gamma \in \mathbb{Z}$ and without any brevity we consider $\|b\|_{CMO^{\max\{r_2, sr_1'/(s-r_1'), \lambda_2\}}(\mathbb{Q}_p^n)} = 1$. Applying Minkowski's inequality to have

$$\begin{aligned}&\left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} \left|H_\Omega^{p,b} f(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r} \\ &= \left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} \left|\frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} \Omega(|\mathbf{t}|_p \mathbf{t}) f(\mathbf{t})(b(\mathbf{x}) - b(\mathbf{t})) d\mathbf{t}\right|^r d\mathbf{x}\right)^{1/r} \\ &\leq \left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} \left|\frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} \Omega(|\mathbf{t}|_p \mathbf{t}) f(\mathbf{t})\left(b(\mathbf{x}) - b_{B_\gamma}\right) d\mathbf{t}\right|^r d\mathbf{x}\right)^{1/r} \\ &\quad + \left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} \left|\frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} \Omega(|\mathbf{t}|_p \mathbf{t}) f(\mathbf{t})\left(b(\mathbf{t}) - b_{B_\gamma}\right) d\mathbf{t}\right|^r d\mathbf{x}\right)^{1/r} \\ &= I + II.\end{aligned} \tag{27}$$

For the evaluation of I, we make use of Lemma (5) which shows that H_{Ω}^{p} is bounded from $L^{r}(\mathbb{Q}_p^n)$ to $L^{r}(\mathbb{Q}_p^n)$, $(1<r<\infty)$. By Hölder's inequality $(1=r/r_1+r/r_2)$, we have

$$\begin{aligned} I &\le |B_\gamma|_H^{-1/r-\lambda}\left(\int_{B_\gamma}\left|b(\mathbf{x})-b_{B_\gamma}\right|^{r_2}d\mathbf{x}\right)^{1/r_2}\left(\int_{B_\gamma}\left|H_{\Omega}^{p}f(\mathbf{x})\right|^{r_1}d\mathbf{x}\right)^{1/r_1}\\ &\le |B_\gamma|_H^{-1/r-\lambda}\left(\int_{B_\gamma}\left|b(\mathbf{x})-b_{B_\gamma}\right|^{r_2}d\mathbf{x}\right)^{1/r_2}\left(\int_{B_\gamma}|f(\mathbf{x})|^{r_1}d\mathbf{x}\right)^{1/r_1}\\ &= C\|f\|_{\dot{B}^{r_1,\lambda_1}(\mathbb{Q}_p^n)}. \end{aligned} \tag{28}$$

In order to estimate II, we proceed as follows

$$\begin{aligned} II^r &\le \frac{1}{|B_\gamma|_H^{1+\lambda r}}\int_{B_\gamma}\left|\frac{1}{|\mathbf{x}|_p^n}\int_{B(\mathbf{0},|\mathbf{x}_p)}\Omega(|\mathbf{t}|_p\mathbf{t})f(\mathbf{t})\left(b(\mathbf{t})-b_{B_\gamma}\right)d\mathbf{t}\right|^r d\mathbf{x}\\ &\le \frac{1}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}\int_{S_k}p^{-knr}\left(\int_{B(\mathbf{0},p^k)}|\Omega\left(|\mathbf{t}|_p\mathbf{t}\right)f(\mathbf{t})\left(b(\mathbf{t})-b_{B_\gamma}\right)d\mathbf{t}|\right)^r d\mathbf{x}\\ &= \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left(\sum_{j=-\infty}^{k}\int_{S_j}|\Omega(p^j\mathbf{t})f(\mathbf{t})\left(b(\mathbf{t})-b_{B_\gamma}\right)d\mathbf{t}|\right)^r\\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left(\sum_{j=-\infty}^{k}\int_{S_j}|\Omega(p^j\mathbf{t})f(\mathbf{t})\left(b(\mathbf{t})-b_{B_j}\right)d\mathbf{t}|\right)^r\\ &\quad+\frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left(\sum_{j=-\infty}^{k}\int_{S_j}|\Omega(p^j\mathbf{t})f(\mathbf{t})\left(b_{B_j}-b_{B_\gamma}\right)d\mathbf{t}|\right)^r\\ &= II_1+II_2. \end{aligned} \tag{29}$$

For $j,k\in\mathbb{Z}$ with $j\le k$, we have

$$\int_{S_j}\left|\Omega(p^j\mathbf{t})\right|^s d\mathbf{t}=\int_{|\mathbf{z}|_p=1}|\Omega(\mathbf{z})|^s p^{jn}d\mathbf{z}\le Cp^{kn}. \tag{30}$$

To evaluate II_1, we apply Hölder's inequality together with (30) to get

$$\begin{aligned} II_1 &\le \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left[\sum_{j=-\infty}^{k}\left(\int_{S_j}\left|\Omega(p^j\mathbf{t})\right|^s d\mathbf{t}\right)^{1/s}\right.\\ &\quad\times\left.\left(\int_{S_j}|f(\mathbf{t})|^{r_1}d\mathbf{t}\right)^{1/r_1}\left(\int_{S_j}\left|b(\mathbf{t})-b_{B_j}\right|^{r_2}d\mathbf{t}\right)^{1/r_2}\right]^r\\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}p^{kn(1-r+r/s)}\left\{\sum_{j=-\infty}^{k}|B_j|^{1/r_1+\lambda_1+1/r_2+\lambda_2}\right\}^r\\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}p^{kn(1+\lambda r)}=\frac{C}{|B_\gamma|_H^{1+\lambda r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}p^{\gamma n(1+\lambda r)}\\ &= C\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}. \end{aligned} \tag{31}$$

The convergence of above series is eminent from $\lambda_1+\lambda_2+1/r_1+1/r_2\ge\lambda_1+1-1/s>-1/r+1-1/s=1/r_1{}'-1/s>0$.

For II_2, we use Lemma 4, inequality (30) and Hölder's inequality to obtain

$$\begin{aligned} II_2 &= \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left(\sum_{j=-\infty}^{k}\int_{S_j}|\Omega(p^j\mathbf{t})f(\mathbf{t})\left(b_{B_j}-b_{B_\gamma}\right)d\mathbf{t}|\right)^r\\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left[\sum_{j=-\infty}^{k}\int_{S_j}\Omega(p^j\mathbf{t})f(\mathbf{t})(\gamma-j)|B_\gamma|_H^{\lambda_2}d\mathbf{t}\right]^r\\ &= \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left[\sum_{j=-\infty}^{k}(\gamma-j)\int_{S_j}\Omega(p^j\mathbf{t})f(t)dt\right]^r\\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda_1 r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left[\sum_{j=-\infty}^{k}(\gamma-j)\left(\int_{S_j}\left|\Omega(p^j\mathbf{t})\right|^s d\mathbf{t}\right)^{1/s}\right.\\ &\quad\times\left.\left(\int_{S_j}|f(\mathbf{t})|^{r_1}d\mathbf{t}\right)^{1/r_1}\left(\int_{S_j}d\mathbf{t}\right)^{1/r_1{}'-1/s}\right]^r\\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda_1 r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}p^{kn(1-r+r/s)}\left[\sum_{j=-\infty}^{k}(\gamma-j)|B_j|^{\lambda_1+1-1/s}\right]^r\\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda_1 r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}p^{kn(1-r+r/s)}(\gamma-k)^r|B_k|^{(\lambda_1+1-1/s)r}\\ &= \frac{C}{|B_\gamma|_H^{1+\lambda_1 r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}(\gamma-k)^r p^{knr(1/r+\lambda_1)}\\ &= \frac{C}{|B_\gamma|_H^{1+\lambda_1 r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}p^{\gamma nr(1/r+\lambda_1)}=C\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}, \end{aligned} \tag{32}$$

where we notice that $0<\lambda_1+1-1/s$ together with $\lambda_1+1/r_1+1/r_2>1/r_2>0=\lambda_1+1/r$. From (28), (31) and (32), we get

$$\left\|H_{\Omega}^{p,b}f\right\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\le C\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}. \tag{33}$$

3. Conclusion

We mainly focused on the boundedness for commutators of rough p-adic Hardy operator on p-adic central Morrey spaces. Besides, we also obtained the boundedness of rough p-adic Hardy operator on p-adic Lebesgue spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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References

[1] G. H. Hardy, "Note on a theorem of Hilbert," *Mathematische Zeitschrift*, vol. 6, no. 3-4, pp. 314–317, 1920.

[2] W. G. Faris, "Weak Lebesgue spaces and quantum mechanical binding," *Duke Mathematical Journal*, vol. 43, pp. 365–373, 1976.

[3] Z. W. Fu, S. Z. Lu, and S. Shi, "Two characterizations of central BMO space via the commutators of Hardy operators," *Forum Mathematicum*, vol. 33, no. 2, pp. 505–529, 2021.

[4] A. Hussain, N. Sarfraz, I. Khan, and A. M. Alqahtani, "Estimates for Commutators of Bilinear Fractional -Adic Hardy Operator on Herz- Type Spaces," *Journal of Function Spaces*, vol. 2021, Article ID 6615604, 7 pages, 2021.

[5] M. Kian, "On a Hardy operator inequality," *Positivity*, vol. 22, no. 3, pp. 773–781, 2018.

[6] S. Lu, "Function characterizations via commutators of Hardy operator," *Front. Math. China.*, vol. 16, no. 1, pp. 1–12, 2021.

[7] Z. Niu, S. Guo, and W. Li, "Hardy operators and the commutators on Hardy spaces," *J. Inequal. Appl.*, vol. 2020, no. 1, 2020.

[8] M. Christ and L. Grafakos, "Best constants for two nonconvolution inequalities," *Proceedings of the American Mathematical Society*, vol. 123, no. 6, pp. 1687–1693, 1995.

[9] Z. W. Fu, Z. G. Liu, S. Z. Lu, and H. B. Wang, "Characterization for commutators of n-dimensional fractional Hardy Operators," *Science in China (Scientia Sinica)*, vol. 50, no. 10, pp. 1418–1426, 2007.

[10] Z. W. Fu, S. Z. Lu, and F. Y. Zhao, "Commutators of n-dimensional rough Hardy operators," *Science China Mathematics*, vol. 54, no. 1, pp. 95–104, 2011.

[11] G. Gao, "Boundedness for commutators of n-dimensional rough Hardy operators on Morrey- Herz spaces," *Computers & Mathematcs with Applications*, vol. 64, no. 4, pp. 544–549, 2012.

[12] G. Gao and Y. Zhong, "Some estimates of Hardy operators and their commutators on Morrey-Herz spaces," *Journal of Mathematical Inequalities*, vol. 11, no. 1, pp. 49–58, 2007.

[13] V. S. Vladimirov, "Tables of integrals for complex-valued Functions of p-adic Arguments," *Proceedings of the Steklov Institute of Mathematics*, vol. 284, no. S2, pp. 1–59, 2014.

[14] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*, World Scientific, Singapore, 1994.

[15] A. Hussain and N. Sarfraz, "The Hausdorff operator on weighted p-adic Morrey and Herz type Spaces," *p-Adic Numbers, Ultrametric Analysis and Applications*, vol. 11, no. 2, article 2295, pp. 151–162, 2019.

[16] A. Hussain and N. Sarfraz, "Optimal Weak Type Estimates for p-Adic Hardy Operators," *p-Adic Numbers, Ultrametric Analysis and Applications*, vol. 12, no. 1, pp. 29–38, 2020.

[17] A. Hussain, N. Sarfraz, and F. Gürbüz, "Sharp Weak Bounds for p-adic Hardy operators on p-adic Linear Spaces," https://arxiv.org/abs/2002.08045.

[18] S. V. Kozyrev, "Methods and applications of ultrametric and p -adic analysis: From wavelet theory to biophysics," *Proceedings of the Steklov Institute of Mathematics*, vol. 274, no. S1, pp. 1–84, 2011.

[19] N. Sarfraz and M. Aslam, "Some weighted estimates for the commutators of *p*-adic Hardy operator on two weighted *p*-adic Herz-type spaces," *AIMS Mathematics*, vol. 6, no. 9, pp. 9633–9646, 2021.

[20] N. Sarfraz, D. Filali, A. Hussain, and F. Jarad, "Weighted Estimates for Commutator of Rough *p*-Adic Fractional Hardy Operator on Weighted *p*-Adic Herz–Morrey Spaces," *Journal of Mathematics*, vol. 2021, Article ID 5559815, 14 pages, 2021.

[21] N. Sarfraz and F. Gürbüz, "Weak and strong boundedness for *p*-adic fractional Hausdorff operator and its commutator," *International Journal of Nonlinear Sciences and Numerical Simulation*, 2021.

[22] N. Sarfraz and A. Hussain, "Estimates for the commutators of p-adic Hausdorff operator on Herz-Morrey spaces," *Mathematics.*, vol. 7, no. 2, p. 127, 2019.

[23] S. S. Volosivets, "Weak and strong estimates for rough Hausdorff type operator defined on p-adic linear space," *p-Adic Numbers, Ultrametric Analysis and Applications*, vol. 9, no. 3, pp. 236–241, 2017.

[24] V. S. Vladimirov and I. V. Volovich, "p-adic quantum mechanics," *Communications in Mathematical Physics*, vol. 123, no. 4, pp. 659–676, 1989.

[25] I. Y. Arefa, B. Dragovich, P. Frampton, and I. V. Volovich, "The wave function of the universe and p-adic gravity," *International Journal of Modern Physics A*, vol. 6, no. 24, pp. 4341–4358, 1991.

[26] L. Brekke and P. G. O. Frued, "*p*-adic numbers in physics," *Physics Reports*, vol. 233, no. 1, pp. 1–66, 1993.

[27] V. A. Avestisov, A. H. Bikulov, and S. V. Kozyrev, "Application of p-adic analysis to models of sponteneous breaking of relica symmetry," *Journal of Physics A: Mathematical and General*, vol. 32150, pp. 8785–8791, 1999.

[28] G. Parisi and N. Sourlas, "p-adic numbers and replica symmetry breaking," *The European Physical Journal B*, vol. 14, no. 3, pp. 535–542, 2000.

[29] Z. W. Fu, Q. Y. Wu, and S. Z. Lu, "Sharp estimates of p-adic Hardy and Hardy-Littlewood-Pólya Operators," *Acta Mathematica Sinica*, vol. 29, no. 1, pp. 137–150, 2013.

[30] R. H. Liu and J. Zhou, "Sharp estimates for the p-adic Hardy type operators on higher-dimensional product spaces," *Journal of Inequalities and Applications*, vol. 2017, no. 1, 2017.

[31] Q. Y. Wu, "Boundedness for Commutators of fractional p-adic Hardy operators," *Journal of Inequalities and Applications*, vol. 2012, no. 1, 2012.

[32] Q. Y. Wu, L. Mi, and Z. W. Fu, "Boundedness of -Adic Hardy Operators and Their Commutators on -Adic Central Morrey and BMO Spaces," *Journal of Function Spaces and Applications*, vol. 2013, article 359193, pp. 1–10, 2013.

[33] A. Hussain, N. Sarfraz, I. Khan, A. Alsuble, and N. N. Hamadnehs, "The boundedness of commutators of rough *p*-adic fractional Hardy type operators on Herz-type spaces," *Journal of Inequalities and Applications*, vol. 2021, no. 1, p. 13, 2021.

[34] M. Ali and Q. Katatbeh, "_L_ $^{-p-}$ bounds for rough parabolic maximal operators," *Heliyon*, vol. 6, no. 10, p. e05153, 2020.

[35] I. Ekincioglu, C. Keskin, and R. V. Guliyev, "Lipschitz estimates for rough fractional multilinear integral operators on local generalized Morrey spaces," *Tbilisi Mathematical Journal*, vol. 13, no. 1, pp. 47–60, 2020.

[36] G. Wang and W. Chen, "*Lp*-estimates for rough bi-parameter Fourier integral operators," *Forum Mathematicum*, vol. 32, no. 6, pp. 1441–1457, 2020.

[37] X. Zhang and F. Liu, "A note on maximal singular integrals with rough kernels," *J. Inequal. Appl.*, vol. 2020, no. 1, 2020.

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Research Article

Mean Square Integral Inequalities for Generalized Convex Stochastic Processes via Beta Function

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The integral inequalities have become a very popular area of research in recent years. The present paper deals with some important generalizations of convex stochastic processes. Several mean square integral inequalities are derived for this generalization. The involvement of the beta function in the results makes the inequalities more convenient for applied sciences.

1. Introduction

Just as the probability theory is regarded as the study of mathematical models of random phenomena, the theory of stochastic processes plays an important role in the investigation of random phenomena depending on time. A random phenomenon that arises through a process which is developing in time and controlled by some probability law is called a stochastic process. Thus, stochastic processes can be referred to as the dynamic part of the probability theory. We will now give a formal definition of a stochastic process.

Various collections of random variables $X(l,\cdot), l \in J$, have the property in some sense that $X(l)$ is stochastically convex (or $-X(l,\cdot)$ is stochastically concave). The stochastic process with convexity properties has a large number of applications. In [1], the authors demonstrated the use of a stochastically convex function in different areas of probability and statistics.

In queueing theory, the convexity of steady-state waiting time is used in [2]. More in [1], the authors used the convexity of payoff in the success rate to obtain an imperfect repair.

In 1980, Nikodem introduced the study of quadratic and convex stochastic processes (see [3, 4]). In [5, 6], Skowronski explained the properties of the Wright-convex and Jensen-convex stochastic process. Also, Kotrys described results on convex and strongly convex stochastic processes, together with a Hermite-Hadamard-type inequality for convex stochastic processes (see [7–9]).

The Hermite-Hadamard inequality for the convex stochastic process is defined as follows:

Let $X : J \times \Omega \longrightarrow \mathbb{R}$ be a convex and mean square continuous in the interval $T \times \Omega$; then, the inequality holds almost everywhere:

$$X\left(\frac{r+s}{2},\cdot\right) \leq \frac{1}{r-s}\int_r^s X(l,\cdot)dl \leq \frac{X(r,\cdot)+X(s,\cdot)}{2}, \quad (1)$$

for any $r, s \in J$. For more details on Hermite-Hadamard-type inequalities for the stochastic process, we may refer the reader to [10–12].

Definition 1 (see [13]). A stochastic process is a collection of random variables $X(l)$ parameterized by $l \in J$, where $J \subset \mathbb{R}$. When $J = \{1, 2, \cdots\}$, then $X(l)$ is said to be a stochastic process in discrete time (i.e., a sequence of random variables). When J is an interval in $\mathbb{R}(J = [0, \infty))$, then we say that $X(l)$ is a stochastic process in continuous time.

For every $\omega \in \Omega$, the function

$$J \ni l \mapsto X(l, \omega) \quad (2)$$

is said to be a path or sample path of $X(l)$.

Definition 2 (see [13]). A family F_l of α-fields on Ω parametrized by $l \in J$, where $J \subset \mathbb{R}$, is said to be a filtration if

$$F_s \subset F_l \subset F, \tag{3}$$

for any $s, l \in J$ such that $s \leq l$.

Definition 3 (see [13]). A stochastic process $X(l)$ parametrized by $l \in T$ is said to be a martingale (supermartingale, submartingale) with respect to a filtration F_l if

(1) $X(l)$ is integrable for each $l \in J$

(2) $X(l)$ is F_l-measurable for each $l \in J$

(3) $X(s) = E(X(l) \mid F_s)$ (respectively, $\leq$ or $\geq$) for every s, $l \in J$ such that $s \leq l$.

Definition 4 (see [7]). Let (Ω, A, P) be an arbitrary probability space and $J \subset \mathbb{R}$ be an interval. A stochastic process $X : \Omega \longrightarrow \mathbb{R}$ is called as follows:

(1) Stochastically continuous in interval J, if $\forall l_\circ \in J$

$$P - \lim_{l \longrightarrow l_\circ} X(l, \cdot) = X(l_\circ, \cdot), \tag{4}$$

where $P - \lim$ denotes the limit in probability.

(2) Mean square continuous in J, if $\forall l_\circ \in J$

$$P - \lim_{l \longrightarrow l_\circ} \mathbb{E}(X(l, \cdot) - X(l_\circ, \cdot)) = 0, \tag{5}$$

where $\mathbb{E}(X(l, \cdot))$ denotes the expectation value of the random variable $X(l, \cdot)$.

(3) Increasing (decreasing) if $\forall \mu, \nu \in J$ such that

$$X(\mu, \cdot) \leq X(\nu, \cdot), X(\mu, \cdot) \geq X(\nu, \cdot). \tag{6}$$

(4) Monotonic if it is increasing or decreasing

(5) If there exists a random variable $X'(l, \cdot)$: $J \times \Omega \longrightarrow \mathbb{R}$, then we say that it is differentiable at a point $l \in J$, such that

$$X'(l, \cdot) = P - \lim_{l \longrightarrow l_\circ} \frac{X(l, \cdot) - X(l_\circ, \cdot)}{l - l_\circ}. \tag{7}$$

A stochastic process $X : J \times \Omega \longrightarrow \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of interval J.

Definition 5 (see [7, 14]). Suppose that (Ω, A, P) be a probability space and $J \subset \mathbb{R}$ be an interval with $E(X(\vartheta)^2) < \infty \forall \vartheta \in J$. If $[r, s] \subset J$, $r = \vartheta_0 < \vartheta_1 < \vartheta_2 < \cdots < \vartheta_n = s$ is a partition of $[r, s]$ and $\Theta \in [\vartheta_{\kappa-1}, \vartheta_\kappa]$ for $\kappa = 1, 2, \cdots, n$. A random variable $Z : \Omega \longrightarrow \mathbb{R}$ is known as mean square integral of the process $X(\vartheta, \cdot)$ on $[r, s]$ if

$$\lim_{n \longrightarrow \infty} E\left[\sum_{\kappa=1}^{\infty} X(\Theta_\kappa, \cdot)(\vartheta_\kappa, \vartheta_{\kappa-1}) - Z(\cdot)\right]^2 = 0, \tag{8}$$

then, we have

$$\int_r^s X(\vartheta, \cdot) d\vartheta = Z(\cdot)(a.e.). \tag{9}$$

Also, the mean square integral operator is increasing; thus,

$$\int_r^s X(\vartheta, \cdot) d\vartheta \leq \int_r^s Y(\vartheta, \cdot)(a.e.), \tag{10}$$

where $X(\vartheta, \cdot) \leq Y(\vartheta, \cdot)$ in $[r, s]$.

For more details on stochastic processes, we may refer the reader to [15, 16].

Next, we write some basic definitions which will be used in this work:

Definition 6 (see [4]). Let (Ω, A, P) be a probability space and $J \subseteq R$ be an interval. A stochastic process $X : J \times \Omega \longrightarrow R$ is called a convex stochastic process; then, the inequality holds almost everywhere:

$$X(\vartheta r + (1 - \vartheta)s, \cdot) \leq \vartheta X(r, \cdot) + (1 - \vartheta)X(s, \cdot), \tag{11}$$

$\forall r, s \in J$ and $\vartheta \in [0, 1]$.

Definition 7 (see [17]). A process $X : J \times \Omega \longrightarrow \mathbb{R}$ is said to be a p-convex stochastic process, if the following inequality holds:

$$X\left(\left[\vartheta r^p + (1 - \vartheta)s^p\right]^{1/p}, \cdot\right) \leq \vartheta X(r, \cdot) + (1 - \vartheta)X(s, \cdot)(a.e.), \tag{12}$$

for all $r, s \in J$ and $\vartheta \in [0, 1]$.

In [18], Barráez et al. defined the definition of the h-convex stochastic process as follows:

Definition 8 (see [18]). Let $h : (0, 1) \longrightarrow \mathbb{R}$ be a nonnegative function, $h \neq 0$. A stochastic process $X : J \times \Omega \longrightarrow \mathbb{R}$ is a h-convex stochastic process, if the inequality holds:

$$X(\vartheta r + (1 - \vartheta)s, \cdot) \leq h(\vartheta)X(r, \cdot) + h(1 - \vartheta)X(s, \cdot)(a.e.), \tag{13}$$

for every $r, s \in J$ and $\vartheta \in [0, 1]$.

Obviously, by taking $h(\vartheta) = \vartheta$ in (13), then the definition of the h-convex stochastic process reduces to the definition of the convex stochastic process [4].

Definition 9 (see [9]). Let $c : \Omega \longrightarrow \mathbb{R}$ be a positive random variable. A stochastic process $X : J \times \Omega \longrightarrow \mathbb{R}$ is known as strongly convex with modulus $c(\cdot) > 0$, if the following inequality holds:

$$X(\vartheta r + (1-\vartheta)s, \cdot) \le \vartheta X(r, \cdot) + (1-\vartheta)X(s, \cdot) - c(\cdot)\vartheta(1-\vartheta)(r-s)^2 (a.e.), \tag{14}$$

for all $r, s \in J$ and $\vartheta \in [0, 1]$.

For more details on the strongly convex stochastic process, we refer to [9], and for some interesting properties of some special function, see [19, 20]. Obviously, if we omit the term $c(\cdot)\vartheta(1-\vartheta)(r-s)^2$ in (14), then we get the definition of a convex stochastic process (see [4]). On the other hand, if we set $c = 0$, then we get it from (14) in limit case. Also, we use the beta function in this present work which is expressed as

$$\beta(r, s) = \int_0^1 \vartheta^{r-1}(1-\vartheta)^{s-1} d\vartheta, \operatorname{Re}(r) > 0, \operatorname{Re}(s) > 0. \tag{15}$$

2. Main Results

Lemma 10 (see [21]). *Suppose that $X : J \times \Omega \longrightarrow \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Then, the following equality holds almost everywhere:*

$$\int_r^s (\omega - r)^\mu (s-\omega)^\nu X(\omega, \cdot) d\omega = (s-r)^{\mu+\nu+1} \int_0^1 (1-\vartheta)^\mu \vartheta^\nu X(\vartheta r + (1-\vartheta)s, \cdot) d\vartheta, \tag{16}$$

for some fixed $\mu, \nu > 0$.

Lemma 11. *Suppose that $X : J \times \Omega \longrightarrow \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Then, the following equality holds almost everywhere:*

$$\int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega = \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \int_0^1 (1-\vartheta)^\mu \vartheta^\nu X\left(\left[\vartheta r^p + (1-\vartheta)s^p\right]^{1/p}, \cdot\right) d\vartheta, \tag{17}$$

for some fixed $\mu, \nu > 0$.

Proof. Let $\omega = [\vartheta r^p + (1-\vartheta)s^p]^{1/p}$. Then, $\vartheta = (s^p - \omega^p)/(s^p - r^p)$, $1-\vartheta = (\omega^p - r^p)/(s^p - r^p)$, and $d\vartheta = -p/(s^p - r^p)\omega^{1-p} d\omega$, so

$$\int_0^1 (1-\vartheta)^\mu \vartheta^\nu X\left(\left[\vartheta r^p + (1-\vartheta)s^p\right]^{1/p}, \cdot\right) d\vartheta = \frac{p}{(s^p - r^p)^{\mu+\nu+1}} \int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega (a.e.), \tag{18}$$

which completes the proof. □

Remark 12. If we take $p = 1$ in Lemma 11, then we obtain Lemma 3.1 of [21].

The following results are derived for p-convex stochastic processes.

Theorem 13. *Suppose that $X : J \times \Omega \longrightarrow \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|X|$ is p-convex on $[r, s]$, where $r, s \in J$ with $r < s$, and $\mu, \nu > 0$ is taken, then the inequality holds almost everywhere:*

$$\int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega \le \frac{(s^p - r^p)^{\mu+\nu+1}}{p} (\beta(\mu+1, \nu+2)|X(r, \cdot)| + \beta(\mu+2, \nu+1)|X(s, \cdot)|). \tag{19}$$

Proof. By using Lemma 11, the definition of the p-convexity of $|X|$ and the beta function yield that

$$\begin{aligned}
&\int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega \\
&\le \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \int_0^1 (1-\vartheta)^\mu \vartheta^\nu \left| X\left(\left[\vartheta r^p + (1-\vartheta)s^p\right]^{1/p}, \cdot\right) \right| d\vartheta \\
&\le \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \int_0^1 (1-\vartheta)^\mu \vartheta^\nu (\vartheta |X(r, \cdot)| + (1-\vartheta)|X(s, \cdot)|) d\vartheta \\
&\le \frac{(s^p - r^p)^{\mu+\nu+1}}{p} (\beta(\mu+1, \nu+2)|X(r, \cdot)| + \beta(\mu+2, \nu+1)|X(s, \cdot)|)(a.e.),
\end{aligned} \tag{20}$$

which completes the proof. □

Remark 14. If we take $p = 1$ in Theorem 13, then we obtain Theorem 3.1 of [21].

Theorem 15. *Suppose that $X : J \times \Omega \longrightarrow \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|X|^q$ is p-convex on $[r, s]$ for $q > 1$ with $1/\kappa + 1/q = 1$, where $r, s \in J$, $r < s$, and $\mu, \nu > 0$ is taken, then the inequality holds almost everywhere:*

$$\int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega \le \frac{(s^p - r^p)^{\mu+\nu+1}}{p} (\beta(\kappa\mu+1, \kappa\nu+1))^{1/\kappa} \left(\frac{|X(r, \cdot)|^q + |X(s, \cdot)|^q}{2} \right)^{1/q}. \tag{21}$$

Proof. Employing Lemma 11 and Hölder's integral inequality, we have $(a.e.)$

$$\begin{aligned}&\int_{r^p}^{s^p}(\omega^p-r^p)^{\mu}(s^p-\omega^p)^{\nu}\frac{X(\omega,\cdot)}{\omega^{1-p}}d\omega\\&\le\frac{(s^p-r^p)^{\mu+\nu+1}}{p}\int_0^1(1-\vartheta)^{\mu}\vartheta^{\nu}\left|X\left([\vartheta r^p+(1-\vartheta)s^p]^{1/p},\cdot\right)\right|d\vartheta\\&\le\frac{(s^p-r^p)^{\mu+\nu+1}}{p}\left(\int_0^1(1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta\right)^{1/\kappa}\\&\quad\times\left(\int_0^1\left|X\left([\vartheta r^p+(1-\vartheta)s^p]^{1/p},\cdot\right)\right|^q d\vartheta\right)^{1/q}.\end{aligned}\tag{22}$$

Since $|X|^q$ is a p-convex stochastic process, one can yield that

$$\begin{aligned}&\int_0^1\left|X\left([\vartheta r^p+(1-\vartheta)s^p]^{1/p},\cdot\right)\right|^q d\vartheta\\&\le\int_0^1(\vartheta|X(r,\cdot)|^q+(1-\vartheta)|X(s,\cdot)|^q)d\vartheta\\&=\frac{|X(r,\cdot)|^q+|X(s,\cdot)|^q}{2}(a.e.),\end{aligned}\tag{23}$$

and by the definition of the beta function, we can write

$$\int_0^1(1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta=\beta(\kappa\mu+1,\kappa\nu+1).\tag{24}$$

Inserting (23) and (24) in (22) yields the required inequality (21). ☐

Remark 16. If we take $p=1$ in Theorem 15, then we get Theorem 3.2 of [21].

Theorem 17. *Let $X: J\times\Omega\longrightarrow\mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|X|^q$ is p-convex on $[r,s]$ for $q>1$, where $r,s\in J$ with $r<s$, and $\mu,\nu>0$ is taken, then the inequality holds almost everywhere:*

$$\begin{aligned}&\int_{r^p}^{s^p}(\omega^p-r^p)^{\mu}(s^p-\omega^p)^{\nu}\frac{X(\omega,\cdot)}{\omega^{1-p}}d\omega\\&\le\frac{(s^p-r^p)^{\mu+\nu+1}}{p}(\beta(\kappa\mu+1,\kappa\nu+1))^{1-(1/q)}\\&\quad\times(\beta(\mu+1,\nu+2)|X(r,\cdot)|^q+\beta(\mu+2,\nu+1)|X(s,\cdot)|^q)^{1/q}.\end{aligned}\tag{25}$$

Proof. Making use of Lemma 11 and the power-mean integral inequality for $\kappa\ge1$ yields that

$$\begin{aligned}&\int_{r^p}^{s^p}(\omega^p-r^p)^{\mu}(s^p-\omega^p)^{\nu}\frac{X(\omega,\cdot)}{\omega^{1-p}}d\omega\\&\le\frac{(s^p-r^p)^{\mu+\nu+1}}{p}\int_0^1(1-\vartheta)^{\mu}\vartheta^{\nu}\left|X\left([\vartheta r^p+(1-\vartheta)s^p]^{1/p},\cdot\right)\right|d\vartheta(a.e.)\\&\le\frac{(s^p-r^p)^{\mu+\nu+1}}{p}\left(\int_0^1(1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta\right)^{1-(1/q)}\\&\quad\times\left(\int_0^1(1-\vartheta)^{\mu}\vartheta^{\nu}\left|X\left([\vartheta r^p+(1-\vartheta)s^p]^{1/p},\cdot\right)\right|^q d\vartheta\right)^{1/q}(a.e.).\end{aligned}\tag{26}$$

By using the p-convexity of the stochastic process $|X|^q$ and by the definition of the beta function, we have $(a.e.)$

$$\begin{aligned}&\le\frac{(s^p-r^p)^{\mu+\nu+1}}{p}\left(\int_0^1(1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta\right)^{1-(1/q)}\\&\quad\times\left(\int_0^1(1-\vartheta)^{\mu}\vartheta^{\nu}(\vartheta|X(r,\cdot)|^q+(1-\vartheta)|X(s,\cdot)|^q)d\vartheta\right)^{1/q}\\&\le\frac{(s^p-r^p)^{\mu+\nu+1}}{p}(\beta(\kappa\mu+1,\kappa\nu+1))^{1-(1/q)}\\&\quad\times(\beta(\mu+1,\nu+2)|X(r,\cdot)|^q+\beta(\mu+2,\nu+1)|X(s,\cdot)|^q)^{1/q}.\end{aligned}\tag{27}$$

which completes the proof. ☐

Remark 18. If we take $p=1$ in Theorem 17, then we obtain Theorem 3.3 of [21].

The following results are derived for h-convex stochastic processes.

Theorem 19. *Suppose that $X: J\times\Omega\longrightarrow\mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|X|$ is h-convex on $[r,s]$, where $r,s\in J$ with $r<s$, and $\mu,\nu>0$ is taken, then the inequality holds almost everywhere:*

$$\begin{aligned}&\int_r^s(\omega-r)^{\mu}(s-\omega)^{\nu}X(\omega,\cdot)d\omega\\&\le(s-r)^{\mu+\nu+1}(|X(r,\cdot)|\beta_h(\vartheta)+|X(s,\cdot)|\beta_h(1-\vartheta)),\end{aligned}\tag{28}$$

where

$$\beta_h(\vartheta)^{\mu,\nu}=\int_0^1(1-\vartheta)^{\mu}\vartheta^{\nu}h(\vartheta)d\vartheta,\tag{29}$$

$$\beta_h(1-\vartheta)^{\mu,\nu}=\int_0^1(1-\vartheta)^{\mu}\vartheta^{\nu}h)(1-\vartheta)d\vartheta.\tag{30}$$

Proof. By Lemma 10, the definition of the h-convexity of $|X|$ and the beta function yield that

$$\begin{aligned}&\int_r^s(\omega-r)^\mu(s-\omega)^\nu X(\omega,\cdot)d\omega\\&\le(s-r)^{\mu+\nu+1}\int_0^1(1-\vartheta)^\mu\vartheta^\nu|X(\vartheta r+(1-\vartheta)s,\cdot)|d\vartheta\\&\le(s-r)^{\mu+\nu+1}\int_0^1(1-\vartheta)^\mu\vartheta^\nu(h(\vartheta)|X(r,\cdot)|+h(1-\vartheta)|X(s,\cdot)|)d\vartheta\\&\le(s-r)^{\mu+\nu+1}\left(|X(r,\cdot)|\int_0^1(1-\vartheta)^\mu\vartheta^\nu h(\vartheta)d\vartheta+|X(s,\cdot)|\int_0^1(1-\vartheta)^\mu\vartheta^\nu h(1-\vartheta)\right)\\&=(s-r)^{\mu+\nu+1}(|X(r,\cdot)|\beta_h(\vartheta)^{\mu,\nu}+|X(s,\cdot)|\beta_h(1-\vartheta)^{\mu,\nu})(a.e.),\end{aligned}\tag{31}$$

which completes the proof. □

Remark 20. If we take $h(\vartheta)=\vartheta$ in Theorem 19, then we obtain Theorem 3.1 of [21].

Theorem 21. *Suppose that $X:J\times\Omega\longrightarrow\mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|X|^q$ is h-convex on $[r,s]$ for $q>1$ with $1/\kappa+1/q=1$, where $r,s\in J$, $r<s$, and $\mu,\nu>0$ is taken, then the inequality holds almost everywhere:*

$$\begin{aligned}&\int_r^s(\omega-r)^\mu(s-\omega)^\nu X(\omega,\cdot)d\omega\\&\le(s-r)^{\mu+\nu+1}(\beta(\kappa\mu+1,\kappa\nu+1))^{1/\kappa}(|X(r,\cdot)|^q\beta_h(\vartheta)+|X(s,\cdot)|^q\beta_h(1-\vartheta))^{1/q},\end{aligned}\tag{32}$$

where $\beta_h(\vartheta)=\int_0^1h(\vartheta)d\vartheta$ and $\beta_h(1-\vartheta)=\int_0^1h(1-\vartheta)d\vartheta$.

Proof. Employing Lemma 10 and Hölder's integral inequality, we have $(a.e.)$

$$\begin{aligned}&\int_r^s(\omega-r)^\mu(s-\omega)^\nu X(\omega,\cdot)d\omega\\&\le(s-r)^{\mu+\nu+1}\int_0^1(1-\vartheta)^\mu\vartheta^\nu|X(\vartheta r+(1-\vartheta)s,\cdot)|d\vartheta\\&\le(s-r)^{\mu+\nu+1}\left(\int_0^1(1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta\right)^{1/\kappa}\left(\int_0^1|X(\vartheta r+(1-\vartheta)s,\cdot)|^qd\vartheta\right)^{1/q}.\end{aligned}\tag{33}$$

Since $|X|^q$ is an h-convex stochastic process, one can yield that

$$\begin{aligned}&\int_0^1|X(\vartheta r+(1-\vartheta)s,\cdot)|^qd\vartheta\\&\le\int_0^1(h(\vartheta)|X(r,\cdot)|^q+h(1-\vartheta)|X(s,\cdot)|^q)d\vartheta\\&\le|X(r,\cdot)|^q\int_0^1h(\vartheta)d\vartheta+|X(s,\cdot)|^q\int_0^1h(1-\vartheta)d\vartheta\\&\le|X(r,\cdot)|^q\beta_h(\vartheta)+|X(s,\cdot)|^q\beta_h(1-\vartheta)(a.e.),\end{aligned}\tag{34}$$

and by the definition of the beta function, we can write

$$\int_0^1(1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta=\beta(\kappa\mu+1,\kappa\nu+1).\tag{35}$$

Inserting (34) and (35) in (33) yields the desired inequality (32). □

Remark 22. If we take $h(\vartheta)=\vartheta$ in Theorem 21, then we obtain Theorem 3.2 of [21].

Theorem 23. *Let $X:J\times\Omega\longrightarrow\mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|X|^q$ is h-convex on $[r,s]$ for $q>1$, where $r,s\in J$ with $r<s$, and $\mu,\nu>0$ is taken, then the inequality holds almost everywhere:*

$$\begin{aligned}&\int_r^s(\omega-r)^\mu(s-\omega)^\nu X(\omega,\cdot)d\omega\\&\le(s-r)^{\mu+\nu+1}(\beta(\kappa\mu+1,\kappa\nu+1))^{1-(1/q)}\\&\cdot(\beta_h(\vartheta)^{\mu,\nu}|X(r,\cdot)|^q+\beta_h(\theta)^{\mu,\nu}|X(s,\cdot)|^q)^{1/q},\end{aligned}\tag{36}$$

where

$$\beta_h(\vartheta)^{\mu,\nu}=\int_0^1(1-\vartheta)^\mu\vartheta^\nu h(\vartheta)d\vartheta,\tag{37}$$

$$\beta_h(1-\vartheta)^{\mu,\nu}=\int_0^1(1-\vartheta)^\mu\vartheta^\nu h(1-\vartheta)d\vartheta.\tag{38}$$

Proof. By Lemma 10 and the power-mean integral inequality for $\kappa\ge1$, one can yield that

$$\begin{aligned}&\int_r^s(\omega-r)^\mu(s-\omega)^\nu X(\omega,\cdot)d\omega\\&\le(s-r)^{\mu+\nu+1}\int_0^1(1-\vartheta)^\mu\vartheta^\nu|X(\vartheta r+(1-\vartheta)s,\cdot)|d\vartheta(a.e.)\\&\le(s-r)^{\mu+\nu+1}\left(\int_0^1(1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta\right)^{1-(1/q)}\\&\times\left(\int_0^1(1-\vartheta)^\mu\vartheta^\nu|X(\vartheta r+(1-\vartheta)s,\cdot)|^qd\vartheta\right)^{1/q}(a.e.).\end{aligned}\tag{39}$$

By using the h-convexity of the stochastic process $|X|^q$ and by the definition of the beta function, we have $(a.e.)$

$$\begin{aligned}&\le(s-r)^{\mu+\nu+1}\left(\int_0^1(1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta\right)^{1-1/q}\\&\times\left(\int_0^1(1-\vartheta)^\mu\vartheta^\nu(h()\vartheta)|X(r,\cdot)|^q+h(1-\vartheta)|X(s,\cdot)|^q)d\vartheta\right)^{1/q}\\&\le(s-r)^{\mu+\nu+1}\left(\int_0^1(1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta\right)^{1-1/q}\\&\times\left(|X(r,\cdot)|^q\int_0^1(1-\vartheta)^\mu\vartheta^\nu h(\vartheta)+|X(s,\cdot)|^q\int_0^1h(1-\vartheta)d\vartheta\right)^{1/q}\\&\le(s-r)^{\mu+\nu+1}(\beta(\kappa\mu+1,\kappa\nu+1))^{1-(1/q)}(\beta_h(\vartheta)^{\mu,\nu}|X(r,\cdot)|^q\\&+\beta_h(\theta)^{\mu,\nu}|X(s,\cdot)|^q)^{1/q},\end{aligned}\tag{40}$$

which completes the proof. □

Remark 24. If we take $h(\vartheta) = \vartheta$ in Theorem 23, then we obtain Theorem 3.3 of [21].

The following results are derived for strongly convex stochastic processes.

Theorem 25. *Suppose that $X : J \times \Omega \longrightarrow \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|X|$ is strongly convex on $[r, s]$, where $r, s \in J$ with $r < s$, and $\mu, \nu > 0$ is taken, then the inequality holds almost everywhere:*

$$\begin{aligned}&\int_r^s (\omega - r)^\mu (s-\omega)^\nu X(\omega, \cdot) d\omega \\ &\quad \le (s-r)^{\mu+\nu+1} \big(\beta(\mu+1, \nu+2)|X(r,\cdot)| + \beta(\mu+2, \nu+1) \\ &\qquad \cdot |X(s,\cdot)| - c(\cdot)(r-s)^2 \beta(\mu+2,\nu+2)\big).\end{aligned} \tag{41}$$

Proof. From Lemma 10, the definition of the strong convexity of $|X|$ and the beta function yield that

$$\begin{aligned}&\int_r^s (\omega - r)^\mu (s-\omega)^\nu X(\omega, \cdot) d\omega \\ &\quad \le (s-r)^{\mu+\nu+1} \int_0^1 (1-\vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1-\vartheta)s, \cdot)| d\vartheta \\ &\quad \le (s-r)^{\mu+\nu+1} \int_0^1 (1-\vartheta)^\mu \vartheta^\nu \big(\vartheta |X(r,\cdot)| \\ &\qquad + 1 - \vartheta |X(s,\cdot)| - c(\cdot)\vartheta(1-\vartheta)(r-s)^2\big) d\vartheta \\ &\quad \le (s-r)^{\mu+\nu+1} \big(\beta(\mu+1,\nu+2)|X(r,\cdot)| \\ &\qquad + \beta(\mu+2,\nu+1)|X(s,\cdot)| - c(\cdot)(r-s)^2 \\ &\qquad \cdot \beta(\mu+2,\nu+2)\big)(a.e.),\end{aligned} \tag{42}$$

which completes the proof. ☐

Remark 26. If we take $c = 0$ in Theorem 25, then we obtain Theorem 3.1 of [21].

Theorem 27. *Suppose that $X : J \times \Omega \longrightarrow \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|X|^q$ is strongly convex on $[r, s]$ for $q > 1$ with $1/\kappa + 1/q = 1$, where $r, s \in J$ with $r < s$, and $\mu, \nu > 0$ is taken, then the inequality holds almost everywhere:*

$$\begin{aligned}&\int_r^s (\omega - r)^\mu (s-\omega)^\nu X(\omega, \cdot) d\omega \\ &\quad \le (s-r)^{\mu+\nu+1} \big(\beta(\kappa\mu+1, \kappa\nu+1)\big)^{1/\kappa} \\ &\qquad \times \left(\frac{1}{2}\left(|X(r,\cdot)|^q + |X(s,\cdot)|^q\right) - \frac{1}{6} c(\cdot)(r-s)^2\right)^{1/q}.\end{aligned} \tag{43}$$

Proof. By Lemma 10 and Hölder's integral inequality, we have $(a.e.)$

$$\begin{aligned}&\int_r^s (\omega - r)^\mu (s-\omega)^\nu X(\omega, \cdot) d\omega \\ &\quad \le (s-r)^{\mu+\nu+1} \int_0^1 (1-\vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1-\vartheta)s, \cdot)| d\vartheta \\ &\quad \le (s-r)^{\mu+\nu+1} \left(\int_0^1 (1-\vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta\right)^{1/\kappa} \\ &\qquad \cdot \left(\int_0^1 |X(\vartheta r + (1-\vartheta)s, \cdot)|^q d\vartheta\right)^{1/q}.\end{aligned} \tag{44}$$

Since $|X|^q$ is a strongly convex stochastic process, one can yield that

$$\begin{aligned}&\int_0^1 |X(\vartheta r + (1-\vartheta)s, \cdot)|^q d\vartheta \\ &\quad \le \int_0^1 \left(\vartheta |X(r,\cdot)|^q + 1 - \vartheta |X(s,\cdot)|^q - c(\cdot)\vartheta(1-\vartheta)(r-s)^2\right) d\vartheta \\ &\quad \le \frac{1}{2}\left(|X(r,\cdot)|^q + |X(s,\cdot)|^q\right) - \frac{1}{6}\left(c(\cdot)(r-s)^2\right)(a.e.),\end{aligned} \tag{45}$$

and taking the definition of the beta function, we can write

$$\int_0^1 (1-\vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta = \beta(\kappa\mu+1, \kappa\nu+1). \tag{46}$$

Replacing (45) and (46) in (44) yields the desired inequality (43). ☐

Remark 28. If we take $c = 0$ in Theorem 27, then we get Theorem 3.2 of [21].

Theorem 29. *Let $X : J \times \Omega \longrightarrow \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. If $|X|^q$ is strongly convex on $[r, s]$ for $q > 1$, where $r, s \in J$ with $r < s$, and $\mu, \nu > 0$ is taken, then the inequality holds almost everywhere:*

$$\begin{aligned}&\int_r^s (\omega - r)^\mu (s-\omega)^\nu X(\omega, \cdot) d\omega \\ &\quad \le (s-r)^{\mu+\nu+1} \beta(\kappa\mu+1, \kappa\nu+1)^{1-(1/q)} \\ &\qquad \times \big(\beta(\mu+1,\nu+2)|X(r,\cdot)|^q + \beta(\mu+2,\nu+1) \\ &\qquad \cdot |X(s,\cdot)|^q - c(\cdot)(r-s)^2 \beta(\mu+2,\nu+2)\big)^{1/q}.\end{aligned} \tag{47}$$

Proof. By making use of Lemma 11 and the power-mean integral inequality for $\kappa \ge 1$, one can yield that

$$\begin{aligned}
&\int_r^s (\omega - r)^\mu (s-\omega)^\nu X(\omega,\cdot)d\omega \\
&\quad \le (s-r)^{\mu+\nu+1}\int_0^1 (1-\vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1-\vartheta)s,\cdot)|d\vartheta (a.e.) \\
&\quad \le (s-r)^{\mu+\nu+1}\left(\int_0^1 (1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta\right)^{1-(1/q)} \\
&\qquad \times \left(\int_0^1 (1-\vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1-\vartheta)s,\cdot)|^q d\vartheta\right)^{1/q} (a.e.).
\end{aligned} \tag{48}$$

By using the strong convexity of the stochastic process $|X|^q$ and taking the definition of the beta function, we have $(a.e.)$

$$\begin{aligned}
&\le (s-r)^{\mu+\nu+1}\left(\int_0^1 (1-\vartheta)^{\kappa\mu}\vartheta^{\kappa\nu}d\vartheta\right)^{1-(1/q)} \\
&\quad \times \left(\int_0^1 (1-\vartheta)^\mu \vartheta^\nu (\vartheta |X(r,\cdot)|^q + (1-\vartheta)\right. \\
&\quad \left. \cdot |X(s,\cdot)|^q - c(\cdot)\vartheta(1-\vartheta)(r-s)^2)d\vartheta\right)^{1/q} \\
&\le (s-r)^{\mu+\nu+1}(\beta(\kappa\mu+1,\kappa\nu+1))^{1-(1/q)} \\
&\quad \times (\beta(\mu+1,\nu+2)|X(r,\cdot)|^q + \beta(\mu+2,\nu+1) \\
&\quad \cdot |X(s,\cdot)|^q - c(\cdot)(r-s)^2\beta(\mu+2,\nu+2))^{1/q},
\end{aligned} \tag{49}$$

which completes the proof. □

Remark 30. If we take $c = 0$ in Theorem 29, then we obtain Theorem 3.1 of [21].

3. Conclusions

Stochastic processes have applications in many disciplines such as biology, chemistry, ecology, neuroscience, physics, image processing, signal processing, control theory, information theory, computer science, cryptography, and telecommunications. In this paper, we studied the generalized convex stochastic processes via a special function "beta function." We established mean square integral inequalities for these generalized convex stochastic processes.

4. Future Directions

It will be interesting for researchers to work on the generalized convex stochastic processes via different fractional internal operators.

Data Availability

All data required for this research are included within this paper.

Conflicts of Interest

The authors declare that they do not have any competing interests.

Authors' Contributions

All authors contributed equally in this paper.

References

[1] M. Shaked and J. Shanthikumar, "Stochastic convexity and its applications," *Advances in Applied Probability*, vol. 20, no. 2, pp. 427–446, 1988.

[2] C. Chesneau, "Study of a unit power-logarithmic distribution," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 218–235, 2021.

[3] K. Nikodem, "On quadratic stochastic processes," *Aequationes Mathematicae*, vol. 21, no. 1, pp. 192–199, 1980.

[4] K. Nikodem, "On convex stochastic processes," *Aequationes Mathematicae*, vol. 20, no. 1, pp. 184–197, 1980.

[5] A. Skowronski, "On some properties of J-convex stochastic processes," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 249–258, 1992.

[6] A. Skowronski, "On wright-convex stochastic processes," *Annales Mathematicae Silesianae*, vol. 9, pp. 29–32, 1995.

[7] D. Kotrys, "Hermite–Hadamard inequality for convex stochastic processes," *Aequationes Mathematicae*, vol. 83, pp. 143–151, 2012.

[8] D. Kotrys, "Remarks on strongly convex stochastic processes," *Aequationes Mathematicae*, vol. 86, no. 1-2, pp. 91–98, 2013.

[9] D. Kotrys, "Remarks on Jensen, Hermite-Hadamard and Fejer inequalities for strongly convex stochastic processes," *Mathematica Aeterna*, vol. 5, pp. 95–104, 2015.

[10] M. E. Omaba and L. O. Omenyi, "Generalized fractional Hadamard type inequalities for (Q_s)-class functions of the second kind," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 270–278, 2021.

[11] P. Fan, "New-type Hoeffding's inequalities and application in tail bounds," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 248–261, 2021.

[12] M. Tariq and S. I. Butt, "Some Ostrowski type integral inequalities via generalized harmonic convex functions," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 200–208, 2021.

[13] Z. Brzezniak and T. Zastawniak, *Basic Stochastic Processes: a Course through Exercises*, Springer Science and Business Media, 2000.

[14] K. Sobczyk, *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer, Dordrecht, 1991.

[15] A. Bain and D. Crisan, *Fundamentals of Stochastic Filtering*, Springer-Verlag, New York, NY, USA, 2009.

[16] P. Devolder, J. Janssen, and R. Manca, *Basic Stochastic Processes. Mathematics and Statistics Series*, ISTE, London, John Wiley and Sons, Inc, 2015.

[17] N. Okur, I. Iscan, and E. Yuksek Dizdar, "Hermite-Hadamard type inequalities for p-convex stochastic processes," *An International Journal of Optimization and Control: Theories & Applications*, vol. 9, no. 2, pp. 148–153, 2019.

[18] D. Barráez, L. González, N. Merentes, and A. M. Moros, "On h-convex stochastic process," *Mathematica Aeterna*, vol. 5, pp. 571–581, 2015.

[19] D. Ritelli and G. Spaletta, "Trinomial equation: the Hypergeometric way," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 236–247, 2021.

[20] S. Foschi and D. Ritelli, "The Lambert function, the quintic equation and the proactive discovery of the implicit function theorem," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 94–114, 2021.

[21] P. Fan, "New-type Hoeffding's inequalities and application in tail bounds," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 248–261, 2021.

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Research Article

Certain Subclass of *m*-Valent Functions Associated with a New Extended Ruscheweyh Operator Related to Conic Domains

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The main object of the present paper is to introduce certain subclass of m-valent functions associated with a new extended Ruscheweyh linear operator in the open unit disk. Also, we investigate a number of geometric properties including coefficient estimates and the Fekete–Szegö type inequalities for this subclass. Several known consequences of the main results are also pointed out.

1. Introduction

Let $\mathscr{A}(m)$ denote the class of functions of the next form:

$$f(\xi) = \xi^m + \sum_{n=m+1}^{\infty} a_n \xi^n \left(m \in \mathbb{N} = \{1, 2, 3, \cdots\}\right), \quad (1)$$

which are analytic and m-valent in the open unit disc $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$, and let $\mathscr{A}(1) = \mathscr{A}$. Also, let f, g be analytic in $\mathbb{D}$, and the function $f(\xi)$ is said to be subordinate to $g(\xi)$ if there exists a function $\omega(\xi)$ analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(\xi)| < 1, \xi \in \mathbb{D}$, such that $f(\xi) = g(\omega(\xi))$. In such a case, we write $f(\xi) \prec g(\xi)$. If g is univalent function, then $f(\xi) \prec g(\xi)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ (see [1, 2] and [3]).

For functions $f(\xi)$ given by (1) and $g(\xi)$ is defined by

$$g(\xi) = \xi^m + \sum_{n=m+1}^{\infty} b_n \xi^n, \quad (2)$$

and the Hadamard product or convolution of $f(\xi)$ and $g(\xi)$ is defined by

$$(f * g)(\xi) = \xi^m + \sum_{n=m+1}^{\infty} a_n b_n \xi^n. \quad (3)$$

For $\nu \in \mathbb{C}$, $k \in \mathbb{R}$, and $n \in \mathbb{N}$, the Pochhammer k-symbol $(\nu)_{n,k}$ is given by (see [4])

$$(\nu)_{n,k} = \nu(\nu + k)(\nu + 2k) \cdots (\nu + (n-1)k) = \prod_{i=1}^{n} (\nu + (i-1)k). \quad (4)$$

We define the function $\phi_m(\delta, k; \xi)$ by

$$\phi_m(\delta,k;\xi) = \frac{\xi^m}{(1-\xi)^{\delta+mk/k}} = \xi^m + \sum_{n=m+1}^{\infty} \frac{(\delta+mk)_{n-m,k}}{(k)_{n-m,k}} \xi^n \ (\delta>-mk\,;k>0\,;\xi\in\mathbb{D}). \tag{5}$$

Corresponding to the function $\phi_m(\delta, k; \xi)$, we consider a linear operator $\mathscr{D}^{\delta+mk-k} : \mathscr{A}(m) \longrightarrow \mathscr{A}(m)(\delta>-mk, k>0)$ which is defined by means of the following Hadamard product (or convolution):

$$\mathscr{D}^{\delta+mk-k} f(\xi) = \phi_m(\delta,k;\xi) * f(\xi) = \xi^m + \sum_{n=m+1}^{\infty} \frac{(\delta+mk)_{n-m,k}}{(k)_{n-m,k}} a_n \xi^n \ (\xi\in\mathbb{D}). \tag{6}$$

It is easily verified from (6) that

$$k\xi\left(\mathscr{D}^{\delta+mk-k} f(\xi)\right)' = (\delta+mk)\mathscr{D}^{\delta+mk} f(\xi) - \delta\mathscr{D}^{\delta+mk-k} f(\xi) \ (k>0). \tag{7}$$

We note that

(1) For $k=1$, the operator $\mathscr{D}^{\delta+mk-k} f(\xi)$ reduced to the differential operator $\mathscr{D}^{\delta+m-1} f(\xi)$ introduced by Goel and Sohi [5] (see also [6, 7] and [8])

(2) For $m=1$, we obtain the k-Ruscheweyh derivative operator $\mathscr{D}_k^\delta$ ([9]), where

$$\mathscr{D}_k^\delta f(\xi) = \frac{\xi}{(1-\xi)^{\delta+k/k}} * f(\xi) = \xi + \sum_{n=2}^{\infty} \frac{(\delta+k)_{n-1,k}}{(k)_{n-1,k}} a_n \xi^n. \tag{8}$$

(3) For $k=m=1$, the operator $\mathscr{D}^{\delta+mk-k} f(\xi)$ reduced tothe well-familiar Ruscheweyh operator $\mathscr{D}^\delta$ ([10])

(4) For $\delta = k - mk$,we have $\mathscr{D}^0 f(\xi) = f(\xi)$, and for $\delta = 2k - mk$, we get $\mathscr{D}^k f(\xi) = \xi^m(\xi^{1-m} f(\xi))'$

By using the linear operator $\mathscr{D}^{\delta+mk-k} f(\xi)$, we define the subclass $\beta - \mathscr{ST}_m(\delta, k, b)$ of $\mathscr{A}(m)$ as follows:

Definition 1. Let $\beta \ge 0$, $\delta > -mk$, $m \in \mathbb{N}$, $k>0$, $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\xi \in \mathbb{D}$. A function $f \in \mathscr{A}(m)$ is in the class $\beta - \mathscr{ST}_m(\delta, k, b)$, if it satisfies

$$\Re\left\{1 + \frac{1}{b}\left(\frac{\xi\left(\mathscr{D}^{\delta+mk-k} f(\xi)\right)'}{m\mathscr{D}^{\delta+mk-k} f(\xi)} - 1\right)\right\} > \beta\left|\frac{1}{b}\left(\frac{\xi\left(\mathscr{D}^{\delta+mk-k} f(\xi)\right)'}{m\mathscr{D}^{\delta+mk-k} f(\xi)} - 1\right)\right|. \tag{9}$$

Geometrically, a function $f \in \beta - \mathscr{ST}_m(\delta, k, b)$ if and only if

$$1 + \frac{1}{b}\left(\frac{\xi\left(\mathscr{D}^{\delta+mk-k} f(\xi)\right)'}{m\mathscr{D}^{\delta+mk-k} f(\xi)} - 1\right), \tag{10}$$

takes all the values in the conic domain $\Omega_\beta = \psi_\beta(\mathbb{D})$, where

$$\Omega_\beta = \left\{u + iv : u > \beta\sqrt{(u-1)^2 + v^2}\right\}, \tag{11}$$

or equivalently,

$$1 + \frac{1}{b}\left(\frac{\xi\left(\mathscr{D}^{\delta+mk-k} f(\xi)\right)'}{m\mathscr{D}^{\delta+mk-k} f(\xi)} - 1\right) \prec \psi_\beta(\xi), \Omega_\beta = \psi_\beta(\mathbb{D}). \tag{12}$$

The boundary $\partial\Omega_\beta$ of the above set becomes the imaginary axis when $\beta = 0$, a hyperbola when $0 < \beta < 1$, a parabola when $\beta = 1$, and an ellipse when $1 < \beta < \infty$. The functions $\psi_\beta(\xi)$ are defined by

$$\psi_\beta(\xi) = \begin{cases} \dfrac{1+\xi}{1-\xi} & (\beta = 0), \\ 1 + \dfrac{1}{1-\beta^2}\cos\left(\dfrac{2}{\pi}(\cos^{-1}\beta) i \log\left(\dfrac{1+\sqrt{\xi}}{1-\sqrt{\xi}}\right)\right) & (0<\beta<1), \\ 1 + \dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{\xi}}{1-\sqrt{\xi}}\right)^2 & (\beta = 1), \\ 1 + \dfrac{1}{\beta^2-1}\sin\left(\dfrac{\pi}{2R(t)}\displaystyle\int_0^{u(\xi)/\sqrt{t}} \dfrac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right) + \dfrac{\beta^2}{\beta^2-1} & (1<\beta<\infty), \end{cases} \tag{13}$$

with $u(\xi) = \xi - \sqrt{t}/1 - \sqrt{t}\xi (0<t<1, \xi \in \mathbb{D})$, where t is chosen such that $k = \cosh(\pi R'(t)/4R(t))$, and $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ the complementary integral of $R(t)$ (see [11, 12] and [13]).

By taking specific values to the parameters β, m, δ, k, and b in the subclass $\beta - \mathscr{ST}_m(\delta, k, b)$, we obtain

(1) $\beta - \mathscr{ST}_m(\delta, k, (1-\alpha/m)\cos\gamma e^{-i\gamma}) = \beta - \mathscr{ST}_m^\gamma(\delta, k, \alpha)(0 \le \alpha < m\,; |\gamma| < \pi/2) = \{f \in \mathscr{A}(m)\text{: } e^{i\gamma}\xi (\mathscr{D}^{\delta+mk-k} f(\xi))'/\mathscr{D}^{\delta+mk-k} f(\xi) \prec (m-\alpha)\cos\gamma\psi_\beta(\xi) + \alpha\cos\gamma + im\sin\gamma\}$ and $\beta - \mathscr{ST}_m(\delta, k, 1-\alpha/m) = \beta - \mathscr{ST}_m(\delta, k, \alpha)(0 \le \alpha < m) = \{f \in \mathscr{A}(m)\text{: } 1/m - \alpha(\xi (\mathscr{D}^{\delta+mk-k} f(\xi))'/\mathscr{D}^{\delta+mk-k} f(\xi) - \alpha) \prec \psi_\beta(\xi)\}$

(2) $\beta - \mathscr{ST}_m(\delta, 1, b) = \beta - \mathscr{ST}_m(\delta, b) = \{f \in \mathscr{A}(m)\text{: } 1 + 1/b(\xi(\mathscr{D}^{\delta+m-1} f(\xi))'/m\mathscr{D}^{\delta+m-1} f(\xi) - 1) \prec \psi_\beta(\xi)\}$,

$\beta - \mathscr{ST}_m(\delta, 1, (1-\alpha/m)\cos\gamma e^{-i\gamma}) = \beta - \mathscr{ST}_m^\gamma(\delta, \alpha)$

$(0 \le \alpha < m\,; |\gamma| < \pi/2) = \{f \in \mathscr{A}(m): e^{i\gamma}\xi (\mathscr{D}^{\delta+m-1}f(\xi))'/\mathscr{D}^{\delta+m-1}f(\xi) \prec (m-\alpha)\cos\gamma\psi_\beta(\xi) + \alpha\cos\gamma + im\sin\gamma\}$ and $\beta - \mathcal{ST}_m(\delta, 1, 1-\alpha/m) = \beta - \mathcal{ST}_m(\delta,\alpha)(0 \le \alpha < m) = \{f \in \mathscr{A}(m): 1/m - \alpha(\xi(\mathscr{D}^{\delta+m-1}f(\xi))'/\mathscr{D}^{\delta+m-1}f(\xi) - \alpha) \prec \psi_\beta(\xi)\}$

(3) $\beta - \mathcal{ST}_1(\delta, 1, b) = \beta - \mathcal{ST}(\delta, b) = \{f \in \mathscr{A} : 1 + 1/b(\xi(\mathscr{D}^\delta f(\xi))'/\mathscr{D}^\delta f(\xi) - 1) \prec \psi_\beta(\xi)\}$,

$\beta - \mathcal{ST}_1(\delta, 1, (1-\alpha)\cos\gamma e^{-i\gamma}) = \beta - \mathcal{ST}^\gamma(\delta,\alpha)(0 \le \alpha < 1\,; |\gamma| < \pi/2) = \{f \in \mathscr{A} : e^{i\gamma}\xi(\mathscr{D}^\delta f(\xi))'/\mathscr{D}^\delta f(\xi) \prec (1-\alpha)\cos\gamma\psi_\beta(\xi) + \alpha\cos\gamma + i\sin\gamma\}$ and $\beta - \mathcal{ST}_1(\delta, 1, 1-\alpha/m) = \beta - \mathcal{ST}(\delta,\alpha)(0 \le \alpha < 1) = \{f \in \mathscr{A} : 1/1 - \alpha(\xi(\mathscr{D}^\delta f(\xi))'/\mathscr{D}^\delta f(\xi) - \alpha) \prec \psi_\beta(\xi)\}$

(4) $\beta - \mathcal{ST}_1(\delta, k, b) = \beta - \mathcal{ST}(\delta, k, b) = \{f \in \mathscr{A} : 1 + 1/b(\xi(\mathscr{D}_k^\delta f(\xi))'/\mathscr{D}_k^\delta f(\xi) - 1) \prec \psi_\beta(\xi)\}$,

$\beta - \mathcal{ST}_1(\delta, k, (1-\alpha)\cos\gamma e^{-i\gamma}) = \beta - \mathcal{ST}^\gamma(\delta, k, \alpha)(0 \le \alpha < 1\,; |\gamma| < \pi/2) = \{f \in \mathscr{A} : e^{i\gamma}\xi(\mathscr{D}_k^\delta f(\xi))'/\mathscr{D}_k^\delta f(\xi) \prec (1-\alpha)\cos\gamma\psi_\beta(\xi) + \alpha\cos\gamma + i\sin\gamma\}$ and $\beta - \mathcal{ST}_1(\delta, k, 1-\alpha/m) = \beta - \mathcal{ST}(\delta, k, \alpha)(0 \le \alpha < 1) = \{f \in \mathscr{A} : 1/1 - \alpha(\xi(\mathscr{D}_k^\delta f(\xi))'/\mathscr{D}_k^\delta f(\xi) - \alpha) \prec \psi_\beta(\xi)\}$

(5) $\beta - \mathcal{ST}_m(k - mk, k, b) = \beta - \mathcal{ST}_m(b) = \{f \in \mathscr{A}(m): 1 + 1/b(\xi f'(\xi)/mf(\xi) - 1) \prec \psi_\beta(\xi)\}$

$\beta - \mathcal{ST}_m(k - mk, k, (1-\alpha/m)\cos\gamma e^{-i\gamma}) = \beta - \mathcal{ST}_m^\gamma(\alpha)(0 \le \alpha < m\,; |\gamma| < \pi/2) = \{f \in \mathscr{A}(m): e^{i\gamma}\xi f'(\xi)/f(\xi) \prec (m-\alpha)\cos\gamma\psi_\beta(\xi) + \alpha\cos\gamma + im\sin\gamma\}$ and (see [14])

(6) $\beta - \mathcal{ST}_m(k - mk, k, 1-\alpha/m) = \beta - \mathcal{ST}_m(\alpha)(0 \le \alpha < m) = \{f \in \mathscr{A}(m): 1/m - \alpha(\xi f'(\xi)/f(\xi) - \alpha) \prec \psi_\beta(\xi)\}$;

$\beta - \mathcal{ST}_1(0, k, b) = \beta - \mathcal{ST}(b) = \{f \in \mathscr{A} : 1 + 1/b(\xi f'(\xi)/f(\xi) - 1) \prec \psi_\beta(\xi)\}$,

$\beta - \mathcal{ST}_1(0, k, (1-\alpha)\cos\gamma e^{-i\gamma}) = \beta - \mathcal{ST}^\gamma(\alpha)(0 \le \alpha < 1\,; |\gamma| < \pi/2) = \{f \in \mathscr{A}(m): e^{i\gamma}\xi f'(\xi)/f(\xi) \prec (1-\alpha)\cos\gamma\psi_\beta(\xi) + \alpha\cos\gamma + i\sin\gamma\}$ and (see [15]) $\beta - \mathcal{ST}_1(0, k, 1-\alpha) = \beta - \mathcal{ST}(\alpha)(0 \le \alpha < 1) = \{f \in \mathscr{A} : 1/1 - \alpha(\xi f'(\xi)/f(\xi) - \alpha) \prec \psi_\beta(\xi)\}$

(7) $0 - \mathcal{ST}_m(\delta, k, b) = \mathcal{S}_m(\delta, k, b) = \{f \in \mathscr{A}(m): \mathfrak{R}[m + 1/b(\xi(\mathscr{D}^{\delta+mk-k}f(\xi))'/\mathscr{D}^{\delta+mk-k}f(\xi) - m)] > 0\}$,

$0 - \mathcal{ST}_m(\delta, k, (1-\alpha/m)\cos\gamma e^{-i\gamma}) = \mathcal{S}_m^\gamma(\delta, k, \alpha)(0 \le \alpha < m\,; |\gamma| < \pi/2) = \{f \in \mathscr{A}(m): \mathfrak{R}(e^{i\gamma}\xi(\mathscr{D}^{\delta+mk-k}f(\xi))'/\mathscr{D}^{\delta+mk-k}f(\xi)) > \alpha\cos\gamma\}$, (see [16] and [17]) $0 - \mathcal{ST}_m(k - mk, k, b) = \mathcal{S}_m^\gamma(\alpha)(0 \le \alpha < m\,; |\gamma| < \pi/2) = \{f \in \mathscr{A}(m): \mathfrak{R}(e^{i\gamma}\xi f'(\xi)/f(\xi)) > \alpha\cos\gamma\}$,(see [18]) $0 - \mathcal{ST}_m(k - mk, k, b) = \mathcal{S}_m(b) = \{f \in \mathscr{A}(m): \mathfrak{R}[m + 1/b(\xi f'(\xi)/f(\xi) - m)] > 0\}$ and $\mathcal{S}_1(b) = \mathcal{S}(b)$ (see [19, 20])

In order to establish our main results, we need the following lemmas.

Lemma 2 [21]. *Let $\psi_\beta(\xi)(0 \le \beta < \infty)$ be defined by (13). If*

$$\psi_\beta(\xi) = 1 + L_1\xi + L_2\xi^2 + \cdots, \tag{14}$$

then

$$L_1 = \begin{cases} \dfrac{2A^2}{1-\beta^2} & (0 \le \beta < 1), \\ \dfrac{8}{\pi^2} & (\beta = 1), \\ \dfrac{\pi^2}{4\sqrt{t}\left(\beta^2 - 1\right)R^2(t)(1+t)} & (1 < \beta < \infty), \end{cases} \tag{15}$$

$$L_2 = \begin{cases} \dfrac{A^2 + 2}{3}L_1 & (0 \le \beta < 1), \\ \dfrac{2}{3}L_1 & (\beta = 1), \\ \dfrac{4R^2(t)\left(t^2 + 6t + 1\right) - \pi^2}{24\sqrt{t}R^2(t)(1+t)}L_1 & (1 \le \beta < \infty), \end{cases} \tag{16}$$

where

$$A = \frac{2\cos^{-1}\beta}{\pi}, \tag{17}$$

and $t \in (0, 1)$ is chosen such that $\beta = \cosh(\pi R'(t)/R(t))$, and $R(t)$ is the Legendre's complete elliptic integral of the first kind.

Lemma 3 [22]. *Let $h(\xi) = 1 + \sum_{n=1}^{\infty} c_n\xi^n \in \mathscr{P}$, i.e., let h be analytic in $\mathbb{D}$ and satisfy $\mathfrak{R}\{h(\xi)\} > 0$ for ξ in $\mathbb{D}$; then, the following sharp estimate holds*

$$\left|c_2 - \nu c_1^2\right| \le 2\max\{1, |2\nu - 1|\} \text{ for all } \nu \in \mathbb{C}. \tag{18}$$

The result is sharp for the functions given by

$$g(\xi) = \frac{1+\xi^2}{1-\xi^2} \text{ or } g(\xi) = \frac{1+\xi}{1-\xi}. \tag{19}$$

Lemma 4 [22]. *If $h(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n \in \mathcal{P}$, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & \text{if } \nu \leq 0, \\ 2 & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2 & \text{if } \nu \geq 1, \end{cases} \quad (20)$$

and when $\upsilon < 0$ or $\nu > 1$, the equality holds if and only if $h(\xi) = (1+\xi)/(1-\xi)$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $h(\xi) = (1+\xi^2)/(1-\xi^2)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$h(\xi) = \left(\frac{1+\lambda}{2}\right)\frac{1+\xi}{1-\xi} + \left(\frac{1-\lambda}{2}\right)\frac{1-\xi}{1+\xi}\ (0 \leq \lambda \leq 1), \quad (21)$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if g is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$.

Also, the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$\begin{aligned} &\left|c_2 - \nu c_1^2\right| + \nu|c_1|^2 \leq 2\left(0 \leq \nu \leq \frac{1}{2}\right), \\ &\left|c_2 - \nu c_1^2\right| + (1-\nu)|c_1|^2 \leq 2\left(\frac{1}{2} \leq \nu \leq 1\right). \end{aligned} \quad (22)$$

In this paper, we investigate a coefficient estimates and the familiar Fekete–Szegö type inequalities for the subclass $\beta - \mathcal{ST}_m(\delta, k, b)$.

2. Main Results

We will assume throughout our discussion, unless otherwise stated, that $0 \leq \beta < \infty$, $m \in \mathbb{N}$, $\delta > -mk$, $k > 0$, $b \in \mathbb{C}^*$, L_1 is given by (15), L_2 is given by (16), and $\xi \in \mathbb{D}$.

Theorem 5. *Let $f \in \mathcal{A}(m)$ be given by (1). If the inequality*

$$\sum_{n=m+1}^{\infty} \{(\beta+1)(n-m) + m|b|\} \frac{(\delta+mk)_{n-m,k}}{(k)_{n-m,k}} |a_n| \leq m|b|, \quad (23)$$

holds, then the function $f \in \beta - \mathcal{ST}_m(\delta, k, b)$.

Proof. Suppose the inequality (23) holds. Also, let us assume

$$H(\xi) = 1 + \frac{1}{b}\left(\frac{\xi\left(\mathcal{D}^{\delta+mk-k}f(\xi)\right)'}{m\mathcal{D}^{\delta+mk-k}f(\xi)} - 1\right). \quad (24)$$

We have

$$\begin{aligned} |H(\xi) - 1| &= \left|\frac{1}{b}\left(\frac{\xi\left(\mathcal{D}^{\delta+mk-k}f(\xi)\right)'}{m\mathcal{D}^{\delta+mk-k}f(\xi)} - 1\right)\right| \\ &= \frac{1}{m|b|}\left|\left(\frac{\sum_{n=m+1}^{\infty}(n-m)\left((\delta+mk)_{n-m,k}/(k)_{n-m,k}\right)a_n\xi^{n-m}}{1 + \sum_{n=m+1}^{\infty}\left((\delta+mk)_{n-m,k}/(k)_{n-m,k}\right)a_n\xi^{n-m}}\right)\right| \\ &\leq \frac{\sum_{n=m+1}^{\infty}(n-m)\left((\delta+mk)_{n-m,k}/(k)_{n-m,k}\right)|a_n|}{m|b|\left[1 - \sum_{n=m+1}^{\infty}\left((\delta+mk)_{n-m,k}/(k)_{n-m,k}\right)|a_n|\right]}. \end{aligned} \quad (25)$$

Now consider

$$\begin{aligned} &\beta|H(\xi) - 1| - \Re\{H(\xi) - 1\} \\ &\quad \leq (\beta+1)|H(\xi) - 1| \\ &\quad < \frac{(\beta+1)\sum_{n=m+1}^{\infty}(n-m)\left((\delta+mk)_{n-m,k}/(k)_{n-m,k}\right)|a_n|}{m|b|\left[1 - \sum_{n=m+1}^{\infty}\left((\delta+mk)_{n-m,k}/(k)_{n-m,k}\right)|a_n|\right]}. \end{aligned} \quad (26)$$

The last expression is bounded by 1 if (23) holds. This completes the proof of Theorem 5. ☐

Corollary 6. *If $f(\xi) \in \beta - \mathcal{ST}_m(\delta, k, b)$, then*

$$|a_n| \leq \frac{m|b|(k)_{n-m,k}}{\{(\beta+1)(n-m) + m|b|\}(\delta+mk)_{n-m,k}}\ (n \geq m+1). \quad (27)$$

The result is sharp for the function

$$f(\xi) = \xi^m + \frac{m|b|(k)_{n-m,k}}{\{(\beta+1)(n-m) + m|b|\}(\delta+mk)_{n-m,k}}\xi^n\ (n \geq m+1). \quad (28)$$

Putting $m = 1$ in Theorem 5, we obtain the following corollary.

Corollary 7. *Let $f \in \mathcal{A}$ be given by (1) with $m = 1$. If the inequality*

$$\sum_{n=2}^{\infty}\{(\beta+1)(n-1) + |b|\}\frac{(\delta+k)_{n-1,k}}{(k)_{n-1,k}}|a_n| \leq |b|, \quad (29)$$

holds, then the function $f \in \beta - \mathcal{ST}(\delta, k, b)$.

Theorem 8. *If $f \in \beta - \mathcal{ST}_m(\delta, k, b)$, then*

$$|a_{m+1}| \leq \frac{m|b|L_1 k}{\delta + mk}, \quad (30)$$

and for all $n = 3, 4, 5, \cdots$,

$$|a_{n+m-1}| \leq \frac{m|b|L_1(k)_{n-1,k}}{(n-1)(\delta+mk)_{n-1,k}}\prod_{j=1}^{n-2}\left(1 + \frac{m|b|L_1}{j}\right). \quad (31)$$

Proof. Let

$$g(\xi) = 1 + \frac{1}{b}\left(\frac{\xi\left(\mathscr{D}^{\delta+mk-k}f(\xi)\right)'}{m\mathscr{D}^{\delta+mk-k}f(\xi)} - 1\right), \tag{32}$$

where

$$g(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n, \tag{33}$$

is analytic function in $\mathbb{D}$, and it can be written as

$$\sum_{n=m+1}^{\infty}(n-m)\frac{(\delta+mk)_{n-m,k}}{(k)_{n-m,k}}a_n\xi^n = mb\mathscr{D}^{\delta+mk-k}f(\xi)\left(\sum_{n=1}^{\infty}c_n\xi^n\right). \tag{34}$$

Comparing the coefficients of ξ^{n+m-1} on both sides

$$(n-1)\frac{(\delta+mk)_{n-1,k}}{(k)_{n-1,k}}a_{n+m-1} = mb\left[c_{n-1} + \frac{(\delta+mk)_{1,k}}{(k)_{1,k}}c_{n-2}a_{m+1} + \cdots + \frac{(\delta+mk)_{n-2,k}}{(k)_{n-2,k}}c_1 a_{n+m-2}\right]. \tag{35}$$

Taking absolute on both sides and then applying the coefficient estimates $|c_n| \le L_1$ (see [13]), we have

$$|a_{n+m-1}| \le \frac{m|b|L_1(k)_{n-1,k}}{(n-1)(\delta+mk)_{n-1,k}} \cdot \left\{1 + \frac{(\delta+mk)_{1,k}}{(k)_{1,k}}|a_{m+1}| + \cdots + \frac{(\delta+mk)_{n-2,k}}{(k)_{n-2,k}}|a_{n+m-2}|\right\}. \tag{36}$$

We apply the mathematical induction on (36) so for $n = 2$
,

$$|a_{m+1}| \le \frac{m|b|L_1(k)_{1,k}}{(\delta+mk)_{1,k}} = \frac{m|b|L_1 k}{\delta+mk}, \tag{37}$$

and this shows that result is true for $n = 2$. Now for $n = 3$,

$$|a_{m+2}| \le \frac{m|b|L_1(k)_{2,k}}{2(\delta+mk)_{2,k}}\left\{1 + \frac{(\delta+mk)_{1,k}}{(k)_{1,k}}|a_{m+1}|\right\}, \tag{38}$$

and using (37), we obtain

$$|a_{m+2}| \le \frac{m|b|L_1(k)_{2,k}}{2(\delta+mk)_{2,k}}(1 + m|b|L_1), \tag{39}$$

which is true for $n = 3$. Let us assume that (31) is true for $n = t$
, that is,

$$|a_{t+m-1}| \le \frac{m|b|L_1(k)_{t-1,k}}{(t-1)(\delta+mk)_{t-1,k}}\prod_{j=1}^{n-2}\left(1 + \frac{m|b|L_1}{j}\right). \tag{40}$$

Consider

$$\begin{aligned}
|a_{t+m}| &\le \frac{m|b|L_1(k)_{t,k}}{t(\delta+mk)_{t,k}}\left\{1 + \frac{(\delta+mk)_{1,k}}{(k)_{1,k}}|a_{m+1}| + \cdots + \frac{(\delta+mk)_{t-1,k}}{(k)_{t-1,k}}|a_{t+m-1}|\right\} \\
&\le \frac{m|b|L_1(k)_{t,k}}{t(\delta+mk)_{t,k}}\left\{1 + m|b|L_1 + \cdots + \frac{(\delta+mk)_{t-1,k}}{(k)_{t-1,k}}|a_{t+m-1}|\right\} \\
&\le \frac{m|b|L_1(k)_{t,k}}{t(\delta+mk)_{t,k}}\left\{1 + m|b|L_1 + \frac{m|b|L_1}{2}(1 + m|b|L_1) + \cdots + \frac{m|b|L_1}{t-1}\right. \\
&\quad \left.\cdot (1 + m|b|L_1)\left(1 + \frac{m|b|L_1}{2}\right)\cdots\left(1 + \frac{m|b|L_1}{t-2}\right)\right\} \\
&= \frac{m|b|L_1(k)_{t,k}}{t(\delta+mk)_{t,k}}(1 + m|b|L_1)\left(1 + \frac{m|b|L_1}{2}\right)\cdots\left(1 + \frac{m|b|L_1}{t-1}\right) \\
&= \frac{m|b|L_1(k)_{t,k}}{t(\delta+mk)_{t,k}}\prod_{j=1}^{t-1}\left(1 + \frac{m|b|L_1}{j}\right).
\end{aligned} \tag{41}$$

Therefore, the result is true for $n = t + 1$. Consequently, using mathematical induction, we proved that the result holds true for all $n (n \ge 2)$. This completes the proof of Theorem 8. ☐

Putting $m = 1$ in Theorem 8, we obtain the following corollary.

Corollary 9. *If $f \in \beta - \mathscr{ST}(\delta, k, b)$, then*

$$|a_2| \le \frac{|b|L_1 k}{\delta + k}, \tag{42}$$

and for all $n = 3, 4, 5, \cdots$,

$$|a_n| \le \frac{|b|L_1(k)_{n-1,k}}{(n-1)(\delta+k)_{n-1,k}}\prod_{j=1}^{n-2}\left(1 + \frac{|b|L_1}{j}\right). \tag{43}$$

Theorem 10. *Let $f \in \beta - \mathscr{ST}_m(\delta, k, b)$. Then., $f(\mathbb{D})$ contains an open disk of radius*

$$r = \frac{\delta + mk}{(m+1)(\delta+mk) + m|b|L_1 k}. \tag{44}$$

Proof. Let $w_0 \ne 0$ be a complex number such that $f(\xi) \ne w_0$ for $\xi \in \mathbb{D}$. Then, $f_1(\xi) = w_0 f(\xi)/w_0 - f(\xi) = \xi^m + (a_{m+1} + 1/w_0)\xi^{m+1} + \cdots$.

Since f_1 is univalent, so

$$\left|a_{m+1} + \frac{1}{w_0}\right| \le m + 1. \tag{45}$$

Now using Theorem 8, we have

$$\left|\frac{1}{w_0}\right| \le m + 1 + \frac{m|b|L_1 k}{\delta + mk}, \tag{46}$$

and hence

$$|w_0| \geq \frac{\delta + mk}{(m+1)(\delta + mk) + m|b|L_1 k}. \quad (47)$$

This completes the proof of Theorem 10. □

Putting $m = 1$ in Theorem 10, we obtain the following corollary.

Corollary 11. *Let $f \in \beta - \mathcal{ST}(\delta, k, b)$. Then, $f(\mathbb{D})$ contains an open disk of radius*

$$r_1 = \frac{\delta + k}{2(\delta + k) + |b|L_1 k}. \quad (48)$$

Theorem 12. *Let $f \in \beta - \mathcal{ST}_m(\delta, k, b)$ with the form (1). Then, for a complex number μ, we have*

$$\left|a_{m+2} - \mu a_{m+1}^2\right| \leq \frac{mbL_1 k^2}{(\delta + mk)(\delta + mk + k)} \max\left\{1, \left|\frac{L_2}{L_1} + mbL_1\left(1 - \frac{\delta + mk + k}{\delta + mk}\mu\right)\right|\right\}. \quad (49)$$

Proof. If $f \in \beta - \mathcal{ST}_m(\delta, k, b)$, then there exists a Schwarz function w, with $w(0) = 0$ and $|w(\xi)| < 1$ such that

$$1 + \frac{1}{b}\left(\frac{\xi\left(\mathcal{D}^{\delta+mk-k} f(\xi)\right)'}{m\mathcal{D}^{\delta+mk-k} f(\xi)} - 1\right) = \psi_\beta(w(\xi)) \ (\xi \in \mathbb{D}). \quad (50)$$

Let $h \in \mathcal{P}$ be a function defined by

$$h(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} = 1 + c_1\xi + c_2\xi^2 + \cdots (\xi \in \mathbb{D}). \quad (51)$$

This gives

$$w(\xi) = \frac{c_1}{2}\xi + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\xi^2 + \cdots, \quad (52)$$

$$\psi_\beta(w(\xi)) = 1 + \frac{1}{2}c_1 L_1 \xi + \frac{1}{2}\left\{\frac{c_1^2 L_2}{2} + \left(c_2 - \frac{c_1^2}{2}\right)L_1\right\}\xi^2 + \cdots. \quad (53)$$

Using (53) in (50), we obtain

$$a_{m+1} = \frac{mbc_1 L_1 k}{2(\delta + mk)},$$
$$a_{m+2} = \frac{mbk^2}{2(\delta + mk)(\delta + mk + k)}\left\{\frac{c_1^2 L_2}{2} + \left(c_2 - \frac{c_1^2}{2}\right)L_1 + \frac{mbc_1^2 L_1^2}{2}\right\}. \quad (54)$$

For any complex number μ, we have

$$a_{m+2} - \mu a_{m+1}^2 = \frac{mbk^2}{2(\delta + mk)(\delta + mk + k)}\left\{\frac{c_1^2 L_2}{2} + \left(c_2 - \frac{c_1^2}{2}\right)L_1 + \frac{mbc_1^2 L_1^2}{2}\right\} - \mu\frac{m^2 b^2 c_1^2 L_1^2 k^2}{4(\delta + mk)^2}. \quad (55)$$

Then (55) can be written as

$$a_{m+2} - \mu a_{m+1}^2 = \frac{mbL_1 k^2}{2(\delta + mk)(\delta + mk + k)}\left\{c_2 - \nu c_1^2\right\}, \quad (56)$$

where

$$\nu = \frac{1}{2}\left\{1 - \frac{L_2}{L_1} - mbL_1\left(1 - \frac{\delta + mk + k}{\delta + mk}\mu\right)\right\}. \quad (57)$$

Now, taking absolute value on both sides and using Lemma 3, we obtain the required result. □

Putting $m = 1$ in Theorem 12, we obtain the following corollary.

Corollary 13. *Let $f \in \beta - \mathcal{ST}(\delta, k, b)$ with the form (1). Then, for a complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{bL_1 k^2}{(\delta + k)(\delta + 2k)} \max\left\{1, \left|\frac{L_2}{L_1} + bL_1\left(1 - \frac{\delta + 2k}{\delta + k}\mu\right)\right|\right\}. \quad (58)$$

Theorem 14. *Let*

$$\sigma_1 = \frac{\left\{mbL_1^2 + L_2 - L_1\right\}(\delta + mk)}{mb(\delta + mk + k)L_1^2}, \sigma_2 = \frac{\left\{mbL_1^2 + L_2 + L_1\right\}(\delta + mk)}{mb(\delta + mk + k)L_1^2}, \sigma_3 = \frac{\left\{mbL_1^2 + L_2\right\}(\delta + mk)}{mb(\delta + mk + k)L_1^2}. \quad (59)$$

If f given by (1) belongs to $\beta - \mathcal{ST}_m(\delta, k, b)$ with $b > 0$, then

$$\left|a_{m+2} - \mu a_{m+1}^2\right| \leq \begin{cases} \frac{mbL_1 k^2}{(\delta + mk)(\delta + mk + k)}\left\{\frac{L_2}{L_1} + mbL_1\left(1 - \frac{\delta + mk + k}{\delta + mk}\mu\right)\right\} & (\mu \leq \sigma_1), \\ \frac{mbL_1 k^2}{(\delta + mk)(\delta + mk + k)} & (\sigma_1 \leq \mu \leq \sigma_2), \\ -\frac{mbL_1 k^2}{(\delta + mk)(\delta + mk + k)}\left\{\frac{L_2}{L_1} + mbL_1\left(1 - \frac{\delta + mk + k}{\delta + mk}\mu\right)\right\} & (\mu \geq \sigma_2). \end{cases} \quad (60)$$

Further, if $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_{m+2} - \mu a_{m+1}^2\right| + \frac{\delta + mk}{mb(\delta + mk + k)L_1}\left\{1 - \frac{L_2}{L_1} - mbL_1\left(1 - \frac{\delta + mk + k}{\delta + mk}\mu\right)\right\}\left|a_{m+1}\right|^2 \le \frac{mbL_1k^2}{(\delta + mk)(\delta + mk + k)}. \quad (61)$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_{m+2} - \mu a_{m+1}^2\right| + \frac{\delta + mk}{mb(\delta + mk + k)L_1} \times \left\{1 + \frac{L_2}{L_1} + mbL_1\left(1 - \frac{\delta + mk + k}{\delta + mk}\mu\right)\right\}\left|a_{m+1}\right|^2 \le \frac{mbL_1k^2}{(\delta + mk)(\delta + mk + k)}. \quad (62)$$

Proof. Applying Lemma 4 to (56) and (57), respectively, we can obtain our results asserted by Theorem 14. □

Putting $m = 1$ in Theorem 14, we obtain

Corollary 15. *Let*

$$\sigma_4 = \frac{\left\{bL_1^2 + L_2 - L_1\right\}(\delta + k)}{b(\delta + 2k)L_1^2}, \sigma_5 = \frac{\left\{bL_1^2 + L_2 + L_1\right\}(\delta + k)}{b(\delta + 2k)L_1^2}, \sigma_6 = \frac{\left\{bL_1^2 + L_2\right\}(\delta + k)}{b(\delta + 2k)L_1^2}. \quad (63)$$

If f given by (1) belongs to $\beta - \mathcal{ST}(\delta, k, b)$ with $b > 0$, then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{bL_1k^2}{(\delta + k)(\delta + 2k)}\left\{\dfrac{L_2}{L_1} + bL_1\left(1 - \dfrac{\delta + 2k}{\delta + k}\mu\right)\right\} & (\mu \le \sigma_4), \\ \dfrac{bL_1k^2}{(\delta + k)(\delta + 2k)} & (\sigma_4 \le \mu \le \sigma_5), \\ -\dfrac{bL_1k^2}{(\delta + k)(\delta + 2k)}\left\{\dfrac{L_2}{L_1} + bL_1\left(1 - \dfrac{\delta + 2k}{\delta + k}\mu\right)\right\} & (\mu \ge \sigma_5). \end{cases} \quad (64)$$

Further, if $\sigma_4 \le \mu \le \sigma_6$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{\delta + k}{b(\delta + 2k)L_1}\left\{1 - \frac{L_2}{L_1} - bL_1\left(1 - \frac{\delta + 2k}{\delta + k}\mu\right)\right\}\left|a_2\right|^2 \le \frac{bL_1k^2}{(\delta + k)(\delta + 2k)}. \quad (65)$$

If $\sigma_6 \le \mu \le \sigma_5$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{\delta + k}{b(\delta + 2k)L_1}\left\{1 + \frac{L_2}{L_1} + bL_1\left(1 - \frac{\delta + 2k}{\delta + k}\mu\right)\right\}\left|a_2\right|^2 \le \frac{bL_1k^2}{(\delta + k)(\delta + 2k)}. \quad (66)$$

Remark 16. For different choices of the parameters β, m, δ, k, and b in the above theorems, we can obtain the corresponding results for each of the following subclasses $\beta - \mathcal{ST}_m^\gamma(\delta, k, \alpha)$, $\beta - \mathcal{ST}_m(\delta, k, \alpha)$, $\beta - \mathcal{ST}_m(\delta, b)$, $\beta - \mathcal{ST}_m^\gamma(\delta, \alpha)$, $\beta - \mathcal{ST}_m(\delta, \alpha)$, $\beta - \mathcal{ST}(\delta, b)$, $\beta - \mathcal{ST}^\gamma(\delta, \alpha)$, $\beta - \mathcal{ST}(\delta, \alpha)$, $\beta - \mathcal{ST}^\gamma(\delta, k, \alpha)$, $\beta - \mathcal{ST}(\delta, k, \alpha)$, $\beta - \mathcal{ST}_m(b)$, $\beta - \mathcal{ST}_m^\gamma(\alpha)$, $\beta - \mathcal{ST}_m(\alpha)$, $\beta - \mathcal{ST}(b)$, $\beta - \mathcal{ST}^\gamma(\alpha)$, $\beta - \mathcal{ST}(\alpha)$, $\mathcal{S}_m(\delta, k, b)$, $\mathcal{S}_m^\gamma(\delta, k, \alpha)$, $\mathcal{S}_m^\gamma(\alpha)$, $\mathcal{S}_m(b)$, and $\mathcal{S}(b)$ which are defined in Section 1.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] T. Bulboacă, *Differential Subordinations and Superordinations, Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.

[2] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics*, vol. 225, Marcel Dekker, New York and Basel, 2000.

[3] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations," *Complex Variables, Theory and Application: An International Journal*, vol. 48, no. 10, pp. 815–826, 2003.

[4] R. Daz and E. Pariguan, "On hypergeometric functions and Pochhammer k – symbol," *Divulgaciones Matemáticas*, vol. 15, no. 2, pp. 179–192, 2007.

[5] R. M. Goel and N. S. Sohi, "A new criterion for p – valent functions," *Proceedings of American Mathematical Society*, vol. 78, no. 3, pp. 353–357, 1980.

[6] M. K. Aouf, "A generalization of multivalent functions defined by Ruscheweyh derivatives," *Soochow Journal of Mathematics*, vol. 17, pp. 83–97, 1991.

[7] V. Kumar and S. L. Shukla, "Multivalent functions defined by Ruscheweyh derivatives," *Indian Journal of Pure and Applied Mathematics*, vol. 15, no. 11, pp. 489–509, 1984.

[8] M. K. Aouf and H. E. Darwish, "A remark on certain p-valent functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 2, 406 pages, 1996.

[9] T. M. Seoudy, "Some subclasses of univalent functions associated with $k-$Ruscheweyh derivative operator," *Ukrainian Mathematical Journal*, In press.

[10] S. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, no. 1, pp. 109–115, 1975.

[11] S. Kanas and A. Wisniowska, "Conic regions and k-uniform convexity," *Journal of Computational and Applied Mathematics*, vol. 105, no. 1-2, pp. 327–336, 1999.

[12] S. Kanas and A. Wisniowska, "Conic domains and starlike functions," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 45, no. 4, pp. 647–658, 2000.

[13] K. I. Noor, M. Arif, and W. Ul-Haq, "On k-uniformly close-to-convex functions of complex order," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 629–635, 2009.

[14] T. O. Salim, M. S. Marouf, and J. M. Shenan, "A subclass of multivalent uniformly convex functions associated with generalized Salagean and Ruscheweyh differential operators," *Acta Universitatis Apulensis*, vol. 26, pp. 289–300, 2011.

[15] R. Bharati, R. Parvatham, and A. Swaminathan, "On subclasses of uniformly convex functions and corresponding class of starlike functions," *Tamkang Journal of Mathematics*, vol. 28, no. 1, pp. 17–32, 1997.

[16] M. K. Aouf, "On coefficient bounds of a certain class of p -valent λ-spiral functions of order α," *International Journal of Mathematics and Mathematical Sciences*, vol. 10, no. 2, 266 pages, 1987.

[17] H. M. Srivastava, M. K. Aouf, and S. Owa, "Certain classes of multivalent functions of order α and type β," *Bulletin de la Societe Mathematique de Belgique. Serie B*, vol. 42, no. 1, pp. 31–66, 1990.

[18] B. Wongsaijai and N. Sukantamala, "Mapping properties of generalized $q-$integral operator of $p-$valent functions involving the Ruscheweyh derivative and the generalized Salagean operator," *Acta Universitatis Apulensis*, vol. 41, pp. 31–50, 2015.

[19] M. A. Nasr and M. K. Aouf, "Bounded starlike functions of complex order," *Indian Academy of Sciences*, vol. 92, no. 2, pp. 97–102, 1983.

[20] M. A. Nasr and M. K. Aouf, "Starlike function of complex order," *The Journal of Natural Sciences and Mathematics*, vol. 25, no. 1, pp. 1–12, 1985.

[21] S. Kanas, "Coefficient estimates in subclasses of the Carathéodory class related to conical domains," *Acta Mathematica Universitatis Comenianae*, vol. 75, no. 2, pp. 149–161, 2005.

[22] W. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, Z. Li, F. Y. Ren, L. Yang, and S. Y. Zhang, Eds., , pp. 157–169, Int. Press, Cambridge, MA, 1994.

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Research Article

On Extended Convex Functions via Incomplete Gamma Functions

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Convex functions play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. In this paper, firstly we introduce the notion of h-exponential convex functions. This notion can be considered as generalizations of many existing definitions of convex functions. Then, we establish some well-known inequalities for the proposed notion via incomplete gamma functions. Precisely speaking, we established trapezoidal, midpoint, and He's inequalities for h-exponential and harmonically exponential convex functions via incomplete gamma functions. Moreover, we gave several remarks to prove that our results are more generalized than the existing results in the literature.

1. Introduction

Convex optimization contributed largely in many areas of pure and applied mathematics during recent years, and convex analysis provides main foundation for convex optimization [1, 2]. Due to huge applications of convex analysis, the researchers always show interest to generalization the notion of convexity. In literature, there exist many versions of convex functions, for example, h-convex function, see [3], r-convex functions, see [4], harmonic convex function, see [5], exponentially convex functions, see [6], etc. [7, 8].

Since convex function is a class of very important functions which is widely used in pure mathematics, functional analysis, optimization theory, and mathematical economics, so to study properties of certain classes of convex functions and establish different inequalities like trapezoidal, midpoint, He's Hermite-Hadamard, Fejér, etc., type inequality is an important area of research. A lot of work is devoted to establish different kinds of inequalities for different classes of convex functions, for example, Iscan [9] established Hermite-Hadamard type inequalities for harmonically convex functions. Bai et al. [10] presented Hermite-Hadamard type inequalities for the m and (α, m)-logarithmically convex functions. Özdemir et al. [11] developed Hermite–Hadamard-type inequalities via (α, m)-convex functions. Chu et al. [12] gave generalizations of Hermite-Hadamard type inequalities for MT-convex functions.

It is always appreciable to derive more version of inequalities for generalized convexities. For some important generalization, we refer [13, 14]. Fractional calculus also provides some broader variety to deal real-world problems. Just like other fields, fractional calculus also sets new trends in inequalities of convex analysis. For more details on fractional integral inequalities, we refer to the readers [15–18]. Many interesting controversies are also part of history of fractional calculus. Some famous definitions of fractional derivative are Riemann-Liouville [19], Caputo-Fabrizio [20], etc. [21–24]. In the present paper, we will deal with incomplete gamma functions. Firstly, we introduce the notions of h-exponential convex functions and harmonically exponential convex functions. Then, we establish some well-known inequalities for the proposed notions via incomplete gamma functions. Precisely speaking, we established trapezoidal, midpoint, and He's inequalities for h-exponential and harmonically exponential convex functions via incomplete gamma functions. Moreover, we gave several remarks to prove that our results are more generalized than the existing results in the literature.

The breakup of this paper is as follows: In Section 2, we present basic definitions and known results. Section 3 contains trapezoidal type inequalities via incomplete gamma function. Midpoint inequalities via incomplete gamma function are presented in Section 4, and He's inequality via the incomplete gamma functions is presented in Section 5. Last section contains concluding remarks and some future directions.

2. Preliminaries

Before starting the main findings, we review some definitions, notations, and theorems which are necessary to proceed. Throughout this paper, L^1 denotes space of all locally integrable functions.

Definition 1 [19]. For any L^1 function $z(u)$ on an interval $[x, y]$ with $u \in [x, y]k$-th left-RL fractional integral of $z(u)$ is given by

$$ {}^{\mathrm{RL}}J_{a^+}^{k} z(u) = \frac{1}{\Gamma(k)} \int_x^u (u-t)^{k-1} z(t)dt, \tag{1}$$

for Re $(k) > 0$. Also, the k-th right- RL fractional integral of $z(u)$ is given by

$$ {}^{RL}J_{x}^{k} z(u) = \frac{1}{\Gamma(k)} \int_u^y (t-u)^{k-1} z(t)dt. \tag{2}$$

Definition 2 [6]. We say that the function $z: M \subseteq R \longrightarrow R$ is exponential type convex on M if

$$ z(tx + (1-t)y) \le \left(e^t - 1\right) z(x) + \left(e^{1-t} - 1\right) z(y), \tag{3}$$

holds for every $x, y \in M$ and $t \in [0, 1]$.

Definition 3 [3]. We say that the function z: $M \subseteq R \longrightarrow$ is h-convex function on M if

$$ z(tx + (1-t)y) \le h(t) z(x) + h(1-t) z(y), \tag{4}$$

where $x, y \in M$ and $t \in [0, 1]$.

We are now ready to define some new convexity, called as h-exponential convex function.

Definition 4. We say that the function z: $M \subseteq R \longrightarrow$ is h-exponential type convex on M if

$$ z(tx + (1-t)y) \le h\left(e^t - 1\right) z(x) + h\left(e^{1-t} - 1\right) z(y), \tag{5}$$

where $x, y \in M$ and $t \in [0, 1]$.

Remark 5.

(1) By substituting $h(e^t - 1) = 1/(e^t - 1)$, $h(e^{1-t} - 1) = 1/(e^{1-t} - 1$ in Definition 3, we get harmonically exponential convex function

(2) By substituting $h(e^t - 1) = e^t - 1$, $h(e^{1-t} - 1) = e^{1-t} - 1$ in Definition 3, we get Definition 2 of exponential convex function

Now, the integral inequality of Hermite-Hadamard (HH) type for a convex function is give by

$$ z\left(\frac{x+y}{2}\right) \le \frac{1}{y-x} \int_x^y z(x)dx \le \frac{z(x) + z(y)}{2}. \tag{6}$$

Sarikaya et al. [25] generalized the HH-inequality (6) to fractional integrals of RL type which is given by

$$ z\left(\frac{x+y}{2}\right) \le \frac{\Gamma(k+1)}{2(y-x)^k} \left[{}^{RL}J_{x^+}^{k} z(y) + {}^{RL}J_{y^-}^{k} z(x) \right] \le \frac{z(x) + z(y)}{2}, \tag{7}$$

where $k > 0$ and $z[x, y] \longrightarrow R$ is let to be an L^1 convex function. After that, Sarikaya and Yildirim [26] found a new inequality of the above

$$ \begin{aligned} z\left(\frac{x+y}{2}\right) &\le \frac{2^{k-1}\Gamma(k+1)}{(y-x)^k} \left[{}^{\mathrm{RL}}J_{\left(\frac{x+y}{2}\right)^+}^{k} z(y) + {}^{RL}J_{\left(\frac{x+y}{2}\right)^-}^{k} z(x) \right] \\ &\le \frac{z(x+y)}{2}. \end{aligned} \tag{8}$$

The following facts will be needed in establishing our main results:

Remark 6 (21). For Re > 0, the following identities hold:

$$ \begin{gathered} \int_0^1 t^{k-1} e^t dt = (-1)^k \gamma(k,-1)\,; \\ \int_0^1 t^{k-1} e^{1-t} dt = \gamma(k, 1), \\ \gamma(k, x) = \int_0^x t^{k-1} e^{-t} dt, x \in C. \end{gathered} \tag{9}$$

Remark 7 (21). For Re > 0, the following identities hold:

$$ \int_0^1 t^{k-1} e^{t/2} dt = (-2)^k \gamma\left(k, \frac{-1}{2}\right); \tag{10}$$

$$ \int_0^1 t^{k-1} e^{1-(t/2)} dt = e2^k \gamma\left(k, \frac{1}{2}\right). \tag{11}$$

Lemma 8 [25]. *If $z : [x, y] \longrightarrow R$ is $L^1[x, y]$ with $0 < x < y$ and $k > 0$, then we have*

$$\frac{z(x)+z(y)}{2} - \frac{\Gamma(k+1)}{(y-x)^k}\left[{}^{RL}J_{x^+}z(y) + {}^{RL}J^k_{y^-}z(x)\right.$$
$$= \frac{y-x}{2}\int_0^1\left[(1-t)^k - t^k\right]z(tx+(1-t)y)dt. \quad (12)$$

Lemma 9 [26]. *If $z : [x, y] \longrightarrow R$ is $L^1[x, y]$ with $0 < x < y$ and $k > 0$, then we have*

$$\frac{2^{k-1}\Gamma(k+1)}{(y-x)^k}\left[{}^{RL}J^k_{((x+y)/2)^+}z(y) + {}^{RL}J^k_{((x+y)/2)^-}z(x) - z\left(\frac{x+y}{2}\right)\right]$$
$$= \frac{y-x}{4}\left[\int_0^1 e^t z\left(\frac{t}{2}x + \frac{2-t}{2}y\right)dt - \int_0^1 t^k z\left(\frac{2-t}{2}x + \frac{t}{2}y\right)dt\right]. \quad (13)$$

3. Trapezoidal Type Inequalities via Incomplete Gamma Function

In this section, we present trapezoidal type inequalities via incomplete gamma function.

Theorem 10. *Suppose that $z : [x, y] \longrightarrow R$ is $L^1[x, y]$ and h-exponential convex function, then we have for $k > 0$,*

$$z\left(\frac{x+y}{2}\right) \le \frac{kh\left(e^{1/2}-1\right)\Gamma(k+1)}{(y-x)^k}\left[{}^{RL}J^k_x z(y) + {}^{RL}J^k_y z(x)\right]$$
$$\le h\left(e^{1/2}-1\right)M[z(x)+z(y)], \quad (14)$$

where $h(e^t + e^{1-t} - 2) \le M$.

Proof. Let $z : I \longrightarrow R$ is h-exp convex function and $k > 0$ then by definition

$$z\left(\frac{x+y}{2}\right) = z\left[\frac{(tx+(1-t)y)+(1-t)x+ty}{2}\right]$$
$$\le h\left(e^{1/2}-1\right)z(tx+(1-t)y)$$
$$+ h\left(e^{1/2}-1\right)z((1-t)x+ty)). \quad (15)$$

Multiplying t^{k-1} on both sides and then integrating on $[0, 1]$, we get

$$\frac{1}{k}z\left(\frac{x+y}{2}\right) \le h\left(e^{1/2}-1\right)\int_0^1 t^{k-1}z(tx+(1-t)y))dt$$
$$+ h\left(e^{1/2}-1\right) \times \int_0^1 t^{k-1}z((1-t)x+ty))dt. \quad (16)$$

Again by small substitution, we have

$$z\left(\frac{x+y}{2}\right) \le kh\left(e^{(1/2)-1}\right)\frac{1}{x-y}\left[\int_y^x\left(\frac{y-u}{y-x}\right)^{k-1}z(u)d(u)\right.$$
$$\left.+\int_x^y\left(\frac{v-x^{k-1}}{y-x}\right)z(v)d(v)\right]$$
$$\le kh\left(e^{1/2}-1\right)\left[\frac{1}{x-y}\int_y^x\left(\frac{y-u}{y-x}\right)^{k-1}z(u)d(u)\right.$$
$$\left.+\frac{1}{x-y}\int_x^y\left(\frac{v-x}{y-x}\right)^{k-1}z(v)d(v)\right]$$
$$\le \frac{\Gamma(k+1)h\left(e^{1/2}-1\right)}{(x-y)^k}\left[{}^{\mathrm{RL}}J^k_{x^+}z(y) + {}^{\mathrm{RL}}J^k_{y^-}z(x)\right]. \quad (17)$$

For other inequalities, take

$$z(tx+(1-t)y) \le h\left(e^t-1\right)z(x) + h\left(e^{1-t}-1\right)z(y);$$
$$z((1-t)x+ty) \le h\left(e^{1-t}-1\right)z(x) + h\left(e^t-1\right)z(y). \quad (18)$$

Adding both inequalities, we get

$$z(tx+(1-t)y) + z((1-t)x+ty) \le h\left(e^t+e^{1-t}-2\right)[z(x)+z(y)]. \quad (19)$$

Multiplying both sides by t^{k-1} and integrating on [0,1], we have

$$\int_0^1 t^{k-1}z(tx+(1-t)y)dt + \int_0^1 t^{k-1}z((1-t)x+ty)dt$$
$$\le \int_0^1 t^{k-1}h\left(e^t+e^{1-t}-2\right)dt[z(x)+z(y)]. \quad (20)$$

By making the change of variables

$$\frac{1}{x-y}\int_y^x\left(\frac{y-u}{y-x}\right)^{k-1}z(u)d(u) + \frac{1}{x-y}\int_y^x\left(\frac{v-x}{y-x}\right)^{k-1}z(v)d(v)$$
$$\le [z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^t+e^{1-t}-2\right)dt,$$
$$\frac{\Gamma(k)}{(x-y)^k}\left[{}^{\mathrm{RL}}J^k_{x^+}z(y) + {}^{\mathrm{RL}}J^k_{y^-}z(x)\right]$$
$$\le [z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^t+e^{1-t}-2\right)dt. \quad (21)$$

Multiplying by $k>0$ and $h(e^{1/2}-1)$ on both sides, we get.

$$\frac{\Gamma(k+1)h\left(e^{1/2}-1\right)}{(x-y)^k}\left[{}^{\mathrm{RL}}J_{x^+}^{k}z(y)+{}^{\mathrm{RL}}J_{y^-}^{k}z(x)\right] \leq kh\left(e^{1/2}-1\right)[z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^t+t^{1-t}-2\right)dt, \tag{22}$$

$$\frac{\Gamma(k+1)h\left(e^{1/2}-1\right)}{(x-y)^k}\left[{}^{\mathrm{RL}}J_{x^+}^{k}z(y)+{}^{\mathrm{RL}}J_{y^-}^{k}z(x)\right] \leq h\left(e^{1/2}-1\right)M[z(x)+z(y)]. \tag{23}$$

□

Corollary 11. *If we substitute $h(e^t+e^{1-t}-2)=(e^t+e^{1-t}-2)$ in (22) and use Remark 6, then both of inequalities (17) and (22) become (7) of [27].*

Remark 12. For $h(e^t+e^{1-t}-2)=1/(e^t+e^{1-t}-2)$, (22) yields trapezoidal type inequalities via the incomplete gamma function for harmonically exponential convex function.

Theorem 13. *Let $z:[x,y]\longrightarrow R$ be $L^1[x,y]$ with $0<x<y$ and $k>0$. If $|z|$ is an h-exp convex function, then we*

$$\begin{aligned} z\left(\frac{x+y}{2}\right) &\leq \frac{h\left(e^{1/2}-1\right)\Gamma(k+1)}{(x-y)^k}\left[{}^{RL}J_{x^+}^{k}z(y)+{}^{RL}J_{y^-}^{k}z(x)\right] \\ &\leq \frac{y-x}{2}\left([\delta_0(k,h_0,h)]+\delta_1(k,h_1,h)]|z(x)|\right. \\ &\quad+[\delta_0(k,h_1,h)+\delta_1(k,h_0,h)]|z(y)| \\ &\quad+[\delta_0(k,h_0,h)+\delta_0(k,h_1,h)]|z(x)| \\ &\quad+[\delta_1(k,h_0,h)+\delta_0(k,h_0,h)]|z(y)|, \end{aligned} \tag{24}$$

where

$$\begin{cases} \delta_0(k.h_0,h)=\int_0^{\frac{1}{2}}(1-t)^k h\left(e^t-1\right)dt=\int_{\frac{1}{2}}^{1}t^k h\left(e^{1-t}-1\right)dt; \\ \delta_0(k,h_1,h)=\int_0^{\frac{1}{2}}(1-t)^k h\left(e^{1-t}-1\right)dt=\int_{\frac{1}{2}}^{1}t^k h\left(e^{t}-1\right)dt; \\ \delta_1(k,h_0,h)=-\int_0^{\frac{1}{2}}t^k h\left(e^{1-t}-1\right)dt=-\int_{\frac{1}{2}}^{1}(1-t)^k h\left(e^{t}-1\right)dt; \\ \delta_1(k,h_1,h)=-\int_0^{\frac{1}{2}}t^k h\left(e^{t}-1\right)dt=-\int_{\frac{1}{2}}^{1}(1-t)^k h\left(e^{1-t}-1\right)dt. \end{cases} \tag{25}$$

Proof. From Lemma 8, we have

$$\begin{aligned} &\left|\frac{z(x)+z(y)}{2}-\frac{\Gamma(k+1)}{2(y-x)^k}\left[{}^{\mathrm{RL}}J_{x^+}^{k}z(y)+{}^{\mathrm{RL}}J_{y^-}^{k}z(x)\right]\right| \\ &\leq \frac{y-x}{2}\int_0^1 (1-t)^k-t^k\Big)|z(tx+(1-t)y|dt \\ &= \frac{y-x}{2}\left[\int_0^{\frac{1}{2}}\left((1-t)^k-t^k\right)|z(tx+(1-t))y|dt\right] \\ &\quad+\left[\int_{\frac{1}{2}}^{1}\left(t^k-(1-t)^k\right)|z(tx+(1-t))y|dt\right]. \end{aligned} \tag{26}$$

By using the h-exp convexity of $|z|$

$$\begin{aligned} &\left|\frac{z(x)+z(y)}{2}-\frac{\Gamma(k+1)}{2(y-x)^k}\left[{}^{\mathrm{RL}}J_{x^+}^{k}z(y)+{}^{\mathrm{RL}}J_{y^-}^{k}z(x)\right]\right| \\ &\leq \frac{y-x}{2}\left[\int_0^{\frac{1}{2}}\left((1-t)^k-t^k\right)\left[h\left(e^t-1\right)|z(x)|\right.\right. \\ &\quad\left.+h\left(e^{1-t}-1\right)|z(y)|\right]dt \\ &\quad+\int_{\frac{1}{2}}^{1}\left(t^k-(1-t)^k\right)\left[h\left(e^t-1\right)|z(x)|\right. \\ &\quad\left.\left.+h\left(e^{1-t}-1\right)|z(y)|\right]dt\right]. \end{aligned} \tag{27}$$

By using identities (25), we get required result. □

4. Midpoint Inequalities via Incomplete Gamma Function

This section contains midpoint inequalities via incomplete gamma function.

Theorem 14. *Let $z:[x,y]\longrightarrow R$ be $L^1[x,y]$ with $0<x<y$ and $k>0$. If $|z|$ is an h -exp convex function, then we*

$$\begin{aligned} &z\left(\frac{x+y}{2}\right)\frac{h\left(e^{1/2}-1\right)\Gamma(k+1)}{(x-y)^k}\left[{}^{RL}J_{((x+y)/2)^+}^{k}z(y)+{}^{RL}J_{((x+y)/2)^-}^{k}z(x)\right] \\ &\leq h\left(e^{1/2}-1\right)M[z(x)+z(y)], \end{aligned} \tag{28}$$

where $h(e^{t/2}+e^{1-(t/2)}-2)\leq M$.

Proof. Let $z:I\longrightarrow R$ is h-exponential convex function and $k>0$, then by definition

$$\begin{aligned} z\left(\frac{x+y}{2}\right) &= z\left(\frac{[(t/2)x+((2-t)/2)y]+[((2-t)/2)x+(t/2)y]}{2}\right) \\ &\leq h\left(e^{1/2}-1\right)z\left(\frac{t}{2}x+\frac{2-t}{2}y\right)+h\left(e^{1/2}-1\right)z\left(\frac{2-t}{2}x+\frac{t}{2}y\right). \end{aligned} \tag{29}$$

Multiplying by t^{k-1} on both sides and integrating w.r.t "t" from [0,1], we get

$$\frac{1}{k}z\left(\frac{x+y}{2}\right) \leq h\left(e^{1/2}-1\right)\int_0^1 t^{k-1}z\left(\frac{t}{2}x+\left(1-\frac{t}{2}\right)y\right)dt + h\left(e^{1/2}-1\right)\int_0^1 t^{k-1}z\left(\frac{2-t}{2}x+\frac{t}{2}y\right)dt. \tag{30}$$

Again by small substitution, we have

$$\begin{aligned}\frac{1}{k}z\left(\frac{x+y}{2}\right) &\leq h\left(e^{1/2}-1\right)\frac{1}{x-y}\int_y^{\frac{x+y}{2}} 2^k\left(\frac{y-u}{y-x}\right)^{k-1}z(u)d(u) \\ &\quad + h\left(e^{1/2}-1\right)\frac{1}{x-y}\int_{\frac{x+y}{2}}^{x} 2^k\left(\frac{v-x}{y-x}\right)^{k-1}z(v)d(v) \\ &\leq 2^k h\left(e^{1/2}-1\right)\frac{1}{x-y}\int_y^{\frac{x+y}{2}}\left(\frac{y-u}{y-x}\right)^{k-1}z(u)d(u) \\ &\quad + 2^k h\left(e^{1/2}-1\right)\frac{1}{x-y}\int_{\frac{x+y}{2}}^{x}\left(\frac{v-x}{y-x}\right)^{k-1}z(v)d(v).\end{aligned} \tag{31}$$

Implies

$$z\left(\frac{x+y}{2}\right) = \frac{h\left(e^{1/2}-1\right)2^k\Gamma(k+1)}{(x-y)^k}\cdot\left[{}^{\mathrm{RL}}J^k_{((x+y)/2)^+}z(y)+{}^{\mathrm{RL}}J^k_{((x+y)/2)^-}z(x)\right]. \tag{32}$$

For other inequalities, take

$$\begin{aligned} z\left(\frac{t}{2}x+\frac{2-t}{2}y\right) &\leq h\left(e^{t/2}-1\right)z(x)+h\left(e^{1-(t/2)}-1\right)z(y); \\ z\left(\frac{2-t}{2}x+\frac{t}{2}y\right) &\leq h\left(e^{(1-t)/2}-1\right)z(x)+h\left(e^{t/2}-1\right)z(y). \end{aligned} \tag{33}$$

Adding both inequalities, we have

$$\left(\frac{t}{2}x+\frac{2-t}{2}y\right)+z\left(\frac{2-t}{2}x+\frac{t}{2}y\right)z \leq h\left(e^{t/2}+e^{1-(t/2)}-2\right)[z(x)+z(y)]. \tag{34}$$

Multiplying by t^{k-1} on both sides and integrating on [0,1], we get

$$\int_0^1 t^{k-1}z\left(\frac{t}{2}x+\frac{2-t}{2}y\right)dt+\int_0^1 t^{k-1}z\left(\frac{2-t}{2}x+\frac{t}{2}y\right)dt \leq [z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^{t/2}+e^{1-(t/2)}-2\right)dt. \tag{35}$$

By making the change of variables, we get

$$\begin{aligned} &2^k\left[\frac{1}{x-y}\int_y^{\frac{x+y}{2}}\left(\frac{y-u}{y-x}\right)^{k-1}z(u)d(u)+\frac{1}{x-y}\int_{\frac{x+y}{2}}^{x}\left(\frac{v-x}{y-x}\right)^{k-1}z(v)d(v)\right] \\ &\leq [z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^{t/2}+e^{1-(t/2)}-2\right)dt, \\ &\frac{2^k\Gamma(k)}{(x-y)^k}\left[{}^{\mathrm{RL}}J^k_{((x+y)/2)^+}z(y)+{}^{\mathrm{RL}}J^k_{((x+y)/2)^-}z(x)\right] \\ &\leq [z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^{t/2}+e^{1-(t/2)}-2\right)dt. \end{aligned} \tag{36}$$

Multiplying $k>0$ and $h(e^{1/2}-1)>0$ on both sides, we have

$$\begin{aligned} &\frac{h\left(e^{1/2}-1\right)2^k\Gamma(k+1)}{(x-y)^k}\left[{}^{\mathrm{RL}}J^k_{((x+y)/2)^+}z(y)+{}^{\mathrm{RL}}J^k_{((x+y)/2)^-}z(x)\right] \\ &\leq kh\left(e^{1/2}-1\right)[z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^{t/2}+e^{1-(t/2)}-2\right)dt \\ &\leq h\left(e^{1/2}-1\right)M[z(x)+z(y)]. \end{aligned} \tag{37}$$

□

Corollary 15. *When we introduced $h(e^{1/2}+e^{1-(t/2)}-2)=e^t+e^{1-(t/2)}-2$ in (37) using Remark 7 and rearrange both inequalities (32) and (37), we get (10) of [27].*

Remark 16. For $h(e^{t/2}+e^{1-(t/2)}-2)=1/(e^{t/2}+e^{1-(t/2)}-2)$, (37) yields midpoint type inequalities via the incomplete gamma function for harmonically exponential convex function.

5. He's Inequality via the Incomplete Gamma Functions

He's inequality via the incomplete gamma functions is presented in this section.

Definition 17. For any L^1 function z on interval $[0,s]$, the k-th He's fractional derivative of $z(x)$ is defined by

$$\begin{aligned} D^k_s w(s) &= \frac{1}{\Gamma(n-k)}\frac{d^n}{ds^n}\int_0^s (t-s)^{n-k-1}z(t)dt, \\ z\left(\frac{x+y}{2}\right) &= z\left(\frac{[tx+(1-t)y]+[(1-t)x+ty]}{2}\right). \end{aligned} \tag{38}$$

By using h-exponential convex function

$$z\left(\frac{x+y}{2}\right) \leq h\left(e^{1/2}-1\right)z((tx+(1-t)y) + h\left(e^{1/2}-1\right)z((1-t)x+ty). \tag{39}$$

Taking $x = 0$ and $y > 0$ for all $s \in (0, 1)$ and multiplying by $(t-s)^{n-k-1}/\Gamma(n-k)$, we get

$$\begin{aligned}
&\frac{1}{\Gamma(n-k)} z\left(\frac{y}{2}\right)\int_0^s (t-s)^{n-k-1}dt \\
&\leq \frac{h(e^{1/2}-1)}{\Gamma(n-k)}\left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right)dt \right.\\
&\left.+\int_0^s (t-s)^{n-k-1} z(ty)dt\right]\frac{(-1)^{n-k}s^{n-k}}{\Gamma(n-k)} z\left(\frac{y}{2}\right) \\
&\leq \frac{h(e^{1/2}-1)}{\Gamma(n-k)}\left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right)dt \right.\\
&\left.+\int_0^s (t-s)^{n-k-1} z(ty)dt\right].
\end{aligned} \tag{40}$$

Hence

$$\frac{(-1)^{n-k}s^{n-k}}{\Gamma(n-k)} z\left(\frac{y}{2}\right) \leq \frac{h(e^{1/2}-1)}{\Gamma(n-k)}\left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right)dt + \int_0^s (t-s)^{n-k-1} z(ty)dt\right]. \tag{41}$$

After getting the n-th derivatives on both sides of (41) w.r.t to s and using Definition 17, we get

$$(-1)^{n-k} z\left(\frac{y}{2}\right) \leq \left[D^k_{sb} z(sb) + (-1)^{n-k} D^k_{(1-s)b} z((1-s)b)\right]. \tag{42}$$

Remark 18. By putting $h(e^{1/2}-1) = (e^{1/2}-1)$ in (41), we get He's inequality (14) of [27].

5.1. He's Inequality for Harmonically Exponential Convex Function. From Definition 17 and by using definition of h-exponential convex function, we have

$$\begin{aligned}
z\left(\frac{x+y}{2}\right) &\leq h\left(e^{1/2}-1\right) z((tx+(1-t)y) \\
&+ h\left(e^{1/2}-1\right) z((1-t)x+ty).
\end{aligned} \tag{43}$$

By harmonically exponential convex function, we have

$$z\left(\frac{x+y}{2}\right) \leq \frac{1}{e^{1/2}-1} z((tx+(1-t)y) + \frac{1}{e^{1/2}-1} z((1-t)x+ty). \tag{44}$$

Taking $x = 0$ and $y > 0$ for all $s \in (0, 1)$, multiplying by $(t-s)^{n-k-1}/\Gamma(n-k)$

$$\begin{aligned}
&\frac{1}{\Gamma(n-k)} z\left(\frac{y}{2}\right)\int_0^s (t-s)^{n-k-1}dt \\
&\leq \frac{1}{(e^{1/2}-1)\Gamma(n-k)}\left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right)dt \right.\\
&\left.+\int_0^s (t-s)^{n-k-1} z(ty)dt\right]\frac{(-1)^{n-k}s^{n-k}}{\Gamma(n-k)} z\left(\frac{y}{2}\right) \\
&\leq \frac{1}{(e^{1/2}-1)\Gamma(n-k)}\left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right)dt \right.\\
&\left.+\int_0^s (t-s)^{n-k-1} z(ty)dt\right].
\end{aligned} \tag{45}$$

Hence

$$\begin{aligned}
\frac{(-1)^{n-k}s^{n-k}}{\Gamma(n-k)} z\left(\frac{y}{2}\right) &\leq \frac{1}{(e^{1/2}-1)\Gamma(n-k)} \\
&\cdot\left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right)dt \right.\\
&\left.+\int_0^s (t-s)^{n-k-1} z(ty)dt\right].
\end{aligned} \tag{46}$$

After getting the n-th derivatives on both sides of (41) w.r.t to s and using Definition 17, we get

$$(-1)^{n-k} z\left(\frac{y}{2}\right) \leq \left[D^k_{sb} z(sb) + (-1)^{n-k} D^k_{(1-s)b} z((1-s)b)\right]. \tag{47}$$

Remark 19. For $1/(e^{1/2}-1) = h(e^{1/2}-1)$ in (47), we get (41). Again substitute $h(e^{1/2}-1) = e^t - 1$ in (41), we get He's inequality (14) of [27].

6. Conclusion

The inequalities in analysis play a vital role to study qualitative properties of functions and solutions of differential equations; we develop various Hermite-Hadamard type inequalities, midpoint inequalities, and trapezoidal and He's inequalities for$h$-exponential convex functions with appropriate substitutions; we may obtain the inequalities for harmonically$h$-exponential convex functions. To establish same inequalities for h-exponential convex stochastic processes is an interesting problem.

Data Availability

There is no additional data required for the finding of results of this paper.

Conflicts of Interest

It is declared that authors have no competing interests.

Authors' Contributions

All authors have equal contribution in this article.

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References

[1] C. Niculescu and L. E. Persson, *Convex Functions and Their Applications (pp. xvi+-255)*, Springer, New York, 2006.

[2] A. W. Roberts, "Convex Functions," in *Handbook of convex geometry*, pp. 1081–1104, Elsevier, North-Holland, 1993.

[3] M. Z. Sarikaya, A. Saglam, and H. Yildirim, "On some Hadamard-type inequalities for h-convex functions," *Journal of Mathematical Inequalities*, vol. 2, no. 3, pp. 335–341, 2008.

[4] P. M. Gill, C. E. M. Pearce, and J. Pecaric, "Hadamard's inequality for r-convex functions," *Journal of Mathematical Analysis and Applications*, vol. 215, no. 2, pp. 461–470, 1997.

[5] I. Abbas Baloch and Y. M. Chu, "Petrović-type inequalities for Harmonic -convex functions," *Journal of Function Spaces*, vol. 2020, 7 pages, 2020.

[6] M. U. Awan, M. A. Noor, and K. I. Noor, "Hermite-Hadamard inequalities for exponentially convex functions," *Applied Mathematics & Information Sciences*, vol. 12, no. 2, pp. 405–409, 2018.

[7] M. U. Awan, M. A. Noor, M. V. Mihai, K. I. Noor, and N. Akhtar, "On approximately harmonic h-convex functions depending on a given function," *Univerzitet u Nišu*, vol. 33, no. 12, pp. 3783–3793, 2019.

[8] I. A. Baloch, S. S. Dragomir, and S. S. Dragomir, "New inequalities based on harmonic log-convex functions," *Open Journal of Mathematical Analysis*, vol. 3, no. 2, pp. 103–105, 2019.

[9] I. Iscan, "Hermite-Hadamard type inequalities for harmonically convex functions," *Hacettepe Journal of Mathematics and statistics*, vol. 43, no. 6, pp. 935–942, 2014.

[10] R. F. Bai, F. Qi, and B. Y. Xi, "Hermite-Hadamard type inequalities for the m-and (α, m) – logarithmically convex functions," *Univerzitet u Nišu*, vol. 27, no. 1, pp. 1–7, 2013.

[11] M. E. Özdemir, M. Avci, and H. Kavurmaci, "Hermite–Hadamard-type inequalities via (α, m) – convexity," *Computers & Mathematics with Applications*, vol. 61, no. 9, pp. 2614–2620, 2011.

[12] Y. M. Chu, M. A. Khan, T. U. Khan, and T. Ali, "Generalizations of Hermite-Hadamard type inequalities for MT-convex functions," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 5, pp. 4305–4316, 2016.

[13] M. Kadakal and I. Iscan, "Exponential type convexity and some related inequalities," *Journal of Inequalities and Applications*, vol. 2020, Article ID 82, 2020.

[14] Y. Zhao, H. Sang, and W. Xiong, "Hermite-Hadamard-type inequalities involving the Riemann-Liouville fractional integrals via h-convex functions," *Journal of inequalities and Applications*, vol. 2020, 2020.

[15] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, "Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of hadamard inequalities," *IEEE access*, vol. 6, pp. 64946–64953, 2018.

[16] X. Z. Yang, G. Farid, W. Nazeer, Y. M. Chu, and C. F. Dong, "Fractional generalized Hadamard and Fejér-Hadamard inequalities for m-convex function," *AIMS Math*, vol. 5, no. 6, pp. 6325–6340, 2020.

[17] A. Adu-Sackey, F. T. Oduro, F. T. Oduro, and G. O. Fosu, "Inequalities approach in determination of convergence of recurrence sequences," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 65–72, 2021.

[18] G. Farid, K. Mahreen, and Y. M. Chu, "Study of inequalities for unified integral operators of generalized convex functions," *Open Journal of Mathematical Sciences*, vol. 5, pp. 80–93, 2021.

[19] M. Riesz, "L'intégrale de Riemann-Liouville et le problème de Cauchy," *Acta Mathematica*, vol. 81, pp. 1–222, 1949.

[20] M. Caputo and M. Fabrizio, "On the notion of fractional derivative and applications to the hysteresis phenomena," *Meccanica*, vol. 52, no. 13, pp. 3043–3052, 2017.

[21] D. Baleanu, A. Kashuri, P. O. Mohammed, and B. Meftah, "General Raina fractional integral inequalities on coordinates of convex functions," *Advances in Difference Equations*, vol. 2021, Article ID 82, 2021.

[22] M. B. Khan, P. O. Mohammed, M. A. Noor, and Y. S. Hamed, "New Hermite–Hadamard inequalities in fuzzy-interval fractional calculus and related inequalities," *Symmetry*, vol. 13, no. 4, p. 673, 2021.

[23] B. Meftah, A. Souahi, and A. Souahi, "Cebyšev inequalities for co-ordinated \(QC\)-convex and \((s,QC)\)-convex," *Engineering and Applied Science Letters*, vol. 4, no. 1, pp. 14–20, 2021.

[24] G. Farid, A. U. Rehman, S. Bibi, and Y. M. Chu, "Refinements of two fractional versions of Hadamard inequalities for Caputo fractional derivatives and related results," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 1–10, 2021.

[25] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

[26] M. Z. Sarikaya and H. Yildirim, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals," *Miskolc Mathematical Notes*, vol. 17, no. 2, pp. 1049–1059, 2016.

[27] P. O. Mohammed, T. Abdeljawad, D. Baleanu, A. Kashuri, F. Hamasalh, and P. Agarwal, "New fractional inequalities of Hermite–Hadamard type involving the incomplete gamma functions," *Journal of Inequalities and Applications*, vol. 2020, Article ID 263, 2020.

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Research Article

Generating Functions for Some Hypergeometric Functions of Four Variables via Laplace Integral Representations

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Generating functions plays an essential role in the investigation of several useful properties of the sequences which they generate. In this paper, we establish certain generating relations, involving some quadruple hypergeometric functions introduced by Bin-Saad and Younis. Some interesting special cases of our main results are also considered.

1. Introduction

The hypergeometric series is the most useful and important special function, and it has been studied to solve various problems in many areas of mathematics, physics, statistics, and engineering [1–5]. Hypergeometric series in several variables appear in numerous fields of applied mathematics, mathematical physics, and chemistry. Very recently, Bin-Saad and Younis [6] introduced thirty new hypergeometric functions of four variables $X_i^{(4)}(i = 1, 2, \cdots, 30)$, eight of them are defined below

$$X_{11}^{(4)}(a_1, a_1, a_1, a_2, a_1, a_2, a_3, a_3 ; c_1, c_1, c_2, c_3 ; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} ; \quad (1)$$

$$X_{12}^{(4)}(a_1, a_1, a_1, a_2, a_1, a_2, a_3, a_3 ; c_2, c_1, c_1, c_3 ; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} ; \quad (2)$$

$$X_{16}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2 ; c_1, c_1, c_2, c_3 ; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{n+q}(a_3)_p}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} ; \quad (3)$$

$$X_{17}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2 ; c_2, c_1, c_1, c_3 ; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{n+q}(a_3)_p}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} ; \quad (4)$$

$$X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2 ; c_1, c_1, c_2, c_3 ; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{n+p+q}(a_3)_p}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} ; \quad (5)$$

$$X_{22}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2 ; c_2, c_1, c_1, c_3 ; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{n+p+q}(a_3)_p}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} ; \quad (6)$$

$$X_{27}^{(4)}(a_1, a_1, a_3, a_1, a_1, a_2, a_4, a_4 ; c_2, c_1, c_1, c_3 ; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_n(a_3)_p(a_4)_{p+q}}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} ; \quad (7)$$

$$X_{28}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_4 ; c_1, c_1, c_2, c_3 ; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_n(a_3)_p(a_4)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} ; \quad (8)$$

for

$$\left(|x| < \frac{1}{4}, |y| < 1, |z| < 1, |u| < 1\right). \tag{9}$$

Here, $(a)_n$ corresponds to the Pochhammer symbol given as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma a} = a(a+1)(a+2)\cdots(a+n-1), n \in \mathbb{N}, \tag{10}$$

and $(a)_0 = 1$. Given the following integral representations:

$$\begin{aligned} &X_{11}^{(4)}(a_1, a_1, a_1, a_2, a_1, a_2, a_3, a_3; c_1, c_1, c_2, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1} \\ &\times {}_0F_1\left(-;c_1;s^2x+sty\right)\Psi_2(a_3;c_2,c_3;sz,tu)dsdt, \\ &\cdot(\Re(a_1)>0, \Re(a_2)>0); \end{aligned} \tag{11}$$

$$\begin{aligned} &X_{12}^{(4)}(a_1, a_1, a_1, a_2, a_1, a_2, a_3, a_3; c_2, c_1, c_1, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_2)}\frac{1}{\Gamma(a_3)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_2-1}t^{a_3-1} \\ &\times H_7(a_1;c_2,c_1;x,sy+tz)\,{}_0F_1(-;c_3;stu)dsdt, \\ &\cdot(\Re(a_2)>0, \Re(a_3)>0); \end{aligned} \tag{12}$$

$$\begin{aligned} &X_{16}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1} \\ &\times {}_0F_1\left(-;c_1;s^2x+sty\right){}_1F_1(a_3;c_2;sz)\,{}_0F_1(-;c_3;stu)dsdt, \\ &\cdot(\Re(a_1)>0, \Re(a_2)>0); \end{aligned} \tag{13}$$

$$\begin{aligned} &X_{17}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1} \\ &\times \Phi_3(a_3;c_1;sz,sty)\,{}_0F_1\left(-;c_2;s^2x\right){}_0F_1(-;c_3;stu)dsdt, \\ &\cdot(\Re(a_1)>0, \Re(a_2)>0); \end{aligned} \tag{14}$$

$$\begin{aligned} &X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1} \\ &\times {}_0F_1\left(-;c_1;s^2x+sty\right){}_1F_1(a_3;c_2;tz)\,{}_0F_1(-;c_3;stu)dsdt, \\ &\cdot(\Re(a_1)>0, \Re(a_2)>0); \end{aligned} \tag{15}$$

$$\begin{aligned} &X_{22}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_3)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_3-1} \\ &\times \Psi_2(a_2;c_1,c_3;sy+tz,su)\,{}_0F_1\left(-;c_2;s^2x\right)dsdt, \\ &\cdot(\Re(a_1)>0, \Re(a_3)>0); \end{aligned} \tag{16}$$

$$\begin{aligned} &X_{27}^{(4)}(a_1, a_1, a_3, a_1, a_1, a_2, a_4, a_4; c_2, c_1, c_1, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_4)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_4-1} \\ &\times \Phi_2(a_2,a_3;c_1;sy,tz)\,{}_0F_1\left(-;c_2;s^2x\right){}_0F_1(-;c_3;stu)dsdt, \\ &\cdot(\Re(a_1)>0, \Re(a_4)>0); \end{aligned} \tag{17}$$

$$\begin{aligned} &X_{28}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_4; c_1, c_1, c_2, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_1)}\int_0^\infty e^{-s}s^{a_1-1}\times\Phi_3\left(a_2;c_1;sy,s^2x\right){}_1F_1 \\ &\cdot(a_3;c_2;sz)\,{}_1F_1(a_4;c_3;su)ds, (\Re(a_1)>0), \end{aligned} \tag{18}$$

where ${}_0F_1, {}_1F_1$ are Kummer's functions and $\Phi_2, \Phi_3, \Psi_2, H_7$ are Humbert functions defined, respectively, by (see [7])

$${}_0F_1(-;b;x) = \sum_{n=0}^\infty \frac{1}{(b)_n}\frac{x^n}{n!}; \tag{19}$$

$${}_1F_1(a;b;x) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n}\frac{x^n}{n!}; \tag{20}$$

$$\Phi_2(a,b;c;x,y) = \sum_{m,n=0}^\infty \frac{(a)_m(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}; \tag{21}$$

$$\Phi_3(b;c;x,y) = \sum_{m,n=0}^\infty \frac{(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}; \tag{22}$$

$$\Psi_2(a;b,c;x,y) = \sum_{m,n=0}^\infty \frac{(a)_{m+n}}{(b)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!}; \tag{23}$$

$$H_7(a;b,c;x,y) = \sum_{m,n=0}^\infty \frac{(a)_{2m+n}}{(b)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{24}$$

Several families of generating functions have been established in diverse ways. These are playing important roles in the theory of special functions of applied mathematics and mathematical physics. One can refer to the extensive work of Srivastava and Manocha [8] for a systematic introduction and to several interesting and useful applications of the various methods of obtaining linear, bilinear, bilateral, or mixed multilateral generating functions for a fairly wide variety of sequences of hypergeometric functions and polynomials in one, two, or more variables, among much abundant literature. Many authors have been presented various generating functions in many different ways (see, for details, [9–11] and the references cited therein). In this paper, we aim at establishing some generating functions for the quadruple functions $X_i^{(4)}(i=11,12,16,17,21,22,27,28)$.

2. Generating Relations

Here, by using the integral representations in (11)–(18), we give certain generating relations involving hypergeometric functions of three and four variables as follows:

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{11}^{(4)}(a_1+k,a_1+k,a_1+k,a_2+k,a_1+k,a_2+k,c_2,c_2;c_1,c_1,c_2,c_2;x,y,z,u)=(1-z)^{-a_1}(1-u)^{-a_2}\cdot\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1-z)(1-u)}\right)^k\times X_3\left(a_1+k,a_2+k;c_1,c_2;\frac{x}{(1-z)^2},\cdot\frac{y}{(1-z)(1-u)},\frac{zu}{(1-z)(1-u)}\right); \tag{25}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{12}^{(4)}(a_1,a_1,a_1,a_2+k,a_1,a_2+k,a_3+k,a_3+k;c_2,c_1,c_1,c_3;x,y,z,u\tau)=(1+u)^{-a_2}(1+\tau)^{-a_3}\cdot\sum_{k,m=0}^{\infty}\frac{(a_1)_{2m}}{(c_2)_m k!m!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k x^m\times F_{28}^{(4)}\cdot(a_1+2m,a_1+2m,c_3,c_3,a_2+k,a_3+k,a_2+k,a_3+k;c_1,c_1,c_3,c_3;\lambda_1 y,\lambda_2 z,\lambda_1 u,\lambda_2\tau),\cdot\left(\lambda_1=\frac{1}{1+u},\lambda_2=\frac{1}{1+\tau}\right); \tag{26}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{16}^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,a_3,a_2+k;c_1,c_1,c_2,c_3;x,y,z,u)=(1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k\times X_{16}^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,c_2-a_3,a_2+k;c_1,c_1,c_2,c_3;\lambda^2x,\lambda y-\lambda z,\lambda u),\left(\lambda=\frac{1}{1-z}\right); \tag{27}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{17}^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,a_3,a_2+k;c_2,c_1,c_1,c_3;x^2,y,z,u)=(1+2x)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1+2x}\right)^k\times K_8\left(a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,a_2+k,a_3,c_2-\frac{1}{2};c_1,c_3,c_1,2c_2-1;\lambda y,\lambda u,\lambda z,4\lambda x\right),\left(\lambda=\frac{1}{1+2x}\right); \tag{28}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{21}^{(4)}(a_1+k,a_1+k,a_2+k,a_1+k,a_1+k,a_2+k,a_3,a_2+k;c_1,c_1,c_2,c_3;x,y,z,u\tau)=(1+u)^{-a_1}(1+\tau-z)^{-a_2}\cdot\sum_{k,p=0}^{\infty}\frac{(a_2+k)_p(c_2-a_3)_p}{(c_2)_p k!p!}\left(\frac{w}{(1+u)(1+\tau-z)}\right)^k\left(\frac{z}{z-\tau-1}\right)^p\times X_{11}^{(4)}(a_1+,a_1+k,a_1+k,a_2+k+p,a_1+k,a_2+k+p,c_3,c_3;c_1,c_1,c_3,c_3;\lambda_1^2x,\lambda_2 y,\lambda_1 u,\lambda_3\tau),\cdot\left(\lambda_1=\frac{1}{1+u},\lambda_2=\frac{1}{(1+u)(1+\tau-z)},\lambda_3=\frac{1}{1+\tau-z}\right); \tag{29}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{22}^{(4)}(a_1+k,a_1+k,a_2,a_1+k,a_1+k,a_2,a_3+k,a_2;c_2,c_1,c_1,c_3;x^2,y,z,u)=(1+2x)^{-a_1}\cdot\sum_{k,q=0}^{\infty}\frac{(a_1+k)_q(a_2)_q}{(c_3)_q k!q!}\left(\frac{w}{1+2x}\right)^k\left(\frac{u}{1+2x}\right)^q\times F_M\left(c_2-\frac{1}{2},a_2+q,a_2+q,a_1+k+q,a_3+k,a_1+k+q;2c_2-1,c_1,c_1;\frac{4x}{1+2x},z,\frac{y}{1+2x}\right); \tag{30}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{27}^{(4)}(a_1+k,a_1+k,a_3,a_1+k,a_1+k,a_2,a_4+k,a_4+k;c_2,c_1,c_1,c_3;x^2,y,z,u)=(1+2x)^{-a_1}\cdot\sum_{k,q=0}^{\infty}\frac{(a_1+k)_q(a_4+k)_q}{(c_3)_q k!q!}\left(\frac{w}{1+2x}\right)^k\left(\frac{u}{1+2x}\right)^q\times F_N\left(c_2-\frac{1}{2},a_3,a_2,a_1+k+q,a_4+k+q,a_1+k+q;2c_2-1,c_1,c_1;\frac{4x}{1+2x},z,\frac{y}{1+2x}\right); \tag{31}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{28}^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u)=(1-z-u)^{-a_1}\cdot\sum_{k,n=0}^{\infty}\frac{(a_1+k)_n(a_2)_n}{(c_1)_n k!n!}\left(\frac{w}{1-z-u}\right)^k\left(\frac{y}{1-z-u}\right)^n\times X_8\left(a_1+k+n,c_2-a_3,c_3-a_4;c_1+n,c_2,c_3;\cdot\frac{x}{(1-z-u)^2},\frac{z}{z+u-1},\frac{u}{z+u-1}\right); \tag{32}$$

where X_3, X_8 are the Exton functions of three variables [12], Lauricella functions F_M, F_N [13], Exton function of four variables K_8 [14], and Sharma and Parihar function $F_{28}^{(4)}$ [15] are defined, respectively, by

$$X_3(a_1, a_2; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_{n+p}}{(c_1)_{m+n}(c_2)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}; \tag{33}$$

$$X_8(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}; \tag{34}$$

$$F_M(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_{n+p}(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}; \tag{35}$$

$$F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}; \tag{36}$$

$$K_8(a_1, a_1, a_1, a_1, b_1, b_1, b_2, b_3; c_1, c_2, c_1, c_3; x, y, z) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p+q}(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}; \tag{37}$$

$$F_{28}^{(4)}(a_1, a_1, a_2, a_2, b_1, b_2, b_1, b_2; c_1, c_1, c_2, c_3; x, y, z) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{p+q}(b_1)_{m+p}(b_2)_{n+q}}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}. \tag{38}$$

Proof. To prove the above equations, we require the following results (see, e.g., [7, 16, 17]):

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m+n) \frac{x^m}{m!} \frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!}; \tag{39}$$

$$\Gamma(z) = s^z \int_0^{\infty} e^{-st} t^{z-1} dt, \Re(z) > 0; \tag{40}$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma a}, a \neq 0, -1, -2, \cdots; \tag{41}$$

$$(a)_{m+n} = (a)_m (a+m)_n; \tag{42}$$

$${}_0F_1(-; a; x^2) = e^{-2x} {}_1F_1\left(a - \frac{1}{2}; 2a-1; 4x\right); \tag{43}$$

$${}_1F_1(a; b; x) = e^x {}_1F_1(b-a; b; -x); \tag{44}$$

$$\Psi_2(c, c; c, x, y) = \exp(x+y) {}_0F_1(-; c; xy). \tag{45}$$

For convenience and simplicity, by denoting the left-hand side of (25) by δ, and using (11), one gets

$$\delta = \sum_{k=0}^{\infty} \frac{w^k}{k! \Gamma(a_1+k) \Gamma(a_2+k)} \int_0^{\infty} \int_0^{\infty} s^{a_1+k-1} t^{a_2+k-1} \times {}_0F_1(-; c_1; s^2 x + sty) \Psi_2(c_2; c_2, c_2; sz, tu) ds dt. \tag{46}$$

In view of (40) and (46), we have

$$\delta = \sum_{k,m,n=0}^{\infty} \frac{w^k x^m y^n}{(c_1)_{m+n} k! m! n! \Gamma(a_1+k) \Gamma(a_2+k)} \int_0^{\infty} \int_0^{\infty} e^{-s(1-z)} e^{-t(1-u)} \times s^{a_1+k+2m+n-1} t^{a_2+k+n-1} {}_0F_1(-; c_2; stzu) ds dt. \tag{47}$$

The function ${}_0F_1$ which appears in the above equation can be replaced by its series form and then interchanging the order of the summation and integral sign which is permissible here, we get

$$\delta = \sum_{k,m,n,p=0}^{\infty} \frac{w^k x^m y^n (zu)^p}{(c_1)_{m+n} (c_2)_p k! m! n! p! \Gamma(a_1+k) \Gamma(a_2+k)} \times \int_0^{\infty} \int_0^{\infty} e^{-s(1-z)} e^{-t(1-u)} s^{a_1+k+2m+n+p-1} t^{a_2+k+n+p-1} ds dt. \tag{48}$$

Now, the use of (41) and (42) in above equation and the simplification with series manipulation completes the proof of relation (25).

The proof of all remaining relations runs in the same way, considering the appropriate integral representation and Laplace transform during the proof. □

3. Special Cases

Here, we shall consider several interesting special cases of our main results stated in the previous section.

Putting $k = 0$ in equations (25) to (28), we obtain the following relations:

$$X_{11}^{(4)}(a_1, a_1, a_1, a_2, a_1, a_2, c_2, c_2; c_1, c_1, c_2, c_2; x, y, z, u) = (1-z)^{-a_1} (1-u)^{-a_2} \times X_3\left(a_1, a_2; c_1, c_2; \frac{x}{(1-z)^2}; \frac{y}{(1-z)(1-u)}, \frac{zu}{(1-z)(1-u)}\right); \tag{49}$$

$$X_{12}^{(4)}(a_1, a_1, a_1, a_2, a_1, a_2, a_3, a_3; c_2, c_1, c_1, c_3; x, y, z, u\tau) = (1+u)^{-a_2} (1+\tau)^{-a_2} \sum_{m=0}^{\infty} \frac{(a_1)_{2m}}{(c_2)_m m!} x^m \times F_{28}^{(4)}(a_1 + 2m, a_1 + 2m, c_3, c_3, a_2, a_3, a_2, a_3; c_1, c_1, c_3, c_3; \frac{y}{1+u}, \frac{z}{1+\tau}, \frac{u}{1+u}, \frac{\tau}{1+\tau}); \tag{50}$$

$$X_{16}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u) = (1-z)^{-a_1} X_{16}^{(4)} \cdot \left(a_1, a_1, a_1, a_1, a_1, a_2, c_2, -a_3, a_2; c_1, c_1, c_2, c_3; \frac{x}{(1-z)^2}, \frac{y}{1-z}, \frac{z}{z-1}, \frac{u}{1-z}\right); \tag{51}$$

$$X_{17}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x^2, y, z, u) = (1+2x)^{-a_1} \times K_8\left(a_1, a_1, a_1, a_1, a_2, a_2, a_3, c_2 - \frac{1}{2}; c_1, c_3, c_1, 2c_2 - 1; \lambda y, \lambda u, \lambda z, 4\lambda x\right), \cdot \left(\lambda = \frac{1}{1+2x}\right). \tag{52}$$

Equations (49), (51), and (52) with $u=0$ yield the Exton's results (see [12]).

Now, if we take $x=0$ in (25) to (27) and (29), and simplification, we shall obtain the following generating relations:

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}H_B(a_1+k,a_2+k,c_2;c_1,c_3,c_2;y,u,z) = (1-z)^{-a_1}(1-u)^{-a_2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1-z)(1-u)}\right)^k \times F_4\left(a_1+k,a_2+k;c_1,c_2;\frac{y}{(1-z)(1-u)},\frac{zu}{(1-z)(1-u)}\right); \tag{53}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}H_A(a_2+k,a_3+k,a_1;c_3,c_1;u\tau,y,z) = (1+u)^{-a_2}(1+\tau)^{-a_3}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1+\tau)(1+u)}\right)^k \times F_{28}^{(4)}\left(a_1,a_1,c_3,c_3,a_2+k,a_3+k,a_2+k,a_3+k;c_1,c_1,c_3,c_3;\frac{y}{1+u},\frac{z}{1+\tau},\frac{u}{1+u},\frac{\tau}{1+\tau}\right); \tag{54}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}F_E(a_1+k,a_1+k,a_1+k,a_3,a_2+k,a_2+k;c_2,c_1,c_3;z,y,u) = (1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k \times F_E(a_1+k,a_1+k,a_1+k,c_2-a_3,a_2+k,a_2+k;c_2,c_1,c_3;\frac{z}{z-1},\frac{y}{1-z},\frac{u}{1-z}); \tag{55}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}F_E(a_2+k,a_2+k,a_2+k,a_3,a_1+k,a_1+k;c_2,c_1,c_3;z,y,u\tau) = (1+u)^{-a_1}(1+\tau-z)^{-a_2}\sum_{k,p=0}^{\infty}\frac{(a_2+k)_p(c_2-a_3)_p}{(c_2)_p\,k!p!} \cdot\left(\frac{w}{(1+u)(1+\tau-z)}\right)^k\left(\frac{z}{z-\tau-1}\right)^p \times H_B\left(a_1+k,a_2+k+p,c_3;c_1,c_3,c_3;\lambda_1 y,\lambda_2\tau,\frac{u}{u+1}\right), \cdot\left(\lambda_1=\frac{1}{(1+u)(1+\tau-z)},\lambda_2=\frac{1}{1+\tau-z}\right). \tag{56}$$

Note that the special cases of each of the above generating relations can be easily derived by assigning the value zero to k. For example,

$$H_B(a_1,a_2,c_2;c_1,c_2;y,u,z) = (1-z)^{-a_1}(1-u)^{-a_2}F_4 \cdot\left(a_1,a_2;c_1,c_2;\frac{y}{(1-z)(1-u)},\frac{zu}{(1-z)(1-u)}\right), \tag{57}$$

which, when $a_2=c_3$, yields [[7], pp. 309 (125)].

Another interesting special case of (56) occurs when we set $u=0$. We thus find that

$$F_2(a_1,a_2,c_2;c_1,c_2;y,z) = (1-z)^{-a_1}{}_2F_1\left(a_1,a_2;c_1;\frac{y}{(1-z)}\right), \tag{58}$$

which is due to Srivastava and Karlsson [7].

Also,

$$H_A(a_2,a_3,a_1;c_3,c_1;u\tau,y,z) = (1+u)^{-a_2}(1+\tau)^{-a_3}F_{28}^{(4)} \cdot\left(a_1,a_1,c_3,c_3,a_2,a_3,a_2,a_3;c_1,c_1,c_3,c_3;\frac{y}{1+u},\frac{z}{1+\tau},\frac{u}{1+u},\frac{\tau}{1+\tau}\right), \tag{59}$$

which, for $y=z=0$, $x=u/1+u$ and $y=\tau/1+\tau$, yields the well-known result [7].

Furthermore, by setting $k=0$ and $y=0$ in (25) and (32), we obtain the Exton's results [12].

Finally, if in (29), we let $z=0$, we shall obtain generating relation between Exton's series X_3 and the quadruple hypergeometric series $X_{11}^{(4)}$:

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_3(a_1+k,a_2+k;c_1,c_3;x,y,u\tau) = (1+u)^{-a_1}(1+\tau)^{-a_2} \cdot\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k \times X_{11}^{(4)}(a_1+k,a_1+k,a_1+k,a_2+k,a_1+k,a_2+k,c_3,c_3;c_1,c_1,c_3,c_3;\lambda_1^2x,\lambda_1\lambda_2y,\lambda_1u,\lambda_2\tau), \cdot\left(\lambda_1=\frac{1}{1+u},\lambda_2=\frac{1}{1+\tau}\right). \tag{60}$$

Formula (61), with $x=0$, yields the generating relation

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}F_4(a_1+k,a_2+k;c_1,c_3;y,u\tau) = (1+u)^{-a_1}(1+\tau)^{-a_2} \cdot\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k \times H_B\left(a_1+k,a_2+k,c_3;c_1,c_3,c_3; \cdot\frac{y}{(1+u)(1+\tau)},\frac{\tau}{1+\tau},\frac{u}{1+u}\right). \tag{61}$$

For $k=0$, we have the elegant transformation

$$F_4(a_1,a_2;c_1,c_3;y,u\tau) = (1+u)^{-a_1}(1+\tau)^{-a_2}H_B \cdot\left(a_1,a_2,c_3;c_1,c_3,c_3;\frac{y}{(1+u)(1+\tau)},\frac{\tau}{1+\tau},\frac{u}{1+u}\right); \tag{62}$$

where ${}_2F_1$, F_2, F_4, H_A, H_B, and F_E are the Gaussian hypergeometric function, Appell's functions, Srivastava's functions, and Lauricella function defined, respectively, by (see [7])

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}; \tag{63} $$

$$ F_2(a, b, c; d, e; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (c)_n}{(d)_m (e)_n} \frac{x^m}{m!} \frac{y^n}{n!}; \tag{64} $$

$$ F_4(a, b; c, d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!}; \tag{65} $$

$$ H_A(a_1, b_1, b_2; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p} (b_1)_{m+n} (b_2)_{n+p}}{(c_1)_m (c_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}; \tag{66} $$

$$ H_B(a_1, b_1, b_2; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p} (b_1)_{m+n} (b_2)_{n+p}}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}; \tag{67} $$

$$ \begin{aligned} &F_E(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_3; x, y, z) \\ &\quad = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p} (b_1)_m (b_2)_{n+p}}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}. \end{aligned} \tag{68} $$

4. Conclusion

Based on the integral representations for quadruple hypergeometric functions (1)–(8), we obtained certain generating functions for these functions. Some particular cases and the consequences of our main results are also considered. We concluded this investigation by remarking that the scheme suggested in the derivation of the results can be applied to find other new generating functions for other quadruple hypergeometric functions and study their special cases.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

[1] L. C. Andrews, *Special Functions for Engineers and Applied Mathematicians*, Macmillan Publishing Company, New York, 1985.

[2] N. N. Lebedev, *Special Functions and Their Applications*, vol. 18, no. 12, 1965, Prentice-Hall, INC, printed in USA, 1965.

[3] A. Saboor, G. Rahman, Z. Anjum, K. S. Nisar, and S. Araci, "A new extension of Srivastava's triple hypergeometric functions and their associated properties," *Analysis*, vol. 41, no. 1, pp. 13–24, 2021.

[4] J. A. Younis and M. G. Bin-Saad, "Integral representations involving new hypergeometric functions of four variables," *Journal of Fractional Calculus and Applications*, vol. 10, pp. 77–91, 2019.

[5] J. A. Younis and M. G. Bin-Saad, "Integral representations and operational relations involving some quadruple hypergeometric functions," *Journal of Fractional Calculus and Applications*, vol. 11, pp. 62–74, 2020.

[6] M. G. Bin-Saad and J. A. Younis, "On connections between certain class of new quadruple and known triple hypergeometric series," *Tamap Journal of Mathematics and Statistics*, 2019.

[7] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Ltd., Chichester, 1984.

[8] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press, Bristone, London, New York and Toronto, 1985.

[9] P. Agarwal, J. A. Younis, and T. Kim, "Certain generating functions for the quadruple hypergeometric series K_{10}," *Notes on Number Theory and Discrete Mathematics*, vol. 25, no. 4, pp. 16–23, 2019.

[10] J. Choi and P. Agarwal, "Certain generating functions involving Appell series," *Far East Journal of Mathematical Sciences*, vol. 84, pp. 25–32, 2014.

[11] M. Ali Ozarslan and E. Ozergin, "Some generating relations for extended hypergeometric function via generalized fractional derivative operator," *Mathematical and Computer Modelling*, vol. 52, pp. 1825–1833, 2010.

[12] H. Exton, "Hypergeometric functions of three variables," *Indian Academy of Mathematics*, vol. 4, pp. 113–119, 1982.

[13] G. Lauricella, "Sulle funzioni ipergeometriche a piu variabili," *Rendiconti del Circolo Matematico di Palermo*, vol. 7, no. S1, pp. 111–158, 1893.

[14] H. Exton, *Multiple Hypergeometric Functions and Applications*, Halsted Press, New York, London, Sydney and Toronto, 1976.

[15] C. Sharma and C. L. Parihar, "Hypergeometric functions of four variables (I)," *Indian Academy of Mathematics*, vol. 11, pp. 121–133, 1989.

[16] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[17] E. D. Rainville and S. Functions, *The Macmillan Company: New York, NY, USA, 1960*, Reprinted by Chelsea publishing company, Bronx, NY, USA, 1971.

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Research Article

Notes on Solutions for Some Systems of Complex Functional Equations in $\mathbb{C}^2$

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The purpose of this article is to give the details of finding the transcendental entire solutions with finite order for the systems of nonlinear partial differential-difference equations $\begin{cases} (((\partial f_1(z_1,z_2))/\partial z_1) + ((\partial f_1(z_1,z_2))/\partial z_2))^{n_1} + P_1(z) f_2(z_1+c_1, z_2+c_2)^{m_1} = Q_1(z), \\ (((\partial f_1(z_1,z_2))/\partial z_1) + ((\partial f_1(z_1,z_2))/\partial z_2))^{n_2} + P_2(z) f_1(z_1+c_1, z_2+c_2)^{m_2} = Q_2(z), \end{cases}$ where $P_1(z), P_2(z), Q_1(z),$ and $Q_2(z)$ are polynomials in $\mathbb{C}^2$; $n_1, n_2, m_1,$ and m_2 are positive integers, and $c = (c_1, c_2) \in \mathbb{C}^2$. We obtain that there exist some pairs of the transcendental entire solutions of finite order for the above system, which is a very powerful supplement to the previous theorems given by Xu and Cao and Xu and Yang.

1. Introduction

In 1970, Yang [1] proved that the functional equations $f^n + g^m = 1$ have no nonconstant entire solutions, if m, n are positive integers satisfying $(1/m) + (1/n) < 1$. After this result, with the aid of the Nevanlinna theory and the difference analogues of the Nevanlinna theory (see [2–6]), there were rapid developments on complex differential and difference equations in one and several complex variables. Some classical results and topics in different fields are considered in difference versions, for example, difference Riccati equations, difference Painlevé equations, and difference Fermat equations (see [7–14]). Recently, Cao and Xu [15–17] investigated the existence of the entire and meromorphic solutions for some Fermat-type partial differential-difference equations by utilizing the Nevanlinna theory and difference Nevanlinna theory of several complex variables [18, 19] and obtained the following theorems which is an extension of the previous results given by Liu and his collaborators (see [20–24]).

Theorem 1 (see ([16], Theorem 1.1)). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then, the Fermat-type partial differential-difference equation*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^n + f(z_1 + c_1, z_2 + c_2)^m = 1, \quad (1)$$

does not have any transcendental entire solution with finite order, where m and n are two distinct positive integers.

Theorem 2 (see ([15], Theorem 3.2)). *Let $c = (c_1, c_2) \in \mathbb{C} \setminus \{0\}$. Suppose that f is a nontrivial meromorphic solution of the Fermat type partial difference equations*

$$\frac{1}{f(z_1 + c_1, z_2 + c_2)^m} + \frac{1}{f(z_1, z_2)^m} = A(z_1, z_2) f(z_1, z_2)^n, \quad (2)$$

or

$$\frac{1}{f(z_1+c_1,z_2+c_2)^m}+\frac{1}{f(z_1+c_1,z_2)^m}+\frac{1}{f(z_1,z_2+c_2)^m}=A(z_1,z_2)f(z_1,z_2)^n, \quad (3)$$

where $m \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}$, and $A(z_1, z_2)$ is a nonzero meromorphic function on $\mathbb{C}^2$ with respect to the solution f, that is $T(r, A) = o(T(r, f))$. If $\delta_f(\infty) > 0$, then

$$\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{r}>0. \quad (4)$$

Remark 3. Let $n = 0$ and $A(z_1, z_2) = 1$, then the above equations become

$$\frac{1}{f(z_1+c_1,z_2+c_2)^m}+\frac{1}{f(z_1,z_2)^m}=1,$$
$$\frac{1}{f(z_1+c_1,z_2+c_2)^m}+\frac{1}{f(z_1+c_1,z_2)^m}+\frac{1}{f(z_1,z_2+c_2)^m}=1, \quad (5)$$

which can be called as the partial difference equations of Fermat type.

In 2020, the first author and his coauthors discussed the transcendental entire solutions with finite order for the systems of partial differential difference equations and gave the conditions on the existence of the finite-order transcendental entire solutions for the following systems, which are some extension and improvements of the previous results given by Xu and Cao and Gao [16, 25].

Theorem 4 (see ([26], Theorem 1.2)). *Let $c = (c_1, c_2) \in \mathbb{C}^2$, and $m_j, n_j (j = 1, 2)$ be positive integers. If the following system of Fermat-type partial differential-difference equations*

$$\begin{cases}\left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^{n_1}+f_2(z_1+c_1,z_2+c_2)^{m_1}=1,\\ \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^{n_2}+f_1(z_1+c_1,z_2+c_2)^{m_2}=1,\end{cases} \quad (6)$$

satisfies one of the conditions

(i) $m_1 m_2 > n_1 n_2$;

(ii) $m_j > (n_j/(n_j - 1))$ for $n_j \geq 2$, $j = 1, 2$.

Then, system (6) does not have any pair of transcendental entire solution with finite order.

Remark 5. Here, (f, g) is called as a pair of finite-order transcendental entire solutions for system

$$\begin{cases} f^{n_1}+g^{m_1}=1,\\ f^{n_2}+g^{m_2}=1,\end{cases} \quad (7)$$

if f, g are transcendental entire functions and $\rho = \max\{\rho(f), \rho(g)\} < \infty$.

Remark 6. The condition $m_j > (n_j/(n_j - 1))$ implies $m_j > 1$. Thus, a question rises naturally: what will happen on the existence of transcendental entire solutions with finite order when $m_j = 1, j = 1, 2$ in system (6)?

In fact, we give the following example to explain that system (6) has a pair of transcendental entire solutions with finite order when $m_1 = m_2 = 1$ and $n_1 = n_2 = 2$, that is,

$$\begin{cases}\left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^{2}+f_2(z_1+c_1,z_2+c_2)=1,\\ \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^{2}+f_1(z_1+c_1,z_2+c_2)=1[rgb]0.00,0.00,1.00.\end{cases} \quad (8)$$

Example 1. Let

$$f_1(z)=1-\frac{1}{4}\pi^2-\frac{1}{4}z_1^2+\frac{1}{2}z_1z_2-\frac{\pi i}{2}z_2+(z_1-\pi i)e^{z_2}-\left[e^{z_2}+\frac{1}{2}(z_2-\pi i)\right]^2,$$
$$f_2(z)=1-\frac{1}{4}\pi^2-\frac{1}{4}z_1^2+\frac{1}{2}z_1z_2-\frac{\pi i}{2}z_2-(z_1-\pi i)e^{z_2}-\left[e^{z_2}-\frac{1}{2}(z_2-\pi i)\right]^2. \quad (9)$$

Then, $f = (f_1, f_2)$ is a pair of transcendental entire solutions of system (8) with $(c_1, c_2) = (\pi i, \pi i)$ and $\rho(f) = 1$.

Corresponding to system (6), we further consider the following system of the partial differential difference equation

$$\begin{cases}\left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}+\dfrac{\partial f_1(z_1,z_2)}{\partial z_2}\right)^{n_1}+P_1(z)f_2(z_1+c_1,z_2+c_2)^{m_1}=Q_1(z),\\ \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}+\dfrac{\partial f_2(z_1,z_2)}{\partial z_2}\right)^{n_2}+P_2(z)f_1(z_1+c_1,z_2+c_2)^{m_2}=Q_2(z),\end{cases} \quad (10)$$

where $P_1(z), P_2(z)$ are two nonzero polynomials in $\mathbb{C}^2$ and obtained.

Theorem 7. *Let $c=(c_1,c_2)\in\mathbb{C}^2$, $m_j, n_j (j=1,2)$ be positive integers satisfies one of the conditions*

(i) $m_1m_2 > n_1n_2$;

(ii) $m_j > n_j/(n_j-1)$ for $n_j \geq 2$, $j=1,2$.

Then, system (10) does not have any pair of transcendental entire solutions with finite order.

The following example shows that the conditions $m_j > (n_j/n_j-1)$ for $n_j\geq 2$ and $j=1,2$ are precise and the existence of finite-order transcendental entire solutions for the system (10) when $n_1=n_2=2$, $m_1=m_2=1$ and $P_1(z)=P_2(z)=Q_1(z)=Q_2(z)=1$, that is,

$$\begin{cases}\left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}+\dfrac{\partial f_1(z_1,z_2)}{\partial z_2}\right)^2+f_2(z_1+c_1,z_2+c_2)=1,\\ \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}+\dfrac{\partial f_2(z_1,z_2)}{\partial z_2}\right)^2+f_1(z_1+c_1,z_2+c_2)=1.\end{cases}\tag{11}$$

Example 2. Let

$$\begin{aligned}f_1(z)&=1-\frac{1}{4}\pi^2-\frac{1}{4}z_1^2+\frac{1}{2}(z_2-z_1)(z_1-\pi i)+(z_1-\pi i)e^{z_2-z_1}-\left[e^{z_2-z_1}+\frac{1}{2}(z_2-z_1-\pi i)\right]^2,\\ f_2(z)&=1-\frac{1}{4}\pi^2-\frac{1}{4}z_1^2+\frac{1}{2}(z_2-z_1)(z_1-\pi i)-(z_1-\pi i)e^{z_2-z_1}-\left[e^{z_2-z_1}-\frac{1}{2}(z_2-z_1-\pi i)\right]^2.\end{aligned}\tag{12}$$

Then, $f=(f_1,f_2)$ is a pair of transcendental entire solutions of system (11) with $(c_1,c_2)=(\pi i,2\pi i)$ and $\rho(f)=1$.

Remark 8. In Sections 3 and 4, we give the details proceeding for obtaining a class of finite-order transcendental entire solutions for systems (8) and (11).

Next, we continue to discuss the existence of the finite-order transcendental entire solutions for several systems including both the difference operator and the partial differential such as

$$\begin{cases}\left(\dfrac{\partial f_1}{\partial z_1}\right)^{n_1}+[f_2(z_1+c_1,z_2+c_2)-f_1(z_1,z_2)]^{m_1}=1,\\ \left(\dfrac{\partial f_2}{\partial z_1}\right)^{n_1}+[f_1(z_1+c_1,z_2+c_2)-f_2(z_1,z_2)]^{m_1}=1,\end{cases}\tag{13}$$

$$\begin{cases}\left(\dfrac{\partial f_1}{\partial z_1}+\dfrac{\partial f_1}{\partial z_2}\right)^{n_1}+[f_2(z_1+c_1,z_2+c_2)-f_1(z_1,z_2)]^{m_1}=1,\\ \left(\dfrac{\partial f_2}{\partial z_1}+\dfrac{\partial f_2}{\partial z_2}\right)^{n_1}+[f_1(z_1+c_1,z_2+c_2)-f_2(z_1,z_2)]^{m_1}=1,\end{cases}\tag{14}$$

where c_1, c_2 are constants in $\mathbb{C}$. It is easy to find the finite-order transcendental entire solutions for systems (13) and (14). For $c_2\neq 0$, system (5) has a pair of finite-order transcendental entire solutions (f_1,f_2) of the forms

$$\begin{cases}f_1=az_1+\dfrac{b+d-2ac_1}{2c_2}z_2+\dfrac{d}{2}+e^{(\pi i/c_2)z_2},\\ f_2=az_1+\dfrac{b+d-2ac_1}{2c_2}z_2+\dfrac{b}{2}-e^{(\pi i/c_2)z_2},\end{cases}\tag{15}$$

and for $c_2\neq c_1$, system (14) has a pair of finite-order transcendental entire solutions (f_1,f_2) of the forms

$$\begin{cases}f_1=az_1+\dfrac{b+d-2ac_1}{2(c_2-c_1)}(z_2-z_1)+\dfrac{d}{2}+e^{(\pi i/(c_2-c_1))(z_2-z_1)},\\ f_2=az_1+\dfrac{b+d-2ac_1}{2(c_2-c_1)}(z_2-z_1)+\dfrac{b}{2}-e^{(\pi i/(c_2-c_1))(z_2-z_1)},\end{cases}\tag{16}$$

where $a,b,d\in\mathbb{C}$ satisfy $1-a^{n_1}=b^{m_1}$ and $1-a^{n_2}=b^{m_2}$. Furthermore, we can give the finite-order transcendental entire solutions for systems (13) and (14) when $n_1=n_2=2$ and $m_1=m_2=1$ easily.

Example 3. The function

$$f=(f_1,f_2)=\left(z_1-z_2+e^{\pi i z_2}, z_1-z_2-e^{\pi i z_2}\right),\tag{17}$$

is a pair of transcendental entire solutions with $\rho(f)=1$ for system (13) when $(c_1,c_2)=(1,1)$, $n_1=n_2=2$, and $m_1=m_2=1$.

Example 4. The function

$$f=(f_1,f_2)=\left(2z_1-z_2+e^{\pi i(z_2-z_1)},2z_1-z_2-e^{\pi i(z_2-z_1)}\right), \tag{18}$$

is a pair of transcendental entire solutions with $\rho(f)=1$ for system (14) when $(c_1,c_2)=(1,2)$, $n_1=n_2=2$ and $m_1=m_2=1$.

Corresponding to systems (13) and (14), we can also obtain the solutions of the following systems

$$\begin{cases}\left(\dfrac{\partial f_1}{\partial z_1}\right)^{n_1}+[f_2(z_1+c_1,z_2+c_2)-f_2(z_1,z_2)]^{m_1}=1,\\ \left(\dfrac{\partial f_2}{\partial z_1}\right)^{n_1}+[f_1(z_1+c_1,z_2+c_2)-f_1(z_1,z_2)]^{m_1}=1,\end{cases} \tag{19}$$

$$\begin{cases}\left(\dfrac{\partial f_1}{\partial z_1}+\dfrac{\partial f_1}{\partial z_2}\right)^{n_1}+[f_2(z_1+c_1,z_2+c_2)-f_2(z_1,z_2)]^{m_1}=1,\\ \left(\dfrac{\partial f_2}{\partial z_1}+\dfrac{\partial f_2}{\partial z_2}\right)^{n_1}+[f_1(z_1+c_1,z_2+c_2)-f_1(z_1,z_2)]^{m_1}=1,\end{cases} \tag{20}$$

where c_1,c_2 are constants in $\mathbb{C}$. In fact, for $c_2\neq 0$, then systems (19) has a pair of solutions with the forms

$$(f_1,f_2)=\left(a_1z_1+\frac{b_2-a_1c_1}{c_2}z_2+G_1(z_2),a_2z_1+\frac{b_1-a_2c_1}{c_2}z_2+G_2(z_2)\right), \tag{21}$$

where $G_1(z_2),G_2(z_2)$ are two period functions with period c_2, and for $s:=z_2-z_1$ and $s_0:=c_2-c_1\neq 0$, then system (20) has a pair of solutions with the forms

$$(f_1,f_2)=\left(a_1z_1+\frac{b_2-a_1c_1}{c_2-c_1}s+G_1(s),a_2z_1+\frac{b_1-a_2c_1}{c_2-c_1}s+G_2(s)\right), \tag{22}$$

where $G_1(s),G_2(s)$ are two period functions with period s_0, and $a_1,a_2,c_1,c_2,d_1,$ and d_2 are constants satisfying

$$a_1^{n_1}+b_1^{m_1}=1,\ a_2^{n_2}+b_2^{m_2}=1. \tag{23}$$

2. Proof of Theorem 7

The following lemmas will be used in this paper.

Lemma 9 ([27, 28]). *Let f be a nonconstant meromorphic function on $\mathbb{C}^n$ and let $I=(i_1,\cdots,i_n)$ be a multi-index with length $|I|=\sum_{j=1}^n i_j$. Assume that $T(r_0,f)\geq e$ for some r_0. Then,*

$$m\left(r,\frac{\partial^I f}{f}\right)=S(r,f), \tag{24}$$

holds for all $r\geq r_0$ outside a set $E\subset(0,+\infty)$ of finite logarithmic measure $\int_E(dt/t)<\infty$, where $\partial^I f=(\partial^{|I|}f)/(\partial z_1^{i_1}\cdots\partial z_n^{i_n})$.

Lemma 10 ([18, 19]). *Let f be a nonconstant meromorphic function with finite order on $\mathbb{C}^n$ such that $f(0)\neq 0,\infty$, and let $\varepsilon>0$. Then, for $c\in\mathbb{C}^n$,*

$$m\left(r,\frac{f(z)}{f(z+c)}\right)+m\left(r,\frac{f(z+c)}{f(z)}\right)=S(r,f), \tag{25}$$

holds for all $r\geq r_0$ outside a set $E\subset(0,+\infty)$ of finite logarithmic measure $\int_E(dt/t)<\infty$.

Lemma 11 (see [29]). *Let f be a nonconstant meromorphic function on $\mathbb{C}^n$. Take a positive integer m and take polynomials of f and its partial derivatives:*

$$P(f)=\sum_{\mathbf{p}\in I}a_{\mathbf{p}}f^{p_0}\left(\partial^{i_1}f\right)^{p_1}\cdots\left(\partial^{i_l}f\right)^{p_l},\ (p)=(p_0,\cdots,p_l),$$
$$Q(f)=\sum_{\mathbf{q}\in I}c_{\mathbf{q}}f^{q_0}\left(\partial^{j_1}f\right)^{q_1}\cdots\left(\partial^{j_l}f\right)^{q_l},\ (q)=(q_0,\cdots,q_l),$$
$$B(f)=\sum_{k=0}^{m}b_kf^k, \tag{26}$$

where I,J are finite sets of distinct elements and $a_{\mathbf{p}},c_{\mathbf{q}}$, and b_k are meromorphic functions on $\mathbb{C}^n$ with $b_m\equiv 0$. Assume that f satisfies the equation

$$B(f)Q(f)=P(f), \tag{27}$$

such that $P(f),Q(f)$, and $B(f)$ are differential polynomials, that is, their coefficients a satisfy $m(r,a)=S(r,f)$. If $\deg(P(f))\leq m=\deg(B(f))$, then

$$m(r,Q(f))=S(r,f), \tag{28}$$

holds for all r possibly outside of a set E with finite logarithmic measure.

Proof. Let (f_1,f_2) be a pair of transcendental entire functions with finite-order satisfying system (10). Here, we will discuss two following cases.

Case 1. $n_1n_2>m_1m_2$. In view of Lemma 10, the following conclusions that

$$m\left(r,\frac{f_j(z_1,z_2)}{f_j(z_1+c_1,z_2+c_2)}\right)=S\left(r,f_j\right),\quad j=1,2, \tag{29}$$

holds for all $r>0$ outside of a possible exceptional set $E_j\subset[1,+\infty)$ of finite logarithmic measure $\int_{E_j}(dt/t)<\infty$. Thus, we can deduce from (29) that

$$T\left(r,f_j(z_1,z_2)\right)=m\left(r,f_j(z_1,z_2)\right)\le m\left(r,\frac{f_j(z_1,z_2)}{f(z_1+c_1,z_2+c_2)}\right)+m\left(r,f_j(z_1+c_1,z_2+c_2)\right)+\log 2$$
$$=m\left(r,f_j(z_1+c_1,z_2+c_2)\right)+S\left(r,f_j\right)=T\left(r,f_j(z_1+c_1,z_2+c_2)\right)+S\left(r,f_j\right),\quad j=1,2, \tag{30}$$

for all $r\in E=:E_1\cup E_2$. By using Lemma 9 and Lemma 11, it follows from (30) that

$$n_1T(r,f_2(z_1,z_2))\le n_1T(r,f_2(z_1+c_1,z_2+c_2))+S(r,f_2)\le T(r,P_1(z)f_2(z_1+c_1,z_2+c_2)^{n_1})+S(r,f_2)$$
$$=T\left(r,\left(\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)^{m_1}-Q_1(z)\right)+S(r,f_2)=m_1T\left(r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)+O(\log r)+S(r,f_2)+S(r,f_1)$$
$$=m_1m\left[r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)+O(\log r)+S(r,f_2)+S(r,f_1)\le m_1\left[m\left(r,\frac{(\partial f_1/\partial z_1)+(\partial f_1/\partial z_2)}{f_1}\right)+m(r,f_1)\right]$$
$$+O(\log r)+S(r,f_1)+S(r,f_2)=m_1T(r,f_1)+O(\log r)+S(r,f_1)+S(r,f_2), \tag{31}$$

for all $r\in E$. Similarly, we have

$$n_2T(r,f_1)\le m_2T(r,f_2)+O(\log r)+S(r,f_1)+S(r,f_2),\quad r\in E. \tag{32}$$

In view of (31) and (32), it yields

$$(n_1n_2-m_1m_2)T\left(r,f_j\right)\le O(\log r)+S(r,f_1)+S(r,f_2),\quad r\in E. \tag{33}$$

In view of $n_1n_2>m_1m_2$, this is impossible since f_1,f_2 are transcendental entire functions.

Case 2. $m_j>(n_j/(n_j-1))$, $n_j\ge 2$, $j=1,2$. In view of the Nevanlinna second fundamental theorem concerning small functions, Lemma 10, and system (11), we can deduce that

$$m_1T\left(r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)=T\left(r,\left(\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)^{m_1}\right)+S(r,f_1)\le\bar N\left(r,\frac{1}{((\partial f_1/\partial z_1)+(\partial f_1/\partial z_2))^{m_1}}\right)+\bar N\left(r,\frac{1}{((\partial f_1/\partial z_1)+(\partial f_1/\partial z_2))^{m_1}-Q_1(z)}\right)$$
$$+S(r,f_1)\le\bar N\left(r,\frac{1}{((\partial f_1/\partial z_1)+(\partial f_1/\partial z_2))^{m_1}}\right)+\bar N\left(r,\frac{1}{P_1(z)f_2(z_1+c_1,z_2+c_2)^{n_1}}\right)+S(r,f_1)\le\bar N\left(r,\frac{1}{(\partial f_1/\partial z_1)+(\partial f_1/\partial z_2)}\right)$$
$$+\bar N\left(r,\frac{1}{f_2(z_1+c_1,z_2+c_2)}\right)+O(\log r)+S(r,f_1)\le T\left(r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)+T(r,f_2(z_1+c_1,z_2+c_2))+O(\log r)+S(r,f_1)+S(r,f_2), \tag{34}$$

that is,

$$(m_1-1)T\left(r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)\le T(r,f_2(z+c))+O(\log r)+S(r,f_1)+S(r,f_2). \tag{35}$$

Similarly, we have

$$(m_2-1)T\left(r,\frac{\partial f_2}{\partial z_1}+\frac{\partial f_2}{\partial z_2}\right)\le T(r,f_1(z+c))+O(\log r)+S(r,f_1)+S(r,f_2). \tag{36}$$

On the other hand, in view of system (10) and Lemma 10, it follows that

$$n_1 T(r, f_2(z_1+c_1, z_2+c_2)) + O(\log r) = T(r, P_1(z) f_2(z_1+c_1, z_2+c_2)^{n_1}) + S(r, f_2) = T\left(r, \left(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_1}{\partial z_2}\right)^{m_1} - Q_1(z)\right) + S(r, f_2) = m_1 T\left(r, \frac{\partial f_1}{\partial z_1} + \frac{\partial f_1}{\partial z_2}\right) + O(\log r) + S(r, f_1) + S(r, f_2). \tag{37}$$

Similarly, we have

$$n_2 T(r, f_1(z_1+c_1, z_2+c_2)) = m_2 T\left(r, \frac{\partial f_2}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\right) + O(\log r) + S(r, f_1) + S(r, f_2). \tag{38}$$

In view of (35)–(38), we obtain that

$$\begin{aligned} &\left(n_1 - \frac{m_1}{m_1 - 1}\right) T(r, f_2(z_1+c_1, z_2+c_2)) \le O(\log r) + S(r, f_1) + S(r, f_2), \\ &\left(n_2 - \frac{m_2}{m_2 - 1}\right) T(r, f_1(z_1+c_1, z_2+c_2)) \le O(\log r) + S(r, f_1) + S(r, f_2). \end{aligned} \tag{39}$$

The fact that $m_j > (n_j/(n_j - 1))$ can lead to a contradiction since f_1, f_2 are transcendental entire functions.

Therefore, this completes the proof of Theorem 7.

3. Entire Solutions for System (8)

Now, the details that we obtain a pair of finite-order transcendental entire solutions for system (8) will be given below.

Let (f_1, f_2) be a pair of finite-order transcendental entire solutions for system (8). Differentiating both equations in system (8) for z_1, we deduce

$$\begin{cases} 2\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2} + \dfrac{\partial f_2(z_1+c_1, z_2+c_2)}{\partial z_1} = 0, \\ 2\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2} + \dfrac{\partial f_1(z_1+c_1, z_2+c_2)}{\partial z_1} = 0. \end{cases} \tag{40}$$

Let $F_1(z_1, z_2) = (\partial f_1(z_1, z_2))/\partial z_1$ and $F_2(z_1, z_2) = (\partial f_2 (z_1, z_2))/\partial z_1$, then it follows from (18) that

$$\begin{cases} 2F_1(z_1,z_2)\dfrac{\partial F_1(z_1,z_2)}{\partial z_1} = -F_2(z_1+c_1, z_2+c_2), \\ 2F_2(z_1,z_2)\dfrac{\partial F_2(z_1,z_2)}{\partial z_1} = -F_1(z_1+c_1, z_2+c_2)[rgb]0.00,0.00,1.00. \end{cases} \tag{41}$$

By Lemmas 9–11, it yields that $(\partial F_j(z_1, z_2))/\partial z_1 = S(r, f_j)$ for $j = 1, 2$. Thus, we can assume that

$$\frac{\partial F_1(z_1,z_2)}{\partial z_1} = a_1, \frac{\partial F_2(z_1,z_2)}{\partial z_1} = a_2, \tag{42}$$

where $a_1, a_2 \in \mathbb{C}$. Solving Equation (42), we have

$$F_1(z_1,z_2) = a_1 z_1 + \varphi_1(z_2), F_2(z_1,z_2) = a_2 z_1 + \varphi_2(z_2), \tag{43}$$

where $\varphi_1(z_2), \varphi_2(z_2)$ are finite-order transcendental entire functions in z_2. Due to Equations (41) and (42), we obtain that

$$F_1(z) = -\frac{1}{2a_1}F_2(z+c), F_2(z) = -\frac{1}{2a_2}F_1(z+c). \tag{44}$$

Substituting (43) into (44), we can deduce that

$$\begin{cases} a_1 z_1 + \varphi_1(z_2) = -\dfrac{1}{2a_1}(a_2 z_1 + a_2 c_1) - \dfrac{1}{2a_1}\varphi_2(z_2 + c_2), \\ a_2 z_1 + \varphi_2(z_2) = -\dfrac{1}{2a_2}(a_1 z_1 + a_1 c_1) - \dfrac{1}{2a_2}\varphi_1(z_2 + c_2), \end{cases} \tag{45}$$

which implies that $a_1^3 = a_2^3 = -(1/8)$. It would be well if $a_1 = a_2 = -(1/2)$. So, it follows that

$$\begin{aligned} &F_1(z_1,z_2) = -\frac{1}{2}z_1 + \varphi_1(z_2), F_2(z_1,z_2) = -\frac{1}{2}z_1 + \varphi_2(z_2), \\ &\varphi_1(z_2+c_2) = \varphi_2(z_2) + \frac{1}{2}c_1, \varphi_2(z_2+c_2) = \varphi_1(z_2) + \frac{1}{2}c_1. \end{aligned} \tag{46}$$

This means that

$$\varphi_1(z_2 + 2c_2) - \varphi_1(z_2) = c_1, \varphi_1(z_2 + 2c_2) - \varphi_1(z_2) = c_1, \tag{47}$$

which imply

$$\varphi_1(z_2) = G_1(z_2) + \frac{c_1}{2c_2}z_2, \varphi_2(z_2) = G_2(z_2) + \frac{c_1}{2c_2}z_2, \tag{48}$$

where $G_1(z_2), G_2(z_2)$ are finite-order entire period function with period $2c_2$ satisfying $G_2(z_2 + c_2) = G_1(z_2)$.

Solving the following system

$$\begin{cases} \dfrac{\partial f_1(z_1,z_2)}{\partial z_1} = F_1(z_1,z_2) = -\dfrac{1}{2}z_1 + \varphi_1(z_2), \\ \dfrac{\partial f_2(z_1,z_2)}{\partial z_1} = F_2(z_1,z_2) = -\dfrac{1}{2}z_1 + \varphi_2(z_2), \end{cases} \tag{49}$$

we obtain that

$$\begin{cases} f_1(z_1,z_2) = -\dfrac{1}{4}z_1^2 + z_1\varphi_1(z_2) + \psi_1(z_2), \\ f_2(z_1,z_2) = -\dfrac{1}{4}z_1^2 + z_1\varphi_2(z_2) + \psi_2(z_2), \end{cases} \tag{50}$$

where $\psi_1(z_2), \psi_2(z_2)$ are finite-order entire functions in z_2. Substituting (50) into (8), and combining with the periodicity of $\varphi_1(z_2)$ and $\varphi_2(z_2)$, it yields that

$$\begin{aligned} &\left(-\frac{1}{2}z_1+\frac{c_1}{2c_2}z_2+G_1(z_2)\right)^2-\frac{1}{4}(z_1+c_1)^2+(z_1+c_1)\varphi_2(z_2+c_2)+\psi_2(z_2+c_2)=1,\\ &\left(-\frac{1}{2}z_1+\frac{c_1}{2c_2}z_2+G_2(z_2)\right)^2-\frac{1}{4}(z_1+c_1)^2+(z_1+c_1)\varphi_1(z_2+c_2)+\psi_1(z_2+c_2)=1. \end{aligned} \tag{51}$$

Thus, we have

$$\begin{aligned} \psi_2(z_2+c_2)&=1-\frac{1}{4}c_1^2-\frac{c_1^2}{2c_2}z_2-c_1G_1(z_2)-\left(\frac{c_1}{2c_2}z_2+G_1(z_2)\right)^2,\\ \psi_1(z_2+c_2)&=1-\frac{1}{4}c_1^2-\frac{c_1^2}{2c_2}z_2-c_1G_2(z_2)-\left(\frac{c_1}{2c_2}z_2+G_2(z_2)\right)^2, \end{aligned} \tag{52}$$

which mean that

$$\psi_2(z_2)=1+\frac{1}{4}c_1^2-\frac{c_1^2}{2c_2}z_2-c_1G_2(z_2)-\left(\frac{c_1}{2c_2}(z_2-c_2)+G_2(z_2)\right)^2, \tag{53}$$

$$\psi_1(z_2)=1+\frac{1}{4}c_1^2-\frac{c_1^2}{2c_2}z_2-c_1G_1(z_2)-\left(\frac{c_1}{2c_2}(z_2-c_2)+G_1(z_2)\right)^2. \tag{54}$$

In view of (48)–(54), it follows that

$$\begin{aligned} f_1(z_1,z_2)&=1+\frac{1}{4}c_1^2-\frac{1}{4}z_1^2+\frac{c_1}{2c_2}z_1z_2-\frac{c_1^2}{2c_2}z_2+(z_1-c_1)G_1(z_2),\\ &\quad-\left[\frac{c_1}{2c_2}(z_2-c_2)+G_1(z_2)\right]^2,\\ f_2(z_1,z_2)&=1+\frac{1}{4}c_1^2-\frac{1}{4}z_1^2+\frac{c_1}{2c_2}z_1z_2-\frac{c_1^2}{2c_2}z_2+(z_1-c_1)G_2(z_2),\\ &\quad-\left[\frac{c_1}{2c_2}(z_2-c_2)+G_2(z_2)\right]^2, \end{aligned} \tag{55}$$

where $G_1(z_2), G_2(z_2)$ are finite-order transcendental entire period functions with period $2c_2$ satisfying $G_2(z_2+c_2) = G_1(z_2)$. Substituting (f_1, f_2) into system (2), it is easy to confirm that (f_1, f_2) is a solution of system (8).

4. Entire Solutions for System (11)

Let (f_1, f_2) be a pair of finite-order transcendental entire solutions of system (13). Next, the detail that we obtain one form of (f_1, f_2) is listed as follows. Differentiating system (13) for z_1, z_2, respectively, we have

$$\begin{cases} 2F_1(z_1,z_2)\left(\dfrac{\partial F_1(z_1,z_2)}{\partial z_1}+\dfrac{\partial F_1(z_1,z_2)}{\partial z_2}\right)+F_2(z_1+c_1,z_2+c_2)=0,\\ 2F_2(z_1,z_2)\left(\dfrac{\partial F_2(z_1,z_2)}{\partial z_1}+\dfrac{\partial F_2(z_1,z_2)}{\partial z_2}\right)+F_1(z_1+c_1,z_2+c_2)=0, \end{cases} \tag{56}$$

where

$$F_1(z_1,z_2)=\frac{\partial f_1(z_1,z_2)}{\partial z_1}+\frac{\partial f_1(z_1,z_2)}{\partial z_2},\quad \text{for } j=1,2. \tag{57}$$

In view of Lemmas 9–11, it follows that $(\partial F_j(z_1,z_2))/\partial z_1 = S(r, f_j)$ for $j=1,2$. For the convenience, assume that

$$\frac{\partial F_j(z_1,z_2)}{\partial z_1}+\frac{\partial F_j(z_1,z_2)}{\partial z_2}=b_j,\quad j=1,2, \tag{58}$$

where $b_j \in \mathbb{C}$. The characteristic equations for Equation (58) are

$$\frac{dz_1}{dt}=1,\ \frac{dz_2}{dt}=1,\ \frac{dF_j}{dt}=b_j. \tag{59}$$

In view of the initial conditions: $z_1=0, z_2=s$, and $F_j = F_j(0,s) := F_j(s)$ with a parameter s, we thus obtain the following parametric representation for the solutions of the characteristic equations: $z_1=t$, $z_2=t+s$,

$$F_j=\int_0^t b_j dt+\mu_j(s)=b_jt+\mu_j(s), \tag{60}$$

where $\mu_j(s), j=1,2$ are entire functions with finite order in s. Thus, it follows that

$$F_j(z_1,z_2)=b_jz_1+\mu_j(z_2-z_1),\quad j=1,2. \tag{61}$$

In view of (56) and (58), it follows that

$$\begin{cases} 2b_1F_1(z_1,z_2)=-F_2(z_1+c_1,z_2+c_2),\\ 2b_2F_2(z_1,z_2)=-F_1(z_1+c_1,z_2+c_2). \end{cases} \tag{62}$$

Substituting (61) into (62), we have that

$$\begin{cases} 2b_1^2 z_1 + 2b_1\mu_1(s) = -b_2(z_1 + c_1) - \mu_2(s + s_0), \\ 2b_2^2 z_1 + 2b_1\mu_1(s) = -b_1(z_1 + c_1) - \mu_1(s + s_0), \end{cases} \quad (63)$$

where $s = z_2 - z_1$ and $s_0 := c_2 - c_1$. This implies that $b_1^3 = b_2^3 = -(1/8)$. Let us assume that $b_1 = b_2 = -(1/8)$. Thus, it yields that

$$\begin{cases} \mu_2(s + s_0) = \mu_1(s) + \frac{1}{2}c_1, \\ \mu_1(s + s_0) = \mu_2(s) + \frac{1}{2}c_1. \end{cases} \quad (64)$$

This means

$$\mu_j(s) = G_j(s) + \tau s, \quad j = 1, 2, \quad (65)$$

$$G_2(s + s_0) = G_1(s), \quad (66)$$

where $G_1(s), G_2(s)$ are finite-order transcendental entire period functions with period $2s_0$, and $\tau = c_1/(2(c_2 - c_1))$. Then, in view of (61) and (65), we deduce

$$F_j(z_1, z_2) = -\frac{1}{2}z_1 + G_j(z_2 - z_1) + \tau(z_2 - z_1), \quad j = 1, 2, \quad (67)$$

that is,

$$\frac{\partial f_j(z_1, z_2)}{\partial z_1} + \frac{\partial f_j(z_1, z_2)}{\partial z_2} = -\frac{1}{2}z_1 + G_j(z_2 - z_1) + \tau(z_2 - z_1). \quad (68)$$

By making use of the characteristic equations for Equation (68) again, let

$$\frac{dz_1}{dt} = 1, \frac{dz_2}{dt} = 1, \frac{df_j}{dt} = -\frac{1}{2}z_1 + G_j(z_2 - z_1) + \tau(z_2 - z_1). \quad (69)$$

In view of the initial conditions: $z_1 = 0, z_2 = s$, and $f_j = f_j(0, s) := f_j(s)$ with a parameter s, we can deduce that the parametric representation for the solutions of the characteristic equations: $z_1 = t$, $z_2 = t + s$, and

$$f_j = \int_0^t \left(-\frac{1}{2}t + G_j(s) + \tau s\right) dt + \nu_j(s) = -\frac{1}{4}t^2 + t\left(G_j(s) + \tau s\right) + \nu_j(s), \quad j = 1, 2, \quad (70)$$

where $\nu_j(s)$ is an entire function with finite order in s. Substituting $t = z_1$ and $s = z_2 - z_1$ into the above form, we have that

$$f_j(z_1, z_2) = -\frac{1}{4}z_1^2 + z_1\left[G_j(z_2 - z_1) + a_3(z_2 - z_1)\right] + \nu_j(z_2 - z_1), \quad j = 1, 2. \quad (71)$$

Substituting (71) into (13), and combining with the periodicity of $G_j(s)$, it follows that

$$\nu_1(s) = 1 - \frac{1}{4}c_1^2 - c_1 G_2(s - s_0) - \tau c_1(s - s_0) - [G_2(s - s_0) + \tau(s - s_0)]^2, \quad (72)$$

$$\nu_2(s) = 1 - \frac{1}{4}c_1^2 - c_1 G_1(s - s_0) - \tau c_1(s - s_0) - [G_1(s - s_0) + \tau(s - s_0)]^2. \quad (73)$$

Thus, in view of (66) and (71)–(73), we obtain that a pair of entire solutions of system (13) are of the forms

$$\begin{aligned} f_1(z_1, z_2) &= 1 + \frac{1}{4}c_1^2 - \frac{1}{4}z_1^2 + \frac{c_1}{2(c_2 - c_1)}(z_2 - z_1)(z_1 - c_1) + (z_1 - c_1)G_1(z_2 - z_1) - \left[G_1(z_2 - z_1) + \frac{c_1}{2(c_2 - c_1)}(z_2 - z_1 - (c_2 - c_1))\right]^2, \\ f_2(z_1, z_2) &= 1 + \frac{1}{4}c_1^2 - \frac{1}{4}z_1^2 + \frac{c_1}{2(c_2 - c_1)}(z_2 - z_1)(z_1 - c_1) + (z_1 - c_1)G_2(z_2 - z_1) - \left[G_2(z_2 - z_1) + \frac{c_1}{2(c_2 - c_1)}(z_2 - z_1 - (c_2 - c_1))\right]^2, \end{aligned} \quad (74)$$

where $G_1(s), G_2(s)$ are finite-order transcendental entire period functions with period $2s_0$ and satisfy (66). Let

$$G_1(s) = e^{(\pi i/(c_2 - c_1))s}, G_2(s) = -e^{(\pi i/(c_2 - c_1))s}. \quad (75)$$

Thus, (f_1, f_2) is a pair of finite-order transcendental entire solutions of system (13).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

H. Y. Xu is responsible for the conceptualization; H.Y. Xu and H. Li for the writing—original draft preparation; H. Li and H. Y. Xu for the writing—review and editing; and H. Y. Xu and H. Li for the funding acquisition.

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References

[1] C. C. Yang, "A generalization of a theorem of P. Montel on entire functions," *Proceedings of the American Mathematical Society*, vol. 26, pp. 332–334, 1970.

[2] Y. M. Chiang and S. J. Feng, "On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane," *The Ramanujan Journal*, vol. 16, no. 1, pp. 105–129, 2008.

[3] R. G. Halburd and R. J. Korhonen, "Difference analogue of the lemma on the logarithmic derivative with applications to difference equations," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 477–487, 2006.

[4] R. G. Halburd and R. J. Korhonen, "Nevanlinna theory for the difference operator," *Annales Academiae Scientiarum Fennicae Mathematica*, vol. 31, no. 2, pp. 463–478, 2006.

[5] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1993.

[6] M. Ru, "The recent progress in Nevanlinna theory," *Journal of Jiangxi Normal University. Natural Sciences Edition*, vol. 42, pp. 1–11, 2018.

[7] F. Gross, "On the equation $f^n + g^n = 1$," *Bulletin of the American Mathematical Society*, vol. 72, pp. 86–88, 1966.

[8] P. C. Hu, "Malmquist type theorem and factorization of meromorphic solutions of partial differential equations," *Complex Variables and Elliptic Equations*, vol. 27, pp. 269–285, 1995.

[9] H. Y. Xu, S. Y. Liu, and Q. P. Li, "The existence and growth of solutions for several systems of complex nonlinear difference equations," *Mediterranean Journal of Mathematics*, vol. 16, pp. 1–30, 2019.

[10] H. Y. Xu and H. Wang, "Notes on the existence of entire solutions for several partial differential-difference equations," *Bulletin of the Iranian Mathematical Society*, 2020.

[11] H. Y. Xu and J. Tu, "Existence of rational solutions for q-difference Painlevé equations," *Electronic jJournal of dDifferential eEquations*, vol. 2020, pp. 1–14, 2020.

[12] C. C. Yang and P. Li, "On the transcendental solutions of a certain type of nonlinear differential equations," *Archiv der Mathematik*, vol. 82, pp. 442–448, 2004.

[13] Q. Y. Yuan, J. R. Long, and D. Z. Qin, "The growth of solutions of two certain types of q-difference differential equations," *Journal of Jiangxi Normal University(Natural Science)*, vol. 44, pp. 6–11, 2020.

[14] J. F. Zhong and H. F. Liu, "The growth of meromorphic solutions of some type of nonlinear difference equations," *Journal of Jiangxi Normal University(Natural Science)*, vol. 43, pp. 508–512, 2019.

[15] T. B. Cao and L. Xu, "Logarithmic difference lemma in several complex variables and partial difference equations," *Annali di Matematica Pura ed Applicata*, vol. 199, no. 2, pp. 767–794, 2020.

[16] L. Xu and T. B. Cao, "Solutions of complex Fermat-type partial difference and differential-difference equations," *Mediterranean Journal of Mathematics*, vol. 15, pp. 1–14, 2018.

[17] L. Xu and T. B. Cao, "Correction to: solutions of complex Fermat-type partial difference and differential-difference equations," *Mediterranean Journal of Mathematics*, vol. 17, pp. 1–4, 2020.

[18] T. B. Cao and R. J. Korhonen, "A new version of the second main theorem for meromorphic mappings intersecting hyperplanes in several complex variables," *Journal of Mathematical Analysis and Applications*, vol. 444, no. 2, pp. 1114–1132, 2016.

[19] R. J. Korhonen, "A difference Picard theorem for meromorphic functions of several variables," *Computational Methods and Function Theory*, vol. 12, no. 1, pp. 343–361, 2012.

[20] K. Liu, "Meromorphic functions sharing a set with applications to difference equations," *Journal of Mathematical Analysis and Applications*, vol. 359, no. 1, pp. 384–393, 2009.

[21] K. Liu, T. B. Cao, and H. Z. Cao, "Entire solutions of Fermat type differential-difference equations," *Archiv der Mathematik*, vol. 99, no. 2, pp. 147–155, 2012.

[22] K. Liu and L. Z. Yang, "On entire solutions of some differential-difference equations," *Computational Methods and Function Theory*, vol. 13, no. 3, pp. 433–447, 2013.

[23] K. Liu and T. B. Cao, "Entire solutions of Fermat type difference differential equations," *Electronic Journal of Differential Equations*, vol. 2013, no. 59, pp. 1–10, 2013.

[24] K. Liu and C. J. Song, "Meromorphic solutions of complex differential-difference equations," *Results in Mathematics*, vol. 72, no. 4, pp. 1759–1771, 2017.

[25] L. Y. Gao, "Entire solutions of two types of systems of complex differential-difference equations," *Acta Mathematica Scientia*, vol. 59, pp. 677–684, 2016.

[26] H. Y. Xu, S. Y. Liu, and Q. P. Li, "Entire solutions for several systems of nonlinear difference and partial differential-difference equations of Fermat-type," *Journal of Mathematical Analysis and Applications*, vol. 483, no. 123641, pp. 1–22, 2020.

[27] A. Biancofiore and W. Stoll, "Another proof of the lemma of the logarithmic derivative in several complex variables," in *Recent Developments in Several Complex Variables*, J. Fornaess, Ed., pp. 29–45, Princeton University Press, Princeton, 1981.

[28] Z. Ye, "On Nevanlinna's second main theorem in projective space," *Inventiones Mathematicae*, vol. 122, no. 1, pp. 475–507, 1995.

[29] P. C. Hu, P. Li, and C. C. Yang, *Unicity of Meromorphic Mappings, Advances in Complex Analysis and its Applications*, vol. 1, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.